Chapter 1

Sets

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[readtex `foundations/sections/09_invertible-maps.ftl.tex`]

1.1 Sub- and supersets

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Definition 1.1. A proper class is a class that is not a set.

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Definition 1.2. Let A be a class. A subset of A is a subclass of A that is a set.

Let a superset of A stand for a superclass of A that is a set. Let a proper subset of A stand for a proper subclass of A that is a set. Let a proper superset of A stand for a proper superclass of A that is a set.

1.2 Powerclasses

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Definition 1.3. Let A be a class. The powerclass of A is

$$\{x \mid x \text{ is a subset of } A\}.$$

Let $\mathcal{P}(A)$ stand for the powerclass of A.

1.3 Systems of sets

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Definition 1.4. A system of sets is a class X such that every element of X is a set.

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Definition 1.5. A system of nonempty sets is a class X such that every element of X is a nonempty set.

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Definition 1.6. Let A be a class. A system of subsets of A is a class X such that every element of X is a subset of A.

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Proposition 1.7. Let A be a class. Then $\emptyset$ is a system of subsets of A.

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Proposition 1.8. Let A be a class. Then $\mathcal{P}(A)$ is a system of subsets of A.

Proposition 1.9. Let X, Y be systems of sets. Then $X \cup Y$ is a system of sets.

Proposition 1.10. Let X, Y be systems of sets. Then $X \cap Y$ is a system of sets.

Proposition 1.11. Let X, Y be systems of sets. Then $X \setminus Y$ is a system of sets.

1.4 Unions

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Definition 1.12. Let X be a system of sets. The union over X is

$$\{a \mid a \in x \text{ for some } x \in X\}.$$

Let $\bigcup X$ stand for the union over X.

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Proposition 1.13.

$$\bigcup \emptyset = \emptyset.$$

Proof. $\bigcup \emptyset = \{a \mid a \in x \text{ for some } x \in \emptyset\}$. $\emptyset$ has no elements. Hence there is no object a such that $a \in x$ for some $x \in \emptyset$. Thus $\bigcup \emptyset = \emptyset$. □

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Proposition 1.14. Let x, y be sets. Then

$$\bigcup \{x, y\} = x \cup y.$$

Proof. Let us show that $\bigcup \{x, y\} \subseteq x \cup y$. Let $a \in \bigcup \{x, y\}$. Then a is contained in some element of $\{x, y\}$. Hence $a \in x$ or $a \in y$. Thus $a \in x \cup y$. End.

Let us show that $x \cup y \subseteq \bigcup \{x, y\}$. Let $a \in x \cup y$. Then $a \in x$ or $a \in y$. Hence a is contained in some element of $\{x, y\}$. Therefore $a \in \bigcup\{x, y\}$. End. $\square$

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Corollary 1.15. Let x be a set. Then

$$\bigcup\{x\} = x.$$

1.5 Intersections

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Definition 1.16. Let X be a system of sets. The intersection over X is

$$\{a \mid a \in x \text{ for all } x \in X\}.$$

Let $\bigcap X$ stand for the intersection over X.

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Proposition 1.17. $\bigcap \emptyset$ is the class of all objects.

Proof. Define $V = \{x \mid x \text{ is an object }\}$. We have $\bigcap \emptyset \subseteq V$. Indeed every element of $\bigcap \emptyset$ is an object.

Let us show that $V \subseteq \bigcap \emptyset$. Let $a \in V$. Then a is an object. For every $x \in \emptyset$ we have $a \in x$. Indeed $\emptyset$ has no elements. Thus $a \in \bigcap \emptyset$. End. $\square$

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Proposition 1.18. Let x, y be sets. Then

$$\bigcap\{x, y\} = x \cap y.$$

Proof. Let us show that $\bigcap\{x, y\} \subseteq x \cap y$. Let $a \in \bigcap\{x, y\}$. Then a is contained in every element of $\{x, y\}$. Hence $a \in x$ and $a \in y$. Thus $a \in x \cap y$. End.

Let us show that $x \cap y \subseteq \bigcap\{x, y\}$. Let $a \in x \cap y$. Then $a \in x$ and $a \in y$. Hence a is contained in every element of $\{x, y\}$. Therefore $a \in \bigcap\{x, y\}$. End. $\square$

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Corollary 1.19. Let x be a set. Then

$$\bigcap\{x\} = x.$$

1.6 Classes of functions

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Definition 1.20. Let x, y be sets. $[x \to y]$ is the class of all maps from x to y.

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Proposition 1.21. Let x, y be sets. Then every element of $[x \to y]$ is a function.

1.7 Axioms for mathematics

Definition 1.22. Let A be a class and a be an object and f be a map such that $A \subseteq \mathrm{dom}(f)$. A is inductive regarding a and f iff $a \in A$ and for all $x \in A$ we have $f(x) \in A$.

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Axiom 1.23 (Set existence). There exists a set.

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Axiom 1.24 (Separation). Let A be a class. If there exists a set x such that every element of A is contained in x then A is a set.

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Axiom 1.25 (Pairing). Let a, b be objects. Then $\{a, b\}$ is a set.

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Axiom 1.26 (Union). Let X be a system of sets. If X is a set then $\bigcup X$ is a set.

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Axiom 1.27 (Infinity). Let A be a class and $a \in A$ and $f : A \to A$. Then there exists a subset of A that is inductive regarding a and f.

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Axiom 1.28 (Powerset). Let x be a set. Then $\mathcal{P}(x)$ is a set.

Let the powerset of x stand for $\mathcal{P}(x)$.

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Axiom 1.29 (Choice). Let X be a system of nonempty sets. Then there exists a map f such that $\mathrm{dom}(f) = X$ and $f(x) \in x$ for any $x \in X$.

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Axiom 1.30 (Foundation). Let X be a nonempty system of sets. Then X has an element x such that X and x are disjoint.

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Axiom 1.31 (Replacement). Let f be a map and x be a set. Then $f[x]$ is a set.

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Axiom 1.32 (Function). Let f be a map. If $\mathrm{dom}(f)$ is a set then f is a function.

1.8 Consequences of the axioms

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Proposition 1.33. $\emptyset$ is a set.

Proof. Take a set x (by axiom 1.23). Define $A = \{y \in x \mid y \neq y\}$. Then A is a set (by axiom 1.24). We have $A = \emptyset$. Hence $\emptyset$ is a set. □

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Proposition 1.34. Let a be an object. Then $\{a\}$ is a set.

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Corollary 1.35. Let A be a class that has a unique element. Then A is a set.

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Proposition 1.36. Let x, y be sets. Then $x \cup y$ is a set.

Proof. Take $X = \{x, y\}$. Then X is a set. Hence $\bigcup X$ is a set (by axiom 1.26). Indeed X is a system of sets. We have $x \cup y = \bigcup X$. Thus $x \cup y$ is a set. □

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Proposition 1.37. Let x, y be sets. Then $x \cap y$ is a set.

Proof. We have $x \cap y \subseteq x$. Hence $x \cap y$ is a set (by axiom 1.24). □

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Proposition 1.38. Let x, y be sets. Then $x \setminus y$ is a set.

Proof. We have $x \setminus y \subseteq x$. Hence $x \setminus y$ is a set (by axiom 1.24). □

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Proposition 1.39. Let x, y be sets. Then $x \times y$ is a set.

Proof. $\{a\}$ and $\{a, b\}$ are sets for each $a \in x$ and each $b \in y$. Define $P = \{\{\{a\}, \{a, b\}\} \mid a \in x \text{ and } b \in y\}$.

(1) P is a set.
Proof. Let us show that $P \subseteq \mathcal{P}(\mathcal{P}(x \cup y))$. Let $p \in P$. Consider $a \in x$ and $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$. Then $a, b \in x \cup y$. Hence $\{a\}, \{a, b\} \in \mathcal{P}(x \cup y)$. Thus $\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(x \cup y))$. End.

$x \cup y$ is a set. Consequently $\mathcal{P}(\mathcal{P}(x \cup y))$ is a set (by axiom 1.28). Therefore P is a set (by axiom 1.24). Qed.

Define $l(p) =$ "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$ in a" for $p \in P$.
Define $r(p) =$ "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$ in b" for $p \in P$.

Define $f(p) = (l(p), r(p))$ for $p \in P$.

Let us show that for any objects u, u', v, v' if $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$ then $u = u'$ and $v = v'$. Let u, u', v, v' be objects. Assume $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$. Then $(\{u\} = \{u'\}$ or $\{u\} = \{u', v'\})$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. Thus $(\{u\} = \{u'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}))$ or $(\{u\} = \{u', v'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}))$.

Case $\{u\} = \{u'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. We have $\{u\} = \{u'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = u' = v$. Hence $\{\{u\}, \{u, u\}\} = \{\{u\}, \{u, v'\}\}$ (by 1). Thus $\{\{u\}\} = \{\{u\}, \{u, v'\}\}$. Therefore $\{u\} = \{u, v'\}$. Consequently $v' = u = v$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End.

Case $\{u\} = \{u', v'\}$ and $(\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\})$. We have $\{u\} = \{u', v'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = v = u'$. Hence $v = v'$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End. End.

Let us show that for any $a \in x$ and any $b \in y$ we have $f(\{\{a\}, \{a, b\}\}) = (a, b)$. Let $a \in x$ and $b \in y$. Take $p = \{\{a\}, \{a, b\}\}$. Then p is a set. Then we can choose

$a' \in x$ and $b' \in y$ such that $p = \{\{a'\}, \{a', b'\}\}$ and $l(p) = a'$. Then $a = a'$ and $b = b'$. Hence $l(p) = a$. Choose $a'' \in x$ and $b'' \in y$ such that $p = \{\{a''\}, \{a'', b''\}\}$ and $r(p) = b''$. Then $a = a''$ and $b = b''$. Thus $r(p) = b$. Therefore $f(p) = (a, b)$. End.

(2) $x \times y = f[P]$.
Proof. For all $p \in P$ we have $l(p) \in x$ and $r(p) \in y$. Hence $f(p) \in x \times y$ for all $p \in P$. Therefore $f[P] \subseteq x \times y$.

Let us show that $x \times y \subseteq f[P]$. Let $z \in x \times y$. Take $a \in x$ and $b \in y$ such that $z = (a, b)$. Then $(a, b) = f(\{\{a\}, \{a, b\}\})$. Hence there exists a $p \in P$ such that $(a, b) = f(p)$. Thus $(a, b) \in f[P]$. End.

Consequently $x \times y = f[P]$. Qed.

Thus $x \times y$ is the image of some set under some map. Therefore $x \times y$ is a set (by axiom 1.31). □

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Proposition 1.40. Let X be a nonempty system of sets. Then $\bigcap X$ is a set.

Proof. Take an element x of X. Then $\bigcap X \subseteq x$. Hence $\bigcap X$ is a set (by axiom 1.24). □

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Proposition 1.41. Let f be a map such that $\mathrm{dom}(f)$ is a set. Then $\mathrm{range}(f)$ is a set.

Proof. $\mathrm{range}(f) = f_*(\mathrm{dom}(f))$ and $f_*(\mathrm{dom}(f))$ is a set. Hence $\mathrm{range}(f)$ is a set (by axiom 1.31). □

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Proposition 1.42. Let A be a class and x be a set. Assume that there exists an injective map from A to x. Then A is a set.

Proof. Consider an injective map f from A to x. Then f^{-1} is a bijection between $\mathrm{range}(f)$ and A. $\mathrm{range}(f)$ is a set and A is the image of $\mathrm{range}(f)$ under f^{-1}. Thus A is a set (by axiom 1.31). □

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Proposition 1.43. There exist no sets x, y such that $x \in y$ and $y \in x$.

Proof. Assume the contrary. Take sets x, y such that $x \in y$ and $y \in x$. Consider an element z of $\{x, y\}$ such that $\{x, y\}$ and z are disjoint (by axiom 1.30). Indeed $\{x, y\}$ is a nonempty system of sets. Then we have $z = x$ or $z = y$.

Case $z = x$. Then x and $\{x, y\}$ are disjoint. Hence $y \notin x$. Contradiction. End.

Case $z = y$. Then y and $\{x, y\}$ are disjoint. Hence $x \notin y$. Contradiction. End. □

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Corollary 1.44. Let x be a set. Then $x \notin x$.

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Proposition 1.45. Let x, y be sets. Then $[x \to y]$ is a set.

Proof. Define $R = \{F \in \mathcal{P}(x \times y) \mid$ (for all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$) and for all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'\}$.

[prover vampire][timelimit 5] Every element of R is a set. Define $h(F) = \lambda a \in x.$ "choose $b \in y$ such that $(a, b) \in F$ in b" for $F \in R$. [prover eprover][/timelimit]

Let us show that $[x \to y] \subseteq \mathrm{range}(h)$. Let $f \in [x \to y]$. Define $F = \{(a, f(a)) \mid a \in x\}$.

Then $F \in R$.
Proof. Define $g(a) = (a, f(a))$ for $a \in x$. Then $F = \mathrm{range}(g)$. Hence F is a set. Thus $F \in \mathcal{P}(x \times y)$. Indeed $F \subseteq x \times y$.

(1) For all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$.

(2) For all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'$. End.

We have $\mathrm{dom}(f) = x = \mathrm{dom}(h(F))$. For each $a \in x$ we have $h(F)(a) = f(a)$. Hence $f = h(F)$. Thus $f \in \mathrm{range}(h)$. End.

Therefore $[x \to y]$ is a set. Indeed R is a set. □