1 Ordering and multiplication

[readtex `arithmetic/sections/01_arithmetic/03_multiplication.ftl.tex`]

[readtex `arithmetic/sections/02_ordering/02_ordering-and-addition.ftl.tex`]

Let k, l, m, n denote natural numbers.

As we did with addition, we will now examine the behaviour of the ordering with respect to multiplication.

Proposition 1.1. Assume $k \neq 0$. Then for all n, m we have

$$n < m \iff n \cdot k < m \cdot k.$$

Proof. Define

$$P = \{ n \in \mathbb{N} \mid \text{for all natural numbers } m \text{ if } n \cdot k < m \cdot k \text{ then } n < m \}.$$

Let us show that every natural number is contained in P. (BASE CASE) P contains 0.

(INDUCTION STEP) For all natural numbers n we have $n \in P \implies n + 1 \in P$. Proof. Let n be a natural number. Assume $n \in P$.

For all natural numbers m if $(n + 1) \cdot k < m \cdot k$ then $n + 1 < m$.
Proof. Let m be a natural number. Assume $(n + 1) \cdot k < m \cdot k$. Then $(n \cdot k) + k < m \cdot k$. Hence $n \cdot k < m \cdot k$. Thus $n < m$. Then $n + 1 \leq m$. If $n + 1 = m$ then $(n + 1) \cdot k = m \cdot k$. Hence the thesis. Qed. Qed.

Therefore every natural number is contained in P. End.

Let n, m be natural numbers.

Case $n < m$. Take a positive natural number l such that $m = n + l$. Then $m \cdot k = (n + l) \cdot k = (n \cdot k) + (l \cdot k)$. $l \cdot k$ is positive. Hence $n \cdot k < m \cdot k$. End.

Case $n \cdot k < m \cdot k$. Then $n < m$. Indeed n and m are contained in P. End. □

Corollary 1.2. Assume $k \neq 0$. Then

$$n < m \iff k \cdot n < k \cdot m.$$

Proof. We have $k \cdot n = n \cdot k$ and $k \cdot m = m \cdot k$. Hence $k \cdot n < k \cdot m$ iff $n \cdot k < m \cdot k$. □

Proposition 1.3. For all n, m we have

$$n, m > k \implies n \cdot m > k.$$

Proof. Define

$$P = \{ \, n \in \mathbb{N} \mid \text{for all natural numbers } m \text{ if } n, m > k \text{ then } n \cdot m > k \qquad \} .$$

(BASE CASE) P contains 0.

(INDUCTION STEP) For all natural numbers n we have $n \in P \implies n + 1 \in P$.
Proof. Let n be a natural number. Assume $n \in P$.

For all natural numbers m if $n + 1, m > k$ then $(n + 1) \cdot m > k$.
Proof. Let m be a natural number. Assume $n+1, m > k$. Then $(n+1) \cdot m = (n \cdot m) + m$. If $n = 0$ then $(n \cdot m) + m = 0 + m = m > k$. If $n \neq 0$ then $(n \cdot m) + m > m > k$. Indeed if $n \neq 0 then n \cdot m > 0$. Indeed $m > 0$. Hence $(n + 1) \cdot m > k$. Qed. Qed.

Hence every natural number is contained in P. □

Corollary 1.4. We have

$$n \leq m \implies k \cdot n \leq k \cdot m.$$

Corollary 1.5. Assume $k \neq 0$. Then

$$k \cdot n \leq k \cdot m \implies n \leq m.$$

Corollary 1.6. We have

$$n \leq m \implies n \cdot k \leq m \cdot k.$$

Corollary 1.7. Assume $k \neq 0$. Then

$$n \cdot k \leq m \cdot k \implies n \leq m.$$

Proposition 1.8. Let $k > 1$ and $m > 0$. Then $k \cdot m > m$.

Proof. Take a natural number l such that $k = l + 2$. Then

$$\begin{aligned} & k \cdot m \\ &= (l + 2) \cdot m \\ &= (l \cdot m) + (2 \cdot m) \\ &= (l \cdot m) + (m + m) \\ &= ((l \cdot m) + m) + m \\ &= ((l + 1) \cdot m) + m \\ &\geq 1 + m \\ &> m. \end{aligned}$$

□