

Cantor's Theorem

In this document we give a proof of Cantor's Theorem:

Theorem. There is no surjection from a set onto its powerset.

Some basic notions and set-theoretic axioms used to formulate and prove it are taken from:

[readtex `preliminaries.ftl.tex`]

Moreover, we need to provide certain definitions concerning surjective functions and the notion of the powerset.

Definition. Let X be a set. A function of X is a function f such that $\text{dom}(f) = X$.

Definition. Let f be a function and Y be a set. f surjects onto Y iff $Y = \{ f(x) \mid x \in \text{dom}(f) \}$.

Let a function from X onto Y stand for a function of X that surjects onto Y .

Definition. Let X be a set. The powerset of X is the collection of subsets of X .

Axiom. The powerset of any set is a set.

On this basis Cantor's theorem and its proof can be formalized as follows.

Theorem. (Cantor) Let M be a set. No function of M surjects onto the powerset of M .

Proof. Assume the contrary. Take a function f from M onto the powerset of M . The value of f at any element of M is a set. Define

$$N = \{ x \in M \mid x \text{ is not an element of } f(x) \}.$$

N is a subset of M . Consider an element z of M such that $f(z) = N$. Then

$$z \in N \iff z \notin f(z) = N.$$

Contradiction. □