A metalanguage for animating inductive definitions

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Talk outline

1 Motivation—schematic rule-based definitions

2 Brief introduction to MLSOS

3 Translating inductive definitions into MLSOS

4 Conclusions

5 Related & Future work
A relation is just a set of mathematical objects.

We usually define infinite relations using inference rules, as the least set closed under the rules.

This involves schematic patterns which we instantiate somehow to produce the underlying mathematical objects. For example:

\[
\begin{align*}
\text{even 0} & \quad \text{even } n \\
\text{even } n & \quad \text{even } n + 2
\end{align*}
\]

As we can see, this is straightforward in the first-order case.

As usual, things get more complicated when we introduce binders.
Notions of **instantiation** are no longer straightforward.

Suppose we have a schematic term

\[ \lambda x. \lambda y. \text{Var } x \]

where \( x \) and \( y \) are schematic pattern variables.

Given *concrete atoms* \( a \) and \( b \), which of the following are valid instantiations of that pattern?

1. \( \lambda a. \lambda b. \text{Var } a \)
2. \( \lambda a. \lambda a. \text{Var } a \)
We are not concerned with proof, but with animating rule-based inductive definitions.

This means you get an executable prototype (almost) for free when you define your inductive rules.

The prototype does proof-search over the rules, in order to model your programming language.

I will present the metalanguage for defining these prototypes later...

..first I will present a formal model of inductive rule-based definitions, which hopefully models informal practice reasonably well.
We define a language of schematic patterns, $p$:

$$p ::= x \mid () \mid (p_1, \ldots, p_n) \mid K \ p \mid \langle \langle x \rangle \rangle p$$

...which are used to build up formulae, $\varphi$:

$$\varphi ::= R \ p \mid x =/= x' \mid \varphi_1 \land \ldots \land \varphi_n \mid \text{true}.$$
We give a semantics to $N$ in terms of ground instantiations, $\gamma$, of variables in patterns to produce $\alpha$-equivalence classes $[g]_\alpha$ of ground nominal terms:

- $\gamma \circ x = \gamma(x)$
- $\gamma(x) = \{a\}$
- $\gamma \circ p = [g]_\alpha$
- $\gamma \circ (\langle x \rangle p) = [\langle a \rangle g]_\alpha$
- $\gamma \circ () = \{()\}$
- $\gamma \circ p = [g]_\alpha$
- $\gamma \circ (K \ p) = [K \ g]_\alpha$
- $\forall i \in \{1, \ldots, n\}. \gamma \circ p_i = [g_i]_\alpha$
- $\gamma \circ (p_1, \ldots, p_n) = [(g_1, \ldots, g_n)]_\alpha$

Note that, at this point, distinct variables may be instantiated with the same atom—even if in abstraction position.
For a definition which defines relations $R_1, \ldots, R_n$, we say that a ground term model, $\mathcal{H}$, is an $n$-tuple $(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ of models—one per relation symbol.

We define a satisfaction relation $\mathcal{H} \models \gamma \varphi$ as follows:

\[
\begin{align*}
\mathcal{H} &\models \gamma \text{ true} \\
\forall i \in \{1, \ldots, n\}. \mathcal{H} &\models \gamma \varphi_i \\
\mathcal{H} &\models \gamma (\varphi_1 \land \ldots \land \varphi_n)
\end{align*}
\]

\[
\begin{align*}
\gamma \odot p = [g]_\alpha &\; [g]_\alpha \in \mathcal{H}_i \\
\mathcal{H} &\models \gamma (R_i p)
\end{align*}
\]

\[
\begin{align*}
\gamma(x) = \{a\} &\; \gamma(x') = \{a'\} \; \; \; \; a \neq a' \\
\mathcal{H} &\models \gamma (x \neq x')
\end{align*}
\]
To get the set of $\mathcal{H}$ which satisfy a definition $\mathcal{N}$, we close under the schematic rules, as follows.

$$
(\mathcal{H} \models_{\gamma} \varphi) \Rightarrow (\mathcal{H} \models_{\gamma} R \, p)
$$

$$
\mathcal{H} \models_{\gamma} (\varphi \Rightarrow R \, p)
$$

$$
\forall R \in \mathcal{N}. \forall \gamma. P(R, \gamma) \Rightarrow \mathcal{H} \models_{\gamma} R
$$

$$
\mathcal{H} \models \mathcal{N}
$$

The predicate $P(R, \gamma)$ restricts the instantiations that can be required to a particular set of schematic rules.
Restrictions on instantiation

- There are various choices for predicate $P$, e.g.
  1. $P(\mathcal{R}, \gamma) \triangleq \text{true}$
     (any instantiation at all is permitted)
  2. $P(\mathcal{R}, \gamma) \triangleq \forall x, y \in \text{av}(\mathcal{R}). x \neq y \Rightarrow \gamma(x) \neq \gamma(y)$
     ($\gamma$ must be injective on names in abstraction position)
  3. $P(\mathcal{R}, \gamma) \triangleq \forall x, y \in \text{vars}(\mathcal{R})$
     
     $\text{sort}(x) = \text{sort}(y) = \alpha \land x \neq y \Rightarrow \gamma(x) \neq \gamma(y)$
     ($\gamma$ must be injective on all names of atom sort)

- The choice here is largely personal.

- However, if (1) were chosen, then proof-search using nominal matching would probably not be complete (cf $\lambda a. \lambda b. a$ vs $\lambda a. \lambda a. a$).
An example rule

- An example: the $\beta$-rule (using syntactic sugar for substitution).

\[
(\beta) \quad t_1'[t_2'/x] \equiv t_3 \\
\text{beta} ((\text{App} (\text{Lam} \langle x \rangle t_1), t_2), t_3)
\]

- Note that the names and $\lambda$-terms are both drawn represented using the same syntactic class of schematic variables.

- We shall see later how they are actually implemented in the metalanguage.
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1. Motivation—schematic rule-based definitions
2. Brief introduction to MLSOS
3. Translating inductive definitions into MLSOS
4. Conclusions
5. Related & Future work
What is MLSOS?

- A minimal calculus for animating rule-based inductive definitions involving binders.
- A little functional/logic programming language which extends the functionality of FreshML.
- MLSOS offers operations useful for proof-search computation over inductive definitions, e.g.:
  1. support for binders using nominal techniques,
  2. pattern-matching using nominal unification
  3. (with a few extra rules for name inequality),
  4. generation of fresh atoms and metavariables, and
  5. branching constructs for proof-search.
Nominal Arities, $\sigma$ ::= $\alpha$ atom sort,
| $\delta$ data sort,
| $1$ unit type,
| $\sigma_1 \ast \cdots \ast \sigma_n$ $n$-tuple,
| $\langle \langle \alpha \rangle \rangle \sigma$ abstraction type.

Types, $\tau$ ::= $\sigma$ nominal arity,
| ans answer type,
| $\tau \rightarrow \tau'$ function type.
Constraints, $c ::=$

- $v ::= v'$  \hspace{1cm} \text{equality constraint,}
- $a # v$  \hspace{1cm} \text{freshness constraint,}
- $v =/\ = v'$  \hspace{1cm} \text{name inequality constraint.}$

Values, $v ::=$

- $x$  \hspace{1cm} \text{value identifier,}
- $\pi X$  \hspace{1cm} \text{suspension,}
- ()  \hspace{1cm} \text{unit,}
- $(v_1, \ldots, v_n)$  \hspace{1cm} \text{$n$-tuple,}
- $\text{fun } f(x : \tau) : \tau' = e$  \hspace{1cm} \text{recursive function,}
- yes  \hspace{1cm} \text{success,}
- $K v$  \hspace{1cm} \text{data construction,}
- $a$  \hspace{1cm} \text{atom,}
- $\llangle a \rrangle v$  \hspace{1cm} \text{atom abstraction.}
Expressions, \( e \) ::= \( v \) \hspace{1cm} \text{value,}
| let \( x = e \) in \( e' \) \hspace{1cm} \text{let-binding,}
| \( v \, v' \) \hspace{1cm} \text{function application,}
| \text{fresh } a : \alpha \text{ in } e \hspace{1cm} \text{fresh atom,}
| \text{some } x : \sigma \text{ in } e \hspace{1cm} \text{new unification variable,}
| c \hspace{1cm} \text{constraint,}
| e_1 \text{ or } \cdots \text{ or } e_n \hspace{1cm} \text{\( n \)-ary branch.}

Frame Stacks, \( S \) ::= \( Id \) \hspace{1cm} \text{empty frame stack,}
| S \circ (x. e) \hspace{1cm} \text{non-empty frame stack.}

- \textbf{NB: branches introduce non-determinism.}
Operational semantics

MLSOS evaluation contexts are of the form

$$\forall \overline{a} \exists \overline{X} \ (\overline{c}; S(e)).$$

- We define a binary transition relation $$\rightarrow_M$$ between configurations.
- As we will see, this relation is non-deterministic...
- This is necessary to do proof-search.
Operational semantics

A few selected operational rules:

- $\mathcal{N}a \exists \overline{X} (\overline{c}; S(c)) \rightarrow_{M} \mathcal{N}a \exists \overline{X} ((\overline{c} \cup \{c\}); S(\text{yes}))$
  if $\models \overline{c} \cup \{c\}$

- $\mathcal{N}a \exists \overline{X} (\overline{c}; S(\text{fresh } a : \alpha \text{ in } e)) \rightarrow_{M} \mathcal{N}a, a : \alpha \exists \overline{X} (\overline{c}'; S(e))$
  if $a \notin \text{dom}(\overline{a})$ and $\overline{c}' \triangleq \{ a \# X \mid X \in \text{dom}(\overline{X}) \} \cup \overline{c}$

- $\mathcal{N}a \exists \overline{X} (\overline{c}; S(\text{some } x : \sigma \text{ in } e)) \rightarrow_{M} \mathcal{N}a \exists \overline{X}, X : \sigma (\overline{c}; S(e[l X/x]))$
  if $X \notin \text{dom}(\overline{X})$

- $\mathcal{N}a \exists \overline{X} (\overline{c}; S(e_1 \text{ or } \cdots \text{ or } e_n)) \rightarrow_{M} \mathcal{N}a \exists \overline{X} (\overline{c}; S(e_i))$
  where $i \in \{1, \ldots, n\}$
We define two notions of observation on configurations:

1. \( \forall \bar{a} \exists \bar{X} \ (\bar{c}; S(e)) \downarrow \)
   if some branch of execution leads to a terminal configuration (i.e. \( \forall \bar{a}' \exists \bar{X}' \ (\bar{c}'; Id(v)) \)), where \( \bar{c}' \) is a satisfiable set of constraints).

2. \( \forall \bar{a} \exists \bar{X} \ (\bar{c}; S(e)) \) fails
   if all branches of execution leads to a stuck configuration (i.e. \( \forall \bar{a}' \exists \bar{X}' \ (\bar{c}'; S'(c')) \)), where \( \bar{c}' \cup \{c'\} \) is an unsatisfiable set of constraints).

These mirror the \( \longrightarrow_M \) rules from earlier.
We write (closed) operational equivalence as $\simeq$.

Two closed MLSOS expressions $e$ and $e'$ are operationally equivalent if their termination and failure behaviour is the same in any context $\forall \bar{a}' \exists \bar{X}' (\bar{c}'; S'(\bar{e}'))$, i.e.

\[
\forall \bar{a}' \exists \bar{X}' (\bar{c}'; S'(e)) \downarrow \iff \forall \bar{a}' \exists \bar{X}' (\bar{c}'; S'(e')) \downarrow \\
\forall \bar{a}' \exists \bar{X}' (\bar{c}'; S'(e)) \text{ fails} \iff \forall \bar{a}' \exists \bar{X}' (\bar{c}'; S'(e')) \text{ fails}
\]

both hold.

We extend this to a relation $\simeq^\circ$ on open expressions by closing, ground substitutions for free value identifiers.
CIU and data correctness results

- **CIU theorem:** $\cong^\circ$ has various nice properties, including being an equivalence relation and a congruence.

- All fairly straightforward except for **compatibility**, which means that operational equivalence respects the term-formers of the language.

- **Data correctness:** $\alpha$-equivalent ground terms cannot be distinguished operationally. The main theorem:

  $$\overline{a} \vdash g \cong g' : \sigma \iff \overline{a} \vdash g \approx_\alpha g' : \sigma.$$ 

- The proofs both involve drawn-out operational reasoning.

- These are the proofs I am trying to automate using nominal Isabelle.
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We have now established some fundamental correctness results about the metalanguage.

Next job: say something about the expressiveness of the metalanguage.

**Strategy:** define a formal translation of *nominal inductive definitions* into MLSOS, and prove and prove that the implementations are:

- **Adequate:** if MLSOS reports that a given term is in the relation then it actually is in the relation, and

- **Complete:** the system will find all members of the relation (probably doesn’t hold in general—see later).
Translating rules into MLSOS

Consider the following $\beta$-rule:

$$(\beta) \quad \frac{t'_1[t'_2/x] \equiv t_3}{\text{beta } ((\text{App (Lam } \langle \langle x \rangle \rangle t_1), t_2), t_3)}$$

Things to be done to translate this rule into MLSOS:

1. Generate fresh atoms and unification variables to stand for the pattern variables in the rule.
2. Decide which atoms need to be fresh for which unification variables.
3. Match against the pattern from the conclusion.
4. If successful, recursively process the formulae from the premise.

In more detail...
Translating rules into MLSOS

\[
(\beta) \quad \frac{t'_1[t'_2/x] \equiv t_3}{\text{beta } ((\text{App } (\text{Lam } \langle x \rangle t_1), t_2), t_3)}
\]

1. Generate fresh atoms and unification variables to stand for the pattern variables in the rule.

- Any variable that appears in abstraction position (coloured red above) is implemented using a fresh atom.
- Any other variable is implemented using a unification variable (including those of atom sort).
(β) \[
\frac{t'_1[t'_2/x] \equiv t_3}{\text{beta } ((\text{App } (\text{Lam } \langle \langle x \rangle \rangle t_1), t_2), t_3)}
\]

2 Decide which atoms need to be fresh for which unification variables.

- In this case, the name \(x\) must be constrained to be fresh for the variables coloured in red.
- Names bound in the conclusion should be fresh for the conclusion.
- Names only bound in the premises should be fresh for both the premises and the conclusion.
- These freshness constraints should not effect adequacy, but will affect completeness. Not clear yet whether these are strong enough / too strong...
Translating rules into MLSOS

\[
\frac{t_1'[t_2'/x] \equiv t_3}{\text{beta } ((\text{App } (\text{Lam } \langle x \rangle t_1), t_2), t_3)}
\]

3 Match against the pattern from the conclusion.

- We create a nominal pattern consisting of fresh atoms and metavariables, and use an equality constraint to match against it.

4 If successful, recursively process the formulae from the premise.

- This is just a recursive call to the function \( f_N \) which implements the inductive definition \( \mathcal{N} \).
Adequacy and completeness

- An **adequacy** proof is mostly complete...
- However, **completeness** is more tricky.
- One can think of “bad” rules for which proof search using nominal unification would fail, e.g.:

\[
R (x, t, \langle x \rangle t)
\]

which produces the graph of \(\lambda\)-abstraction.
- When encoded into MLSOS, this would fail because of the freshness constraints in the conclusion (which would seem reasonable...)
- A syntactic criterion on schematic rules is needed to rule out such definitions.
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Proof-search over inductive definitions with binders has raised issues similar to those in theorem-proving, e.g. “VC-compatibility”

The status of these rules, which are equivariant yet still somehow “bad”, needs further investigation

However, we hope that our system will permit a reasonable number of interesting programs to be written!
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Related work

- **FreshML**: Shinwell, Pitts, Gabbay
- **αProlog**: Cheney, Urban
- **Abella, λProlog, Bedwyr** etc: Miller, Baelde et al
- **Nominal Isabelle**: everyone here!
- **PLTRedex**: Findler et al
- **Curry**: Hanus et al
- **Twelf** etc: Pfenning et al
Future work

- Finish soundness and completeness proofs
- Implement the system and program some real examples
- Investigate optimisations in the compilation of rules into MLSOS code
- Compare our system with others...
- ... und endlich muß ich eine Dissertation schreiben!