The Cantor-Schröder-Bernstein Theorem

ℕaproche formalization:

Alexander Holz (2019),
Marcel Schütz (2019 - 2021)

This is a formalization of the proof of the *Cantor-Schröder-Bernstein Theorem*, i.e. of the fact that two sets are equipollent if they can be embedded into each other, based on some version of the *Knaster-Tarski Fixed Point Theorem*.

[readtex `set-theory/sections/02_functions/05_functions-and-set-systems.ftl.tex`]

[readtex `set-theory/sections/02_functions/06_equipollency.ftl.tex`]

Theorem 0.1 (Cantor Schroeder Bernstein). Let x, y be sets. x and y are equipollent iff there exists a function from x into y and there exists a function from y into x.

Proof. Case x and y are equipollent. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence f is a function from x into y and f^{-1} is a function from y into x. End.

Case there exists a function from x into y and there exists a function from y into x. Take a function f from x into y. Take a function g from y into x. We have $y \setminus f[a] \subseteq y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) = x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

h is a function from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Indeed $h(a)$ is a subset of x for each subset a of x.

Let us show that h preserves subsets. Let a, b be subsets of x. Assume $a \subseteq b$. Then $f[a] \subseteq f[b]$. Hence $y \setminus f[b] \subseteq y \setminus f[a]$. Thus $g[y \setminus f[b]] \subseteq g[y \setminus f[a]]$ (by 8.28). Indeed $y \setminus f[b]$ and $y \setminus f[a]$ are subsets of y. Therefore $x \setminus g[y \setminus f[a]] \subseteq x \setminus g[y \setminus f[b]]$. Consequently $h[a] \subseteq h[b]$. End.

Hence we can take a fixed point c of h (by 11.5).

(2) Define $F(u) = f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. g^{-1} is a bijection between range(g) and y. Thus $x \setminus c = g[y \setminus f[c]] \subseteq \text{range}(g)$. Therefore $x \setminus c$ is a subset of

$\mathrm{dom}(g^{-1})$.

(3) Define $G(u) = g^{-1}(u)$ for $u \in x \setminus c$.

F is a bijection between c and $\mathrm{range}(F)$. G is a bijection between $x \setminus c$ and $\mathrm{range}(G)$.

Define

$$H(u) = \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for $u \in x$.

Let us show that H is a function to y. $\mathrm{dom}(H)$ is a set and every value of H is an object. Hence H is a function.

Let us show that every value of H lies in y. Let v be a value of H. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End. End.

Let us show that every element of y is a value of H. Let $v \in y$.

Case $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. End.

Case $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. End. End.

Let us show that H is one to one. Let $u, v \in \mathrm{dom}(H)$. Assume $u \neq v$.

Case $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. End.

Case $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. End.

Case $u \in c$ and $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus f[c]]$. We have $G(v) \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. End.

Case $u \notin c$ and $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus f[c]]$. We have $G(u) \in y \setminus f[c]$. Thus $G(u) \neq F(v)$. End. End.

Hence H is a bijection between x and y. End. $\square$