Chapter 1

Surjections, injections and bijections

File: `foundations/sections/08_injections-surjections-bijections.ftl.tex`

[readtex `foundations/sections/06_maps.ftl.tex`]

1.1 Surjective maps

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Definition 1.1. Let f be a map and B be a class. f is surjective onto B iff $\mathrm{range}(f) = B$.

Let f surjects onto B stand for f is surjective onto B. Let a surjective map onto B stand for a map that is surjective onto B.

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Definition 1.2. Let A, B be classes. A surjective map from A to B is a map of A that is surjective onto B.

Let a surjective map from A onto B stand for a surjective map from A to B. Let $f : A \twoheadrightarrow B$ stand for f is a surjective map from A onto B.

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Proposition 1.3. Let B be a class and f be a map to B. f is surjective onto B iff every element of B is a value of f.

Proof. Case f is surjective onto B. Then $B = \mathrm{range}(f)$. Let b be an element of B. Then $b \in \mathrm{range}(f)$. Hence b is a value of f. End.

Case every element of B is a value of f. Let us show that $B \subseteq \mathrm{range}(f)$. Let $b \in B$. Then b is a value of f. Hence $b \in \mathrm{range}(f)$. End.

Let us show that $\mathrm{range}(f) \subseteq B$. Let $b \in \mathrm{range}(f)$. Then b is a value of f. Hence $b \in B$. End. End. □

1.2 Injective maps

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Definition 1.4. Let f be a map. f is injective iff for all $a, a' \in \mathrm{dom}(f)$ if $f(a) = f(a')$ then $a = a'$.

Let $f : A \hookrightarrow B$ stand for f is an injective map from A to B.

1.3 Bijective maps

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Definition 1.5. Let A, B be classes. A bijection between A and B is an injective map of A that is surjective onto B.

Let a bijection from A to B stand for a bijection between A and B.

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Proposition 1.6. Let A, B be classes and $f : A \hookrightarrow B$. Then f is a bijection between A and $\mathrm{range}(f)$.

Proof. f is injective and surjects onto range(f). Hence f is a bijection between A and range(f). □

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Definition 1.7. Let A be a class. A permutation of A is a bijection between A and A.

1.4 Some basic facts

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Proposition 1.8. Let A be a class. Then id_A is a permutation of A.

Proof. (1) id_A is a map on A.

(2) id_A is surjective onto A.
Proof. Let $a \in A$. Then $a = \mathrm{id}_A(a)$. Hence $a \in \mathrm{range}(\mathrm{id}_A)$. Qed.

(3) id_A is injective.
Proof. Let $a, a' \in A$. Assume $\mathrm{id}_A(a) = \mathrm{id}_A(a')$. Then $a = a'$. Qed. □

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Proposition 1.9. Let A, B, C be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$. Then $g \circ f$ is a surjective map from A onto C.

Proof. $g \circ f$ is a map of A.

Let us show that $g \circ f$ is surjective onto C. Let $c \in C$. Take $b \in B$ such that $c = g(b)$. Take $a \in A$ such that $b = f(a)$. Then $c = g(f(a)) = (g \circ f)(a)$. End. □

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Proposition 1.10. Let A, B, C be classes and $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. Then $g \circ f$ is an injective map from A to C.

Proof. $g \circ f$ is a map of A.

Let us show that $g \circ f$ is injective. Let $a, a' \in A$. Assume $(g \circ f)(a) = (g \circ f)(a')$.

Then $g(f(a)) = g(f(a'))$. Hence $f(a) = f(a')$. Indeed $f(a), f(a') \in B$. Thus $a = a'$. End. $\square$

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Corollary 1.11. Let A, B, C be classes. Let f be a bijection between A and B and g be a bijection between B and C. Then $g \circ f$ is a bijection between A and C.

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Proposition 1.12. Let A, B be classes and $f : A \hookrightarrow B$ and $X \subseteq A$. Then $f \upharpoonright X$ is injective.

Proof. Let $a, a' \in X$. Assume $(f \upharpoonright X)(a) = (f \upharpoonright X)(a')$. Then $f(a) = f(a')$. Hence $a = a'$. $\square$

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Proposition 1.13. Let A, B be classes and $f : A \hookrightarrow B$ and $X \subseteq A$. Then $f \upharpoonright X$ is a bijection between X and $f_*(X)$.

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Corollary 1.14. Let A, B be classes and $f : A \hookrightarrow B$. Then f is a bijection between A and $f_*(A)$.