1 The symmetric difference

[readtex `set-theory/sections/01_sets/01_sets.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets.

It is time to indroduce a new operation on sets: The *symmetric difference*.

1.1 Definition

The symmetric difference of two sets is the set of all objects wich are contained in *exactly one* of these sets.

Definition 1.1. $x \triangle y = (x \cup y) \setminus (x \cap y)$.

Let the symmetric difference of x and y stand for $x \triangle y$.

Lemma 1.2. $x \triangle y$ is a set.

Proof. x and y are sets. Hence $x \cup y$ and $x \cap y$ are sets. Thus $(x \cup y) \setminus (x \cap y)$ is a set. Therefore $x \triangle y$ is a set. $\square$

Alternatively, we could have defined the symmetric difference as follows:

Proposition 1.3. $x \triangle y = (x \setminus y) \cup (y \setminus x)$.

Proof. Let us show that $x \triangle y \subseteq (x \setminus y) \cup (y \setminus x)$. Let $u \in x \triangle y$. Then $u \in x \cup y$ and $u \notin x \cap y$. Hence ($u \in x$ or $u \in y$) and not ($u \in x$ and $u \in y$). Thus ($u \in x$ or $u \in y$) and ($u \notin x$ or $u \notin y$). Therefore if $u \in x$ then $u \notin y$. If $u \in y$ then $u \notin x$. Then we have ($u \in x$ and $u \notin y$) or ($u \in y$ and $u \notin x$). Hence $u \in x \setminus y$ or $u \in y \setminus x$. Thus $u \in (x \setminus y) \cup (y \setminus x)$. End.

Let us show that $((x \setminus y) \cup (y \setminus x)) \subseteq x \triangle y$. Let $u \in (x \setminus y) \cup (y \setminus x)$. Then ($u \in x$ and $u \notin y$) or ($u \in y$ and $u \notin x$). If $u \in x$ and $u \notin y$ then $u \in x \cup y$ and $u \notin x \cap y$. If $u \in y$ and $u \notin x$ then $u \in x \cup y$ and $u \notin x \cap y$. Hence $u \in x \cup y$ and $u \notin x \cap y$. Thus $u \in (x \cup y) \setminus (x \cap y) = x \triangle y$. End. $\square$

1.2 Computation laws

As we did with our previously introduced set operations let us prove some of the most important algebraic properties of the symmetric difference.

Commutativity:

Proposition 1.4.

$$x \triangle y = y \triangle x.$$

Proof. $x \triangle y = (x \cup y) \setminus (x \cap y) = (y \cup x) \setminus (y \cap x) = y \triangle x$. $\square$

Associativity:

Proposition 1.5.

$$((x \triangle y) \triangle z) = x \triangle (y \triangle z).$$

Proof. Take $A = (((x \setminus y) \cup (y \setminus x)) \setminus z) \cup (z \setminus ((x \setminus y) \cup (y \setminus x)))$.

Take $B = (x \setminus ((y \setminus z) \cup (z \setminus y))) \cup (((y \setminus z) \cup (z \setminus y)) \setminus x)$.

We have $x \triangle y = (x \setminus y) \cup (y \setminus x)$ and $y \triangle z = (y \setminus z) \cup (z \setminus y)$. Hence $(x \triangle y) \triangle z = A$ and $x \triangle (y \triangle z) = B$.

Let us show that (A) $A \subseteq B$. Let $u \in A$.

(A 1) Case $u \in ((x \setminus y) \cup (y \setminus x)) \setminus z$. Then $u \notin z$.

(A 1a) Case $u \in x \setminus y$. Then $u \notin y \setminus z$ and $u \notin z \setminus y$. $u \in x$. Hence $u \in x \setminus ((y \setminus z) \cup (z \setminus y))$. Thus $u \in B$. End.

(A 1b) Case $u \in y \setminus x$. Then $u \in y \setminus z$. Hence $u \in (y \setminus z) \cup (z \setminus y)$. $u \notin x$. Thus $u \in ((y \setminus z) \cup (z \setminus y)) \setminus x$. Therefore $u \in B$. End. End.

(A 2) Case $u \in z \setminus ((x \setminus y) \cup (y \setminus x))$. Then $u \in z$. $u \notin x \setminus y$ and $u \notin y \setminus x$. Hence not ($u \in x \setminus y$ or $u \in y \setminus x$). Thus not (($u \in x$ and $u \notin y$) or ($u \in y$ and $u \notin x$)). Therefore ($u \notin x$ or $u \in y$) and ($u \notin y$ or $u \in x$).

(A 2a) Case $u \in x$. Then $u \in y$. Hence $u \notin (y \setminus z) \cup (z \setminus y)$. Thus $u \in x \setminus ((y \setminus z) \cup (z \setminus y))$. Therefore $u \in B$. End.

(A 2b) Case $u \notin x$. Then $u \notin y$. Hence $u \in z \setminus y$. Thus $u \in (y \setminus z) \cup (z \setminus y)$. Therefore $u \in ((y \setminus z) \cup (z \setminus y)) \setminus x$. Then we have $u \in B$. End. End. End.

Let us show that (B) $B \subseteq A$. Let $u \in B$.

(B 1) Case $u \in x \setminus ((y \setminus z) \cup (z \setminus y))$. Then $u \in x$. $u \notin y \setminus z$ and $u \notin z \setminus y$. Hence not ($u \in y \setminus z$ or $u \in z \setminus y$). Thus not (($u \in y$ and $u \notin z$) or ($u \in z$ and $u \notin y$)). Therefore ($u \notin y$ or $u \in z$) and ($u \notin z$ or $u \in y$).

(B 1a) Case $u \in y$. Then $u \in z$. $u \notin x \setminus y$ and $u \notin y \setminus x$. Hence $u \notin (x \setminus y) \cup (y \setminus x)$. Thus $u \in z \setminus ((x \setminus y) \cup (y \setminus x))$. Therefore $u \in A$. End.

(B 1b) Case $u \notin y$. Then $u \notin z$. $u \in x \setminus y$. Hence $u \in (x \setminus y) \cup (y \setminus x)$. Thus $u \in ((x \setminus y) \cup (y \setminus x)) \setminus z$. Therefore $u \in A$. End. End.

(B 2) Case $u \in ((y \setminus z) \cup (z \setminus y)) \setminus x$. Then $u \notin x$.

(B 2a) Case $u \in y \setminus z$. Then $u \in y \setminus x$. Hence $u \in (x \setminus y) \cup (y \setminus x)$. Thus $u \in ((x \setminus y) \cup (y \setminus x)) \setminus z$. Therefore $u \in A$. End.

(B 2b) Case $u \in z \setminus y$. Then $u \in z$. $u \notin x \setminus y$ and $u \notin y \setminus x$. Hence $u \notin (x \setminus y) \cup (y \setminus x)$. Thus $u \in z \setminus ((x \setminus y) \cup (y \setminus x))$. Therefore $u \in A$. End. End. End. $\square$

Distributivity of intersection and symmetric difference:

Proposition 1.6.

$$x \cap (y \triangle z) = (x \cap y) \triangle (x \cap z).$$

Proof. $x \cap (y \triangle z) = x \cap ((y \setminus z) \cup (z \setminus y)) = (x \cap (y \setminus z)) \cup (x \cap (z \setminus y))$.

$x \cap (y \setminus z) = (x \cap y) \setminus (x \cap z)$. $x \cap (z \setminus y) = (x \cap z) \setminus (x \cap y)$.

Hence $x \cap (y \triangle z) = ((x \cap y) \setminus (x \cap z)) \cup ((x \cap z) \setminus (x \cap y)) = (x \cap y) \triangle (x \cap z)$. □

Miscellaneous rules:

Proposition 1.7.

$$x \subseteq y \iff x \triangle y = y \setminus x.$$

Proof. Case $x \subseteq y$. Then $x \cup y = y$ and $x \cap y = x$. Hence the thesis. End.

Case $x \triangle y = y \setminus x$. Let $u \in x$. Then $u \notin y \setminus x$. Hence $u \notin x \triangle y$. Thus $u \notin x \cup y$ or $u \in x \cap y$. Indeed $x \triangle y = (x \cup y) \setminus (x \cap y)$. If $u \notin x \cup y$ then we have a contradiction. Therefore $u \in x \cap y$. Then we have the thesis. End. □

Proposition 1.8.

$$x \triangle y = x \triangle z \iff y = z.$$

Proof. Case $x \triangle y = x \triangle z$.

Let us show that $y \subseteq z$. Let $u \in y$.

Case $u \in x$. Then $u \notin x \triangle y$. Hence $u \notin x \triangle z$. Therefore $u \in x \cap z$. Indeed $x \triangle z = (x \cup z) \setminus (x \cap z)$. Hence $u \in z$. End.

Case $u \notin x$. Then $u \in x \triangle y$. Indeed $u \in x \cup y$ and $u \notin x \cap y$. Hence $u \in x \triangle z$. Thus $u \in x \cup z$ and $u \notin x \cap z$. Therefore $u \in x$ or $u \in z$. Then we have the thesis. End. End.

Let us show that $z \subseteq y$. Let $u \in z$.

Case $u \in x$. Then $u \notin x \triangle z$. Hence $u \notin x \triangle y$. Therefore $u \in x \cap y$. Indeed $u \notin x \cup y$ or $u \in x \cap y$. Hence $u \in y$. End.

Case $u \notin x$. Then $u \in x \triangle z$. Indeed $u \in x \cup z$ and $u \notin x \cap z$. Hence $u \in x \triangle y$. Thus $u \in x \cup y$ and $u \notin x \cap y$. Therefore $u \in x$ or $u \in y$. Then we have the thesis. End. End. End. □

Proposition 1.9.

$$x \triangle x = \emptyset.$$

Proof. $x \triangle x = (x \cup x) \setminus (x \cap x) = x \setminus x = \emptyset$. □

Proposition 1.10.

$$x \triangle \emptyset = x.$$

Proof. $x \triangle \emptyset = (x \cup \emptyset) \setminus (x \cap \emptyset) = x \setminus \emptyset = x.$ □

Proposition 1.11.

$$x = y \iff x \triangle y = \emptyset.$$

Proof. Case $x = y$. Then $x \triangle y = (x \cup x) \setminus (x \cap x) = x \setminus x = \emptyset$. Hence the thesis. End.

Case $x \triangle y = \emptyset$. Then $(x \cup y) \setminus (x \cap y)$ is empty. Hence every element of $x \cup y$ is an element of $x \cap y$. Thus for all elements u if $u \in x$ or $u \in y$ then $u \in x$ and $u \in y$. Therefore every element of x is an element of y. Every element of y is an element of x. Then we have the thesis. End. □