1 Addition

[readtex `arithmetic/sections/01_arithmetic/01_peano-axioms.ftl.tex`]

Let k, l, m, n denote natural numbers.

1.1 Axioms

Up to now our arithmetic – if it already deserves this name – is very primitive. In this section we change this by inductively defining an addition operation.

Signature 1.1. $n + m$ is a natural number.

Let the sum of n and m stand for $n + m$.

Axiom 1.2 (1$^{\text{st}}$ addition axiom). $n + 0 = n$.

Axiom 1.3 (2$^{\text{nd}}$ addition axiom). $n + \mathrm{succ}(m) = \mathrm{succ}(n + m)$.

1.2 Immediate consequences

Having this characterization of addition at hand, the successor operation turns out to coincide with the "+1" operation.

Lemma 1.4. $\mathrm{succ}(n) = n + 1$.

This enables us to restate all previous axioms purely in terms of addition.

Corollary 1.5 (1$^{\text{st}}$ Peano axiom). If $n + 1 = m + 1$ then $n = m$.

Corollary 1.6 (2$^{\text{nd}}$ Peano axiom). For no n we have $n + 1 = 0$.

Corollary 1.7 (3$^{\text{rd}}$ Peano axiom). Let P be a class. Assume $0 \in P$ and for all n: $n \in P \implies n + 1 \in P$. Then every natural number is an element of P.

Corollary 1.8 (2$^{\text{nd}}$ addition axiom). $n + (m + 1) = (n + m) + 1$.

1.3 Computation laws

Let us now prove the common computation laws for addition: Associativity, commutativity and the cancellation laws.

Associativity:

Proposition 1.9. For all n, m, k we have

$$n + (m + k) = (n + m) + k.$$

Proof. Define

$$P = \{\, k \in \mathbb{N} \mid \text{for all } n, m\text{: } n + (m + k) = (n + m) + k \,\}.$$

(BASE CASE) 0 is contained in P. Indeed $n+(m+0) = n+m = (n+m)+0$ for all natural numbers n, m.

(INDUCTION STEP) For all k we have $k \in P \implies k + 1 \in P$.
Proof. Let k be a natural number. Assume $k \in P$.

Let us show that $n + (m + (k + 1)) = (n + m) + (k + 1)$ for all natural numbers n, m.
Let n, m be natural numbers. Then $n + m$ is a natural number.

$$\begin{aligned} &n + (m + (k + 1)) \\ &= n + ((m + k) + 1) \\ &= (n + (m + k)) + 1 \\ &= ((n + m) + k) + 1 \\ &= (n + m) + (k + 1). \end{aligned}$$

Hence the thesis. End.

Therefore we have $k + 1 \in P$. Qed.

Thus every natural number is an element of P. □

Commutativity:

Proposition 1.10. For all n, m we have

$$n + m = m + n.$$

Proof. Define

$$P = \{\, m \in \mathbb{N} \mid n + m = m + n \text{ for all natural numbers } n \,\}.$$

(BASE CASE 1) 0 is an element of P.
Proof. Define

$$Q = \{\, n \in \mathbb{N} \mid n + 0 = 0 + n \,\}.$$

0 belongs to Q.

For all n we have $n \in Q \implies n+1 \in Q$.
Proof. Let n be a natural number. Assume $n \in Q$.

$$\begin{aligned}
&(n+1)+0 \\
&= n+1 \\
&= (n+0)+1 \\
&= (0+n)+1 \\
&= 0+(n+1).
\end{aligned}$$

Qed.

Thus every natural number belongs to Q. Therefore 0 is an element of P.
Qed.

(BASE CASE 2) 1 is contained in P.
Proof. Define

$$Q = \{\, n \in \mathbb{N} \mid n+1 = 1+n \,\}.$$

0 is an element of Q.

For all natural numbers n we have $n \in Q \implies n+1 \in Q$.
Proof. Let n be a natural number. Assume that n is contained in Q.

$$\begin{aligned}
&(n+1)+1 \\
&= (1+n)+1 \\
&= 1+(n+1).
\end{aligned}$$

Qed.

Thus every natural number belongs to Q. Therefore 1 is an element of P.
Qed.

(INDUCTION STEP) For all natural numbers n we have $n \in P \implies n+1 \in P$.
Proof. Let n be an natural number. Assume $n \in P$.

$(n+1)+m = m+(n+1)$ for all natural numbers m.
Proof. Let m be a natural number.

$$\begin{aligned}
&(n+1)+m \\
&= n+(1+m) \\
&= (1+m)+n \\
&= (m+1)+n
\end{aligned}$$

$$= m + (n+1).$$

Qed. Qed.

Hence every natural number is an element of P. $\square$

Cancellation:

Proposition 1.11. For all natural numbers n, m, k we have

$$n + k = m + k \implies n = m.$$

Proof. Define

$$P = \{\, k \in \mathbb{N} \mid \text{for all natural numbers } n, m \text{ if } n + k = m + k \text{ then } n = m \,\}.$$

(BASE CASE) 0 is an element of P.

(INDUCTION STEP) For all k we have $k \in P \implies k + 1 \in P$.
Proof. Let k be a natural number. Assume $k \in P$.

For all natural numbers n, m we have $n + (k+1) = m + (k+1) \implies n = m$.
Proof. Let n, m be natural numbers. Assume $n + (k+1) = m + (k+1)$.
Then $(n + k) + 1 = (m + k) + 1$. Hence $n + k = m + k$. Thus $n = m$. Qed.

Hence the thesis (by 3rd Peano axiom). Qed.

Therefore every natural number is an element of P. $\square$

Corollary 1.12. For all n, m, k we have

$$k + n = k + m \implies n = m.$$

Proof. Let n, m, k be natural numbers. Assume $k + n = k + m$. We have $k + n = n + k$ and $k + m = m + k$. Hence $n + k = m + k$. Thus $n = m$. $\square$