Zermelo's Well-ordering Theorem

$\mathbb{N}$aproche formalization:

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This is a formalization of *Zermelo's Well-ordering Theorem*, i.e. of the assertion that under the assumption of the axiom of choice every set is equinumerous to some ordinal number, where an ordinal number is regarded as a transitive set whose elements are transitive sets as well. The proof of this theorem presented here is oriented on [1].

On mid-range hardware $\mathbb{N}$aproche needs approximately 4 Minutes to verify this formalization plus approximately 15 minutes to verify the library files it depends on.

[readtex `foundations/sections/13_equinumerosity.ftl.tex`]

[readtex `set-theory/sections/04_recursion.ftl.tex`]

Definition. Let X be a system of nonempty sets. A choice function for X is a map g such that $\mathrm{dom}(g) = X$ and $g(x) \in x$ for any $x \in X$.

Axiom (Choice). Let X be a system of nonempty sets. Then there exists a choice function for X.

In the following, for any class A, we write $A^{<\infty}$ to denote the collection of all maps $f : \alpha \to A$ for some ordinal α. Moreover, for any map $G : A^{<\infty} \to A$ we say that a map $F : \mathbf{Ord} \to A$, where $\mathbf{Ord}$ denotes the class of all ordinals, is recursive regarding G if $F(\alpha) = G(F \upharpoonright \alpha)$ for all $\alpha \in \mathbf{Ord}$.

Theorem (Zermelo). Every set is equinumerous to some ordinal.

Proof. Let x be a set. Consider a choice function g for $\mathcal{P}(x) \setminus \{\emptyset\}$. For any $F \in (x \cup \{x\})^{<\infty}$ if $x \setminus \mathrm{range}(F) \neq \emptyset$ then $x \setminus \mathrm{range}(F) \in \mathrm{dom}(g)$. Indeed $x \setminus \mathrm{range}(F)$ is a subset of x for any $F \in (x \cup \{x\})^{<\infty}$. Define

$$G(F) = \begin{cases} g(x \setminus \mathrm{range}(F)) & : x \setminus \mathrm{range}(F) \neq \emptyset \\ x & : x \setminus \mathrm{range}(F) = \emptyset \end{cases}$$

for $F \in (x \cup \{x\})^{<\infty}$. Then for any $F \in (x \cup \{x\})^{<\infty}$ if $x \setminus \mathrm{range}(F) \neq \emptyset$ then $G(F) \in x \setminus \mathrm{range}(F)$. G is a map from $(x \cup \{x\})^{<\infty}$ to $x \cup \{x\}$.

Indeed we can show that for any $F \in (x \cup \{x\})^{<\infty}$ we have $G(F) \in x \cup \{x\}$. Let $F \in (x \cup \{x\})^{<\infty}$. If $x \setminus \mathrm{range}(F) \neq \emptyset$ then $G(F) \in x \setminus \mathrm{range}(F)$. If $x \setminus \mathrm{range}(F) = \emptyset$ then $G(F) = x$. Hence $G(F) \in x \cup \{x\}$. End. Hence we can take a map F from **Ord** to $x \cup \{x\}$ that is recursive regarding G. For any ordinal α we have $F \restriction \alpha \in (x \cup \{x\})^{<\infty}$.

For any $\alpha \in \mathbf{Ord}$ we have

$$x \setminus F[\alpha] \neq \emptyset \implies F(\alpha) \in x \setminus F[\alpha]$$

and

$$x \setminus F[\alpha] = \emptyset \implies F(\alpha) = x.$$

Proof. Let $\alpha \in \mathbf{Ord}$. We have $F[\alpha] = \{F(\beta) \mid \beta \in \alpha\}$. Hence $F[\alpha] = \{G(F \restriction \beta) \mid \beta \in \alpha\}$. We have $\mathrm{range}(F \restriction \alpha) = \{F(\beta) \mid \beta \in \alpha\}$. Thus $\mathrm{range}(F \restriction \alpha) = F[\alpha]$.

Case $x \setminus F[\alpha] \neq \emptyset$. Then $x \setminus \mathrm{range}(F \restriction \alpha) \neq \emptyset$. Hence $F(\alpha) = G(F \restriction \alpha) \in x \setminus \mathrm{range}(F \restriction \alpha) = x \setminus F[\alpha]$. End.

Case $x \setminus F[\alpha] \neq \emptyset$. Then $x \setminus \mathrm{range}(F \restriction \alpha) = \emptyset$. Hence $F(\alpha) = G(F \restriction \alpha) = x$. End. Qed.

(1) For any ordinals α, β such that $\alpha < \beta$ and $F(\beta) \neq x$ we have $F(\alpha), F(\beta) \in x$ and $F(\alpha) \neq F(\beta)$.
Proof. Let $\alpha, \beta \in \mathbf{Ord}$. Assume $\alpha < \beta$ and $F(\beta) \neq x$. Then $x \setminus F[\beta] \neq \emptyset$. (a) Hence $F(\beta) \in x \setminus F[\beta]$. We have $F[\alpha] \subseteq F[\beta]$. Thus $x \setminus F[\alpha] \neq \emptyset$. (b) Therefore $F(\alpha) \in x \setminus F[\alpha]$. Consequently $F(\alpha), F(\beta) \in x$ (by a, b). We have $F(\alpha) \in F[\beta]$ and $F(\beta) \notin F[\beta]$. Thus $F(\alpha) \neq F(\beta)$. Qed.

(2) There exists an ordinal α such that $F(\alpha) = x$.
Proof. Assume the contrary. Then F is a map from **Ord** to x.

Let us show that F is injective. Let $\alpha, \beta \in \mathbf{Ord}$. Assume $\alpha \neq \beta$. Then $\alpha < \beta$ or $\beta < \alpha$. Hence $F(\alpha) \neq F(\beta)$ (by 1). Indeed $F(\alpha), F(\beta) \neq x$. End.

Thus F is an injective map from some proper class to some set. Contradiction. Qed.

Define $\Phi = \{\alpha \in \mathbf{Ord} \mid F(\alpha) = x\}$. Φ is nonempty. Hence we can take a least element α of Φ regarding $\in$. Take $f = F \restriction \alpha$. Then f is a map from α to x. Indeed for no $\beta \in \alpha$ we have $F(\beta) = x$. Indeed for all $\beta \in \alpha$ we have $(\beta, \alpha) \in \in$.

(3) f is surjective onto x.
Proof. $x \setminus F[\alpha] = \emptyset$. Hence $\mathrm{range}(f) = f[\alpha] = F[\alpha] = x$. Qed.

(4) f is injective.
Proof. Let $\beta, \gamma \in \alpha$. Assume $\beta \neq \gamma$. We have $f(\beta), f(\gamma) \neq x$. Hence $f(\beta) \neq f(\gamma)$ (by 1). Indeed $\beta < \gamma$ or $\gamma < \beta$. Qed.

Therefore f is a bijection between α and x. Consequently x and α are equinumerous. □

References

[1] Peter Koepke, *Set Theory*; lecture notes, winter 2018/19, University of Bonn