1 Image and preimage

[readtex `set-theory/sections/02_functions/01_functions.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets. Let f, g, h denote functions.

1.1 The image

Given an arbitrary set z we can ask ourselves where its elements are mapped to under a function f. The resulting set of such an application of f to all elements of z is called the *image* of z under f.

Lemma 1.1. Let f be a function. There exists a set y such that $y = \{ f(u) \mid u \in \mathrm{dom}(f) \cap z \}$.

Proof. Take $y = \mathrm{range}(f \restriction (\mathrm{dom}(f) \cap z))$. Then

$$y = \{ (f \restriction (\mathrm{dom}(f) \cap z))(u) \mid u \in \mathrm{dom}(f) \cap z \} .$$

Hence $y = \{ f(u) \mid u \in \mathrm{dom}(f) \cap z \}$. □

Definition 1.2. Let f be a function. $f[z]$ is the set y such that $y = \{ f(u) \mid u \in \mathrm{dom}(f) \cap z \}$.

Let the image of z under f stand for $f[z]$. Let the direct image of z under f stand for $f[z]$.

Proposition 1.3. Let f be a function from x to y and $a \subseteq x$. Then $f[a] = \{ f(u) \mid u \in a \}$.

Proof. $f[a] = \{ f(u) \mid u \in \mathrm{dom}(f) \cap a \}$. $\mathrm{dom}(f) \cap a = x \cap a = a$. Hence the thesis. □

Corollary 1.4. Let f be a function from x to y. Then $f[x] = \mathrm{range}(f)$.

Proof. We have $f[x] = \{ f(u) \mid u \in x \}$. Hence $f[x] = \mathrm{range}(f)$. □

Corollary 1.5. Let f be a function from x to y and $a \subseteq x$. Then $f[a] = \mathrm{range}(f \restriction a)$.

Proof. We have $f[a] = \{ f(u) \mid u \in a \}$. Hence $f[a] = \mathrm{range}(f \restriction a)$. □

Proposition 1.6. Let $a \subseteq x$. Then $\mathrm{id}_x[a] = a$.

Proof. $\mathrm{id}_x[a] = \{ \mathrm{id}_x(u) \mid u \in a \}$. We have $\mathrm{id}_x(u) = u$ for all $u \in a$. Hence $\mathrm{id}_x[a] = \{ u \mid u \in a \}$. Thus $\mathrm{id}_x[a] = a$. □

Proposition 1.7. Let $a \subseteq x$ and v be an element. Assume that a is nonempty. Then $\mathrm{const}_{x,v}[a] = \{v\}$.

Proof. Let us show that $\mathrm{const}_{x,v}[a] \subseteq \{v\}$. Let $w \in \mathrm{const}_{x,v}[a]$. Take $u \in a$ such that $w = \mathrm{const}_{x,v}(u)$. Then $w = v$. Hence $w \in \{v\}$. End.

Let us show that $\{v\} \subseteq \mathrm{const}_{x,v}[a]$. Let $w \in \{v\}$. Then $w = v$. Take $u \in a$. Then $\mathrm{const}_{x,v}(u) = v = w$. Hence $w \in \mathrm{const}_{x,v}[a]$. End. □

Proposition 1.8. Let f be a function from x into y and $a \subseteq x$. Then $f \restriction a$ is a bijection between a and $f[a]$.

Proof. (1) $f \restriction a$ is a function of a.

(2) $f \restriction a$ is one to one.

(3) $\mathrm{range}(f \restriction a) = f[a]$. Proof. Let us show that $\mathrm{range}(f \restriction a) \subseteq f[a]$. Let $v \in \mathrm{range}(f \restriction a)$. Take $u \in a$ such that $v = (f \restriction a)(u)$. Then $v = f(u)$. Hence $v \in f[a]$. End.

Let us show that $f[a] \subseteq \mathrm{range}(f \restriction a)$. Let $v \in f[a]$. Take $u \in a$ such that $v = f(u)$. Then $v = (f \restriction a)(u)$. Hence $v \in \mathrm{range}(f \restriction a)$. End. Qed.

Thus $f \restriction a$ is an one to one function from a onto $f[a]$. Therefore $f \restriction a$ is a bijection between a and $f[a]$. □

1.2 The preimage

Similar to the construction of the image of a set under a function, we can consider a set z and ask ourselves which elements of a function f are mapped into z. This yields the so-called *preimage* of z under f.

Lemma 1.9. Let f be a function. There exists a set y such that $y = \{\, u \in \mathrm{dom}(f) \mid f(u) \in z \,\}$.

Proof. Case $f(u) \in z$ for all $u \in \mathrm{dom}(f)$. Obvious.

Case $f(u) \notin z$ for some $u \in \mathrm{dom}(f)$. Take $w \in \mathrm{dom}(f)$ such that $f(w) \notin z$. (1) Define

$$g(u) = \begin{cases} u & : f(u) \in z \\ w & : f(u) \notin z \end{cases}$$

for $u \in \mathrm{dom}(f)$. $\mathrm{range}(g) = \{\, g(u) \mid u \in \mathrm{dom}(f) \,\}$. Hence $\mathrm{range}(g) = \{\, u \in \mathrm{dom}(f) \mid f(u) \in z \text{ or } u = w \,\}$ (by 1). Take $y = \mathrm{range}(g) \setminus \{w\}$. Then $y = \{\, u \in \mathrm{dom}(f) \mid f(u) \in z \,\}$. End. □

Definition 1.10. Let f be a function. $f^{-}[z]$ is the set y such that $y = \{\, u \in \mathrm{dom}(f) \mid f(u) \in z \,\}$.

Let the preimage of z under f stand for $f^{-}[z]$. Let the inverse image of z under f stand for $f^{-}[z]$.

Proposition 1.11. Let $b \subseteq y$. Then $\mathrm{id}_y^{-}[b] = b$.

Proof. $\mathrm{id}_y^-[b] = \{ u \in y \mid \mathrm{id}_y(u) \in b \}$. $\mathrm{id}_y(u) = u$ for all $u \in y$. Hence $\mathrm{id}_y^-[b] = \{ u \in y \mid u \in b \}$. Thus $\mathrm{id}_y^-[b] = b$. □

Proposition 1.12. Let v be an element and z be a set that contains v. Then $\mathrm{const}_{x,v}^-[z] = x$.

Proof. $\mathrm{const}_{x,v}^-[z] = \{ u \in x \mid \mathrm{const}_{x,v}(u) \in z \}$. $\mathrm{const}_{x,v}(u) = v$ for every $u \in x$. Hence $\mathrm{const}_{x,v}^-[z] = \{ u \in x \mid v \in z \}$. We have $v \in z$. Thus $\mathrm{const}_{x,v}^-[z] = x$. □

Proposition 1.13. Let v be an element and z be a set that does not contain v. Then $\mathrm{const}_{x,v}^-[z] = \emptyset$.

Proof. $\mathrm{const}_{x,v}^-[z] = \{ u \in x \mid \mathrm{const}_{x,v}(u) \in z \}$. $\mathrm{const}_{x,v}(u) = v$ for every $u \in x$. Hence $\mathrm{const}_{x,v}^-[z] = \{ u \in x \mid v \in z \}$. It is wrong that $v \in z$. Thus $\mathrm{const}_{x,v}^-[z] = \emptyset$. □

1.3 Computation rules

To conclude this paragraph let us prove some facts about the image and preimage.

Proposition 1.14. Let f be a function from x to y and $a \subseteq x$ and $u \in x$. Then $u \in a \implies f(u) \in f[a]$.

Proof. Assume $u \in a$. We have $f[a] = \{ f(u') \mid u' \in a \}$. Hence $f(u) \in f[a]$. □

Proposition 1.15. Let f be a function from x to y and $b \subseteq y$ and $u \in x$. Then $f(u) \in b \iff u \in f^-[b]$.

Proof. We have $f^-[b] = \{ u' \in x \mid f(u') \in b \}$. Hence $u \in f^-[b]$ iff $u \in x$ and $f(u) \in b$. Then we have the thesis. □

Proposition 1.16. Let f be a function from x to y. Then $f[x] \subseteq y$.

Proof. $f[x] = f[\mathrm{dom}(f)] = \mathrm{range}(f) \subseteq y$. □

Proposition 1.17. Let f be a function from x to y. Then $f^-[y] = x$.

Proof. We have $f^-[y] = \{ u \in x \mid f(u) \in y \}$. $f(u)$ is an element of y for all $u \in x$. Hence the thesis. □

Proposition 1.18. Let f be a function from x to y. Then $f[f^-[y]] = f[x]$.

Proof. Let us show that $f[f^-[y]] \subseteq f[x]$. Let $v \in f[f^-[y]]$. Take $u \in f^-[y] \cap x$ such that $v = f(u)$. Then $u \in x$. Hence $v \in f[x]$. End.

Let us show that $f[x] \subseteq f[f^{-}[y]]$. Let $v \in f[x]$. Take $u \in x$ such that $v = f(u)$. We have $v \in y$. Hence $u \in f^{-}[y]$. Thus $f(u) \in f[f^{-}[y]]$. Indeed $f^{-}[y] \subseteq x$. Therefore $v \in f[f^{-}[y]]$. End. □

Proposition 1.19. Let f be a function from x to y. Then $f^{-}[f[x]] = x$.

Proof. $f^{-}[f[x]] = \{ u \in x \mid f(u) \in f[x] \}$. For all $u \in x$ we have $f(u) \in f[x]$. Hence every element of $f^{-}[f[x]]$ is contained in x and every element of x is contained in $f^{-}[f[x]]$. Thus $f^{-}[f[x]] = x$. □

Proposition 1.20. Let f be a function from x to y and $b \subseteq y$. Then $f[f^{-}[b]] = b \cap f[x]$.

Proof. Let us show that $f[f^{-}[b]] \subseteq b \cap f[x]$. Let $v \in f[f^{-}[b]]$. Take $u \in f^{-}[b]$ such that $v = f(u)$. Then $f(u) \in b \cap f[x]$. Hence we have $v \in b \cap f[x]$. End.

Let us show that $b \cap f[x] \subseteq f[f^{-}[b]]$. Let $v \in b \cap f[x]$. Take $u \in x$ such that $v = f(u)$. Then $u \in f^{-}[b]$. Hence $f(u) \in f[f^{-}[b]]$. End. □

Corollary 1.21. Let f be a function from x to y and $b \subseteq y$. Then $f[f^{-}[b]] \subseteq b$.

Proof. We have $f[f^{-}[b]] = b \cap f[x] \subseteq b$. Hence $f[f^{-}[b]] \subseteq b$. □

Proposition 1.22. Let f be a function from x to y and $a \subseteq x$. Then $f^{-}[f[a]] \supseteq a$.

Proof. Let $u \in a$. Then $f(u) \in f[a]$. Hence $u \in f^{-}[f[a]]$. Indeed $f[a] \subseteq y$. □

Proposition 1.23. Let f be a function from x to y and $a \subseteq x$. Then $f[a] = \emptyset \iff a = \emptyset$.

Proof. Case $f[a] = \emptyset$. Then there is no $u \in a$ such that $f(u) \in f[a]$. For all $u \in a$ we have $f(u) \in f[a]$. Hence a is empty. End.

Case $a = \emptyset$. For all $v \in f[a]$ we have $v = f(u)$ for some $u \in a$. There is no $u \in a$. Hence $f[a]$ is empty. End. □

Proposition 1.24. Let f be a function from x to y and $b \subseteq y$. Then $f^{-}[b] = \emptyset \iff b \subseteq y \setminus f[x]$.

Proof. Case $f^{-}[b] = \emptyset$. Let $v \in b$. Then $v \in y$.

There is no $u \in x$ such that $v = f(u)$.
Proof. Assume the contrary. Take $u \in x$ such that $v = f(u)$. Then $u \in f^{-}[b]$. Contradiction. Qed.

Hence $v \notin f[x]$. Therefore $v \in y \setminus f[x]$. End.

Case $b \subseteq y \setminus f[x]$. Then no element of b is an element of $f[x]$. Assume that $f^{-}[b]$ is nonempty. Take $u \in f^{-}[b]$. Then $f(u) \in b$ and $f(u) \in f[x]$. Contradiction. End. □

Proposition 1.25. Let f be a function from x to y and $a \subseteq x$ and $b \subseteq y$. Then $f[a] \cap b = \emptyset \iff a \cap f^{-}[b] = \emptyset$.

Proof. Case $f[a] \cap b = \emptyset$. Assume that $a \cap f^{-}[b]$ is nonempty. Take $u \in a \cap f^{-}[b]$. Then $f(u) \in f[a]$ and $f(u) \in b$. Hence $f(u) \in f[a] \cap b$. Contradiction. End.

Case $a \cap f^{-}[b] = \emptyset$. Assume that $f[a] \cap b$ is nonempty. Take $v \in f[a] \cap b$. Consider a $u \in a$ such that $v = f(u)$. Then $u \in f^{-}[b]$. Indeed $v \in b$. Hence $u \in a \cap f^{-}[b]$. Contradiction. End. □

Proposition 1.26. Let f be a function from x to y and g be a function from y to z and $a \subseteq x$. Then $(g \circ f)[a] = g[f[a]]$.

Proof. $((g \circ f)[a]) = \{ g(f(u)) \mid u \in a \}$. $f[a]$ is a subset of y. We have $g[f[a]] = \{ g(v) \mid v \in f[a] \}$ and $f[a] = \{ f(u) \mid u \in a \}$. Thus $g[f[a]] = \{ g(f(u)) \mid u \in a \}$. Therefore $(g \circ f)[a] = g[f[a]]$. Indeed $((g \circ f)[a])$ and $g[f[a]]$ are sets. □

Proposition 1.27. Let f be a function from x to y and g be a function from y to z and $c \subseteq z$. Then $(g \circ f)^{-}[z] = f^{-}[g^{-}[z]]$.

Proof. $((g \circ f)^{-}[z]) = \{ u \in x \mid g(f(u)) \in z \}$. We have $g^{-}[z] = \{ v \in y \mid g(v) \in z \}$ and $f^{-}[g^{-}[z]] = \{ u \in x \mid f(u) \in g^{-}[z] \}$. Hence $f^{-}[g^{-}[z]] = \{ u \in x \mid g(f(u)) \in z \}$. Thus $(g \circ f)^{-}[z] = f^{-}[g^{-}[z]]$. □

Proposition 1.28. Let f be a function from x to y and $a, a' \subseteq x$. Then $a \subseteq a' \implies f[a] \subseteq f[a']$.

Proof. Assume $a \subseteq a'$. Let $v \in f[a]$. Take $u \in a$ such that $f(u) = v$. Then $u \in a'$. Hence $v = f(u) \in f[a']$. □

Proposition 1.29. Let f be a function from x to y and $b, b' \subseteq y$. Then $b \subseteq b' \implies f^{-}[b] \subseteq f^{-}[b']$.

Proof. Assume $b \subseteq b'$. Let $u \in f^{-}[b]$. Then $f(u) \in b$. Hence $f(u) \in b'$. Thus $u \in f^{-}[b']$. □

Proposition 1.30. Let f be a function from x to y and $a, a' \subseteq x$. Then $f[a \cup a'] = f[a] \cup f[a']$.

Proof. Let us show that $f[a \cup a'] \subseteq f[a] \cup f[a']$. Let $v \in f[a \cup a']$. Take $u \in a \cup a'$ such that $v = f(u)$. Then $u \in a$ or $u \in a'$. Hence $f(u) \in f[a]$ or $f(u) \in f[a']$. Thus $v = f(u) \in f[a] \cup f[a']$. End.

Let us show that $f[a] \cup f[a'] \subseteq f[a \cup a']$. Let $v \in f[a] \cup f[a']$.

Case $v \in f[a]$. Take $u \in a$ such that $v = f(u)$. Then $u \in a \cup a'$. Hence $v \in f[a \cup a']$. End.

Case $v \in f[a']$. Take $u \in a'$ such that $v = f(u)$. Then $u \in a \cup a'$. Hence $v \in f[a \cup a']$. End. End. $\square$

Proposition 1.31. Let f be a function from x to y and $b, b' \subseteq y$. Then $f^{-}[b \cup b'] = f^{-}[b] \cup f^{-}[b']$.

Proof. Let us show that $f^{-}[b \cup b'] \subseteq f^{-}[b] \cup f^{-}[b']$. Let $u \in f^{-}[b \cup b']$. Then $f(u) \in b \cup b'$. Hence $f(u) \in b$ or $f(u) \in b'$. If $f(u) \in b$ then $u \in f^{-}[b]$. If $f(u) \in b'$ then $u \in f^{-}[b']$. Thus $u \in f^{-}[b] \cup f^{-}[b']$. End.

Let us show that $f^{-}[b] \cup f^{-}[b'] \subseteq f^{-}[b \cup b']$. Let $u \in f^{-}[b] \cup f^{-}[b']$. Then $u \in f^{-}[b]$ or $u \in f^{-}[b']$. If $u \in f^{-}[b]$ then $f(u) \in b$. If $u \in f^{-}[b']$ then $f(u) \in b'$. Hence $f(u) \in b \cup b'$. Thus $u \in f^{-}[b \cup b']$. End. $\square$

Proposition 1.32. Let f be a function from x to y and $a, a' \subseteq x$. Then $f[a \cap a'] \subseteq f[a] \cap f[a']$.

Proof. Let $v \in f[a \cap a']$. Take $u \in a \cap a'$ such that $v = f(u)$. Then $u \in a$ and $u \in a'$. Hence $f(u) \in f[a]$ and $f(u) \in f[a']$. Thus $v \in f[a] \cap f[a]$. $\square$

Proposition 1.33. Let f be a function from x to y and $b, b' \subseteq y$. Then $f^{-}[b \cap b'] = f^{-}[b] \cap f^{-}[b']$.

Proof. Let us show that $f^{-}[b \cap b'] \subseteq f^{-}[b] \cap f^{-}[b']$. Let $u \in f^{-}[b \cap b']$. Then $f(u) \in b \cap b'$. Hence $f(u) \in b$ and $f(u) \in b'$. Thus $u \in f^{-}[b]$ and $u \in f^{-}[b']$. Therefore $u \in f^{-}[b] \cap f^{-}[b']$. End.

Let us show that $f^{-}[b] \cap f^{-}[b'] \subseteq f^{-}[b \cap b']$. Let $u \in f^{-}[b] \cap f^{-}[b']$. Then $u \in f^{-}[b]$ and $u \in f^{-}[b']$. Hence $f(u) \in b$ and $f(u) \in b'$. Thus $f(u) \in b \cap b'$. Therefore $u \in f^{-}[b \cap b']$. End. $\square$

Proposition 1.34. Let f be a function from x to y and $a, a' \subseteq x$. Then $f[a \setminus a'] \supseteq f[a] \setminus f[a']$.

Proof. Let $v \in f[a] \setminus f[a']$. Then $v \in f[a]$ and $v \notin f[a']$. Take $u \in a$ such that $v = f(u)$. If $u \in a'$ then $v \in f[a']$. Hence $u \notin a'$. Thus $u \in a \setminus a'$. Therefore $v = f(u) \in f[a \setminus a']$. $\square$

Proposition 1.35. Let f be a function from x to y and $b, b' \subseteq y$. Then $f^{-}[b \setminus b'] = f^{-}[b] \setminus f^{-}[b']$.

Proof. Let us show that $f^{-}[b \setminus b'] \subseteq f^{-}[b] \setminus f^{-}[b']$. Let $u \in f^{-}[b \setminus b']$. Then $f(u) \in b \setminus b'$. Hence $f(u) \in b$ and $f(u) \notin b'$. Thus $u \in f^{-}[b]$ and $u \notin f^{-}[b']$. Therefore $u \in f^{-}[b] \setminus f^{-}[b']$. End.

Let us show that $f^{-}[b] \setminus f^{-}[b'] \subseteq f^{-}[b \setminus b']$. Let $u \in f^{-}[b] \setminus f^{-}[b']$. Then $u \in f^{-}[b]$ and $u \notin f^{-}[b']$. Hence $f(u) \in b$ and $f(u) \notin b'$. Thus $f(u) \in b \setminus b'$. Therefore $u \in f^{-}[b \setminus b']$. End. $\square$