Maximum modulus principle

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1 Preliminaries

[synonym number/-s]
Let the domain of f stand for $\mathrm{dom}(f)$. Let z is in M stand for z is an element of M. Let M contains z stand for z is in M.
Let f denote a function. Let M denote a set.

Definition 1. A subset of M is a set N such that every element of N is an element of M.

Definition 2. Assume M is a subset of the domain of f. $f^{\to}[M] = \{f(x)|x \in M\}$.

Axiom 3. Assume M is a subset of the domain of f. $f^{\to}[M]$ is a set.

2 Real and complex numbers

Signature 4. A complex number is a notion.

Let z, w denote complex numbers.

Axiom 5. Every element of $\mathrm{dom}(f)$ is a complex number and for every element z of $\mathrm{dom}(f)$ $f(z)$ is a complex number.

Axiom 6. Every element of M is a complex number.

Signature 7. A real number is a notion.

Let x, y denote real numbers.

Axiom 8. x is setsized.

Signature 9. $|z|$ is a real number.

Signature 10. x is positive is an atom.

Let ε, δ denote positive real numbers.

Signature 11. $x < y$ is an atom.

Let $x \leq y$ stand for $x = y$ or $x < y$.

Axiom 12. If $x < y$ then not $y < x$.

3 Properties of functions and open balls

Signature 13. f is holomorphic is an atom.

Signature 14. $B_\varepsilon(z)$ is a set that contains z.

Axiom 15. Assume x is element of $B_\varepsilon(z)$. Then x is a real number.

Lemma 16. Assume x is an element of $B_\varepsilon(z)$. Then x is setsized.

Axiom 17. $|z| < |w|$ for some element w of $B_\varepsilon(z)$.

Definition 18. M is open iff for every element z of M there exists ε such that $B_\varepsilon(z)$ is a subset of M.

Axiom 19. $B_\varepsilon(z)$ is open.

Definition 20. A local maximal point of f is an element z of the domain of f such that there exists ε such that $B_\varepsilon(z)$ is a subset of the domain of f and $|f(w)| \leq |f(z)|$ for every element w of $B_\varepsilon(z)$.

Definition 21. Let U be a subset of the domain of f. f is constant on U iff there exists z such that $f(w) = z$ for every element w of U.

Let f is constant stand for f is constant on the domain of f.

Axiom 22. Assume f is holomorphic and $B_\varepsilon(z)$ is a subset of the domain of f. If f is not constant on $B_\varepsilon(z)$ then $f^{\rightarrow}[B_\varepsilon(z)]$ is open.

4 Maximum modulus principle

Signature 23. A region is an open set.

Axiom 24. **(Identity theorem)** Assume f is holomorphic and the domain of f is a region. Assume that $B_\varepsilon(z)$ is a subset of the domain of f. If f is constant on $B_\varepsilon(z)$ then f is constant.

Proposition 25. **(Maximum modulus principle)** Assume f is holomorphic and the domain of f is a region. If f has a local maximal point then f is constant.

Proof. Let z be a local maximal point of f. Take ε such that $B_\varepsilon(z)$ is a subset of $\mathrm{dom}(f)$ and $|f(w)| \leq |f(z)|$ for every element w of $B_\varepsilon(z)$.

Let us show that f is constant on $B_\varepsilon(z)$. Proof by contradiction. Assume the contrary. Then $f^{\rightarrow}[B_\varepsilon(z)]$ is open. We can take δ such that $B_\delta(f(z))$ is a subset of $f^{\rightarrow}[B_\varepsilon(z)]$. Therefore there exists an element w of $B_\varepsilon(z)$ such that $|f(z)| < |f(w)|$. Contradiction. End.

Hence f is constant. $\square$