The Hahn-Banach Theorem
for Real Vector Spaces

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Abstract
The Hahn-Banach Theorem is one of the most fundamental results in
functional analysis. We present a fully formal proof of two versions of the
theorem, one for general linear spaces and another for normed spaces. This
development is based on simply-typed classical set-theory, as provided by
Isabelle/HOL.

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1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser’s textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.
Part I

Basic Notions

2 Bounds

theory Bounds
imports Main HOL-Analysis.Continuum-Not-Denumerable
begin

locale lub =
  fixes A and x
  assumes least [intro?]: (∀a. a ∈ A ⟹ a ≤ b) ⟹ x ≤ b
  and upper [intro?]: a ∈ A ⟹ a ≤ x

lemmas [elim?] = lub.least lub.upper

definition the-lub :: 'a::order set ⇒ 'a (⨆) where
  the-lub A = The (lub A)

lemma the-lub-equality [elim?):
  assumes lub A x
  shows ⨆ A = (x::'a::order)
⟨proof⟩

lemma the-lubI-ex:
  assumes ex: ∃x. lub A x
  shows lub A (⨆ A)
⟨proof⟩

lemma real-complete: ∃a::real. a ∈ A ⟹ ∃y. ∀a ∈ A. a ≤ y ⟹ ∃x. lub A x
⟨proof⟩
end

3 Vector spaces

theory Vector-Space
imports Complex-Main Bounds
begin

3.1 Signature

For the definition of real vector spaces a type 'a of the sort {plus, minus, zero}
is considered, on which a real scalar multiplication · is declared.

consts
  prod :: real ⇒ 'a::{plus,minus,zero} ⇒ 'a (infixr · 70)
3.2 Vector space laws

A vector space is a non-empty set $V$ of elements from a with the following vector space laws: The set $V$ is closed under addition and scalar multiplication, addition is associative and commutative; $-x$ is the inverse of $x$ wrt. addition and $0$ is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number $1$ is the neutral element of scalar multiplication.

locale vectorspace =
  fixes $V$
  assumes non-empty [iff, intro?!]: $V \neq \{\}$
  and add-closed [iff]: $x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V$
  and mult-closed [iff]: $x \in V \Longrightarrow a \cdot x \in V$
  and add-assoc: $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z)$
  and add-commute: $x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x$
  and diff-self [simp]: $x \in V \Longrightarrow x - x = 0$
  and add-zero-left [simp]: $x \in V \Longrightarrow 0 + x = x$
  and add-mult-distrib1: $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$
  and add-mult-distrib2: $x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$
  and mult-assoc: $x \in V \Longrightarrow (a \cdot b) \cdot x = a \cdot (b \cdot x)$
  and mult-1 [simp]: $x \in V \Longrightarrow 1 \cdot x = x$
  and negate-eq1: $x \in V \Longrightarrow -1 \cdot x = (-1) \cdot x$
  and diff-eq1: $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + (-y)$

begin

lemma negate-eq2: $x \in V \Longrightarrow (-1) \cdot x = -x$
  ⟨proof⟩

lemma negate-eq2a: $x \in V \Longrightarrow -1 \cdot x = -x$
  ⟨proof⟩

lemma diff-eq2: $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x - y$
  ⟨proof⟩

lemma diff-closed [iff]: $x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V$
  ⟨proof⟩

lemma neg-closed [iff]: $x \in V \Longrightarrow -x \in V$
  ⟨proof⟩

lemma add-left-commute:
  $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow x + (y + z) = y + (x + z)$
  ⟨proof⟩

lemmas add-ac = add-assoc add-commute add-left-commute

The existence of the zero element of a vector space follows from the non-
emptiness of carrier set.

lemma zero [iff]: $0 \in V$
  ⟨proof⟩

lemma add-zero-right [simp]: $x \in V \Longrightarrow x + 0 = x$
  ⟨proof⟩
### 3.2 Vector space laws

**lemma** mult-assoc2: \( x \in V \implies a \cdot b \cdot x = (a \ast b) \cdot x \)

(proof)

**lemma** diff-mult-distrib1: \( x \in V \implies y \in V \implies a \cdot (x - y) = a \cdot x - a \cdot y \)

(proof)

**lemma** diff-mult-distrib2: \( x \in V \implies (a - b) \cdot x = a \cdot x - (b \cdot x) \)

(proof)

**lemmas** distrib =
  add-mult-distrib1 add-mult-distrib2
diff-mult-distrib1 diff-mult-distrib2

Further derived laws:

**lemma** mult-zero-left [simp]: \( x \in V \implies 0 \cdot x = 0 \)

(proof)

**lemma** mult-zero-right [simp]: \( a \cdot 0 = (0::\'a) \)

(proof)

**lemma** minus-mult-cancel [simp]: \( x \in V \implies (- a) \cdot - x = a \cdot x \)

(proof)

**lemma** add-minus-left-eq-diff: \( x \in V \implies y \in V \implies - x + y = y - x \)

(proof)

**lemma** add-minus [simp]: \( x \in V \implies x + - x = 0 \)

(proof)

**lemma** add-minus-left [simp]: \( x \in V \implies - x + x = 0 \)

(proof)

**lemma** minus-minus [simp]: \( x \in V \implies -(- x) = x \)

(proof)

**lemma** minus-zero [simp]: \( - (0::\'a) = 0 \)

(proof)

**lemma** minus-zero-iff [simp]:
  assumes \( x: x \in V \)
  shows \( (- x = 0) = (x = 0) \)

(proof)

**lemma** add-minus-cancel [simp]: \( x \in V \implies y \in V \implies x + (- x + y) = y \)

(proof)

**lemma** minus-add-cancel [simp]: \( x \in V \implies y \in V \implies - x + (x + y) = y \)

(proof)

**lemma** minus-add-distrib [simp]: \( x \in V \implies y \in V \implies -(x + y) = -x - y \)

(proof)

**lemma** diff-zero [simp]: \( x \in V \implies x - 0 = x \)
Lemma \textbf{diff-zero-right [simp]}: \( x \in V \implies 0 - x = - x \)

\begin{proof}
\end{proof}

Lemma \textbf{add-left-cancel}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( z : z \in V \)
  \item \textbf{shows}: \((x + y = x + z) = (y = z)\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{add-right-cancel}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( z : z \in V \)
  \item \textbf{shows}: \((y = z) = (y + x = z + x)\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{add-assoc-cong}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( x' : x' \in V \) and \( y' : y' \in V \) and \( z : z \in V \)
  \item \textbf{shows}: \((x + y = x' + y') = (x + (y + z) = x' + (y' + z))\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{mult-left-commute}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( a : a \neq 0 \)
  \item \textbf{shows}: \((a \cdot x = a \cdot y) = (x = y)\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{mult-zero-uniq}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( ax : a \cdot x = 0 \) and \( ax' : a \cdot x' = 0 \)
  \item \textbf{shows}: \(a = 0\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{mult-left-cancel}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( a : a \neq 0 \)
  \item \textbf{shows}: \((a \cdot x = a \cdot y) = (x = y)\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{mult-right-cancel}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( a \neq 0 \)
  \item \textbf{shows}: \((a \cdot x = b \cdot x) = (a = b)\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{eq-diff-eq}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( z : z \in V \)
  \item \textbf{shows}: \((x = z - y) = (x + y = z)\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{add-minus-eq-minus}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( xy : x + y = 0 \)
  \item \textbf{shows}: \(x = - y\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{add-minus-eq}:
\begin{itemize}
  \item \textbf{assumes}: \( x : x \in V \) and \( y : y \in V \) and \( xy : x - y = 0 \)
  \item \textbf{shows}: \(x = y\)
\end{itemize}

\begin{proof}
\end{proof}

Lemma \textbf{add-diff-swap}:
\begin{itemize}
  \item \textbf{assumes}: \( vs : a \in V \) and \( b : b \in V \) and \( c : c \in V \) and \( d : d \in V \)
\end{itemize}

\begin{proof}
\end{proof}
\[\begin{align*}
\text{and eq: } & a + b = c + d \\
\text{shows } & a - c = d - b \\
\end{align*}\]

(proof)

lemma vs-add-cancel-21:
assumes vs: \( x \in V \ y \in V \ z \in V \ u \in V \)
shows \((x + (y + z)) = y + u) = (x + z = u)\)
(proof)

lemma add-cancel-end:
assumes vs: \( x \in V \ y \in V \ z \in V \)
shows \((x + (y + z) = y) = (x = -z)\)
(proof)

end

4 Subspaces

theory Subspace

imports Vector-Space HOL-Library.Set-Algebras

begin

4.1 Definition

A non-empty subset \( U \) of a vector space \( V \) is a \textit{subspace} of \( V \), iff \( U \) is closed under addition and scalar multiplication.

locale subspace =
fixes U :: \( 'a::{minus, plus, zero, uminus} \) set and V
assumes non-empty [iff, intro]: \( U \neq \{\} \)
and subset [iff]: \( U \subseteq V \)
and add-closed [iff]: \( x \in U \implies y \in U \implies x + y \in U \)
and mult-closed [iff]: \( x \in U \implies a \cdot x \in U \)

notation (symbols)
subspace (infix \( \leq \) 50)

declare vectorspace.intro [intro?] subspace.intro [intro?]

lemma subspace-subset [elim]: \( U \leq V \implies U \subseteq V \)
(proof)

lemma (in subspace) subsetD [iff]: \( x \in U \implies x \in V \)
(proof)

lemma subspaceD [elim]: \( U \leq V \implies x \in U \implies x \in V \)
(proof)

lemma rev-subspaceD [elim?]: \( x \in U \implies U \leq V \implies x \in V \)
(proof)

lemma (in subspace) diff-closed [iff]:
assumes vectorspace $V$
assumes $x$: $x \in U$ and $y$: $y \in U$
shows $x - y \in U$
(proof)

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

lemma (in subspace) zero [intro]:
assumes vectorspace $V$
shows $0 \in U$
(proof)

lemma (in subspace) neg-closed [iff]:
assumes vectorspace $V$
assumes $x$: $x \in U$
shows $-x \in U$
(proof)

Further derived laws: every subspace is a vector space.

lemma (in subspace) vectorspace [iff]:
assumes vectorspace $V$
shows vectorspace $U$
(proof)

The subspace relation is reflexive.

lemma (in vectorspace) subspace-refl [intro]: $V \subseteq V$
(proof)

The subspace relation is transitive.

lemma (in vectorspace) subspace-trans [trans]:
$U \subseteq V \Rightarrow V \subseteq W \Rightarrow U \subseteq W$
(proof)

4.2 Linear closure

The linear closure of a vector $x$ is the set of all scalar multiples of $x$.

definition lin :: ("a::\{minus, plus, zero\}) \Rightarrow 'a set
where lin $x = \{a \cdot x | a. \text{True}\}$

lemma linI [intro]: $y = a \cdot x \Rightarrow y \in \text{lin } x$
(proof)

lemma linI' [iff]: $a \cdot x \in \text{lin } x$
(proof)

lemma linE [elim]:
assumes $x \in \text{lin } v$
obtains $a :: \text{real}$ where $x = a \cdot v$
(proof)

Every vector is contained in its linear closure.
4.3 Sum of two vectorspaces

The sum of two vectorspaces $U$ and $V$ is the set of all sums of elements from $U$ and $V$.

**Lemma** (in vectorspace) sum-def: $U + V = \{ u + v \mid u \in U \land v \in V \}$ (proof)

**Lemma** sumE [elim]:
\[
x \in U + V \implies (\bigwedge u v. x = u + v \implies u \in U \implies v \in V \implies C) \implies C
\]
(proof)

**Lemma** sumI [intro]:
\[
u \in U \implies v \in V \implies x = u + v \implies x \in U + V
\]
(proof)

**Lemma** sumI’ [intro]:
\[
u \in U \implies v \in V \implies u + v \in U + V
\]
(proof)

$U$ is a subspace of $U + V$.

**Lemma** subspace-sum1 [iff]:
\[
\text{assumes vectorspace } U \text{ vectorspace } V
\]
\[
\text{shows } U \subseteq U + V
\]
(proof)

The sum of two subspaces is again a subspace.

**Lemma** sum-subspace [iff]:
\[
\text{assumes subspace } U \subseteq E \text{ vectorspace } E \subseteq V \subseteq E
\]
\[
\text{shows } U + V \subseteq E
\]
(proof)

The sum of two subspaces is a vectorspace.

**Lemma** sum-vs [intro]?:
\[
U \subseteq E \implies V \subseteq E \implies \text{vectorspace } E \implies \text{vectorspace } (U + V)
\]
(proof)
4.4 Direct sums

The sum of \( U \) and \( V \) is called direct, iff the zero element is the only common element of \( U \) and \( V \). For every element \( x \) of the direct sum of \( U \) and \( V \) the decomposition in \( x = u + v \) with \( u \in U \) and \( v \in V \) is unique.

**lemma decomp:**

- **assumes** vectorspace \( E \) subspace \( U \) \( E \) subspace \( V \)
- **assumes** direct: \( U \cap V = \{ 0 \} \)
  - and \( u1: u1 \in U \) and \( u2: u2 \in U \)
  - and \( v1: v1 \in V \) and \( v2: v2 \in V \)
- **sum** \( u1 + v1 = u2 + v2 \)
- **shows** \( u1 = u2 \land v1 = v2 \)

⟨ proof ⟩

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page ??): for any element \( y + a \cdot x_0 \) of the direct sum of a vectorspace \( H \) and the linear closure of \( x_0 \) the components \( y \in H \) and \( a \) are uniquely determined.

**lemma decomp-H′:**

- **assumes** vectorspace \( E \) subspace \( H \) \( E \) subspace \( H \)
  - and \( y1: y1 \in H \) and \( y2: y2 \in H \)
  - and \( x′: x′ \notin H \) \( x′ \in E \) \( x′ \neq 0 \)
  - and \( eq: y1 + a1 \cdot x′ = y2 + a2 \cdot x′ \)
- **shows** \( y1 = y2 \land a1 = a2 \)

⟨ proof ⟩

Since for any element \( y + a \cdot x′ \) of the direct sum of a vectorspace \( H \) and the linear closure of \( x′ \) the components \( y \in H \) and \( a \) are unique, it follows from \( y \in H \) that \( a = 0 \).

**lemma decomp-H′-H:**

- **assumes** vectorspace \( E \) subspace \( H \) \( E \) subspace \( H \)
  - and \( t: t \in H \)
  - and \( x′: x′ \notin H \) \( x′ \in E \) \( x′ \neq 0 \)
- **shows** \( (SOME (y, a). t = y + a \cdot x′ \land y \in H) = (t, 0) \)

⟨ proof ⟩

The components \( y \in H \) and \( a \) in \( y + a \cdot x′ \) are unique, so the function \( h′ \) defined by \( h′(y + a \cdot x′) = h y + a \cdot ξ \) is definite.

**lemma h′-definite:**

- **fixes** \( H \)
- **assumes** \( h′-def: \)
  - \( \forall x. h′ x = \)
  - \( (\text{let } (y, a) = \text{SOME } (y, a). (x = y + a \cdot x′ \land y \in H)) \)
  - \( \text{in } (h y) + a * xi) \)
- **assumes** vectorspace \( E \) subspace \( H \) \( E \) subspace \( H \)
- **assumes** \( y: y \in H \)
  - and \( x′: x′ \notin H \) \( x′ \in E \) \( x′ \neq 0 \)
- **shows** \( h′ x = h y + a * xi \)

⟨ proof ⟩

end
5 Normed vector spaces

theory Normed-Space
imports Subspace
begin

5.1 Quasinorms

A *seminorm* $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogeneous and subadditive.

locale seminorm =
  fixes V :: 'a::{minus, plus, zero, uminus} set
  fixes norm :: 'a => real (\|\-\|)
assumes ge-zero [iff]: \( x \in V \implies 0 \leq \|x\| \)
and abs-homogenous [iff]: \( x \in V \implies \|a \cdot x\| = |a| \cdot \|x\| \)
and subadditive [iff]: \( x \in V \implies y \in V \implies \|x + y\| \leq \|x\| + \|y\| \)
declare seminorm.intro [intro?]

lemma (in seminorm) diff-subadditive:
  assumes vectorspace V
  shows \( x \in V \implies y \in V \implies \|x - y\| \leq \|x\| + \|y\| \)
 ⟨proof⟩

lemma (in seminorm) minus:
  assumes vectorspace V
  shows \( x \in V \implies \|-x\| = \|x\| \)
 ⟨proof⟩

5.2 Norms

A *norm* $\|\cdot\|$ is a seminorm that maps only the 0 vector to 0.

locale norm = seminorm +
  assumes zero-iff [iff]: \( x \in V \implies (\|x\| = 0) = (x = 0) \)

5.3 Normed vector spaces

A vector space together with a norm is called a normed space.

locale normed-vectorspace = vectorspace + norm
declare normed-vectorspace.intro [intro?]

lemma (in normed-vectorspace) gt-zero [intro?):
  assumes \( x : x \in V \text{ and } \neg q: x \neq 0 \)
  shows \( \theta < \|x\| \)
 ⟨proof⟩

Any subspace of a normed vector space is again a normed vectorspace.

lemma subspace-normed-vs [intro?]:
  fixes F E norm
  assumes subspace F E normed-vectorspace E norm
shows normed-vectorspace F norm
⟨proof⟩
end

6 Linearforms

theory Linearform
imports Vector-Space
begin
A linear form is a function on a vector space into the reals that is additive and multiplicative.

locale linearform =
  fixes V :: 'a::{minus, plus, zero, uminus} set and f
  assumes add [iff]: \( x \in V \implies y \in V \implies f(x + y) = f(x) + f(y) \)
  and mult [iff]: \( x \in V \implies f(a \cdot x) = a \cdot f(x) \)

declare linearform.intro [intro?]

lemma (in linearform) neg [iff]:
  assumes vectorspace V
  shows \( x \in V \implies f(-x) = -f(x) \)
⟨proof⟩

lemma (in linearform) diff [iff]:
  assumes vectorspace V
  shows \( x \in V \implies y \in V \implies f(x - y) = f(x) - f(y) \)
⟨proof⟩

Every linear form yields 0 for the 0 vector.

lemma (in linearform) zero [iff]:
  assumes vectorspace V
  shows \( f(0) = 0 \)
⟨proof⟩
end

7 An order on functions

theory Function-Order
imports Subspace Linearform
begin

7.1 The graph of a function

We define the graph of a (real) function \( f \) with domain \( F \) as the set

\[ \{(x, f(x)) \mid x \in F\} \]

So we are modeling partial functions by specifying the domain and the mapping function. We use the term “function” also for its graph.
7.2 Functions ordered by domain extension

A function $h'$ is an extension of $h$, if the graph of $h$ is a subset of the graph of $h'$.

**Lemma** graph-extI:

$(\forall x. x \in H \implies h x = h' x) \implies H \subseteq H'$

**Lemma** graph-extD1 [dest?]:

$graph H h \subseteq graph H' h' \implies x \in H \implies h x = h' x$

**Lemma** graph-extD2 [dest?]:

$graph H h \subseteq graph H' h' \implies H \subseteq H'$

7.3 Domain and function of a graph

The inverse functions to $graph$ are $domain$ and $funct$.

**Definition** domain :: 'a graph ⇒ 'a set

**Definition** funct :: 'a graph ⇒ ('a ⇒ real)

The following lemma states that $g$ is the graph of a function if the relation induced by $g$ is unique.

**Lemma** graph-domain-funct:

$\forall x \forall y \forall z. (x, y) \in g \implies (x, z) \in g \implies z = y$

7.4 Norm-preserving extensions of a function

Given a linear form $f$ on the space $F$ and a seminorm $p$ on $E$. The set of all linear extensions of $f$, to superspaces $H$ of $F$, which are bounded by $p$, is defined as follows.
8 The norm of a function

theory Function-Norm
imports Normed-Space Function-Order
begin

8.1 Continuous linear forms

A linear form \( f \) on a normed vector space \( (V, \| \cdot \|) \) is continuous, if it is bounded, i.e.

\[
\exists c \in \mathbb{R}. \forall x \in V. \| f x \| \leq c \cdot \| x \|
\]
In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

locale continuous = linearform +
fixes norm :: - ⇒ real (∥-∥)
assumes bounded: ∃c. ∀x ∈ V. |f x| ≤ c ∗ ∥x∥
declare continuous.intro [intro?] continuous-axioms.intro [intro?]

lemma continuousI [intro]:
fixes norm :: - ⇒ real (∥-∥)
assumes linearform V f
assumes r: ∀x ∈ V. |f x| ≤ c ∗ ∥x∥
shows continuous V f norm
⟨proof⟩

8.2 The norm of a linear form

The least real number c for which holds
∀x ∈ V. |f x| ≤ c · ∥x∥
is called the norm of f.

For non-trivial vector spaces V ≠ {0} the norm can be defined as
∥f∥ = sup x ≠ 0. |f x| / ∥x∥

For the case V = {0} the supremum would be taken from an empty set. Since \( \mathbb{R} \) is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be \( \{ \} \geq 0 \) so that fn-norm has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be 0, as all other elements are \( \{ \} \geq 0 \).

Thus we define the set B where the supremum is taken from as follows:

\{ 0 \} ∪ \{ |f x| / ∥x∥. x ≠ 0 ∧ x ∈ F \}

fn-norm is equal to the supremum of B, if the supremum exists (otherwise it is undefined).

locale fn-norm =
fixes norm :: - ⇒ real (∥-∥)
fixes B defines B V f ≡ \{ 0 \} ∪ \{ |f x| / ∥x∥. x ≠ 0 ∧ x ∈ V \}
fixes fn-norm (∥-∥) defines ∥f∥ = sup (B V f)

locale normed-vectorspace-with-fn-norm = normed-vectorspace + fn-norm

lemma (in fn-norm) B-not-empty [intro]: 0 ∈ B V f
⟨proof⟩

The following lemma states that every continuous linear form on a normed space \((V, ∥-∥)\) has a function norm.

lemma (in normed-vectorspace-with-fn-norm) fn-norm-works:
assumes continuous \( V f \) norm
shows \( \text{lub} \ (B \ V f \ (\|f\| \cdot V)) \)
(proof)

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ub [iff]:
assumes continuous \( V f \) norm
assumes \( b: \ b \in B \ V f \)
shows \( b \leq \|f\| \cdot V \)
(proof)

lemma (in normed-vectorspace-with-fn-norm) fn-norm-leastB:
assumes continuous \( V f \) norm
assumes \( b: \ \forall b. \ b \in B \ V f \implies b \leq y \)
shows \( \|f\| \cdot V \leq y \)
(proof)

The norm of a continuous function is always \( \geq 0 \).

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ge-zero [iff]:
assumes continuous \( V f \) norm
shows \( 0 \leq \|f\| \cdot V \)
(proof)

The fundamental property of function norms is:
\[
|f \ x| \leq \|f\| \cdot \|x\|
\]

lemma (in normed-vectorspace-with-fn-norm) fn-norm-le-cong:
assumes continuous \( V f \) norm linearform \( V f \)
assumes \( x: \ x \in V \)
shows \( |f \ x| \leq \|f\| \cdot V \ast \|x\| \)
(proof)

The function norm is the least positive real number for which the following
inequality holds:
\[
|f \ x| \leq c \cdot \|x\|
\]

lemma (in normed-vectorspace-with-fn-norm) fn-norm-least [intro]:
assumes continuous \( V f \) norm
assumes ineq: \( \forall x. \ x \in V \implies |f \ x| \leq c \ast \|x\| \) and ge: \( 0 \leq c \)
shows \( \|f\| \cdot V \leq c \)
(proof)

end

9 Zorn’s Lemma

theory Zorn-Lemma
imports Main
begin

Zorn’s Lemmas states: if every linear ordered subset of an ordered set \( S \) has an
upper bound in \( S \), then there exists a maximal element in \( S \). In our application,
$S$ is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn’s lemma can be modified: if $S$ is non-empty, it suffices to show that for every non-empty chain $c$ in $S$ the union of $c$ also lies in $S$.

**Theorem Zorn’s Lemma:**

**Assumes** $r: \bigwedge c. c \in \text{chains } S \implies \exists x. x \in c \implies \bigcup c \in S$

**And** $aS: a \in S$

**Shows** $\exists y \in S. \forall z \in S. y \subseteq z \implies z = y$

(proof)

end
Part II

Lemmas for the Proof

10 The supremum wrt. the function order

theory Hahn-Banach-Sup-Lemmas
imports Function-Norm Zorn-Lemma
begin

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let $E$ be a real vector space with a seminorm $p$ on $E$, $F$ is a subspace of $E$ and $f$ a linear form on $F$. We consider a chain $c$ of norm-preserving extensions of $f$, such that $\bigcup c = \text{graph } H \cdot h$. We will show some properties about the limit function $h$, i.e. the supremum of the chain $c$.

Let $c$ be a chain of norm-preserving extensions of the function $f$ and let $\text{graph } H \cdot h$ be the supremum of $c$. Every element in $H$ is member of one of the elements of the chain.

lemma [dest?] = chainsD
lemma chainsE2 [elim?] = chainsD2 [elim-format]

lemma some-H' h':
  assumes $M$: $M = \text{norm-pres-extensions } E \cdot p \cdot F \cdot f$
  and $cM$: $c \in \text{chains } M$
  and $u$: $\text{graph } H \cdot h = \bigcup c$
  and $x$: $x \in H$
  shows $\exists H' \cdot h', \text{graph } H' \cdot h' \subseteq c$
    $\land (x, h \cdot x) \in \text{graph } H' \cdot h'$
    $\land \text{linearform } H' \cdot h' \land H' \subseteq E$
    $\land F \subseteq H' \land \text{graph } F \cdot f \subseteq \text{graph } H' \cdot h'$
    $\land (\forall x \in H'. h' \cdot x \leq p \cdot x)$
⟨proof⟩

Let $c$ be a chain of norm-preserving extensions of the function $f$ and let $\text{graph } H \cdot h$ be the supremum of $c$. Every element in the domain $H$ of the supremum function is member of the domain $H'$ of some function $h'$, such that $h$ extends $h'$.

lemma some-H' h':
  assumes $M$: $M = \text{norm-pres-extensions } E \cdot p \cdot F \cdot f$
  and $cM$: $c \in \text{chains } M$
  and $u$: $\text{graph } H \cdot h = \bigcup c$
  and $x$: $x \in H$
  shows $\exists H' \cdot h', x \in H' \land \text{graph } H' \cdot h' \subseteq \text{graph } H \cdot h$
    $\land \text{linearform } H' \cdot h' \land H' \subseteq E \land F \subseteq H'$
    $\land \text{graph } F \cdot f \subseteq \text{graph } H' \cdot h' \land (\forall x \in H'. h' \cdot x \leq p \cdot x)$
⟨proof⟩

Any two elements $x$ and $y$ in the domain $H$ of the supremum function $h$ are both in the domain $H'$ of some function $h'$, such that $h$ extends $h'$.
lemma some-H′h′2:
assumes M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and u: graph H h = ⋃ c
and x: x ∈ H
and y: y ∈ H
shows ∃ H′ h′. x ∈ H′ ∧ y ∈ H′
∧ graph H′ h′ ⋒ graph H h
∧ linearform H′ h′ ∧ H′ ⋒ E ∧ F ⋒ H′
∧ graph F f ⋒ graph H′ h′ ∧ (∀ x ∈ H′. h′ x ≤ p x)
⟨proof⟩
The relation induced by the graph of the supremum of a chain c is definite, i.e. it is the graph of a function.

lemma sup-definite:
assumes M-def: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and xy: (x, y) ∈ ⋃ c
and xz: (x, z) ∈ ⋃ c
shows z = y
⟨proof⟩
The limit function h is linear. Every element x in the domain of h is in the domain of a function h′ in the chain of norm preserving extensions. Furthermore, h is an extension of h′ so the function values of x are identical for h′ and h. Finally, the function h′ is linear by construction of M.

lemma sup-lf:
assumes M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and u: graph H h = ⋃ c
shows linearform H h
⟨proof⟩
The limit of a non-empty chain of norm preserving extensions of f is an extension of f, since every element of the chain is an extension of f and the supremum is an extension for every element of the chain.

lemma sup-ext:
assumes graph: graph H h = ⋃ c
and M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and ex: ∃ x. x ∈ c
shows graph F f ⋒ graph H h
⟨proof⟩
The domain H of the limit function is a superspace of F, since F is a subset of H. The existence of the 0 element in F and the closure properties follow from the fact that F is a vector space.

lemma sup-supF:
assumes graph: graph H h = ⋃ c
and M: M = norm-pres-extensions E p F f
and \( cM: c \in \text{chains } M \)
and \( ex: \exists x. x \in c \)
and \( FE: F \subseteq E \)
shows \( F \subseteq H \)
(proof)

The domain \( H \) of the limit function is a subspace of \( E \).

**lemma** sup-subE:
assumes graph: \( \text{graph } H h = \bigcup c \)
and \( M: M = \text{norm-pres-extensions } E p F f \)
and \( cM: c \in \text{chains } M \)
and \( ex: \exists x. x \in c \)
and \( FE: F \subseteq E \)
and \( E: \text{vectorspace } E \)
shows \( H \subseteq E \)
(proof)

The limit function is bounded by the norm \( p \) as well, since all elements in the chain are bounded by \( p \).

**lemma** sup-norm-pres:
assumes graph: \( \text{graph } H h = \bigcup c \)
and \( M: M = \text{norm-pres-extensions } E p F f \)
and \( cM: c \in \text{chains } M \)
shows \( \forall x \in H. h x \leq p x \)
(proof)

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma \textit{abs-Hahn-Banach} (see page 24). For real vector spaces the following inequality are equivalent:

\[
\forall x \in H. |h x| \leq p x \quad \text{and} \quad \forall x \in H. h x \leq p x
\]

**lemma** abs-ineq-iff:
assumes subspace \( H E \) and \( \text{vectorspace } E \) and \( \text{seminorm } E p \)
and \( \text{linearform } H h \)
shows \( (\forall x \in H. |h x| \leq p x) = (\forall x \in H. h x \leq p x) \) (is \( ?L = ?R \))
(proof)

end

# 11 Extending non-maximal functions

**theory** Hahn-Banach-Ext-Lemmas
**imports** Function-Norm
**begin**

In this section the following context is presumed. Let \( E \) be a real vector space with a seminorm \( q \) on \( E \). \( F \) is a subspace of \( E \) and \( f \) a linear function on \( F \). We consider a subspace \( H \) of \( E \) that is a superspace of \( F \) and a linear form \( h \) on \( H \). \( H \) is a not equal to \( E \) and \( x_0 \) is an element in \( E - H \). \( H \) is extended to the direct sum \( H' = H + \text{lin } x_0 \), so for any \( x \in H' \) the decomposition of \( x = y + \).
a \cdot x \text{ with } y \in H \text{ is unique. } h' \text{ is defined on } H' \text{ by } h' x = h y + a \cdot \xi \text{ for a certain } \xi.

Subsequently we show some properties of this extension \( h' \) of \( h \).

This lemma will be used to show the existence of a linear extension of \( f \) (see page ??). It is a consequence of the completeness of \( \mathbb{R} \). To show

\[ \exists \xi. \forall y \in F. \ a y \leq \xi \land \xi \leq b y \]

it suffices to show that

\[ \forall u \in F. \forall v \in F. \ a u \leq b v \]

**lemma ex-xi:**

- **assumes** vectorspace \( F \)
- **assumes** \( r: \exists u v. u \in F \Rightarrow v \in F \Rightarrow a u \leq b v \)
- **shows** \( \exists \xi::\text{real}. \forall y \in F. \ a y \leq \xi \land \xi \leq b y \)

\( \langle \text{proof} \rangle \)

The function \( h' \) is defined as a \( h' x = h y + a \cdot \xi \) where \( x = y + a \cdot \xi \) is a linear extension of \( h \) to \( H' \).

**lemma h'-lf:**

- **assumes** \( h'\)-def: \( \forall x. \ h' x = (\text{let } (y, a) = \text{SOME } (y, a). \ x = y + a \cdot x_0 \land y \in H \ \text{in } h y + a \cdot \xi) \)
- **and** \( H'\)-def: \( H' = H + \text{lin } x_0 \)
- **assumes** \( H: \text{linearform } H \)
- **assumes** \( x_0: x_0 \notin H \ \ x_0 \in E \ \ x_0 \neq 0 \)
- **assumes** \( E: \text{vectorspace } E \)
- **shows** \( \text{linearform } H' \ h' \)

\( \langle \text{proof} \rangle \)

The linear extension \( h' \) of \( h \) is bounded by the seminorm \( p \).

**lemma h'-norm-pres:**

- **assumes** \( h'\)-def: \( \forall x. \ h' x = (\text{let } (y, a) = \text{SOME } (y, a). \ x = y + a \cdot x_0 \land y \in H \ \text{in } h y + a \cdot \xi) \)
- **and** \( H'\)-def: \( H' = H + \text{lin } x_0 \)
- **assumes** \( E: \text{vectorspace } E \) and \( H: \text{subspace } H E \)
- **and** \( \text{seminorm } E \ p \) and **linearform** \( H \ h \)
- **assumes** \( a: \forall y \in H. \ h y \leq p y \)
- **and** \( a': \forall y \in H. \ - p (y + x_0) - h y \leq \xi \land \xi \leq p (y + x_0) - h y \)
- **shows** \( \forall x \in H'. \ h' x \leq p x \)

\( \langle \text{proof} \rangle \)

end
Part III
The Main Proof

12 The Hahn-Banach Theorem

theory Hahn-Banach
imports Hahn-Banach-Lemmas
begin

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

12.1 The Hahn-Banach Theorem for vector spaces

Hahn-Banach Theorem. Let $F$ be a subspace of a real vector space $E$, let $p$ be a semi-norm on $E$, and $f$ be a linear form defined on $F$ such that $f$ is bounded by $p$, i.e. $\forall x \in F. f(x) \leq p(x)$. Then $f$ can be extended to a linear form $h$ on $E$ such that $h$ is norm-preserving, i.e. $h$ is also bounded by $p$.

Proof Sketch.

1. Define $M$ as the set of norm-preserving extensions of $f$ to subspaces of $E$. The linear forms in $M$ are ordered by domain extension.
2. We show that every non-empty chain in $M$ has an upper bound in $M$.
3. With Zorn’s Lemma we conclude that there is a maximal function $g$ in $M$.
4. The domain $H$ of $g$ is the whole space $E$, as shown by classical contradiction:
   - Assuming $g$ is not defined on whole $E$, it can still be extended in a norm-preserving way to a super-space $H'$ of $H$.
   - Thus $g$ can not be maximal. Contradiction!

theorem Hahn-Banach:
  assumes $E$: vectorspace $E$ and subspace $F E$
  and seminorm $E p$ and linearform $F f$
  assumes $p: \forall x \in F. f(x) \leq p(x)$
  shows $\exists h. linearform E h \land (\forall x \in F. h(x) = f(x)) \land (\forall x \in E. h(x) \leq p(x))$
  — Let $E$ be a vector space, $F$ a subspace of $E$, $p$ a seminorm on $E$,
  — and $f$ a linear form on $F$ such that $f$ is bounded by $p$,
  — then $f$ can be extended to a linear form $h$ on $E$ in a norm-preserving way.
⟨proof⟩

12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form $f$ and a seminorm $p$ the following inequality are equivalent:1

1This was shown in lemma abs-ineq-iff (see page 22).
12.3 The Hahn-Banach Theorem for normed spaces

\[ \forall x \in H. \ |h x| \leq p x \quad \text{and} \quad \forall x \in H. h x \leq p x \]

**Theorem abs-Hahn-Banach:**

**Assumes** \( E: \text{vectorspace } E \) and \( FE: \text{subspace } F \ E \)

and \( lf: \text{linearform } F f \) and \( sn: \text{seminorm } E p \)

**Assumes** \( fp: \forall x \in F. \ |f x| \leq p x \)

**Shows** \( \exists g. \ \text{linearform } E g \)

\( \land (\forall x \in F. \ g x = f x) \)

\( \land (\forall x \in E. \ |g x| \leq p x) \)

(proof)

**12.3 The Hahn-Banach Theorem for normed spaces**

Every continuous linear form \( f \) on a subspace \( F \) of a norm space \( E \), can be extended to a continuous linear form \( g \) on \( E \) such that \( \|f\| = \|g\| \).

**Theorem norm-Hahn-Banach:**

fixes \( V \) and \( \text{norm } (\|\cdot\|) \)

fixes \( B \) defines \( \bigvee V f. B V f \equiv \{0\} \cup \{|f x| / \|x\| | x \neq 0 \land x \in V\} \)

fixes \( fn-norm \) \((\|\cdot\|_r-\{0, 1000\}, 999)\)

defines \( \bigvee V f. \|f\|_V \equiv \biguplus (B V f) \)

**Assumes** \( E-norm. \ \text{normed-vectorspace } E \) norm and \( FE: \text{subspace } F \ E \)

and \( \text{linearform: linearform } E f \) and \( \text{continuous } F f \) norm

**Shows** \( \exists g. \ \text{linearform } E g \)

\( \land \text{continuous } E g \) norm

\( \land (\forall x \in F. \ g x = f x) \)

\( \land \|g\|_E = \|f\|_F \)

(proof)

end

**References**

