

# Chapter 1

## Ordering

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### 1.1 Definitions and immediate consequences

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**Definition 1.1.** Let  $n, m$  be natural numbers.  $n < m$  iff there exists a nonzero natural number  $k$  such that  $m = n + k$ .

Let  $n$  is less than  $m$  stand for  $n < m$ . Let  $n > m$  stand for  $m < n$ . Let  $n$  is greater than  $m$  stand for  $n > m$ . Let  $n \not< m$  stand for  $n$  is not less than  $m$ . Let  $n \not> m$  stand for  $n$  is not greater than  $m$ .

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**Definition 1.2.** Let  $n$  be a natural number.  $\mathbb{N}_{< n} = \{k \in \mathbb{N} \mid k < n\}$ .

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**Definition 1.3.** Let  $n$  be a natural number.  $\mathbb{N}_{>n} = \{k \in \mathbb{N} \mid k > n\}$ .

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**Definition 1.4.** Let  $n$  be a natural number.  $n$  is positive iff  $n > 0$ .

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**Definition 1.5.** Let  $n, m$  be natural numbers.  $n \leq m$  iff there exists a natural number  $k$  such that  $m = n + k$ .

Let  $n$  is less than or equal to  $m$  stand for  $n \leq m$ . Let  $n \geq m$  stand for  $m \leq n$ . Let  $n$  is greater than or equal to  $m$  stand for  $n \geq m$ . Let  $n \not\leq m$  stand for  $n$  is not less than or equal to  $m$ . Let  $n \not\geq m$  stand for  $n$  is not greater than or equal to  $m$ .

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**Definition 1.6.** Let  $n$  be a natural number.  $\mathbb{N}_{\leq n} = \{k \in \mathbb{N} \mid k \leq n\}$ .

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**Definition 1.7.** Let  $n$  be a natural number.  $\mathbb{N}_{\geq n} = \{k \in \mathbb{N} \mid k \geq n\}$ .

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**Proposition 1.8.** Let  $n, m$  be natural numbers.  $n \leq m$  iff  $n < m$  or  $n = m$ .

*Proof.* Case  $n \leq m$ . Take a natural number  $k$  such that  $m = n + k$ . If  $k = 0$  then  $n = m$ . If  $k \neq 0$  then  $n < m$ . End.

Case  $n < m$  or  $n = m$ . If  $n < m$  then there is a positive natural number  $k$  such that  $m = n + k$ . If  $n = m$  then  $m = n + 0$ . Thus if  $n < m$  then there is a natural number  $k$  such that  $m = n + k$ . End.  $\square$

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**Definition 1.9.** Let  $n$  be a natural number. A predecessor of  $n$  is a natural number that is less than  $n$ .

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**Definition 1.10.** Let  $n$  be a natural number. A successor of  $n$  is a natural number that is greater than  $n$ .

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**Proposition 1.11.** Let  $n$  be a natural number. Then  $n$  is positive iff  $n$  is nonzero.

*Proof.* Case  $n$  is positive. Take a positive natural number  $k$  such that  $n = 0 + k = k$ . Then we have  $n \neq 0$ . End.

Case  $n$  is nonzero. Take a natural number  $k$  such that  $n = k + 1$ . Then  $n = 0 + (k + 1)$ .  $k + 1$  is positive. Hence  $0 < n$ . End.  $\square$

## 1.2 Basic properties

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**Proposition 1.12.** Let  $n$  be a natural number. Then

$$n \not< n.$$

*Proof.* Assume the contrary. Then we can take a positive natural number  $k$  such that  $n = n + k$ . Then we have  $0 = k$ . Contradiction.  $\square$

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**Proposition 1.13.** Let  $n, m$  be natural numbers. Then

$$n < m \text{ implies } n \neq m.$$

*Proof.* Assume  $n < m$ . Take a positive natural number  $k$  such that  $m = n + k$ . If  $n = m$  then  $k = 0$ . Hence  $n \neq m$ .  $\square$

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**Proposition 1.14.** Let  $n, m$  be natural numbers. Then

$$(n \leq m \text{ and } m \leq n) \text{ implies } n = m.$$

*Proof.* Assume  $n \leq m$  and  $m \leq n$ . Take natural numbers  $k, l$  such that  $m = n + k$  and  $n = m + l$ . Then  $m = (m + l) + k = m + (l + k)$ . Hence  $l + k = 0$ . Thus  $l = 0 = k$ . Indeed if  $l \neq 0$  or  $k \neq 0$  then  $l + k$  is the direct successor of some natural number. Therefore  $m = n$ .  $\square$

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**Proposition 1.15.** Let  $n, m, k$  be natural numbers. Then

$$n < m < k \text{ implies } n < k.$$

*Proof.* Assume  $n < m < k$ . Take a positive natural number  $a$  such that  $m = n + a$ . Take a positive natural number  $b$  such that  $k = m + b$ . Then  $k = (n + a) + b = n + (a + b)$ .  $a + b$  is positive. Hence  $n < k$ .  $\square$

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**Proposition 1.16.** Let  $n, m, k$  be natural numbers. Then

$$n \leq m \leq k \text{ implies } n \leq k.$$

*Proof.* Assume  $n \leq m \leq k$ . Case  $n = m = k$ . Obvious. Case  $n = m < k$ . Obvious. Case  $n < m = k$ . Obvious. Case  $n < m < k$ . Obvious.  $\square$

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**Proposition 1.17.** Let  $n, m, k$  be natural numbers. Then

$$n \leq m < k \text{ implies } n < k.$$

*Proof.* Assume  $n \leq m < k$ . If  $n = m$  then  $n < k$ . If  $n < m$  then  $n < k$ .  $\square$

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**Proposition 1.18.** Let  $n, m, k$  be natural numbers. Then

$$n < m \leq k \text{ implies } n < k.$$

*Proof.* Assume  $n < m \leq k$ . If  $m = k$  then  $n < k$ . If  $m < k$  then  $n < k$ .  $\square$

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**Proposition 1.19.** Let  $n, m$  be natural numbers. Then

$$n < m \quad \text{implies} \quad n + 1 \leq m.$$

*Proof.* Assume  $n < m$ . Take a positive natural number  $k$  such that  $m = n + k$ .

Case  $k = 1$ . Then  $m = n + 1$ . Hence  $n + 1 \leq m$ . End.

Case  $k \neq 1$ . Then we can take a natural number  $l$  such that  $k = l + 1$ . Then  $m = n + (l + 1) = (n + l) + 1 = (n + 1) + l$ .  $l$  is positive. Thus  $n + 1 < m$ . End.  $\square$

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**Proposition 1.20.** Let  $n, m$  be natural numbers. Then  $n < m$  or  $n = m$  or  $n > m$ .

*Proof.* Define  $\Phi = \{m' \in \mathbb{N} \mid n < m' \text{ or } n = m' \text{ or } n > m'\}$ .

(1)  $\Phi$  contains 0.

(2) For all  $m' \in \Phi$  we have  $m' + 1 \in \Phi$ .

*Proof.* Let  $m' \in \Phi$ .

Case  $n < m'$ . Obvious.

Case  $n = m'$ . Obvious.

Case  $n > m'$ . Take a positive natural number  $k$  such that  $n = m' + k$ .

Case  $k = 1$ . Obvious.

Case  $k \neq 1$ . Take a natural number  $l$  such that  $n = (m' + 1) + l$ . Hence  $n > m' + 1$ . Indeed  $l$  is positive. End. Qed. Qed.

Thus every natural number is contained in  $\Phi$ . Therefore  $n < m$  or  $n = m$  or  $n > m$ .  $\square$

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**Proposition 1.21.** Let  $n, m$  be natural numbers. Then

$$n \not< m \quad \text{iff} \quad n \geq m.$$

*Proof.* Case  $n \not< m$ . Then  $n = m$  or  $n > m$ . Hence  $n \geq m$ . End.

Case  $n \geq m$ . Assume  $n < m$ . Then  $n \leq m$ . Hence  $n = m$ . Contradiction. End.  $\square$

### 1.3 Ordering and successors

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**Proposition 1.22.** Let  $n, m$  be natural numbers. Then

$$n < m \leq n + 1 \quad \text{implies} \quad m = n + 1.$$

*Proof.* Assume  $n < m \leq n + 1$ . Take a positive natural number  $k$  such that  $m = n + k$ . Take a natural number  $l$  such that  $n + 1 = m + l$ . Then  $n + 1 = m + l = (n + k) + l = n + (k + l)$ . Hence  $k + l = 1$ .

We have  $l = 0$ .

Proof. Assume the contrary. Then  $k, l > 0$ .

Case  $k, l = 1$ . Then  $k + l = 2 \neq 1$ . Contradiction. End.

Case  $k = 1$  and  $l \neq 1$ . Then  $l > 1$ . Hence  $k + l > 1 + l > 1$ . Contradiction. End.

Case  $k \neq 1$  and  $l = 1$ . Then  $k > 1$ . Hence  $k + l > k + 1 > 1$ . Contradiction. End.

Case  $k, l \neq 1$ . Take natural numbers  $a, b$  such that  $k = a + 1$  and  $l = b + 1$ . Indeed  $k, l \neq 0$ . Hence  $k = a + 1$  and  $l = b + 1$ . Thus  $k, l > 1$ . Indeed  $a, b$  are positive. End. Qed.

Then we have  $n + 1 = m + l = m + 0 = m$ .  $\square$

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**Proposition 1.23.** Let  $n, m$  be natural numbers. Then

$$n \leq m < n + 1 \quad \text{implies} \quad n = m.$$

*Proof.* Assume  $n \leq m < n + 1$ .

Case  $n = m$ . Obvious.

Case  $n < m$ . Then  $n < m \leq n + 1$ . Hence  $n = m$ . End.  $\square$

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**Corollary 1.24.** Let  $n$  be a natural number. There is no natural number  $m$  such that  $n < m < n + 1$ .

*Proof.* Assume the contrary. Take a natural number  $m$  such that  $n < m < n + 1$ . Then  $n < m \leq n + 1$  and  $n \leq m < n + 1$ . Hence  $m = n + 1$  and  $m = n$ . Hence  $n = n + 1$ . Contradiction.  $\square$

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**Proposition 1.25.** Let  $n$  be a natural number. Then

$$n + 1 \geq 1.$$

*Proof.* Case  $n = 0$ . Obvious.

Case  $n \neq 0$ . Then  $n > 0$ . Hence  $n + 1 > 0 + 1 = 1$ . End.  $\square$

## 1.4 Ordering and addition

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**Proposition 1.26.** Let  $n, m, k$  be natural numbers. Then

$$n < m \quad \text{iff} \quad n + k < m + k.$$

*Proof.* Case  $n < m$ . Take a positive natural number  $l$  such that  $m = n + l$ . Then  $m + k = (n + l) + k = (n + k) + l$ . Hence  $n + k < m + k$ . End.

Case  $n + k < m + k$ . Take a positive natural number  $l$  such that  $m + k = (n + k) + l$ .  $(n + k) + l = n + (k + l) = n + (l + k) = (n + l) + k$ . Hence  $m + k = (n + l) + k$ . Thus  $m = n + l$ . Therefore  $n < m$ . End.  $\square$

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**Corollary 1.27.** Let  $n, m, k$  be natural numbers. Then

$$n < m \quad \text{iff} \quad k + n < k + m.$$

*Proof.* We have  $k + n = n + k$  and  $k + m = m + k$ . Hence  $k + n < k + m$  iff  $n + k < m + k$ .  $\square$

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**Corollary 1.28.** Let  $n, m, k$  be natural numbers. Then

$$n \leq m \quad \text{iff} \quad k + n \leq k + m.$$

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**Corollary 1.29.** Let  $n, m, k$  be natural numbers. Then

$$n \leq m \quad \text{iff} \quad n + k \leq m + k.$$

## 1.5 The natural numbers are well-ordered

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**Definition 1.30.**

$$< = \{(n, m) \mid n \text{ and } m \text{ are natural numbers such that } n < m\}.$$

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**Proposition 1.31.** Let  $A$  be a nonempty subclass of  $\mathbb{N}$ . Let  $n, m$  be least elements of  $A$  regarding  $<$ . Then  $n = m$ .

*Proof.* Assume  $n \neq m$ . Then  $n < m$  or  $m < n$ . If  $n < m$  then  $n \notin A$ . If  $m < n$  then  $m \notin A$ . Hence  $n, m \notin A$ . Contradiction. Therefore  $n = m$ .  $\square$

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**Proposition 1.32.** Let  $A$  be a nonempty subclass of  $\mathbb{N}$ . Then  $A$  has a least element regarding  $<$ .

*Proof.* Assume the contrary.

Let us show that for each  $n \in A$  there exists a  $m \in A$  such that  $m < n$ . Let  $n \in A$ .  $A$  has no least element regarding  $<$ . Assume that there exists no  $m \in A$  such that  $m < n$ . Then  $n \leq m$  for all  $m \in A$ . Hence  $n$  is a least element of  $A$  regarding  $<$ . Contradiction. End.

Define  $\Phi = \{n \in \mathbb{N} \mid n \text{ is less than any element of } A\}$ .



(1)  $\Phi$  contains 0.

Proof.  $0 \notin A$ . Hence 0 is less than every element of  $A$ . Thus  $0 \in \Phi$ . Qed.

(2) For all  $n \in \Phi$  we have  $n + 1 \in \Phi$ .

Proof. Let  $n \in \Phi$ . Then  $n$  is less than any element of  $A$ . Assume that  $\Phi$  does not contain  $n + 1$ . Then we can take an  $m \in A$  such that  $n + 1 \not\leq m$ . Then  $n < m \leq n + 1$ . Hence  $m = n + 1$ . Thus  $n + 1$  is a least element of  $A$  regarding  $<$ . Contradiction. Qed.

Then  $\Phi$  contains every natural number. Therefore every natural number is less than any element of  $A$ . Consequently  $A$  is empty. Contradiction.  $\square$

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**Corollary 1.33.**  $<$  is a wellorder on every nonempty subclass of  $\mathbb{N}$ .

*Proof.* Let  $A$  be a nonempty subclass of  $\mathbb{N}$ . For any  $n, m \in A$  we have  $(n, m) \in <$  iff  $n < m$ .

(1)  $<$  is irreflexive on  $A$ . Indeed for any  $n \in A$  we have  $n \not\leq n$ .

(2)  $<$  is transitive on  $A$ . Indeed for any  $n, m, k \in A$  if  $n < m$  and  $m < k$  then  $n < k$ .

(3)  $<$  is connected on  $A$ . Indeed for any distinct  $n, m \in A$  we have  $n < m$  or  $m < n$ .

Hence  $<$  is a strict linear order on  $A$ .  $<$  is wellfounded on  $A$ . Indeed every nonempty subclass of  $A$  has a least element regarding  $<$ . Thus  $<$  is a wellorder on  $A$ .  $\square$

## 1.6 Induction revisited

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**Theorem 1.34.** Let  $A$  be a class. Assume for all  $n \in \mathbb{N}$  if  $A$  contains all predecessors of  $n$  then  $A$  contains  $n$ . Then  $A$  contains every natural number.

*Proof.* Assume the contrary. Take a natural number  $n$  that is not contained in  $A$ . Then  $n$  is contained in  $\mathbb{N} \setminus A$ . Hence we can take a least element  $m$  of  $\mathbb{N} \setminus A$  regarding  $<$ . Then  $\mathbb{N} \setminus A$  does not contain any predecessor of  $m$ . Therefore  $A$  contains all predecessors of  $m$ . Consequently  $A$  contains  $m$ . Contradiction.  $\square$

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**Theorem 1.35.** Let  $A$  be a class. Let  $k$  be a natural number such that  $k \in A$ . Assume that for all  $n \in \mathbb{N}_{\geq k}$  if  $n \in A$  then  $n + 1 \in A$ . Then for all  $n \in \mathbb{N}_{\geq k}$  we have  $n \in A$ .

*Proof.* Define  $\Phi = \{n \in \mathbb{N} \mid \text{if } n \geq k \text{ then } n \in A\}$ .

(1)  $\Phi$  contains 0. Indeed if  $0 \geq k$  then  $0 = k \in A$ .

(2) For all  $n \in \Phi$  we have  $n + 1 \in \Phi$ .

*Proof.* Let  $n \in \Phi$ .

Let us show that if  $n + 1 \geq k$  then  $n + 1 \in A$ . Assume  $n + 1 \geq k$ .

Case  $n < k$ . Then  $n + 1 = k$ . Hence  $n + 1 \in A$ . End.

Case  $n \geq k$ . Then  $n \in A$ . Hence  $n + 1 \in A$ . End. End.

Therefore  $n + 1 \in \Phi$ . Qed.

Thus  $\Phi$  contains every natural number. Consequently for all  $n \in \mathbb{N}_{\geq k}$  we have  $n \in A$ .  $\square$