Maximum modulus principle

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1 Introduction

We formalize that the maximum modulus principle in complex analysis is implied by the open mapping theorem and the identity theorem.

We use general set theoretic preliminaries:

[readtex preliminaries.ftl.tex]

2 Real and complex numbers

Signature 1. A complex number is a mathematical object.

Let z, w denote complex numbers.

Definition 2. $\mathbb{C}$ is the collection of all complex numbers.

Signature 3. A real number is a complex number.

Let x, y denote real numbers.

Signature 4. $|z|$ is a real number.

Signature 5. x is positive is an atom.

Let ε, δ denote positive real numbers.

Signature 6. $x < y$ is an atom.

Let $x > y$ stand for $y < x$. Let $x \leq y$ stand for $x = y$ or $x < y$.

Axiom 7. If $x < y$ then not $y < x$.

3 Open balls

Signature 8. $B_\varepsilon(z)$ is a subset of $\mathbb{C}$ that contains z.

Axiom 9. $|z| < |w|$ for some element w of $B_\varepsilon(z)$.

Let M denote a subset of $\mathbb{C}$.

Definition 10. M is open iff for every element z of M there exists ε such that $B_\varepsilon(z)$ is a subset of M.

Axiom 11. $B_\varepsilon(z)$ is open.

Signature 12. A region is an open subset of $\mathbb{C}$.

Signature 13. Let M be a region. M is simply connected is an atom.

4 Holomorphic functions

Signature 14. A holomorphic function is a function f such that $Dom(f) \subseteq \mathbb{C}$ and $f[Dom(f)] \subseteq \mathbb{C}$.

Let f denote a holomorphic function.

Definition 15. A local maximal point of f is an element z of the domain of f such that there exists ε such that $B_\varepsilon(z)$ is a subset of the domain of f and $|f(w)| \leq |f(z)|$ for every element w of $B_\varepsilon(z)$.

Definition 16. Let U be a subset of the domain of f. f is constant on U iff there exists z such that $f(w) = z$ for every element w of U.

Let f is constant stand for f is constant on the domain of f.

Axiom 17. **(Open Mapping Theorem)** Assume f is a holomorphic function and $B_\varepsilon(z)$ is a subset of the domain of f. If f is not constant on $B_\varepsilon(z)$ then $f[B_\varepsilon(z)]$ is open.

Axiom 18. **(Identity theorem)** Assume f is a holomorphic function and the domain of f is a region. Assume that $B_\varepsilon(z)$ is a subset of the domain of f. If f is constant on $B_\varepsilon(z)$ then f is constant.

Proposition 19. **(Maximum modulus principle)** Assume f is a holomorphic function and the domain of f is a region. If f has a local maximal point then f is constant.

Proof. Let z be a local maximal point of f. Take ε such that $B_\varepsilon(z)$ is a subset of $\mathrm{dom}(f)$ and $|f(w)| \leq |f(z)|$ for every element w of $B_\varepsilon(z)$.

Let us show that f is constant on $B_\varepsilon(z)$. Proof by contradiction. Assume the contrary. Then $f[B_\varepsilon(z)]$ is open. We can take δ such that $B_\delta(f(z))$ is a subset of $f[B_\varepsilon(z)]$. Therefore there exists an element w of $B_\varepsilon(z)$ such that $|f(z)| < |f(w)|$. Contradiction. End.

Hence f is constant. $\square$