1 Ordering

[readtex `arithmetic/sections/01_arithmetic/02_addition.ftl.tex`]

Let k, l, m, n denote natural numbers.

In this section we will establish an order on the natural numbers.

1.1 Definitions and immediate consequences

A natural number m should be greater than a natural number n if m can be reached from n by iteratively applying the successor operation to n. Or, in other words, if m can be reached by adding a nonzero natural number to m.

Definition 1.1. $n < m$ iff there exists a nonzero natural number k such that $m = n + k$.

Let n is less than m stand for $n < m$. Let $n > m$ stand for $m < n$. Let n is greater than m stand for $n > m$. Let $n \not< m$ stand for n is not less than m. Let $n \not> m$ stand for n is not greater than m. Let n is positive stand for $n > 0$.

Definition 1.2. $n \leq m$ iff there exists a natural number k such that $m = n + k$.

Let n is less than or equal to m stand for $n \leq m$. Let $n \geq m$ stand for $m \leq n$. Let n is greater than or equal to m stand for $n \geq m$. Let $n \nleq m$ stand for n is not less than or equal to m. Let $n \ngeq m$ stand for n is not greater than or equal to m.

Proposition 1.3. $n \leq m$ iff $n < m$ or $n = m$.

Proof. Case $n \leq m$. Take a natural number k such that $m = n + k$. If $k = 0$ then $n = m$. If $k \neq 0$ then $n < m$. End.

Case $n < m$ or $n = m$. If $n < m$ then there is a positive natural number k such that $m = n + k$. If $n = m$ then $m = n + 0$. Thus if $n < m$ then there is a natural number k such that $m = n + k$. Hence the thesis. End. $\square$

This relation enables us to to generalize the notions of direct predecessors and successors:

Definition 1.4. A predecessor of n is a natural number that is less than n.

Definition 1.5. A successor of n is a natural number that is greater than n.

A direct consequence of the definition of our ordering relation is that the terms "positive" and "non-zero"coincide on the natural numbers.

Proposition 1.6. n is positive iff n is nonzero.

Proof. Case n is positive. Take a positive natural number k such that $n = 0 + k = k$. Then we have $n \neq 0$. End.

Case n is nonzero. Take a natural number k such that $n = k + 1$. Then $n = 0 + (k + 1)$. $k + 1$ is positive. Hence $0 < n$. End. ☐

1.2 Basic properties

Let us now prove some basic relational properties of the ordering.

Proposition 1.7. $n \not< n$.

Proof. Assume the contrary. Then we can take a positive natural number k such that $n = n + k$. Then we have $0 = k$. Contradiction. ☐

Proposition 1.8. If $n < m$ then $n \neq m$.

Proof. Assume $n < m$. Take a positive natural number k such that $m = n + k$. If $n = m$ then $k = 0$. Hence $n \neq m$. ☐

Proposition 1.9. If $n \leq m$ and $m \leq n$ then $n = m$.

Proof. Assume $n \leq m$ and $m \leq n$. Take natural numbers k, l such that $m = n + k$ and $n = m + l$. Then $m = (m + l) + k = m + (l + k)$. Hence $l + k = 0$. Therefore $l = 0 = k$. Then we have the thesis. ☐

Proposition 1.10. If $n < m < k$ then $n < k$.

Proof. Assume $n < m < k$. Take a positive natural number a such that $m = n + a$. Take a positive natural number b such that $k = m + b$. Then $k = (n + a) + b = n + (a + b)$. $a + b$ is positive. Hence $n < k$. ☐

Proposition 1.11. If $n \leq m \leq k$ then $n \leq k$.

Proof. Case $n = m = k$. Obvious.

Case $n = m < k$. Obvious.

Case $n < m = k$. Obvious.

Case $n < m < k$. Obvious. ☐

Proposition 1.12. If $n \leq m < k$ then $n < k$.

Proof. Assume $n \leq m < k$. If $n = m$ then $n < k$. If $n < m$ then $n < k$. ☐

Proposition 1.13. If $n < m \leq k$ then $n < k$.

Proof. Assume $n < m \leq k$. If $m = k$ then $n < k$. If $m < k$ then $n < k$. ☐

Proposition 1.14. If $n < m$ then $n + 1 \leq m$.

Proof. Assume $n < m$. Take a positive natural number k such that $m = n + k$.

Case $k = 1$. Then $m = n + 1$. Hence $n + 1 \leq m$. End.

Case $k \neq 1$. Then we can take a natural number l such that $k = l + 1$. Then $m = n + (l + 1) = (n + l) + 1 = (n + 1) + l$. l is positive. Thus $n + 1 < m$. End. $\square$

Proposition 1.15. For all n, m we have $n < m$ or $n = m$ or $n > m$.

Proof. Define

$$P = \left\{ m \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{for all natural numbers } n \text{ we have } n < m \text{ or } n = m \text{ or} \\ n > m \end{array} \right\}.$$

(BASE CASE) P contains 0.

(INDUCTION STEP) For all natural numbers m we have $m \in P \implies m + 1 \in P$.
Proof. Let m be a natural number. Assume $m \in P$.

For all natural numbers n we have $n < m + 1$ or $n = m + 1$ or $n > m + 1$.
Proof. Let n be a natural number.

Case $n < m$. Obvious.

Case $n = m$. Obvious.

Case $n > m$. Take a positive natural number k such that $n = m + k$.

Case $k = 1$. Obvious.

Case$k \neq 1$. Take a natural number l such that $n = (m + 1) + l$. Hence $n > m + 1$. Indeed l is positive. End. End. Qed. Qed.

Thus every natural number is contained in P. $\square$

Proposition 1.16. $n \nless m$ iff $n \geq m$.

Proof. Case $n \nless m$. Then $n = m$ or $n > m$. Hence $n \geq m$. End.

Case $n \geq m$. Assume $n < m$. Then $n \leq m$. Hence $n = m$. Contradiction. End. $\square$

1.3 Ordering and successors

We end this section by showing that there are no natural numbers between n and $n + 1$.

Proposition 1.17. If $n < m \leq n + 1$ then $m = n + 1$.

Proof. Assume $n < m \leq n + 1$. Take a positive natural number k such that $m = n + k$. Take a natural number l such that $n + 1 = m + l$. Then $n + 1 = m + l = (n + k) + l = n + (k + l)$. Hence $k + l = 1$.

We have $l = 0$.
Proof. Assume the contrary. Then $k, l > 0$.

Case $k, l = 1$. Then $k + l = 2 \neq 1$. Contradiction. End.

Case $k = 1 and l \neq 1$. Then $l > 1$. Hence $k + l > 1 + l > 1$. Contradiction. End.

Case $k \neq 1 and l = 1$. Then $k > 1$. Hence $k + l > k + 1 > 1$. Contradiction. End.

Case $k, l \neq 1$. Take natural numbers a, b such that $k = a + 1$ and $l = b + 1$. Indeed $k, l \neq 0$. Hence $k = a + 1$ and $l = b + 1$. Thus $k, l > 1$. Indeed a, b are positive. End. Qed.

Then we have $n + 1 = m + l = m + 0 = m$. $\square$

Proposition 1.18. If $n \leq m < n + 1$ then $n = m$.

Proof. Assume $n \leq m < n + 1$.

Case $n = m$. Obvious.

Case $n < m$. Then $n < m \leq n + 1$. Hence $n = m$. End. $\square$

Corollary 1.19. There is no natural number m such that $n < m < n + 1$.

Proof. Assume the contrary. Take a natural number m such that $n < m < n + 1$. Then $n < m \leq n + 1$ and $n \leq m < n + 1$. Hence $m = n + 1$ and $m = n$ (by 1.17, 1.18). Hence $n = n + 1$. Contradiction. $\square$

Proposition 1.20. $n + 1 \geq 1$.

Proof. Case $n = 0$. Obvious.

Case $n \neq 0$. Then $n > 0$. Hence $n + 1 > 0 + 1 = 1$. End. $\square$