1 Functions and the symmetric difference

[readtex `set-theory/sections/01_sets/04_symmetric-difference.ftl.tex`]

[readtex `set-theory/sections/02_functions/02_image-and-preimage.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets. Let f, g, h denote functions.

In this paragraph we will briefly examine the behaviour of the image and preimage of a function with respect to the symmetric difference.

Proposition 1.1. Let f be a function from x to y and $a, a' \subseteq x$. Then

$$f[a \triangle a'] \supseteq f[a] \triangle f[a'].$$

Proof. Let $v \in f[a] \triangle f[a']$. We have $f[a] \triangle f[a'] = (f[a] \cup f[a']) \setminus (f[a] \cap f[a'])$. Hence $v \in f[a] \cup f[a']$ and $v \notin f[a] \cap f[a']$. We have $f[a] \cup f[a'] = f[a \cup a']$ (by **??**).

Thus we can take $u \in a \cup a'$ such that $v = f(u)$.

Let us show that $u \notin a \cap a'$. Assume the contrary. Then $v = f(u) \in f[a \cap a']$. We have $f[a \cap a'] \subseteq f[a] \cap f[a']$. Hence $v \in f[a] \cap f[a']$. Contradiction. End.

Thus $u \in a \triangle a'$. Therefore $v \in f[a \triangle a']$. □

Proposition 1.2. Let f be a function from x to y and $b, b' \subseteq y$. Then

$$f^{-}[b \triangle b'] \supseteq f^{-}[b] \triangle f^{-}[b'].$$

Proof. Let $u \in f^{-}[b] \triangle f^{-}[b']$. Then $u \in f^{-}[b] \cup f^{-}[b']$ and $u \notin f^{-}[b] \cap f^{-}[b']$. We have $f^{-}[b] \cup f^{-}[b'] = f^{-}[b \cup b']$. Hence we can take $v \in b \cup b'$ such that $f(u) = v$.

Let us show that $v \notin b \cap b'$. Assume the contrary. Then $v = f(u) \in b \cap b'$. Hence $u \in f^{-}[b \cap b'] = f^{-}[b] \cap f^{-}[b']$. Contradiction. End.

Therefore $v \in b \triangle b'$. Hence $u \in f^{-}[b \triangle b']$. □