1 The powerset

[readtex `set-theory/sections/01_sets/01_sets.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets.

In this paragraph we consider collections of subsets of a given set. To ensure that these are sets themselves, we need another axiom.

Axiom 1.1 (Powerset). There exists a set z such that $z = \{ y \mid y \subseteq x \}$.

Definition 1.2. $\mathcal{P}(x)$ is the set z such that $z = \{ y \mid y \subseteq x \}$.

Let the powerset of x stand for $\mathcal{P}(x)$.

Proposition 1.3. $\emptyset$ and x are elements of $\mathcal{P}(x)$.

Proof. We have $\emptyset, x \subseteq x$. Hence the thesis. □

Corollary 1.4. $\mathcal{P}(x)$ is nonempty.

Proposition 1.5. $\mathcal{P}(x)$ is a system of subsets of x.

Proposition 1.6. $\bigcup \mathcal{P}(x) = x$.

Proof. Every element of $\mathcal{P}(x)$ is a subset of x. Hence $\bigcup \mathcal{P}(x) \subseteq x$.

We have $x \in \mathcal{P}(x)$. Hence every element of x is an element of some element of $\mathcal{P}(x)$. Thus every element of x belongs to $\bigcup \mathcal{P}(x)$. Therefore $x \subseteq \bigcup \mathcal{P}(x)$.

Then we have the thesis. □

Proposition 1.7. $\bigcap \mathcal{P}(x) = \emptyset$.

Proof. We have $\emptyset \in \mathcal{P}(x)$. Hence every element of $\bigcap \mathcal{P}(x)$ is an element of $\emptyset$. Thus $\bigcap \mathcal{P}(x)$ is empty. Therefore $\bigcap \mathcal{P}(x) = \emptyset$. □