The mutilated checkerboard

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1 Introduction

Max Black proposed this problem in his book *Critical Thinking* (1946). It was later discussed by Martin Gardner in his *Scientific American* column, *Mathematical Games*. John McCarthy, one of the founders of Artificial Intelligence described it as a *Tough Nut for Proof Procedures* and discussed fully automatic or interactive proofs of the solution.

There have been several formalization of the Checkerboard problem before. A survey article by Manfred Kerber and Martin Pollet called *A Tough Nut for Mathematical Knowledge Management* lists a couple of formalizations.

2 Setting up the checkerboard

We introduce *types* (or *notions*) and constants to model checkerboards as a Cartesian product of *ranks* $1, 2, \ldots, 8$ and *files* $\mathrm{a}, \mathrm{b}, \ldots, \mathrm{h}$. In future versions these signature declarations will be grouped together as a single declaration of an inductively defined set.

$\mathbb{N}$aproche allows us to group elements into *classes* and *sets* as long as they are *setsized* (informally also called *small*).

Signature 1. A rank is a notion.

Let r, s denote ranks.

Axiom 2. r is setsized.

Signature 3. 1 is a rank.

Signature 4. 2 is a rank.

Signature 5. 3 is a rank.

Signature 6. 4 is a rank.

Signature 7. 5 is a rank.

Signature 8. 6 is a rank.

Signature 9. 7 is a rank.

Signature 10. 8 is a rank.

Definition 11. $\mathbf{R} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Signature 12. A file is a notion.

Let f, g denote files.

Axiom 13. f is setsized.

Signature 14. a is a file.

Signature 15. b is a file.

Signature 16. c is a file.

Signature 17. d is a file.

Signature 18. e is a file.

Signature 19. f is a file.

Signature 20. g is a file.

Signature 21. h is a file.

Definition 22. $\mathbf{F} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}\}$.

Signature 23. A square is a notion.

Axiom 24. (f, r) is a square.

Let v, w, x, y, z denote squares.

Is there a set of all squares? This may not be true for an arbitrary notion, but it is true for squares, so we assume it as an axiom. Note that we can always form the class C of all inhabitants of a notion as long as $x \in C$ can only be true for setsized x. Morse and Kelley [2, 3] use the same approach in their axiomatization of set theory.

Definition 25. $\mathbf{C}$ is the class of squares x such that $x = (f, r)$ for some element f of $\mathbf{F}$ and some element r of $\mathbf{R}$.

Axiom 26. $\mathbf{C}$ is a set.

3 Preliminaries about sets and functions

We enrich the small built-in set theory by further properties and axioms that will be used in the course of our argument. To keep the document fully self-contained we formulate the necessary definitions and axioms ourselves. Note that there are many degrees of freedom in picking an axiomatic setting.

Let A, B, C denote sets.

Definition 27. A subset of B is a set A such that every element of A is an element of B.

Axiom 28. **(Extensionality)** If A is a subset of B and B is a subset of A then $A = B$.

Definition 29. A proper subset of B is a subset A of B such that $A \neq B$.

Definition 30. A is disjoint from B iff there is no element of A that is an element of B.

Definition 31. A family is a set F such that every element of F is a set.

Definition 32. A disjoint family is a family F such that A is disjoint from B for all nonequal elements A, B of F.

Definition 33. $B \cap C = \{x \in B \mid x \in C\}$.

Definition 34. $B \setminus C = \{x \in B \mid x \notin C\}$.

The notion of *object* is the built-in *largest notion*, containing all other notions. Also note that the proof of the lemma below really is omitted and not merely hidden: with the help of automated theorem provers such as the E theorem prover [1], Naproche can accept some theorems without any additional argumentation.

Lemma 35. Every set is an object.

The built-in ordered pair notation that we already used in the first subsection does not include the universal property of ordered pairs, so we postulate it as an axiom.

Axiom 36. Let $\alpha, \beta, \gamma, \delta$ be objects. If $(\alpha, \beta) = (\gamma, \delta)$ then $\alpha = \gamma$ and $\beta = \delta$.

(Unary) functions are built into Naproche; $F(t)$ denotes the application of a function F to an argument t and $\mathrm{dom}(F)$ stands for the domain of F. In our exposition we shall use functions to compare cardinalities of black and white squares. As with sets, we introduce some further properties of functions.

Let F, G denote functions.

Definition 37. $F : A \to B$ iff $\mathrm{dom}(F) = A$ and $F(x)$ is an element of B for all elements x of A.

Bijective functions are the basis of the modern theory of cardinalities; sets have the same cardinality iff there is a bijection between them.

Definition 38. $F : A \leftrightarrow B$ iff $F : A \to B$ and there exists G such that $G : B \to A$ and (for all elements x of A we have $G(F(x)) = x$) and (for all elements y of B we have $F(G(y)) = y$).

4 Cardinalities of Finite Sets

Definition 39. A is equinumerous with B iff there is F such that $F : A \leftrightarrow B$.

Lemma 40. Assume that A is equinumerous with B. Then B is equinumerous with A.

Lemma 41. Assume that A is equinumerous with B and B is equinumerous with C. Then A is equinumerous with C.

Proof. Take a function F such that $F : A \leftrightarrow B$. Take a function G such that $G : B \to A$ and (for all elements x of A we have $G(F(x)) = x$) and (for all elements y of B we have $F(G(y)) = y$). Take a function H such that $H : B \leftrightarrow C$. Take a function I such that $I : C \to B$ and (for all elements x of B we have $I(H(x)) = x$) and (for all elements y of C we have $H(I(y)) = y$). Define $J(x) = H(F(x))$ for x in A. $J : A \leftrightarrow C$. Indeed define $K(y) = G(I(y))$ for y in C. $\square$

For the finite checkerboard problem we only need to consider finite sets. Intuitively we can thus assume that all sets considered are finite, and then we have the following finiteness axiom:

Axiom 42. If A is a proper subset of B then A is not equinumerous with B.

5 The Mutilated Checkerboard

Defining the mutilated checkerboard is straightforward: we simply remove the two corners.

Definition 43. $\mathbf{C}' = \{(\mathrm{a}, 1), (\mathrm{h}, 8)\}$.

Definition 44. $\mathbf{M} = \mathbf{C} \setminus \mathbf{C}'$.

Let the mutilated checkerboard stand for $\mathbf{M}$.

6 Dominoes

To define dominoes, we introduce concepts of adjacency by first declaring new relations and then axiomatizing them. As usual, chaining of relation symbols indicates a conjunction.

Signature 45. r is vertically adjacent to s is a relation.

Let $r \sim s$ stand for r is vertically adjacent to s.

Axiom 46. If $r \sim s$ then $s \sim r$.

Axiom 47. $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 6 \sim 7 \sim 8$.

Signature 48. f is horizontally adjacent to g is a relation.

Let $f \sim' g$ stand for f is horizontally adjacent to g.

Axiom 49. If $f \sim' g$ then $g \sim' f$.

Axiom 50. $\mathrm{a} \sim' \mathrm{b} \sim' \mathrm{c} \sim' \mathrm{d} \sim' \mathrm{e} \sim' \mathrm{f} \sim' \mathrm{g} \sim' \mathrm{h}$.

Definition 51. x is adjacent to y iff there exist f, r, g, s such that $x = (f, r)$ and $y = (g, s)$ and (($f = g$ and r is vertically adjacent to s) or ($r = s$ and f is horizontally adjacent to g)).

Definition 52. A domino is a set D such that $D = \{x, y\}$ for some adjacent squares x, y.

7 Domino Tilings

Definition 53. A domino tiling is a disjoint family T such that every element of T is a domino.

Let A denote a subset of $\mathbf{C}$.

Definition 54. A domino tiling of A is a domino tiling T such that for every square x x is an element of A iff x is an element of some element of T.

We shall prove:

Theorem. The mutilated checkerboard has no domino tiling.

8 Colours

We shall solve the mutilated checkerboard problem by a cardinality argument. Squares on an actual checkerboard are coloured black and white and we can count colours on dominoes and on the mutilated checkerboard $\mathbf{M}$.

The introduction of colours can be viewed as a creative move typical of mathematics: changing perspectives and introducing aspects that are not part of the original problem. The mutilated checkerboard was first discussed under a cognition-theoretic perspective: can one solve the problem *without* inventing new concepts and completely stay within the realm of squares, subsets of the checkerboard and dominoes.

Signature 55. x is black is a relation.

Signature 56. x is white is a relation.

Axiom 57. x is black iff x is not white.

Axiom 58. If x is adjacent to y then x is black iff y is white.

Axiom 59. $(\mathrm{a}, 1)$ is black.

Axiom 60. $(\mathrm{h}, 8)$ is black.

Definition 61. **B** is the class of black elements of **C**.

Definition 62. **W** is the class of white elements of **C**.

Lemma 63. **B** is a set.

Lemma 64. **W** is a set.

9 Counting Colours on Checkerboards

The original checkerboard has an equal number of black and white squares. Since our setup does not include numbers for counting, we rather work with equinumerosity. The following argument formalizes that we can invert the colours of a checkerboard by swapping the files a and b, c and d, etc.. We formalize swapping by a first-order function symbol Swap.

Signature 65. Let x be an element of **C**. $\mathrm{Swap}\, x$ is an element of **C**.

Let t denote an element of **R**.

Axiom 66. $\mathrm{Swap}(\mathrm{a}, t) = (\mathrm{b}, t)$ and $\mathrm{Swap}(\mathrm{b}, t) = (\mathrm{a}, t)$.

Axiom 67. $\mathrm{Swap}(\mathrm{c}, t) = (\mathrm{d}, t)$ and $\mathrm{Swap}(\mathrm{d}, t) = (\mathrm{c}, t)$.

Axiom 68. $\mathrm{Swap}(\mathrm{e}, t) = (\mathrm{f}, t)$ and $\mathrm{Swap}(\mathrm{f}, t) = (\mathrm{e}, t)$.

Axiom 69. $\mathrm{Swap}(\mathrm{g}, t) = (\mathrm{h}, t)$ and $\mathrm{Swap}(\mathrm{h}, t) = (\mathrm{g}, t)$.

Lemma 70. Let x be an element of **C**. $\mathrm{Swap}\, x$ is adjacent to x.

Proof. Take f, r such that $x = (f, r)$. r is an element of **R**. Case $f = \mathrm{a}$. End. Case $f = \mathrm{b}$. End. Case $f = \mathrm{c}$. End. Case $f = \mathrm{d}$. End. Case $f = \mathrm{e}$. End. Case $f = \mathrm{f}$. End. Case $f = \mathrm{g}$. End. □

Swap is an involution.

Lemma 71. Let x be an element of **C**. $\mathrm{Swap}(\mathrm{Swap}\, x) = x$.

Proof. Take f, r such that $x = (f, r)$. r is an element of **R**. Case $f = \mathrm{a}$. End. Case $f = \mathrm{b}$. End. Case $f = \mathrm{c}$. End. Case $f = \mathrm{d}$. End. Case $f = \mathrm{e}$. End. Case $f = \mathrm{f}$. End. Case $f = \mathrm{g}$. End. □

Lemma 72. Let x be an element of **C**. x is black iff $\mathrm{Swap}\, x$ is white.

Using Swap we can define a witness of $\mathbf{B} \leftrightarrow \mathbf{W}$.

Lemma 73. **B** is equinumerous with **W**.

Proof. Define $F(x) = \mathrm{Swap}\, x$ for x in **B**. Define $G(x) = \mathrm{Swap}\, x$ for x in **W**. Then $F : \mathbf{B} \to \mathbf{W}$ and $G : \mathbf{W} \to \mathbf{B}$. For all elements x of **B** we have $G(F(x)) = x$. For all elements x of **W** we have $F(G(x)) = x$. $F : \mathbf{B} \leftrightarrow \mathbf{W}$. □

Given a domino tiling one can also swap the squares of each dominoes, leading to similar properties.

Signature 74. Assume that T is a domino tiling of A. Let x be an element of A. $\mathrm{Swap}_T^A(x)$ is a square y such that there is an element D of T such that $D = x, y$.

Lemma 75. Assume that T is a domino tiling of A. Let x be an element of A. Then $\mathrm{Swap}_T^A(x)$ is an element of A.

Proof. Let $y = \mathrm{Swap}_T^A(x)$. Take an element D of T such that $D = x, y$. □

Swapping dominoes is also an involution.

Lemma 76. Assume that T is a domino tiling of A. Let x be an element of A. Then $\mathrm{Swap}_T^A(\mathrm{Swap}_T^A(x)) = x$.

Proof. Let $y = \mathrm{Swap}_T^A(x)$. Take an element Y of T such that $Y = x, y$. Let $z = \mathrm{Swap}_T^A(y)$. Take an element Z of T such that $Z = y, z$. Then $x = z$. □

Lemma 77. Assume that T is a domino tiling of A. Let x be a black element of A. Then $\mathrm{Swap}_T^A(x)$ is white.

Proof. Let $y = \mathrm{Swap}_T^A(x)$. Take an element Y of T such that $Y = x, y$. □

10 The Theorem

We can easily show that a domino tiling involves as many black as white squares.

Lemma 78. Let T be a domino tiling of A. Then $A \cap \mathbf{B}$ is equinumerous with $A \cap \mathbf{W}$.

Proof. Define $F(x) = \mathrm{Swap}_T^A(x)$ for x in $A \cap \mathbf{B}$. Define $G(x) = \mathrm{Swap}_T^A(x)$ for x in $A \cap \mathbf{W}$. Then $F : A \cap \mathbf{B} \to A \cap \mathbf{W}$ and $G : A \cap \mathbf{W} \to A \cap \mathbf{B}$. For all elements x of $A \cap \mathbf{B}$ we have $G(F(x)) = x$. For all elements x of $A \cap \mathbf{W}$ we have $F(G(x)) = x$. $F : A \cap \mathbf{B} \leftrightarrow A \cap \mathbf{W}$. □

In mutilating the checkerboard, one only removes black squares

Lemma 79. $\mathbf{M} \cap \mathbf{W} = \mathbf{W}$.

Proof. $\mathbf{M} \cap \mathbf{W}$ is a subset of $\mathbf{W}$. $\mathbf{W}$ is a subset of $\mathbf{M}$. Proof. Let x be an element of $\mathbf{W}$. $x \neq (\mathrm{a}, 1)$ and $x \neq (\mathrm{h}, 8)$. Indeed $(\mathrm{h}, 8)$ is black. End. □

Now the theorem follows by putting together the previous cardinality properties. Note that the phrasing *[...] has no domino tiling* in the theorem is automatically derived from the definition of *a domino tiling of [...]*.

Theorem 80. The mutilated checkerboard has no domino tiling.

Proof. Proof by contradiction. Assume T is a domino tiling of $\mathbf{M}$. $\mathbf{M} \cap \mathbf{B}$ is equinumerous with $\mathbf{M} \cap \mathbf{W}$. Indeed $\mathbf{M}$ is a subset of $\mathbf{C}$. $\mathbf{M} \cap \mathbf{B}$ is equinumerous with $\mathbf{W}$. $\mathbf{M} \cap \mathbf{B}$ is equinumerous with $\mathbf{B}$. Contradiction. Indeed $\mathbf{M} \cap \mathbf{B}$ is a proper subset of $\mathbf{B}$. $\square$

References

[1] Stephan Schulz et al., *E*, `https://eprover.org`

[2] John L. Kelley, *General Topology*, Springer, 1975

[3] Anthony Perry Morse, Trevor J McMinn, *A theory of sets*, Academic press, 1965