1 Exponentiation

[readtex `arithmetic/sections/01_arithmetic/03_multiplication.ftl.tex`]

Let k, l, m, n denote natural numbers.

1.1 Axioms

Another common operation on the natural numbers is exponentiation. Again, we introduce it as an inductively defined operation.

Signature 1.1. n^m is a natural number.

Let the square of n stand for n^2. Let the cube of n stand for n^3.

Axiom 1.2 (1st exponentiation axiom). $n^0 = 1$.

Axiom 1.3 (2nd exponentiation axiom). $n^{m+1} = n^m \cdot n$.

1.2 Computation laws

As in the previous sections let us prove some basic arithmetical properties of our new operation.

Exponentiation with 0, 1 and 2

Proposition 1.4. Assume that $n \neq 0$. Then

$$0^n = 0.$$

Proof. Take a natural number m such that $n = m + 1$. Then

$$0^n = 0^{m+1} = 0^m \cdot 0 = 0.$$

□

Proposition 1.5. For all natural numbers n we have

$$1^n = 1.$$

Proof. Define

$$P = \{ n \in \mathbb{N} \mid 1^n = 1 \}.$$

(BASE CASE) P contains 0.

(INDUCTION STEP) For all natural numbers n we have $n \in P \implies n + 1 \in P$.

Proof. Let n be a natural number. Assume $n \in P$. Then

$1^{n+1} = 1^n \cdot 1 = 1 \cdot 1 = 1$. Qed.

Hence every natural number is contained in P . ☐

Proposition 1.6. $n^1 = n$.

Proof. $n^1 = n^{0+1} = n^0 \cdot n = 1 \cdot n = n$. ☐

Proposition 1.7. $n^2 = n \cdot n$.

Proof. $n^2 = n^{1+1} = n^1 \cdot n = n \cdot n$. ☐

Sums as exponents:

Proposition 1.8. For all n, m, k we have

$$k^{n+m} = k^n \cdot k^m.$$

Proof. Define

$P = \{ \, k \in \mathbb{N} \mid n^{m+k} = n^m \cdot n^k \text{ for all natural numbers } n, m \quad \}$.

(BASE CASE) P contains 0.
Proof. Let us show that for all natural numbers n, m we have $n^{m+0} = n^m \cdot n^0$. Let n, m be natural numbers. Then

$n^{m+0} = n^m = n^m \cdot 1 = n^m \cdot n^0$. End. Qed.

(INDUCTION STEP) For all natural numbers k we have $k \in P \implies k + 1 \in P$.
Proof. Let k be a natural number. Assume $k \in P$.

Let us show that for all natural numbers n, m we have $n^{m+(k+1)} = n^m \cdot n^{k+1}$. Let n, m be natural numbers. Then

$$\begin{aligned}
&n^{m+(k+1)} \\
&= n^{(m+k)+1} \\
&= n^{m+k} \cdot n \\
&= (n^m \cdot n^k) \cdot n \\
&= n^m \cdot (n^k \cdot n) \\
&= n^m \cdot n^{k+1}.
\end{aligned}$$

End. Qed.

Hence every natural number is contained in P. ☐

Products as exponents:

Proposition 1.9. For all n, m, k we have

$$k^{n \cdot m} = (k^n)^m.$$

Proof. Define

$$P = \left\{ k \in \mathbb{N} \;\middle|\; n^{m \cdot k} = (n^m)^k \text{ for all natural numbers } n, m \right\}.$$

(BASE CASE) P contains 0. Indeed $(n^m)^0 = 1 = n^0 = n^{m \cdot 0}$ for all natural numbers n, m.

(INDUCTION STEP) For all natural numbers k we have $k \in P \implies k + 1 \in P$.
Proof. Let k be a natural number. Assume $k \in P$.

For all natural numbers n, m we have $(n^m)^{k+1} = n^{m \cdot (k+1)}$.
Proof. Let n, m be natural numbers. Then

$$\begin{aligned} &(n^m)^{k+1} \\ &= (n^m)^k \cdot n^m \\ &= n^{m \cdot k} \cdot n^m \\ &= n^{(m \cdot k)+m} \\ &= n^{m \cdot (k+1)}. \end{aligned}$$

Qed. Qed.

Therefore every natural number is contained in P. $\square$

Products as base:

Proposition 1.10. For all natural numbers n, m, k we have

$$((n \cdot m)^k) = n^k \cdot m^k.$$

Proof. Define

$$P = \{ k \in \mathbb{N} \mid (n \cdot m)^k = n^k \cdot m^k \text{ for all natural numbers } n, m \quad \}.$$

(BASE CASE) P contains 0. Indeed $((n \cdot m)^0) = 1 = 1 \cdot 1 = n^0 \cdot m^0$ for all natural numbers n, m.

(INDUCTION STEP) For all natural numbers k we have $k \in P \implies k + 1 \in P$.
Proof. Let k be a natural number. Assume $k \in P$.

$((n \cdot m)^{k+1}) = n^{k+1} \cdot m^{k+1}$ for all natural numbers n, m.
Proof. Let n, m be natural numbers.

(Claim) We have

$$\begin{aligned}
&(n^k \cdot m^k) \cdot (n \cdot m) \\
&= ((n^k \cdot m^k) \cdot n) \cdot m \\
&= (n^k \cdot (m^k \cdot n)) \cdot m \\
&= (n^k \cdot (n \cdot m^k)) \cdot m \\
&= ((n^k \cdot n) \cdot m^k) \cdot m \\
&= (n^k \cdot n) \cdot (m^k \cdot m).
\end{aligned}$$

Hence

$$\begin{aligned}
&(n \cdot m)^{k+1} \\
&= (n \cdot m)^k \cdot (n \cdot m) \\
&= (n^k \cdot m^k) \cdot (n \cdot m) \\
&= (n^k \cdot n) \cdot (m^k \cdot m) \\
&= n^{k+1} \cdot m^{k+1}.
\end{aligned}$$

Qed. Qed.

Therefore every natural number is contained in P. □

Zeroes of exponentiation:

Proposition 1.11. For all n, m we have

$$n^m = 0 \iff (n = 0 \text{ and } m \neq 0).$$

Proof. (1) For all n, m if $n^m = 0$ then $n = 0$ and $m \neq 0$.
Proof. Define

$$P = \left\{ m \in \mathbb{N} \;\middle|\; \text{for all natural numbers } n \text{ if } n^m = 0 \text{ then } n = 0 \text{ and } m \neq 0 \right\}.$$

(BASE CASE) P contains 0. Indeed for all natural numbers n if $n^0 = 0$ then we have a contradiction.

(INDUCTION STEP) For all natural numbers m we have $m \in P \implies m + 1 \in P$.
Proof. Let m be a natural number. Assume $m \in P$.

For all natural numbers n if $n^{m+1} = 0$ then $n = 0$ and $m + 1 \neq 0$.
Proof. Let n be a natural number. Assume $n^{m+1} = 0$. Then $0 = n^{m+1} = n^m \cdot n$. Hence $n^m = 0$ or $n = 0$. We have $m + 1 \neq 0$ and if $n^m = 0$ then $n = 0$. Hence the thesis. Qed. Qed.

Thus every natural number is contained in P. Qed.

(2) For all n, m if $n = 0$ and $m \neq 0$ then $n^m = 0$.
Proof. Let n, m be natural numbers. Assume $n = 0$ and $m \neq 0$. Take a natural number k such that $m = k + 1$. Then

$$
\begin{aligned}
& n^m \\
&= n^{k+1} \\
&= n^k \cdot n \\
&= 0^k \cdot 0 \\
&= 0.
\end{aligned}
$$

Qed. $\square$