Knaster-Tarski fixed point theorem

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The Knaster-Tarski theorem is a result from lattice theory about fixed points of monotone functions. Bronisław Knaster and Alfred Tarski established it in 1928 for the special case of power set lattices (Knaster, B. with Tarski, A.: Un théorème sur les fonctions d'ensembles, Ann. Soc. Polon. Math. vol. 6 (1928), 133-134). This more general result was stated by Tarski in 1955 (Tarski, A.: A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics vol. 5 (1955), no. 2, 285-309).

The formulation and proof of the theorem require a minimal axiomatic setup on posets and functions.

1 ForTheL Setup

[synonym set/-s] [synonym subset/-s] [synonym element/-s]
[synonym belong/-s] [synonym bound/-s] [synonym supremum/suprema]
[synonym infimum/infima] [synonym lattice/-s] [synonym function/-s]
[synonym point/-s]

2 Sets and Elements

Signature 1. An element is a notion.

Let S, T denote sets. Let x, y, z, u, v, w denote elements.

Axiom 2. S is setsized.

Axiom 3. x is setsized.

Axiom 4. Every element of S is an element.

Let x belongs to S denote $x \in S$.

Definition 5. **(DefEmpty)** S is empty iff S has no elements.

Definition 6. **(DefSub)** A subset of S is a class T such that every $x \in T$ belongs to S.

Axiom 7. Let C be a subset of T. Then C is a set.

3 Partial Order, Supremum and Infimum

Signature 8. **(LessRel)** $x \leq y$ is an atom.

Axiom 9. **(ARefl)** $x \leq x$.

Axiom 10. **(ASymm)** $x \leq y \leq x \implies x = y$.

Axiom 11. **(ATrans)** $x \leq y \leq z \implies x \leq z$.

Definition 12. **(DefLB)** Let S be a subset of T. A lower bound of S in T is an element u of T such that $u \leq x$ for every $x \in S$.

Definition 13. **(DefUB)** Let S be a subset of T. An upper bound of S in T is an element u of T such that $x \leq u$ for every $x \in S$.

Definition 14. **(DefInf)** Let S be a subset of T. An infimum of S in T is an element u of T such that u is a lower bound of S in T and for every lower bound v of S in T we have $v \leq u$.

Definition 15. **(DefSup)** Let S be a subset of T. A supremum of S in T is an element u of T such that u is a upper bound of S in T and for every upper bound v of S in T we have $u \leq v$.

Lemma 16. **(SupUn)** Let S be a subset of T. Let u, v be suprema of S in T. Then $u = v$.

Lemma 17. **(InfUn)** Let S be a subset of T. Let u, v be infima of S in T. Then $u = v$.

Definition 18. **(DefCLat)** A complete lattice is a set S such that every subset of S has an infimum in S and a supremum in S.

4 Functions

Let f stand for a function.

Axiom 19. $Dom(f)$ is a set.

Signature 20. **(RanSort)** $Ran(f)$ is a set.

Definition 21. **(DefDom)** f is on S iff $Dom(f) = Ran(f) = S$.

Axiom 22. **(ImgSort)** Let x belong to $Dom(f)$. $f(x)$ is an element of $Ran(f)$.

Definition 23. **(DefFix)** A fixed point of f is an element x of $Dom(f)$ such that $f(x) = x$.

Definition 24. **(DefMonot)** f is monotone iff for all $x, y \in Dom(f)$ $x \leq y \implies f(x) \leq f(y)$.

5 Knaster-Tarski theorem

Theorem 25. **(KnasterTarski)** Let U be a complete lattice and f be a monotone function on U. Let S be the class of fixed points of f. Then S is a complete lattice.

Proof. Let T be a subset of S. Let us show that T has a supremum in S. Define $P = \{x \text{ in } U | f(x) \leq x \text{ and } x \text{ is an upper bound of } T \text{ in } U\}$. Take an infimum p of P in U. $f(p)$ is a lower bound of P in U and an upper bound of T in U. Hence p is a fixed point of f and a supremum of T in S. end.
Let us show that T has an infimum in S.
Define $Q = \{x \text{ in } U | x \leq f(x) \text{ and } x \text{ is a lower bound of } T \text{ in } U\}$. Take a supremum q of Q in U. $f(q)$ is an upper bound of Q in U and a lower bound of T in U. Hence q is a fixed point of f and an infimum of T in S. end. $\square$