1 Peano Arithmetic

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1.1 The Peano axioms

This arithmetic is based on the notion of natural numbers. These are introduced as some sort of elements that is equiped with a a unary function succ (which maps any natural number to its direct successor) and that contains a constant 0 (the unique least natural number).

Signature 1.1. A natural number is an element.

Let k, l, m, n denote natural numbers.

Definition 1.2. $\mathbb{N}$ is the class of natural numbers.

Signature 1.3. 0 is a natural number.

Let n is nonzero stand for $n \neq 0$.

Signature 1.4. $\mathrm{succ}(n)$ is a natural number.

Let the direct successor of n stand for $\mathrm{succ}(n)$.

The natural numbers are characterized by the following so-called Peano axioms.

Axiom 1.5 (1$^{\mathrm{st}}$ Peano axiom). If $\mathrm{succ}(n) = \mathrm{succ}(m)$ then $n = m$.

Axiom 1.6 (2$^{\mathrm{nd}}$ Peano axiom). 0 is not the direct successor of any natural number.

Axiom 1.7 (3$^{\mathrm{rd}}$ Peano axiom). Let P be a class. Assume $0 \in P$ and for all natural numbers n we have $n \in P \implies \mathrm{succ}(n) \in P$. Then every natural number is an element of P.

1.2 Immediate consequences

The 3$^{\mathrm{rd}}$ Peano axiom (also called the *induction axiom*) allows us to prove that the signature $(0, \mathrm{succ})$ captures the whole class of natural numbers in the sense that every natural number is either zero or a successor:

Proposition 1.8. For all n we have $n = 0$ or $n = \mathrm{succ}(m)$ for some natural number m.

Proof. Define

$P = \{\, n \in \mathbb{N} \mid n = 0 \text{ or } n = \mathrm{succ}(m) \text{ for some natural number } m \,\}$.

$0 \in P$ and for all natural numbers n we have $n \in P \implies \mathrm{succ}(n) \in P$.

Hence the thesis (by 3rd Peano axiom). $\square$

This allows us to define the direct predecessor of a non-zero natural number as follows.

Definition 1.9. Let n be nonzero. $\mathrm{pred}(n)$ is the natural number m such that $\mathrm{succ}(m) = n$.

Let the direct predecessor of n stand for $\mathrm{pred}(n)$.

Note that direct predecessors must be unique by the 2nd Peano axiom. Moreover, we can show that no natural number is its own successor.

Proposition 1.10. For no natural number n we have $n = \mathrm{succ}(n)$.

Proof. Define
$$P = \{ n \in \mathbb{N} \mid n \neq \mathrm{succ}(n) \} .$$

(BASE CASE) 0 belongs to P.

(INDUCTION STEP) For all n we have $n \in P \implies \mathrm{succ}(n) \in P$.
Proof. Let n be a natural number. Assume that $n \in P$. Then $n \neq \mathrm{succ}(n)$. If $\mathrm{succ}(n) = \mathrm{succ}(\mathrm{succ}(n))$ then $n = \mathrm{succ}(n)$. Thus it is wrong that $\mathrm{succ}(n) = \mathrm{succ}(\mathrm{succ}(n))$. Hence $\mathrm{succ}(n) \in P$. Qed.

Therefore every natural number is an element of P. Then we have the thesis. $\square$

1.3 Additional constants

Let us end this section by introducing new constant symbols for the first few successors of 0.

Definition 1.11. $1 = \mathrm{succ}(0)$.

Definition 1.12. $2 = \mathrm{succ}(1)$.

Definition 1.13. $3 = \mathrm{succ}(2)$.

Definition 1.14. $4 = \mathrm{succ}(3)$.

Definition 1.15. $5 = \mathrm{succ}(4)$.

Definition 1.16. $6 = \mathrm{succ}(5)$.

Definition 1.17. $7 = \mathrm{succ}(6)$.

Definition 1.18. $8 = \mathrm{succ}(7)$.

Definition 1.19. $9 = \mathrm{succ}(8)$.