Regularity of successor cardinals

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1 Preliminaries

[synonym cardinal/-s] [synonym ordinal/-s]
Let x, y, X, Y denote sets.

Axiom 1. For any objects x, y, x', y' if $(x, y) = (x', y')$ then $x' = x$ and $y' = y$.

Axiom 2. Let X be a set. Let x be an element of X. Then x is setsized.

Axiom 3. Let x, y be setsized objects. Then (x, y) is setsized.

Axiom 4. Let f be a function. Let X be a set. Assume $\mathrm{dom}(f) = X$. Let x be an element of X. Then $f(x)$ is setsized.

Definition 5. $X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$.

Axiom 6. $X \times Y$ is a set.

Lemma 7. Let x, y be objects. If (x, y) is an element of $X \times Y$ then x is an element of X and y is an element of Y.

Let f denote a function.

Definition 8. A subset of X is a set Y such that every element of Y is an element of X.

Let $Y \subseteq X$ stand for Y is a subset of X.

Axiom 9. **(Extensionality)** If $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

Definition 10. Assume $X \subseteq \mathrm{dom}(f)$. $f^{\rightarrow}[X] = \{f(x) \mid x \in X\}$.

Axiom 11. Assume $X \subseteq \mathrm{dom}(f)$. $f^{\rightarrow}[X]$ is a set.

Let f is surjective from X onto Y stand for $\mathrm{dom}(f) = X$ and $f^{\rightarrow}[X] = Y$.
Let $f : X \twoheadrightarrow Y$ stand for f is surjective from X onto Y.

2 Ordinals

Signature 12. An ordinal is a set.

Let α, β denote ordinals.

Axiom 13. Every element of α is an ordinal.

Axiom 14. **(Transitivity)** Let $x \in y \in \alpha$. Then $x \in \alpha$.

Signature 15. $\alpha < \beta$ is an atom.

Axiom 16. $\alpha < \beta$ or $\beta < \alpha$ or $\beta = \alpha$.

Axiom 17. If $\alpha < \beta$ then α is an element of β.

Let $a \leq b$ stand for $a = b$ or $a < b$.

3 Cardinals

Signature 18. A cardinal is an ordinal.

Let κ, μ, ν denote cardinals.

Signature 19. **(Cardinality)** $|X|$ is a cardinal.

Axiom 20. **(Existence of surjection)** Assume X has an element. $|X| \leq |Y|$ iff there exists a function f such that $\mathrm{dom}(f) = Y$ and $f^{\rightarrow}[Y] = X$.

Axiom 21. $|X \times X| = |X|$.

Axiom 22. $|\kappa| = \kappa$.

Axiom 23. Let Y be a subset of X. $|Y| \leq |X|$.

Signature 24. κ^+ is a cardinal.

Axiom 25. $\kappa < \kappa^+$.

Axiom 26. $|\alpha| \leq \kappa$ for every element α of κ^+.

Axiom 27. For no cardinals μ, ν we have $\mu < \nu$ and $\nu < \mu$.

Axiom 28. There is no cardinal ν such that $\kappa < \nu < \kappa^+$.

Definition 29. The empty set is a cardinal η such that η is an element of (every ordinal that has an element).

Definition 30. The constant zero on X is a function f such that $\mathrm{dom}(f) = X$ and $f(x)$ is the empty set for every element x of X.

Let 0^X stand for the constant zero on X.

4 Cofinality and regular cardinals

Definition 31. **(Cofinality)** Let κ be a cardinal. Let Y be a subset of κ. Y is cofinal in κ iff for every element x of κ there exists an element y of Y such that $x < y$.

Let a cofinal subset of κ stand for a subset of κ that is cofinal in κ.

Definition 32. κ is regular iff $|x| = \kappa$ for every cofinal subset x of κ.

5 Hausdorff's theorem

The following result appears in [1, p. 443], where Hausdorff mentions that the proof is *"ganz einfach"* (*"very simple"*) and can be skipped.

Theorem 33. **(Hausdorff)** κ^+ is regular.

Proof. Proof by contradiction. Assume the contrary. Take a cofinal subset x of κ^+ such that $|x| \neq \kappa^+$. Then $|x| \leq \kappa$. Take a function f that is surjective from κ onto x (by existence of surjection). Indeed x has an element and $|\kappa| = \kappa$. Define

$$g(z) = \begin{cases} \text{choose a function } h \text{ such that } h : \kappa \twoheadrightarrow z \text{ in } h & z \text{ has an element} \\ 0^\kappa & z \text{ has no element} \end{cases}$$

for z in κ^+.

Define $h((\xi, \zeta)) = g(f(\xi))(\zeta)$ for (ξ, ζ) in $\kappa \times \kappa$.

Let us show that h is surjective from $\kappa \times \kappa$ onto κ^+. $\mathrm{dom}(h) = \kappa \times \kappa$.

Every element of κ^+ is an element of $h^\to[\kappa \times \kappa]$. Proof. Let n be an element of κ^+. Take an element ξ of κ such that $n < f(\xi)$. Take an element ζ of κ such that $g(f(\xi))(\zeta) = n$. Then $n = h((\xi, \zeta))$. Therefore the thesis. Indeed (ξ, ζ) is an element of $\kappa \times \kappa$. End.

Every element of $h^\to[\kappa \times \kappa]$ is an element of κ^+. proof. Let n be an element of $h^\to[\kappa \times \kappa]$. We can take elements a, b of κ such that $n = h((a, b))$. Then $n = g(f(a))(b)$. $f(a)$ is an element of κ^+. Every element of $f(a)$ is an element of κ^+.

Case $f(a)$ has an element. Obvious.

Case $f(a)$ has no element. Obvious. End. End.

Therefore $\kappa^+ \leq \kappa$. Contradiction. $\square$

References

[1] Felix Hausdorff (1908), *Grundzüge einer Theorie der geordneten Mengen*; Teubner, *Mathematische Annalen*, vol. 65, p. 435–505