Chapter 1

Ordinal numbers

File: `set-theory/sections/02_ordinals.ftl.tex`

[readtex `foundations/sections/11_binary-relations.ftl.tex`]

[readtex `set-theory/sections/01_transitive-classes.ftl.tex`]

SET_THEORY_02_229593678086144

Definition 1.1. An ordinal number is a transitive set α such that every element of α is a transitive set.

Let an ordinal stand for an ordinal number.

SET_THEORY_02_5852994258075648

Definition 1.2. **Ord** is the class of all ordinals.

SET_THEORY_02_2358097091756032

Proposition 1.3. Let α be an ordinal. Then every element of α is an ordinal.

Proof. Let x be an element of α. Then x is transitive.

Let us show that every element of x is a subset of x. Let y be an element of x. Then y is a subset of x. Let z be an element of y. Every element of y is an element of x. Hence z is an element of x. End.

Thus every element of x is transitive. Therefore x is an ordinal. □

SET_THEORY_02_7202164443185152

Proposition 1.4. Let α be an ordinal and $x \subseteq \alpha$. Then $\bigcup x$ is an ordinal.

Proof. (1) $\bigcup x$ is transitive.
Proof. Let $y \in \bigcup x$ and $z \in y$. Take $w \in x$ such that $y \in w$. Then $w \in \alpha$. Hence w is transitive. Thus $z \in w$. Therefore $z \in \bigcup x$. Qed.

(2) Every element of $\bigcup x$ is transitive.
Proof. Let $y \in \bigcup x$. Let $z \in y$ and $v \in z$. Take $w \in x$ such that $y \in w$. We have $w \in \alpha$. Hence w is an ordinal. Thus y is an ordinal. Therefore y is transitive. Consequently $v \in y$. Qed. □

1.1 Zero and successors

SET_THEORY_02_8385964858671104

Definition 1.5. $0 = \emptyset$.

Let α is nonzero stand for $\alpha \neq 0$.

SET_THEORY_02_8166925802668032

Definition 1.6. Let α be an ordinal. $\mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$.

SET_THEORY_02_8483196888940544

Proposition 1.7. 0 is an ordinal.

Proof. Every element of 0 is a transitive set and every element of 0 is a subset of 0. □

SET_THEORY_02_1624410224066560

Proposition 1.8. Let α be an ordinal. Then $\mathrm{succ}(\alpha)$ is an ordinal.

Proof. (1) $\mathrm{succ}(\alpha)$ is transitive.
Proof. Let $x \in \mathrm{succ}(\alpha)$ and $y \in x$. Then $x \in \alpha$ or $x = \alpha$. Hence $y \in \alpha$. Thus

$y \in \mathrm{succ}(\alpha)$. Qed.

(2) Every element of $\mathrm{succ}(\alpha)$ is transitive.
Proof. Let $x \in \mathrm{succ}(\alpha)$. Then $x \in \alpha$ or $x = \alpha$. Hence x is transitive. Indeed α is transitive and every element of α is transitive. Qed. □

SET_THEORY_02_8651096763400192

Proposition 1.9. Let α, β be ordinals. If $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$ then $\alpha = \beta$.

Proof. Assume $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$.

(1) $\alpha \subseteq \beta$.
Proof. Let $\gamma \in \alpha$. Then $\gamma \in \alpha \cup \{\alpha\} = \mathrm{succ}(\alpha) = \mathrm{succ}(\beta) = \beta \cup \{\beta\}$. Hence $\gamma \in \beta$ or $\gamma = \beta$. Assume $\gamma = \beta$. Then $\beta \in \alpha$. Hence $\beta = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\alpha \setminus \{\gamma\}) \cup \{\alpha\}$. Therefore $\alpha \in \beta$. Consequently $\alpha \in \beta \in \alpha$. Contradiction. Qed.

(2) $\beta \subseteq \alpha$.
Proof. Let $\gamma \in \beta$. Then $\gamma \in \beta \cup \{\beta\} = \mathrm{succ}(\beta) = \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\gamma \in \alpha$ or $\gamma = \alpha$. Assume $\gamma = \alpha$. Then $\alpha \in \beta$. Hence $\alpha = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\beta \setminus \{\gamma\}) \cup \{\beta\}$. Therefore $\beta \in \alpha$. Consequently $\beta \in \alpha \in \beta$. Contradiction. Qed. □

1.2 The standard ordering of the ordinals

SET_THEORY_02_6654252130762752

Definition 1.10. Let α, β be ordinals. α is less than β iff $\alpha \in \beta$.

Let $\alpha < \beta$ stand for α is less than β. Let $\alpha \not< \beta$ stand for not $\alpha < \beta$.

Let α is greater than β stand for $\beta < \alpha$. Let $\alpha > \beta$ stand for $\beta < \alpha$. Let $\alpha \not> \beta$ stand for not $\alpha > \beta$.

SET_THEORY_02_2639956210089984

Definition 1.11. Let α, β be ordinals. α is less than or equal to β iff $\alpha < \beta$ or $\alpha = \beta$.

Let $\alpha \leq \beta$ stand for α is less than or equal to β. Let $\alpha \nleq \beta$ stand for not $\alpha \leq \beta$.

Let α is greater than or equal to β stand for $\beta \leq \alpha$. Let $\alpha \geq \beta$ stand for $\beta \leq \alpha$. Let $\alpha \ngeq \beta$ stand for not $\alpha \geq \beta$.

SET_THEORY_02_3089369577553920

Proposition 1.12. Let α, β be ordinals. Then

$$\alpha \leq \beta \quad \text{implies} \quad \alpha \subseteq \beta.$$

Proof. Case $\alpha \leq \beta$. Then $\alpha < \beta$ or $\alpha = \beta$. Let $x \in \alpha$. If $\alpha < \beta$ then $x \in \alpha \in \beta$. Hence if $\alpha < \beta$ then $x \in \beta$. If $\alpha = \beta$ then $x \in \beta$. Thus $x \in \beta$. End. □

SET_THEORY_02_6229364135952384

Proposition 1.13. Let α be an ordinal. Then

$$\alpha \not< \alpha.$$

Proof. Assume $\alpha < \alpha$. Then $\alpha \in \alpha$. Contradiction. □

SET_THEORY_02_7098683017396224

Proposition 1.14. Let α, β, γ be ordinals. Then

$$(\alpha < \beta \text{ and } \beta < \gamma) \quad \text{implies} \quad \alpha < \gamma.$$

Proof. Assume $\alpha < \beta$ and $\beta < \gamma$. Then $\alpha \in \beta \in \gamma$. Hence $\alpha \in \gamma$. Thus $\alpha < \gamma$. □

SET_THEORY_02_1718825707896832

Proposition 1.15. Let α, β be ordinals. Then $\alpha < \beta$ or $\alpha = \beta$ or $\alpha > \beta$.

Proof. Assume the contrary. Define

$$A = \left\{ \alpha' \in \mathbf{Ord} \;\middle|\; \begin{array}{l} \text{there exists an ordinal } \beta' \text{ such that neither } \alpha' < \beta' \text{ nor } \alpha' = \beta' \\ \text{nor } \alpha' > \beta' \end{array} \right\}.$$

A is nonempty. Hence we can take a least element α' of A regarding $\in$. Define

$$B = \{\beta' \in \mathbf{Ord} \mid \text{neither } \alpha' < \beta' \text{ nor } \alpha' = \beta' \text{ nor } \alpha' > \beta'\}.$$

B is nonempty. Hence we can take a least element β' of B regarding $\in$.

Let us show that $\alpha' \subseteq \beta'$. Let $a \in \alpha'$. Then $a < \beta'$ or $a = \beta'$ or $a > \beta'$. Indeed if neither $a < \beta'$ nor $a = \beta'$ nor $a > \beta'$ then $a \in A$. If $a = \beta'$ then $\beta' < \alpha'$. If $a > \beta'$ then $\beta' < \alpha'$. Hence $a < \beta'$. Thus $a \in \beta'$. End.

Let us show that $\beta' \subseteq \alpha'$. Let $b \in \beta'$. Then $b < \alpha'$ or $b = \alpha'$ or $b > \alpha'$. If $b = \alpha'$ then $\alpha' < \beta'$. If $b > \alpha'$ then $\alpha' < \beta'$. Hence $b < \alpha'$. Thus $b \in \alpha'$. End.

Hence $\alpha' = \beta'$. Contradiction. □

SET_THEORY_02_610496856195072

Proposition 1.16. Let α, β be ordinals. Then

$$\alpha \subseteq \beta \quad \text{implies} \quad \alpha \leq \beta.$$

Proof. Assume $\alpha \subseteq \beta$.

Case $\alpha = \beta$. Trivial.

Case $\alpha \neq \beta$. Then $\alpha < \beta$ or $\alpha > \beta$. Assume $\alpha > \beta$. Then $\beta \in \alpha$. Hence $\beta \in \beta$. Contradiction. End. □

SET_THEORY_02_5689190964527104

Proposition 1.17. Let α be an ordinal. Then

$$\alpha < \mathrm{succ}(\alpha).$$

SET_THEORY_02_4064972025888768

Proposition 1.18. Let α, β be ordinals. Then

$$\beta < \mathrm{succ}(\alpha) \quad \text{implies} \quad \beta \leq \alpha.$$

Proof. Assume $\beta < \mathrm{succ}(\alpha)$. Then $\beta \in \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\beta \in \alpha$ or $\beta \in \{\alpha\}$. Thus $\beta < \alpha$ or $\beta = \alpha$. Therefore $\beta \leq \alpha$. □

SET_THEORY_02_8242798790705152

Proposition 1.19. Let α be an ordinal. There exists no ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$.

Proof. Assume the contrary. Consider an ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$. Then $\beta < \alpha$ or $\beta = \alpha$. Hence $\alpha < \alpha$. Contradiction. □

1.3 Successor and limit ordinals

SET_THEORY_02_7129712109289472

Definition 1.20. A successor ordinal is an ordinal α such that $\alpha = \operatorname{succ}(\beta)$ for some ordinal β.

SET_THEORY_02_4240355610329088

Proposition 1.21. Let α be an ordinal. There exists no ordinal β such that $\alpha < \beta < \operatorname{succ}(\alpha)$.

Proof. Assume the contrary. Choose an ordinal β such that $\alpha < \beta < \operatorname{succ}(\alpha)$. Then $\alpha \in \beta \in \alpha \cup \{\alpha\}$. Hence $\beta \in \alpha$ or $\beta = \alpha$. Then $\alpha \in \alpha$. Contradiction. □

SET_THEORY_02_735071524880384

Definition 1.22. Let α be a successor ordinal. $\operatorname{pred}(\alpha)$ is the ordinal β such that $\alpha = \operatorname{succ}(\beta)$.

SET_THEORY_02_7678388934279168

Definition 1.23. A limit ordinal is an ordinal λ such that neither λ is a successor ordinal nor $\lambda = 0$.

SET_THEORY_02_4659024620421120

Proposition 1.24. Let λ be a limit ordinal and $\alpha \in \lambda$. Then λ contains $\operatorname{succ}(\alpha)$.

Proof. If $\operatorname{succ}(\alpha) \notin \lambda$ then $\alpha < \lambda < \operatorname{succ}(\alpha)$. □

SET_THEORY_03_2217148434874368

Theorem 1.25 (Burali-Forti). **Ord** is a proper class.

Proof. Assume that **Ord** is a set. **Ord** is transitive and every element of **Ord** is transitive. Hence **Ord** is an ordinal. Thus **Ord** $\in$ **Ord**. Contradiction. □

1.4 Transfinite induction

SET_THEORY_02_4059354166722560

Definition 1.26.

$$< = \{(\alpha, \beta) \mid \alpha \text{ and } \beta \text{ are ordinals such that } \alpha < \beta\}.$$

SET_THEORY_02_4859038791630848

Proposition 1.27. $<$ is a strong wellorder on **Ord**.

Proof. For any ordinals α, β we have $(\alpha, \beta) \in <$ iff $\alpha < \beta$.

(1) $<$ is irreflexive on **Ord**. Indeed for any ordinal α we have $\alpha \nless \alpha$.

(2) $<$ is transitive on **Ord**. Indeed for any ordinals α, β, γ if $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.

(3) $<$ is connected on **Ord**. Indeed for any distinct ordinals α, β we have $\alpha < \beta$ or $\beta < \alpha$.

Hence $<$ is a strict linear order on **Ord**.

(4) $<$ is wellfounded on **Ord**.
Proof. Let A be a nonempty subclass of **Ord**. Then we can take a least element α of A regarding $\in$. Then α is a least element of A regarding $<$. Qed.

Hence $<$ is strongly wellfounded on **Ord**. Indeed for any $\beta \in$ **Ord** we have $\beta = \{\alpha \in \mathbf{Ord} \mid (\alpha, \beta) \in <\}$. Thus $<$ is a strong wellorder on **Ord**. $\square$

SET_THEORY_02_1042046129274880

Corollary 1.28. Let A be a subclass of **Ord**. If A is nonempty then A has a least element regarding $<$.

SET_THEORY_02_1991423647809536

Corollary 1.29. Let A be a subclass of **Ord**. If A is nonempty then A has a least element regarding $\in$.

SET_THEORY_03_8114657499807744

Proposition 1.30. $<$ is a strong wellorder on any ordinal.

Proof. Let α be an ordinal. For all $\beta, \gamma \in \alpha$ we have $(\beta, \gamma) \in <$ iff $\beta < \gamma$.

(1) $<$ is irreflexive on α. Indeed for all $\beta \in \alpha$ we have $\alpha \nless \alpha$.

(2) $<$ is transitive on α. Indeed for all $\beta, \gamma, \delta \in \alpha$ if $\beta < \gamma$ and $\gamma < \delta$ then $\beta < \delta$.

(3) $<$ is connected on α. Indeed for any distinct $\beta, \gamma \in \alpha$ we have $\beta < \gamma$ or $\gamma < \beta$.

Hence $<$ is a strict linear order on α.

(4) $<$ is wellfounded on α.
Proof. Let A be a nonempty subclass of α. Then we can take a least element β of A regarding $<$. Indeed A is a subclass of **Ord**. Qed.

Hence $<$ is strongly wellfounded on α. Indeed for any $\gamma \in \alpha$ we have $\gamma = \{\beta \in \mathbf{Ord} \mid (\beta, \gamma) \in <\}$. Thus $<$ is a strong wellorder on α. [unfold off] ☐

Note: In the proof below 11.24 refers to the Foundations library!

SET_THEORY_02_8493935460614144

Theorem 1.31. Let Φ be a class. Assume that for all ordinals α if Φ contains all ordinals less than α then Φ contains α. Then Φ contains every ordinal.

Proof. Define $B = \{x \mid x$ is a set and if $x \in \mathbf{Ord}$ then $x \in \Phi\}$.

Let us show that for all sets x if B contains every element of x that is a set then B contains x. Let x be a set. Assume that every element of x that is a set is contained in B.

Case $x \notin \mathbf{Ord}$. Trivial.

Case $x \in \mathbf{Ord}$. Then Φ contains all ordinals less than x. Hence Φ contains x. Thus $x \in B$. End. End.

[prover vampire] Hence B contains every set (by 11.24). Thus Φ contains every ordinal. ☐

SET_THEORY_02_7892040431960064

Theorem 1.32. Let Φ be a class. (Initial case) Assume that Φ contains 0. (Successor step) Assume that for all ordinals α if $\alpha \in \Phi$ then $\mathrm{succ}(\alpha) \in \Phi$. (Limit step) Assume that for all limit ordinals λ if every ordinals less than λ is contained in Φ then $\lambda \in \Phi$. Then Φ contains every ordinal.

Proof. Let us show that for all ordinals α if Φ contains all ordinals less than α then

Φ contains α. Let α be an ordinal. Then $\alpha = 0$ or α is a successor ordinal or α is a limit ordinal. Assume that Φ contains all ordinals less than α.

Case $\alpha = 0$. Trivial.

Case α is a successor ordinal. Take an ordinal β such that $\alpha = \mathrm{succ}(\beta)$. Then $\beta \in \Phi$. Hence $\alpha \in \Phi$ (by successor step). End.

Case α is a limit ordinal. Then $\beta \in \Phi$ for all ordinals β less than α. Hence $\alpha \in \Phi$ (by limit step). End. End.

[prover vampire] Thus Φ contains every ordinal (by 1.31). $\square$