1 Sets

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We base our formalization on the notions of classes, sets and objects which are hardcoded into $\mathbb{N}$aproche.

Let x, y, z denote sets.

Sets are regarded as classes which are also objects, the latter being entities that are in some sense small enough to be contained in classes.

Axiom 1.1. x is a class.

Axiom 1.2. x is an objects.

Axiom 1.3. Let u be an element of x. Then u is an object.

1.1 Subsets

Let us continue with the notion of *subsets*, i.e. sets which are included in some other set.

Definition 1.4. A subset of x is a set y such that every element of y is an element of x.

Let $y \subseteq x$ stand for y is a subset of x. Let $y \subset x$ stand for $y \subseteq x$. Let a superset of x stand for a set y such that $x \subseteq y$. Let $y \supseteq x$ stand for y is a superset of x. Let $y \supset x$ stand for $y \supseteq x$. Let x includes y stand for $y \subseteq x$. Let y is included in x stand for x includes y.

Definition 1.5. A proper subset of x is a subset of x that is not equal to x.

Let $y \subsetneq x$ stand for x is a proper subset of x. Let a proper superset of x stand for a set y such that $x \subsetneq y$. Let $y \supsetneq x$ stand for y is a proper superset of x.

Proposition 1.6. $x \subseteq x$.

Proposition 1.7. If $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$.

1.2 Set extensionality

Since the only distinguishing feature of a set should be its elements, let us add the following *extensionality axiom* to our theory.

Axiom 1.8 (Set extensionality). If $x \subseteq y$ and $y \subseteq x$ then $x = y$.

1.3 Separation

Our next axiom ensures that the universe of sets is closed under taking subcollections. This means that any subcollection of a given set is itself a set.

Axiom 1.9 (Separation). Let C be a collection and x be a set. Assume that every element of C is contained in x. Then C is a set.

1.4 Set existence

Up to now our theory does not admit the existence of a single set. This is changed by the following axiom.

Axiom 1.10 (Set existence). There exists a set.

1.5 The empty set

The last to axioms allow us now to show that there exists a unique set that does not contain any element – the *empty* set.

Definition 1.11. x is empty iff x has no elements.

Let x is nonempty stand for x is not empty.

Lemma 1.12. There exists an empty set.

Proof. Define $C = \{ u \mid \text{contradiction} \}$. Take a set x (by Set existence). Then every element of C is contained in x. Hence C is a set (by Separation). C has no element. Hence the thesis. $\square$

Lemma 1.13. If x and y are empty then $x = y$.

Proof. Assume that x and y are empty. Then every element of x is an element of y and every element of y is an element of x. Hence $x \subseteq y$ and $y \subseteq x$. Thus $x = y$. $\square$

Definition 1.14. $\emptyset$ is the empty set.

Let $\{\}$ stand for $\emptyset$. Let the empty set stand for $\emptyset$.

Proposition 1.15. $\emptyset$ is a subset of every set.

Proof. Let x be a set. Then every element of $\emptyset$ is an element of x. Indeed $\emptyset$ has no element. Hence $\emptyset \subseteq x$. $\square$

1.6 Pairing

Let us now consider an axiom which allows us to collect two given objects into a set which contains exactly these two ones.

Axiom 1.16 (Pairing). Let u, v be objects. There exists a set z such that $z = \{ w \mid w = u \text{ or } w = v \}$.

Definition 1.17. Let u, v be elements. $\{u, v\}$ is the set z such that $z = \{ w \mid w = u \text{ or } w = v \}$.

Let the unordered pair of u and v stand for $\{u, v\}$.

Lemma 1.18. Let u be an element. There exists a set z such that $z = \{ w \mid w = u \}$.

Proof. Take $z = \{u, u\}$. Then $z = \{ w \mid w = u \}$. □

Definition 1.19. Let u be an element. $\{u\}$ is the set z such that $z = \{ w \mid w = u \}$.

Let the singleton set of u stand for $\{u\}$.

Definition 1.20. A singleton set is a set x such that $x = \{u\}$ for some element u.

1.7 Set-systems

Sets whose elements are all sets as well are called *set- systems* or *systems of sets*.

Definition 1.21. A system of sets is a set X such that every element of X is a set.

Let X, Y, Z denote systems of sets.
Let a set of X stand for an element of X.

Definition 1.22. A system of nonempty sets is a system of sets X such that every set of X is nonempty.

Proposition 1.23. $\{x\}$ is a system of sets.

Proposition 1.24. $\{x, y\}$ is a system of sets.

Definition 1.25. A system of subsets of x is a set X such that every set of X is a subset of x.

Proposition 1.26. Every system of subsets of x is a system of sets.

1.8 Intersections

Considering a set-system X we can extract all objects which are contained in *every* member of X into a new set, called the *intersection* over X.

Lemma 1.27. Let x be a nonempty system of sets. Then there exists a set z such that $z = \{ u \mid u \text{ is contained in every member of } x \}$.

Proof. Take an element y of x. Then y is a set. (1) Define $z =$

$\{ u \mid u$ is contained in every element of $x \}$. Every element of z is contained in y. Hence z is a set. Therefore the thesis (by 1). $\square$

Definition 1.28. Let x be a nonempty system of sets. $\bigcap x$ is the set z such that $z = \{ u \mid u$ is contained in every member of $x \}$.

Let the intersection over x stand for $\bigcap x$.

The notion of the intersection over a set-system can be used to provide an operation which maps two sets to the set of all elements they have in common.

Lemma 1.29. Let x, y be sets. Then there exists a set z such that $z = \{ u \mid u \in x$ and $u \in y \}$.

Proof. Take $z = \bigcap \{x, y\}$. Then

$$z = \{ u \mid u \text{ is contained in every element of } \{x, y\} \} .$$

Hence $z = \{ u \mid u \in x$ and $u \in y \}$. $\square$

Definition 1.30. $x \cap y$ is the set z such that $z = \{ u \mid u \in x$ and $u \in y \}$.

Let the intersection of x and y stand for $x \cap y$.

Proposition 1.31. $\bigcap \{x, y\} = x \cap y$.

Proof. Let us show that $\bigcap \{x, y\} \subseteq x \cap y$. Let $u \in \bigcap \{x, y\}$. Then u is an element of every element of $\{x, y\}$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$. End.

Let us show that $x \cap y \subseteq \bigcap \{x, y\}$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence u is an element of every element of $\{x, y\}$. Thus $u \in \bigcap \{x, y\}$. End. $\square$

Corollary 1.32. $\bigcap \{x\} = x$.

Proof. $\bigcap \{x\} = \bigcap \{x, x\} = x \cap x = x$. $\square$

Proposition 1.33. Let x be a nonempty system of sets. Then $y \subseteq \bigcap x$ iff y is a subset of every element of x.

Proof. Case $y \subseteq \bigcap x$. Let z be an element of x. Let $u \in y$. Then $u \in \bigcap x$. Hence $u \in z$. End.

Case y is a subset of every element of x. Let $u \in y$. Then $u \in z$ for all sets z such that $z \in x$. Hence $u \in \bigcap x$. End. $\square$

An important notion is that of *disjoint* sets, i.e. sets wich do not have any elements in common.

Definition 1.34. x and y are disjoint iff $x \cap y = \emptyset$.

Obviously this yields a symmetric relation on the universe of sets.

Proposition 1.35. If x and y are disjoint then y and x are disjoint.

Proof. Assume that x and y are disjoint. Then $x \cap y$ is empty. Hence there is no element u such that $u \in x$ and $u \in y$. Thus $y \cap x$ is empty. Therefore y and x are disjoint. □

1.9 Unions

Analogous to the definition of the intersection over a set-system we now want to consider for a given set-system X the collection of all elements which lie in *some* member of X. To ensure that this collection is a set we need an additional axiom.

Axiom 1.36 (Union). Let x be a system of sets. Then there exists a set z such that $z = \{ u \mid u \text{ is contained in some element of } x \}$.

Definition 1.37. Let x be a system of sets. $\bigcup x$ is the set z such that $z = \{ u \mid u \text{ is contained in some element of } x \}$.

Let the union over x stand for $\bigcup x$.

Lemma 1.38. Let x, y be sets. Then there exists a set z such that $z = \{ u \mid u \in x \text{ or } u \in y \}$.

Proof. Take $z = \bigcup\{x, y\}$. Then

$$z = \{ u \mid u \text{ is contained in some element of } \{x, y\} \} .$$

Hence $z = \{ u \mid u \in x \text{ or } u \in y \}$. □

Definition 1.39. $x \cup y$ is the set z such that $z = \{ w \mid w \in x \text{ or } w \in y \}$.

Let the union of x and y stand for $x \cup y$.

Proposition 1.40. $\bigcup\{x, y\} = x \cup y$.

Proof. Let us show that $\bigcup\{x, y\} \subseteq x \cup y$. Let $u \in \bigcup\{x, y\}$. Then u is an element of some element of $\{x, y\}$. Hence $u \in x$ or $u \in y$. Thus $u \in x \cup y$. End.

Let us show that $x \cup y \subseteq \bigcup\{x, y\}$. Let $u \in x \cup y$. Then $u \in x$ or $u \in y$. Hence u is an element of some element of $\{x, y\}$. Thus $u \in \bigcup\{x, y\}$. End. □

Corollary 1.41. $\bigcup\{x\} = x$.

Proof. $\bigcup\{x\} = \bigcup\{x, x\} = x \cup x = x$. □

Proposition 1.42. Let x be a system of sets. Then $\bigcup x \subseteq y$ iff every element of x is a subset of y.

Proof. Case $\bigcup x \subseteq y$. Let z be an element of x. Let $u \in z$. Then u is an element of some element of x. Hence $u \in \bigcup x$. Thus $u \in y$. End.

Case every element of x is a subset of y. Let $u \in \bigcup x$. Take a set z such that $z \in x$ and $u \in z$. Then z is a subset of y. Hence $u \in y$. End. $\square$

Proposition 1.43. $\bigcup \emptyset = \emptyset$.

Proof. $\emptyset$ has no elements. Hence there is no $x \in \emptyset$ that has an element. Thus $\bigcup \emptyset$ is empty. Therefore $\bigcup \emptyset = \emptyset$. $\square$

1.10 Partitions

Another important notion is that of a *partition* of a set x. i.e. a set which splits x into pairwise disjoint subsets.

Definition 1.44. A partition of x is a system of sets P such that every element of P is a subset of x and every element of x is contained in some member of P and all distinct sets A, B of P are pairwise disjoint.

Proposition 1.45. Let P be a partition of x. Then $x = \bigcup P$.

Proof. Let us show that $x \subseteq \bigcup P$. Let $u \in x$. Take a set A of P such that $u \in A$. Then we have $u \in \bigcup P$. End.

Let us show that $\bigcup P \subseteq x$. Let $u \in \bigcup P$. Then we can take a set A of P such that $u \in A$. A is a subset of x. Hence $u \in x$. End. $\square$

1.11 Complements

Let us define another operation on sets: The *(relative) complement*.

Lemma 1.46. Let x, y be sets. There exists a set z such that $z = \{ w \mid w \in x \text{ and } w \notin y \}$.

Proof. Define $z = \{ w \mid w \in x \text{ and } w \notin y \}$. Then every element of z is contained in x. Hence z is a set (by Separation). $\square$

Definition 1.47. $x \setminus y$ is the set such that $x \setminus y = \{ w \mid w \in x \text{ and } w \notin y \}$.

Let the complement of y in x stand for $x \setminus y$.

1.12 Computation laws

Now that we are provided with the most common operations on sets let us have a look on their algebraic properties.

Commutativity of union and intersection:

Proposition 1.48.

$$x \cup y = y \cup x.$$

Proof. Let us show that $x \cup y \subseteq y \cup x$. Let $u \in x \cup y$. Then $u \in x$ or $u \in y$. Hence $u \in y$ or $u \in x$. Thus $u \in y \cup x$. End.

Let us show that $y \cup x \subseteq x \cup y$. Let $u \in y \cup x$. Then $u \in y$ or $u \in x$. Hence $u \in x$ or $u \in y$. Thus $u \in x \cup y$. End. □

Proposition 1.49.

$$x \cap y = y \cap x.$$

Proof. Let us show that $x \cap y \subseteq y \cap x$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \in y$ and $u \in x$. Thus $u \in y \cap x$. End.

Let us show that $y \cap x \subseteq x \cap y$. Let $u \in y \cap x$. Then $u \in y$ and $u \in x$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$. End. □

Associativity of union and intersection:

Proposition 1.50.

$$((x \cup y) \cup z) = x \cup (y \cup z).$$

Proof. Let us show that $((x \cup y) \cup z) \subseteq x \cup (y \cup z)$. Let $u \in (x \cup y) \cup z$. Then $u \in x \cup y$ or $u \in z$. Hence $u \in x$ or $u \in y$ or $u \in z$. Thus $u \in x$ or $u \in (y \cup z)$. Therefore $u \in x \cup (y \cup z)$. End.

Let us show that $x \cup (y \cup z) \subseteq (x \cup y) \cup z$. Let $u \in x \cup (y \cup z)$. Then $u \in x$ or $u \in y \cup z$. Hence $u \in x$ or $u \in y$ or $u \in z$. Thus $u \in x \cup y$ or $u \in z$. Therefore $u \in (x \cup y) \cup z$. End. □

Proposition 1.51.

$$((x \cap y) \cap z) = x \cap (y \cap z).$$

Proof. Let us show that $((x \cap y) \cap z) \subseteq x \cap (y \cap z)$. Let $u \in (x \cap y) \cap z$. Then $u \in x \cap y$ and $u \in z$. Hence $u \in x$ and $u \in y$ and $u \in z$. Thus $u \in x$ and $u \in (y \cap z)$. Therefore$u \in x \cap (y \cap z)$. End.

Let us show that $x \cap (y \cap z) \subseteq (x \cap y) \cap z$. Let $u \in x \cap (y \cap z)$. Then $u \in x$ and $u \in y \cap z$. Hence $u \in x$ and $u \in y$ and $u \in z$. Thus $u \in x \cap y$ and $u \in z$. Therefore$u \in (x \cap y) \cap z$. End. □

Distributivity of union and intersection:

Proposition 1.52.

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z).$$

Proof. Let us show that $x \cap (y \cup z) \subseteq (x \cap y) \cup (x \cap z)$. Let $u \in x \cap (y \cup z)$. Then $u \in x$ and $u \in y \cup z$. Hence $u \in x$ and ($u \in y$ or $u \in z$). Thus ($u \in x$ and $u \in y$) or ($u \in x$ and $u \in z$). Therefore $u \in x \cap y$ or $u \in x \cap z$. Hence $u \in (x \cap y) \cup (x \cap z)$. End.

Let us show that $((x \cap y) \cup (x \cap z)) \subseteq x \cap (y \cup z)$. Let $u \in (x \cap y) \cup (x \cap z)$. Then $u \in x \cap y$ or $u \in x \cap z$. Hence ($u \in x$ and $u \in y$) or ($u \in x$ and $u \in z$). Thus $u \in x$ and ($u \in y$ or $u \in z$). Therefore $u \in x$ and $u \in y \cup z$. Hence$u \in x \cap (y \cup z)$. End. □

Proposition 1.53.

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z).$$

Proof. Let us show that $x \cup (y \cap z) \subseteq (x \cup y) \cap (x \cup z)$. Let $u \in x \cup (y \cap z)$. Then $u \in x$ or $u \in y \cap z$. Hence $u \in x$ or ($u \in y$ and $u \in z$). Thus ($u \in x$ or $u \in y$) and ($u \in$ x or $u \in z$). Therefore $u \in x \cup y$ and $u \in x \cup z$. Hence $u \in (x \cup y) \cap (x \cup z)$. End.

Let us show that $((x \cup y) \cap (x \cup z)) \subseteq x \cup (y \cap z)$. Let $u \in (x \cup y) \cap (x \cup z)$. Then $u \in x \cup y$ and $u \in x \cup z$. Hence ($u \in x$ or $u \in y$) and ($u \in x$ or $u \in z$). Thus $u \in x$ or ($u \in y$ and $u \in z$). Therefore $u \in x$ or $u \in y \cap z$. Hence $u \in x \cup (y \cap z)$. End. □

Idempocy laws for union and intersection:

Proposition 1.54.

$$x \cup x = x.$$

Proof. $x \cup x = \{ u \mid u \in x \text{ or } u \in x \}$. Hence $x \cup x = \{ u \mid u \in x \}$. Thus $x \cup x = x$. □

Proposition 1.55.

$$x \cap x = x.$$

Proof. $x \cap x = \{ u \mid u \in x \text{ and } u \in x \}$. Hence $x \cap x = \{ u \mid u \in x \}$. Thus $x \cap x = x$. □

Distributivity of complement wrt. union and intersection:

Proposition 1.56.

$$x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z).$$

Proof. Let us show that $x \setminus (y \cap z) \subseteq (x \setminus y) \cup (x \setminus z)$. Let $u \in x \setminus (y \cap z)$. Then $u \in x$ and $u \notin y \cap z$. Hence it is wrong that ($u \in y$ and $u \in z$). Thus $u \notin y$ or $u \notin z$. Therefore $u \in x$ and ($u \notin y$ or $u \notin z$). Then ($u \in x$ and $u \notin y$) or ($u \in x$ and $u \notin z$). Hence $u \in x \setminus y$ or $u \in x \setminus z$. Thus $u \in (x \setminus y) \cup (x \setminus z)$. End.

Let us show that $((x \setminus y) \cup (x \setminus z)) \subseteq x \setminus (y \cap z)$. Let $u \in (x \setminus y) \cup (x \setminus z)$. Then $u \in x \setminus y$ or $u \in x \setminus z$. Hence ($u \in x$ and $u \notin y$) or ($u \in x$ and $u \notin z$). Thus $u \in x$ and ($u \notin y$ or $u \notin z$). Therefore $u \in x$ and not ($u \in y$ and $u \in z$). Then $u \in x$ and not $u \in y \cap z$. Hence $u \in x \setminus (y \cap z)$. End. $\square$

Proposition 1.57.

$$x \setminus (y \cup z) = (x \setminus y) \cap (x \setminus z).$$

Proof. Let us show that $x \setminus (y \cup z) \subseteq (x \setminus y) \cap (x \setminus z)$. Let $u \in x \setminus (y \cup z)$. Then $u \in x$ and $u \notin y \cup z$. Hence it is wrong that ($u \in y$ or $u \in z$). Thus $u \notin y$ and $u \notin z$. Therefore $u \in x$ and ($u \notin y$ and $u \notin z$). Then ($u \in x$ and $u \notin y$) and ($u \in x$ and $u \notin z$). Hence $u \in x \setminus y$ and $u \in x \setminus z$. Thus $u \in (x \setminus y) \cap (x \setminus z)$. End.

Let us show that $((x \setminus y) \cap (x \setminus z)) \subseteq x \setminus (y \cup z)$. Let $u \in (x \setminus y) \cap (x \setminus z)$. Then $u \in x \setminus y$ and $u \in x \setminus z$. Hence ($u \in x$ and $u \notin y$) and ($u \in x$ and $u \notin z$). Thus $u \in x$ and ($u \notin y$ and $u \notin z$). Therefore $u \in x$ and not ($u \in y$ or $u \in z$). Then $u \in x$ and not $u \in y \cup z$. Hence $u \in x \setminus (y \cup z)$. End. $\square$

Subset laws:

Proposition 1.58.

$$x \subseteq x \cup y.$$

Proof. Let $u \in x$. Then $u \in x$ or $u \in y$. Hence $u \in x \cup y$. $\square$

Proposition 1.59.

$$x \cap y \subseteq x.$$

Proof. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \in x$. $\square$

Proposition 1.60.

$$x \subseteq y \iff x \cup y = y.$$

Proof. Case $x \subseteq y$.

Let us show that $x \cup y \subseteq y$. Let $u \in x \cup y$. Then $u \in x$ or $u \in y$. If $u \in x$ then $u \in y$. Hence $u \in y$. End.

Let us show that $y \subseteq x \cup y$. Let $u \in y$. Then $u \in x$ or $u \in y$. Hence $u \in x \cup y$. End. End.

Case $x \cup y = y$. Let $u \in x$. Then $u \in x$ or $u \in y$. Hence $u \in x \cup y = y$. End. $\square$

Proposition 1.61.

$$x \subseteq y \iff x \cap y = x.$$

Proof. Case $x \subseteq y$.

Let us show that $x \cap y \subseteq x$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \in x$. End.

Let us show that $x \subseteq x \cap y$. Let $u \in x$. Then $u \in y$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$. End. End.

Case $x \cap y = x$. Let $u \in x$. Then $u \in x \cap y$. Hence $u \in x$ and $u \in y$. Thus $u \in y$. End. $\square$

Complement laws:

Proposition 1.62.

$$x \setminus x = \emptyset.$$

Proof. $x \setminus x$ has no elements. Indeed $x \setminus x = \{ u \mid u \in x \text{ and } u \notin x \}$. Hence the thesis. $\square$

Proposition 1.63.

$$x \setminus \emptyset = x.$$

Proof. $x \setminus \emptyset = \{ u \mid u \in x \text{ and } u \notin \emptyset \}$. No element is an element of $\emptyset$. Hence $x \setminus \emptyset = \{ u \mid u \in x \}$. Then we have the thesis. $\square$

Proposition 1.64.

$$x \setminus (x \setminus y) = x \cap y.$$

Proof. Let us show that $x \setminus (x \setminus y) \subseteq x \cap y$. Let $u \in x \setminus (x \setminus y)$. Then $u \in x$ and $u \notin x \setminus y$. Hence $u \notin x$ or $u \in y$. Thus $u \in y$. Therefore $u \in x \cap y$. End.

Let us show that $x \cap y \subseteq x \setminus (x \setminus y)$. Let $u \in x \cap y$. Then $u \in x$ and $u \in y$. Hence $u \notin x$ or $u \in y$. Thus $u \notin x \setminus y$. Therefore $u \in x \setminus (x \setminus y)$. End. $\square$

Proposition 1.65.

$$y \subseteq x \iff x \setminus (x \setminus y) = y.$$

Proof. Case $y \subseteq x$. Obvious.

Case $x \setminus (x \setminus y) = y$. Then every element of y is an element of $x \setminus (x \setminus y)$. Thus every element of y is an element of x. Then we have the thesis. End. $\square$

Proposition 1.66.

$$x \cap (y \setminus z) = (x \cap y) \setminus (x \cap z).$$

Proof. Let us show that $x \cap (y \setminus z) \subseteq (x \cap y) \setminus (x \cap z)$. Let $u \in x \cap (y \setminus z)$. Then $u \in x$ and $u \in y \setminus z$. Hence $u \in x$ and $u \in y$. Thus $u \in x \cap y$ and $u \notin z$. Therefore $u \notin x \cap z$. Then we have $u \in (x \cap y) \setminus (x \cap z)$. End.

Let us show that $((x \cap y) \setminus (x \cap z)) \subseteq x \cap (y \setminus z)$. Let $u \in (x \cap y) \setminus (x \cap z)$. Then $u \in x$ and $u \in y$. $u \notin x \cap z$. Hence $u \notin z$. Thus $u \in y \setminus z$. Therefore $u \in x \cap (y \setminus z)$. End. $\square$