# Chapter 1

# Finite ordinals

**File:** `set-theory/sections/03_finite-ordinals.ftl.tex`

[readtex `set-theory/sections/02_ordinals.ftl.tex`]

SET_THEORY_03_4310076227584000

**Definition 1.1.**

$$\omega = \left\{ n \in \mathbf{Ord} \;\middle|\; \begin{array}{l} n \in X \text{ for every } X \subseteq \mathbf{Ord} \text{ such that } 0 \in X \text{ and for all} \\ x \in X \text{ we have } \operatorname{succ}(x) \in X \end{array} \right\}.$$

SET_THEORY_03_3576717620805632

**Proposition 1.2.** $0 \in \omega$.

SET_THEORY_03_8807317141192704

**Proposition 1.3.** Let $n \in \omega$. Then $\operatorname{succ}(n) \in \omega$.

SET_THEORY_03_344585425387520

**Proposition 1.4.** Let $\Phi \subseteq \omega$. Assume that $0 \in \Phi$ and for every $x \in \Phi$ we have $\operatorname{succ}(x) \in \Phi$. Then $\Phi = \omega$.

*Proof.* Suppose $\Phi \neq \omega$. Consider an element $n$ of $\omega$ that is not contained in $\Phi$. Take $\Phi' = \Phi \setminus \{n\}$.

(1) $0 \in \Phi'$. Indeed $0 \in \Phi$ and $0 \neq n$.

(2) For each $x \in \Phi'$ we have $\mathrm{succ}(x) \in \Phi'$.
Proof. Let $x \in \Phi'$. Then $\mathrm{succ}(x) \in \Phi$.

Let us show that $\mathrm{succ}(x) \neq n$. Assume $\mathrm{succ}(x) = n$. Then $x \notin \Phi$. Indeed $n \notin \Phi$ and if $x \in \Phi$ then $n = \mathrm{succ}(x) \in \Phi$. Contradiction. End.

Thus $\mathrm{succ}(x) \in \Phi'$. Qed.

Therefore every element of $\omega$ lies in $\Phi'$. Indeed $\Phi' \subseteq \mathbf{Ord}$. Consequently $n \in \Phi'$. Contradiction. □

SET_THEORY_03_4847727433220096

**Corollary 1.5.** $\omega$ is a set.

*Proof.* Define $f(n) = \mathrm{succ}(n)$ for $n \in \omega$. Take a subset $X$ of $\omega$ that is inductive regarding 0 and $f$. Indeed $f$ is a map from $\omega$ to $\omega$. Then we have $0 \in X$ and for each $n \in X$ we have $\mathrm{succ}(n) \in X$. Thus $X = \omega$. Therefore $\omega$ is a set. □

SET_THEORY_03_5885789275684864

**Proposition 1.6.** Let $n \in \omega$. Then $n = 0$ or $n = \mathrm{succ}(m)$ for some $m \in \omega$.

*Proof.* Assume the contrary. Consider a $k \in \omega$ such that neither $k = 0$ nor $k = \mathrm{succ}(m)$ for some $m \in \omega$. Take a class $\omega'$ such that $\omega' = \omega \setminus \{k\}$. Then $\omega'$ is a set.

(1) $0 \in \omega'$. Indeed $k \neq 0$.

(2) For all $m \in \omega'$ we have $\mathrm{succ}(m) \in \omega'$.
Proof. Let $m \in \omega'$. Then $\mathrm{succ}(m) \neq k$. Hence $\mathrm{succ}(m) \in \omega'$. Qed.

Thus every element of $\omega$ is contained in $\omega'$. Therefore $k \in \omega'$. Contradiction. □

SET_THEORY_03_5057540872208384

**Proposition 1.7.** Every element of $\omega$ is an ordinal.

SET_THEORY_03_764451995254784

**Proposition 1.8.** $\omega$ is a limit ordinal.

*Proof.* $\omega$ is transitive.
Proof. Define $\Phi = \{n \in \omega \mid \text{for all } m \in n \text{ we have } m \in \omega\}$.

(1) $0 \in \Phi$.

(2) For all $n \in \Phi$ we have $\mathrm{succ}(n) \in \Phi$.
Proof. Let $n \in \Phi$. Then every element of $n$ is contained in $\omega$. Hence every element of $\mathrm{succ}(n)$ is contained in $\omega$. Thus $\mathrm{succ}(n) \in \Phi$. Qed.

Therefore $\omega \subseteq \Phi$. Consequently $\omega$ is transitive. Qed.

Every element of $\omega$ is an ordinal. Hence every element of $\omega$ is transitive. Thus $\omega$ is an ordinal.

$\omega$ is a limit ordinal.
Proof. Assume the contrary. We have $\omega \neq 0$. Hence $\omega$ is a successor ordinal. Take an ordinal $\alpha$ such that $\mathrm{succ}(\alpha) = \omega$. Then $\alpha \in \omega$. Thus $\omega = \mathrm{succ}(\alpha) \in \omega$. Contradiction. Qed. □

SET_THEORY_03_5517271459954688

**Proposition 1.9.** Let $\lambda$ be a limit ordinal. Then

$$\omega \leq \lambda.$$

*Proof.* Assume the contrary. Then $\lambda < \omega$. Consequently $\lambda \in \omega$. Hence $\lambda = 0$ or $\lambda = \mathrm{succ}(n)$ for some $n \in \omega$. Thus $\lambda$ is not a limit ordinal. Contradiction. □

SET_THEORY_03_1991057988386816

**Definition 1.10.** $1 = \mathrm{succ}(0)$.

SET_THEORY_03_5809204518453248

**Definition 1.11.** $2 = \mathrm{succ}(1)$.

SET_THEORY_03_4388003120152576

**Proposition 1.12.** $1 = \{0\}$.

SET_THEORY_03_930896899211264

**Proposition 1.13.** $2 = \{0, 1\}$.