# 1 Standard exercises

[readtex arithmetic/sections/01_arithmetic/04_exponentiation.ftl.tex]

[readtex arithmetic/sections/01_arithmetic/05_factorial.ftl.tex]

[readtex arithmetic/sections/02_ordering/04_ordering-and-exponentiation.ftl.tex]

[readtex arithmetic/sections/02_ordering/05_induction.ftl.tex]

Let $k, l, m, n$ denote natural numbers.

In this section we will have a look some standard text book exercises on induction and prove them within our arithmetic.

**Proposition 1.1.** We have

$$(n+1)^2 = (n^2 + (2 \cdot n)) + 1.$$

*Proof.* We have

$$\begin{aligned}
&(n+1)^2 \\
&= (n+1) \cdot (n+1) \\
&= ((n+1) \cdot n) + (n+1) \\
&= ((n \cdot n) + n) + (n+1) \\
&= (n^2 + n) + (n+1) \\
&= ((n^2 + n) + n) + 1 \\
&= (n^2 + (n+n)) + 1 \\
&= (n^2 + (2 \cdot n)) + 1.
\end{aligned}$$

□

**Proposition 1.2.** For all $n$ if $n \geq 3$ then

$$n^2 > (2 \cdot n) + 1.$$

*Proof.* Define

$$P = \left\{ n \in \mathbb{N} \mid n^2 > (2 \cdot n) + 1 \right\}.$$

(BASE CASE) $P$ contains 3.

(INDUCTION STEP) For all natural numbers $n$ such that $n \geq 3$ we have $n \in P \implies n+1 \in P$.
Proof. Let $n$ be a natural number. Suppose $n \geq 3$. Assume $n \in P$.

$(n^2 + (2 \cdot n)) + 1 > (((2 \cdot n) + 1) + (2 \cdot n)) + 1$. Indeed $n^2 + (2 \cdot n) > ((2 \cdot n) + 1) + (2 \cdot n)$.

$(2 \cdot (n + n)) + 1 > (2 \cdot (n + 1)) + 1$. Indeed $2 \cdot (n + n) > 2 \cdot (n + 1)$. Indeed $n + n > n + 1$ and $2 \neq 0$.

Hence

$$\begin{gathered}(n+1)^2 \\ = (n^2 + (2 \cdot n)) + 1 \\ > (((2 \cdot n) + 1) + (2 \cdot n)) + 1 \\ > ((2 \cdot n) + (2 \cdot n)) + 1 \\ = (2 \cdot (n + n)) + 1 \\ > (2 \cdot (n + 1)) + 1.\end{gathered}$$

Thus $(n + 1)^2 > (2 \cdot (n + 1)) + 1$ (by **??**). Qed.

Therefore $P$ contains every natural number $n$ such that $n \geq 3$ (by **??**). $\square$

**Proposition 1.3.** For all $n$ if $n \geq 5$ then

$$2^n > n^2.$$

*Proof.* Define

$$P = \left\{ n \in \mathbb{N} \;\middle|\; 2^n > n^2 \right\}.$$

(BASE CASE) $P$ contains 5. Indeed $2^5 = 2 \cdot (2 \cdot (2 \cdot (2 \cdot 2))) = (5 \cdot 5) + 7 > 5 \cdot 5 = 5^2$. Indeed $((5 \cdot 5) + 7) > 5 \cdot 5$.

(INDUCTION STEP) For all natural numbers $n$ such that $n \geq 5$ we have $n \in P \implies n + 1 \in P$.

Proof. Let $n$ be a natural number. Suppose $n \geq 5$. Assume $n \in P$. Then $2^n > n^2$.

(1) $2^n \cdot 2 > n^2 \cdot 2$ (by **??**). Indeed $2 \neq 0$.

(2) $n^2 \cdot 2 = n^2 + n^2$.

(3) $n^2 + n^2 > n^2 + ((2 \cdot n) + 1)$ (by **??**). Indeed $n^2 > (2 \cdot n) + 1$.

(4) $n^2 + ((2 \cdot n) + 1) = (n + 1)^2$.

Hence

$$\begin{gathered}2^{n+1} \\ = 2^n \cdot 2 \\ > n^2 \cdot 2 \\ = n^2 + n^2 \\ > n^2 + ((2 \cdot n) + 1)\end{gathered}$$

$$= (n+1)^2.$$

Thus $2^{n+1} > (n+1)^2$. Qed.

Therefore $P$ contains every natural number $n$ such that $n \geq 5$ (by **??**). $\square$

**Proposition 1.4.** For all $n$ if $n \geq 2$ then

$$n^n > n!.$$

*Proof.* Define

$$P = \{ n \in \mathbb{N} \mid n^n > n! \}.$$

(BASE CASE) $P$ contains 2.

(INDUCTION STEP) For all natural numbers $n$ such that $n \geq 2$ we have $n \in P \implies n+1 \in P$.
Proof. Let $n$ be a natural number. Suppose $n \geq 2$. Assume $n \in P$.

(1) $(n+1)^n \cdot (n+1) > n^n \cdot (n+1)$.
Proof. We have $n+1 > n$ and $n \neq 0$. Thus $(n+1)^n > n^n$ (by **??**). $n+1$ is nonzero. Hence the thesis (by **??**). Qed.

(2) $n^n \cdot (n+1) > n! \cdot (n+1)$ (by **??**). Indeed $n^n > n!$ and $n+1 \neq 0$.

Hence

$$\begin{aligned}
&(n+1)^{n+1} \\
&= (n+1)^n \cdot (n+1) \\
&> n^n \cdot (n+1) \\
&> n! \cdot (n+1) \\
&= (n+1)!.
\end{aligned}$$

Thus $(n+1)^{n+1} > (n+1)!$. Qed.

Therefore $P$ contains every natural number $n$ such that $n \geq 2$ (by **??**). $\square$

**Proposition 1.5.** For all $n$ if $n \geq 4$ then

$$n! > 2^n.$$

*Proof.* Define

$$P = \{ n \in \mathbb{N} \mid n! > 2^n \}.$$

(BASE CASE) $P$ contains 4.
Proof.

$$\begin{aligned}
&(4!) \\
&= 4 \cdot (3 \cdot 2)
\end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot (2 \cdot (3 \cdot 2)) \\
&= 3 \cdot (2 \cdot (2 \cdot 2)) \\
&> 2 \cdot (2 \cdot (2 \cdot 2)) \\
&= 2^4.
\end{aligned}
$$

Qed.

(INDUCTION STEP) For all natural numbers $n$ such that $n \geq 4$ we have $n \in P \implies n+1 \in P$.

Proof. Let $n$ be a natural number. Suppose $n \geq 4$. Assume $n \in P$. Then $n! > 2^n$.

(1) $0 \neq n+1 > 2$. Indeed $n > 1$.

(2) $n! \cdot (n+1) > 2^n \cdot (n+1)$ (by **??**).

(3) $2^n \cdot (n+1) > 2^n \cdot 2$ (by **??**). Indeed $2^n \neq 0$.

Hence

$$
\begin{aligned}
&((n+1)!) \\
&= n! \cdot (n+1) \\
&> 2^n \cdot (n+1) \\
&> 2^n \cdot 2 \\
&= 2^{n+1}.
\end{aligned}
$$

Thus $(n+1)! > 2^{n+1}$. Qed.

Therefore $P$ contains every natural number $n$ such that $n \geq 4$ (by **??**). $\square$