1 Ordering and exponentiation

[readtex `arithmetic/sections/01_arithmetic/04_exponentiation.ftl.tex`]

[readtex `arithmetic/sections/02_ordering/03_ordering-and-multiplication.ftl.tex`]

Let k, l, m, n denote natural numbers.

To conclude our investigations about the interplay between the ordering and our arithmetical operations, let us have a look at exponentiation.

Proposition 1.1. Assume $k \neq 0$. Then for all n, m we have

$$n < m \iff n^k < m^k.$$

Proof. Define

$$P = \left\{ k' \in \mathbb{N} \,\middle|\, \begin{array}{l} \text{for all natural numbers } n, m \text{ if } n < m \text{ and } k' > 1 \text{ then} \\ n^{k'} < m^{k'} \end{array} \right\}.$$

Let us show that every natural number is contained in P. (BASE CASE 1) P contains 0.

(BASE CASE 2) P contains 1.

(BASE CASE 3) P contains 2.
Proof. Let us show that for all natural numbers n, m if $n < m$ then $n^2 < m^2$. Let n, m be natural numbers. Assume $n < m$.

Case $n = 0$ or $m = 0$. Obvious.

Case $n, m \neq 0$. Then $n \cdot n < n \cdot m < m \cdot m$ (by **??**, **??**). Hence $n^2 = n \cdot n < n \cdot m < m \cdot m = m^2$. End. End. Qed.

(INDUCTION STEP) For all natural numbers k' we have $k' \in P \implies k' + 1 \in P$.
Proof. Let k' be a natural number. Assume $k' \in P$.

For all natural numbers n, m if $n < m$ and $k' + 1 > 1$ then $n^{k'+1} < m^{k'+1}$.
Proof. Let n, m be natural numbers. Assume $n < m$ and $k' + 1 > 1$. Then $n^{k'} < m^{k'}$. Indeed $k' \neq 0$ and $if k' = 1$ then $n^{k'} < m^{k'}$.

Case $k' \leq 1$. Then $k' = 0$ or $k' = 1$. Hence $k' + 1 = 1$ or $k' + 1 = 2$. Thus $k' + 1 \in P$. Therefore $n^{k'+1} < m^{k'+1}$. End.

Case $k' > 1$. Case $n = 0$. Then $m \neq 0$. Hence $n^{k'+1} = 0 < m^{k'} \cdot m = m^{k'+1}$. Thus $n^{k'+1} < m^{k'+1}$. End.

Case $n \neq 0$. Then $n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m$ (by **??**, **??**). Indeed

$n^{k'} < m^{k'} \neq 0$. Hence $n^{k'+1} = n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m = m^{k'+1}$. Thus $n^{k'+1} < m^{k'+1}$ (by **??**). End. End.

Hence the thesis. Indeed $k' \leq 1$ or $k' > 1$. Qed.

$k' + 1 \in P$. Qed.

Therefore every natural number is contained in P. End.

Define

$$Q = \left\{ k' \in \mathbb{N} \;\middle|\; \text{for all natural numbers } n, m \text{ if } n \geq m \text{ then } n^{k'} \geq m^{k'} \right\}.$$

Let us show that every natural number is contained in Q. (BASE CASE) Q contains 0.

(INDUCTION STEP) For all natural numbers k' we have $k' \in Q \implies k' + 1 \in Q$.
Proof. Let k' be a natural number. Assume $k' \in Q$.

For all natural numbers n, m if $n \geq m$ then $n^{k'+1} \geq m^{k'+1}$.
Proof. Let n, m be natural numbers. Assume $n \geq m$. Then $n^{k'} \geq m^{k'}$. Hence $n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m$. Thus $n^{k'+1} = n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m = m^{k'+1}$. Therefore $n^{k'+1} \geq m^{k'+1}$ (by **??**). Qed.

Hence the thesis. Indeed $k' + 1$ is a natural number. Qed.

Thus every natural number is contained in Q. End.

Let n, m be natural numbers.

Case $n < m$. Case $k = 1$. Obvious.

Case$k \neq 1$. Then $k > 1$. Indeed $k < 1$ or $k = 1$ or $k > 1$. Hence $n^k < m^k$. Indeed n and m belong to P. End. End.

Case $n^k < m^k$. Then $n^k \ngeq m^k$. Hence $n \ngeq m$. Indeed n and m are contained in Q. Thus $n < m$. End. ☐

Corollary 1.2. Assume $k \neq 0$. Then

$$n^k = m^k \implies n = m.$$

Proof. Assume $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $n^k < m^k$ (by 1.1). If $m < n$ then $m^k < n^k$. Thus $n^k \neq m^k$. Hence the thesis. ☐

Corollary 1.3. Assume $k \neq 0$. Then

$$n^k \leq m^k \iff n \leq m.$$

Proof. If $n^k < m^k$ then $n < m$. If $n^k = m^k$ then $n = m$.

If $n < m$ then $n^k < m^k$ (by 1.1). If $n = m$ then $n^k = m^k$. ☐

Proposition 1.4. Assume $k > 1$. Then for all n, m we have

$$n < m \iff k^n < k^m.$$

Proof. Define

$$P = \left\{ m \in \mathbb{N} \;\middle|\; \text{for all natural numbers } n \text{ if } k > 1 \text{ and } n < m \text{ then } k^n < k^m \right\}.$$

Let us show that every natural number is contained in P.

(BASE CASE) P contains 0.

(INDUCTION STEP) For all natural numbers m we have $m \in P \implies m + 1 \in P$.
Proof. Let m be a natural number. Assume $m \in P$.

For all natural numbers n if $k > 1$ and $n < m + 1$ then $k^n < k^{m+1}$.
Proof. Let n be natural numbers such that $k > 1$ and $n < m + 1$. Then $n \leq m$. We have $k^m \cdot 1 < k^m \cdot k$. Indeed $k^m \neq 0$. Case $n = m$. Then $k^n = k^m < k^m \cdot k = k^{m+1}$. End.

Case $n < m$. Then $k^n < k^m < k^m \cdot k = k^{m+1}$. End. Qed. Qed.

Hence every natural number is contained in P. End.

Define

$$Q = \left\{ n \in \mathbb{N} \;\middle|\; \text{for all natural numbers } m \text{ if } n \geq m \text{ then } k^n \geq k^m \text{ or } k \leq 1 \right\}.$$

Let us show that every natural number is contained in Q.

(BASE CASE) $0 \in Q$.

(INDUCTION STEP) For all natural numbers n we have $n \in Q \implies n + 1 \in Q$.
Proof. Let n be a natural number. Assume $n \in Q$.

For all natural numbers m if $n + 1 \geq m$ then $k^{n+1} \geq k^m$ or $k \leq 1$.
Proof. Let m be natural numbers. Assume $n + 1 \geq m$.

Case $n + 1 = m$. Obvious.

Case $n + 1 > m$. Then $n \geq m$. Hence $k^n \geq k^m$ or $k \leq 1$.

Case $k \leq 1$. Obvious.

Case $k^n \geq k^m$. We have $k^n \cdot 1 \leq k^n \cdot k$. Indeed $1 \leq k$ and $k^n \neq 0$. Hence $k^m \leq k^n = k^n \cdot 1 \leq k^n \cdot k = k^{n+1}$. End. End. Qed. Qed.

Thus every natural number is contained in Q. End.

Let n, m be natural numbers.

Case $n < m$. Then $k^n < k^m$. Indeed n and m are contained in P. End.

Case $k^n < k^m$. Then it is wrong that $k^n \geq k^m$ or $k \leq 1$. Hence $n \ngeq m$. Indeed n and m are contained in Q. Thus $n < m$. End. $\square$

Corollary 1.5. Assume $k > 1$. Then

$$k^n = k^m \implies n = m.$$

Proof. Assume $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $k^n < k^m$. If $m < n$ then $k^m < k^n$. Thus $k^n \neq k^m$. Hence the thesis. $\square$

Corollary 1.6. Assume $k > 1$. Then

$$n \leq m \iff k^n \leq k^m.$$