1 Functions and set-systems

[readtex `set-theory/sections/01_sets/02_powerset.ftl.tex`]

[readtex `set-theory/sections/02_functions/01_functions.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets. Let f, g, h denote functions.

When dealing with set-systems, we might want to consider functions wich preserve the order given by the $\subseteq$-relation on these set-systems.

Definition 1.1. A function between systems of sets is a function f such that f is a function from X to Y for some systems of sets X, Y.

Definition 1.2. Let f be a function between systems of sets. f preserves subsets iff for all $x, y \in \mathrm{dom}(f)$ if $x \subseteq y$ then $f(x) \subseteq f(y)$.

Definition 1.3. Let f be a function between systems of sets. f preserves supersets iff for all $x, y \in \mathrm{dom}(f)$ if $x \supseteq y$ then $f(x) \supseteq f(y)$.

Lemma 1.4. Let f be a function between systems of sets. Then f preserves subsets iff f preserves supersets.

Proof. Case f preserves subsets. Let $x, y \in \mathrm{dom}(f)$. Assume $x \supseteq y$. Then $y \subseteq x$. Hence $f(y) \subseteq f(x)$. Thus $f(x) \supseteq f(y)$. End.

Case f preserves supersets. Let $x, y \in \mathrm{dom}(f)$. Assume $x \subseteq y$. Then $y \supseteq x$. Hence $f(y) \supseteq f(x)$. Thus $f(x) \subseteq f(y)$. End. $\square$

A famous result about order-preserving functions is the *Knaster-Tarski fixed point theorem*:

Theorem 1.5 (Knaster-Tarski). Let h be a function from $\mathcal{P}(x)$ to $\mathcal{P}(x)$ that preserves subsets. Then h has a fixed point.

Proof. (1) Define $A = \{ y \mid y \subseteq x \text{ and } y \subseteq h(y) \}$. Then A is a subset of $\mathcal{P}(x)$ (by **??**). We have $\bigcup A \in \mathcal{P}(x)$.

Let us show that (2) $\bigcup A \subseteq h(\bigcup A)$. Let $u \in \bigcup A$. Take $y \in A$ such that $u \in y$. Then $u \in h(y)$. We have $y \subseteq \bigcup A$. Hence $h(y) \subseteq h(\bigcup A)$. Thus $h(y) \subseteq h(\bigcup A)$. Therefore $u \in h(\bigcup A)$. End.

Then $h(\bigcup A) \in A$ (by 1). Indeed $h(\bigcup A) \subseteq x$. (3) Hence $h(\bigcup A) \subseteq \bigcup A$. Indeed every element of $h(\bigcup A)$ is an element of some element of A.

Thus $h(\bigcup A) = \bigcup A$ (by 2, 3). $\square$