1 Ordered pairs

[readtex `set-theory/sections/01_sets/01_sets.ftl.tex`]

Let u, v, w, u', v', w' denote objects. Let x, y, z, x', y', z' denote sets.

In this paragraph we introduce the *ordered pair* of two objects, following the definition proposed by Kuratowski.

Note that Naproche has ordered pairs already built in. Thus we have to formulate the definition of them as an axiom.

Axiom 1.1. $(u, v) = \{\{u\}, \{u, v\}\}$.

Proposition 1.2. Let u, v be objects. Then (u, v) is an object.

Proof. $\{u\}$ and $\{u, v\}$ are objects. Hence $\{\{u\}, \{u, v\}\}$ is an object. We have $(u, v) = \{\{u\}, \{u, v\}\}$. Thus (u, v) is an object. □

The central property of ordered pairs is that two of them agree if they agree on each component.

Proposition 1.3. If $(u, v) = (u', v')$ then $u = u'$ and $v = v'$.

Proof. Assume $(u, v) = (u', v')$. (1) Then $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$. Hence $(\{u\} = \{u'\}$ or $\{u\} = \{u', v'\})$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. Thus $(\{u\} = \{u'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}))$ or $(\{u\} = \{u', v'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}))$.

Case $\{u\} = \{u'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. We have $\{u\} = \{u'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = u' = v$. Hence $\{\{u\}, \{u, u\}\} = \{\{u\}, \{u, v'\}\}$ (by 1). Thus $\{\{u\}\} = \{\{u\}, \{u, v'\}\}$. Therefore $\{u\} = \{u, v'\}$. Consequently $v' = u = v$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End.

Case $\{u\} = \{u', v'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. We have $\{u\} = \{u', v'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = v = u'$. Hence $v = v'$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End. □

Definition 1.4. A pair is an object x such that $x = (u, v)$ for some objects u, v.

Let an ordered pair stand for a pair.

Definition 1.5. Let x be a pair. The first component of x is the object u such that $x = (u, v)$ for some object v.

Let the first entry of x stand for the first component of x.

Definition 1.6. Let x be a pair. The second component of x is the object v such that $x = (u, v)$ for some object u.

Let the second entry of x stand for the second component of x.

Lemma 1.7. Let x be a pair. Let u be the first component of x and v be the second component of x. Then $x = (u, v)$.

Lemma 1.8. Let x, y be pairs. Assume that the first component of x agrees with the first component of y and the second component of x agrees with the second component of y. Then $x = y$.