1 Cartesian products

[readtex `set-theory/sections/01_sets/02_powerset.ftl.tex`]

[readtex `set-theory/sections/01_sets/05_ordered-pairs.ftl.tex`]

Let u, v, w, u', v', w' denote objects. Let x, y, z, x', y', z' denote sets.

Let us now consider collections of ordered pairs. We can show that for any given sets x, y the collection of all pairs whose first component lies in x and whose second component lies in y is a set. This set is called the *Cartesian product* of x and y.

Lemma 1.1. There exists a set z such that

$$z = \{ (u, v) \mid u \in x \text{ and } v \in y \} .$$

Proof. (1) Define $z = \{ (u, v) \mid u \in x \text{ and } v \in y \}$. Take $z' = \mathcal{P}(\mathcal{P}(x \cup y))$. Then z' is a set.

Let us show that every element of z is contained in z'. Let $w \in z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and $v \in y$. Hence $\{u\}$ and $\{u, v\}$ are subsets of $x \cup y$. Thus $\{u\}$ and $\{u, v\}$ are elements of $\mathcal{P}(x \cup y)$. Therefore $w = \{\{u\}, \{u, v\}\} \subseteq \mathcal{P}(x \cup y)$. Consequently $w \in \mathcal{P}(\mathcal{P}(x \cup y)) = z'$. End.

Hence z is a set (by **??**). Therefore the thesis (by 1). □

Definition 1.2. $x \times y$ is the set z such that $z = \{ (u, v) \mid u \in x \text{ and } v \in y \}$.

Let the Cartesian product of x and y stand for $x \times y$.

Proposition 1.3. $(u, v) \in x \times y$ iff $u \in x$ and $v \in y$.

Proof. Case $(u, v) \in x \times y$. We can take $u' \in x$ and $v' \in y$ such that $(u, v) = (u', v')$. Then $u = u'$ and $v = v'$. Hence $u \in x$ and $v \in y$. End.

Case $u \in x$ and $v \in y$. u and v are elements. Hence (u, v) is an element. Therefore $(u, v) \in x \times y$. Indeed $x \times y = \{ (u', v') \mid u' \in x \text{ and } v' \in y \}$. End. □

Proposition 1.4. $x \times y$ is empty iff x is empty or y is empty.

Proof. Case $x \times y$ is empty. Assume that x and y are nonempty. Thus we can take an element u of x and an element v of y. Then (u, v) is an element of $x \times y$. Contradiction. End.

Case x is empty or y is empty. Assume that $x \times y$ is nonempty. Then we can take an element z of $x \times y$. Then $z = (u, v)$ for some $u \in x$ and some $v \in y$. Hence x and y are nonempty. Contradiction. End. □

Proposition 1.5. $\{u\} \times \{v\} = \{(u, v)\}$.

Proof. Let us show that $\{u\} \times \{v\} \subseteq \{(u, v)\}$. Let $w \in \{u\} \times \{v\}$. Take $a \in \{u\}$ and $b \in \{v\}$ such that $w = (a, b)$. We have $a = u$ and $b = v$. Hence $w = (u, v)$. Thus $w \in \{(u, v)\}$. End.

Let us show that $\{(u, v)\} \subseteq \{u\} \times \{v\}$. Let $w \in \{(u, v)\}$. Then $w = (u, v)$. We have $u \in \{u\}$ and $v \in \{v\}$. Hence $w \in \{u\} \times \{v\}$. End. ☐

1.1 Computation laws

As always let us have a look at the algebraic properties of our new operation.

Subset laws:

Proposition 1.6.

$$x \subseteq y \implies x \times z \subseteq y \times z.$$

Proof. Assume $x \subseteq y$. Let $w \in x \times z$. Take $u \in x$ and $v \in z$ such that $w = (u, v)$. Then $u \in y$. Hence $(u, v) \in y \times z$. ☐

Proposition 1.7. Assume that x and x' are nonempty.

$$(x \times x') \subseteq (y \times y') \iff (x \subseteq y \text{ and } x' \subseteq y').$$

Proof. Case $(x \times x') \subseteq (y \times y')$. Let us show that for all $u \in x$ and all $v \in x'$ we have $u \in y$ and $v \in y'$. Let $u \in x$ and $v \in x'$. Then $(u, v) \in x \times x'$. Hence $(u, v) \in y \times y'$. Thus $u \in y$ and $v \in y'$. End. End.

Case $x \subseteq y$ and $x' \subseteq y'$. Let $w \in x \times x'$. Take $u \in x$ and $v \in x'$ such that $w = (u, v)$. Then $u \in y$ and $v \in y'$. Hence $(u, v) \in y \times y'$. End. ☐

Distributivity of product and union:

Proposition 1.8.

$$((x \cup y) \times z) = (x \times z) \cup (y \times z).$$

Proof. Let us show that $((x \cup y) \times z) \subseteq (x \times z) \cup (y \times z)$. Let $w \in (x \cup y) \times z$. Take $u \in x \cup y$ and $v \in z$ such that $w = (u, v)$. Then $u \in x$ or $u \in y$. If $u \in x$ then $w \in x \times z$ and if $u \in y$ then $w \in y \times z$. Hence $w \in x \times z$ or $w \in y \times z$. Thus $w \in (x \times z) \cup (y \times z)$. End.

Let us show that $((x \times z) \cup (y \times z)) \subseteq (x \cup y) \times z$. Let $w \in (x \times z) \cup (y \times z)$. Then $w \in x \times z$ or $w \in y \times z$. Take elements u, v such that $w = (u, v)$. Then ($u \in x$ or $u \in y$) and $v \in z$. Hence $u \in x \cup y$. Thus $w \in (x \cup y) \times z$. End. ☐

Proposition 1.9.

$$x \times (y \cup z) = (x \times y) \cup (x \times z).$$

Proof. Let us show that $x \times (y \cup z) \subseteq (x \times y) \cup (x \times z)$. Let $w \in x \times (y \cup z)$. Take $u \in x$ and $v \in y \cup z$ such that $w = (u, v)$. Then $v \in y$ or $v \in z$. Hence $w \in x \times y$ or $w \in x \times z$. Indeed if $v \in y$ then $w \in x \times y$ and if $v \in z$ then $w \in x \times z$. Thus $w \in (x \times y) \cup (x \times z)$. End.

Let us show that $((x \times y) \cup (x \times z)) \subseteq x \times (y \cup z)$. Let $w \in (x \times y) \cup (x \times z)$. Then $w \in x \times y$ or $w \in x \times z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and ($v \in y$ or $v \in z$). Hence $w \in x \times (y \cup z)$. End. □

Distributivity of product and intersection:

Proposition 1.10.

$$((x \cap y) \times z) = (x \times z) \cap (y \times z).$$

Proof. Let us show that $((x \cap y) \times z) \subseteq (x \times z) \cap (y \times z)$. Let $w \in (x \cap y) \times z$. Take $u \in x \cap y$ and $v \in z$ such that $w = (u, v)$. Then $u \in x$ and $u \in y$. Hence $w \in x \times z$ and $w \in y \times z$. Thus $w \in (x \times z) \cap (y \times z)$. End.

Let us show that $((x \times z) \cap (y \times z)) \subseteq (x \cap y) \times z$. Let $w \in (x \times z) \cap (y \times z)$. Then $w \in x \times z$ and $w \in y \times z$. Take elements u, v such that $w = (u, v)$. Then ($u \in x$ and $u \in y$) and $v \in z$. Hence $u \in x \cap y$. Thus $w \in (x \cap y) \times z$. End. □

Proposition 1.11.

$$x \times (y \cap z) = (x \times y) \cap (x \times z).$$

Proof. Let us show that $x \times (y \cap z) \subseteq (x \times y) \cap (x \times z)$. Let $w \in x \times (y \cap z)$. Take $u \in x$ and $v \in y \cap z$ such that $w = (u, v)$. Then $v \in y$ and $v \in z$. Hence $w \in x \times y$ and $w \in x \times z$. Thus $w \in (x \times y) \cap (x \times z)$. End.

Let us show that $((x \times y) \cap (x \times z)) \subseteq x \times (y \cap z)$. Let $w \in (x \times y) \cap (x \times z)$. Then $w \in x \times y$ and $w \in x \times z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and ($v \in y$ and $v \in z$). Hence $w \in x \times (y \cap z)$. End. □

Distributivity of product and complement:

Proposition 1.12.

$$((x \setminus y) \times z) = (x \times z) \setminus (y \times z).$$

Proof. Let us show that $((x \setminus y) \times z) \subseteq (x \times z) \setminus (y \times z)$. Let $w \in (x \setminus y) \times z$.

Take $u \in x \setminus y$ and $v \in z$ such that $w = (u, v)$. Then $u \in x$ and $u \notin y$. Hence $w \in x \times z$ and $w \notin y \times z$. Thus $w \in (x \times z) \setminus (y \times z)$. End.

Let us show that $((x \times z) \setminus (y \times z)) \subseteq (x \setminus y) \times z$. Let $w \in (x \times z) \setminus (y \times z)$. Then $w \in x \times z$ and $w \notin y \times z$. Take $u \in x$ and $v \in z$ such that $w = (u, v)$. Then $u \notin y$. Indeed if $u \in y$ then $w \in y \times z$. Hence $u \in x \setminus y$. Thus $w \in (x \setminus y) \times z$. End. □

Proposition 1.13.

$$x \times (y \setminus z) = (x \times y) \setminus (x \times z).$$

Proof. Let us show that $x \times (y \setminus z) \subseteq (x \times y) \setminus (x \times z)$. Let $w \in x \times (y \setminus z)$. Take $u \in x$ and $v \in y \setminus z$ such that $w = (u, v)$. Then $v \in y$ and $v \notin z$. Hence $w \in x \times y$ and $w \notin x \times z$. Thus $w \in (x \times y) \setminus (x \times z)$. End.

Let us show that $((x \times y) \setminus (x \times z)) \subseteq x \times (y \setminus z)$. Let $w \in (x \times y) \setminus (x \times z)$. Then $w \in x \times y$ and $w \notin x \times z$. Take elements u, v such that $w = (u, v)$. Then $u \in x$ and ($v \in y$ and $v \notin z$). Hence $w \in x \times (y \setminus z)$. End. □

Equality law:

Proposition 1.14. Assume that x and x' are nonempty or y and y' are nonempty. Then

$$(x \times x') = (y \times y') \iff (x = y \text{ and } x' = y').$$

Proof. Case $x \times x' = y \times y'$. Then x and x' are nonempty iff y and y' are nonempty.

Let us show that for all $u \in x$ and all $v \in x'$ we have $u \in y$ and $v \in y'$. Let $u \in x$ and $v \in x'$. Then $(u, v) \in x \times x'$. Hence we can take $w \in y \times y'$ such that $w = (u, v)$. Thus $u \in y$ and $v \in y'$. End.

Therefore $x \subseteq y$ and $x' \subseteq y'$. Indeed x and x' are nonempty.

Let us show that for all $u \in y$ and all $v \in y'$ we have $u \in x$ and $v \in x'$. Let $u \in y$ and $v \in y'$. Then $(u, v) \in y \times y'$. Hence we can take $w \in x \times x'$ such that $w = (u, v)$. Thus $(u, v) \in x \times x'$. End.

Therefore $y \subseteq x$ and $y' \subseteq x'$. Indeed y and y' are nonempty. End.

Case $x = y$ and $x' = y'$. Trivial. □

Intersection of products:

Proposition 1.15.

$$((x \times y) \cap (x' \times y')) = (x \cap x') \times (y \cap y').$$

Proof. Let us show that $((x \times y) \cap (x' \times y')) \subseteq (x \cap x') \times (y \cap y')$. Let $w \in (x \times y) \cap (x' \times y')$. Then $w \in x \times y$ and $w \in x' \times y'$. Take elements u, v such that $w = (u, v)$. Then $u \in x, x'$ and $v \in y, y'$. Hence $u \in x \cap x'$ and $v \in y \cap y'$. Thus $w \in (x \cap x') \times (y \cap y')$. End.

Let us show that $(x \cap x') \times (y \cap y') \subseteq (x \times y) \cap (x' \times y')$. Let $w \in (x \cap x') \times (y \cap y')$. Take elements u, v such that $w = (u, v)$. Then $u \in x \cap x'$ and $v \in y \cap y'$ (by 1.3). Hence $u \in x, x'$ and $v \in y, y'$. Thus $w \in x \times y$ and $w \in x' \times y'$. Therefore $w \in (x \times y) \cap (x' \times y')$. End. $\square$

Union of products:

Proposition 1.16.

$$((x \times y) \cup (x' \times y')) \subseteq (x \cup x') \times (y \cup y').$$

Proof. Let $w \in (x \times y) \cup (x' \times y')$. Then $w \in x \times y$ or $w \in x' \times y'$. Take elements u, v such that $w = (u, v)$. Then ($u \in x$ or $u \in x'$) and ($v \in y$ or $v \in y'$). Hence $u \in x \cup x'$ and $v \in y \cup y'$. Thus $w \in (x \cup x') \times (y \cup y')$. $\square$

Complement of products:

Proposition 1.17.

$$((x \times y) \setminus (x' \times y')) = (x \times (y \setminus y')) \cup ((x \setminus x') \times y).$$

Proof. Let us show that $((x \times y) \setminus (x' \times y')) \subseteq (x \times (y \setminus y')) \cup ((x \setminus x') \times y)$. Let $w \in (x \times y) \setminus (x' \times y')$. Then $w \in x \times y$ and $w \notin x' \times y'$. Take $u \in x$ and $v \in y$ such that $w = (u, v)$. Then it is wrong that $u \in x'$ and $v \in y'$. Hence $u \notin x'$ or $v \notin y'$. Thus $u \in x \setminus x'$ or $v \in y \setminus y'$. Therefore $w \in x \times (y \setminus y')$ or $w \in (x \setminus x') \times y$. Hence we have $w \in (x \times (y \setminus y')) \cup ((x \setminus x') \times y)$. End.

Let us show that $(x \times (y \setminus y')) \cup ((x \setminus x') \times y) \subseteq (x \times y) \setminus (x' \times y')$. Let $w \in (x \times (y \setminus y')) \cup ((x \setminus x') \times y)$. Then $w \in (x \times (y \setminus y'))$ or $w \in ((x \setminus x') \times y)$. Take elements u, v such that $w = (u, v)$. Then ($u \in x$ and $v \in y \setminus y'$) or ($u \in x \setminus x'$ and $v \in y$) (by 1.3).

Case $u \in x$ and $v \in y \setminus y'$. Then $u \in x$ and $v \in y$. Hence $w \in x \times y$. We have $v \notin y'$. Thus $w \notin x' \times y'$. Therefore $w \in (x \times y) \setminus (x' \times y')$. End.

Case $u \in x \setminus x'$ and $v \in y$. Then $u \in x$ and $v \in y$. Hence $w \in x \times y$. We

have $u \notin x'$. Thus $w \notin x' \times y'$. Therefore $w \in (x \times y) \setminus (x' \times y')$. End.
End. $\square$