1 Functions

[readtex `set-theory/sections/01_sets/01_sets.ftl.tex`]

Let u, v, w denote objects. Let x, y, z denote sets. Let f, g, h denote functions.

In this section we introduce the notion of *functions* as some kind of “small” maps, i.e. maps whose domains are sets and whose values are objects.

1.1 Function axioms

Definition 1.1. Let f be a map. A value of f is an object v such that $v = f(u)$ for some $u \in \mathrm{dom}(f)$.

Definition 1.2. A fixed point of f is an element u of the domain of f such that $f(u) = u$.

As with sets we give an *extentionality axiom* for functions, which asserts that two functions are identical if their domains and values agree.

Axiom 1.3 (Function extensionality). Let f, g be functions. If $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(u) = g(u)$ for all $u \in \mathrm{dom}(f)$ then $f = g$.

Since functions are already built-in notions of $\mathbb{N}$aproche we cannot introduce them via a definition such as the following:

Definition. A function is a map f such that $\mathrm{dom}(f)$ is a set and every value of f is an object.

Instead we have to describe them axiomatically.

Axiom 1.4. Let f be a map. Assume that $\mathrm{dom}(f)$ is a set. Assume that every value of f is an object. Then f is a function.

Axiom 1.5. Let f be a function. Then f is a map.

Axiom 1.6. Let f be a function. Then $\mathrm{dom}(f)$ is a set.

Axiom 1.7. Let f be a function. Let x be an element of $\mathrm{dom}(f)$. Then $f(x)$ is an object.

The next axiom we introduce does not just fulfil definitional purposes. Instead it ensures that the image of any set under an arbitrary *mapping* is also a set. It plays an important role in the construction of certain infinite sets.

Axiom 1.8 (Replacement). Let f be a map and x be a set. There exists a set y such that $y = \{ f(u) \mid u \in \mathrm{dom}(f) \text{ and } u \in x \}$.

Corollary 1.9. Let f be a function. There exists a set y such that $y = \{ f(u) \mid u \in \mathrm{dom}(f) \}$.

Proof. Take $x = \mathrm{dom}(f)$. Then x is a set. Hence we can take a set y such that $y = \{ f(u) \mid u \in \mathrm{dom}(f) \}$ (by Replacement). Indeed f is a map. $\square$

1.2 The range

Using the replacement axiom we can easily define the *range* of a function as the set of all its values.

Definition 1.10. Let f be a function. $\mathrm{range}(f)$ is the set y such that $y = \{ f(u) \mid u \in \mathrm{dom}(f) \}$.

Let the range of f stand for $\mathrm{range}(f)$.

Proposition 1.11. v is a value of f iff $v \in \mathrm{range}(f)$.

Proof. Case v is a value of f. Take $u \in \mathrm{dom}(f)$ such that $v = f(u)$. v is an element. Hence $v \in \mathrm{range}(f)$. End.

Case $v \in \mathrm{range}(f)$. Then $v = f(u)$ for some $u \in \mathrm{dom}(f)$. Hence v is a value of f. End. $\square$

1.3 Functions between sets

In the following we mostly want to consider functions *between* two sets x and y, i.e. functions whose domain is x and which maps all elements of x into y.

Definition 1.12. A function of x is a function f such that $\mathrm{dom}(f) = x$.

Definition 1.13. A function to y is a function f such that $f(u) \in y$ for all $u \in \mathrm{dom}(f)$.

Let a function from x to y stand for a function f of x such that f is a function to y. Let $f : x \to y$ stand for f is a function from x to y.

Proposition 1.14. Let f be a function from x to y. Then $\mathrm{range}(f) \subseteq y$.

Proof. Let $v \in \mathrm{range}(f)$. Take $u \in x$ such that $v = f(u)$. Then $v \in y$. $\square$

Definition 1.15. A function on x is a function from x to x.

There are three important types of functions: Functions which are *injective*, i.e. one-to-one correspondences between their domain and range, functions which are *surjrective*, i.e. whose values match all elements of a given set, and functions which are *bijective*, i.e. both *injective* and *surjective*.

Definition 1.16. A function onto y is a function f such that $y = \mathrm{range}(f)$.

Let f surjects onto y stand for $y = \mathrm{range}(f)$.

Definition 1.17. A function from x onto y is a function f of x such that f is a function onto y.

Let $f : x \twoheadrightarrow y$ stand for f is a function from x onto y.

Proposition 1.18. f is a function onto range(f).

Proposition 1.19. Let f be a function onto y. Then f is a function to y.

Proof. Let $u \in \mathrm{dom}(f)$. Then $f(u) \in \mathrm{range}(f)$. Hence $f(u) \in y$. □

Definition 1.20. f is one to one iff for all $u, v \in \mathrm{dom}(f)$ if $f(u) = f(v)$ then $u = v$.

Definition 1.21. A function into y is an one to one function to y.

Definition 1.22. A function from x into y is a function f of x such that f is a function into y.

Let $f : x \hookrightarrow y$ stand for f is a function from x into y.

Definition 1.23. A bijection between x and y is a one to one function f from x onto y.

Let a bijection from x to y stand for a bijection between x and y.

Proposition 1.24. Let f be a function from x into y. Then f is a bijection between x and range(f).

Proof. f is one to one and f is a function from x onto range(f). Hence f is a bijection between x and range(f). □

Definition 1.25. A permutation of x is a bijection between x and x.

1.4 The identity function

Let us consider some special function: The *identity* function, which just maps any element of its domain to itself.

Lemma 1.26. There is a function ι of x such that $\iota(u) = u$ for all $u \in x$.

Proof. Define $\iota(u) = u$ for $u \in x$. □

Definition 1.27. id_x is the function of x such that $\mathrm{id}_x(u) = u$ for all $u \in x$.

Let the identity function on x stand for id_x.

Proposition 1.28. id_x is a permutation of x.

Proof. (1) id_x is a function of x.

(2) id_x is a function onto x. Proof. Let $v \in x$. Then $v = \mathrm{id}_x(v)$. Hence $v \in \mathrm{range}(\mathrm{id}_x)$. Qed.

(3) id_x is a function into x. Proof. Let $v, v' \in x$. Assume $\mathrm{id}_x(v) = \mathrm{id}_x(v')$. Then $v = v'$. Qed. □

1.5 Constant functions

Another importat class of functions is that of *constant* functions. Such functions map every element of their domain to the same value.

Lemma 1.29. Let x be a set and v be an element. There is a function c of x such that $c(u) = v$ for all $u \in x$.

Proof. Define $c(u) = v$ for $u \in x$. $\square$

Definition 1.30. $\mathrm{const}_{x,v}$ is the function of x such that $\mathrm{const}_{x,v}(u) = v$ for all $u \in x$.

Let the constant function on x with value v stand for $\mathrm{const}_{x,v}$.

Proposition 1.31. Assume $v \in y$. Then $\mathrm{const}_{x,v}$ is a function from x to y.

Proof. We have $\mathrm{dom}(\mathrm{const}_{x,v}) = x$ and $\mathrm{const}_{x,v}(u) = v$ for all $u \in x$. Hence $\mathrm{const}_{x,v}(u)$ is an element of y for all $u \in x$. Thus $\mathrm{range}(\mathrm{const}_{x,v}) \subseteq y$. Therefore $\mathrm{const}_{x,v}$ is a function from x to y. $\square$

Definition 1.32. Let f be a function. f is constant iff there exists an object v such that $f(u) = v$ for all $u \in \mathrm{dom}(f)$.

Proposition 1.33. $\mathrm{const}_{x,v}$ is constant.

Proof. We have $\mathrm{const}_{x,v}(u) = v$ for all $u \in x$. Hence the thesis. $\square$

1.6 Composition

Let us now consider some operations on functions. The first one, called *composition*, allows us to combine two functions to a new one by applying them one after another.

Lemma 1.34. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. Then there is a function h such that $\mathrm{dom}(h) = \mathrm{dom}(f)$ and $h(u) = g(f(u))$ for all $u \in \mathrm{dom}(h)$.

Proof. Define $h(u) = g(f(u))$ for $u \in \mathrm{dom}(f)$. $\square$

Definition 1.35. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. $g \circ f$ is the function h such that $\mathrm{dom}(h) = \mathrm{dom}(f)$ and $h(u) = g(f(u))$ for all $u \in \mathrm{dom}(h)$.

Let the composition of g and f stand for $g \circ f$.

Lemma 1.36. Let f be a function from x to y and g be a function from y to z. Then $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$.

Proposition 1.37. Let f be a function from x to y and g be a function from y to z. Then $g \circ f$ is a function from x to z.

Proof. (1) $g \circ f$ is a function of x. Indeed $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = x$.

(2) $\mathrm{range}(g \circ f) \subseteq z$. Proof. Let $w \in \mathrm{range}(g \circ f)$. Take $u \in x$ such that $(g \circ f)(u) = w$. Then $w = g(f(u))$. We have $f(u) \in y$. Hence $w \in z$. Qed. $\square$

Lemma 1.38. Let f be a function from x to y and g be a function from y to z. Then $\mathrm{dom}(g \circ f) = x$ and $\mathrm{range}(g \circ f) \subseteq z$.

Proposition 1.39. Let f be a function from x to y. Then $f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f$.

Proof. x is the domain of $f \circ \mathrm{id}_x$ and the domain of f and the domain of $\mathrm{id}_y \circ f$. $(f \circ \mathrm{id}_x)(u) = f(\mathrm{id}_x(u)) = f(u) = \mathrm{id}_y(f(u)) = (\mathrm{id}_y \circ f)(u)$ for all $u \in x$. Hence the thesis (by Function extensionality). $\square$

Proposition 1.40. Let f be a function from x to y and v be an element. Then $\mathrm{const}_{y,v} \circ f = \mathrm{const}_{x,v}$.

Proof. We have $\mathrm{dom}(\mathrm{const}_{y,v} \circ f) = \mathrm{dom}(f) = x = \mathrm{dom}(\mathrm{const}_{x,v})$. $(\mathrm{const}_{y,v} \circ f)(u) = \mathrm{const}_{y,v}(f(u)) = v = \mathrm{const}_{x,v}(u)$ for all $u \in x$. Hence the thesis (by Function extensionality). $\square$

Proposition 1.41. Let f be a function from y to z and $v \in y$. Then $f \circ \mathrm{const}_{x,v} = \mathrm{const}_{x,f(v)}$.

Proof. We have

$$\mathrm{dom}(f \circ \mathrm{const}_{x,v}) = \mathrm{dom}(\mathrm{const}_{x,v}) = x = \mathrm{dom}(\mathrm{const}_{x,f(v)}).$$

For all $u \in x$ we have

$$(f \circ \mathrm{const}_{x,v})(u) = f(\mathrm{const}_{x,v}(u)) = f(v) = \mathrm{const}_{x,f(v)}(u).$$

Hence the thesis (by Function extensionality). $\square$

Proposition 1.42. Let f be a function from x onto y and g be a function from y onto z. Then $g \circ f$ is a function from x onto z.

Proof. $g \circ f$ is a function of x.

Let us show that $g \circ f$ is a function onto z. Let $w \in z$. Take $v \in y$ such that $w = g(v)$. Take $u \in x$ such that $v = f(u)$. Then $w = g(f(u)) = (g \circ f)(u)$. End. $\square$

Proposition 1.43. Let f be a function from x into y and g be a function from y into z. Then $g \circ f$ is a function from x into z.

Proof. $g \circ f$ is a function of x.

Let us show that $g \circ f$ is one to one. Let $u, u' \in x$. Assume $(g \circ f)(u) = (g \circ f)(u')$. Then $g(f(u)) = g(f(u'))$. Hence $f(u) = f(u')$. Indeed $f(u), f(u') \in y$. Thus $u = u'$. End. $\square$

Corollary 1.44. Let f be a bijection between x and y and g be a bijection between y and z. Then $g \circ f$ is a bijection between x and z.

Proof. $g \circ f$ is a function from x onto z and a function into z. Hence the thesis. $\square$

1.7 Restriction

Another operation on functions is the *restriction* to a subset of their domain.

Lemma 1.45. Let $a \subseteq \mathrm{dom}(f)$. Then there is a function h of a such that $h(u) = f(u)$ for all $u \in a$.

Proof. Define $h(u) = f(u)$ for $u \in a$. $\square$

Definition 1.46. Let $a \subseteq \mathrm{dom}(f)$. $f \upharpoonright a$ is the function h of a such that $h(u) = f(u)$ for all $u \in a$.

Let the restriction of f to a stand for $f \upharpoonright a$.

Proposition 1.47. Let f be a function from x to y and $a \subseteq x$. Then $f \upharpoonright a$ is a function from a to y.

Proof. We have $\mathrm{dom}(f \upharpoonright a) = a$. Then $(f \upharpoonright a)(u) = f(u) \in y$ for all $u \in a$. Hence $f \upharpoonright a$ is a function from a to y. $\square$

Proposition 1.48. Let $a \subseteq x$. Then $\mathrm{id}_x \upharpoonright a = \mathrm{id}_a$.

Proof. We have $\mathrm{dom}(\mathrm{id}_x \upharpoonright a) = a = \mathrm{dom}(\mathrm{id}_a)$. $(\mathrm{id}_x \upharpoonright a)(u) = \mathrm{id}_x(u) = u = \mathrm{id}_a(u)$ for all $u \in a$. Hence the thesis (by Function extensionality). $\square$

Proposition 1.49. Let v be an element and $a \subseteq x$. Then $\mathrm{const}_{x,v} \upharpoonright a = \mathrm{const}_{a,v}$.

Proof. We have $\mathrm{dom}(\mathrm{const}_{x,v} \upharpoonright a) = a = \mathrm{dom}(\mathrm{const}_{a,v})$. $(\mathrm{const}_{x,v} \upharpoonright a)(u) = \mathrm{const}_{x,v}(u) = v = \mathrm{const}_{a,v}(u)$ for all $u \in a$. Hence the thesis (by Function extensionality). $\square$

Proposition 1.50. Let f be an one to one function from x to y and $a \subseteq x$. Then $f \upharpoonright a$ is one to one.

Proof. Let $u, u' \in a$. Assume $(f \upharpoonright a)(u) = (f \upharpoonright a)(u')$. Then $f(u) = f(u')$. Hence $u = u'$. $\square$