König's Theorem

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König's Theorem is an important set-theoretical result about the arithmetic of cardinals. It was proved by Julius König in 1905 (König, Julius: Zum Kontinuumsproblem, Mathematische Annalen 60 (1905), 177–180). The proof is reminiscent of Cantor's diagonal argument for proving that $\kappa < 2^\kappa$.

The formalization begins with a minimal axiomatic setup on sets, functions and cardinals, sufficient for the proof of the theorem.

On an older laptop with an Intel Pentium N3710 processor ℕaproche takes around 35 seconds to check this text.

1 ForTheL Setup

[synonym cardinal/-s] [synonym sequence/-s] [read ZFC.ftl]

2 Sets and Functions

Let M, N denote sets.

Definition 1. A subset of M is a set N such that every element of N is an element of M.

Definition 2. $M \setminus N = \{x \in M | x$ is not an element of $N\}$.

Let f denote a function. Let the domain of f stand for $\mathrm{dom}(f)$.

Definition 3. Assume M is a subset of the domain of f. $f[M] = \{f(x) | x$ is an element of $M\}$.

Let the image of f stand for $f[\mathrm{dom}(f)]$.

Axiom 4. Assume M is a subset of the domain of f. Then $f[M]$ is a set.

Axiom 5. f is setsized.

3 Cardinals

Signature 6. A cardinal is a set.

Let A, B, C denote cardinals.

Signature 7. $A < B$ is an atom.

Let A is less than B stand for $A < B$. Let $A \leq B$ stand for $A = B$ or $A < B$.

Axiom 8. $A < B < C \implies A < C$.

Axiom 9. $A < B$ or $B < A$ or $B = A$.

Signature 10. The cardinality of M is a cardinal.

Let $\mathrm{card}(M)$ denote the cardinality of M.

Axiom 11. $\mathrm{card}(A) = A$.

Axiom 12. **(ImageCard)** Assume M is a subset of $Dom(f)$. $\mathrm{card}(f[M]) \leq \mathrm{card}(M)$.

Axiom 13. Assume the cardinality of N is less than the cardinality of M. Then $M \setminus N$ has an element.

Axiom 14. **(SurjExi)** Assume $\mathrm{card}(M) \leq \mathrm{card}(N)$. Assume M has an element. There exists a function f such that N is the domain of f and M is the image of f.

4 Sums and Products of cardinals

Let D denote a set.

Definition 15. A sequence of cardinals on D is a function κ such that $\mathrm{dom}(\kappa) = D$ and $\kappa(i)$ is a cardinal for every element i of D.

Let f_i stand for $f(i)$.

Signature 16. Let κ be a sequence of cardinals on D. $\dot{\bigcup}_{i \in D} \kappa_i$ is a set.

Axiom 17. **(SumDef)** Let κ be a sequence of cardinals on D. $\dot{\bigcup}_{i \in D} \kappa_i = \{(n, i) | i \text{ is an element of } D \text{ and } n \text{ is an element of } \kappa_i\}$.

Axiom 18. Let κ be a sequence of cardinals on D. Then $\dot{\bigcup}_{i \in D} \kappa_i$ is a set.

Definition 19. Let κ be a sequence of cardinals on D. $\sum_{i \in D} \kappa_i = \mathrm{card}(\dot{\bigcup}_{i \in D} \kappa_i)$.

Signature 20. Let κ be a sequence of cardinals on D. $\bigtimes_{i \in D} \kappa_i$ is a set.

Axiom 21. **(ProdDef)** Let κ be a sequence of cardinals on D. $\bigtimes_{i \in D} \kappa_i = \{$ function $f | \mathrm{dom}(f) = D \wedge (f(i)$ is an element of κ_i for every element i of D)}.

Axiom 22. Let κ be a sequence of cardinals on D. Then $\bigtimes_{i\in D}\kappa_i$ is a set.

Definition 23. Let κ be a sequence of cardinals on D. $\prod_{i\in D}\kappa_i = \mathrm{card}(\bigtimes_{i\in D}\kappa_i)$.

König's Theorem requires some form of the axiom of choice. Currently choice is built into Naproche by the *choose* construct in function definitions.

The axiom of choice is also required to show that products of non-empty factors are themselves non-empty:

Lemma 24. **(Choice)** Let λ be a sequence of cardinals on D. Assume that λ_i has an element for every element i of D. Then $\bigtimes_{i\in D}\lambda_i$ has an element.

Proof. Define $f(i) =$ choose an element v of λ_i in v for i in D. Then f is an element of $\bigtimes_{i\in D}\lambda_i$. □

5 König's theorem

Theorem 25. Let κ, λ be sequences of cardinals on D. Assume that for every element i of D $\kappa_i < \lambda_i$. Then

$$\sum_{i\in D}\kappa_i < \prod_{i\in D}\lambda_i.$$

Proof. Proof by contradiction. Assume the contrary. Then

$$\prod_{i\in D}\lambda_i \leq \sum_{i\in D}\kappa_i.$$

Take a function G such that $\dot\bigcup_{i\in D}\kappa_i$ is the domain of G and $\bigtimes_{i\in D}\lambda_i$ is the image of G. Indeed $\bigtimes_{i\in D}\lambda_i$ has an element.

Define

$\Delta(i) = \{G((n,i))(i) | n$ is an element of $\kappa_i\}$

for i in D.

For every element f of $\bigtimes_{i\in D}\lambda_i$ for every element i of D $f(i)$ is an element of λ_i. For every element i of D λ_i is a set. For every element i of D for every element d of $\Delta(i)$ we have $d \in \lambda_i$. For every element i of D $\Delta(i)$ is a set.

(1) For every element i of D $\mathrm{card}(\Delta(i)) < \lambda_i$.

Proof. Let i be an element of D. Define $F(n) = G((n,i))(i)$ for n in κ_i. Then $F[\kappa(i)] = \Delta(i)$. end.

Define $f(i) =$ choose an element v of $\lambda_i \setminus \Delta(i)$ in v for i in D. Then f is an element of $\bigtimes_{i \in D} \lambda_i$. Take an element j of D and an element m of κ_j such that $G((m, j)) = f$. $G((m, j))(j)$ is an element of $\Delta(j)$ and $f(j)$ is not an element of $\Delta(j)$. Contradiction. $\square$