1 Factorial

[readtex `arithmetic/sections/01_arithmetic/03_multiplication.ftl.tex`]

Let k, l, m, n denote natural numbers.

An operation rather rarely mentioned together with (formal) Peano arithmetic is the factorial operation which we are going to define now.

Signature 1.1. $n!$ is a natural number.

Axiom 1.2 (1^{st} factorial axiom). $(0!) = 1$.

Axiom 1.3 (2^{nd} factorial axiom). $((n+1)!) = n! \cdot (n+1)$.

Note that we have to put the LHS of any expression of the form "$x! = y$" in parentheses, because such an expression can either be understood as "x factorial is equal to y" or as "x is not equal to y" by Naproche since it treats the combination of an exclamation mark followed by an equality sign as a synonym for "$\neq$".

Proposition 1.4. $n!$ is nonzero for any natural number n.

Proof. Define
$$P = \{\, n \in \mathbb{N} \mid n! \neq 0 \,\}.$$

(BASE CASE) P contains 0. Indeed $(0!) = 1 \neq 0$.

(INDUCTION STEP) For every natural number n we have $n \in P \implies n+1 \in P$.
Proof. Let n be a natural number. Assume $n \in P$. We have $((n+1)!) = (n+1) \cdot (n!)$. $n+1$ and $n!$ are nonzero. Hence $(n+1)!$ is nonzero. Qed.

Thus P contains every natural number. $\square$