Isabelle/HOL Exercises
Trees, Inductive Data Types

Representation of Propositional Formulae by Polynomials

Let the following data type for propositional formulae be given:

\[
\text{datatype } \text{form} = T | \text{Var} \text{ nat} | \text{And} \text{ form form} | \text{Xor} \text{ form form}
\]

Here \( T \) denotes a formula that is always true, \( \text{Var} \text{ n} \) denotes a propositional variable, its name given by a natural number, \( \text{And} \text{ f1 f2} \) denotes the AND combination, and \( \text{Xor} \text{ f1 f2} \) the XOR (exclusive or) combination of two formulae. A constructor \( F \) for a formula that is always false is not necessary, since \( F \) can be expressed by \( \text{Xor} \text{ T T} \).

**Exercise 1:** Define a function

\[
\text{evalf} :: (\text{nat} \Rightarrow \text{bool}) \Rightarrow \text{form} \Rightarrow \text{bool}
\]
that evaluates a formula under a given variable assignment.

Propositional formulae can be represented by so-called polynomials. A polynomial is a list of lists of propositional variables, i.e. an element of type \( \text{nat list list} \). The inner lists (the so-called monomials) are interpreted as conjunctive combination of variables, whereas the outer list is interpreted as exclusive-or combination of the inner lists.

**Exercise 2:** Define two functions

\[
\text{evalm} :: (\text{nat} \Rightarrow \text{bool}) \Rightarrow \text{nat list} \Rightarrow \text{bool}
\]
\[
\text{evalp} :: (\text{nat} \Rightarrow \text{bool}) \Rightarrow \text{nat list list} \Rightarrow \text{bool}
\]
for evaluation of monomials and polynomials under a given variable assignment. In particular think about how empty lists have to be evaluated.

**Exercise 3:** Define a function

\[
\text{poly} :: \text{form} \Rightarrow \text{nat list list}
\]
that turns a formula into a polynomial. You will need an auxiliary function

\[
\text{mulpp} :: \text{nat list list} \Rightarrow \text{nat list list} \Rightarrow \text{nat list list}
\]
to “multiply” two polynomials, i.e. to compute

\[
((v_1^1 \odot \cdots \odot v_{m_1}^1) \oplus \cdots \oplus (v_k^k \odot \cdots \odot v_{m_k}^k)) \odot ((w_1^1 \odot \cdots \odot w_{n_1}^1) \oplus \cdots \oplus (w_j^j \odot \cdots \odot w_{n_j}^j))
\]
where $\oplus$ denotes “exclusive or”, and $\odot$ denotes “and”. This is done using the usual calculation rules for addition and multiplication.

**Exercise 4:** Now show correctness of your function $poly$:

**Theorem** $poly\_correct$: \(\text{evalf } e \ f = \text{evalp } e \ (\text{poly } f)\)"

It is useful to prove a similar correctness theorem for $mulpp$ first.