

# A Hilbert-style calculus embedded into Naproche

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## 1 The implicational propositional calculus

[synonym formula/-s/-e]

**Signature 1.** A formula is an object.

Let  $P, Q, R, S$  denote formulas.

**Signature 2.**  $P \supset Q$  is a formula.

**Signature 3.**  $\vdash P$  is a relation.

Let  $P$  is deducible stand for  $\vdash P$ .

**Axiom 4. (Detachment)** Suppose  $\vdash P$  and  $\vdash P \supset Q$ . Then  $\vdash Q$ .

**Axiom 5. (Implosion)**  $\vdash P \supset (Q \supset P)$ .

**Axiom 6. (Chain)**  $\vdash (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$ .

**Proposition 7. (Weakening)** (0) Assume  $\vdash Q$ . Then  $\vdash P \supset Q$ .

*Proof.* (1)  $\vdash Q \supset (P \supset Q)$  (by Implosion).

(2)  $\vdash P \supset Q$  (by Detachment, 0, 1). □

**Proposition 8. (Weakening of implication)** (0) Assume  $\vdash Q \supset R$ . Then  $\vdash (P \supset Q) \supset (P \supset R)$ .

*Proof.* (1)  $\vdash P \supset (Q \supset R)$  (by Weakening, 0).

(2)  $\vdash (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$  (by Chain).

(3)  $\vdash (P \supset Q) \supset (P \supset R)$  (by Detachment, 1, 2). □

**Proposition 9. (Transitivity)** (0) Suppose  $\vdash P \supset Q$  and  $\vdash Q \supset R$ . Then  $\vdash P \supset R$ .

*Proof.* (1)  $\vdash (P \supset Q) \supset (P \supset R)$  (by Weakening of implication, 0).

(2)  $\vdash P \supset R$  (by Detachment, 0, 1). □

**Proposition 10. (Identity)**  $\vdash P \supset P$ .

*Proof.* (1)  $\vdash (P \supset ((P \supset P) \supset P)) \supset ((P \supset (P \supset P)) \supset (P \supset P))$  (by Chain).

(2)  $\vdash P \supset ((P \supset P) \supset P)$  (by Implosion).

(3)  $\vdash (P \supset (P \supset P)) \supset (P \supset P)$  (by 1, 2, Detachment).

(4)  $\vdash P \supset (P \supset P)$  (by Implosion).

(5)  $\vdash P \supset P$  (by 3, 4, Detachment).  $\square$

**Proposition 11.** (0) Suppose  $\vdash P \supset (P \supset Q)$ . Then  $\vdash P \supset Q$ .

*Proof.* (1)  $\vdash (P \supset (P \supset Q)) \supset ((P \supset P) \supset (P \supset Q))$  (by Chain).

(2)  $\vdash (P \supset P) \supset (P \supset Q)$  (by 0, 1, Detachment).

(3)  $\vdash P \supset P$  (by Identity).

(4)  $\vdash P \supset Q$  (by 2, 3, Detachment).  $\square$

## 2 Adding negation

**Signature 12. (Falsum)**  $\perp$  is a formula.

Let  $\neg P$  stand for  $P \supset \perp$ .

**Axiom 13. (Double negation)**  $\vdash \neg\neg P \supset P$ .

**Proposition 14. (Explosion)**  $\vdash \perp \supset P$ .