1 Induction

[readtex `arithmetic/sections/02_ordering/01_ordering.ftl.tex`]

Let k, l, m, n denote natural numbers.

When we introduced the Peano axioms we came across an induction axiom which gives us a method to prove universal assertions about the natural numbers. In this section we will give some reformulations of this induction principle.

1.1 Least natural numbers

As a first example of such a reformulation we will show in this paragraph that every collection of natural numbers admits a smallest element.

Let P denote a class.

Definition 1.1. A least natural number of P is a natural number n such that $n \in P$ and no natural number that is less than n belongs to P.

Lemma 1.2. Let n, m be least natural numbers of P. Then $n = m$.

Proof. Assume $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $n \notin P$ and if $m < n$ then $m \notin P$. Contradiction. Therefore $n = m$. $\square$

Theorem 1.3. Assume that P contains some natural number. Then P has a least natural number.

Proof. Assume the contrary. Define

$$Q = \{\, n \in \mathbb{N} \mid \text{n is less than any natural number } m \text{ such that } m \in P \,\}.$$

Let us show that every natural number belongs to Q.

(BASE CASE) Q contains 0.
Proof. If P contains 0 then 0 is the least natural number of P. Hence 0 is less than any natural number m such that $m \in P$. Therefore Q contains 0. Qed.

For all natural numbers n we have $n \in Q \implies n + 1 \in Q$.
Proof. Let n be a natural number. Assume $n \in Q$. Then n is less than any natural number m such that $m \in P$. Assume that Q does not contain $n+1$. Then we can take a natural number m such that $m \in P$ and $n + 1 \not< m$. Hence $n < m \leq n + 1$. Thus $m = n + 1$. Then $n + 1$ is the least natural number of P. Contradiction. Qed. End.

Then every natural number is less than any natural number n such that $n \in P$. Hence there is no natural number n such that $n \in P$. Contradiction. $\square$

1.2 Induction via predecessors

Next we will see how to merge the base and induction step of a proof by induction into a single step. This yields a new induction principle.

Theorem 1.4. Assume for all natural numbers n if P contains all predecessors of n then P contains n. Then P contains every natural number.

Proof. Assume the contrary. Take a natural number n such that P does not contain n. Define $Q = \{ k \in \mathbb{N} \mid k \notin P \}$. Then Q contains n. Thus we can take a least natural number m of Q. Hence Q does not contain any predecessor of m. Therefore P contains all predecessors of m. Thus P contains m. Contradiction. □

1.3 Induction above a certain number

In our induction principle given by the 3^{rd} Peano axiom we considered the number 0 as the starting point of an inductive proof. But we can as well start at any arbitrary number k to prove that a statement holds for all natural numbers from k on.

Theorem 1.5. Let k be a natural number such that $k \in P$. Suppose that for all natural numbers n such that $n \geq k$ we have $n \in P \implies n+1 \in P$. Then for every natural number n such that $n \geq k$ we have $n \in P$.

Proof. Define

$$Q = \{ n \in \mathbb{N} \mid \text{if } n \geq k \text{ then } n \in P \}.$$

Let us show that every natural number belongs to Q.

(BASE CASE) We have $0 \in Q$.

(INDUCTION STEP) For all natural numbers n we have $n \in Q \implies n+1 \in Q$.
Proof. Let n be a natural number. Assume $n \in Q$.

If $n+1 \geq k$ then $n+1 \in P$.
Proof. Assume $n+1 \geq k$.

Case $n < k$. Then $n+1 = k$. Hence $n+1 \in P$. End.

Case $n \geq k$. Then $n \in P$. Hence $n+1 \in P$. End. Qed.

Thus we have $n+1 \in Q$. Qed. End.

Therefore Q contains every natural number. Hence the thesis. □