
Overview of Isabelle/HOL

System Architecture

<i>Isabelle</i>	generic theorem prover

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<i>Isabelle/HOL</i>	Isabelle instance for HOL
<i>Isabelle</i>	generic theorem prover

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<i>ProofGeneral</i>	(X)Emacs based interface
<i>Isabelle/HOL</i>	Isabelle instance for HOL
<i>Isabelle</i>	generic theorem prover

HOL

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Higher-order = functions are values, too!

Formulae

Syntax (in decreasing priority):

$$\begin{array}{l} form ::= (form) \quad | \quad term = term \quad | \quad \neg form \\ \quad | \quad form \wedge form \quad | \quad form \vee form \quad | \quad form \longrightarrow form \\ \quad | \quad \forall x. form \quad | \quad \exists x. form \end{array}$$

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Scope of quantifiers: as far to the right as possible

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- $A \longrightarrow B \longrightarrow C \equiv A \longrightarrow (B \longrightarrow C) \not\equiv (A \longrightarrow B) \longrightarrow C$!

Warning

Quantifiers have low priority and need to be parenthesized:

$$! \quad P \wedge \forall x. Q x \quad \rightsquigarrow \quad P \wedge (\forall x. Q x) \quad !$$

Types and Terms

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Parentheses: $T1 \Rightarrow T2 \Rightarrow T3 \equiv T1 \Rightarrow (T2 \Rightarrow T3)$

Terms: Basic syntax

Syntax:

$term$	$::=$	$(term)$	
		a	constant or variable (identifier)
		$term\ term$	function application
		$\lambda x. term$	function “abstraction”

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Parantheses: $f\ a_1\ a_2\ a_3 \equiv ((f\ a_1)\ a_2)\ a_3$

λ -calculus on one slide

Informal notation: $t[x]$

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- **Computation:**
Replace formal by actual parameter (“ β -reduction”):
 $(\lambda x.t[x]) a \longrightarrow_{\beta} t[a]$

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Example: $(\lambda x. x + 5) 3 \longrightarrow_{\beta} (3 + 5)$

\longrightarrow_{β} *in Isabelle: Don't worry, be happy*

Isabelle performs β -reduction automatically

Isabelle considers $(\lambda x.t[x])a$ and $t[a]$ equivalent

Terms and Types

Terms must be well-typed

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Notation: $t :: \tau$ means t is a well-typed term of type τ .

Type inference

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User can help with **type annotations** inside the term.

Example: $f (x::nat)$

Currying

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
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Advantage: *partial application* $f a_1$ with $a_1 :: \tau_1$

Terms: Syntactic sugar

Some predefined syntactic sugar:

- *Infix*: $+$, $-$, $*$, $\#$, $@$, ...
- *Mixfix*: *if _ then _ else _*, *case _ of*, ...

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Prefix binds more strongly than infix:

$$! \quad f x + y \equiv (f x) + y \neq f (x + y) \quad !$$

Base types: bool, nat, list

Type bool

Formulae = terms of type *bool*

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True :: *bool*

False :: *bool*

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if-and-only-if: =

Type nat

$0 :: \text{nat}$

$\text{Suc} :: \text{nat} \Rightarrow \text{nat}$

$+, *, \dots :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$

\vdots

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! Numbers and arithmetic operations are overloaded:

$0, 1, 2, \dots :: 'a, \quad + :: 'a \Rightarrow 'a \Rightarrow 'a$

You need type annotations: $1 :: \text{nat}, x + (y::\text{nat})$

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... unless the context is unambiguous: $\text{Suc } z$

Type list

- `[]`: empty list
- `x # xs`: list with first element `x` ("*head*")
and rest `xs` ("*tail*")
- Syntactic sugar: `[x1, ..., xn]`

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Large library:

`hd`, `tl`, `map`, `length`, `filter`, `set`, `nth`, `take`, `drop`, `distinct`, ...

Don't reinvent, reuse!

~> `HOL/List.thy`

Isabelle Theories

Theory = Module

Syntax: `theory` *MyTh*
`imports` *ImpTh*₁ ... *ImpTh*_{*n*}
`begin`
(declarations, definitions, theorems, proofs, ...)*
`end`

- *MyTh*: name of theory. Must live in file *MyTh.thy*
- *ImpTh*_{*i*}: name of *imported* theories. Import transitive.

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Usually: `theory` *MyTh*
`imports` Main
⋮

Proof General



An Isabelle Interface

by David Aspinall

Proof General

Customized version of (x)emacs:

- all of emacs (info: C-h i)
- Isabelle aware (when editing .thy files)
- mathematical symbols (“x-symbols”)

X-Symbols

Input of funny symbols in Proof General

- via menu (“X-Symbol”)
- via ascii encoding (similar to $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$): `\<and>`, `\<or>`, ...
- via abbreviation: `/\`, `\/`, `-->`, ...

x-symbol	\forall	\exists	λ	\neg	\wedge	\vee	\longrightarrow	\Rightarrow
ascii (1)	<code>\<forall></code>	<code>\<exists></code>	<code>\<lambda></code>	<code>\<not></code>	<code>/\</code>	<code>\/</code>	<code>--></code>	<code>=></code>
ascii (2)	ALL	EX	%	~	&			

(1) is converted to x-symbol, (2) stays ascii.

Finding theorems

1. Click on **Find** button
2. Input search pattern (e.g. “_ & *True*”)

Demo: terms and types