Square roots of primes are irrational

Rational numbers

[synonym number/-s] [synonym divide/-s]

Signature 1. A rational number is a notion.

Let s, r, q stand for rational numbers.

Signature 2. $r \cdot q$ is a rational number.

Axiom 3. $r \cdot q = q \cdot r$.

Axiom 4. $r \cdot (q \cdot s) = (r \cdot q) \cdot s$.

Axiom 5. If $q \cdot s = q \cdot r$ then $s = r$.

Natural numbers

Signature 6. A natural number is a rational number.

Let n, m, k denote natural numbers.

Axiom 7. $n \cdot m$ is a natural number.

Definition 8. $n \mid q$ iff there exists k such that $k \cdot n = q$.

Let n divides m stand for $n \mid m$. Let a divisor of m stand for a natural number that divides m.

Definition 9. n and m are coprime iff n and m have no common divisor.

Axiom 10. There exist coprime m, n such that $m \cdot q = n$.

Prime numbers

Signature 11. A prime number is a natural number.

Let p denote a prime number.

Axiom 12. If $p \mid n \cdot m$ then $p \mid n$ or $p \mid m$.

Let q^2 stand for $q \cdot q$.

Proposition 13. $q^2 = p$ for no rational number q.

Proof. Proof by contradiction. Assume the contrary. Take a rational number q such that $p = q^2$. Take coprime m, n such that $m \cdot q = n$. Then $p \cdot m^2 = n^2$. Therefore p divides n. Take a natural number k such that $n = k \cdot p$. Then $p \cdot m^2 = p \cdot (k \cdot n)$. Therefore $m \cdot m$ is equal to $p \cdot k^2$. Hence p divides m. Contradiction. $\square$