1 Subtraction

[readtex `arithmetic/sections/01_arithmetic/03_multiplication.ftl.tex`]

[readtex `arithmetic/sections/02_ordering/01_ordering.ftl.tex`]

Let k, l, m, n denote natural numbers.

The notion of an ordering on the natural numbers enables us to (partially) define an inverse operation of addition, namely the subtraction operation.

Definition 1.1. Let $n \geq m$. $n - m$ is the natural number k such that $n = m + k$.

Let the difference of n and m stand for $n - m$.

As we did for the previously introduced operations let us prove some basic facts about subtraction.

Proposition 1.2. Let $n \geq m$. Then $n - m = 0$ iff $n = m$.

Proof. Case $n - m = 0$. Then $n = (n - m) + m = 0 + m = m$. End.

Case $n = m$. We have $(n - m) + m = n = m = 0 + m$. Hence $n - m = 0$. End. ☐

Corollary 1.3. $n - n = 0$.

Proposition 1.4. $n - 0 = n$.

Proof. We have $n = (n - 0) + 0 = n - 0$. ☐

Proposition 1.5. Let $n \geq m$. Then $n - m \leq n$.

Proof. We have $(n - m) + m = n$. Hence $n - m \leq n$. ☐

Proposition 1.6. Let n be nonzero. $n - 1$ is the direct predecessor of n.

Proof. We have $(n-1)+1 = n = \mathrm{pred}(n)+1$. Hence $n-1 = \mathrm{pred}(n)$. ☐

Proposition 1.7. Let $n > m$. Assume $m \neq 0$. Then $n - m < n$.

Proof. We have $(n - m) + m = n$. Assume $n - m = n$. Then $n + m = (n - m) + m = n = n + 0$. Hence $m = 0$. Contradiction. ☐

Proposition 1.8. Assume $n \geq m$. Then

$$(n - m) + k = (n + k) - m.$$

Proof. Assume $n \geq m$. We have

$$\begin{aligned} &((n-m)+k)+m \\ &= ((n-m)+m)+k \\ &= n+k \\ &= ((n+k)-m)+m. \end{aligned}$$

Hence $(n-m)+k = (n+k)-m$. ∎

Proposition 1.9. Assume $n \geq m+k$. Then

$$(n-m)-k = n-(m+k).$$

Proof. We have

$$\begin{aligned} &((n-m)-k)+(m+k) \\ &= (((n-m)-k)+k)+m \\ &= (n-m)+m \\ &= n \\ &= (n-(m+k))+(m+k). \end{aligned}$$

Hence $(n-m)-k = n-(m+k)$. ∎

Proposition 1.10. Let $n \geq m$. Then

$$(n-m)\cdot k = (n\cdot k)-(m\cdot k).$$

Proof. We have

$$\begin{aligned} &((n-m)\cdot k)+(m\cdot k) \\ &= ((n-m)+m)\cdot k \\ &= n\cdot k \\ &= ((n\cdot k)-(m\cdot k))+(m\cdot k). \end{aligned}$$

Hence $(n-m)\cdot k = (n\cdot k)-(m\cdot k)$. ∎