1 The axiom of regularity

[readtex `set-theory/sections/01_sets/01_sets.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets.

The *axiom of regularity* (or *axiom of foundation*) states that every non-empty set has a $\in$-minimal element.

Axiom 1.1 (Regularity). Every nonempty set x that contains some set contains some set y such that x and y are disjoint.

As a consequence we get that no set can contain itself. Moreover, this allows us to show that there exists no universal set, i.e. that "the set of all sets" does not exist.

Proposition 1.2. No set x is an element of x.

Proof. Assume the contrary. Take a set x such that $x \in x$. We can take an element y of $\{x\}$ such that $\{x\}$ and y are disjoint (by Regularity). Indeed $\{x\}$ contains some set. Then $y = x$. Hence $\{x\}$ and x are disjoint. Contradiction. Indeed $x \in \{x\}$ and $x \in x$. $\square$

Corollary 1.3. There is no set that contains every set.

Proof. Assume the contrary. Take a set V that contains every set. Then V is an element of V. Contradiction. $\square$

Proposition 1.4. There exist no sets x, y such that $x \in y$ and $y \in x$.

Proof. Assume the contrary. Take sets x, y such that $x \in y$ and $y \in x$. Consider an element z of $\{x, y\}$ such that $\{x, y\}$ and z are disjoint (by Regularity). Indeed $\{x, y\}$ contains some set. We have $z = x$ or $z = y$.

Case $z = x$. Then x and $\{x, y\}$ are disjoint. Hence $y \notin x$. Contradiction. End.

Case $z = y$. Then y and $\{x, y\}$ are disjoint. Hence $x \notin y$. Contradiction. End. $\square$