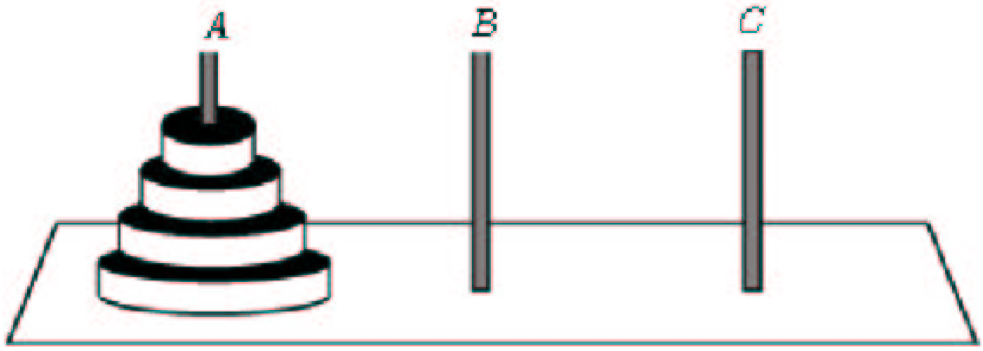
Isabelle/HOL Exercises
Projects

The Towers of Hanoi

We are given 3 pegs A, B and C, and n disks with a hole, such that no two disks have the same diameter. Initially all n disks rest on peg A, ordered according to their size, with the largest one at the bottom. The aim is to transfer all n disks from A to C by a sequence of single-disk moves such that we never place a larger disk on top of a smaller one. Peg B may be used for intermediate storage.



The pegs and moves can be modelled as follows:

datatype *peg* = *A* | *B* | *C*

type_synonym *move* = *"peg * peg"*

Define a primitive recursive function

consts
  move :: "nat => peg => peg => move list"

such that `move` n a b returns a list of (legal) moves that transfer n disks from peg a to peg c.

Show that this requires $2^n - 1$ moves:

theorem *"length (move n a b) = 2^n - 1"*

Hint: You need to strengthen the theorem for the induction to go through. Beware: subtraction on natural numbers behaves oddly: $n - m = 0$ if $n \leq m$.

Correctness

In the last section we introduced the towers of Hanoi and defined a function `move` to generate the moves to solve the puzzle. Now it is time to show that `move` is correct. This means that

- when executing the list of moves, the result is indeed the intended one, i.e. all disks are moved from one peg to another, and
- all of the moves are legal, i.e. never is a larger disk placed on top of a smaller one.

Hint: This is a non-trivial undertaking. The complexity of your proofs will depend crucially on your choice of model, and you may have to revise your model as you proceed with the proof.