Isabelle/HOL — Higher-Order Logic

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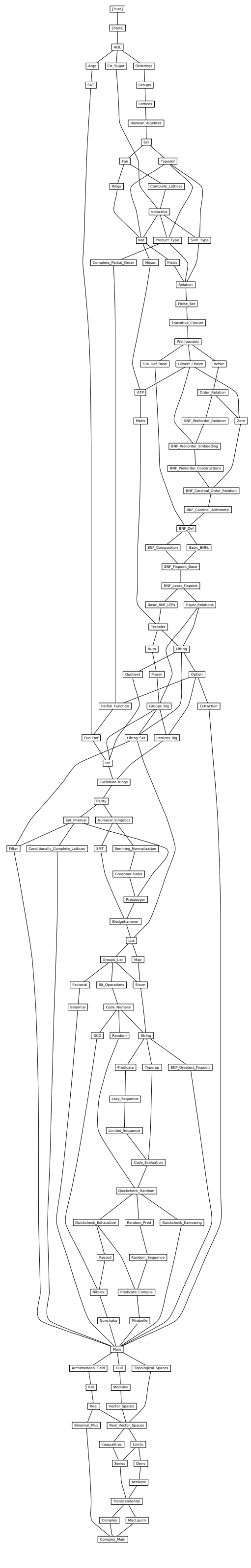
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1 Loading the code generator and related modules

theory *Code-Generator*
imports *Pure*
keywords
print-codeproc code-thms code-deps :: *diag* **and**
export-code code-identifier code-printing code-reserved
code-monad code-reflect :: *thy-decl* **and**
checking **and**
datatypes functions module-name file file-prefix
constant type-constructor type-class class-relation class-instance code-module
:: *quasi-command*
begin

ML-file ‹~~/src/Tools/cache-io.ML›
ML-file ‹~~/src/Tools/Code/code-preproc.ML›
ML-file ‹~~/src/Tools/Code/code-symbol.ML›
ML-file ‹~~/src/Tools/Code/code-thingol.ML›
ML-file ‹~~/src/Tools/Code/code-simp.ML›
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ML-file ‹~~/src/Tools/Code/code-namespace.ML›
ML-file ‹~~/src/Tools/Code/code-ml.ML›
ML-file ‹~~/src/Tools/Code/code-haskell.ML›
ML-file ‹~~/src/Tools/Code/code-scala.ML›

code-datatype *TYPE*('*a*::{})

definition *holds* :: *prop* **where**
holds ≡ ((λ*x*::*prop*. *x*) ≡ (λ*x*. *x*))

lemma *holds*: *PROP holds*
by (*unfold holds-def*) (*rule reflexive*)

code-datatype *holds*

lemma *implies-code* [*code*]:
(*PROP holds* ⟹ *PROP P*) ≡ *PROP P*
(*PROP P* ⟹ *PROP holds*) ≡ *PROP holds*
proof −
show (*PROP holds* ⟹ *PROP P*) ≡ *PROP P*
proof
assume *PROP holds* ⟹ *PROP P*
then show *PROP P* **using** *holds* **.**
next
assume *PROP P*
then show *PROP P* **.**
qed
next

show (*PROP P* ⟹ *PROP holds*) ≡ *PROP holds*
by *rule* (*rule holds*)+
qed

ML-file ‹~~/*src*/*Tools*/*Code*/*code-runtime.ML*›
ML-file ‹~~/*src*/*Tools*/*nbe.ML*›

hide-const (**open**) *holds*

end

2 The basis of Higher-Order Logic

theory *HOL*
imports *Pure Tools.Code-Generator*
keywords
try solve-direct quickcheck print-coercions print-claset
print-induct-rules :: *diag* **and**
quickcheck-params :: *thy-decl*
abbrevs *?<* = $\exists_{\leq 1}$
begin

ML-file ‹~~/*src*/*Tools*/*misc-legacy.ML*›
ML-file ‹~~/*src*/*Tools*/*try.ML*›
ML-file ‹~~/*src*/*Tools*/*quickcheck.ML*›
ML-file ‹~~/*src*/*Tools*/*solve-direct.ML*›
ML-file ‹~~/*src*/*Tools*/*IsaPlanner*/*zipper.ML*›
ML-file ‹~~/*src*/*Tools*/*IsaPlanner*/*isand.ML*›
ML-file ‹~~/*src*/*Tools*/*IsaPlanner*/*rw-inst.ML*›
ML-file ‹~~/*src*/*Provers*/*hypsubst.ML*›
ML-file ‹~~/*src*/*Provers*/*splitter.ML*›
ML-file ‹~~/*src*/*Provers*/*classical.ML*›
ML-file ‹~~/*src*/*Provers*/*blast.ML*›
ML-file ‹~~/*src*/*Provers*/*clasimp.ML*›
ML-file ‹~~/*src*/*Tools*/*eqsubst.ML*›
ML-file ‹~~/*src*/*Provers*/*quantifier1.ML*›
ML-file ‹~~/*src*/*Tools*/*atomize-elim.ML*›
ML-file ‹~~/*src*/*Tools*/*cong-tac.ML*›
ML-file ‹~~/*src*/*Tools*/*intuitionistic.ML*› **setup** ‹*Intuitionistic.method-setup* ***binding*** ‹*iprover*››
ML-file ‹~~/*src*/*Tools*/*project-rule.ML*›
ML-file ‹~~/*src*/*Tools*/*subtyping.ML*›
ML-file ‹~~/*src*/*Tools*/*case-product.ML*›

ML ‹*Plugin-Name.declare-setup* ***binding*** ‹*extraction*››

ML ‹
Plugin-Name.declare-setup ***binding*** ‹*quickcheck-random*›;

```
  Plugin-Name.declare-setup binding ‹quickcheck-exhaustive›;
  Plugin-Name.declare-setup binding ‹quickcheck-bounded-forall›;
  Plugin-Name.declare-setup binding ‹quickcheck-full-exhaustive›;
  Plugin-Name.declare-setup binding ‹quickcheck-narrowing›;
›
ML ‹
  Plugin-Name.define-setup binding ‹quickcheck›
   [plugin ‹quickcheck-exhaustive›,
    plugin ‹quickcheck-random›,
    plugin ‹quickcheck-bounded-forall›,
    plugin ‹quickcheck-full-exhaustive›,
    plugin ‹quickcheck-narrowing›]
›
```

2.1 Primitive logic

The definition of the logic is based on Mike Gordon's technical report [2] that describes the first implementation of HOL. However, there are a number of differences. In particular, we start with the definite description operator and introduce Hilbert's ε operator only much later. Moreover, axiom $(P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \longrightarrow (P = Q)$ is derived from the other axioms. The fact that this axiom is derivable was first noticed by Bruno Barras (for Mike Gordon's line of HOL systems) and later independently by Alexander Maletzky (for Isabelle/HOL).

2.1.1 Core syntax

```
setup ‹Axclass.class-axiomatization (binding ‹type›, [])›
default-sort type
setup ‹Object-Logic.add-base-sort sort ‹type››

setup ‹Proofterm.set-preproc (Proof-Rewrite-Rules.standard-preproc [])›

axiomatization where fun-arity: OFCLASS('a ⇒ 'b, type-class)
instance fun :: (type, type) type by (rule fun-arity)

axiomatization where itself-arity: OFCLASS('a itself, type-class)
instance itself :: (type) type by (rule itself-arity)

typedecl bool

judgment Trueprop :: bool ⇒ prop  (‹(‹notation=judgment›-)› 5)

axiomatization implies :: [bool, bool] ⇒ bool  (infixr ‹⟶› 25)
  and eq :: ['a, 'a] ⇒ bool
  and The :: ('a ⇒ bool) ⇒ 'a

notation (input)
```

eq (**infixl** ‹=› *50*)
notation (**output**)
eq (**infix** ‹=› *50*)

The input syntax for *eq* is more permissive than the output syntax because of the large amount of material that relies on infixl.

2.1.2 Defined connectives and quantifiers

definition *True* :: *bool*
where *True* ≡ ((λ*x*::*bool*. *x*) = (λ*x*. *x*))

definition *All* :: (*'a* ⇒ *bool*) ⇒ *bool* (**binder** ‹∀› *10*)
where *All P* ≡ (*P* = (λ*x*. *True*))

definition *Ex* :: (*'a* ⇒ *bool*) ⇒ *bool* (**binder** ‹∃› *10*)
where *Ex P* ≡ ∀ *Q*. (∀ *x*. *P x* ⟶ *Q*) ⟶ *Q*

definition *False* :: *bool*
where *False* ≡ (∀ *P*. *P*)

definition *Not* :: *bool* ⇒ *bool* (‹(‹*open-block notation*=‹*prefix* ¬››¬ -)› [*40*] *40*)
where *not-def*: ¬ *P* ≡ *P* ⟶ *False*

definition *conj* :: [*bool*, *bool*] ⇒ *bool* (**infixr** ‹∧› *35*)
where *and-def*: *P* ∧ *Q* ≡ ∀ *R*. (*P* ⟶ *Q* ⟶ *R*) ⟶ *R*

definition *disj* :: [*bool*, *bool*] ⇒ *bool* (**infixr** ‹∨› *30*)
where *or-def*: *P* ∨ *Q* ≡ ∀ *R*. (*P* ⟶ *R*) ⟶ (*Q* ⟶ *R*) ⟶ *R*

definition *Uniq* :: (*'a* ⇒ *bool*) ⇒ *bool*
where *Uniq P* ≡ (∀ *x y*. *P x* ⟶ *P y* ⟶ *y* = *x*)

definition *Ex1* :: (*'a* ⇒ *bool*) ⇒ *bool*
where *Ex1 P* ≡ ∃ *x*. *P x* ∧ (∀ *y*. *P y* ⟶ *y* = *x*)

2.1.3 Additional concrete syntax

syntax (*ASCII*) *-Uniq* :: *pttrn* ⇒ *bool* ⇒ *bool* (‹(‹*indent*=*4* *notation*=‹*binder* ?<››?< -./ -)› [*0*, *10*] *10*)
syntax *-Uniq* :: *pttrn* ⇒ *bool* ⇒ *bool* (‹(‹*indent*=*2* *notation*=‹*binder* $\exists_{\leq 1}$››$\exists_{\leq 1}$ -./ -)› [*0*, *10*] *10*)

syntax-consts *-Uniq* ⇌ *Uniq*

translations $\exists_{\leq 1}x$. *P* ⇌ *CONST Uniq* (λ*x*. *P*)

typed-print-translation ‹
[(***const-syntax*** ‹*Uniq*›, *Syntax-Trans.preserve-binder-abs-tr'* ***syntax-const*** ‹*-Uniq*›)]

› — to avoid eta-contraction of body

syntax (*ASCII*)
 -Ex1 :: *pttrn* ⇒ *bool* ⇒ *bool* (‹(‹*indent=3 notation=*‹*binder EX!*››*EX!* -./ -)› [*0*, *10*] *10*)
syntax (*input*)
 -Ex1 :: *pttrn* ⇒ *bool* ⇒ *bool* (‹(‹*indent=3 notation=*‹*binder ?!*››*?!* -./ -)› [*0*, *10*] *10*)
syntax *-Ex1* :: *pttrn* ⇒ *bool* ⇒ *bool* (‹(‹*indent=3 notation=*‹*binder* ∃!››∃!-./ -)› [*0*, *10*] *10*)

syntax-consts *-Ex1* ⇌ *Ex1*

translations ∃!*x*. *P* ⇌ *CONST Ex1* (λ*x*. *P*)

typed-print-translation ‹
 [(***const-syntax***‹*Ex1*›, *Syntax-Trans.preserve-binder-abs-tr′* ***syntax-const***‹*-Ex1*›)]
› — to avoid eta-contraction of body

syntax
 -Not-Ex :: *idts* ⇒ *bool* ⇒ *bool* (‹(‹*indent=3 notation=*‹*binder* ∄››∄-./ -)› [*0*, *10*] *10*)
 -Not-Ex1 :: *pttrn* ⇒ *bool* ⇒ *bool* (‹(‹*indent=3 notation=*‹*binder* ∄!››∄!-./ -)› [*0*, *10*] *10*)
syntax-consts
 -Not-Ex ⇌ *Ex* **and**
 -Not-Ex1 ⇌ *Ex1*
translations
 ∄*x*. *P* ⇌ ¬ (∃*x*. *P*)
 ∄!*x*. *P* ⇌ ¬ (∃!*x*. *P*)

abbreviation *not-equal* :: [′*a*, ′*a*] ⇒ *bool* (**infix** ‹≠› *50*)
 where *x* ≠ *y* ≡ ¬ (*x* = *y*)

notation (*ASCII*)
 Not (‹(‹*open-block notation=*‹*prefix* ~››~ -)› [*40*] *40*) **and**
 conj (**infixr** ‹&› *35*) **and**
 disj (**infixr** ‹|› *30*) **and**
 implies (**infixr** ‹-->› *25*) **and**
 not-equal (**infix** ‹~=› *50*)

abbreviation (*iff*)
 iff :: [*bool*, *bool*] ⇒ *bool* (**infixr** ‹⟷› *25*)
 where *A* ⟷ *B* ≡ *A* = *B*

syntax *-The* :: [*pttrn*, *bool*] ⇒ ′*a* (‹(‹*indent=3 notation=*‹*binder THE*››*THE* -./

-)› [0, 10] 10)
syntax-consts -The ⇌ The
translations THE x. P ⇌ CONST The (λx. P)
print-translation ‹
  [(***const-syntax*** ‹The›, fn ctxt => fn [Abs abs] =>
      let val (x, t) = Syntax-Trans.atomic-abs-tr′ ctxt abs
      in Syntax.const ***syntax-const*** ‹-The› $ x $ t end)]
› — To avoid eta-contraction of body

nonterminal case-syn **and** cases-syn
syntax
  -case-syntax :: ['a, cases-syn] ⇒ 'b (‹(‹notation=‹mixfix case expression››case -
of/ -)› 10)
  -case1 :: ['a, 'b] ⇒ case-syn
  (‹(‹indent=2 notation=‹mixfix case clause››(‹open-block notation=‹pattern case››-)
⇒/ -)› 10)
  :: case-syn ⇒ cases-syn (‹-›)
  -case2 :: [case-syn, cases-syn] ⇒ cases-syn (‹-/ | -›)
syntax (ASCII)
  -case1 :: ['a, 'b] ⇒ case-syn
  (‹(‹indent=2 notation=‹mixfix case clause››(‹open-block notation=‹pattern case››-)
=>/ -)› 10)

notation (ASCII)
  All (**binder** ‹ALL › 10) **and**
  Ex (**binder** ‹EX › 10)

notation (input)
  All (**binder** ‹! › 10) **and**
  Ex (**binder** ‹? › 10)

2.1.4 Axioms and basic definitions

axiomatization where
  refl: t = (t::'a) **and**
  subst: s = t ⟹ P s ⟹ P t **and**
  ext: (⋀x::'a. (f x ::'b) = g x) ⟹ (λx. f x) = (λx. g x)
  — Extensionality is built into the meta-logic, and this rule expresses a related property. It is an eta-expanded version of the traditional rule, and similar to the ABS rule of HOL **and**

  the-eq-trivial: (THE x. x = a) = (a::'a)

axiomatization where
  impI: (P ⟹ Q) ⟹ P ⟶ Q **and**
  mp: ⟦P ⟶ Q; P⟧ ⟹ Q **and**

  True-or-False: (P = True) ∨ (P = False)

definition *If* :: *bool* ⇒ *'a* ⇒ *'a* ⇒ *'a* (‹(‹*notation*=‹*mixfix if expression*››*if* (-)/ *then* (-)/ *else* (-))› [*0*, *0*, *10*] *10*)
  where *If P x y* ≡ (*THE z*::*'a*. (*P* = *True* ⟶ *z* = *x*) ∧ (*P* = *False* ⟶ *z* = *y*))

definition *Let* :: *'a* ⇒ (*'a* ⇒ *'b*) ⇒ *'b*
  where *Let s f* ≡ *f s*

nonterminal *letbinds* **and** *letbind*

open-bundle *let-syntax*
begin

syntax
  -bind :: [*pttrn*, *'a*] ⇒ *letbind* (‹(‹*indent*=*2* *notation*=‹*mixfix let binding*››- =/ -)› *10*)
  :: *letbind* ⇒ *letbinds* (‹-›)
  -binds :: [*letbind*, *letbinds*] ⇒ *letbinds* (‹-;/ -›)
  -Let :: [*letbinds*, *'a*] ⇒ *'a* (‹(‹*notation*=‹*mixfix let expression*››*let* (-)/ *in* (-))› [*0*, *10*] *10*)
syntax-consts
  -bind -binds -Let ⇌ *Let*
translations
  -Let (*-binds b bs*) *e* ⇌ *-Let b* (*-Let bs e*)
  let x = *a in e* ⇌ *CONST Let a* (λ*x*. *e*)

end

axiomatization *undefined* :: *'a*

class *default* = **fixes** *default* :: *'a*

2.2 Fundamental rules

2.2.1 Equality

lemma *sym*: *s* = *t* ⟹ *t* = *s*
  by (*erule subst*) (*rule refl*)

lemma *ssubst*: *t* = *s* ⟹ *P s* ⟹ *P t*
  by (*drule sym*) (*erule subst*)

lemma *trans*: ⟦*r* = *s*; *s* = *t*⟧ ⟹ *r* = *t*
  by (*erule subst*)

lemma *trans-sym* [*Pure.elim?*]: *r* = *s* ⟹ *t* = *s* ⟹ *r* = *t*
  by (*rule trans* [*OF* - *sym*])

lemma *meta-eq-to-obj-eq*:
  assumes *A* ≡ *B*
  shows *A* = *B*

unfolding *assms* **by** (*rule refl*)

Useful with *erule* for proving equalities from known equalities.

lemma *box-equals*: $\llbracket a = b;\ a = c;\ b = d \rrbracket \Longrightarrow c = d$
by (*iprover intro*: *sym trans*)

For calculational reasoning:

lemma *forw-subst*: $a = b \Longrightarrow P\ b \Longrightarrow P\ a$
by (*rule ssubst*)

lemma *back-subst*: $P\ a \Longrightarrow a = b \Longrightarrow P\ b$
by (*rule subst*)

2.2.2 Congruence rules for application

Similar to *AP-THM* in Gordon's HOL.

lemma *fun-cong*: $(f :: 'a \Rightarrow 'b) = g \Longrightarrow f\ x = g\ x$
by (*iprover intro*: *refl elim*: *subst*)

Similar to *AP-TERM* in Gordon's HOL and FOL's *subst-context*.

lemma *arg-cong*: $x = y \Longrightarrow f\ x = f\ y$
by (*iprover intro*: *refl elim*: *subst*)

lemma *arg-cong2*: $\llbracket a = b;\ c = d \rrbracket \Longrightarrow f\ a\ c = f\ b\ d$
by (*iprover intro*: *refl elim*: *subst*)

lemma *cong*: $\llbracket f = g;\ (x{::}'a) = y \rrbracket \Longrightarrow f\ x = g\ y$
by (*iprover intro*: *refl elim*: *subst*)

ML ‹*fun cong-tac ctxt* = *Cong-Tac.cong-tac ctxt* @{*thm cong*}›

2.2.3 Equality of booleans – iff

lemma *iffD2*: $\llbracket P = Q;\ Q \rrbracket \Longrightarrow P$
by (*erule ssubst*)

lemma *rev-iffD2*: $\llbracket Q;\ P = Q \rrbracket \Longrightarrow P$
by (*erule iffD2*)

lemma *iffD1*: $Q = P \Longrightarrow Q \Longrightarrow P$
by (*drule sym*) (*rule iffD2*)

lemma *rev-iffD1*: $Q \Longrightarrow Q = P \Longrightarrow P$
by (*drule sym*) (*rule rev-iffD2*)

lemma *iffE*:
assumes *major*: $P = Q$
and *minor*: $\llbracket P \longrightarrow Q;\ Q \longrightarrow P \rrbracket \Longrightarrow R$

shows *R*
by (*iprover intro*: *minor impI major* [*THEN iffD2*] *major* [*THEN iffD1*])

2.2.4 True (1)

lemma *TrueI*: *True*
unfolding *True-def* **by** (*rule refl*)

lemma *eqTrueE*: $P = True \Longrightarrow P$
by (*erule iffD2*) (*rule TrueI*)

2.2.5 Universal quantifier (1)

lemma *spec*: $\forall x::'a.\ P\ x \Longrightarrow P\ x$
unfolding *All-def* **by** (*iprover intro*: *eqTrueE fun-cong*)

lemma *allE*:
assumes *major*: $\forall x.\ P\ x$ **and** *minor*: $P\ x \Longrightarrow R$
shows *R*
by (*iprover intro*: *minor major* [*THEN spec*])

lemma *all-dupE*:
assumes *major*: $\forall x.\ P\ x$ **and** *minor*: $[\![P\ x; \forall x.\ P\ x]\!] \Longrightarrow R$
shows *R*
by (*iprover intro*: *minor major major* [*THEN spec*])

2.2.6 False

Depends upon *spec*; it is impossible to do propositional logic before quantifiers!

lemma *FalseE*: $False \Longrightarrow P$
unfolding *False-def* **by** (*erule spec*)

lemma *False-neq-True*: $False = True \Longrightarrow P$
by (*erule eqTrueE* [*THEN FalseE*])

2.2.7 Negation

lemma *notI*:
assumes $P \Longrightarrow False$
shows $\neg\ P$
unfolding *not-def* **by** (*iprover intro*: *impI assms*)

lemma *False-not-True*: $False \neq True$
by (*iprover intro*: *notI elim*: *False-neq-True*)

lemma *True-not-False*: $True \neq False$
by (*iprover intro*: *notI dest*: *sym elim*: *False-neq-True*)

lemma *notE*: $[\![\neg\ P;\ P]\!] \Longrightarrow R$
  unfolding *not-def*
  by (*iprover intro*: *mp* [*THEN FalseE*])

2.2.8 Implication

lemma *impE*:
  assumes $P \longrightarrow Q$ P $Q \Longrightarrow R$
  shows R
  by (*iprover intro*: *assms mp*)

Reduces Q to $P \longrightarrow Q$, allowing substitution in P.

lemma *rev-mp*: $[\![P;\ P \longrightarrow Q]\!] \Longrightarrow Q$
  by (*rule mp*)

lemma *contrapos-nn*:
  assumes *major*: $\neg\ Q$
    and *minor*: $P \Longrightarrow Q$
  shows $\neg\ P$
  by (*iprover intro*: *notI minor major* [*THEN notE*])

Not used at all, but we already have the other 3 combinations.

lemma *contrapos-pn*:
  assumes *major*: Q
    and *minor*: $P \Longrightarrow \neg\ Q$
  shows $\neg\ P$
  by (*iprover intro*: *notI minor major notE*)

lemma *not-sym*: $t \neq s \Longrightarrow s \neq t$
  by (*erule contrapos-nn*) (*erule sym*)

lemma *eq-neq-eq-imp-neq*: $[\![x = a;\ a \neq b;\ b = y]\!] \Longrightarrow x \neq y$
  by (*erule subst*, *erule ssubst*, *assumption*)

2.2.9 Disjunction (1)

lemma *disjE*:
  assumes *major*: $P \vee Q$
    and *minorP*: $P \Longrightarrow R$
    and *minorQ*: $Q \Longrightarrow R$
  shows R
  by (*iprover intro*: *minorP minorQ impI*
      major [*unfolded or-def*, *THEN spec*, *THEN mp*, *THEN mp*])

2.2.10 Derivation of *iffI*

In an intuitionistic version of HOL *iffI* needs to be an axiom.

lemma *iffI*:
  assumes $P \Longrightarrow Q$ **and** $Q \Longrightarrow P$

```
  shows P = Q
proof (rule disjE[OF True-or-False[of P]])
  assume 1: P = True
  note Q = assms(1)[OF eqTrueE[OF this]]
  from 1 show ?thesis
  proof (rule ssubst)
    from True-or-False[of Q] show True = Q
    proof (rule disjE)
      assume Q = True
      thus ?thesis by(rule sym)
    next
      assume Q = False
      with Q have False by (rule rev-iffD1)
      thus ?thesis by (rule FalseE)
    qed
  qed
next
  assume 2: P = False
  thus ?thesis
  proof (rule ssubst)
    from True-or-False[of Q] show False = Q
    proof (rule disjE)
      assume Q = True
      from 2 assms(2)[OF eqTrueE[OF this]] have False by (rule iffD1)
      thus ?thesis by (rule FalseE)
    next
      assume Q = False
      thus ?thesis by(rule sym)
    qed
  qed
qed
```

2.2.11 True (2)

```
lemma eqTrueI: P ⟹ P = True
  by (iprover intro: iffI TrueI)
```

2.2.12 Universal quantifier (2)

```
lemma allI:
  assumes ⋀x::'a. P x
  shows ∀x. P x
  unfolding All-def by (iprover intro: ext eqTrueI assms)
```

2.2.13 Existential quantifier

```
lemma exI: P x ⟹ ∃x::'a. P x
  unfolding Ex-def by (iprover intro: allI allE impI mp)

lemma exE:
```

assumes *major*: $\exists x{::}'a.\ P\ x$
and *minor*: $\bigwedge x.\ P\ x \Longrightarrow Q$
shows Q
by (*rule major* [*unfolded Ex-def*, *THEN spec*, *THEN mp*]) (*iprover intro*: *impI* [*THEN allI*] *minor*)

2.2.14 Conjunction

lemma *conjI*: $[\![P;\ Q]\!] \Longrightarrow P \wedge Q$
unfolding *and-def* **by** (*iprover intro*: *impI* [*THEN allI*] *mp*)

lemma *conjunct1*: $[\![P \wedge Q]\!] \Longrightarrow P$
unfolding *and-def* **by** (*iprover intro*: *impI dest*: *spec mp*)

lemma *conjunct2*: $[\![P \wedge Q]\!] \Longrightarrow Q$
unfolding *and-def* **by** (*iprover intro*: *impI dest*: *spec mp*)

lemma *conjE*:
assumes *major*: $P \wedge Q$
and *minor*: $[\![P;\ Q]\!] \Longrightarrow R$
shows R
proof (*rule minor*)
show P **by** (*rule major* [*THEN conjunct1*])
show Q **by** (*rule major* [*THEN conjunct2*])
qed

lemma *context-conjI*:
assumes $P\ P \Longrightarrow Q$
shows $P \wedge Q$
by (*iprover intro*: *conjI assms*)

2.2.15 Disjunction (2)

lemma *disjI1*: $P \Longrightarrow P \vee Q$
unfolding *or-def* **by** (*iprover intro*: *allI impI mp*)

lemma *disjI2*: $Q \Longrightarrow P \vee Q$
unfolding *or-def* **by** (*iprover intro*: *allI impI mp*)

2.2.16 Classical logic

lemma *classical*:
assumes $\neg\ P \Longrightarrow P$
shows P
proof (*rule True-or-False* [*THEN disjE*])
show P **if** $P = \mathit{True}$
using *that* **by** (*iprover intro*: *eqTrueE*)
show P **if** $P = \mathit{False}$
proof (*intro notI assms*)
assume P

with *that* **show** *False*
by (*iprover elim*: *subst*)
qed
qed

lemmas *ccontr* = *FalseE* [*THEN classical*]

notE with premises exchanged; it discharges $\neg R$ so that it can be used to make elimination rules.

lemma *rev-notE*:
assumes *premp*: P
and *premnot*: $\neg R \Longrightarrow \neg P$
shows R
by (*iprover intro*: *ccontr notE* [*OF premnot premp*])

Double negation law.

lemma *notnotD*: $\neg\neg P \Longrightarrow P$
by (*iprover intro*: *ccontr notE*)

lemma *contrapos-pp*:
assumes *p1*: Q
and *p2*: $\neg P \Longrightarrow \neg Q$
shows P
by (*iprover intro*: *classical p1 p2 notE*)

2.2.17 Unique existence

lemma *Uniq-I* [*intro?*]:
assumes $\bigwedge x\ y.\ [\![P\ x;\ P\ y]\!] \Longrightarrow y = x$
shows *Uniq P*
unfolding *Uniq-def* **by** (*iprover intro*: *assms allI impI*)

lemma *Uniq-D* [*dest?*]: $[\![\mathit{Uniq}\ P;\ P\ a;\ P\ b]\!] \Longrightarrow a{=}b$
unfolding *Uniq-def* **by** (*iprover dest*: *spec mp*)

lemma *ex1I*:
assumes $P\ a\ \bigwedge x.\ P\ x \Longrightarrow x = a$
shows $\exists!x.\ P\ x$
unfolding *Ex1-def* **by** (*iprover intro*: *assms exI conjI allI impI*)

Sometimes easier to use: the premises have no shared variables. Safe!

lemma *ex-ex1I*:
assumes *ex-prem*: $\exists x.\ P\ x$
and *eq*: $\bigwedge x\ y.\ [\![P\ x;\ P\ y]\!] \Longrightarrow x = y$
shows $\exists!x.\ P\ x$
by (*iprover intro*: *ex-prem* [*THEN exE*] *ex1I eq*)

lemma *ex1E*:
assumes *major*: $\exists!x.\ P\ x$ **and** *minor*: $\bigwedge x.\ [\![P\ x;\ \forall y.\ P\ y \longrightarrow y = x]\!] \Longrightarrow R$

shows R
proof (*rule major* [*unfolded Ex1-def*, *THEN exE*])
show $\bigwedge x.\ P\ x \wedge (\forall\, y.\ P\ y \longrightarrow y = x) \Longrightarrow R$
by (*iprover intro*: *minor elim*: *conjE*)
qed

lemma *ex1-implies-ex*: $\exists!x.\ P\ x \Longrightarrow \exists\, x.\ P\ x$
by (*iprover intro*: *exI elim*: *ex1E*)

2.2.18 Classical intro rules for disjunction and existential quantifiers

lemma *disjCI*:
assumes $\neg\ Q \Longrightarrow P$
shows $P \vee Q$
by (*rule classical*) (*iprover intro*: *assms disjI1 disjI2 notI elim*: *notE*)

lemma *excluded-middle*: $\neg\ P \vee P$
by (*iprover intro*: *disjCI*)

case distinction as a natural deduction rule. Note that $\neg\ P$ is the second case, not the first.

lemma *case-split* [*case-names True False*]:
assumes $P \Longrightarrow Q\ \neg\ P \Longrightarrow Q$
shows Q
using *excluded-middle* [*of P*]
by (*iprover intro*: *assms elim*: *disjE*)

Classical implies ($\longrightarrow$) elimination.

lemma *impCE*:
assumes *major*: $P \longrightarrow Q$
and *minor*: $\neg\ P \Longrightarrow R\ Q \Longrightarrow R$
shows R
using *excluded-middle* [*of P*]
by (*iprover intro*: *minor major* [*THEN mp*] *elim*: *disjE*)+

This version of $\longrightarrow$ elimination works on Q before P. It works best for those cases in which P holds "almost everywhere". Can't install as default: would break old proofs.

lemma *impCE'*:
assumes *major*: $P \longrightarrow Q$
and *minor*: $Q \Longrightarrow R\ \neg\ P \Longrightarrow R$
shows R
using *assms* **by** (*elim impCE*)

The analogous introduction rule for conjunction, above, is even constructive

lemma *context-disjE*:
assumes *major*: $P \vee Q$ **and** *minor*: $P \Longrightarrow R\ \neg P \Longrightarrow Q \Longrightarrow R$

shows R
by (*iprover intro*: *disjE* [*OF major*] *disjE* [*OF excluded-middle*] *assms*)

Classical $\longleftrightarrow$ elimination.

lemma *iffCE*:
assumes *major*: $P = Q$
and *minor*: $[\![P;\ Q]\!] \Longrightarrow R$ $[\![\neg\ P;\ \neg\ Q]\!] \Longrightarrow R$
shows R
by (*rule major* [*THEN iffE*]) (*iprover intro*: *minor elim*: *impCE notE*)

lemma *exCI*:
assumes $\forall\, x.\ \neg\ P\ x \Longrightarrow P\ a$
shows $\exists\, x.\ P\ x$
by (*rule ccontr*) (*iprover intro*: *assms exI allI notI notE* [*of* $\exists\, x.\ P\ x$])

2.2.19 Intuitionistic Reasoning

lemma *impE'*:
assumes *1*: $P \longrightarrow Q$
and *2*: $Q \Longrightarrow R$
and *3*: $P \longrightarrow Q \Longrightarrow P$
shows R
proof −
from *3* **and** *1* **have** P **.**
with *1* **have** Q **by** (*rule impE*)
with *2* **show** R **.**
qed

lemma *allE'*:
assumes *1*: $\forall\, x.\ P\ x$
and *2*: $P\ x \Longrightarrow \forall\, x.\ P\ x \Longrightarrow Q$
shows Q
proof −
from *1* **have** $P\ x$ **by** (*rule spec*)
from *this* **and** *1* **show** Q **by** (*rule 2*)
qed

lemma *notE'*:
assumes *1*: $\neg\ P$
and *2*: $\neg\ P \Longrightarrow P$
shows R
proof −
from *2* **and** *1* **have** P **.**
with *1* **show** R **by** (*rule notE*)
qed

lemma *TrueE*: $True \Longrightarrow P \Longrightarrow P$ **.**
lemma *notFalseE*: $\neg\ False \Longrightarrow P \Longrightarrow P$ **.**

lemmas [*Pure.elim*!] = *disjE iffE FalseE conjE exE TrueE notFalseE*
and [*Pure.intro*!] = *iffI conjI impI TrueI notI allI refl*
and [*Pure.elim 2*] = *allE notE′ impE′*
and [*Pure.intro*] = *exI disjI2 disjI1*

lemmas [*trans*] = *trans*
and [*sym*] = *sym not-sym*
and [*Pure.elim?*] = *iffD1 iffD2 impE*

2.2.20 Atomizing meta-level connectives

axiomatization where
eq-reflection: $x = y \Longrightarrow x \equiv y$ — admissible axiom

lemma *atomize-all* [*atomize*]: $(\bigwedge x.\ P\ x) \equiv Trueprop\ (\forall x.\ P\ x)$
proof
assume $\bigwedge x.\ P\ x$
then show $\forall x.\ P\ x$ **..**
next
assume $\forall x.\ P\ x$
then show $\bigwedge x.\ P\ x$ **by** (*rule allE*)
qed

lemma *atomize-imp* [*atomize*]: $(A \Longrightarrow B) \equiv Trueprop\ (A \longrightarrow B)$
proof
assume *r*: $A \Longrightarrow B$
show $A \longrightarrow B$ **by** (*rule impI*) (*rule r*)
next
assume $A \longrightarrow B$ **and** A
then show B **by** (*rule mp*)
qed

lemma *atomize-not*: $(A \Longrightarrow False) \equiv Trueprop\ (\neg\ A)$
proof
assume *r*: $A \Longrightarrow False$
show $\neg\ A$ **by** (*rule notI*) (*rule r*)
next
assume $\neg\ A$ **and** A
then show *False* **by** (*rule notE*)
qed

lemma *atomize-eq* [*atomize, code*]: $(x \equiv y) \equiv Trueprop\ (x = y)$
proof
assume $x \equiv y$
show $x = y$ **by** (*unfold* ‹$x \equiv y$›) (*rule refl*)
next
assume $x = y$
then show $x \equiv y$ **by** (*rule eq-reflection*)
qed

lemma *atomize-conj* [*atomize*]: $(A \;\&\&\&\; B) \equiv \mathit{Trueprop}\ (A \wedge B)$
proof
  assume *conj*: $A \;\&\&\&\; B$
  show $A \wedge B$
  proof (*rule conjI*)
    from *conj* **show** A **by** (*rule conjunctionD1*)
    from *conj* **show** B **by** (*rule conjunctionD2*)
  qed
next
  assume *conj*: $A \wedge B$
  show $A \;\&\&\&\; B$
  proof −
    from *conj* **show** A **..**
    from *conj* **show** B **..**
  qed
qed

lemmas [*symmetric*, *rulify*] = *atomize-all atomize-imp*
  and [*symmetric*, *defn*] = *atomize-all atomize-imp atomize-eq*

2.2.21 Atomizing elimination rules

lemma *atomize-exL*[*atomize-elim*]: $(\bigwedge x.\ P\ x \Longrightarrow Q) \equiv ((\exists x.\ P\ x) \Longrightarrow Q)$
  by (*rule equal-intr-rule*) *iprover+*

lemma *atomize-conjL*[*atomize-elim*]: $(A \Longrightarrow B \Longrightarrow C) \equiv (A \wedge B \Longrightarrow C)$
  by (*rule equal-intr-rule*) *iprover+*

lemma *atomize-disjL*[*atomize-elim*]: $((A \Longrightarrow C) \Longrightarrow (B \Longrightarrow C) \Longrightarrow C) \equiv ((A \vee B \Longrightarrow C) \Longrightarrow C)$
  by (*rule equal-intr-rule*) *iprover+*

lemma *atomize-elimL*[*atomize-elim*]: $(\bigwedge B.\ (A \Longrightarrow B) \Longrightarrow B) \equiv \mathit{Trueprop}\ A$ **..**

2.3 Package setup

ML-file ‹*Tools/hologic.ML*›
ML-file ‹*Tools/rewrite-hol-proof.ML*›

setup ‹*Proofterm.set-preproc* (*Proof-Rewrite-Rules.standard-preproc Rewrite-HOL-Proof.rews*)›

2.3.1 Sledgehammer setup

Theorems blacklisted to Sledgehammer. These theorems typically produce clauses that are prolific (match too many equality or membership literals) and relate to seldom-used facts. Some duplicate other rules.

named-theorems *no-atp theorems that should be filtered out by Sledgehammer*

2.3.2 Classical Reasoner setup

lemma *imp-elim*: $P \longrightarrow Q \Longrightarrow (\neg\ R \Longrightarrow P) \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
  by (*rule classical*) *iprover*

lemma *swap*: $\neg\ P \Longrightarrow (\neg\ R \Longrightarrow P) \Longrightarrow R$
  by (*rule classical*) *iprover*

lemma *thin-refl*: $[\![x = x;\ PROP\ W]\!] \Longrightarrow PROP\ W$ **.**

ML ‹
```
structure Hypsubst = Hypsubst
(
  val dest-eq = HOLogic.dest-eq
  val dest-Trueprop = HOLogic.dest-Trueprop
  val dest-imp = HOLogic.dest-imp
  val eq-reflection = @{thm eq-reflection}
  val rev-eq-reflection = @{thm meta-eq-to-obj-eq}
  val imp-intr = @{thm impI}
  val rev-mp = @{thm rev-mp}
  val subst = @{thm subst}
  val sym = @{thm sym}
  val thin-refl = @{thm thin-refl};
);
open Hypsubst;

structure Classical = Classical
(
  val imp-elim = @{thm imp-elim}
  val not-elim = @{thm notE}
  val swap = @{thm swap}
  val classical = @{thm classical}
  val sizef = Drule.size-of-thm
  val hyp-subst-tacs = [Hypsubst.hyp-subst-tac]
);

structure Basic-Classical: BASIC-CLASSICAL = Classical;
open Basic-Classical;
```
›

setup ‹
```
  (*prevent substitution on bool*)
  let
    fun non-bool-eq Const- ‹HOL.eq T› = T <> Type ‹bool›
      | non-bool-eq - = false;
    fun hyp-subst-tac' ctxt =
      SUBGOAL (fn (goal, i) =>
        if Term.exists-subterm non-bool-eq goal
        then Hypsubst.hyp-subst-tac ctxt i
        else no-tac);
```

in
Context-Rules.addSWrapper (*fn ctxt => fn tac => hyp-subst-tac′ ctxt ORELSE′ tac*)
end
›

declare *iffI* [*intro*!]
and *notI* [*intro*!]
and *impI* [*intro*!]
and *disjCI* [*intro*!]
and *conjI* [*intro*!]
and *TrueI* [*intro*!]
and *refl* [*intro*!]

declare *iffCE* [*elim*!]
and *FalseE* [*elim*!]
and *impCE* [*elim*!]
and *disjE* [*elim*!]
and *conjE* [*elim*!]

declare *ex-ex1I* [*intro*!]
and *allI* [*intro*!]
and *exI* [*intro*]

declare *exE* [*elim*!]
allE [*elim*]

ML ‹*val HOL-cs = claset-of* ***context***›

lemma *contrapos-np*: $\neg Q \Longrightarrow (\neg P \Longrightarrow Q) \Longrightarrow P$
by (*erule swap*)

declare *ex-ex1I* [*rule del*, *intro*! *2*]
and *ex1I* [*intro*]

declare *ext* [*intro*]

lemmas [*intro?*] = *ext*
and [*elim?*] = *ex1-implies-ex*

Better than *ex1E* for classical reasoner: needs no quantifier duplication!

lemma *alt-ex1E* [*elim*!]:
assumes *major*: $\exists!x.\ P\ x$
and *minor*: $\bigwedge x.\ [\![P\ x;\ \forall\, y\ y'.\ P\ y \wedge P\ y' \longrightarrow y = y']\!] \Longrightarrow R$
shows R
proof (*rule ex1E* [*OF major minor*])
show $\forall\, y\ y'.\ P\ y \wedge P\ y' \longrightarrow y = y'$ **if** $P\ x$ **and** §: $\forall\, y.\ P\ y \longrightarrow y = x$ **for** x
using ‹*P x*› § § **by** *fast*
qed *assumption*

And again using Uniq

lemma *alt-ex1E'*:
assumes $\exists!x.\ P\ x$ $\bigwedge x.\ [\![P\ x; \exists_{\leq 1}x.\ P\ x]\!] \Longrightarrow R$
shows R
using *assms* **unfolding** *Uniq-def* **by** *fast*

lemma *ex1-iff-ex-Uniq*: $(\exists!x.\ P\ x) \longleftrightarrow (\exists x.\ P\ x) \wedge (\exists_{\leq 1}x.\ P\ x)$
unfolding *Uniq-def* **by** *fast*

```
ML ‹
  structure Blast = Blast
  (
    structure Classical = Classical
    val Trueprop-const = dest-Const Const ‹Trueprop›
    val equality-name = const-name ‹HOL.eq›
    val not-name = const-name ‹Not›
    val notE = @{thm notE}
    val ccontr = @{thm ccontr}
    val hyp-subst-tac = Hypsubst.blast-hyp-subst-tac
  );
  val blast-tac = Blast.blast-tac;
›
```

2.3.3 THE: definite description operator

lemma *the-equality* [*intro*]:
assumes $P\ a$
and $\bigwedge x.\ P\ x \Longrightarrow x = a$
shows $(THE\ x.\ P\ x) = a$
by (*blast intro*: *assms trans* [*OF arg-cong* [**where** *f*=*The*] *the-eq-trivial*])

lemma *theI*:
assumes $P\ a$
and $\bigwedge x.\ P\ x \Longrightarrow x = a$
shows $P\ (THE\ x.\ P\ x)$
by (*iprover intro*: *assms the-equality* [*THEN ssubst*])

lemma *theI'*: $\exists!x.\ P\ x \Longrightarrow P\ (THE\ x.\ P\ x)$
by (*blast intro*: *theI*)

Easier to apply than *theI*: only one occurrence of P.

lemma *theI2*:
assumes $P\ a$ $\bigwedge x.\ P\ x \Longrightarrow x = a$ $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $Q\ (THE\ x.\ P\ x)$
by (*iprover intro*: *assms theI*)

lemma *the1I2*:
assumes $\exists!x.\ P\ x$ $\bigwedge x.\ P\ x \Longrightarrow Q\ x$

shows *Q* (*THE x. P x*)
by (*iprover intro*: *assms*(*2*) *theI2*[**where** *P*=*P* **and** *Q*=*Q*] *ex1E*[*OF assms*(*1*)] *elim*: *allE impE*)

lemma *the1-equality* [*elim?*]: ⟦∃!*x*. *P x*; *P a*⟧ ⟹ (*THE x*. *P x*) = *a*
by *blast*

lemma *the1-equality'*: ⟦$\exists_{\leq 1}$*x*. *P x*; *P a*⟧ ⟹ (*THE x*. *P x*) = *a*
unfolding *Uniq-def* **by** *blast*

lemma *the-sym-eq-trivial*: (*THE y*. *x* = *y*) = *x*
by *blast*

2.3.4 Simplifier

lemma *eta-contract-eq*: (λ*s*. *f s*) = *f* **..**

```
lemma subst-all:
  ‹(⋀x. x = a ⟹ PROP P x) ≡ PROP P a›
  ‹(⋀x. a = x ⟹ PROP P x) ≡ PROP P a›
proof −
  show ‹(⋀x. x = a ⟹ PROP P x) ≡ PROP P a›
  proof (rule equal-intr-rule)
    assume ∗: ‹⋀x. x = a ⟹ PROP P x›
    show ‹PROP P a›
      by (rule ∗) (rule refl)
  next
    fix x
    assume ‹PROP P a› and ‹x = a›
    from ‹x = a› have ‹x ≡ a›
      by (rule eq-reflection)
    with ‹PROP P a› show ‹PROP P x›
      by simp
  qed
  show ‹(⋀x. a = x ⟹ PROP P x) ≡ PROP P a›
  proof (rule equal-intr-rule)
    assume ∗: ‹⋀x. a = x ⟹ PROP P x›
    show ‹PROP P a›
      by (rule ∗) (rule refl)
  next
    fix x
    assume ‹PROP P a› and ‹a = x›
    from ‹a = x› have ‹a ≡ x›
      by (rule eq-reflection)
    with ‹PROP P a› show ‹PROP P x›
      by simp
  qed
qed
```

lemma *simp-thms*:
shows *not-not*: $(\neg\ \neg\ P) = P$
and *Not-eq-iff*: $((\neg\ P) = (\neg\ Q)) = (P = Q)$
and
$(P \neq Q) = (P = (\neg\ Q))$
$(P \vee \neg P) = \mathit{True} \quad (\neg\ P \vee P) = \mathit{True}$
$(x = x) = \mathit{True}$
and *not-True-eq-False* [*code*]: $(\neg\ \mathit{True}) = \mathit{False}$
and *not-False-eq-True* [*code*]: $(\neg\ \mathit{False}) = \mathit{True}$
and
$(\neg\ P) \neq P \;\; P \neq (\neg\ P)$
$(\mathit{True} = P) = P$
and *eq-True*: $(P = \mathit{True}) = P$
and $(\mathit{False} = P) = (\neg\ P)$
and *eq-False*: $(P = \mathit{False}) = (\neg\ P)$
and
$(\mathit{True} \longrightarrow P) = P \;\; (\mathit{False} \longrightarrow P) = \mathit{True}$
$(P \longrightarrow \mathit{True}) = \mathit{True} \;\; (P \longrightarrow P) = \mathit{True}$
$(P \longrightarrow \mathit{False}) = (\neg\ P) \;\; (P \longrightarrow \neg\ P) = (\neg\ P)$
$(P \wedge \mathit{True}) = P \;\; (\mathit{True} \wedge P) = P$
$(P \wedge \mathit{False}) = \mathit{False} \;\; (\mathit{False} \wedge P) = \mathit{False}$
$(P \wedge P) = P \;\; (P \wedge (P \wedge Q)) = (P \wedge Q)$
$(P \wedge \neg\ P) = \mathit{False} \quad (\neg\ P \wedge P) = \mathit{False}$
$(P \vee \mathit{True}) = \mathit{True} \;\; (\mathit{True} \vee P) = \mathit{True}$
$(P \vee \mathit{False}) = P \;\; (\mathit{False} \vee P) = P$
$(P \vee P) = P \;\; (P \vee (P \vee Q)) = (P \vee Q)$ **and**
$(\forall x.\ P) = P \;\; (\exists x.\ P) = P \;\; \exists x.\ x = t \;\; \exists x.\ t = x$
and
$\bigwedge P.\ (\exists x.\ x = t \wedge P\ x) = P\ t$
$\bigwedge P.\ (\exists x.\ t = x \wedge P\ x) = P\ t$
$\bigwedge P.\ (\forall x.\ x = t \longrightarrow P\ x) = P\ t$
$\bigwedge P.\ (\forall x.\ t = x \longrightarrow P\ x) = P\ t$
$(\forall x.\ x \neq t) = \mathit{False} \;\; (\forall x.\ t \neq x) = \mathit{False}$
by (*blast*, *blast*, *blast*, *blast*, *blast*, *iprover*+)

lemma *disj-absorb*: $A \vee A \longleftrightarrow A$
by *blast*

lemma *disj-left-absorb*: $A \vee (A \vee B) \longleftrightarrow A \vee B$
by *blast*

lemma *conj-absorb*: $A \wedge A \longleftrightarrow A$
by *blast*

lemma *conj-left-absorb*: $A \wedge (A \wedge B) \longleftrightarrow A \wedge B$
by *blast*

lemma *eq-ac*:
shows *eq-commute*: $a = b \longleftrightarrow b = a$

and *iff-left-commute*: $(P \longleftrightarrow (Q \longleftrightarrow R)) \longleftrightarrow (Q \longleftrightarrow (P \longleftrightarrow R))$
and *iff-assoc*: $((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R))$
by (*iprover*, *blast*+)

lemma *neq-commute*: $a \neq b \longleftrightarrow b \neq a$ **by** *iprover*

lemma *conj-comms*:
shows *conj-commute*: $P \wedge Q \longleftrightarrow Q \wedge P$
and *conj-left-commute*: $P \wedge (Q \wedge R) \longleftrightarrow Q \wedge (P \wedge R)$ **by** *iprover*+
lemma *conj-assoc*: $(P \wedge Q) \wedge R \longleftrightarrow P \wedge (Q \wedge R)$ **by** *iprover*

lemmas *conj-ac* = *conj-commute conj-left-commute conj-assoc*

lemma *disj-comms*:
shows *disj-commute*: $P \vee Q \longleftrightarrow Q \vee P$
and *disj-left-commute*: $P \vee (Q \vee R) \longleftrightarrow Q \vee (P \vee R)$ **by** *iprover*+
lemma *disj-assoc*: $(P \vee Q) \vee R \longleftrightarrow P \vee (Q \vee R)$ **by** *iprover*

lemmas *disj-ac* = *disj-commute disj-left-commute disj-assoc*

lemma *conj-disj-distribL*: $P \wedge (Q \vee R) \longleftrightarrow P \wedge Q \vee P \wedge R$ **by** *iprover*
lemma *conj-disj-distribR*: $(P \vee Q) \wedge R \longleftrightarrow P \wedge R \vee Q \wedge R$ **by** *iprover*

lemma *disj-conj-distribL*: $P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R)$ **by** *iprover*
lemma *disj-conj-distribR*: $(P \wedge Q) \vee R \longleftrightarrow (P \vee R) \wedge (Q \vee R)$ **by** *iprover*

lemma *imp-conjR*: $(P \longrightarrow (Q \wedge R)) = ((P \longrightarrow Q) \wedge (P \longrightarrow R))$ **by** *iprover*
lemma *imp-conjL*: $((P \wedge Q) \longrightarrow R) = (P \longrightarrow (Q \longrightarrow R))$ **by** *iprover*
lemma *imp-disjL*: $((P \vee Q) \longrightarrow R) = ((P \longrightarrow R) \wedge (Q \longrightarrow R))$ **by** *iprover*

These two are specialized, but *imp-disj-not1* is useful in *Auth/Yahalom*.

lemma *imp-disj-not1*: $(P \longrightarrow Q \vee R) \longleftrightarrow (\neg Q \longrightarrow P \longrightarrow R)$ **by** *blast*
lemma *imp-disj-not2*: $(P \longrightarrow Q \vee R) \longleftrightarrow (\neg R \longrightarrow P \longrightarrow Q)$ **by** *blast*

lemma *imp-disj1*: $((P \longrightarrow Q) \vee R) \longleftrightarrow (P \longrightarrow Q \vee R)$ **by** *blast*
lemma *imp-disj2*: $(Q \vee (P \longrightarrow R)) \longleftrightarrow (P \longrightarrow Q \vee R)$ **by** *blast*

lemma *imp-cong*: $(P = P') \Longrightarrow (P' \Longrightarrow (Q = Q')) \Longrightarrow ((P \longrightarrow Q) \longleftrightarrow (P' \longrightarrow Q'))$
by *iprover*

lemma *de-Morgan-disj*: $\neg (P \vee Q) \longleftrightarrow \neg P \wedge \neg Q$ **by** *iprover*
lemma *de-Morgan-conj*: $\neg (P \wedge Q) \longleftrightarrow \neg P \vee \neg Q$ **by** *blast*
lemma *not-imp*: $\neg (P \longrightarrow Q) \longleftrightarrow P \wedge \neg Q$ **by** *blast*
lemma *not-iff*: $P \neq Q \longleftrightarrow (P \longleftrightarrow \neg Q)$ **by** *blast*
lemma *disj-not1*: $\neg P \vee Q \longleftrightarrow (P \longrightarrow Q)$ **by** *blast*
lemma *disj-not2*: $P \vee \neg Q \longleftrightarrow (Q \longrightarrow P)$ **by** *blast* — changes orientation :-(
lemma *imp-conv-disj*: $(P \longrightarrow Q) \longleftrightarrow (\neg P) \vee Q$ **by** *blast*
lemma *disj-imp*: $P \vee Q \longleftrightarrow \neg P \longrightarrow Q$ **by** *blast*

lemma *iff-conv-conj-imp*: $(P \longleftrightarrow Q) \longleftrightarrow (P \longrightarrow Q) \wedge (Q \longrightarrow P)$ **by** *iprover*

lemma *cases-simp*: $(P \longrightarrow Q) \wedge (\neg P \longrightarrow Q) \longleftrightarrow Q$
— Avoids duplication of subgoals after *if-split*, when the true and false
— cases boil down to the same thing.
by *blast*

lemma *not-all*: $\neg (\forall x. P\ x) \longleftrightarrow (\exists x. \neg P\ x)$ **by** *blast*
lemma *imp-all*: $((\forall x. P\ x) \longrightarrow Q) \longleftrightarrow (\exists x. P\ x \longrightarrow Q)$ **by** *blast*
lemma *not-ex*: $\neg (\exists x. P\ x) \longleftrightarrow (\forall x. \neg P\ x)$ **by** *iprover*
lemma *imp-ex*: $((\exists x. P\ x) \longrightarrow Q) \longleftrightarrow (\forall x. P\ x \longrightarrow Q)$ **by** *iprover*
lemma *all-not-ex*: $(\forall x. P\ x) \longleftrightarrow \neg (\exists x. \neg P\ x)$ **by** *blast*

declare *All-def* [*no-atp*]

lemma *ex-disj-distrib*: $(\exists x. P\ x \vee Q\ x) \longleftrightarrow (\exists x. P\ x) \vee (\exists x. Q\ x)$ **by** *iprover*
lemma *all-conj-distrib*: $(\forall x. P\ x \wedge Q\ x) \longleftrightarrow (\forall x. P\ x) \wedge (\forall x. Q\ x)$ **by** *iprover*
lemma *all-imp-conj-distrib*: $(\forall x. P\ x \longrightarrow Q\ x \wedge R\ x) \longleftrightarrow (\forall x. P\ x \longrightarrow Q\ x) \wedge (\forall x. P\ x \longrightarrow R\ x)$
by *iprover*

The $\wedge$ congruence rule: not included by default! May slow rewrite proofs down by as much as 50%

lemma *conj-cong*: $P = P' \Longrightarrow (P' \Longrightarrow Q = Q') \Longrightarrow (P \wedge Q) = (P' \wedge Q')$
by *iprover*

lemma *rev-conj-cong*: $Q = Q' \Longrightarrow (Q' \Longrightarrow P = P') \Longrightarrow (P \wedge Q) = (P' \wedge Q')$
by *iprover*

The | congruence rule: not included by default!

lemma *disj-cong*: $P = P' \Longrightarrow (\neg P' \Longrightarrow Q = Q') \Longrightarrow (P \vee Q) = (P' \vee Q')$
by *blast*

if-then-else rules

lemma *if-True* [*code*]: $(\textit{if True then } x \textit{ else } y) = x$
unfolding *If-def* **by** *blast*

lemma *if-False* [*code*]: $(\textit{if False then } x \textit{ else } y) = y$
unfolding *If-def* **by** *blast*

lemma *if-P*: $P \Longrightarrow (\textit{if } P \textit{ then } x \textit{ else } y) = x$
unfolding *If-def* **by** *blast*

lemma *if-not-P*: $\neg P \Longrightarrow (\textit{if } P \textit{ then } x \textit{ else } y) = y$
unfolding *If-def* **by** *blast*

lemma *if-split*: P (*if* Q *then* x *else* y) = (($Q \longrightarrow P\ x$) $\wedge$ ($\neg\ Q \longrightarrow P\ y$))
proof (*rule case-split* [*of Q*])
  show *?thesis* **if** Q
    using *that* **by** (*simplesubst if-P*) *blast+*
  show *?thesis* **if** $\neg\ Q$
    using *that* **by** (*simplesubst if-not-P*) *blast+*
qed

lemma *if-split-asm*: P (*if* Q *then* x *else* y) = ($\neg$ (($Q \wedge \neg\ P\ x$) $\vee$ ($\neg\ Q \wedge \neg\ P\ y$)))
  by (*simplesubst if-split*) *blast*

lemmas *if-splits* [*no-atp*] = *if-split if-split-asm*

lemma *if-cancel*: (*if* c *then* x *else* x) = x
  by (*simplesubst if-split*) *blast*

lemma *if-eq-cancel*: (*if* $x = y$ *then* y *else* x) = x
  by (*simplesubst if-split*) *blast*

lemma *if-bool-eq-conj*: (*if* P *then* Q *else* R) = (($P \longrightarrow Q$) $\wedge$ ($\neg\ P \longrightarrow R$))
  — This form is useful for expanding *if*s on the RIGHT of the $\Longrightarrow$ symbol.
  by (*rule if-split*)

lemma *if-bool-eq-disj*: (*if* P *then* Q *else* R) = (($P \wedge Q$) $\vee$ ($\neg\ P \wedge R$))
  — And this form is useful for expanding *if*s on the LEFT.
  by (*simplesubst if-split*) *blast*

lemma *Eq-TrueI*: $P \Longrightarrow P \equiv$ *True* **unfolding** *atomize-eq* **by** *iprover*
lemma *Eq-FalseI*: $\neg\ P \Longrightarrow P \equiv$ *False* **unfolding** *atomize-eq* **by** *iprover*

let rules for simproc

lemma *Let-folded*: $f\ x \equiv g\ x \Longrightarrow$ *Let* $x\ f \equiv$ *Let* $x\ g$
  by (*unfold Let-def*)

lemma *Let-unfold*: $f\ x \equiv g \Longrightarrow$ *Let* $x\ f \equiv g$
  by (*unfold Let-def*)

The following copy of the implication operator is useful for fine-tuning congruence rules. It instructs the simplifier to simplify its premise.

definition *simp-implies* :: *prop* ⇒ *prop* ⇒ *prop* (**infixr** ‹=*simp*=>› *1*)
  where *simp-implies* ≡ (⟹)

lemma *simp-impliesI*:
  assumes *PQ*: (*PROP P* ⟹ *PROP Q*)
  shows *PROP P* =*simp*=> *PROP Q*
  unfolding *simp-implies-def*
  by (*iprover intro*: *PQ*)

lemma *simp-impliesE*:
assumes *PQ*: *PROP P =simp=> PROP Q*
and *P*: *PROP P*
and *QR*: $PROP\ Q \Longrightarrow PROP\ R$
shows *PROP R*
by (*iprover intro*: *QR P PQ* [*unfolded simp-implies-def*])

lemma *simp-implies-cong*:
assumes PP' :$PROP\ P \equiv PROP\ P'$
and $P'QQ'$: $PROP\ P' \Longrightarrow (PROP\ Q \equiv PROP\ Q')$
shows (*PROP P =simp=> PROP Q*) ≡ (*PROP P′ =simp=> PROP Q′*)
unfolding *simp-implies-def*
proof (*rule equal-intr-rule*)
assume *PQ*: $PROP\ P \Longrightarrow PROP\ Q$
and P': $PROP\ P'$
from PP' [*symmetric*] **and** P' **have** *PROP P*
by (*rule equal-elim-rule1*)
then have *PROP Q* **by** (*rule PQ*)
with $P'QQ'$ [*OF P′*] **show** $PROP\ Q'$ **by** (*rule equal-elim-rule1*)
next
assume $P'Q'$: $PROP\ P' \Longrightarrow PROP\ Q'$
and *P*: *PROP P*
from PP' **and** *P* **have** P': $PROP\ P'$ **by** (*rule equal-elim-rule1*)
then have $PROP\ Q'$ **by** (*rule P′Q′*)
with $P'QQ'$ [*OF P′*, *symmetric*] **show** *PROP Q*
by (*rule equal-elim-rule1*)
qed

lemma *uncurry*:
assumes $P \longrightarrow Q \longrightarrow R$
shows $P \wedge Q \longrightarrow R$
using *assms* **by** *blast*

lemma *iff-allI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\forall x.\ P\ x) = (\forall x.\ Q\ x)$
using *assms* **by** *blast*

lemma *iff-exI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\exists x.\ P\ x) = (\exists x.\ Q\ x)$
using *assms* **by** *blast*

lemma *all-comm*: $(\forall x\ y.\ P\ x\ y) = (\forall y\ x.\ P\ x\ y)$
by *blast*

lemma *ex-comm*: $(\exists x\ y.\ P\ x\ y) = (\exists y\ x.\ P\ x\ y)$
by *blast*

ML-file ‹*Tools/simpdata.ML*›
ML ‹*open Simpdata*›

```
setup ‹
  map-theory-simpset (put-simpset HOL-basic-ss) #>
  Simplifier.method-setup Splitter.split-modifiers
›
```

simproc-setup *defined-Ex* (∃ *x*. *P x*) = ‹*K Quantifier1.rearrange-Ex*›
simproc-setup *defined-All* (∀ *x*. *P x*) = ‹*K Quantifier1.rearrange-All*›
simproc-setup *defined-all*(⋀*x*. *PROP P x*) = ‹*K Quantifier1.rearrange-all*›

Simproc for proving $(y = x) \equiv False$ from premise $\neg (x = y)$:

```
simproc-setup neq (x = y) = ‹
  let
    val neq-to-EQ-False = @{thm not-sym} RS @{thm Eq-FalseI};
    fun is-neq eq lhs rhs thm =
      (case Thm.prop-of thm of
        - $ (Not $ (eq' $ l' $ r')) =>
          Not = HOLogic.Not andalso eq' = eq andalso
          r' aconv lhs andalso l' aconv rhs
      | - => false);
    fun proc ss ct =
      (case Thm.term-of ct of
        eq $ lhs $ rhs =>
          (case find-first (is-neq eq lhs rhs) (Simplifier.prems-of ss) of
            SOME thm => SOME (thm RS neq-to-EQ-False)
          | NONE => NONE)
       | - => NONE);
  in K proc end
›
```

```
simproc-setup let-simp (Let x f) = ‹
  let
    fun count-loose (Bound i) k = if i >= k then 1 else 0
      | count-loose (s $ t) k = count-loose s k + count-loose t k
      | count-loose (Abs (-, -, t)) k = count-loose  t (k + 1)
      | count-loose - - = 0;
    fun is-trivial-let Const- ‹Let - - for x t› =
      (case t of
        Abs (-, -, t') => count-loose t' 0 <= 1
      | - => true);
  in
    K (fn ctxt => fn ct =>
      if is-trivial-let (Thm.term-of ct)
      then SOME @{thm Let-def} (*no or one ocurrence of bound variable*)
      else
        let (*Norbert Schirmer's case*)
          val t = Thm.term-of ct;
```

```
    val (t', ctxt') = yield-singleton (Variable.import-terms false) t ctxt;
  in
    Option.map (hd o Variable.export ctxt' ctxt o single)
      (case t' of Const- ‹Let - - for x f› => (* x and f are already in normal form *)
        if is-Free x orelse is-Bound x orelse is-Const x
        then SOME @{thm Let-def}
        else
          let
            val n = case f of (Abs (x, -, -)) => x | - => x;
            val cx = Thm.cterm-of ctxt x;
            val xT = Thm.typ-of-cterm cx;
            val cf = Thm.cterm-of ctxt f;
            val fx-g = Simplifier.rewrite ctxt (Thm.apply cf cx);
            val (- $ - $ g) = Thm.prop-of fx-g;
            val g' = abstract-over (x, g);
            val abs-g'= Abs (n, xT, g');
          in
            if g aconv g' then
              let
                val rl =
                  infer-instantiate ctxt [((f, 0), cf), ((x, 0), cx)] @{thm Let-unfold};
              in SOME (rl OF [fx-g]) end
              else if (Envir.beta-eta-contract f) aconv (Envir.beta-eta-contract abs-g')
            then NONE (*avoid identity conversion*)
            else
              let
                val g'x = abs-g' $ x;
                val g-g'x = Thm.symmetric (Thm.beta-conversion false (Thm.cterm-of ctxt g'x));
                val rl =
                  @{thm Let-folded} |> infer-instantiate ctxt
                    [((f, 0), Thm.cterm-of ctxt f),
                     ((x, 0), cx),
                     ((g, 0), Thm.cterm-of ctxt abs-g')];
              in SOME (rl OF [Thm.transitive fx-g g-g'x]) end
          end
      | - => NONE)
  end)
end
›
```

lemma *True-implies-equals*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
proof
  assume $True \Longrightarrow PROP\ P$
  from *this* [*OF TrueI*] **show** *PROP P* **.**
next
  assume *PROP P*

then show *PROP P* **.**
qed

lemma *implies-True-equals*: $(PROP\ P \Longrightarrow True) \equiv Trueprop\ True$
by *standard* (*intro TrueI*)

lemma *False-implies-equals*: $(False \Longrightarrow P) \equiv Trueprop\ True$
by *standard simp-all*

lemma *implies-False-swap*:
$(False \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow False \Longrightarrow PROP\ Q)$
by (*rule swap-prems-eq*)

simproc-setup *eliminate-false-implies* (*False* ⟹ *PROP P*) = ‹

```
let
  fun conv n =
    if n > 1 then
      Conv.rewr-conv @{thm Pure.swap-prems-eq}
      then-conv Conv.arg-conv (conv (n − 1))
      then-conv Conv.rewr-conv @{thm HOL.implies-True-equals}
    else
      Conv.rewr-conv @{thm HOL.False-implies-equals}
in
  fn - => fn - => fn ct =>
    let
      val t = Thm.term-of ct
      val n = length (Logic.strip-imp-prems t)
    in
      (case Logic.strip-imp-concl t of
        Const- ‹Trueprop for -› => SOME (conv n ct)
      | - => NONE)
    end
end
›
```

lemma *ex-simps*:
$\bigwedge P\ Q.\ (\exists x.\ P\ x \land Q) = ((\exists x.\ P\ x) \land Q)$
$\bigwedge P\ Q.\ (\exists x.\ P \land Q\ x) = (P \land (\exists x.\ Q\ x))$
$\bigwedge P\ Q.\ (\exists x.\ P\ x \lor Q) = ((\exists x.\ P\ x) \lor Q)$
$\bigwedge P\ Q.\ (\exists x.\ P \lor Q\ x) = (P \lor (\exists x.\ Q\ x))$
$\bigwedge P\ Q.\ (\exists x.\ P\ x \longrightarrow Q) = ((\forall x.\ P\ x) \longrightarrow Q)$
$\bigwedge P\ Q.\ (\exists x.\ P \longrightarrow Q\ x) = (P \longrightarrow (\exists x.\ Q\ x))$
— Miniscoping: pushing in existential quantifiers.
by (*iprover* | *blast*)+

lemma *all-simps*:
$\bigwedge P\ Q.\ (\forall x.\ P\ x \land Q) = ((\forall x.\ P\ x) \land Q)$

$\bigwedge P\ Q.\ (\forall x.\ P \wedge Q\ x) \quad = (P \wedge (\forall x.\ Q\ x))$
$\bigwedge P\ Q.\ (\forall x.\ P\ x \vee Q) \quad = ((\forall x.\ P\ x) \vee Q)$
$\bigwedge P\ Q.\ (\forall x.\ P \vee Q\ x) \quad = (P \vee (\forall x.\ Q\ x))$
$\bigwedge P\ Q.\ (\forall x.\ P\ x \longrightarrow Q) = ((\exists x.\ P\ x) \longrightarrow Q)$
$\bigwedge P\ Q.\ (\forall x.\ P \longrightarrow Q\ x) = (P \longrightarrow (\forall x.\ Q\ x))$
— Miniscoping: pushing in universal quantifiers.
by (*iprover* | *blast*)+

lemmas [*simp*] =
triv-forall-equality — prunes params
True-implies-equals implies-True-equals — prune *True* in asms
False-implies-equals — prune *False* in asms
if-True
if-False
if-cancel
if-eq-cancel
imp-disjL — In general it seems wrong to add distributive laws by default: they might cause exponential blow-up. But *imp-disjL* has been in for a while and cannot be removed without affecting existing proofs. Moreover, rewriting by $(P \vee Q \longrightarrow R) = ((P \longrightarrow R) \wedge (Q \longrightarrow R))$ might be justified on the grounds that it allows simplification of R in the two cases.
conj-assoc
disj-assoc
de-Morgan-conj
de-Morgan-disj
imp-disj1
imp-disj2
not-imp
disj-not1
not-all
not-ex
cases-simp
the-eq-trivial
the-sym-eq-trivial
ex-simps
all-simps
simp-thms
subst-all

lemmas [*cong*] = *imp-cong simp-implies-cong*
lemmas [*split*] = *if-split*

ML ‹*val HOL-ss = simpset-of* ***context***›

Simplifies x assuming c and y assuming $\neg\ c$.

lemma *if-cong*:
assumes $b = c$
and $c \Longrightarrow x = u$
and $\neg\ c \Longrightarrow y = v$

shows $(\textit{if } b \textit{ then } x \textit{ else } y) = (\textit{if } c \textit{ then } u \textit{ else } v)$
using *assms* **by** *simp*

Prevents simplification of x and y: faster and allows the execution of functional programs.

lemma *if-weak-cong* [*cong*]:
assumes $b = c$
shows $(\textit{if } b \textit{ then } x \textit{ else } y) = (\textit{if } c \textit{ then } x \textit{ else } y)$
using *assms* **by** (*rule arg-cong*)

Prevents simplification of t: much faster

lemma *let-weak-cong*:
assumes $a = b$
shows $(\textit{let } x = a \textit{ in } t\ x) = (\textit{let } x = b \textit{ in } t\ x)$
using *assms* **by** (*rule arg-cong*)

To tidy up the result of a simproc. Only the RHS will be simplified.

lemma *eq-cong2*:
assumes $u = u'$
shows $(t \equiv u) \equiv (t \equiv u')$
using *assms* **by** *simp*

lemma *if-distrib*: $f\ (\textit{if } c \textit{ then } x \textit{ else } y) = (\textit{if } c \textit{ then } f\ x \textit{ else } f\ y)$
by *simp*

lemma *if-distribR*: $(\textit{if } b \textit{ then } f \textit{ else } g)\ x = (\textit{if } b \textit{ then } f\ x \textit{ else } g\ x)$
by *simp*

lemma *all-if-distrib*: $(\forall x.\ \textit{if } x = a \textit{ then } P\ x \textit{ else } Q\ x) \longleftrightarrow P\ a \wedge (\forall x.\ x \neq a \longrightarrow Q\ x)$
by *auto*

lemma *ex-if-distrib*: $(\exists x.\ \textit{if } x = a \textit{ then } P\ x \textit{ else } Q\ x) \longleftrightarrow P\ a \vee (\exists x.\ x \neq a \wedge Q\ x)$
by *auto*

lemma *if-if-eq-conj*: $(\textit{if } P \textit{ then if } Q \textit{ then } x \textit{ else } y \textit{ else } y) = (\textit{if } P \wedge Q \textit{ then } x \textit{ else } y)$
by *simp*

As a simplification rule, it replaces all function equalities by first-order equalities.

lemma *fun-eq-iff*: $f = g \longleftrightarrow (\forall x.\ f\ x = g\ x)$
by *auto*

2.3.5 Generic cases and induction

Rule projections:

ML ‹

structure Project-Rule = *Project-Rule*
(
val conjunct1 = @{*thm conjunct1*}
val conjunct2 = @{*thm conjunct2*}
val mp = @{*thm mp*}
);
›

context
begin

qualified definition *induct-forall P* ≡ ∀ *x*. *P x*
qualified definition *induct-implies A B* ≡ *A* ⟶ *B*
qualified definition *induct-equal x y* ≡ *x* = *y*
qualified definition *induct-conj A B* ≡ *A* ∧ *B*
qualified definition *induct-true* ≡ *True*
qualified definition *induct-false* ≡ *False*

lemma *induct-forall-eq*: (⋀*x*. *P x*) ≡ *Trueprop* (*induct-forall* (λ*x*. *P x*))
by (*unfold atomize-all induct-forall-def*)

lemma *induct-implies-eq*: (*A* ⟹ *B*) ≡ *Trueprop* (*induct-implies A B*)
by (*unfold atomize-imp induct-implies-def*)

lemma *induct-equal-eq*: (*x* ≡ *y*) ≡ *Trueprop* (*induct-equal x y*)
by (*unfold atomize-eq induct-equal-def*)

lemma *induct-conj-eq*: (*A* &&& *B*) ≡ *Trueprop* (*induct-conj A B*)
by (*unfold atomize-conj induct-conj-def*)

lemmas *induct-atomize′* = *induct-forall-eq induct-implies-eq induct-conj-eq*
lemmas *induct-atomize* = *induct-atomize′ induct-equal-eq*
lemmas *induct-rulify′* [*symmetric*] = *induct-atomize′*
lemmas *induct-rulify* [*symmetric*] = *induct-atomize*
lemmas *induct-rulify-fallback* =
induct-forall-def induct-implies-def induct-equal-def induct-conj-def
induct-true-def induct-false-def

lemma *induct-forall-conj*: *induct-forall* (λ*x*. *induct-conj* (*A x*) (*B x*)) =
induct-conj (*induct-forall A*) (*induct-forall B*)
by (*unfold induct-forall-def induct-conj-def*) *iprover*

lemma *induct-implies-conj*: *induct-implies C* (*induct-conj A B*) =
induct-conj (*induct-implies C A*) (*induct-implies C B*)
by (*unfold induct-implies-def induct-conj-def*) *iprover*

lemma *induct-conj-curry*: (*induct-conj A B* ⟹ *PROP C*) ≡ (*A* ⟹ *B* ⟹ *PROP C*)
proof

assume *r*: *induct-conj A B* $\Longrightarrow$ *PROP C*
assume *ab*: *A B*
show *PROP C* **by** (*rule r*) (*simp add*: *induct-conj-def ab*)
next
assume *r*: *A* $\Longrightarrow$ *B* $\Longrightarrow$ *PROP C*
assume *ab*: *induct-conj A B*
show *PROP C* **by** (*rule r*) (*simp-all add*: *ab* [*unfolded induct-conj-def*])
qed

lemmas *induct-conj* = *induct-forall-conj induct-implies-conj induct-conj-curry*

lemma *induct-trueI*: *induct-true*
by (*simp add*: *induct-true-def*)

Method setup.

ML-file ‹~~/src/Tools/induct.ML›
ML ‹

```
structure Induct = Induct
(
  val cases-default = @{thm case-split}
  val atomize = @{thms induct-atomize}
  val rulify = @{thms induct-rulify'}
  val rulify-fallback = @{thms induct-rulify-fallback}
  val equal-def = @{thm induct-equal-def}
  fun dest-def Const- ‹induct-equal - for t u› = SOME (t, u)
    | dest-def - = NONE
  fun trivial-tac ctxt = match-tac ctxt @{thms induct-trueI}
)
```
›

ML-file ‹~~/src/Tools/induction.ML›

simproc-setup passive *swap-induct-false* (*induct-false* $\Longrightarrow$ *PROP P* $\Longrightarrow$ *PROP Q*) =

```
‹fn - => fn - => fn ct =>
  (case Thm.term-of ct of
    - $ (P as - $ Const- ‹induct-false›) $ (- $ Q $ -) =>
      if P <> Q then SOME Drule.swap-prems-eq else NONE
  | - => NONE)›
```

simproc-setup passive *induct-equal-conj-curry* (*induct-conj P Q* $\Longrightarrow$ *PROP R*) =

```
‹fn - => fn - => fn ct =>
  (case Thm.term-of ct of
    - $ (- $ P) $ - =>
      let
        fun is-conj Const- ‹induct-conj for P Q› = is-conj P andalso is-conj Q
          | is-conj Const- ‹induct-equal - for - -› = true
          | is-conj Const- ‹induct-true› = true
```

```
        | is-conj Const- ‹induct-false› = true
        | is-conj - = false
      in if is-conj P then SOME @{thm induct-conj-curry} else NONE end
    | - => NONE)›
```

declaration ‹
```
  K (Induct.map-simpset
    (Simplifier.add-proc simproc ‹swap-induct-false› #>
     Simplifier.add-proc simproc ‹induct-equal-conj-curry› #>
     Simplifier.set-mksimps (fn ctxt =>
       Simpdata.mksimps Simpdata.mksimps-pairs ctxt #>
     map (rewrite-rule ctxt (map Thm.symmetric @{thms induct-rulify-fallback})))))
›
```

Pre-simplification of induction and cases rules

lemma [*induct-simp*]: ($\bigwedge x$. *induct-equal x t* ⟹ *PROP P x*) ≡ *PROP P t*
  unfolding *induct-equal-def*
proof
  assume *r*: $\bigwedge x$. *x = t* ⟹ *PROP P x*
  show *PROP P t* **by** (*rule r* [*OF refl*])
next
  fix *x*
  assume *PROP P t x = t*
  then show *PROP P x* **by** *simp*
qed

lemma [*induct-simp*]: ($\bigwedge x$. *induct-equal t x* ⟹ *PROP P x*) ≡ *PROP P t*
  unfolding *induct-equal-def*
proof
  assume *r*: $\bigwedge x$. *t = x* ⟹ *PROP P x*
  show *PROP P t* **by** (*rule r* [*OF refl*])
next
  fix *x*
  assume *PROP P t t = x*
  then show *PROP P x* **by** *simp*
qed

lemma [*induct-simp*]: (*induct-false* ⟹ *P*) ≡ *Trueprop induct-true*
  unfolding *induct-false-def induct-true-def*
  by (*iprover intro*: *equal-intr-rule*)

lemma [*induct-simp*]: (*induct-true* ⟹ *PROP P*) ≡ *PROP P*
  unfolding *induct-true-def*
proof
  assume *True* ⟹ *PROP P*
  then show *PROP P* **using** *TrueI* **.**
next
  assume *PROP P*
  then show *PROP P* **.**

qed

lemma [*induct-simp*]: $(PROP\ P \Longrightarrow induct\text{-}true) \equiv Trueprop\ induct\text{-}true$
unfolding *induct-true-def*
by (*iprover intro*: *equal-intr-rule*)

lemma [*induct-simp*]: $(\bigwedge x::'a::\{\}.\ induct\text{-}true) \equiv Trueprop\ induct\text{-}true$
unfolding *induct-true-def*
by (*iprover intro*: *equal-intr-rule*)

lemma [*induct-simp*]: $induct\text{-}implies\ induct\text{-}true\ P \equiv P$
by (*simp add*: *induct-implies-def induct-true-def*)

lemma [*induct-simp*]: $x = x \longleftrightarrow True$
by (*rule simp-thms*)

end

ML-file ‹~~/*src*/*Tools*/*induct-tacs.ML*›

2.3.6 Coherent logic

ML-file ‹~~/*src*/*Tools*/*coherent.ML*›
ML ‹
```
structure Coherent = Coherent
(
  val atomize-elimL = @{thm atomize-elimL};
  val atomize-exL = @{thm atomize-exL};
  val atomize-conjL = @{thm atomize-conjL};
  val atomize-disjL = @{thm atomize-disjL};
 val operator-names = [const-name ‹HOL.disj›, const-name ‹HOL.conj›, const-name ‹Ex›];
);
```
›

2.3.7 Reorienting equalities

ML ‹
```
signature REORIENT-PROC =
sig
  val add : (term −> bool) −> theory −> theory
  val proc : Simplifier.proc
end;

structure Reorient-Proc : REORIENT-PROC =
struct
  structure Data = Theory-Data
  (
    type T = ((term −> bool) ∗ stamp) list;
    val empty = [];
    fun merge data : T = Library.merge (eq-snd (op =)) data;
```

```
  );
  fun add m = Data.map (cons (m, stamp ()));
  fun matches thy t = exists (fn (m, -) => m t) (Data.get thy);

  val meta-reorient = @{thm eq-commute [THEN eq-reflection]};
  fun proc ctxt ct =
    let
      val thy = Proof-Context.theory-of ctxt;
    in
      case Thm.term-of ct of
        (- $ t $ u) => if matches thy u then NONE else SOME meta-reorient
      | - => NONE
    end;
end;
›
```

2.4 Other simple lemmas and lemma duplicates

lemma *eq-iff-swap*: $(x = y \longleftrightarrow P) \Longrightarrow (y = x \longleftrightarrow P)$
by *blast*

lemma *all-cong1*: $(\bigwedge x.\ P\ x = P'\ x) \Longrightarrow (\forall x.\ P\ x) = (\forall x.\ P'\ x)$
  by *auto*

lemma *ex-cong1*: $(\bigwedge x.\ P\ x = P'\ x) \Longrightarrow (\exists x.\ P\ x) = (\exists x.\ P'\ x)$
  by *auto*

lemma *all-cong*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x = P'\ x) \Longrightarrow (\forall x.\ Q\ x \longrightarrow P\ x) = (\forall x.\ Q\ x \longrightarrow P'\ x)$
  by *auto*

lemma *ex-cong*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x = P'\ x) \Longrightarrow (\exists x.\ Q\ x \wedge P\ x) = (\exists x.\ Q\ x \wedge P'\ x)$
  by *auto*

lemma *ex1-eq* [*iff*]: $\exists!x.\ x = t$ $\exists!x.\ t = x$
  by *blast+*

lemma *choice-eq*: $(\forall x.\ \exists!y.\ P\ x\ y) = (\exists!f.\ \forall x.\ P\ x\ (f\ x))$ (**is** *?lhs* = *?rhs*)
proof (*intro iffI allI*)
  assume *L*: *?lhs*
  then have §: $\forall x.\ P\ x\ (THE\ y.\ P\ x\ y)$
    by (*best intro*: *theI′*)
  show *?rhs*
    by (*rule ex1I*) (*use L* § **in** ‹*fast+*›)
next
  fix *x*
  assume *R*: *?rhs*
  then obtain *f* **where** *f*: $\forall x.\ P\ x\ (f\ x)$ **and** *f1*: $\bigwedge y.\ (\forall x.\ P\ x\ (y\ x)) \Longrightarrow y = f$

```
    by (blast elim: ex1E)
  show ∃!y. P x y
  proof (rule ex1I)
    show P x (f x)
      using f by blast
    show y = f x if P x y for y
    proof −
      have P z (if z = x then y else f z) for z
        using f that by (auto split: if-split)
      with f1 [of λz. if z = x then y else f z] f
      show ?thesis
        by (auto simp add: split: if-split-asm dest: fun-cong)
    qed
  qed
qed
```

lemmas *eq-sym-conv* = *eq-commute*

```
lemma nnf-simps:
  (¬ (P ∧ Q)) = (¬ P ∨ ¬ Q)
  (¬ (P ∨ Q)) = (¬ P ∧ ¬ Q)
  (P ⟶ Q) = (¬ P ∨ Q)
  (P = Q) = ((P ∧ Q) ∨ (¬ P ∧ ¬ Q))
  (¬ (P = Q)) = ((P ∧ ¬ Q) ∨ (¬ P ∧ Q))
  (¬ ¬ P) = P
  by blast+
```

2.5 Basic ML bindings

```
ML ‹
val FalseE = @{thm FalseE}
val Let-def = @{thm Let-def}
val TrueI = @{thm TrueI}
val allE = @{thm allE}
val allI = @{thm allI}
val all-dupE = @{thm all-dupE}
val arg-cong = @{thm arg-cong}
val box-equals = @{thm box-equals}
val ccontr = @{thm ccontr}
val classical = @{thm classical}
val conjE = @{thm conjE}
val conjI = @{thm conjI}
val conjunct1 = @{thm conjunct1}
val conjunct2 = @{thm conjunct2}
val disjCI = @{thm disjCI}
val disjE = @{thm disjE}
val disjI1 = @{thm disjI1}
val disjI2 = @{thm disjI2}
val eq-reflection = @{thm eq-reflection}
```

val ex1E = @{*thm ex1E*}
val ex1I = @{*thm ex1I*}
val ex1-implies-ex = @{*thm ex1-implies-ex*}
val exE = @{*thm exE*}
val exI = @{*thm exI*}
val excluded-middle = @{*thm excluded-middle*}
val ext = @{*thm ext*}
val fun-cong = @{*thm fun-cong*}
val iffD1 = @{*thm iffD1*}
val iffD2 = @{*thm iffD2*}
val iffI = @{*thm iffI*}
val impE = @{*thm impE*}
val impI = @{*thm impI*}
val meta-eq-to-obj-eq = @{*thm meta-eq-to-obj-eq*}
val mp = @{*thm mp*}
val notE = @{*thm notE*}
val notI = @{*thm notI*}
val not-all = @{*thm not-all*}
val not-ex = @{*thm not-ex*}
val not-iff = @{*thm not-iff*}
val not-not = @{*thm not-not*}
val not-sym = @{*thm not-sym*}
val refl = @{*thm refl*}
val rev-mp = @{*thm rev-mp*}
val spec = @{*thm spec*}
val ssubst = @{*thm ssubst*}
val subst = @{*thm subst*}
val sym = @{*thm sym*}
val trans = @{*thm trans*}
›

locale *cnf*
begin

lemma *clause2raw-notE*: $[\![P; \neg P]\!] \Longrightarrow$ *False* **by** *auto*
lemma *clause2raw-not-disj*: $[\![\neg P; \neg Q]\!] \Longrightarrow \neg (P \vee Q)$ **by** *auto*
lemma *clause2raw-not-not*: $P \Longrightarrow \neg\neg P$ **by** *auto*

lemma *iff-refl*: $(P{::}bool) = P$ **by** *auto*
lemma *iff-trans*: [| $(P{::}bool) = Q$; $Q = R$ |] ==> $P = R$ **by** *auto*
lemma *conj-cong*: [| $P = P'$; $Q = Q'$ |] ==> $(P \wedge Q) = (P' \wedge Q')$ **by** *auto*
lemma *disj-cong*: [| $P = P'$; $Q = Q'$ |] ==> $(P \vee Q) = (P' \vee Q')$ **by** *auto*

lemma *make-nnf-imp*: [| $(\neg P) = P'$; $Q = Q'$ |] ==> $(P \longrightarrow Q) = (P' \vee Q')$ **by** *auto*
lemma *make-nnf-iff*: [| $P = P'$; $(\neg P) = NP$; $Q = Q'$; $(\neg Q) = NQ$ |] ==> $(P = Q) = ((P' \vee NQ) \wedge (NP \vee Q'))$ **by** *auto*
lemma *make-nnf-not-false*: $(\neg$*False*$)$ = *True* **by** *auto*
lemma *make-nnf-not-true*: $(\neg$*True*$)$ = *False* **by** *auto*

lemma *make-nnf-not-conj*: $[| (\neg P) = P'; (\neg Q) = Q' |] ==> (\neg(P \wedge Q)) = (P' \vee Q')$ **by** *auto*
lemma *make-nnf-not-disj*: $[| (\neg P) = P'; (\neg Q) = Q' |] ==> (\neg(P \vee Q)) = (P' \wedge Q')$ **by** *auto*
lemma *make-nnf-not-imp*: $[| P = P'; (\neg Q) = Q' |] ==> (\neg(P \longrightarrow Q)) = (P' \wedge Q')$ **by** *auto*
lemma *make-nnf-not-iff*: $[| P = P'; (\neg P) = NP; Q = Q'; (\neg Q) = NQ |] ==> (\neg(P = Q)) = ((P' \vee Q') \wedge (NP \vee NQ))$ **by** *auto*
lemma *make-nnf-not-not*: $P = P' ==> (\neg\neg P) = P'$ **by** *auto*

lemma *simp-TF-conj-True-l*: $[| P = True; Q = Q' |] ==> (P \wedge Q) = Q'$ **by** *auto*
lemma *simp-TF-conj-True-r*: $[| P = P'; Q = True |] ==> (P \wedge Q) = P'$ **by** *auto*
lemma *simp-TF-conj-False-l*: $P = False ==> (P \wedge Q) = False$ **by** *auto*
lemma *simp-TF-conj-False-r*: $Q = False ==> (P \wedge Q) = False$ **by** *auto*
lemma *simp-TF-disj-True-l*: $P = True ==> (P \vee Q) = True$ **by** *auto*
lemma *simp-TF-disj-True-r*: $Q = True ==> (P \vee Q) = True$ **by** *auto*
lemma *simp-TF-disj-False-l*: $[| P = False; Q = Q' |] ==> (P \vee Q) = Q'$ **by** *auto*
lemma *simp-TF-disj-False-r*: $[| P = P'; Q = False |] ==> (P \vee Q) = P'$ **by** *auto*

lemma *make-cnf-disj-conj-l*: $[| (P \vee R) = PR; (Q \vee R) = QR |] ==> ((P \wedge Q) \vee R) = (PR \wedge QR)$ **by** *auto*
lemma *make-cnf-disj-conj-r*: $[| (P \vee Q) = PQ; (P \vee R) = PR |] ==> (P \vee (Q \wedge R)) = (PQ \wedge PR)$ **by** *auto*

lemma *make-cnfx-disj-ex-l*: $((\exists (x::bool). P\ x) \vee Q) = (\exists x. P\ x \vee Q)$ **by** *auto*
lemma *make-cnfx-disj-ex-r*: $(P \vee (\exists (x::bool). Q\ x)) = (\exists x. P \vee Q\ x)$ **by** *auto*
lemma *make-cnfx-newlit*: $(P \vee Q) = (\exists x. (P \vee x) \wedge (Q \vee \neg x))$ **by** *auto*
lemma *make-cnfx-ex-cong*: $(\forall (x::bool). P\ x = Q\ x) \Longrightarrow (\exists x. P\ x) = (\exists x. Q\ x)$ **by** *auto*

lemma *weakening-thm*: $[| P; Q |] ==> Q$ **by** *auto*

lemma *cnftac-eq-imp*: $[| P = Q; P |] ==> Q$ **by** *auto*

end

ML-file ‹*Tools/cnf.ML*›

3 *NO-MATCH* simproc

The simplification procedure can be used to avoid simplification of terms of a certain form.

definition *NO-MATCH* :: $'a \Rightarrow 'b \Rightarrow bool$
  where *NO-MATCH pat val* $\equiv$ *True*

lemma *NO-MATCH-cong*[*cong*]: *NO-MATCH pat val* = *NO-MATCH pat val*
  by (*rule refl*)

declare [[*coercion-args NO-MATCH* $-$ $-$]]

simproc-setup *NO-MATCH* (*NO-MATCH pat val*) = ‹*K* (*fn ctxt* => *fn ct* =>
let
val thy = *Proof-Context.theory-of ctxt*
val dest-binop = *Term.dest-comb* #> *apfst* (*Term.dest-comb* #> *snd*)
val m = *Pattern.matches thy* (*dest-binop* (*Thm.term-of ct*))
in if m then NONE else SOME @{*thm NO-MATCH-def*} *end*)
›

This setup ensures that a rewrite rule of the form *NO-MATCH pat val* $\Longrightarrow$ *t* is only applied, if the pattern *pat* does not match the value *val*.

Tagging a premise of a simp rule with ASSUMPTION forces the simplifier not to simplify the argument and to solve it by an assumption.

definition *ASSUMPTION* :: *bool* $\Rightarrow$ *bool*
where *ASSUMPTION A* $\equiv$ *A*

lemma *ASSUMPTION-cong*[*cong*]: *ASSUMPTION A* = *ASSUMPTION A*
by (*rule refl*)

lemma *ASSUMPTION-I*: *A* $\Longrightarrow$ *ASSUMPTION A*
by (*simp add*: *ASSUMPTION-def*)

lemma *ASSUMPTION-D*: *ASSUMPTION A* $\Longrightarrow$ *A*
by (*simp add*: *ASSUMPTION-def*)

setup ‹
let
val asm-sol = *mk-solver ASSUMPTION* (*fn ctxt* =>
resolve-tac ctxt [@{*thm ASSUMPTION-I*}] *THEN'*
resolve-tac ctxt (*Simplifier.prems-of ctxt*))
in
map-theory-simpset (*fn ctxt* => *Simplifier.addSolver* (*ctxt*,*asm-sol*))
end
›

3.1 Code generator setup

3.1.1 Generic code generator preprocessor setup

lemma *conj-left-cong*: $P \longleftrightarrow Q \Longrightarrow P \wedge R \longleftrightarrow Q \wedge R$
by (*fact arg-cong*)

lemma *disj-left-cong*: $P \longleftrightarrow Q \Longrightarrow P \vee R \longleftrightarrow Q \vee R$
by (*fact arg-cong*)

setup ‹
Code-Preproc.map-pre (*put-simpset HOL-basic-ss*) #>
Code-Preproc.map-post (*put-simpset HOL-basic-ss*) #>

```
  Code-Simp.map-ss (put-simpset HOL-basic-ss #>
  Simplifier.add-cong @{thm conj-left-cong} #>
  Simplifier.add-cong @{thm disj-left-cong})
›
```

3.1.2 Equality

class *equal* =
  fixes *equal* :: $'a \Rightarrow 'a \Rightarrow bool$
  assumes *equal-eq*: *equal x y* ⟷ *x* = *y*
begin

lemma *equal*: *equal* = (=)
  by (*rule ext equal-eq*)+

lemma *equal-refl*: *equal x x* ⟷ *True*
  unfolding *equal* **by** (*rule iffI TrueI refl*)+

lemma *eq-equal*: (=) ≡ *equal*
  by (*rule eq-reflection*) (*rule ext*, *rule ext*, *rule sym*, *rule equal-eq*)

end

declare *eq-equal* [*symmetric*, *code-post*]
declare *eq-equal* [*code*]

simproc-setup passive *equal* (*HOL.eq*) =
```
  ‹fn - => fn - => fn ct =>
    (case Thm.term-of ct of
      Const- ‹HOL.eq T› => if is-Type T then SOME @{thm eq-equal} else NONE
    | - => NONE)›
```

setup ‹*Code-Preproc.map-pre* (*Simplifier.add-proc* ***simproc*** ‹*equal*›)›

3.1.3 Generic code generator foundation

Datatype *bool*

code-datatype *True False*

lemma [*code*]:
  shows *False* ∧ *P* ⟷ *False*
    and *True* ∧ *P* ⟷ *P*
    and *P* ∧ *False* ⟷ *False*
    and *P* ∧ *True* ⟷ *P*
  by *simp-all*

lemma [*code*]:
  shows *False* ∨ *P* ⟷ *P*
    and *True* ∨ *P* ⟷ *True*

and $P \vee False \longleftrightarrow P$
and $P \vee True \longleftrightarrow True$
by *simp-all*

lemma [*code*]:
shows $(False \longrightarrow P) \longleftrightarrow True$
and $(True \longrightarrow P) \longleftrightarrow P$
and $(P \longrightarrow False) \longleftrightarrow \neg P$
and $(P \longrightarrow True) \longleftrightarrow True$
by *simp-all*

More about *prop*

lemma [*code nbe*]:
shows $(True \Longrightarrow PROP\ Q) \equiv PROP\ Q$
and $(PROP\ Q \Longrightarrow True) \equiv Trueprop\ True$
and $(P \Longrightarrow R) \equiv Trueprop\ (P \longrightarrow R)$
by (*auto intro*!: *equal-intr-rule*)

lemma *Trueprop-code* [*code*]: *Trueprop True* ≡ *Code-Generator.holds*
by (*auto intro*!: *equal-intr-rule holds*)

declare *Trueprop-code* [*symmetric*, *code-post*]

Equality

declare *simp-thms*(*6*) [*code nbe*]

instantiation *itself* :: (*type*) *equal*
begin

definition *equal-itself* :: *'a itself* ⇒ *'a itself* ⇒ *bool*
where *equal-itself x y* ⟷ *x* = *y*

instance
by *standard* (*fact equal-itself-def*)

end

lemma *equal-itself-code* [*code*]: *equal TYPE*(*'a*) *TYPE*(*'a*) ⟷ *True*
by (*simp add*: *equal*)

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*equal*›, *SOME* ***typ*** ‹*'a*::*type* ⇒ *'a* ⇒ *bool*›)›

lemma *equal-alias-cert*: *OFCLASS*(*'a*, *equal-class*) ≡ (((=) :: *'a* ⇒ *'a* ⇒ *bool*) ≡ *equal*)
(**is** *?ofclass* ≡ *?equal*)
proof
assume *PROP ?ofclass*
show *PROP ?equal*

by (*tactic* ‹*ALLGOALS* (*resolve-tac* ***context*** [*Thm.unconstrainT* @{*thm eq-equal*}])›)
(*fact* ‹*PROP ?ofclass*›)
next
assume *PROP ?equal*
show *PROP ?ofclass* **proof**
qed (*simp add*: ‹*PROP ?equal*›)
qed

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*equal*›, *SOME* ***typ*** ‹*'a*::*equal* ⇒ *'a* ⇒ *bool*›)›

setup ‹*Nbe.add-const-alias* @{*thm equal-alias-cert*}›

Cases

lemma *Let-case-cert*:
assumes *CASE* ≡ (λ*x*. *Let x f*)
shows *CASE x* ≡ *f x*
using *assms* **by** *simp-all*

setup ‹
Code.declare-case-global @{*thm Let-case-cert*} #>
Code.declare-undefined-global ***const-name*** ‹*undefined*›
›

declare [[*code abort*: *undefined*]]

3.1.4 Generic code generator target languages

type *bool*

code-printing
type-constructor *bool* ⇀
(*SML*) *bool* **and** (*OCaml*) *bool* **and** (*Haskell*) *Bool* **and** (*Scala*) *Boolean*
| **constant** *True* ⇀
(*SML*) *true* **and** (*OCaml*) *true* **and** (*Haskell*) *True* **and** (*Scala*) *true*
| **constant** *False* ⇀
(*SML*) *false* **and** (*OCaml*) *false* **and** (*Haskell*) *False* **and** (*Scala*) *false*

code-reserved
(*SML*) *bool true false*
and (*OCaml*) *bool*
and (*Scala*) *Boolean*

code-printing
constant *Not* ⇀
(*SML*) *not* **and** (*OCaml*) *not* **and** (*Haskell*) *not* **and** (*Scala*) '! -
| **constant** *HOL.conj* ⇀
(*SML*) **infixl** *1 andalso* **and** (*OCaml*) **infixl** *3* && **and** (*Haskell*) **infixr** *3* && **and** (*Scala*) **infixl** *3* &&
| **constant** *HOL.disj* ⇀

(*SML*) **infixl** *0 orelse* **and** (*OCaml*) **infixl** *2* || **and** (*Haskell*) **infixl** *2* || **and** (*Scala*) **infixl** *1* ||
| **constant** *HOL.implies* ⇀
(*SML*) !(*if* (-)/ *then* (-)/ *else true*)
and (*OCaml*) !(*if* (-)/ *then* (-)/ *else true*)
and (*Haskell*) !(*if* (-)/ *then* (-)/ *else True*)
and (*Scala*) !((-) *match* {/ *case true* => (-)/ *case false* => *true*/ })
| **constant** *If* ⇀
(*SML*) !(*if* (-)/ *then* (-)/ *else* (-))
and (*OCaml*) !(*if* (-)/ *then* (-)/ *else* (-))
and (*Haskell*) !(*if* (-)/ *then* (-)/ *else* (-))
and (*Scala*) !((-) *match* {/ *case true* => (-)/ *case false* => (-)/ })

code-reserved
(*SML*) *not*
and (*OCaml*) *not*

code-identifier
code-module *Pure* ⇀
(*SML*) *HOL* **and** (*OCaml*) *HOL* **and** (*Haskell*) *HOL* **and** (*Scala*) *HOL*

Using built-in Haskell equality.

code-printing
type-class *equal* ⇀ (*Haskell*) *Eq*
| **constant** *HOL.equal* ⇀ (*Haskell*) **infix** *4* ==
| **constant** *HOL.eq* ⇀ (*Haskell*) **infix** *4* ==

undefined

code-printing
constant *undefined* ⇀
(*SML*) !(*raise*/ *Fail*/ *undefined*)
and (*OCaml*) *failwith*/ *undefined*
and (*Haskell*) *error*/ *undefined*
and (*Scala*) !*sys.error*(*undefined*)

3.1.5 Evaluation and normalization by evaluation

method-setup *eval* = ‹
```
let
  fun eval-tac ctxt =
    let val conv = Code-Runtime.dynamic-holds-conv
    in
      CONVERSION (Conv.params-conv ~1 (Conv.concl-conv ~1 o conv) ctxt)
THEN'
      resolve-tac ctxt [TrueI]
    end
in
  Scan.succeed (SIMPLE-METHOD' o eval-tac)
end
```

› *solve goal by evaluation*

method-setup *normalization* = ‹
  Scan.succeed (*fn ctxt* =>
    SIMPLE-METHOD'
      (*CHANGED-PROP o*
        (*CONVERSION* (*Nbe.dynamic-conv ctxt*)
          THEN-ALL-NEW (*TRY o resolve-tac ctxt* [*TrueI*]))))
› *solve goal by normalization*

3.2 Counterexample Search Units

3.2.1 Quickcheck

quickcheck-params [*size* = *5*, *iterations* = *50*]

3.2.2 Nitpick setup

named-theorems *nitpick-unfold alternative definitions of constants as needed by Nitpick*
  and *nitpick-simp equational specification of constants as needed by Nitpick*
  and *nitpick-psimp partial equational specification of constants as needed by Nitpick*
  and *nitpick-choice-spec choice specification of constants as needed by Nitpick*

declare *if-bool-eq-conj* [*nitpick-unfold*, *no-atp*]
  and *if-bool-eq-disj* [*no-atp*]

3.3 Preprocessing for the predicate compiler

named-theorems *code-pred-def alternative definitions of constants for the Predicate Compiler*
  and *code-pred-inline inlining definitions for the Predicate Compiler*
  and *code-pred-simp simplification rules for the optimisations in the Predicate Compiler*

3.4 Legacy tactics and ML bindings

ML ‹
  (∗ *combination of* (*spec RS spec RS* ...(*j times*) ... *spec RS mp*) ∗)
  local
    fun wrong-prem ***Const-*** ‹*All* - *for* ‹*Abs* (-, -, *t*)›› = *wrong-prem t*
      | *wrong-prem* (*Bound* -) = *true*
      | *wrong-prem* - = *false*;
    val filter-right = *filter* (*not o wrong-prem o HOLogic.dest-Trueprop o hd o Thm.take-prems-of 1*);
    fun smp i = *funpow i* (*fn m* => *filter-right* ([*spec*] *RL m*)) [*mp*];
  in
    fun smp-tac ctxt j = *EVERY'* [*dresolve-tac ctxt* (*smp j*), *assume-tac ctxt*];
  end;

```
  local
    val nnf-ss =
      simpset-of (put-simpset HOL-basic-ss context addsimps @{thms simp-thms
nnf-simps});
  in
    fun nnf-conv ctxt = Simplifier.rewrite (put-simpset nnf-ss ctxt);
  end
›
```

hide-const (**open**) *eq equal*

end

4 Abstract orderings

theory *Orderings*
imports *HOL*
keywords *print-orders* :: *diag*
begin

4.1 Abstract ordering

locale *partial-preordering* =
  fixes *less-eq* :: ‹$'a \Rightarrow 'a \Rightarrow bool$› (**infix** ‹≤› *50*)
  assumes *refl*: ‹$a \leq a$› — not *iff*: makes problems due to multiple (dual) interpretations
    and *trans*: ‹$a \leq b \Longrightarrow b \leq c \Longrightarrow a \leq c$›

locale *preordering* = *partial-preordering* +
  fixes *less* :: ‹$'a \Rightarrow 'a \Rightarrow bool$› (**infix** ‹<› *50*)
  assumes *strict-iff-not*: ‹$a < b \longleftrightarrow a \leq b \wedge \neg\, b \leq a$›
begin

lemma *strict-implies-order*:
  ‹$a < b \Longrightarrow a \leq b$›
  by (*simp add*: *strict-iff-not*)

lemma *irrefl*: — not *iff*: makes problems due to multiple (dual) interpretations
  ‹$\neg\, a < a$›
  by (*simp add*: *strict-iff-not*)

lemma *asym*:
  ‹$a < b \Longrightarrow b < a \Longrightarrow False$›
  by (*auto simp add*: *strict-iff-not*)

lemma *strict-trans1*:
  ‹$a \leq b \Longrightarrow b < c \Longrightarrow a < c$›
  by (*auto simp add*: *strict-iff-not intro*: *trans*)

lemma *strict-trans2*:
  ‹$a < b \Longrightarrow b \leq c \Longrightarrow a < c$›
  by (*auto simp add*: *strict-iff-not intro*: *trans*)

lemma *strict-trans*:
  ‹$a < b \Longrightarrow b < c \Longrightarrow a < c$›
  by (*auto intro*: *strict-trans1 strict-implies-order*)

end

lemma *preordering-strictI*: — Alternative introduction rule with bias towards strict order
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes *less-eq-less*: ‹$\bigwedge a\ b.\ a \leq b \longleftrightarrow a < b \lor a = b$›
    assumes *asym*: ‹$\bigwedge a\ b.\ a < b \Longrightarrow \neg\ b < a$›
  assumes *irrefl*: ‹$\bigwedge a.\ \neg\ a < a$›
  assumes *trans*: ‹$\bigwedge a\ b\ c.\ a < b \Longrightarrow b < c \Longrightarrow a < c$›
  shows ‹*preordering* (≤) (<)›
proof
  fix *a b*
  show ‹$a < b \longleftrightarrow a \leq b \land \neg\ b \leq a$›
    by (*auto simp add*: *less-eq-less asym irrefl*)
next
  fix *a*
  show ‹$a \leq a$›
    by (*auto simp add*: *less-eq-less*)
next
  fix *a b c*
  assume ‹$a \leq b$› **and** ‹$b \leq c$› **then show** ‹$a \leq c$›
    by (*auto simp add*: *less-eq-less intro*: *trans*)
qed

lemma *preordering-dualI*:
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes ‹*preordering* ($\lambda a\ b.\ b \leq a$) ($\lambda a\ b.\ b < a$)›
  shows ‹*preordering* (≤) (<)›
proof –
  from *assms* **interpret** *preordering* ‹$\lambda a\ b.\ b \leq a$› ‹$\lambda a\ b.\ b < a$› **.**
  show *?thesis*
    by *standard* (*auto simp*: *strict-iff-not refl intro*: *trans*)
qed

locale *ordering* = *partial-preordering* +
  fixes *less* :: ‹$'a \Rightarrow 'a \Rightarrow bool$› (**infix** ‹<› *50*)
  assumes *strict-iff-order*: ‹$a < b \longleftrightarrow a \leq b \land a \neq b$›
  assumes *antisym*: ‹$a \leq b \Longrightarrow b \leq a \Longrightarrow a = b$›
begin

sublocale *preordering* ‹(≤)› ‹(<)›
proof
  show ‹$a < b \longleftrightarrow a \leq b \wedge \neg\ b \leq a$› **for** *a b*
    by (*auto simp add*: *strict-iff-order intro*: *antisym*)
qed

lemma *strict-implies-not-eq*:
  ‹$a < b \Longrightarrow a \neq b$›
  by (*simp add*: *strict-iff-order*)

lemma *not-eq-order-implies-strict*:
  ‹$a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$›
  by (*simp add*: *strict-iff-order*)

lemma *order-iff-strict*:
  ‹$a \leq b \longleftrightarrow a < b \vee a = b$›
  by (*auto simp add*: *strict-iff-order refl*)

lemma *eq-iff*: ‹$a = b \longleftrightarrow a \leq b \wedge b \leq a$›
  by (*auto simp add*: *refl intro*: *antisym*)

end

lemma *ordering-strictI*: — Alternative introduction rule with bias towards strict order
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes *less-eq-less*: ‹$\bigwedge a\ b.\ a \leq b \longleftrightarrow a < b \vee a = b$›
    assumes *asym*: ‹$\bigwedge a\ b.\ a < b \Longrightarrow \neg\ b < a$›
  assumes *irrefl*: ‹$\bigwedge a.\ \neg\ a < a$›
  assumes *trans*: ‹$\bigwedge a\ b\ c.\ a < b \Longrightarrow b < c \Longrightarrow a < c$›
  shows ‹*ordering* (≤) (<)›
proof
  fix *a b*
  show ‹$a < b \longleftrightarrow a \leq b \wedge a \neq b$›
    by (*auto simp add*: *less-eq-less asym irrefl*)
next
  fix *a*
  show ‹$a \leq a$›
    by (*auto simp add*: *less-eq-less*)
next
  fix *a b c*
  assume ‹$a \leq b$› **and** ‹$b \leq c$› **then show** ‹$a \leq c$›
    by (*auto simp add*: *less-eq-less intro*: *trans*)
next
  fix *a b*
  assume ‹$a \leq b$› **and** ‹$b \leq a$› **then show** ‹$a = b$›
    by (*auto simp add*: *less-eq-less asym*)

qed

lemma *ordering-dualI*:
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes ‹*ordering* (λ*a b*. *b* ≤ *a*) (λ*a b*. *b* < *a*)›
  shows ‹*ordering* (≤) (<)›
proof −
  from *assms* **interpret** *ordering* ‹λ*a b*. *b* ≤ *a*› ‹λ*a b*. *b* < *a*› **.**
  show *?thesis*
    by *standard* (*auto simp*: *strict-iff-order refl intro*: *antisym trans*)
qed

locale *ordering-top* = *ordering* +
  fixes *top* :: ‹′*a*› (‹⊤›)
  assumes *extremum* [*simp*]: ‹*a* ≤ ⊤›
begin

lemma *extremum-uniqueI*:
  ‹⊤ ≤ *a* ⟹ *a* = ⊤›
  by (*rule antisym*) *auto*

lemma *extremum-unique*:
  ‹⊤ ≤ *a* ⟷ *a* = ⊤›
  by (*auto intro*: *antisym*)

lemma *extremum-strict* [*simp*]:
  ‹¬ (⊤ < *a*)›
  using *extremum* [*of a*] **by** (*auto simp add*: *order-iff-strict intro*: *asym irrefl*)

lemma *not-eq-extremum*:
  ‹*a* ≠ ⊤ ⟷ *a* < ⊤›
  by (*auto simp add*: *order-iff-strict intro*: *not-eq-order-implies-strict extremum*)

end

4.2 Syntactic orders

class *ord* =
  fixes *less-eq* :: ′*a* ⇒ ′*a* ⇒ *bool*
    and *less* :: ′*a* ⇒ ′*a* ⇒ *bool*
begin

notation
  less-eq (‹′(≤′)›) **and**
  less-eq (‹(‹*notation*=‹*infix* ≤››-/ ≤ -)› [*51*, *51*] *50*) **and**
  less (‹′(<′)›) **and**
  less (‹(‹*notation*=‹*infix* <››-/ < -)› [*51*, *51*] *50*)

abbreviation (*input*)
greater-eq (**infix** ‹$\geq$› *50*)
where $x \geq y \equiv y \leq x$

abbreviation (*input*)
greater (**infix** ‹$>$› *50*)
where $x > y \equiv y < x$

notation (*ASCII*)
less-eq (‹′(<=′)›) **and**
less-eq (‹(‹*notation*=‹*infix* <=››-/ <= -)› [*51*, *51*] *50*)

notation (*input*)
greater-eq (**infix** ‹>=› *50*)

end

4.3 Quasi orders

class *preorder* = *ord* +
assumes *less-le-not-le*: $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$
and *order-refl* [*iff*]: $x \leq x$
and *order-trans*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
begin

sublocale *order*: *preordering less-eq less* + *dual-order*: *preordering greater-eq greater*
proof −
interpret *preordering less-eq less*
by *standard* (*auto intro*: *order-trans simp add*: *less-le-not-le*)
show ‹*preordering less-eq less*›
by (*fact preordering-axioms*)
then show ‹*preordering greater-eq greater*›
by (*rule preordering-dualI*)
qed

Reflexivity.

lemma *eq-refl*: $x = y \Longrightarrow x \leq y$
— This form is useful with the classical reasoner.
by (*erule ssubst*) (*rule order-refl*)

lemma *less-irrefl* [*iff*]: $\neg\, x < x$
by (*simp add*: *less-le-not-le*)

lemma *less-imp-le*: $x < y \Longrightarrow x \leq y$
by (*simp add*: *less-le-not-le*)

Asymmetry.

lemma *less-not-sym*: $x < y \Longrightarrow \neg (y < x)$
by (*simp add*: *less-le-not-le*)

lemma *less-asym*: $x < y \Longrightarrow (\neg\ P \Longrightarrow y < x) \Longrightarrow P$
by (*drule less-not-sym*, *erule contrapos-np*) *simp*

Transitivity.

lemma *less-trans*: $x < y \Longrightarrow y < z \Longrightarrow x < z$
by (*auto simp add*: *less-le-not-le intro*: *order-trans*)

lemma *le-less-trans*: $x \leq y \Longrightarrow y < z \Longrightarrow x < z$
by (*auto simp add*: *less-le-not-le intro*: *order-trans*)

lemma *less-le-trans*: $x < y \Longrightarrow y \leq z \Longrightarrow x < z$
by (*auto simp add*: *less-le-not-le intro*: *order-trans*)

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-less*: $x < y \Longrightarrow (\neg\ y < x) \longleftrightarrow True$
by (*blast elim*: *less-asym*)

lemma *less-imp-triv*: $x < y \Longrightarrow (y < x \longrightarrow P) \longleftrightarrow True$
by (*blast elim*: *less-asym*)

Transitivity rules for calculational reasoning

lemma *less-asym'*: $a < b \Longrightarrow b < a \Longrightarrow P$
by (*rule less-asym*)

Dual order

lemma *dual-preorder*:
‹*class.preorder* (≥) (>)›
by *standard* (*auto simp add*: *less-le-not-le intro*: *order-trans*)

end

lemma *preordering-preorderI*:
‹*class.preorder* (≤) (<)› **if** ‹*preordering* (≤) (<)›
for *less-eq* (**infix** ‹≤› *50*) **and** *less* (**infix** ‹<› *50*)
proof −
from *that* **interpret** *preordering* ‹(≤)› ‹(<)› **.**
show *?thesis*
by *standard* (*auto simp add*: *strict-iff-not refl intro*: *trans*)
qed

4.4 Partial orders

class *order* = *preorder* +
assumes *order-antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
begin

lemma *less-le*: $x < y \longleftrightarrow x \leq y \wedge x \neq y$
by (*auto simp add*: *less-le-not-le intro*: *order-antisym*)

sublocale *order*: *ordering less-eq less* + *dual-order*: *ordering greater-eq greater*
proof −
  interpret *ordering less-eq less*
    by *standard* (*auto intro*: *order-antisym order-trans simp add*: *less-le*)
  show *ordering less-eq less*
    by (*fact ordering-axioms*)
  then show *ordering greater-eq greater*
    by (*rule ordering-dualI*)
qed

Reflexivity.

lemma *le-less*: $x \leq y \longleftrightarrow x < y \vee x = y$
    — NOT suitable for iff, since it can cause PROOF FAILED.
by (*fact order.order-iff-strict*)

lemma *le-imp-less-or-eq*: $x \leq y \Longrightarrow x < y \vee x = y$
by (*simp add*: *less-le*)

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-eq*: $x < y \Longrightarrow (x = y) \longleftrightarrow False$
by *auto*

lemma *less-imp-not-eq2*: $x < y \Longrightarrow (y = x) \longleftrightarrow False$
by *auto*

Transitivity rules for calculational reasoning

lemma *neq-le-trans*: $a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$
by (*fact order.not-eq-order-implies-strict*)

lemma *le-neq-trans*: $a \leq b \Longrightarrow a \neq b \Longrightarrow a < b$
by (*rule order.not-eq-order-implies-strict*)

Asymmetry.

lemma *order-eq-iff*: $x = y \longleftrightarrow x \leq y \wedge y \leq x$
  by (*fact order.eq-iff*)

lemma *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x = y$
  by (*simp add*: *order.eq-iff*)

lemma *less-imp-neq*: $x < y \Longrightarrow x \neq y$
  by (*fact order.strict-implies-not-eq*)

lemma *antisym-conv1*: $\neg\, x < y \Longrightarrow x \leq y \longleftrightarrow x = y$
  by (*simp add*: *local.le-less*)

lemma *antisym-conv2*: $x \leq y \Longrightarrow \neg\, x < y \longleftrightarrow x = y$
  by (*simp add*: *local.less-le*)

```
lemma leD: y ≤ x ⟹ ¬ x < y
  by (auto simp: less-le order.antisym)
```

Least value operator

```
definition (in ord)
  Least :: ('a ⇒ bool) ⇒ 'a (binder ‹LEAST › 10) where
  Least P = (THE x. P x ∧ (∀ y. P y ⟶ x ≤ y))

lemma Least-equality:
  assumes P x
    and ⋀y. P y ⟹ x ≤ y
  shows Least P = x
unfolding Least-def by (rule the-equality)
  (blast intro: assms order.antisym)+

lemma LeastI2-order:
  assumes P x
    and ⋀y. P y ⟹ x ≤ y
    and ⋀x. P x ⟹ ∀ y. P y ⟶ x ≤ y ⟹ Q x
  shows Q (Least P)
unfolding Least-def by (rule theI2)
  (blast intro: assms order.antisym)+

lemma Least-ex1:
  assumes   ∃!x. P x ∧ (∀ y. P y ⟶ x ≤ y)
  shows      Least1I: P (Least P) and Least1-le: P z ⟹ Least P ≤ z
  using      theI'[OF assms]
  unfolding Least-def
  by         auto
```

Greatest value operator

```
definition Greatest :: ('a ⇒ bool) ⇒ 'a (binder ‹GREATEST › 10) where
Greatest P = (THE x. P x ∧ (∀ y. P y ⟶ x ≥ y))

lemma GreatestI2-order:
  ⟦ P x;
    ⋀y. P y ⟹ x ≥ y;
    ⋀x. ⟦ P x; ∀ y. P y ⟶ x ≥ y ⟧ ⟹ Q x ⟧
  ⟹ Q (Greatest P)
unfolding Greatest-def
by (rule theI2) (blast intro: order.antisym)+

lemma Greatest-equality:
  ⟦ P x;  ⋀y. P y ⟹ x ≥ y ⟧ ⟹ Greatest P = x
unfolding Greatest-def
by (rule the-equality) (blast intro: order.antisym)+

end
```

lemma *ordering-orderI*:
fixes *less-eq* (**infix** ‹≤› *50*)
and *less* (**infix** ‹<› *50*)
assumes *ordering less-eq less*
shows *class.order less-eq less*
proof −
from *assms* **interpret** *ordering less-eq less* **.**
show *?thesis*
by *standard* (*auto intro*: *antisym trans simp add*: *refl strict-iff-order*)
qed

lemma *order-strictI*:
fixes *less* (**infix** ‹<› *50*)
and *less-eq* (**infix** ‹≤› *50*)
assumes $\bigwedge a\ b.\ a \leq b \longleftrightarrow a < b \lor a = b$
assumes $\bigwedge a\ b.\ a < b \Longrightarrow \neg\ b < a$
assumes $\bigwedge a.\ \neg\ a < a$
assumes $\bigwedge a\ b\ c.\ a < b \Longrightarrow b < c \Longrightarrow a < c$
shows *class.order less-eq less*
by (*rule ordering-orderI*) (*rule ordering-strictI*, (*fact assms*)+)

context *order*
begin

Dual order

lemma *dual-order*:
class.order (≥) (>)
using *dual-order.ordering-axioms* **by** (*rule ordering-orderI*)

end

4.5 Linear (total) orders

class *linorder* = *order* +
assumes *linear*: $x \leq y \lor y \leq x$
begin

lemma *less-linear*: $x < y \lor x = y \lor y < x$
unfolding *less-le* **using** *less-le linear* **by** *blast*

lemma *le-less-linear*: $x \leq y \lor y < x$
by (*simp add*: *le-less less-linear*)

lemma *le-cases* [*case-names le ge*]:
$(x \leq y \Longrightarrow P) \Longrightarrow (y \leq x \Longrightarrow P) \Longrightarrow P$
using *linear* **by** *blast*

lemma (**in** *linorder*) *le-cases3*:

$[\![[\![x \le y; y \le z]\!] \Longrightarrow P; [\![y \le x; x \le z]\!] \Longrightarrow P; [\![x \le z; z \le y]\!] \Longrightarrow P;$
$[\![z \le y; y \le x]\!] \Longrightarrow P; [\![y \le z; z \le x]\!] \Longrightarrow P; [\![z \le x; x \le y]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*blast intro*: *le-cases*)

lemma *linorder-cases* [*case-names less equal greater*]:
$(x < y \Longrightarrow P) \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow (y < x \Longrightarrow P) \Longrightarrow P$
using *less-linear* **by** *blast*

lemma *linorder-wlog*[*case-names le sym*]:
$(\bigwedge a\ b.\ a \le b \Longrightarrow P\ a\ b) \Longrightarrow (\bigwedge a\ b.\ P\ b\ a \Longrightarrow P\ a\ b) \Longrightarrow P\ a\ b$
by (*cases rule*: *le-cases*[*of a b*]) *blast+*

lemma *not-less*: $\neg\ x < y \longleftrightarrow y \le x$
unfolding *less-le*
using *linear* **by** (*blast intro*: *order.antisym*)

lemma *not-less-iff-gr-or-eq*: $\neg(x < y) \longleftrightarrow (x > y \vee x = y)$
by (*auto simp add*:*not-less le-less*)

lemma *not-le*: $\neg\ x \le y \longleftrightarrow y < x$
unfolding *less-le*
using *linear* **by** (*blast intro*: *order.antisym*)

lemma *neq-iff*: $x \ne y \longleftrightarrow x < y \vee y < x$
by (*cut-tac* $x = x$ **and** $y = y$ **in** *less-linear*, *auto*)

lemma *neqE*: $x \ne y \Longrightarrow (x < y \Longrightarrow R) \Longrightarrow (y < x \Longrightarrow R) \Longrightarrow R$
by (*simp add*: *neq-iff*) *blast*

lemma *antisym-conv3*: $\neg\ y < x \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
by (*blast intro*: *order.antisym dest*: *not-less* [*THEN iffD1*])

lemma *leI*: $\neg\ x < y \Longrightarrow y \le x$
unfolding *not-less* **.**

lemma *not-le-imp-less*: $\neg\ y \le x \Longrightarrow x < y$
unfolding *not-le* **.**

lemma *linorder-less-wlog*[*case-names less refl sym*]:
$[\![\bigwedge a\ b.\ a < b \Longrightarrow P\ a\ b;\ \bigwedge a.\ P\ a\ a;\ \bigwedge a\ b.\ P\ b\ a \Longrightarrow P\ a\ b]\!] \Longrightarrow P\ a\ b$
using *antisym-conv3* **by** *blast*

Dual order

lemma *dual-linorder*:
class.linorder $(\ge)$ $(>)$
by (*rule class.linorder.intro*, *rule dual-order*) (*unfold-locales*, *rule linear*)

end

Alternative introduction rule with bias towards strict order

lemma *linorder-strictI*:
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes *class.order less-eq less*
  assumes *trichotomy*: $\bigwedge a\ b.\ a < b \lor a = b \lor b < a$
  shows *class.linorder less-eq less*
proof −
  interpret *order less-eq less*
    by (*fact* ‹*class.order less-eq less*›)
  show *?thesis*
  proof
    fix *a b*
    show $a \leq b \lor b \leq a$
      using *trichotomy* **by** (*auto simp add*: *le-less*)
  qed
qed

4.6 Reasoning tools setup

ML-file ‹~~/*src*/*Provers*/*order-procedure.ML*›
ML-file ‹~~/*src*/*Provers*/*order-tac.ML*›

ML ‹
```
structure Logic-Signature : LOGIC-SIGNATURE = struct
  val mk-Trueprop = HOLogic.mk-Trueprop
  val dest-Trueprop = HOLogic.dest-Trueprop
  val Trueprop-conv = HOLogic.Trueprop-conv
  val Not = HOLogic.Not
  val conj = HOLogic.conj
  val disj = HOLogic.disj

  val notI = @{thm notI}
  val ccontr = @{thm ccontr}
  val conjI = @{thm conjI}
  val conjE = @{thm conjE}
  val disjE = @{thm disjE}

  val not-not-conv = Conv.rewr-conv @{thm eq-reflection[OF not-not]}
  val de-Morgan-conj-conv = Conv.rewr-conv @{thm eq-reflection[OF de-Morgan-conj]}
  val de-Morgan-disj-conv = Conv.rewr-conv @{thm eq-reflection[OF de-Morgan-disj]}
  val conj-disj-distribL-conv = Conv.rewr-conv @{thm eq-reflection[OF conj-disj-distribL]}
  val conj-disj-distribR-conv = Conv.rewr-conv @{thm eq-reflection[OF conj-disj-distribR]}
end

structure HOL-Base-Order-Tac = Base-Order-Tac(
  structure Logic-Sig = Logic-Signature;
  (* Exclude types with specialised solvers. *)
  val excluded-types = [HOLogic.natT, HOLogic.intT, HOLogic.realT]
)
```

```
structure HOL-Order-Tac = Order-Tac(structure Base-Tac = HOL-Base-Order-Tac)

fun print-orders ctxt0 =
  let
    val ctxt = Config.put show-sorts true ctxt0
    val orders = HOL-Order-Tac.Data.get (Context.Proof ctxt)
    fun pretty-term t = Pretty.block
      [Pretty.quote (Syntax.pretty-term ctxt t), Pretty.brk 1,
        Pretty.str ::, Pretty.brk 1,
        Pretty.quote (Syntax.pretty-typ ctxt (type-of t)), Pretty.brk 1]
    fun pretty-order ({kind = kind, ops = ops, ...}, -) =
      Pretty.block ([Pretty.str (@{make-string} kind), Pretty.str :, Pretty.brk 1]
                  @ map pretty-term [#eq ops, #le ops, #lt ops])
  in
    Pretty.writeln (Pretty.big-list order structures: (map pretty-order orders))
  end

val - =
  Outer-Syntax.command command-keyword ‹print-orders›
    print order structures available to order reasoner
    (Scan.succeed (Toplevel.keep (print-orders o Toplevel.context-of)))

›

method-setup order = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (HOL-Order-Tac.tac [] ctxt))
› partial and linear order reasoner
```

The method *order* allows one to use the order tactic located in `../Provers/order_tac.ML` in a standalone fashion.

The tactic rearranges the goal to prove *False*, then retrieves order literals of partial and linear orders (i.e. $x = y$, $x \leq y$, $x < y$, and their negated versions) from the premises and finally tries to derive a contradiction. Its main use case is as a solver to *simp* (see below), where it e.g. solves premises of conditional rewrite rules.

The tactic has two configuration attributes that control its behaviour:

- *order-trace* toggles tracing for the solver.
- *order-split-limit* limits the number of order literals of the form $\neg\ x < y$ that are passed to the tactic. This is helpful since these literals lead to case splitting and thus exponential runtime. This only applies to partial orders.

  We setup the solver for HOL with the structure `HOL_Order_Tac` here but the prover is agnostic to the object logic. It is possible to register orders with the prover using the functions `HOL_Order_Tac.declare_order` and `HOL_Order_Tac.declare_linorder`, which we do below for the type classes *order* and *linorder*. If possible, one should instantiate these type classes instead of registering new orders with the solver. One can also interpret the type class locales *order* and *linorder*. An example can be seen in `Library/Sublist.thy`, which contains e.g. the prefix order on lists.

The diagnostic command **print-orders** shows all orders known to the tactic in the current context.

Declarations to set up transitivity reasoner of partial and linear orders.

context *order*
begin

lemma *nless-le*: $(\neg\ a < b) \longleftrightarrow (\neg\ a \leq b) \vee a = b$
using *local.dual-order.order-iff-strict* **by** *blast*

local-setup ‹
HOL-Order-Tac.declare-order {
ops = {*eq* = @{*term* ‹(=) :: 'a ⇒ 'a ⇒ bool›}, *le* = @{*term* ‹(≤)›}, *lt* = @{*term* ‹(<)›}},
thms = {*trans* = @{*thm order-trans*}, *refl* = @{*thm order-refl*}, *eqD1* = @{*thm eq-refl*},
eqD2 = @{*thm eq-refl*[*OF sym*]}, *antisym* = @{*thm order-antisym*}, *contr* = @{*thm notE*}},
conv-thms = {*less-le* = @{*thm eq-reflection*[*OF less-le*]},
nless-le = @{*thm eq-reflection*[*OF nless-le*]}}
}
›

end

context *linorder*
begin

lemma *nle-le*: $(\neg\ a \leq b) \longleftrightarrow b \leq a \wedge b \neq a$
using *not-le less-le* **by** *simp*

local-setup ‹
HOL-Order-Tac.declare-linorder {
ops = {*eq* = @{*term* ‹(=) :: 'a ⇒ 'a ⇒ bool›}, *le* = @{*term* ‹(≤)›}, *lt* = @{*term* ‹(<)›}},
thms = {*trans* = @{*thm order-trans*}, *refl* = @{*thm order-refl*}, *eqD1* = @{*thm eq-refl*},
eqD2 = @{*thm eq-refl*[*OF sym*]}, *antisym* = @{*thm order-antisym*}, *contr* = @{*thm notE*}},
conv-thms = {*less-le* = @{*thm eq-reflection*[*OF less-le*]},
nless-le = @{*thm eq-reflection*[*OF not-less*]},

```
              nle-le = @{thm eq-reflection[OF nle-le]}}
  }
›

end

setup ‹
  map-theory-simpset (fn ctxt0 => ctxt0 addSolver
   mk-solver partial and linear orders (fn ctxt => HOL-Order-Tac.tac (Simplifier.prems-of
ctxt) ctxt))
›

ML ‹
local
  fun prp t thm = Thm.prop-of thm = t;  (* FIXME proper aconv!? *)
in

fun antisym-le-simproc ctxt ct =
  (case Thm.term-of ct of
    (le as Const (-, T)) $ r $ s =>
     (let
        val prems = Simplifier.prems-of ctxt;
        val less = Const (const-name ‹less›, T);
        val t = HOLogic.mk-Trueprop(le $ s $ r);
      in
        (case find-first (prp t) prems of
          NONE =>
            let val t = HOLogic.mk-Trueprop(HOLogic.Not $ (less $ r $ s)) in
              (case find-first (prp t) prems of
                NONE => NONE
              | SOME thm => SOME(mk-meta-eq(thm RS @{thm antisym-conv1})))
              end
      | SOME thm => SOME (mk-meta-eq (thm RS @{thm order-class.antisym-conv})))
      end handle THM - => NONE)
  | - => NONE);

fun antisym-less-simproc ctxt ct =
  (case Thm.term-of ct of
    NotC $ ((less as Const(-,T)) $ r $ s) =>
     (let
       val prems = Simplifier.prems-of ctxt;
       val le = Const (const-name ‹less-eq›, T);
       val t = HOLogic.mk-Trueprop(le $ r $ s);
      in
        (case find-first (prp t) prems of
          NONE =>
            let val t = HOLogic.mk-Trueprop (NotC $ (less $ s $ r)) in
              (case find-first (prp t) prems of
                NONE => NONE
```

| *SOME thm* => *SOME* (*mk-meta-eq*(*thm RS* @{*thm linorder-class.antisym-conv3*}))
end
| *SOME thm* => *SOME* (*mk-meta-eq* (*thm RS* @{*thm antisym-conv2*})))
end handle THM - => *NONE*)
| - => *NONE*);

end;
›

simproc-setup *antisym-le* ((*x*::'*a*::*order*) ≤ *y*) = *K antisym-le-simproc*
simproc-setup *antisym-less* (¬ (*x*::'*a*::*linorder*) < *y*) = *K antisym-less-simproc*

4.7 Bounded quantifiers

syntax (*ASCII*)
-*All-less* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ALL››ALL -<-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-less* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder EX››EX -<-./ -)› [*0*, *0*, *10*] *10*)
-*All-less-eq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ALL››ALL -<=-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-less-eq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder EX››EX -<=-./ -)› [*0*, *0*, *10*] *10*)

-*All-greater* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ALL››ALL ->-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-greater* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder EX››EX ->-./ -)› [*0*, *0*, *10*] *10*)
-*All-greater-eq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ALL››ALL ->=-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-greater-eq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder EX››EX ->=-./ -)› [*0*, *0*, *10*] *10*)

-*All-neq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ALL››ALL -~=-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-neq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder EX››EX -~=-./ -)› [*0*, *0*, *10*] *10*)

syntax
-*All-less* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ∀››∀ -<-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-less* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ∃››∃ -<-./ -)› [*0*, *0*, *10*] *10*)
-*All-less-eq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ∀››∀ -≤-./ -)› [*0*, *0*, *10*] *10*)
-*Ex-less-eq* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ∃››∃ -≤-./ -)› [*0*, *0*, *10*] *10*)

-*All-greater* :: [*idt*, '*a*, *bool*] => *bool* (‹(‹indent=3 notation=‹binder ∀››∀ ->-./

-)› [0, 0, 10] 10)
  -Ex-greater :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ∃››∃ ->-./
-)› [0, 0, 10] 10)
  -All-greater-eq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ∀››∀ -≥-./
-)› [0, 0, 10] 10)
  -Ex-greater-eq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ∃››∃ -≥-./
-)› [0, 0, 10] 10)

  -All-neq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ∀››∀ -≠-./ -)›
[0, 0, 10] 10)
  -Ex-neq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ∃››∃ -≠-./ -)›
[0, 0, 10] 10)

syntax (*input*)
  -All-less :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder !››! -<-./ -)›
[0, 0, 10] 10)
  -Ex-less :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ?››? -<-./ -)›
[0, 0, 10] 10)
  -All-less-eq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder !››! -<=-./
-)› [0, 0, 10] 10)
  -Ex-less-eq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ?››? -<=-./
-)› [0, 0, 10] 10)
  -All-neq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder !››! -~=-./ -)›
[0, 0, 10] 10)
  -Ex-neq :: [idt, 'a, bool] => bool (‹(‹indent=3 notation=‹binder ?››? -~=-./
-)› [0, 0, 10] 10)

syntax-consts
  -All-less -All-less-eq -All-greater -All-greater-eq -All-neq ⇌ All **and**
  -Ex-less -Ex-less-eq -Ex-greater -Ex-greater-eq -Ex-neq ⇌ Ex

translations
  ∀x<y. P ⇀ ∀x. x < y ⟶ P
  ∃x<y. P ⇀ ∃x. x < y ∧ P
  ∀x≤y. P ⇀ ∀x. x ≤ y ⟶ P
  ∃x≤y. P ⇀ ∃x. x ≤ y ∧ P
  ∀x>y. P ⇀ ∀x. x > y ⟶ P
  ∃x>y. P ⇀ ∃x. x > y ∧ P
  ∀x≥y. P ⇀ ∀x. x ≥ y ⟶ P
  ∃x≥y. P ⇀ ∃x. x ≥ y ∧ P
  ∀x≠y. P ⇀ ∀x. x ≠ y ⟶ P
  ∃x≠y. P ⇀ ∃x. x ≠ y ∧ P

print-translation ‹
let
  val All-binder = Mixfix.binder-name ***const-syntax*** ‹All›;
  val Ex-binder = Mixfix.binder-name ***const-syntax*** ‹Ex›;
  val impl = ***const-syntax*** ‹HOL.implies›;
  val conj = ***const-syntax*** ‹HOL.conj›;

```
  val less = const-syntax ‹less›;
  val less-eq = const-syntax ‹less-eq›;

  val trans =
   [((All-binder, impl, less),
    (syntax-const ‹-All-less›, syntax-const ‹-All-greater›)),
    ((All-binder, impl, less-eq),
    (syntax-const ‹-All-less-eq›, syntax-const ‹-All-greater-eq›)),
    ((Ex-binder, conj, less),
    (syntax-const ‹-Ex-less›, syntax-const ‹-Ex-greater›)),
    ((Ex-binder, conj, less-eq),
    (syntax-const ‹-Ex-less-eq›, syntax-const ‹-Ex-greater-eq›))];

  fun matches-bound v t =
    (case t of
      Const (syntax-const ‹-bound›, -) $ Free (v', -) => v = v'
    | - => false);
  fun contains-var v = Term.exists-subterm (fn Free (x, -) => x = v | - => false);
  fun mk x c n P = Syntax.const c $ Syntax-Trans.mark-bound-body x $ n $ P;

  fun tr' q = (q, fn - =>
    (fn [Const (syntax-const ‹-bound›, -) $ Free (v, T),
        Const (c, -) $ (Const (d, -) $ t $ u) $ P] =>
        (case AList.lookup (=) trans (q, c, d) of
          NONE => raise Match
        | SOME (l, g) =>
            if matches-bound v t andalso not (contains-var v u) then mk (v, T) l u P
            else if matches-bound v u andalso not (contains-var v t) then mk (v, T)
g t P
            else raise Match)
      | - => raise Match));
in [tr' All-binder, tr' Ex-binder] end
›
```

4.8 Transitivity reasoning

context *ord*
begin

lemma *ord-le-eq-trans*: $a \leq b \Longrightarrow b = c \Longrightarrow a \leq c$
 by (*rule subst*)

lemma *ord-eq-le-trans*: $a = b \Longrightarrow b \leq c \Longrightarrow a \leq c$
 by (*rule ssubst*)

lemma *ord-less-eq-trans*: $a < b \Longrightarrow b = c \Longrightarrow a < c$
 by (*rule subst*)

lemma *ord-eq-less-trans*: $a = b \Longrightarrow b < c \Longrightarrow a < c$

by (*rule ssubst*)

end

lemma *order-less-subst2*: $(a::'a::order) < b \Longrightarrow f\ b < (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
proof −
assume r: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a < b$ **hence** $f\ a < f\ b$ **by** (*rule r*)
also assume $f\ b < c$
finally (*less-trans*) **show** *?thesis* **.**
qed

lemma *order-less-subst1*: $(a::'a::order) < f\ b \Longrightarrow (b::'b::order) < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
proof −
assume r: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a < f\ b$
also assume $b < c$ **hence** $f\ b < f\ c$ **by** (*rule r*)
finally (*less-trans*) **show** *?thesis* **.**
qed

lemma *order-le-less-subst2*: $(a::'a::order) <= b \Longrightarrow f\ b < (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a < c$
proof −
assume r: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= b$ **hence** $f\ a <= f\ b$ **by** (*rule r*)
also assume $f\ b < c$
finally (*le-less-trans*) **show** *?thesis* **.**
qed

lemma *order-le-less-subst1*: $(a::'a::order) <= f\ b \Longrightarrow (b::'b::order) < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
proof −
assume r: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a <= f\ b$
also assume $b < c$ **hence** $f\ b < f\ c$ **by** (*rule r*)
finally (*le-less-trans*) **show** *?thesis* **.**
qed

lemma *order-less-le-subst2*: $(a::'a::order) < b \Longrightarrow f\ b <= (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
proof −
assume r: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a < b$ **hence** $f\ a < f\ b$ **by** (*rule r*)
also assume $f\ b <= c$
finally (*less-le-trans*) **show** *?thesis* **.**
qed

lemma *order-less-le-subst1*: $(a::'a::order) < f\ b \Longrightarrow (b::'b::order) <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a < f\ c$
proof –
assume r: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a < f\ b$
also assume $b <= c$ **hence** $f\ b <= f\ c$ **by** (*rule r*)
finally (*less-le-trans*) **show** *?thesis* **.**
qed

lemma *order-subst1*: $(a::'a::order) <= f\ b \Longrightarrow (b::'b::order) <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a <= f\ c$
proof –
assume r: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= f\ b$
also assume $b <= c$ **hence** $f\ b <= f\ c$ **by** (*rule r*)
finally (*order-trans*) **show** *?thesis* **.**
qed

lemma *order-subst2*: $(a::'a::order) <= b \Longrightarrow f\ b <= (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a <= c$
proof –
assume r: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= b$ **hence** $f\ a <= f\ b$ **by** (*rule r*)
also assume $f\ b <= c$
finally (*order-trans*) **show** *?thesis* **.**
qed

lemma *ord-le-eq-subst*: $a <= b \Longrightarrow f\ b = c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a <= c$
proof –
assume r: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= b$ **hence** $f\ a <= f\ b$ **by** (*rule r*)
also assume $f\ b = c$
finally (*ord-le-eq-trans*) **show** *?thesis* **.**
qed

lemma *ord-eq-le-subst*: $a = f\ b \Longrightarrow b <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a <= f\ c$
proof –
assume r: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a = f\ b$
also assume $b <= c$ **hence** $f\ b <= f\ c$ **by** (*rule r*)
finally (*ord-eq-le-trans*) **show** *?thesis* **.**
qed

lemma *ord-less-eq-subst*: $a < b \Longrightarrow f\ b = c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
proof –
assume r: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$

assume $a < b$ **hence** $f\ a < f\ b$ **by** (*rule r*)
also assume $f\ b = c$
finally (*ord-less-eq-trans*) **show** *?thesis* **.**
qed

lemma *ord-eq-less-subst*: $a = f\ b \Longrightarrow b < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
proof −
assume r: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a = f\ b$
also assume $b < c$ **hence** $f\ b < f\ c$ **by** (*rule r*)
finally (*ord-eq-less-trans*) **show** *?thesis* **.**
qed

Note that this list of rules is in reverse order of priorities.

lemmas [*trans*] =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp

lemmas (**in** *order*) [*trans*] =
neq-le-trans
le-neq-trans

lemmas (**in** *preorder*) [*trans*] =
less-trans
less-asym'
le-less-trans
less-le-trans
order-trans

lemmas (**in** *order*) [*trans*] =
order.antisym

lemmas (**in** *ord*) [*trans*] =
ord-le-eq-trans

ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans

lemmas [*trans*] =
trans

lemmas *order-trans-rules* =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp
neq-le-trans
le-neq-trans
less-trans
less-asym'
le-less-trans
less-le-trans
order-trans
order.antisym
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans
trans

These support proving chains of decreasing inequalities a ≥ b ≥ c ... in Isar proofs.

lemma *xt1* [*no-atp*]:
$a = b \Longrightarrow b > c \Longrightarrow a > c$
$a > b \Longrightarrow b = c \Longrightarrow a > c$
$a = b \Longrightarrow b \geq c \Longrightarrow a \geq c$
$a \geq b \Longrightarrow b = c \Longrightarrow a \geq c$
$(x::'a::order) \geq y \Longrightarrow y \geq x \Longrightarrow x = y$
$(x::'a::order) \geq y \Longrightarrow y \geq z \Longrightarrow x \geq z$
$(x::'a::order) > y \Longrightarrow y \geq z \Longrightarrow x > z$
$(x::'a::order) \geq y \Longrightarrow y > z \Longrightarrow x > z$

$(a::'a::order) > b \Longrightarrow b > a \Longrightarrow P$
$(x::'a::order) > y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) \geq b \Longrightarrow a \neq b \Longrightarrow a > b$
$(a::'a::order) \neq b \Longrightarrow a \geq b \Longrightarrow a > b$
$a = f\ b \Longrightarrow b > c \Longrightarrow (\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
$a > b \Longrightarrow f\ b = c \Longrightarrow (\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
$a = f\ b \Longrightarrow b \geq c \Longrightarrow (\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y) \Longrightarrow a \geq f\ c$
$a \geq b \Longrightarrow f\ b = c \Longrightarrow (\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y) \Longrightarrow f\ a \geq c$
by *auto*

lemma *xt2* [*no-atp*]:
assumes $(a::'a::order) \geq f\ b$
and $b \geq c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $a \geq f\ c$
using *assms* **by** *force*

lemma *xt3* [*no-atp*]:
assumes $(a::'a::order) \geq b$
and $(f\ b::'b::order) \geq c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $f\ a \geq c$
using *assms* **by** *force*

lemma *xt4* [*no-atp*]:
assumes $(a::'a::order) > f\ b$
and $(b::'b::order) \geq c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $a > f\ c$
using *assms* **by** *force*

lemma *xt5* [*no-atp*]:
assumes $(a::'a::order) > b$
and $(f\ b::'b::order) \geq c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $f\ a > c$
using *assms* **by** *force*

lemma *xt6* [*no-atp*]:
assumes $(a::'a::order) \geq f\ b$
and $b > c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $a > f\ c$
using *assms* **by** *force*

lemma *xt7* [*no-atp*]:
assumes $(a::'a::order) \geq b$
and $(f\ b::'b::order) > c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$

shows $f\ a > c$
using *assms* **by** *force*

lemma *xt8* [*no-atp*]:
assumes $(a{::}'a{::}order) > f\ b$
and $(b{::}'b{::}order) > c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $a > f\ c$
using *assms* **by** *force*

lemma *xt9* [*no-atp*]:
assumes $(a{::}'a{::}order) > b$
and $(f\ b{::}'b{::}order) > c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $f\ a > c$
using *assms* **by** *force*

lemmas *xtrans* = *xt1 xt2 xt3 xt4 xt5 xt6 xt7 xt8 xt9*

4.9 min and max – fundamental

definition (**in** *ord*) *min* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
$min\ a\ b = (if\ a \leq b\ then\ a\ else\ b)$

definition (**in** *ord*) *max* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
$max\ a\ b = (if\ a \leq b\ then\ b\ else\ a)$

lemma *min-absorb1*: $x \leq y \Longrightarrow min\ x\ y = x$
by (*simp add*: *min-def*)

lemma *max-absorb2*: $x \leq y \Longrightarrow max\ x\ y = y$
by (*simp add*: *max-def*)

lemma *min-absorb2*: $(y{::}'a{::}order) \leq x \Longrightarrow min\ x\ y = y$
by (*simp add*:*min-def*)

lemma *max-absorb1*: $(y{::}'a{::}order) \leq x \Longrightarrow max\ x\ y = x$
by (*simp add*: *max-def*)

lemma *max-min-same* [*simp*]:
fixes $x\ y :: 'a :: linorder$
shows $max\ x\ (min\ x\ y) = x\ max\ (min\ x\ y)\ x = x\ max\ (min\ x\ y)\ y = y\ max\ y\ (min\ x\ y) = y$
by(*auto simp add*: *max-def min-def*)

4.10 (Unique) top and bottom elements

class *bot* =
fixes *bot* :: $'a$ (‹⊥›)

class *order-bot* = *order* + *bot* +
assumes *bot-least*: $\bot \leq a$
begin

sublocale *bot*: *ordering-top greater-eq greater bot*
by *standard* (*fact bot-least*)

lemma *le-bot*:
$a \leq \bot \Longrightarrow a = \bot$
by (*fact bot.extremum-uniqueI*)

lemma *bot-unique*:
$a \leq \bot \longleftrightarrow a = \bot$
by (*fact bot.extremum-unique*)

lemma *not-less-bot*:
$\neg\ a < \bot$
by (*fact bot.extremum-strict*)

lemma *bot-less*:
$a \neq \bot \longleftrightarrow \bot < a$
by (*fact bot.not-eq-extremum*)

lemma *max-bot*[*simp*]: *max bot x* = *x*
by(*simp add*: *max-def bot-unique*)

lemma *max-bot2*[*simp*]: *max x bot* = *x*
by(*simp add*: *max-def bot-unique*)

lemma *min-bot*[*simp*]: *min bot x* = *bot*
by(*simp add*: *min-def bot-unique*)

lemma *min-bot2*[*simp*]: *min x bot* = *bot*
by(*simp add*: *min-def bot-unique*)

end

class *top* =
fixes *top* :: *'a* (‹⊤›)

class *order-top* = *order* + *top* +
assumes *top-greatest*: $a \leq \top$
begin

sublocale *top*: *ordering-top less-eq less top*
by *standard* (*fact top-greatest*)

lemma *top-le*:
$\top \leq a \Longrightarrow a = \top$

by (*fact top.extremum-uniqueI*)

lemma *top-unique*:
$\top \leq a \longleftrightarrow a = \top$
by (*fact top.extremum-unique*)

lemma *not-top-less*:
$\neg \top < a$
by (*fact top.extremum-strict*)

lemma *less-top*:
$a \neq \top \longleftrightarrow a < \top$
by (*fact top.not-eq-extremum*)

lemma *max-top*[*simp*]: *max top x* = *top*
by(*simp add*: *max-def top-unique*)

lemma *max-top2*[*simp*]: *max x top* = *top*
by(*simp add*: *max-def top-unique*)

lemma *min-top*[*simp*]: *min top x* = *x*
by(*simp add*: *min-def top-unique*)

lemma *min-top2*[*simp*]: *min x top* = *x*
by(*simp add*: *min-def top-unique*)

end

4.11 Dense orders

class *dense-order* = *order* +
assumes *dense*: $x < y \Longrightarrow (\exists z.\ x < z \land z < y)$

class *dense-linorder* = *linorder* + *dense-order*
begin

lemma *dense-le*:
fixes *y z* :: *'a*
assumes $\bigwedge x.\ x < y \Longrightarrow x \leq z$
shows $y \leq z$
proof (*rule ccontr*)
assume ¬ *?thesis*
hence $z < y$ **by** *simp*
from *dense*[*OF this*]
obtain *x* **where** $x < y$ **and** $z < x$ **by** *safe*
moreover have $x \leq z$ **using** *assms*[*OF* ‹$x < y$›] **.**
ultimately show *False* **by** *auto*
qed

```
lemma dense-le-bounded:
  fixes x y z :: 'a
  assumes x < y
  assumes *: ⋀w. ⟦ x < w ; w < y ⟧ ⟹ w ≤ z
  shows y ≤ z
proof (rule dense-le)
  fix w assume w < y
  from dense[OF ‹x < y›] obtain u where x < u u < y by safe
  from linear[of u w]
  show w ≤ z
  proof (rule disjE)
    assume u ≤ w
    from less-le-trans[OF ‹x < u› ‹u ≤ w›] ‹w < y›
    show w ≤ z by (rule *)
  next
    assume w ≤ u
    from ‹w ≤ u› *[OF ‹x < u› ‹u < y›]
    show w ≤ z by (rule order-trans)
  qed
qed

lemma dense-ge:
  fixes y z :: 'a
  assumes ⋀x. z < x ⟹ y ≤ x
  shows y ≤ z
proof (rule ccontr)
  assume ¬ ?thesis
  hence z < y by simp
  from dense[OF this]
  obtain x where x < y and z < x by safe
  moreover have y ≤ x using assms[OF ‹z < x›] .
  ultimately show False by auto
qed

lemma dense-ge-bounded:
  fixes x y z :: 'a
  assumes z < x
  assumes *: ⋀w. ⟦ z < w ; w < x ⟧ ⟹ y ≤ w
  shows y ≤ z
proof (rule dense-ge)
  fix w assume z < w
  from dense[OF ‹z < x›] obtain u where z < u u < x by safe
  from linear[of u w]
  show y ≤ w
  proof (rule disjE)
    assume w ≤ u
    from ‹z < w› le-less-trans[OF ‹w ≤ u› ‹u < x›]
    show y ≤ w by (rule *)
  next
```

assume $u \leq w$
from $*[OF \langle z < u\rangle \ \langle u < x\rangle] \ \langle u \leq w\rangle$
show $y \leq w$ **by** (*rule order-trans*)
qed
qed

end

class *no-top* = *order* +
assumes *gt-ex*: $\exists\, y.\ x < y$

class *no-bot* = *order* +
assumes *lt-ex*: $\exists\, y.\ y < x$

class *unbounded-dense-linorder* = *dense-linorder* + *no-top* + *no-bot*

4.12 Wellorders

class *wellorder* = *linorder* +
assumes *less-induct* [*case-names less*]: $(\bigwedge x.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow$
$P\ a$
begin

lemma *wellorder-Least-lemma*:
fixes $k :: 'a$
assumes $P\ k$
shows *LeastI*: $P\ (LEAST\ x.\ P\ x)$ **and** *Least-le*: $(LEAST\ x.\ P\ x) \leq k$
proof −
have $P\ (LEAST\ x.\ P\ x) \wedge (LEAST\ x.\ P\ x) \leq k$
using *assms* **proof** (*induct k rule*: *less-induct*)
case (*less x*) **then have** $P\ x$ **by** *simp*
show *?case* **proof** (*rule classical*)
assume *assm*: $\neg\ (P\ (LEAST\ a.\ P\ a) \wedge (LEAST\ a.\ P\ a) \leq x)$
have $\bigwedge y.\ P\ y \Longrightarrow x \leq y$
proof (*rule classical*)
fix y
assume $P\ y$ **and** $\neg\ x \leq y$
with *less* **have** $P\ (LEAST\ a.\ P\ a)$ **and** $(LEAST\ a.\ P\ a) \leq y$
by (*auto simp add*: *not-le*)
with *assm* **have** $x < (LEAST\ a.\ P\ a)$ **and** $(LEAST\ a.\ P\ a) \leq y$
by *auto*
then show $x \leq y$ **by** *auto*
qed
with $\langle P\ x\rangle$ **have** *Least*: $(LEAST\ a.\ P\ a) = x$
by (*rule Least-equality*)
with $\langle P\ x\rangle$ **show** *?thesis* **by** *simp*
qed
qed
then show $P\ (LEAST\ x.\ P\ x)$ **and** $(LEAST\ x.\ P\ x) \leq k$ **by** *auto*

qed

— The following 3 lemmas are due to Brian Huffman
lemma *LeastI-ex*: $\exists x.\ P\ x \Longrightarrow P\ (Least\ P)$
by (*erule exE*) (*erule LeastI*)

lemma *LeastI2*:
$P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
by (*blast intro*: *LeastI*)

lemma *LeastI2-ex*:
$\exists a.\ P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
by (*blast intro*: *LeastI-ex*)

lemma *LeastI2-wellorder*:
assumes $P\ a$
and $\bigwedge a.\ [\![\ P\ a;\ \forall b.\ P\ b \longrightarrow a \le b\ ]\!] \Longrightarrow Q\ a$
shows $Q\ (Least\ P)$
proof (*rule LeastI2-order*)
show $P\ (Least\ P)$ **using** ‹$P\ a$› **by** (*rule LeastI*)
next
fix y **assume** $P\ y$ **thus** $Least\ P \le y$ **by** (*rule Least-le*)
next
fix x **assume** $P\ x\ \forall y.\ P\ y \longrightarrow x \le y$ **thus** $Q\ x$ **by** (*rule assms(2)*)
qed

lemma *LeastI2-wellorder-ex*:
assumes $\exists x.\ P\ x$
and $\bigwedge a.\ [\![\ P\ a;\ \forall b.\ P\ b \longrightarrow a \le b\ ]\!] \Longrightarrow Q\ a$
shows $Q\ (Least\ P)$
using *assms* **by** *clarify* (*blast intro*!: *LeastI2-wellorder*)

lemma *not-less-Least*: $k < (LEAST\ x.\ P\ x) \Longrightarrow \neg\ P\ k$
apply (*simp add*: *not-le* [*symmetric*])
apply (*erule contrapos-nn*)
apply (*erule Least-le*)
done

lemma *exists-least-iff*: $(\exists n.\ P\ n) \longleftrightarrow (\exists n.\ P\ n \land (\forall m < n.\ \neg\ P\ m))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?rhs* **thus** *?lhs* **by** *blast*
next
assume *H*: *?lhs* **then obtain** n **where** n: $P\ n$ **by** *blast*
let $?x = Least\ P$
{ fix m **assume** m: $m < ?x$
from *not-less-Least*[*OF m*] **have** $\neg\ P\ m$ **. }**
with *LeastI-ex*[*OF H*] **show** *?rhs* **by** *blast*
qed

end

4.13 Order on *bool*

instantiation *bool* :: {*order-bot*, *order-top*, *linorder*}
begin

definition
le-bool-def [*simp*]: $P \leq Q \longleftrightarrow P \longrightarrow Q$

definition
[*simp*]: $(P::bool) < Q \longleftrightarrow \neg P \wedge Q$

definition
[*simp*]: $\bot \longleftrightarrow False$

definition
[*simp*]: $\top \longleftrightarrow True$

instance proof
qed *auto*

end

lemma *le-boolI*: $(P \Longrightarrow Q) \Longrightarrow P \leq Q$
by *simp*

lemma *le-boolI'*: $P \longrightarrow Q \Longrightarrow P \leq Q$
by *simp*

lemma *le-boolE*: $P \leq Q \Longrightarrow P \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
by *simp*

lemma *le-boolD*: $P \leq Q \Longrightarrow P \longrightarrow Q$
by *simp*

lemma *bot-boolE*: $\bot \Longrightarrow P$
by *simp*

lemma *top-boolI*: $\top$
by *simp*

lemma [*code*]:
$False \leq b \longleftrightarrow True$
$True \leq b \longleftrightarrow b$
$False < b \longleftrightarrow b$
$True < b \longleftrightarrow False$
by *simp-all*

4.14 Order on - ⇒ -

instantiation *fun* :: (*type*, *ord*) *ord*
begin

definition
  $le\text{-}fun\text{-}def$: $f \le g \longleftrightarrow (\forall x.\ f\ x \le g\ x)$

definition
  $(f{::}'a \Rightarrow 'b) < g \longleftrightarrow f \le g \wedge \neg (g \le f)$

instance ..

end

instance *fun* :: (*type*, *preorder*) *preorder* **proof**
qed (*auto simp add*: *le-fun-def less-fun-def*
  intro: *order-trans order.antisym*)

instance *fun* :: (*type*, *order*) *order* **proof**
qed (*auto simp add*: *le-fun-def intro*: *order.antisym*)

instantiation *fun* :: (*type*, *bot*) *bot*
begin

definition
  $\bot = (\lambda x.\ \bot)$

instance ..

end

instantiation *fun* :: (*type*, *order-bot*) *order-bot*
begin

lemma *bot-apply* [*simp*, *code*]:
  $\bot\ x = \bot$
  by (*simp add*: *bot-fun-def*)

instance proof
qed (*simp add*: *le-fun-def*)

end

instantiation *fun* :: (*type*, *top*) *top*
begin

definition
  [*no-atp*]: $\top = (\lambda x.\ \top)$

instance ..

end

instantiation *fun* :: (*type*, *order-top*) *order-top*
begin

lemma *top-apply* [*simp*, *code*]:
  $\top\ x = \top$
  by (*simp add*: *top-fun-def*)

instance proof
qed (*simp add*: *le-fun-def*)

end

lemma *le-funI*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow f \leq g$
  unfolding *le-fun-def* **by** *simp*

lemma *le-funE*: $f \leq g \Longrightarrow (f\ x \leq g\ x \Longrightarrow P) \Longrightarrow P$
  unfolding *le-fun-def* **by** *simp*

lemma *le-funD*: $f \leq g \Longrightarrow f\ x \leq g\ x$
  by (*rule le-funE*)

4.15 Order on unary and binary predicates

lemma *predicate1I*:
  assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
  shows $P \leq Q$
  apply (*rule le-funI*)
  apply (*rule le-boolI*)
  apply (*rule PQ*)
  apply *assumption*
  done

lemma *predicate1D*:
  $P \leq Q \Longrightarrow P\ x \Longrightarrow Q\ x$
  apply (*erule le-funE*)
  apply (*erule le-boolE*)
  apply *assumption+*
  done

lemma *rev-predicate1D*:
  $P\ x \Longrightarrow P \leq Q \Longrightarrow Q\ x$
  by (*rule predicate1D*)

lemma *predicate2I*:
  assumes *PQ*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$

shows $P \le Q$
apply (*rule le-funI*)+
apply (*rule le-boolI*)
apply (*rule PQ*)
apply *assumption*
done

lemma *predicate2D*:
$P \le Q \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
apply (*erule le-funE*)+
apply (*erule le-boolE*)
apply *assumption*+
done

lemma *rev-predicate2D*:
$P\ x\ y \Longrightarrow P \le Q \Longrightarrow Q\ x\ y$
by (*rule predicate2D*)

lemma *bot1E* [*no-atp*]: $\bot\ x \Longrightarrow P$
by (*simp add*: *bot-fun-def*)

lemma *bot2E*: $\bot\ x\ y \Longrightarrow P$
by (*simp add*: *bot-fun-def*)

lemma *top1I*: $\top\ x$
by (*simp add*: *top-fun-def*)

lemma *top2I*: $\top\ x\ y$
by (*simp add*: *top-fun-def*)

4.16 Name duplicates

lemmas *antisym* = *order.antisym*
lemmas *eq-iff* = *order.eq-iff*

lemmas *order-eq-refl* = *preorder-class.eq-refl*
lemmas *order-less-irrefl* = *preorder-class.less-irrefl*
lemmas *order-less-imp-le* = *preorder-class.less-imp-le*
lemmas *order-less-not-sym* = *preorder-class.less-not-sym*
lemmas *order-less-asym* = *preorder-class.less-asym*
lemmas *order-less-trans* = *preorder-class.less-trans*
lemmas *order-le-less-trans* = *preorder-class.le-less-trans*
lemmas *order-less-le-trans* = *preorder-class.less-le-trans*
lemmas *order-less-imp-not-less* = *preorder-class.less-imp-not-less*
lemmas *order-less-imp-triv* = *preorder-class.less-imp-triv*
lemmas *order-less-asym'* = *preorder-class.less-asym'*

lemmas *order-less-le* = *order-class.less-le*
lemmas *order-le-less* = *order-class.le-less*

lemmas *order-le-imp-less-or-eq* = *order-class.le-imp-less-or-eq*
lemmas *order-less-imp-not-eq* = *order-class.less-imp-not-eq*
lemmas *order-less-imp-not-eq2* = *order-class.less-imp-not-eq2*
lemmas *order-neq-le-trans* = *order-class.neq-le-trans*
lemmas *order-le-neq-trans* = *order-class.le-neq-trans*
lemmas *order-eq-iff* = *order-class.order.eq-iff*
lemmas *order-antisym-conv* = *order-class.antisym-conv*

lemmas *linorder-linear* = *linorder-class.linear*
lemmas *linorder-less-linear* = *linorder-class.less-linear*
lemmas *linorder-le-less-linear* = *linorder-class.le-less-linear*
lemmas *linorder-le-cases* = *linorder-class.le-cases*
lemmas *linorder-not-less* = *linorder-class.not-less*
lemmas *linorder-not-le* = *linorder-class.not-le*
lemmas *linorder-neq-iff* = *linorder-class.neq-iff*
lemmas *linorder-neqE* = *linorder-class.neqE*

end

5 Groups, also combined with orderings

theory *Groups*
imports *Orderings*
begin

5.1 Dynamic facts

named-theorems *ac-simps associativity and commutativity simplification rules*
and *algebra-simps algebra simplification rules for rings*
and *algebra-split-simps algebra simplification rules for rings*, *with potential goal splitting*
and *field-simps algebra simplification rules for fields*
and *field-split-simps algebra simplification rules for fields*, *with potential goal splitting*

The rewrites accumulated in *algebra-simps* deal with the classical algebraic structures of groups, rings and family. They simplify terms by multiplying everything out (in case of a ring) and bringing sums and products into a canonical form (by ordered rewriting). As a result it decides group and ring equalities but also helps with inequalities.

Of course it also works for fields, but it knows nothing about multiplicative inverses or division. This is catered for by *field-simps*.

Facts in *field-simps* multiply with denominators in (in)equations if they can be proved to be non-zero (for equations) or positive/negative (for inequalities). Can be too aggressive and is therefore separate from the more benign *algebra-simps*.

Collections *algebra-split-simps* and *field-split-simps* correspond to *algebra-simps* and *field-simps* but contain more aggresive rules that may lead to goal splitting.

5.2 Abstract structures

These locales provide basic structures for interpretation into bigger structures; extensions require careful thinking, otherwise undesired effects may occur due to interpretation.

```
locale semigroup =
  fixes f :: 'a ⇒ 'a ⇒ 'a  (infixl ‹∗› 70)
  assumes assoc [ac-simps]: a ∗ b ∗ c = a ∗ (b ∗ c)

locale abel-semigroup = semigroup +
  assumes commute [ac-simps]: a ∗ b = b ∗ a
begin

lemma left-commute [ac-simps]: b ∗ (a ∗ c) = a ∗ (b ∗ c)
proof −
  have (b ∗ a) ∗ c = (a ∗ b) ∗ c
    by (simp only: commute)
  then show ?thesis
    by (simp only: assoc)
qed

end

locale monoid = semigroup +
  fixes z :: 'a (‹1›)
  assumes left-neutral [simp]: 1 ∗ a = a
  assumes right-neutral [simp]: a ∗ 1 = a

locale comm-monoid = abel-semigroup +
  fixes z :: 'a (‹1›)
  assumes comm-neutral: a ∗ 1 = a
begin

sublocale monoid
  by standard (simp-all add: commute comm-neutral)

end

locale group = semigroup +
  fixes z :: 'a (‹1›)
  fixes inverse :: 'a ⇒ 'a
  assumes group-left-neutral: 1 ∗ a = a
  assumes left-inverse [simp]:  inverse a ∗ a = 1
begin
```

lemma *left-cancel*: $a * b = a * c \longleftrightarrow b = c$
proof
  assume $a * b = a * c$
  then have *inverse* $a * (a * b) =$ *inverse* $a * (a * c)$ **by** *simp*
  then have (*inverse* $a * a) * b =$ (*inverse* $a * a) * c$
    by (*simp only*: *assoc*)
  then show $b = c$ **by** (*simp add*: *group-left-neutral*)
qed *simp*

sublocale *monoid*
proof
  fix a
  have *inverse* $a * a = \mathbf{1}$ **by** *simp*
  then have *inverse* $a * (a * \mathbf{1}) =$ *inverse* $a * a$
    by (*simp add*: *group-left-neutral assoc* [*symmetric*])
  with *left-cancel* **show** $a * \mathbf{1} = a$
    by (*simp only*: *left-cancel*)
qed (*fact group-left-neutral*)

lemma *inverse-unique*:
  assumes $a * b = \mathbf{1}$
  shows *inverse* $a = b$
proof −
  from *assms* **have** *inverse* $a * (a * b) =$ *inverse* a
    by *simp*
  then show *?thesis*
    by (*simp add*: *assoc* [*symmetric*])
qed

lemma *inverse-neutral* [*simp*]: *inverse* $\mathbf{1} = \mathbf{1}$
  by (*rule inverse-unique*) *simp*

lemma *inverse-inverse* [*simp*]: *inverse* (*inverse* a) $= a$
  by (*rule inverse-unique*) *simp*

lemma *right-inverse* [*simp*]: $a *$ *inverse* $a = \mathbf{1}$
proof −
  have $a *$ *inverse* $a =$ *inverse* (*inverse* a) $*$ *inverse* a
    by *simp*
  also have $\ldots = \mathbf{1}$
    by (*rule left-inverse*)
  then show *?thesis* **by** *simp*
qed

lemma *inverse-distrib-swap*: *inverse* $(a * b) =$ *inverse* $b *$ *inverse* a
proof (*rule inverse-unique*)
  have $a * b *$ (*inverse* $b *$ *inverse* a) $=$
    $a * (b *$ *inverse* $b) *$ *inverse* a
    by (*simp only*: *assoc*)

```
  also have ... = 1
    by simp
  finally show a * b * (inverse b * inverse a) = 1 .
qed

lemma right-cancel: b * a = c * a ⟷ b = c
proof
  assume b * a = c * a
  then have b * a * inverse a= c * a * inverse a
    by simp
  then show b = c
    by (simp add: assoc)
qed simp

end
```

5.3 Generic operations

```
class zero =
  fixes zero :: 'a (‹0›)

class one =
  fixes one  :: 'a (‹1›)

hide-const (open) zero one

lemma Let-0 [simp]: Let 0 f = f 0
  unfolding Let-def ..

lemma Let-1 [simp]: Let 1 f = f 1
  unfolding Let-def ..

setup ‹
  Reorient-Proc.add
    (fn Const(const-name ‹Groups.zero›, -) => true
      | Const(const-name ‹Groups.one›, -) => true
      | - => false)
›

simproc-setup reorient-zero (0 = x) = ‹K Reorient-Proc.proc›
simproc-setup reorient-one (1 = x) = ‹K Reorient-Proc.proc›

typed-print-translation ‹
  let
    fun tr' c = (c, fn ctxt => fn T => fn ts =>
      if null ts andalso Printer.type-emphasis ctxt T then
        Syntax.const syntax-const ‹-constrain› $ Syntax.const c $
          Syntax-Phases.term-of-typ ctxt T
      else raise Match);
```

in map tr′ [***const-syntax*** ‹*Groups.one*›, ***const-syntax*** ‹*Groups.zero*›] *end*
› — show types that are presumably too general

class *plus* =
fixes *plus* :: ′*a* ⇒ ′*a* ⇒ ′*a* (**infixl** ‹+› *65*)

class *minus* =
fixes *minus* :: ′*a* ⇒ ′*a* ⇒ ′*a* (**infixl** ‹−› *65*)

class *uminus* =
fixes *uminus* :: ′*a* ⇒ ′*a* (‹(‹*open-block notation*=‹*prefix* −››− -)› [*81*] *80*)

class *times* =
fixes *times* :: ′*a* ⇒ ′*a* ⇒ ′*a* (**infixl** ‹∗› *70*)

bundle *uminus-syntax*
begin
notation *uminus* (‹(‹*open-block notation*=‹*prefix* −››− -)› [*81*] *80*)
end

5.4 Semigroups and Monoids

class *semigroup-add* = *plus* +
assumes *add-assoc*: $(a + b) + c = a + (b + c)$
begin

sublocale *add*: *semigroup plus*
by *standard* (*fact add-assoc*)

declare *add.assoc* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]

end

hide-fact *add-assoc*

class *ab-semigroup-add* = *semigroup-add* +
assumes *add-commute*: $a + b = b + a$
begin

sublocale *add*: *abel-semigroup plus*
by *standard* (*fact add-commute*)

declare *add.commute* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]
add.left-commute [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]

lemmas *add-ac* = *add.assoc add.commute add.left-commute*

end

hide-fact *add-commute*

lemmas *add-ac* = *add.assoc add.commute add.left-commute*

class *semigroup-mult* = *times* +
assumes *mult-assoc*: $(a * b) * c = a * (b * c)$
begin

sublocale *mult*: *semigroup times*
by *standard* (*fact mult-assoc*)

declare *mult.assoc* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]

end

hide-fact *mult-assoc*

class *ab-semigroup-mult* = *semigroup-mult* +
assumes *mult-commute*: $a * b = b * a$
begin

sublocale *mult*: *abel-semigroup times*
by *standard* (*fact mult-commute*)

declare *mult.commute* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]
mult.left-commute [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]

lemmas *mult-ac* = *mult.assoc mult.commute mult.left-commute*

end

hide-fact *mult-commute*

lemmas *mult-ac* = *mult.assoc mult.commute mult.left-commute*

class *monoid-add* = *zero* + *semigroup-add* +
assumes *add-0-left*: $0 + a = a$
and *add-0-right*: $a + 0 = a$
begin

sublocale *add*: *monoid plus 0*
by *standard* (*fact add-0-left add-0-right*)+

end

lemma *zero-reorient*: $0 = x \longleftrightarrow x = 0$
by (*fact eq-commute*)

class *comm-monoid-add* = *zero* + *ab-semigroup-add* +

assumes *add-0*: *0* + *a* = *a*
begin

subclass *monoid-add*
by *standard* (*simp-all add*: *add-0 add.commute* [*of* - *0*])

sublocale *add*: *comm-monoid plus 0*
by *standard* (*simp add*: *ac-simps*)

end

class *monoid-mult* = *one* + *semigroup-mult* +
assumes *mult-1-left*: *1* ∗ *a* = *a*
and *mult-1-right*: *a* ∗ *1* = *a*
begin

sublocale *mult*: *monoid times 1*
by *standard* (*fact mult-1-left mult-1-right*)+

end

lemma *one-reorient*: *1* = *x* ⟷ *x* = *1*
by (*fact eq-commute*)

class *comm-monoid-mult* = *one* + *ab-semigroup-mult* +
assumes *mult-1*: *1* ∗ *a* = *a*
begin

subclass *monoid-mult*
by *standard* (*simp-all add*: *mult-1 mult.commute* [*of* - *1*])

sublocale *mult*: *comm-monoid times 1*
by *standard* (*simp add*: *ac-simps*)

end

class *cancel-semigroup-add* = *semigroup-add* +
assumes *add-left-imp-eq*: *a* + *b* = *a* + *c* ⟹ *b* = *c*
assumes *add-right-imp-eq*: *b* + *a* = *c* + *a* ⟹ *b* = *c*
begin

lemma *add-left-cancel* [*simp*]: *a* + *b* = *a* + *c* ⟷ *b* = *c*
by (*blast dest*: *add-left-imp-eq*)

lemma *add-right-cancel* [*simp*]: *b* + *a* = *c* + *a* ⟷ *b* = *c*
by (*blast dest*: *add-right-imp-eq*)

end

class *cancel-ab-semigroup-add* = *ab-semigroup-add* + *minus* +
assumes *add-diff-cancel-left'* [*simp*]: $(a + b) - a = b$
assumes *diff-diff-add* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - b - c = a - (b + c)$
begin

lemma *add-diff-cancel-right'* [*simp*]: $(a + b) - b = a$
using *add-diff-cancel-left'* [*of b a*] **by** (*simp add*: *ac-simps*)

subclass *cancel-semigroup-add*
proof
fix *a b c* :: *'a*
assume $a + b = a + c$
then have $a + b - a = a + c - a$
by *simp*
then show $b = c$
by *simp*
next
fix *a b c* :: *'a*
assume $b + a = c + a$
then have $b + a - a = c + a - a$
by *simp*
then show $b = c$
by *simp*
qed

lemma *add-diff-cancel-left* [*simp*]: $(c + a) - (c + b) = a - b$
unfolding *diff-diff-add* [*symmetric*] **by** *simp*

lemma *add-diff-cancel-right* [*simp*]: $(a + c) - (b + c) = a - b$
using *add-diff-cancel-left* [*symmetric*] **by** (*simp add*: *ac-simps*)

lemma *diff-right-commute*: $a - c - b = a - b - c$
by (*simp add*: *diff-diff-add add.commute*)

end

class *cancel-comm-monoid-add* = *cancel-ab-semigroup-add* + *comm-monoid-add*
begin

lemma *diff-zero* [*simp*]: $a - 0 = a$
using *add-diff-cancel-right'* [*of a 0*] **by** *simp*

lemma *diff-cancel* [*simp*]: $a - a = 0$
proof −
have $(a + 0) - (a + 0) = 0$
by (*simp only*: *add-diff-cancel-left diff-zero*)
then show *?thesis* **by** *simp*
qed

lemma *add-implies-diff*:
  assumes $c + b = a$
  shows $c = a - b$
proof −
  from *assms* **have** $(b + c) - (b + 0) = a - b$
    by (*simp add*: *add.commute*)
  then show $c = a - b$ **by** *simp*
qed

lemma *add-cancel-right-right* [*simp*]: $a = a + b \longleftrightarrow b = 0$
  (**is** $?P \longleftrightarrow ?Q$)
proof
  assume $?Q$
  then show $?P$ **by** *simp*
next
  assume $?P$
  then have $a - a = a + b - a$ **by** *simp*
  then show $?Q$ **by** *simp*
qed

lemma *add-cancel-right-left* [*simp*]: $a = b + a \longleftrightarrow b = 0$
  using *add-cancel-right-right* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *add-cancel-left-right* [*simp*]: $a + b = a \longleftrightarrow b = 0$
  by (*auto dest*: *sym*)

lemma *add-cancel-left-left* [*simp*]: $b + a = a \longleftrightarrow b = 0$
  by (*auto dest*: *sym*)

end

class *comm-monoid-diff* = *cancel-comm-monoid-add* +
  assumes *zero-diff* [*simp*]: $0 - a = 0$
begin

lemma *diff-add-zero* [*simp*]: $a - (a + b) = 0$
proof −
  have $a - (a + b) = (a + 0) - (a + b)$
    by *simp*
  also have $\ldots = 0$
    by (*simp only*: *add-diff-cancel-left zero-diff*)
  finally show *?thesis* **.**
qed

end

5.5 Groups

class *group-add* = *minus* + *uminus* + *monoid-add* +
  assumes *left-minus*: $- a + a = 0$
  assumes *add-uminus-conv-diff* [*simp*]: $a + (- b) = a - b$
begin

lemma *diff-conv-add-uminus*: $a - b = a + (- b)$
  by *simp*

sublocale *add*: *group plus 0 uminus*
  by *standard* (*simp-all add*: *left-minus*)

lemma *minus-unique*: $a + b = 0 \Longrightarrow - a = b$
  by (*fact add.inverse-unique*)

lemma *minus-zero*: $- 0 = 0$
  by (*fact add.inverse-neutral*)

lemma *minus-minus*: $- (- a) = a$
  by (*fact add.inverse-inverse*)

lemma *right-minus*: $a + - a = 0$
  by (*fact add.right-inverse*)

lemma *diff-self* [*simp*]: $a - a = 0$
  using *right-minus* [*of a*] **by** *simp*

subclass *cancel-semigroup-add*
  by *standard* (*simp-all add*: *add.left-cancel add.right-cancel*)

lemma *minus-add-cancel* [*simp*]: $- a + (a + b) = b$
  by (*simp add*: *add.assoc* [*symmetric*])

lemma *add-minus-cancel* [*simp*]: $a + (- a + b) = b$
  by (*simp add*: *add.assoc* [*symmetric*])

lemma *diff-add-cancel* [*simp*]: $a - b + b = a$
  by (*simp only*: *diff-conv-add-uminus add.assoc*) *simp*

lemma *add-diff-cancel* [*simp*]: $a + b - b = a$
  by (*simp only*: *diff-conv-add-uminus add.assoc*) *simp*

lemma *minus-add*: $- (a + b) = - b + - a$
  by (*fact add.inverse-distrib-swap*)

lemma *right-minus-eq* [*simp*]: $a - b = 0 \longleftrightarrow a = b$
proof
  assume $a - b = 0$
  have $a = (a - b) + b$ **by** (*simp add*: *add.assoc*)

also have ... $= b$ **using** ‹$a - b = 0$› **by** *simp*
finally show $a = b$ **.**
next
assume $a = b$
then show $a - b = 0$ **by** *simp*
qed

lemma *eq-iff-diff-eq-0*: $a = b \longleftrightarrow a - b = 0$
by (*fact right-minus-eq* [*symmetric*])

lemma *diff-0* [*simp*]: $0 - a = - a$
by (*simp only*: *diff-conv-add-uminus add-0-left*)

lemma *diff-0-right* [*simp*]: $a - 0 = a$
by (*simp only*: *diff-conv-add-uminus minus-zero add-0-right*)

lemma *diff-minus-eq-add* [*simp*]: $a - - b = a + b$
by (*simp only*: *diff-conv-add-uminus minus-minus*)

lemma *neg-equal-iff-equal* [*simp*]: $- a = - b \longleftrightarrow a = b$
proof
assume $- a = - b$
then have $- (- a) = - (- b)$ **by** *simp*
then show $a = b$ **by** *simp*
next
assume $a = b$
then show $- a = - b$ **by** *simp*
qed

lemma *neg-equal-0-iff-equal* [*simp*]: $- a = 0 \longleftrightarrow a = 0$
by (*subst neg-equal-iff-equal* [*symmetric*]) *simp*

lemma *neg-0-equal-iff-equal* [*simp*]: $0 = - a \longleftrightarrow 0 = a$
by (*subst neg-equal-iff-equal* [*symmetric*]) *simp*

The next two equations can make the simplifier loop!

lemma *equation-minus-iff*: $a = - b \longleftrightarrow b = - a$
proof –
have $- (- a) = - b \longleftrightarrow - a = b$
by (*rule neg-equal-iff-equal*)
then show *?thesis*
by (*simp add*: *eq-commute*)
qed

lemma *minus-equation-iff*: $- a = b \longleftrightarrow - b = a$
proof –
have $- a = - (- b) \longleftrightarrow a = -b$
by (*rule neg-equal-iff-equal*)
then show *?thesis*

by (*simp add*: *eq-commute*)
qed

lemma *eq-neg-iff-add-eq-0*: $a = - b \longleftrightarrow a + b = 0$
proof
assume $a = - b$
then show $a + b = 0$ **by** *simp*
next
assume $a + b = 0$
moreover have $a + (b + - b) = (a + b) + - b$
by (*simp only*: *add.assoc*)
ultimately show $a = - b$
by *simp*
qed

lemma *add-eq-0-iff2*: $a + b = 0 \longleftrightarrow a = - b$
by (*fact eq-neg-iff-add-eq-0* [*symmetric*])

lemma *neg-eq-iff-add-eq-0*: $- a = b \longleftrightarrow a + b = 0$
by (*auto simp add*: *add-eq-0-iff2*)

lemma *add-eq-0-iff*: $a + b = 0 \longleftrightarrow b = - a$
by (*auto simp add*: *neg-eq-iff-add-eq-0* [*symmetric*])

lemma *minus-diff-eq* [*simp*]: $- (a - b) = b - a$
by (*simp only*: *neg-eq-iff-add-eq-0 diff-conv-add-uminus add.assoc minus-add-cancel*) *simp*

lemma *add-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a + (b - c) = (a + b) - c$
by (*simp only*: *diff-conv-add-uminus add.assoc*)

lemma *diff-add-eq-diff-diff-swap*: $a - (b + c) = a - c - b$
by (*simp only*: *diff-conv-add-uminus add.assoc minus-add*)

lemma *diff-eq-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - b = c \longleftrightarrow a = c + b$
by *auto*

lemma *eq-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a = c - b \longleftrightarrow a + b = c$
by *auto*

lemma *diff-diff-eq2* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - (b - c) = (a + c) - b$
by (*simp only*: *diff-conv-add-uminus add.assoc*) *simp*

lemma *diff-eq-diff-eq*: $a - b = c - d \Longrightarrow a = b \longleftrightarrow c = d$
by (*simp only*: *eq-iff-diff-eq-0* [*of a b*] *eq-iff-diff-eq-0* [*of c d*])

```
end

class ab-group-add = minus + uminus + comm-monoid-add +
  assumes ab-left-minus: - a + a = 0
  assumes ab-diff-conv-add-uminus: a - b = a + (- b)
begin

subclass group-add
  by standard (simp-all add: ab-left-minus ab-diff-conv-add-uminus)

subclass cancel-comm-monoid-add
proof
  fix a b c :: 'a
  have b + a - a = b
    by simp
  then show a + b - a = b
    by (simp add: ac-simps)
  show a - b - c = a - (b + c)
    by (simp add: algebra-simps)
qed

lemma uminus-add-conv-diff [simp]: - a + b = b - a
  by (simp add: add.commute)

lemma minus-add-distrib [simp]: - (a + b) = - a + - b
  by (simp add: algebra-simps)

lemma diff-add-eq [algebra-simps, algebra-split-simps, field-simps, field-split-simps]:
  (a - b) + c = (a + c) - b
  by (simp add: algebra-simps)

lemma minus-diff-commute:
  - b - a = - a - b
  by (simp only: diff-conv-add-uminus add.commute)

end
```

5.6 (Partially) Ordered Groups

The theory of partially ordered groups is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society, 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press, 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer

class *ordered-ab-semigroup-add* = *order* + *ab-semigroup-add* +
  assumes *add-left-mono*: $a \le b \Longrightarrow c + a \le c + b$
begin

lemma *add-right-mono*: $a \le b \Longrightarrow a + c \le b + c$
  by (*simp add*: *add.commute* [*of* - *c*] *add-left-mono*)

non-strict, in both arguments

lemma *add-mono*: $a \le b \Longrightarrow c \le d \Longrightarrow a + c \le b + d$
  by (*simp add*: *add.commute add-left-mono add-right-mono* [*THEN order-trans*])

end

Strict monotonicity in both arguments

class *strict-ordered-ab-semigroup-add* = *ordered-ab-semigroup-add* +
  assumes *add-strict-mono*: $a < b \Longrightarrow c < d \Longrightarrow a + c < b + d$

class *ordered-cancel-ab-semigroup-add* =
  ordered-ab-semigroup-add + *cancel-ab-semigroup-add*
begin

lemma *add-strict-left-mono*: $a < b \Longrightarrow c + a < c + b$
  by (*auto simp add*: *less-le add-left-mono*)

lemma *add-strict-right-mono*: $a < b \Longrightarrow a + c < b + c$
  by (*simp add*: *add.commute* [*of* - *c*] *add-strict-left-mono*)

subclass *strict-ordered-ab-semigroup-add*
proof
  show $\bigwedge a\ b\ c\ d.\ [\![a < b;\ c < d]\!] \Longrightarrow a + c < b + d$
    by (*iprover intro*: *add-strict-left-mono add-strict-right-mono less-trans*)
qed

lemma *add-less-le-mono*: $a < b \Longrightarrow c \le d \Longrightarrow a + c < b + d$
  by (*iprover intro*: *add-left-mono add-strict-right-mono less-le-trans*)

lemma *add-le-less-mono*: $a \le b \Longrightarrow c < d \Longrightarrow a + c < b + d$
  by (*iprover intro*: *add-strict-left-mono add-right-mono less-le-trans*)

end

class *ordered-ab-semigroup-add-imp-le* = *ordered-cancel-ab-semigroup-add* +
  assumes *add-le-imp-le-left*: $c + a \le c + b \Longrightarrow a \le b$
begin

lemma *add-less-imp-less-left*:
  assumes *less*: $c + a < c + b$
  shows $a < b$
proof –
  from *less* **have** *le*: $c + a \leq c + b$
    by (*simp add*: *order-le-less*)
  have $a \leq b$
    using *add-le-imp-le-left* [*OF le*] **.**
  moreover have $a \neq b$
  proof (*rule ccontr*)
    assume ¬ *?thesis*
    then have $a = b$ **by** *simp*
    then have $c + a = c + b$ **by** *simp*
    with *less* **show** *False* **by** *simp*
  qed
  ultimately show $a < b$
    by (*simp add*: *order-le-less*)
qed

lemma *add-less-imp-less-right*: $a + c < b + c \Longrightarrow a < b$
  by (*rule add-less-imp-less-left* [*of c*]) (*simp add*: *add.commute*)

lemma *add-less-cancel-left* [*simp*]: $c + a < c + b \longleftrightarrow a < b$
  by (*blast intro*: *add-less-imp-less-left add-strict-left-mono*)

lemma *add-less-cancel-right* [*simp*]: $a + c < b + c \longleftrightarrow a < b$
  by (*blast intro*: *add-less-imp-less-right add-strict-right-mono*)

lemma *add-le-cancel-left* [*simp*]: $c + a \leq c + b \longleftrightarrow a \leq b$
  by (*auto simp*: *dest*: *add-le-imp-le-left add-left-mono*)

lemma *add-le-cancel-right* [*simp*]: $a + c \leq b + c \longleftrightarrow a \leq b$
  by (*simp add*: *add.commute* [*of a c*] *add.commute* [*of b c*])

lemma *add-le-imp-le-right*: $a + c \leq b + c \Longrightarrow a \leq b$
  by *simp*

lemma *max-add-distrib-left*: *max* x y + z = *max* $(x + z)$ $(y + z)$
  unfolding *max-def* **by** *auto*

lemma *min-add-distrib-left*: *min* x y + z = *min* $(x + z)$ $(y + z)$
  unfolding *min-def* **by** *auto*

lemma *max-add-distrib-right*: x + *max* y z = *max* $(x + y)$ $(x + z)$
  unfolding *max-def* **by** *auto*

lemma *min-add-distrib-right*: x + *min* y z = *min* $(x + y)$ $(x + z)$
  unfolding *min-def* **by** *auto*

end

5.7 Support for reasoning about signs

class *ordered-comm-monoid-add* = *comm-monoid-add* + *ordered-ab-semigroup-add*
begin

lemma *add-nonneg-nonneg* [*simp*]: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a + b$
using *add-mono*[*of 0 a 0 b*] **by** *simp*

lemma *add-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow a + b \leq 0$
using *add-mono*[*of a 0 b 0*] **by** *simp*

lemma *add-nonneg-eq-0-iff*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
using *add-left-mono*[*of 0 y x*] *add-right-mono*[*of 0 x y*] **by** *auto*

lemma *add-nonpos-eq-0-iff*: $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
using *add-left-mono*[*of y 0 x*] *add-right-mono*[*of x 0 y*] **by** *auto*

lemma *add-increasing*: $0 \leq a \Longrightarrow b \leq c \Longrightarrow b \leq a + c$
using *add-mono* [*of 0 a b c*] **by** *simp*

lemma *add-increasing2*: $0 \leq c \Longrightarrow b \leq a \Longrightarrow b \leq a + c$
by (*simp add*: *add-increasing add.commute* [*of a*])

lemma *add-decreasing*: $a \leq 0 \Longrightarrow c \leq b \Longrightarrow a + c \leq b$
using *add-mono* [*of a 0 c b*] **by** *simp*

lemma *add-decreasing2*: $c \leq 0 \Longrightarrow a \leq b \Longrightarrow a + c \leq b$
using *add-mono*[*of a b c 0*] **by** *simp*

lemma *add-pos-nonneg*: $0 < a \Longrightarrow 0 \leq b \Longrightarrow 0 < a + b$
using *less-le-trans*[*of 0 a a + b*] **by** (*simp add*: *add-increasing2*)

lemma *add-pos-pos*: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a + b$
by (*intro add-pos-nonneg less-imp-le*)

lemma *add-nonneg-pos*: $0 \leq a \Longrightarrow 0 < b \Longrightarrow 0 < a + b$
using *add-pos-nonneg*[*of b a*] **by** (*simp add*: *add-commute*)

lemma *add-neg-nonpos*: $a < 0 \Longrightarrow b \leq 0 \Longrightarrow a + b < 0$
using *le-less-trans*[*of a + b a 0*] **by** (*simp add*: *add-decreasing2*)

lemma *add-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow a + b < 0$
by (*intro add-neg-nonpos less-imp-le*)

lemma *add-nonpos-neg*: $a \leq 0 \Longrightarrow b < 0 \Longrightarrow a + b < 0$
using *add-neg-nonpos*[*of b a*] **by** (*simp add*: *add-commute*)

lemmas *add-sign-intros* =
add-pos-nonneg add-pos-pos add-nonneg-pos add-nonneg-nonneg
add-neg-nonpos add-neg-neg add-nonpos-neg add-nonpos-nonpos

end

class *strict-ordered-comm-monoid-add* = *comm-monoid-add* + *strict-ordered-ab-semigroup-add*
begin

lemma *pos-add-strict*: $0 < a \Longrightarrow b < c \Longrightarrow b < a + c$
using *add-strict-mono* [*of 0 a b c*] **by** *simp*

end

class *ordered-cancel-comm-monoid-add* = *ordered-comm-monoid-add* + *cancel-ab-semigroup-add*
begin

subclass *ordered-cancel-ab-semigroup-add* **..**
subclass *strict-ordered-comm-monoid-add* **..**

lemma *add-strict-increasing*: $0 < a \Longrightarrow b \leq c \Longrightarrow b < a + c$
using *add-less-le-mono* [*of 0 a b c*] **by** *simp*

lemma *add-strict-increasing2*: $0 \leq a \Longrightarrow b < c \Longrightarrow b < a + c$
using *add-le-less-mono* [*of 0 a b c*] **by** *simp*

end

class *ordered-ab-semigroup-monoid-add-imp-le* = *monoid-add* + *ordered-ab-semigroup-add-imp-le*
begin

lemma *add-less-same-cancel1* [*simp*]: $b + a < b \longleftrightarrow a < 0$
using *add-less-cancel-left* [*of - - 0*] **by** *simp*

lemma *add-less-same-cancel2* [*simp*]: $a + b < b \longleftrightarrow a < 0$
using *add-less-cancel-right* [*of - - 0*] **by** *simp*

lemma *less-add-same-cancel1* [*simp*]: $a < a + b \longleftrightarrow 0 < b$
using *add-less-cancel-left* [*of - 0*] **by** *simp*

lemma *less-add-same-cancel2* [*simp*]: $a < b + a \longleftrightarrow 0 < b$
using *add-less-cancel-right* [*of 0*] **by** *simp*

lemma *add-le-same-cancel1* [*simp*]: $b + a \leq b \longleftrightarrow a \leq 0$
using *add-le-cancel-left* [*of - - 0*] **by** *simp*

lemma *add-le-same-cancel2* [*simp*]: $a + b \leq b \longleftrightarrow a \leq 0$
using *add-le-cancel-right* [*of - - 0*] **by** *simp*

lemma *le-add-same-cancel1* [*simp*]: $a \leq a + b \longleftrightarrow 0 \leq b$
  using *add-le-cancel-left* [*of - 0*] **by** *simp*

lemma *le-add-same-cancel2* [*simp*]: $a \leq b + a \longleftrightarrow 0 \leq b$
  using *add-le-cancel-right* [*of 0*] **by** *simp*

subclass *cancel-comm-monoid-add*
  by *standard auto*

subclass *ordered-cancel-comm-monoid-add*
  by *standard*

end

class *ordered-ab-group-add* = *ab-group-add* + *ordered-ab-semigroup-add*
begin

subclass *ordered-cancel-ab-semigroup-add* **..**

subclass *ordered-ab-semigroup-monoid-add-imp-le*
proof
  fix $a\ b\ c :: 'a$
  assume $c + a \leq c + b$
  then have $(-c) + (c + a) \leq (-c) + (c + b)$
    by (*rule add-left-mono*)
  then have $((-c) + c) + a \leq ((-c) + c) + b$
    by (*simp only*: *add.assoc*)
  then show $a \leq b$ **by** *simp*
qed

lemma *max-diff-distrib-left*: $max\ x\ y - z = max\ (x - z)\ (y - z)$
  using *max-add-distrib-left* [*of* $x\ y - z$] **by** *simp*

lemma *min-diff-distrib-left*: $min\ x\ y - z = min\ (x - z)\ (y - z)$
  using *min-add-distrib-left* [*of* $x\ y - z$] **by** *simp*

lemma *le-imp-neg-le*:
  assumes $a \leq b$
  shows $- b \leq - a$
proof −
  from *assms* **have** $- a + a \leq - a + b$
    by (*rule add-left-mono*)
  then have $0 \leq - a + b$
    by *simp*
  then have $0 + (- b) \leq (- a + b) + (- b)$
    by (*rule add-right-mono*)
  then show *?thesis*
    by (*simp add*: *algebra-simps*)
qed

lemma *neg-le-iff-le* [*simp*]: $- b \le - a \longleftrightarrow a \le b$
proof
  assume $- b \le - a$
  then have $- (- a) \le - (- b)$
    by (*rule le-imp-neg-le*)
  then show $a \le b$
    by *simp*
next
  assume $a \le b$
  then show $- b \le - a$
    by (*rule le-imp-neg-le*)
qed

lemma *neg-le-0-iff-le* [*simp*]: $- a \le 0 \longleftrightarrow 0 \le a$
  by (*subst neg-le-iff-le* [*symmetric*]) *simp*

lemma *neg-0-le-iff-le* [*simp*]: $0 \le - a \longleftrightarrow a \le 0$
  by (*subst neg-le-iff-le* [*symmetric*]) *simp*

lemma *neg-less-iff-less* [*simp*]: $- b < - a \longleftrightarrow a < b$
  by (*auto simp add*: *less-le*)

lemma *neg-less-0-iff-less* [*simp*]: $- a < 0 \longleftrightarrow 0 < a$
  by (*subst neg-less-iff-less* [*symmetric*]) *simp*

lemma *neg-0-less-iff-less* [*simp*]: $0 < - a \longleftrightarrow a < 0$
  by (*subst neg-less-iff-less* [*symmetric*]) *simp*

The next several equations can make the simplifier loop!

lemma *less-minus-iff*: $a < - b \longleftrightarrow b < - a$
proof –
  have $- (- a) < - b \longleftrightarrow b < - a$
    by (*rule neg-less-iff-less*)
  then show *?thesis* **by** *simp*
qed

lemma *minus-less-iff*: $- a < b \longleftrightarrow - b < a$
proof –
  have $- a < - (- b) \longleftrightarrow - b < a$
    by (*rule neg-less-iff-less*)
  then show *?thesis* **by** *simp*
qed

lemma *le-minus-iff*: $a \le - b \longleftrightarrow b \le - a$
  by (*auto simp*: *order.order-iff-strict less-minus-iff*)

lemma *minus-le-iff*: $- a \le b \longleftrightarrow - b \le a$
  by (*auto simp add*: *le-less minus-less-iff*)

lemma *diff-less-0-iff-less* [*simp*]: $a - b < 0 \longleftrightarrow a < b$
proof –
  have $a - b < 0 \longleftrightarrow a + (- b) < b + (- b)$
    by *simp*
  also have $\ldots \longleftrightarrow a < b$
    by (*simp only*: *add-less-cancel-right*)
  finally show *?thesis* **.**
qed

lemmas *less-iff-diff-less-0* = *diff-less-0-iff-less* [*symmetric*]

lemma *diff-less-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
  $a - b < c \longleftrightarrow a < c + b$
proof (*subst less-iff-diff-less-0* [*of a*])
  show $(a - b < c) = (a - (c + b) < 0)$
    by (*simp add*: *algebra-simps less-iff-diff-less-0* [*of - c*])
qed

lemma *less-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
  $a < c - b \longleftrightarrow a + b < c$
proof (*subst less-iff-diff-less-0* [*of a + b*])
  show $(a < c - b) = (a + b - c < 0)$
    by (*simp add*: *algebra-simps less-iff-diff-less-0* [*of a*])
qed

lemma *diff-gt-0-iff-gt* [*simp*]: $a - b > 0 \longleftrightarrow a > b$
  by (*simp add*: *less-diff-eq*)

lemma *diff-le-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
  $a - b \leq c \longleftrightarrow a \leq c + b$
  by (*auto simp add*: *le-less diff-less-eq*)

lemma *le-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
  $a \leq c - b \longleftrightarrow a + b \leq c$
  by (*auto simp add*: *le-less less-diff-eq*)

lemma *diff-le-0-iff-le* [*simp*]: $a - b \leq 0 \longleftrightarrow a \leq b$
  by (*simp add*: *algebra-simps*)

lemmas *le-iff-diff-le-0* = *diff-le-0-iff-le* [*symmetric*]

lemma *diff-ge-0-iff-ge* [*simp*]: $a - b \geq 0 \longleftrightarrow a \geq b$
  by (*simp add*: *le-diff-eq*)

lemma *diff-eq-diff-less*: $a - b = c - d \Longrightarrow a < b \longleftrightarrow c < d$
  by (*auto simp only*: *less-iff-diff-less-0* [*of a b*] *less-iff-diff-less-0* [*of c d*])

lemma *diff-eq-diff-less-eq*: $a - b = c - d \Longrightarrow a \leq b \longleftrightarrow c \leq d$

by (*auto simp only*: *le-iff-diff-le-0* [*of a b*] *le-iff-diff-le-0* [*of c d*])

lemma *diff-mono*: $a \le b \Longrightarrow d \le c \Longrightarrow a - c \le b - d$
by (*simp add*: *field-simps add-mono*)

lemma *diff-left-mono*: $b \le a \Longrightarrow c - a \le c - b$
by (*simp add*: *field-simps*)

lemma *diff-right-mono*: $a \le b \Longrightarrow a - c \le b - c$
by (*simp add*: *field-simps*)

lemma *diff-strict-mono*: $a < b \Longrightarrow d < c \Longrightarrow a - c < b - d$
by (*simp add*: *field-simps add-strict-mono*)

lemma *diff-strict-left-mono*: $b < a \Longrightarrow c - a < c - b$
by (*simp add*: *field-simps*)

lemma *diff-strict-right-mono*: $a < b \Longrightarrow a - c < b - c$
by (*simp add*: *field-simps*)

end

locale *group-cancel*
begin

lemma *add1*: $(A::'a::comm\text{-}monoid\text{-}add) \equiv k + a \Longrightarrow A + b \equiv k + (a + b)$
by (*simp only*: *ac-simps*)

lemma *add2*: $(B::'a::comm\text{-}monoid\text{-}add) \equiv k + b \Longrightarrow a + B \equiv k + (a + b)$
by (*simp only*: *ac-simps*)

lemma *sub1*: $(A::'a::ab\text{-}group\text{-}add) \equiv k + a \Longrightarrow A - b \equiv k + (a - b)$
by (*simp only*: *add-diff-eq*)

lemma *sub2*: $(B::'a::ab\text{-}group\text{-}add) \equiv k + b \Longrightarrow a - B \equiv - k + (a - b)$
by (*simp only*: *minus-add diff-conv-add-uminus ac-simps*)

lemma *neg1*: $(A::'a::ab\text{-}group\text{-}add) \equiv k + a \Longrightarrow - A \equiv - k + - a$
by (*simp only*: *minus-add-distrib*)

lemma *rule0*: $(a::'a::comm\text{-}monoid\text{-}add) \equiv a + 0$
by (*simp only*: *add-0-right*)

end

ML-file ‹*Tools/group-cancel.ML*›

simproc-setup *group-cancel-add* ($a + b::'a::ab\text{-}group\text{-}add$) =
‹*fn phi => fn ss => try Group-Cancel.cancel-add-conv*›

simproc-setup *group-cancel-diff* $(a - b::'a::ab\text{-}group\text{-}add) =$
‹*fn phi => fn ss => try Group-Cancel.cancel-diff-conv*›

simproc-setup *group-cancel-eq* $(a = (b::'a::ab\text{-}group\text{-}add)) =$
‹*fn phi => fn ss => try Group-Cancel.cancel-eq-conv*›

simproc-setup *group-cancel-le* $(a \leq (b::'a::ordered\text{-}ab\text{-}group\text{-}add)) =$
‹*fn phi => fn ss => try Group-Cancel.cancel-le-conv*›

simproc-setup *group-cancel-less* $(a < (b::'a::ordered\text{-}ab\text{-}group\text{-}add)) =$
‹*fn phi => fn ss => try Group-Cancel.cancel-less-conv*›

class *linordered-ab-semigroup-add* =
linorder + *ordered-ab-semigroup-add*

class *linordered-cancel-ab-semigroup-add* =
linorder + *ordered-cancel-ab-semigroup-add*
begin

subclass *linordered-ab-semigroup-add* **..**

subclass *ordered-ab-semigroup-add-imp-le*
proof
  fix $a\ b\ c :: 'a$
  assume *le1*: $c + a \leq c + b$
  show $a \leq b$
  proof (*rule ccontr*)
    assume ∗: ¬ *?thesis*
    then have $b \leq a$ **by** (*simp add*: *linorder-not-le*)
    then have $c + b \leq c + a$ **by** (*rule add-left-mono*)
    then have $c + a = c + b$
      using *le1* **by** (*iprover intro*: *order.antisym*)
    then have $a = b$
      by *simp*
    with ∗ **show** *False*
      by (*simp add*: *linorder-not-le* [*symmetric*])
  qed
qed

end

class *linordered-ab-group-add* = *linorder* + *ordered-ab-group-add*
begin

subclass *linordered-cancel-ab-semigroup-add* **..**

lemma *equal-neg-zero* [*simp*]: $a = - a \longleftrightarrow a = 0$
proof

assume $a = 0$
then show $a = - a$ **by** *simp*
next
assume *A*: $a = - a$
show $a = 0$
proof (*cases* $0 \leq a$)
case *True*
with *A* **have** $0 \leq - a$ **by** *auto*
with *le-minus-iff* **have** $a \leq 0$ **by** *simp*
with *True* **show** *?thesis* **by** (*auto intro*: *order-trans*)
next
case *False*
then have *B*: $a \leq 0$ **by** *auto*
with *A* **have** $- a \leq 0$ **by** *auto*
with *B* **show** *?thesis* **by** (*auto intro*: *order-trans*)
qed
qed

lemma *neg-equal-zero* [*simp*]: $- a = a \longleftrightarrow a = 0$
by (*auto dest*: *sym*)

lemma *neg-less-eq-nonneg* [*simp*]: $- a \leq a \longleftrightarrow 0 \leq a$
proof
assume $*$: $- a \leq a$
show $0 \leq a$
proof (*rule classical*)
assume $\neg$ *?thesis*
then have $a < 0$ **by** *auto*
with $*$ **have** $- a < 0$ **by** (*rule le-less-trans*)
then show *?thesis* **by** *auto*
qed
next
assume $*$: $0 \leq a$
then have $- a \leq 0$ **by** (*simp add*: *minus-le-iff*)
from *this* $*$ **show** $- a \leq a$ **by** (*rule order-trans*)
qed

lemma *neg-less-pos* [*simp*]: $- a < a \longleftrightarrow 0 < a$
by (*auto simp add*: *less-le*)

lemma *less-eq-neg-nonpos* [*simp*]: $a \leq - a \longleftrightarrow a \leq 0$
using *neg-less-eq-nonneg* [*of* $- a$] **by** *simp*

lemma *less-neg-neg* [*simp*]: $a < - a \longleftrightarrow a < 0$
using *neg-less-pos* [*of* $- a$] **by** *simp*

lemma *double-zero* [*simp*]: $a + a = 0 \longleftrightarrow a = 0$
proof
assume $a + a = 0$

then have *a*: − *a* = *a* **by** (*rule minus-unique*)
then show *a* = *0* **by** (*simp only*: *neg-equal-zero*)
next
assume *a* = *0*
then show *a* + *a* = *0* **by** *simp*
qed

lemma *double-zero-sym* [*simp*]: *0* = *a* + *a* ⟷ *a* = *0*
using *double-zero* [*of a*] **by** (*simp only*: *eq-commute*)

lemma *zero-less-double-add-iff-zero-less-single-add* [*simp*]: *0* < *a* + *a* ⟷ *0* < *a*
proof
assume *0* < *a* + *a*
then have *0* − *a* < *a* **by** (*simp only*: *diff-less-eq*)
then have − *a* < *a* **by** *simp*
then show *0* < *a* **by** *simp*
next
assume *0* < *a*
with *this* **have** *0* + *0* < *a* + *a*
by (*rule add-strict-mono*)
then show *0* < *a* + *a* **by** *simp*
qed

lemma *zero-le-double-add-iff-zero-le-single-add* [*simp*]: *0* ≤ *a* + *a* ⟷ *0* ≤ *a*
by (*auto simp add*: *le-less*)

lemma *double-add-less-zero-iff-single-add-less-zero* [*simp*]: *a* + *a* < *0* ⟷ *a* < *0*
proof −
have ¬ *a* + *a* < *0* ⟷ ¬ *a* < *0*
by (*simp add*: *not-less*)
then show *?thesis* **by** *simp*
qed

lemma *double-add-le-zero-iff-single-add-le-zero* [*simp*]: *a* + *a* ≤ *0* ⟷ *a* ≤ *0*
proof −
have ¬ *a* + *a* ≤ *0* ⟷ ¬ *a* ≤ *0*
by (*simp add*: *not-le*)
then show *?thesis* **by** *simp*
qed

lemma *minus-max-eq-min*: − *max x y* = *min* (− *x*) (− *y*)
by (*auto simp add*: *max-def min-def*)

lemma *minus-min-eq-max*: − *min x y* = *max* (− *x*) (− *y*)
by (*auto simp add*: *max-def min-def*)

end

class *abs* =

```
  fixes abs :: 'a ⇒ 'a  (‹(‹open-block notation=‹mixfix abs››|-|)›)

bundle abs-syntax
begin
notation abs  (‹(‹open-block notation=‹mixfix abs››|-|)›)
end

class sgn =
  fixes sgn :: 'a ⇒ 'a

class ordered-ab-group-add-abs = ordered-ab-group-add + abs +
  assumes abs-ge-zero [simp]: |a| ≥ 0
    and abs-ge-self: a ≤ |a|
    and abs-leI: a ≤ b ⟹ − a ≤ b ⟹ |a| ≤ b
    and abs-minus-cancel [simp]: |−a| = |a|
    and abs-triangle-ineq: |a + b| ≤ |a| + |b|
begin

lemma abs-minus-le-zero: − |a| ≤ 0
  unfolding neg-le-0-iff-le by simp

lemma abs-of-nonneg [simp]:
  assumes nonneg: 0 ≤ a
  shows |a| = a
proof (rule order.antisym)
  show a ≤ |a| by (rule abs-ge-self)
  from nonneg le-imp-neg-le have − a ≤ 0 by simp
  from this nonneg have − a ≤ a by (rule order-trans)
  then show |a| ≤ a by (auto intro: abs-leI)
qed

lemma abs-idempotent [simp]: ||a|| = |a|
  by (rule order.antisym) (auto intro!: abs-ge-self abs-leI order-trans [of − |a| 0
|a|])

lemma abs-eq-0 [simp]: |a| = 0 ⟷ a = 0
proof −
  have |a| = 0 ⟹ a = 0
  proof (rule order.antisym)
    assume zero: |a| = 0
    with abs-ge-self show a ≤ 0 by auto
    from zero have |−a| = 0 by simp
    with abs-ge-self [of − a] have − a ≤ 0 by auto
    with neg-le-0-iff-le show 0 ≤ a by auto
  qed
  then show ?thesis by auto
qed

lemma abs-zero [simp]: |0| = 0
```

by *simp*

lemma *abs-0-eq* [*simp*]: $0 = |a| \longleftrightarrow a = 0$
proof −
  have $0 = |a| \longleftrightarrow |a| = 0$ **by** (*simp only*: *eq-ac*)
  then show *?thesis* **by** *simp*
qed

lemma *abs-le-zero-iff* [*simp*]: $|a| \leq 0 \longleftrightarrow a = 0$
proof
  assume $|a| \leq 0$
  then have $|a| = 0$ **by** (*rule order.antisym*) *simp*
  then show $a = 0$ **by** *simp*
next
  assume $a = 0$
  then show $|a| \leq 0$ **by** *simp*
qed

lemma *abs-le-self-iff* [*simp*]: $|a| \leq a \longleftrightarrow 0 \leq a$
proof −
  have $0 \leq |a|$
    using *abs-ge-zero* **by** *blast*
  then have $|a| \leq a \Longrightarrow 0 \leq a$
    using *order.trans* **by** *blast*
  then show *?thesis*
    using *abs-of-nonneg eq-refl* **by** *blast*
qed

lemma *zero-less-abs-iff* [*simp*]: $0 < |a| \longleftrightarrow a \neq 0$
  by (*simp add*: *less-le*)

lemma *abs-not-less-zero* [*simp*]: $\neg\ |a| < 0$
proof −
  have $x \leq y \Longrightarrow \neg\ y < x$ **for** $x\ y$ **by** *auto*
  then show *?thesis* **by** *simp*
qed

lemma *abs-ge-minus-self*: $-\ a \leq |a|$
proof −
  have $-\ a \leq |-a|$ **by** (*rule abs-ge-self*)
  then show *?thesis* **by** *simp*
qed

lemma *abs-minus-commute*: $|a - b| = |b - a|$
proof −
  have $|a - b| = |-\ (a - b)|$
    by (*simp only*: *abs-minus-cancel*)
  also have $\ldots = |b - a|$ **by** *simp*
  finally show *?thesis* **.**

qed

lemma *abs-of-pos*: $0 < a \Longrightarrow |a| = a$
  by (*rule abs-of-nonneg*) (*rule less-imp-le*)

lemma *abs-of-nonpos* [*simp*]:
  assumes $a \leq 0$
  shows $|a| = - a$
proof −
  let $?b = - a$
  have $- ?b \leq 0 \Longrightarrow |- ?b| = - (- ?b)$
    unfolding *abs-minus-cancel* [*of ?b*]
    unfolding *neg-le-0-iff-le* [*of ?b*]
    unfolding *minus-minus* **by** (*erule abs-of-nonneg*)
  then show *?thesis* **using** *assms* **by** *auto*
qed

lemma *abs-of-neg*: $a < 0 \Longrightarrow |a| = - a$
  by (*rule abs-of-nonpos*) (*rule less-imp-le*)

lemma *abs-le-D1*: $|a| \leq b \Longrightarrow a \leq b$
  using *abs-ge-self* **by** (*blast intro*: *order-trans*)

lemma *abs-le-D2*: $|a| \leq b \Longrightarrow - a \leq b$
  using *abs-le-D1* [*of* − *a*] **by** *simp*

lemma *abs-le-iff*: $|a| \leq b \longleftrightarrow a \leq b \land - a \leq b$
  by (*blast intro*: *abs-leI dest*: *abs-le-D1 abs-le-D2*)

lemma *abs-triangle-ineq2*: $|a| - |b| \leq |a - b|$
proof −
  have $|a| = |b + (a - b)|$
    by (*simp add*: *algebra-simps*)
  then have $|a| \leq |b| + |a - b|$
    by (*simp add*: *abs-triangle-ineq*)
  then show *?thesis*
    by (*simp add*: *algebra-simps*)
qed

lemma *abs-triangle-ineq2-sym*: $|a| - |b| \leq |b - a|$
  by (*simp only*: *abs-minus-commute* [*of b*] *abs-triangle-ineq2*)

lemma *abs-triangle-ineq3*: $||a| - |b|| \leq |a - b|$
  by (*simp add*: *abs-le-iff abs-triangle-ineq2 abs-triangle-ineq2-sym*)

lemma *abs-triangle-ineq4*: $|a - b| \leq |a| + |b|$
proof −
  have $|a - b| = |a + - b|$
    by (*simp add*: *algebra-simps*)

```
  also have ... ≤ |a| + |− b|
    by (rule abs-triangle-ineq)
  finally show ?thesis by simp
qed

lemma abs-diff-triangle-ineq: |a + b − (c + d)| ≤ |a − c| + |b − d|
proof −
  have |a + b − (c + d)| = |(a − c) + (b − d)|
    by (simp add: algebra-simps)
  also have ... ≤ |a − c| + |b − d|
    by (rule abs-triangle-ineq)
  finally show ?thesis .
qed

lemma abs-add-abs [simp]: ||a| + |b|| = |a| + |b|
  (is ?L = ?R)
proof (rule order.antisym)
  show ?L ≥ ?R by (rule abs-ge-self)
  have ?L ≤ ||a|| + ||b|| by (rule abs-triangle-ineq)
  also have ... = ?R by simp
  finally show ?L ≤ ?R .
qed

end

lemma dense-eq0-I:
  fixes x::'a::{dense-linorder,ordered-ab-group-add-abs}
  assumes ⋀e. 0 < e ⟹ |x| ≤ e
  shows x = 0
proof (cases |x| = 0)
  case False
  then have |x| > 0
    by simp
  then obtain z where 0 < z z < |x|
    using dense by force
  then show ?thesis
    using assms by (simp flip: not-less)
qed auto

hide-fact (open) ab-diff-conv-add-uminus add-0 mult-1 ab-left-minus

lemmas add-0 = add-0-left
lemmas mult-1 = mult-1-left
lemmas ab-left-minus = left-minus
lemmas diff-diff-eq = diff-diff-add
```

5.8 Canonically ordered monoids

Canonically ordered monoids are never groups.

class *canonically-ordered-monoid-add* = *comm-monoid-add* + *order* +
assumes *le-iff-add*: $a \leq b \longleftrightarrow (\exists\, c.\ b = a + c)$
begin

lemma *zero-le*[*simp*]: $0 \leq x$
by (*auto simp*: *le-iff-add*)

lemma *le-zero-eq*[*simp*]: $n \leq 0 \longleftrightarrow n = 0$
by (*auto intro*: *order.antisym*)

lemma *not-less-zero*[*simp*]: $\neg\ n < 0$
by (*auto simp*: *less-le*)

lemma *zero-less-iff-neq-zero*: $0 < n \longleftrightarrow n \neq 0$
by (*auto simp*: *less-le*)

This theorem is useful with *blast*

lemma *gr-zeroI*: $(n = 0 \Longrightarrow \mathit{False}) \Longrightarrow 0 < n$
by (*rule zero-less-iff-neq-zero*[*THEN iffD2*]) *iprover*

lemma *not-gr-zero*[*simp*]: $\neg\ 0 < n \longleftrightarrow n = 0$
by (*simp add*: *zero-less-iff-neq-zero*)

subclass *ordered-comm-monoid-add*
proof qed (*auto simp*: *le-iff-add add-ac*)

lemma *gr-implies-not-zero*: $m < n \Longrightarrow n \neq 0$
by *auto*

lemma *add-eq-0-iff-both-eq-0*[*simp*]: $x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
by (*intro add-nonneg-eq-0-iff zero-le*)

lemma *zero-eq-add-iff-both-eq-0*[*simp*]: $0 = x + y \longleftrightarrow x = 0 \wedge y = 0$
using *add-eq-0-iff-both-eq-0*[*of x y*] **unfolding** *eq-commute*[*of 0*] **.**

lemma *less-eqE*:
assumes ‹$a \leq b$›
obtains *c* **where** ‹$b = a + c$›
using *assms* **by** (*auto simp add*: *le-iff-add*)

lemma *lessE*:
assumes ‹$a < b$›
obtains *c* **where** ‹$b = a + c$› **and** ‹$c \neq 0$›
proof −
from *assms* **have** ‹$a \leq b$› ‹$a \neq b$›
by *simp-all*
from ‹$a \leq b$› **obtain** *c* **where** ‹$b = a + c$›
by (*rule less-eqE*)
moreover have ‹$c \neq 0$› **using** ‹$a \neq b$› ‹$b = a + c$›

by *auto*
ultimately show *?thesis*
by (*rule that*)
qed

lemmas *zero-order* = *zero-le le-zero-eq not-less-zero zero-less-iff-neq-zero not-gr-zero*
— This should be attributed with [*iff*], but then *blast* fails in *Set*.

end

class *ordered-cancel-comm-monoid-diff* =
canonically-ordered-monoid-add + *comm-monoid-diff* + *ordered-ab-semigroup-add-imp-le*
begin

context
fixes $a\ b :: 'a$
assumes *le*: $a \leq b$
begin

lemma *add-diff-inverse*: $a + (b - a) = b$
using *le* **by** (*auto simp add*: *le-iff-add*)

lemma *add-diff-assoc*: $c + (b - a) = c + b - a$
using *le* **by** (*auto simp add*: *le-iff-add add.left-commute* [*of c*])

lemma *add-diff-assoc2*: $b - a + c = b + c - a$
using *le* **by** (*auto simp add*: *le-iff-add add.assoc*)

lemma *diff-add-assoc*: $c + b - a = c + (b - a)$
using *le* **by** (*simp add*: *add.commute add-diff-assoc*)

lemma *diff-add-assoc2*: $b + c - a = b - a + c$
using *le* **by** (*simp add*: *add.commute add-diff-assoc*)

lemma *diff-diff-right*: $c - (b - a) = c + a - b$
by (*simp add*: *add-diff-inverse add-diff-cancel-left* [*of* a c $b - a$, *symmetric*] *add.commute*)

lemma *diff-add*: $b - a + a = b$
by (*simp add*: *add.commute add-diff-inverse*)

lemma *le-add-diff*: $c \leq b + c - a$
by (*auto simp add*: *add.commute diff-add-assoc2 le-iff-add*)

lemma *le-imp-diff-is-add*: $a \leq b \Longrightarrow b - a = c \longleftrightarrow b = c + a$
by (*auto simp add*: *add.commute add-diff-inverse*)

lemma *le-diff-conv2*: $c \leq b - a \longleftrightarrow c + a \leq b$
(**is** *?P* $\longleftrightarrow$ *?Q*)

proof
  assume *?P*
  then have $c + a \le b - a + a$
    by (*rule add-right-mono*)
  then show *?Q*
    by (*simp add*: *add-diff-inverse add.commute*)
next
  assume *?Q*
  then have $a + c \le a + (b - a)$
    by (*simp add*: *add-diff-inverse add.commute*)
  then show *?P* **by** *simp*
qed

end

end

5.9 Tools setup

lemma *add-mono-thms-linordered-semiring*:
  fixes *i j k* :: *'a*::*ordered-ab-semigroup-add*
  shows $i \le j \wedge k \le l \Longrightarrow i + k \le j + l$
    and $i = j \wedge k \le l \Longrightarrow i + k \le j + l$
    and $i \le j \wedge k = l \Longrightarrow i + k \le j + l$
    and $i = j \wedge k = l \Longrightarrow i + k = j + l$
  by (*rule add-mono*, *clarify*+)+

lemma *add-mono-thms-linordered-field*:
  fixes *i j k* :: *'a*::*ordered-cancel-ab-semigroup-add*
  shows $i < j \wedge k = l \Longrightarrow i + k < j + l$
    and $i = j \wedge k < l \Longrightarrow i + k < j + l$
    and $i < j \wedge k \le l \Longrightarrow i + k < j + l$
    and $i \le j \wedge k < l \Longrightarrow i + k < j + l$
    and $i < j \wedge k < l \Longrightarrow i + k < j + l$
  by (*auto intro*: *add-strict-right-mono add-strict-left-mono*
      add-less-le-mono add-le-less-mono add-strict-mono)

code-identifier
  code-module *Groups* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

6 Abstract lattices

theory *Lattices*
imports *Groups*
begin

6.1 Abstract semilattice

These locales provide a basic structure for interpretation into bigger structures; extensions require careful thinking, otherwise undesired effects may occur due to interpretation.

```
locale semilattice = abel-semigroup +
  assumes idem [simp]: a ∗ a = a
begin

lemma left-idem [simp]: a ∗ (a ∗ b) = a ∗ b
  by (simp add: assoc [symmetric])

lemma right-idem [simp]: (a ∗ b) ∗ b = a ∗ b
  by (simp add: assoc)

end

locale semilattice-neutr = semilattice + comm-monoid

locale semilattice-order = semilattice +
  fixes less-eq :: 'a ⇒ 'a ⇒ bool (infix ‹≤› 50)
    and less :: 'a ⇒ 'a ⇒ bool (infix ‹<› 50)
  assumes order-iff: a ≤ b ⟷ a = a ∗ b
    and strict-order-iff: a < b ⟷ a = a ∗ b ∧ a ≠ b
begin

lemma orderI: a = a ∗ b ⟹ a ≤ b
  by (simp add: order-iff)

lemma orderE:
  assumes a ≤ b
  obtains a = a ∗ b
  using assms by (unfold order-iff)

sublocale ordering less-eq less
proof
  show a < b ⟷ a ≤ b ∧ a ≠ b for a b
    by (simp add: order-iff strict-order-iff)
next
  show a ≤ a for a
    by (simp add: order-iff)
next
  fix a b
  assume a ≤ b b ≤ a
  then have a = a ∗ b a ∗ b = b
    by (simp-all add: order-iff commute)
  then show a = b by simp
next
  fix a b c
```

assume $a \leq b$ $b \leq c$
then have $a = a * b$ $b = b * c$
by (*simp-all add*: *order-iff commute*)
then have $a = a * (b * c)$
by *simp*
then have $a = (a * b) * c$
by (*simp add*: *assoc*)
with ‹$a = a * b$› [*symmetric*] **have** $a = a * c$ **by** *simp*
then show $a \leq c$ **by** (*rule orderI*)
qed

lemma *cobounded1* [*simp*]: $a * b \leq a$
by (*simp add*: *order-iff commute*)

lemma *cobounded2* [*simp*]: $a * b \leq b$
by (*simp add*: *order-iff*)

lemma *boundedI*:
assumes $a \leq b$ **and** $a \leq c$
shows $a \leq b * c$
proof (*rule orderI*)
from *assms* **obtain** $a * b = a$ **and** $a * c = a$
by (*auto elim*!: *orderE*)
then show $a = a * (b * c)$
by (*simp add*: *assoc* [*symmetric*])
qed

lemma *boundedE*:
assumes $a \leq b * c$
obtains $a \leq b$ **and** $a \leq c$
using *assms* **by** (*blast intro*: *trans cobounded1 cobounded2*)

lemma *bounded-iff* [*simp*]: $a \leq b * c \longleftrightarrow a \leq b \wedge a \leq c$
by (*blast intro*: *boundedI elim*: *boundedE*)

lemma *strict-boundedE*:
assumes $a < b * c$
obtains $a < b$ **and** $a < c$
using *assms* **by** (*auto simp add*: *commute strict-iff-order elim*: *orderE intro*!: *that*)+

lemma *coboundedI1*: $a \leq c \Longrightarrow a * b \leq c$
by (*rule trans*) *auto*

lemma *coboundedI2*: $b \leq c \Longrightarrow a * b \leq c$
by (*rule trans*) *auto*

lemma *strict-coboundedI1*: $a < c \Longrightarrow a * b < c$
using *irrefl*

by (*auto intro*: *not-eq-order-implies-strict coboundedI1 strict-implies-order elim*: *strict-boundedE*)

lemma *strict-coboundedI2*: $b < c \Longrightarrow a * b < c$
using *strict-coboundedI1* [*of* b c a] **by** (*simp add*: *commute*)

lemma *mono*: $a \leq c \Longrightarrow b \leq d \Longrightarrow a * b \leq c * d$
by (*blast intro*: *boundedI coboundedI1 coboundedI2*)

lemma *absorb1*: $a \leq b \Longrightarrow a * b = a$
by (*rule antisym*) (*auto simp*: *refl*)

lemma *absorb2*: $b \leq a \Longrightarrow a * b = b$
by (*rule antisym*) (*auto simp*: *refl*)

lemma *absorb3*: $a < b \Longrightarrow a * b = a$
by (*rule absorb1*) (*rule strict-implies-order*)

lemma *absorb4*: $b < a \Longrightarrow a * b = b$
by (*rule absorb2*) (*rule strict-implies-order*)

lemma *absorb-iff1*: $a \leq b \longleftrightarrow a * b = a$
using *order-iff* **by** *auto*

lemma *absorb-iff2*: $b \leq a \longleftrightarrow a * b = b$
using *order-iff* **by** (*auto simp add*: *commute*)

end

locale *semilattice-neutr-order* = *semilattice-neutr* + *semilattice-order*
begin

sublocale *ordering-top less-eq less* **1**
by *standard* (*simp add*: *order-iff*)

lemma *eq-neutr-iff* [*simp*]: ‹$a * b = \mathbf{1} \longleftrightarrow a = \mathbf{1} \wedge b = \mathbf{1}$›
by (*simp add*: *eq-iff*)

lemma *neutr-eq-iff* [*simp*]: ‹$\mathbf{1} = a * b \longleftrightarrow a = \mathbf{1} \wedge b = \mathbf{1}$›
by (*simp add*: *eq-iff*)

end

Interpretations for boolean operators

interpretation *conj*: *semilattice-neutr* ‹($\wedge$)› *True*
by *standard auto*

interpretation *disj*: *semilattice-neutr* ‹($\vee$)› *False*
by *standard auto*

declare *conj-assoc* [*ac-simps del*] *disj-assoc* [*ac-simps del*] — already simp by default

6.2 Syntactic infimum and supremum operations

class *inf* =
  fixes *inf* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** ‹⊓› *70*)

class *sup* =
  fixes *sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** ‹⊔› *65*)

6.3 Concrete lattices

class *semilattice-inf* = *order* + *inf* +
  assumes *inf-le1* [*simp*]: $x \sqcap y \leq x$
  and *inf-le2* [*simp*]: $x \sqcap y \leq y$
  and *inf-greatest*: $x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \sqcap z$

class *semilattice-sup* = *order* + *sup* +
  assumes *sup-ge1* [*simp*]: $x \leq x \sqcup y$
  and *sup-ge2* [*simp*]: $y \leq x \sqcup y$
  and *sup-least*: $y \leq x \Longrightarrow z \leq x \Longrightarrow y \sqcup z \leq x$
begin

Dual lattice.

lemma *dual-semilattice*: *class.semilattice-inf sup greater-eq greater*
  by (*rule class.semilattice-inf.intro*, *rule dual-order*)
    (*unfold-locales*, *simp-all add*: *sup-least*)

end

class *lattice* = *semilattice-inf* + *semilattice-sup*

6.3.1 Intro and elim rules

context *semilattice-inf*
begin

lemma *le-infI1*: $a \leq x \Longrightarrow a \sqcap b \leq x$
  by (*rule order-trans*) *auto*

lemma *le-infI2*: $b \leq x \Longrightarrow a \sqcap b \leq x$
  by (*rule order-trans*) *auto*

lemma *le-infI*: $x \leq a \Longrightarrow x \leq b \Longrightarrow x \leq a \sqcap b$
  by (*fact inf-greatest*)

lemma *le-infE*: $x \leq a \sqcap b \Longrightarrow (x \leq a \Longrightarrow x \leq b \Longrightarrow P) \Longrightarrow P$
  by (*blast intro*: *order-trans inf-le1 inf-le2*)

lemma *le-inf-iff*: $x \leq y \sqcap z \longleftrightarrow x \leq y \wedge x \leq z$
by (*blast intro*: *le-infI elim*: *le-infE*)

lemma *le-iff-inf*: $x \leq y \longleftrightarrow x \sqcap y = x$
by (*auto intro*: *le-infI1 order.antisym dest*: *order.eq-iff* [*THEN iffD1*] *simp add*: *le-inf-iff*)

lemma *inf-mono*: $a \leq c \Longrightarrow b \leq d \Longrightarrow a \sqcap b \leq c \sqcap d$
by (*fast intro*: *inf-greatest le-infI1 le-infI2*)

end

context *semilattice-sup*
begin

lemma *le-supI1*: $x \leq a \Longrightarrow x \leq a \sqcup b$
by (*rule order-trans*) *auto*

lemma *le-supI2*: $x \leq b \Longrightarrow x \leq a \sqcup b$
by (*rule order-trans*) *auto*

lemma *le-supI*: $a \leq x \Longrightarrow b \leq x \Longrightarrow a \sqcup b \leq x$
by (*fact sup-least*)

lemma *le-supE*: $a \sqcup b \leq x \Longrightarrow (a \leq x \Longrightarrow b \leq x \Longrightarrow P) \Longrightarrow P$
by (*blast intro*: *order-trans sup-ge1 sup-ge2*)

lemma *le-sup-iff*: $x \sqcup y \leq z \longleftrightarrow x \leq z \wedge y \leq z$
by (*blast intro*: *le-supI elim*: *le-supE*)

lemma *le-iff-sup*: $x \leq y \longleftrightarrow x \sqcup y = y$
by (*auto intro*: *le-supI2 order.antisym dest*: *order.eq-iff* [*THEN iffD1*] *simp add*: *le-sup-iff*)

lemma *sup-mono*: $a \leq c \Longrightarrow b \leq d \Longrightarrow a \sqcup b \leq c \sqcup d$
by (*fast intro*: *sup-least le-supI1 le-supI2*)

end

6.3.2 Equational laws

context *semilattice-inf*
begin

sublocale *inf*: *semilattice inf*
proof
fix $a\ b\ c$
show $(a \sqcap b) \sqcap c = a \sqcap (b \sqcap c)$

by (*rule order.antisym*) (*auto intro*: *le-infI1 le-infI2 simp add*: *le-inf-iff*)
show $a \sqcap b = b \sqcap a$
by (*rule order.antisym*) (*auto simp add*: *le-inf-iff*)
show $a \sqcap a = a$
by (*rule order.antisym*) (*auto simp add*: *le-inf-iff*)
qed

sublocale *inf*: *semilattice-order inf less-eq less*
by *standard* (*auto simp add*: *le-iff-inf less-le*)

lemma *inf-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
by (*fact inf.assoc*)

lemma *inf-commute*: $(x \sqcap y) = (y \sqcap x)$
by (*fact inf.commute*)

lemma *inf-left-commute*: $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$
by (*fact inf.left-commute*)

lemma *inf-idem*: $x \sqcap x = x$
by (*fact inf.idem*)

lemma *inf-left-idem*: $x \sqcap (x \sqcap y) = x \sqcap y$
by (*fact inf.left-idem*)

lemma *inf-right-idem*: $(x \sqcap y) \sqcap y = x \sqcap y$
by (*fact inf.right-idem*)

lemma *inf-absorb1*: $x \leq y \Longrightarrow x \sqcap y = x$
by (*rule order.antisym*) *auto*

lemma *inf-absorb2*: $y \leq x \Longrightarrow x \sqcap y = y$
by (*rule order.antisym*) *auto*

lemmas *inf-aci* = *inf-commute inf-assoc inf-left-commute inf-left-idem*

end

context *semilattice-sup*
begin

sublocale *sup*: *semilattice sup*
proof
fix *a b c*
show $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$
by (*rule order.antisym*) (*auto intro*: *le-supI1 le-supI2 simp add*: *le-sup-iff*)
show $a \sqcup b = b \sqcup a$
by (*rule order.antisym*) (*auto simp add*: *le-sup-iff*)
show $a \sqcup a = a$

by (*rule order.antisym*) (*auto simp add*: *le-sup-iff*)
qed

sublocale *sup*: *semilattice-order sup greater-eq greater*
by *standard* (*auto simp add*: *le-iff-sup sup.commute less-le*)

lemma *sup-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
by (*fact sup.assoc*)

lemma *sup-commute*: $(x \sqcup y) = (y \sqcup x)$
by (*fact sup.commute*)

lemma *sup-left-commute*: $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
by (*fact sup.left-commute*)

lemma *sup-idem*: $x \sqcup x = x$
by (*fact sup.idem*)

lemma *sup-left-idem* [*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
by (*fact sup.left-idem*)

lemma *sup-absorb1*: $y \leq x \Longrightarrow x \sqcup y = x$
by (*rule order.antisym*) *auto*

lemma *sup-absorb2*: $x \leq y \Longrightarrow x \sqcup y = y$
by (*rule order.antisym*) *auto*

lemmas *sup-aci* = *sup-commute sup-assoc sup-left-commute sup-left-idem*

end

context *lattice*
begin

lemma *dual-lattice*: *class.lattice sup* ($\geq$) ($>$) *inf*
by (*rule class.lattice.intro*,
rule dual-semilattice,
rule class.semilattice-sup.intro,
rule dual-order)
(*unfold-locales*, *auto*)

lemma *inf-sup-absorb* [*simp*]: $x \sqcap (x \sqcup y) = x$
by (*blast intro*: *order.antisym inf-le1 inf-greatest sup-ge1*)

lemma *sup-inf-absorb* [*simp*]: $x \sqcup (x \sqcap y) = x$
by (*blast intro*: *order.antisym sup-ge1 sup-least inf-le1*)

lemmas *inf-sup-aci* = *inf-aci sup-aci*

lemmas *inf-sup-ord* = *inf-le1 inf-le2 sup-ge1 sup-ge2*

Towards distributivity.

lemma *distrib-sup-le*: $x \sqcup (y \sqcap z) \leq (x \sqcup y) \sqcap (x \sqcup z)$
by (*auto intro*: *le-infI1 le-infI2 le-supI1 le-supI2*)

lemma *distrib-inf-le*: $(x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
by (*auto intro*: *le-infI1 le-infI2 le-supI1 le-supI2*)

If you have one of them, you have them all.

lemma *distrib-imp1*:
assumes *distrib*: $\bigwedge x\ y\ z.\ x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
shows $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
proof−
have $x \sqcup (y \sqcap z) = (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z)$
by *simp*
also have $\ldots = x \sqcup (z \sqcap (x \sqcup y))$
by (*simp add*: *distrib inf-commute sup-assoc del*: *sup-inf-absorb*)
also have $\ldots = ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z)$
by (*simp add*: *inf-commute*)
also have $\ldots = (x \sqcup y) \sqcap (x \sqcup z)$ **by**(*simp add*:*distrib*)
finally show *?thesis* **.**
qed

lemma *distrib-imp2*:
assumes *distrib*: $\bigwedge x\ y\ z.\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
proof−
have $x \sqcap (y \sqcup z) = (x \sqcap (x \sqcup z)) \sqcap (y \sqcup z)$
by *simp*
also have $\ldots = x \sqcap (z \sqcup (x \sqcap y))$
by (*simp add*: *distrib sup-commute inf-assoc del*: *inf-sup-absorb*)
also have $\ldots = ((x \sqcap y) \sqcup x) \sqcap ((x \sqcap y) \sqcup z)$
by (*simp add*: *sup-commute*)
also have $\ldots = (x \sqcap y) \sqcup (x \sqcap z)$ **by** (*simp add*:*distrib*)
finally show *?thesis* **.**
qed

end

6.3.3 Strict order

context *semilattice-inf*
begin

lemma *less-infI1*: $a < x \Longrightarrow a \sqcap b < x$
by (*auto simp add*: *less-le inf-absorb1 intro*: *le-infI1*)

lemma *less-infI2*: $b < x \Longrightarrow a \sqcap b < x$

by (*auto simp add*: *less-le inf-absorb2 intro*: *le-infI2*)

end

context *semilattice-sup*
begin

lemma *less-supI1*: $x < a \Longrightarrow x < a \sqcup b$
using *dual-semilattice*
by (*rule semilattice-inf.less-infI1*)

lemma *less-supI2*: $x < b \Longrightarrow x < a \sqcup b$
using *dual-semilattice*
by (*rule semilattice-inf.less-infI2*)

end

6.4 Distributive lattices

class *distrib-lattice* = *lattice* +
assumes *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

context *distrib-lattice*
begin

lemma *sup-inf-distrib2*: $(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
by (*simp add*: *sup-commute sup-inf-distrib1*)

lemma *inf-sup-distrib1*: $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
by (*rule distrib-imp2* [*OF sup-inf-distrib1*])

lemma *inf-sup-distrib2*: $(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
by (*simp add*: *inf-commute inf-sup-distrib1*)

lemma *dual-distrib-lattice*: *class.distrib-lattice sup* $(\geq)$ $(>)$ *inf*
by (*rule class.distrib-lattice.intro*, *rule dual-lattice*)
(*unfold-locales*, *fact inf-sup-distrib1*)

lemmas *sup-inf-distrib* = *sup-inf-distrib1 sup-inf-distrib2*

lemmas *inf-sup-distrib* = *inf-sup-distrib1 inf-sup-distrib2*

lemmas *distrib* = *sup-inf-distrib1 sup-inf-distrib2 inf-sup-distrib1 inf-sup-distrib2*

end

6.5 Bounded lattices

class *bounded-semilattice-inf-top* = *semilattice-inf* + *order-top*
begin

sublocale *inf-top*: *semilattice-neutr inf top*
  + *inf-top*: *semilattice-neutr-order inf top less-eq less*
proof
  show $x \sqcap \top = x$ **for** x
    by (*rule inf-absorb1*) *simp*
qed

lemma *inf-top-left*: $\top \sqcap x = x$
  by (*fact inf-top.left-neutral*)

lemma *inf-top-right*: $x \sqcap \top = x$
  by (*fact inf-top.right-neutral*)

lemma *inf-eq-top-iff*: $x \sqcap y = \top \longleftrightarrow x = \top \wedge y = \top$
  by (*fact inf-top.eq-neutr-iff*)

lemma *top-eq-inf-iff*: $\top = x \sqcap y \longleftrightarrow x = \top \wedge y = \top$
  by (*fact inf-top.neutr-eq-iff*)

end

class *bounded-semilattice-sup-bot* = *semilattice-sup* + *order-bot*
begin

sublocale *sup-bot*: *semilattice-neutr sup bot*
  + *sup-bot*: *semilattice-neutr-order sup bot greater-eq greater*
proof
  show $x \sqcup \bot = x$ **for** x
    by (*rule sup-absorb1*) *simp*
qed

lemma *sup-bot-left*: $\bot \sqcup x = x$
  by (*fact sup-bot.left-neutral*)

lemma *sup-bot-right*: $x \sqcup \bot = x$
  by (*fact sup-bot.right-neutral*)

lemma *sup-eq-bot-iff*: $x \sqcup y = \bot \longleftrightarrow x = \bot \wedge y = \bot$
  by (*fact sup-bot.eq-neutr-iff*)

lemma *bot-eq-sup-iff*: $\bot = x \sqcup y \longleftrightarrow x = \bot \wedge y = \bot$
  by (*fact sup-bot.neutr-eq-iff*)

end

class *bounded-lattice-bot* = *lattice* + *order-bot*
begin

subclass *bounded-semilattice-sup-bot* **..**

lemma *inf-bot-left* [*simp*]: $\bot \sqcap x = \bot$
by (*rule inf-absorb1*) *simp*

lemma *inf-bot-right* [*simp*]: $x \sqcap \bot = \bot$
by (*rule inf-absorb2*) *simp*

end

class *bounded-lattice-top* = *lattice* + *order-top*
begin

subclass *bounded-semilattice-inf-top* **..**

lemma *sup-top-left* [*simp*]: $\top \sqcup x = \top$
by (*rule sup-absorb1*) *simp*

lemma *sup-top-right* [*simp*]: $x \sqcup \top = \top$
by (*rule sup-absorb2*) *simp*

end

class *bounded-lattice* = *lattice* + *order-bot* + *order-top*
begin

subclass *bounded-lattice-bot* **..**
subclass *bounded-lattice-top* **..**

lemma *dual-bounded-lattice*: *class.bounded-lattice sup greater-eq greater inf* $\top$ $\bot$
by *unfold-locales* (*auto simp add*: *less-le-not-le*)

end

6.6 *min/max* as special case of lattice

context *linorder*
begin

sublocale *min*: *semilattice-order min less-eq less*
+ *max*: *semilattice-order max greater-eq greater*
by *standard* (*auto simp add*: *min-def max-def*)

declare *min.absorb1* [*simp*] *min.absorb2* [*simp*]
min.absorb3 [*simp*] *min.absorb4* [*simp*]
max.absorb1 [*simp*] *max.absorb2* [*simp*]
max.absorb3 [*simp*] *max.absorb4* [*simp*]

lemma *min-le-iff-disj*: $min\ x\ y \leq z \longleftrightarrow x \leq z \lor y \leq z$

unfolding *min-def* **using** *linear* **by** (*auto intro*: *order-trans*)

lemma *le-max-iff-disj*: $z \leq max\ x\ y \longleftrightarrow z \leq x \lor z \leq y$
unfolding *max-def* **using** *linear* **by** (*auto intro*: *order-trans*)

lemma *min-less-iff-disj*: $min\ x\ y < z \longleftrightarrow x < z \lor y < z$
unfolding *min-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *less-max-iff-disj*: $z < max\ x\ y \longleftrightarrow z < x \lor z < y$
unfolding *max-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *min-less-iff-conj* [*simp*]: $z < min\ x\ y \longleftrightarrow z < x \land z < y$
unfolding *min-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *max-less-iff-conj* [*simp*]: $max\ x\ y < z \longleftrightarrow x < z \land y < z$
unfolding *max-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *min-max-distrib1*: $min\ (max\ b\ c)\ a = max\ (min\ b\ a)\ (min\ c\ a)$
by (*auto simp add*: *min-def max-def not-le dest*: *le-less-trans less-trans intro*: *antisym*)

lemma *min-max-distrib2*: $min\ a\ (max\ b\ c) = max\ (min\ a\ b)\ (min\ a\ c)$
by (*auto simp add*: *min-def max-def not-le dest*: *le-less-trans less-trans intro*: *antisym*)

lemma *max-min-distrib1*: $max\ (min\ b\ c)\ a = min\ (max\ b\ a)\ (max\ c\ a)$
by (*auto simp add*: *min-def max-def not-le dest*: *le-less-trans less-trans intro*: *antisym*)

lemma *max-min-distrib2*: $max\ a\ (min\ b\ c) = min\ (max\ a\ b)\ (max\ a\ c)$
by (*auto simp add*: *min-def max-def not-le dest*: *le-less-trans less-trans intro*: *antisym*)

lemmas *min-max-distribs* = *min-max-distrib1 min-max-distrib2 max-min-distrib1 max-min-distrib2*

lemma *split-min* [*no-atp*]: $P\ (min\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ i) \land (\neg\ i \leq j \longrightarrow P\ j)$
by (*simp add*: *min-def*)

lemma *split-max* [*no-atp*]: $P\ (max\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ j) \land (\neg\ i \leq j \longrightarrow P\ i)$
by (*simp add*: *max-def*)

lemma *split-min-lin* [*no-atp*]:
‹$P\ (min\ a\ b) \longleftrightarrow (b = a \longrightarrow P\ a) \land (a < b \longrightarrow P\ a) \land (b < a \longrightarrow P\ b)$›
by (*cases a b rule*: *linorder-cases*) *auto*

lemma *split-max-lin* [*no-atp*]:
‹$P\ (max\ a\ b) \longleftrightarrow (b = a \longrightarrow P\ a) \land (a < b \longrightarrow P\ b) \land (b < a \longrightarrow P\ a)$›
by (*cases a b rule*: *linorder-cases*) *auto*

end

lemma *inf-min*: $inf = (min :: 'a::\{semilattice\text{-}inf,linorder\} \Rightarrow 'a \Rightarrow 'a)$
by (*auto intro*: *antisym simp add*: *min-def fun-eq-iff*)

lemma *sup-max*: $sup = (max :: 'a::\{semilattice\text{-}sup,linorder\} \Rightarrow 'a \Rightarrow 'a)$
by (*auto intro*: *antisym simp add*: *max-def fun-eq-iff*)

6.7 Uniqueness of inf and sup

lemma (**in** *semilattice-inf*) *inf-unique*:
fixes *f* (**infixl** ‹△› *70*)
assumes *le1*: $\bigwedge x\ y.\ x \triangle y \le x$
and *le2*: $\bigwedge x\ y.\ x \triangle y \le y$
and *greatest*: $\bigwedge x\ y\ z.\ x \le y \Longrightarrow x \le z \Longrightarrow x \le y \triangle z$
shows $x \sqcap y = x \triangle y$
proof (*rule order.antisym*)
show $x \triangle y \le x \sqcap y$
by (*rule le-infI*) (*rule le1*, *rule le2*)
have *leI*: $\bigwedge x\ y\ z.\ x \le y \Longrightarrow x \le z \Longrightarrow x \le y \triangle z$
by (*blast intro*: *greatest*)
show $x \sqcap y \le x \triangle y$
by (*rule leI*) *simp-all*
qed

lemma (**in** *semilattice-sup*) *sup-unique*:
fixes *f* (**infixl** ‹∇› *70*)
assumes *ge1* [*simp*]: $\bigwedge x\ y.\ x \le x \nabla y$
and *ge2*: $\bigwedge x\ y.\ y \le x \nabla y$
and *least*: $\bigwedge x\ y\ z.\ y \le x \Longrightarrow z \le x \Longrightarrow y \nabla z \le x$
shows $x \sqcup y = x \nabla y$
proof (*rule order.antisym*)
show $x \sqcup y \le x \nabla y$
by (*rule le-supI*) (*rule ge1*, *rule ge2*)
have *leI*: $\bigwedge x\ y\ z.\ x \le z \Longrightarrow y \le z \Longrightarrow x \nabla y \le z$
by (*blast intro*: *least*)
show $x \nabla y \le x \sqcup y$
by (*rule leI*) *simp-all*
qed

6.8 Lattice on - ⇒ -

instantiation *fun* :: (*type*, *semilattice-sup*) *semilattice-sup*
begin

definition $f \sqcup g = (\lambda x.\ f\ x \sqcup g\ x)$

lemma *sup-apply* [*simp*, *code*]: $(f \sqcup g)\ x = f\ x \sqcup g\ x$
by (*simp add*: *sup-fun-def*)

instance
  by *standard* (*simp-all add*: *le-fun-def*)

end

instantiation *fun* :: (*type*, *semilattice-inf*) *semilattice-inf*
begin

definition $f \sqcap g = (\lambda x.\ f\ x \sqcap g\ x)$

lemma *inf-apply* [*simp*, *code*]: $(f \sqcap g)\ x = f\ x \sqcap g\ x$
  by (*simp add*: *inf-fun-def*)

instance by *standard* (*simp-all add*: *le-fun-def*)

end

instance *fun* :: (*type*, *lattice*) *lattice* **..**

instance *fun* :: (*type*, *distrib-lattice*) *distrib-lattice*
  by *standard* (*rule ext*, *simp add*: *sup-inf-distrib1*)

instance *fun* :: (*type*, *bounded-lattice*) *bounded-lattice* **..**

instantiation *fun* :: (*type*, *uminus*) *uminus*
begin

definition *fun-Compl-def*: $-\ A = (\lambda x.\ -\ A\ x)$

lemma *uminus-apply* [*simp*, *code*]: $(-\ A)\ x = -\ (A\ x)$
  by (*simp add*: *fun-Compl-def*)

instance ..

end

instantiation *fun* :: (*type*, *minus*) *minus*
begin

definition *fun-diff-def*: $A - B = (\lambda x.\ A\ x - B\ x)$

lemma *minus-apply* [*simp*, *code*]: $(A - B)\ x = A\ x - B\ x$
  by (*simp add*: *fun-diff-def*)

instance ..

end

end

7 Boolean Algebras

theory *Boolean-Algebras*
 imports *Lattices*
begin

7.1 Abstract boolean algebra

locale *abstract-boolean-algebra* = *conj*: *abel-semigroup* ‹(⊓)› + *disj*: *abel-semigroup* ‹(⊔)›
 for *conj* :: ‹'a ⇒ 'a ⇒ 'a› (**infixr** ‹⊓› *70*)
  and *disj* :: ‹'a ⇒ 'a ⇒ 'a› (**infixr** ‹⊔› *65*) +
 fixes *compl* :: ‹'a ⇒ 'a› (‹(‹open-block notation=‹prefix −››− -)› [*81*] *80*)
  and *zero* :: ‹'a› (‹**0**›)
  and *one* :: ‹'a› (‹**1**›)
 assumes *conj-disj-distrib*: ‹x ⊓ (y ⊔ z) = (x ⊓ y) ⊔ (x ⊓ z)›
  and *disj-conj-distrib*: ‹x ⊔ (y ⊓ z) = (x ⊔ y) ⊓ (x ⊔ z)›
  and *conj-one-right*: ‹x ⊓ **1** = x›
  and *disj-zero-right*: ‹x ⊔ **0** = x›
  and *conj-cancel-right* [*simp*]: ‹x ⊓ − x = **0**›
  and *disj-cancel-right* [*simp*]: ‹x ⊔ − x = **1**›
begin

sublocale *conj*: *semilattice-neutr* ‹(⊓)› ‹**1**›
proof
 show x ⊓ **1** = x **for** x
  by (*fact conj-one-right*)
 show x ⊓ x = x **for** x
 proof −
  have x ⊓ x = (x ⊓ x) ⊔ **0**
   by (*simp add*: *disj-zero-right*)
  also have ... = (x ⊓ x) ⊔ (x ⊓ − x)
   by *simp*
  also have ... = x ⊓ (x ⊔ − x)
   by (*simp only*: *conj-disj-distrib*)
  also have ... = x ⊓ **1**
   by *simp*
  also have ... = x
   by (*simp add*: *conj-one-right*)
  finally show *?thesis* **.**
 qed
qed

sublocale *disj*: *semilattice-neutr* ‹(⊔)› ‹**0**›
proof
 show x ⊔ **0** = x **for** x
  by (*fact disj-zero-right*)

show $x \sqcup x = x$ **for** x
proof −
have $x \sqcup x = (x \sqcup x) \sqcap \mathbf{1}$
by *simp*
also have $\ldots = (x \sqcup x) \sqcap (x \sqcup - x)$
by *simp*
also have $\ldots = x \sqcup (x \sqcap - x)$
by (*simp only*: *disj-conj-distrib*)
also have $\ldots = x \sqcup \mathbf{0}$
by *simp*
also have $\ldots = x$
by (*simp add*: *disj-zero-right*)
finally show *?thesis* **.**
qed
qed

7.1.1 Complement

lemma *complement-unique*:
assumes *1*: $a \sqcap x = \mathbf{0}$
assumes *2*: $a \sqcup x = \mathbf{1}$
assumes *3*: $a \sqcap y = \mathbf{0}$
assumes *4*: $a \sqcup y = \mathbf{1}$
shows $x = y$
proof −
from *1 3* **have** $(a \sqcap x) \sqcup (x \sqcap y) = (a \sqcap y) \sqcup (x \sqcap y)$
by *simp*
then have $(x \sqcap a) \sqcup (x \sqcap y) = (y \sqcap a) \sqcup (y \sqcap x)$
by (*simp add*: *ac-simps*)
then have $x \sqcap (a \sqcup y) = y \sqcap (a \sqcup x)$
by (*simp add*: *conj-disj-distrib*)
with *2 4* **have** $x \sqcap \mathbf{1} = y \sqcap \mathbf{1}$
by *simp*
then show $x = y$
by *simp*
qed

lemma *compl-unique*: $x \sqcap y = \mathbf{0} \Longrightarrow x \sqcup y = \mathbf{1} \Longrightarrow - x = y$
by (*rule complement-unique* [*OF conj-cancel-right disj-cancel-right*])

lemma *double-compl* [*simp*]: $- (- x) = x$
proof (*rule compl-unique*)
show $- x \sqcap x = \mathbf{0}$
by (*simp only*: *conj-cancel-right conj.commute*)
show $- x \sqcup x = \mathbf{1}$
by (*simp only*: *disj-cancel-right disj.commute*)
qed

lemma *compl-eq-compl-iff* [*simp*]:

‹− x = − y ⟷ x = y› (**is** ‹?P ⟷ ?Q›)
proof
assume ‹?Q›
then show *?P* **by** *simp*
next
assume ‹?P›
then have ‹− (− x) = − (− y)›
by *simp*
then show *?Q*
by *simp*
qed

7.1.2 Conjunction

lemma *conj-zero-right* [*simp*]: $x \sqcap \mathbf{0} = \mathbf{0}$
using *conj.left-idem conj-cancel-right* **by** *fastforce*

lemma *compl-one* [*simp*]: $- \mathbf{1} = \mathbf{0}$
by (*rule compl-unique* [*OF conj-zero-right disj-zero-right*])

lemma *conj-zero-left* [*simp*]: $\mathbf{0} \sqcap x = \mathbf{0}$
by (*subst conj.commute*) (*rule conj-zero-right*)

lemma *conj-cancel-left* [*simp*]: $- x \sqcap x = \mathbf{0}$
by (*subst conj.commute*) (*rule conj-cancel-right*)

lemma *conj-disj-distrib2*: $(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
by (*simp only*: *conj.commute conj-disj-distrib*)

lemmas *conj-disj-distribs* = *conj-disj-distrib conj-disj-distrib2*

7.1.3 Disjunction

context
begin

interpretation *dual*: *abstract-boolean-algebra* ‹(⊔)› ‹(⊓)› *compl* ‹**1**› ‹**0**›
apply *standard*
apply (*rule disj-conj-distrib*)
apply (*rule conj-disj-distrib*)
apply *simp-all*
done

lemma *disj-one-right* [*simp*]: $x \sqcup \mathbf{1} = \mathbf{1}$
by (*fact dual.conj-zero-right*)

lemma *compl-zero* [*simp*]: $- \mathbf{0} = \mathbf{1}$
by (*fact dual.compl-one*)

lemma *disj-one-left* [*simp*]: $\mathbf{1} \sqcup x = \mathbf{1}$

```
  by (fact dual.conj-zero-left)

lemma disj-cancel-left [simp]: − x ⊔ x = 1
  by (fact dual.conj-cancel-left)

lemma disj-conj-distrib2: (y ⊓ z) ⊔ x = (y ⊔ x) ⊓ (z ⊔ x)
  by (fact dual.conj-disj-distrib2)

lemmas disj-conj-distribs = disj-conj-distrib disj-conj-distrib2

end
```

7.1.4 De Morgan's Laws

```
lemma de-Morgan-conj [simp]: − (x ⊓ y) = − x ⊔ − y
proof (rule compl-unique)
  have (x ⊓ y) ⊓ (− x ⊔ − y) = ((x ⊓ y) ⊓ − x) ⊔ ((x ⊓ y) ⊓ − y)
    by (rule conj-disj-distrib)
  also have ... = (y ⊓ (x ⊓ − x)) ⊔ (x ⊓ (y ⊓ − y))
    by (simp only: ac-simps)
  finally show (x ⊓ y) ⊓ (− x ⊔ − y) = 0
    by (simp only: conj-cancel-right conj-zero-right disj-zero-right)
next
  have (x ⊓ y) ⊔ (− x ⊔ − y) = (x ⊔ (− x ⊔ − y)) ⊓ (y ⊔ (− x ⊔ − y))
    by (rule disj-conj-distrib2)
  also have ... = (− y ⊔ (x ⊔ − x)) ⊓ (− x ⊔ (y ⊔ − y))
    by (simp only: ac-simps)
  finally show (x ⊓ y) ⊔ (− x ⊔ − y) = 1
    by (simp only: disj-cancel-right disj-one-right conj-one-right)
qed

context
begin

interpretation dual: abstract-boolean-algebra ‹(⊔)› ‹(⊓)› compl ‹1› ‹0›
  apply standard
       apply (rule disj-conj-distrib)
      apply (rule conj-disj-distrib)
     apply simp-all
  done

lemma de-Morgan-disj [simp]: − (x ⊔ y) = − x ⊓ − y
  by (fact dual.de-Morgan-conj)

end

end
```

7.2 Symmetric Difference

locale *abstract-boolean-algebra-sym-diff* = *abstract-boolean-algebra* +
  fixes *xor* :: ‹′*a* ⇒ ′*a* ⇒ ′*a*› (**infixr** ‹⊖› *65*)
  assumes *xor-def* : ‹*x* ⊖ *y* = (*x* ⊓ − *y*) ⊔ (− *x* ⊓ *y*)›
begin

sublocale *xor*: *comm-monoid xor* ‹**0**›
proof
  fix *x y z* :: ′*a*
  let *?t* = (*x* ⊓ *y* ⊓ *z*) ⊔ (*x* ⊓ − *y* ⊓ − *z*) ⊔ (− *x* ⊓ *y* ⊓ − *z*) ⊔ (− *x* ⊓ − *y* ⊓ *z*)
  have *?t* ⊔ (*z* ⊓ *x* ⊓ − *x*) ⊔ (*z* ⊓ *y* ⊓ − *y*) = *?t* ⊔ (*x* ⊓ *y* ⊓ − *y*) ⊔ (*x* ⊓ *z* ⊓ − *z*)
    by (*simp only*: *conj-cancel-right conj-zero-right*)
  then show (*x* ⊖ *y*) ⊖ *z* = *x* ⊖ (*y* ⊖ *z*)
    by (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
      (*simp only*: *conj-disj-distribs conj-ac ac-simps*)
  show *x* ⊖ *y* = *y* ⊖ *x*
    by (*simp only*: *xor-def ac-simps*)
  show *x* ⊖ **0** = *x*
    by (*simp add*: *xor-def*)
qed

lemma *xor-def2*:
  ‹*x* ⊖ *y* = (*x* ⊔ *y*) ⊓ (− *x* ⊔ − *y*)›
proof −
  note *xor-def* [*of x y*]
  also have ‹*x* ⊓ − *y* ⊔ − *x* ⊓ *y* = ((*x* ⊔ − *x*) ⊓ (− *y* ⊔ − *x*)) ⊓ (*x* ⊔ *y*) ⊓ (− *y* ⊔ *y*)›
    by (*simp add*: *ac-simps disj-conj-distribs*)
  also have ‹. . . = (*x* ⊔ *y*) ⊓ (− *x* ⊔ − *y*)›
    by (*simp add*: *ac-simps*)
  finally show *?thesis* **.**
qed

lemma *xor-one-right* [*simp*]: *x* ⊖ **1** = − *x*
  by (*simp only*: *xor-def compl-one conj-zero-right conj-one-right disj.left-neutral*)

lemma *xor-one-left* [*simp*]: **1** ⊖ *x* = − *x*
  using *xor-one-right* [*of x*] **by** (*simp add*: *ac-simps*)

lemma *xor-self* [*simp*]: *x* ⊖ *x* = **0**
  by (*simp only*: *xor-def conj-cancel-right conj-cancel-left disj-zero-right*)

lemma *xor-left-self* [*simp*]: *x* ⊖ (*x* ⊖ *y*) = *y*
  by (*simp only*: *xor.assoc* [*symmetric*] *xor-self xor.left-neutral*)

lemma *xor-compl-left* [*simp*]: − *x* ⊖ *y* = − (*x* ⊖ *y*)
  by (*simp add*: *ac-simps flip*: *xor-one-left*)

lemma *xor-compl-right* [*simp*]: $x \ominus - y = - (x \ominus y)$
  using *xor.commute xor-compl-left* **by** *auto*

lemma *xor-cancel-right* [*simp*]: $x \ominus - x = \mathbf{1}$
  by (*simp only*: *xor-compl-right xor-self compl-zero*)

lemma *xor-cancel-left* [*simp*]: $- x \ominus x = \mathbf{1}$
  by (*simp only*: *xor-compl-left xor-self compl-zero*)

lemma *conj-xor-distrib*: $x \sqcap (y \ominus z) = (x \sqcap y) \ominus (x \sqcap z)$
proof −
  have $*$: $(x \sqcap y \sqcap - z) \sqcup (x \sqcap - y \sqcap z) =$
      $(y \sqcap x \sqcap - x) \sqcup (z \sqcap x \sqcap - x) \sqcup (x \sqcap y \sqcap - z) \sqcup (x \sqcap - y \sqcap z)$
    by (*simp only*: *conj-cancel-right conj-zero-right disj.left-neutral*)
  then show $x \sqcap (y \ominus z) = (x \sqcap y) \ominus (x \sqcap z)$
    by (*simp* (*no-asm-use*) *only*:
      xor-def de-Morgan-disj de-Morgan-conj double-compl
      conj-disj-distribs ac-simps)
qed

lemma *conj-xor-distrib2*: $(y \ominus z) \sqcap x = (y \sqcap x) \ominus (z \sqcap x)$
  by (*simp add*: *conj.commute conj-xor-distrib*)

lemmas *conj-xor-distribs* = *conj-xor-distrib conj-xor-distrib2*

end

7.3 Type classes

class *boolean-algebra* = *distrib-lattice* + *bounded-lattice* + *minus* + *uminus* +
  assumes *inf-compl-bot*: ‹$x \sqcap - x = \bot$›
    and *sup-compl-top*: ‹$x \sqcup - x = \top$›
  assumes *diff-eq*: ‹$x - y = x \sqcap - y$›
begin

sublocale *boolean-algebra*: *abstract-boolean-algebra* ‹($\sqcap$)› ‹($\sqcup$)› *uminus* $\bot$ $\top$
  apply *standard*
      apply (*rule inf-sup-distrib1*)
     apply (*rule sup-inf-distrib1*)
    apply (*simp-all add*: *ac-simps inf-compl-bot sup-compl-top*)
  done

lemma *compl-inf-bot*: $- x \sqcap x = \bot$
  by (*fact boolean-algebra.conj-cancel-left*)

lemma *compl-sup-top*: $- x \sqcup x = \top$
  by (*fact boolean-algebra.disj-cancel-left*)

lemma *compl-unique*:
  assumes $x \sqcap y = \bot$
    and $x \sqcup y = \top$
  shows $- x = y$
  using *assms* **by** (*rule boolean-algebra.compl-unique*)

lemma *double-compl*: $- (- x) = x$
  by (*fact boolean-algebra.double-compl*)

lemma *compl-eq-compl-iff*: $- x = - y \longleftrightarrow x = y$
  by (*fact boolean-algebra.compl-eq-compl-iff*)

lemma *compl-bot-eq*: $- \bot = \top$
  by (*fact boolean-algebra.compl-zero*)

lemma *compl-top-eq*: $- \top = \bot$
  by (*fact boolean-algebra.compl-one*)

lemma *compl-inf*: $- (x \sqcap y) = - x \sqcup - y$
  by (*fact boolean-algebra.de-Morgan-conj*)

lemma *compl-sup*: $- (x \sqcup y) = - x \sqcap - y$
  by (*fact boolean-algebra.de-Morgan-disj*)

lemma *compl-mono*:
  assumes $x \leq y$
  shows $- y \leq - x$
proof −
  from *assms* **have** $x \sqcup y = y$ **by** (*simp only*: *le-iff-sup*)
  then have $- (x \sqcup y) = - y$ **by** *simp*
  then have $- x \sqcap - y = - y$ **by** *simp*
  then have $- y \sqcap - x = - y$ **by** (*simp only*: *inf-commute*)
  then show *?thesis* **by** (*simp only*: *le-iff-inf*)
qed

lemma *compl-le-compl-iff* [*simp*]: $- x \leq - y \longleftrightarrow y \leq x$
  by (*auto dest*: *compl-mono*)

lemma *compl-le-swap1*:
  assumes $y \leq - x$
  shows $x \leq -y$
proof −
  from *assms* **have** $- (- x) \leq - y$ **by** (*simp only*: *compl-le-compl-iff*)
  then show *?thesis* **by** *simp*
qed

lemma *compl-le-swap2*:
  assumes $- y \leq x$
  shows $- x \leq y$

proof −
  from *assms* **have** $- x \leq - (- y)$ **by** (*simp only*: *compl-le-compl-iff*)
  then show *?thesis* **by** *simp*
qed

lemma *compl-less-compl-iff* [*simp*]: $- x < - y \longleftrightarrow y < x$
  by (*auto simp add*: *less-le*)

lemma *compl-less-swap1*:
  assumes $y < - x$
  shows $x < - y$
proof −
  from *assms* **have** $- (- x) < - y$ **by** (*simp only*: *compl-less-compl-iff*)
  then show *?thesis* **by** *simp*
qed

lemma *compl-less-swap2*:
  assumes $- y < x$
  shows $- x < y$
proof −
  from *assms* **have** $- x < - (- y)$
    by (*simp only*: *compl-less-compl-iff*)
  then show *?thesis* **by** *simp*
qed

lemma *sup-cancel-left1*: ‹$x \sqcup a \sqcup (- x \sqcup b) = \top$›
  by (*simp add*: *ac-simps*)

lemma *sup-cancel-left2*: ‹$- x \sqcup a \sqcup (x \sqcup b) = \top$›
  by (*simp add*: *ac-simps*)

lemma *inf-cancel-left1*: ‹$x \sqcap a \sqcap (- x \sqcap b) = \bot$›
  by (*simp add*: *ac-simps*)

lemma *inf-cancel-left2*: ‹$- x \sqcap a \sqcap (x \sqcap b) = \bot$›
  by (*simp add*: *ac-simps*)

lemma *sup-compl-top-left1* [*simp*]: ‹$- x \sqcup (x \sqcup y) = \top$›
  by (*simp add*: *sup-assoc* [*symmetric*])

lemma *sup-compl-top-left2* [*simp*]: ‹$x \sqcup (- x \sqcup y) = \top$›
  using *sup-compl-top-left1* [*of* $- x$ *y*] **by** *simp*

lemma *inf-compl-bot-left1* [*simp*]: ‹$- x \sqcap (x \sqcap y) = \bot$›
  by (*simp add*: *inf-assoc* [*symmetric*])

lemma *inf-compl-bot-left2* [*simp*]: ‹$x \sqcap (- x \sqcap y) = \bot$›
  using *inf-compl-bot-left1* [*of* $- x$ *y*] **by** *simp*

lemma *inf-compl-bot-right* [*simp*]: ‹$x \sqcap (y \sqcap - x) = \bot$›
by (*subst inf-left-commute*) *simp*

end

7.4 Lattice on *bool*

instantiation *bool* :: *boolean-algebra*
begin

definition *bool-Compl-def* [*simp*]: $uminus = Not$

definition *bool-diff-def* [*simp*]: $A - B \longleftrightarrow A \wedge \neg B$

definition [*simp*]: $P \sqcap Q \longleftrightarrow P \wedge Q$

definition [*simp*]: $P \sqcup Q \longleftrightarrow P \vee Q$

instance by *standard auto*

end

lemma *sup-boolI1*: $P \Longrightarrow P \sqcup Q$
by *simp*

lemma *sup-boolI2*: $Q \Longrightarrow P \sqcup Q$
by *simp*

lemma *sup-boolE*: $P \sqcup Q \Longrightarrow (P \Longrightarrow R) \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
by *auto*

instance *fun* :: (*type*, *boolean-algebra*) *boolean-algebra*
by *standard* (*rule ext*, *simp-all add*: *inf-compl-bot sup-compl-top diff-eq*)+

7.5 Lattice on unary and binary predicates

lemma *inf1I*: $A\ x \Longrightarrow B\ x \Longrightarrow (A \sqcap B)\ x$
by (*simp add*: *inf-fun-def*)

lemma *inf2I*: $A\ x\ y \Longrightarrow B\ x\ y \Longrightarrow (A \sqcap B)\ x\ y$
by (*simp add*: *inf-fun-def*)

lemma *inf1E*: $(A \sqcap B)\ x \Longrightarrow (A\ x \Longrightarrow B\ x \Longrightarrow P) \Longrightarrow P$
by (*simp add*: *inf-fun-def*)

lemma *inf2E*: $(A \sqcap B)\ x\ y \Longrightarrow (A\ x\ y \Longrightarrow B\ x\ y \Longrightarrow P) \Longrightarrow P$
by (*simp add*: *inf-fun-def*)

lemma *inf1D1*: $(A \sqcap B)\ x \Longrightarrow A\ x$
by (*rule inf1E*)

lemma *inf2D1*: $(A \sqcap B)\ x\ y \Longrightarrow A\ x\ y$
by (*rule inf2E*)

lemma *inf1D2*: $(A \sqcap B)\ x \Longrightarrow B\ x$
by (*rule inf1E*)

lemma *inf2D2*: $(A \sqcap B)\ x\ y \Longrightarrow B\ x\ y$
by (*rule inf2E*)

lemma *sup1I1*: $A\ x \Longrightarrow (A \sqcup B)\ x$
by (*simp add*: *sup-fun-def*)

lemma *sup2I1*: $A\ x\ y \Longrightarrow (A \sqcup B)\ x\ y$
by (*simp add*: *sup-fun-def*)

lemma *sup1I2*: $B\ x \Longrightarrow (A \sqcup B)\ x$
by (*simp add*: *sup-fun-def*)

lemma *sup2I2*: $B\ x\ y \Longrightarrow (A \sqcup B)\ x\ y$
by (*simp add*: *sup-fun-def*)

lemma *sup1E*: $(A \sqcup B)\ x \Longrightarrow (A\ x \Longrightarrow P) \Longrightarrow (B\ x \Longrightarrow P) \Longrightarrow P$
by (*simp add*: *sup-fun-def*) *iprover*

lemma *sup2E*: $(A \sqcup B)\ x\ y \Longrightarrow (A\ x\ y \Longrightarrow P) \Longrightarrow (B\ x\ y \Longrightarrow P) \Longrightarrow P$
by (*simp add*: *sup-fun-def*) *iprover*

Classical introduction rule: no commitment to A vs B.

lemma *sup1CI*: $(\neg\ B\ x \Longrightarrow A\ x) \Longrightarrow (A \sqcup B)\ x$
by (*auto simp add*: *sup-fun-def*)

lemma *sup2CI*: $(\neg\ B\ x\ y \Longrightarrow A\ x\ y) \Longrightarrow (A \sqcup B)\ x\ y$
by (*auto simp add*: *sup-fun-def*)

7.6 Simproc setup

locale *boolean-algebra-cancel*
begin

lemma *sup1*: $(A::'a::semilattice\text{-}sup) \equiv sup\ k\ a \Longrightarrow sup\ A\ b \equiv sup\ k\ (sup\ a\ b)$
by (*simp only*: *ac-simps*)

lemma *sup2*: $(B::'a::semilattice\text{-}sup) \equiv sup\ k\ b \Longrightarrow sup\ a\ B \equiv sup\ k\ (sup\ a\ b)$
by (*simp only*: *ac-simps*)

lemma *sup0*: $(a::'a::bounded\text{-}semilattice\text{-}sup\text{-}bot) \equiv sup\ a\ bot$
by *simp*

lemma *inf1*: $(A::'a::semilattice\text{-}inf) \equiv inf\ k\ a \Longrightarrow inf\ A\ b \equiv inf\ k\ (inf\ a\ b)$
  by (*simp only*: *ac-simps*)

lemma *inf2*: $(B::'a::semilattice\text{-}inf) \equiv inf\ k\ b \Longrightarrow inf\ a\ B \equiv inf\ k\ (inf\ a\ b)$
  by (*simp only*: *ac-simps*)

lemma *inf0*: $(a::'a::bounded\text{-}semilattice\text{-}inf\text{-}top) \equiv inf\ a\ top$
  by *simp*

end

ML-file ‹*Tools/boolean-algebra-cancel.ML*›

simproc-setup *boolean-algebra-cancel-sup* (*sup a b*::'*a*::*boolean-algebra*) =
  ‹*K* (*K* (*try Boolean-Algebra-Cancel.cancel-sup-conv*))›

simproc-setup *boolean-algebra-cancel-inf* (*inf a b*::'*a*::*boolean-algebra*) =
  ‹*K* (*K* (*try Boolean-Algebra-Cancel.cancel-inf-conv*))›

context *boolean-algebra*
begin

lemma *shunt1*: $(x \sqcap y \leq z) \longleftrightarrow (x \leq -y \sqcup z)$
proof
  assume $x \sqcap y \leq z$
  hence $-y \sqcup (x \sqcap y) \leq -y \sqcup z$
    using *sup.mono* **by** *blast*
  hence $-y \sqcup x \leq -y \sqcup z$
    by (*simp add*: *sup-inf-distrib1*)
  thus $x \leq -y \sqcup z$
    by *simp*
next
  assume $x \leq -y \sqcup z$
  hence $x \sqcap y \leq (-y \sqcup z) \sqcap y$
    using *inf-mono* **by** *auto*
  thus $x \sqcap y \leq z$
    using *inf.boundedE inf-sup-distrib2* **by** *auto*
qed

lemma *shunt2*: $(x \sqcap -y \leq z) \longleftrightarrow (x \leq y \sqcup z)$
  by (*simp add*: *shunt1*)

lemma *inf-shunt*: $(x \sqcap y = \bot) \longleftrightarrow (x \leq -\ y)$
  by (*simp add*: *order.eq-iff shunt1*)

lemma *sup-shunt*: $(x \sqcup y = \top) \longleftrightarrow (-\ x \leq y)$
  using *inf-shunt* [*of* ‹− *x*› ‹− *y*›, *symmetric*]
  by (*simp flip*: *compl-sup compl-top-eq*)

lemma *diff-shunt-var*[*simp*]: $(x - y = \bot) \longleftrightarrow (x \le y)$
by (*simp add*: *diff-eq inf-shunt*)

lemma *diff-shunt*[*simp*]: $(\bot = x - y) \longleftrightarrow (x \le y)$
by (*auto simp flip*: *diff-shunt-var*)

lemma *sup-neg-inf*:
‹$p \le q \sqcup r \longleftrightarrow p \sqcap -q \le r$› (**is** ‹$?P \longleftrightarrow ?Q$›)
proof
assume *?P*
then have ‹$p \sqcap - q \le (q \sqcup r) \sqcap - q$›
by (*rule inf-mono*) *simp*
then show *?Q*
by (*simp add*: *inf-sup-distrib2*)
next
assume *?Q*
then have ‹$p \sqcap - q \sqcup q \le r \sqcup q$›
by (*rule sup-mono*) *simp*
then show *?P*
by (*simp add*: *sup-inf-distrib ac-simps*)
qed

end

end

8 Set theory for higher-order logic

theory *Set*
imports *Lattices Boolean-Algebras*
begin

8.1 Sets as predicates

typedecl *'a set*

axiomatization *Collect* :: $('a \Rightarrow bool) \Rightarrow 'a\ set$ — comprehension
and *member* :: $'a \Rightarrow 'a\ set \Rightarrow bool$ — membership
where *mem-Collect-eq* [*iff*, *code-unfold*]: $member\ a\ (Collect\ P) = P\ a$
and *Collect-mem-eq* [*simp*]: $Collect\ (\lambda x.\ member\ x\ A) = A$

notation
member (‹'(∈')›) **and**
member (‹(‹notation=‹infix ∈››-/ ∈ -)› [*51*, *51*] *50*)

abbreviation *not-member*
where $not\text{-}member\ x\ A \equiv \neg\ (x \in A)$ — non-membership
notation

not-member (‹′(∉′)›) **and**
not-member (‹(‹notation=‹infix ∉››-/ ∉ -)› [51, 51] 50)

open-bundle *member-ASCII-syntax*
begin
notation (*ASCII*)
member (‹′(:′)›) **and**
member (‹(‹notation=‹infix :››-/ : -)› [51, 51] 50) **and**
not-member (‹′(~:′)›) **and**
not-member (‹(‹notation=‹infix ~:››-/ ~: -)› [51, 51] 50)
end

Set comprehensions

syntax
-Coll :: *pttrn* ⇒ *bool* ⇒ *′a set* (‹(‹indent=1 notation=‹mixfix set comprehension››{-./ -})›)
syntax-consts
-Coll ⇌ *Collect*
translations
{*x*. *P*} ⇌ *CONST Collect* (λ*x*. *P*)

syntax (*ASCII*)
-Collect :: *pttrn* ⇒ *′a set* ⇒ *bool* ⇒ *′a set* (‹(‹indent=1 notation=‹mixfix set comprehension››{(-/: -)./ -})›)
syntax
-Collect :: *pttrn* ⇒ *′a set* ⇒ *bool* ⇒ *′a set* (‹(‹indent=1 notation=‹mixfix set comprehension››{(-/ ∈ -)./ -})›)
translations
{*p*:*A*. *P*} ⇀ *CONST Collect* (λ*p*. *p* ∈ *A* ∧ *P*)

ML ‹
fun Collect-binder-tr′ c [*Abs* (*x*, *T*, *t*), *Const* (***const-syntax***‹*Collect*›, -) $ *Abs* (*y*, -, *P*)] =
    if x = *y then*
      let
        val x′ = *Syntax-Trans.mark-bound-body* (*x*, *T*);
        val t′ = *subst-bound* (*x′*, *t*);
        val P′ = *subst-bound* (*x′*, *P*);
      in Syntax.const c $ *Syntax-Trans.mark-bound-abs* (*x*, *T*) $ *P′* $ *t′ end*
    else raise Match
  | *Collect-binder-tr′* - - = *raise Match*
›

lemma *CollectI*: *P a* ⟹ *a* ∈ {*x*. *P x*}
by *simp*

lemma *CollectD*: *a* ∈ {*x*. *P x*} ⟹ *P a*
by *simp*

lemma *Collect-cong*: $(\bigwedge x.\ P\ x = Q\ x) \Longrightarrow \{x.\ P\ x\} = \{x.\ Q\ x\}$
  by *simp*

Simproc for pulling $x = t$ in $\{x.\ \dots\ \wedge\ x = t\ \wedge\ \dots\}$ to the front (and similarly for $t = x$):

simproc-setup *defined-Collect* $(\{x.\ P\ x \wedge Q\ x\})$ = ‹
  K (*Quantifier1.rearrange-Collect*
    (*fn ctxt* =>
      resolve-tac ctxt @{*thms Collect-cong*} *1 THEN*
      resolve-tac ctxt @{*thms iffI*} *1 THEN*
      ALLGOALS
        (*EVERY'* [*REPEAT-DETERM o eresolve-tac ctxt* @{*thms conjE*},
          DEPTH-SOLVE-1 o (*assume-tac ctxt ORELSE' resolve-tac ctxt* @{*thms conjI*})]))))
›

lemmas *CollectE* = *CollectD* [*elim-format*]

lemma *set-eqI*:
  assumes $\bigwedge x.\ x \in A \longleftrightarrow x \in B$
  shows $A = B$
proof −
  from *assms* **have** $\{x.\ x \in A\} = \{x.\ x \in B\}$
    by *simp*
  then show *?thesis* **by** *simp*
qed

lemma *set-eq-iff*: $A = B \longleftrightarrow (\forall x.\ x \in A \longleftrightarrow x \in B)$
  by (*auto intro*:*set-eqI*)

lemma *Collect-eqI*:
  assumes $\bigwedge x.\ P\ x = Q\ x$
  shows *Collect P* = *Collect Q*
  using *assms* **by** (*auto intro*: *set-eqI*)

Lifting of predicate class instances

instantiation *set* :: (*type*) *boolean-algebra*
begin

definition *less-eq-set*
  where $A \leq B \longleftrightarrow (\lambda x.\ member\ x\ A) \leq (\lambda x.\ member\ x\ B)$

definition *less-set*
  where $A < B \longleftrightarrow (\lambda x.\ member\ x\ A) < (\lambda x.\ member\ x\ B)$

definition *inf-set*
  where $A \sqcap B = Collect\ ((\lambda x.\ member\ x\ A) \sqcap (\lambda x.\ member\ x\ B))$

definition *sup-set*

where $A \sqcup B = Collect\ ((\lambda x.\ member\ x\ A) \sqcup (\lambda x.\ member\ x\ B))$

definition *bot-set*
where $\bot = Collect\ \bot$

definition *top-set*
where $\top = Collect\ \top$

definition *uminus-set*
where $-\ A = Collect\ (-\ (\lambda x.\ member\ x\ A))$

definition *minus-set*
where $A - B = Collect\ ((\lambda x.\ member\ x\ A) - (\lambda x.\ member\ x\ B))$

instance
by *standard*
(*simp-all add*: *less-eq-set-def less-set-def inf-set-def sup-set-def*
bot-set-def top-set-def uminus-set-def minus-set-def
less-le-not-le sup-inf-distrib1 diff-eq set-eqI fun-eq-iff
del: *inf-apply sup-apply bot-apply top-apply minus-apply uminus-apply*)

end

Set enumerations

abbreviation *empty* :: *'a set* (‹{}›)
where $\{\} \equiv bot$

definition *insert* :: $'a \Rightarrow 'a\ set \Rightarrow 'a\ set$
where *insert-compr*: $insert\ a\ B = \{x.\ x = a \vee x \in B\}$

open-bundle *set-enumeration-syntax*
begin

syntax
-Finset :: $args \Rightarrow 'a\ set$ (‹(‹*indent=1 notation=*‹*mixfix set enumeration*››{-})›)
syntax-consts
-Finset $\rightleftharpoons$ *insert*
translations
$\{x, xs\} \rightleftharpoons CONST\ insert\ x\ \{xs\}$
$\{x\} \rightleftharpoons CONST\ insert\ x\ \{\}$

end

8.2 Subsets and bounded quantifiers

abbreviation *subset* :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$
where $subset \equiv less$

abbreviation *subset-eq* :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$

where *subset-eq* ≡ *less-eq*

notation
subset (‹'(⊂')›) **and**
subset (‹(‹notation=‹infix ⊂››-/ ⊂ -)› [51, 51] 50) **and**
subset-eq (‹'(⊆')›) **and**
subset-eq (‹(‹notation=‹infix ⊆››-/ ⊆ -)› [51, 51] 50)

abbreviation (*input*)
supset :: 'a set ⇒ 'a set ⇒ bool **where**
supset ≡ *greater*

abbreviation (*input*)
supset-eq :: 'a set ⇒ 'a set ⇒ bool **where**
supset-eq ≡ *greater-eq*

notation
supset (‹'(⊃')›) **and**
supset (‹(‹notation=‹infix ⊃››-/ ⊃ -)› [51, 51] 50) **and**
supset-eq (‹'(⊇')›) **and**
supset-eq (‹(‹notation=‹infix ⊇››-/ ⊇ -)› [51, 51] 50)

notation (*ASCII* **output**)
subset (‹'(<')›) **and**
subset (‹(‹notation=‹infix <››-/ < -)› [51, 51] 50) **and**
subset-eq (‹'(<=')›) **and**
subset-eq (‹(‹notation=‹infix <=››-/ <= -)› [51, 51] 50)

definition *Ball* :: 'a set ⇒ ('a ⇒ bool) ⇒ bool
where *Ball A P* ⟷ (∀ x. x ∈ A ⟶ P x) — bounded universal quantifiers

definition *Bex* :: 'a set ⇒ ('a ⇒ bool) ⇒ bool
where *Bex A P* ⟷ (∃ x. x ∈ A ∧ P x) — bounded existential quantifiers

syntax (*ASCII*)
-Ball :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ALL :››ALL (-/:-)./ -)› [0, 0, 10] 10)
-Bex :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder EX :››EX (-/:-)./ -)› [0, 0, 10] 10)
-Bex1 :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder EX! :››EX! (-/:-)./ -)› [0, 0, 10] 10)
-Bleast :: id ⇒ 'a set ⇒ bool ⇒ 'a (‹(‹indent=3 notation=‹binder LEAST :››LEAST (-/:-)./ -)› [0, 0, 10] 10)

syntax (*input*)
-Ball :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ! :››! (-/:-)./ -)› [0, 0, 10] 10)
-Bex :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ? :››? (-/:-)./ -)› [0, 0, 10] 10)

-Bex1 :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ?! :››?! (-/:-)./ -)› [0, 0, 10] 10)

syntax
-Ball :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ∀››∀(-/∈-)./ -)› [0, 0, 10] 10)
-Bex :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ∃››∃(-/∈-)./ -)› [0, 0, 10] 10)
-Bex1 :: pttrn ⇒ 'a set ⇒ bool ⇒ bool (‹(‹indent=3 notation=‹binder ∃!››∃!(-/∈-)./ -)› [0, 0, 10] 10)
-Bleast :: id ⇒ 'a set ⇒ bool ⇒ 'a (‹(‹indent=3 notation=‹binder LEAST››LEAST(-/∈-)./ -)› [0, 0, 10] 10)

syntax-consts
-Ball ⇌ Ball **and**
-Bex ⇌ Bex **and**
-Bex1 ⇌ Ex1 **and**
-Bleast ⇌ Least

translations
∀x∈A. P ⇌ CONST Ball A (λx. P)
∃x∈A. P ⇌ CONST Bex A (λx. P)
∃!x∈A. P ⇀ ∃!x. x ∈ A ∧ P
LEAST x:A. P ⇀ LEAST x. x ∈ A ∧ P

syntax (*ASCII* **output**)
-setlessAll :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ALL››ALL -<-./ -)› [0, 0, 10] 10)
-setlessEx :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder EX››EX -<-./ -)› [0, 0, 10] 10)
-setleAll :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ALL››ALL -<=-./ -)› [0, 0, 10] 10)
-setleEx :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder EX››EX -<=-./ -)› [0, 0, 10] 10)
-setleEx1 :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder EX!››EX! -<=-./ -)› [0, 0, 10] 10)

syntax
-setlessAll :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ∀››∀ -⊂-./ -)› [0, 0, 10] 10)
-setlessEx :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ∃››∃ -⊂-./ -)› [0, 0, 10] 10)
-setleAll :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ∀››∀ -⊆-./ -)› [0, 0, 10] 10)
-setleEx :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ∃››∃ -⊆-./ -)› [0, 0, 10] 10)
-setleEx1 :: [idt, 'a, bool] ⇒ bool (‹(‹indent=3 notation=‹binder ∃!››∃!-⊆-./ -)› [0, 0, 10] 10)

syntax-consts
-setlessAll -setleAll ⇌ *All* **and**
-setlessEx -setleEx ⇌ *Ex* **and**
-setleEx1 ⇌ *Ex1*

translations
$\forall A{\subset}B.\ P \rightharpoonup \forall A.\ A \subset B \longrightarrow P$
$\exists A{\subset}B.\ P \rightharpoonup \exists A.\ A \subset B \wedge P$
$\forall A{\subseteq}B.\ P \rightharpoonup \forall A.\ A \subseteq B \longrightarrow P$
$\exists A{\subseteq}B.\ P \rightharpoonup \exists A.\ A \subseteq B \wedge P$
$\exists!A{\subseteq}B.\ P \rightharpoonup \exists!A.\ A \subseteq B \wedge P$

print-translation ‹
```
let
  val All-binder = Mixfix.binder-name const-syntax ‹All›;
  val Ex-binder = Mixfix.binder-name const-syntax ‹Ex›;
  val impl = const-syntax ‹HOL.implies›;
  val conj = const-syntax ‹HOL.conj›;
  val sbset = const-syntax ‹subset›;
  val sbset-eq = const-syntax ‹subset-eq›;

  val trans =
   [((All-binder, impl, sbset), syntax-const ‹-setlessAll›),
    ((All-binder, impl, sbset-eq), syntax-const ‹-setleAll›),
    ((Ex-binder, conj, sbset), syntax-const ‹-setlessEx›),
    ((Ex-binder, conj, sbset-eq), syntax-const ‹-setleEx›)];

  fun mk v (v', T) c n P =
    if v = v' andalso not (Term.exists-subterm (fn Free (x, -) => x = v | - =>
false) n)
    then Syntax.const c $ Syntax-Trans.mark-bound-body (v', T) $ n $ P
    else raise Match;

  fun tr' q = (q, fn - =>
    (fn [Const (syntax-const ‹-bound›, -) $ Free (v, Type ‹set -›),
        Const (c, -) $
          (Const (d, -) $ (Const (syntax-const ‹-bound›, -) $ Free (v', T)) $ n)
$ P] =>
        (case AList.lookup (=) trans (q, c, d) of
          NONE => raise Match
        | SOME l => mk v (v', T) l n P)
      | - => raise Match));
in
  [tr' All-binder, tr' Ex-binder]
end
```
›

Translate between $\{e \mid x1\ldots xn.\ P\}$ and $\{u.\ \exists x1\ldots xn.\ u = e \wedge P\}$; $\{y.\ \exists x1\ldots xn.\ y = e \wedge P\}$ is only translated if $[0..n] \subseteq bvs\ e$.

syntax
```
  -Setcompr :: 'a ⇒ idts ⇒ bool ⇒ 'a set
    (‹(‹indent=1 notation=‹mixfix set comprehension››{- |/-./ -})›)
```
syntax-consts
```
  -Setcompr ⇌ Collect
```

parse-translation ‹
```
  let
    val ex-tr = snd (Syntax-Trans.mk-binder-tr (EX , const-syntax ‹Ex›));

    fun nvars (Const (syntax-const ‹-idts›, -) $ - $ idts) = nvars idts + 1
      | nvars - = 1;

    fun setcompr-tr ctxt [e, idts, b] =
      let
        val eq = Syntax.const const-syntax ‹HOL.eq› $ Bound (nvars idts) $ e;
        val P = Syntax.const const-syntax ‹HOL.conj› $ eq $ b;
        val exP = ex-tr ctxt [idts, P];
      in Syntax.const const-syntax ‹Collect› $ absdummy dummyT exP end;

  in [(syntax-const ‹-Setcompr›, setcompr-tr)] end
›
```

typed-print-translation ‹
```
 [(const-syntax ‹Ball›, Syntax-Trans.preserve-binder-abs2-tr' syntax-const ‹-Ball›),
  (const-syntax ‹Bex›, Syntax-Trans.preserve-binder-abs2-tr' syntax-const ‹-Bex›)]
```
› — to avoid eta-contraction of body

print-translation ‹
```
let
  val ex-tr' = snd (Syntax-Trans.mk-binder-tr' (const-syntax ‹Ex›, DUMMY));

  fun setcompr-tr' ctxt [Abs (abs as (-, -, P))] =
    let
      fun check (Const (const-syntax ‹Ex›, -) $ Abs (-, -, P), n) = check (P, n +
1)
        | check (Const (const-syntax ‹HOL.conj›, -) $
              (Const (const-syntax ‹HOL.eq›, -) $ Bound m $ e) $ P, n) =
            n > 0 andalso m = n andalso not (loose-bvar1 (P, n)) andalso
            subset (=) (0 upto (n − 1), add-loose-bnos (e, 0, []))
        | check - = false;

        fun tr' (- $ abs) =
          let val - $ idts $ (- $ (- $ - $ e) $ Q) = ex-tr' ctxt [abs]
          in Syntax.const syntax-const ‹-Setcompr› $ e $ idts $ Q end;
    in
      if check (P, 0) then tr' P
      else
        let
```

```
      val (x as - $ Free(xN, -), t) = Syntax-Trans.atomic-abs-tr' ctxt abs;
      val M = Syntax.const syntax-const ‹-Coll› $ x $ t;
    in
      case t of
        Const (const-syntax ‹HOL.conj›, -) $
          (Const (const-syntax ‹Set.member›, -) $
            (Const (syntax-const ‹-bound›, -) $ Free (yN, -)) $ A) $ P =>
        if xN = yN then Syntax.const syntax-const ‹-Collect› $ x $ A $ P else
M
      | - => M
    end
  end;
  in [(const-syntax ‹Collect›, setcompr-tr')] end
›
```

simproc-setup *defined-Bex* ($\exists x\in A.\ P\ x \wedge Q\ x$) = ‹
  K (*Quantifier1.rearrange-Bex* (*fn ctxt* => *unfold-tac ctxt* @{*thms Bex-def*}))
›

simproc-setup *defined-All* ($\forall x\in A.\ P\ x \longrightarrow Q\ x$) = ‹
  K (*Quantifier1.rearrange-Ball* (*fn ctxt* => *unfold-tac ctxt* @{*thms Ball-def*}))
›

lemma *ballI* [*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall x\in A.\ P\ x$
  by (*simp add*: *Ball-def*)

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: $\forall x\in A.\ P\ x \Longrightarrow x \in A \Longrightarrow P\ x$
  by (*simp add*: *Ball-def*)

Gives better instantiation for bound:

setup ‹
  map-theory-claset (*fn ctxt* =>
    ctxt addbefore (*bspec*, *fn ctxt'* => *dresolve-tac ctxt'* @{*thms bspec*} *THEN'* *assume-tac ctxt'*))
›

ML ‹

```
structure Simpdata =
struct
  open Simpdata;
  val mksimps-pairs = [(const-name ‹Ball›, @{thms bspec})] @ mksimps-pairs;
end;

open Simpdata;
```

›

declaration ‹*fn* - => *Simplifier.map-ss* (*Simplifier.set-mksimps* (*mksimps mk-*

simps-pairs))›

lemma *ballE* [*elim*]: $\forall x \in A.\ P\ x \Longrightarrow (P\ x \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
unfolding *Ball-def* **by** *blast*

lemma *bexI* [*intro*]: $P\ x \Longrightarrow x \in A \Longrightarrow \exists x \in A.\ P\ x$
— Normally the best argument order: $P\ x$ constrains the choice of $x \in A$.
unfolding *Bex-def* **by** *blast*

lemma *rev-bexI* [*intro?*]: $x \in A \Longrightarrow P\ x \Longrightarrow \exists x \in A.\ P\ x$
— The best argument order when there is only one $x \in A$.
unfolding *Bex-def* **by** *blast*

lemma *bexCI*: $(\forall x \in A.\ \neg\ P\ x \Longrightarrow P\ a) \Longrightarrow a \in A \Longrightarrow \exists x \in A.\ P\ x$
unfolding *Bex-def* **by** *blast*

lemma *bexE* [*elim!*]: $\exists x \in A.\ P\ x \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow P\ x \Longrightarrow Q) \Longrightarrow Q$
unfolding *Bex-def* **by** *blast*

lemma *ball-triv* [*simp*]: $(\forall x \in A.\ P) \longleftrightarrow ((\exists x.\ x \in A) \longrightarrow P)$
— trivial rewrite rule.
by (*simp add*: *Ball-def*)

lemma *bex-triv* [*simp*]: $(\exists x \in A.\ P) \longleftrightarrow ((\exists x.\ x \in A) \wedge P)$
— Dual form for existentials.
by (*simp add*: *Bex-def*)

lemma *bex-triv-one-point1* [*simp*]: $(\exists x \in A.\ x = a) \longleftrightarrow a \in A$
by *blast*

lemma *bex-triv-one-point2* [*simp*]: $(\exists x \in A.\ a = x) \longleftrightarrow a \in A$
by *blast*

lemma *bex-one-point1* [*simp*]: $(\exists x \in A.\ x = a \wedge P\ x) \longleftrightarrow a \in A \wedge P\ a$
by *blast*

lemma *bex-one-point2* [*simp*]: $(\exists x \in A.\ a = x \wedge P\ x) \longleftrightarrow a \in A \wedge P\ a$
by *blast*

lemma *ball-one-point1* [*simp*]: $(\forall x \in A.\ x = a \longrightarrow P\ x) \longleftrightarrow (a \in A \longrightarrow P\ a)$
by *blast*

lemma *ball-one-point2* [*simp*]: $(\forall x \in A.\ a = x \longrightarrow P\ x) \longleftrightarrow (a \in A \longrightarrow P\ a)$
by *blast*

lemma *ball-conj-distrib*: $(\forall x \in A.\ P\ x \wedge Q\ x) \longleftrightarrow (\forall x \in A.\ P\ x) \wedge (\forall x \in A.\ Q\ x)$
by *blast*

lemma *bex-disj-distrib*: $(\exists x \in A.\ P\ x \vee Q\ x) \longleftrightarrow (\exists x \in A.\ P\ x) \vee (\exists x \in A.\ Q\ x)$

by *blast*

Congruence rules

lemma *ball-cong*:
$[\![A = B; \bigwedge x.\ x \in B \Longrightarrow P\ x \longleftrightarrow Q\ x]\!] \Longrightarrow$
$(\forall x{\in}A.\ P\ x) \longleftrightarrow (\forall x{\in}B.\ Q\ x)$
by (*simp add*: *Ball-def*)

lemma *ball-cong-simp* [*cong*]:
$[\![A = B; \bigwedge x.\ x \in B =simp=> P\ x \longleftrightarrow Q\ x]\!] \Longrightarrow$
$(\forall x{\in}A.\ P\ x) \longleftrightarrow (\forall x{\in}B.\ Q\ x)$
by (*simp add*: *simp-implies-def Ball-def*)

lemma *bex-cong*:
$[\![A = B; \bigwedge x.\ x \in B \Longrightarrow P\ x \longleftrightarrow Q\ x]\!] \Longrightarrow$
$(\exists x{\in}A.\ P\ x) \longleftrightarrow (\exists x{\in}B.\ Q\ x)$
by (*simp add*: *Bex-def cong*: *conj-cong*)

lemma *bex-cong-simp* [*cong*]:
$[\![A = B; \bigwedge x.\ x \in B =simp=> P\ x \longleftrightarrow Q\ x]\!] \Longrightarrow$
$(\exists x{\in}A.\ P\ x) \longleftrightarrow (\exists x{\in}B.\ Q\ x)$
by (*simp add*: *simp-implies-def Bex-def cong*: *conj-cong*)

lemma *bex1-def*: $(\exists!x{\in}X.\ P\ x) \longleftrightarrow (\exists x{\in}X.\ P\ x) \wedge (\forall x{\in}X.\ \forall y{\in}X.\ P\ x \longrightarrow P\ y \longrightarrow x = y)$
by *auto*

8.3 Basic operations

8.3.1 Subsets

lemma *subsetI* [*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$
by (*simp add*: *less-eq-set-def le-fun-def*)

Map the type *'a set* ⇒ *anything* to just *'a*; for overloading constants whose first argument has type *'a set*.

lemma *subsetD* [*elim*, *intro?*]: $A \subseteq B \Longrightarrow c \in A \Longrightarrow c \in B$
by (*simp add*: *less-eq-set-def le-fun-def*)
— Rule in Modus Ponens style.

lemma *rev-subsetD* [*intro?*,*no-atp*]: $c \in A \Longrightarrow A \subseteq B \Longrightarrow c \in B$
— The same, with reversed premises for use with *erule* – cf. $[\![?P;\ ?P \longrightarrow ?Q]\!] \Longrightarrow ?Q$.
by (*rule subsetD*)

lemma *subsetCE* [*elim*,*no-atp*]: $A \subseteq B \Longrightarrow (c \notin A \Longrightarrow P) \Longrightarrow (c \in B \Longrightarrow P) \Longrightarrow P$
— Classical elimination rule.
by (*auto simp add*: *less-eq-set-def le-fun-def*)

lemma *subset-eq*: $A \subseteq B \longleftrightarrow (\forall x{\in}A.\ x \in B)$
by *blast*

lemma *contra-subsetD* [*no-atp*]: $A \subseteq B \Longrightarrow c \notin B \Longrightarrow c \notin A$
by *blast*

lemma *subset-refl*: $A \subseteq A$
by (*fact order-refl*)

lemma *subset-trans*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow A \subseteq C$
by (*fact order-trans*)

lemma *subset-not-subset-eq* [*code*]: $A \subset B \longleftrightarrow A \subseteq B \wedge \neg B \subseteq A$
by (*fact less-le-not-le*)

lemma *eq-mem-trans*: $a = b \Longrightarrow b \in A \Longrightarrow a \in A$
by *simp*

lemmas *basic-trans-rules* [*trans*] =
order-trans-rules rev-subsetD subsetD eq-mem-trans

8.3.2 Equality

lemma *subset-antisym* [*intro!*]: $A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow A = B$
— Anti-symmetry of the subset relation.
by (*iprover intro*: *set-eqI subsetD*)

Equality rules from ZF set theory – are they appropriate here?

lemma *equalityD1*: $A = B \Longrightarrow A \subseteq B$
by *simp*

lemma *equalityD2*: $A = B \Longrightarrow B \subseteq A$
by *simp*

Be careful when adding this to the claset as *subset-empty* is in the simpset: $A = \{\}$ goes to $\{\} \subseteq A$ and $A \subseteq \{\}$ and then back to $A = \{\}$!

lemma *equalityE*: $A = B \Longrightarrow (A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow P) \Longrightarrow P$
by *simp*

lemma *equalityCE* [*elim*]: $A = B \Longrightarrow (c \in A \Longrightarrow c \in B \Longrightarrow P) \Longrightarrow (c \notin A \Longrightarrow c \notin B \Longrightarrow P) \Longrightarrow P$
by *blast*

lemma *eqset-imp-iff*: $A = B \Longrightarrow x \in A \longleftrightarrow x \in B$
by *simp*

lemma *eqelem-imp-iff*: $x = y \Longrightarrow x \in A \longleftrightarrow y \in A$
by *simp*

8.3.3 The empty set

lemma *empty-def*: $\{\} = \{x.\ \mathit{False}\}$
by (*simp add*: *bot-set-def bot-fun-def*)

lemma *empty-iff* [*simp*]: $c \in \{\} \longleftrightarrow \mathit{False}$
by (*simp add*: *empty-def*)

lemma *emptyE* [*elim*!]: $a \in \{\} \Longrightarrow P$
by *simp*

lemma *empty-subsetI* [*iff*]: $\{\} \subseteq A$
— One effect is to delete the ASSUMPTION $\{\} \subseteq A$
by *blast*

lemma *equals0I*: $(\bigwedge y.\ y \in A \Longrightarrow \mathit{False}) \Longrightarrow A = \{\}$
by *blast*

lemma *equals0D*: $A = \{\} \Longrightarrow a \notin A$
— Use for reasoning about disjointness: $A \cap B = \{\}$
by *blast*

lemma *ball-empty* [*simp*]: $\mathit{Ball}\ \{\}\ P \longleftrightarrow \mathit{True}$
by (*simp add*: *Ball-def*)

lemma *bex-empty* [*simp*]: $\mathit{Bex}\ \{\}\ P \longleftrightarrow \mathit{False}$
by (*simp add*: *Bex-def*)

8.3.4 The universal set – UNIV

abbreviation $\mathit{UNIV} :: {}'a\ \mathit{set}$
where $\mathit{UNIV} \equiv \mathit{top}$

lemma *UNIV-def*: $\mathit{UNIV} = \{x.\ \mathit{True}\}$
by (*simp add*: *top-set-def top-fun-def*)

lemma *UNIV-I* [*simp*]: $x \in \mathit{UNIV}$
by (*simp add*: *UNIV-def*)

declare *UNIV-I* [*intro*] — unsafe makes it less likely to cause problems

lemma *UNIV-witness* [*intro?*]: $\exists x.\ x \in \mathit{UNIV}$
by *simp*

lemma *subset-UNIV*: $A \subseteq \mathit{UNIV}$
by (*fact top-greatest*)

Eta-contracting these two rules (to remove P) causes them to be ignored because of their interaction with congruence rules.

lemma *ball-UNIV* [*simp*]: *Ball UNIV P* $\longleftrightarrow$ *All P*
by (*simp add*: *Ball-def*)

lemma *bex-UNIV* [*simp*]: *Bex UNIV P* $\longleftrightarrow$ *Ex P*
by (*simp add*: *Bex-def*)

lemma *UNIV-eq-I*: $(\bigwedge x.\ x \in A) \Longrightarrow UNIV = A$
by *auto*

lemma *UNIV-not-empty* [*iff*]: $UNIV \neq \{\}$
by (*blast elim*: *equalityE*)

lemma *empty-not-UNIV*[*simp*]: $\{\} \neq UNIV$
by *blast*

8.3.5 The Powerset operator – Pow

definition *Pow* :: $'a\ set \Rightarrow 'a\ set\ set$
where *Pow-def*: $Pow\ A = \{B.\ B \subseteq A\}$

lemma *Pow-iff* [*iff*]: $A \in Pow\ B \longleftrightarrow A \subseteq B$
by (*simp add*: *Pow-def*)

lemma *PowI*: $A \subseteq B \Longrightarrow A \in Pow\ B$
by (*simp add*: *Pow-def*)

lemma *PowD*: $A \in Pow\ B \Longrightarrow A \subseteq B$
by (*simp add*: *Pow-def*)

lemma *Pow-bottom*: $\{\} \in Pow\ B$
by *simp*

lemma *Pow-top*: $A \in Pow\ A$
by *simp*

lemma *Pow-not-empty*: $Pow\ A \neq \{\}$
using *Pow-top* **by** *blast*

8.3.6 Set complement

lemma *Compl-iff* [*simp*]: $c \in -\ A \longleftrightarrow c \notin A$
by (*simp add*: *fun-Compl-def uminus-set-def*)

lemma *ComplI* [*intro*!]: $(c \in A \Longrightarrow False) \Longrightarrow c \in -\ A$
by (*simp add*: *fun-Compl-def uminus-set-def*) *blast*

This form, with negated conclusion, works well with the Classical prover. Negated assumptions behave like formulae on the right side of the notional turnstile . . .

lemma *ComplD* [*dest*!]: $c \in -\ A \Longrightarrow c \notin A$
by *simp*

lemmas *ComplE* = *ComplD* [*elim-format*]

lemma *Compl-eq*: $-\ A = \{x.\ \neg\ x \in A\}$
by *blast*

8.3.7 Binary intersection

abbreviation *inter* :: $'a\ set \Rightarrow\ 'a\ set \Rightarrow\ 'a\ set$ (**infixl** ‹∩› *70*)
where $(\cap) \equiv inf$

notation (*ASCII*)
inter (**infixl** ‹*Int*› *70*)

lemma *Int-def*: $A \cap B = \{x.\ x \in A \wedge x \in B\}$
by (*simp add*: *inf-set-def inf-fun-def*)

lemma *Int-iff* [*simp*]: $c \in A \cap B \longleftrightarrow c \in A \wedge c \in B$
unfolding *Int-def* **by** *blast*

lemma *IntI* [*intro*!]: $c \in A \Longrightarrow c \in B \Longrightarrow c \in A \cap B$
by *simp*

lemma *IntD1*: $c \in A \cap B \Longrightarrow c \in A$
by *simp*

lemma *IntD2*: $c \in A \cap B \Longrightarrow c \in B$
by *simp*

lemma *IntE* [*elim*!]: $c \in A \cap B \Longrightarrow (c \in A \Longrightarrow c \in B \Longrightarrow P) \Longrightarrow P$
by *simp*

8.3.8 Binary union

abbreviation *union* :: $'a\ set \Rightarrow\ 'a\ set \Rightarrow\ 'a\ set$ (**infixl** ‹∪› *65*)
where $union \equiv sup$

notation (*ASCII*)
union (**infixl** ‹*Un*› *65*)

lemma *Un-def*: $A \cup B = \{x.\ x \in A \vee x \in B\}$
by (*simp add*: *sup-set-def sup-fun-def*)

lemma *Un-iff* [*simp*]: $c \in A \cup B \longleftrightarrow c \in A \vee c \in B$
unfolding *Un-def* **by** *blast*

lemma *UnI1* [*elim?*]: $c \in A \Longrightarrow c \in A \cup B$
by *simp*

lemma *UnI2* [*elim?*]: $c \in B \Longrightarrow c \in A \cup B$
by *simp*

Classical introduction rule: no commitment to A vs. B.

lemma *UnCI* [*intro*!]: $(c \notin B \Longrightarrow c \in A) \Longrightarrow c \in A \cup B$
by *auto*

lemma *UnE* [*elim*!]: $c \in A \cup B \Longrightarrow (c \in A \Longrightarrow P) \Longrightarrow (c \in B \Longrightarrow P) \Longrightarrow P$
unfolding *Un-def* **by** *blast*

lemma *insert-def*: *insert* a $B = \{x.\ x = a\} \cup B$
by (*simp add*: *insert-compr Un-def*)

8.3.9 Set difference

lemma *Diff-iff* [*simp*]: $c \in A - B \longleftrightarrow c \in A \wedge c \notin B$
by (*simp add*: *minus-set-def fun-diff-def*)

lemma *DiffI* [*intro*!]: $c \in A \Longrightarrow c \notin B \Longrightarrow c \in A - B$
by *simp*

lemma *DiffD1*: $c \in A - B \Longrightarrow c \in A$
by *simp*

lemma *DiffD2*: $c \in A - B \Longrightarrow c \in B \Longrightarrow P$
by *simp*

lemma *DiffE* [*elim*!]: $c \in A - B \Longrightarrow (c \in A \Longrightarrow c \notin B \Longrightarrow P) \Longrightarrow P$
by *simp*

lemma *set-diff-eq*: $A - B = \{x.\ x \in A \wedge x \notin B\}$
by *blast*

lemma *Compl-eq-Diff-UNIV*: $- A = (UNIV - A)$
by *blast*

abbreviation *sym-diff* :: $'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set$ **where**
sym-diff A $B \equiv ((A - B) \cup (B - A))$

8.3.10 Augmenting a set — *insert*

lemma *insert-iff* [*simp*]: $a \in$ *insert* b $A \longleftrightarrow a = b \vee a \in A$
unfolding *insert-def* **by** *blast*

lemma *insertI1*: $a \in$ *insert* a B
by *simp*

lemma *insertI2*: $a \in B \Longrightarrow a \in$ *insert* b B

by *simp*

lemma *insertE* [*elim*!]: $a \in insert\ b\ A \Longrightarrow (a = b \Longrightarrow P) \Longrightarrow (a \in A \Longrightarrow P) \Longrightarrow P$
unfolding *insert-def* **by** *blast*

lemma *insertCI* [*intro*!]: $(a \notin B \Longrightarrow a = b) \Longrightarrow a \in insert\ b\ B$
— Classical introduction rule.
by *auto*

lemma *subset-insert-iff*: $A \subseteq insert\ x\ B \longleftrightarrow (if\ x \in A\ then\ A - \{x\} \subseteq B\ else\ A \subseteq B)$
by *auto*

lemma *set-insert*:
assumes $x \in A$
obtains B **where** $A = insert\ x\ B$ **and** $x \notin B$
proof
show $A = insert\ x\ (A - \{x\})$ **using** *assms* **by** *blast*
show $x \notin A - \{x\}$ **by** *blast*
qed

lemma *insert-ident*: $x \notin A \Longrightarrow x \notin B \Longrightarrow insert\ x\ A = insert\ x\ B \longleftrightarrow A = B$
by *auto*

lemma *insert-eq-iff*:
assumes $a \notin A\ b \notin B$
shows $insert\ a\ A = insert\ b\ B \longleftrightarrow$
$(if\ a = b\ then\ A = B\ else\ \exists C.\ A = insert\ b\ C \wedge b \notin C \wedge B = insert\ a\ C \wedge a \notin C)$
(**is** $?L \longleftrightarrow ?R$)
proof
show $?R$ **if** $?L$
proof (*cases* $a = b$)
case *True*
with *assms* ‹$?L$› **show** $?R$
by (*simp add*: *insert-ident*)
next
case *False*
let $?C = A - \{b\}$
have $A = insert\ b\ ?C \wedge b \notin ?C \wedge B = insert\ a\ ?C \wedge a \notin ?C$
using *assms* ‹$?L$› ‹$a \neq b$› **by** *auto*
then show $?R$ **using** ‹$a \neq b$› **by** *auto*
qed
show $?L$ **if** $?R$
using *that* **by** (*auto split*: *if-splits*)
qed

lemma *insert-UNIV*[*simp*]: $insert\ x\ UNIV = UNIV$

by *auto*

8.3.11 Singletons, using insert

lemma *singletonI* [*intro*!]: $a \in \{a\}$
— Redundant? But unlike *insertCI*, it proves the subgoal immediately!
by (*rule insertI1*)

lemma *singletonD* [*dest*!]: $b \in \{a\} \Longrightarrow b = a$
by *blast*

lemmas *singletonE* = *singletonD* [*elim-format*]

lemma *singleton-iff*: $b \in \{a\} \longleftrightarrow b = a$
by *blast*

lemma *singleton-inject* [*dest*!]: $\{a\} = \{b\} \Longrightarrow a = b$
by *blast*

lemma *singleton-insert-inj-eq* [*iff*]: $\{b\} = insert\ a\ A \longleftrightarrow a = b \wedge A \subseteq \{b\}$
by *blast*

lemma *singleton-insert-inj-eq'* [*iff*]: $insert\ a\ A = \{b\} \longleftrightarrow a = b \wedge A \subseteq \{b\}$
by *blast*

lemma *subset-singletonD*: $A \subseteq \{x\} \Longrightarrow A = \{\} \vee A = \{x\}$
by *fast*

lemma *subset-singleton-iff*: $X \subseteq \{a\} \longleftrightarrow X = \{\} \vee X = \{a\}$
by *blast*

lemma *subset-singleton-iff-Uniq*: $(\exists a.\ A \subseteq \{a\}) \longleftrightarrow (\exists_{\leq 1} x.\ x \in A)$
unfolding *Uniq-def* **by** *blast*

lemma *singleton-conv* [*simp*]: $\{x.\ x = a\} = \{a\}$
by *blast*

lemma *singleton-conv2* [*simp*]: $\{x.\ a = x\} = \{a\}$
by *blast*

lemma *Diff-single-insert*: $A - \{x\} \subseteq B \Longrightarrow A \subseteq insert\ x\ B$
by *blast*

lemma *subset-Diff-insert*: $A \subseteq B - insert\ x\ C \longleftrightarrow A \subseteq B - C \wedge x \notin A$
by *blast*

lemma *doubleton-eq-iff*: $\{a, b\} = \{c, d\} \longleftrightarrow a = c \wedge b = d \vee a = d \wedge b = c$
by (*blast elim*: *equalityE*)

lemma *Un-singleton-iff*: $A \cup B = \{x\} \longleftrightarrow A = \{\} \wedge B = \{x\} \vee A = \{x\} \wedge B = \{\} \vee A = \{x\} \wedge B = \{x\}$
by *auto*

lemma *singleton-Un-iff*: $\{x\} = A \cup B \longleftrightarrow A = \{\} \wedge B = \{x\} \vee A = \{x\} \wedge B = \{\} \vee A = \{x\} \wedge B = \{x\}$
by *auto*

8.3.12 Image of a set under a function

Frequently b does not have the syntactic form of $f\,x$.

definition *image* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'b\ set$ (**infixr** ‹'› *90*)
where $f\ `\ A = \{y.\ \exists x{\in}A.\ y = f\,x\}$

lemma *image-eqI* [*simp*, *intro*]: $b = f\,x \Longrightarrow x \in A \Longrightarrow b \in f\ `\ A$
unfolding *image-def* **by** *blast*

lemma *imageI*: $x \in A \Longrightarrow f\,x \in f\ `\ A$
by (*rule image-eqI*) (*rule refl*)

lemma *rev-image-eqI*: $x \in A \Longrightarrow b = f\,x \Longrightarrow b \in f\ `\ A$
— This version's more effective when we already have the required x.
by (*rule image-eqI*)

lemma *imageE* [*elim*!]:
assumes $b \in (\lambda x.\ f\,x)\ `\ A$ — The eta-expansion gives variable-name preservation.
obtains x **where** $b = f\,x$ **and** $x \in A$
using *assms* **unfolding** *image-def* **by** *blast*

lemma *Compr-image-eq*: $\{x \in f\ `\ A.\ P\ x\} = f\ `\ \{x \in A.\ P\ (f\,x)\}$
by *auto*

lemma *image-Un*: $f\ `\ (A \cup B) = f\ `\ A \cup f\ `\ B$
by *blast*

lemma *image-iff*: $z \in f\ `\ A \longleftrightarrow (\exists x{\in}A.\ z = f\,x)$
by *blast*

lemma *image-subsetI*: $(\bigwedge x.\ x \in A \Longrightarrow f\,x \in B) \Longrightarrow f\ `\ A \subseteq B$
— Replaces the three steps *subsetI*, *imageE*, *hypsubst*, but breaks too many existing proofs.
by *blast*

lemma *image-subset-iff*: $f\ `\ A \subseteq B \longleftrightarrow (\forall x{\in}A.\ f\,x \in B)$
— This rewrite rule would confuse users if made default.
by *blast*

lemma *subset-imageE*:
assumes $B \subseteq f\ `\ A$

obtains C **where** $C \subseteq A$ **and** $B = f ` C$
proof −
from *assms* **have** $B = f ` \{a \in A.\ f\ a \in B\}$ **by** *fast*
moreover have $\{a \in A.\ f\ a \in B\} \subseteq A$ **by** *blast*
ultimately show *thesis* **by** (*blast intro*: *that*)
qed

lemma *subset-image-iff*: $B \subseteq f ` A \longleftrightarrow (\exists AA \subseteq A.\ B = f ` AA)$
by (*blast elim*: *subset-imageE*)

lemma *image-ident* [*simp*]: $(\lambda x.\ x) ` Y = Y$
by *blast*

lemma *image-empty* [*simp*]: $f ` \{\} = \{\}$
by *blast*

lemma *image-insert* [*simp*]: $f ` insert\ a\ B = insert\ (f\ a)\ (f ` B)$
by *blast*

lemma *image-constant*: $x \in A \Longrightarrow (\lambda x.\ c) ` A = \{c\}$
by *auto*

lemma *image-constant-conv*: $(\lambda x.\ c) ` A = (if\ A = \{\}\ then\ \{\}\ else\ \{c\})$
by *auto*

lemma *image-image*: $f ` (g ` A) = (\lambda x.\ f\ (g\ x)) ` A$
by *blast*

lemma *insert-image* [*simp*]: $x \in A \Longrightarrow insert\ (f\ x)\ (f ` A) = f ` A$
by *blast*

lemma *image-is-empty* [*iff*]: $f ` A = \{\} \longleftrightarrow A = \{\}$
by *blast*

lemma *empty-is-image* [*iff*]: $\{\} = f ` A \longleftrightarrow A = \{\}$
by *blast*

lemma *image-Collect*: $f ` \{x.\ P\ x\} = \{f\ x \mid x.\ P\ x\}$
— NOT suitable as a default simp rule: the RHS isn't simpler than the LHS, with its implicit quantifier and conjunction. Also image enjoys better equational properties than does the RHS.
by *blast*

lemma *if-image-distrib* [*simp*]:
$(\lambda x.\ if\ P\ x\ then\ f\ x\ else\ g\ x) ` S = f ` (S \cap \{x.\ P\ x\}) \cup g ` (S \cap \{x.\ \neg\ P\ x\})$
by *auto*

lemma *image-cong*:
$f ` M = g ` N$ **if** $M = N$ $\bigwedge x.\ x \in N \Longrightarrow f\ x = g\ x$

using *that* **by** (*simp add*: *image-def*)

lemma *image-cong-simp* [*cong*]:
f ‘ $M = g$ ‘ N **if** $M = N$ $\bigwedge x.\ x \in N$ =*simp*=> $f\ x = g\ x$
using *that image-cong* [*of M N f g*] **by** (*simp add*: *simp-implies-def*)

lemma *image-Int-subset*: f ‘ $(A \cap B) \subseteq f$ ‘ $A \cap f$ ‘ B
by *blast*

lemma *image-diff-subset*: f ‘ $A - f$ ‘ $B \subseteq f$ ‘ $(A - B)$
by *blast*

lemma *Setcompr-eq-image*: $\{f\ x\ |x.\ x \in A\} = f$ ‘ A
by *blast*

lemma *setcompr-eq-image*: $\{f\ x\ |x.\ P\ x\} = f$ ‘ $\{x.\ P\ x\}$
by *auto*

lemma *ball-imageD*: $\forall x \in f$ ‘ $A.\ P\ x \Longrightarrow \forall x \in A.\ P\ (f\ x)$
by *simp*

lemma *bex-imageD*: $\exists x \in f$ ‘ $A.\ P\ x \Longrightarrow \exists x \in A.\ P\ (f\ x)$
by *auto*

lemma *image-add-0* [*simp*]: (+) (*0*::*'a*::*comm-monoid-add*) ‘ $S = S$
by *auto*

theorem *Cantors-theorem*: $\nexists f.\ f$ ‘ $A = Pow\ A$
proof
assume $\exists f.\ f$ ‘ $A = Pow\ A$
then obtain f **where** f: f ‘ $A = Pow\ A$ **..**
let $?X = \{a \in A.\ a \notin f\ a\}$
have $?X \in Pow\ A$ **by** *blast*
then have $?X \in f$ ‘ A **by** (*simp only*: *f*)
then obtain x **where** $x \in A$ **and** $f\ x = ?X$ **by** *blast*
then show *False* **by** *blast*
qed

Range of a function – just an abbreviation for image!

abbreviation *range* :: $('a \Rightarrow 'b) \Rightarrow 'b\ set$ — of function
where *range f* $\equiv f$ ‘ *UNIV*

lemma *range-eqI*: $b = f\ x \Longrightarrow b \in range\ f$
by *simp*

lemma *rangeI*: $f\ x \in range\ f$
by *simp*

lemma *rangeE* [*elim?*]: $b \in range\ (\lambda x.\ f\ x) \Longrightarrow (\bigwedge x.\ b = f\ x \Longrightarrow P) \Longrightarrow P$

by (*rule imageE*)

lemma *range-subsetD*: $range\ f \subseteq B \Longrightarrow f\ i \in B$
by *blast*

lemma *full-SetCompr-eq*: $\{u.\ \exists x.\ u = f\ x\} = range\ f$
by *auto*

lemma *range-composition*: $range\ (\lambda x.\ f\ (g\ x)) = f\ `\ range\ g$
by *auto*

lemma *range-constant* [*simp*]: $range\ (\lambda\text{-}.\ x) = \{x\}$
by (*simp add*: *image-constant*)

lemma *range-eq-singletonD*: $range\ f = \{a\} \Longrightarrow f\ x = a$
by *auto*

8.3.13 Some rules with *if*

Elimination of $\{x.\ \ldots \wedge x = t \wedge \ldots\}$.

lemma *Collect-conv-if*: $\{x.\ x = a \wedge P\ x\} = (if\ P\ a\ then\ \{a\}\ else\ \{\})$
by *auto*

lemma *Collect-conv-if2*: $\{x.\ a = x \wedge P\ x\} = (if\ P\ a\ then\ \{a\}\ else\ \{\})$
by *auto*

Rewrite rules for boolean case-splitting: faster than *if-split* [*split*].

lemma *if-split-eq1*: $(if\ Q\ then\ x\ else\ y) = b \longleftrightarrow (Q \longrightarrow x = b) \wedge (\neg\ Q \longrightarrow y = b)$
by (*rule if-split*)

lemma *if-split-eq2*: $a = (if\ Q\ then\ x\ else\ y) \longleftrightarrow (Q \longrightarrow a = x) \wedge (\neg\ Q \longrightarrow a = y)$
by (*rule if-split*)

Split ifs on either side of the membership relation. Not for [*simp*] – can cause goals to blow up!

lemma *if-split-mem1*: $(if\ Q\ then\ x\ else\ y) \in b \longleftrightarrow (Q \longrightarrow x \in b) \wedge (\neg\ Q \longrightarrow y \in b)$
by (*rule if-split*)

lemma *if-split-mem2*: $(a \in (if\ Q\ then\ x\ else\ y)) \longleftrightarrow (Q \longrightarrow a \in x) \wedge (\neg\ Q \longrightarrow a \in y)$
by (*rule if-split* [**where** $P = \lambda S.\ a \in S$])

lemmas *split-ifs* = *if-bool-eq-conj if-split-eq1 if-split-eq2 if-split-mem1 if-split-mem2*

8.4 Further operations and lemmas

8.4.1 The "proper subset" relation

lemma *psubsetI* [*intro*!]: $A \subseteq B \Longrightarrow A \neq B \Longrightarrow A \subset B$
unfolding *less-le* **by** *blast*

lemma *psubsetE* [*elim*!]: $A \subset B \Longrightarrow (A \subseteq B \Longrightarrow \neg\ B \subseteq A \Longrightarrow R) \Longrightarrow R$
unfolding *less-le* **by** *blast*

lemma *psubset-insert-iff*:
$A \subset insert\ x\ B \longleftrightarrow (if\ x \in B\ then\ A \subset B\ else\ if\ x \in A\ then\ A - \{x\} \subset B\ else\ A \subseteq B)$
by (*auto simp add*: *less-le subset-insert-iff*)

lemma *psubset-eq*: $A \subset B \longleftrightarrow A \subseteq B \land A \neq B$
by (*simp only*: *less-le*)

lemma *psubset-imp-subset*: $A \subset B \Longrightarrow A \subseteq B$
by (*simp add*: *psubset-eq*)

lemma *psubset-trans*: $A \subset B \Longrightarrow B \subset C \Longrightarrow A \subset C$
unfolding *less-le* **by** (*auto dest*: *subset-antisym*)

lemma *psubsetD*: $A \subset B \Longrightarrow c \in A \Longrightarrow c \in B$
unfolding *less-le* **by** (*auto dest*: *subsetD*)

lemma *psubset-subset-trans*: $A \subset B \Longrightarrow B \subseteq C \Longrightarrow A \subset C$
by (*auto simp add*: *psubset-eq*)

lemma *subset-psubset-trans*: $A \subseteq B \Longrightarrow B \subset C \Longrightarrow A \subset C$
by (*auto simp add*: *psubset-eq*)

lemma *psubset-imp-ex-mem*: $A \subset B \Longrightarrow \exists\, b.\ b \in B - A$
unfolding *less-le* **by** *blast*

lemma *atomize-ball*: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \equiv Trueprop\ (\forall\, x{\in}A.\ P\ x)$
by (*simp only*: *Ball-def atomize-all atomize-imp*)

lemmas [*symmetric*, *rulify*] = *atomize-ball*
and [*symmetric*, *defn*] = *atomize-ball*

lemma *image-Pow-mono*: $f\ `\ A \subseteq B \Longrightarrow image\ f\ `\ Pow\ A \subseteq Pow\ B$
by *blast*

lemma *image-Pow-surj*: $f\ `\ A = B \Longrightarrow image\ f\ `\ Pow\ A = Pow\ B$
by (*blast elim*: *subset-imageE*)

8.4.2 Derived rules involving subsets.

insert.

lemma *subset-insertI*: $B \subseteq insert\ a\ B$
by (*rule subsetI*) (*erule insertI2*)

lemma *subset-insertI2*: $A \subseteq B \Longrightarrow A \subseteq insert\ b\ B$
by *blast*

lemma *subset-insert*: $x \notin A \Longrightarrow A \subseteq insert\ x\ B \longleftrightarrow A \subseteq B$
by *blast*

Finite Union – the least upper bound of two sets.

lemma *Un-upper1*: $A \subseteq A \cup B$
by (*fact sup-ge1*)

lemma *Un-upper2*: $B \subseteq A \cup B$
by (*fact sup-ge2*)

lemma *Un-least*: $A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow A \cup B \subseteq C$
by (*fact sup-least*)

Finite Intersection – the greatest lower bound of two sets.

lemma *Int-lower1*: $A \cap B \subseteq A$
by (*fact inf-le1*)

lemma *Int-lower2*: $A \cap B \subseteq B$
by (*fact inf-le2*)

lemma *Int-greatest*: $C \subseteq A \Longrightarrow C \subseteq B \Longrightarrow C \subseteq A \cap B$
by (*fact inf-greatest*)

Set difference.

lemma *Diff-subset*[*simp*]: $A - B \subseteq A$
by *blast*

lemma *Diff-subset-conv*: $A - B \subseteq C \longleftrightarrow A \subseteq B \cup C$
by *blast*

8.4.3 Equalities involving union, intersection, inclusion, etc.

{}.

lemma *Collect-const* [*simp*]: $\{s.\ P\} = (if\ P\ then\ UNIV\ else\ \{\})$
— supersedes *Collect-False-empty*
by *auto*

lemma *subset-empty* [*simp*]: $A \subseteq \{\} \longleftrightarrow A = \{\}$
by (*fact bot-unique*)

lemma *not-psubset-empty* [*iff*]: $\neg (A < \{\})$
by (*fact not-less-bot*)

lemma *Collect-subset* [*simp*]: $\{x{\in}A.\ P\ x\} \subseteq A$ **by** *auto*

lemma *Collect-empty-eq* [*simp*]: $Collect\ P = \{\} \longleftrightarrow (\forall x.\ \neg\ P\ x)$
by *blast*

lemma *empty-Collect-eq* [*simp*]: $\{\} = Collect\ P \longleftrightarrow (\forall x.\ \neg\ P\ x)$
by *blast*

lemma *Collect-neg-eq*: $\{x.\ \neg\ P\ x\} = -\ \{x.\ P\ x\}$
by *blast*

lemma *Collect-disj-eq*: $\{x.\ P\ x \lor Q\ x\} = \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
by *blast*

lemma *Collect-imp-eq*: $\{x.\ P\ x \longrightarrow Q\ x\} = -\ \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
by *blast*

lemma *Collect-conj-eq*: $\{x.\ P\ x \land Q\ x\} = \{x.\ P\ x\} \cap \{x.\ Q\ x\}$
by *blast*

lemma *Collect-mono-iff*: $Collect\ P \subseteq Collect\ Q \longleftrightarrow (\forall x.\ P\ x \longrightarrow Q\ x)$
by *blast*

insert.

lemma *insert-is-Un*: $insert\ a\ A = \{a\} \cup A$
— NOT SUITABLE FOR REWRITING since $\{a\} \equiv insert\ a\ \{\}$
by *blast*

lemma *insert-not-empty* [*simp*]: $insert\ a\ A \neq \{\}$
and *empty-not-insert* [*simp*]: $\{\} \neq insert\ a\ A$
by *blast+*

lemma *insert-absorb*: $a \in A \Longrightarrow insert\ a\ A = A$
— [*simp*] causes recursive calls when there are nested inserts
— with *quadratic* running time
by *blast*

lemma *insert-absorb2* [*simp*]: $insert\ x\ (insert\ x\ A) = insert\ x\ A$
by *blast*

lemma *insert-commute*: $insert\ x\ (insert\ y\ A) = insert\ y\ (insert\ x\ A)$
by *blast*

lemma *insert-subset* [*simp*]: $insert\ x\ A \subseteq B \longleftrightarrow x \in B \wedge A \subseteq B$
by *blast*

lemma *mk-disjoint-insert*: $a \in A \Longrightarrow \exists B.\ A = insert\ a\ B \wedge a \notin B$
— use new B rather than $A - \{a\}$ to avoid infinite unfolding
by (*rule exI* [**where** $x = A - \{a\}$]) *blast*

lemma *insert-Collect*: $insert\ a\ (Collect\ P) = \{u.\ u \neq a \longrightarrow P\ u\}$
by *auto*

lemma *insert-inter-insert* [*simp*]: $insert\ a\ A \cap insert\ a\ B = insert\ a\ (A \cap B)$
by *blast*

lemma *insert-disjoint* [*simp*]:
$insert\ a\ A \cap B = \{\} \longleftrightarrow a \notin B \wedge A \cap B = \{\}$
$\{\} = insert\ a\ A \cap B \longleftrightarrow a \notin B \wedge \{\} = A \cap B$
by *auto*

lemma *disjoint-insert* [*simp*]:
$B \cap insert\ a\ A = \{\} \longleftrightarrow a \notin B \wedge B \cap A = \{\}$
$\{\} = A \cap insert\ b\ B \longleftrightarrow b \notin A \wedge \{\} = A \cap B$
by *auto*

Int

lemma *Int-absorb*: $A \cap A = A$
by (*fact inf-idem*)

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
by (*fact inf-left-idem*)

lemma *Int-commute*: $A \cap B = B \cap A$
by (*fact inf-commute*)

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
by (*fact inf-left-commute*)

lemma *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$
by (*fact inf-assoc*)

lemmas *Int-ac* = *Int-assoc Int-left-absorb Int-commute Int-left-commute*
— Intersection is an AC-operator

lemma *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
by (*fact inf-absorb2*)

lemma *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
by (*fact inf-absorb1*)

lemma *Int-empty-left*: $\{\} \cap B = \{\}$

by (*fact inf-bot-left*)

lemma *Int-empty-right*: $A \cap \{\} = \{\}$
by (*fact inf-bot-right*)

lemma *disjoint-eq-subset-Compl*: $A \cap B = \{\} \longleftrightarrow A \subseteq - B$
by *blast*

lemma *disjoint-iff*: $A \cap B = \{\} \longleftrightarrow (\forall x.\ x{\in}A \longrightarrow x \notin B)$
by *blast*

lemma *disjoint-iff-not-equal*: $A \cap B = \{\} \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}B.\ x \neq y)$
by *blast*

lemma *Int-UNIV-left*: $UNIV \cap B = B$
by (*fact inf-top-left*)

lemma *Int-UNIV-right*: $A \cap UNIV = A$
by (*fact inf-top-right*)

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
by (*fact inf-sup-distrib1*)

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
by (*fact inf-sup-distrib2*)

lemma *Int-UNIV*: $A \cap B = UNIV \longleftrightarrow A = UNIV \wedge B = UNIV$
by (*fact inf-eq-top-iff*)

lemma *Int-subset-iff*: $C \subseteq A \cap B \longleftrightarrow C \subseteq A \wedge C \subseteq B$
by (*fact le-inf-iff*)

lemma *Int-Collect*: $x \in A \cap \{x.\ P\ x\} \longleftrightarrow x \in A \wedge P\ x$
by *blast*

Un.

lemma *Un-absorb*: $A \cup A = A$
by (*fact sup-idem*)

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
by (*fact sup-left-idem*)

lemma *Un-commute*: $A \cup B = B \cup A$
by (*fact sup-commute*)

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
by (*fact sup-left-commute*)

lemma *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$

by (*fact sup-assoc*)

lemmas *Un-ac* = *Un-assoc Un-left-absorb Un-commute Un-left-commute*
— Union is an AC-operator

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
by (*fact sup-absorb2*)

lemma *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
by (*fact sup-absorb1*)

lemma *Un-empty-left*: $\{\} \cup B = B$
by (*fact sup-bot-left*)

lemma *Un-empty-right*: $A \cup \{\} = A$
by (*fact sup-bot-right*)

lemma *Un-UNIV-left*: $UNIV \cup B = UNIV$
by (*fact sup-top-left*)

lemma *Un-UNIV-right*: $A \cup UNIV = UNIV$
by (*fact sup-top-right*)

lemma *Un-insert-left* [*simp*]: (*insert a B*) $\cup$ *C* = *insert a* ($B \cup C$)
by *blast*

lemma *Un-insert-right* [*simp*]: $A \cup$ (*insert a B*) = *insert a* ($A \cup B$)
by *blast*

lemma *Int-insert-left*: (*insert a B*) $\cap$ *C* = (*if* $a \in C$ *then insert a* ($B \cap C$) *else* $B \cap C$)
by *auto*

lemma *Int-insert-left-if0* [*simp*]: $a \notin C \Longrightarrow$ (*insert a B*) $\cap C = B \cap C$
by *auto*

lemma *Int-insert-left-if1* [*simp*]: $a \in C \Longrightarrow$ (*insert a B*) $\cap$ *C* = *insert a* ($B \cap C$)
by *auto*

lemma *Int-insert-right*: $A \cap$ (*insert a B*) = (*if* $a \in A$ *then insert a* ($A \cap B$) *else* $A \cap B$)
by *auto*

lemma *Int-insert-right-if0* [*simp*]: $a \notin A \Longrightarrow A \cap$ (*insert a B*) $= A \cap B$
by *auto*

lemma *Int-insert-right-if1* [*simp*]: $a \in A \Longrightarrow A \cap$ (*insert a B*) = *insert a* ($A \cap B$)
by *auto*

lemma *Un-Int-distrib*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
by (*fact sup-inf-distrib1*)

lemma *Un-Int-distrib2*: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$
by (*fact sup-inf-distrib2*)

lemma *Un-Int-crazy*: $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
by *blast*

lemma *subset-Un-eq*: $A \subseteq B \longleftrightarrow A \cup B = B$
by (*fact le-iff-sup*)

lemma *Un-empty* [*iff*]: $A \cup B = \{\} \longleftrightarrow A = \{\} \wedge B = \{\}$
by (*fact sup-eq-bot-iff*)

lemma *Un-subset-iff*: $A \cup B \subseteq C \longleftrightarrow A \subseteq C \wedge B \subseteq C$
by (*fact le-sup-iff*)

lemma *Un-Diff-Int*: $(A - B) \cup (A \cap B) = A$
by *blast*

lemma *Diff-Int2*: $A \cap C - B \cap C = A \cap C - B$
by *blast*

lemma *subset-UnE*:
assumes $C \subseteq A \cup B$
obtains A' B' **where** $A' \subseteq A$ $B' \subseteq B$ $C = A' \cup B'$
proof
show $C \cap A \subseteq A$ $C \cap B \subseteq B$ $C = (C \cap A) \cup (C \cap B)$
using *assms* **by** *blast*+
qed

lemma *Un-Int-eq* [*simp*]: $(S \cup T) \cap S = S$ $(S \cup T) \cap T = T$ $S \cap (S \cup T) = S$ $T \cap (S \cup T) = T$
by *auto*

lemma *Int-Un-eq* [*simp*]: $(S \cap T) \cup S = S$ $(S \cap T) \cup T = T$ $S \cup (S \cap T) = S$ $T \cup (S \cap T) = T$
by *auto*

Set complement

lemma *Compl-disjoint* [*simp*]: $A \cap - A = \{\}$
by (*fact inf-compl-bot*)

lemma *Compl-disjoint2* [*simp*]: $- A \cap A = \{\}$
by (*fact compl-inf-bot*)

lemma *Compl-partition*: $A \cup - A = UNIV$
by (*fact sup-compl-top*)

lemma *Compl-partition2*: $- A \cup A = UNIV$
by (*fact compl-sup-top*)

lemma *double-complement*: $- (-A) = A$ **for** A :: *'a set*
by (*fact double-compl*)

lemma *Compl-Un*: $- (A \cup B) = (- A) \cap (- B)$
by (*fact compl-sup*)

lemma *Compl-Int*: $- (A \cap B) = (- A) \cup (- B)$
by (*fact compl-inf*)

lemma *subset-Compl-self-eq*: $A \subseteq - A \longleftrightarrow A = \{\}$
by *blast*

lemma *Un-Int-assoc-eq*: $(A \cap B) \cup C = A \cap (B \cup C) \longleftrightarrow C \subseteq A$
— Halmos, Naive Set Theory, page 16.
by *blast*

lemma *Compl-UNIV-eq*: $- UNIV = \{\}$
by (*fact compl-top-eq*)

lemma *Compl-empty-eq*: $- \{\} = UNIV$
by (*fact compl-bot-eq*)

lemma *Compl-subset-Compl-iff* [*iff*]: $- A \subseteq - B \longleftrightarrow B \subseteq A$
by (*fact compl-le-compl-iff*)

lemma *Compl-eq-Compl-iff* [*iff*]: $- A = - B \longleftrightarrow A = B$
for A B :: *'a set*
by (*fact compl-eq-compl-iff*)

lemma *Compl-insert*: $-$ *insert* x $A = (- A) - \{x\}$
by *blast*

Bounded quantifiers.

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall x \in A \cup B.\ P\ x) \longleftrightarrow (\forall x{\in}A.\ P\ x) \wedge (\forall x{\in}B.\ P\ x)$
by *blast*

lemma *bex-Un*: $(\exists x \in A \cup B.\ P\ x) \longleftrightarrow (\exists x{\in}A.\ P\ x) \vee (\exists x{\in}B.\ P\ x)$
by *blast*

Set difference.

lemma *Diff-eq*: $A - B = A \cap (- B)$
by(*rule boolean-algebra-class.diff-eq*)

lemma *Diff-eq-empty-iff*: $A - B = \{\} \longleftrightarrow A \subseteq B$
by(*rule boolean-algebra-class.diff-shunt-var*)

lemma *Diff-cancel* [*simp*]: $A - A = \{\}$
by *blast*

lemma *Diff-idemp* [*simp*]: $(A - B) - B = A - B$
for $A\ B$ :: *'a set*
by *blast*

lemma *Diff-triv*: $A \cap B = \{\} \Longrightarrow A - B = A$
by (*blast elim*: *equalityE*)

lemma *empty-Diff* [*simp*]: $\{\} - A = \{\}$
by *blast*

lemma *Diff-empty* [*simp*]: $A - \{\} = A$
by *blast*

lemma *Diff-UNIV* [*simp*]: $A - UNIV = \{\}$
by *blast*

lemma *Diff-insert0* [*simp*]: $x \notin A \Longrightarrow A - insert\ x\ B = A - B$
by *blast*

lemma *Diff-insert*: $A - insert\ a\ B = A - B - \{a\}$
— NOT SUITABLE FOR REWRITING since $\{a\} \equiv insert\ a\ 0$
by *blast*

lemma *Diff-insert2*: $A - insert\ a\ B = A - \{a\} - B$
— NOT SUITABLE FOR REWRITING since $\{a\} \equiv insert\ a\ 0$
by *blast*

lemma *insert-Diff-if*: $insert\ x\ A - B = (if\ x \in B\ then\ A - B\ else\ insert\ x\ (A - B))$
by *auto*

lemma *insert-Diff1* [*simp*]: $x \in B \Longrightarrow insert\ x\ A - B = A - B$
by *blast*

lemma *insert-Diff-single*[*simp*]: $insert\ a\ (A - \{a\}) = insert\ a\ A$
by *blast*

lemma *insert-Diff*: $a \in A \Longrightarrow insert\ a\ (A - \{a\}) = A$
by *blast*

lemma *Diff-insert-absorb*: $x \notin A \Longrightarrow (insert\ x\ A) - \{x\} = A$
by *auto*

lemma *Diff-disjoint* [*simp*]: $A \cap (B - A) = \{\}$
by *blast*

lemma *Diff-partition*: $A \subseteq B \Longrightarrow A \cup (B - A) = B$
by *blast*

lemma *double-diff*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow B - (C - A) = A$
by *blast*

lemma *Un-Diff-cancel* [*simp*]: $A \cup (B - A) = A \cup B$
by *blast*

lemma *Un-Diff-cancel2* [*simp*]: $(B - A) \cup A = B \cup A$
by *blast*

lemma *Diff-Un*: $A - (B \cup C) = (A - B) \cap (A - C)$
by *blast*

lemma *Diff-Int*: $A - (B \cap C) = (A - B) \cup (A - C)$
by *blast*

lemma *Diff-Diff-Int*: $A - (A - B) = A \cap B$
by *blast*

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
by *blast*

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
by *blast*

lemma *Diff-Int-distrib*: $C \cap (A - B) = (C \cap A) - (C \cap B)$
by *blast*

lemma *Diff-Int-distrib2*: $(A - B) \cap C = (A \cap C) - (B \cap C)$
by *blast*

lemma *Diff-Compl* [*simp*]: $A - (- B) = A \cap B$
by *auto*

lemma *Compl-Diff-eq* [*simp*]: $- (A - B) = - A \cup B$
by *blast*

lemma *subset-Compl-singleton* [*simp*]: $A \subseteq - \{b\} \longleftrightarrow b \notin A$
by *blast*

Quantification over type *bool*.

lemma *bool-induct*: $P\ True \Longrightarrow P\ False \Longrightarrow P\ x$
by (*cases x*) *auto*

lemma *all-bool-eq*: $(\forall b.\ P\ b) \longleftrightarrow P\ True \wedge P\ False$
by (*auto intro*: *bool-induct*)

lemma *bool-contrapos*: $P\ x \Longrightarrow \neg\ P\ False \Longrightarrow P\ True$
by (*cases x*) *auto*

lemma *ex-bool-eq*: $(\exists b.\ P\ b) \longleftrightarrow P\ True \vee P\ False$
by (*auto intro*: *bool-contrapos*)

lemma *UNIV-bool*: $UNIV = \{False, True\}$
by (*auto intro*: *bool-induct*)

Pow

lemma *Pow-empty* [*simp*]: $Pow\ \{\} = \{\{\}\}$
by (*auto simp add*: *Pow-def*)

lemma *Pow-singleton-iff* [*simp*]: $Pow\ X = \{Y\} \longleftrightarrow X = \{\} \wedge Y = \{\}$
by *blast*

lemma *Pow-insert*: $Pow\ (insert\ a\ A) = Pow\ A \cup (insert\ a\ `\ Pow\ A)$
by (*blast intro*: *image-eqI* [**where** $?x = u - \{a\}$ **for** u])

lemma *Pow-Compl*: $Pow\ (-\ A) = \{-\ B \mid B.\ A \in Pow\ B\}$
by (*blast intro*: *exI* [**where** $?x = -\ u$ **for** u])

lemma *Pow-UNIV* [*simp*]: $Pow\ UNIV = UNIV$
by *blast*

lemma *Un-Pow-subset*: $Pow\ A \cup Pow\ B \subseteq Pow\ (A \cup B)$
by *blast*

lemma *Pow-Int-eq* [*simp*]: $Pow\ (A \cap B) = Pow\ A \cap Pow\ B$
by *blast*

Miscellany.

lemma *Int-Diff-disjoint*: $A \cap B \cap (A - B) = \{\}$
by *blast*

lemma *Int-Diff-Un*: $A \cap B \cup (A - B) = A$
by *blast*

lemma *set-eq-subset*: $A = B \longleftrightarrow A \subseteq B \wedge B \subseteq A$
by *blast*

lemma *subset-iff*: $A \subseteq B \longleftrightarrow (\forall t.\ t \in A \longrightarrow t \in B)$

by *blast*

lemma *subset-iff-psubset-eq*: $A \subseteq B \longleftrightarrow A \subset B \vee A = B$
unfolding *less-le* **by** *blast*

lemma *all-not-in-conv* [*simp*]: $(\forall x.\ x \notin A) \longleftrightarrow A = \{\}$
by *blast*

lemma *ex-in-conv*: $(\exists x.\ x \in A) \longleftrightarrow A \neq \{\}$
by *blast*

lemma *ball-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P\ x \vee Q) \longleftrightarrow ((\forall x{\in}A.\ P\ x) \vee Q)$
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P \vee Q\ x) \longleftrightarrow (P \vee (\forall x{\in}A.\ Q\ x))$
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P \longrightarrow Q\ x) \longleftrightarrow (P \longrightarrow (\forall x{\in}A.\ Q\ x))$
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P\ x \longrightarrow Q) \longleftrightarrow ((\exists x{\in}A.\ P\ x) \longrightarrow Q)$
$\bigwedge P.\ (\forall x{\in}\{\}.\ P\ x) \longleftrightarrow True$
$\bigwedge P.\ (\forall x{\in}UNIV.\ P\ x) \longleftrightarrow (\forall x.\ P\ x)$
$\bigwedge a\ B\ P.\ (\forall x{\in}insert\ a\ B.\ P\ x) \longleftrightarrow (P\ a \wedge (\forall x{\in}B.\ P\ x))$
$\bigwedge P\ Q.\ (\forall x{\in}Collect\ Q.\ P\ x) \longleftrightarrow (\forall x.\ Q\ x \longrightarrow P\ x)$
$\bigwedge A\ P\ f.\ (\forall x{\in}f`A.\ P\ x) \longleftrightarrow (\forall x{\in}A.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg\ (\forall x{\in}A.\ P\ x)) \longleftrightarrow (\exists x{\in}A.\ \neg\ P\ x)$
by *auto*

lemma *bex-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ (\exists x{\in}A.\ P\ x \wedge Q) \longleftrightarrow ((\exists x{\in}A.\ P\ x) \wedge Q)$
$\bigwedge A\ P\ Q.\ (\exists x{\in}A.\ P \wedge Q\ x) \longleftrightarrow (P \wedge (\exists x{\in}A.\ Q\ x))$
$\bigwedge P.\ (\exists x{\in}\{\}.\ P\ x) \longleftrightarrow False$
$\bigwedge P.\ (\exists x{\in}UNIV.\ P\ x) \longleftrightarrow (\exists x.\ P\ x)$
$\bigwedge a\ B\ P.\ (\exists x{\in}insert\ a\ B.\ P\ x) \longleftrightarrow (P\ a \vee (\exists x{\in}B.\ P\ x))$
$\bigwedge P\ Q.\ (\exists x{\in}Collect\ Q.\ P\ x) \longleftrightarrow (\exists x.\ Q\ x \wedge P\ x)$
$\bigwedge A\ P\ f.\ (\exists x{\in}f`A.\ P\ x) \longleftrightarrow (\exists x{\in}A.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg(\exists x{\in}A.\ P\ x)) \longleftrightarrow (\forall x{\in}A.\ \neg\ P\ x)$
by *auto*

lemma *ex-image-cong-iff* [*simp*, *no-atp*]:
$(\exists x.\ x{\in}f`A) \longleftrightarrow A \neq \{\}$ $(\exists x.\ x{\in}f`A \wedge P\ x) \longleftrightarrow (\exists x{\in}A.\ P\ (f\ x))$
by *auto*

8.4.4 Monotonicity of various operations

lemma *image-mono*: $A \subseteq B \Longrightarrow f\ `\ A \subseteq f\ `\ B$
by *blast*

lemma *Pow-mono*: $A \subseteq B \Longrightarrow Pow\ A \subseteq Pow\ B$
by *blast*

lemma *insert-mono*: $C \subseteq D \Longrightarrow insert\ a\ C \subseteq insert\ a\ D$
by *blast*

lemma *Un-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cup B \subseteq C \cup D$
by (*fact sup-mono*)

lemma *Int-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cap B \subseteq C \cap D$
by (*fact inf-mono*)

lemma *Diff-mono*: $A \subseteq C \Longrightarrow D \subseteq B \Longrightarrow A - B \subseteq C - D$
by *blast*

lemma *Compl-anti-mono*: $A \subseteq B \Longrightarrow - B \subseteq - A$
by (*fact compl-mono*)

Monotonicity of implications.

lemma *in-mono*: $A \subseteq B \Longrightarrow x \in A \longrightarrow x \in B$
by (*rule impI*) (*erule subsetD*)

lemma *conj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \wedge P2) \longrightarrow (Q1 \wedge Q2)$
by *iprover*

lemma *disj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \vee P2) \longrightarrow (Q1 \vee Q2)$
by *iprover*

lemma *imp-mono*: $Q1 \longrightarrow P1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \longrightarrow P2) \longrightarrow (Q1 \longrightarrow Q2)$
by *iprover*

lemma *imp-refl*: $P \longrightarrow P$ **..**

lemma *not-mono*: $Q \longrightarrow P \Longrightarrow \neg P \longrightarrow \neg Q$
by *iprover*

lemma *ex-mono*: $(\bigwedge x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (\exists\, x.\ P\ x) \longrightarrow (\exists\, x.\ Q\ x)$
by *iprover*

lemma *all-mono*: $(\bigwedge x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (\forall\, x.\ P\ x) \longrightarrow (\forall\, x.\ Q\ x)$
by *iprover*

lemma *Collect-mono*: $(\bigwedge x.\ P\ x \longrightarrow Q\ x) \Longrightarrow$ *Collect* $P \subseteq$ *Collect* Q
by *blast*

lemma *Int-Collect-mono*: $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow P\ x \longrightarrow Q\ x) \Longrightarrow A \cap$ *Collect* $P \subseteq B \cap$ *Collect* Q
by *blast*

lemmas *basic-monos* =
subset-refl imp-refl disj-mono conj-mono ex-mono Collect-mono in-mono

lemma *eq-to-mono*: $a = b \Longrightarrow c = d \Longrightarrow b \longrightarrow d \Longrightarrow a \longrightarrow c$

by *iprover*

8.4.5 Inverse image of a function

definition *vimage* :: $('a \Rightarrow 'b) \Rightarrow 'b\ set \Rightarrow 'a\ set$ (**infixr** ‹−'› *90*)
where $f\ -`\ B \equiv \{x.\ f\ x \in B\}$

lemma *vimage-eq* [*simp*]: $a \in f\ -`\ B \longleftrightarrow f\ a \in B$
unfolding *vimage-def* **by** *blast*

lemma *vimage-singleton-eq*: $a \in f\ -`\ \{b\} \longleftrightarrow f\ a = b$
by *simp*

lemma *vimageI* [*intro*]: $f\ a = b \Longrightarrow b \in B \Longrightarrow a \in f\ -`\ B$
unfolding *vimage-def* **by** *blast*

lemma *vimageI2*: $f\ a \in A \Longrightarrow a \in f\ -`\ A$
unfolding *vimage-def* **by** *fast*

lemma *vimageE* [*elim*!]: $a \in f\ -`\ B \Longrightarrow (\bigwedge x.\ f\ a = x \Longrightarrow x \in B \Longrightarrow P) \Longrightarrow P$
unfolding *vimage-def* **by** *blast*

lemma *vimageD*: $a \in f\ -`\ A \Longrightarrow f\ a \in A$
unfolding *vimage-def* **by** *fast*

lemma *vimage-empty* [*simp*]: $f\ -`\ \{\} = \{\}$
by *blast*

lemma *vimage-Compl*: $f\ -`\ (-\ A) = -\ (f\ -`\ A)$
by *blast*

lemma *vimage-Un* [*simp*]: $f\ -`\ (A \cup B) = (f\ -`\ A) \cup (f\ -`\ B)$
by *blast*

lemma *vimage-Int* [*simp*]: $f\ -`\ (A \cap B) = (f\ -`\ A) \cap (f\ -`\ B)$
by *fast*

lemma *vimage-Collect-eq* [*simp*]: $f\ -`\ Collect\ P = \{y.\ P\ (f\ y)\}$
by *blast*

lemma *vimage-Collect*: $(\bigwedge x.\ P\ (f\ x) = Q\ x) \Longrightarrow f\ -`\ (Collect\ P) = Collect\ Q$
by *blast*

lemma *vimage-insert*: $f\ -`\ (insert\ a\ B) = (f\ -`\ \{a\}) \cup (f\ -`\ B)$
— NOT suitable for rewriting because of the recurrence of $\{a\}$.
by *blast*

lemma *vimage-Diff*: $f\ -`\ (A - B) = (f\ -`\ A) - (f\ -`\ B)$
by *blast*

lemma *vimage-UNIV* [*simp*]: f −' *UNIV* = *UNIV*
by *blast*

lemma *vimage-mono*: $A \subseteq B \Longrightarrow f$ −' $A \subseteq f$ −' B
— monotonicity
by *blast*

lemma *vimage-image-eq*: f −' (f ' A) = $\{y.\ \exists x{\in}A.\ f\ x = f\ y\}$
by (*blast intro*: *sym*)

lemma *image-vimage-subset*: f ' (f −' A) $\subseteq A$
by *blast*

lemma *image-vimage-eq* [*simp*]: f ' (f −' A) = $A \cap$ *range* f
by *blast*

lemma *image-subset-iff-subset-vimage*: f ' $A \subseteq B \longleftrightarrow A \subseteq f$ −' B
by *blast*

lemma *subset-vimage-iff*: $A \subseteq f$ −' $B \longleftrightarrow (\forall x{\in}A.\ f\ x \in B)$
by *auto*

lemma *vimage-const* [*simp*]: (($\lambda x.\ c$) −' A) = (*if* $c \in A$ *then UNIV else* {})
by *auto*

lemma *vimage-if* [*simp*]: (($\lambda x.$ *if* $x \in B$ *then* c *else* d) −' A) =
(*if* $c \in A$ *then* (*if* $d \in A$ *then UNIV else* B)
else if $d \in A$ *then* − B *else* {})
by (*auto simp add*: *vimage-def*)

lemma *vimage-inter-cong*: ($\bigwedge w.\ w \in S \Longrightarrow f\ w = g\ w$) $\Longrightarrow f$ −' $y \cap S = g$ −' $y \cap S$
by *auto*

lemma *vimage-ident* [*simp*]: ($\lambda x.\ x$) −' $Y = Y$
by *blast*

8.4.6 Singleton sets

definition *is-singleton* :: *'a set* ⇒ *bool*
where *is-singleton* $A \longleftrightarrow (\exists x.\ A = \{x\})$

lemma *is-singletonI* [*simp*, *intro*!]: *is-singleton* $\{x\}$
unfolding *is-singleton-def* **by** *simp*

lemma *is-singletonI'*: $A \neq \{\} \Longrightarrow (\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x = y) \Longrightarrow$ *is-singleton* A
unfolding *is-singleton-def* **by** *blast*

lemma *is-singletonE*: *is-singleton A* $\Longrightarrow$ $(\bigwedge x.\ A = \{x\} \Longrightarrow P) \Longrightarrow P$
unfolding *is-singleton-def* **by** *blast*

8.4.7 Getting the contents of a singleton set

definition *the-elem* :: *'a set* $\Rightarrow$ *'a*
where *the-elem X* = (*THE x. X* = $\{x\}$)

lemma *the-elem-eq* [*simp*]: *the-elem* $\{x\} = x$
by (*simp add*: *the-elem-def*)

lemma *is-singleton-the-elem*: *is-singleton A* $\longleftrightarrow$ $A = \{$*the-elem A*$\}$
by (*auto simp*: *is-singleton-def*)

lemma *the-elem-image-unique*:
assumes $A \neq \{\}$
and $*$: $\bigwedge y.\ y \in A \Longrightarrow f\ y = a$
shows *the-elem* $(f\ `\ A) = a$
unfolding *the-elem-def*
proof (*rule the1-equality*)
from ‹$A \neq \{\}$› **obtain** *y* **where** $y \in A$ **by** *auto*
with $*$ ‹$y \in A$› **have** $a \in f\ `\ A$ **by** *blast*
with $*$ **show** $f\ `\ A = \{a\}$ **by** *auto*
then show $\exists!x.\ f\ `\ A = \{x\}$ **by** *auto*
qed

8.4.8 Monad operation

definition *bind* :: *'a set* $\Rightarrow$ (*'a* $\Rightarrow$ *'b set*) $\Rightarrow$ *'b set*
where *bind A f* = $\{x.\ \exists B \in f`A.\ x \in B\}$

hide-const (**open**) *bind*

lemma *bind-bind*: *Set.bind* (*Set.bind A B*) *C* = *Set.bind A* (λx. *Set.bind* (*B x*) *C*)
for *A* :: *'a set*
by (*auto simp*: *bind-def*)

lemma *empty-bind* [*simp*]: *Set.bind* $\{\}$ *f* = $\{\}$
by (*simp add*: *bind-def*)

lemma *nonempty-bind-const*: $A \neq \{\} \Longrightarrow$ *Set.bind A* (λ-. *B*) = *B*
by (*auto simp*: *bind-def*)

lemma *bind-const*: *Set.bind A* (λ-. *B*) = (*if* $A = \{\}$ *then* $\{\}$ *else B*)
by (*auto simp*: *bind-def*)

lemma *bind-singleton-conv-image*: *Set.bind A* (λx. $\{f\ x\}$) = $f\ `\ A$
by (*auto simp*: *bind-def*)

8.4.9 Operations for execution

definition *is-empty* :: *'a set* ⇒ *bool*
  where [*code-abbrev*]: *is-empty A* ⟷ *A* = {}

hide-const (**open**) *is-empty*

definition *remove* :: *'a* ⇒ *'a set* ⇒ *'a set*
  where [*code-abbrev*]: *remove x A* = *A* − {*x*}

hide-const (**open**) *remove*

lemma *member-remove* [*simp*]: *x* ∈ *Set.remove y A* ⟷ *x* ∈ *A* ∧ *x* ≠ *y*
  by (*simp add*: *remove-def*)

definition *filter* :: (*'a* ⇒ *bool*) ⇒ *'a set* ⇒ *'a set*
  where [*code-abbrev*]: *filter P A* = {*a* ∈ *A*. *P a*}

hide-const (**open**) *filter*

lemma *member-filter* [*simp*]: *x* ∈ *Set.filter P A* ⟷ *x* ∈ *A* ∧ *P x*
  by (*simp add*: *filter-def*)

instantiation *set* :: (*equal*) *equal*
begin

definition *HOL.equal A B* ⟷ *A* ⊆ *B* ∧ *B* ⊆ *A*

instance by *standard* (*auto simp add*: *equal-set-def*)

end

Misc

definition *pairwise* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a set* ⇒ *bool*
  where *pairwise R S* ⟷ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *x* ≠ *y* ⟶ *R x y*)

lemma *pairwise-alt*: *pairwise R S* ⟷ (∀ *x*∈*S*. ∀ *y*∈*S*−{*x*}. *R x y*)
by (*auto simp add*: *pairwise-def*)

lemma *pairwise-trivial* [*simp*]: *pairwise* (λ*i j*. *j* ≠ *i*) *I*
  by (*auto simp*: *pairwise-def*)

lemma *pairwiseI* [*intro?*]:
  pairwise R S **if** ⋀*x y*. *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *x* ≠ *y* ⟹ *R x y*
  using *that* **by** (*simp add*: *pairwise-def*)

lemma *pairwiseD*:
  R x y **and** *R y x*
  if *pairwise R S x* ∈ *S* **and** *y* ∈ *S* **and** *x* ≠ *y*
  using *that* **by** (*simp-all add*: *pairwise-def*)

lemma *pairwise-empty* [*simp*]: *pairwise* P {}
by (*simp add*: *pairwise-def*)

lemma *pairwise-singleton* [*simp*]: *pairwise* P $\{A\}$
by (*simp add*: *pairwise-def*)

lemma *pairwise-insert*:
pairwise r (*insert* x s) $\longleftrightarrow$ ($\forall y.\ y \in s \wedge y \neq x \longrightarrow r\ x\ y \wedge r\ y\ x$) $\wedge$ *pairwise* r s
by (*force simp*: *pairwise-def*)

lemma *pairwise-subset*: *pairwise* P $S \Longrightarrow T \subseteq S \Longrightarrow$ *pairwise* P T
by (*force simp*: *pairwise-def*)

lemma *pairwise-mono*: ⟦*pairwise* P A; $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$; $B \subseteq A$⟧ $\Longrightarrow$ *pairwise* Q B
by (*fastforce simp*: *pairwise-def*)

lemma *pairwise-imageI*:
pairwise P (f ‘ A)
if $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow f\ x \neq f\ y \Longrightarrow P\ (f\ x)\ (f\ y)$
using *that* **by** (*auto intro*: *pairwiseI*)

lemma *pairwise-image*: *pairwise* r (f ‘ s) $\longleftrightarrow$ *pairwise* ($\lambda x\ y.\ (f\ x \neq f\ y) \longrightarrow r\ (f\ x)\ (f\ y)$) s
by (*force simp*: *pairwise-def*)

definition *disjnt* :: $'a$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *bool*
where *disjnt* A $B \longleftrightarrow A \cap B = \{\}$

lemma *disjnt-self-iff-empty* [*simp*]: *disjnt* S $S \longleftrightarrow S = \{\}$
by (*auto simp*: *disjnt-def*)

lemma *disjnt-commute*: *disjnt* A B = *disjnt* B A
by (*auto simp*: *disjnt-def*)

lemma *disjnt-iff*: *disjnt* A $B \longleftrightarrow (\forall x.\ \neg\ (x \in A \wedge x \in B))$
by (*force simp*: *disjnt-def*)

lemma *disjnt-sym*: *disjnt* A $B \Longrightarrow$ *disjnt* B A
using *disjnt-iff* **by** *blast*

lemma *disjnt-empty1* [*simp*]: *disjnt* {} A **and** *disjnt-empty2* [*simp*]: *disjnt* A {}
by (*auto simp*: *disjnt-def*)

lemma *disjnt-insert1* [*simp*]: *disjnt* (*insert* a X) $Y \longleftrightarrow a \notin Y \wedge$ *disjnt* X Y
by (*simp add*: *disjnt-def*)

lemma *disjnt-insert2* [*simp*]: *disjnt* Y (*insert* a X) $\longleftrightarrow a \notin Y \wedge$ *disjnt* Y X

by (*simp add*: *disjnt-def*)

lemma *disjnt-subset1* : $[\![disjnt\ X\ Y;\ Z \subseteq X]\!] \Longrightarrow disjnt\ Z\ Y$
by (*auto simp*: *disjnt-def*)

lemma *disjnt-subset2* : $[\![disjnt\ X\ Y;\ Z \subseteq Y]\!] \Longrightarrow disjnt\ X\ Z$
by (*auto simp*: *disjnt-def*)

lemma *disjnt-Un1* [*simp*]: $disjnt\ (A \cup B)\ C \longleftrightarrow disjnt\ A\ C \wedge disjnt\ B\ C$
by (*auto simp*: *disjnt-def*)

lemma *disjnt-Un2* [*simp*]: $disjnt\ C\ (A \cup B) \longleftrightarrow disjnt\ C\ A \wedge disjnt\ C\ B$
by (*auto simp*: *disjnt-def*)

lemma *disjnt-Diff1*: $disjnt\ (X-Y)\ (U-V)$ **and** *disjnt-Diff2*: $disjnt\ (U-V)\ (X-Y)$
if $X \subseteq V$
using *that* **by** (*auto simp*: *disjnt-def*)

lemma *disjoint-image-subset*: $[\![pairwise\ disjnt\ \mathcal{A};\ \bigwedge X.\ X \in \mathcal{A} \Longrightarrow f\ X \subseteq X]\!] \Longrightarrow$
$pairwise\ disjnt\ (f\ `\mathcal{A})$
unfolding *disjnt-def pairwise-def* **by** *fast*

lemma *pairwise-disjnt-iff*: $pairwise\ disjnt\ \mathcal{A} \longleftrightarrow (\forall x.\ \exists_{\leq 1}\ X.\ X \in \mathcal{A} \wedge x \in X)$
by (*auto simp*: *Uniq-def disjnt-iff pairwise-def*)

lemma *disjnt-insert*:
‹$disjnt\ (insert\ x\ M)\ N$› **if** ‹$x \notin N$› ‹$disjnt\ M\ N$›
using *that* **by** (*simp add*: *disjnt-def*)

lemma *Int-emptyI*: $(\bigwedge x.\ x \in A \Longrightarrow x \in B \Longrightarrow False) \Longrightarrow A \cap B = \{\}$
by *blast*

lemma *in-image-insert-iff*:
assumes $\bigwedge C.\ C \in B \Longrightarrow x \notin C$
shows $A \in insert\ x\ `\ B \longleftrightarrow x \in A \wedge A - \{x\} \in B$ (**is** $?P \longleftrightarrow ?Q$)
proof
assume $?P$ **then show** $?Q$
using *assms* **by** *auto*
next
assume $?Q$
then have $x \in A$ **and** $A - \{x\} \in B$
by *simp-all*
from ‹$A - \{x\} \in B$› **have** $insert\ x\ (A - \{x\}) \in insert\ x\ `\ B$
by (*rule imageI*)
also from ‹$x \in A$›
have $insert\ x\ (A - \{x\}) = A$
by *auto*
finally show $?P$ **.**
qed

hide-const (**open**) *member not-member*

lemmas *equalityI* = *subset-antisym*
lemmas *set-mp* = *subsetD*
lemmas *set-rev-mp* = *rev-subsetD*

ML ‹
val Ball-def = @{*thm Ball-def*}
val Bex-def = @{*thm Bex-def*}
val CollectD = @{*thm CollectD*}
val CollectE = @{*thm CollectE*}
val CollectI = @{*thm CollectI*}
val Collect-conj-eq = @{*thm Collect-conj-eq*}
val Collect-mem-eq = @{*thm Collect-mem-eq*}
val IntD1 = @{*thm IntD1*}
val IntD2 = @{*thm IntD2*}
val IntE = @{*thm IntE*}
val IntI = @{*thm IntI*}
val Int-Collect = @{*thm Int-Collect*}
val UNIV-I = @{*thm UNIV-I*}
val UNIV-witness = @{*thm UNIV-witness*}
val UnE = @{*thm UnE*}
val UnI1 = @{*thm UnI1*}
val UnI2 = @{*thm UnI2*}
val ballE = @{*thm ballE*}
val ballI = @{*thm ballI*}
val bexCI = @{*thm bexCI*}
val bexE = @{*thm bexE*}
val bexI = @{*thm bexI*}
val bex-triv = @{*thm bex-triv*}
val bspec = @{*thm bspec*}
val contra-subsetD = @{*thm contra-subsetD*}
val equalityCE = @{*thm equalityCE*}
val equalityD1 = @{*thm equalityD1*}
val equalityD2 = @{*thm equalityD2*}
val equalityE = @{*thm equalityE*}
val equalityI = @{*thm equalityI*}
val imageE = @{*thm imageE*}
val imageI = @{*thm imageI*}
val image-Un = @{*thm image-Un*}
val image-insert = @{*thm image-insert*}
val insert-commute = @{*thm insert-commute*}
val insert-iff = @{*thm insert-iff*}
val mem-Collect-eq = @{*thm mem-Collect-eq*}
val rangeE = @{*thm rangeE*}
val rangeI = @{*thm rangeI*}
val range-eqI = @{*thm range-eqI*}
val subsetCE = @{*thm subsetCE*}

val subsetD = @{*thm subsetD*}
val subsetI = @{*thm subsetI*}
val subset-refl = @{*thm subset-refl*}
val subset-trans = @{*thm subset-trans*}
val vimageD = @{*thm vimageD*}
val vimageE = @{*thm vimageE*}
val vimageI = @{*thm vimageI*}
val vimageI2 = @{*thm vimageI2*}
val vimage-Collect = @{*thm vimage-Collect*}
val vimage-Int = @{*thm vimage-Int*}
val vimage-Un = @{*thm vimage-Un*}
›

end

9 HOL type definitions

theory *Typedef*
imports *Set*
keywords
  typedef :: *thy-goal-defn* **and**
  morphisms :: *quasi-command*
begin

locale *type-definition* =
  fixes *Rep* **and** *Abs* **and** *A*
  assumes *Rep*: *Rep x* ∈ *A*
    and *Rep-inverse*: *Abs* (*Rep x*) = *x*
    and *Abs-inverse*: *y* ∈ *A* ⟹ *Rep* (*Abs y*) = *y*
  — This will be axiomatized for each typedef!
begin

lemma *Rep-inject*: *Rep x* = *Rep y* ⟷ *x* = *y*
proof
  assume *Rep x* = *Rep y*
  then have *Abs* (*Rep x*) = *Abs* (*Rep y*) **by** (*simp only*:)
  also have *Abs* (*Rep x*) = *x* **by** (*rule Rep-inverse*)
  also have *Abs* (*Rep y*) = *y* **by** (*rule Rep-inverse*)
  finally show *x* = *y* **.**
next
  show *x* = *y* ⟹ *Rep x* = *Rep y* **by** (*simp only*:)
qed

lemma *Abs-inject*:
  assumes *x* ∈ *A* **and** *y* ∈ *A*
  shows *Abs x* = *Abs y* ⟷ *x* = *y*
proof
  assume *Abs x* = *Abs y*
  then have *Rep* (*Abs x*) = *Rep* (*Abs y*) **by** (*simp only*:)

also from ‹$x \in A$› **have** $Rep\ (Abs\ x) = x$ **by** (*rule Abs-inverse*)
also from ‹$y \in A$› **have** $Rep\ (Abs\ y) = y$ **by** (*rule Abs-inverse*)
finally show $x = y$ **.**
next
show $x = y \Longrightarrow Abs\ x = Abs\ y$ **by** (*simp only*:)
qed

lemma *Rep-cases* [*cases set*]:
assumes $y \in A$
and *hyp*: $\bigwedge x.\ y = Rep\ x \Longrightarrow P$
shows P
proof (*rule hyp*)
from ‹$y \in A$› **have** $Rep\ (Abs\ y) = y$ **by** (*rule Abs-inverse*)
then show $y = Rep\ (Abs\ y)$ **..**
qed

lemma *Abs-cases* [*cases type*]:
assumes r: $\bigwedge y.\ x = Abs\ y \Longrightarrow y \in A \Longrightarrow P$
shows P
proof (*rule r*)
have $Abs\ (Rep\ x) = x$ **by** (*rule Rep-inverse*)
then show $x = Abs\ (Rep\ x)$ **..**
show $Rep\ x \in A$ **by** (*rule Rep*)
qed

lemma *Rep-induct* [*induct set*]:
assumes y: $y \in A$
and *hyp*: $\bigwedge x.\ P\ (Rep\ x)$
shows $P\ y$
proof −
have $P\ (Rep\ (Abs\ y))$ **by** (*rule hyp*)
also from y **have** $Rep\ (Abs\ y) = y$ **by** (*rule Abs-inverse*)
finally show $P\ y$ **.**
qed

lemma *Abs-induct* [*induct type*]:
assumes r: $\bigwedge y.\ y \in A \Longrightarrow P\ (Abs\ y)$
shows $P\ x$
proof −
have $Rep\ x \in A$ **by** (*rule Rep*)
then have $P\ (Abs\ (Rep\ x))$ **by** (*rule r*)
also have $Abs\ (Rep\ x) = x$ **by** (*rule Rep-inverse*)
finally show $P\ x$ **.**
qed

lemma *Rep-range*: $range\ Rep = A$
proof
show $range\ Rep \subseteq A$ **using** *Rep* **by** (*auto simp add*: *image-def*)
show $A \subseteq range\ Rep$

proof
fix x **assume** $x \in A$
then have $x = Rep\ (Abs\ x)$ **by** (*rule Abs-inverse* [*symmetric*])
then show $x \in range\ Rep$ **by** (*rule range-eqI*)
qed
qed

lemma *Abs-image*: $Abs\ `\ A = UNIV$
proof
show $Abs\ `\ A \subseteq UNIV$ **by** (*rule subset-UNIV*)
show $UNIV \subseteq Abs\ `\ A$
proof
show $x \in Abs\ `\ A$ **for** x
proof (*rule image-eqI*)
show $x = Abs\ (Rep\ x)$ **by** (*rule Rep-inverse* [*symmetric*])
show $Rep\ x \in A$ **by** (*rule Rep*)
qed
qed
qed

end

ML-file ‹*Tools/typedef.ML*›

end

10 Notions about functions

theory *Fun*
imports *Set*
keywords *functor* :: *thy-goal-defn*
begin

lemma *apply-inverse*: $f\ x = u \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow g\ (f\ x) = x) \Longrightarrow P\ x \Longrightarrow x = g\ u$
by *auto*

Uniqueness, so NOT the axiom of choice.

lemma *uniq-choice*: $\forall x.\ \exists !y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$
by (*force intro*: *theI'*)

lemma *b-uniq-choice*: $\forall x \in S.\ \exists !y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x \in S.\ Q\ x\ (f\ x)$
by (*force intro*: *theI'*)

10.1 The Identity Function *id*

definition $id :: 'a \Rightarrow 'a$
where $id = (\lambda x.\ x)$

lemma *id-apply* [*simp*]: *id* $x = x$
by (*simp add*: *id-def*)

lemma *image-id* [*simp*]: *image id* = *id*
by (*simp add*: *id-def fun-eq-iff*)

lemma *vimage-id* [*simp*]: *vimage id* = *id*
by (*simp add*: *id-def fun-eq-iff*)

lemma *eq-id-iff*: $(\forall x.\ f\ x = x) \longleftrightarrow f = id$
by *auto*

code-printing
constant *id* ⇀ (*Haskell*) *id*

10.2 The Composition Operator $f \circ g$

definition *comp* :: $('b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ (**infixl** ‹∘› *55*)
where $f \circ g = (\lambda x.\ f\ (g\ x))$

notation (*ASCII*)
comp (**infixl** ‹*o*› *55*)

lemma *comp-apply* [*simp*]: $(f \circ g)\ x = f\ (g\ x)$
by (*simp add*: *comp-def*)

lemma *comp-assoc*: $(f \circ g) \circ h = f \circ (g \circ h)$
by (*simp add*: *fun-eq-iff*)

lemma *id-comp* [*simp*]: $id \circ g = g$
by (*simp add*: *fun-eq-iff*)

lemma *comp-id* [*simp*]: $f \circ id = f$
by (*simp add*: *fun-eq-iff*)

lemma *comp-eq-dest*: $a \circ b = c \circ d \Longrightarrow a\ (b\ v) = c\ (d\ v)$
by (*simp add*: *fun-eq-iff*)

lemma *comp-eq-elim*: $a \circ b = c \circ d \Longrightarrow ((\bigwedge v.\ a\ (b\ v) = c\ (d\ v)) \Longrightarrow R) \Longrightarrow R$
by (*simp add*: *fun-eq-iff*)

lemma *comp-eq-dest-lhs*: $a \circ b = c \Longrightarrow a\ (b\ v) = c\ v$
by *clarsimp*

lemma *comp-eq-id-dest*: $a \circ b = id \circ c \Longrightarrow a\ (b\ v) = c\ v$
by *clarsimp*

lemma *image-comp*: f ‘ $(g$ ‘ $r) = (f \circ g)$ ‘ r
by *auto*

lemma *vimage-comp*: $f -` (g -` x) = (g \circ f) -` x$
by *auto*

lemma *image-eq-imp-comp*: $f ` A = g ` B \Longrightarrow (h \circ f) ` A = (h \circ g) ` B$
by (*auto simp*: *comp-def elim*!: *equalityE*)

lemma *image-bind*: $f ` (Set.bind\ A\ g) = Set.bind\ A\ ((`)\ f \circ g)$
by (*auto simp add*: *Set.bind-def*)

lemma *bind-image*: $Set.bind\ (f ` A)\ g = Set.bind\ A\ (g \circ f)$
by (*auto simp add*: *Set.bind-def*)

lemma (**in** *group-add*) *minus-comp-minus* [*simp*]: $uminus \circ uminus = id$
by (*simp add*: *fun-eq-iff*)

lemma (**in** *boolean-algebra*) *minus-comp-minus* [*simp*]: $uminus \circ uminus = id$
by (*simp add*: *fun-eq-iff*)

code-printing
constant *comp* ⇀ (*SML*) **infixl** *5 o* **and** (*Haskell*) **infixr** *9* .

10.3 The Forward Composition Operator *fcomp*

definition *fcomp* :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'c$ (**infixl** ‹∘>› *60*)
where $f \circ\!> g = (\lambda x.\ g\ (f\ x))$

lemma *fcomp-apply* [*simp*]: $(f \circ\!> g)\ x = g\ (f\ x)$
by (*simp add*: *fcomp-def*)

lemma *fcomp-assoc*: $(f \circ\!> g) \circ\!> h = f \circ\!> (g \circ\!> h)$
by (*simp add*: *fcomp-def*)

lemma *id-fcomp* [*simp*]: $id \circ\!> g = g$
by (*simp add*: *fcomp-def*)

lemma *fcomp-id* [*simp*]: $f \circ\!> id = f$
by (*simp add*: *fcomp-def*)

lemma *fcomp-comp*: $fcomp\ f\ g = comp\ g\ f$
by (*simp add*: *ext*)

code-printing
constant *fcomp* ⇀ (*Eval*) **infixl** *1* #>

no-notation *fcomp* (**infixl** ‹∘>› *60*)

10.4 Mapping functions

definition *map-fun* :: $('c \Rightarrow 'a) \Rightarrow ('b \Rightarrow 'd) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'c \Rightarrow 'd$

where $map\text{-}fun\ f\ g\ h = g \circ h \circ f$

lemma *map-fun-apply* [*simp*]: $map\text{-}fun\ f\ g\ h\ x = g\ (h\ (f\ x))$
by (*simp add*: *map-fun-def*)

10.5 Injectivity and Bijectivity

definition $inj\text{-}on :: ('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow bool$ — injective
where $inj\text{-}on\ f\ A \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}A.\ f\ x = f\ y \longrightarrow x = y)$

definition $bij\text{-}betw :: ('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow bool$ — bijective
where $bij\text{-}betw\ f\ A\ B \longleftrightarrow inj\text{-}on\ f\ A \wedge f\ `\ A = B$

A common special case: functions injective, surjective or bijective over the entire domain type.

abbreviation $inj :: ('a \Rightarrow 'b) \Rightarrow bool$
where $inj\ f \equiv inj\text{-}on\ f\ UNIV$

abbreviation $surj :: ('a \Rightarrow 'b) \Rightarrow bool$
where $surj\ f \equiv range\ f = UNIV$

translations — The negated case:
$\neg\ CONST\ surj\ f \leftharpoondown CONST\ range\ f \neq CONST\ UNIV$

abbreviation $bij :: ('a \Rightarrow 'b) \Rightarrow bool$
where $bij\ f \equiv bij\text{-}betw\ f\ UNIV\ UNIV$

lemma *inj-def*: $inj\ f \longleftrightarrow (\forall x\ y.\ f\ x = f\ y \longrightarrow x = y)$
unfolding *inj-on-def* **by** *blast*

lemma *injI*: $(\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y) \Longrightarrow inj\ f$
unfolding *inj-def* **by** *blast*

theorem *range-ex1-eq*: $inj\ f \Longrightarrow b \in range\ f \longleftrightarrow (\exists!x.\ b = f\ x)$
unfolding *inj-def* **by** *blast*

lemma *injD*: $inj\ f \Longrightarrow f\ x = f\ y \Longrightarrow x = y$
by (*simp add*: *inj-def*)

lemma *inj-on-eq-iff*: $inj\text{-}on\ f\ A \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \longleftrightarrow x = y$
by (*auto simp*: *inj-on-def*)

lemma *inj-on-cong*: $(\bigwedge a.\ a \in A \Longrightarrow f\ a = g\ a) \Longrightarrow inj\text{-}on\ f\ A \longleftrightarrow inj\text{-}on\ g\ A$
by (*auto simp*: *inj-on-def*)

lemma *image-strict-mono*: $inj\text{-}on\ f\ B \Longrightarrow A \subset B \Longrightarrow f\ `\ A \subset f\ `\ B$
unfolding *inj-on-def* **by** *blast*

lemma *inj-compose*: $inj\ f \Longrightarrow inj\ g \Longrightarrow inj\ (f \circ g)$

by (*simp add*: *inj-def*)

lemma *inj-fun*: *inj f* $\Longrightarrow$ *inj* ($\lambda x\ y.\ f\ x$)
by (*simp add*: *inj-def fun-eq-iff*)

lemma *inj-eq*: *inj f* $\Longrightarrow f\ x = f\ y \longleftrightarrow x = y$
by (*simp add*: *inj-on-eq-iff*)

lemma *inj-on-iff-Uniq*: *inj-on f A* $\longleftrightarrow (\forall x{\in}A.\ \exists_{\leq 1}y.\ y{\in}A \land f\ x = f\ y)$
by (*auto simp*: *Uniq-def inj-on-def*)

lemma *inj-on-id*[*simp*]: *inj-on id A*
by (*simp add*: *inj-on-def*)

lemma *inj-on-id2*[*simp*]: *inj-on* ($\lambda x.\ x$) *A*
by (*simp add*: *inj-on-def*)

lemma *inj-on-Int*: *inj-on f A* $\lor$ *inj-on f B* $\Longrightarrow$ *inj-on f* $(A \cap B)$
unfolding *inj-on-def* **by** *blast*

lemma *surj-id*: *surj id*
by *simp*

lemma *bij-id*[*simp*]: *bij id*
by (*simp add*: *bij-betw-def*)

lemma *bij-uminus*: *bij* (*uminus* :: *'a* $\Rightarrow$ *'a*::*group-add*)
unfolding *bij-betw-def inj-on-def*
by (*force intro*: *minus-minus* [*symmetric*])

lemma *bij-betwE*: *bij-betw f A B* $\Longrightarrow \forall a{\in}A.\ f\ a \in B$
unfolding *bij-betw-def* **by** *auto*

lemma *inj-onI* [*intro?*]: $(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \Longrightarrow x = y) \Longrightarrow$ *inj-on f A*
by (*simp add*: *inj-on-def*)

For those frequent proofs by contradiction

lemma *inj-onCI*: $(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \Longrightarrow x \neq y \Longrightarrow False)$ $\Longrightarrow$ *inj-on f A*
by (*force simp*: *inj-on-def*)

lemma *inj-on-inverseI*: $(\bigwedge x.\ x \in A \Longrightarrow g\ (f\ x) = x) \Longrightarrow$ *inj-on f A*
by (*auto dest*: *arg-cong* [*of concl*: *g*] *simp add*: *inj-on-def*)

lemma *inj-onD*: *inj-on f A* $\Longrightarrow f\ x = f\ y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x = y$
unfolding *inj-on-def* **by** *blast*

lemma *inj-on-subset*:

assumes *inj-on f A*
and $B \subseteq A$
shows *inj-on f B*
proof (*rule inj-onI*)
fix *a b*
assume $a \in B$ **and** $b \in B$
with *assms* **have** $a \in A$ **and** $b \in A$
by *auto*
moreover assume $f\ a = f\ b$
ultimately show $a = b$
using *assms* **by** (*auto dest*: *inj-onD*)
qed

lemma *comp-inj-on*: *inj-on f A* $\Longrightarrow$ *inj-on g* $(f\ `\ A) \Longrightarrow$ *inj-on* $(g \circ f)$ *A*
by (*simp add*: *comp-def inj-on-def*)

lemma *inj-on-imageI*: *inj-on* $(g \circ f)$ $A \Longrightarrow$ *inj-on g* $(f\ `\ A)$
by (*auto simp add*: *inj-on-def*)

lemma *inj-on-image-iff*:
$\forall x \in A.\ \forall y \in A.\ g\ (f\ x) = g\ (f\ y) \longleftrightarrow g\ x = g\ y \Longrightarrow$ *inj-on f A* $\Longrightarrow$ *inj-on g* $(f\ `\ A) \longleftrightarrow$ *inj-on g A*
unfolding *inj-on-def* **by** *blast*

lemma *inj-on-contraD*: *inj-on f A* $\Longrightarrow x \neq y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow f\ x \neq f\ y$
unfolding *inj-on-def* **by** *blast*

lemma *inj-singleton* [*simp*]: *inj-on* $(\lambda x.\ \{x\})$ *A*
by (*simp add*: *inj-on-def*)

lemma *inj-on-empty*[*iff*]: *inj-on f* $\{\}$
by (*simp add*: *inj-on-def*)

lemma *subset-inj-on*: *inj-on f B* $\Longrightarrow A \subseteq B \Longrightarrow$ *inj-on f A*
unfolding *inj-on-def* **by** *blast*

lemma *inj-on-Un*: *inj-on f* $(A \cup B) \longleftrightarrow$ *inj-on f A* $\wedge$ *inj-on f B* $\wedge f\ `\ (A - B) \cap f\ `\ (B - A) = \{\}$
unfolding *inj-on-def* **by** (*blast intro*: *sym*)

lemma *inj-on-insert* [*iff*]: *inj-on f* (*insert a A*) $\longleftrightarrow$ *inj-on f A* $\wedge f\ a \notin f\ `\ (A - \{a\})$
unfolding *inj-on-def* **by** (*blast intro*: *sym*)

lemma *inj-on-diff*: *inj-on f A* $\Longrightarrow$ *inj-on f* $(A - B)$
unfolding *inj-on-def* **by** *blast*

lemma *comp-inj-on-iff*: *inj-on f A* $\Longrightarrow$ *inj-on* $f'\ (f\ `\ A) \longleftrightarrow$ *inj-on* $(f' \circ f)$ *A*
by (*auto simp*: *comp-inj-on inj-on-def*)

lemma *inj-on-imageI2*: *inj-on* $(f' \circ f)$ $A \Longrightarrow$ *inj-on* f A
  by (*auto simp*: *comp-inj-on inj-on-def*)

lemma *inj-img-insertE*:
  assumes *inj-on* f A
  assumes $x \notin B$
    and *insert* x $B = f$ ‘ A
  obtains x' A' **where** $x' \notin A'$ **and** $A =$ *insert* x' A' **and** $x = f\ x'$ **and** $B = f$ ‘ A'
proof −
  from *assms* **have** $x \in f$ ‘ A **by** *auto*
  then obtain x' **where** ∗: $x' \in A$ $x = f\ x'$ **by** *auto*
  then have A: $A =$ *insert* x' $(A - \{x'\})$ **by** *auto*
  with *assms* ∗ **have** B: $B = f$ ‘ $(A - \{x'\})$ **by** (*auto dest*: *inj-on-contraD*)
  have $x' \notin A - \{x'\}$ **by** *simp*
  from *this* A ‹$x = f\ x'$› B **show** *?thesis* **..**
qed

lemma *linorder-inj-onI*:
  fixes A :: *'a::order set*
  assumes *ne*: $\bigwedge x\ y.\ [\![x < y;\ x{\in}A;\ y{\in}A]\!] \Longrightarrow f\ x \neq f\ y$ **and** *lin*: $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A]\!] \Longrightarrow x{\le}y \vee y{\le}x$
  shows *inj-on* f A
proof (*rule inj-onI*)
  fix x y
  assume *eq*: $f\ x = f\ y$ **and** $x{\in}A$ $y{\in}A$
  then show $x = y$
    using *lin* [*of* x y] *ne* **by** (*force simp*: *dual-order.order-iff-strict*)
qed

lemma *linorder-inj-onI'*:
  fixes A :: *'a* :: *linorder set*
  assumes $\bigwedge i\ j.\ i \in A \Longrightarrow j \in A \Longrightarrow i < j \Longrightarrow f\ i \neq f\ j$
  shows *inj-on* f A
  by (*intro linorder-inj-onI*) (*auto simp add*: *assms*)

lemma *linorder-injI*:
  assumes $\bigwedge x\ y$::*'a*::*linorder*. $x < y \Longrightarrow f\ x \neq f\ y$
  shows *inj* f
    — Courtesy of Stephan Merz
using *assms* **by** (*simp add*: *linorder-inj-onI'*)

lemma *inj-on-image-Pow*: *inj-on* f $A \Longrightarrow$ *inj-on* (*image* f) (*Pow* A)
  unfolding *Pow-def inj-on-def* **by** *blast*

lemma *bij-betw-image-Pow*: *bij-betw* f A $B \Longrightarrow$ *bij-betw* (*image* f) (*Pow* A) (*Pow* B)
  by (*auto simp add*: *bij-betw-def inj-on-image-Pow image-Pow-surj*)

lemma *surj-def*: *surj f* $\longleftrightarrow$ ($\forall$ *y*. $\exists$ *x*. *y* = *f x*)
by *auto*

lemma *surjI*:
assumes $\bigwedge x.\ g\ (f\ x) = x$
shows *surj g*
using *assms* [*symmetric*] **by** *auto*

lemma *surjD*: *surj f* $\Longrightarrow \exists x.\ y = f\ x$
by (*simp add*: *surj-def*)

lemma *surjE*: *surj f* $\Longrightarrow (\bigwedge x.\ y = f\ x \Longrightarrow C) \Longrightarrow C$
by (*simp add*: *surj-def*) *blast*

lemma *comp-surj*: *surj f* $\Longrightarrow$ *surj g* $\Longrightarrow$ *surj* ($g \circ f$)
using *image-comp* [*of g f UNIV*] **by** *simp*

lemma *bij-betw-imageI*: *inj-on f A* $\Longrightarrow$ *f* ‘ *A* = *B* $\Longrightarrow$ *bij-betw f A B*
unfolding *bij-betw-def* **by** *clarify*

lemma *bij-betw-imp-surj-on*: *bij-betw f A B* $\Longrightarrow$ *f* ‘ *A* = *B*
unfolding *bij-betw-def* **by** *clarify*

lemma *bij-betw-imp-surj*: *bij-betw f A UNIV* $\Longrightarrow$ *surj f*
unfolding *bij-betw-def* **by** *auto*

lemma *bij-betw-empty1*: *bij-betw f* {} *A* $\Longrightarrow$ *A* = {}
unfolding *bij-betw-def* **by** *blast*

lemma *bij-betw-empty2*: *bij-betw f A* {} $\Longrightarrow$ *A* = {}
unfolding *bij-betw-def* **by** *blast*

lemma *inj-on-imp-bij-betw*: *inj-on f A* $\Longrightarrow$ *bij-betw f A* (*f* ‘ *A*)
unfolding *bij-betw-def* **by** *simp*

lemma *bij-betw-DiffI*:
assumes *bij-betw f A B bij-betw f C D* $C \subseteq A$ $D \subseteq B$
shows *bij-betw f* ($A - C$) ($B - D$)
using *assms* **unfolding** *bij-betw-def inj-on-def* **by** *auto*

lemma *bij-betw-singleton-iff* [*simp*]: *bij-betw f* {*x*} {*y*} $\longleftrightarrow$ *f x* = *y*
by (*auto simp*: *bij-betw-def*)

lemma *bij-betw-singletonI* [*intro*]: *f x* = *y* $\Longrightarrow$ *bij-betw f* {*x*} {*y*}
by *auto*

lemma *bij-betw-apply*: ⟦*bij-betw f A B*; $a \in A$⟧ $\Longrightarrow f\ a \in B$
unfolding *bij-betw-def* **by** *auto*

lemma *bij-def*: $bij\ f \longleftrightarrow inj\ f \wedge surj\ f$
  by (*rule bij-betw-def*)

lemma *bijI*: $inj\ f \Longrightarrow surj\ f \Longrightarrow bij\ f$
  by (*rule bij-betw-imageI*)

lemma *bij-is-inj*: $bij\ f \Longrightarrow inj\ f$
  by (*simp add*: *bij-def*)

lemma *bij-is-surj*: $bij\ f \Longrightarrow surj\ f$
  by (*simp add*: *bij-def*)

lemma *bij-betw-imp-inj-on*: *bij-betw* $f\ A\ B \Longrightarrow$ *inj-on* $f\ A$
  by (*simp add*: *bij-betw-def*)

lemma *bij-betw-trans*: *bij-betw* $f\ A\ B \Longrightarrow$ *bij-betw* $g\ B\ C \Longrightarrow$ *bij-betw* $(g \circ f)\ A\ C$
  by (*auto simp add*:*bij-betw-def comp-inj-on*)

lemma *bij-comp*: $bij\ f \Longrightarrow bij\ g \Longrightarrow bij\ (g \circ f)$
  by (*rule bij-betw-trans*)

lemma *bij-betw-comp-iff*: *bij-betw* $f\ A\ A' \Longrightarrow$ *bij-betw* $f'\ A'\ A'' \longleftrightarrow$ *bij-betw* $(f' \circ f)\ A\ A''$
  by (*auto simp add*: *bij-betw-def inj-on-def*)

lemma *bij-betw-Collect*:
  assumes *bij-betw* $f\ A\ B$ $\bigwedge x.\ x \in A \Longrightarrow Q\ (f\ x) \longleftrightarrow P\ x$
  shows *bij-betw* $f\ \{x{\in}A.\ P\ x\}\ \{y{\in}B.\ Q\ y\}$
  using *assms* **by** (*auto simp add*: *bij-betw-def inj-on-def*)

lemma *bij-betw-comp-iff2*:
  assumes *bij*: *bij-betw* $f'\ A'\ A''$
    and *img*: $f\ `\ A \le A'$
  shows *bij-betw* $f\ A\ A' \longleftrightarrow$ *bij-betw* $(f' \circ f)\ A\ A''$ (**is** $?L \longleftrightarrow ?R$)
proof
  assume $?L$
  then show $?R$
    using *assms* **by** (*auto simp add*: *bij-betw-comp-iff*)
  next
    assume $*$: $?R$
    have *inj-on* $(f' \circ f)\ A \Longrightarrow$ *inj-on* $f\ A$
      using *inj-on-imageI2* **by** *blast*
    moreover have $A' \subseteq f\ `\ A$
    proof
      fix a'
      assume $**$: $a' \in A'$
      with *bij* **have** $f'\ a' \in A''$
        unfolding *bij-betw-def* **by** *auto*

```
      with ∗ obtain a where 1: a ∈ A ∧ f′ (f a) = f′ a′
        unfolding bij-betw-def by force
      with img have f a ∈ A′ by auto
      with bij ∗∗ 1 have f a = a′
        unfolding bij-betw-def inj-on-def by auto
      with 1 show a′ ∈ f ‘ A by auto
    qed
    ultimately show ?L
      using img ∗ by (auto simp add: bij-betw-def)
qed

lemma bij-betw-inv:
  assumes bij-betw f A B
  shows ∃ g. bij-betw g B A
proof −
  have i: inj-on f A and s: f ‘ A = B
    using assms by (auto simp: bij-betw-def)
  let ?P = λb a. a ∈ A ∧ f a = b
  let ?g = λb. The (?P b)
  have g: ?g b = a if P: ?P b a for a b
  proof −
    from that s have ex1: ∃ a. ?P b a by blast
    then have uex1: ∃!a. ?P b a by (blast dest:inj-onD[OF i])
    then show ?thesis
      using the1-equality[OF uex1, OF P] P by simp
  qed
  have inj-on ?g B
  proof (rule inj-onI)
    fix x y
    assume x ∈ B y ∈ B ?g x = ?g y
    from s ‹x ∈ B› obtain a1 where a1: ?P x a1 by blast
    from s ‹y ∈ B› obtain a2 where a2: ?P y a2 by blast
    from g [OF a1] a1 g [OF a2] a2 ‹?g x = ?g y› show x = y by simp
  qed
  moreover have ?g ‘ B = A
  proof safe
    fix b
    assume b ∈ B
    with s obtain a where P: ?P b a by blast
    with g[OF P] show ?g b ∈ A by auto
  next
    fix a
    assume a ∈ A
    with s obtain b where P: ?P b a by blast
    with s have b ∈ B by blast
    with g[OF P] have ∃ b∈B. a = ?g b by blast
    then show a ∈ ?g ‘ B
      by auto
  qed
```

ultimately show *?thesis*
by (*auto simp*: *bij-betw-def*)
qed

lemma *bij-betw-cong*: ($\bigwedge a.\ a \in A \Longrightarrow f\ a = g\ a$) $\Longrightarrow$ *bij-betw f A A'* = *bij-betw g A A'*
unfolding *bij-betw-def inj-on-def* **by** *safe force+*

lemma *bij-betw-id*[*intro*, *simp*]: *bij-betw id A A*
unfolding *bij-betw-def id-def* **by** *auto*

lemma *bij-betw-id-iff*: *bij-betw id A B* $\longleftrightarrow$ $A = B$
by (*auto simp add*: *bij-betw-def*)

lemma *bij-betw-combine*:
bij-betw f A B $\Longrightarrow$ *bij-betw f C D* $\Longrightarrow$ $B \cap D = \{\}$ $\Longrightarrow$ *bij-betw f* $(A \cup C)$ $(B \cup D)$
unfolding *bij-betw-def inj-on-Un image-Un* **by** *auto*

lemma *bij-betw-subset*: *bij-betw f A A'* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ f ‘ $B = B'$ $\Longrightarrow$ *bij-betw f B B'*
by (*auto simp add*: *bij-betw-def inj-on-def*)

lemma *bij-betw-ball*: *bij-betw f A B* $\Longrightarrow$ ($\forall\, b \in B.\ phi\ b$) = ($\forall\, a \in A.\ phi\ (f\ a)$)
unfolding *bij-betw-def inj-on-def* **by** *blast*

lemma *bij-pointE*:
assumes *bij f*
obtains x **where** $y = f\ x$ **and** $\bigwedge x'.\ y = f\ x' \Longrightarrow x' = x$
proof −
from *assms* **have** *inj f* **by** (*rule bij-is-inj*)
moreover from *assms* **have** *surj f* **by** (*rule bij-is-surj*)
then have $y \in$ *range f* **by** *simp*
ultimately have $\exists!x.\ y = f\ x$ **by** (*simp add*: *range-ex1-eq*)
with *that* **show** *thesis* **by** *blast*
qed

lemma *bij-iff*:
‹*bij f* $\longleftrightarrow$ ($\forall\, x.\ \exists!y.\ f\ y = x$)› (**is** ‹*?P* $\longleftrightarrow$ *?Q*›)
proof
assume *?P*
then have ‹*inj f*› ‹*surj f*›
by (*simp-all add*: *bij-def*)
show *?Q*
proof
fix y
from ‹*surj f*› **obtain** x **where** ‹$y = f\ x$›
by (*auto simp add*: *surj-def*)
with ‹*inj f*› **show** ‹$\exists!x.\ f\ x = y$›

```
      by (auto simp add: inj-def)
  qed
next
  assume ?Q
  then have ‹inj f›
    by (auto simp add: inj-def)
  moreover have ‹∃ x. y = f x› for y
  proof −
    from ‹?Q› obtain x where ‹f x = y›
      by blast
    then have ‹y = f x›
      by simp
    then show ?thesis ..
  qed
  then have ‹surj f›
    by (auto simp add: surj-def)
  ultimately show ?P
    by (rule bijI)
qed

lemma bij-betw-partition:
  ‹bij-betw f A B›
  if ‹bij-betw f (A ∪ C) (B ∪ D)› ‹bij-betw f C D› ‹A ∩ C = {}› ‹B ∩ D = {}›
proof −
  from that have ‹inj-on f (A ∪ C)› ‹inj-on f C› ‹f ‘ (A ∪ C) = B ∪ D› ‹f ‘ C
= D›
    by (simp-all add: bij-betw-def)
  then have ‹inj-on f A› and ‹f ‘ (A − C) ∩ f ‘ (C − A) = {}›
    by (simp-all add: inj-on-Un)
  with ‹A ∩ C = {}› have ‹f ‘ A ∩ f ‘ C = {}›
    by auto
  with ‹f ‘ (A ∪ C) = B ∪ D› ‹f ‘ C = D›  ‹B ∩ D = {}›
  have ‹f ‘ A = B›
    by blast
  with ‹inj-on f A› show ?thesis
    by (simp add: bij-betw-def)
qed

lemma surj-image-vimage-eq: surj f ⟹ f ‘ (f −‘ A) = A
  by simp

lemma surj-vimage-empty:
  assumes surj f
  shows f −‘ A = {} ⟷ A = {}
  using surj-image-vimage-eq [OF ‹surj f›, of A]
  by (intro iffI) fastforce+

lemma inj-vimage-image-eq: inj f ⟹ f −‘ (f ‘ A) = A
  unfolding inj-def by blast
```

lemma *vimage-subsetD*: $surj\ f \Longrightarrow f\ -`\ B \subseteq A \Longrightarrow B \subseteq f\ `\ A$
by (*blast intro*: *sym*)

lemma *vimage-subsetI*: $inj\ f \Longrightarrow B \subseteq f\ `\ A \Longrightarrow f\ -`\ B \subseteq A$
unfolding *inj-def* **by** *blast*

lemma *vimage-subset-eq*: $bij\ f \Longrightarrow f\ -`\ B \subseteq A \longleftrightarrow B \subseteq f\ `\ A$
unfolding *bij-def* **by** (*blast del*: *subsetI intro*: *vimage-subsetI vimage-subsetD*)

lemma *inj-on-image-eq-iff*: $inj\text{-}on\ f\ C \Longrightarrow A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow f\ `\ A = f\ `\ B \longleftrightarrow A = B$
by (*fastforce simp*: *inj-on-def*)

lemma *inj-on-Un-image-eq-iff*: $inj\text{-}on\ f\ (A \cup B) \Longrightarrow f\ `\ A = f\ `\ B \longleftrightarrow A = B$
by (*erule inj-on-image-eq-iff*) *simp-all*

lemma *inj-on-image-Int*: $inj\text{-}on\ f\ C \Longrightarrow A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow f\ `\ (A \cap B) = f\ `\ A \cap f\ `\ B$
unfolding *inj-on-def* **by** *blast*

lemma *inj-on-image-set-diff*: $inj\text{-}on\ f\ C \Longrightarrow A - B \subseteq C \Longrightarrow B \subseteq C \Longrightarrow f\ `\ (A - B) = f\ `\ A - f\ `\ B$
unfolding *inj-on-def* **by** *blast*

lemma *image-Int*: $inj\ f \Longrightarrow f\ `\ (A \cap B) = f\ `\ A \cap f\ `\ B$
unfolding *inj-def* **by** *blast*

lemma *image-set-diff*: $inj\ f \Longrightarrow f\ `\ (A - B) = f\ `\ A - f\ `\ B$
unfolding *inj-def* **by** *blast*

lemma *inj-on-image-mem-iff*: $inj\text{-}on\ f\ B \Longrightarrow a \in B \Longrightarrow A \subseteq B \Longrightarrow f\ a \in f\ `\ A \longleftrightarrow a \in A$
by (*auto simp*: *inj-on-def*)

lemma *inj-image-mem-iff*: $inj\ f \Longrightarrow f\ a \in f\ `\ A \longleftrightarrow a \in A$
by (*blast dest*: *injD*)

lemma *inj-image-subset-iff*: $inj\ f \Longrightarrow f\ `\ A \subseteq f\ `\ B \longleftrightarrow A \subseteq B$
by (*blast dest*: *injD*)

lemma *inj-image-eq-iff*: $inj\ f \Longrightarrow f\ `\ A = f\ `\ B \longleftrightarrow A = B$
by (*blast dest*: *injD*)

lemma *surj-Compl-image-subset*: $surj\ f \Longrightarrow -\ (f\ `\ A) \subseteq f\ `\ (-\ A)$
by *auto*

lemma *inj-image-Compl-subset*: $inj\ f \Longrightarrow f\ `\ (-\ A) \subseteq -\ (f\ `\ A)$
by (*auto simp*: *inj-def*)

lemma *bij-image-Compl-eq*: *bij f* $\Longrightarrow$ *f* ‘ (− *A*) = − (*f* ‘ *A*)
by (*simp add*: *bij-def inj-image-Compl-subset surj-Compl-image-subset equalityI*)

lemma *inj-vimage-singleton*: *inj f* $\Longrightarrow$ *f* −‘ {*a*} ⊆ {*THE x*. *f x* = *a*}
— The inverse image of a singleton under an injective function is included in a singleton.
by (*simp add*: *inj-def*) (*blast intro*: *the-equality* [*symmetric*])

lemma *inj-on-vimage-singleton*: *inj-on f A* $\Longrightarrow$ *f* −‘ {*a*} ∩ *A* ⊆ {*THE x*. *x* ∈ *A* ∧ *f x* = *a*}
by (*auto simp add*: *inj-on-def intro*: *the-equality* [*symmetric*])

lemma *bij-betw-byWitness*:
assumes *left*: ∀ *a* ∈ *A*. *f′* (*f a*) = *a*
and *right*: ∀ *a′* ∈ *A′*. *f* (*f′ a′*) = *a′*
and *f* ‘ *A* ⊆ *A′*
and *img2*: *f′* ‘ *A′* ⊆ *A*
shows *bij-betw f A A′*
using *assms*
unfolding *bij-betw-def inj-on-def*
proof *safe*
fix *a b*
assume *a* ∈ *A* *b* ∈ *A*
with *left* **have** *a* = *f′* (*f a*) ∧ *b* = *f′* (*f b*) **by** *simp*
moreover assume *f a* = *f b*
ultimately show *a* = *b* **by** *simp*
next
fix *a′* **assume** ∗: *a′* ∈ *A′*
with *img2* **have** *f′ a′* ∈ *A* **by** *blast*
moreover from ∗ *right* **have** *a′* = *f* (*f′ a′*) **by** *simp*
ultimately show *a′* ∈ *f* ‘ *A* **by** *blast*
qed

corollary *notIn-Un-bij-betw*:
assumes *b* ∉ *A*
and *f b* ∉ *A′*
and *bij-betw f A A′*
shows *bij-betw f* (*A* ∪ {*b*}) (*A′* ∪ {*f b*})
proof −
have *bij-betw f* {*b*} {*f b*}
unfolding *bij-betw-def inj-on-def* **by** *simp*
with *assms* **show** *?thesis*
using *bij-betw-combine*[*of f A A′* {*b*} {*f b*}] **by** *blast*
qed

lemma *notIn-Un-bij-betw3*:
assumes *b* ∉ *A*
and *f b* ∉ *A′*

shows *bij-betw f A A'* = *bij-betw f* (*A* ∪ {*b*}) (*A'* ∪ {*f b*})
proof
assume *bij-betw f A A'*
then show *bij-betw f* (*A* ∪ {*b*}) (*A'* ∪ {*f b*})
using *assms notIn-Un-bij-betw* [*of b A f A'*] **by** *blast*
next
assume ∗: *bij-betw f* (*A* ∪ {*b*}) (*A'* ∪ {*f b*})
have *f ' A* = *A'*
proof *safe*
fix *a*
assume ∗∗: *a* ∈ *A*
then have *f a* ∈ *A'* ∪ {*f b*}
using ∗ **unfolding** *bij-betw-def* **by** *blast*
moreover
have *False* **if** *f a* = *f b*
proof −
have *a* = *b*
using ∗ ∗∗ *that* **unfolding** *bij-betw-def inj-on-def* **by** *blast*
with ‹*b* ∉ *A*› ∗∗ **show** *?thesis* **by** *blast*
qed
ultimately show *f a* ∈ *A'* **by** *blast*
next
fix *a'*
assume ∗∗: *a'* ∈ *A'*
then have *a'* ∈ *f '* (*A* ∪ {*b*})
using ∗ **by** (*auto simp add*: *bij-betw-def*)
then obtain *a* **where** *1*: *a* ∈ *A* ∪ {*b*} ∧ *f a* = *a'* **by** *blast*
moreover
have *False* **if** *a* = *b* **using** *1* ∗∗ ‹*f b* ∉ *A'*› *that* **by** *blast*
ultimately have *a* ∈ *A* **by** *blast*
with *1* **show** *a'* ∈ *f ' A* **by** *blast*
qed
then show *bij-betw f A A'*
using ∗ *bij-betw-subset*[*of f A* ∪ {*b*} - *A*] **by** *blast*
qed

lemma *inj-on-disjoint-Un*:
assumes *inj-on f A* **and** *inj-on g B*
and *f ' A* ∩ *g ' B* = {}
shows *inj-on* (λ*x*. *if x* ∈ *A then f x else g x*) (*A* ∪ *B*)
using *assms* **by** (*simp add*: *inj-on-def disjoint-iff*) (*blast*)

lemma *bij-betw-disjoint-Un*:
assumes *bij-betw f A C* **and** *bij-betw g B D*
and *A* ∩ *B* = {}
and *C* ∩ *D* = {}
shows *bij-betw* (λ*x*. *if x* ∈ *A then f x else g x*) (*A* ∪ *B*) (*C* ∪ *D*)
using *assms* **by** (*auto simp*: *inj-on-disjoint-Un bij-betw-def*)

lemma *involuntory-imp-bij*:
  ‹*bij f*› **if** ‹$\bigwedge x.\ f\ (f\ x) = x$›
proof (*rule bijI*)
  from *that* **show** ‹*surj f*›
    by (*rule surjI*)
  show ‹*inj f*›
  proof (*rule injI*)
    fix *x y*
    assume ‹$f\ x = f\ y$›
    then have ‹$f\ (f\ x) = f\ (f\ y)$›
      by *simp*
    then show ‹$x = y$›
      by (*simp add*: *that*)
  qed
qed

10.5.1 Inj/surj/bij of Algebraic Operations

context *cancel-semigroup-add*
begin

lemma *inj-on-add* [*simp*]:
  inj-on ((+) *a*) *A*
  by (*rule inj-onI*) *simp*

lemma *inj-on-add′* [*simp*]:
  inj-on ($\lambda b.\ b + a$) *A*
  by (*rule inj-onI*) *simp*

lemma *bij-betw-add* [*simp*]:
  bij-betw ((+) *a*) *A B* $\longleftrightarrow$ (+) *a* ‘ *A* = *B*
  by (*simp add*: *bij-betw-def*)

end

context *group-add*
begin

lemma *diff-left-imp-eq*: $a - b = a - c \Longrightarrow b = c$
unfolding *add-uminus-conv-diff*[*symmetric*]
by(*drule local.add-left-imp-eq*) *simp*

lemma *inj-uminus*[*simp*, *intro*]: *inj-on uminus A*
  by (*auto intro*!: *inj-onI*)

lemma *surj-uminus*[*simp*]: *surj uminus*
using *surjI minus-minus* **by** *blast*

lemma *surj-plus* [*simp*]:

surj ((+) *a*)
proof (*standard*, *simp*, *standard*, *simp*)
fix *x*
have $x = a + (-a + x)$ **by** (*simp add*: *add.assoc*)
thus $x \in$ *range* ((+) *a*) **by** *blast*
qed

lemma *surj-plus-right* [*simp*]:
surj ($\lambda b.\ b+a$)
proof (*standard*, *simp*, *standard*, *simp*)
fix *b* **show** $b \in$ *range* ($\lambda b.\ b+a$)
using *diff-add-cancel*[*of b a*, *symmetric*] **by** *blast*
qed

lemma *inj-on-diff-left* [*simp*]:
‹*inj-on* ((−) *a*) *A*›
by (*auto intro*: *inj-onI dest*!: *diff-left-imp-eq*)

lemma *inj-on-diff-right* [*simp*]:
‹*inj-on* ($\lambda b.\ b - a$) *A*›
by (*auto intro*: *inj-onI simp add*: *algebra-simps*)

lemma *surj-diff* [*simp*]:
surj ((−) *a*)
proof (*standard*, *simp*, *standard*, *simp*)
fix *x*
have $x = a - (-\ x + a)$ **by** (*simp add*: *algebra-simps*)
thus $x \in$ *range* ((−) *a*) **by** *blast*
qed

lemma *surj-diff-right* [*simp*]:
surj ($\lambda x.\ x - a$)
proof (*standard*, *simp*, *standard*, *simp*)
fix *x*
have $x = x + a - a$ **by** *simp*
thus $x \in$ *range* ($\lambda x.\ x - a$) **by** *fast*
qed

lemma shows *bij-plus*: *bij* ((+) *a*) **and** *bij-plus-right*: *bij* ($\lambda x.\ x + a$)
and *bij-uminus*: *bij uminus*
and *bij-diff*: *bij* ((−) *a*) **and** *bij-diff-right*: *bij* ($\lambda x.\ x - a$)
by(*simp-all add*: *bij-def*)

lemma *translation-subtract-Compl*:
($\lambda x.\ x - a$) ‘ (− *t*) = − (($\lambda x.\ x - a$) ‘ *t*)
by(*rule bij-image-Compl-eq*)
(*auto simp add*: *bij-def surj-def inj-def diff-eq-eq intro*!: *add-diff-cancel*[*symmetric*])

lemma *translation-diff*:

(+) *a* ‘ (*s* − *t*) = ((+) *a* ‘ *s*) − ((+) *a* ‘ *t*)
by *auto*

lemma *translation-subtract-diff*:
(λ*x*. *x* − *a*) ‘ (*s* − *t*) = ((λ*x*. *x* − *a*) ‘ *s*) − ((λ*x*. *x* − *a*) ‘ *t*)
by(*rule image-set-diff*)(*simp add*: *inj-on-def diff-eq-eq*)

lemma *translation-Int*:
(+) *a* ‘ (*s* ∩ *t*) = ((+) *a* ‘ *s*) ∩ ((+) *a* ‘ *t*)
by *auto*

lemma *translation-subtract-Int*:
(λ*x*. *x* − *a*) ‘ (*s* ∩ *t*) = ((λ*x*. *x* − *a*) ‘ *s*) ∩ ((λ*x*. *x* − *a*) ‘ *t*)
by(*rule image-Int*)(*simp add*: *inj-on-def diff-eq-eq*)

end

context *ab-group-add*
begin

lemma *translation-Compl*:
(+) *a* ‘ (− *t*) = − ((+) *a* ‘ *t*)
proof (*rule set-eqI*)
fix *b*
show *b* ∈ (+) *a* ‘ (− *t*) ⟷ *b* ∈ − (+) *a* ‘ *t*
by (*auto simp*: *image-iff algebra-simps intro*!: *bexI* [*of* - *b* − *a*])
qed

end

10.6 Function Updating

definition *fun-upd* :: (′*a* ⇒ ′*b*) ⇒ ′*a* ⇒ ′*b* ⇒ (′*a* ⇒ ′*b*)
where *fun-upd f a b* = (λ*x*. *if x* = *a then b else f x*)

nonterminal *updbinds* **and** *updbind*

open-bundle *update-syntax*
begin

syntax
-*updbind* :: ′*a* ⇒ ′*a* ⇒ *updbind* (‹(‹*indent*=*2 notation*=‹*mixfix update*››- :=/ -)›)
:: *updbind* ⇒ *updbinds* (‹-›)
-*updbinds*:: *updbind* ⇒ *updbinds* ⇒ *updbinds* (‹-,/ -›)
-*Update* :: ′*a* ⇒ *updbinds* ⇒ ′*a*
(‹(‹*open-block notation*=‹*mixfix function update*››-/′((*2*-)′))› [*1000*, *0*] *900*)
syntax-consts

-Update ⇌ *fun-upd*
translations
-Update f (*-updbinds b bs*) ⇌ *-Update* (*-Update f b*) *bs*
f(*x*:=*y*) ⇌ *CONST fun-upd f x y*

end

lemma *fun-upd-idem-iff*: $f(x{:=}y) = f \longleftrightarrow f\ x = y$
unfolding *fun-upd-def*
apply *safe*
apply (*erule subst*)
apply *auto*
done

lemma *fun-upd-idem*: $f\ x = y \Longrightarrow f(x := y) = f$
by (*simp only*: *fun-upd-idem-iff*)

lemma *fun-upd-triv* [*iff*]: $f(x := f\ x) = f$
by (*simp only*: *fun-upd-idem*)

lemma *fun-upd-apply* [*simp*]: $(f(x := y))\ z = (\mathit{if}\ z = x\ \mathit{then}\ y\ \mathit{else}\ f\ z)$
by (*simp add*: *fun-upd-def*)

lemma *fun-upd-same*: $(f(x := y))\ x = y$
by *simp*

lemma *fun-upd-other*: $z \neq x \Longrightarrow (f(x := y))\ z = f\ z$
by *simp*

lemma *fun-upd-upd* [*simp*]: $f(x := y, x := z) = f(x := z)$
by (*simp add*: *fun-eq-iff*)

lemma *fun-upd-twist*: $a \neq c \Longrightarrow (m(a := b))(c := d) = (m(c := d))(a := b)$
by *auto*

lemma *inj-on-fun-updI*: $\mathit{inj\text{-}on}\ f\ A \Longrightarrow y \notin f\ `\ A \Longrightarrow \mathit{inj\text{-}on}\ (f(x := y))\ A$
by (*auto simp*: *inj-on-def*)

lemma *fun-upd-image*: $f(x := y)\ `\ A = (\mathit{if}\ x \in A\ \mathit{then}\ \mathit{insert}\ y\ (f\ `\ (A - \{x\}))\ \mathit{else}\ f\ `\ A)$
by *auto*

lemma *fun-upd-comp*: $f \circ (g(x := y)) = (f \circ g)(x := f\ y)$
by *auto*

lemma *fun-upd-eqD*: $f(x := y) = g(x := z) \Longrightarrow y = z$

by (*simp add*: *fun-eq-iff split*: *if-split-asm*)

10.7 *override-on*

definition *override-on* :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'b$
where *override-on f g A* = $(\lambda a.\ if\ a \in A\ then\ g\ a\ else\ f\ a)$

lemma *override-on-emptyset*[*simp*]: *override-on f g* {} = *f*
by (*simp add*: *override-on-def*)

lemma *override-on-apply-notin*[*simp*]: $a \notin A \Longrightarrow$ (*override-on f g A*) *a* = *f a*
by (*simp add*: *override-on-def*)

lemma *override-on-apply-in*[*simp*]: $a \in A \Longrightarrow$ (*override-on f g A*) *a* = *g a*
by (*simp add*: *override-on-def*)

lemma *override-on-insert*: *override-on f g* (*insert x X*) = (*override-on f g X*)(*x*:=*g x*)
by (*simp add*: *override-on-def fun-eq-iff*)

lemma *override-on-insert'*: *override-on f g* (*insert x X*) = (*override-on* (*f*(*x*:=*g x*)) *g X*)
by (*simp add*: *override-on-def fun-eq-iff*)

10.8 Inversion of injective functions

definition *the-inv-into* :: $'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a)$
where *the-inv-into A f* = $(\lambda x.\ THE\ y.\ y \in A \wedge f\ y = x)$

lemma *the-inv-into-f-f*: *inj-on f A* $\Longrightarrow x \in A \Longrightarrow$ *the-inv-into A f* (*f x*) = *x*
unfolding *the-inv-into-def inj-on-def* **by** *blast*

lemma *f-the-inv-into-f*: *inj-on f A* $\Longrightarrow y \in f$ ‘ $A \Longrightarrow f$ (*the-inv-into A f y*) = *y*
unfolding *the-inv-into-def*
by (*rule the1I2*; *blast dest*: *inj-onD*)

lemma *f-the-inv-into-f-bij-betw*:
bij-betw f A B $\Longrightarrow$ (*bij-betw f A B* $\Longrightarrow x \in B$) $\Longrightarrow f$ (*the-inv-into A f x*) = *x*
unfolding *bij-betw-def* **by** (*blast intro*: *f-the-inv-into-f*)

lemma *the-inv-into-into*: *inj-on f A* $\Longrightarrow x \in f$ ‘ $A \Longrightarrow A \subseteq B \Longrightarrow$ *the-inv-into A f x* $\in B$
unfolding *the-inv-into-def*
by (*rule the1I2*; *blast dest*: *inj-onD*)

lemma *the-inv-into-onto* [*simp*]: *inj-on f A* $\Longrightarrow$ *the-inv-into A f* ‘ (*f* ‘ *A*) = *A*
by (*fast intro*: *the-inv-into-into the-inv-into-f-f* [*symmetric*])

lemma *the-inv-into-f-eq*: *inj-on f A* $\Longrightarrow f\ x = y \Longrightarrow x \in A \Longrightarrow$ *the-inv-into A f y* = *x*

by (*force simp add*: *the-inv-into-f-f*)

lemma *the-inv-into-comp*:
inj-on f (*g* ‘ *A*) ⟹ *inj-on g A* ⟹ *x* ∈ *f* ‘ *g* ‘ *A* ⟹
the-inv-into A (*f* ∘ *g*) *x* = (*the-inv-into A g* ∘ *the-inv-into* (*g* ‘ *A*) *f*) *x*
apply (*rule the-inv-into-f-eq*)
apply (*fast intro*: *comp-inj-on*)
apply (*simp add*: *f-the-inv-into-f the-inv-into-into*)
apply (*simp add*: *the-inv-into-into*)
done

lemma *inj-on-the-inv-into*: *inj-on f A* ⟹ *inj-on* (*the-inv-into A f*) (*f* ‘ *A*)
by (*auto intro*: *inj-onI simp*: *the-inv-into-f-f*)

lemma *bij-betw-the-inv-into*: *bij-betw f A B* ⟹ *bij-betw* (*the-inv-into A f*) *B A*
by (*auto simp add*: *bij-betw-def inj-on-the-inv-into the-inv-into-into*)

lemma *bij-betw-iff-bijections*:
bij-betw f A B ⟷ (∃ *g*. (∀ *x* ∈ *A*. *f x* ∈ *B* ∧ *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *B*. *g y* ∈ *A* ∧ *f*(*g y*) = *y*))
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*auto simp*: *bij-betw-def f-the-inv-into-f the-inv-into-f-f the-inv-into-into exI*[**where** *?x*=*the-inv-into A f*])
next
show *?rhs* ⟹ *?lhs*
by (*force intro*: *bij-betw-byWitness*)
qed

abbreviation *the-inv* :: (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*)
where *the-inv f* ≡ *the-inv-into UNIV f*

lemma *the-inv-f-f*: *the-inv f* (*f x*) = *x* **if** *inj f*
using *that UNIV-I* **by** (*rule the-inv-into-f-f*)

10.9 Monotonicity

definition *monotone-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'b* ⇒ *'b* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *monotone-on A orda ordb f* ⟷ (∀ *x*∈*A*. ∀ *y*∈*A*. *orda x y* ⟶ *ordb* (*f x*) (*f y*))

abbreviation *monotone* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'b* ⇒ *'b* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *monotone* ≡ *monotone-on UNIV*

lemma *monotone-def*[*no-atp*]: *monotone orda ordb f* ⟷ (∀ *x y*. *orda x y* ⟶ *ordb* (*f x*) (*f y*))

by (*simp add*: *monotone-on-def*)

Lemma *monotone-def* is provided for backward compatibility.

lemma *monotone-onI*:
($\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow orda\ x\ y \Longrightarrow ordb\ (f\ x)\ (f\ y)) \Longrightarrow monotone\text{-}on\ A\ orda\ ordb\ f$
by (*simp add*: *monotone-on-def*)

lemma *monotoneI*[*intro?*]: ($\bigwedge x\ y.\ orda\ x\ y \Longrightarrow ordb\ (f\ x)\ (f\ y)) \Longrightarrow monotone\ orda\ ordb\ f$
by (*rule monotone-onI*)

lemma *monotone-onD*:
$monotone\text{-}on\ A\ orda\ ordb\ f \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow orda\ x\ y \Longrightarrow ordb\ (f\ x)\ (f\ y)$
by (*simp add*: *monotone-on-def*)

lemma *monotoneD*[*dest?*]: $monotone\ orda\ ordb\ f \Longrightarrow orda\ x\ y \Longrightarrow ordb\ (f\ x)\ (f\ y)$
by (*rule monotone-onD*[*of UNIV*, *simplified*])

lemma *monotone-on-subset*: $monotone\text{-}on\ A\ orda\ ordb\ f \Longrightarrow B \subseteq A \Longrightarrow monotone\text{-}on\ B\ orda\ ordb\ f$
by (*auto intro*: *monotone-onI dest*: *monotone-onD*)

lemma *monotone-on-empty*[*simp*]: *monotone-on* {} *orda ordb f*
by (*auto intro*: *monotone-onI dest*: *monotone-onD*)

lemma *monotone-on-o*:
assumes
mono-f: *monotone-on A orda ordb f* **and**
mono-g: *monotone-on B ordc orda g* **and**
$g\ `\ B \subseteq A$
shows *monotone-on B ordc ordb* $(f \circ g)$
proof (*rule monotone-onI*)
fix *x y* **assume** $x \in B$ **and** $y \in B$ **and** *ordc x y*
hence *orda* (*g x*) (*g y*)
by (*rule mono-g*[*THEN monotone-onD*])
moreover from ‹$g\ `\ B \subseteq A$› ‹$x \in B$› ‹$y \in B$› **have** $g\ x \in A$ **and** $g\ y \in A$
unfolding *image-subset-iff* **by** *simp-all*
ultimately show *ordb* $((f \circ g)\ x)\ ((f \circ g)\ y)$
using *mono-f*[*THEN monotone-onD*] **by** *simp*
qed

10.9.1 Specializations For *ord* Type Class And More

context *ord* **begin**

abbreviation *mono-on* :: $'a\ set \Rightarrow ('a \Rightarrow 'b :: ord) \Rightarrow bool$
where $mono\text{-}on\ A \equiv monotone\text{-}on\ A\ (\leq)\ (\leq)$

abbreviation *strict-mono-on* :: $'a$ *set* $\Rightarrow$ ($'a \Rightarrow 'b$ :: *ord*) $\Rightarrow$ *bool*
  where *strict-mono-on* $A \equiv$ *monotone-on* A $(<)$ $(<)$

abbreviation *antimono-on* :: $'a$ *set* $\Rightarrow$ ($'a \Rightarrow 'b$ :: *ord*) $\Rightarrow$ *bool*
  where *antimono-on* $A \equiv$ *monotone-on* A $(\leq)$ $(\geq)$

abbreviation *strict-antimono-on* :: $'a$ *set* $\Rightarrow$ ($'a \Rightarrow 'b$ :: *ord*) $\Rightarrow$ *bool*
  where *strict-antimono-on* $A \equiv$ *monotone-on* A $(<)$ $(>)$

lemma *mono-on-def*[*no-atp*]: *mono-on* A $f \longleftrightarrow (\forall r\ s.\ r \in A \wedge s \in A \wedge r \leq s \longrightarrow f\ r \leq f\ s)$
  by (*auto simp add*: *monotone-on-def*)

lemma *strict-mono-on-def*[*no-atp*]:
  strict-mono-on A $f \longleftrightarrow (\forall r\ s.\ r \in A \wedge s \in A \wedge r < s \longrightarrow f\ r < f\ s)$
  by (*auto simp add*: *monotone-on-def*)

Lemmas *mono-on-def* and *strict-mono-on-def* are provided for backward compatibility.

lemma *mono-onI*:
  $(\bigwedge r\ s.\ r \in A \Longrightarrow s \in A \Longrightarrow r \leq s \Longrightarrow f\ r \leq f\ s) \Longrightarrow$ *mono-on* A f
  by (*rule monotone-onI*)

lemma *strict-mono-onI*:
  $(\bigwedge r\ s.\ r \in A \Longrightarrow s \in A \Longrightarrow r < s \Longrightarrow f\ r < f\ s) \Longrightarrow$ *strict-mono-on* A f
  by (*rule monotone-onI*)

lemma *mono-onD*: $[\![$*mono-on* A f; $r \in A$; $s \in A$; $r \leq s]\!] \Longrightarrow f\ r \leq f\ s$
  by (*rule monotone-onD*)

lemma *strict-mono-onD*: $[\![$*strict-mono-on* A f; $r \in A$; $s \in A$; $r < s]\!] \Longrightarrow f\ r < f\ s$
  by (*rule monotone-onD*)

lemma *mono-on-subset*: *mono-on* A $f \Longrightarrow B \subseteq A \Longrightarrow$ *mono-on* B f
  by (*rule monotone-on-subset*)

end

lemma *mono-on-greaterD*:
  fixes g :: $'a$::*linorder* $\Rightarrow$ $'b$::*linorder*
  assumes *mono-on* A g $x \in A$ $y \in A$ $g\ x > g\ y$
  shows $x > y$
proof (*rule ccontr*)
  assume $\neg x > y$
  hence $x \leq y$ **by** (*simp add*: *not-less*)
  from *assms*(*1–3*) **and** *this* **have** $g\ x \leq g\ y$ **by** (*rule mono-onD*)
  with *assms*(*4*) **show** *False* **by** *simp*
qed

context *order* **begin**

abbreviation *mono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$
where *mono* $\equiv$ *mono-on UNIV*

abbreviation *strict-mono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$
where *strict-mono* $\equiv$ *strict-mono-on UNIV*

abbreviation *antimono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$
where *antimono* $\equiv$ *monotone* $(\leq)$ $(\lambda x\ y.\ y \leq x)$

lemma *mono-def*[*no-atp*]: *mono f* $\longleftrightarrow$ $(\forall x\ y.\ x \leq y \longrightarrow f\ x \leq f\ y)$
by (*simp add*: *monotone-on-def*)

lemma *strict-mono-def*[*no-atp*]: *strict-mono f* $\longleftrightarrow$ $(\forall x\ y.\ x < y \longrightarrow f\ x < f\ y)$
by (*simp add*: *monotone-on-def*)

lemma *antimono-def*[*no-atp*]: *antimono f* $\longleftrightarrow$ $(\forall x\ y.\ x \leq y \longrightarrow f\ x \geq f\ y)$
by (*simp add*: *monotone-on-def*)

Lemmas *mono-def*, *strict-mono-def*, and *antimono-def* are provided for backward compatibility.

lemma *monoI* [*intro?*]: $(\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y) \Longrightarrow$ *mono f*
by (*rule monotoneI*)

lemma *strict-monoI* [*intro?*]: $(\bigwedge x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow$ *strict-mono f*
by (*rule monotoneI*)

lemma *antimonoI* [*intro?*]: $(\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \geq f\ y) \Longrightarrow$ *antimono f*
by (*rule monotoneI*)

lemma *monoD* [*dest?*]: *mono f* $\Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
by (*rule monotoneD*)

lemma *strict-monoD* [*dest?*]: *strict-mono f* $\Longrightarrow x < y \Longrightarrow f\ x < f\ y$
by (*rule monotoneD*)

lemma *antimonoD* [*dest?*]: *antimono f* $\Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
by (*rule monotoneD*)

lemma *monoE*:
assumes *mono f*
assumes $x \leq y$
obtains $f\ x \leq f\ y$
proof
from *assms* **show** $f\ x \leq f\ y$ **by** (*simp add*: *mono-def*)
qed

lemma *antimonoE*:
fixes *f* :: *'a* ⇒ *'b*::*order*
assumes *antimono f*
assumes $x \leq y$
obtains $f\ x \geq f\ y$
proof
from *assms* **show** $f\ x \geq f\ y$ **by** (*simp add*: *antimono-def*)
qed

lemma *mono-imp-mono-on*: *mono f* ⟹ *mono-on A f*
by (*rule monotone-on-subset*[*OF* - *subset-UNIV*])

lemma *strict-mono-mono* [*dest?*]:
assumes *strict-mono f*
shows *mono f*
proof (*rule monoI*)
fix *x y*
assume $x \leq y$
show $f\ x \leq f\ y$
proof (*cases* $x = y$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False* **with** ‹$x \leq y$› **have** $x < y$ **by** *simp*
with *assms strict-monoD* **have** $f\ x < f\ y$ **by** *auto*
then show *?thesis* **by** *simp*

qed
qed

lemma *mono-on-ident*: *mono-on S* (λ*x*. *x*)
by (*simp add*: *monotone-on-def*)

lemma *strict-mono-on-ident*: *strict-mono-on S* (λ*x*. *x*)
by (*simp add*: *monotone-on-def*)

lemma *mono-on-const*:
fixes *a* :: *'b*::*order* **shows** *mono-on S* (λ*x*. *a*)
by (*simp add*: *mono-on-def*)

lemma *antimono-on-const*:
fixes *a* :: *'b*::*order* **shows** *antimono-on S* (λ*x*. *a*)
by (*simp add*: *monotone-on-def*)

end

context *linorder* **begin**

lemma *mono-invE*:
fixes *f* :: *'a* ⇒ *'b*::*order*

```
  assumes mono f
  assumes f x < f y
  obtains x ≤ y
proof
  show x ≤ y
  proof (rule ccontr)
    assume ¬ x ≤ y
    then have y ≤ x by simp
    with ‹mono f› obtain f y ≤ f x by (rule monoE)
    with ‹f x < f y› show False by simp
  qed
qed

lemma mono-strict-invE:
  fixes f :: 'a ⇒ 'b::order
  assumes mono f
  assumes f x < f y
  obtains x < y
proof
  show x < y
  proof (rule ccontr)
    assume ¬ x < y
    then have y ≤ x by simp
    with ‹mono f› obtain f y ≤ f x by (rule monoE)
    with ‹f x < f y› show False by simp
  qed
qed

lemma strict-mono-eq:
  assumes strict-mono f
  shows f x = f y ⟷ x = y
proof
  assume f x = f y
  show x = y proof (cases x y rule: linorder-cases)
    case less with assms strict-monoD have f x < f y by auto
    with ‹f x = f y› show ?thesis by simp
  next
    case equal then show ?thesis .
  next
    case greater with assms strict-monoD have f y < f x by auto
    with ‹f x = f y› show ?thesis by simp
  qed
qed simp

lemma strict-mono-less-eq:
  assumes strict-mono f
  shows f x ≤ f y ⟷ x ≤ y
proof
  assume x ≤ y
```

with *assms strict-mono-mono monoD* **show** $f\ x \le f\ y$ **by** *auto*
next
assume $f\ x \le f\ y$
show $x \le y$ **proof** (*rule ccontr*)
assume $\neg\ x \le y$ **then have** $y < x$ **by** *simp*
with *assms strict-monoD* **have** $f\ y < f\ x$ **by** *auto*
with ‹$f\ x \le f\ y$› **show** *False* **by** *simp*
qed
qed

lemma *strict-mono-less*:
assumes *strict-mono f*
shows $f\ x < f\ y \longleftrightarrow x < y$
using *assms*
by (*auto simp add*: *less-le Orderings.less-le strict-mono-eq strict-mono-less-eq*)

end

lemma *strict-mono-inv*:
fixes f :: ($'a$::*linorder*) $\Rightarrow$ ($'b$::*linorder*)
assumes *strict-mono f* **and** *surj f* **and** *inv*: $\bigwedge x.\ g\ (f\ x) = x$
shows *strict-mono g*
proof
fix $x\ y$ :: $'b$ **assume** $x < y$
from ‹*surj f*› **obtain** $x'\ y'$ **where** [*simp*]: $x = f\ x'\ y = f\ y'$ **by** *blast*
with ‹$x < y$› **and** ‹*strict-mono f*› **have** $x' < y'$ **by** (*simp add*: *strict-mono-less*)
with *inv* **show** $g\ x < g\ y$ **by** *simp*
qed

lemma *strict-mono-on-imp-inj-on*:
fixes f :: $'a$::*linorder* $\Rightarrow$ $'b$::*preorder*
assumes *strict-mono-on A f*
shows *inj-on f A*
proof (*rule inj-onI*)
fix $x\ y$ **assume** $x \in A\ y \in A\ f\ x = f\ y$
thus $x = y$
by (*cases x y rule*: *linorder-cases*)
(*auto dest*: *strict-mono-onD*[*OF assms*, *of x y*] *strict-mono-onD*[*OF assms*, *of y x*])
qed

lemma *strict-mono-on-leD*:
fixes f :: $'a$::*linorder* $\Rightarrow$ $'b$::*preorder*
assumes *strict-mono-on A f* $x \in A\ y \in A\ x \le y$
shows $f\ x \le f\ y$
proof (*cases* $x = y$)
case *True*
then show *?thesis* **by** *simp*
next

case *False*
with *assms* **have** $f\ x < f\ y$
using *strict-mono-onD*[*OF assms*(*1*)] **by** *simp*
then show *?thesis* **by** (*rule less-imp-le*)
qed

lemma *strict-mono-on-eqD*:
fixes f :: ′*c*::*linorder* ⇒ ′*d*::*preorder*
assumes *strict-mono-on A f* $f\ x = f\ y$ $x \in A$ $y \in A$
shows $y = x$
using *assms* **by** (*cases rule*: *linorder-cases*) (*auto dest*: *strict-mono-onD*)

lemma *strict-mono-on-imp-mono-on*: *strict-mono-on A f* ⟹ *mono-on A f*
for f :: ′*a*::*linorder* ⇒ ′*b*::*preorder*
by (*rule mono-onI*, *rule strict-mono-on-leD*)

lemma *mono-imp-strict-mono*:
fixes f :: ′*a*::*order* ⇒ ′*b*::*order*
shows ⟦*mono-on S f*; *inj-on f S*⟧ ⟹ *strict-mono-on S f*
by (*auto simp add*: *monotone-on-def order-less-le inj-on-eq-iff*)

lemma *strict-mono-iff-mono*:
fixes f :: ′*a*::*linorder* ⇒ ′*b*::*order*
shows *strict-mono-on S f* ⟷ *mono-on S f* ∧ *inj-on f S*
proof
show *strict-mono-on S f* ⟹ *mono-on S f* ∧ *inj-on f S*
by (*simp add*: *strict-mono-on-imp-inj-on strict-mono-on-imp-mono-on*)
qed (*auto intro*: *mono-imp-strict-mono*)

lemma *antimono-imp-strict-antimono*:
fixes f :: ′*a*::*order* ⇒ ′*b*::*order*
shows ⟦*antimono-on S f*; *inj-on f S*⟧ ⟹ *strict-antimono-on S f*
by (*auto simp add*: *monotone-on-def order-less-le inj-on-eq-iff*)

lemma *strict-antimono-iff-antimono*:
fixes f :: ′*a*::*linorder* ⇒ ′*b*::*order*
shows *strict-antimono-on S f* ⟷ *antimono-on S f* ∧ *inj-on f S*
proof
show *strict-antimono-on S f* ⟹ *antimono-on S f* ∧ *inj-on f S*
by (*force simp add*: *monotone-on-def intro*: *linorder-inj-onI*)
qed (*auto intro*: *antimono-imp-strict-antimono*)

lemma *mono-compose*: *mono Q* ⟹ *mono* ($\lambda i\ x.\ Q\ i\ (f\ x)$)
unfolding *mono-def le-fun-def* **by** *auto*

lemma *mono-add*:
fixes a :: ′*a*::*ordered-ab-semigroup-add*
shows *mono* ((+) a)
by (*simp add*: *add-left-mono monoI*)

lemma (**in** *semilattice-inf*) *mono-inf*: *mono f* $\Longrightarrow$ *f* $(A \sqcap B) \leq f\ A \sqcap f\ B$
for *f* :: *'a* $\Rightarrow$ *'b::semilattice-inf*
by (*auto simp add*: *mono-def intro*: *Lattices.inf-greatest*)

lemma (**in** *semilattice-sup*) *mono-sup*: *mono f* $\Longrightarrow f\ A \sqcup f\ B \leq f\ (A \sqcup B)$
for *f* :: *'a* $\Rightarrow$ *'b::semilattice-sup*
by (*auto simp add*: *mono-def intro*: *Lattices.sup-least*)

lemma (**in** *linorder*) *min-of-mono*: *mono f* $\Longrightarrow$ *min* (*f m*) (*f n*) = *f* (*min m n*)
by (*auto simp*: *mono-def Orderings.min-def min-def intro*: *Orderings.antisym*)

lemma (**in** *linorder*) *max-of-mono*: *mono f* $\Longrightarrow$ *max* (*f m*) (*f n*) = *f* (*max m n*)
by (*auto simp*: *mono-def Orderings.max-def max-def intro*: *Orderings.antisym*)

lemma (**in** *linorder*)
max-of-antimono: *antimono f* $\Longrightarrow$ *max* (*f x*) (*f y*) = *f* (*min x y*) **and**
min-of-antimono: *antimono f* $\Longrightarrow$ *min* (*f x*) (*f y*) = *f* (*max x y*)
by (*auto simp*: *antimono-def Orderings.max-def max-def Orderings.min-def min-def intro*!: *antisym*)

lemma (**in** *linorder*) *strict-mono-imp-inj-on*: *strict-mono f* $\Longrightarrow$ *inj-on f A*
by (*auto intro*!: *inj-onI dest*: *strict-mono-eq*)

lemma *mono-Int*: *mono f* $\Longrightarrow f\ (A \cap B) \subseteq f\ A \cap f\ B$
by (*fact mono-inf*)

lemma *mono-Un*: *mono f* $\Longrightarrow f\ A \cup f\ B \subseteq f\ (A \cup B)$
by (*fact mono-sup*)

10.9.2 Least value operator

lemma *Least-mono*: *mono f* $\Longrightarrow \exists x \in S.\ \forall y \in S.\ x \leq y \Longrightarrow (LEAST\ y.\ y \in f\ `\ S) = f\ (LEAST\ x.\ x \in S)$
for *f* :: *'a::order* $\Rightarrow$ *'b::order*
— Courtesy of Stephan Merz
apply *clarify*
apply (*erule-tac* $P = \lambda x.\ x \in S$ **in** *LeastI2-order*)
apply *fast*
apply (*rule LeastI2-order*)
apply (*auto elim*: *monoD intro*!: *order-antisym*)
done

10.10 Setup

10.10.1 Proof tools

Simplify terms of the form $f(\ldots,x{:=}y,\ldots,x{:=}z,\ldots)$ to $f(\ldots,x{:=}z,\ldots)$

simproc-setup *fun-upd2* $(f(v := w,\ x := y)) = \langle$

```
  let
    fun gen-fun-upd - - - - NONE = NONE
      | gen-fun-upd A B x y (SOME f) = SOME Const‹fun-upd A B for f x y›
    fun find-double (t as Const- ‹fun-upd A B for f x y›) =
      let
        fun find Const- ‹fun-upd - - for g v w› =
              if v aconv x then SOME g
              else gen-fun-upd A B v w (find g)
          | find t = NONE
      in gen-fun-upd A B x y (find f) end

    val ss = simpset-of context
  in
    fn - => fn ctxt => fn ct =>
      let val t = Thm.term-of ct in
        find-double t |> Option.map (fn rhs =>
          Goal.prove ctxt [] [] (Logic.mk-equals (t, rhs))
            (fn - =>
              resolve-tac ctxt [eq-reflection] 1 THEN
              resolve-tac ctxt @{thms ext} 1 THEN
              simp-tac (put-simpset ss ctxt) 1))
      end
  end
›
```

10.10.2 Functorial structure of types

ML-file ‹*Tools/functor.ML*›

functor *map-fun*: *map-fun*
  by (*simp-all add*: *fun-eq-iff*)

functor *vimage*
  by (*simp-all add*: *fun-eq-iff vimage-comp*)

Legacy theorem names

lemmas *o-def* = *comp-def*
lemmas *o-apply* = *comp-apply*
lemmas *o-assoc* = *comp-assoc* [*symmetric*]
lemmas *id-o* = *id-comp*
lemmas *o-id* = *comp-id*
lemmas *o-eq-dest* = *comp-eq-dest*
lemmas *o-eq-elim* = *comp-eq-elim*
lemmas *o-eq-dest-lhs* = *comp-eq-dest-lhs*
lemmas *o-eq-id-dest* = *comp-eq-id-dest*

end

11 Complete lattices

theory *Complete-Lattices*
imports *Fun*
begin

11.1 Syntactic infimum and supremum operations

class *Inf* =
fixes *Inf* :: *'a set* ⇒ *'a* (‹(‹*open-block notation*=‹*prefix* ⨅››⨅ -)› [*900*] *900*)

class *Sup* =
fixes *Sup* :: *'a set* ⇒ *'a* (‹(‹*open-block notation*=‹*prefix* ⨆››⨆ -)› [*900*] *900*)

syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder INF*››*INF* -./ -)› [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder INF*››*INF* -∈-./ -)› [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder SUP*››*SUP* -./ -)› [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder SUP*››*SUP* -∈-./ -)› [*0*, *0*, *10*] *10*)

syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨅››⨅-./ -)› [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨅››⨅-∈-./ -)› [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨆››⨆-./ -)› [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨆››⨆-∈-./ -)› [*0*, *0*, *10*] *10*)

syntax-consts
-INF1 -INF ⇌ *Inf* **and**
-SUP1 -SUP ⇌ *Sup*

translations
⨅*x y*. *f* ⇌ ⨅*x*. ⨅*y*. *f*
⨅*x*. *f* ⇌ ⨅(*CONST range* (λ*x*. *f*))
⨅*x*∈*A*. *f* ⇌ *CONST Inf* ((λ*x*. *f*) ‘ *A*)
⨆*x y*. *f* ⇌ ⨆*x*. ⨆*y*. *f*
⨆*x*. *f* ⇌ ⨆(*CONST range* (λ*x*. *f*))
⨆*x*∈*A*. *f* ⇌ *CONST Sup* ((λ*x*. *f*) ‘ *A*)

context *Inf*
begin

lemma *INF-image*: ⨅ (*g* ‘ *f* ‘ *A*) = ⨅ ((*g* ∘ *f*) ‘ *A*)

by (*simp add*: *image-comp*)

lemma *INF-identity-eq* [*simp*]: $(\bigsqcap x{\in}A.\ x) = \bigsqcap A$
by *simp*

lemma *INF-id-eq* [*simp*]: $\bigsqcap(id\ `\ A) = \bigsqcap A$
by *simp*

lemma *INF-cong*: $A = B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow C\ x = D\ x) \Longrightarrow \bigsqcap(C\ `\ A) = \bigsqcap(D\ `\ B)$
by (*simp add*: *image-def*)

lemma *INF-cong-simp*:
$A = B \Longrightarrow (\bigwedge x.\ x \in B =simp=> C\ x = D\ x) \Longrightarrow \bigsqcap(C\ `\ A) = \bigsqcap(D\ `\ B)$
unfolding *simp-implies-def* **by** (*fact INF-cong*)

end

context *Sup*
begin

lemma *SUP-image*: $\bigsqcup\ (g\ `\ f\ `\ A) = \bigsqcup\ ((g \circ f)\ `\ A)$
by(*fact Inf.INF-image*)

lemma *SUP-identity-eq* [*simp*]: $(\bigsqcup x{\in}A.\ x) = \bigsqcup A$
by(*fact Inf.INF-identity-eq*)

lemma *SUP-id-eq* [*simp*]: $\bigsqcup(id\ `\ A) = \bigsqcup A$
by(*fact Inf.INF-id-eq*)

lemma *SUP-cong*: $A = B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow C\ x = D\ x) \Longrightarrow \bigsqcup(C\ `\ A) = \bigsqcup(D\ `\ B)$
by (*fact Inf.INF-cong*)

lemma *SUP-cong-simp*:
$A = B \Longrightarrow (\bigwedge x.\ x \in B =simp=> C\ x = D\ x) \Longrightarrow \bigsqcup(C\ `\ A) = \bigsqcup(D\ `\ B)$
by (*fact Inf.INF-cong-simp*)

end

11.2 Abstract complete lattices

A complete lattice always has a bottom and a top, so we include them into the following type class, along with assumptions that define bottom and top in terms of infimum and supremum.

class *complete-lattice* = *lattice* + *Inf* + *Sup* + *bot* + *top* +
assumes *Inf-lower*: $x \in A \Longrightarrow \bigsqcap A \le x$
and *Inf-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow z \le x) \Longrightarrow z \le \bigsqcap A$
and *Sup-upper*: $x \in A \Longrightarrow x \le \bigsqcup A$

and *Sup-least*: $(\bigwedge x.\ x \in A \Longrightarrow x \leq z) \Longrightarrow \bigsqcup A \leq z$
and *Inf-empty* [*simp*]: $\bigsqcap\{\} = \top$
and *Sup-empty* [*simp*]: $\bigsqcup\{\} = \bot$
begin

subclass *bounded-lattice*
proof
fix a
show $\bot \leq a$
by (*auto intro*: *Sup-least simp only*: *Sup-empty* [*symmetric*])
show $a \leq \top$
by (*auto intro*: *Inf-greatest simp only*: *Inf-empty* [*symmetric*])
qed

lemma *dual-complete-lattice*: *class.complete-lattice Sup Inf sup* $(\geq)$ $(>)$ *inf* $\top$ $\bot$
by (*auto intro*!: *class.complete-lattice.intro dual-lattice*)
(*unfold-locales*, (*fact Inf-empty Sup-empty Sup-upper Sup-least Inf-lower Inf-greatest*)+)

end

context *complete-lattice*
begin

lemma *Sup-eqI*:
$(\bigwedge y.\ y \in A \Longrightarrow y \leq x) \Longrightarrow (\bigwedge y.\ (\bigwedge z.\ z \in A \Longrightarrow z \leq y) \Longrightarrow x \leq y) \Longrightarrow \bigsqcup A = x$
by (*blast intro*: *order.antisym Sup-least Sup-upper*)

lemma *Inf-eqI*:
$(\bigwedge i.\ i \in A \Longrightarrow x \leq i) \Longrightarrow (\bigwedge y.\ (\bigwedge i.\ i \in A \Longrightarrow y \leq i) \Longrightarrow y \leq x) \Longrightarrow \bigsqcap A = x$
by (*blast intro*: *order.antisym Inf-greatest Inf-lower*)

lemma *SUP-eqI*:
$(\bigwedge i.\ i \in A \Longrightarrow f\ i \leq x) \Longrightarrow (\bigwedge y.\ (\bigwedge i.\ i \in A \Longrightarrow f\ i \leq y) \Longrightarrow x \leq y) \Longrightarrow (\bigsqcup i{\in}A.\ f\ i) = x$
using *Sup-eqI* [*of f ' A x*] **by** *auto*

lemma *INF-eqI*:
$(\bigwedge i.\ i \in A \Longrightarrow x \leq f\ i) \Longrightarrow (\bigwedge y.\ (\bigwedge i.\ i \in A \Longrightarrow f\ i \geq y) \Longrightarrow x \geq y) \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) = x$
using *Inf-eqI* [*of f ' A x*] **by** *auto*

lemma *INF-lower*: $i \in A \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) \leq f\ i$
using *Inf-lower* [*of - f ' A*] **by** *simp*

lemma *INF-greatest*: $(\bigwedge i.\ i \in A \Longrightarrow u \leq f\ i) \Longrightarrow u \leq (\bigsqcap i{\in}A.\ f\ i)$
using *Inf-greatest* [*of f ' A*] **by** *auto*

lemma *SUP-upper*: $i \in A \Longrightarrow f\ i \leq (\bigsqcup i{\in}A.\ f\ i)$

using *Sup-upper* [*of* - *f* ' *A*] **by** *simp*

lemma *SUP-least*: $(\bigwedge i.\ i \in A \Longrightarrow f\ i \leq u) \Longrightarrow (\bigsqcup i{\in}A.\ f\ i) \leq u$
using *Sup-least* [*of f* ' *A*] **by** *auto*

lemma *Inf-lower2*: $u \in A \Longrightarrow u \leq v \Longrightarrow \bigsqcap A \leq v$
using *Inf-lower* [*of u A*] **by** *auto*

lemma *INF-lower2*: $i \in A \Longrightarrow f\ i \leq u \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) \leq u$
using *INF-lower* [*of i A f*] **by** *auto*

lemma *Sup-upper2*: $u \in A \Longrightarrow v \leq u \Longrightarrow v \leq \bigsqcup A$
using *Sup-upper* [*of u A*] **by** *auto*

lemma *SUP-upper2*: $i \in A \Longrightarrow u \leq f\ i \Longrightarrow u \leq (\bigsqcup i{\in}A.\ f\ i)$
using *SUP-upper* [*of i A f*] **by** *auto*

lemma *le-Inf-iff*: $b \leq \bigsqcap A \longleftrightarrow (\forall a{\in}A.\ b \leq a)$
by (*auto intro*: *Inf-greatest dest*: *Inf-lower*)

lemma *le-INF-iff*: $u \leq (\bigsqcap i{\in}A.\ f\ i) \longleftrightarrow (\forall i{\in}A.\ u \leq f\ i)$
using *le-Inf-iff* [*of* - *f* ' *A*] **by** *simp*

lemma *Sup-le-iff*: $\bigsqcup A \leq b \longleftrightarrow (\forall a{\in}A.\ a \leq b)$
by (*auto intro*: *Sup-least dest*: *Sup-upper*)

lemma *SUP-le-iff*: $(\bigsqcup i{\in}A.\ f\ i) \leq u \longleftrightarrow (\forall i{\in}A.\ f\ i \leq u)$
using *Sup-le-iff* [*of f* ' *A*] **by** *simp*

lemma *Inf-insert* [*simp*]: $\bigsqcap(insert\ a\ A) = a \sqcap \bigsqcap A$
by (*auto intro*: *le-infI le-infI1 le-infI2 order.antisym Inf-greatest Inf-lower*)

lemma *INF-insert*: $(\bigsqcap x{\in}insert\ a\ A.\ f\ x) = f\ a \sqcap \bigsqcap(f\ `\ A)$
by *simp*

lemma *Sup-insert* [*simp*]: $\bigsqcup(insert\ a\ A) = a \sqcup \bigsqcup A$
by (*auto intro*: *le-supI le-supI1 le-supI2 order.antisym Sup-least Sup-upper*)

lemma *SUP-insert*: $(\bigsqcup x{\in}insert\ a\ A.\ f\ x) = f\ a \sqcup \bigsqcup(f\ `\ A)$
by *simp*

lemma *INF-empty*: $(\bigsqcap x{\in}\{\}.\ f\ x) = \top$
by *simp*

lemma *SUP-empty*: $(\bigsqcup x{\in}\{\}.\ f\ x) = \bot$
by *simp*

lemma *Inf-UNIV* [*simp*]: $\bigsqcap UNIV = \bot$
by (*auto intro*!: *order.antisym Inf-lower*)

lemma *Sup-UNIV* [*simp*]: $\bigsqcup$ *UNIV* = $\top$
by (*auto intro*!: *order.antisym Sup-upper*)

lemma *Inf-eq-Sup*: $\bigsqcap A = \bigsqcup \{b. \forall a \in A. b \le a\}$
by (*auto intro*: *order.antisym Inf-lower Inf-greatest Sup-upper Sup-least*)

lemma *Sup-eq-Inf*: $\bigsqcup A = \bigsqcap \{b. \forall a \in A. a \le b\}$
by (*auto intro*: *order.antisym Inf-lower Inf-greatest Sup-upper Sup-least*)

lemma *Inf-superset-mono*: $B \subseteq A \Longrightarrow \bigsqcap A \le \bigsqcap B$
by (*auto intro*: *Inf-greatest Inf-lower*)

lemma *Sup-subset-mono*: $A \subseteq B \Longrightarrow \bigsqcup A \le \bigsqcup B$
by (*auto intro*: *Sup-least Sup-upper*)

lemma *Inf-mono*:
assumes $\bigwedge b.\ b \in B \Longrightarrow \exists a{\in}A.\ a \le b$
shows $\bigsqcap A \le \bigsqcap B$
proof (*rule Inf-greatest*)
fix *b* **assume** $b \in B$
with *assms* **obtain** *a* **where** $a \in A$ **and** $a \le b$ **by** *blast*
from ‹$a \in A$› **have** $\bigsqcap A \le a$ **by** (*rule Inf-lower*)
with ‹$a \le b$› **show** $\bigsqcap A \le b$ **by** *auto*
qed

lemma *INF-mono*: $(\bigwedge m.\ m \in B \Longrightarrow \exists n{\in}A.\ f\ n \le g\ m) \Longrightarrow (\bigsqcap n{\in}A.\ f\ n) \le (\bigsqcap n{\in}B.\ g\ n)$
using *Inf-mono* [*of g ‘ B f ‘ A*] **by** *auto*

lemma *INF-mono'*: $(\bigwedge x.\ f\ x \le g\ x) \Longrightarrow (\bigsqcap x{\in}A.\ f\ x) \le (\bigsqcap x{\in}A.\ g\ x)$
by (*rule INF-mono*) *auto*

lemma *Sup-mono*:
assumes $\bigwedge a.\ a \in A \Longrightarrow \exists b{\in}B.\ a \le b$
shows $\bigsqcup A \le \bigsqcup B$
proof (*rule Sup-least*)
fix *a* **assume** $a \in A$
with *assms* **obtain** *b* **where** $b \in B$ **and** $a \le b$ **by** *blast*
from ‹$b \in B$› **have** $b \le \bigsqcup B$ **by** (*rule Sup-upper*)
with ‹$a \le b$› **show** $a \le \bigsqcup B$ **by** *auto*
qed

lemma *SUP-mono*: $(\bigwedge n.\ n \in A \Longrightarrow \exists m{\in}B.\ f\ n \le g\ m) \Longrightarrow (\bigsqcup n{\in}A.\ f\ n) \le (\bigsqcup n{\in}B.\ g\ n)$
using *Sup-mono* [*of f ‘ A g ‘ B*] **by** *auto*

lemma *SUP-mono'*: $(\bigwedge x.\ f\ x \le g\ x) \Longrightarrow (\bigsqcup x{\in}A.\ f\ x) \le (\bigsqcup x{\in}A.\ g\ x)$
by (*rule SUP-mono*) *auto*

lemma *INF-superset-mono*: $B \subseteq A \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow f\ x \le g\ x) \Longrightarrow (\bigsqcap x{\in}A.\ f\ x) \le (\bigsqcap x{\in}B.\ g\ x)$
— The last inclusion is POSITIVE!
by (*blast intro*: *INF-mono dest*: *subsetD*)

lemma *SUP-subset-mono*: $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \le g\ x) \Longrightarrow (\bigsqcup x{\in}A.\ f\ x) \le (\bigsqcup x{\in}B.\ g\ x)$
by (*blast intro*: *SUP-mono dest*: *subsetD*)

lemma *Inf-less-eq*:
assumes $\bigwedge v.\ v \in A \Longrightarrow v \le u$
and $A \neq \{\}$
shows $\bigsqcap A \le u$
proof −
from ‹$A \neq \{\}$› **obtain** v **where** $v \in A$ **by** *blast*
moreover from ‹$v \in A$› *assms*(*1*) **have** $v \le u$ **by** *blast*
ultimately show *?thesis* **by** (*rule Inf-lower2*)
qed

lemma *less-eq-Sup*:
assumes $\bigwedge v.\ v \in A \Longrightarrow u \le v$
and $A \neq \{\}$
shows $u \le \bigsqcup A$
proof −
from ‹$A \neq \{\}$› **obtain** v **where** $v \in A$ **by** *blast*
moreover from ‹$v \in A$› *assms*(*1*) **have** $u \le v$ **by** *blast*
ultimately show *?thesis* **by** (*rule Sup-upper2*)
qed

lemma *INF-eq*:
assumes $\bigwedge i.\ i \in A \Longrightarrow \exists j{\in}B.\ f\ i \ge g\ j$
and $\bigwedge j.\ j \in B \Longrightarrow \exists i{\in}A.\ g\ j \ge f\ i$
shows $\bigsqcap(f\ `\ A) = \bigsqcap(g\ `\ B)$
by (*intro order.antisym INF-greatest*) (*blast intro*: *INF-lower2 dest*: *assms*)+

lemma *SUP-eq*:
assumes $\bigwedge i.\ i \in A \Longrightarrow \exists j{\in}B.\ f\ i \le g\ j$
and $\bigwedge j.\ j \in B \Longrightarrow \exists i{\in}A.\ g\ j \le f\ i$
shows $\bigsqcup(f\ `\ A) = \bigsqcup(g\ `\ B)$
by (*intro order.antisym SUP-least*) (*blast intro*: *SUP-upper2 dest*: *assms*)+

lemma *less-eq-Inf-inter*: $\bigsqcap A \sqcup \bigsqcap B \le \bigsqcap(A \cap B)$
by (*auto intro*: *Inf-greatest Inf-lower*)

lemma *Sup-inter-less-eq*: $\bigsqcup(A \cap B) \le \bigsqcup A \sqcap \bigsqcup B$
by (*auto intro*: *Sup-least Sup-upper*)

lemma *Inf-union-distrib*: $\bigsqcap(A \cup B) = \bigsqcap A \sqcap \bigsqcap B$

by (*rule order.antisym*) (*auto intro*: *Inf-greatest Inf-lower le-infI1 le-infI2*)

lemma *INF-union*: $(\bigsqcap i \in A \cup B.\ M\ i) = (\bigsqcap i \in A.\ M\ i) \sqcap (\bigsqcap i{\in}B.\ M\ i)$
by (*auto intro*!: *order.antisym INF-mono intro*: *le-infI1 le-infI2 INF-greatest INF-lower*)

lemma *Sup-union-distrib*: $\bigsqcup(A \cup B) = \bigsqcup A \sqcup \bigsqcup B$
by (*rule order.antisym*) (*auto intro*: *Sup-least Sup-upper le-supI1 le-supI2*)

lemma *SUP-union*: $(\bigsqcup i \in A \cup B.\ M\ i) = (\bigsqcup i \in A.\ M\ i) \sqcup (\bigsqcup i{\in}B.\ M\ i)$
by (*auto intro*!: *order.antisym SUP-mono intro*: *le-supI1 le-supI2 SUP-least SUP-upper*)

lemma *INF-inf-distrib*: $(\bigsqcap a{\in}A.\ f\ a) \sqcap (\bigsqcap a{\in}A.\ g\ a) = (\bigsqcap a{\in}A.\ f\ a \sqcap g\ a)$
by (*rule order.antisym*) (*rule INF-greatest*, *auto intro*: *le-infI1 le-infI2 INF-lower INF-mono*)

lemma *SUP-sup-distrib*: $(\bigsqcup a{\in}A.\ f\ a) \sqcup (\bigsqcup a{\in}A.\ g\ a) = (\bigsqcup a{\in}A.\ f\ a \sqcup g\ a)$
(**is** *?L* = *?R*)
proof (*rule order.antisym*)
show $?L \le ?R$
by (*auto intro*: *le-supI1 le-supI2 SUP-upper SUP-mono*)
show $?R \le ?L$
by (*rule SUP-least*) (*auto intro*: *le-supI1 le-supI2 SUP-upper*)
qed

lemma *Inf-top-conv* [*simp*]:
$\bigsqcap A = \top \longleftrightarrow (\forall x{\in}A.\ x = \top)$
$\top = \bigsqcap A \longleftrightarrow (\forall x{\in}A.\ x = \top)$
proof −
show $\bigsqcap A = \top \longleftrightarrow (\forall x{\in}A.\ x = \top)$
proof
assume $\forall x{\in}A.\ x = \top$
then have $A = \{\} \vee A = \{\top\}$ **by** *auto*
then show $\bigsqcap A = \top$ **by** *auto*
next
assume $\bigsqcap A = \top$
show $\forall x{\in}A.\ x = \top$
proof (*rule ccontr*)
assume $\neg (\forall x{\in}A.\ x = \top)$
then obtain *x* **where** $x \in A$ **and** $x \neq \top$ **by** *blast*
then obtain *B* **where** *A* = *insert x B* **by** *blast*
with ‹$\bigsqcap A = \top$› ‹$x \neq \top$› **show** *False* **by** *simp*
qed
qed
then show $\top = \bigsqcap A \longleftrightarrow (\forall x{\in}A.\ x = \top)$ **by** *auto*
qed

lemma *INF-top-conv* [*simp*]:
$(\bigsqcap x{\in}A.\ B\ x) = \top \longleftrightarrow (\forall x{\in}A.\ B\ x = \top)$

$\top = (\bigsqcap x{\in}A.\ B\ x) \longleftrightarrow (\forall\, x{\in}A.\ B\ x = \top)$
using *Inf-top-conv* [*of B ' A*] **by** *simp-all*

lemma *Sup-bot-conv* [*simp*]:
$\bigsqcup A = \bot \longleftrightarrow (\forall\, x{\in}A.\ x = \bot)$
$\bot = \bigsqcup A \longleftrightarrow (\forall\, x{\in}A.\ x = \bot)$
using *dual-complete-lattice*
by (*rule complete-lattice.Inf-top-conv*)+

lemma *SUP-bot-conv* [*simp*]:
$(\bigsqcup x{\in}A.\ B\ x) = \bot \longleftrightarrow (\forall\, x{\in}A.\ B\ x = \bot)$
$\bot = (\bigsqcup x{\in}A.\ B\ x) \longleftrightarrow (\forall\, x{\in}A.\ B\ x = \bot)$
using *Sup-bot-conv* [*of B ' A*] **by** *simp-all*

lemma *INF-constant*: $(\bigsqcap y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \top\ \mathit{else}\ c)$
by (*auto intro*: *order.antisym INF-lower INF-greatest*)

lemma *SUP-constant*: $(\bigsqcup y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \bot\ \mathit{else}\ c)$
by (*auto intro*: *order.antisym SUP-upper SUP-least*)

lemma *INF-const* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcap i{\in}A.\ f) = f$
by (*simp add*: *INF-constant*)

lemma *SUP-const* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcup i{\in}A.\ f) = f$
by (*simp add*: *SUP-constant*)

lemma *INF-top* [*simp*]: $(\bigsqcap x{\in}A.\ \top) = \top$
by (*cases* $A = \{\}$) *simp-all*

lemma *SUP-bot* [*simp*]: $(\bigsqcup x{\in}A.\ \bot) = \bot$
by (*cases* $A = \{\}$) *simp-all*

lemma *INF-commute*: $(\bigsqcap i{\in}A.\ \bigsqcap j{\in}B.\ f\ i\ j) = (\bigsqcap j{\in}B.\ \bigsqcap i{\in}A.\ f\ i\ j)$
by (*iprover intro*: *INF-lower INF-greatest order-trans order.antisym*)

lemma *SUP-commute*: $(\bigsqcup i{\in}A.\ \bigsqcup j{\in}B.\ f\ i\ j) = (\bigsqcup j{\in}B.\ \bigsqcup i{\in}A.\ f\ i\ j)$
by (*iprover intro*: *SUP-upper SUP-least order-trans order.antisym*)

lemma *INF-absorb*:
assumes $k \in I$
shows $A\ k \sqcap (\bigsqcap i{\in}I.\ A\ i) = (\bigsqcap i{\in}I.\ A\ i)$
proof −
from *assms* **obtain** J **where** $I = \mathit{insert}\ k\ J$ **by** *blast*
then show *?thesis* **by** *simp*
qed

lemma *SUP-absorb*:
assumes $k \in I$
shows $A\ k \sqcup (\bigsqcup i{\in}I.\ A\ i) = (\bigsqcup i{\in}I.\ A\ i)$

proof –
  from *assms* **obtain** J **where** $I = insert\ k\ J$ **by** *blast*
  then show *?thesis* **by** *simp*
qed

lemma *INF-inf-const1*: $I \neq \{\} \Longrightarrow (\bigsqcap i{\in}I.\ inf\ x\ (f\ i)) = inf\ x\ (\bigsqcap i{\in}I.\ f\ i)$
  by (*intro order.antisym INF-greatest inf-mono order-refl INF-lower*)
    (*auto intro*: *INF-lower2 le-infI2 intro*!: *INF-mono*)

lemma *INF-inf-const2*: $I \neq \{\} \Longrightarrow (\bigsqcap i{\in}I.\ inf\ (f\ i)\ x) = inf\ (\bigsqcap i{\in}I.\ f\ i)\ x$
  using *INF-inf-const1*[*of I x f*] **by** (*simp add*: *inf-commute*)

lemma *less-INF-D*:
  assumes $y < (\bigsqcap i{\in}A.\ f\ i)\ i \in A$
  shows $y < f\ i$
proof –
  note ‹$y < (\bigsqcap i{\in}A.\ f\ i)$›
  also have $(\bigsqcap i{\in}A.\ f\ i) \leq f\ i$ **using** ‹$i \in A$›
    by (*rule INF-lower*)
  finally show $y < f\ i$ **.**
qed

lemma *SUP-lessD*:
  assumes $(\bigsqcup i{\in}A.\ f\ i) < y\ i \in A$
  shows $f\ i < y$
proof –
  have $f\ i \leq (\bigsqcup i{\in}A.\ f\ i)$
    using ‹$i \in A$› **by** (*rule SUP-upper*)
  also note ‹$(\bigsqcup i{\in}A.\ f\ i) < y$›
  finally show $f\ i < y$ **.**
qed

lemma *INF-UNIV-bool-expand*: $(\bigsqcap b.\ A\ b) = A\ True \sqcap A\ False$
  by (*simp add*: *UNIV-bool inf-commute*)

lemma *SUP-UNIV-bool-expand*: $(\bigsqcup b.\ A\ b) = A\ True \sqcup A\ False$
  by (*simp add*: *UNIV-bool sup-commute*)

lemma *Inf-le-Sup*: $A \neq \{\} \Longrightarrow Inf\ A \leq Sup\ A$
  by (*blast intro*: *Sup-upper2 Inf-lower ex-in-conv*)

lemma *INF-le-SUP*: $A \neq \{\} \Longrightarrow \bigsqcap(f\ `\ A) \leq \bigsqcup(f\ `\ A)$
  using *Inf-le-Sup* [*of f ‘ A*] **by** *simp*

lemma *INF-eq-const*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i = x) \Longrightarrow \bigsqcap(f\ `\ I) = x$
  by (*auto intro*: *INF-eqI*)

lemma *SUP-eq-const*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i = x) \Longrightarrow \bigsqcup(f\ `\ I) = x$
  by (*auto intro*: *SUP-eqI*)

lemma *INF-eq-iff*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i \leq c) \Longrightarrow \bigsqcap(f\ `\ I) = c \longleftrightarrow (\forall i{\in}I.\ f\ i = c)$
  by (*auto intro*: *INF-eq-const INF-lower order.antisym*)

lemma *SUP-eq-iff*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow c \leq f\ i) \Longrightarrow \bigsqcup(f\ `\ I) = c \longleftrightarrow (\forall i{\in}I.\ f\ i = c)$
  by (*auto intro*: *SUP-eq-const SUP-upper order.antisym*)

end

context *complete-lattice*
begin
lemma *Sup-Inf-le*: $Sup\ (Inf\ `\ \{f\ `\ A \mid f\ .\ (\forall\ Y \in A\ .\ f\ Y \in Y)\}) \leq Inf\ (Sup\ `\ A)$
  by (*rule SUP-least*, *clarify*, *rule INF-greatest*, *simp add*: *INF-lower2 Sup-upper*)
end

class *complete-distrib-lattice* = *complete-lattice* +
  assumes *Inf-Sup-le*: $Inf\ (Sup\ `\ A) \leq Sup\ (Inf\ `\ \{f\ `\ A \mid f\ .\ (\forall\ Y \in A\ .\ f\ Y \in Y)\})$
begin

lemma *Inf-Sup*: $Inf\ (Sup\ `\ A) = Sup\ (Inf\ `\ \{f\ `\ A \mid f\ .\ (\forall\ Y \in A\ .\ f\ Y \in Y)\})$
  by (*rule order.antisym*, *rule Inf-Sup-le*, *rule Sup-Inf-le*)

subclass *distrib-lattice*
proof
  fix $a\ b\ c$
  show $a \sqcup b \sqcap c = (a \sqcup b) \sqcap (a \sqcup c)$
  proof (*rule order.antisym*, *simp-all*, *safe*)
    show $b \sqcap c \leq a \sqcup b$
      by (*rule le-infI1*, *simp*)
    show $b \sqcap c \leq a \sqcup c$
      by (*rule le-infI2*, *simp*)
    have [*simp*]: $a \sqcap c \leq a \sqcup b \sqcap c$
      by (*rule le-infI1*, *simp*)
    have [*simp*]: $b \sqcap a \leq a \sqcup b \sqcap c$
      by (*rule le-infI2*, *simp*)
    have $\bigsqcap(Sup\ `\ \{\{a, b\}, \{a, c\}\}) =$
      $\bigsqcup(Inf\ `\ \{f\ `\ \{\{a, b\}, \{a, c\}\} \mid f.\ \forall Y{\in}\{\{a, b\}, \{a, c\}\}.\ f\ Y \in Y\})$
      by (*rule Inf-Sup*)
    from *this* **show** $(a \sqcup b) \sqcap (a \sqcup c) \leq a \sqcup b \sqcap c$
      apply *simp*
      by (*rule SUP-least*, *safe*, *simp-all*)
  qed
qed
end

context *complete-lattice*

begin
context
fixes *f* :: *'a* ⇒ *'b*::*complete-lattice*
assumes *mono f*
begin

lemma *mono-Inf*: *f* (⨅*A*) ≤ (⨅*x*∈*A*. *f x*)
using ‹*mono f*› **by** (*auto intro*: *complete-lattice-class.INF-greatest Inf-lower dest*: *monoD*)

lemma *mono-Sup*: (⨆*x*∈*A*. *f x*) ≤ *f* (⨆*A*)
using ‹*mono f*› **by** (*auto intro*: *complete-lattice-class.SUP-least Sup-upper dest*: *monoD*)

lemma *mono-INF*: *f* (⨅*i*∈*I*. *A i*) ≤ (⨅*x*∈*I*. *f* (*A x*))
by (*intro complete-lattice-class.INF-greatest monoD*[*OF* ‹*mono f*›] *INF-lower*)

lemma *mono-SUP*: (⨆*x*∈*I*. *f* (*A x*)) ≤ *f* (⨆*i*∈*I*. *A i*)
by (*intro complete-lattice-class.SUP-least monoD*[*OF* ‹*mono f*›] *SUP-upper*)

end

end

class *complete-boolean-algebra* = *boolean-algebra* + *complete-distrib-lattice*
begin

lemma *uminus-Inf*: − (⨅*A*) = ⨆(*uminus* ‘ *A*)
proof (*rule order.antisym*)
show − ⨅*A* ≤ ⨆(*uminus* ‘ *A*)
by (*rule compl-le-swap2*, *rule Inf-greatest*, *rule compl-le-swap2*, *rule Sup-upper*) *simp*
show ⨆(*uminus* ‘ *A*) ≤ − ⨅*A*
by (*rule Sup-least*, *rule compl-le-swap1*, *rule Inf-lower*) *auto*
qed

lemma *uminus-INF*: − (⨅*x*∈*A*. *B x*) = (⨆*x*∈*A*. − *B x*)
by (*simp add*: *uminus-Inf image-image*)

lemma *uminus-Sup*: − (⨆*A*) = ⨅(*uminus* ‘ *A*)
proof −
have ⨆*A* = − ⨅(*uminus* ‘ *A*)
by (*simp add*: *image-image uminus-INF*)
then show *?thesis* **by** *simp*
qed

lemma *uminus-SUP*: − (⨆*x*∈*A*. *B x*) = (⨅*x*∈*A*. − *B x*)
by (*simp add*: *uminus-Sup image-image*)

end

class *complete-linorder* = *linorder* + *complete-lattice*
begin

lemma *dual-complete-linorder*:
  class.complete-linorder Sup Inf sup (≥) (>) *inf* ⊤ ⊥
 by (*rule class.complete-linorder.intro*, *rule dual-complete-lattice*, *rule dual-linorder*)

lemma *complete-linorder-inf-min*: *inf* = *min*
  by (*auto intro*: *order.antisym simp add*: *min-def fun-eq-iff*)

lemma *complete-linorder-sup-max*: *sup* = *max*
  by (*auto intro*: *order.antisym simp add*: *max-def fun-eq-iff*)

lemma *Inf-less-iff*: $\bigsqcap S < a \longleftrightarrow (\exists x{\in}S.\ x < a)$
  by (*simp add*: *not-le* [*symmetric*] *le-Inf-iff*)

lemma *INF-less-iff*: $(\bigsqcap i{\in}A.\ f\ i) < a \longleftrightarrow (\exists x{\in}A.\ f\ x < a)$
  by (*simp add*: *Inf-less-iff* [*of f ‘ A*])

lemma *less-Sup-iff*: $a < \bigsqcup S \longleftrightarrow (\exists x{\in}S.\ a < x)$
  by (*simp add*: *not-le* [*symmetric*] *Sup-le-iff*)

lemma *less-SUP-iff*: $a < (\bigsqcup i{\in}A.\ f\ i) \longleftrightarrow (\exists x{\in}A.\ a < f\ x)$
  by (*simp add*: *less-Sup-iff* [*of - f ‘ A*])

lemma *Sup-eq-top-iff* [*simp*]: $\bigsqcup A = \top \longleftrightarrow (\forall x{<}\top.\ \exists i{\in}A.\ x < i)$
proof
  assume ∗: $\bigsqcup A = \top$
  show $(\forall x{<}\top.\ \exists i{\in}A.\ x < i)$
    unfolding ∗ [*symmetric*]
  proof (*intro allI impI*)
    fix *x*
    assume $x < \bigsqcup A$
    then show $\exists i{\in}A.\ x < i$
      by (*simp add*: *less-Sup-iff*)
  qed
next
  assume ∗: $\forall x{<}\top.\ \exists i{\in}A.\ x < i$
  show $\bigsqcup A = \top$
  proof (*rule ccontr*)
    assume $\bigsqcup A \neq \top$
    with *top-greatest* [*of* $\bigsqcup A$] **have** $\bigsqcup A < \top$
      unfolding *le-less* **by** *auto*
    with ∗ **have** $\bigsqcup A < \bigsqcup A$
      unfolding *less-Sup-iff* **by** *auto*
    then show *False* **by** *auto*
  qed

qed

lemma *SUP-eq-top-iff* [*simp*]: $(\bigsqcup i\in A.\ f\ i) = \top \longleftrightarrow (\forall x<\top.\ \exists i\in A.\ x < f\ i)$
  using *Sup-eq-top-iff* [*of f ' A*] **by** *simp*

lemma *Inf-eq-bot-iff* [*simp*]: $\bigsqcap A = \bot \longleftrightarrow (\forall x>\bot.\ \exists i\in A.\ i < x)$
  using *dual-complete-linorder*
  by (*rule complete-linorder.Sup-eq-top-iff*)

lemma *INF-eq-bot-iff* [*simp*]: $(\bigsqcap i\in A.\ f\ i) = \bot \longleftrightarrow (\forall x>\bot.\ \exists i\in A.\ f\ i < x)$
  using *Inf-eq-bot-iff* [*of f ' A*] **by** *simp*

lemma *Inf-le-iff*: $\bigsqcap A \le x \longleftrightarrow (\forall y>x.\ \exists a\in A.\ y > a)$
proof *safe*
  fix y
  assume $x \ge \bigsqcap A$ $y > x$
  then have $y > \bigsqcap A$ **by** *auto*
  then show $\exists a\in A.\ y > a$
    unfolding *Inf-less-iff* **.**
qed (*auto elim*!: *allE*[*of* - $\bigsqcap A$] *simp add*: *not-le*[*symmetric*] *Inf-lower*)

lemma *INF-le-iff*: $\bigsqcap(f\ `\ A) \le x \longleftrightarrow (\forall y>x.\ \exists i\in A.\ y > f\ i)$
  using *Inf-le-iff* [*of f ' A*] **by** *simp*

lemma *le-Sup-iff*: $x \le \bigsqcup A \longleftrightarrow (\forall y<x.\ \exists a\in A.\ y < a)$
proof *safe*
  fix y
  assume $x \le \bigsqcup A$ $y < x$
  then have $y < \bigsqcup A$ **by** *auto*
  then show $\exists a\in A.\ y < a$
    unfolding *less-Sup-iff* **.**
qed (*auto elim*!: *allE*[*of* - $\bigsqcup A$] *simp add*: *not-le*[*symmetric*] *Sup-upper*)

lemma *le-SUP-iff*: $x \le \bigsqcup(f\ `\ A) \longleftrightarrow (\forall y<x.\ \exists i\in A.\ y < f\ i)$
  using *le-Sup-iff* [*of - f ' A*] **by** *simp*

end

11.3 Complete lattice on *bool*

instantiation *bool* :: *complete-lattice*
begin

definition [*simp*, *code*]: $\bigsqcap A \longleftrightarrow False \notin A$

definition [*simp*, *code*]: $\bigsqcup A \longleftrightarrow True \in A$

instance
  by *standard* (*auto intro*: *bool-induct*)

end

lemma *not-False-in-image-Ball* [*simp*]: $False \notin P \text{ ` } A \longleftrightarrow Ball\ A\ P$
by *auto*

lemma *True-in-image-Bex* [*simp*]: $True \in P \text{ ` } A \longleftrightarrow Bex\ A\ P$
by *auto*

lemma *INF-bool-eq* [*simp*]: $(\lambda A\ f.\ \bigsqcap(f \text{ ` } A)) = Ball$
by (*simp add*: *fun-eq-iff*)

lemma *SUP-bool-eq* [*simp*]: $(\lambda A\ f.\ \bigsqcup(f \text{ ` } A)) = Bex$
by (*simp add*: *fun-eq-iff*)

instance *bool* :: *complete-boolean-algebra*
by (*standard*, *fastforce*)

11.4 Complete lattice on - ⇒ -

instantiation *fun* :: (*type*, *Inf*) *Inf*
begin

definition $\bigsqcap A = (\lambda x.\ \bigsqcap f{\in}A.\ f\ x)$

lemma *Inf-apply* [*simp*, *code*]: $(\bigsqcap A)\ x = (\bigsqcap f{\in}A.\ f\ x)$
by (*simp add*: *Inf-fun-def*)

instance ..

end

instantiation *fun* :: (*type*, *Sup*) *Sup*
begin

definition $\bigsqcup A = (\lambda x.\ \bigsqcup f{\in}A.\ f\ x)$

lemma *Sup-apply* [*simp*, *code*]: $(\bigsqcup A)\ x = (\bigsqcup f{\in}A.\ f\ x)$
by (*simp add*: *Sup-fun-def*)

instance ..

end

instantiation *fun* :: (*type*, *complete-lattice*) *complete-lattice*
begin

instance
by *standard* (*auto simp add*: *le-fun-def intro*: *INF-lower INF-greatest SUP-upper*

SUP-least)

end

lemma *INF-apply* [*simp*]: $(\bigsqcap y{\in}A.\ f\ y)\ x = (\bigsqcap y{\in}A.\ f\ y\ x)$
by (*simp add*: *image-comp*)

lemma *SUP-apply* [*simp*]: $(\bigsqcup y{\in}A.\ f\ y)\ x = (\bigsqcup y{\in}A.\ f\ y\ x)$
by (*simp add*: *image-comp*)

11.5 Complete lattice on unary and binary predicates

lemma *Inf1-I*: $(\bigwedge P.\ P \in A \Longrightarrow P\ a) \Longrightarrow (\bigsqcap A)\ a$
by *auto*

lemma *INF1-I*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x\ b) \Longrightarrow (\bigsqcap x{\in}A.\ B\ x)\ b$
by *simp*

lemma *INF2-I*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x\ b\ c) \Longrightarrow (\bigsqcap x{\in}A.\ B\ x)\ b\ c$
by *simp*

lemma *Inf2-I*: $(\bigwedge r.\ r \in A \Longrightarrow r\ a\ b) \Longrightarrow (\bigsqcap A)\ a\ b$
by *auto*

lemma *Inf1-D*: $(\bigsqcap A)\ a \Longrightarrow P \in A \Longrightarrow P\ a$
by *auto*

lemma *INF1-D*: $(\bigsqcap x{\in}A.\ B\ x)\ b \Longrightarrow a \in A \Longrightarrow B\ a\ b$
by *simp*

lemma *Inf2-D*: $(\bigsqcap A)\ a\ b \Longrightarrow r \in A \Longrightarrow r\ a\ b$
by *auto*

lemma *INF2-D*: $(\bigsqcap x{\in}A.\ B\ x)\ b\ c \Longrightarrow a \in A \Longrightarrow B\ a\ b\ c$
by *simp*

lemma *Inf1-E*:
assumes $(\bigsqcap A)\ a$
obtains $P\ a \mid P \notin A$
using *assms* **by** *auto*

lemma *INF1-E*:
assumes $(\bigsqcap x{\in}A.\ B\ x)\ b$
obtains $B\ a\ b \mid a \notin A$
using *assms* **by** *auto*

lemma *Inf2-E*:
assumes $(\bigsqcap A)\ a\ b$
obtains $r\ a\ b \mid r \notin A$

using *assms* **by** *auto*

lemma *INF2-E*:
assumes $(\bigsqcap x{\in}A.\ B\ x)\ b\ c$
obtains $B\ a\ b\ c \mid a \notin A$
using *assms* **by** *auto*

lemma *Sup1-I*: $P \in A \Longrightarrow P\ a \Longrightarrow (\bigsqcup A)\ a$
by *auto*

lemma *SUP1-I*: $a \in A \Longrightarrow B\ a\ b \Longrightarrow (\bigsqcup x{\in}A.\ B\ x)\ b$
by *auto*

lemma *Sup2-I*: $r \in A \Longrightarrow r\ a\ b \Longrightarrow (\bigsqcup A)\ a\ b$
by *auto*

lemma *SUP2-I*: $a \in A \Longrightarrow B\ a\ b\ c \Longrightarrow (\bigsqcup x{\in}A.\ B\ x)\ b\ c$
by *auto*

lemma *Sup1-E*:
assumes $(\bigsqcup A)\ a$
obtains P **where** $P \in A$ **and** $P\ a$
using *assms* **by** *auto*

lemma *SUP1-E*:
assumes $(\bigsqcup x{\in}A.\ B\ x)\ b$
obtains x **where** $x \in A$ **and** $B\ x\ b$
using *assms* **by** *auto*

lemma *Sup2-E*:
assumes $(\bigsqcup A)\ a\ b$
obtains r **where** $r \in A\ r\ a\ b$
using *assms* **by** *auto*

lemma *SUP2-E*:
assumes $(\bigsqcup x{\in}A.\ B\ x)\ b\ c$
obtains x **where** $x \in A\ B\ x\ b\ c$
using *assms* **by** *auto*

11.6 Complete lattice on - *set*

instantiation *set* :: (*type*) *complete-lattice*
begin

definition $\bigsqcap A = \{x.\ \bigsqcap((\lambda B.\ x \in B)\ `\ A)\}$

definition $\bigsqcup A = \{x.\ \bigsqcup((\lambda B.\ x \in B)\ `\ A)\}$

instance

by *standard* (*auto simp add*: *less-eq-set-def Inf-set-def Sup-set-def le-fun-def*)

end

11.6.1 Inter

abbreviation *Inter* :: $'a\ set\ set \Rightarrow 'a\ set$ (‹$\bigcap$›)
where $\bigcap S \equiv \prod S$

lemma *Inter-eq*: $\bigcap A = \{x.\ \forall B \in A.\ x \in B\}$
proof (*rule set-eqI*)
fix x
have $(\forall Q \in \{P.\ \exists B \in A.\ P \longleftrightarrow x \in B\}.\ Q) \longleftrightarrow (\forall B \in A.\ x \in B)$
by *auto*
then show $x \in \bigcap A \longleftrightarrow x \in \{x.\ \forall B \in A.\ x \in B\}$
by (*simp add*: *Inf-set-def image-def*)
qed

lemma *Inter-iff* [*simp*]: $A \in \bigcap C \longleftrightarrow (\forall X \in C.\ A \in X)$
by (*unfold Inter-eq*) *blast*

lemma *InterI* [*intro!*]: $(\bigwedge X.\ X \in C \Longrightarrow A \in X) \Longrightarrow A \in \bigcap C$
by (*simp add*: *Inter-eq*)

A "destruct" rule – every X in C contains A as an element, but $A \in X$ can hold when $X \in C$ does not! This rule is analogous to *spec*.

lemma *InterD* [*elim*, *Pure.elim*]: $A \in \bigcap C \Longrightarrow X \in C \Longrightarrow A \in X$
by *auto*

lemma *InterE* [*elim*]: $A \in \bigcap C \Longrightarrow (X \notin C \Longrightarrow R) \Longrightarrow (A \in X \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination rule – does not require proving $X \in C$.
unfolding *Inter-eq* **by** *blast*

lemma *Inter-lower*: $B \in A \Longrightarrow \bigcap A \subseteq B$
by (*fact Inf-lower*)

lemma *Inter-subset*: $(\bigwedge X.\ X \in A \Longrightarrow X \subseteq B) \Longrightarrow A \neq \{\} \Longrightarrow \bigcap A \subseteq B$
by (*fact Inf-less-eq*)

lemma *Inter-greatest*: $(\bigwedge X.\ X \in A \Longrightarrow C \subseteq X) \Longrightarrow C \subseteq \bigcap A$
by (*fact Inf-greatest*)

lemma *Inter-empty*: $\bigcap \{\} = UNIV$
by (*fact Inf-empty*)

lemma *Inter-UNIV*: $\bigcap UNIV = \{\}$
by (*fact Inf-UNIV*)

lemma *Inter-insert*: $\bigcap (insert\ a\ B) = a \cap \bigcap B$

by (*fact Inf-insert*)

lemma *Inter-Un-subset*: $\bigcap A \cup \bigcap B \subseteq \bigcap(A \cap B)$
by (*fact less-eq-Inf-inter*)

lemma *Inter-Un-distrib*: $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$
by (*fact Inf-union-distrib*)

lemma *Inter-UNIV-conv* [*simp*]:
$\bigcap A = UNIV \longleftrightarrow (\forall x{\in}A.\ x = UNIV)$
$UNIV = \bigcap A \longleftrightarrow (\forall x{\in}A.\ x = UNIV)$
by (*fact Inf-top-conv*)+

lemma *Inter-anti-mono*: $B \subseteq A \Longrightarrow \bigcap A \subseteq \bigcap B$
by (*fact Inf-superset-mono*)

11.6.2 Intersections of families

syntax (*ASCII*)
-*INTER1* :: *pttrns* ⇒ ′*b set* ⇒ ′*b set* (‹(‹indent=3 notation=‹binder INT››INT -./ -)› [0, 10] 10)
-*INTER* :: *pttrn* ⇒ ′*a set* ⇒ ′*b set* ⇒ ′*b set* (‹(‹indent=3 notation=‹binder INT››INT -:-./ -)› [0, 0, 10] 10)

syntax
-*INTER1* :: *pttrns* ⇒ ′*b set* ⇒ ′*b set* (‹(‹indent=3 notation=‹binder ⋂››⋂-./ -)› [0, 10] 10)
-*INTER* :: *pttrn* ⇒ ′*a set* ⇒ ′*b set* ⇒ ′*b set* (‹(‹indent=3 notation=‹binder ⋂››⋂-∈-./ -)› [0, 0, 10] 10)

syntax (*latex* **output**)
-*INTER1* :: *pttrns* ⇒ ′*b set* ⇒ ′*b set* (‹(3⋂(‹unbreakable›-)/ -)› [0, 10] 10)
-*INTER* :: *pttrn* ⇒ ′*a set* ⇒ ′*b set* ⇒ ′*b set* (‹(3⋂(‹unbreakable›-∈-)/ -)› [0, 0, 10] 10)

syntax-consts
-*INTER1* -*INTER* ⇌ *Inter*

translations
$\bigcap x\ y.\ f \rightleftharpoons \bigcap x.\ \bigcap y.\ f$
$\bigcap x.\ f \rightleftharpoons \bigcap(CONST\ range\ (\lambda x.\ f))$
$\bigcap x{\in}A.\ f \rightleftharpoons CONST\ Inter\ ((\lambda x.\ f)\ `\ A)$

lemma *INTER-eq*: $(\bigcap x{\in}A.\ B\ x) = \{y.\ \forall x{\in}A.\ y \in B\ x\}$
by (*auto intro*!: *INF-eqI*)

lemma *INT-iff* [*simp*]: $b \in (\bigcap x{\in}A.\ B\ x) \longleftrightarrow (\forall x{\in}A.\ b \in B\ x)$
using *Inter-iff* [*of* - *B* ‘ *A*] **by** *simp*

lemma *INT-I* [*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow b \in B\ x) \Longrightarrow b \in (\bigcap x{\in}A.\ B\ x)$
by *auto*

lemma *INT-D* [*elim*, *Pure.elim*]: $b \in (\bigcap x{\in}A.\ B\ x) \Longrightarrow a \in A \Longrightarrow b \in B\ a$
by *auto*

lemma *INT-E* [*elim*]: $b \in (\bigcap x{\in}A.\ B\ x) \Longrightarrow (b \in B\ a \Longrightarrow R) \Longrightarrow (a \notin A \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination – by the Excluded Middle on $a \in A$.
by *auto*

lemma *Collect-ball-eq*: $\{x.\ \forall\, y{\in}A.\ P\ x\ y\} = (\bigcap y{\in}A.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *Collect-all-eq*: $\{x.\ \forall\, y.\ P\ x\ y\} = (\bigcap y.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *INT-lower*: $a \in A \Longrightarrow (\bigcap x{\in}A.\ B\ x) \subseteq B\ a$
by (*fact INF-lower*)

lemma *INT-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow C \subseteq B\ x) \Longrightarrow C \subseteq (\bigcap x{\in}A.\ B\ x)$
by (*fact INF-greatest*)

lemma *INT-empty*: $(\bigcap x{\in}\{\}.\ B\ x) = UNIV$
by (*fact INF-empty*)

lemma *INT-absorb*: $k \in I \Longrightarrow A\ k \cap (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ A\ i)$
by (*fact INF-absorb*)

lemma *INT-subset-iff*: $B \subseteq (\bigcap i{\in}I.\ A\ i) \longleftrightarrow (\forall\, i{\in}I.\ B \subseteq A\ i)$
by (*fact le-INF-iff*)

lemma *INT-insert* [*simp*]: $(\bigcap x \in insert\ a\ A.\ B\ x) = B\ a \cap \bigcap\ (B\ `\ A)$
by (*fact INF-insert*)

lemma *INT-Un*: $(\bigcap i \in A \cup B.\ M\ i) = (\bigcap i \in A.\ M\ i) \cap (\bigcap i{\in}B.\ M\ i)$
by (*fact INF-union*)

lemma *INT-insert-distrib*: $u \in A \Longrightarrow (\bigcap x{\in}A.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcap x{\in}A.\ B\ x)$
by *blast*

lemma *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (if\ A = \{\}\ then\ UNIV\ else\ c)$
by (*fact INF-constant*)

lemma *INTER-UNIV-conv*:
$(UNIV = (\bigcap x{\in}A.\ B\ x)) = (\forall\, x{\in}A.\ B\ x = UNIV)$
$((\bigcap x{\in}A.\ B\ x) = UNIV) = (\forall\, x{\in}A.\ B\ x = UNIV)$

by (*fact INF-top-conv*)+

lemma *INT-bool-eq*: $(\bigcap b.\ A\ b) = A\ True \cap A\ False$
by (*fact INF-UNIV-bool-expand*)

lemma *INT-anti-mono*: $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow (\bigcap x\in B.\ f\ x) \subseteq (\bigcap x\in A.\ g\ x)$
— The last inclusion is POSITIVE!
by (*fact INF-superset-mono*)

lemma *Pow-INT-eq*: $Pow\ (\bigcap x\in A.\ B\ x) = (\bigcap x\in A.\ Pow\ (B\ x))$
by *blast*

lemma *vimage-INT*: $f\ -`\ (\bigcap x\in A.\ B\ x) = (\bigcap x\in A.\ f\ -`\ B\ x)$
by *blast*

11.6.3 Union

abbreviation *Union* :: $'a\ set\ set \Rightarrow 'a\ set$ $(\langle\bigcup\rangle)$
where $\bigcup S \equiv \bigsqcup S$

lemma *Union-eq*: $\bigcup A = \{x.\ \exists B \in A.\ x \in B\}$
proof (*rule set-eqI*)
fix x
have $(\exists Q\in\{P.\ \exists B\in A.\ P \longleftrightarrow x \in B\}.\ Q) \longleftrightarrow (\exists B\in A.\ x \in B)$
by *auto*
then show $x \in \bigcup A \longleftrightarrow x \in \{x.\ \exists B\in A.\ x \in B\}$
by (*simp add*: *Sup-set-def image-def*)
qed

lemma *Union-iff* [*simp*]: $A \in \bigcup C \longleftrightarrow (\exists X\in C.\ A\in X)$
by (*unfold Union-eq*) *blast*

lemma *UnionI* [*intro*]: $X \in C \Longrightarrow A \in X \Longrightarrow A \in \bigcup C$
— The order of the premises presupposes that C is rigid; A may be flexible.
by *auto*

lemma *UnionE* [*elim!*]: $A \in \bigcup C \Longrightarrow (\bigwedge X.\ A \in X \Longrightarrow X \in C \Longrightarrow R) \Longrightarrow R$
by *auto*

lemma *Union-upper*: $B \in A \Longrightarrow B \subseteq \bigcup A$
by (*fact Sup-upper*)

lemma *Union-least*: $(\bigwedge X.\ X \in A \Longrightarrow X \subseteq C) \Longrightarrow \bigcup A \subseteq C$
by (*fact Sup-least*)

lemma *Union-empty*: $\bigcup\{\} = \{\}$
by (*fact Sup-empty*)

lemma *Union-UNIV*: $\bigcup UNIV = UNIV$
by (*fact Sup-UNIV*)

lemma *Union-insert*: $\bigcup (insert\ a\ B) = a \cup \bigcup B$
by (*fact Sup-insert*)

lemma *Union-Un-distrib* [*simp*]: $\bigcup (A \cup B) = \bigcup A \cup \bigcup B$
by (*fact Sup-union-distrib*)

lemma *Union-Int-subset*: $\bigcup (A \cap B) \subseteq \bigcup A \cap \bigcup B$
by (*fact Sup-inter-less-eq*)

lemma *Union-empty-conv*: $(\bigcup A = \{\}) \longleftrightarrow (\forall x \in A.\ x = \{\})$
by (*fact Sup-bot-conv*)

lemma *empty-Union-conv*: $(\{\} = \bigcup A) \longleftrightarrow (\forall x \in A.\ x = \{\})$
by (*fact Sup-bot-conv*)

lemma *subset-Pow-Union*: $A \subseteq Pow\ (\bigcup A)$
by *blast*

lemma *Union-Pow-eq* [*simp*]: $\bigcup (Pow\ A) = A$
by *blast*

lemma *Union-mono*: $A \subseteq B \Longrightarrow \bigcup A \subseteq \bigcup B$
by (*fact Sup-subset-mono*)

lemma *Union-subsetI*: $(\bigwedge x.\ x \in A \Longrightarrow \exists y.\ y \in B \wedge x \subseteq y) \Longrightarrow \bigcup A \subseteq \bigcup B$
by *blast*

lemma *disjnt-inj-on-iff*:
$[\![inj\text{-}on\ f\ (\bigcup \mathcal{A});\ X \in \mathcal{A};\ Y \in \mathcal{A}]\!] \Longrightarrow disjnt\ (f\ `\ X)\ (f\ `\ Y) \longleftrightarrow disjnt\ X\ Y$
unfolding *disjnt-def*
by *safe* (*use inj-on-eq-iff* **in** ‹*fastforce+*›)

lemma *disjnt-Union1* [*simp*]: $disjnt\ (\bigcup \mathcal{A})\ B \longleftrightarrow (\forall A \in \mathcal{A}.\ disjnt\ A\ B)$
by (*auto simp*: *disjnt-def*)

lemma *disjnt-Union2* [*simp*]: $disjnt\ B\ (\bigcup \mathcal{A}) \longleftrightarrow (\forall A \in \mathcal{A}.\ disjnt\ B\ A)$
by (*auto simp*: *disjnt-def*)

11.6.4 Unions of families

syntax (*ASCII*)
-UNION1 :: *pttrns* => *'b set* => *'b set* (‹(‹*indent=3 notation=*‹*binder UN*››*UN -./ -*)› [*0*, *10*] *10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (‹(‹*indent=3 notation=*‹*binder UN*››*UN -:-./ -*)› [*0*, *0*, *10*] *10*)

syntax
-UNION1 :: *pttrns => 'b set => 'b set* (‹(‹indent=3 notation=‹binder ⋃››⋃ -./ -)› [0, 10] 10)
-UNION :: *pttrn => 'a set => 'b set => 'b set* (‹(‹indent=3 notation=‹binder ⋃››⋃ -∈-./ -)› [0, 0, 10] 10)

syntax (*latex* **output**)
-UNION1 :: *pttrns => 'b set => 'b set* (‹(3⋃ (‹unbreakable›-)/ -)› [0, 10] 10)
-UNION :: *pttrn => 'a set => 'b set => 'b set* (‹(3⋃ (‹unbreakable›-∈-)/ -)› [0, 0, 10] 10)

syntax-consts
-UNION1 -UNION ⇌ *Union*

translations
⋃ *x y. f* ⇌ ⋃ *x.* ⋃ *y. f*
⋃ *x. f* ⇌ ⋃ (*CONST range* (λ*x. f*))
⋃ *x*∈*A. f* ⇌ *CONST Union* ((λ*x. f*) ' *A*)

Note the difference between ordinary syntax of indexed unions and intersections (e.g. $\bigcup a_1 \in A_1.\ B$) and their LaTeX rendition: $\bigcup_{a_1 \in A_1} B$.

lemma *disjoint-UN-iff*: *disjnt A* (⋃ *i*∈*I. B i*) ⟷ (∀ *i*∈*I. disjnt A* (*B i*))
by (*auto simp*: *disjnt-def*)

lemma *UNION-eq*: (⋃ *x*∈*A. B x*) = {*y.* ∃ *x*∈*A. y* ∈ *B x*}
by (*auto intro*!: *SUP-eqI*)

lemma *bind-UNION* [*code*]: *Set.bind A f* = ⋃ (*f* ' *A*)
by (*simp add*: *bind-def UNION-eq*)

lemma *member-bind* [*simp*]: *x* ∈ *Set.bind A f* ⟷ *x* ∈ ⋃ (*f* ' *A*)
by (*simp add*: *bind-UNION*)

lemma *Union-SetCompr-eq*: ⋃ {*f x*| *x. P x*} = {*a.* ∃ *x. P x* ∧ *a* ∈ *f x*}
by *blast*

lemma *UN-iff* [*simp*]: *b* ∈ (⋃ *x*∈*A. B x*) ⟷ (∃ *x*∈*A. b* ∈ *B x*)
using *Union-iff* [*of - B* ' *A*] **by** *simp*

lemma *UN-I* [*intro*]: *a* ∈ *A* ⟹ *b* ∈ *B a* ⟹ *b* ∈ (⋃ *x*∈*A. B x*)
— The order of the premises presupposes that *A* is rigid; *b* may be flexible.
by *auto*

lemma *UN-E* [*elim*!]: *b* ∈ (⋃ *x*∈*A. B x*) ⟹ (⋀*x. x*∈*A* ⟹ *b* ∈ *B x* ⟹ *R*) ⟹ *R*
by *auto*

lemma *UN-upper*: *a* ∈ *A* ⟹ *B a* ⊆ (⋃ *x*∈*A. B x*)

by (*fact SUP-upper*)

lemma *UN-least*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C) \Longrightarrow (\bigcup x{\in}A.\ B\ x) \subseteq C$
by (*fact SUP-least*)

lemma *Collect-bex-eq*: $\{x.\ \exists\, y{\in}A.\ P\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *UN-insert-distrib*: $u \in A \Longrightarrow (\bigcup x{\in}A.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcup x{\in}A.\ B\ x)$
by *blast*

lemma *UN-empty*: $(\bigcup x{\in}\{\}.\ B\ x) = \{\}$
by (*fact SUP-empty*)

lemma *UN-empty2*: $(\bigcup x{\in}A.\ \{\}) = \{\}$
by (*fact SUP-bot*)

lemma *UN-absorb*: $k \in I \Longrightarrow A\ k \cup (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ A\ i)$
by (*fact SUP-absorb*)

lemma *UN-insert* [*simp*]: $(\bigcup x{\in}insert\ a\ A.\ B\ x) = B\ a \cup \bigcup (B\ `\ A)$
by (*fact SUP-insert*)

lemma *UN-Un* [*simp*]: $(\bigcup i \in A \cup B.\ M\ i) = (\bigcup i{\in}A.\ M\ i) \cup (\bigcup i{\in}B.\ M\ i)$
by (*fact SUP-union*)

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B\ y).\ C\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}B\ y.\ C\ x)$
by *blast*

lemma *UN-subset-iff*: $((\bigcup i{\in}I.\ A\ i) \subseteq B) = (\forall\, i{\in}I.\ A\ i \subseteq B)$
by (*fact SUP-le-iff*)

lemma *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (if\ A = \{\}\ then\ \{\}\ else\ c)$
by (*fact SUP-constant*)

lemma *UNION-singleton-eq-range*: $(\bigcup x{\in}A.\ \{f\ x\}) = f\ `\ A$
by *blast*

lemma *image-Union*: $f\ `\ \bigcup S = (\bigcup x{\in}S.\ f\ `\ x)$
by *blast*

lemma *UNION-empty-conv*:
$\{\} = (\bigcup x{\in}A.\ B\ x) \longleftrightarrow (\forall\, x{\in}A.\ B\ x = \{\})$
$(\bigcup x{\in}A.\ B\ x) = \{\} \longleftrightarrow (\forall\, x{\in}A.\ B\ x = \{\})$
by (*fact SUP-bot-conv*)+

lemma *Collect-ex-eq*: $\{x.\ \exists\, y.\ P\ x\ y\} = (\bigcup y.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *ball-UN*: $(\forall z \in \bigcup (B \text{ ` } A).\ P\ z) \longleftrightarrow (\forall x{\in}A.\ \forall z \in B\ x.\ P\ z)$
by *blast*

lemma *bex-UN*: $(\exists z \in \bigcup (B \text{ ` } A).\ P\ z) \longleftrightarrow (\exists x{\in}A.\ \exists z{\in}B\ x.\ P\ z)$
by *blast*

lemma *Un-eq-UN*: $A \cup B = (\bigcup b.\ \textit{if } b \textit{ then } A \textit{ else } B)$
by *safe* (*auto simp add*: *if-split-mem2*)

lemma *UN-bool-eq*: $(\bigcup b.\ A\ b) = (A\ \textit{True} \cup A\ \textit{False})$
by (*fact SUP-UNIV-bool-expand*)

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ \textit{Pow}\ (B\ x)) \subseteq \textit{Pow}\ (\bigcup x{\in}A.\ B\ x)$
by *blast*

lemma *UN-mono*:
$A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow$
$(\bigcup x{\in}A.\ f\ x) \subseteq (\bigcup x{\in}B.\ g\ x)$
by (*fact SUP-subset-mono*)

lemma *vimage-Union*: $f\ -\text{`}\ (\bigcup A) = (\bigcup X{\in}A.\ f\ -\text{`}\ X)$
by *blast*

lemma *vimage-UN*: $f\ -\text{`}\ (\bigcup x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ f\ -\text{`}\ B\ x)$
by *blast*

lemma *vimage-eq-UN*: $f\ -\text{`}\ B = (\bigcup y{\in}B.\ f\ -\text{`}\ \{y\})$
— NOT suitable for rewriting
by *blast*

lemma *image-UN*: $f \text{ ` } \bigcup (B \text{ ` } A) = (\bigcup x{\in}A.\ f \text{ ` } B\ x)$
by *blast*

lemma *UN-singleton* [*simp*]: $(\bigcup x{\in}A.\ \{x\}) = A$
by *blast*

lemma *inj-on-image*: $\textit{inj-on}\ f\ (\bigcup A) \Longrightarrow \textit{inj-on}\ ((\text{`})\ f)\ A$
unfolding *inj-on-def* **by** *blast*

11.6.5 Distributive laws

lemma *Int-Union*: $A \cap \bigcup B = (\bigcup C{\in}B.\ A \cap C)$
by *blast*

lemma *Un-Inter*: $A \cup \bigcap B = (\bigcap C{\in}B.\ A \cup C)$
by *blast*

lemma *Int-Union2*: $\bigcup B \cap A = (\bigcup C{\in}B.\ C \cap A)$

by *blast*

lemma *INT-Int-distrib*: $(\bigcap i{\in}I.\ A\ i \cap B\ i) = (\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i)$
by (*rule sym*) (*rule INF-inf-distrib*)

lemma *UN-Un-distrib*: $(\bigcup i{\in}I.\ A\ i \cup B\ i) = (\bigcup i{\in}I.\ A\ i) \cup (\bigcup i{\in}I.\ B\ i)$
by (*rule sym*) (*rule SUP-sup-distrib*)

lemma *Int-Inter-image*: $(\bigcap x{\in}C.\ A\ x \cap B\ x) = \bigcap(A\ `\ C) \cap \bigcap(B\ `\ C)$
by (*simp add*: *INT-Int-distrib*)

lemma *Int-Inter-eq*: $A \cap \bigcap\mathcal{B} = (\mathit{if}\ \mathcal{B}{=}\{\}\ \mathit{then}\ A\ \mathit{else}\ (\bigcap B{\in}\mathcal{B}.\ A \cap B))$
$\bigcap\mathcal{B} \cap A = (\mathit{if}\ \mathcal{B}{=}\{\}\ \mathit{then}\ A\ \mathit{else}\ (\bigcap B{\in}\mathcal{B}.\ B \cap A))$
by *auto*

lemma *Un-Union-image*: $(\bigcup x{\in}C.\ A\ x \cup B\ x) = \bigcup(A\ `\ C) \cup \bigcup(B\ `\ C)$
— Devlin, Fundamentals of Contemporary Set Theory, page 12, exercise 5:
— Union of a family of unions
by (*simp add*: *UN-Un-distrib*)

lemma *Un-INT-distrib*: $B \cup (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ B \cup A\ i)$
by *blast*

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ B \cap A\ i)$
— Halmos, Naive Set Theory, page 35.
by *blast*

lemma *Int-UN-distrib2*: $(\bigcup i{\in}I.\ A\ i) \cap (\bigcup j{\in}J.\ B\ j) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A\ i \cap B\ j)$
by *blast*

lemma *Un-INT-distrib2*: $(\bigcap i{\in}I.\ A\ i) \cup (\bigcap j{\in}J.\ B\ j) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A\ i \cup B\ j)$
by *blast*

lemma *Union-disjoint*: $(\bigcup C \cap A = \{\}) \longleftrightarrow (\forall B{\in}C.\ B \cap A = \{\})$
by *blast*

lemma *SUP-UNION*: $(\bigsqcup x{\in}(\bigcup y{\in}A.\ g\ y).\ f\ x) = (\bigsqcup y{\in}A.\ \bigsqcup x{\in}g\ y.\ f\ x :: - :: \mathit{complete\text{-}lattice})$
by (*rule order-antisym*) (*blast intro*: *SUP-least SUP-upper2*)+

11.7 Injections and bijections

lemma *inj-on-Inter*: $S \neq \{\} \Longrightarrow (\bigwedge A.\ A \in S \Longrightarrow \mathit{inj\text{-}on}\ f\ A) \Longrightarrow \mathit{inj\text{-}on}\ f\ (\bigcap S)$
unfolding *inj-on-def* **by** *blast*

lemma *inj-on-INTER*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow \mathit{inj\text{-}on}\ f\ (A\ i)) \Longrightarrow \mathit{inj\text{-}on}\ f\ (\bigcap i \in I.\ A\ i)$

unfolding *inj-on-def* **by** *safe simp*

lemma *inj-on-UNION-chain*:
assumes *chain*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow A\ i \leq A\ j \vee A\ j \leq A\ i$
and *inj*: $\bigwedge i.\ i \in I \Longrightarrow$ *inj-on* $f\ (A\ i)$
shows *inj-on* $f\ (\bigcup i \in I.\ A\ i)$
proof −
have $x = y$
if ∗: $i \in I\ j \in I$
and ∗∗: $x \in A\ i\ y \in A\ j$
and ∗∗∗: $f\ x = f\ y$
for $i\ j\ x\ y$
using *chain* [*OF* ∗]
proof
assume $A\ i \leq A\ j$
with ∗∗ **have** $x \in A\ j$ **by** *auto*
with *inj* ∗ ∗∗ ∗∗∗ **show** *?thesis*
by (*auto simp add*: *inj-on-def*)
next
assume $A\ j \leq A\ i$
with ∗∗ **have** $y \in A\ i$ **by** *auto*
with *inj* ∗ ∗∗ ∗∗∗ **show** *?thesis*
by (*auto simp add*: *inj-on-def*)
qed
then show *?thesis*
by (*unfold inj-on-def UNION-eq*) *auto*
qed

lemma *bij-betw-UNION-chain*:
assumes *chain*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow A\ i \leq A\ j \vee A\ j \leq A\ i$
and *bij*: $\bigwedge i.\ i \in I \Longrightarrow$ *bij-betw* $f\ (A\ i)\ (A'\ i)$
shows *bij-betw* $f\ (\bigcup i \in I.\ A\ i)\ (\bigcup i \in I.\ A'\ i)$
unfolding *bij-betw-def*
proof *safe*
have $\bigwedge i.\ i \in I \Longrightarrow$ *inj-on* $f\ (A\ i)$
using *bij bij-betw-def*[*of f*] **by** *auto*
then show *inj-on* $f\ (\bigcup (A\ `\ I))$
using *chain inj-on-UNION-chain*[*of I A f*] **by** *auto*
next
fix $i\ x$
assume ∗: $i \in I\ x \in A\ i$
with *bij* **have** $f\ x \in A'\ i$
by (*auto simp*: *bij-betw-def*)
with ∗ **show** $f\ x \in \bigcup (A'\ `\ I)$ **by** *blast*
next
fix $i\ x'$
assume ∗: $i \in I\ x' \in A'\ i$
with *bij* **have** $\exists x \in A\ i.\ x' = f\ x$
unfolding *bij-betw-def* **by** *blast*

with $*$ **have** $\exists j \in I.\ \exists x \in A\ j.\ x' = f\ x$
  by *blast*
then show $x' \in f\ `\ \bigcup(A\ `\ I)$
  by *blast*
qed

lemma *image-INT*: *inj-on f C* $\Longrightarrow \forall x{\in}A.\ B\ x \subseteq C \Longrightarrow j \in A \Longrightarrow f\ `\ (\bigcap(B\ `\ A)) = (\bigcap x{\in}A.\ f\ `\ B\ x)$
  by (*auto simp add*: *inj-on-def*) *blast*

lemma *bij-image-INT*: *bij f* $\Longrightarrow f\ `\ (\bigcap(B\ `\ A)) = (\bigcap x{\in}A.\ f\ `\ B\ x)$
  by (*auto simp*: *bij-def inj-def surj-def*) *blast*

lemma *UNION-fun-upd*: $\bigcup(A(i := B)\ `\ J) = \bigcup(A\ `\ (J - \{i\})) \cup$ (*if* $i \in J$ *then* B *else* $\{\}$)
  by (*auto simp add*: *set-eq-iff*)

lemma *bij-betw-Pow*:
  assumes *bij-betw f A B*
  shows *bij-betw* (*image f*) (*Pow A*) (*Pow B*)
proof −
  from *assms* **have** *inj-on f A*
    by (*rule bij-betw-imp-inj-on*)
  then have *inj-on f* ($\bigcup$(*Pow A*))
    by *simp*
  then have *inj-on* (*image f*) (*Pow A*)
    by (*rule inj-on-image*)
  then have *bij-betw* (*image f*) (*Pow A*) (*image f ` Pow A*)
    by (*rule inj-on-imp-bij-betw*)
  moreover from *assms* **have** $f\ `\ A = B$
    by (*rule bij-betw-imp-surj-on*)
  then have *image f ` Pow A = Pow B*
    by (*rule image-Pow-surj*)
  ultimately show *?thesis* **by** *simp*
qed

11.7.1 Complement

lemma *Compl-INT* [*simp*]: $-\ (\bigcap x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ -B\ x)$
  by *blast*

lemma *Compl-UN* [*simp*]: $-\ (\bigcup x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ -B\ x)$
  by *blast*

11.7.2 Miniscoping and maxiscoping

Miniscoping: pushing in quantifiers and big Unions and Intersections.

lemma *UN-simps* [*simp*]:

$\bigwedge a\ B\ C.\ (\bigcup x{\in}C.\ insert\ a\ (B\ x)) = (if\ C{=}\{\}\ then\ \{\}\ else\ insert\ a\ (\bigcup x{\in}C.\ B\ x))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x \cup B) = ((if\ C{=}\{\}\ then\ \{\}\ else\ (\bigcup x{\in}C.\ A\ x) \cup B))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A \cup B\ x) = ((if\ C{=}\{\}\ then\ \{\}\ else\ A \cup (\bigcup x{\in}C.\ B\ x)))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x \cap B) = ((\bigcup x{\in}C.\ A\ x) \cap B)$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A \cap B\ x) = (A \cap(\bigcup x{\in}C.\ B\ x))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x - B) = ((\bigcup x{\in}C.\ A\ x) - B)$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A - B\ x) = (A - (\bigcap x{\in}C.\ B\ x))$
$\bigwedge A\ B.\ (\bigcup x{\in}\bigcup A.\ B\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}y.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcup z{\in}(\bigcup (B\ `\ A)).\ C\ z) = (\bigcup x{\in}A.\ \bigcup z{\in}B\ x.\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcup x{\in}f`A.\ B\ x) = (\bigcup a{\in}A.\ B\ (f\ a))$
by *auto*

lemma *INT-simps* [*simp*]:
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x \cap B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (\bigcap x{\in}C.\ A\ x) \cap B)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A \cap B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A \cap(\bigcap x{\in}C.\ B\ x))$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x - B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (\bigcap x{\in}C.\ A\ x) - B)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A - B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A - (\bigcup x{\in}C.\ B\ x))$
$\bigwedge a\ B\ C.\ (\bigcap x{\in}C.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcap x{\in}C.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x \cup B) = ((\bigcap x{\in}C.\ A\ x) \cup B)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A \cup B\ x) = (A \cup (\bigcap x{\in}C.\ B\ x))$
$\bigwedge A\ B.\ (\bigcap x{\in}\bigcup A.\ B\ x) = (\bigcap y{\in}A.\ \bigcap x{\in}y.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcap z{\in}(\bigcup (B\ `\ A)).\ C\ z) = (\bigcap x{\in}A.\ \bigcap z{\in}B\ x.\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcap x{\in}f`A.\ B\ x) = (\bigcap a{\in}A.\ B\ (f\ a))$
by *auto*

lemma *UN-ball-bex-simps* [*simp*]:
$\bigwedge A\ P.\ (\forall x{\in}\bigcup A.\ P\ x) \longleftrightarrow (\forall y{\in}A.\ \forall x{\in}y.\ P\ x)$
$\bigwedge A\ B\ P.\ (\forall x{\in}(\bigcup (B\ `\ A)).\ P\ x) = (\forall a{\in}A.\ \forall x{\in}\ B\ a.\ P\ x)$
$\bigwedge A\ P.\ (\exists x{\in}\bigcup A.\ P\ x) \longleftrightarrow (\exists y{\in}A.\ \exists x{\in}y.\ P\ x)$
$\bigwedge A\ B\ P.\ (\exists x{\in}(\bigcup (B\ `\ A)).\ P\ x) \longleftrightarrow (\exists a{\in}A.\ \exists x{\in}B\ a.\ P\ x)$
by *auto*

Maxiscoping: pulling out big Unions and Intersections.

lemma *UN-extend-simps*:
$\bigwedge a\ B\ C.\ insert\ a\ (\bigcup x{\in}C.\ B\ x) = (if\ C{=}\{\}\ then\ \{a\}\ else\ (\bigcup x{\in}C.\ insert\ a\ (B\ x)))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x) \cup B = (if\ C{=}\{\}\ then\ B\ else\ (\bigcup x{\in}C.\ A\ x \cup B))$
$\bigwedge A\ B\ C.\ A \cup (\bigcup x{\in}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (\bigcup x{\in}C.\ A \cup B\ x))$
$\bigwedge A\ B\ C.\ ((\bigcup x{\in}C.\ A\ x) \cap B) = (\bigcup x{\in}C.\ A\ x \cap B)$
$\bigwedge A\ B\ C.\ (A \cap (\bigcup x{\in}C.\ B\ x)) = (\bigcup x{\in}C.\ A \cap B\ x)$
$\bigwedge A\ B\ C.\ ((\bigcup x{\in}C.\ A\ x) - B) = (\bigcup x{\in}C.\ A\ x - B)$
$\bigwedge A\ B\ C.\ (A - (\bigcap x{\in}C.\ B\ x)) = (\bigcup x{\in}C.\ A - B\ x)$
$\bigwedge A\ B.\ (\bigcup y{\in}A.\ \bigcup x{\in}y.\ B\ x) = (\bigcup x{\in}\bigcup A.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}A.\ \bigcup z{\in}B\ x.\ C\ z) = (\bigcup z{\in}(\bigcup (B\ `\ A)).\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcup a{\in}A.\ B\ (f\ a)) = (\bigcup x{\in}f`A.\ B\ x)$
by *auto*

lemma *INT-extend-simps*:

$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x) \cap B = (\textit{if}\ C{=}\{\}\ \textit{then}\ B\ \textit{else}\ (\bigcap x{\in}C.\ A\ x \cap B))$
$\bigwedge A\ B\ C.\ A \cap (\bigcap x{\in}C.\ B\ x) = (\textit{if}\ C{=}\{\}\ \textit{then}\ A\ \textit{else}\ (\bigcap x{\in}C.\ A \cap B\ x))$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x) - B = (\textit{if}\ C{=}\{\}\ \textit{then}\ UNIV - B\ \textit{else}\ (\bigcap x{\in}C.\ A\ x - B))$
$\bigwedge A\ B\ C.\ A - (\bigcup x{\in}C.\ B\ x) = (\textit{if}\ C{=}\{\}\ \textit{then}\ A\ \textit{else}\ (\bigcap x{\in}C.\ A - B\ x))$
$\bigwedge a\ B\ C.\ \textit{insert}\ a\ (\bigcap x{\in}C.\ B\ x) = (\bigcap x{\in}C.\ \textit{insert}\ a\ (B\ x))$
$\bigwedge A\ B\ C.\ ((\bigcap x{\in}C.\ A\ x) \cup B) = (\bigcap x{\in}C.\ A\ x \cup B)$
$\bigwedge A\ B\ C.\ A \cup (\bigcap x{\in}C.\ B\ x) = (\bigcap x{\in}C.\ A \cup B\ x)$
$\bigwedge A\ B.\ (\bigcap y{\in}A.\ \bigcap x{\in}y.\ B\ x) = (\bigcap x{\in}\bigcup A.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}A.\ \bigcap z{\in}B\ x.\ C\ z) = (\bigcap z{\in}(\bigcup (B\ `\ A)).\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcap a{\in}A.\ B\ (f\ a)) = (\bigcap x{\in}f`A.\ B\ x)$
by *auto*

Finally

lemmas *mem-simps* =
insert-iff empty-iff Un-iff Int-iff Compl-iff Diff-iff
mem-Collect-eq UN-iff Union-iff INT-iff Inter-iff
— Each of these has ALREADY been added [*simp*] above.

end

12 Wrapping Existing Freely Generated Type's Constructors

theory *Ctr-Sugar*
imports *HOL*
keywords
print-case-translations :: *diag* **and**
free-constructors :: *thy-goal*
begin

consts
case-guard :: $bool \Rightarrow {'a} \Rightarrow ({'a} \Rightarrow {'b}) \Rightarrow {'b}$
case-nil :: ${'a} \Rightarrow {'b}$
case-cons :: $({'a} \Rightarrow {'b}) \Rightarrow ({'a} \Rightarrow {'b}) \Rightarrow {'a} \Rightarrow {'b}$
case-elem :: ${'a} \Rightarrow {'b} \Rightarrow {'a} \Rightarrow {'b}$
case-abs :: $({'c} \Rightarrow {'b}) \Rightarrow {'b}$

declare [[*coercion-args case-guard* − + −]]
declare [[*coercion-args case-cons* − −]]
declare [[*coercion-args case-abs* −]]
declare [[*coercion-args case-elem* − +]]

ML-file ‹*Tools/Ctr-Sugar/case-translation.ML*›

lemma *iffI-np*: $[\![x \Longrightarrow \neg\ y;\ \neg\ x \Longrightarrow y]\!] \Longrightarrow \neg\ x \longleftrightarrow y$
by (*erule iffI*) (*erule contrapos-pn*)

lemma *iff-contradict*:
$\neg P \Longrightarrow P \longleftrightarrow Q \Longrightarrow Q \Longrightarrow R$
$\neg Q \Longrightarrow P \longleftrightarrow Q \Longrightarrow P \Longrightarrow R$
by *blast+*

ML-file ‹*Tools/Ctr-Sugar/ctr-sugar-util.ML*›
ML-file ‹*Tools/Ctr-Sugar/ctr-sugar-tactics.ML*›
ML-file ‹*Tools/Ctr-Sugar/ctr-sugar-code.ML*›
ML-file ‹*Tools/Ctr-Sugar/ctr-sugar.ML*›

Coinduction method that avoids some boilerplate compared with coinduct.

ML-file ‹*Tools/coinduction.ML*›

end

13 Knaster-Tarski Fixpoint Theorem and inductive definitions

theory *Inductive*
imports *Complete-Lattices Ctr-Sugar*
keywords
inductive coinductive inductive-cases inductive-simps :: *thy-defn* **and**
monos **and**
print-inductives :: *diag* **and**
old-rep-datatype :: *thy-goal* **and**
primrec :: *thy-defn*
begin

13.1 Least fixed points

context *complete-lattice*
begin

definition *lfp* :: $('a \Rightarrow 'a) \Rightarrow 'a$
where $lfp\ f = Inf\ \{u.\ f\ u \leq u\}$

lemma *lfp-lowerbound*: $f\ A \leq A \Longrightarrow lfp\ f \leq A$
unfolding *lfp-def* **by** (*rule Inf-lower*) *simp*

lemma *lfp-greatest*: $(\bigwedge u.\ f\ u \leq u \Longrightarrow A \leq u) \Longrightarrow A \leq lfp\ f$
unfolding *lfp-def* **by** (*rule Inf-greatest*) *simp*

end

lemma *lfp-fixpoint*:
assumes *mono f*
shows $f\ (lfp\ f) = lfp\ f$
unfolding *lfp-def*

```
proof (rule order-antisym)
  let ?H = {u. f u ≤ u}
  let ?a = ⨅ ?H
  show f ?a ≤ ?a
  proof (rule Inf-greatest)
    fix x
    assume x ∈ ?H
    then have ?a ≤ x by (rule Inf-lower)
    with ‹mono f› have f ?a ≤ f x ..
    also from ‹x ∈ ?H› have f x ≤ x ..
    finally show f ?a ≤ x .
  qed
  show ?a ≤ f ?a
  proof (rule Inf-lower)
    from ‹mono f› and ‹f ?a ≤ ?a› have f (f ?a) ≤ f ?a ..
    then show f ?a ∈ ?H ..
  qed
qed

lemma lfp-unfold: mono f ⟹ lfp f = f (lfp f)
  by (rule lfp-fixpoint [symmetric])

lemma lfp-const: lfp (λx. t) = t
  by (rule lfp-unfold) (simp add: mono-def)

lemma lfp-eqI: mono F ⟹ F x = x ⟹ (⋀z. F z = z ⟹ x ≤ z) ⟹ lfp F = x
  by (rule antisym) (simp-all add: lfp-lowerbound lfp-unfold[symmetric])
```

13.2 General induction rules for least fixed points

```
lemma lfp-ordinal-induct [case-names mono step union]:
  fixes f :: 'a::complete-lattice ⇒ 'a
  assumes mono: mono f
    and P-f: ⋀S. P S ⟹ S ≤ lfp f ⟹ P (f S)
    and P-Union: ⋀M. ∀S∈M. P S ⟹ P (Sup M)
  shows P (lfp f)
proof -
  let ?M = {S. S ≤ lfp f ∧ P S}
  from P-Union have P (Sup ?M) by simp
  also have Sup ?M = lfp f
  proof (rule antisym)
    show Sup ?M ≤ lfp f
      by (blast intro: Sup-least)
    then have f (Sup ?M) ≤ f (lfp f)
      by (rule mono [THEN monoD])
    then have f (Sup ?M) ≤ lfp f
      using mono [THEN lfp-unfold] by simp
    then have f (Sup ?M) ∈ ?M
      using P-Union by simp (intro P-f Sup-least, auto)
```

then have *f* (*Sup ?M*) ≤ *Sup ?M*
by (*rule Sup-upper*)
then show *lfp f* ≤ *Sup ?M*
by (*rule lfp-lowerbound*)
qed
finally show *?thesis* **.**
qed

theorem *lfp-induct*:
assumes *mono*: *mono f*
and *ind*: *f* (*inf* (*lfp f*) *P*) ≤ *P*
shows *lfp f* ≤ *P*
proof (*induct rule*: *lfp-ordinal-induct*)
case *mono*
show *?case* **by** *fact*
next
case (*step S*)
then show *?case*
by (*intro order-trans*[*OF - ind*] *monoD*[*OF mono*]) *auto*
next
case (*union M*)
then show *?case*
by (*auto intro*: *Sup-least*)
qed

lemma *lfp-induct-set*:
assumes *lfp*: *a* ∈ *lfp f*
and *mono*: *mono f*
and *hyp*: ⋀*x*. *x* ∈ *f* (*lfp f* ∩ {*x*. *P x*}) ⟹ *P x*
shows *P a*
by (*rule lfp-induct* [*THEN subsetD*, *THEN CollectD*, *OF mono - lfp*]) (*auto intro*: *hyp*)

lemma *lfp-ordinal-induct-set*:
assumes *mono*: *mono f*
and *P-f*: ⋀*S*. *P S* ⟹ *P* (*f S*)
and *P-Union*: ⋀*M*. ∀ *S*∈*M*. *P S* ⟹ *P* (⋃ *M*)
shows *P* (*lfp f*)
using *assms* **by** (*rule lfp-ordinal-induct*)

Definition forms of *lfp-unfold* and *lfp-induct*, to control unfolding.

lemma *def-lfp-unfold*: *h* ≡ *lfp f* ⟹ *mono f* ⟹ *h* = *f h*
by (*auto intro*!: *lfp-unfold*)

lemma *def-lfp-induct*: *A* ≡ *lfp f* ⟹ *mono f* ⟹ *f* (*inf A P*) ≤ *P* ⟹ *A* ≤ *P*
by (*blast intro*: *lfp-induct*)

lemma *def-lfp-induct-set*:
A ≡ *lfp f* ⟹ *mono f* ⟹ *a* ∈ *A* ⟹ (⋀*x*. *x* ∈ *f* (*A* ∩ {*x*. *P x*}) ⟹ *P x*) ⟹

P a
  by (*blast intro*: *lfp-induct-set*)

Monotonicity of *lfp*!

lemma *lfp-mono*: $(\bigwedge Z.\ f\ Z \le g\ Z) \Longrightarrow lfp\ f \le lfp\ g$
  by (*rule lfp-lowerbound* [*THEN lfp-greatest*]) (*blast intro*: *order-trans*)

13.3 Greatest fixed points

context *complete-lattice*
begin

definition *gfp* :: $('a \Rightarrow 'a) \Rightarrow 'a$
  where $gfp\ f = Sup\ \{u.\ u \le f\ u\}$

lemma *gfp-upperbound*: $X \le f\ X \Longrightarrow X \le gfp\ f$
  by (*auto simp add*: *gfp-def intro*: *Sup-upper*)

lemma *gfp-least*: $(\bigwedge u.\ u \le f\ u \Longrightarrow u \le X) \Longrightarrow gfp\ f \le X$
  by (*auto simp add*: *gfp-def intro*: *Sup-least*)

end

lemma *lfp-le-gfp*: $mono\ f \Longrightarrow lfp\ f \le gfp\ f$
  by (*rule gfp-upperbound*) (*simp add*: *lfp-fixpoint*)

lemma *gfp-fixpoint*:
  assumes *mono f*
  shows $f\ (gfp\ f) = gfp\ f$
  unfolding *gfp-def*
proof (*rule order-antisym*)
  let $?H = \{u.\ u \le f\ u\}$
  let $?a = \bigsqcup ?H$
  show $?a \le f\ ?a$
  proof (*rule Sup-least*)
    fix *x*
    assume $x \in ?H$
    then have $x \le f\ x$ **..**
    also from ‹$x \in ?H$› **have** $x \le ?a$ **by** (*rule Sup-upper*)
    with ‹*mono f*› **have** $f\ x \le f\ ?a$ **..**
    finally show $x \le f\ ?a$ **.**
  qed
  show $f\ ?a \le ?a$
  proof (*rule Sup-upper*)
    from ‹*mono f*› **and** ‹$?a \le f\ ?a$› **have** $f\ ?a \le f\ (f\ ?a)$ **..**
    then show $f\ ?a \in ?H$ **..**
  qed
qed

lemma *gfp-unfold*: *mono f* $\Longrightarrow$ *gfp f* = *f* (*gfp f*)
  by (*rule gfp-fixpoint* [*symmetric*])

lemma *gfp-const*: *gfp* ($\lambda x.\ t$) = *t*
  by (*rule gfp-unfold*) (*simp add*: *mono-def*)

lemma *gfp-eqI*: *mono F* $\Longrightarrow$ *F x* = *x* $\Longrightarrow$ ($\bigwedge z.\ F\ z = z \Longrightarrow z \le x$) $\Longrightarrow$ *gfp F* = *x*
  by (*rule antisym*) (*simp-all add*: *gfp-upperbound gfp-unfold*[*symmetric*])

13.4 Coinduction rules for greatest fixed points

Weak version.

lemma *weak-coinduct*: $a \in X \Longrightarrow X \subseteq f\ X \Longrightarrow a \in$ *gfp f*
  by (*rule gfp-upperbound* [*THEN subsetD*]) *auto*

lemma *weak-coinduct-image*: $a \in X \Longrightarrow g`X \subseteq f\ (g`X) \Longrightarrow g\ a \in$ *gfp f*
  apply (*erule gfp-upperbound* [*THEN subsetD*])
  apply (*erule imageI*)
  done

lemma *coinduct-lemma*: $X \le f$ (*sup X* (*gfp f*)) $\Longrightarrow$ *mono f* $\Longrightarrow$ *sup X* (*gfp f*) $\le$ *f* (*sup X* (*gfp f*))
  apply (*frule gfp-unfold* [*THEN eq-refl*])
  apply (*drule mono-sup*)
  apply (*rule le-supI*)
   apply *assumption*
  apply (*rule order-trans*)
   apply (*rule order-trans*)
    apply *assumption*
   apply (*rule sup-ge2*)
  apply *assumption*
  done

Strong version, thanks to Coen and Frost.

lemma *coinduct-set*: *mono f* $\Longrightarrow a \in X \Longrightarrow X \subseteq f\ (X \cup$ *gfp f*) $\Longrightarrow a \in$ *gfp f*
  by (*rule weak-coinduct*[*rotated*], *rule coinduct-lemma*) *blast+*

lemma *gfp-fun-UnI2*: *mono f* $\Longrightarrow a \in$ *gfp f* $\Longrightarrow a \in f\ (X \cup$ *gfp f*)
  by (*blast dest*: *gfp-fixpoint mono-Un*)

lemma *gfp-ordinal-induct*[*case-names mono step union*]:
  fixes *f* :: *'a::complete-lattice* $\Rightarrow$ *'a*
  assumes *mono*: *mono f*
    and *P-f*: $\bigwedge S.\ P\ S \Longrightarrow$ *gfp f* $\le S \Longrightarrow P\ (f\ S)$
    and *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P$ (*Inf M*)
  shows *P* (*gfp f*)
proof −
  let *?M* = {*S*. *gfp f* $\le S \wedge P\ S$}

```
  from P-Union have P (Inf ?M) by simp
  also have Inf ?M = gfp f
  proof (rule antisym)
    show gfp f ≤ Inf ?M
      by (blast intro: Inf-greatest)
    then have f (gfp f) ≤ f (Inf ?M)
      by (rule mono [THEN monoD])
    then have gfp f ≤ f (Inf ?M)
      using mono [THEN gfp-unfold] by simp
    then have f (Inf ?M) ∈ ?M
      using P-Union by simp (intro P-f Inf-greatest, auto)
    then have Inf ?M ≤ f (Inf ?M)
      by (rule Inf-lower)
    then show Inf ?M ≤ gfp f
      by (rule gfp-upperbound)
  qed
  finally show ?thesis .
qed

lemma coinduct:
  assumes mono: mono f
    and ind: X ≤ f (sup X (gfp f))
  shows X ≤ gfp f
proof (induct rule: gfp-ordinal-induct)
  case mono
  then show ?case by fact
next
  case (step S)
  then show ?case
    by (intro order-trans[OF ind -] monoD[OF mono]) auto
next
  case (union M)
  then show ?case
    by (auto intro: mono Inf-greatest)
qed
```

13.5 Even Stronger Coinduction Rule, by Martin Coen

Weakens the condition $X \subseteq f\ X$ to one expressed using both *lfp* and *gfp*

```
lemma coinduct3-mono-lemma: mono f ⟹ mono (λx. f x ∪ X ∪ B)
  by (iprover intro: subset-refl monoI Un-mono monoD)

lemma coinduct3-lemma:
  X ⊆ f (lfp (λx. f x ∪ X ∪ gfp f)) ⟹ mono f ⟹
    lfp (λx. f x ∪ X ∪ gfp f) ⊆ f (lfp (λx. f x ∪ X ∪ gfp f))
  apply (rule subset-trans)
   apply (erule coinduct3-mono-lemma [THEN lfp-unfold [THEN eq-refl]])
  apply (rule Un-least [THEN Un-least])
    apply (rule subset-refl, assumption)
```

apply (*rule gfp-unfold* [*THEN equalityD1*, *THEN subset-trans*], *assumption*)
apply (*rule monoD*, *assumption*)
apply (*subst coinduct3-mono-lemma* [*THEN lfp-unfold*], *auto*)
done

lemma *coinduct3*: $mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (lfp\ (\lambda x.\ f\ x \cup X \cup gfp\ f)) \Longrightarrow a \in gfp\ f$
apply (*rule coinduct3-lemma* [*THEN* [*2*] *weak-coinduct*])
apply (*rule coinduct3-mono-lemma* [*THEN lfp-unfold*, *THEN ssubst*])
apply *simp-all*
done

Definition forms of *gfp-unfold* and *coinduct*, to control unfolding.

lemma *def-gfp-unfold*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow A = f\ A$
by (*auto intro*!: *gfp-unfold*)

lemma *def-coinduct*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow X \le f\ (sup\ X\ A) \Longrightarrow X \le A$
by (*iprover intro*!: *coinduct*)

lemma *def-coinduct-set*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (X \cup A) \Longrightarrow a \in A$
by (*auto intro*!: *coinduct-set*)

lemma *def-Collect-coinduct*:
$A \equiv gfp\ (\lambda w.\ Collect\ (P\ w)) \Longrightarrow mono\ (\lambda w.\ Collect\ (P\ w)) \Longrightarrow a \in X \Longrightarrow$
$(\bigwedge z.\ z \in X \Longrightarrow P\ (X \cup A)\ z) \Longrightarrow a \in A$
by (*erule def-coinduct-set*) *auto*

lemma *def-coinduct3*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (lfp\ (\lambda x.\ f\ x \cup X \cup A)) \Longrightarrow a \in A$
by (*auto intro*!: *coinduct3*)

Monotonicity of *gfp*!

lemma *gfp-mono*: $(\bigwedge Z.\ f\ Z \le g\ Z) \Longrightarrow gfp\ f \le gfp\ g$
by (*rule gfp-upperbound* [*THEN gfp-least*]) (*blast intro*: *order-trans*)

13.6 Rules for fixed point calculus

lemma *lfp-rolling*:
assumes *mono g mono f*
shows $g\ (lfp\ (\lambda x.\ f\ (g\ x))) = lfp\ (\lambda x.\ g\ (f\ x))$
proof (*rule antisym*)
have $*$: $mono\ (\lambda x.\ f\ (g\ x))$
using *assms* **by** (*auto simp*: *mono-def*)
show $lfp\ (\lambda x.\ g\ (f\ x)) \le g\ (lfp\ (\lambda x.\ f\ (g\ x)))$
by (*rule lfp-lowerbound*) (*simp add*: *lfp-unfold*[*OF* $*$, *symmetric*])
show $g\ (lfp\ (\lambda x.\ f\ (g\ x))) \le lfp\ (\lambda x.\ g\ (f\ x))$
proof (*rule lfp-greatest*)
fix *u*

assume *u*: $g\ (f\ u) \leq u$
then have $g\ (lfp\ (\lambda x.\ f\ (g\ x))) \leq g\ (f\ u)$
by (*intro assms*[*THEN monoD*] *lfp-lowerbound*)
with *u* **show** $g\ (lfp\ (\lambda x.\ f\ (g\ x))) \leq u$
by *auto*
qed
qed

lemma *lfp-lfp*:
assumes *f*: $\bigwedge x\ y\ w\ z.\ x \leq y \Longrightarrow w \leq z \Longrightarrow f\ x\ w \leq f\ y\ z$
shows $lfp\ (\lambda x.\ lfp\ (f\ x)) = lfp\ (\lambda x.\ f\ x\ x)$
proof (*rule antisym*)
have ∗: $mono\ (\lambda x.\ f\ x\ x)$
by (*blast intro*: *monoI f*)
show $lfp\ (\lambda x.\ lfp\ (f\ x)) \leq lfp\ (\lambda x.\ f\ x\ x)$
by (*intro lfp-lowerbound*) (*simp add*: *lfp-unfold*[*OF* ∗, *symmetric*])
show $lfp\ (\lambda x.\ lfp\ (f\ x)) \geq lfp\ (\lambda x.\ f\ x\ x)$ (**is** $?F \geq$ -)
proof (*intro lfp-lowerbound*)
have ∗: $?F = lfp\ (f\ ?F)$
by (*rule lfp-unfold*) (*blast intro*: *monoI lfp-mono f*)
also have $\ldots = f\ ?F\ (lfp\ (f\ ?F))$
by (*rule lfp-unfold*) (*blast intro*: *monoI lfp-mono f*)
finally show $f\ ?F\ ?F \leq ?F$
by (*simp add*: ∗[*symmetric*])
qed
qed

lemma *gfp-rolling*:
assumes *mono g mono f*
shows $g\ (gfp\ (\lambda x.\ f\ (g\ x))) = gfp\ (\lambda x.\ g\ (f\ x))$
proof (*rule antisym*)
have ∗: $mono\ (\lambda x.\ f\ (g\ x))$
using *assms* **by** (*auto simp*: *mono-def*)
show $g\ (gfp\ (\lambda x.\ f\ (g\ x))) \leq gfp\ (\lambda x.\ g\ (f\ x))$
by (*rule gfp-upperbound*) (*simp add*: *gfp-unfold*[*OF* ∗, *symmetric*])
show $gfp\ (\lambda x.\ g\ (f\ x)) \leq g\ (gfp\ (\lambda x.\ f\ (g\ x)))$
proof (*rule gfp-least*)
fix *u*
assume *u*: $u \leq g\ (f\ u)$
then have $g\ (f\ u) \leq g\ (gfp\ (\lambda x.\ f\ (g\ x)))$
by (*intro assms*[*THEN monoD*] *gfp-upperbound*)
with *u* **show** $u \leq g\ (gfp\ (\lambda x.\ f\ (g\ x)))$
by *auto*
qed
qed

lemma *gfp-gfp*:
assumes *f*: $\bigwedge x\ y\ w\ z.\ x \leq y \Longrightarrow w \leq z \Longrightarrow f\ x\ w \leq f\ y\ z$
shows $gfp\ (\lambda x.\ gfp\ (f\ x)) = gfp\ (\lambda x.\ f\ x\ x)$

proof (*rule antisym*)
  have ∗: *mono* (λ*x*. *f x x*)
    by (*blast intro*: *monoI f*)
  show *gfp* (λ*x*. *f x x*) ≤ *gfp* (λ*x*. *gfp* (*f x*))
    by (*intro gfp-upperbound*) (*simp add*: *gfp-unfold*[*OF* ∗, *symmetric*])
  show *gfp* (λ*x*. *gfp* (*f x*)) ≤ *gfp* (λ*x*. *f x x*) (**is** *?F* ≤ -)
  proof (*intro gfp-upperbound*)
    have ∗: *?F* = *gfp* (*f ?F*)
      by (*rule gfp-unfold*) (*blast intro*: *monoI gfp-mono f*)
    also have ... = *f ?F* (*gfp* (*f ?F*))
      by (*rule gfp-unfold*) (*blast intro*: *monoI gfp-mono f*)
    finally show *?F* ≤ *f ?F ?F*
      by (*simp add*: ∗[*symmetric*])
  qed
qed

13.7 Inductive predicates and sets

Package setup.

lemmas *basic-monos* =
  subset-refl imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
  Collect-mono in-mono vimage-mono

lemma *le-rel-bool-arg-iff*: *X* ≤ *Y* ⟷ *X False* ≤ *Y False* ∧ *X True* ≤ *Y True*
  unfolding *le-fun-def le-bool-def* **using** *bool-induct* **by** *auto*

lemma *imp-conj-iff*: ((*P* ⟶ *Q*) ∧ *P*) = (*P* ∧ *Q*)
  by *blast*

lemma *meta-fun-cong*: *P* ≡ *Q* ⟹ *P a* ≡ *Q a*
  by *auto*

ML-file ‹*Tools/inductive.ML*›

lemmas [*mono*] =
  imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
  imp-mono not-mono
  Ball-def Bex-def
  induct-rulify-fallback

13.8 The Schroeder-Bernstein Theorem

See also:

- `$ISABELLE_HOME/src/HOL/ex/Set_Theory.thy`
- http://planetmath.org/proofofschroederbernsteintheoremusingtarskiknastertheorem
- Springer LNCS 828 (cover page)

```
theorem Schroeder-Bernstein:
  fixes f :: 'a ⇒ 'b and g :: 'b ⇒ 'a
    and A :: 'a set and B :: 'b set
  assumes inj1: inj-on f A and sub1: f ` A ⊆ B
    and inj2: inj-on g B and sub2: g ` B ⊆ A
  shows ∃ h. bij-betw h A B
proof (rule exI, rule bij-betw-imageI)
  define X where X = lfp (λX. A − (g ` (B − (f ` X))))
  define g' where g' = the-inv-into (B − (f ` X)) g
  let ?h = λz. if z ∈ X then f z else g' z

  have X: X = A − (g ` (B − (f ` X)))
    unfolding X-def by (rule lfp-unfold) (blast intro: monoI)
  then have X-compl: A − X = g ` (B − (f ` X))
    using sub2 by blast

  from inj2 have inj2': inj-on g (B − (f ` X))
    by (rule inj-on-subset) auto
  with X-compl have *: g' ` (A − X) = B − (f ` X)
    by (simp add: g'-def)

  from X have X-sub: X ⊆ A by auto
  from X sub1 have fX-sub: f ` X ⊆ B by auto

  show ?h ` A = B
  proof −
    from X-sub have ?h ` A = ?h ` (X ∪ (A − X)) by auto
    also have ... = ?h ` X ∪ ?h ` (A − X) by (simp only: image-Un)
    also have ?h ` X = f ` X by auto
    also from * have ?h ` (A − X) = B − (f ` X) by auto
    also from fX-sub have f ` X ∪ (B − f ` X) = B by blast
    finally show ?thesis .
  qed
  show inj-on ?h A
  proof −
    from inj1 X-sub have on-X: inj-on f X
      by (rule subset-inj-on)

    have on-X-compl: inj-on g' (A − X)
      unfolding g'-def X-compl
      by (rule inj-on-the-inv-into) (rule inj2')

    have impossible: False if eq: f a = g' b and a: a ∈ X and b: b ∈ A − X for a b
    proof −
      from a have fa: f a ∈ f ` X by (rule imageI)
      from b have g' b ∈ g' ` (A − X) by (rule imageI)
      with * have g' b ∈ − (f ` X) by simp
      with eq fa show False by simp
```

```
    qed

    show ?thesis
    proof (rule inj-onI)
      fix a b
      assume h: ?h a = ?h b
      assume a ∈ A and b ∈ A
      then consider a ∈ X b ∈ X | a ∈ A − X b ∈ A − X
        | a ∈ X b ∈ A − X | a ∈ A − X b ∈ X
        by blast
      then show a = b
      proof cases
        case 1
        with h on-X show ?thesis by (simp add: inj-on-eq-iff)
      next
        case 2
        with h on-X-compl show ?thesis by (simp add: inj-on-eq-iff)
      next
        case 3
        with h impossible [of a b] have False by simp
        then show ?thesis ..
      next
        case 4
        with h impossible [of b a] have False by simp
        then show ?thesis ..
      qed
    qed
  qed
qed
```

13.9 Inductive datatypes and primitive recursion

Package setup.

```
ML-file ‹Tools/Old-Datatype/old-datatype-aux.ML›
ML-file ‹Tools/Old-Datatype/old-datatype-prop.ML›
ML-file ‹Tools/Old-Datatype/old-datatype-data.ML›
ML-file ‹Tools/Old-Datatype/old-rep-datatype.ML›
ML-file ‹Tools/Old-Datatype/old-datatype-codegen.ML›
ML-file ‹Tools/BNF/bnf-fp-rec-sugar-util.ML›
ML-file ‹Tools/Old-Datatype/old-primrec.ML›
ML-file ‹Tools/BNF/bnf-lfp-rec-sugar.ML›
```

Lambda-abstractions with pattern matching:

```
syntax (ASCII)
  -lam-pats-syntax :: cases-syn ⇒ 'a ⇒ 'b  (‹(‹notation=abstraction›%-)› 10)
syntax
  -lam-pats-syntax :: cases-syn ⇒ 'a ⇒ 'b  (‹(‹notation=abstraction›λ-)› 10)
parse-translation ‹
  let
```

```
  fun fun-tr ctxt [cs] =
    let
    val x = Syntax.free (#1 (Name.variant x (Name.build-context (Term.declare-free-names
cs))));
      val ft = Case-Translation.case-tr true ctxt [x, cs];
    in lambda x ft end
  in [(syntax-const ‹-lam-pats-syntax›, fun-tr)] end
›
```

end

14 Cartesian products

theory *Product-Type*
imports *Typedef Inductive Fun*
keywords *inductive-set coinductive-set* :: *thy-defn*
begin

14.1 *bool* is a datatype

free-constructors (*discs-sels*) *case-bool* **for** *True* | *False*
by *auto*

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

setup ‹*Sign.mandatory-path old*›

old-rep-datatype *True False* **by** (*auto intro*: *bool-induct*)

setup ‹*Sign.parent-path*›

But erase the prefix for properties that are not generated by *free-constructors*.

setup ‹*Sign.mandatory-path bool*›

lemmas *induct* = *old.bool.induct*
lemmas *inducts* = *old.bool.inducts*
lemmas *rec* = *old.bool.rec*
lemmas *simps* = *bool.distinct bool.case bool.rec*

setup ‹*Sign.parent-path*›

declare *case-split* [*cases type*: *bool*]
— prefer plain propositional version

lemma [*code*]: *HOL.equal False P* ⟷ ¬ *P*
and [*code*]: *HOL.equal True P* ⟷ *P*
and [*code*]: *HOL.equal P False* ⟷ ¬ *P*
and [*code*]: *HOL.equal P True* ⟷ *P*
and [*code nbe*]: *HOL.equal P P* ⟷ *True*

by (*simp-all add*: *equal*)

lemma *If-case-cert*:
assumes *CASE* ≡ (λ*b*. *If b f g*)
shows (*CASE True* ≡ *f*) &&& (*CASE False* ≡ *g*)
using *assms* **by** *simp-all*

setup ‹*Code.declare-case-global* @{*thm If-case-cert*}›

code-printing
constant *HOL.equal* :: *bool* ⇒ *bool* ⇒ *bool* ⇀ (*Haskell*) **infix** *4* ==
| **class-instance** *bool* :: *equal* ⇀ (*Haskell*) −

14.2 The *unit* type

typedef *unit* = {*True*}
by *auto*

definition *Unity* :: *unit* (‹'(')›)
where () = *Abs-unit True*

lemma *unit-eq* [*no-atp*]: *u* = ()
by (*induct u*) (*simp add*: *Unity-def*)

Simplification procedure for *unit-eq*. Cannot use this rule directly — it loops!

simproc-setup *unit-eq* (*x*::*unit*) = ‹
K (*K* (*fn ct* =>
if HOLogic.is-unit (*Thm.term-of ct*) *then NONE*
else SOME (*mk-meta-eq* @{*thm unit-eq*})))
›

free-constructors *case-unit* **for** ()
by *auto*

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

setup ‹*Sign.mandatory-path old*›

old-rep-datatype () **by** *simp*

setup ‹*Sign.parent-path*›

But erase the prefix for properties that are not generated by *free-constructors*.

setup ‹*Sign.mandatory-path unit*›

lemmas *induct* = *old.unit.induct*
lemmas *inducts* = *old.unit.inducts*
lemmas *rec* = *old.unit.rec*
lemmas *simps* = *unit.case unit.rec*

setup ‹*Sign.parent-path*›

lemma *unit-all-eq1*: ($\bigwedge x$::*unit*. *PROP P x*) ≡ *PROP P* ()
  by *simp*

lemma *unit-all-eq2*: ($\bigwedge x$::*unit*. *PROP P*) ≡ *PROP P*
  by (*rule triv-forall-equality*)

This rewrite counters the effect of simproc *unit-eq* on λu::*unit*. *f u*, replacing it by *f* rather than by λu. *f* ().

lemma *unit-abs-eta-conv* [*simp*]: (λu::*unit*. *f* ()) = *f*
  by (*rule ext*) *simp*

lemma *UNIV-unit*: *UNIV* = {()}
  by *auto*

instantiation *unit* :: *default*
begin

definition *default* = ()

instance ..

end

instantiation *unit* :: {*complete-boolean-algebra*,*complete-linorder*,*wellorder*}
begin

definition *less-eq-unit* :: *unit* ⇒ *unit* ⇒ *bool*
  where (-::*unit*) ≤ - ⟷ *True*

lemma *less-eq-unit* [*iff*]: *u* ≤ *v* **for** *u v* :: *unit*
  by (*simp add*: *less-eq-unit-def*)

definition *less-unit* :: *unit* ⇒ *unit* ⇒ *bool*
  where (-::*unit*) < - ⟷ *False*

lemma *less-unit* [*iff*]: ¬ *u* < *v* **for** *u v* :: *unit*
  by (*simp-all add*: *less-eq-unit-def less-unit-def*)

definition *bot-unit* :: *unit*
  where [*code-unfold*]: ⊥ = ()

definition *top-unit* :: *unit*
  where [*code-unfold*]: ⊤ = ()

definition *inf-unit* :: *unit* ⇒ *unit* ⇒ *unit*
  where [*simp*]: - ⊓ - = ()

definition *sup-unit* :: *unit* ⇒ *unit* ⇒ *unit*
where [*simp*]: - ⊔ - = ()

definition *Inf-unit* :: *unit set* ⇒ *unit*
where [*simp*]: ⨅ - = ()

definition *Sup-unit* :: *unit set* ⇒ *unit*
where [*simp*]: ⨆ - = ()

definition *uminus-unit* :: *unit* ⇒ *unit*
where [*simp*]: − - = ()

declare *less-eq-unit-def* [*abs-def*, *code-unfold*]
less-unit-def [*abs-def*, *code-unfold*]
inf-unit-def [*abs-def*, *code-unfold*]
sup-unit-def [*abs-def*, *code-unfold*]
Inf-unit-def [*abs-def*, *code-unfold*]
Sup-unit-def [*abs-def*, *code-unfold*]
uminus-unit-def [*abs-def*, *code-unfold*]

instance
by *intro-classes auto*

end

lemma [*code*]: *HOL.equal u v* ⟷ *True* **for** *u v* :: *unit*
unfolding *equal unit-eq* [*of u*] *unit-eq* [*of v*] **by** (*rule iffI TrueI refl*)+

code-printing
type-constructor *unit* ⇀
(*SML*) *unit*
and (*OCaml*) *unit*
and (*Haskell*) ()
and (*Scala*) *Unit*
| **constant** *Unity* ⇀
(*SML*) ()
and (*OCaml*) ()
and (*Haskell*) ()
and (*Scala*) ()
| **class-instance** *unit* :: *equal* ⇀
(*Haskell*) −
| **constant** *HOL.equal* :: *unit* ⇒ *unit* ⇒ *bool* ⇀
(*Haskell*) **infix** *4* ==

code-reserved
(*SML*) *unit*
and (*OCaml*) *unit*
and (*Scala*) *Unit*

14.3 The product type

14.3.1 Type definition

definition *Pair-Rep* :: *'a* ⇒ *'b* ⇒ *'a* ⇒ *'b* ⇒ *bool*
  where *Pair-Rep a b* = (λ*x y*. *x* = *a* ∧ *y* = *b*)

definition *prod* = {*f*. ∃ *a b*. *f* = *Pair-Rep* (*a*::*'a*) (*b*::*'b*)}

typedef (*'a*, *'b*) *prod* (‹(‹*notation*=‹*infix* ×››- ×/ -)› [*21*, *20*] *20*) = *prod* :: (*'a* ⇒ *'b* ⇒ *bool*) *set*
  unfolding *prod-def* **by** *auto*

type-notation (*ASCII*)
  prod (**infixr** ‹∗› *20*)

definition *Pair* :: *'a* ⇒ *'b* ⇒ *'a* × *'b*
  where *Pair a b* = *Abs-prod* (*Pair-Rep a b*)

lemma *prod-cases*: (⋀*a b*. *P* (*Pair a b*)) ⟹ *P p*
  by (*cases p*) (*auto simp add*: *prod-def Pair-def Pair-Rep-def*)

free-constructors *case-prod* **for** *Pair fst snd*
proof −
  fix *P* :: *bool* **and** *p* :: *'a* × *'b*
  show (⋀*x1 x2*. *p* = *Pair x1 x2* ⟹ *P*) ⟹ *P*
    by (*cases p*) (*auto simp add*: *prod-def Pair-def Pair-Rep-def*)
next
  fix *a c* :: *'a* **and** *b d* :: *'b*
  have *Pair-Rep a b* = *Pair-Rep c d* ⟷ *a* = *c* ∧ *b* = *d*
    by (*auto simp add*: *Pair-Rep-def fun-eq-iff*)
  moreover have *Pair-Rep a b* ∈ *prod* **and** *Pair-Rep c d* ∈ *prod*
    by (*auto simp add*: *prod-def*)
  ultimately show *Pair a b* = *Pair c d* ⟷ *a* = *c* ∧ *b* = *d*
    by (*simp add*: *Pair-def Abs-prod-inject*)
qed

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

setup ‹*Sign.mandatory-path old*›

old-rep-datatype *Pair*
  by (*erule prod-cases*) (*rule prod.inject*)

setup ‹*Sign.parent-path*›

But erase the prefix for properties that are not generated by *free-constructors*.

setup ‹*Sign.mandatory-path prod*›

declare *old.prod.inject* [*iff del*]

lemmas *induct* = *old.prod.induct*
lemmas *inducts* = *old.prod.inducts*
lemmas *rec* = *old.prod.rec*
lemmas *simps* = *prod.inject prod.case prod.rec*

setup ‹*Sign.parent-path*›

declare *prod.case* [*nitpick-simp del*]
declare *old.prod.case-cong-weak* [*cong del*]
declare *prod.case-eq-if* [*mono*]
declare *prod.split* [*no-atp*]
declare *prod.split-asm* [*no-atp*]

prod.split could be declared as [*split*] done after the Splitter has been speeded up significantly; precompute the constants involved and don't do anything unless the current goal contains one of those constants.

14.3.2 Tuple syntax

Patterns – extends pre-defined type *pttrn* used in abstractions.

nonterminal *tuple-args* **and** *patterns*

open-bundle *tuple-syntax*
begin

syntax
-tuple :: $'a \Rightarrow$ *tuple-args* $\Rightarrow 'a \times 'b$ (‹(‹*indent=1 notation=*‹*mixfix tuple*››'(-,/ -'))›)
-tuple-arg :: $'a \Rightarrow$ *tuple-args* (‹-›)
-tuple-args :: $'a \Rightarrow$ *tuple-args* $\Rightarrow$ *tuple-args* (‹-,/ -›)
-pattern :: *pttrn* $\Rightarrow$ *patterns* $\Rightarrow$ *pttrn* (‹(‹*open-block notation=*‹*pattern tuple*››'(-,/ -'))›)
:: *pttrn* $\Rightarrow$ *patterns* (‹-›)
-patterns :: *pttrn* $\Rightarrow$ *patterns* $\Rightarrow$ *patterns* (‹-,/ -›)
-unit :: *pttrn* (‹(‹*open-block notation=*‹*pattern unit*››'(')›)
syntax-consts
-pattern -patterns ⇌ *case-prod* **and**
-unit ⇌ *case-unit*
translations
(x, y) ⇌ *CONST Pair x y*
-pattern x y ⇌ *CONST Pair x y*
-patterns x y ⇌ *CONST Pair x y*
-tuple x (*-tuple-args y z*) ⇌ *-tuple x* (*-tuple-arg* (*-tuple y z*))
$\lambda(x, y, zs)$. b ⇌ *CONST case-prod* (λx (y, zs). b)
$\lambda(x, y)$. b ⇌ *CONST case-prod* (λx y. b)
-abs (*CONST Pair x y*) t ⇀ $\lambda(x, y)$. t
— This rule accommodates tuples in *case C* ... (x, y) ... $\Rightarrow$ ...: The (x, y) is parsed as *Pair x y* because it is *logic*, not *pttrn*.

λ(). b ⇌ CONST case-unit b
-abs (CONST Unity) t ⇀ λ(). t

end

print *case-prod f* as *case-prod f* and *case-prod f* as *case-prod f*

typed-print-translation ‹

```
let
  fun case-prod-guess-names-tr' - T [Abs (x, -, Abs -)] = raise Match
    | case-prod-guess-names-tr' ctxt T [Abs (x, xT, t)] =
        (case (head-of t) of
          Const (const-syntax ‹case-prod›, -) => raise Match
        | - =>
          let
            val (- :: yT :: -) = binder-types (domain-type T) handle Bind => raise
Match;
            val (y, t') = Syntax-Trans.atomic-abs-tr' ctxt (y, yT, incr-boundvars 1
t $ Bound 0);
            val (x', t'') = Syntax-Trans.atomic-abs-tr' ctxt (x, xT, t');
          in
            Syntax.const syntax-const ‹-abs› $
              (Syntax.const syntax-const ‹-pattern› $ x' $ y) $ t''
          end)
    | case-prod-guess-names-tr' ctxt T [t] =
        (case head-of t of
          Const (const-syntax ‹case-prod›, -) => raise Match
        | - =>
          let
            val (xT :: yT :: -) = binder-types (domain-type T) handle Bind =>
raise Match;
            val (y, t') =
              Syntax-Trans.atomic-abs-tr' ctxt (y, yT, incr-boundvars 2 t $ Bound
1 $ Bound 0);
            val (x', t'') = Syntax-Trans.atomic-abs-tr' ctxt (x, xT, t');
          in
            Syntax.const syntax-const ‹-abs› $
              (Syntax.const syntax-const ‹-pattern› $ x' $ y) $ t''
          end)
    | case-prod-guess-names-tr' - - - = raise Match;
  in [(const-syntax ‹case-prod›, case-prod-guess-names-tr')] end
›
```

Reconstruct pattern from (nested) *case-prod*s, avoiding eta-contraction of body; required for enclosing "let", if "let" does not avoid eta-contraction, which has been observed to occur.

print-translation ‹

```
let
  fun case-prod-tr' ctxt [Abs (x, T, t as (Abs abs))] =
        (* case-prod (λx y. t) ⇒ λ(x, y) t *)
```

```
      let
        val (y, t′) = Syntax-Trans.atomic-abs-tr′ ctxt abs;
        val (x′, t″) = Syntax-Trans.atomic-abs-tr′ ctxt (x, T, t′);
      in
        Syntax.const syntax-const‹-abs› $
          (Syntax.const syntax-const‹-pattern› $ x′ $ y) $ t″
      end
    | case-prod-tr′ ctxt [Abs (x, T, (s as Const (const-syntax‹case-prod›, -) $
t))] =
      (∗ case-prod (λx. (case-prod (λy z. t))) ⇒ λ(x, y, z). t ∗)
      let
        val Const (syntax-const‹-abs›, -) $
          (Const (syntax-const‹-pattern›, -) $ y $ z) $ t′ =
            case-prod-tr′ ctxt [t];
        val (x′, t″) = Syntax-Trans.atomic-abs-tr′ ctxt (x, T, t′);
      in
        Syntax.const syntax-const‹-abs› $
          (Syntax.const syntax-const‹-pattern› $ x′ $
            (Syntax.const syntax-const‹-patterns› $ y $ z)) $ t″
      end
    | case-prod-tr′ ctxt [Const (const-syntax‹case-prod›, -) $ t] =
      (∗ case-prod (case-prod (λx y z. t)) ⇒ λ((x, y), z). t ∗)
      case-prod-tr′ ctxt [(case-prod-tr′ ctxt [t])]
        (∗ inner case-prod-tr′ creates next pattern ∗)
    | case-prod-tr′ ctxt [Const (syntax-const‹-abs›, -) $ x-y $ Abs abs] =
      (∗ case-prod (λpttrn z. t) ⇒ λ(pttrn, z). t ∗)
      let val (z, t) = Syntax-Trans.atomic-abs-tr′ ctxt abs in
        Syntax.const syntax-const‹-abs› $
          (Syntax.const syntax-const‹-pattern› $ x-y $ z) $ t
      end
    | case-prod-tr′ - - = raise Match;
  in [(const-syntax‹case-prod›, case-prod-tr′)] end
›
```

14.3.3 Code generator setup

```
code-printing
  type-constructor prod ⇀
    (SML) infix 2 ∗
    and (OCaml) infix 2 ∗
    and (Haskell) !((-),/ (-))
    and (Scala) ((-),/ (-))
| constant Pair ⇀
    (SML) !((-),/ (-))
    and (OCaml) !((-),/ (-))
    and (Haskell) !((-),/ (-))
    and (Scala) !((-),/ (-))
| class-instance prod :: equal ⇀
    (Haskell) −
```

| **constant** *HOL.equal* :: $'a \times 'b \Rightarrow 'a \times 'b \Rightarrow bool \rightharpoonup$
(*Haskell*) **infix** *4* ==
| **constant** *fst* $\rightharpoonup$ (*Haskell*) *fst*
| **constant** *snd* $\rightharpoonup$ (*Haskell*) *snd*

14.3.4 Fundamental operations and properties

lemma *Pair-inject*: $(a, b) = (a', b') \Longrightarrow (a = a' \Longrightarrow b = b' \Longrightarrow R) \Longrightarrow R$
by *simp*

lemma *surj-pair* [*simp*]: $\exists\, x\ y.\ p = (x, y)$
by (*cases p*) *simp*

lemma *fst-eqD*: $fst\ (x, y) = a \Longrightarrow x = a$
by *simp*

lemma *snd-eqD*: $snd\ (x, y) = a \Longrightarrow y = a$
by *simp*

lemma *case-prod-unfold* [*nitpick-unfold*]: $case\text{-}prod = (\lambda c\ p.\ c\ (fst\ p)\ (snd\ p))$
by (*simp add*: *fun-eq-iff split*: *prod.split*)

lemma *case-prod-conv* [*simp*, *code*]: $(case\ (a, b)\ of\ (c, d) \Rightarrow f\ c\ d) = f\ a\ b$
by (*fact prod.case*)

lemmas *surjective-pairing* = *prod.collapse* [*symmetric*]

lemma *prod-eq-iff*: $s = t \longleftrightarrow fst\ s = fst\ t \wedge snd\ s = snd\ t$
by (*cases s*, *cases t*) *simp*

lemma *prod-eqI* [*intro?*]: $fst\ p = fst\ q \Longrightarrow snd\ p = snd\ q \Longrightarrow p = q$
by (*simp add*: *prod-eq-iff*)

lemma *case-prodI*: $f\ a\ b \Longrightarrow case\ (a, b)\ of\ (c, d) \Rightarrow f\ c\ d$
by (*rule prod.case* [*THEN iffD2*])

lemma *case-prodD*: $(case\ (a, b)\ of\ (c, d) \Rightarrow f\ c\ d) \Longrightarrow f\ a\ b$
by (*rule prod.case* [*THEN iffD1*])

lemma *case-prod-Pair* [*simp*]: *case-prod Pair* = *id*
by (*simp add*: *fun-eq-iff split*: *prod.split*)

lemma *case-prod-eta*: $(\lambda(x, y).\ f\ (x, y)) = f$
— Subsumes the old *split-Pair* when *f* is the identity function.
by (*simp add*: *fun-eq-iff split*: *prod.split*)

lemma *case-prod-comp*: $(case\ x\ of\ (a, b) \Rightarrow (f \circ g)\ a\ b) = f\ (g\ (fst\ x))\ (snd\ x)$

by (*cases x*) *simp*

lemma *The-case-prod*: *The* (*case-prod P*) = (*THE xy*. *P* (*fst xy*) (*snd xy*))
by (*simp add*: *case-prod-unfold*)

lemma *cond-case-prod-eta*: ($\bigwedge$*x y*. *f x y* = *g* (*x*, *y*)) $\Longrightarrow$ (λ(*x*, *y*). *f x y*) = *g*
by (*simp add*: *case-prod-eta*)

lemma *split-paired-all* [*no-atp*]: ($\bigwedge$*x*. *PROP P x*) ≡ ($\bigwedge$*a b*. *PROP P* (*a*, *b*))
proof
fix *a b*
assume $\bigwedge$*x*. *PROP P x*
then show *PROP P* (*a*, *b*) **.**
next
fix *x*
assume $\bigwedge$*a b*. *PROP P* (*a*, *b*)
from ‹*PROP P* (*fst x*, *snd x*)› **show** *PROP P x* **by** *simp*
qed

The rule *split-paired-all* does not work with the Simplifier because it also affects premises in congrence rules, where this can lead to premises of the form $\bigwedge$*a b*. ... = *?P*(*a*, *b*) which cannot be solved by reflexivity.

lemmas *split-tupled-all* = *split-paired-all unit-all-eq2*

ML ‹
```
(* replace parameters of product type by individual component parameters *)
local (* filtering with exists-paired-all is an essential optimization *)
  fun exists-paired-all (Const (const-name ‹Pure.all›, -) $ Abs (-, T, t)) =
        can HOLogic.dest-prodT T orelse exists-paired-all t
    | exists-paired-all (t $ u) = exists-paired-all t orelse exists-paired-all u
    | exists-paired-all (Abs (-, -, t)) = exists-paired-all t
    | exists-paired-all - = false;
  val ss =
    simpset-of
     (put-simpset HOL-basic-ss context
    addsimps [@{thm split-paired-all}, @{thm unit-all-eq2}, @{thm unit-abs-eta-conv}]
      |> Simplifier.add-proc simproc ‹unit-eq›);
in
  fun split-all-tac ctxt = SUBGOAL (fn (t, i) =>
   if exists-paired-all t then safe-full-simp-tac (put-simpset ss ctxt) i else no-tac);

  fun unsafe-split-all-tac ctxt = SUBGOAL (fn (t, i) =>
    if exists-paired-all t then full-simp-tac (put-simpset ss ctxt) i else no-tac);

  fun split-all ctxt th =
    if exists-paired-all (Thm.prop-of th)
    then full-simplify (put-simpset ss ctxt) th else th;
end;
```
›

setup ‹*map-theory-claset* (*fn ctxt* => *ctxt addSbefore* (*split-all-tac*, *split-all-tac*))›

lemma *split-paired-All* [*simp*, *no-atp*]: $(\forall x.\ P\ x) \longleftrightarrow (\forall a\ b.\ P\ (a,\ b))$
— [*iff*] is not a good idea because it makes *blast* loop
by *fast*

lemma *split-paired-Ex* [*simp*, *no-atp*]: $(\exists x.\ P\ x) \longleftrightarrow (\exists a\ b.\ P\ (a,\ b))$
by *fast*

lemma *split-paired-The* [*no-atp*]: $(THE\ x.\ P\ x) = (THE\ (a,\ b).\ P\ (a,\ b))$
— Can't be added to simpset: loops!
by (*simp add*: *case-prod-eta*)

Simplification procedure for *cond-case-prod-eta*. Using *case-prod-eta* as a rewrite rule is not general enough, and using *cond-case-prod-eta* directly would render some existing proofs very inefficient; similarly for *prod.case-eq-if*.

ML ‹
```
local
  val cond-case-prod-eta-ss =
   simpset-of (put-simpset HOL-basic-ss context addsimps @{thms cond-case-prod-eta});
  fun Pair-pat k 0 (Bound m) = (m = k)
    | Pair-pat k i (Const (const-name ‹Pair›, -) $ Bound m $ t) =
        i > 0 andalso m = k + i andalso Pair-pat k (i − 1) t
    | Pair-pat - - - = false;
  fun no-args k i (Abs (-, -, t)) = no-args (k + 1) i t
    | no-args k i (t $ u) = no-args k i t andalso no-args k i u
    | no-args k i (Bound m) = m < k orelse m > k + i
    | no-args - - - = true;
  fun split-pat tp i (Abs  (-, -, t)) = if tp 0 i t then SOME (i, t) else NONE
     | split-pat tp i (Const (const-name ‹case-prod›, -) $ Abs (-, -, t)) = split-pat
tp (i + 1) t
    | split-pat tp i - = NONE;
  fun metaeq ctxt lhs rhs = mk-meta-eq (Goal.prove ctxt [] []
        (HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs, rhs)))
        (K (simp-tac (put-simpset cond-case-prod-eta-ss ctxt) 1)));

  fun beta-term-pat k i (Abs (-, -, t)) = beta-term-pat (k + 1) i t
    | beta-term-pat k i (t $ u) =
        Pair-pat k i (t $ u) orelse (beta-term-pat k i t andalso beta-term-pat k i u)
    | beta-term-pat k i t = no-args k i t;
  fun eta-term-pat k i (f $ arg) = no-args k i f andalso Pair-pat k i arg
    | eta-term-pat - - - = false;
  fun subst arg k i (Abs (x, T, t)) = Abs (x, T, subst arg (k+1) i t)
    | subst arg k i (t $ u) =
        if Pair-pat k i (t $ u) then incr-boundvars k arg
        else (subst arg k i t $ subst arg k i u)
    | subst arg k i t = t;
in
```

```
  fun beta-proc ctxt (s as Const (const-name ‹case-prod›, -) $ Abs (-, -, t) $ arg)
=
        (case split-pat beta-term-pat 1 t of
          SOME (i, f) => SOME (metaeq ctxt s (subst arg 0 i f))
        | NONE => NONE)
    | beta-proc - - = NONE;
  fun eta-proc ctxt (s as Const (const-name ‹case-prod›, -) $ Abs (-, -, t)) =
        (case split-pat eta-term-pat 1 t of
          SOME (-, ft) => SOME (metaeq ctxt s (let val f $ - = ft in f end))
        | NONE => NONE)
    | eta-proc - - = NONE;
end;
›
simproc-setup case-prod-beta (case-prod f z) =
  ‹K (fn ctxt => fn ct => beta-proc ctxt (Thm.term-of ct))›
simproc-setup case-prod-eta (case-prod f) =
  ‹K (fn ctxt => fn ct => eta-proc ctxt (Thm.term-of ct))›

lemma case-prod-beta': (λ(x,y). f x y) = (λx. f (fst x) (snd x))
  by (auto simp: fun-eq-iff)
```

case-prod used as a logical connective or set former.

These rules are for use with *blast*; could instead call *simp* using *prod.split* as rewrite.

```
lemma case-prodI2:
  ⋀p. (⋀a b. p = (a, b) ⟹ c a b) ⟹ case p of (a, b) ⇒ c a b
  by (simp add: split-tupled-all)

lemma case-prodI2':
  ⋀p. (⋀a b. (a, b) = p ⟹ c a b x) ⟹ (case p of (a, b) ⇒ c a b) x
  by (simp add: split-tupled-all)

lemma case-prodE [elim!]:
  (case p of (a, b) ⇒ c a b) ⟹ (⋀x y. p = (x, y) ⟹ c x y ⟹ Q) ⟹ Q
  by (induct p) simp

lemma case-prodE' [elim!]:
  (case p of (a, b) ⇒ c a b) z ⟹ (⋀x y. p = (x, y) ⟹ c x y z ⟹ Q) ⟹ Q
  by (induct p) simp

lemma case-prodE2:
  assumes q: Q (case z of (a, b) ⇒ P a b)
    and r: ⋀x y. z = (x, y) ⟹ Q (P x y) ⟹ R
  shows R
proof (rule r)
  show z = (fst z, snd z) by simp
  then show Q (P (fst z) (snd z))
    using q by (simp add: case-prod-unfold)
```

qed

lemma *case-prodD'*: (*case* (*a*, *b*) *of* (*c*, *d*) ⇒ *R c d*) *c* ⟹ *R a b c*
 by *simp*

lemma *mem-case-prodI*: *z* ∈ *c a b* ⟹ *z* ∈ (*case* (*a*, *b*) *of* (*d*, *e*) ⇒ *c d e*)
 by *simp*

lemma *mem-case-prodI2* [*intro*!]:
 ⋀*p*. (⋀*a b*. *p* = (*a*, *b*) ⟹ *z* ∈ *c a b*) ⟹ *z* ∈ (*case p of* (*a*, *b*) ⇒ *c a b*)
 by (*simp only*: *split-tupled-all*) *simp*

declare *mem-case-prodI* [*intro*!] — postponed to maintain traditional declaration order!
declare *case-prodI2'* [*intro*!] — postponed to maintain traditional declaration order!
declare *case-prodI2* [*intro*!] — postponed to maintain traditional declaration order!
declare *case-prodI* [*intro*!] — postponed to maintain traditional declaration order!

lemma *mem-case-prodE* [*elim*!]:
 assumes *z* ∈ *case-prod c p*
 obtains *x y* **where** *p* = (*x*, *y*) **and** *z* ∈ *c x y*
 using *assms* **by** (*rule case-prodE2*)

```
ML ‹
local (* filtering with exists-p-split is an essential optimization *)
 fun exists-p-split (Const (const-name ‹case-prod›,-) $ - $ (Const (const-name ‹Pair›,-)$-$-))
= true
    | exists-p-split (t $ u) = exists-p-split t orelse exists-p-split u
    | exists-p-split (Abs (-, -, t)) = exists-p-split t
    | exists-p-split - = false;
in
  fun split-conv-tac ctxt = SUBGOAL (fn (t, i) =>
    if exists-p-split t
   then safe-full-simp-tac (put-simpset HOL-basic-ss ctxt addsimps @{thms case-prod-conv})
i
    else no-tac);
end;
›
```

setup ‹*map-theory-claset* (*fn ctxt* => *ctxt addSbefore* (*split-conv-tac*, *split-conv-tac*))›

lemma *split-eta-SetCompr* [*simp*, *no-atp*]: (λ*u*. ∃ *x y*. *u* = (*x*, *y*) ∧ *P* (*x*, *y*)) = *P*
 by (*rule ext*) *fast*

lemma *split-eta-SetCompr2* [*simp*, *no-atp*]: (λ*u*. ∃ *x y*. *u* = (*x*, *y*) ∧ *P x y*) = *case-prod P*
 by (*rule ext*) *fast*

lemma *split-part* [*simp*]: $(\lambda(a,b).\ P \wedge Q\ a\ b) = (\lambda ab.\ P \wedge \mathit{case\text{-}prod}\ Q\ ab)$
— Allows simplifications of nested splits in case of independent predicates.
by (*rule ext*) *blast*

lemma *split-comp-eq*:
fixes $f :: {'a} \Rightarrow {'b} \Rightarrow {'c}$
and $g :: {'d} \Rightarrow {'a}$
shows $(\lambda u.\ f\ (g\ (\mathit{fst}\ u))\ (\mathit{snd}\ u)) = \mathit{case\text{-}prod}\ (\lambda x.\ f\ (g\ x))$
by (*rule ext*) *auto*

lemma *pair-imageI* [*intro*]: $(a,\ b) \in A \Longrightarrow f\ a\ b \in (\lambda(a,\ b).\ f\ a\ b)\ `\ A$
by (*rule image-eqI* [**where** $x = (a,\ b)$]) *auto*

lemma *Collect-const-case-prod*[*simp*]: $\{(a,b).\ P\} = (\mathit{if}\ P\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ \{\})$
by *auto*

lemma *The-split-eq* [*simp*]: $(\mathit{THE}\ (x',y').\ x = x' \wedge y = y') = (x,\ y)$
by *blast*

lemma *case-prod-beta*: $\mathit{case\text{-}prod}\ f\ p = f\ (\mathit{fst}\ p)\ (\mathit{snd}\ p)$
by (*fact prod.case-eq-if*)

lemma *prod-cases3* [*cases type*]:
obtains (*fields*) $a\ b\ c$ **where** $y = (a,\ b,\ c)$
proof (*cases y*)
case (*Pair a b*)
with *that* **show** *?thesis*
by (*cases b*) *blast*
qed

lemma *prod-induct3* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c.\ P\ (a,\ b,\ c)) \Longrightarrow P\ x$
by (*cases x*) *blast*

lemma *prod-cases4* [*cases type*]:
obtains (*fields*) $a\ b\ c\ d$ **where** $y = (a,\ b,\ c,\ d)$
proof (*cases y*)
case (*fields a b c*)
with *that* **show** *?thesis*
by (*cases c*) *blast*
qed

lemma *prod-induct4* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c\ d.\ P\ (a,\ b,\ c,\ d)) \Longrightarrow P\ x$

by (*cases x*) *blast*

lemma *prod-cases5* [*cases type*]:
obtains (*fields*) *a b c d e* **where** $y = (a, b, c, d, e)$
proof (*cases y*)
case (*fields a b c d*)
with *that* **show** *?thesis*
by (*cases d*) *blast*
qed

lemma *prod-induct5* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c\ d\ e.\ P\ (a, b, c, d, e)) \Longrightarrow P\ x$
by (*cases x*) *blast*

lemma *prod-cases6* [*cases type*]:
obtains (*fields*) *a b c d e f* **where** $y = (a, b, c, d, e, f)$
proof (*cases y*)
case (*fields a b c d e*)
with *that* **show** *?thesis*
by (*cases e*) *blast*
qed

lemma *prod-induct6* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c\ d\ e\ f.\ P\ (a, b, c, d, e, f)) \Longrightarrow P\ x$
by (*cases x*) *blast*

lemma *prod-cases7* [*cases type*]:
obtains (*fields*) *a b c d e f g* **where** $y = (a, b, c, d, e, f, g)$
proof (*cases y*)
case (*fields a b c d e f*)
with *that* **show** *?thesis*
by (*cases f*) *blast*
qed

lemma *prod-induct7* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c\ d\ e\ f\ g.\ P\ (a, b, c, d, e, f, g)) \Longrightarrow P\ x$
by (*cases x*) *blast*

definition *internal-case-prod* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'c$
where *internal-case-prod* ≡ *case-prod*

lemma *internal-case-prod-conv*: *internal-case-prod c* $(a, b) = c\ a\ b$
by (*simp only*: *internal-case-prod-def case-prod-conv*)

ML-file ‹*Tools/split-rule.ML*›

hide-const *internal-case-prod*

14.3.5 Derived operations

definition *curry* :: ($'a \times 'b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'b \Rightarrow 'c$
where *curry* = ($\lambda c\ x\ y.\ c\ (x,\ y)$)

lemma *curry-conv* [*simp*, *code*]: *curry f a b* = *f* (*a*, *b*)
by (*simp add*: *curry-def*)

lemma *curryI* [*intro*!]: *f* (*a*, *b*) $\Longrightarrow$ *curry f a b*
by (*simp add*: *curry-def*)

lemma *curryD* [*dest*!]: *curry f a b* $\Longrightarrow$ *f* (*a*, *b*)
by (*simp add*: *curry-def*)

lemma *curryE*: *curry f a b* $\Longrightarrow$ (*f* (*a*, *b*) $\Longrightarrow$ *Q*) $\Longrightarrow$ *Q*
by (*simp add*: *curry-def*)

lemma *curry-case-prod* [*simp*]: *curry* (*case-prod f*) = *f*
by (*simp add*: *curry-def case-prod-unfold*)

lemma *case-prod-curry* [*simp*]: *case-prod* (*curry f*) = *f*
by (*simp add*: *curry-def case-prod-unfold*)

lemma *curry-K*: *curry* ($\lambda x.\ c$) = ($\lambda x\ y.\ c$)
by (*simp add*: *fun-eq-iff*)

The composition-uncurry combinator.

definition *scomp* :: ($'a \Rightarrow 'b \times 'c) \Rightarrow ('b \Rightarrow 'c \Rightarrow 'd) \Rightarrow 'a \Rightarrow 'd$ (**infixl** ‹∘→› *60*)
where *f* ∘→ *g* = ($\lambda x.$ *case-prod g* (*f x*))

no-notation *scomp* (**infixl** ‹∘→› *60*)

bundle *state-combinator-syntax*
begin
notation *fcomp* (**infixl** ‹∘>› *60*)
notation *scomp* (**infixl** ‹∘→› *60*)
end

context
includes *state-combinator-syntax*
begin

lemma *scomp-unfold*: (∘→) = ($\lambda f\ g\ x.$ *g* (*fst* (*f x*)) (*snd* (*f x*)))
by (*simp add*: *fun-eq-iff scomp-def case-prod-unfold*)

lemma *scomp-apply* [*simp*]: (*f* ∘→ *g*) *x* = *case-prod g* (*f x*)
by (*simp add*: *scomp-unfold case-prod-unfold*)

lemma *Pair-scomp*: *Pair x* ∘→ *f* = *f x*

by (*simp add*: *fun-eq-iff*)

lemma *scomp-Pair*: $x \circ\!\!\rightarrow Pair = x$
by (*simp add*: *fun-eq-iff*)

lemma *scomp-scomp*: $(f \circ\!\!\rightarrow g) \circ\!\!\rightarrow h = f \circ\!\!\rightarrow (\lambda x.\ g\ x \circ\!\!\rightarrow h)$
by (*simp add*: *fun-eq-iff scomp-unfold*)

lemma *scomp-fcomp*: $(f \circ\!\!\rightarrow g) \circ\!> h = f \circ\!\!\rightarrow (\lambda x.\ g\ x \circ\!> h)$
by (*simp add*: *fun-eq-iff scomp-unfold fcomp-def*)

lemma *fcomp-scomp*: $(f \circ\!> g) \circ\!\!\rightarrow h = f \circ\!> (g \circ\!\!\rightarrow h)$
by (*simp add*: *fun-eq-iff scomp-unfold*)

end

code-printing
constant *scomp* ⇀ (*Eval*) **infixl** *3 #−>*

map-prod — action of the product functor upon functions.

definition *map-prod* :: $('a \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'd) \Rightarrow 'a \times 'b \Rightarrow 'c \times 'd$
where *map-prod f g* $= (\lambda(x, y).\ (f\ x,\ g\ y))$

lemma *map-prod-simp* [*simp*, *code*]: *map-prod f g* $(a, b) = (f\ a,\ g\ b)$
by (*simp add*: *map-prod-def*)

functor *map-prod*: *map-prod*
by (*auto simp add*: *split-paired-all*)

lemma *fst-map-prod* [*simp*]: *fst* (*map-prod f g x*) = *f* (*fst x*)
by (*cases x*) *simp-all*

lemma *snd-map-prod* [*simp*]: *snd* (*map-prod f g x*) = *g* (*snd x*)
by (*cases x*) *simp-all*

lemma *fst-comp-map-prod* [*simp*]: *fst* ∘ *map-prod f g* = *f* ∘ *fst*
by (*rule ext*) *simp-all*

lemma *snd-comp-map-prod* [*simp*]: *snd* ∘ *map-prod f g* = *g* ∘ *snd*
by (*rule ext*) *simp-all*

lemma *map-prod-compose*: *map-prod* (*f1* ∘ *f2*) (*g1* ∘ *g2*) = (*map-prod f1 g1* ∘ *map-prod f2 g2*)
by (*rule ext*) (*simp add*: *map-prod.compositionality comp-def*)

lemma *map-prod-ident* [*simp*]: *map-prod* $(\lambda x.\ x)\ (\lambda y.\ y) = (\lambda z.\ z)$
by (*rule ext*) (*simp add*: *map-prod.identity*)

lemma *map-prod-imageI* [*intro*]: $(a, b) \in R \Longrightarrow (f\ a,\ g\ b) \in$ *map-prod f g* ‘ R

by (*rule image-eqI*) *simp-all*

lemma *prod-fun-imageE* [*elim*!]:
assumes *major*: $c \in$ *map-prod f g* ‘ *R*
and *cases*: $\bigwedge x\ y.\ c = (f\ x,\ g\ y) \Longrightarrow (x,\ y) \in R \Longrightarrow P$
shows *P*
proof (*rule major* [*THEN imageE*])
fix *x*
assume $c =$ *map-prod f g x* $x \in R$
then show *P*
using *cases* **by** (*cases x*) *simp*
qed

definition *apfst* :: $('a \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'c \times 'b$
where *apfst f* = *map-prod f id*

definition *apsnd* :: $('b \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'a \times 'c$
where *apsnd f* = *map-prod id f*

lemma *apfst-conv* [*simp*, *code*]: *apfst f* $(x,\ y) = (f\ x,\ y)$
by (*simp add*: *apfst-def*)

lemma *apsnd-conv* [*simp*, *code*]: *apsnd f* $(x,\ y) = (x,\ f\ y)$
by (*simp add*: *apsnd-def*)

lemma *fst-apfst* [*simp*]: *fst* (*apfst f x*) = *f* (*fst x*)
by (*cases x*) *simp*

lemma *fst-comp-apfst* [*simp*]: *fst* ∘ *apfst f* = *f* ∘ *fst*
by (*simp add*: *fun-eq-iff*)

lemma *fst-apsnd* [*simp*]: *fst* (*apsnd f x*) = *fst x*
by (*cases x*) *simp*

lemma *fst-comp-apsnd* [*simp*]: *fst* ∘ *apsnd f* = *fst*
by (*simp add*: *fun-eq-iff*)

lemma *snd-apfst* [*simp*]: *snd* (*apfst f x*) = *snd x*
by (*cases x*) *simp*

lemma *snd-comp-apfst* [*simp*]: *snd* ∘ *apfst f* = *snd*
by (*simp add*: *fun-eq-iff*)

lemma *snd-apsnd* [*simp*]: *snd* (*apsnd f x*) = *f* (*snd x*)
by (*cases x*) *simp*

lemma *snd-comp-apsnd* [*simp*]: *snd* ∘ *apsnd f* = *f* ∘ *snd*
by (*simp add*: *fun-eq-iff*)

lemma *apfst-compose*: *apfst f* (*apfst g x*) = *apfst* (*f* ∘ *g*) *x*
by (*cases x*) *simp*

lemma *apsnd-compose*: *apsnd f* (*apsnd g x*) = *apsnd* (*f* ∘ *g*) *x*
by (*cases x*) *simp*

lemma *apfst-apsnd* [*simp*]: *apfst f* (*apsnd g x*) = (*f* (*fst x*), *g* (*snd x*))
by (*cases x*) *simp*

lemma *apsnd-apfst* [*simp*]: *apsnd f* (*apfst g x*) = (*g* (*fst x*), *f* (*snd x*))
by (*cases x*) *simp*

lemma *apfst-id* [*simp*]: *apfst id* = *id*
by (*simp add*: *fun-eq-iff*)

lemma *apsnd-id* [*simp*]: *apsnd id* = *id*
by (*simp add*: *fun-eq-iff*)

lemma *apfst-eq-conv* [*simp*]: *apfst f x* = *apfst g x* ⟷ *f* (*fst x*) = *g* (*fst x*)
by (*cases x*) *simp*

lemma *apsnd-eq-conv* [*simp*]: *apsnd f x* = *apsnd g x* ⟷ *f* (*snd x*) = *g* (*snd x*)
by (*cases x*) *simp*

lemma *apsnd-apfst-commute*: *apsnd f* (*apfst g p*) = *apfst g* (*apsnd f p*)
by *simp*

context
begin

local-setup ‹*Local-Theory.map-background-naming* (*Name-Space.mandatory-path prod*)›

definition *swap* :: ′*a* × ′*b* ⇒ ′*b* × ′*a*
where *swap p* = (*snd p*, *fst p*)

end

lemma *swap-simp* [*simp*]: *prod.swap* (*x*, *y*) = (*y*, *x*)
by (*simp add*: *prod.swap-def*)

lemma *swap-swap* [*simp*]: *prod.swap* (*prod.swap p*) = *p*
by (*cases p*) *simp*

lemma *swap-comp-swap* [*simp*]: *prod.swap* ∘ *prod.swap* = *id*
by (*simp add*: *fun-eq-iff*)

lemma *pair-in-swap-image* [*simp*]: (*y*, *x*) ∈ *prod.swap* ‘ *A* ⟷ (*x*, *y*) ∈ *A*
by (*auto intro*!: *image-eqI*)

lemma *inj-swap* [*simp*]: *inj-on prod.swap A*
  by (*rule inj-onI*) *auto*

lemma *swap-inj-on*: *inj-on* (λ(*i*, *j*). (*j*, *i*)) *A*
  by (*rule inj-onI*) *auto*

lemma *surj-swap* [*simp*]: *surj prod.swap*
  by (*rule surjI* [*of - prod.swap*]) *simp*

lemma *bij-swap* [*simp*]: *bij prod.swap*
  by (*simp add*: *bij-def*)

lemma *case-swap* [*simp*]: (*case prod.swap p of* (*y*, *x*) $\Rightarrow$ *f x y*) = (*case p of* (*x*, *y*) $\Rightarrow$ *f x y*)
  by (*cases p*) *simp*

lemma *fst-swap* [*simp*]: *fst* (*prod.swap x*) = *snd x*
  by (*cases x*) *simp*

lemma *snd-swap* [*simp*]: *snd* (*prod.swap x*) = *fst x*
  by (*cases x*) *simp*

lemma *split-pairs*: (*A*,*B*) = *X* $\longleftrightarrow$ *fst X* = *A* $\wedge$ *snd X* = *B*
  and *split-pairs2*: *X* = (*A*,*B*) $\longleftrightarrow$ *fst X* = *A* $\wedge$ *snd X* = *B*
  by *auto*

Disjoint union of a family of sets – Sigma.

definition *Sigma* :: *'a set* $\Rightarrow$ (*'a* $\Rightarrow$ *'b set*) $\Rightarrow$ (*'a* $\times$ *'b*) *set*
  where *Sigma A B* $\equiv$ $\bigcup$*x*$\in$*A*. $\bigcup$*y*$\in$*B x*. {*Pair x y*}

context
begin
qualified abbreviation *Times* :: *'a set* $\Rightarrow$ *'b set* $\Rightarrow$ (*'a* $\times$ *'b*) *set* (**infixr** ‹×› *80*)
  where *A* $\times$ *B* $\equiv$ *Sigma A* (λ-. *B*)
end

bundle *set-product-syntax*
begin
notation *Product-Type.Times* (**infixr** ‹×› *80*)
end

syntax
  -Sigma :: *pttrn* $\Rightarrow$ *'a set* $\Rightarrow$ *'b set* $\Rightarrow$ (*'a* $\times$ *'b*) *set*
    (‹(‹*indent=3 notation=*‹*binder SIGMA*››*SIGMA* -:-./ -)› [*0*, *0*, *10*] *10*)
syntax-consts
  -Sigma ⇌ *Sigma*
translations
  SIGMA x:*A*. *B* ⇌ *CONST Sigma A* (λ*x*. *B*)

lemma *SigmaI* [*intro*!]: $a \in A \Longrightarrow b \in B\ a \Longrightarrow (a,\ b) \in Sigma\ A\ B$
unfolding *Sigma-def* **by** *blast*

lemma *SigmaE* [*elim*!]: $c \in Sigma\ A\ B \Longrightarrow (\bigwedge x\ y.\ x \in A \Longrightarrow y \in B\ x \Longrightarrow c = (x,\ y) \Longrightarrow P) \Longrightarrow P$
— The general elimination rule.
unfolding *Sigma-def* **by** *blast*

Elimination of $(a,\ b) \in A \times B$ – introduces no eigenvariables.

lemma *SigmaD1*: $(a,\ b) \in Sigma\ A\ B \Longrightarrow a \in A$
by *blast*

lemma *SigmaD2*: $(a,\ b) \in Sigma\ A\ B \Longrightarrow b \in B\ a$
by *blast*

lemma *SigmaE2*: $(a,\ b) \in Sigma\ A\ B \Longrightarrow (a \in A \Longrightarrow b \in B\ a \Longrightarrow P) \Longrightarrow P$
by *blast*

lemma *Sigma-cong*: $A = B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow C\ x = D\ x) \Longrightarrow (SIGMA\ x{:}A.\ C\ x) = (SIGMA\ x{:}B.\ D\ x)$
by *auto*

lemma *Sigma-mono*: $A \subseteq C \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq D\ x) \Longrightarrow Sigma\ A\ B \subseteq Sigma\ C\ D$
by *blast*

lemma *Sigma-empty1* [*simp*]: $Sigma\ \{\}\ B = \{\}$
by *blast*

lemma *Sigma-empty2* [*simp*]: $A \times \{\} = \{\}$
by *blast*

lemma *UNIV-Times-UNIV* [*simp*]: $UNIV \times UNIV = UNIV$
by *auto*

lemma *Compl-Times-UNIV1* [*simp*]: $-\ (UNIV \times A) = UNIV \times (-A)$
by *auto*

lemma *Compl-Times-UNIV2* [*simp*]: $-\ (A \times UNIV) = (-A) \times UNIV$
by *auto*

lemma *mem-Sigma-iff* [*iff*]: $(a,\ b) \in Sigma\ A\ B \longleftrightarrow a \in A \wedge b \in B\ a$
by *blast*

lemma *mem-Times-iff*: $x \in A \times B \longleftrightarrow fst\ x \in A \wedge snd\ x \in B$
by (*induct x*) *simp*

lemma *Sigma-empty-iff*: $(SIGMA\ i{:}I.\ X\ i) = \{\} \longleftrightarrow (\forall\, i{\in}I.\ X\ i = \{\})$

by *auto*

lemma *Times-subset-cancel2*: $x \in C \Longrightarrow A \times C \subseteq B \times C \longleftrightarrow A \subseteq B$
by *blast*

lemma *Times-eq-cancel2*: $x \in C \Longrightarrow A \times C = B \times C \longleftrightarrow A = B$
by (*blast elim*: *equalityE*)

lemma *Collect-case-prod-Sigma*: $\{(x, y). P\ x \wedge Q\ x\ y\} = (SIGMA\ x{:}Collect\ P.\ Collect\ (Q\ x))$
by *blast*

lemma *Collect-case-prod* [*simp*]: $\{(a, b). P\ a \wedge Q\ b\} = Collect\ P \times Collect\ Q$
by (*fact Collect-case-prod-Sigma*)

lemma *Collect-case-prodD*: $x \in Collect\ (case\text{-}prod\ A) \Longrightarrow A\ (fst\ x)\ (snd\ x)$
by *auto*

lemma *Collect-case-prod-mono*: $A \le B \Longrightarrow Collect\ (case\text{-}prod\ A) \subseteq Collect\ (case\text{-}prod\ B)$
by *auto* (*auto elim*!: *le-funE*)

lemma *Collect-split-mono-strong*:
$X = fst\ `\ A \Longrightarrow Y = snd\ `\ A \Longrightarrow \forall a{\in}X.\ \forall b \in Y.\ P\ a\ b \longrightarrow Q\ a\ b$
$\Longrightarrow A \subseteq Collect\ (case\text{-}prod\ P) \Longrightarrow A \subseteq Collect\ (case\text{-}prod\ Q)$
by *fastforce*

lemma *UN-Times-distrib*: $(\bigcup (a, b){\in}A \times B.\ E\ a \times F\ b) = \bigcup (E\ `\ A) \times \bigcup (F\ `\ B)$
— Suggested by Pierre Chartier
by *blast*

lemma *split-paired-Ball-Sigma* [*simp*, *no-atp*]: $(\forall z{\in}Sigma\ A\ B.\ P\ z) \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}B\ x.\ P\ (x, y))$
by *blast*

lemma *split-paired-Bex-Sigma* [*simp*, *no-atp*]: $(\exists z{\in}Sigma\ A\ B.\ P\ z) \longleftrightarrow (\exists x{\in}A.\ \exists y{\in}B\ x.\ P\ (x, y))$
by *blast*

lemma *Sigma-Un-distrib1*: $Sigma\ (I \cup J)\ C = Sigma\ I\ C \cup Sigma\ J\ C$
by *blast*

lemma *Sigma-Un-distrib2*: $(SIGMA\ i{:}I.\ A\ i \cup B\ i) = Sigma\ I\ A \cup Sigma\ I\ B$
by *blast*

lemma *Sigma-Int-distrib1*: $Sigma\ (I \cap J)\ C = Sigma\ I\ C \cap Sigma\ J\ C$
by *blast*

lemma *Sigma-Int-distrib2*: $(SIGMA\ i{:}I.\ A\ i \cap B\ i) = Sigma\ I\ A \cap Sigma\ I\ B$

by *blast*

lemma *Sigma-Diff-distrib1*: *Sigma* $(I - J)$ C = *Sigma* I C − *Sigma* J C
by *blast*

lemma *Sigma-Diff-distrib2*: (*SIGMA* $i{:}I.$ A i − B i) = *Sigma* I A − *Sigma* I B
by *blast*

lemma *Sigma-Union*: *Sigma* $(\bigcup X)$ B = $(\bigcup A{\in}X.$ *Sigma* A $B)$
by *blast*

lemma *Pair-vimage-Sigma*: *Pair* x −‘ *Sigma* A f = (*if* $x \in A$ *then* f x *else* {})
by *auto*

Non-dependent versions are needed to avoid the need for higher-order matching, especially when the rules are re-oriented.

lemma *Times-Un-distrib1*: $(A \cup B) \times C = A \times C \cup B \times C$
by (*fact Sigma-Un-distrib1*)

lemma *Times-Int-distrib1*: $(A \cap B) \times C = A \times C \cap B \times C$
by (*fact Sigma-Int-distrib1*)

lemma *Times-Diff-distrib1*: $(A - B) \times C = A \times C - B \times C$
by (*fact Sigma-Diff-distrib1*)

lemma *Times-empty* [*simp*]: $A \times B = \{\} \longleftrightarrow A = \{\} \vee B = \{\}$
by *auto*

lemma *times-subset-iff*: $A \times C \subseteq B \times D \longleftrightarrow A{=}\{\} \vee C{=}\{\} \vee A \subseteq B \wedge C \subseteq D$
by *blast*

lemma *times-eq-iff*: $A \times B = C \times D \longleftrightarrow A = C \wedge B = D \vee (A = \{\} \vee B = \{\}) \wedge (C = \{\} \vee D = \{\})$
by *auto*

lemma *fst-image-times* [*simp*]: *fst* ‘ $(A \times B)$ = (*if* $B = \{\}$ *then* {} *else* A)
by *force*

lemma *snd-image-times* [*simp*]: *snd* ‘ $(A \times B)$ = (*if* $A = \{\}$ *then* {} *else* B)
by *force*

lemma *fst-image-Sigma*: *fst* ‘ (*Sigma* A B) = $\{x \in A.\ B(x) \neq \{\}\}$
by *force*

lemma *snd-image-Sigma*: *snd* ‘ (*Sigma* A B) = $(\bigcup\ x \in A.\ B\ x)$
by *force*

lemma *vimage-fst*: *fst* −‘ $A = A \times$ *UNIV*
by *auto*

lemma *vimage-snd*: $snd -` A = UNIV \times A$
by *auto*

lemma *insert-Times-insert* [*simp*]:
$insert\ a\ A \times insert\ b\ B = insert\ (a,b)\ (A \times insert\ b\ B \cup \{a\} \times B)$
by *blast*

lemma *sing-Times-sing*: $\{x\}\times\{y\} = \{(x,y)\}$
by *simp*

lemma *vimage-Times*: $f -` (A \times B) = (fst \circ f) -` A \cap (snd \circ f) -` B$
proof (*rule set-eqI*)
show $x \in f -` (A \times B) \longleftrightarrow x \in (fst \circ f) -` A \cap (snd \circ f) -` B$ **for** x
by (*cases f x*) (*auto split*: *prod.split*)
qed

lemma *Times-Int-Times*: $A \times B \cap C \times D = (A \cap C) \times (B \cap D)$
by *auto*

lemma *image-paired-Times*:
$(\lambda(x,y).\ (f\ x,\ g\ y))$ ` $(A \times B) = (f$ ` $A) \times (g$ ` $B)$
by *auto*

lemma *Times-insert-right*: $A \times insert\ y\ B = (\lambda x.\ (x,\ y))$ ` $A \cup A \times B$
by *auto*

lemma *Times-insert-left*: $insert\ x\ A \times B = (\lambda y.\ (x,\ y))$ ` $B \cup A \times B$
by *auto*

lemma *product-swap*: *prod.swap* ` $(A \times B) = B \times A$
by (*auto simp add*: *set-eq-iff*)

lemma *swap-product*: $(\lambda(i,\ j).\ (j,\ i))$ ` $(A \times B) = B \times A$
by (*auto simp add*: *set-eq-iff*)

lemma *image-split-eq-Sigma*: $(\lambda x.\ (f\ x,\ g\ x))$ ` $A = Sigma\ (f$ ` $A)\ (\lambda x.\ g$ ` $(f -`$
$\{x\} \cap A))$
proof (*safe intro*!: *imageI*)
fix $a\ b$
assume $*$: $a \in A\ b \in A$ **and** *eq*: $f\ a = f\ b$
show $(f\ b,\ g\ a) \in (\lambda x.\ (f\ x,\ g\ x))$ ` A
using $*$ *eq*[*symmetric*] **by** *auto*
qed *simp-all*

lemma *subset-fst-snd*: $A \subseteq (fst$ ` $A \times snd$ ` $A)$
by *force*

lemma *inj-on-apfst* [*simp*]: *inj-on* (*apfst f*) $(A \times UNIV) \longleftrightarrow$ *inj-on f A*

by (*auto simp add*: *inj-on-def*)

lemma *inj-apfst* [*simp*]: *inj* (*apfst f*) ⟷ *inj f*
using *inj-on-apfst*[*of f UNIV*] **by** *simp*

lemma *inj-on-apsnd* [*simp*]: *inj-on* (*apsnd f*) (*UNIV* × *A*) ⟷ *inj-on f A*
by (*auto simp add*: *inj-on-def*)

lemma *inj-apsnd* [*simp*]: *inj* (*apsnd f*) ⟷ *inj f*
using *inj-on-apsnd*[*of f UNIV*] **by** *simp*

context
begin

qualified definition *product* :: *'a set* ⇒ *'b set* ⇒ (*'a* × *'b*) *set*
where [*code-abbrev*]: *product A B* = *A* × *B*

lemma *member-product*: *x* ∈ *Product-Type.product A B* ⟷ *x* ∈ *A* × *B*
by (*simp add*: *product-def*)

end

The following *map-prod* lemmas are due to Joachim Breitner:

lemma *map-prod-inj-on*:
assumes *inj-on f A*
and *inj-on g B*
shows *inj-on* (*map-prod f g*) (*A* × *B*)
proof (*rule inj-onI*)
fix *x* :: *'a* × *'c*
fix *y* :: *'a* × *'c*
assume *x* ∈ *A* × *B*
then have *fst x* ∈ *A* **and** *snd x* ∈ *B* **by** *auto*
assume *y* ∈ *A* × *B*
then have *fst y* ∈ *A* **and** *snd y* ∈ *B* **by** *auto*
assume *map-prod f g x* = *map-prod f g y*
then have *fst* (*map-prod f g x*) = *fst* (*map-prod f g y*) **by** *auto*
then have *f* (*fst x*) = *f* (*fst y*) **by** (*cases x*, *cases y*) *auto*
with ‹*inj-on f A*› **and** ‹*fst x* ∈ *A*› **and** ‹*fst y* ∈ *A*› **have** *fst x* = *fst y*
by (*auto dest*: *inj-onD*)
moreover from ‹*map-prod f g x* = *map-prod f g y*›
have *snd* (*map-prod f g x*) = *snd* (*map-prod f g y*) **by** *auto*
then have *g* (*snd x*) = *g* (*snd y*) **by** (*cases x*, *cases y*) *auto*
with ‹*inj-on g B*› **and** ‹*snd x* ∈ *B*› **and** ‹*snd y* ∈ *B*› **have** *snd x* = *snd y*
by (*auto dest*: *inj-onD*)
ultimately show *x* = *y* **by** (*rule prod-eqI*)
qed

lemma *map-prod-surj*:
fixes *f* :: *'a* ⇒ *'b*

and g :: $'c \Rightarrow 'd$
assumes *surj f* **and** *surj g*
shows *surj* (*map-prod f g*)
unfolding *surj-def*
proof
fix y :: $'b \times 'd$
from ‹*surj f*› **obtain** a **where** *fst* $y = f\ a$
by (*auto elim*: *surjE*)
moreover
from ‹*surj g*› **obtain** b **where** *snd* $y = g\ b$
by (*auto elim*: *surjE*)
ultimately have (*fst y*, *snd y*) = *map-prod f g* (*a*,*b*)
by *auto*
then show $\exists x.\ y =$ *map-prod f g x*
by *auto*
qed

lemma *map-prod-surj-on*:
assumes f ‘ $A = A'$ **and** g ‘ $B = B'$
shows *map-prod f g* ‘ $(A \times B) = A' \times B'$
unfolding *image-def*
proof (*rule set-eqI*, *rule iffI*)
fix x :: $'a \times 'c$
assume $x \in \{y::'a \times 'c.\ \exists x::'b \times 'd \in A \times B.\ y =$ *map-prod f g x*$\}$
then obtain y **where** $y \in A \times B$ **and** $x =$ *map-prod f g y*
by *blast*
from ‹*image f A* = A'› **and** ‹$y \in A \times B$› **have** f (*fst y*) $\in A'$
by *auto*
moreover from ‹*image g B* = B'› **and** ‹$y \in A \times B$› **have** g (*snd y*) $\in B'$
by *auto*
ultimately have (f (*fst y*), g (*snd y*)) $\in (A' \times B')$
by *auto*
with ‹$x =$ *map-prod f g y*› **show** $x \in A' \times B'$
by (*cases y*) *auto*
next
fix x :: $'a \times 'c$
assume $x \in A' \times B'$
then have *fst* $x \in A'$ **and** *snd* $x \in B'$
by *auto*
from ‹*image f A* = A'› **and** ‹*fst* $x \in A'$› **have** *fst* $x \in$ *image f A*
by *auto*
then obtain a **where** $a \in A$ **and** *fst* $x = f\ a$
by (*rule imageE*)
moreover from ‹*image g B* = B'› **and** ‹*snd* $x \in B'$› **obtain** b **where** $b \in B$
and *snd* $x = g\ b$
by *auto*
ultimately have (*fst x*, *snd x*) = *map-prod f g* (*a*, *b*)
by *auto*
moreover from ‹$a \in A$› **and** ‹$b \in B$› **have** $(a\ ,\ b) \in A \times B$

```
    by auto
  ultimately have ∃ y ∈ A × B. x = map-prod f g y
    by auto
  then show x ∈ {x. ∃ y ∈ A × B. x = map-prod f g y}
    by auto
qed
```

14.4 Simproc for rewriting a set comprehension into a point-free expression

ML-file ‹*Tools/set-comprehension-pointfree.ML*›

simproc-setup passive *set-comprehension* (*Collect P*) =
  ‹*K Set-Comprehension-Pointfree.code-proc*›

setup ‹*Code-Preproc.map-pre* (*Simplifier.add-proc* ***simproc*** ‹*set-comprehension*›)›

14.5 Lemmas about disjointness

lemma *disjnt-Times1-iff* [*simp*]: *disjnt* (*C* × *A*) (*C* × *B*) ⟷ *C* = {} ∨ *disjnt A B*
  by (*auto simp*: *disjnt-def*)

lemma *disjnt-Times2-iff* [*simp*]: *disjnt* (*A* × *C*) (*B* × *C*) ⟷ *C* = {} ∨ *disjnt A B*
  by (*auto simp*: *disjnt-def*)

lemma *disjnt-Sigma-iff*: *disjnt* (*Sigma A C*) (*Sigma B C*) ⟷ (∀ *i* ∈ *A*∩*B*. *C i* = {}) ∨ *disjnt A B*
  by (*auto simp*: *disjnt-def*)

14.6 Inductively defined sets

```
simproc-setup Collect-mem (Collect t) = ‹
  K (fn ctxt => fn ct =>
    (case Thm.term-of ct of
      S as Const- ‹Collect A for t› =>
        let val (u, -, ps) = HOLogic.strip-ptupleabs t in
          (case u of
            (c as Const- ‹Set.member - for q S'›) =>
              (case try (HOLogic.strip-ptuple ps) q of
                NONE => NONE
              | SOME ts =>
                  if not (Term.is-open S') andalso
                    ts = map Bound (length ps downto 0)
                  then
                    let
                      val simp =
                        full-simp-tac (put-simpset HOL-basic-ss ctxt
                          addsimps [@{thm split-paired-all}, @{thm case-prod-conv}]) 1
```

```
                  in
                    SOME (Goal.prove ctxt [] [] Const ‹Pure.eq Type ‹set A› for S
S'›
                      (K (EVERY
                        [resolve-tac ctxt [eq-reflection] 1,
                         resolve-tac ctxt @{thms subset-antisym} 1,
                         resolve-tac ctxt @{thms subsetI} 1,
                         dresolve-tac ctxt @{thms CollectD} 1, simp,
                         resolve-tac ctxt @{thms subsetI} 1,
                         resolve-tac ctxt @{thms CollectI} 1, simp])))
                  end
                else NONE)
        | - => NONE)
        end
    | - => NONE))
›
```

ML-file ‹*Tools/inductive-set.ML*›

14.7 Legacy theorem bindings and duplicates

lemmas *fst-conv* = *prod.sel*(*1*)
lemmas *snd-conv* = *prod.sel*(*2*)
lemmas *split-def* = *case-prod-unfold*
lemmas *split-beta'* = *case-prod-beta'*
lemmas *split-beta* = *prod.case-eq-if*
lemmas *split-conv* = *case-prod-conv*
lemmas *split* = *case-prod-conv*

hide-const (**open**) *prod*

end

15 The Disjoint Sum of Two Types

theory *Sum-Type*
  imports *Typedef Inductive Fun*
begin

15.1 Construction of the sum type and its basic abstract operations

definition *Inl-Rep* :: $'a \Rightarrow 'a \Rightarrow 'b \Rightarrow bool \Rightarrow bool$
  where *Inl-Rep* $a\ x\ y\ p \longleftrightarrow x = a \wedge p$

definition *Inr-Rep* :: $'b \Rightarrow 'a \Rightarrow 'b \Rightarrow bool \Rightarrow bool$
  where *Inr-Rep* $b\ x\ y\ p \longleftrightarrow y = b \wedge \neg\ p$

definition *sum* = $\{f.\ (\exists\, a.\ f = \textit{Inl-Rep}\ (a{::}'a)) \vee (\exists\, b.\ f = \textit{Inr-Rep}\ (b{::}'b))\}$

typedef ($'a$, $'b$) *sum* (**infixr** ‹+› *10*) = *sum* :: ($'a \Rightarrow 'b \Rightarrow$ *bool* $\Rightarrow$ *bool*) *set*
unfolding *sum-def* **by** *auto*

lemma *Inl-RepI*: *Inl-Rep a* ∈ *sum*
by (*auto simp add*: *sum-def*)

lemma *Inr-RepI*: *Inr-Rep b* ∈ *sum*
by (*auto simp add*: *sum-def*)

lemma *inj-on-Abs-sum*: $A \subseteq$ *sum* $\Longrightarrow$ *inj-on Abs-sum A*
by (*rule inj-on-inverseI*, *rule Abs-sum-inverse*) *auto*

lemma *Inl-Rep-inject*: *inj-on Inl-Rep A*
proof (*rule inj-onI*)
show $\bigwedge a\ c$. *Inl-Rep a* = *Inl-Rep c* $\Longrightarrow a = c$
by (*auto simp add*: *Inl-Rep-def fun-eq-iff*)
qed

lemma *Inr-Rep-inject*: *inj-on Inr-Rep A*
proof (*rule inj-onI*)
show $\bigwedge b\ d$. *Inr-Rep b* = *Inr-Rep d* $\Longrightarrow b = d$
by (*auto simp add*: *Inr-Rep-def fun-eq-iff*)
qed

lemma *Inl-Rep-not-Inr-Rep*: *Inl-Rep a* ≠ *Inr-Rep b*
by (*auto simp add*: *Inl-Rep-def Inr-Rep-def fun-eq-iff*)

definition *Inl* :: $'a \Rightarrow 'a + 'b$
where *Inl* = *Abs-sum* ∘ *Inl-Rep*

definition *Inr* :: $'b \Rightarrow 'a + 'b$
where *Inr* = *Abs-sum* ∘ *Inr-Rep*

lemma *inj-Inl* [*simp*]: *inj-on Inl A*
by (*auto simp add*: *Inl-def intro*!: *comp-inj-on Inl-Rep-inject inj-on-Abs-sum Inl-RepI*)

lemma *Inl-inject*: *Inl x* = *Inl y* $\Longrightarrow x = y$
using *inj-Inl* **by** (*rule injD*)

lemma *inj-Inr* [*simp*]: *inj-on Inr A*
by (*auto simp add*: *Inr-def intro*!: *comp-inj-on Inr-Rep-inject inj-on-Abs-sum Inr-RepI*)

lemma *Inr-inject*: *Inr x* = *Inr y* $\Longrightarrow x = y$
using *inj-Inr* **by** (*rule injD*)

lemma *Inl-not-Inr*: *Inl a* ≠ *Inr b*

proof –
  have {*Inl-Rep a*, *Inr-Rep b*} ⊆ *sum*
    using *Inl-RepI* [*of a*] *Inr-RepI* [*of b*] **by** *auto*
  with *inj-on-Abs-sum* **have** *inj-on Abs-sum* {*Inl-Rep a*, *Inr-Rep b*} **.**
  with *Inl-Rep-not-Inr-Rep* [*of a b*] *inj-on-contraD* **have** *Abs-sum* (*Inl-Rep a*) ≠ *Abs-sum* (*Inr-Rep b*)
    by *auto*
  then show *?thesis*
    by (*simp add*: *Inl-def Inr-def*)
qed

lemma *Inr-not-Inl*: *Inr b* ≠ *Inl a*
  using *Inl-not-Inr* **by** (*rule not-sym*)

lemma *sumE*:
  assumes ⋀*x*::′*a*. *s* = *Inl x* ⟹ *P*
    and ⋀*y*::′*b*. *s* = *Inr y* ⟹ *P*
  shows *P*
proof (*rule Abs-sum-cases* [*of s*])
  fix *f*
  assume *s* = *Abs-sum f* **and** *f* ∈ *sum*
  with *assms* **show** *P*
    by (*auto simp add*: *sum-def Inl-def Inr-def*)
qed

free-constructors *case-sum* **for**
  isl: *Inl projl*
| *Inr projr*
  by (*erule sumE*, *assumption*) (*auto dest*: *Inl-inject Inr-inject simp add*: *Inl-not-Inr*)

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

setup ‹*Sign.mandatory-path old*›

old-rep-datatype *Inl Inr*
proof –
  fix *P*
  fix *s* :: ′*a* + ′*b*
  assume *x*: ⋀*x*::′*a*. *P* (*Inl x*) **and** *y*: ⋀*y*::′*b*. *P* (*Inr y*)
  then show *P s* **by** (*auto intro*: *sumE* [*of s*])
qed (*auto dest*: *Inl-inject Inr-inject simp add*: *Inl-not-Inr*)

setup ‹*Sign.parent-path*›

But erase the prefix for properties that are not generated by *free-constructors*.

setup ‹*Sign.mandatory-path sum*›

declare
  old.sum.inject[*iff del*]
  old.sum.distinct(*1*)[*simp del*, *induct-simp del*]

lemmas *induct* = *old.sum.induct*
lemmas *inducts* = *old.sum.inducts*
lemmas *rec* = *old.sum.rec*
lemmas *simps* = *sum.inject sum.distinct sum.case sum.rec*

setup ‹*Sign.parent-path*›

primrec *map-sum* :: $('a \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'd) \Rightarrow 'a + 'b \Rightarrow 'c + 'd$
  where
    map-sum f1 f2 (*Inl a*) = *Inl* (*f1 a*)
  | *map-sum f1 f2* (*Inr a*) = *Inr* (*f2 a*)

functor *map-sum*: *map-sum*
proof −
  show *map-sum f g* ∘ *map-sum h i* = *map-sum* (*f* ∘ *h*) (*g* ∘ *i*) **for** *f g h i*
  proof
    show (*map-sum f g* ∘ *map-sum h i*) *s* = *map-sum* (*f* ∘ *h*) (*g* ∘ *i*) *s* **for** *s*
      by (*cases s*) *simp-all*
  qed
  show *map-sum id id* = *id*
  proof
    show *map-sum id id s* = *id s* **for** *s*
      by (*cases s*) *simp-all*
  qed
qed

lemma *split-sum-all*: $(\forall x.\ P\ x) \longleftrightarrow (\forall x.\ P\ (Inl\ x)) \land (\forall x.\ P\ (Inr\ x))$
  by (*auto intro*: *sum.induct*)

lemma *split-sum-ex*: $(\exists x.\ P\ x) \longleftrightarrow (\exists x.\ P\ (Inl\ x)) \lor (\exists x.\ P\ (Inr\ x))$
  using *split-sum-all*[*of* $\lambda x.\ \neg P\ x$] **by** *blast*

15.2 Projections

lemma *case-sum-KK* [*simp*]: *case-sum* $(\lambda x.\ a)\ (\lambda x.\ a) = (\lambda x.\ a)$
  by (*rule ext*) (*simp split*: *sum.split*)

lemma *surjective-sum*: *case-sum* $(\lambda x::'a.\ f\ (Inl\ x))\ (\lambda y::'b.\ f\ (Inr\ y)) = f$
proof
  fix *s* :: $'a + 'b$
  show (*case s of Inl* $(x::'a) \Rightarrow f\ (Inl\ x)$ | *Inr* $(y::'b) \Rightarrow f\ (Inr\ y)) = f\ s$
    by (*cases s*) *simp-all*
qed

lemma *case-sum-inject*:
  assumes *a*: *case-sum f1 f2* = *case-sum g1 g2*
    and *r*: $f1 = g1 \Longrightarrow f2 = g2 \Longrightarrow P$
  shows *P*

```
proof (rule r)
  show f1 = g1
  proof
    fix x :: 'a
    from a have case-sum f1 f2 (Inl x) = case-sum g1 g2 (Inl x) by simp
    then show f1 x = g1 x by simp
  qed
  show f2 = g2
  proof
    fix y :: 'b
    from a have case-sum f1 f2 (Inr y) = case-sum g1 g2 (Inr y) by simp
    then show f2 y = g2 y by simp
  qed
qed

primrec Suml :: ('a ⇒ 'c) ⇒ 'a + 'b ⇒ 'c
  where Suml f (Inl x) = f x

primrec Sumr :: ('b ⇒ 'c) ⇒ 'a + 'b ⇒ 'c
  where Sumr f (Inr x) = f x

lemma Suml-inject:
  assumes Suml f = Suml g
  shows f = g
proof
  fix x :: 'a
  let ?s = Inl x :: 'a + 'b
  from assms have Suml f ?s = Suml g ?s by simp
  then show f x = g x by simp
qed

lemma Sumr-inject:
  assumes Sumr f = Sumr g
  shows f = g
proof
  fix x :: 'b
  let ?s = Inr x :: 'a + 'b
  from assms have Sumr f ?s = Sumr g ?s by simp
  then show f x = g x by simp
qed
```

15.3 The Disjoint Sum of Sets

```
definition Plus :: 'a set ⇒ 'b set ⇒ ('a + 'b) set  (infixr ‹<+>› 65)
  where A <+> B = Inl ` A ∪ Inr ` B

hide-const (open) Plus — Valuable identifier

lemma InlI [intro!]: a ∈ A ⟹ Inl a ∈ A <+> B
```

by (*simp add*: *Plus-def*)

lemma *InrI* [*intro*!]: $b \in B \Longrightarrow Inr\ b \in A <+> B$
by (*simp add*: *Plus-def*)

Exhaustion rule for sums, a degenerate form of induction

lemma *PlusE* [*elim*!]:
$u \in A <+> B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow u = Inl\ x \Longrightarrow P) \Longrightarrow (\bigwedge y.\ y \in B \Longrightarrow u = Inr\ y \Longrightarrow P) \Longrightarrow P$
by (*auto simp add*: *Plus-def*)

lemma *Plus-eq-empty-conv* [*simp*]: $A <+> B = \{\} \longleftrightarrow A = \{\} \wedge B = \{\}$
by *auto*

lemma *UNIV-Plus-UNIV* [*simp*]: $UNIV <+> UNIV = UNIV$
proof (*rule set-eqI*)
fix $u :: 'a + 'b$
show $u \in UNIV <+> UNIV \longleftrightarrow u \in UNIV$ **by** (*cases u*) *auto*
qed

lemma *UNIV-sum*: $UNIV = Inl\ `\ UNIV \cup Inr\ `\ UNIV$
proof −
have $x \in range\ Inl$ **if** $x \notin range\ Inr$ **for** $x :: 'a + 'b$
using *that* **by** (*cases x*) *simp-all*
then show *?thesis* **by** *auto*
qed

hide-const (**open**) *Suml Sumr sum*

end

16 Rings

theory *Rings*
imports *Groups Set Fun*
begin

16.1 Semirings and rings

class *semiring* = *ab-semigroup-add* + *semigroup-mult* +
assumes *distrib-right* [*algebra-simps*, *algebra-split-simps*]: $(a + b) * c = a * c + b * c$
assumes *distrib-left* [*algebra-simps*, *algebra-split-simps*]: $a * (b + c) = a * b + a * c$
begin

For the *combine-numerals* simproc

lemma *combine-common-factor*: $a * e + (b * e + c) = (a + b) * e + c$

by (*simp add*: *distrib-right ac-simps*)

end

class *mult-zero* = *times* + *zero* +
assumes *mult-zero-left* [*simp*]: $0 * a = 0$
assumes *mult-zero-right* [*simp*]: $a * 0 = 0$
begin

lemma *mult-not-zero*: $a * b \neq 0 \Longrightarrow a \neq 0 \wedge b \neq 0$
by *auto*

end

class *semiring-0* = *semiring* + *comm-monoid-add* + *mult-zero*

class *semiring-0-cancel* = *semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0*
proof
fix $a :: 'a$
have $0 * a + 0 * a = 0 * a + 0$
by (*simp add*: *distrib-right* [*symmetric*])
then show $0 * a = 0$
by (*simp only*: *add-left-cancel*)
have $a * 0 + a * 0 = a * 0 + 0$
by (*simp add*: *distrib-left* [*symmetric*])
then show $a * 0 = 0$
by (*simp only*: *add-left-cancel*)
qed

end

class *comm-semiring* = *ab-semigroup-add* + *ab-semigroup-mult* +
assumes *distrib*: $(a + b) * c = a * c + b * c$
begin

subclass *semiring*
proof
fix $a\ b\ c :: 'a$
show $(a + b) * c = a * c + b * c$
by (*simp add*: *distrib*)
have $a * (b + c) = (b + c) * a$
by (*simp add*: *ac-simps*)
also have $\ldots = b * a + c * a$
by (*simp only*: *distrib*)
also have $\ldots = a * b + a * c$
by (*simp add*: *ac-simps*)

finally show $a * (b + c) = a * b + a * c$
by *blast*
qed

end

class *comm-semiring-0* = *comm-semiring* + *comm-monoid-add* + *mult-zero*
begin

subclass *semiring-0* **..**

end

class *comm-semiring-0-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* **..**

subclass *comm-semiring-0* **..**

end

class *zero-neq-one* = *zero* + *one* +
assumes *zero-neq-one* [*simp*]: $0 \neq 1$
begin

lemma *one-neq-zero* [*simp*]: $1 \neq 0$
by (*rule not-sym*) (*rule zero-neq-one*)

definition *of-bool* :: *bool* ⇒ *'a*
where *of-bool p* = (*if p then 1 else 0*)

lemma *of-bool-eq* [*simp*, *code*]:
of-bool False = *0*
of-bool True = *1*
by (*simp-all add*: *of-bool-def*)

lemma *of-bool-eq-iff*: *of-bool p* = *of-bool q* ⟷ *p* = *q*
by (*simp add*: *of-bool-def*)

lemma *split-of-bool* [*split*]: *P* (*of-bool p*) ⟷ (*p* ⟶ *P 1*) ∧ (¬ *p* ⟶ *P 0*)
by (*cases p*) *simp-all*

lemma *split-of-bool-asm*: *P* (*of-bool p*) ⟷ ¬ (*p* ∧ ¬ *P 1* ∨ ¬ *p* ∧ ¬ *P 0*)
by (*cases p*) *simp-all*

lemma *of-bool-eq-0-iff* [*simp*]:
‹*of-bool P* = *0* ⟷ ¬ *P*›
by *simp*

lemma *of-bool-eq-1-iff* [*simp*]:
‹*of-bool* P = *1* ⟷ P›
by *simp*

end

class *semiring-1* = *zero-neq-one* + *semiring-0* + *monoid-mult*
begin

lemma *of-bool-conj*:
of-bool ($P \wedge Q$) = *of-bool* P ∗ *of-bool* Q
by *auto*

end

lemma *lambda-zero*: (λ*h*::′*a*::*mult-zero*. *0*) = (∗) *0*
by *auto*

lemma *lambda-one*: (λ*x*::′*a*::*monoid-mult*. *x*) = (∗) *1*
by *auto*

16.2 Abstract divisibility

class *dvd* = *times*
begin

definition *dvd* :: ′*a* ⇒ ′*a* ⇒ *bool* (**infix** ‹*dvd*› *50*)
where *b dvd a* ⟷ (∃ *k*. *a* = *b* ∗ *k*)

lemma *dvdI* [*intro?*]: *a* = *b* ∗ *k* ⟹ *b dvd a*
unfolding *dvd-def* **..**

lemma *dvdE* [*elim*]: *b dvd a* ⟹ (⋀*k*. *a* = *b* ∗ *k* ⟹ *P*) ⟹ *P*
unfolding *dvd-def* **by** *blast*

end

context *comm-monoid-mult*
begin

subclass *dvd* **.**

lemma *dvd-refl* [*simp*]: *a dvd a*
proof
show *a* = *a* ∗ *1* **by** *simp*
qed

lemma *dvd-trans* [*trans*]:

assumes *a dvd b* **and** *b dvd c*
shows *a dvd c*
proof –
from *assms* **obtain** *v* **where** $b = a * v$
by *auto*
moreover from *assms* **obtain** *w* **where** $c = b * w$
by *auto*
ultimately have $c = a * (v * w)$
by (*simp add*: *mult.assoc*)
then show *?thesis* **..**
qed

lemma *subset-divisors-dvd*: {*c*. *c dvd a*} ⊆ {*c*. *c dvd b*} ⟷ *a dvd b*
by (*auto simp add*: *subset-iff intro*: *dvd-trans*)

lemma *strict-subset-divisors-dvd*: {*c*. *c dvd a*} ⊂ {*c*. *c dvd b*} ⟷ *a dvd b* ∧ ¬ *b dvd a*
by (*auto simp add*: *subset-iff intro*: *dvd-trans*)

lemma *one-dvd* [*simp*]: *1 dvd a*
by (*auto intro*: *dvdI*)

lemma *dvd-mult* [*simp*]: *a dvd* (*b* ∗ *c*) **if** *a dvd c*
using *that* **by** (*auto intro*: *mult.left-commute dvdI*)

lemma *dvd-mult2* [*simp*]: *a dvd* (*b* ∗ *c*) **if** *a dvd b*
using *that dvd-mult* [*of a b c*] **by** (*simp add*: *ac-simps*)

lemma *dvd-triv-right* [*simp*]: *a dvd b* ∗ *a*
by (*rule dvd-mult*) (*rule dvd-refl*)

lemma *dvd-triv-left* [*simp*]: *a dvd a* ∗ *b*
by (*rule dvd-mult2*) (*rule dvd-refl*)

lemma *mult-dvd-mono*:
assumes *a dvd b*
and *c dvd d*
shows *a* ∗ *c dvd b* ∗ *d*
proof –
from ‹*a dvd b*› **obtain** b' **where** $b = a * b'$ **..**
moreover from ‹*c dvd d*› **obtain** d' **where** $d = c * d'$ **..**
ultimately have $b * d = (a * c) * (b' * d')$
by (*simp add*: *ac-simps*)
then show *?thesis* **..**
qed

lemma *dvd-mult-left*: *a* ∗ *b dvd c* ⟹ *a dvd c*
by (*simp add*: *dvd-def mult.assoc*) *blast*

lemma *dvd-mult-right*: $a * b$ *dvd* $c \Longrightarrow b$ *dvd* c
using *dvd-mult-left* [*of* b a c] **by** (*simp add*: *ac-simps*)

end

class *comm-semiring-1* = *zero-neq-one* + *comm-semiring-0* + *comm-monoid-mult*
begin

subclass *semiring-1* **..**

lemma *dvd-0-left-iff* [*simp*]: *0 dvd a* $\longleftrightarrow$ $a = 0$
by *auto*

lemma *dvd-0-right* [*iff*]: *a dvd 0*
proof
show $0 = a * 0$ **by** *simp*
qed

lemma *dvd-0-left*: *0 dvd a* $\Longrightarrow a = 0$
by *simp*

lemma *dvd-add* [*simp*]:
assumes *a dvd b* **and** *a dvd c*
shows *a dvd* $(b + c)$
proof −
from ‹*a dvd b*› **obtain** b' **where** $b = a * b'$ **..**
moreover from ‹*a dvd c*› **obtain** c' **where** $c = a * c'$ **..**
ultimately have $b + c = a * (b' + c')$
by (*simp add*: *distrib-left*)
then show *?thesis* **..**
qed

end

class *semiring-1-cancel* = *semiring* + *cancel-comm-monoid-add*
+ *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-0-cancel* **..**

subclass *semiring-1* **..**

end

class *comm-semiring-1-cancel* =
comm-semiring + *cancel-comm-monoid-add* + *zero-neq-one* + *comm-monoid-mult*
+
assumes *right-diff-distrib'* [*algebra-simps*, *algebra-split-simps*]:
$a * (b - c) = a * b - a * c$

begin

subclass *semiring-1-cancel* **..**
subclass *comm-semiring-0-cancel* **..**
subclass *comm-semiring-1* **..**

lemma *left-diff-distrib′* [*algebra-simps*, *algebra-split-simps*]:
  $(b - c) * a = b * a - c * a$
  by (*simp add*: *algebra-simps*)

lemma *dvd-add-times-triv-left-iff* [*simp*]: a *dvd* $c * a + b \longleftrightarrow a$ *dvd* b
proof −
  have a *dvd* $a * c + b \longleftrightarrow a$ *dvd* b (**is** *?P* ⟷ *?Q*)
  proof
    assume *?Q*
    then show *?P* **by** *simp*
  next
    assume *?P*
    then obtain d **where** $a * c + b = a * d$ **..**
    then have $a * c + b - a * c = a * d - a * c$ **by** *simp*
    then have $b = a * d - a * c$ **by** *simp*
    then have $b = a * (d - c)$ **by** (*simp add*: *algebra-simps*)
    then show *?Q* **..**
  qed
  then show a *dvd* $c * a + b \longleftrightarrow a$ *dvd* b **by** (*simp add*: *ac-simps*)
qed

lemma *dvd-add-times-triv-right-iff* [*simp*]: a *dvd* $b + c * a \longleftrightarrow a$ *dvd* b
  using *dvd-add-times-triv-left-iff* [*of a c b*] **by** (*simp add*: *ac-simps*)

lemma *dvd-add-triv-left-iff* [*simp*]: a *dvd* $a + b \longleftrightarrow a$ *dvd* b
  using *dvd-add-times-triv-left-iff* [*of a 1 b*] **by** *simp*

lemma *dvd-add-triv-right-iff* [*simp*]: a *dvd* $b + a \longleftrightarrow a$ *dvd* b
  using *dvd-add-times-triv-right-iff* [*of a b 1*] **by** *simp*

lemma *dvd-add-right-iff*:
  assumes a *dvd* b
  shows a *dvd* $b + c \longleftrightarrow a$ *dvd* c (**is** *?P* ⟷ *?Q*)
proof
  assume *?P*
  then obtain d **where** $b + c = a * d$ **..**
  moreover from ‹*a dvd b*› **obtain** e **where** $b = a * e$ **..**
  ultimately have $a * e + c = a * d$ **by** *simp*
  then have $a * e + c - a * e = a * d - a * e$ **by** *simp*
  then have $c = a * d - a * e$ **by** *simp*
  then have $c = a * (d - e)$ **by** (*simp add*: *algebra-simps*)
  then show *?Q* **..**
next

assume *?Q*
with *assms* **show** *?P* **by** *simp*
qed

lemma *dvd-add-left-iff*: $a\ dvd\ c \Longrightarrow a\ dvd\ b + c \longleftrightarrow a\ dvd\ b$
using *dvd-add-right-iff* [*of a c b*] **by** (*simp add*: *ac-simps*)

end

class *ring* = *semiring* + *ab-group-add*
begin

subclass *semiring-0-cancel* **..**

Distribution rules

lemma *minus-mult-left*: $- (a * b) = - a * b$
by (*rule minus-unique*) (*simp add*: *distrib-right* [*symmetric*])

lemma *minus-mult-right*: $- (a * b) = a * - b$
by (*rule minus-unique*) (*simp add*: *distrib-left* [*symmetric*])

Extract signs from products

lemmas *mult-minus-left* [*simp*] = *minus-mult-left* [*symmetric*]
lemmas *mult-minus-right* [*simp*] = *minus-mult-right* [*symmetric*]

lemma *minus-mult-minus* [*simp*]: $- a * - b = a * b$
by *simp*

lemma *minus-mult-commute*: $- a * b = a * - b$
by *simp*

lemma *right-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$a * (b - c) = a * b - a * c$
using *distrib-left* [*of a b* $-c$] **by** *simp*

lemma *left-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$(a - b) * c = a * c - b * c$
using *distrib-right* [*of* $a - b$ *c*] **by** *simp*

lemmas *ring-distribs* = *distrib-left distrib-right left-diff-distrib right-diff-distrib*

lemma *eq-add-iff1*: $a * e + c = b * e + d \longleftrightarrow (a - b) * e + c = d$
by (*simp add*: *algebra-simps*)

lemma *eq-add-iff2*: $a * e + c = b * e + d \longleftrightarrow c = (b - a) * e + d$
by (*simp add*: *algebra-simps*)

end

lemmas *ring-distribs* = *distrib-left distrib-right left-diff-distrib right-diff-distrib*

class *comm-ring* = *comm-semiring* + *ab-group-add*
begin

subclass *ring* **..**
subclass *comm-semiring-0-cancel* **..**

lemma *square-diff-square-factored*: *x* ∗ *x* − *y* ∗ *y* = (*x* + *y*) ∗ (*x* − *y*)
  by (*simp add*: *algebra-simps*)

end

class *ring-1* = *ring* + *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-1-cancel* **..**

lemma *of-bool-not-iff*:
  ‹*of-bool* (¬ *P*) = *1* − *of-bool P*›
  by *simp*

lemma *square-diff-one-factored*: *x* ∗ *x* − *1* = (*x* + *1*) ∗ (*x* − *1*)
  by (*simp add*: *algebra-simps*)

end

class *comm-ring-1* = *comm-ring* + *zero-neq-one* + *comm-monoid-mult*
begin

subclass *ring-1* **..**
subclass *comm-semiring-1-cancel*
  by *standard* (*simp add*: *algebra-simps*)

lemma *dvd-minus-iff* [*simp*]: *x dvd* − *y* ⟷ *x dvd y*
proof
  assume *x dvd* − *y*
  then have *x dvd* − *1* ∗ − *y* **by** (*rule dvd-mult*)
  then show *x dvd y* **by** *simp*
next
  assume *x dvd y*
  then have *x dvd* − *1* ∗ *y* **by** (*rule dvd-mult*)
  then show *x dvd* − *y* **by** *simp*
qed

lemma *minus-dvd-iff* [*simp*]: − *x dvd y* ⟷ *x dvd y*
proof
  assume − *x dvd y*
  then obtain *k* **where** *y* = − *x* ∗ *k* **..**

then have $y = x * - k$ **by** *simp*
then show x *dvd* y **..**
next
assume x *dvd* y
then obtain k **where** $y = x * k$ **..**
then have $y = - x * - k$ **by** *simp*
then show $- x$ *dvd* y **..**
qed

lemma *dvd-diff* [*simp*]: x *dvd* $y \Longrightarrow x$ *dvd* $z \Longrightarrow x$ *dvd* $(y - z)$
using *dvd-add* [*of* x y $- z$] **by** *simp*

end

16.3 Towards integral domains

class *semiring-no-zero-divisors* = *semiring-0* +
assumes *no-zero-divisors*: $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$
begin

lemma *divisors-zero*:
assumes $a * b = 0$
shows $a = 0 \vee b = 0$
proof (*rule classical*)
assume $\neg$ *?thesis*
then have $a \neq 0$ **and** $b \neq 0$ **by** *auto*
with *no-zero-divisors* **have** $a * b \neq 0$ **by** *blast*
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *mult-eq-0-iff* [*simp*]: $a * b = 0 \longleftrightarrow a = 0 \vee b = 0$
proof (*cases* $a = 0 \vee b = 0$)
case *False*
then have $a \neq 0$ **and** $b \neq 0$ **by** *auto*
then show *?thesis* **using** *no-zero-divisors* **by** *simp*
next
case *True*
then show *?thesis* **by** *auto*
qed

end

class *semiring-1-no-zero-divisors* = *semiring-1* + *semiring-no-zero-divisors*

class *semiring-no-zero-divisors-cancel* = *semiring-no-zero-divisors* +
assumes *mult-cancel-right* [*simp*]: $a * c = b * c \longleftrightarrow c = 0 \vee a = b$
and *mult-cancel-left* [*simp*]: $c * a = c * b \longleftrightarrow c = 0 \vee a = b$
begin

lemma *mult-left-cancel*: $c \neq 0 \Longrightarrow c * a = c * b \longleftrightarrow a = b$
  by *simp*

lemma *mult-right-cancel*: $c \neq 0 \Longrightarrow a * c = b * c \longleftrightarrow a = b$
  by *simp*

end

class *ring-no-zero-divisors* = *ring* + *semiring-no-zero-divisors*
begin

subclass *semiring-no-zero-divisors-cancel*
proof
  fix *a b c*
  have $a * c = b * c \longleftrightarrow (a - b) * c = 0$
    by (*simp add*: *algebra-simps*)
  also have $\ldots \longleftrightarrow c = 0 \vee a = b$
    by *auto*
  finally show $a * c = b * c \longleftrightarrow c = 0 \vee a = b$ **.**
  have $c * a = c * b \longleftrightarrow c * (a - b) = 0$
    by (*simp add*: *algebra-simps*)
  also have $\ldots \longleftrightarrow c = 0 \vee a = b$
    by *auto*
  finally show $c * a = c * b \longleftrightarrow c = 0 \vee a = b$ **.**
qed

end

class *ring-1-no-zero-divisors* = *ring-1* + *ring-no-zero-divisors*
begin

subclass *semiring-1-no-zero-divisors* **..**

lemma *square-eq-1-iff*: $x * x = 1 \longleftrightarrow x = 1 \vee x = - 1$
proof −
  have $(x - 1) * (x + 1) = x * x - 1$
    by (*simp add*: *algebra-simps*)
  then have $x * x = 1 \longleftrightarrow (x - 1) * (x + 1) = 0$
    by *simp*
  then show *?thesis*
    by (*simp add*: *eq-neg-iff-add-eq-0*)
qed

lemma *mult-cancel-right1* [*simp*]: $c = b * c \longleftrightarrow c = 0 \vee b = 1$
  using *mult-cancel-right* [*of 1 c b*] **by** *auto*

lemma *mult-cancel-right2* [*simp*]: $a * c = c \longleftrightarrow c = 0 \vee a = 1$
  using *mult-cancel-right* [*of a c 1*] **by** *simp*

```
lemma mult-cancel-left1 [simp]: c = c ∗ b ⟷ c = 0 ∨ b = 1
  using mult-cancel-left [of c 1 b] by force

lemma mult-cancel-left2 [simp]: c ∗ a = c ⟷ c = 0 ∨ a = 1
  using mult-cancel-left [of c a 1] by simp

end

class semidom = comm-semiring-1-cancel + semiring-no-zero-divisors
begin

subclass semiring-1-no-zero-divisors ..

end

class idom = comm-ring-1 + semiring-no-zero-divisors
begin

subclass semidom ..

subclass ring-1-no-zero-divisors ..

lemma dvd-mult-cancel-right [simp]:
  a ∗ c dvd b ∗ c ⟷ c = 0 ∨ a dvd b
proof −
  have a ∗ c dvd b ∗ c ⟷ (∃ k. b ∗ c = (a ∗ k) ∗ c)
    by (auto simp add: ac-simps)
  also have (∃ k. b ∗ c = (a ∗ k) ∗ c) ⟷ c = 0 ∨ a dvd b
    by auto
  finally show ?thesis .
qed

lemma dvd-mult-cancel-left [simp]:
  c ∗ a dvd c ∗ b ⟷ c = 0 ∨ a dvd b
  using dvd-mult-cancel-right [of a c b] by (simp add: ac-simps)

lemma square-eq-iff: a ∗ a = b ∗ b ⟷ a = b ∨ a = − b
proof
  assume a ∗ a = b ∗ b
  then have (a − b) ∗ (a + b) = 0
    by (simp add: algebra-simps)
  then show a = b ∨ a = − b
    by (simp add: eq-neg-iff-add-eq-0)
next
  assume a = b ∨ a = − b
  then show a ∗ a = b ∗ b by auto
qed

lemma inj-mult-left [simp]: ‹inj ((∗) a) ⟷ a ≠ 0› (is ‹?P ⟷ ?Q›)
```

```
proof
  assume ?P
  show ?Q
  proof
    assume ‹a = 0›
    with ‹?P› have inj ((∗) 0)
      by simp
    moreover have 0 ∗ 0 = 0 ∗ 1
      by simp
    ultimately have 0 = 1
      by (rule injD)
    then show False
      by simp
  qed
next
  assume ?Q then show ?P
    by (auto intro: injI)
qed

end

class idom-abs-sgn = idom + abs + sgn +
  assumes sgn-mult-abs: sgn a ∗ |a| = a
    and sgn-sgn [simp]: sgn (sgn a) = sgn a
    and abs-abs [simp]: ||a|| = |a|
    and abs-0 [simp]: |0| = 0
    and sgn-0 [simp]: sgn 0 = 0
    and sgn-1 [simp]: sgn 1 = 1
    and sgn-minus-1: sgn (− 1) = − 1
    and sgn-mult: sgn (a ∗ b) = sgn a ∗ sgn b
begin

lemma sgn-eq-0-iff:
  sgn a = 0 ⟷ a = 0
proof −
  { assume sgn a = 0
    then have sgn a ∗ |a| = 0
      by simp
    then have a = 0
      by (simp add: sgn-mult-abs)
  } then show ?thesis
    by auto
qed

lemma abs-eq-0-iff:
  |a| = 0 ⟷ a = 0
proof −
  { assume |a| = 0
    then have sgn a ∗ |a| = 0
```

```
      by simp
    then have a = 0
      by (simp add: sgn-mult-abs)
  } then show ?thesis
    by auto
qed

lemma abs-mult-sgn:
  |a| * sgn a = a
  using sgn-mult-abs [of a] by (simp add: ac-simps)

lemma abs-1 [simp]:
  |1| = 1
  using sgn-mult-abs [of 1] by simp

lemma sgn-abs [simp]:
  |sgn a| = of-bool (a ≠ 0)
  using sgn-mult-abs [of sgn a] mult-cancel-left [of sgn a |sgn a| 1]
  by (auto simp add: sgn-eq-0-iff)

lemma abs-sgn [simp]:
  sgn |a| = of-bool (a ≠ 0)
  using sgn-mult-abs [of |a|] mult-cancel-right [of sgn |a| |a| 1]
  by (auto simp add: abs-eq-0-iff)

lemma abs-mult:
  |a * b| = |a| * |b|
proof (cases a = 0 ∨ b = 0)
  case True
  then show ?thesis
    by auto
next
  case False
  then have *: sgn (a * b) ≠ 0
    by (simp add: sgn-eq-0-iff)
  from abs-mult-sgn [of a * b] abs-mult-sgn [of a] abs-mult-sgn [of b]
  have |a * b| * sgn (a * b) = |a| * sgn a * |b| * sgn b
    by (simp add: ac-simps)
  then have |a * b| * sgn (a * b) = |a| * |b| * sgn (a * b)
    by (simp add: sgn-mult ac-simps)
  with * show ?thesis
    by simp
qed

lemma sgn-minus [simp]:
  sgn (− a) = − sgn a
proof −
  from sgn-minus-1 have sgn (− 1 * a) = − 1 * sgn a
    by (simp only: sgn-mult)
```

then show *?thesis*
by *simp*
qed

lemma *abs-minus* [*simp*]:
$|- a| = |a|$
proof –
have [*simp*]: $|- 1| = 1$
using *sgn-mult-abs* [*of* − *1*] **by** *simp*
then have $|- 1 * a| = 1 * |a|$
by (*simp only*: *abs-mult*)
then show *?thesis*
by *simp*
qed

end

16.4 (Partial) Division

class *divide* =
fixes *divide* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** ‹*div*› *70*)

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*divide*›, *SOME* ***typ*** ‹$'a \Rightarrow 'a \Rightarrow 'a$›)›

context *semiring*
begin

lemma [*field-simps*, *field-split-simps*]:
shows *distrib-left-NO-MATCH*: *NO-MATCH* (*x div y*) $a \Longrightarrow a * (b + c) = a * b + a * c$
and *distrib-right-NO-MATCH*: *NO-MATCH* (*x div y*) $c \Longrightarrow (a + b) * c = a * c + b * c$
by (*rule distrib-left distrib-right*)+

end

context *ring*
begin

lemma [*field-simps*, *field-split-simps*]:
shows *left-diff-distrib-NO-MATCH*: *NO-MATCH* (*x div y*) $c \Longrightarrow (a - b) * c = a * c - b * c$
and *right-diff-distrib-NO-MATCH*: *NO-MATCH* (*x div y*) $a \Longrightarrow a * (b - c) = a * b - a * c$
by (*rule left-diff-distrib right-diff-distrib*)+

end

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*divide*›, *SOME* ***typ*** ‹'*a*::*divide* ⇒ '*a* ⇒ '*a*›)›

```
class divide-trivial = zero + one + divide +
  assumes div-by-0 [simp]: ‹a div 0 = 0›
    and div-by-1 [simp]: ‹a div 1 = a›
    and div-0 [simp]: ‹0 div a = 0›
```

Algebraic classes with division

```
class semidom-divide = semidom + divide +
  assumes nonzero-mult-div-cancel-right [simp]: ‹b ≠ 0 ⟹ (a ∗ b) div b = a›
  assumes semidom-div-by-0: ‹a div 0 = 0›
begin

lemma nonzero-mult-div-cancel-left [simp]: ‹a ≠ 0 ⟹ (a ∗ b) div a = b›
  using nonzero-mult-div-cancel-right [of a b] by (simp add: ac-simps)

subclass divide-trivial
proof
  show [simp]: ‹a div 0 = 0› for a
    by (fact semidom-div-by-0)
  show ‹a div 1 = a› for a
    using nonzero-mult-div-cancel-right [of 1 a] by simp
  show ‹0 div a = 0› for a
    using nonzero-mult-div-cancel-right [of a 0] by (cases ‹a = 0›) simp-all
qed

subclass semiring-no-zero-divisors-cancel
proof
  show ∗: a ∗ c = b ∗ c ⟷ c = 0 ∨ a = b for a b c
  proof (cases c = 0)
    case True
    then show ?thesis by simp
  next
    case False
    have a = b if a ∗ c = b ∗ c
    proof −
      from that have a ∗ c div c = b ∗ c div c
        by simp
      with False show ?thesis
        by simp
    qed
    then show ?thesis by auto
  qed
  show c ∗ a = c ∗ b ⟷ c = 0 ∨ a = b for a b c
    using ∗ [of a c b] by (simp add: ac-simps)
qed

lemma div-self [simp]: a ≠ 0 ⟹ a div a = 1
```

```
  using nonzero-mult-div-cancel-left [of a 1] by simp

lemma dvd-div-eq-0-iff:
  assumes b dvd a
  shows a div b = 0 ⟷ a = 0
  using assms by (elim dvdE, cases b = 0) simp-all

lemma dvd-div-eq-cancel:
  a div c = b div c ⟹ c dvd a ⟹ c dvd b ⟹ a = b
  by (elim dvdE, cases c = 0) simp-all

lemma dvd-div-eq-iff:
  c dvd a ⟹ c dvd b ⟹ a div c = b div c ⟷ a = b
  by (elim dvdE, cases c = 0) simp-all

lemma inj-on-mult:
  inj-on ((∗) a) A if a ≠ 0
proof (rule inj-onI)
  fix b c
  assume a ∗ b = a ∗ c
  then have a ∗ b div a = a ∗ c div a
    by (simp only:)
  with that show b = c
    by simp
qed

end

class idom-divide = idom + semidom-divide
begin

lemma dvd-neg-div:
  assumes b dvd a
  shows − a div b = − (a div b)
proof (cases b = 0)
  case True
  then show ?thesis by simp
next
  case False
  from assms obtain c where a = b ∗ c ..
  then have − a div b = (b ∗ − c) div b
    by simp
  from False also have ... = − c
    by (rule nonzero-mult-div-cancel-left)
  with False ‹a = b ∗ c› show ?thesis
    by simp
qed

lemma dvd-div-neg:
```

```
  assumes b dvd a
  shows a div − b = − (a div b)
proof (cases b = 0)
  case True
  then show ?thesis by simp
next
  case False
  then have − b ≠ 0
    by simp
  from assms obtain c where a = b ∗ c ..
  then have a div − b = (− b ∗ − c) div − b
    by simp
  from ‹− b ≠ 0› also have . . . = − c
    by (rule nonzero-mult-div-cancel-left)
  with False ‹a = b ∗ c› show ?thesis
    by simp
qed

end

class algebraic-semidom = semidom-divide
begin
```

Class *algebraic-semidom* enriches a integral domain by notions from algebra, like units in a ring. It is a separate class to avoid spoiling fields with notions which are degenerated there.

```
lemma dvd-times-left-cancel-iff [simp]:
  assumes a ≠ 0
  shows a ∗ b dvd a ∗ c ⟷ b dvd c
    (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then obtain d where a ∗ c = a ∗ b ∗ d ..
  with assms have c = b ∗ d by (simp add: ac-simps)
  then show ?rhs ..
next
  assume ?rhs
  then obtain d where c = b ∗ d ..
  then have a ∗ c = a ∗ b ∗ d by (simp add: ac-simps)
  then show ?lhs ..
qed

lemma dvd-times-right-cancel-iff [simp]:
  assumes a ≠ 0
  shows b ∗ a dvd c ∗ a ⟷ b dvd c
  using dvd-times-left-cancel-iff [of a b c] assms by (simp add: ac-simps)

lemma div-dvd-iff-mult:
  assumes b ≠ 0 and b dvd a
```

shows *a div b dvd c* ⟷ *a dvd c* ∗ *b*
proof −
from ‹*b dvd a*› **obtain** *d* **where** *a* = *b* ∗ *d* **..**
with ‹*b* ≠ *0*› **show** *?thesis* **by** (*simp add*: *ac-simps*)
qed

lemma *dvd-div-iff-mult*:
assumes *c* ≠ *0* **and** *c dvd b*
shows *a dvd b div c* ⟷ *a* ∗ *c dvd b*
proof −
from ‹*c dvd b*› **obtain** *d* **where** *b* = *c* ∗ *d* **..**
with ‹*c* ≠ *0*› **show** *?thesis* **by** (*simp add*: *mult.commute* [*of a*])
qed

lemma *div-dvd-div* [*simp*]:
assumes *a dvd b* **and** *a dvd c*
shows *b div a dvd c div a* ⟷ *b dvd c*
proof (*cases a* = *0*)
case *True*
with *assms* **show** *?thesis* **by** *simp*
next
case *False*
moreover from *assms* **obtain** *k l* **where** *b* = *a* ∗ *k* **and** *c* = *a* ∗ *l*
by *blast*
ultimately show *?thesis* **by** *simp*
qed

lemma *div-add* [*simp*]:
assumes *c dvd a* **and** *c dvd b*
shows (*a* + *b*) *div c* = *a div c* + *b div c*
proof (*cases c* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
moreover from *assms* **obtain** *k l* **where** *a* = *c* ∗ *k* **and** *b* = *c* ∗ *l*
by *blast*
moreover have *c* ∗ *k* + *c* ∗ *l* = *c* ∗ (*k* + *l*)
by (*simp add*: *algebra-simps*)
ultimately show *?thesis*
by *simp*
qed

lemma *div-mult-div-if-dvd*:
assumes *b dvd a* **and** *d dvd c*
shows (*a div b*) ∗ (*c div d*) = (*a* ∗ *c*) *div* (*b* ∗ *d*)
proof (*cases b* = *0* ∨ *c* = *0*)
case *True*
with *assms* **show** *?thesis* **by** *auto*

next
  case *False*
  moreover from *assms* **obtain** $k\ l$ **where** $a = b * k$ **and** $c = d * l$
    by *blast*
  moreover have $b * k * (d * l)\ div\ (b * d) = (b * d) * (k * l)\ div\ (b * d)$
    by (*simp add*: *ac-simps*)
  ultimately show *?thesis* **by** *simp*
qed

lemma *dvd-div-eq-mult*:
  assumes $a \neq 0$ **and** $a\ dvd\ b$
  shows $b\ div\ a = c \longleftrightarrow b = c * a$
    (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
  assume *?rhs*
  then show *?lhs* **by** (*simp add*: *assms*)
next
  assume *?lhs*
  then have $b\ div\ a * a = c * a$ **by** *simp*
  moreover from *assms* **have** $b\ div\ a * a = b$
    by (*auto simp add*: *ac-simps*)
  ultimately show *?rhs* **by** *simp*
qed

lemma *dvd-div-mult-self* [*simp*]: $a\ dvd\ b \Longrightarrow b\ div\ a * a = b$
  by (*cases* $a = 0$) (*auto simp add*: *ac-simps*)

lemma *dvd-mult-div-cancel* [*simp*]: $a\ dvd\ b \Longrightarrow a * (b\ div\ a) = b$
  using *dvd-div-mult-self* [*of* $a\ b$] **by** (*simp add*: *ac-simps*)

lemma *div-mult-swap*:
  assumes $c\ dvd\ b$
  shows $a * (b\ div\ c) = (a * b)\ div\ c$
proof (*cases* $c = 0$)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  from *assms* **obtain** d **where** $b = c * d$ **..**
  moreover from *False* **have** $a * divide\ (d * c)\ c = ((a * d) * c)\ div\ c$
    by *simp*
  ultimately show *?thesis* **by** (*simp add*: *ac-simps*)
qed

lemma *dvd-div-mult*: $c\ dvd\ b \Longrightarrow b\ div\ c * a = (b * a)\ div\ c$
  using *div-mult-swap* [*of* $c\ b\ a$] **by** (*simp add*: *ac-simps*)

lemma *dvd-div-mult2-eq*:
  assumes $b * c\ dvd\ a$

```
  shows a div (b * c) = a div b div c
proof -
  from assms obtain k where a = b * c * k ..
  then show ?thesis
    by (cases b = 0 ∨ c = 0) (auto, simp add: ac-simps)
qed

lemma dvd-div-div-eq-mult:
  assumes a ≠ 0 c ≠ 0 and a dvd b c dvd d
  shows b div a = d div c ⟷ b * c = a * d
    (is ?lhs ⟷ ?rhs)
proof -
  from assms have a * c ≠ 0 by simp
  then have ?lhs ⟷ b div a * (a * c) = d div c * (a * c)
    by simp
  also have ... ⟷ (a * (b div a)) * c = (c * (d div c)) * a
    by (simp add: ac-simps)
  also have ... ⟷ (a * b div a) * c = (c * d div c) * a
    using assms by (simp add: div-mult-swap)
  also have ... ⟷ ?rhs
    using assms by (simp add: ac-simps)
  finally show ?thesis .
qed

lemma dvd-mult-imp-div:
  assumes a * c dvd b
  shows a dvd b div c
proof (cases c = 0)
  case True then show ?thesis by simp
next
  case False
  from ‹a * c dvd b› obtain d where b = a * c * d ..
  with False show ?thesis
    by (simp add: mult.commute [of a] mult.assoc)
qed

lemma div-div-eq-right:
  assumes c dvd b b dvd a
  shows  a div (b div c) = a div b * c
proof (cases c = 0 ∨ b = 0)
  case True
  then show ?thesis
    by auto
next
  case False
  from assms obtain r s where b = c * r and a = c * r * s
    by blast
  moreover with False have r ≠ 0
    by auto
```

```
  ultimately show ?thesis using False
    by simp (simp add: mult.commute [of - r] mult.assoc mult.commute [of c])
qed

lemma div-div-div-same:
  assumes d dvd b b dvd a
  shows   (a div d) div (b div d) = a div b
proof (cases b = 0 ∨ d = 0)
  case True
  with assms show ?thesis
    by auto
next
  case False
  from assms obtain r s
    where a = d * r * s and b = d * r
    by blast
  with False show ?thesis
    by simp (simp add: ac-simps)
qed
```

Units: invertible elements in a ring

```
abbreviation is-unit :: 'a ⇒ bool
  where is-unit a ≡ a dvd 1

lemma not-is-unit-0 [simp]: ¬ is-unit 0
  by simp

lemma unit-imp-dvd [dest]: is-unit b ⟹ b dvd a
  by (rule dvd-trans [of - 1]) simp-all

lemma unit-dvdE:
  assumes is-unit a
  obtains c where a ≠ 0 and b = a * c
proof -
  from assms have a dvd b by auto
  then obtain c where b = a * c ..
  moreover from assms have a ≠ 0 by auto
  ultimately show thesis using that by blast
qed

lemma dvd-unit-imp-unit: a dvd b ⟹ is-unit b ⟹ is-unit a
  by (rule dvd-trans)

lemma unit-div-1-unit [simp, intro]:
  assumes is-unit a
  shows is-unit (1 div a)
proof -
  from assms have 1 = 1 div a * a by simp
  then show is-unit (1 div a) by (rule dvdI)
```

qed

lemma *is-unitE* [*elim?*]:
  assumes *is-unit a*
  obtains *b* **where** $a \neq 0$ **and** $b \neq 0$
    and *is-unit b* **and** *1 div a* = *b* **and** *1 div b* = *a*
    and $a * b = 1$ **and** *c div a* = $c * b$
proof (*rule that*)
  define *b* **where** *b* = *1 div a*
  then show *1 div a* = *b* **by** *simp*
  from *assms b-def* **show** *is-unit b* **by** *simp*
  with *assms* **show** $a \neq 0$ **and** $b \neq 0$ **by** *auto*
  from *assms b-def* **show** $a * b = 1$ **by** *simp*
  then have $1 = a * b$ **..**
  with *b-def* ‹$b \neq 0$› **show** *1 div b* = *a* **by** *simp*
  from *assms* **have** *a dvd c* **..**
  then obtain *d* **where** $c = a * d$ **..**
  with ‹$a \neq 0$› ‹$a * b = 1$› **show** *c div a* = $c * b$
    by (*simp add*: *mult.assoc mult.left-commute* [*of a*])
qed

lemma *unit-prod* [*intro*]: *is-unit a* ⟹ *is-unit b* ⟹ *is-unit* ($a * b$)
  by (*subst mult-1-left* [*of 1*, *symmetric*]) (*rule mult-dvd-mono*)

lemma *is-unit-mult-iff*: *is-unit* ($a * b$) ⟷ *is-unit a* ∧ *is-unit b*
  by (*auto dest*: *dvd-mult-left dvd-mult-right*)

lemma *unit-div* [*intro*]: *is-unit a* ⟹ *is-unit b* ⟹ *is-unit* (*a div b*)
  by (*erule is-unitE* [*of b a*]) (*simp add*: *ac-simps unit-prod*)

lemma *mult-unit-dvd-iff*:
  assumes *is-unit b*
  shows $a * b$ *dvd c* ⟷ *a dvd c*
proof
  assume $a * b$ *dvd c*
  with *assms* **show** *a dvd c*
    by (*simp add*: *dvd-mult-left*)
next
  assume *a dvd c*
  then obtain *k* **where** $c = a * k$ **..**
  with *assms* **have** $c = (a * b) * (1 \; div \; b * k)$
    by (*simp add*: *mult-ac*)
  then show $a * b$ *dvd c* **by** (*rule dvdI*)
qed

lemma *mult-unit-dvd-iff'*: *is-unit a* ⟹ ($a * b$) *dvd c* ⟷ *b dvd c*
  using *mult-unit-dvd-iff* [*of a b c*] **by** (*simp add*: *ac-simps*)

lemma *dvd-mult-unit-iff*:

assumes *is-unit b*
shows *a dvd c ∗ b ⟷ a dvd c*
proof
assume *a dvd c ∗ b*
with *assms* **have** *c ∗ b dvd c ∗ (b ∗ (1 div b))*
by (*subst mult-assoc* [*symmetric*]) *simp*
also from *assms* **have** *b ∗ (1 div b) = 1*
by (*rule is-unitE*) *simp*
finally have *c ∗ b dvd c* **by** *simp*
with ‹*a dvd c ∗ b*› **show** *a dvd c* **by** (*rule dvd-trans*)
next
assume *a dvd c*
then show *a dvd c ∗ b* **by** *simp*
qed

lemma *dvd-mult-unit-iff ′*: *is-unit b ⟹ a dvd b ∗ c ⟷ a dvd c*
using *dvd-mult-unit-iff* [*of b a c*] **by** (*simp add*: *ac-simps*)

lemma *div-unit-dvd-iff*: *is-unit b ⟹ a div b dvd c ⟷ a dvd c*
by (*erule is-unitE* [*of - a*]) (*auto simp add*: *mult-unit-dvd-iff*)

lemma *dvd-div-unit-iff*: *is-unit b ⟹ a dvd c div b ⟷ a dvd c*
by (*erule is-unitE* [*of - c*]) (*simp add*: *dvd-mult-unit-iff*)

lemmas *unit-dvd-iff = mult-unit-dvd-iff mult-unit-dvd-iff ′*
dvd-mult-unit-iff dvd-mult-unit-iff ′
div-unit-dvd-iff dvd-div-unit-iff

lemma *unit-mult-div-div* [*simp*]: *is-unit a ⟹ b ∗ (1 div a) = b div a*
by (*erule is-unitE* [*of - b*]) *simp*

lemma *unit-div-mult-self* [*simp*]: *is-unit a ⟹ b div a ∗ a = b*
by (*rule dvd-div-mult-self*) *auto*

lemma *unit-div-1-div-1* [*simp*]: *is-unit a ⟹ 1 div (1 div a) = a*
by (*erule is-unitE*) *simp*

lemma *unit-div-mult-swap*: *is-unit c ⟹ a ∗ (b div c) = (a ∗ b) div c*
by (*erule unit-dvdE* [*of - b*]) (*simp add*: *mult.left-commute* [*of - c*])

lemma *unit-div-commute*: *is-unit b ⟹ (a div b) ∗ c = (a ∗ c) div b*
using *unit-div-mult-swap* [*of b c a*] **by** (*simp add*: *ac-simps*)

lemma *unit-eq-div1*: *is-unit b ⟹ a div b = c ⟷ a = c ∗ b*
by (*auto elim*: *is-unitE*)

lemma *unit-eq-div2*: *is-unit b ⟹ a = c div b ⟷ a ∗ b = c*
using *unit-eq-div1* [*of b c a*] **by** *auto*

lemma *unit-mult-left-cancel*: *is-unit a* $\Longrightarrow$ *a* $*$ *b* = *a* $*$ *c* $\longleftrightarrow$ *b* = *c*
  using *mult-cancel-left* [*of a b c*] **by** *auto*

lemma *unit-mult-right-cancel*: *is-unit a* $\Longrightarrow$ *b* $*$ *a* = *c* $*$ *a* $\longleftrightarrow$ *b* = *c*
  using *unit-mult-left-cancel* [*of a b c*] **by** (*auto simp add*: *ac-simps*)

lemma *unit-div-cancel*:
  assumes *is-unit a*
  shows *b div a* = *c div a* $\longleftrightarrow$ *b* = *c*
proof −
  from *assms* **have** *is-unit* (*1 div a*) **by** *simp*
  then have *b* $*$ (*1 div a*) = *c* $*$ (*1 div a*) $\longleftrightarrow$ *b* = *c*
    by (*rule unit-mult-right-cancel*)
  with *assms* **show** *?thesis* **by** *simp*
qed

lemma *is-unit-div-mult2-eq*:
  assumes *is-unit b* **and** *is-unit c*
  shows *a div* (*b* $*$ *c*) = *a div b div c*
proof −
  from *assms* **have** *is-unit* (*b* $*$ *c*)
    by (*simp add*: *unit-prod*)
  then have *b* $*$ *c dvd a*
    by (*rule unit-imp-dvd*)
  then show *?thesis*
    by (*rule dvd-div-mult2-eq*)
qed

lemma *is-unit-div-mult-cancel-left*:
  assumes *a* $\neq$ *0* **and** *is-unit b*
  shows *a div* (*a* $*$ *b*) = *1 div b*
proof −
  from *assms* **have** *a div* (*a* $*$ *b*) = *a div a div b*
    by (*simp add*: *mult-unit-dvd-iff dvd-div-mult2-eq*)
  with *assms* **show** *?thesis* **by** *simp*
qed

lemma *is-unit-div-mult-cancel-right*:
  assumes *a* $\neq$ *0* **and** *is-unit b*
  shows *a div* (*b* $*$ *a*) = *1 div b*
  using *assms is-unit-div-mult-cancel-left* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *unit-div-eq-0-iff*:
  assumes *is-unit b*
  shows *a div b* = *0* $\longleftrightarrow$ *a* = *0*
  using *assms* **by** (*simp add*: *dvd-div-eq-0-iff unit-imp-dvd*)

lemma *div-mult-unit2*:
  is-unit c $\Longrightarrow$ *b dvd a* $\Longrightarrow$ *a div* (*b* $*$ *c*) = *a div b div c*

by (*rule dvd-div-mult2-eq*) (*simp-all add*: *mult-unit-dvd-iff*)

Coprimality

definition *coprime* :: $'a \Rightarrow 'a \Rightarrow bool$
where *coprime a b* $\longleftrightarrow$ ($\forall$ *c. c dvd a* $\longrightarrow$ *c dvd b* $\longrightarrow$ *is-unit c*)

lemma *coprimeI*:
assumes $\bigwedge$*c. c dvd a* $\Longrightarrow$ *c dvd b* $\Longrightarrow$ *is-unit c*
shows *coprime a b*
using *assms* **by** (*auto simp*: *coprime-def*)

lemma *not-coprimeI*:
assumes *c dvd a* **and** *c dvd b* **and** ¬ *is-unit c*
shows ¬ *coprime a b*
using *assms* **by** (*auto simp*: *coprime-def*)

lemma *coprime-common-divisor*:
is-unit c **if** *coprime a b* **and** *c dvd a* **and** *c dvd b*
using *that* **by** (*auto simp*: *coprime-def*)

lemma *not-coprimeE*:
assumes ¬ *coprime a b*
obtains *c* **where** *c dvd a* **and** *c dvd b* **and** ¬ *is-unit c*
using *assms* **by** (*auto simp*: *coprime-def*)

lemma *coprime-imp-coprime*:
coprime a b **if** *coprime c d*
and $\bigwedge$*e.* ¬ *is-unit e* $\Longrightarrow$ *e dvd a* $\Longrightarrow$ *e dvd b* $\Longrightarrow$ *e dvd c*
and $\bigwedge$*e.* ¬ *is-unit e* $\Longrightarrow$ *e dvd a* $\Longrightarrow$ *e dvd b* $\Longrightarrow$ *e dvd d*
proof (*rule coprimeI*)
fix *e*
assume *e dvd a* **and** *e dvd b*
with *that* **have** *e dvd c* **and** *e dvd d*
by (*auto intro*: *dvd-trans*)
with ‹*coprime c d*› **show** *is-unit e*
by (*rule coprime-common-divisor*)
qed

lemma *coprime-divisors*:
coprime a b **if** *a dvd c b dvd d* **and** *coprime c d*
using ‹*coprime c d*› **proof** (*rule coprime-imp-coprime*)
fix *e*
assume *e dvd a* **then show** *e dvd c*
using ‹*a dvd c*› **by** (*rule dvd-trans*)
assume *e dvd b* **then show** *e dvd d*
using ‹*b dvd d*› **by** (*rule dvd-trans*)
qed

lemma *coprime-self* [*simp*]:

coprime a a ⟷ *is-unit a* (**is** *?P* ⟷ *?Q*)
proof
assume *?P*
then show *?Q*
by (*rule coprime-common-divisor*) *simp-all*
next
assume *?Q*
show *?P*
by (*rule coprimeI*) (*erule dvd-unit-imp-unit*, *rule* ‹*?Q*›)
qed

lemma *coprime-commute* [*ac-simps*]:
coprime b a ⟷ *coprime a b*
unfolding *coprime-def* **by** *auto*

lemma *is-unit-left-imp-coprime*:
coprime a b **if** *is-unit a*
proof (*rule coprimeI*)
fix *c*
assume *c dvd a*
with *that* **show** *is-unit c*
by (*auto intro*: *dvd-unit-imp-unit*)
qed

lemma *is-unit-right-imp-coprime*:
coprime a b **if** *is-unit b*
using *that is-unit-left-imp-coprime* [*of b a*] **by** (*simp add*: *ac-simps*)

lemma *coprime-1-left* [*simp*]:
coprime 1 a
by (*rule coprimeI*)

lemma *coprime-1-right* [*simp*]:
coprime a 1
by (*rule coprimeI*)

lemma *coprime-0-left-iff* [*simp*]:
coprime 0 a ⟷ *is-unit a*
by (*auto intro*: *coprimeI dvd-unit-imp-unit coprime-common-divisor* [*of 0 a a*])

lemma *coprime-0-right-iff* [*simp*]:
coprime a 0 ⟷ *is-unit a*
using *coprime-0-left-iff* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *coprime-mult-self-left-iff* [*simp*]:
coprime (*c* ∗ *a*) (*c* ∗ *b*) ⟷ *is-unit c* ∧ *coprime a b*
by (*auto intro*: *coprime-common-divisor*)
(*rule coprimeI*, *auto intro*: *coprime-common-divisor simp add*: *dvd-mult-unit-iff'*)+

lemma *coprime-mult-self-right-iff* [*simp*]:
coprime $(a * c)$ $(b * c)$ $\longleftrightarrow$ *is-unit* $c \wedge$ *coprime* a b
using *coprime-mult-self-left-iff* [*of* c a b] **by** (*simp add*: *ac-simps*)

lemma *coprime-absorb-left*:
assumes x *dvd* y
shows *coprime* x y $\longleftrightarrow$ *is-unit* x
using *assms coprime-common-divisor is-unit-left-imp-coprime* **by** *auto*

lemma *coprime-absorb-right*:
assumes y *dvd* x
shows *coprime* x y $\longleftrightarrow$ *is-unit* y
using *assms coprime-common-divisor is-unit-right-imp-coprime* **by** *auto*

end

class *unit-factor* =
fixes *unit-factor* :: $'a \Rightarrow 'a$

class *semidom-divide-unit-factor* = *semidom-divide* + *unit-factor* +
assumes *unit-factor-0* [*simp*]: *unit-factor* $0 = 0$
and *is-unit-unit-factor*: a *dvd* $1 \Longrightarrow$ *unit-factor* $a = a$
and *unit-factor-is-unit*: $a \neq 0 \Longrightarrow$ *unit-factor* a *dvd* 1
and *unit-factor-mult-unit-left*: a *dvd* $1 \Longrightarrow$ *unit-factor* $(a * b) = a *$ *unit-factor* b
— This fine-grained hierarchy will later on allow lean normalization of polynomials
begin

lemma *unit-factor-mult-unit-right*: a *dvd* $1 \Longrightarrow$ *unit-factor* $(b * a) =$ *unit-factor* $b * a$
using *unit-factor-mult-unit-left*[*of* a b] **by** (*simp add*: *mult-ac*)

lemmas [*simp*] = *unit-factor-mult-unit-left unit-factor-mult-unit-right*

end

class *normalization-semidom* = *algebraic-semidom* + *semidom-divide-unit-factor* +
fixes *normalize* :: $'a \Rightarrow 'a$
assumes *unit-factor-mult-normalize* [*simp*]: *unit-factor* $a *$ *normalize* $a = a$
and *normalize-0* [*simp*]: *normalize* $0 = 0$
begin

Class *normalization-semidom* cultivates the idea that each integral domain can be split into equivalence classes whose representants are associated, i.e. divide each other. *normalize* specifies a canonical representant for each equivalence class. The rationale behind this is that it is easier to reason about equality than equivalences, hence we prefer to think about equality of normalized values rather than associated elements.

declare *unit-factor-is-unit* [*iff*]

lemma *unit-factor-dvd* [*simp*]: $a \neq 0 \Longrightarrow$ *unit-factor a dvd b*
by (*rule unit-imp-dvd*) *simp*

lemma *unit-factor-self* [*simp*]: *unit-factor a dvd a*
by (*cases a* = *0*) *simp-all*

lemma *normalize-mult-unit-factor* [*simp*]: *normalize a* ∗ *unit-factor a* = *a*
using *unit-factor-mult-normalize* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *normalize-eq-0-iff* [*simp*]: *normalize a* = *0* ⟷ *a* = *0*
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
moreover have *unit-factor a* ∗ *normalize a* = *a* **by** *simp*
ultimately show *?rhs* **by** *simp*
next
assume *?rhs*
then show *?lhs* **by** *simp*
qed

lemma *unit-factor-eq-0-iff* [*simp*]: *unit-factor a* = *0* ⟷ *a* = *0*
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
moreover have *unit-factor a* ∗ *normalize a* = *a* **by** *simp*
ultimately show *?rhs* **by** *simp*
next
assume *?rhs*
then show *?lhs* **by** *simp*
qed

lemma *div-unit-factor* [*simp*]: *a div unit-factor a* = *normalize a*
proof (*cases a* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have *unit-factor a* ≠ *0*
by *simp*
with *nonzero-mult-div-cancel-left*
have *unit-factor a* ∗ *normalize a div unit-factor a* = *normalize a*
by *blast*
then show *?thesis* **by** *simp*
qed

lemma *normalize-div* [*simp*]: *normalize a div a* = *1 div unit-factor a*
proof (*cases a* = *0*)

```
  case True
  then show ?thesis by simp
next
  case False
  have normalize a div a = normalize a div (unit-factor a * normalize a)
    by simp
  also have ... = 1 div unit-factor a
    using False by (subst is-unit-div-mult-cancel-right) simp-all
  finally show ?thesis .
qed

lemma is-unit-normalize:
  assumes is-unit a
  shows normalize a = 1
proof -
  from assms have unit-factor a = a
    by (rule is-unit-unit-factor)
  moreover from assms have a ≠ 0
    by auto
  moreover have normalize a = a div unit-factor a
    by simp
  ultimately show ?thesis
    by simp
qed

lemma unit-factor-1 [simp]: unit-factor 1 = 1
  by (rule is-unit-unit-factor) simp

lemma normalize-1 [simp]: normalize 1 = 1
  by (rule is-unit-normalize) simp

lemma normalize-1-iff: normalize a = 1 ⟷ is-unit a
  (is ?lhs ⟷ ?rhs)
proof
  assume ?rhs
  then show ?lhs by (rule is-unit-normalize)
next
  assume ?lhs
  then have unit-factor a * normalize a = unit-factor a * 1
    by simp
  then have unit-factor a = a
    by simp
  moreover
  from ‹?lhs› have a ≠ 0 by auto
  then have is-unit (unit-factor a) by simp
  ultimately show ?rhs by simp
qed

lemma div-normalize [simp]: a div normalize a = unit-factor a
```

proof (*cases a = 0*)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  then have *normalize a ≠ 0* **by** *simp*
  with *nonzero-mult-div-cancel-right*
  have *unit-factor a ∗ normalize a div normalize a = unit-factor a* **by** *blast*
  then show *?thesis* **by** *simp*
qed

lemma *mult-one-div-unit-factor* [*simp*]: *a ∗ (1 div unit-factor b) = a div unit-factor b*
  by (*cases b = 0*) *simp-all*

lemma *inv-unit-factor-eq-0-iff* [*simp*]:
  1 div unit-factor a = 0 ⟷ a = 0
  (**is** *?lhs ⟷ ?rhs*)
proof
  assume *?lhs*
  then have *a ∗ (1 div unit-factor a) = a ∗ 0*
    by *simp*
  then show *?rhs*
    by *simp*
next
  assume *?rhs*
  then show *?lhs* **by** *simp*
qed

lemma *unit-factor-idem* [*simp*]: *unit-factor (unit-factor a) = unit-factor a*
  by (*cases a = 0*) (*auto intro*: *is-unit-unit-factor*)

lemma *normalize-unit-factor* [*simp*]: *a ≠ 0 ⟹ normalize (unit-factor a) = 1*
  by (*rule is-unit-normalize*) *simp*

lemma *normalize-mult-unit-left* [*simp*]:
  assumes *a dvd 1*
  shows *normalize (a ∗ b) = normalize b*
proof (*cases b = 0*)
  case *False*
  have *a ∗ unit-factor b ∗ normalize (a ∗ b) = unit-factor (a ∗ b) ∗ normalize (a ∗ b)*
    using *assms* **by** (*subst unit-factor-mult-unit-left*) *auto*
  also have *. . . = a ∗ b* **by** *simp*
  also have *b = unit-factor b ∗ normalize b* **by** *simp*
  hence *a ∗ b = a ∗ unit-factor b ∗ normalize b*
    by (*simp only*: *mult-ac*)
  finally show *?thesis*
    using *assms False* **by** *auto*

qed *auto*

lemma *normalize-mult-unit-right* [*simp*]:
assumes *b dvd 1*
shows *normalize* (*a* ∗ *b*) = *normalize a*
using *assms* **by** (*subst mult.commute*) *auto*

lemma *normalize-idem* [*simp*]: *normalize* (*normalize a*) = *normalize a*
proof (*cases a* = *0*)
case *False*
have *normalize a* = *normalize* (*unit-factor a* ∗ *normalize a*)
by *simp*
also from *False* **have** ... = *normalize* (*normalize a*)
by (*subst normalize-mult-unit-left*) *auto*
finally show *?thesis* **..**
qed *auto*

lemma *unit-factor-normalize* [*simp*]:
assumes *a* ≠ *0*
shows *unit-factor* (*normalize a*) = *1*
proof −
from *assms* **have** ∗: *normalize a* ≠ *0*
by *simp*
have *unit-factor* (*normalize a*) ∗ *normalize* (*normalize a*) = *normalize a*
by (*simp only*: *unit-factor-mult-normalize*)
then have *unit-factor* (*normalize a*) ∗ *normalize a* = *normalize a*
by *simp*
with ∗ **have** *unit-factor* (*normalize a*) ∗ *normalize a div normalize a* = *normalize a div normalize a*
by *simp*
with ∗ **show** *?thesis*
by *simp*
qed

lemma *normalize-dvd-iff* [*simp*]: *normalize a dvd b* ⟷ *a dvd b*
proof −
have *normalize a dvd b* ⟷ *unit-factor a* ∗ *normalize a dvd b*
using *mult-unit-dvd-iff* [*of unit-factor a normalize a b*]
by (*cases a* = *0*) *simp-all*
then show *?thesis* **by** *simp*
qed

lemma *dvd-normalize-iff* [*simp*]: *a dvd normalize b* ⟷ *a dvd b*
proof −
have *a dvd normalize b* ⟷ *a dvd normalize b* ∗ *unit-factor b*
using *dvd-mult-unit-iff* [*of unit-factor b a normalize b*]
by (*cases b* = *0*) *simp-all*
then show *?thesis* **by** *simp*
qed

lemma *normalize-idem-imp-unit-factor-eq*:
  assumes *normalize a* = *a*
  shows *unit-factor a* = *of-bool* (*a* ≠ *0*)
proof (*cases a* = *0*)
  case *True*
  then show *?thesis*
    by *simp*
next
  case *False*
  then show *?thesis*
    using *assms unit-factor-normalize* [*of a*] **by** *simp*
qed

lemma *normalize-idem-imp-is-unit-iff*:
  assumes *normalize a* = *a*
  shows *is-unit a* ⟷ *a* = *1*
  using *assms* **by** (*cases a* = *0*) (*auto dest*: *is-unit-normalize*)

lemma *coprime-normalize-left-iff* [*simp*]:
  coprime (*normalize a*) *b* ⟷ *coprime a b*
  by (*rule iffI*; *rule coprimeI*) (*auto intro*: *coprime-common-divisor*)

lemma *coprime-normalize-right-iff* [*simp*]:
  coprime a (*normalize b*) ⟷ *coprime a b*
  using *coprime-normalize-left-iff* [*of b a*] **by** (*simp add*: *ac-simps*)

We avoid an explicit definition of associated elements but prefer explicit normalisation instead. In theory we could define an abbreviation like *associated a b* = (*normalize a* = *normalize b*) but this is counterproductive without suggestive infix syntax, which we do not want to sacrifice for this purpose here.

lemma *associatedI*:
  assumes *a dvd b* **and** *b dvd a*
  shows *normalize a* = *normalize b*
proof (*cases a* = *0* ∨ *b* = *0*)
  case *True*
  with *assms* **show** *?thesis* **by** *auto*
next
  case *False*
  from ‹*a dvd b*› **obtain** *c* **where** *b*: *b* = *a* ∗ *c* **..**
  moreover from ‹*b dvd a*› **obtain** *d* **where** *a*: *a* = *b* ∗ *d* **..**
  ultimately have *b* ∗ *1* = *b* ∗ (*c* ∗ *d*)
    by (*simp add*: *ac-simps*)
  with *False* **have** *1* = *c* ∗ *d*
    unfolding *mult-cancel-left* **by** *simp*
  then have *is-unit c* **and** *is-unit d*
    by *auto*
  with *a b* **show** *?thesis*

```
    by (simp add: is-unit-normalize)
qed

lemma associatedD1: normalize a = normalize b ⟹ a dvd b
  using dvd-normalize-iff [of - b, symmetric] normalize-dvd-iff [of a -, symmetric]
  by simp

lemma associatedD2: normalize a = normalize b ⟹ b dvd a
  using dvd-normalize-iff [of - a, symmetric] normalize-dvd-iff [of b -, symmetric]
  by simp

lemma associated-unit: normalize a = normalize b ⟹ is-unit a ⟹ is-unit b
  using dvd-unit-imp-unit by (auto dest!: associatedD1 associatedD2)

lemma associated-iff-dvd: normalize a = normalize b ⟷ a dvd b ∧ b dvd a
  (is ?lhs ⟷ ?rhs)
proof
  assume ?rhs
  then show ?lhs by (auto intro!: associatedI)
next
  assume ?lhs
  then have unit-factor a ∗ normalize a = unit-factor a ∗ normalize b
    by simp
  then have ∗: normalize b ∗ unit-factor a = a
    by (simp add: ac-simps)
  show ?rhs
  proof (cases a = 0 ∨ b = 0)
    case True
    with ‹?lhs› show ?thesis by auto
  next
    case False
    then have b dvd normalize b ∗ unit-factor a and normalize b ∗ unit-factor a
dvd b
      by (simp-all add: mult-unit-dvd-iff dvd-mult-unit-iff)
    with ∗ show ?thesis by simp
  qed
qed

lemma associated-eqI:
  assumes a dvd b and b dvd a
  assumes normalize a = a and normalize b = b
  shows a = b
proof −
  from assms have normalize a = normalize b
    unfolding associated-iff-dvd by simp
  with ‹normalize a = a› have a = normalize b
    by simp
  with ‹normalize b = b› show a = b
    by simp
```

qed

lemma *normalize-unit-factor-eqI*:
  assumes *normalize a = normalize b*
    and *unit-factor a = unit-factor b*
  shows *a = b*
proof –
  from *assms* **have** *unit-factor a ∗ normalize a = unit-factor b ∗ normalize b*
    by *simp*
  then show *?thesis*
    by *simp*
qed

lemma *normalize-mult-normalize-left* [*simp*]: *normalize* (*normalize a ∗ b*) = *normalize* (*a ∗ b*)
  by (*rule associated-eqI*) (*auto intro*!: *mult-dvd-mono*)

lemma *normalize-mult-normalize-right* [*simp*]: *normalize* (*a ∗ normalize b*) = *normalize* (*a ∗ b*)
  by (*rule associated-eqI*) (*auto intro*!: *mult-dvd-mono*)

end

class *normalization-semidom-multiplicative* = *normalization-semidom* +
  assumes *unit-factor-mult*: *unit-factor* (*a ∗ b*) = *unit-factor a ∗ unit-factor b*
begin

lemma *normalize-mult*: *normalize* (*a ∗ b*) = *normalize a ∗ normalize b*
proof (*cases a = 0 ∨ b = 0*)
  case *True*
  then show *?thesis* **by** *auto*
next
  case *False*
  have *unit-factor* (*a ∗ b*) ∗ *normalize* (*a ∗ b*) = *a ∗ b*
    by (*rule unit-factor-mult-normalize*)
  then have *normalize* (*a ∗ b*) = *a ∗ b div unit-factor* (*a ∗ b*)
    by *simp*
  also have . . . = *a ∗ b div unit-factor* (*b ∗ a*)
    by (*simp add*: *ac-simps*)
  also have . . . = *a ∗ b div unit-factor b div unit-factor a*
    using *False* **by** (*simp add*: *unit-factor-mult is-unit-div-mult2-eq* [*symmetric*])
  also have . . . = *a ∗* (*b div unit-factor b*) *div unit-factor a*
    using *False* **by** (*subst unit-div-mult-swap*) *simp-all*
  also have . . . = *normalize a ∗ normalize b*
    using *False*
   by (*simp add*: *mult.commute* [*of a*] *mult.commute* [*of normalize a*] *unit-div-mult-swap* [*symmetric*])
  finally show *?thesis* **.**

qed

lemma *dvd-unit-factor-div*:
assumes *b dvd a*
shows *unit-factor* (*a div b*) = *unit-factor a div unit-factor b*
proof −
from *assms* **have** *a* = *a div b* ∗ *b*
by *simp*
then have *unit-factor a* = *unit-factor* (*a div b* ∗ *b*)
by *simp*
then show *?thesis*
by (*cases b* = *0*) (*simp-all add*: *unit-factor-mult*)
qed

lemma *dvd-normalize-div*:
assumes *b dvd a*
shows *normalize* (*a div b*) = *normalize a div normalize b*
proof −
from *assms* **have** *a* = *a div b* ∗ *b*
by *simp*
then have *normalize a* = *normalize* (*a div b* ∗ *b*)
by *simp*
then show *?thesis*
by (*cases b* = *0*) (*simp-all add*: *normalize-mult*)
qed

end

Syntactic division remainder operator

class *modulo* = *dvd* + *divide* +
fixes *modulo* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ‹*mod*› *70*)

Arbitrary quotient and remainder partitions

class *semiring-modulo* = *comm-semiring-1-cancel* + *divide* + *modulo* +
assumes *div-mult-mod-eq*: ‹*a div b* ∗ *b* + *a mod b* = *a*›
begin

lemma *mod-div-decomp*:
fixes *a b*
obtains *q r* **where** *q* = *a div b* **and** *r* = *a mod b*
and *a* = *q* ∗ *b* + *r*
proof −
from *div-mult-mod-eq* **have** *a* = *a div b* ∗ *b* + *a mod b* **by** *simp*
moreover have *a div b* = *a div b* **..**
moreover have *a mod b* = *a mod b* **..**
note *that* **ultimately show** *thesis* **by** *blast*
qed

lemma *mult-div-mod-eq*: *b* ∗ (*a div b*) + *a mod b* = *a*

using *div-mult-mod-eq* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *mod-div-mult-eq*: *a mod b* + *a div b* ∗ *b* = *a*
using *div-mult-mod-eq* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *mod-mult-div-eq*: *a mod b* + *b* ∗ (*a div b*) = *a*
using *div-mult-mod-eq* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *minus-div-mult-eq-mod*: *a* − *a div b* ∗ *b* = *a mod b*
by (*rule add-implies-diff* [*symmetric*]) (*fact mod-div-mult-eq*)

lemma *minus-mult-div-eq-mod*: *a* − *b* ∗ (*a div b*) = *a mod b*
by (*rule add-implies-diff* [*symmetric*]) (*fact mod-mult-div-eq*)

lemma *minus-mod-eq-div-mult*: *a* − *a mod b* = *a div b* ∗ *b*
by (*rule add-implies-diff* [*symmetric*]) (*fact div-mult-mod-eq*)

lemma *minus-mod-eq-mult-div*: *a* − *a mod b* = *b* ∗ (*a div b*)
by (*rule add-implies-diff* [*symmetric*]) (*fact mult-div-mod-eq*)

lemma *mod-0-imp-dvd* [*dest*!]:
b dvd a **if** *a mod b* = *0*
proof −
have *b dvd* (*a div b*) ∗ *b* **by** *simp*
also have (*a div b*) ∗ *b* = *a*
using *div-mult-mod-eq* [*of a b*] **by** (*simp add*: *that*)
finally show *?thesis* **.**
qed

lemma [*nitpick-unfold*]:
a mod b = *a* − *a div b* ∗ *b*
by (*fact minus-div-mult-eq-mod* [*symmetric*])

end

class *semiring-modulo-trivial* = *semiring-modulo* + *divide-trivial*
begin

lemma *mod-0* [*simp*]:
‹*0 mod a* = *0*›
using *div-mult-mod-eq* [*of 0 a*] **by** *simp*

lemma *mod-by-0* [*simp*]:
‹*a mod 0* = *a*›
using *div-mult-mod-eq* [*of a 0*] **by** *simp*

lemma *mod-by-1* [*simp*]:
‹*a mod 1* = *0*›
proof −

```
  have ‹a + a mod 1 = a›
    using div-mult-mod-eq [of a 1] by simp
  then have ‹a + a mod 1 = a + 0›
    by simp
  then show ?thesis
    by (rule add-left-imp-eq)
qed

end
```

16.5 Quotient and remainder in integral domains

```
class semidom-modulo = algebraic-semidom + semiring-modulo
begin

subclass semiring-modulo-trivial ..

lemma mod-self [simp]:
  a mod a = 0
  using div-mult-mod-eq [of a a] by simp

lemma dvd-imp-mod-0 [simp]:
  b mod a = 0 if a dvd b
  using that minus-div-mult-eq-mod [of b a] by simp

lemma mod-eq-0-iff-dvd:
  a mod b = 0 ⟷ b dvd a
  by (auto intro: mod-0-imp-dvd)

lemma dvd-eq-mod-eq-0 [nitpick-unfold, code]:
  a dvd b ⟷ b mod a = 0
  by (simp add: mod-eq-0-iff-dvd)

lemma dvd-mod-iff:
  assumes c dvd b
  shows c dvd a mod b ⟷ c dvd a
proof −
  from assms have (c dvd a mod b) ⟷ (c dvd ((a div b) ∗ b + a mod b))
    by (simp add: dvd-add-right-iff)
  also have (a div b) ∗ b + a mod b = a
    using div-mult-mod-eq [of a b] by simp
  finally show ?thesis .
qed

lemma dvd-mod-imp-dvd:
  assumes c dvd a mod b and c dvd b
  shows c dvd a
  using assms dvd-mod-iff [of c b a] by simp
```

lemma *dvd-minus-mod* [*simp*]:
  b dvd a − a mod b
  by (*simp add*: *minus-mod-eq-div-mult*)

lemma *cancel-div-mod-rules*:
  ((*a div b*) ∗ *b* + *a mod b*) + *c* = *a* + *c*
  (*b* ∗ (*a div b*) + *a mod b*) + *c* = *a* + *c*
  by (*simp-all add*: *div-mult-mod-eq mult-div-mod-eq*)

end

class *idom-modulo* = *idom* + *semidom-modulo*
begin

subclass *idom-divide* **..**

lemma *div-diff* [*simp*]:
  c dvd a ⟹ *c dvd b* ⟹ (*a* − *b*) *div c* = *a div c* − *b div c*
  using *div-add* [*of* - - − *b*] **by** (*simp add*: *dvd-neg-div*)

end

16.6 Interlude: basic tool support for algebraic and arithmetic calculations

named-theorems *arith arith facts* −− *only ground formulas*
ML-file ‹*Tools/arith-data.ML*›

ML-file ‹~~/*src*/*Provers*/*Arith*/*cancel-div-mod.ML*›

ML ‹
structure Cancel-Div-Mod-Ring = *Cancel-Div-Mod*
(
  val div-name = ***const-name*** ‹*divide*›;
  val mod-name = ***const-name*** ‹*modulo*›;
  val mk-binop = *HOLogic.mk-binop*;
  val mk-sum = *Arith-Data.mk-sum*;
  val dest-sum = *Arith-Data.dest-sum*;

  val div-mod-eqs = *map mk-meta-eq* @{*thms cancel-div-mod-rules*};

  val prove-eq-sums = *Arith-Data.prove-conv2 all-tac* (*Arith-Data.simp-all-tac*
    @{*thms diff-conv-add-uminus add-0-left add-0-right ac-simps*})
)
›

simproc-setup *cancel-div-mod-int* ((*a*::′*a*::*semidom-modulo*) + *b*) =
  ‹*K Cancel-Div-Mod-Ring.proc*›

16.7 Ordered semirings and rings

The theory of partially ordered rings is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society, 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press, 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer

class *ordered-semiring* = *semiring* + *ordered-comm-monoid-add* +
  assumes *mult-left-mono*: $a \le b \Longrightarrow 0 \le c \Longrightarrow c * a \le c * b$
  assumes *mult-right-mono*: $a \le b \Longrightarrow 0 \le c \Longrightarrow a * c \le b * c$
begin

lemma *mult-mono*: $a \le b \Longrightarrow c \le d \Longrightarrow 0 \le b \Longrightarrow 0 \le c \Longrightarrow a * c \le b * d$
  apply (*erule* (*1*) *mult-right-mono* [*THEN order-trans*])
  apply (*erule* (*1*) *mult-left-mono*)
  done

lemma *mult-mono'*: $a \le b \Longrightarrow c \le d \Longrightarrow 0 \le a \Longrightarrow 0 \le c \Longrightarrow a * c \le b * d$
  by (*rule mult-mono*) (*fast intro*: *order-trans*)+

end

lemma *mono-mult*:
  fixes *a* :: *'a*::*ordered-semiring*
  shows $a \ge 0 \Longrightarrow$ *mono* ((∗) *a*)
  by (*simp add*: *mono-def mult-left-mono*)

class *ordered-semiring-0* = *semiring-0* + *ordered-semiring*
begin

lemma *mult-nonneg-nonneg* [*simp*]: $0 \le a \Longrightarrow 0 \le b \Longrightarrow 0 \le a * b$
  using *mult-left-mono* [*of 0 b a*] **by** *simp*

lemma *mult-nonneg-nonpos*: $0 \le a \Longrightarrow b \le 0 \Longrightarrow a * b \le 0$
  using *mult-left-mono* [*of b 0 a*] **by** *simp*

lemma *mult-nonpos-nonneg*: $a \le 0 \Longrightarrow 0 \le b \Longrightarrow a * b \le 0$
  using *mult-right-mono* [*of a 0 b*] **by** *simp*

Legacy – use *mult-nonpos-nonneg*.

lemma *mult-nonneg-nonpos2*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow b * a \leq 0$
by (*drule mult-right-mono* [*of b 0*]) *auto*

lemma *split-mult-neg-le*: $(0 \leq a \wedge b \leq 0) \vee (a \leq 0 \wedge 0 \leq b) \Longrightarrow a * b \leq 0$
by (*auto simp add*: *mult-nonneg-nonpos mult-nonneg-nonpos2*)

end

class *ordered-cancel-semiring* = *ordered-semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* **..**

subclass *ordered-semiring-0* **..**

end

class *linordered-semiring* = *ordered-semiring* + *linordered-cancel-ab-semigroup-add*
begin

subclass *ordered-cancel-semiring* **..**

subclass *ordered-cancel-comm-monoid-add* **..**

subclass *ordered-ab-semigroup-monoid-add-imp-le* **..**

lemma *mult-left-less-imp-less*: $c * a < c * b \Longrightarrow 0 \leq c \Longrightarrow a < b$
by (*force simp add*: *mult-left-mono not-le* [*symmetric*])

lemma *mult-right-less-imp-less*: $a * c < b * c \Longrightarrow 0 \leq c \Longrightarrow a < b$
by (*force simp add*: *mult-right-mono not-le* [*symmetric*])

end

class *zero-less-one* = *order* + *zero* + *one* +
assumes *zero-less-one* [*simp*]: $0 < 1$
begin

subclass *zero-neq-one*
by *standard* (*simp add*: *less-imp-neq*)

lemma *zero-le-one* [*simp*]:
‹$0 \leq 1$› **by** (*rule less-imp-le*) *simp*

end

class *linordered-semiring-1* = *linordered-semiring* + *semiring-1* + *zero-less-one*
begin

lemma *convex-bound-le*:
assumes $x \leq a$ $y \leq a$ $0 \leq u$ $0 \leq v$ $u + v = 1$
shows $u * x + v * y \leq a$
proof−
from *assms* **have** $u * x + v * y \leq u * a + v * a$
by (*simp add*: *add-mono mult-left-mono*)
with *assms* **show** *?thesis*
unfolding *distrib-right*[*symmetric*] **by** *simp*
qed

end

class *linordered-semiring-strict* = *semiring* + *comm-monoid-add* + *linordered-cancel-ab-semigroup-add* +
assumes *mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
assumes *mult-strict-right-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow a * c < b * c$
begin

subclass *semiring-0-cancel* **..**

subclass *linordered-semiring*
proof
fix a b c :: $'a$
assume ∗: $a \leq b$ $0 \leq c$
then show $c * a \leq c * b$
unfolding *le-less*
using *mult-strict-left-mono* **by** (*cases* $c = 0$) *auto*
from ∗ **show** $a * c \leq b * c$
unfolding *le-less*
using *mult-strict-right-mono* **by** (*cases* $c = 0$) *auto*
qed

lemma *mult-left-le-imp-le*: $c * a \leq c * b \Longrightarrow 0 < c \Longrightarrow a \leq b$
by (*auto simp add*: *mult-strict-left-mono -not-less* [*symmetric*])

lemma *mult-right-le-imp-le*: $a * c \leq b * c \Longrightarrow 0 < c \Longrightarrow a \leq b$
by (*auto simp add*: *mult-strict-right-mono not-less* [*symmetric*])

lemma *mult-pos-pos*[*simp*]: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a * b$
using *mult-strict-left-mono* [*of 0 b a*] **by** *simp*

lemma *mult-pos-neg*: $0 < a \Longrightarrow b < 0 \Longrightarrow a * b < 0$
using *mult-strict-left-mono* [*of b 0 a*] **by** *simp*

lemma *mult-neg-pos*: $a < 0 \Longrightarrow 0 < b \Longrightarrow a * b < 0$
using *mult-strict-right-mono* [*of a 0 b*] **by** *simp*

Legacy – use *mult-neg-pos*.

lemma *mult-pos-neg2*: $0 < a \Longrightarrow b < 0 \Longrightarrow b * a < 0$

by (*drule mult-strict-right-mono* [*of b 0*]) *auto*

lemma *zero-less-mult-pos*:
assumes $0 < a * b$ $0 < a$ **shows** $0 < b$
proof (*cases* $b \leq 0$)
case *True*
then show *?thesis*
using *assms* **by** (*auto simp*: *le-less dest*: *less-not-sym mult-pos-neg* [*of a b*])
qed (*auto simp add*: *le-less not-less*)

lemma *zero-less-mult-pos2*:
assumes $0 < b * a$ $0 < a$ **shows** $0 < b$
proof (*cases* $b \leq 0$)
case *True*
then show *?thesis*
using *assms* **by** (*auto simp*: *le-less dest*: *less-not-sym mult-pos-neg2* [*of a b*])
qed (*auto simp add*: *le-less not-less*)

Strict monotonicity in both arguments

lemma *mult-strict-mono*:
assumes $a < b$ $c < d$ $0 < b$ $0 \leq c$
shows $a * c < b * d$
proof (*cases* $c = 0$)
case *True*
with *assms* **show** *?thesis*
by *simp*
next
case *False*
with *assms* **have** $a*c < b*c$
by (*simp add*: *mult-strict-right-mono* [*OF* ‹$a < b$›])
also have $\ldots < b*d$
by (*simp add*: *assms mult-strict-left-mono*)
finally show *?thesis* **.**
qed

This weaker variant has more natural premises

lemma *mult-strict-mono'*:
assumes $a < b$ **and** $c < d$ **and** $0 \leq a$ **and** $0 \leq c$
shows $a * c < b * d$
using *assms* **by** (*auto simp add*: *mult-strict-mono*)

lemma *mult-less-le-imp-less*:
assumes $a < b$ **and** $c \leq d$ **and** $0 \leq a$ **and** $0 < c$
shows $a * c < b * d$
proof −
have $a * c < b * c$
by (*simp add*: *assms mult-strict-right-mono*)
also have ... $\leq b * d$

by (*intro mult-left-mono*) (*use assms* **in** *auto*)
finally show *?thesis* **.**
qed

lemma *mult-le-less-imp-less*:
assumes $a \leq b$ **and** $c < d$ **and** $0 < a$ **and** $0 \leq c$
shows $a * c < b * d$
proof −
have $a * c \leq b * c$
by (*simp add*: *assms mult-right-mono*)
also have ... $< b * d$
by (*intro mult-strict-left-mono*) (*use assms* **in** *auto*)
finally show *?thesis* **.**
qed

end

class *linordered-semiring-1-strict* = *linordered-semiring-strict* + *semiring-1* + *zero-less-one*
begin

subclass *linordered-semiring-1* **..**

lemma *convex-bound-lt*:
assumes $x < a$ $y < a$ $0 \leq u$ $0 \leq v$ $u + v = 1$
shows $u * x + v * y < a$
proof −
from *assms* **have** $u * x + v * y < u * a + v * a$
by (*cases* $u = 0$) (*auto intro*!: *add-less-le-mono mult-strict-left-mono mult-left-mono*)
with *assms* **show** *?thesis*
unfolding *distrib-right*[*symmetric*] **by** *simp*
qed

end

class *ordered-comm-semiring* = *comm-semiring-0* + *ordered-ab-semigroup-add* +
assumes *comm-mult-left-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
begin

subclass *ordered-semiring*
proof
fix a b c :: $'a$
assume $a \leq b$ $0 \leq c$
then show $c * a \leq c * b$ **by** (*rule comm-mult-left-mono*)
then show $a * c \leq b * c$ **by** (*simp only*: *mult.commute*)
qed

end

class *ordered-cancel-comm-semiring* = *ordered-comm-semiring* + *cancel-comm-monoid-add*

begin

subclass *comm-semiring-0-cancel* **..**
subclass *ordered-comm-semiring* **..**
subclass *ordered-cancel-semiring* **..**

end

class *linordered-comm-semiring-strict* = *comm-semiring-0* + *linordered-cancel-ab-semigroup-add* +
  assumes *comm-mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
begin

subclass *linordered-semiring-strict*
proof
  fix $a\ b\ c :: 'a$
  assume $a < b\ 0 < c$
  then show $c * a < c * b$
    by (*rule comm-mult-strict-left-mono*)
  then show $a * c < b * c$
    by (*simp only*: *mult.commute*)
qed

subclass *ordered-cancel-comm-semiring*
proof
  fix $a\ b\ c :: 'a$
  assume $a \leq b\ 0 \leq c$
  then show $c * a \leq c * b$
    unfolding *le-less*
    using *mult-strict-left-mono* **by** (*cases* $c = 0$) *auto*
qed

end

class *ordered-ring* = *ring* + *ordered-cancel-semiring*
begin

subclass *ordered-ab-group-add* **..**

lemma *less-add-iff1*: $a * e + c < b * e + d \longleftrightarrow (a - b) * e + c < d$
  by (*simp add*: *algebra-simps*)

lemma *less-add-iff2*: $a * e + c < b * e + d \longleftrightarrow c < (b - a) * e + d$
  by (*simp add*: *algebra-simps*)

lemma *le-add-iff1*: $a * e + c \leq b * e + d \longleftrightarrow (a - b) * e + c \leq d$
  by (*simp add*: *algebra-simps*)

lemma *le-add-iff2*: $a * e + c \leq b * e + d \longleftrightarrow c \leq (b - a) * e + d$

by (*simp add*: *algebra-simps*)

lemma *mult-left-mono-neg*: $b \leq a \Longrightarrow c \leq 0 \Longrightarrow c * a \leq c * b$
by (*auto dest*: *mult-left-mono* [*of - - - c*])

lemma *mult-right-mono-neg*: $b \leq a \Longrightarrow c \leq 0 \Longrightarrow a * c \leq b * c$
by (*auto dest*: *mult-right-mono* [*of - - - c*])

lemma *mult-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a * b$
using *mult-right-mono-neg* [*of a 0 b*] **by** *simp*

lemma *split-mult-pos-le*: $(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a * b$
by (*auto simp add*: *mult-nonpos-nonpos*)

end

class *abs-if* = *minus* + *uminus* + *ord* + *zero* + *abs* +
assumes *abs-if*: $|a| = (\textit{if } a < 0 \textit{ then } - a \textit{ else } a)$

class *linordered-ring* = *ring* + *linordered-semiring* + *linordered-ab-group-add* + *abs-if*
begin

subclass *ordered-ring* **..**

subclass *ordered-ab-group-add-abs*
proof
fix *a b*
show $|a + b| \leq |a| + |b|$
by (*auto simp add*: *abs-if not-le not-less algebra-simps*
simp del: *add.commute dest*: *add-neg-neg add-nonneg-nonneg*)
qed (*auto simp*: *abs-if*)

lemma *zero-le-square* [*simp*]: $0 \leq a * a$
using *linear* [*of 0 a*] **by** (*auto simp add*: *mult-nonpos-nonpos*)

lemma *not-square-less-zero* [*simp*]: $\neg (a * a < 0)$
by (*simp add*: *not-less*)

proposition *abs-eq-iff*: $|x| = |y| \longleftrightarrow x = y \vee x = -y$
by (*auto simp add*: *abs-if split*: *if-split-asm*)

lemma *abs-eq-iff'*:
$|a| = b \longleftrightarrow b \geq 0 \wedge (a = b \vee a = - b)$
by (*cases* $a \geq 0$) *auto*

lemma *eq-abs-iff'*:
$a = |b| \longleftrightarrow a \geq 0 \wedge (b = a \vee b = - a)$
using *abs-eq-iff'* [*of b a*] **by** *auto*

lemma *sum-squares-ge-zero*: $0 \le x * x + y * y$
  by (*intro add-nonneg-nonneg zero-le-square*)

lemma *not-sum-squares-lt-zero*: $\neg\ x * x + y * y < 0$
  by (*simp add*: *not-less sum-squares-ge-zero*)

end

class *linordered-ring-strict* = *ring* + *linordered-semiring-strict*
  + *ordered-ab-group-add* + *abs-if*
begin

subclass *linordered-ring* ..

lemma *mult-strict-left-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow c * a < c * b$
  using *mult-strict-left-mono* [*of* $b\ a\ -\ c$] **by** *simp*

lemma *mult-strict-right-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow a * c < b * c$
  using *mult-strict-right-mono* [*of* $b\ a\ -\ c$] **by** *simp*

lemma *mult-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow 0 < a * b$
  using *mult-strict-right-mono-neg* [*of* $a\ 0\ b$] **by** *simp*

subclass *ring-no-zero-divisors*
proof
  fix *a b*
  assume $a \neq 0$
  then have *a*: $a < 0 \lor 0 < a$ **by** (*simp add*: *neq-iff*)
  assume $b \neq 0$
  then have *b*: $b < 0 \lor 0 < b$ **by** (*simp add*: *neq-iff*)
  have $a * b < 0 \lor 0 < a * b$
  proof (*cases* $a < 0$)
    case *True*
    show *?thesis*
    proof (*cases* $b < 0$)
      case *True*
      with ‹$a < 0$› **show** *?thesis* **by** (*auto dest*: *mult-neg-neg*)
    next
      case *False*
      with *b* **have** $0 < b$ **by** *auto*
      with ‹$a < 0$› **show** *?thesis* **by** (*auto dest*: *mult-strict-right-mono*)
    qed
  next
    case *False*
    with *a* **have** $0 < a$ **by** *auto*
    show *?thesis*
    proof (*cases* $b < 0$)
      case *True*

with ‹$0 < a$› **show** *?thesis*
by (*auto dest*: *mult-strict-right-mono-neg*)
next
case *False*
with *b* **have** $0 < b$ **by** *auto*
with ‹$0 < a$› **show** *?thesis* **by** *auto*
qed
qed
then show $a * b \neq 0$
by (*simp add*: *neq-iff*)
qed

lemma *zero-less-mult-iff* [*algebra-split-simps*, *field-split-simps*]:
$0 < a * b \longleftrightarrow 0 < a \land 0 < b \lor a < 0 \land b < 0$
by (*cases a 0 b 0 rule*: *linorder-cases*[*case-product linorder-cases*])
(*auto simp add*: *mult-neg-neg not-less le-less dest*: *zero-less-mult-pos zero-less-mult-pos2*)

lemma *zero-le-mult-iff* [*algebra-split-simps*, *field-split-simps*]:
$0 \leq a * b \longleftrightarrow 0 \leq a \land 0 \leq b \lor a \leq 0 \land b \leq 0$
by (*auto simp add*: *eq-commute* [*of 0*] *le-less not-less zero-less-mult-iff*)

lemma *mult-less-0-iff* [*algebra-split-simps*, *field-split-simps*]:
$a * b < 0 \longleftrightarrow 0 < a \land b < 0 \lor a < 0 \land 0 < b$
using *zero-less-mult-iff* [*of* $-$ *a b*] **by** *auto*

lemma *mult-le-0-iff* [*algebra-split-simps*, *field-split-simps*]:
$a * b \leq 0 \longleftrightarrow 0 \leq a \land b \leq 0 \lor a \leq 0 \land 0 \leq b$
using *zero-le-mult-iff* [*of* $-$ *a b*] **by** *auto*

Cancellation laws for $c * a < c * b$ and $a * c < b * c$, also with the relations $\leq$ and equality.

These "disjunction" versions produce two cases when the comparison is an assumption, but effectively four when the comparison is a goal.

lemma *mult-less-cancel-right-disj*: $a * c < b * c \longleftrightarrow 0 < c \land a < b \lor c < 0 \land b < a$
proof (*cases* $c = 0$)
case *False*
show *?thesis* (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then have $c < 0 \Longrightarrow b < a$ $c > 0 \Longrightarrow b > a$
by (*auto simp flip*: *not-le intro*: *mult-right-mono mult-right-mono-neg*)
with *False* **show** *?rhs*
by (*auto simp add*: *neq-iff*)
next
assume *?rhs*
with *False* **show** *?lhs*
by (*auto simp add*: *mult-strict-right-mono mult-strict-right-mono-neg*)

qed
qed *auto*

lemma *mult-less-cancel-left-disj*: $c * a < c * b \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
proof (*cases* $c = 0$)
case *False*
show *?thesis* (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then have $c < 0 \Longrightarrow b < a$ $c > 0 \Longrightarrow b > a$
by (*auto simp flip*: *not-le intro*: *mult-left-mono mult-left-mono-neg*)
with *False* **show** *?rhs*
by (*auto simp add*: *neq-iff*)
next
assume *?rhs*
with *False* **show** *?lhs*
by (*auto simp add*: *mult-strict-left-mono mult-strict-left-mono-neg*)
qed
qed *auto*

The “conjunction of implication” lemmas produce two cases when the comparison is a goal, but give four when the comparison is an assumption.

lemma *mult-less-cancel-right*: $a * c < b * c \longleftrightarrow (0 \le c \longrightarrow a < b) \wedge (c \le 0 \longrightarrow b < a)$
using *mult-less-cancel-right-disj* [*of a c b*] **by** *auto*

lemma *mult-less-cancel-left*: $c * a < c * b \longleftrightarrow (0 \le c \longrightarrow a < b) \wedge (c \le 0 \longrightarrow b < a)$
using *mult-less-cancel-left-disj* [*of c a b*] **by** *auto*

lemma *mult-le-cancel-right*: $a * c \le b * c \longleftrightarrow (0 < c \longrightarrow a \le b) \wedge (c < 0 \longrightarrow b \le a)$
by (*simp add*: *not-less* [*symmetric*] *mult-less-cancel-right-disj*)

lemma *mult-le-cancel-left*: $c * a \le c * b \longleftrightarrow (0 < c \longrightarrow a \le b) \wedge (c < 0 \longrightarrow b \le a)$
by (*simp add*: *not-less* [*symmetric*] *mult-less-cancel-left-disj*)

lemma *mult-le-cancel-left-pos*: $0 < c \Longrightarrow c * a \le c * b \longleftrightarrow a \le b$
by (*auto simp*: *mult-le-cancel-left*)

lemma *mult-le-cancel-left-neg*: $c < 0 \Longrightarrow c * a \le c * b \longleftrightarrow b \le a$
by (*auto simp*: *mult-le-cancel-left*)

lemma *mult-less-cancel-left-pos*: $0 < c \Longrightarrow c * a < c * b \longleftrightarrow a < b$
by (*auto simp*: *mult-less-cancel-left*)

lemma *mult-less-cancel-left-neg*: $c < 0 \Longrightarrow c * a < c * b \longleftrightarrow b < a$

by (*auto simp*: *mult-less-cancel-left*)

lemma *mult-le-cancel-right-pos*: $0 < c \Longrightarrow a * c \leq b * c \longleftrightarrow a \leq b$
by (*auto simp*: *mult-le-cancel-right*)

lemma *mult-le-cancel-right-neg*: $c < 0 \Longrightarrow a * c \leq b * c \longleftrightarrow b \leq a$
by (*auto simp*: *mult-le-cancel-right*)

lemma *mult-less-cancel-right-pos*: $0 < c \Longrightarrow a * c < b * c \longleftrightarrow a < b$
by (*auto simp*: *mult-less-cancel-right*)

lemma *mult-less-cancel-right-neg*: $c < 0 \Longrightarrow a * c < b * c \longleftrightarrow b < a$
by (*auto simp*: *mult-less-cancel-right*)

end

lemmas *mult-sign-intros* =
mult-nonneg-nonneg mult-nonneg-nonpos
mult-nonpos-nonneg mult-nonpos-nonpos
mult-pos-pos mult-pos-neg
mult-neg-pos mult-neg-neg

class *ordered-comm-ring* = *comm-ring* + *ordered-comm-semiring*
begin

subclass *ordered-ring* **..**
subclass *ordered-cancel-comm-semiring* **..**

end

class *linordered-nonzero-semiring* = *ordered-comm-semiring* + *monoid-mult* + *linorder* + *zero-less-one* +
assumes *add-mono1*: $a < b \Longrightarrow a + 1 < b + 1$
begin

subclass *zero-neq-one*
by *standard*

subclass *comm-semiring-1*
by *standard* (*rule mult-1-left*)

lemma *zero-le-one* [*simp*]: $0 \leq 1$
by (*rule zero-less-one* [*THEN less-imp-le*])

lemma *not-one-le-zero* [*simp*]: $\neg\ 1 \leq 0$
by (*simp add*: *not-le*)

lemma *not-one-less-zero* [*simp*]: $\neg\ 1 < 0$
by (*simp add*: *not-less*)

lemma *of-bool-less-eq-iff* [*simp*]:
‹*of-bool P* ≤ *of-bool Q* ⟷ (*P* ⟶ *Q*)›
by *auto*

lemma *of-bool-less-iff* [*simp*]:
‹*of-bool P* < *of-bool Q* ⟷ ¬ *P* ∧ *Q*›
by *auto*

lemma *mult-left-le*: *c* ≤ *1* ⟹ *0* ≤ *a* ⟹ *a* ∗ *c* ≤ *a*
using *mult-left-mono*[*of c 1 a*] **by** *simp*

lemma *mult-le-one*: *a* ≤ *1* ⟹ *0* ≤ *b* ⟹ *b* ≤ *1* ⟹ *a* ∗ *b* ≤ *1*
using *mult-mono*[*of a 1 b 1*] **by** *simp*

lemma *zero-less-two*: *0* < *1* + *1*
using *add-pos-pos*[*OF zero-less-one zero-less-one*] **.**

end

class *linordered-semidom* = *semidom* + *linordered-comm-semiring-strict* + *zero-less-one* +
assumes *le-add-diff-inverse2* [*simp*]: *b* ≤ *a* ⟹ *a* − *b* + *b* = *a*
begin

subclass *linordered-nonzero-semiring*
proof
show *a* + *1* < *b* + *1* **if** *a* < *b* **for** *a b*
proof (*rule ccontr*)
assume ¬ *a* + *1* < *b* + *1*
moreover with *that* **have** *a* + *1* < *b* + *1*
by *simp*
ultimately show *False*
by *contradiction*
qed
qed

lemma *zero-less-eq-of-bool* [*simp*]:
‹*0* ≤ *of-bool P*›
by *simp*

lemma *zero-less-of-bool-iff* [*simp*]:
‹*0* < *of-bool P* ⟷ *P*›
by *simp*

lemma *of-bool-less-eq-one* [*simp*]:
‹*of-bool P* ≤ *1*›
by *simp*

lemma *of-bool-less-one-iff* [*simp*]:
  ‹*of-bool* $P < 1 \longleftrightarrow \neg P$›
  by *simp*

lemma *of-bool-or-iff* [*simp*]:
  ‹*of-bool* $(P \vee Q)$ = *max* (*of-bool* P) (*of-bool* Q)›
  by (*simp add*: *max-def*)

Addition is the inverse of subtraction.

lemma *le-add-diff-inverse* [*simp*]: $b \leq a \Longrightarrow b + (a - b) = a$
  by (*frule le-add-diff-inverse2*) (*simp add*: *add.commute*)

lemma *add-diff-inverse*: $\neg\ a < b \Longrightarrow b + (a - b) = a$
  by *simp*

lemma *add-le-imp-le-diff*:
  assumes $i + k \leq n$ **shows** $i \leq n - k$
proof −
  have $n - (i + k) + i + k = n$
    by (*simp add*: *assms add.assoc*)
  with *assms add-implies-diff* **have** $i + k \leq n - k + k$
    by *fastforce*
  then show *?thesis*
    by *simp*
qed

lemma *add-le-add-imp-diff-le*:
  assumes *1*: $i + k \leq n$
    and *2*: $n \leq j + k$
  shows $i + k \leq n \Longrightarrow n \leq j + k \Longrightarrow n - k \leq j$
proof −
  have $n - (i + k) + i + k = n$
    using *1* **by** (*simp add*: *add.assoc*)
  moreover have $n - k = n - k - i + i$
    using *1* **by** (*simp add*: *add-le-imp-le-diff*)
  ultimately show *?thesis*
    using *2 add-le-imp-le-diff* [*of* $n{-}k$ k $j + k$]
    by (*simp add*: *add.commute diff-diff-add*)
qed

lemma *less-1-mult*: $1 < m \Longrightarrow 1 < n \Longrightarrow 1 < m * n$
  using *mult-strict-mono* [*of 1 m 1 n*] **by** (*simp add*: *less-trans* [*OF zero-less-one*])

end

class *linordered-idom* = *comm-ring-1* + *linordered-comm-semiring-strict* +
  ordered-ab-group-add + *abs-if* + *sgn* +
  assumes *sgn-if*: *sgn* x = (*if* $x = 0$ *then* 0 *else if* $0 < x$ *then* 1 *else* $-\ 1$)
begin

subclass *linordered-ring-strict* **..**

subclass *linordered-semiring-1-strict*
proof
  have *0* ≤ *1* ∗ *1*
    by (*fact zero-le-square*)
  then show *0* < *1*
    by (*simp add*: *le-less*)
qed

subclass *ordered-comm-ring* **..**
subclass *idom* **..**

subclass *linordered-semidom*
  by *standard simp*

subclass *idom-abs-sgn*
  by *standard*
    (*auto simp add*: *sgn-if abs-if zero-less-mult-iff*)

lemma *abs-bool-eq* [*simp*]:
  ‹|*of-bool P*| = *of-bool P*›
  by *simp*

lemma *linorder-neqE-linordered-idom*:
  assumes *x* ≠ *y*
  obtains *x* < *y* | *y* < *x*
  using *assms* **by** (*rule neqE*)

These cancellation simp rules also produce two cases when the comparison is a goal.

lemma *mult-le-cancel-right1*: *c* ≤ *b* ∗ *c* ⟷ (*0* < *c* ⟶ *1* ≤ *b*) ∧ (*c* < *0* ⟶ *b* ≤ *1*)
  using *mult-le-cancel-right* [*of 1 c b*] **by** *simp*

lemma *mult-le-cancel-right2*: *a* ∗ *c* ≤ *c* ⟷ (*0* < *c* ⟶ *a* ≤ *1*) ∧ (*c* < *0* ⟶ *1* ≤ *a*)
  using *mult-le-cancel-right* [*of a c 1*] **by** *simp*

lemma *mult-le-cancel-left1*: *c* ≤ *c* ∗ *b* ⟷ (*0* < *c* ⟶ *1* ≤ *b*) ∧ (*c* < *0* ⟶ *b* ≤ *1*)
  using *mult-le-cancel-left* [*of c 1 b*] **by** *simp*

lemma *mult-le-cancel-left2*: *c* ∗ *a* ≤ *c* ⟷ (*0* < *c* ⟶ *a* ≤ *1*) ∧ (*c* < *0* ⟶ *1* ≤ *a*)
  using *mult-le-cancel-left* [*of c a 1*] **by** *simp*

lemma *mult-less-cancel-right1*: *c* < *b* ∗ *c* ⟷ (*0* ≤ *c* ⟶ *1* < *b*) ∧ (*c* ≤ *0* ⟶

$b < 1$)
using *mult-less-cancel-right* [*of 1 c b*] **by** *simp*

lemma *mult-less-cancel-right2*: $a * c < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
using *mult-less-cancel-right* [*of a c 1*] **by** *simp*

lemma *mult-less-cancel-left1*: $c < c * b \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
using *mult-less-cancel-left* [*of c 1 b*] **by** *simp*

lemma *mult-less-cancel-left2*: $c * a < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
using *mult-less-cancel-left* [*of c a 1*] **by** *simp*

lemma *sgn-0-0*: $sgn\ a = 0 \longleftrightarrow a = 0$
by (*fact sgn-eq-0-iff*)

lemma *sgn-1-pos*: $sgn\ a = 1 \longleftrightarrow a > 0$
unfolding *sgn-if* **by** *simp*

lemma *sgn-1-neg*: $sgn\ a = -\ 1 \longleftrightarrow a < 0$
unfolding *sgn-if* **by** *auto*

lemma *sgn-pos* [*simp*]: $0 < a \Longrightarrow sgn\ a = 1$
by (*simp only*: *sgn-1-pos*)

lemma *sgn-neg* [*simp*]: $a < 0 \Longrightarrow sgn\ a = -\ 1$
by (*simp only*: *sgn-1-neg*)

lemma *abs-sgn*: $|k| = k * sgn\ k$
unfolding *sgn-if abs-if* **by** *auto*

lemma *sgn-greater* [*simp*]: $0 < sgn\ a \longleftrightarrow 0 < a$
unfolding *sgn-if* **by** *auto*

lemma *sgn-less* [*simp*]: $sgn\ a < 0 \longleftrightarrow a < 0$
unfolding *sgn-if* **by** *auto*

lemma *abs-sgn-eq-1* [*simp*]:
$a \neq 0 \Longrightarrow |sgn\ a| = 1$
by *simp*

lemma *abs-sgn-eq*: $|sgn\ a| = (if\ a = 0\ then\ 0\ else\ 1)$
by (*simp add*: *sgn-if*)

lemma *sgn-mult-self-eq* [*simp*]:
$sgn\ a * sgn\ a = of\text{-}bool\ (a \neq 0)$
by (*cases* $a > 0$) *simp-all*

lemma *left-sgn-mult-self-eq* [*simp*]:
‹*sgn a* ∗ (*sgn a* ∗ *b*) = *of-bool* (*a* ≠ *0*) ∗ *b*›
by (*simp flip*: *mult.assoc*)

lemma *abs-mult-self-eq* [*simp*]:
|*a*| ∗ |*a*| = *a* ∗ *a*
by (*cases a* > *0*) *simp-all*

lemma *same-sgn-sgn-add*:
sgn (*a* + *b*) = *sgn a* **if** *sgn b* = *sgn a*
proof (*cases a 0 rule*: *linorder-cases*)
case *equal*
with *that* **show** *?thesis*
by *simp*
next
case *less*
with *that* **have** *b* < *0*
by (*simp add*: *sgn-1-neg*)
with ‹*a* < *0*› **have** *a* + *b* < *0*
by (*rule add-neg-neg*)
with ‹*a* < *0*› **show** *?thesis*
by *simp*
next
case *greater*
with *that* **have** *b* > *0*
by (*simp add*: *sgn-1-pos*)
with ‹*a* > *0*› **have** *a* + *b* > *0*
by (*rule add-pos-pos*)
with ‹*a* > *0*› **show** *?thesis*
by *simp*
qed

lemma *same-sgn-abs-add*:
|*a* + *b*| = |*a*| + |*b*| **if** *sgn b* = *sgn a*
proof −
have *a* + *b* = *sgn a* ∗ |*a*| + *sgn b* ∗ |*b*|
by (*simp add*: *sgn-mult-abs*)
also have . . . = *sgn a* ∗ (|*a*| + |*b*|)
using *that* **by** (*simp add*: *algebra-simps*)
finally show *?thesis*
by (*auto simp add*: *abs-mult*)
qed

lemma *sgn-not-eq-imp*:
sgn a = − *sgn b* **if** *sgn b* ≠ *sgn a* **and** *sgn a* ≠ *0* **and** *sgn b* ≠ *0*
using *that* **by** (*cases a* < *0*) (*auto simp add*: *sgn-0-0 sgn-1-pos sgn-1-neg*)

lemma *abs-dvd-iff* [*simp*]: |*m*| *dvd k* ⟷ *m dvd k*

by (*simp add*: *abs-if*)

lemma *dvd-abs-iff* [*simp*]: $m\ dvd\ |k| \longleftrightarrow m\ dvd\ k$
by (*simp add*: *abs-if*)

lemma *dvd-if-abs-eq*: $|l| = |k| \Longrightarrow l\ dvd\ k$
by (*subst abs-dvd-iff* [*symmetric*]) *simp*

The following lemmas can be proven in more general structures, but are dangerous as simp rules in absence of $(- ?a = ?a) = (?a = 0)$, $(- ?a < ?a) = (0 < ?a)$, $(- ?a \leq ?a) = (0 \leq ?a)$.

lemma *equation-minus-iff-1* [*simp*, *no-atp*]: $1 = - a \longleftrightarrow a = - 1$
by (*fact equation-minus-iff*)

lemma *minus-equation-iff-1* [*simp*, *no-atp*]: $- a = 1 \longleftrightarrow a = - 1$
by (*subst minus-equation-iff*, *auto*)

lemma *le-minus-iff-1* [*simp*, *no-atp*]: $1 \leq - b \longleftrightarrow b \leq - 1$
by (*fact le-minus-iff*)

lemma *minus-le-iff-1* [*simp*, *no-atp*]: $- a \leq 1 \longleftrightarrow - 1 \leq a$
by (*fact minus-le-iff*)

lemma *less-minus-iff-1* [*simp*, *no-atp*]: $1 < - b \longleftrightarrow b < - 1$
by (*fact less-minus-iff*)

lemma *minus-less-iff-1* [*simp*, *no-atp*]: $- a < 1 \longleftrightarrow - 1 < a$
by (*fact minus-less-iff*)

lemma *add-less-zeroD*:
shows $x+y < 0 \Longrightarrow x<0 \lor y<0$
by (*auto simp*: *not-less intro*: *le-less-trans* [*of* - $x+y$])

Is this really better than just rewriting with *abs-if*?

lemma *abs-split* [*no-atp*]: ‹$P\ |a| \longleftrightarrow (0 \leq a \longrightarrow P\ a) \land (a < 0 \longrightarrow P\ (- a))$›
by (*force dest*: *order-less-le-trans simp add*: *abs-if linorder-not-less*)

end

class *discrete-linordered-semidom* = *linordered-semidom* +
assumes *less-iff-succ-less-eq*: ‹$a < b \longleftrightarrow a + 1 \leq b$›
begin

lemma *less-eq-iff-succ-less*:
‹$a \leq b \longleftrightarrow a < b + 1$›
using *less-iff-succ-less-eq* [*of a* ‹$b + 1$›] **by** *simp*

end

Reasoning about inequalities with division

context *linordered-semidom*
begin

lemma *less-add-one*: $a < a + 1$
proof −
  have $a + 0 < a + 1$
    by (*blast intro*: *zero-less-one add-strict-left-mono*)
  then show *?thesis* **by** *simp*
qed

end

context *linordered-idom*
begin

lemma *mult-right-le-one-le*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow y \leq 1 \Longrightarrow x * y \leq x$
  by (*rule mult-left-le*)

lemma *mult-left-le-one-le*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow y \leq 1 \Longrightarrow y * x \leq x$
  by (*auto simp add*: *mult-le-cancel-right2*)

end

Absolute Value

context *linordered-idom*
begin

lemma *mult-sgn-abs*: $sgn\ x * |x| = x$
  by (*fact sgn-mult-abs*)

lemma *abs-one*: $|1| = 1$
  by (*fact abs-1*)

end

class *ordered-ring-abs* = *ordered-ring* + *ordered-ab-group-add-abs* +
  assumes *abs-eq-mult*:
    $(0 \leq a \vee a \leq 0) \wedge (0 \leq b \vee b \leq 0) \Longrightarrow |a * b| = |a| * |b|$

context *linordered-idom*
begin

subclass *ordered-ring-abs*
  by *standard* (*auto simp*: *abs-if not-less mult-less-0-iff*)

lemma *abs-mult-self*: $|a| * |a| = a * a$
  by (*fact abs-mult-self-eq*)

lemma *abs-mult-less*:
  assumes *ac*: $|a| < c$
    and *bd*: $|b| < d$
  shows $|a| * |b| < c * d$
proof −
  from *ac* **have** $0 < c$
    by (*blast intro*: *le-less-trans abs-ge-zero*)
  with *bd* **show** *?thesis* **by** (*simp add*: *ac mult-strict-mono*)
qed

lemma *abs-less-iff*: $|a| < b \longleftrightarrow a < b \wedge - a < b$
  by (*simp add*: *less-le abs-le-iff*) (*auto simp add*: *abs-if*)

lemma *abs-mult-pos*: $0 \leq x \Longrightarrow |y| * x = |y * x|$
  by (*simp add*: *abs-mult*)

lemma *abs-mult-pos'*: $0 \leq x \Longrightarrow x * |y| = |x * y|$
  by (*simp add*: *abs-mult*)

lemma *abs-diff-less-iff*: $|x - a| < r \longleftrightarrow a - r < x \wedge x < a + r$
  by (*auto simp add*: *diff-less-eq ac-simps abs-less-iff*)

lemma *abs-diff-le-iff*: $|x - a| \leq r \longleftrightarrow a - r \leq x \wedge x \leq a + r$
  by (*auto simp add*: *diff-le-eq ac-simps abs-le-iff*)

lemma *abs-add-one-gt-zero*: $0 < 1 + |x|$
  by (*auto simp*: *abs-if not-less intro*: *zero-less-one add-strict-increasing less-trans*)

end

16.8 Dioids

Dioids are the alternative extensions of semirings, a semiring can either be a ring or a dioid but never both.

class *dioid* = *semiring-1* + *canonically-ordered-monoid-add*
begin

subclass *ordered-semiring*
  by *standard* (*auto simp*: *le-iff-add distrib-left distrib-right*)

end

hide-fact (**open**) *comm-mult-left-mono comm-mult-strict-left-mono distrib*

code-identifier
  code-module *Rings* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

17 Natural numbers

theory *Nat*
imports *Inductive Typedef Fun Rings*
begin

17.1 Type *ind*

typedecl *ind*

axiomatization *Zero-Rep* :: *ind* **and** *Suc-Rep* :: *ind* ⇒ *ind*
— The axiom of infinity in 2 parts:
where *Suc-Rep-inject*: *Suc-Rep x* = *Suc-Rep y* ⟹ *x* = *y*
and *Suc-Rep-not-Zero-Rep*: *Suc-Rep x* ≠ *Zero-Rep*

17.2 Type nat

Type definition

inductive *Nat* :: *ind* ⇒ *bool*
where
Zero-RepI: *Nat Zero-Rep*
| *Suc-RepI*: *Nat i* ⟹ *Nat* (*Suc-Rep i*)

typedef *nat* = {*n*. *Nat n*}
morphisms *Rep-Nat Abs-Nat*
using *Nat.Zero-RepI* **by** *auto*

lemma *Nat-Rep-Nat*: *Nat* (*Rep-Nat n*)
using *Rep-Nat* **by** *simp*

lemma *Nat-Abs-Nat-inverse*: *Nat n* ⟹ *Rep-Nat* (*Abs-Nat n*) = *n*
using *Abs-Nat-inverse* **by** *simp*

lemma *Nat-Abs-Nat-inject*: *Nat n* ⟹ *Nat m* ⟹ *Abs-Nat n* = *Abs-Nat m* ⟷ *n* = *m*
using *Abs-Nat-inject* **by** *simp*

instantiation *nat* :: *zero*
begin

definition *Zero-nat-def*: *0* = *Abs-Nat Zero-Rep*

instance ..

end

definition *Suc* :: *nat* ⇒ *nat*
where *Suc n* = *Abs-Nat* (*Suc-Rep* (*Rep-Nat n*))

lemma *Suc-not-Zero*: *Suc m* ≠ *0*
by (*simp add*: *Zero-nat-def Suc-def Suc-RepI Zero-RepI*
Nat-Abs-Nat-inject Suc-Rep-not-Zero-Rep Nat-Rep-Nat)

lemma *Zero-not-Suc*: *0* ≠ *Suc m*
by (*rule not-sym*) (*rule Suc-not-Zero*)

lemma *Suc-Rep-inject'*: *Suc-Rep x* = *Suc-Rep y* ⟷ *x* = *y*
by (*rule iffI*, *rule Suc-Rep-inject*) *simp-all*

lemma *nat-induct0*:
assumes *P 0* **and** ⋀*n*. *P n* ⟹ *P* (*Suc n*)
shows *P n*
proof −
have *P* (*Abs-Nat* (*Rep-Nat n*))
using *assms* **unfolding** *Zero-nat-def Suc-def*
by (*iprover intro*: *Nat-Rep-Nat* [*THEN Nat.induct*] *elim*: *Nat-Abs-Nat-inverse*
[*THEN subst*])
then show *?thesis*
by (*simp add*: *Rep-Nat-inverse*)
qed

free-constructors *case-nat* **for** *0* :: *nat* | *Suc pred*
where *pred* (*0* :: *nat*) = (*0* :: *nat*)
proof *atomize-elim*
fix *n*
show *n* = *0* ∨ (∃ *m*. *n* = *Suc m*)
by (*induction n rule*: *nat-induct0*) *auto*
next
fix *n m*
show (*Suc n* = *Suc m*) = (*n* = *m*)
by (*simp add*: *Suc-def Nat-Abs-Nat-inject Nat-Rep-Nat Suc-RepI Suc-Rep-inject'*
Rep-Nat-inject)
next
fix *n*
show *0* ≠ *Suc n*
by (*simp add*: *Suc-not-Zero*)
qed

— Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.
setup ‹*Sign.mandatory-path old*›

old-rep-datatype *0* :: *nat Suc*
by (*erule nat-induct0*) *auto*

setup ‹*Sign.parent-path*›

— But erase the prefix for properties that are not generated by *free-constructors*.

setup ‹*Sign.mandatory-path nat*›

declare *old.nat.inject*[*iff del*]
  and *old.nat.distinct*(*1*)[*simp del*, *induct-simp del*]

lemmas *induct* = *old.nat.induct*
lemmas *inducts* = *old.nat.inducts*
lemmas *rec* = *old.nat.rec*
lemmas *simps* = *nat.inject nat.distinct nat.case nat.rec*

setup ‹*Sign.parent-path*›

abbreviation *rec-nat* :: *'a* ⇒ (*nat* ⇒ *'a* ⇒ *'a*) ⇒ *nat* ⇒ *'a*
  where *rec-nat* ≡ *old.rec-nat*

declare *nat.sel*[*code del*]

hide-const (**open**) *Nat.pred* — hide everything related to the selector

lemma *nat-exhaust* [*case-names 0 Suc*, *cases type*: *nat*]:
  (*y* = *0* ⟹ *P*) ⟹ (⋀*nat*. *y* = *Suc nat* ⟹ *P*) ⟹ *P*
  — for backward compatibility – names of variables differ
  by (*rule old.nat.exhaust*)

lemma *nat-induct* [*case-names 0 Suc*, *induct type*: *nat*]:
  fixes *n*
  assumes *P 0* **and** ⋀*n*. *P n* ⟹ *P* (*Suc n*)
  shows *P n*
  — for backward compatibility – names of variables differ
  using *assms* **by** (*rule nat.induct*)

hide-fact
  nat-exhaust
  nat-induct0

ML ‹
val nat-basic-lfp-sugar =
  let
    val ctr-sugar = *the* (*Ctr-Sugar.ctr-sugar-of-global* ***theory*** ***type-name*** ‹*nat*›);
    val recx = *Logic.varify-types-global* ***term*** ‹*rec-nat*›;
    val C = *body-type* (*fastype-of recx*);
  in
   {*T* = *HOLogic.natT*, *fp-res-index* = *0*, *C* = *C*, *fun-arg-Tsss* = [[], [[*HOLogic.natT*, *C*]]],
     ctr-sugar = *ctr-sugar*, *recx* = *recx*, *rec-thms* = @{*thms nat.rec*}}
  end;
›

setup ‹

let
  *fun basic-lfp-sugars-of - [**typ** ‹nat›] - - ctxt =*
    ([], [0], [nat-basic-lfp-sugar], [], [], [], TrueI (∗dummy∗), [], false, ctxt)
   | basic-lfp-sugars-of bs arg-Ts callers callssss ctxt =
    BNF-LFP-Rec-Sugar.default-basic-lfp-sugars-of bs arg-Ts callers callssss ctxt;
in
  BNF-LFP-Rec-Sugar.register-lfp-rec-extension
   {nested-simps = [], special-endgame-tac = K (K (K (K no-tac))), is-new-datatype = K (K true),
    basic-lfp-sugars-of = basic-lfp-sugars-of, rewrite-nested-rec-call = NONE}
end
›

Injectiveness and distinctness lemmas

lemma *inj-Suc* [*simp*]:
  inj-on Suc N
  by (*simp add*: *inj-on-def*)

lemma *bij-betw-Suc* [*simp*]:
  $bij\text{-}betw\ Suc\ M\ N \longleftrightarrow Suc\ `\ M = N$
  by (*simp add*: *bij-betw-def*)

lemma *Suc-neq-Zero*: $Suc\ m = 0 \Longrightarrow R$
  by (*rule notE*) (*rule Suc-not-Zero*)

lemma *Zero-neq-Suc*: $0 = Suc\ m \Longrightarrow R$
  by (*rule Suc-neq-Zero*) (*erule sym*)

lemma *Suc-inject*: $Suc\ x = Suc\ y \Longrightarrow x = y$
  by (*rule inj-Suc* [*THEN injD*])

lemma *n-not-Suc-n*: $n \neq Suc\ n$
  by (*induct n*) *simp-all*

lemma *Suc-n-not-n*: $Suc\ n \neq n$
  by (*rule not-sym*) (*rule n-not-Suc-n*)

A special form of induction for reasoning about $m < n$ and $m - n$.

lemma *diff-induct*:
  assumes $\bigwedge x.\ P\ x\ 0$
    and $\bigwedge y.\ P\ 0\ (Suc\ y)$
    and $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (Suc\ x)\ (Suc\ y)$
  shows *P m n*
proof (*induct n arbitrary*: *m*)
  case *0*
  show *?case* **by** (*rule assms*(*1*))
next
  case (*Suc n*)
  show *?case*

```
  proof (induct m)
    case 0
    show ?case by (rule assms(2))
  next
    case (Suc m)
    from ‹P m n› show ?case by (rule assms(3))
  qed
qed
```

17.3 Arithmetic operators

```
instantiation nat :: comm-monoid-diff
begin

primrec plus-nat
  where
    add-0: 0 + n = (n::nat)
  | add-Suc: Suc m + n = Suc (m + n)

lemma add-0-right [simp]: m + 0 = m
  for m :: nat
  by (induct m) simp-all

lemma add-Suc-right [simp]: m + Suc n = Suc (m + n)
  by (induct m) simp-all

declare add-0 [code]

lemma add-Suc-shift [code]: Suc m + n = m + Suc n
  by simp

primrec minus-nat
  where
    diff-0 [code]: m − 0 = (m::nat)
  | diff-Suc: m − Suc n = (case m − n of 0 ⇒ 0 | Suc k ⇒ k)

declare diff-Suc [simp del]

lemma diff-0-eq-0 [simp, code]: 0 − n = 0
  for n :: nat
  by (induct n) (simp-all add: diff-Suc)

lemma diff-Suc-Suc [simp, code]: Suc m − Suc n = m − n
  by (induct n) (simp-all add: diff-Suc)

instance
proof
  fix n m q :: nat
  show (n + m) + q = n + (m + q) by (induct n) simp-all
```

```
  show n + m = m + n by (induct n) simp-all
  show m + n − m = n by (induct m) simp-all
  show n − m − q = n − (m + q) by (induct q) (simp-all add: diff-Suc)
  show 0 + n = n by simp
  show 0 − n = 0 by simp
qed

end

hide-fact (open) add-0 add-0-right diff-0

instantiation nat :: comm-semiring-1-cancel
begin

definition One-nat-def [simp]: 1 = Suc 0

primrec times-nat
  where
    mult-0: 0 ∗ n = (0::nat)
  | mult-Suc: Suc m ∗ n = n + (m ∗ n)

lemma mult-0-right [simp]: m ∗ 0 = 0
  for m :: nat
  by (induct m) simp-all

lemma mult-Suc-right [simp]: m ∗ Suc n = m + (m ∗ n)
  by (induct m) (simp-all add: add.left-commute)

lemma add-mult-distrib: (m + n) ∗ k = (m ∗ k) + (n ∗ k)
  for m n k :: nat
  by (induct m) (simp-all add: add.assoc)

instance
proof
  fix k n m q :: nat
  show 0 ≠ (1::nat)
    by simp
  show 1 ∗ n = n
    by simp
  show n ∗ m = m ∗ n
    by (induct n) simp-all
  show (n ∗ m) ∗ q = n ∗ (m ∗ q)
    by (induct n) (simp-all add: add-mult-distrib)
  show (n + m) ∗ q = n ∗ q + m ∗ q
    by (rule add-mult-distrib)
  show k ∗ (m − n) = (k ∗ m) − (k ∗ n)
    by (induct m n rule: diff-induct) simp-all
qed
```

end

17.3.1 Addition

Reasoning about $m + 0 = 0$, etc.

lemma *add-is-0* [*iff*]: $m + n = 0 \longleftrightarrow m = 0 \wedge n = 0$
for *m n* :: *nat*
by (*cases m*) *simp-all*

lemma *add-is-1*: $m + n = Suc\ 0 \longleftrightarrow m = Suc\ 0 \wedge n = 0 \vee m = 0 \wedge n = Suc\ 0$
by (*cases m*) *simp-all*

lemma *one-is-add*: $Suc\ 0 = m + n \longleftrightarrow m = Suc\ 0 \wedge n = 0 \vee m = 0 \wedge n = Suc\ 0$
by (*rule trans*, *rule eq-commute*, *rule add-is-1*)

lemma *add-eq-self-zero*: $m + n = m \Longrightarrow n = 0$
for *m n* :: *nat*
by (*induct m*) *simp-all*

lemma *plus-1-eq-Suc*:
plus 1 = *Suc*
by (*simp add*: *fun-eq-iff*)

lemma *Suc-eq-plus1*: $Suc\ n = n + 1$
by *simp*

lemma *Suc-eq-plus1-left*: $Suc\ n = 1 + n$
by *simp*

17.3.2 Difference

lemma *Suc-diff-diff* [*simp*]: $(Suc\ m - n) - Suc\ k = m - n - k$
by (*simp add*: *diff-diff-add*)

lemma *diff-Suc-1*: $Suc\ n - 1 = n$
by *simp*

lemma *diff-Suc-1'* [*simp*]: $Suc\ n - Suc\ 0 = n$
by *simp*

17.3.3 Multiplication

lemma *mult-is-0* [*simp*]: $m * n = 0 \longleftrightarrow m = 0 \vee n = 0$ **for** *m n* :: *nat*
by (*induct m*) *auto*

lemma *mult-eq-1-iff* [*simp*]: $m * n = Suc\ 0 \longleftrightarrow m = Suc\ 0 \wedge n = Suc\ 0$
proof (*induct m*)
case *0*

then show *?case* **by** *simp*
next
case (*Suc m*)
then show *?case* **by** (*induct n*) *auto*
qed

lemma *one-eq-mult-iff* [*simp*]: $Suc\ 0 = m * n \longleftrightarrow m = Suc\ 0 \wedge n = Suc\ 0$
by (*simp add*: *eq-commute flip*: *mult-eq-1-iff*)

lemma *nat-mult-eq-1-iff* [*simp*]: $m * n = 1 \longleftrightarrow m = 1 \wedge n = 1$
and *nat-1-eq-mult-iff* [*simp*]: $1 = m * n \longleftrightarrow m = 1 \wedge n = 1$ **for** *m n* :: *nat*
by *auto*

lemma *mult-cancel1* [*simp*]: $k * m = k * n \longleftrightarrow m = n \vee k = 0$
for *k m n* :: *nat*
proof −
have $k \neq 0 \Longrightarrow k * m = k * n \Longrightarrow m = n$
proof (*induct n arbitrary*: *m*)
case *0*
then show $m = 0$ **by** *simp*
next
case (*Suc n*)
then show $m = Suc\ n$
by (*cases m*) (*simp-all add*: *eq-commute* [*of 0*])
qed
then show *?thesis* **by** *auto*
qed

lemma *mult-cancel2* [*simp*]: $m * k = n * k \longleftrightarrow m = n \vee k = 0$
for *k m n* :: *nat*
by (*simp add*: *mult.commute*)

lemma *Suc-mult-cancel1*: $Suc\ k * m = Suc\ k * n \longleftrightarrow m = n$
by (*subst mult-cancel1*) *simp*

17.4 Orders on *nat*

17.4.1 Operation definition

instantiation *nat* :: *linorder*
begin

primrec *less-eq-nat*
where
$(0{::}nat) \leq n \longleftrightarrow True$
| $Suc\ m \leq n \longleftrightarrow (case\ n\ of\ 0 \Rightarrow False \mid Suc\ n \Rightarrow m \leq n)$

declare *less-eq-nat.simps* [*simp del*]

lemma *le0* [*iff*]: $0 \leq n$ **for**

n :: *nat*
by (*simp add*: *less-eq-nat.simps*)

lemma [*code*]: $0 \leq n \longleftrightarrow True$
for *n* :: *nat*
by *simp*

definition *less-nat*
where *less-eq-Suc-le*: $n < m \longleftrightarrow Suc\ n \leq m$

lemma *Suc-le-mono* [*iff*]: $Suc\ n \leq Suc\ m \longleftrightarrow n \leq m$
by (*simp add*: *less-eq-nat.simps*(*2*))

lemma *Suc-le-eq* [*code*]: $Suc\ m \leq n \longleftrightarrow m < n$
unfolding *less-eq-Suc-le* **..**

lemma *le-0-eq* [*iff*]: $n \leq 0 \longleftrightarrow n = 0$
for *n* :: *nat*
by (*induct n*) (*simp-all add*: *less-eq-nat.simps*(*2*))

lemma *not-less0* [*iff*]: $\neg\ n < 0$
for *n* :: *nat*
by (*simp add*: *less-eq-Suc-le*)

lemma *less-nat-zero-code* [*code*]: $n < 0 \longleftrightarrow False$
for *n* :: *nat*
by *simp*

lemma *Suc-less-eq* [*iff*]: $Suc\ m < Suc\ n \longleftrightarrow m < n$
by (*simp add*: *less-eq-Suc-le*)

lemma *less-Suc-eq-le* [*code*]: $m < Suc\ n \longleftrightarrow m \leq n$
by (*simp add*: *less-eq-Suc-le*)

lemma *Suc-less-eq2*: $Suc\ n < m \longleftrightarrow (\exists\, m'.\ m = Suc\ m' \wedge n < m')$
by (*cases m*) *auto*

lemma *le-SucI*: $m \leq n \Longrightarrow m \leq Suc\ n$
by (*induct m arbitrary*: *n*) (*simp-all add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*)

lemma *Suc-leD*: $Suc\ m \leq n \Longrightarrow m \leq n$
by (*cases n*) (*auto intro*: *le-SucI*)

lemma *less-SucI*: $m < n \Longrightarrow m < Suc\ n$
by (*simp add*: *less-eq-Suc-le*) (*erule Suc-leD*)

lemma *Suc-lessD*: $Suc\ m < n \Longrightarrow m < n$
by (*simp add*: *less-eq-Suc-le*) (*erule Suc-leD*)

```
instance
proof
  fix n m q :: nat
  show n < m ⟷ n ≤ m ∧ ¬ m ≤ n
  proof (induct n arbitrary: m)
    case 0
    then show ?case
      by (cases m) (simp-all add: less-eq-Suc-le)
  next
    case (Suc n)
    then show ?case
      by (cases m) (simp-all add: less-eq-Suc-le)
  qed
  show n ≤ n
    by (induct n) simp-all
  then show n = m if n ≤ m and m ≤ n
    using that by (induct n arbitrary: m)
      (simp-all add: less-eq-nat.simps(2) split: nat.splits)
  show n ≤ q if n ≤ m and m ≤ q
    using that
  proof (induct n arbitrary: m q)
    case 0
    show ?case by simp
  next
    case (Suc n)
    then show ?case
      by (simp-all (no-asm-use) add: less-eq-nat.simps(2) split: nat.splits, clarify,
        simp-all (no-asm-use) add: less-eq-nat.simps(2) split: nat.splits, clarify,
        simp-all (no-asm-use) add: less-eq-nat.simps(2) split: nat.splits)
  qed
  show n ≤ m ∨ m ≤ n
    by (induct n arbitrary: m)
      (simp-all add: less-eq-nat.simps(2) split: nat.splits)
qed

end

instantiation nat :: order-bot
begin

definition bot-nat :: nat
  where bot-nat = 0

instance
  by standard (simp add: bot-nat-def)

end

instance nat :: no-top
```

by *standard* (*auto intro*: *less-Suc-eq-le* [*THEN iffD2*])

17.4.2 Introduction properties

lemma *lessI* [*iff*]: $n < Suc\ n$
by (*simp add*: *less-Suc-eq-le*)

lemma *zero-less-Suc* [*iff*]: $0 < Suc\ n$
by (*simp add*: *less-Suc-eq-le*)

17.4.3 Elimination properties

lemma *less-not-refl*: $\neg\ n < n$
for n :: *nat*
by (*rule order-less-irrefl*)

lemma *less-not-refl2*: $n < m \Longrightarrow m \neq n$
for $m\ n$ :: *nat*
by (*rule not-sym*) (*rule less-imp-neq*)

lemma *less-not-refl3*: $s < t \Longrightarrow s \neq t$
for $s\ t$ :: *nat*
by (*rule less-imp-neq*)

lemma *less-irrefl-nat*: $n < n \Longrightarrow R$
for n :: *nat*
by (*rule notE*, *rule less-not-refl*)

lemma *less-zeroE*: $n < 0 \Longrightarrow R$
for n :: *nat*
by (*rule notE*) (*rule not-less0*)

lemma *less-Suc-eq*: $m < Suc\ n \longleftrightarrow m < n \lor m = n$
unfolding *less-Suc-eq-le le-less* **..**

lemma *less-Suc0* [*iff*]: $(n < Suc\ 0) = (n = 0)$
by (*simp add*: *less-Suc-eq*)

lemma *less-one* [*iff*]: $n < 1 \longleftrightarrow n = 0$
for n :: *nat*
unfolding *One-nat-def* **by** (*rule less-Suc0*)

lemma *Suc-mono*: $m < n \Longrightarrow Suc\ m < Suc\ n$
by *simp*

"Less than" is antisymmetric, sort of.

lemma *less-antisym*: $\neg\ n < m \Longrightarrow n < Suc\ m \Longrightarrow m = n$
unfolding *not-less less-Suc-eq-le* **by** (*rule antisym*)

lemma *nat-neq-iff*: $m \neq n \longleftrightarrow m < n \lor n < m$

for *m n* :: *nat*
by (*rule linorder-neq-iff*)

17.4.4 Inductive (?) properties

lemma *Suc-lessI*: $m < n \Longrightarrow Suc\ m \neq n \Longrightarrow Suc\ m < n$
unfolding *less-eq-Suc-le* [*of m*] *le-less* **by** *simp*

lemma *lessE*:
assumes *major*: $i < k$
and *1*: $k = Suc\ i \Longrightarrow P$
and *2*: $\bigwedge j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$
shows *P*
proof −
from *major* **have** $\exists j.\ i \leq j \wedge k = Suc\ j$
unfolding *less-eq-Suc-le* **by** (*induct k*) *simp-all*
then have $(\exists j.\ i < j \wedge k = Suc\ j) \vee k = Suc\ i$
by (*auto simp add*: *less-le*)
with *1 2* **show** *P* **by** *auto*
qed

lemma *less-SucE*:
assumes *major*: $m < Suc\ n$
and *less*: $m < n \Longrightarrow P$
and *eq*: $m = n \Longrightarrow P$
shows *P*
proof (*rule major* [*THEN lessE*])
show $Suc\ n = Suc\ m \Longrightarrow P$
using *eq* **by** *blast*
show $\bigwedge j.\ [\![m < j;\ Suc\ n = Suc\ j]\!] \Longrightarrow P$
by (*blast intro*: *less*)
qed

lemma *Suc-lessE*:
assumes *major*: $Suc\ i < k$
and *minor*: $\bigwedge j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$
shows *P*
proof (*rule major* [*THEN lessE*])
show $k = Suc\ (Suc\ i) \Longrightarrow P$
using *lessI minor* **by** *iprover*
show $\bigwedge j.\ [\![Suc\ i < j;\ k = Suc\ j]\!] \Longrightarrow P$
using *Suc-lessD minor* **by** *iprover*
qed

lemma *Suc-less-SucD*: $Suc\ m < Suc\ n \Longrightarrow m < n$
by *simp*

lemma *less-trans-Suc*:
assumes *le*: $i < j$

shows $j < k \Longrightarrow Suc\ i < k$
proof (*induct k*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc k*)
with *le* **show** *?case*
by *simp* (*auto simp add*: *less-Suc-eq dest*: *Suc-lessD*)
qed

Can be used with *less-Suc-eq* to get $n = m \vee n < m$.

lemma *not-less-eq*: $\neg\ m < n \longleftrightarrow n < Suc\ m$
by (*simp only*: *not-less less-Suc-eq-le*)

lemma *not-less-eq-eq*: $\neg\ m \le n \longleftrightarrow Suc\ n \le m$
by (*simp only*: *not-le Suc-le-eq*)

Properties of "less than or equal".

lemma *le-imp-less-Suc*: $m \le n \Longrightarrow m < Suc\ n$
by (*simp only*: *less-Suc-eq-le*)

lemma *Suc-n-not-le-n*: $\neg\ Suc\ n \le n$
by (*simp add*: *not-le less-Suc-eq-le*)

lemma *le-Suc-eq*: $m \le Suc\ n \longleftrightarrow m \le n \vee m = Suc\ n$
by (*simp add*: *less-Suc-eq-le* [*symmetric*] *less-Suc-eq*)

lemma *le-SucE*: $m \le Suc\ n \Longrightarrow (m \le n \Longrightarrow R) \Longrightarrow (m = Suc\ n \Longrightarrow R) \Longrightarrow R$
by (*drule le-Suc-eq* [*THEN iffD1*], *iprover+*)

lemma *Suc-leI*: $m < n \Longrightarrow Suc\ m \le n$
by (*simp only*: *Suc-le-eq*)

Stronger version of *Suc-leD*.

lemma *Suc-le-lessD*: $Suc\ m \le n \Longrightarrow m < n$
by (*simp only*: *Suc-le-eq*)

lemma *less-imp-le-nat*: $m < n \Longrightarrow m \le n$ **for** $m\ n :: nat$
unfolding *less-eq-Suc-le* **by** (*rule Suc-leD*)

For instance, $(Suc\ m < Suc\ n) = (Suc\ m \le n) = (m < n)$

lemmas *le-simps* = *less-imp-le-nat less-Suc-eq-le Suc-le-eq*

Equivalence of $m \le n$ and $m < n \vee m = n$

lemma *less-or-eq-imp-le*: $m < n \vee m = n \Longrightarrow m \le n$
for $m\ n :: nat$
unfolding *le-less* **.**

lemma *le-eq-less-or-eq*: $m \leq n \longleftrightarrow m < n \lor m = n$
for $m\ n$ :: *nat*
by (*rule le-less*)

Useful with *blast*.

lemma *eq-imp-le*: $m = n \Longrightarrow m \leq n$
for $m\ n$ :: *nat*
by *auto*

lemma *le-refl*: $n \leq n$
for n :: *nat*
by *simp*

lemma *le-trans*: $i \leq j \Longrightarrow j \leq k \Longrightarrow i \leq k$
for $i\ j\ k$ :: *nat*
by (*rule order-trans*)

lemma *le-antisym*: $m \leq n \Longrightarrow n \leq m \Longrightarrow m = n$
for $m\ n$ :: *nat*
by (*rule antisym*)

lemma *nat-less-le*: $m < n \longleftrightarrow m \leq n \land m \neq n$
for $m\ n$ :: *nat*
by (*rule less-le*)

lemma *le-neq-implies-less*: $m \leq n \Longrightarrow m \neq n \Longrightarrow m < n$
for $m\ n$ :: *nat*
unfolding *less-le* **..**

lemma *nat-le-linear*: $m \leq n \lor n \leq m$
for $m\ n$ :: *nat*
by (*rule linear*)

lemmas *linorder-neqE-nat* = *linorder-neqE* [**where** *'a* = *nat*]

lemma *le-less-Suc-eq*: $m \leq n \Longrightarrow n < Suc\ m \longleftrightarrow n = m$
unfolding *less-Suc-eq-le* **by** *auto*

lemma *not-less-less-Suc-eq*: $\neg\ n < m \Longrightarrow n < Suc\ m \longleftrightarrow n = m$
unfolding *not-less* **by** (*rule le-less-Suc-eq*)

lemmas *not-less-simps* = *not-less-less-Suc-eq le-less-Suc-eq*

lemma *not0-implies-Suc*: $n \neq 0 \Longrightarrow \exists m.\ n = Suc\ m$
by (*cases n*) *simp-all*

lemma *gr0-implies-Suc*: $n > 0 \Longrightarrow \exists m.\ n = Suc\ m$
by (*cases n*) *simp-all*

lemma *gr-implies-not0*: $m < n \Longrightarrow n \neq 0$
  for *m n* :: *nat*
  by (*cases n*) *simp-all*

lemma *neq0-conv*[*iff*]: $n \neq 0 \longleftrightarrow 0 < n$
  for *n* :: *nat*
  by (*cases n*) *simp-all*

This theorem is useful with *blast*

lemma *gr0I*: $(n = 0 \Longrightarrow False) \Longrightarrow 0 < n$
  for *n* :: *nat*
  by (*rule neq0-conv*[*THEN iffD1*]) *iprover*

lemma *gr0-conv-Suc*: $0 < n \longleftrightarrow (\exists m.\ n = Suc\ m)$
  by (*fast intro*: *not0-implies-Suc*)

lemma *not-gr0* [*iff*]: $\neg\ 0 < n \longleftrightarrow n = 0$
  for *n* :: *nat*
  using *neq0-conv* **by** *blast*

lemma *Suc-le-D*: $Suc\ n \leq m' \Longrightarrow \exists m.\ m' = Suc\ m$
  by (*induct m'*) *simp-all*

Useful in certain inductive arguments

lemma *less-Suc-eq-0-disj*: $m < Suc\ n \longleftrightarrow m = 0 \lor (\exists j.\ m = Suc\ j \land j < n)$
  by (*cases m*) *simp-all*

lemma *All-less-Suc*: $(\forall i < Suc\ n.\ P\ i) = (P\ n \land (\forall i < n.\ P\ i))$
  by (*auto simp*: *less-Suc-eq*)

lemma *All-less-Suc2*: $(\forall i < Suc\ n.\ P\ i) = (P\ 0 \land (\forall i < n.\ P(Suc\ i)))$
  by (*auto simp*: *less-Suc-eq-0-disj*)

lemma *Ex-less-Suc*: $(\exists i < Suc\ n.\ P\ i) = (P\ n \lor (\exists i < n.\ P\ i))$
  by (*auto simp*: *less-Suc-eq*)

lemma *Ex-less-Suc2*: $(\exists i < Suc\ n.\ P\ i) = (P\ 0 \lor (\exists i < n.\ P(Suc\ i)))$
  by (*auto simp*: *less-Suc-eq-0-disj*)

mono (non-strict) doesn't imply increasing, as the function could be constant

lemma *strict-mono-imp-increasing*:
  fixes *n*::*nat*
  assumes *strict-mono f* **shows** $f\ n \geq n$
proof (*induction n*)
  case *0*
  then show *?case*
    by *auto*
next
  case (*Suc n*)

then show *?case*
unfolding *not-less-eq-eq* [*symmetric*]
using *Suc-n-not-le-n assms order-trans strict-mono-less-eq* **by** *blast*
qed

17.4.5 Monotonicity of Addition

lemma *Suc-pred* [*simp*]: $n > 0 \Longrightarrow Suc\ (n - Suc\ 0) = n$
by (*simp add*: *diff-Suc split*: *nat.split*)

lemma *Suc-diff-1* [*simp*]: $0 < n \Longrightarrow Suc\ (n - 1) = n$
unfolding *One-nat-def* **by** (*rule Suc-pred*)

lemma *nat-add-left-cancel-le* [*simp*]: $k + m \leq k + n \longleftrightarrow m \leq n$
for *k m n* :: *nat*
by (*induct k*) *simp-all*

lemma *nat-add-left-cancel-less* [*simp*]: $k + m < k + n \longleftrightarrow m < n$
for *k m n* :: *nat*
by (*induct k*) *simp-all*

lemma *add-gr-0* [*iff*]: $m + n > 0 \longleftrightarrow m > 0 \lor n > 0$
for *m n* :: *nat*
by (*auto dest*: *gr0-implies-Suc*)

strict, in 1st argument

lemma *add-less-mono1*: $i < j \Longrightarrow i + k < j + k$
for *i j k* :: *nat*
by (*induct k*) *simp-all*

strict, in both arguments

lemma *add-less-mono*:
fixes *i j k l* :: *nat*
assumes $i < j$ $k < l$ **shows** $i + k < j + l$
proof −
have $i + k < j + k$
by (*simp add*: *add-less-mono1 assms*)
also have ... $< j + l$
using ‹$i < j$› **by** (*induction j*) (*auto simp*: *assms*)
finally show *?thesis* **.**
qed

lemma *less-imp-Suc-add*: $m < n \Longrightarrow \exists k.\ n = Suc\ (m + k)$
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case*

  by (*simp add*: *order-le-less*)
  (*blast elim*!: *less-SucE intro*!: *Nat.add-0-right* [*symmetric*] *add-Suc-right* [*symmetric*])
qed

lemma *le-Suc-ex*: $k \leq l \Longrightarrow (\exists n.\ l = k + n)$
 for *k l* :: *nat*
 by (*auto simp*: *less-Suc-eq-le*[*symmetric*] *dest*: *less-imp-Suc-add*)

lemma *less-natE*:
 assumes ‹$m < n$›
 obtains *q* **where** ‹$n = Suc\ (m + q)$›
 using *assms* **by** (*auto dest*: *less-imp-Suc-add intro*: *that*)

strict, in 1st argument; proof is by induction on $k > 0$

lemma *mult-less-mono2*:
 fixes *i j* :: *nat*
 assumes $i < j$ **and** $0 < k$
 shows $k * i < k * j$
 using ‹$0 < k$›
proof (*induct k*)
 case *0*
 then show *?case* **by** *simp*
next
 case (*Suc k*)
 with ‹$i < j$› **show** *?case*
  by (*cases k*) (*simp-all add*: *add-less-mono*)
qed

Addition is the inverse of subtraction: if $n \leq m$ then $n + (m - n) = m$.

lemma *add-diff-inverse-nat*: $\neg\ m < n \Longrightarrow n + (m - n) = m$
 for *m n* :: *nat*
 by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *nat-le-iff-add*: $m \leq n \longleftrightarrow (\exists k.\ n = m + k)$
 for *m n* :: *nat*
 using *nat-add-left-cancel-le*[*of m 0*] **by** (*auto dest*: *le-Suc-ex*)

The naturals form an ordered *semidom* and a *dioid*.

instance *nat* :: *discrete-linordered-semidom*
proof
 fix *m n q* :: *nat*
 show ‹$0 < (1::nat)$›
  by *simp*
 show ‹$m \leq n \Longrightarrow q + m \leq q + n$›
  by *simp*
 show ‹$m < n \Longrightarrow 0 < q \Longrightarrow q * m < q * n$›
  by (*simp add*: *mult-less-mono2*)
 show ‹$m \neq 0 \Longrightarrow n \neq 0 \Longrightarrow m * n \neq 0$›
  by *simp*

show ‹*n* ≤ *m* ⟹ (*m* − *n*) + *n* = *m*›
by (*simp add*: *add-diff-inverse-nat add.commute linorder-not-less*)
show ‹*m* < *n* ⟷ *m* + *1* ≤ *n*›
by (*simp add*: *Suc-le-eq*)
qed

instance *nat* :: *dioid*
by *standard* (*rule nat-le-iff-add*)

declare *le0*[*simp del*] — This is now *0* ≤ *?x*
declare *le-0-eq*[*simp del*] — This is now (*?n* ≤ *0*) = (*?n* = *0*)
declare *not-less0*[*simp del*] — This is now ¬ *?n* < *0*
declare *not-gr0*[*simp del*] — This is now (¬ *0* < *?n*) = (*?n* = *0*)

instance *nat* :: *ordered-cancel-comm-monoid-add* **..**
instance *nat* :: *ordered-cancel-comm-monoid-diff* **..**

17.4.6 *min* and *max*

global-interpretation *bot-nat-0*: *ordering-top* ‹(≥)› ‹(>)› ‹*0*::*nat*›
by *standard simp*

global-interpretation *max-nat*: *semilattice-neutr-order max* ‹*0*::*nat*› ‹(≥)› ‹(>)›
by *standard* (*simp add*: *max-def*)

lemma *mono-Suc*: *mono Suc*
by (*rule monoI*) *simp*

lemma *min-0L* [*simp*]: *min 0 n* = *0*
for *n* :: *nat*
by (*rule min-absorb1*) *simp*

lemma *min-0R* [*simp*]: *min n 0* = *0*
for *n* :: *nat*
by (*rule min-absorb2*) *simp*

lemma *min-Suc-Suc* [*simp*]: *min* (*Suc m*) (*Suc n*) = *Suc* (*min m n*)
by (*simp add*: *mono-Suc min-of-mono*)

lemma *min-Suc1*: *min* (*Suc n*) *m* = (*case m of 0* ⇒ *0* | *Suc m′* ⇒ *Suc*(*min n m′*))
by (*simp split*: *nat.split*)

lemma *min-Suc2*: *min m* (*Suc n*) = (*case m of 0* ⇒ *0* | *Suc m′* ⇒ *Suc*(*min m′ n*))
by (*simp split*: *nat.split*)

lemma *max-0L* [*simp*]: *max 0 n* = *n*
for *n* :: *nat*

by (*fact max-nat.left-neutral*)

lemma *max-0R* [*simp*]: *max n 0* = *n*
for *n* :: *nat*
by (*fact max-nat.right-neutral*)

lemma *max-Suc-Suc* [*simp*]: *max* (*Suc m*) (*Suc n*) = *Suc* (*max m n*)
by (*simp add*: *mono-Suc max-of-mono*)

lemma *max-Suc1*: *max* (*Suc n*) *m* = (*case m of 0* ⇒ *Suc n* | *Suc m'* ⇒ *Suc* (*max n m'*))
by (*simp split*: *nat.split*)

lemma *max-Suc2*: *max m* (*Suc n*) = (*case m of 0* ⇒ *Suc n* | *Suc m'* ⇒ *Suc* (*max m' n*))
by (*simp split*: *nat.split*)

lemma *nat-mult-min-left*: *min m n* ∗ *q* = *min* (*m* ∗ *q*) (*n* ∗ *q*)
for *m n q* :: *nat*
by (*simp add*: *min-def not-le*)
(*auto dest*: *mult-right-le-imp-le mult-right-less-imp-less le-less-trans*)

lemma *nat-mult-min-right*: *m* ∗ *min n q* = *min* (*m* ∗ *n*) (*m* ∗ *q*)
for *m n q* :: *nat*
by (*simp add*: *min-def not-le*)
(*auto dest*: *mult-left-le-imp-le mult-left-less-imp-less le-less-trans*)

lemma *nat-add-max-left*: *max m n* + *q* = *max* (*m* + *q*) (*n* + *q*)
for *m n q* :: *nat*
by (*simp add*: *max-def*)

lemma *nat-add-max-right*: *m* + *max n q* = *max* (*m* + *n*) (*m* + *q*)
for *m n q* :: *nat*
by (*simp add*: *max-def*)

lemma *nat-mult-max-left*: *max m n* ∗ *q* = *max* (*m* ∗ *q*) (*n* ∗ *q*)
for *m n q* :: *nat*
by (*simp add*: *max-def not-le*)
(*auto dest*: *mult-right-le-imp-le mult-right-less-imp-less le-less-trans*)

lemma *nat-mult-max-right*: *m* ∗ *max n q* = *max* (*m* ∗ *n*) (*m* ∗ *q*)
for *m n q* :: *nat*
by (*simp add*: *max-def not-le*)
(*auto dest*: *mult-left-le-imp-le mult-left-less-imp-less le-less-trans*)

17.4.7 Additional theorems about (≤)

Complete induction, aka course-of-values induction

instance *nat* :: *wellorder*

```
proof
  fix P and n :: nat
  assume step: (⋀m. m < n ⟹ P m) ⟹ P n for n :: nat
  have ⋀q. q ≤ n ⟹ P q
  proof (induct n)
    case (0 n)
    have P 0 by (rule step) auto
    with 0 show ?case by auto
  next
    case (Suc m n)
    then have n ≤ m ∨ n = Suc m
      by (simp add: le-Suc-eq)
    then show ?case
    proof
      assume n ≤ m
      then show P n by (rule Suc(1))
    next
      assume n: n = Suc m
      show P n by (rule step) (rule Suc(1), simp add: n le-simps)
    qed
  qed
  then show P n by auto
qed


lemma Least-eq-0[simp]: P 0 ⟹ Least P = 0
  for P :: nat ⇒ bool
  by (rule Least-equality[OF - le0])

lemma Least-Suc:
  assumes P n ¬ P 0
  shows (LEAST n. P n) = Suc (LEAST m. P (Suc m))
proof (cases n)
  case (Suc m)
  show ?thesis
  proof (rule antisym)
    show (LEAST x. P x) ≤ Suc (LEAST x. P (Suc x))
      using assms Suc by (force intro: LeastI Least-le)
    have §: P (LEAST x. P x)
      by (blast intro: LeastI assms)
    show Suc (LEAST m. P (Suc m)) ≤ (LEAST n. P n)
    proof (cases (LEAST n. P n))
      case 0
      then show ?thesis
        using § by (simp add: assms)
    next
      case Suc
      with § show ?thesis
        by (auto simp: Least-le)
```

qed
qed
qed (*use assms* **in** *auto*)

lemma *Least-Suc2*: $P\ n \Longrightarrow Q\ m \Longrightarrow \neg\ P\ 0 \Longrightarrow \forall k.\ P\ (Suc\ k) = Q\ k \Longrightarrow Least\ P = Suc\ (Least\ Q)$
by (*erule* (*1*) *Least-Suc* [*THEN ssubst*]) *simp*

lemma *ex-least-nat-le*:
fixes $P :: nat \Rightarrow bool$
assumes $P\ n \ \neg\ P\ 0$
shows $\exists k \le n.\ (\forall i<k.\ \neg\ P\ i) \wedge P\ k$
proof (*cases n*)
case (*Suc m*)
with *assms* **show** *?thesis*
by (*blast intro*: *Least-le LeastI-ex dest*: *not-less-Least*)
qed (*use assms* **in** *auto*)

lemma *ex-least-nat-less*:
fixes $P :: nat \Rightarrow bool$
assumes $P\ n \ \neg\ P\ 0$
shows $\exists k<n.\ (\forall i \le k.\ \neg\ P\ i) \wedge P\ (Suc\ k)$
proof (*cases n*)
case (*Suc m*)
then obtain *k* **where** *k*: $k \le n\ \forall i<k.\ \neg\ P\ i\ P\ k$
using *ex-least-nat-le* [*OF assms*] **by** *blast*
show *?thesis*
by (*cases k*) (*use assms k less-eq-Suc-le* **in** *auto*)
qed (*use assms* **in** *auto*)

lemma *nat-less-induct*:
fixes $P :: nat \Rightarrow bool$
assumes $\bigwedge n.\ \forall m.\ m < n \longrightarrow P\ m \Longrightarrow P\ n$
shows $P\ n$
using *assms less-induct* **by** *blast*

lemma *measure-induct-rule* [*case-names less*]:
fixes $f :: 'a \Rightarrow 'b::wellorder$
assumes *step*: $\bigwedge x.\ (\bigwedge y.\ f\ y < f\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows $P\ a$
by (*induct* $m \equiv f\ a$ *arbitrary*: *a rule*: *less-induct*) (*auto intro*: *step*)

old style induction rules:

lemma *measure-induct*:
fixes $f :: 'a \Rightarrow 'b::wellorder$
shows $(\bigwedge x.\ \forall y.\ f\ y < f\ x \longrightarrow P\ y \Longrightarrow P\ x) \Longrightarrow P\ a$
by (*rule measure-induct-rule* [*of f P a*]) *iprover*

lemma *full-nat-induct*:
assumes *step*: $\bigwedge n.\ (\forall m.\ Suc\ m \leq n \longrightarrow P\ m) \Longrightarrow P\ n$
shows $P\ n$
by (*rule less-induct*) (*auto intro*: *step simp*:*le-simps*)

An induction rule for establishing binary relations

lemma *less-Suc-induct* [*consumes 1*]:
assumes *less*: $i < j$
and *step*: $\bigwedge i.\ P\ i\ (Suc\ i)$
and *trans*: $\bigwedge i\ j\ k.\ i < j \Longrightarrow j < k \Longrightarrow P\ i\ j \Longrightarrow P\ j\ k \Longrightarrow P\ i\ k$
shows $P\ i\ j$
proof −
from *less* **obtain** k **where** *j*: $j = Suc\ (i + k)$
by (*auto dest*: *less-imp-Suc-add*)
have $P\ i\ (Suc\ (i + k))$
proof (*induct k*)
case *0*
show *?case* **by** (*simp add*: *step*)
next
case (*Suc k*)
have $0 + i < Suc\ k + i$ **by** (*rule add-less-mono1*) *simp*
then have $i < Suc\ (i + k)$ **by** (*simp add*: *add.commute*)
from *trans*[*OF this lessI Suc step*]
show *?case* **by** *simp*
qed
then show $P\ i\ j$ **by** (*simp add*: *j*)
qed

The method of infinite descent, frequently used in number theory. Provided by Roelof Oosterhuis. $P\ n$ is true for all natural numbers if

- case "0": given $n = 0$ prove $P\ n$
- case "smaller": given $n > 0$ and $\neg\ P\ n$ prove there exists a smaller natural number m such that $\neg\ P\ m$.

lemma *infinite-descent*: $(\bigwedge n.\ \neg\ P\ n \Longrightarrow \exists m<n.\ \neg\ P\ m) \Longrightarrow P\ n$ **for** $P :: nat \Rightarrow bool$
— compact version without explicit base case
by (*induct n rule*: *less-induct*) *auto*

lemma *infinite-descent0* [*case-names 0 smaller*]:
fixes $P :: nat \Rightarrow bool$
assumes $P\ 0$
and $\bigwedge n.\ n > 0 \Longrightarrow \neg\ P\ n \Longrightarrow \exists m.\ m < n \wedge \neg\ P\ m$
shows $P\ n$
proof (*rule infinite-descent*)
fix n
show $\neg\ P\ n \Longrightarrow \exists m<n.\ \neg\ P\ m$

using *assms* **by** $(cases\ n > 0)$ *auto*
qed

Infinite descent using a mapping to *nat*: $P\ x$ is true for all $x \in D$ if there exists a $V \in D \Rightarrow nat$ and

- case "0": given $V\ x = 0$ prove $P\ x$
- "smaller": given $V\ x > 0$ and $\neg\ P\ x$ prove there exists a $y \in D$ such that $V\ y < V\ x$ and $\neg\ P\ y$.

corollary *infinite-descent0-measure* [*case-names 0 smaller*]:
fixes $V :: 'a \Rightarrow nat$
assumes *1*: $\bigwedge x.\ V\ x = 0 \Longrightarrow P\ x$
and *2*: $\bigwedge x.\ V\ x > 0 \Longrightarrow \neg\ P\ x \Longrightarrow \exists\, y.\ V\ y < V\ x \wedge \neg\ P\ y$
shows $P\ x$
proof −
obtain n **where** $n = V\ x$ **by** *auto*
moreover have $\bigwedge x.\ V\ x = n \Longrightarrow P\ x$
proof (*induct n rule*: *infinite-descent0*)
case *0*
with *1* **show** $P\ x$ **by** *auto*
next
case (*smaller n*)
then obtain x **where** $*$: $V\ x = n$ **and** $V\ x > 0 \wedge \neg\ P\ x$ **by** *auto*
with *2* **obtain** y **where** $V\ y < V\ x \wedge \neg\ P\ y$ **by** *auto*
with $*$ **obtain** m **where** $m = V\ y \wedge m < n \wedge \neg\ P\ y$ **by** *auto*
then show *?case* **by** *auto*
qed
ultimately show $P\ x$ **by** *auto*
qed

Again, without explicit base case:

lemma *infinite-descent-measure*:
fixes $V :: 'a \Rightarrow nat$
assumes $\bigwedge x.\ \neg\ P\ x \Longrightarrow \exists\, y.\ V\ y < V\ x \wedge \neg\ P\ y$
shows $P\ x$
proof −
from *assms* **obtain** n **where** $n = V\ x$ **by** *auto*
moreover have $\bigwedge x.\ V\ x = n \Longrightarrow P\ x$
proof −
have $\exists\, m < V\ x.\ \exists\, y.\ V\ y = m \wedge \neg\ P\ y$ **if** $\neg\ P\ x$ **for** x
using *assms* **and** *that* **by** *auto*
then show $\bigwedge x.\ V\ x = n \Longrightarrow P\ x$
by (*induct n rule*: *infinite-descent*, *auto*)
qed
ultimately show $P\ x$ **by** *auto*
qed

A (clumsy) way of lifting $<$ monotonicity to $\leq$ monotonicity

lemma *less-mono-imp-le-mono*:
fixes $f :: nat \Rightarrow nat$
and $i\ j :: nat$
assumes $\bigwedge i\ j{::}nat.\ i < j \Longrightarrow f\ i < f\ j$
and $i \leq j$
shows $f\ i \leq f\ j$
using *assms* **by** (*auto simp add*: *order-le-less*)

non-strict, in 1st argument

lemma *add-le-mono1*: $i \leq j \Longrightarrow i + k \leq j + k$
for $i\ j\ k :: nat$
by (*rule add-right-mono*)

non-strict, in both arguments

lemma *add-le-mono*: $i \leq j \Longrightarrow k \leq l \Longrightarrow i + k \leq j + l$
for $i\ j\ k\ l :: nat$
by (*rule add-mono*)

lemma *le-add2*: $n \leq m + n$
for $m\ n :: nat$
by *simp*

lemma *le-add1*: $n \leq n + m$
for $m\ n :: nat$
by *simp*

lemma *less-add-Suc1*: $i < Suc\ (i + m)$
by (*rule le-less-trans*, *rule le-add1*, *rule lessI*)

lemma *less-add-Suc2*: $i < Suc\ (m + i)$
by (*rule le-less-trans*, *rule le-add2*, *rule lessI*)

lemma *less-iff-Suc-add*: $m < n \longleftrightarrow (\exists k.\ n = Suc\ (m + k))$
by (*iprover intro*!: *less-add-Suc1 less-imp-Suc-add*)

lemma *trans-le-add1*: $i \leq j \Longrightarrow i \leq j + m$
for $i\ j\ m :: nat$
by (*rule le-trans*, *assumption*, *rule le-add1*)

lemma *trans-le-add2*: $i \leq j \Longrightarrow i \leq m + j$
for $i\ j\ m :: nat$
by (*rule le-trans*, *assumption*, *rule le-add2*)

lemma *trans-less-add1*: $i < j \Longrightarrow i < j + m$
for $i\ j\ m :: nat$
by (*rule less-le-trans*, *assumption*, *rule le-add1*)

lemma *trans-less-add2*: $i < j \Longrightarrow i < m + j$

for *i j m* :: *nat*
by (*rule less-le-trans*, *assumption*, *rule le-add2*)

lemma *add-lessD1*: $i + j < k \Longrightarrow i < k$
for *i j k* :: *nat*
by (*rule le-less-trans* [*of* - *i+j*]) (*simp-all add*: *le-add1*)

lemma *not-add-less1* [*iff*]: $\neg\ i + j < i$
for *i j* :: *nat*
by *simp*

lemma *not-add-less2* [*iff*]: $\neg\ j + i < i$
for *i j* :: *nat*
by *simp*

lemma *add-leD1*: $m + k \leq n \Longrightarrow m \leq n$
for *k m n* :: *nat*
by (*rule order-trans* [*of* - *m* + *k*]) (*simp-all add*: *le-add1*)

lemma *add-leD2*: $m + k \leq n \Longrightarrow k \leq n$
for *k m n* :: *nat*
by (*force simp add*: *add.commute dest*: *add-leD1*)

lemma *add-leE*: $m + k \leq n \Longrightarrow (m \leq n \Longrightarrow k \leq n \Longrightarrow R) \Longrightarrow R$
for *k m n* :: *nat*
by (*blast dest*: *add-leD1 add-leD2*)

needs $\bigwedge k$ for *ac-simps* to work

lemma *less-add-eq-less*: $\bigwedge k.\ k < l \Longrightarrow m + l = k + n \Longrightarrow m < n$
for *l m n* :: *nat*
by (*force simp del*: *add-Suc-right simp add*: *less-iff-Suc-add add-Suc-right* [*symmetric*] *ac-simps*)

17.4.8 More results about difference

lemma *Suc-diff-le*: $n \leq m \Longrightarrow Suc\ m - n = Suc\ (m - n)$
by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *diff-less-Suc*: $m - n < Suc\ m$
by (*induct m n rule*: *diff-induct*) (*auto simp*: *less-Suc-eq*)

lemma *diff-le-self* [*simp*]: $m - n \leq m$
for *m n* :: *nat*
by (*induct m n rule*: *diff-induct*) (*simp-all add*: *le-SucI*)

lemma *less-imp-diff-less*: $j < k \Longrightarrow j - n < k$
for *j k n* :: *nat*
by (*rule le-less-trans*, *rule diff-le-self*)

lemma *diff-Suc-less* [*simp*]: $0 < n \Longrightarrow n - Suc\ i < n$
  by (*cases n*) (*auto simp add*: *le-simps*)

lemma *diff-add-assoc*: $k \leq j \Longrightarrow (i + j) - k = i + (j - k)$
  for *i j k* :: *nat*
  by (*fact ordered-cancel-comm-monoid-diff-class.diff-add-assoc*)

lemma *add-diff-assoc* [*simp*]: $k \leq j \Longrightarrow i + (j - k) = i + j - k$
  for *i j k* :: *nat*
  by (*fact ordered-cancel-comm-monoid-diff-class.add-diff-assoc*)

lemma *diff-add-assoc2*: $k \leq j \Longrightarrow (j + i) - k = (j - k) + i$
  for *i j k* :: *nat*
  by (*fact ordered-cancel-comm-monoid-diff-class.diff-add-assoc2*)

lemma *add-diff-assoc2* [*simp*]: $k \leq j \Longrightarrow j - k + i = j + i - k$
  for *i j k* :: *nat*
  by (*fact ordered-cancel-comm-monoid-diff-class.add-diff-assoc2*)

lemma *le-imp-diff-is-add*: $i \leq j \Longrightarrow (j - i = k) = (j = k + i)$
  for *i j k* :: *nat*
  by *auto*

lemma *diff-is-0-eq* [*simp*]: $m - n = 0 \longleftrightarrow m \leq n$
  for *m n* :: *nat*
  by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *diff-is-0-eq'* [*simp*]: $m \leq n \Longrightarrow m - n = 0$
  for *m n* :: *nat*
  by *simp*

lemma *zero-less-diff* [*simp*]: $0 < n - m \longleftrightarrow m < n$
  for *m n* :: *nat*
  by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *less-imp-add-positive*:
  assumes $i < j$
  shows $\exists k{::}nat.\ 0 < k \wedge i + k = j$
proof
  from *assms* **show** $0 < j - i \wedge i + (j - i) = j$
    by (*simp add*: *order-less-imp-le*)
qed

a nice rewrite for bounded subtraction

lemma *nat-minus-add-max*: $n - m + m = max\ n\ m$
  for *m n* :: *nat*
  by (*simp add*: *max-def not-le order-less-imp-le*)

lemma *nat-diff-split*: $P\ (a - b) \longleftrightarrow (a < b \longrightarrow P\ 0) \wedge (\forall d.\ a = b + d \longrightarrow P\ d)$

for $a\ b :: nat$
— elimination of $-$ on *nat*
by $(cases\ a < b)\ (auto\ simp\ add: not\text{-}less\ le\text{-}less\ dest!: add\text{-}eq\text{-}self\text{-}zero\ [OF\ sym])$

lemma *nat-diff-split-asm*: $P\ (a - b) \longleftrightarrow \neg\ (a < b \wedge \neg\ P\ 0 \vee (\exists\, d.\ a = b + d \wedge \neg\ P\ d))$
for $a\ b :: nat$
— elimination of $-$ on *nat* in assumptions
by $(auto\ split: nat\text{-}diff\text{-}split)$

lemmas *nat-diff-splits* = *nat-diff-split nat-diff-split-asm*

lemma *Suc-pred'*: $0 < n \Longrightarrow n = Suc(n - 1)$
by *simp*

lemma *add-eq-if*: $m + n = (if\ m = 0\ then\ n\ else\ Suc\ ((m - 1) + n))$
unfolding *One-nat-def* **by** $(cases\ m)\ simp\text{-}all$

lemma *mult-eq-if*: $m * n = (if\ m = 0\ then\ 0\ else\ n + ((m - 1) * n))$
for $m\ n :: nat$
by $(cases\ m)\ simp\text{-}all$

lemma *Suc-diff-eq-diff-pred*: $0 < n \Longrightarrow Suc\ m - n = m - (n - 1)$
by $(cases\ n)\ simp\text{-}all$

lemma *diff-Suc-eq-diff-pred*: $m - Suc\ n = (m - 1) - n$
by $(cases\ m)\ simp\text{-}all$

lemma *Let-Suc* [*simp*]: $Let\ (Suc\ n)\ f \equiv f\ (Suc\ n)$
by $(fact\ Let\text{-}def)$

17.4.9 Monotonicity of multiplication

lemma *mult-le-mono1*: $i \le j \Longrightarrow i * k \le j * k$
for $i\ j\ k :: nat$
by $(simp\ add: mult\text{-}right\text{-}mono)$

lemma *mult-le-mono2*: $i \le j \Longrightarrow k * i \le k * j$
for $i\ j\ k :: nat$
by $(simp\ add: mult\text{-}left\text{-}mono)$

$\le$ monotonicity, BOTH arguments

lemma *mult-le-mono*: $i \le j \Longrightarrow k \le l \Longrightarrow i * k \le j * l$
for $i\ j\ k\ l :: nat$
by $(simp\ add: mult\text{-}mono)$

lemma *mult-less-mono1*: $i < j \Longrightarrow 0 < k \Longrightarrow i * k < j * k$
for $i\ j\ k :: nat$
by $(simp\ add: mult\text{-}strict\text{-}right\text{-}mono)$

Differs from the standard *zero-less-mult-iff* in that there are no negative numbers.

```
lemma nat-0-less-mult-iff [simp]: 0 < m * n ⟷ 0 < m ∧ 0 < n
  for m n :: nat
proof (induct m)
  case 0
  then show ?case by simp
next
  case (Suc m)
  then show ?case by (cases n) simp-all
qed

lemma one-le-mult-iff [simp]: Suc 0 ≤ m * n ⟷ Suc 0 ≤ m ∧ Suc 0 ≤ n
proof (induct m)
  case 0
  then show ?case by simp
next
  case (Suc m)
  then show ?case by (cases n) simp-all
qed

lemma mult-less-cancel2 [simp]: m * k < n * k ⟷ 0 < k ∧ m < n
  for k m n :: nat
proof (intro iffI conjI)
  assume m: m * k < n * k
  then show 0 < k
    by (cases k) auto
  show m < n
  proof (cases k)
    case 0
    then show ?thesis
      using m by auto
  next
    case (Suc k')
    then show ?thesis
      using m
      by (simp flip: linorder-not-le) (blast intro: add-mono mult-le-mono1)
  qed
next
  assume 0 < k ∧ m < n
  then show m * k < n * k
    by (blast intro: mult-less-mono1)
qed

lemma mult-less-cancel1 [simp]: k * m < k * n ⟷ 0 < k ∧ m < n
  for k m n :: nat
  by (simp add: mult.commute [of k])

lemma mult-le-cancel1 [simp]: k * m ≤ k * n ⟷ (0 < k ⟶ m ≤ n)
```

for *k m n* :: *nat*
by (*simp add*: *linorder-not-less* [*symmetric*], *auto*)

lemma *mult-le-cancel2* [*simp*]: $m * k \leq n * k \longleftrightarrow (0 < k \longrightarrow m \leq n)$
for *k m n* :: *nat*
by (*simp add*: *linorder-not-less* [*symmetric*], *auto*)

lemma *Suc-mult-less-cancel1*: $Suc\ k * m < Suc\ k * n \longleftrightarrow m < n$
by (*subst mult-less-cancel1*) *simp*

lemma *Suc-mult-le-cancel1*: $Suc\ k * m \leq Suc\ k * n \longleftrightarrow m \leq n$
by (*subst mult-le-cancel1*) *simp*

lemma *le-square*: $m \leq m * m$
for *m* :: *nat*
by (*cases m*) (*auto intro*: *le-add1*)

lemma *le-cube*: $m \leq m * (m * m)$
for *m* :: *nat*
by (*cases m*) (*auto intro*: *le-add1*)

Lemma for *gcd*

lemma *mult-eq-self-implies-10*:
fixes *m n* :: *nat*
assumes $m = m * n$ **shows** $n = 1 \lor m = 0$
proof (*rule disjCI*)
assume $m \neq 0$
show $n = 1$
proof (*cases n 1*::*nat rule*: *linorder-cases*)
case *greater*
show *?thesis*
using *assms mult-less-mono2* [*OF greater*, *of m*] ‹$m \neq 0$› **by** *auto*
qed (*use assms* ‹$m \neq 0$› **in** *auto*)
qed

lemma *mono-times-nat*:
fixes *n* :: *nat*
assumes $n > 0$
shows *mono* (*times n*)
proof
fix *m q* :: *nat*
assume $m \leq q$
with *assms* **show** $n * m \leq n * q$ **by** *simp*
qed

The lattice order on *nat*.

instantiation *nat* :: *distrib-lattice*
begin

definition (*inf* :: *nat* ⇒ *nat* ⇒ *nat*) = *min*

definition (*sup* :: *nat* ⇒ *nat* ⇒ *nat*) = *max*

instance
  by *intro-classes*
    (*auto simp add*: *inf-nat-def sup-nat-def max-def not-le min-def*
      intro: *order-less-imp-le antisym elim*!: *order-trans order-less-trans*)

end

17.5 Natural operation of natural numbers on functions

We use the same logical constant for the power operations on functions and relations, in order to share the same syntax.

consts *compow* :: *nat* ⇒ *'a* ⇒ *'a*

abbreviation *compower* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ‹^^› *80*)
  where *f* ^^ *n* ≡ *compow n f*

notation (*latex* **output**)
  compower (‹(-⁻)› [*1000*] *1000*)

f ^^ *n* = *f* ∘ . . . ∘ *f*, the *n*-fold composition of *f*

overloading
  funpow ≡ *compow* :: *nat* ⇒ (*'a* ⇒ *'a*) ⇒ (*'a* ⇒ *'a*)
begin

primrec *funpow* :: *nat* ⇒ (*'a* ⇒ *'a*) ⇒ *'a* ⇒ *'a*
  where
    funpow 0 f = *id*
  | *funpow* (*Suc n*) *f* = *f* ∘ *funpow n f*

end

lemma *funpow-0* [*simp*]: (*f* ^^ *0*) *x* = *x*
  by *simp*

lemma *funpow-Suc-right*: *f* ^^ *Suc n* = *f* ^^ *n* ∘ *f*
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  fix *n*
  assume *f* ^^ *Suc n* = *f* ^^ *n* ∘ *f*
  then show *f* ^^ *Suc* (*Suc n*) = *f* ^^ *Suc n* ∘ *f*
    by (*simp add*: *o-assoc*)
qed

lemmas *funpow-simps-right* = *funpow.simps*(*1*) *funpow-Suc-right*

For code generation.

context
begin

qualified definition *funpow* :: *nat* ⇒ ('*a* ⇒ '*a*) ⇒ '*a* ⇒ '*a*
  where *funpow-code-def* [*code-abbrev*]: *funpow* = *compow*

lemma [*code*]:
  funpow (*Suc n*) *f* = *f* ∘ *funpow n f*
  funpow 0 f = *id*
  by (*simp-all add*: *funpow-code-def*)

end

lemma *funpow-add*: *f* ^^ (*m* + *n*) = *f* ^^ *m* ∘ *f* ^^ *n*
  by (*induct m*) *simp-all*

lemma *funpow-mult*: (*f* ^^ *m*) ^^ *n* = *f* ^^ (*m* ∗ *n*)
  for *f* :: '*a* ⇒ '*a*
  by (*induct n*) (*simp-all add*: *funpow-add*)

lemma *funpow-swap1*: *f* ((*f* ^^ *n*) *x*) = (*f* ^^ *n*) (*f x*)
proof −
  have *f* ((*f* ^^ *n*) *x*) = (*f* ^^ (*n* + *1*)) *x* **by** *simp*
  also have ... = (*f* ^^ *n* ∘ *f* ^^ *1*) *x* **by** (*simp only*: *funpow-add*)
  also have ... = (*f* ^^ *n*) (*f x*) **by** *simp*
  finally show *?thesis* **.**
qed

lemma *comp-funpow*: *comp f* ^^ *n* = *comp* (*f* ^^ *n*)
  for *f* :: '*a* ⇒ '*a*
  by (*induct n*) *simp-all*

lemma *Suc-funpow*[*simp*]: *Suc* ^^ *n* = ((+) *n*)
  by (*induct n*) *simp-all*

lemma *id-funpow*[*simp*]: *id* ^^ *n* = *id*
  by (*induct n*) *simp-all*

lemma *funpow-mono*: *mono f* ⟹ *A* ≤ *B* ⟹ (*f* ^^ *n*) *A* ≤ (*f* ^^ *n*) *B*
  for *f* :: '*a* ⇒ ('*a*::*order*)
  by (*induct n*) (*auto simp*: *mono-def*)

lemma *funpow-mono2*:
  assumes *mono f*
    and *i* ≤ *j*

and $x \le y$
and $x \le f\ x$
shows $(f\ \hat{}\hat{}\ i)\ x \le (f\ \hat{}\hat{}\ j)\ y$
using *assms(2,3)*
proof (*induct j arbitrary*: *y*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc j*)
show *?case*
proof(*cases i = Suc j*)
case *True*
with *assms(1) Suc* **show** *?thesis*
by (*simp del*: *funpow.simps add*: *funpow-simps-right monoD funpow-mono*)
next
case *False*
with *assms(1,4) Suc* **show** *?thesis*
by (*simp del*: *funpow.simps add*: *funpow-simps-right le-eq-less-or-eq less-Suc-eq-le*)
(*simp add*: *Suc.hyps monoD order-subst1*)
qed
qed

lemma *inj-fn*[*simp*]:
fixes *f*::*'a ⇒ 'a*
assumes *inj f*
shows *inj* $(f\hat{}\hat{}n)$
proof (*induction n*)
case *Suc* **thus** *?case* **using** *inj-compose*[*OF assms Suc.IH*] **by** (*simp del*: *comp-apply*)
qed *simp*

lemma *surj-fn*[*simp*]:
fixes *f*::*'a ⇒ 'a*
assumes *surj f*
shows *surj* $(f\hat{}\hat{}n)$
proof (*induction n*)
case *Suc* **thus** *?case* **by** (*simp add*: *comp-surj*[*OF Suc.IH assms*] *del*: *comp-apply*)
qed *simp*

lemma *bij-fn*[*simp*]:
fixes *f*::*'a ⇒ 'a*
assumes *bij f*
shows *bij* $(f\hat{}\hat{}n)$
by (*rule bijI*[*OF inj-fn*[*OF bij-is-inj*[*OF assms*]] *surj-fn*[*OF bij-is-surj*[*OF assms*]]])

lemma *bij-betw-funpow*:
assumes *bij-betw f S S* **shows** *bij-betw* $(f\ \hat{}\hat{}\ n)$ *S S*
proof (*induct n*)
case *0* **then show** *?case* **by** (*auto simp*: *id-def*[*symmetric*])
next

```
  case (Suc n)
  then show ?case unfolding funpow.simps using assms by (rule bij-betw-trans)
qed
```

17.6 Kleene iteration

```
lemma Kleene-iter-lpfp:
  fixes f :: 'a::order-bot ⇒ 'a
  assumes mono f
    and f p ≤ p
  shows (f ^^ k) bot ≤ p
proof (induct k)
  case 0
  show ?case by simp
next
  case Suc
  show ?case
    using monoD[OF assms(1) Suc] assms(2) by simp
qed

lemma lfp-Kleene-iter:
  assumes mono f
    and (f ^^ Suc k) bot = (f ^^ k) bot
  shows lfp f = (f ^^ k) bot
proof (rule antisym)
  show lfp f ≤ (f ^^ k) bot
  proof (rule lfp-lowerbound)
    show f ((f ^^ k) bot) ≤ (f ^^ k) bot
      using assms(2) by simp
  qed
  show (f ^^ k) bot ≤ lfp f
    using Kleene-iter-lpfp[OF assms(1)] lfp-unfold[OF assms(1)] by simp
qed

lemma mono-pow: mono f ⟹ mono (f ^^ n)
  for f :: 'a ⇒ 'a::order
  by (induct n) (auto simp: mono-def)

lemma lfp-funpow:
  assumes f: mono f
  shows lfp (f ^^ Suc n) = lfp f
proof (rule antisym)
  show lfp f ≤ lfp (f ^^ Suc n)
  proof (rule lfp-lowerbound)
    have f (lfp (f ^^ Suc n)) = lfp (λx. f ((f ^^ n) x))
      unfolding funpow-Suc-right by (simp add: lfp-rolling f mono-pow comp-def)
    then show f (lfp (f ^^ Suc n)) ≤ lfp (f ^^ Suc n)
      by (simp add: comp-def)
  qed
```

have $(f \frown\frown n)$ $(lfp\ f) = lfp\ f$ **for** n
  by (*induct n*) (*auto intro*: *f lfp-fixpoint*)
then show $lfp\ (f \frown\frown Suc\ n) \leq lfp\ f$
  by (*intro lfp-lowerbound*) (*simp del*: *funpow.simps*)
qed

lemma *gfp-funpow*:
assumes *f*: *mono f*
shows $gfp\ (f \frown\frown Suc\ n) = gfp\ f$
proof (*rule antisym*)
show $gfp\ f \geq gfp\ (f \frown\frown Suc\ n)$
proof (*rule gfp-upperbound*)
have $f\ (gfp\ (f \frown\frown Suc\ n)) = gfp\ (\lambda x.\ f\ ((f \frown\frown n)\ x))$
unfolding *funpow-Suc-right* **by** (*simp add*: *gfp-rolling f mono-pow comp-def*)
then show $f\ (gfp\ (f \frown\frown Suc\ n)) \geq gfp\ (f \frown\frown Suc\ n)$
by (*simp add*: *comp-def*)
qed
have $(f \frown\frown n)\ (gfp\ f) = gfp\ f$ **for** n
by (*induct n*) (*auto intro*: *f gfp-fixpoint*)
then show $gfp\ (f \frown\frown Suc\ n) \geq gfp\ f$
by (*intro gfp-upperbound*) (*simp del*: *funpow.simps*)
qed

lemma *Kleene-iter-gpfp*:
fixes f :: *'a::order-top* ⇒ *'a*
assumes *mono f*
and $p \leq f\ p$
shows $p \leq (f \frown\frown k)\ top$
proof (*induct k*)
case *0*
show *?case* **by** *simp*
next
case *Suc*
show *?case*
using *monoD*[*OF assms*(*1*) *Suc*] *assms*(*2*) **by** *simp*
qed

lemma *gfp-Kleene-iter*:
assumes *mono f*
and $(f \frown\frown Suc\ k)\ top = (f \frown\frown k)\ top$
shows $gfp\ f = (f \frown\frown k)\ top$
(**is** *?lhs* = *?rhs*)
proof (*rule antisym*)
have $?rhs \leq f\ ?rhs$
using *assms*(*2*) **by** *simp*
then show $?rhs \leq ?lhs$
by (*rule gfp-upperbound*)
show $?lhs \leq ?rhs$
using *Kleene-iter-gpfp*[*OF assms*(*1*)] *gfp-unfold*[*OF assms*(*1*)] **by** *simp*

qed

17.7 Embedding of the naturals into any *semiring-1*: *of-nat*

context *semiring-1*
begin

definition *of-nat* :: *nat* ⇒ *'a*
  where *of-nat n* = (*plus 1* ^^ *n*) *0*

lemma *of-nat-simps* [*simp*]:
  shows *of-nat-0*: *of-nat 0* = *0*
    and *of-nat-Suc*: *of-nat* (*Suc m*) = *1* + *of-nat m*
  by (*simp-all add*: *of-nat-def*)

lemma *of-nat-1* [*simp*]: *of-nat 1* = *1*
  by (*simp add*: *of-nat-def*)

lemma *of-nat-add* [*simp*]: *of-nat* (*m* + *n*) = *of-nat m* + *of-nat n*
  by (*induct m*) (*simp-all add*: *ac-simps*)

lemma *of-nat-mult* [*simp*]: *of-nat* (*m* ∗ *n*) = *of-nat m* ∗ *of-nat n*
  by (*induct m*) (*simp-all add*: *ac-simps distrib-right*)

lemma *mult-of-nat-commute*: *of-nat x* ∗ *y* = *y* ∗ *of-nat x*
  by (*induct x*) (*simp-all add*: *algebra-simps*)

primrec *of-nat-aux* :: (*'a* ⇒ *'a*) ⇒ *nat* ⇒ *'a* ⇒ *'a*
  where
    of-nat-aux inc 0 i = *i*
  | *of-nat-aux inc* (*Suc n*) *i* = *of-nat-aux inc n* (*inc i*) — tail recursive

lemma *of-nat-code*: *of-nat n* = *of-nat-aux* (λ*i*. *i* + *1*) *n 0*
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  have ⋀*i*. *of-nat-aux* (λ*i*. *i* + *1*) *n* (*i* + *1*) = *of-nat-aux* (λ*i*. *i* + *1*) *n i* + *1*
    by (*induct n*) *simp-all*
  from *this* [*of 0*] **have** *of-nat-aux* (λ*i*. *i* + *1*) *n 1* = *of-nat-aux* (λ*i*. *i* + *1*) *n 0*
+ *1*
    by *simp*
  with *Suc* **show** *?case*
    by (*simp add*: *add.commute*)
qed

lemma *of-nat-of-bool* [*simp*]:
  of-nat (*of-bool P*) = *of-bool P*

by *auto*

end

declare *of-nat-code* [*code*]

context *semiring-1-cancel*
begin

lemma *of-nat-diff* [*simp*]:
‹*of-nat* (*m* − *n*) = *of-nat m* − *of-nat n*› **if** ‹*n* ≤ *m*›
proof −
from *that* **obtain** *q* **where** ‹*m* = *n* + *q*›
by (*blast dest*: *le-Suc-ex*)
then show *?thesis*
by *simp*
qed

lemma *of-nat-diff-if*: ‹*of-nat* (*m* − *n*) = (*if n≤m then of-nat m* − *of-nat n else 0*)›
by (*simp add*: *not-le less-imp-le*)

end

Class for unital semirings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *semiring-char-0* = *semiring-1* +
assumes *inj-of-nat*: *inj of-nat*
begin

lemma *of-nat-eq-iff* [*simp*]: *of-nat m* = *of-nat n* ⟷ *m* = *n*
by (*auto intro*: *inj-of-nat injD*)

Special cases where either operand is zero

lemma *of-nat-0-eq-iff* [*simp*]: *0* = *of-nat n* ⟷ *0* = *n*
by (*fact of-nat-eq-iff* [*of 0 n*, *unfolded of-nat-0*])

lemma *of-nat-eq-0-iff* [*simp*]: *of-nat m* = *0* ⟷ *m* = *0*
by (*fact of-nat-eq-iff* [*of m 0*, *unfolded of-nat-0*])

lemma *of-nat-1-eq-iff* [*simp*]: *1* = *of-nat n* ⟷ *n=1*
using *of-nat-eq-iff* **by** *fastforce*

lemma *of-nat-eq-1-iff* [*simp*]: *of-nat n* = *1* ⟷ *n=1*
using *of-nat-eq-iff* **by** *fastforce*

lemma *of-nat-neq-0* [*simp*]: *of-nat* (*Suc n*) ≠ *0*
unfolding *of-nat-eq-0-iff* **by** *simp*

lemma *of-nat-0-neq* [*simp*]: $0 \neq$ *of-nat* (*Suc n*)
unfolding *of-nat-0-eq-iff* **by** *simp*

end

class *ring-char-0* = *ring-1* + *semiring-char-0*

context *linordered-nonzero-semiring*
begin

lemma *of-nat-0-le-iff* [*simp*]: $0 \leq$ *of-nat n*
by (*induct n*) *simp-all*

lemma *of-nat-less-0-iff* [*simp*]: $\neg$ *of-nat m* < 0
by (*simp add*: *not-less*)

lemma *of-nat-mono*[*simp*]: $i \leq j \Longrightarrow$ *of-nat i* $\leq$ *of-nat j*
by (*auto simp*: *le-iff-add intro*!: *add-increasing2*)

lemma *of-nat-less-iff* [*simp*]: *of-nat m* $<$ *of-nat n* $\longleftrightarrow m < n$
proof(*induct m n rule*: *diff-induct*)
case (*1 m*) **then show** *?case*
by *auto*
next
case (*2 n*) **then show** *?case*
by (*simp add*: *add-pos-nonneg*)
next
case (*3 m n*)
then show *?case*
by (*auto simp*: *add-commute* [*of 1*] *add-mono1 not-less add-right-mono leD*)
qed

lemma *of-nat-le-iff* [*simp*]: *of-nat m* $\leq$ *of-nat n* $\longleftrightarrow m \leq n$
by (*simp add*: *not-less* [*symmetric*] *linorder-not-less* [*symmetric*])

lemma *less-imp-of-nat-less*: $m < n \Longrightarrow$ *of-nat m* $<$ *of-nat n*
by *simp*

lemma *of-nat-less-imp-less*: *of-nat m* $<$ *of-nat n* $\Longrightarrow m < n$
by *simp*

Every *linordered-nonzero-semiring* has characteristic zero.

subclass *semiring-char-0*
by *standard* (*auto intro*!: *injI simp add*: *order.eq-iff*)

Special cases where either operand is zero

lemma *of-nat-le-0-iff* [*simp*]: *of-nat m* $\leq 0 \longleftrightarrow m = 0$
by (*rule of-nat-le-iff* [*of - 0*, *simplified*])

lemma *of-nat-0-less-iff* [*simp*]: *0* < *of-nat n* ⟷ *0* < *n*
by (*rule of-nat-less-iff* [*of 0*, *simplified*])

end

context *linordered-nonzero-semiring*
begin

lemma *of-nat-max*: *of-nat* (*max x y*) = *max* (*of-nat x*) (*of-nat y*)
by (*auto simp*: *max-def ord-class.max-def*)

lemma *of-nat-min*: *of-nat* (*min x y*) = *min* (*of-nat x*) (*of-nat y*)
by (*auto simp*: *min-def ord-class.min-def*)

end

context *linordered-semidom*
begin

subclass *linordered-nonzero-semiring* **..**

subclass *semiring-char-0* **..**

end

context *linordered-idom*
begin

lemma *abs-of-nat* [*simp*]:
|*of-nat n*| = *of-nat n*
by (*simp add*: *abs-if*)

lemma *sgn-of-nat* [*simp*]:
sgn (*of-nat n*) = *of-bool* (*n* > *0*)
by *simp*

end

lemma *of-nat-id* [*simp*]: *of-nat n* = *n*
by (*induct n*) *simp-all*

lemma *of-nat-eq-id* [*simp*]: *of-nat* = *id*
by (*auto simp add*: *fun-eq-iff*)

17.8 The set of natural numbers

context *semiring-1*
begin

definition *Nats* :: *'a set* (‹ℕ›)
  where ℕ = *range of-nat*

lemma *of-nat-in-Nats* [*simp*]: *of-nat n* ∈ ℕ
  by (*simp add*: *Nats-def*)

lemma *Nats-0* [*simp*]: *0* ∈ ℕ
  using *of-nat-0* [*symmetric*] **unfolding** *Nats-def*
  by (*rule range-eqI*)

lemma *Nats-1* [*simp*]: *1* ∈ ℕ
  using *of-nat-1* [*symmetric*] **unfolding** *Nats-def*
  by (*rule range-eqI*)

lemma *Nats-add* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* + *b* ∈ ℕ
  unfolding *Nats-def* **using** *of-nat-add* [*symmetric*]
  by (*blast intro*: *range-eqI*)

lemma *Nats-mult* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* ∗ *b* ∈ ℕ
  unfolding *Nats-def* **using** *of-nat-mult* [*symmetric*]
  by (*blast intro*: *range-eqI*)

lemma *Nats-cases* [*cases set*: *Nats*]:
  assumes *x* ∈ ℕ
  obtains (*of-nat*) *n* **where** *x* = *of-nat n*
  unfolding *Nats-def*
proof −
  from ‹*x* ∈ ℕ› **have** *x* ∈ *range of-nat* **unfolding** *Nats-def* **.**
  then obtain *n* **where** *x* = *of-nat n* **..**
  then show *thesis* **..**
qed

lemma *Nats-induct* [*case-names of-nat*, *induct set*: *Nats*]: *x* ∈ ℕ ⟹ (⋀*n*. *P* (*of-nat n*)) ⟹ *P x*
  by (*rule Nats-cases*) *auto*

lemma *Nats-nonempty* [*simp*]: ℕ ≠ {}
  unfolding *Nats-def* **by** *auto*

end

lemma *Nats-diff* [*simp*]:
  fixes *a*:: *'a*::*linordered-idom*
  assumes *a* ∈ ℕ *b* ∈ ℕ *b* ≤ *a* **shows** *a* − *b* ∈ ℕ
proof −
  obtain *i* **where** *i*: *a* = *of-nat i*
    using *Nats-cases assms* **by** *blast*
  obtain *j* **where** *j*: *b* = *of-nat j*
    using *Nats-cases assms* **by** *blast*

have $j \le i$
using ‹$b \le a$› *i j of-nat-le-iff* **by** *blast*
then have $*$: *of-nat* $i -$ *of-nat* $j = ($*of-nat* $(i-j)$ :: $'a)$
by (*simp add*: *of-nat-diff*)
then show *?thesis*
by (*simp add*: $*$ *i j*)
qed

17.9 Further arithmetic facts concerning the natural numbers

lemma *subst-equals*:
assumes $t = s$ **and** $u = t$
shows $u = s$
using *assms(2,1)* **by** (*rule trans*)

locale *nat-arith*
begin

lemma *add1*: $(A::'a::comm\text{-}monoid\text{-}add) \equiv k + a \Longrightarrow A + b \equiv k + (a + b)$
by (*simp only*: *ac-simps*)

lemma *add2*: $(B::'a::comm\text{-}monoid\text{-}add) \equiv k + b \Longrightarrow a + B \equiv k + (a + b)$
by (*simp only*: *ac-simps*)

lemma *suc1*: $A == k + a \Longrightarrow Suc\ A \equiv k + Suc\ a$
by (*simp only*: *add-Suc-right*)

lemma *rule0*: $(a::'a::comm\text{-}monoid\text{-}add) \equiv a + 0$
by (*simp only*: *add-0-right*)

end

ML-file ‹*Tools/nat-arith.ML*›

simproc-setup *nateq-cancel-sums*
$((l::nat) + m = n \mid (l::nat) = m + n \mid Suc\ m = n \mid m = Suc\ n) =$
‹*K* (*try o Nat-Arith.cancel-eq-conv*)›

simproc-setup *natless-cancel-sums*
$((l::nat) + m < n \mid (l::nat) < m + n \mid Suc\ m < n \mid m < Suc\ n) =$
‹*K* (*try o Nat-Arith.cancel-less-conv*)›

simproc-setup *natle-cancel-sums*
$((l::nat) + m \le n \mid (l::nat) \le m + n \mid Suc\ m \le n \mid m \le Suc\ n) =$
‹*K* (*try o Nat-Arith.cancel-le-conv*)›

simproc-setup *natdiff-cancel-sums*
$((l::nat) + m - n \mid (l::nat) - (m + n) \mid Suc\ m - n \mid m - Suc\ n) =$

```
  ‹K (try o Nat-Arith.cancel-diff-conv)›

context order
begin

lemma lift-Suc-mono-le:
  assumes mono: ⋀n. f n ≤ f (Suc n)
    and n ≤ n′
  shows f n ≤ f n′
proof (cases n < n′)
  case True
  then show ?thesis
    by (induct n n′ rule: less-Suc-induct) (auto intro: mono)
next
  case False
  with ‹n ≤ n′› show ?thesis by auto
qed

lemma lift-Suc-antimono-le:
  assumes mono: ⋀n. f n ≥ f (Suc n)
    and n ≤ n′
  shows f n ≥ f n′
proof (cases n < n′)
  case True
  then show ?thesis
    by (induct n n′ rule: less-Suc-induct) (auto intro: mono)
next
  case False
  with ‹n ≤ n′› show ?thesis by auto
qed

lemma lift-Suc-mono-less:
  assumes mono: ⋀n. f n < f (Suc n)
    and n < n′
  shows f n < f n′
  using ‹n < n′› by (induct n n′ rule: less-Suc-induct) (auto intro: mono)

lemma lift-Suc-mono-less-iff: (⋀n. f n < f (Suc n)) ⟹ f n < f m ⟷ n < m
  by (blast intro: less-asym′ lift-Suc-mono-less [of f]
    dest: linorder-not-less[THEN iffD1] le-eq-less-or-eq [THEN iffD1])

end

lemma mono-iff-le-Suc: mono f ⟷ (∀ n. f n ≤ f (Suc n))
  unfolding mono-def by (auto intro: lift-Suc-mono-le [of f])

lemma antimono-iff-le-Suc: antimono f ⟷ (∀ n. f (Suc n) ≤ f n)
  unfolding antimono-def by (auto intro: lift-Suc-antimono-le [of f])
```

lemma *strict-mono-Suc-iff*: *strict-mono f* $\longleftrightarrow$ ($\forall$ *n*. *f n* < *f* (*Suc n*))
proof (*intro iffI strict-monoI*)
  assume ∗: $\forall$ *n*. *f n* < *f* (*Suc n*)
  fix *m n* :: *nat* **assume** *m* < *n*
  thus *f m* < *f n*
    by (*induction rule*: *less-Suc-induct*) (*use* ∗ **in** *auto*)
qed (*auto simp*: *strict-mono-def*)

lemma *strict-mono-add*: *strict-mono* (λ*n*::'*a*::*linordered-semidom*. *n* + *k*)
  by (*auto simp*: *strict-mono-def*)

lemma *mono-nat-linear-lb*:
  fixes *f* :: *nat* $\Rightarrow$ *nat*
  assumes $\bigwedge$*m n*. *m* < *n* $\Longrightarrow$ *f m* < *f n*
  shows *f m* + *k* $\le$ *f* (*m* + *k*)
proof (*induct k*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc k*)
  then have *Suc* (*f m* + *k*) $\le$ *Suc* (*f* (*m* + *k*)) **by** *simp*
  also from *assms* [*of m* + *k Suc* (*m* + *k*)] **have** *Suc* (*f* (*m* + *k*)) $\le$ *f* (*Suc* (*m* +
k))
    by (*simp add*: *Suc-le-eq*)
  finally show *?case* **by** *simp*
qed

lemma *bex-const1-if-mono-below-diag*: **fixes** *f* :: *nat* $\Rightarrow$ *nat* **assumes** *mono f*
shows *f n* < *n* $\Longrightarrow$ $\exists$ *i*<*n*. *f*(*Suc i*) = *f i*
proof(*induction n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  have ∗: *f n* $\le$ *f*(*Suc n*) **using** *assms*[*simplified mono-iff-le-Suc*] **by** *blast*
  from *Suc.prems*[*simplified less-Suc-eq*]
  show *?case*
  proof
    assume *f*(*Suc n*) < *n*
    from *order.strict-trans1* [*OF* ∗ *this*]
    show *?thesis* **using** *Suc.IH less-SucI* **by** *blast*
  next
    assume *f*(*Suc n*) = *n*
    from *order.strict-trans1* [*OF* ∗ *Suc.prems*, *simplified less-Suc-eq*]
    show *?case*
    proof
      assume *f n* < *n*
      thus *?thesis* **using** *Suc.IH less-SucI* **by** *blast*
    next

assume *f n* = *n*
with ‹*f*(*Suc n*) = *n*› **show** *?thesis* **by** *auto*
qed
qed
qed

lemma *bex-const1-if-mono-below-diag-Suc*:
fixes *f* :: *nat* ⇒ *nat* **assumes** *mono f f*(*Suc m*) ≤ *m*
shows ∃ *i*≤*m*. *f* (*Suc i*) = *f i*
using *bex-const1-if-mono-below-diag*[*OF assms*(*1*), *of Suc m*] *assms*(*2*) *less-Suc-eq-le*
by *blast*

Subtraction laws, mostly by Clemens Ballarin

lemma *diff-less-mono*:
fixes *a b c* :: *nat*
assumes *a* < *b* **and** *c* ≤ *a*
shows *a* − *c* < *b* − *c*
proof −
from *assms* **obtain** *d e* **where** *b* = *c* + (*d* + *e*) **and** *a* = *c* + *e* **and** *d* > *0*
by (*auto dest*!: *le-Suc-ex less-imp-Suc-add simp add*: *ac-simps*)
then show *?thesis* **by** *simp*
qed

lemma *less-diff-conv*: *i* < *j* − *k* ⟷ *i* + *k* < *j*
for *i j k* :: *nat*
by (*cases k* ≤ *j*) (*auto simp add*: *not-le dest*: *less-imp-Suc-add le-Suc-ex*)

lemma *less-diff-conv2*: *k* ≤ *j* ⟹ *j* − *k* < *i* ⟷ *j* < *i* + *k*
for *j k i* :: *nat*
by (*auto dest*: *le-Suc-ex*)

lemma *le-diff-conv*: *j* − *k* ≤ *i* ⟷ *j* ≤ *i* + *k*
for *j k i* :: *nat*
by (*cases k* ≤ *j*) (*auto simp add*: *not-le dest*!: *less-imp-Suc-add le-Suc-ex*)

lemma *diff-diff-cancel* [*simp*]: *i* ≤ *n* ⟹ *n* − (*n* − *i*) = *i*
for *i n* :: *nat*
by (*auto dest*: *le-Suc-ex*)

lemma *diff-less* [*simp*]: *0* < *n* ⟹ *0* < *m* ⟹ *m* − *n* < *m*
for *i n* :: *nat*
by (*auto dest*: *less-imp-Suc-add*)

Simplification of relational expressions involving subtraction

lemma *diff-diff-eq*: *k* ≤ *m* ⟹ *k* ≤ *n* ⟹ *m* − *k* − (*n* − *k*) = *m* − *n*
for *m n k* :: *nat*
by (*auto dest*!: *le-Suc-ex*)

hide-fact (**open**) *diff-diff-eq*

lemma *eq-diff-iff*: $k \le m \Longrightarrow k \le n \Longrightarrow m - k = n - k \longleftrightarrow m = n$
for *m n k* :: *nat*
by (*auto dest*: *le-Suc-ex*)

lemma *less-diff-iff*: $k \le m \Longrightarrow k \le n \Longrightarrow m - k < n - k \longleftrightarrow m < n$
for *m n k* :: *nat*
by (*auto dest*!: *le-Suc-ex*)

lemma *le-diff-iff*: $k \le m \Longrightarrow k \le n \Longrightarrow m - k \le n - k \longleftrightarrow m \le n$
for *m n k* :: *nat*
by (*auto dest*!: *le-Suc-ex*)

lemma *le-diff-iff'*: $a \le c \Longrightarrow b \le c \Longrightarrow c - a \le c - b \longleftrightarrow b \le a$
for *a b c* :: *nat*
by (*force dest*: *le-Suc-ex*)

(Anti)Monotonicity of subtraction – by Stephan Merz

lemma *diff-le-mono*: $m \le n \Longrightarrow m - l \le n - l$
for *m n l* :: *nat*
by (*auto dest*: *less-imp-le less-imp-Suc-add split*: *nat-diff-split*)

lemma *diff-le-mono2*: $m \le n \Longrightarrow l - n \le l - m$
for *m n l* :: *nat*
by (*auto dest*: *less-imp-le le-Suc-ex less-imp-Suc-add less-le-trans split*: *nat-diff-split*)

lemma *diff-less-mono2*: $m < n \Longrightarrow m < l \Longrightarrow l - n < l - m$
for *m n l* :: *nat*
by (*auto dest*: *less-imp-Suc-add split*: *nat-diff-split*)

lemma *diffs0-imp-equal*: $m - n = 0 \Longrightarrow n - m = 0 \Longrightarrow m = n$
for *m n* :: *nat*
by (*simp split*: *nat-diff-split*)

lemma *min-diff*: $min\ (m - i)\ (n - i) = min\ m\ n - i$
for *m n i* :: *nat*
by (*cases m n rule*: *le-cases*)
(*auto simp add*: *not-le min.absorb1 min.absorb2 min.absorb-iff1* [*symmetric*] *diff-le-mono*)

lemma *inj-on-diff-nat*:
fixes *k* :: *nat*
assumes $\bigwedge n.\ n \in N \Longrightarrow k \le n$
shows *inj-on* $(\lambda n.\ n - k)\ N$
proof (*rule inj-onI*)
fix *x y*
assume *a*: $x \in N$ $y \in N$ $x - k = y - k$
with *assms* **have** $x - k + k = y - k + k$ **by** *auto*
with *a assms* **show** $x = y$ **by** (*auto simp add*: *eq-diff-iff*)

qed

Rewriting to pull differences out

lemma *diff-diff-right* [*simp*]: $k \leq j \Longrightarrow i - (j - k) = i + k - j$
  for *i j k* :: *nat*
  by (*fact diff-diff-right*)

lemma *diff-Suc-diff-eq1* [*simp*]:
  assumes $k \leq j$
  shows $i - \mathit{Suc}\ (j - k) = i + k - \mathit{Suc}\ j$
proof −
  from *assms* **have** ∗: $\mathit{Suc}\ (j - k) = \mathit{Suc}\ j - k$
    by (*simp add*: *Suc-diff-le*)
  from *assms* **have** $k \leq \mathit{Suc}\ j$
    by (*rule order-trans*) *simp*
  with *diff-diff-right* [*of k Suc j i*] ∗ **show** *?thesis*
    by *simp*
qed

lemma *diff-Suc-diff-eq2* [*simp*]:
  assumes $k \leq j$
  shows $\mathit{Suc}\ (j - k) - i = \mathit{Suc}\ j - (k + i)$
proof −
  from *assms* **obtain** *n* **where** $j = k + n$
    by (*auto dest*: *le-Suc-ex*)
  moreover have $\mathit{Suc}\ n - i = (k + \mathit{Suc}\ n) - (k + i)$
    using *add-diff-cancel-left* [*of k Suc n i*] **by** *simp*
  ultimately show *?thesis* **by** *simp*
qed

lemma *Suc-diff-Suc*:
  assumes $n < m$
  shows $\mathit{Suc}\ (m - \mathit{Suc}\ n) = m - n$
proof −
  from *assms* **obtain** *q* **where** $m = n + \mathit{Suc}\ q$
    by (*auto dest*: *less-imp-Suc-add*)
  moreover define *r* **where** $r = \mathit{Suc}\ q$
  ultimately have $\mathit{Suc}\ (m - \mathit{Suc}\ n) = r$ **and** $m = n + r$
    by *simp-all*
  then show *?thesis* **by** *simp*
qed

lemma *one-less-mult*: $\mathit{Suc}\ 0 < n \Longrightarrow \mathit{Suc}\ 0 < m \Longrightarrow \mathit{Suc}\ 0 < m * n$
  using *less-1-mult* [*of n m*] **by** (*simp add*: *ac-simps*)

lemma *n-less-m-mult-n*: $0 < n \Longrightarrow \mathit{Suc}\ 0 < m \Longrightarrow n < m * n$
  using *mult-strict-right-mono* [*of 1 m n*] **by** *simp*

lemma *n-less-n-mult-m*: $0 < n \Longrightarrow \mathit{Suc}\ 0 < m \Longrightarrow n < n * m$

using *mult-strict-left-mono* [*of 1 m n*] **by** *simp*

Induction starting beyond zero

lemma *nat-induct-at-least* [*consumes 1*, *case-names base Suc*]:
P n **if** $n \geq m$ *P m* $\bigwedge n.\ n \geq m \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$
proof −
define *q* **where** $q = n - m$
with ‹$n \geq m$› **have** $n = m + q$
by *simp*
moreover have $P\ (m + q)$
by (*induction q*) (*use that* **in** *simp-all*)
ultimately show *P n*
by *simp*
qed

lemma *nat-induct-non-zero* [*consumes 1*, *case-names 1 Suc*]:
P n **if** $n > 0$ *P 1* $\bigwedge n.\ n > 0 \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$
proof −
from ‹$n > 0$› **have** $n \geq 1$
by (*cases n*) *simp-all*
moreover note ‹*P 1*›
moreover have $\bigwedge n.\ n \geq 1 \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$
using ‹$\bigwedge n.\ n > 0 \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$›
by (*simp add*: *Suc-le-eq*)
ultimately show *P n*
by (*rule nat-induct-at-least*)
qed

Specialized induction principles that work "backwards":

lemma *inc-induct* [*consumes 1*, *case-names base step*]:
assumes *less*: $i \leq j$
and *base*: *P j*
and *step*: $\bigwedge n.\ i \leq n \Longrightarrow n < j \Longrightarrow P\ (Suc\ n) \Longrightarrow P\ n$
shows *P i*
using *less step*
proof (*induct* $j - i$ *arbitrary*: *i*)
case (*0 i*)
then have $i = j$ **by** *simp*
with *base* **show** *?case* **by** *simp*
next
case (*Suc d n*)
from *Suc.hyps* **have** $n \neq j$ **by** *auto*
with *Suc* **have** $n < j$ **by** (*simp add*: *less-le*)
from ‹$Suc\ d = j - n$› **have** $d + 1 = j - n$ **by** *simp*
then have $d + 1 - 1 = j - n - 1$ **by** *simp*
then have $d = j - n - 1$ **by** *simp*
then have $d = j - (n + 1)$ **by** (*simp add*: *diff-diff-eq*)
then have $d = j - Suc\ n$ **by** *simp*
moreover from ‹$n < j$› **have** $Suc\ n \leq j$ **by** (*simp add*: *Suc-le-eq*)

ultimately have P $(Suc\ n)$
proof (*rule Suc.hyps*)
fix q
assume $Suc\ n \le q$
then have $n \le q$ **by** (*simp add*: *Suc-le-eq less-imp-le*)
moreover assume $q < j$
moreover assume P $(Suc\ q)$
ultimately show $P\ q$ **by** (*rule Suc.prems*)
qed
with *order-refl* ‹$n < j$› **show** $P\ n$ **by** (*rule Suc.prems*)
qed

lemma *strict-inc-induct* [*consumes 1*, *case-names base step*]:
assumes *less*: $i < j$
and *base*: $\bigwedge i.\ j = Suc\ i \Longrightarrow P\ i$
and *step*: $\bigwedge i.\ i < j \Longrightarrow P\ (Suc\ i) \Longrightarrow P\ i$
shows $P\ i$
using *less* **proof** (*induct* $j - i - 1$ *arbitrary*: i)
case $(0\ i)$
from ‹$i < j$› **obtain** n **where** $j = i + n$ **and** $n > 0$
by (*auto dest*!: *less-imp-Suc-add*)
with *0* **have** $j = Suc\ i$
by (*auto intro*: *order-antisym simp add*: *Suc-le-eq*)
with *base* **show** *?case* **by** *simp*
next
case $(Suc\ d\ i)$
from ‹$Suc\ d = j - i - 1$› **have** $*$: $Suc\ d = j - Suc\ i$
by (*simp add*: *diff-diff-add*)
then have $Suc\ d - 1 = j - Suc\ i - 1$ **by** *simp*
then have $d = j - Suc\ i - 1$ **by** *simp*
moreover from $*$ **have** $j - Suc\ i \ne 0$ **by** *auto*
then have $Suc\ i < j$ **by** (*simp add*: *not-le*)
ultimately have $P\ (Suc\ i)$ **by** (*rule Suc.hyps*)
with ‹$i < j$› **show** $P\ i$ **by** (*rule step*)
qed

lemma *zero-induct-lemma*: $P\ k \Longrightarrow (\bigwedge n.\ P\ (Suc\ n) \Longrightarrow P\ n) \Longrightarrow P\ (k - i)$
using *inc-induct*[*of* $k - i\ k\ P$, *simplified*] **by** *blast*

lemma *zero-induct*: $P\ k \Longrightarrow (\bigwedge n.\ P\ (Suc\ n) \Longrightarrow P\ n) \Longrightarrow P\ 0$
using *inc-induct*[*of* $0\ k\ P$] **by** *blast*

Further induction rule similar to $[\![?i \le ?j;\ ?P\ ?j;\ \bigwedge n.\ [\![?i \le n;\ n < ?j;\ ?P\ (Suc\ n)]\!] \Longrightarrow ?P\ n]\!] \Longrightarrow ?P\ ?i$.

lemma *dec-induct* [*consumes 1*, *case-names base step*]:
$i \le j \Longrightarrow P\ i \Longrightarrow (\bigwedge n.\ i \le n \Longrightarrow n < j \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)) \Longrightarrow P\ j$
proof (*induct* j *arbitrary*: i)
case *0*
then show *?case* **by** *simp*

next
  case (*Suc j*)
  from *Suc.prems* **consider** $i \le j \mid i = Suc\ j$
    by (*auto simp add*: *le-Suc-eq*)
  then show *?case*
  proof *cases*
    case *1*
    moreover have $j < Suc\ j$ **by** *simp*
    moreover have $P\ j$ **using** ‹$i \le j$› ‹$P\ i$›
    proof (*rule Suc.hyps*)
      fix q
      assume $i \le q$
      moreover assume $q < j$ **then have** $q < Suc\ j$
        by (*simp add*: *less-Suc-eq*)
      moreover assume $P\ q$
      ultimately show $P\ (Suc\ q)$ **by** (*rule Suc.prems*)
    qed
    ultimately show $P\ (Suc\ j)$ **by** (*rule Suc.prems*)
  next
    case *2*
    with ‹$P\ i$› **show** $P\ (Suc\ j)$ **by** *simp*
  qed
qed

lemma *transitive-stepwise-le*:
  assumes $m \le n$ $\bigwedge x.\ R\ x\ x$ $\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$ **and** $\bigwedge n.\ R\ n\ (Suc\ n)$
  shows $R\ m\ n$
using ‹$m \le n$›
  by (*induction rule*: *dec-induct*) (*use assms* **in** *blast*)+

17.9.1 Greatest operator

lemma *ex-has-greatest-nat*:
  $P\ (k{::}nat) \Longrightarrow \forall y.\ P\ y \longrightarrow y \le b \Longrightarrow \exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow y \le x)$
proof (*induction* $b-k$ *arbitrary*: $b\ k$ *rule*: *less-induct*)
  case *less*
  show *?case*
  proof *cases*
    assume $\exists n>k.\ P\ n$
    then obtain n **where** $n>k\ P\ n$ **by** *blast*
    have $n \le b$ **using** ‹$P\ n$› *less.prems*(*2*) **by** *auto*
    hence $b-n < b-k$
      by(*rule diff-less-mono2*[*OF* ‹$k<n$› *less-le-trans*[*OF* ‹$k<n$›]])
    from *less.hyps*[*OF this* ‹$P\ n$› *less.prems*(*2*)]
    show *?thesis* **.**
  next
    assume $\neg\ (\exists n>k.\ P\ n)$
    hence $\forall y.\ P\ y \longrightarrow y \le k$ **by** (*auto simp*: *not-less*)

thus *?thesis* **using** *less.prems(1)* **by** *auto*
qed
qed

lemma
fixes *k::nat*
assumes $P\ k$ **and** *minor*: $\bigwedge y.\ P\ y \Longrightarrow y \leq b$
shows *GreatestI-nat*: $P\ (Greatest\ P)$
and *Greatest-le-nat*: $k \leq Greatest\ P$
proof −
obtain x **where** $P\ x\ \bigwedge y.\ P\ y \Longrightarrow y \leq x$
using *assms ex-has-greatest-nat* **by** *blast*
with ‹$P\ k$› **show** $P\ (Greatest\ P)\ k \leq Greatest\ P$
using *GreatestI2-order* **by** *blast+*
qed

lemma *GreatestI-ex-nat*:
$\llbracket \exists k::nat.\ P\ k;\ \bigwedge y.\ P\ y \Longrightarrow y \leq b \rrbracket \Longrightarrow P\ (Greatest\ P)$
by (*blast intro*: *GreatestI-nat*)

17.10 Monotonicity of *funpow*

lemma *funpow-increasing*: $m \leq n \Longrightarrow mono\ f \Longrightarrow (f\ \widehat{}\widehat{}\ n)\ \top \leq (f\ \widehat{}\widehat{}\ m)\ \top$
for f :: $'a::order\text{-}top \Rightarrow\ 'a$
by (*induct rule*: *inc-induct*)
(*auto simp del*: *funpow.simps(2)* *simp add*: *funpow-Suc-right*
intro: *order-trans[OF - funpow-mono]*)

lemma *funpow-decreasing*: $m \leq n \Longrightarrow mono\ f \Longrightarrow (f\ \widehat{}\widehat{}\ m)\ \bot \leq (f\ \widehat{}\widehat{}\ n)\ \bot$
for f :: $'a::order\text{-}bot \Rightarrow\ 'a$
by (*induct rule*: *dec-induct*)
(*auto simp del*: *funpow.simps(2)* *simp add*: *funpow-Suc-right*
intro: *order-trans[OF - funpow-mono]*)

lemma *mono-funpow*: $mono\ Q \Longrightarrow mono\ (\lambda i.\ (Q\ \widehat{}\widehat{}\ i)\ \bot)$
for Q :: $'a::order\text{-}bot \Rightarrow\ 'a$
by (*auto intro!*: *funpow-decreasing simp*: *mono-def*)

lemma *antimono-funpow*: $mono\ Q \Longrightarrow antimono\ (\lambda i.\ (Q\ \widehat{}\widehat{}\ i)\ \top)$
for Q :: $'a::order\text{-}top \Rightarrow\ 'a$
by (*auto intro!*: *funpow-increasing simp*: *antimono-def*)

17.11 The divides relation on *nat*

lemma *dvd-1-left* [*iff*]: *Suc 0 dvd k*
by (*simp add*: *dvd-def*)

lemma *dvd-1-iff-1* [*simp*]: $m\ dvd\ Suc\ 0 \longleftrightarrow m = Suc\ 0$
by (*simp add*: *dvd-def*)

lemma *nat-dvd-1-iff-1* [*simp*]: *m dvd 1* ⟷ *m* = *1*
  for *m* :: *nat*
  by (*simp add*: *dvd-def*)

lemma *dvd-antisym*: *m dvd n* ⟹ *n dvd m* ⟹ *m* = *n*
  for *m n* :: *nat*
  unfolding *dvd-def* **by** (*force dest*: *mult-eq-self-implies-10 simp add*: *mult.assoc*)

lemma *dvd-diff-nat* [*simp*]: *k dvd m* ⟹ *k dvd n* ⟹ *k dvd* (*m* − *n*)
  for *k m n* :: *nat*
  unfolding *dvd-def* **by** (*blast intro*: *right-diff-distrib′* [*symmetric*])

lemma *dvd-diffD*:
  fixes *k m n* :: *nat*
  assumes *k dvd m* − *n k dvd n n* ≤ *m*
  shows *k dvd m*
proof −
  have *k dvd n* + (*m* − *n*)
    using *assms* **by** (*blast intro*: *dvd-add*)
  with *assms* **show** *?thesis*
    by *simp*
qed

lemma *dvd-diffD1*: *k dvd m* − *n* ⟹ *k dvd m* ⟹ *n* ≤ *m* ⟹ *k dvd n*
  for *k m n* :: *nat*
  by (*drule-tac m* = *m* **in** *dvd-diff-nat*) *auto*

lemma *dvd-mult-cancel*:
  fixes *m n k* :: *nat*
  assumes *k* ∗ *m dvd k* ∗ *n* **and** *0* < *k*
  shows *m dvd n*
proof −
  from *assms*(*1*) **obtain** *q* **where** *k* ∗ *n* = (*k* ∗ *m*) ∗ *q* **..**
  then have *k* ∗ *n* = *k* ∗ (*m* ∗ *q*) **by** (*simp add*: *ac-simps*)
  with ‹*0* < *k*› **have** *n* = *m* ∗ *q* **by** (*auto simp add*: *mult-left-cancel*)
  then show *?thesis* **..**
qed

lemma *dvd-mult-cancel1*:
  fixes *m n* :: *nat*
  assumes *0* < *m*
  shows *m* ∗ *n dvd m* ⟷ *n* = *1*
proof
  assume *m* ∗ *n dvd m*
  then have *m* ∗ *n dvd m* ∗ *1*
    by *simp*
  then have *n dvd 1*
    by (*iprover intro*: *assms dvd-mult-cancel*)
  then show *n* = *1*

by *auto*
qed *auto*

lemma *dvd-mult-cancel2*: $0 < m \Longrightarrow n * m$ *dvd* $m \longleftrightarrow n = 1$
for $m\ n$ :: *nat*
using *dvd-mult-cancel1* [*of m n*] **by** (*simp add*: *ac-simps*)

lemma *dvd-imp-le*: k *dvd* $n \Longrightarrow 0 < n \Longrightarrow k \leq n$
for $k\ n$ :: *nat*
by (*auto elim*!: *dvdE*) (*auto simp add*: *gr0-conv-Suc*)

lemma *nat-dvd-not-less*: $0 < m \Longrightarrow m < n \Longrightarrow \neg\ n$ *dvd* m
for $m\ n$ :: *nat*
by (*auto elim*!: *dvdE*) (*auto simp add*: *gr0-conv-Suc*)

lemma *less-eq-dvd-minus*:
fixes $m\ n$ :: *nat*
assumes $m \leq n$
shows m *dvd* $n \longleftrightarrow m$ *dvd* $n - m$
proof −
from *assms* **have** $n = m + (n - m)$ **by** *simp*
then obtain q **where** $n = m + q$ **..**
then show *?thesis* **by** (*simp add*: *add.commute* [*of m*])
qed

lemma *dvd-minus-self*: m *dvd* $n - m \longleftrightarrow n < m \vee m$ *dvd* n
for $m\ n$ :: *nat*
by (*cases* $n < m$) (*auto elim*!: *dvdE simp add*: *not-less le-imp-diff-is-add dest*: *less-imp-le*)

lemma *dvd-minus-add*:
fixes $m\ n\ q\ r$ :: *nat*
assumes $q \leq n\ q \leq r * m$
shows m *dvd* $n - q \longleftrightarrow m$ *dvd* $n + (r * m - q)$
proof −
have m *dvd* $n - q \longleftrightarrow m$ *dvd* $r * m + (n - q)$
using *dvd-add-times-triv-left-iff* [*of m r*] **by** *simp*
also from *assms* **have** ... $\longleftrightarrow m$ *dvd* $r * m + n - q$ **by** *simp*
also from *assms* **have** ... $\longleftrightarrow m$ *dvd* $(r * m - q) + n$ **by** *simp*
also have ... $\longleftrightarrow m$ *dvd* $n + (r * m - q)$ **by** (*simp add*: *add.commute*)
finally show *?thesis* **.**
qed

17.12 Aliasses

lemma *nat-mult-1*: $1 * n = n$
for n :: *nat*
by (*fact mult-1-left*)

lemma *nat-mult-1-right*: $n * 1 = n$
for n :: *nat*
by (*fact mult-1-right*)

lemma *diff-mult-distrib*: $(m - n) * k = (m * k) - (n * k)$
for $k\ m\ n$ :: *nat*
by (*fact left-diff-distrib′*)

lemma *diff-mult-distrib2*: $k * (m - n) = (k * m) - (k * n)$
for $k\ m\ n$ :: *nat*
by (*fact right-diff-distrib′*)

lemma *le-diff-conv2*: $k \leq j \Longrightarrow (i \leq j - k) = (i + k \leq j)$
for $i\ j\ k$ :: *nat*
by (*fact le-diff-conv2*)

lemma *diff-self-eq-0* [*simp*]: $m - m = 0$
for m :: *nat*
by (*fact diff-cancel*)

lemma *diff-diff-left* [*simp*]: $i - j - k = i - (j + k)$
for $i\ j\ k$ :: *nat*
by (*fact diff-diff-add*)

lemma *diff-commute*: $i - j - k = i - k - j$
for $i\ j\ k$ :: *nat*
by (*fact diff-right-commute*)

lemma *diff-add-inverse*: $(n + m) - n = m$
for $m\ n$ :: *nat*
by (*fact add-diff-cancel-left′*)

lemma *diff-add-inverse2*: $(m + n) - n = m$
for $m\ n$ :: *nat*
by (*fact add-diff-cancel-right′*)

lemma *diff-cancel*: $(k + m) - (k + n) = m - n$
for $k\ m\ n$ :: *nat*
by (*fact add-diff-cancel-left*)

lemma *diff-cancel2*: $(m + k) - (n + k) = m - n$
for $k\ m\ n$ :: *nat*
by (*fact add-diff-cancel-right*)

lemma *diff-add-0*: $n - (n + m) = 0$
for $m\ n$ :: *nat*
by (*fact diff-add-zero*)

lemma *add-mult-distrib2*: $k * (m + n) = (k * m) + (k * n)$
for *k m n* :: *nat*
by (*fact distrib-left*)

lemmas *nat-distrib* =
add-mult-distrib distrib-left diff-mult-distrib diff-mult-distrib2

17.13 Size of a datatype value

class *size* =
fixes *size* :: $'a \Rightarrow nat$ — see further theory *Wellfounded*

instantiation *nat* :: *size*
begin

definition *size-nat* **where** [*simp*, *code*]: *size* (*n*::*nat*) = *n*

instance ..

end

lemmas *size-nat* = *size-nat-def*

lemma *size-neq-size-imp-neq*: *size* $x \neq$ *size* $y \Longrightarrow x \neq y$
by (*erule contrapos-nn*) (*rule arg-cong*)

17.14 Code module namespace

code-identifier
code-module *Nat* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

hide-const (**open**) *of-nat-aux*

end

18 Fields

theory *Fields*
imports *Nat*
begin

18.1 Division rings

A division ring is like a field, but without the commutativity requirement.

class *inverse* = *divide* +
fixes *inverse* :: $'a \Rightarrow 'a$
begin

abbreviation *inverse-divide* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ‹'/› *70*)
where
inverse-divide ≡ *divide*

end

Setup for linear arithmetic prover

ML-file ‹*~~/src/Provers/Arith/fast-lin-arith.ML*›
ML-file ‹*Tools/lin-arith.ML*›
setup ‹*Lin-Arith.global-setup*›
declaration ‹*K* (
Lin-Arith.init-arith-data
#> Lin-Arith.add-discrete-type ***type-name*** ‹*nat*›
#> Lin-Arith.add-lessD @{*thm Suc-leI*}
#> Lin-Arith.add-simps @{*thms simp-thms ring-distribs if-True if-False*
minus-diff-eq
add-0-left add-0-right order-less-irrefl
zero-neq-one zero-less-one zero-le-one
zero-neq-one [*THEN not-sym*] *not-one-le-zero not-one-less-zero*
add-Suc add-Suc-right nat.inject
Suc-le-mono Suc-less-eq Zero-not-Suc
Suc-not-Zero le-0-eq One-nat-def}
#> Lin-Arith.add-simprocs [***simproc*** ‹*group-cancel-add*›, ***simproc*** ‹*group-cancel-diff*›,
simproc ‹*group-cancel-eq*›, ***simproc*** ‹*group-cancel-le*›,
simproc ‹*group-cancel-less*›,
simproc ‹*nateq-cancel-sums*›,***simproc*** ‹*natless-cancel-sums*›,
simproc ‹*natle-cancel-sums*›])›

simproc-setup *fast-arith-nat* ((*m*::*nat*) < *n* | (*m*::*nat*) ≤ *n* | (*m*::*nat*) = *n*) =
‹*K Lin-Arith.simproc*› — Because of this simproc, the arithmetic solver is really only useful to detect inconsistencies among the premises for subgoals which are *not* themselves (in)equalities, because the latter activate *fast-nat-arith-simproc* anyway. However, it seems cheaper to activate the solver all the time rather than add the additional check.

lemmas [*linarith-split*] = *nat-diff-split split-min split-max abs-split*

Lemmas *divide-simps* move division to the outside and eliminates them on (in)equalities.

named-theorems *divide-simps rewrite rules to eliminate divisions*

class *division-ring* = *ring-1* + *inverse* +
assumes *left-inverse* [*simp*]: *a* ≠ *0* ⟹ *inverse a* ∗ *a* = *1*
assumes *right-inverse* [*simp*]: *a* ≠ *0* ⟹ *a* ∗ *inverse a* = *1*
assumes *divide-inverse*: *a* / *b* = *a* ∗ *inverse b*
assumes *inverse-zero* [*simp*]: *inverse 0* = *0*
begin

subclass *ring-1-no-zero-divisors*

proof
  fix *a b* :: *'a*
  assume *a*: $a \neq 0$ **and** *b*: $b \neq 0$
  show $a * b \neq 0$
  proof
    assume *ab*: $a * b = 0$
    hence $0 = inverse\ a * (a * b) * inverse\ b$ **by** *simp*
    also have $\ldots = (inverse\ a * a) * (b * inverse\ b)$
      by (*simp only*: *mult.assoc*)
    also have $\ldots = 1$ **using** *a b* **by** *simp*
    finally show *False* **by** *simp*
  qed
qed

lemma *nonzero-imp-inverse-nonzero*:
  $a \neq 0 \Longrightarrow inverse\ a \neq 0$
proof
  assume *ianz*: $inverse\ a = 0$
  assume $a \neq 0$
  hence $1 = a * inverse\ a$ **by** *simp*
  also have ... $= 0$ **by** (*simp add*: *ianz*)
  finally have $1 = 0$ **.**
  thus *False* **by** (*simp add*: *eq-commute*)
qed

lemma *inverse-zero-imp-zero*:
  assumes $inverse\ a = 0$ **shows** $a = 0$
proof (*rule ccontr*)
  assume $a \neq 0$
  then have $inverse\ a \neq 0$
    by (*simp add*: *nonzero-imp-inverse-nonzero*)
  with *assms* **show** *False*
    by *auto*
qed

lemma *inverse-unique*:
  assumes *ab*: $a * b = 1$
  shows $inverse\ a = b$
proof −
  have $a \neq 0$ **using** *ab* **by** (*cases* $a = 0$) *simp-all*
  moreover have $inverse\ a * (a * b) = inverse\ a$ **by** (*simp add*: *ab*)
  ultimately show *?thesis* **by** (*simp add*: *mult.assoc* [*symmetric*])
qed

lemma *nonzero-inverse-minus-eq*:
  $a \neq 0 \Longrightarrow inverse\ (-\ a) = -\ inverse\ a$
by (*rule inverse-unique*) *simp*

lemma *nonzero-inverse-inverse-eq*:

a ≠ *0* ⟹ *inverse* (*inverse a*) = *a*
by (*rule inverse-unique*) *simp*

lemma *nonzero-inverse-eq-imp-eq*:
assumes *inverse a* = *inverse b* **and** *a* ≠ *0* **and** *b* ≠ *0*
shows *a* = *b*
proof −
from ‹*inverse a* = *inverse b*›
have *inverse* (*inverse a*) = *inverse* (*inverse b*) **by** (*rule arg-cong*)
with ‹*a* ≠ *0*› **and** ‹*b* ≠ *0*› **show** *a* = *b*
by (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *inverse-1* [*simp*]: *inverse 1* = *1*
by (*rule inverse-unique*) *simp*

subclass *divide-trivial*
by *standard* (*simp-all add*: *divide-inverse*)

lemma *nonzero-inverse-mult-distrib*:
assumes *a* ≠ *0* **and** *b* ≠ *0*
shows *inverse* (*a* ∗ *b*) = *inverse b* ∗ *inverse a*
proof −
have *a* ∗ (*b* ∗ *inverse b*) ∗ *inverse a* = *1* **using** *assms* **by** *simp*
hence *a* ∗ *b* ∗ (*inverse b* ∗ *inverse a*) = *1* **by** (*simp only*: *mult.assoc*)
thus *?thesis* **by** (*rule inverse-unique*)
qed

lemma *division-ring-inverse-add*:
a ≠ *0* ⟹ *b* ≠ *0* ⟹ *inverse a* + *inverse b* = *inverse a* ∗ (*a* + *b*) ∗ *inverse b*
by (*simp add*: *algebra-simps*)

lemma *division-ring-inverse-diff*:
a ≠ *0* ⟹ *b* ≠ *0* ⟹ *inverse a* − *inverse b* = *inverse a* ∗ (*b* − *a*) ∗ *inverse b*
by (*simp add*: *algebra-simps*)

lemma *right-inverse-eq*: *b* ≠ *0* ⟹ *a* / *b* = *1* ⟷ *a* = *b*
proof
assume *neq*: *b* ≠ *0*
{
hence *a* = (*a* / *b*) ∗ *b* **by** (*simp add*: *divide-inverse mult.assoc*)
also assume *a* / *b* = *1*
finally show *a* = *b* **by** *simp*
next
assume *a* = *b*
with *neq* **show** *a* / *b* = *1* **by** (*simp add*: *divide-inverse*)
}
qed

lemma *nonzero-inverse-eq-divide*: $a \neq 0 \Longrightarrow$ *inverse* $a = 1 / a$
by (*simp add*: *divide-inverse*)

lemma *divide-self* [*simp*]: $a \neq 0 \Longrightarrow a / a = 1$
by (*simp add*: *divide-inverse*)

lemma *inverse-eq-divide* [*field-simps*, *field-split-simps*, *divide-simps*]: *inverse* $a = 1 / a$
by (*simp add*: *divide-inverse*)

lemma *add-divide-distrib*: $(a+b) / c = a/c + b/c$
by (*simp add*: *divide-inverse algebra-simps*)

lemma *times-divide-eq-right* [*simp*]: $a * (b / c) = (a * b) / c$
by (*simp add*: *divide-inverse mult.assoc*)

lemma *minus-divide-left*: $- (a / b) = (-a) / b$
by (*simp add*: *divide-inverse*)

lemma *nonzero-minus-divide-right*: $b \neq 0 \Longrightarrow - (a / b) = a / (- b)$
by (*simp add*: *divide-inverse nonzero-inverse-minus-eq*)

lemma *nonzero-minus-divide-divide*: $b \neq 0 \Longrightarrow (-a) / (-b) = a / b$
by (*simp add*: *divide-inverse nonzero-inverse-minus-eq*)

lemma *divide-minus-left* [*simp*]: $(-a) / b = - (a / b)$
by (*simp add*: *divide-inverse*)

lemma *diff-divide-distrib*: $(a - b) / c = a / c - b / c$
using *add-divide-distrib* [*of* $a - b$ c] **by** *simp*

lemma *nonzero-eq-divide-eq* [*field-simps*]: $c \neq 0 \Longrightarrow a = b / c \longleftrightarrow a * c = b$
proof −
assume [*simp*]: $c \neq 0$
have $a = b / c \longleftrightarrow a * c = (b / c) * c$ **by** *simp*
also have ... $\longleftrightarrow a * c = b$ **by** (*simp add*: *divide-inverse mult.assoc*)
finally show *?thesis* **.**
qed

lemma *nonzero-divide-eq-eq* [*field-simps*]: $c \neq 0 \Longrightarrow b / c = a \longleftrightarrow b = a * c$
proof −
assume [*simp*]: $c \neq 0$
have $b / c = a \longleftrightarrow (b / c) * c = a * c$ **by** *simp*
also have ... $\longleftrightarrow b = a * c$ **by** (*simp add*: *divide-inverse mult.assoc*)
finally show *?thesis* **.**
qed

lemma *nonzero-neg-divide-eq-eq* [*field-simps*]: $b \neq 0 \Longrightarrow - (a / b) = c \longleftrightarrow - a = c * b$

using *nonzero-divide-eq-eq*[*of b* −*a c*] **by** *simp*

lemma *nonzero-neg-divide-eq-eq2* [*field-simps*]: $b \neq 0 \Longrightarrow c = - (a / b) \longleftrightarrow c * b = - a$
using *nonzero-neg-divide-eq-eq*[*of b a c*] **by** *auto*

lemma *divide-eq-imp*: $c \neq 0 \Longrightarrow b = a * c \Longrightarrow b / c = a$
by (*simp add*: *divide-inverse mult.assoc*)

lemma *eq-divide-imp*: $c \neq 0 \Longrightarrow a * c = b \Longrightarrow a = b / c$
by (*drule sym*) (*simp add*: *divide-inverse mult.assoc*)

lemma *add-divide-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x + y / z = (x * z + y) / z$
by (*simp add*: *add-divide-distrib nonzero-eq-divide-eq*)

lemma *divide-add-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x / z + y = (x + y * z) / z$
by (*simp add*: *add-divide-distrib nonzero-eq-divide-eq*)

lemma *diff-divide-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x - y / z = (x * z - y) / z$
by (*simp add*: *diff-divide-distrib nonzero-eq-divide-eq eq-diff-eq*)

lemma *minus-divide-add-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow - (x / z) + y = (- x + y * z) / z$
by (*simp add*: *add-divide-distrib diff-divide-eq-iff*)

lemma *divide-diff-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x / z - y = (x - y * z) / z$
by (*simp add*: *field-simps*)

lemma *minus-divide-diff-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow - (x / z) - y = (- x - y * z) / z$
by (*simp add*: *divide-diff-eq-iff*[*symmetric*])

lemma *division-ring-divide-zero*:
$a / 0 = 0$
by (*fact div-by-0*)

lemma *divide-self-if* [*simp*]:
$a / a = (\textit{if } a = 0 \textit{ then } 0 \textit{ else } 1)$
by *simp*

lemma *inverse-nonzero-iff-nonzero* [*simp*]:
$\textit{inverse } a = 0 \longleftrightarrow a = 0$
by (*rule iffI*) (*fact inverse-zero-imp-zero*, *simp*)

lemma *inverse-minus-eq* [*simp*]:

inverse (− *a*) = − *inverse a*
proof *cases*
assume *a=0* **thus** *?thesis* **by** *simp*
next
assume *a≠0*
thus *?thesis* **by** (*simp add*: *nonzero-inverse-minus-eq*)
qed

lemma *inverse-inverse-eq* [*simp*]:
inverse (*inverse a*) = *a*
proof *cases*
assume *a=0* **thus** *?thesis* **by** *simp*
next
assume *a≠0*
thus *?thesis* **by** (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *inverse-eq-imp-eq*:
inverse a = *inverse b* ⟹ *a* = *b*
by (*drule arg-cong* [**where** *f=inverse*], *simp*)

lemma *inverse-eq-iff-eq* [*simp*]:
inverse a = *inverse b* ⟷ *a* = *b*
by (*force dest*!: *inverse-eq-imp-eq*)

lemma *mult-commute-imp-mult-inverse-commute*:
assumes *y* ∗ *x* = *x* ∗ *y*
shows *inverse y* ∗ *x* = *x* ∗ *inverse y*
proof (*cases y=0*)
case *False*
hence *x* ∗ *inverse y* = *inverse y* ∗ *y* ∗ *x* ∗ *inverse y*
by *simp*
also have ... = *inverse y* ∗ (*x* ∗ *y* ∗ *inverse y*)
by (*simp add*: *mult.assoc assms*)
finally show *?thesis* **by** (*simp add*: *mult.assoc False*)
qed *simp*

lemmas *mult-inverse-of-nat-commute* =
mult-commute-imp-mult-inverse-commute[*OF mult-of-nat-commute*]

lemma *divide-divide-eq-left'*:
(*a* / *b*) / *c* = *a* / (*c* ∗ *b*)
by (*cases b* = *0* ∨ *c* = *0*)
(*auto simp*: *divide-inverse mult.assoc nonzero-inverse-mult-distrib*)

lemma *add-divide-eq-if-simps* [*field-split-simps*, *divide-simps*]:
a + *b* / *z* = (*if z* = *0 then a else* (*a* ∗ *z* + *b*) / *z*)
a / *z* + *b* = (*if z* = *0 then b else* (*a* + *b* ∗ *z*) / *z*)
− (*a* / *z*) + *b* = (*if z* = *0 then b else* (−*a* + *b* ∗ *z*) / *z*)

a − b / z = (if z = 0 then a else (a ∗ z − b) / z)
a / z − b = (if z = 0 then −b else (a − b ∗ z) / z)
− (a / z) − b = (if z = 0 then −b else (− a − b ∗ z) / z)
by (*simp-all add*: *add-divide-eq-iff divide-add-eq-iff diff-divide-eq-iff divide-diff-eq-iff minus-divide-diff-eq-iff*)

lemma [*field-split-simps*, *divide-simps*]:
shows *divide-eq-eq*: *b / c = a ⟷ (if c ≠ 0 then b = a ∗ c else a = 0)*
and *eq-divide-eq*: *a = b / c ⟷ (if c ≠ 0 then a ∗ c = b else a = 0)*
and *minus-divide-eq-eq*: *− (b / c) = a ⟷ (if c ≠ 0 then − b = a ∗ c else a = 0)*
and *eq-minus-divide-eq*: *a = − (b / c) ⟷ (if c ≠ 0 then a ∗ c = − b else a = 0)*
by (*auto simp add*: *field-simps*)

end

18.2 Fields

class *field = comm-ring-1 + inverse +*
assumes *field-inverse*: *a ≠ 0 ⟹ inverse a ∗ a = 1*
assumes *field-divide-inverse*: *a / b = a ∗ inverse b*
assumes *field-inverse-zero*: *inverse 0 = 0*
begin

subclass *division-ring*
proof
fix *a :: 'a*
assume *a ≠ 0*
thus *inverse a ∗ a = 1* **by** (*rule field-inverse*)
thus *a ∗ inverse a = 1* **by** (*simp only*: *mult.commute*)
next
fix *a b :: 'a*
show *a / b = a ∗ inverse b* **by** (*rule field-divide-inverse*)
next
show *inverse 0 = 0*
by (*fact field-inverse-zero*)
qed

subclass *idom-divide*
proof
fix *b a*
assume *b ≠ 0*
then show *a ∗ b / b = a*
by (*simp add*: *divide-inverse ac-simps*)
next
fix *a*
show *a / 0 = 0*
by (*simp add*: *divide-inverse*)

qed

There is no slick version using division by zero.

lemma *inverse-add*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *inverse* a + *inverse* b = $(a + b) *$ *inverse* $a *$ *inverse* b
by (*simp add*: *division-ring-inverse-add ac-simps*)

lemma *nonzero-mult-divide-mult-cancel-left* [*simp*]:
assumes [*simp*]: $c \neq 0$
shows $(c * a) / (c * b) = a / b$
proof (*cases* $b = 0$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then have $(c*a)/(c*b) = c * a *$ (*inverse* b $*$ *inverse* c)
by (*simp add*: *divide-inverse nonzero-inverse-mult-distrib*)
also have ... = $a *$ *inverse* $b *$ (*inverse* $c * c$)
by (*simp only*: *ac-simps*)
also have ... = $a *$ *inverse* b **by** *simp*
finally show *?thesis* **by** (*simp add*: *divide-inverse*)
qed

lemma *nonzero-mult-divide-mult-cancel-right* [*simp*]:
$c \neq 0 \Longrightarrow (a * c) / (b * c) = a / b$
using *nonzero-mult-divide-mult-cancel-left* [*of c a b*] **by** (*simp add*: *ac-simps*)

lemma *times-divide-eq-left* [*simp*]: $(b / c) * a = (b * a) / c$
by (*simp add*: *divide-inverse ac-simps*)

lemma *divide-inverse-commute*: a / b = *inverse* $b * a$
by (*simp add*: *divide-inverse mult.commute*)

lemma *add-frac-eq*:
assumes $y \neq 0$ **and** $z \neq 0$
shows $x / y + w / z = (x * z + w * y) / (y * z)$
proof −
have $x / y + w / z = (x * z) / (y * z) + (y * w) / (y * z)$
using *assms* **by** *simp*
also have ... $= (x * z + y * w) / (y * z)$
by (*simp only*: *add-divide-distrib*)
finally show *?thesis*
by (*simp only*: *mult.commute*)
qed

Special Cancellation Simprules for Division

lemma *nonzero-divide-mult-cancel-right* [*simp*]:
$b \neq 0 \Longrightarrow b / (a * b) = 1 / a$
using *nonzero-mult-divide-mult-cancel-right* [*of b 1 a*] **by** *simp*

lemma *nonzero-divide-mult-cancel-left* [*simp*]:
$a \neq 0 \Longrightarrow a / (a * b) = 1 / b$
using *nonzero-mult-divide-mult-cancel-left* [*of a 1 b*] **by** *simp*

lemma *nonzero-mult-divide-mult-cancel-left2* [*simp*]:
$c \neq 0 \Longrightarrow (c * a) / (b * c) = a / b$
using *nonzero-mult-divide-mult-cancel-left* [*of c a b*] **by** (*simp add*: *ac-simps*)

lemma *nonzero-mult-divide-mult-cancel-right2* [*simp*]:
$c \neq 0 \Longrightarrow (a * c) / (c * b) = a / b$
using *nonzero-mult-divide-mult-cancel-right* [*of b c a*] **by** (*simp add*: *ac-simps*)

lemma *diff-frac-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x / y - w / z = (x * z - w * y) / (y * z)$
by (*simp add*: *field-simps*)

lemma *frac-eq-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (x / y = w / z) = (x * z = w * y)$
by (*simp add*: *field-simps*)

lemma *divide-minus1* [*simp*]: $x / - 1 = - x$
using *nonzero-minus-divide-right* [*of 1 x*] **by** *simp*

This version builds in division by zero while also re-orienting the right-hand side.

lemma *inverse-mult-distrib* [*simp*]:
inverse $(a * b) =$ *inverse* $a *$ *inverse* b
proof *cases*
assume $a \neq 0 \wedge b \neq 0$
thus *?thesis* **by** (*simp add*: *nonzero-inverse-mult-distrib ac-simps*)
next
assume $\neg (a \neq 0 \wedge b \neq 0)$
thus *?thesis* **by** *force*
qed

lemma *inverse-divide* [*simp*]:
inverse $(a / b) = b / a$
by (*simp add*: *divide-inverse mult.commute*)

Calculations with fractions

There is a whole bunch of simp-rules just for class *field* but none for class *field* and *nonzero-divides* because the latter are covered by a simproc.

lemmas *mult-divide-mult-cancel-left* = *nonzero-mult-divide-mult-cancel-left*

lemmas *mult-divide-mult-cancel-right* = *nonzero-mult-divide-mult-cancel-right*

lemma *divide-divide-eq-right* [*simp*]:
$a / (b / c) = (a * c) / b$

by (*simp add*: *divide-inverse ac-simps*)

lemma *divide-divide-eq-left* [*simp*]:
$(a\ /\ b)\ /\ c = a\ /\ (b * c)$
by (*simp add*: *divide-inverse mult.assoc*)

lemma *divide-divide-times-eq*:
$(x\ /\ y)\ /\ (z\ /\ w) = (x * w)\ /\ (y * z)$
by *simp*

Special Cancellation Simprules for Division

lemma *mult-divide-mult-cancel-left-if* [*simp*]:
shows $(c * a)\ /\ (c * b) = ($*if* $c = 0$ *then* 0 *else* $a\ /\ b)$
by *simp*

Division and Unary Minus

lemma *minus-divide-right*:
$-\ (a\ /\ b) = a\ /\ -\ b$
by (*simp add*: *divide-inverse*)

lemma *divide-minus-right* [*simp*]:
$a\ /\ -\ b = -\ (a\ /\ b)$
by (*simp add*: *divide-inverse*)

lemma *minus-divide-divide*:
$(-\ a)\ /\ (-\ b) = a\ /\ b$
by (*cases b=0*) (*simp-all add*: *nonzero-minus-divide-divide*)

lemma *inverse-eq-1-iff* [*simp*]:
inverse $x = 1 \longleftrightarrow x = 1$
using *inverse-eq-iff-eq* [*of x 1*] **by** *simp*

lemma *divide-eq-0-iff* [*simp*]:
$a\ /\ b = 0 \longleftrightarrow a = 0 \vee b = 0$
by (*simp add*: *divide-inverse*)

lemma *divide-cancel-right* [*simp*]:
$a\ /\ c = b\ /\ c \longleftrightarrow c = 0 \vee a = b$
by (*cases c=0*) (*simp-all add*: *divide-inverse*)

lemma *divide-cancel-left* [*simp*]:
$c\ /\ a = c\ /\ b \longleftrightarrow c = 0 \vee a = b$
by (*cases c=0*) (*simp-all add*: *divide-inverse*)

lemma *divide-eq-1-iff* [*simp*]:
$a\ /\ b = 1 \longleftrightarrow b \neq 0 \wedge a = b$
by (*cases b=0*) (*simp-all add*: *right-inverse-eq*)

lemma *one-eq-divide-iff* [*simp*]:

1 = *a* / *b* ⟷ *b* ≠ *0* ∧ *a* = *b*
by (*simp add*: *eq-commute* [*of 1*])

lemma *divide-eq-minus-1-iff*:
(*a* / *b* = − *1*) ⟷ *b* ≠ *0* ∧ *a* = − *b*
using *divide-eq-1-iff* **by** *fastforce*

lemma *times-divide-times-eq*:
(*x* / *y*) ∗ (*z* / *w*) = (*x* ∗ *z*) / (*y* ∗ *w*)
by *simp*

lemma *add-frac-num*:
y ≠ *0* ⟹ *x* / *y* + *z* = (*x* + *z* ∗ *y*) / *y*
by (*simp add*: *add-divide-distrib*)

lemma *add-num-frac*:
y ≠ *0* ⟹ *z* + *x* / *y* = (*x* + *z* ∗ *y*) / *y*
by (*simp add*: *add-divide-distrib add.commute*)

lemma *dvd-field-iff*:
a dvd b ⟷ (*a* = *0* ⟶ *b* = *0*)
proof (*cases a* = *0*)
case *False*
then have *b* = *a* ∗ (*b* / *a*)
by (*simp add*: *field-simps*)
then have *a dvd b* **..**
with *False* **show** *?thesis*
by *simp*
qed *simp*

lemma *inj-divide-right* [*simp*]:
inj (λ*b*. *b* / *a*) ⟷ *a* ≠ *0*
proof −
have (λ*b*. *b* / *a*) = (∗) (*inverse a*)
by (*simp add*: *field-simps fun-eq-iff*)
then have *inj* (λ*y*. *y* / *a*) ⟷ *inj* ((∗) (*inverse a*))
by *simp*
also have ... ⟷ *inverse a* ≠ *0*
by *simp*
also have ... ⟷ *a* ≠ *0*
by *simp*
finally show *?thesis*
by *simp*
qed

end

class *field-char-0* = *field* + *ring-char-0*

18.3 Ordered fields

class *field-abs-sgn* = *field* + *idom-abs-sgn*
begin

lemma *sgn-inverse* [*simp*]:
  sgn (*inverse a*) = *inverse* (*sgn a*)
proof (*cases a* = *0*)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False*
  then have *a* ∗ *inverse a* = *1*
    by *simp*
  then have *sgn* (*a* ∗ *inverse a*) = *sgn 1*
    by *simp*
  then have *sgn a* ∗ *sgn* (*inverse a*) = *1*
    by (*simp add*: *sgn-mult*)
  then have *inverse* (*sgn a*) ∗ (*sgn a* ∗ *sgn* (*inverse a*)) = *inverse* (*sgn a*) ∗ *1*
    by *simp*
  then have (*inverse* (*sgn a*) ∗ *sgn a*) ∗ *sgn* (*inverse a*) = *inverse* (*sgn a*)
    by (*simp add*: *ac-simps*)
  with *False* **show** *?thesis*
    by (*simp add*: *sgn-eq-0-iff*)
qed

lemma *abs-inverse* [*simp*]:
  |*inverse a*| = *inverse* |*a*|
proof −
  from *sgn-mult-abs* [*of inverse a*] *sgn-mult-abs* [*of a*]
  have *inverse* (*sgn a*) ∗ |*inverse a*| = *inverse* (*sgn a* ∗ |*a*|)
    by *simp*
  then show *?thesis* **by** (*auto simp add*: *sgn-eq-0-iff*)
qed

lemma *sgn-divide* [*simp*]:
  sgn (*a* / *b*) = *sgn a* / *sgn b*
  unfolding *divide-inverse sgn-mult* **by** *simp*

lemma *abs-divide* [*simp*]:
  |*a* / *b*| = |*a*| / |*b*|
  unfolding *divide-inverse abs-mult* **by** *simp*

end

class *linordered-field* = *field* + *linordered-idom*
begin

lemma *positive-imp-inverse-positive*:
  assumes *a-gt-0*: *0* < *a*
  shows *0* < *inverse a*

proof −
  have $0 < a * inverse\ a$
    by (*simp add*: *a-gt-0* [*THEN less-imp-not-eq2*])
  thus $0 < inverse\ a$
    by (*simp add*: *a-gt-0* [*THEN less-not-sym*] *zero-less-mult-iff*)
qed

lemma *negative-imp-inverse-negative*:
  $a < 0 \Longrightarrow inverse\ a < 0$
  using *positive-imp-inverse-positive* [*of* $-a$]
  by (*simp add*: *nonzero-inverse-minus-eq less-imp-not-eq*)

lemma *inverse-le-imp-le*:
  assumes *invle*: $inverse\ a \leq inverse\ b$ **and** *apos*: $0 < a$
  shows $b \leq a$
proof (*rule classical*)
  assume $\neg\ b \leq a$
  hence $a < b$ **by** (*simp add*: *linorder-not-le*)
  hence *bpos*: $0 < b$ **by** (*blast intro*: *apos less-trans*)
  hence $a * inverse\ a \leq a * inverse\ b$
    by (*simp add*: *apos invle less-imp-le mult-left-mono*)
  hence $(a * inverse\ a) * b \leq (a * inverse\ b) * b$
    by (*simp add*: *bpos less-imp-le mult-right-mono*)
  thus $b \leq a$ **by** (*simp add*: *mult.assoc apos bpos less-imp-not-eq2*)
qed

lemma *inverse-positive-imp-positive*:
  assumes *inv-gt-0*: $0 < inverse\ a$ **and** *nz*: $a \neq 0$
  shows $0 < a$
proof −
  have $0 < inverse\ (inverse\ a)$
    using *inv-gt-0* **by** (*rule positive-imp-inverse-positive*)
  thus $0 < a$
    using *nz* **by** (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *inverse-negative-imp-negative*:
  assumes *inv-less-0*: $inverse\ a < 0$ **and** *nz*: $a \neq 0$
  shows $a < 0$
proof −
  have $inverse\ (inverse\ a) < 0$
    using *inv-less-0* **by** (*rule negative-imp-inverse-negative*)
  thus $a < 0$ **using** *nz* **by** (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *linordered-field-no-lb*:
  $\forall x.\ \exists y.\ y < x$
proof
  fix $x::'a$

```
  have m1: − (1::'a) < 0 by simp
  from add-strict-right-mono[OF m1, where c=x]
  have (− 1) + x < x by simp
  thus ∃ y. y < x by blast
qed

lemma linordered-field-no-ub:
  ∀ x. ∃ y. y > x
proof
  fix x::'a
  have m1: (1::'a) > 0 by simp
  from add-strict-right-mono[OF m1, where c=x]
  have 1 + x > x by simp
  thus ∃ y. y > x by blast
qed

lemma less-imp-inverse-less:
  assumes less: a < b and apos: 0 < a
  shows inverse b < inverse a
proof (rule ccontr)
  assume ¬ inverse b < inverse a
  hence inverse a ≤ inverse b by simp
  hence ¬ (a < b)
    by (simp add: not-less inverse-le-imp-le [OF - apos])
  thus False by (rule notE [OF - less])
qed

lemma inverse-less-imp-less:
  assumes inverse a < inverse b 0 < a
  shows b < a
proof −
  have a ≠ b
    using assms by (simp add: less-le)
  moreover have b ≤ a
    using assms by (force simp: less-le dest: inverse-le-imp-le)
  ultimately show ?thesis
    by (simp add: less-le)
qed
```

Both premises are essential. Consider -1 and 1.

```
lemma inverse-less-iff-less [simp]:
  0 < a ⟹ 0 < b ⟹ inverse a < inverse b ⟷ b < a
  by (blast intro: less-imp-inverse-less dest: inverse-less-imp-less)

lemma le-imp-inverse-le:
  a ≤ b ⟹ 0 < a ⟹ inverse b ≤ inverse a
  by (force simp add: le-less less-imp-inverse-less)

lemma inverse-le-iff-le [simp]:
```

  0 < a ⟹ 0 < b ⟹ inverse a ≤ inverse b ⟷ b ≤ a
  by (*blast intro*: *le-imp-inverse-le dest*: *inverse-le-imp-le*)

These results refer to both operands being negative. The opposite-sign case is trivial, since inverse preserves signs.

lemma *inverse-le-imp-le-neg*:
  assumes *inverse a ≤ inverse b b < 0*
  shows *b ≤ a*
proof (*rule classical*)
  assume ¬ *b ≤ a*
  with ‹*b < 0*› **have** *a < 0*
    by *force*
  with *assms* **show** *b ≤ a*
    using *inverse-le-imp-le* [*of −b −a*] **by** (*simp add*: *nonzero-inverse-minus-eq*)
qed

lemma *less-imp-inverse-less-neg*:
  assumes *a < b b < 0*
  shows *inverse b < inverse a*
proof −
  have *a < 0*
    using *assms* **by** (*blast intro*: *less-trans*)
  with *less-imp-inverse-less* [*of −b −a*] **show** *?thesis*
    by (*simp add*: *nonzero-inverse-minus-eq assms*)
qed

lemma *inverse-less-imp-less-neg*:
  assumes *inverse a < inverse b b < 0*
  shows *b < a*
proof (*rule classical*)
  assume ¬ *b < a*
  with ‹*b < 0*› **have** *a < 0*
    by *force*
  with *inverse-less-imp-less* [*of −b −a*] **show** *?thesis*
    by (*simp add*: *nonzero-inverse-minus-eq assms*)
qed

lemma *inverse-less-iff-less-neg* [*simp*]:
  a < 0 ⟹ b < 0 ⟹ inverse a < inverse b ⟷ b < a
  using *inverse-less-iff-less* [*of −b −a*]
  by (*simp del*: *inverse-less-iff-less add*: *nonzero-inverse-minus-eq*)

lemma *le-imp-inverse-le-neg*:
  a ≤ b ⟹ b < 0 ⟹ inverse b ≤ inverse a
  by (*force simp add*: *le-less less-imp-inverse-less-neg*)

lemma *inverse-le-iff-le-neg* [*simp*]:
  a < 0 ⟹ b < 0 ⟹ inverse a ≤ inverse b ⟷ b ≤ a
  by (*blast intro*: *le-imp-inverse-le-neg dest*: *inverse-le-imp-le-neg*)

lemma *one-less-inverse*:
  $0 < a \Longrightarrow a < 1 \Longrightarrow 1 < inverse\ a$
  using *less-imp-inverse-less* [*of a 1*, *unfolded inverse-1*] **.**

lemma *one-le-inverse*:
  $0 < a \Longrightarrow a \le 1 \Longrightarrow 1 \le inverse\ a$
  using *le-imp-inverse-le* [*of a 1*, *unfolded inverse-1*] **.**

lemma *pos-le-divide-eq* [*field-simps*]:
  assumes $0 < c$
  shows $a \le b / c \longleftrightarrow a * c \le b$
proof −
  from *assms* **have** $a \le b / c \longleftrightarrow a * c \le (b / c) * c$
    using *mult-le-cancel-right* [*of a c b * inverse c*] **by** (*auto simp add*: *field-simps*)
  also have ... $\longleftrightarrow a * c \le b$
    by (*simp add*: *less-imp-not-eq2* [*OF assms*] *divide-inverse mult.assoc*)
  finally show *?thesis* **.**
qed

lemma *pos-less-divide-eq* [*field-simps*]:
  assumes $0 < c$
  shows $a < b / c \longleftrightarrow a * c < b$
proof −
  from *assms* **have** $a < b / c \longleftrightarrow a * c < (b / c) * c$
    using *mult-less-cancel-right* [*of a c b / c*] **by** *auto*
  also have ... $= (a*c < b)$
    by (*simp add*: *less-imp-not-eq2* [*OF assms*] *divide-inverse mult.assoc*)
  finally show *?thesis* **.**
qed

lemma *neg-less-divide-eq* [*field-simps*]:
  assumes $c < 0$
  shows $a < b / c \longleftrightarrow b < a * c$
proof −
  from *assms* **have** $a < b / c \longleftrightarrow (b / c) * c < a * c$
    using *mult-less-cancel-right* [*of b / c c a*] **by** *auto*
  also have ... $\longleftrightarrow b < a * c$
    by (*simp add*: *less-imp-not-eq* [*OF assms*] *divide-inverse mult.assoc*)
  finally show *?thesis* **.**
qed

lemma *neg-le-divide-eq* [*field-simps*]:
  assumes $c < 0$
  shows $a \le b / c \longleftrightarrow b \le a * c$
proof −
  from *assms* **have** $a \le b / c \longleftrightarrow (b / c) * c \le a * c$
    using *mult-le-cancel-right* [*of b * inverse c c a*] **by** (*auto simp add*: *field-simps*)
  also have ... $\longleftrightarrow b \le a * c$

```
    by (simp add: less-imp-not-eq [OF assms] divide-inverse mult.assoc)
  finally show ?thesis .
qed

lemma pos-divide-le-eq [field-simps]:
  assumes 0 < c
  shows b / c ≤ a ⟷ b ≤ a * c
proof –
  from assms have b / c ≤ a ⟷ (b / c) * c ≤ a * c
    using mult-le-cancel-right [of b / c c a] by auto
  also have ... ⟷ b ≤ a * c
    by (simp add: less-imp-not-eq2 [OF assms] divide-inverse mult.assoc)
  finally show ?thesis .
qed

lemma pos-divide-less-eq [field-simps]:
  assumes 0 < c
  shows b / c < a ⟷ b < a * c
proof –
  from assms have b / c < a ⟷ (b / c) * c < a * c
    using mult-less-cancel-right [of b / c c a] by auto
  also have ... ⟷ b < a * c
    by (simp add: less-imp-not-eq2 [OF assms] divide-inverse mult.assoc)
  finally show ?thesis .
qed

lemma neg-divide-le-eq [field-simps]:
  assumes c < 0
  shows b / c ≤ a ⟷ a * c ≤ b
proof –
  from assms have b / c ≤ a ⟷ a * c ≤ (b / c) * c
    using mult-le-cancel-right [of a c b / c] by auto
  also have ... ⟷ a * c ≤ b
    by (simp add: less-imp-not-eq [OF assms] divide-inverse mult.assoc)
  finally show ?thesis .
qed

lemma neg-divide-less-eq [field-simps]:
  assumes c < 0
  shows b / c < a ⟷ a * c < b
proof –
  from assms have b / c < a ⟷ a * c < b / c * c
    using mult-less-cancel-right [of a c b / c] by auto
  also have ... ⟷ a * c < b
    by (simp add: less-imp-not-eq [OF assms] divide-inverse mult.assoc)
  finally show ?thesis .
qed
```

The following *field-simps* rules are necessary, as minus is always moved atop of division but we want to get rid of division.

lemma *pos-le-minus-divide-eq* [*field-simps*]: $0 < c \Longrightarrow a \le - (b / c) \longleftrightarrow a * c \le - b$
unfolding *minus-divide-left* **by** (*rule pos-le-divide-eq*)

lemma *neg-le-minus-divide-eq* [*field-simps*]: $c < 0 \Longrightarrow a \le - (b / c) \longleftrightarrow - b \le a * c$
unfolding *minus-divide-left* **by** (*rule neg-le-divide-eq*)

lemma *pos-less-minus-divide-eq* [*field-simps*]: $0 < c \Longrightarrow a < - (b / c) \longleftrightarrow a * c < - b$
unfolding *minus-divide-left* **by** (*rule pos-less-divide-eq*)

lemma *neg-less-minus-divide-eq* [*field-simps*]: $c < 0 \Longrightarrow a < - (b / c) \longleftrightarrow - b < a * c$
unfolding *minus-divide-left* **by** (*rule neg-less-divide-eq*)

lemma *pos-minus-divide-less-eq* [*field-simps*]: $0 < c \Longrightarrow - (b / c) < a \longleftrightarrow - b < a * c$
unfolding *minus-divide-left* **by** (*rule pos-divide-less-eq*)

lemma *neg-minus-divide-less-eq* [*field-simps*]: $c < 0 \Longrightarrow - (b / c) < a \longleftrightarrow a * c < - b$
unfolding *minus-divide-left* **by** (*rule neg-divide-less-eq*)

lemma *pos-minus-divide-le-eq* [*field-simps*]: $0 < c \Longrightarrow - (b / c) \le a \longleftrightarrow - b \le a * c$
unfolding *minus-divide-left* **by** (*rule pos-divide-le-eq*)

lemma *neg-minus-divide-le-eq* [*field-simps*]: $c < 0 \Longrightarrow - (b / c) \le a \longleftrightarrow a * c \le - b$
unfolding *minus-divide-left* **by** (*rule neg-divide-le-eq*)

lemma *frac-less-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x / y < w / z \longleftrightarrow (x * z - w * y) / (y * z) < 0$
by (*subst less-iff-diff-less-0*) (*simp add*: *diff-frac-eq*)

lemma *frac-le-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x / y \le w / z \longleftrightarrow (x * z - w * y) / (y * z) \le 0$
by (*subst le-iff-diff-le-0*) (*simp add*: *diff-frac-eq*)

lemma *divide-pos-pos*[*simp*]:
$0 < x \Longrightarrow 0 < y \Longrightarrow 0 < x / y$
by(*simp add*:*field-simps*)

lemma *divide-nonneg-pos*:
$0 \le x \Longrightarrow 0 < y \Longrightarrow 0 \le x / y$
by(*simp add*:*field-simps*)

lemma *divide-neg-pos*:
$x < 0 \Longrightarrow 0 < y \Longrightarrow x / y < 0$
by(*simp add*:*field-simps*)

lemma *divide-nonpos-pos*:
$x \leq 0 \Longrightarrow 0 < y \Longrightarrow x / y \leq 0$
by(*simp add*:*field-simps*)

lemma *divide-pos-neg*:
$0 < x \Longrightarrow y < 0 \Longrightarrow x / y < 0$
by(*simp add*:*field-simps*)

lemma *divide-nonneg-neg*:
$0 \leq x \Longrightarrow y < 0 \Longrightarrow x / y \leq 0$
by(*simp add*:*field-simps*)

lemma *divide-neg-neg*:
$x < 0 \Longrightarrow y < 0 \Longrightarrow 0 < x / y$
by(*simp add*:*field-simps*)

lemma *divide-nonpos-neg*:
$x \leq 0 \Longrightarrow y < 0 \Longrightarrow 0 \leq x / y$
by(*simp add*:*field-simps*)

lemma *divide-strict-right-mono*:
$[\![a < b; 0 < c]\!] \Longrightarrow a / c < b / c$
by (*simp add*: *less-imp-not-eq2 divide-inverse mult-strict-right-mono positive-imp-inverse-positive*)

lemma *divide-strict-right-mono-neg*:
assumes $b < a$ $c < 0$ **shows** $a / c < b / c$
proof –
have $b / - c < a / - c$
by (*rule divide-strict-right-mono*) (*use assms* **in** *auto*)
then show *?thesis*
by (*simp add*: *less-imp-not-eq*)
qed

The last premise ensures that a and b have the same sign

lemma *divide-strict-left-mono*:
$[\![b < a; 0 < c; 0 < a*b]\!] \Longrightarrow c / a < c / b$
by (*auto simp*: *field-simps zero-less-mult-iff mult-strict-right-mono*)

lemma *divide-left-mono*:
$[\![b \leq a; 0 \leq c; 0 < a*b]\!] \Longrightarrow c / a \leq c / b$
by (*auto simp*: *field-simps zero-less-mult-iff mult-right-mono*)

lemma *divide-strict-left-mono-neg*:

$[\![a < b;\ c < 0;\ 0 < a*b]\!] \Longrightarrow c\ /\ a < c\ /\ b$
by (*auto simp*: *field-simps zero-less-mult-iff mult-strict-right-mono-neg*)

lemma *mult-imp-div-pos-le*: $0 < y \Longrightarrow x \le z * y \Longrightarrow x\ /\ y \le z$
by (*subst pos-divide-le-eq*, *assumption+*)

lemma *mult-imp-le-div-pos*: $0 < y \Longrightarrow z * y \le x \Longrightarrow z \le x\ /\ y$
by(*simp add*:*field-simps*)

lemma *mult-imp-div-pos-less*: $0 < y \Longrightarrow x < z * y \Longrightarrow x\ /\ y < z$
by(*simp add*:*field-simps*)

lemma *mult-imp-less-div-pos*: $0 < y \Longrightarrow z * y < x \Longrightarrow z < x\ /\ y$
by(*simp add*:*field-simps*)

lemma *frac-le*:
assumes $0 \le y$ $x \le y$ $0 < w$ $w \le z$
shows $x\ /\ z \le y\ /\ w$
proof (*rule mult-imp-div-pos-le*)
show $z > 0$
using *assms* **by** *simp*
have $x \le y * z\ /\ w$
proof (*rule mult-imp-le-div-pos* [*OF* ‹$0 < w$›])
show $x * w \le y * z$
using *assms* **by** (*auto intro*: *mult-mono*)
qed
also have ... $= y\ /\ w * z$
by *simp*
finally show $x \le y\ /\ w * z$ **.**
qed

lemma *frac-less*:
assumes $0 \le x$ $x < y$ $0 < w$ $w \le z$
shows $x\ /\ z < y\ /\ w$
proof (*rule mult-imp-div-pos-less*)
show $z > 0$
using *assms* **by** *simp*
have $x < y * z\ /\ w$
proof (*rule mult-imp-less-div-pos* [*OF* ‹$0 < w$›])
show $x * w < y * z$
using *assms* **by** (*auto intro*: *mult-less-le-imp-less*)
qed
also have ... $= y\ /\ w * z$
by *simp*
finally show $x < y\ /\ w * z$ **.**
qed

lemma *frac-less2*:
assumes $0 < x$ $x \le y$ $0 < w$ $w < z$

shows $x\ /\ z < y\ /\ w$
proof (*rule mult-imp-div-pos-less*)
show $z > 0$
using *assms* **by** *simp*
show $x < y\ /\ w * z$
using *assms* **by** (*force intro*: *mult-imp-less-div-pos mult-le-less-imp-less*)
qed

As above, with a better name

lemma *divide-mono*:
$[\![b \leq a;\ c \leq d;\ 0 < b;\ 0 \leq c]\!] \Longrightarrow c\ /\ a \leq d\ /\ b$
by (*simp add*: *frac-le*)

lemma *less-half-sum*: $a < b \Longrightarrow a < (a+b)\ /\ (1+1)$
by (*simp add*: *field-simps zero-less-two*)

lemma *gt-half-sum*: $a < b \Longrightarrow (a+b)/(1+1) < b$
by (*simp add*: *field-simps zero-less-two*)

subclass *unbounded-dense-linorder*
proof
fix $x\ y$ :: $'a$
from *less-add-one* **show** $\exists\, y.\ x < y$ **..**
from *less-add-one* **have** $x + (-\ 1) < (x + 1) + (-\ 1)$ **by** (*rule add-strict-right-mono*)
then have $x - 1 < x + 1 - 1$ **by** *simp*
then have $x - 1 < x$ **by** (*simp add*: *algebra-simps*)
then show $\exists\, y.\ y < x$ **..**
show $x < y \Longrightarrow \exists\, z{>}x.\ z < y$ **by** (*blast intro*!: *less-half-sum gt-half-sum*)
qed

subclass *field-abs-sgn* **..**

lemma *inverse-sgn* [*simp*]:
inverse (*sgn a*) = *sgn a*
by (*cases a 0 rule*: *linorder-cases*) *simp-all*

lemma *divide-sgn* [*simp*]:
$a\ /\ sgn\ b = a * sgn\ b$
by (*cases b 0 rule*: *linorder-cases*) *simp-all*

lemma *nonzero-abs-inverse*:
$a \neq 0 \Longrightarrow |inverse\ a| = inverse\ |a|$
by (*rule abs-inverse*)

lemma *nonzero-abs-divide*:
$b \neq 0 \Longrightarrow |a\ /\ b| = |a|\ /\ |b|$
by (*rule abs-divide*)

lemma *field-le-epsilon*:

assumes *e*: $\bigwedge e.\ 0 < e \Longrightarrow x \leq y + e$
shows $x \leq y$
proof (*rule dense-le*)
fix *t* **assume** $t < x$
hence $0 < x - t$ **by** (*simp add*: *less-diff-eq*)
from *e* [*OF this*] **have** $x + 0 \leq x + (y - t)$ **by** (*simp add*: *algebra-simps*)
then have $0 \leq y - t$ **by** (*simp only*: *add-le-cancel-left*)
then show $t \leq y$ **by** (*simp add*: *algebra-simps*)
qed

lemma *inverse-positive-iff-positive* [*simp*]: $(0 < \mathit{inverse}\ a) = (0 < a)$
proof (*cases* $a = 0$)
case *False*
then show *?thesis*
by (*blast intro*: *inverse-positive-imp-positive positive-imp-inverse-positive*)
qed *auto*

lemma *inverse-negative-iff-negative* [*simp*]: $(\mathit{inverse}\ a < 0) = (a < 0)$
proof (*cases* $a = 0$)
case *False*
then show *?thesis*
by (*blast intro*: *inverse-negative-imp-negative negative-imp-inverse-negative*)
qed *auto*

lemma *inverse-nonnegative-iff-nonnegative* [*simp*]: $0 \leq \mathit{inverse}\ a \longleftrightarrow 0 \leq a$
by (*simp add*: *not-less* [*symmetric*])

lemma *inverse-nonpositive-iff-nonpositive* [*simp*]: $\mathit{inverse}\ a \leq 0 \longleftrightarrow a \leq 0$
by (*simp add*: *not-less* [*symmetric*])

lemma *one-less-inverse-iff*: $1 < \mathit{inverse}\ x \longleftrightarrow 0 < x \wedge x < 1$
using *less-trans*[*of 1 x 0* **for** *x*]
by (*cases x 0 rule*: *linorder-cases*) (*auto simp add*: *field-simps*)

lemma *one-le-inverse-iff*: $1 \leq \mathit{inverse}\ x \longleftrightarrow 0 < x \wedge x \leq 1$
proof (*cases* $x = 1$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False* **then have** $\mathit{inverse}\ x \neq 1$ **by** *simp*
then have $1 \neq \mathit{inverse}\ x$ **by** *blast*
then have $1 \leq \mathit{inverse}\ x \longleftrightarrow 1 < \mathit{inverse}\ x$ **by** (*simp add*: *le-less*)
with *False* **show** *?thesis* **by** (*auto simp add*: *one-less-inverse-iff*)
qed

lemma *inverse-less-1-iff*: $\mathit{inverse}\ x < 1 \longleftrightarrow x \leq 0 \vee 1 < x$
by (*simp add*: *not-le* [*symmetric*] *one-le-inverse-iff*)

lemma *inverse-le-1-iff*: $\mathit{inverse}\ x \leq 1 \longleftrightarrow x \leq 0 \vee 1 \leq x$
by (*simp add*: *not-less* [*symmetric*] *one-less-inverse-iff*)

lemma [*field-split-simps*, *divide-simps*]:
shows *le-divide-eq*: $a \leq b \ / \ c \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ a * c \leq b\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ b \leq a * c\ \mathit{else}\ a \leq 0)$
and *divide-le-eq*: $b \ / \ c \leq a \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ b \leq a * c\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ a * c \leq b\ \mathit{else}\ 0 \leq a)$
and *less-divide-eq*: $a < b \ / \ c \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ a * c < b\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ b < a * c\ \mathit{else}\ a < 0)$
and *divide-less-eq*: $b \ / \ c < a \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ b < a * c\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ a * c < b\ \mathit{else}\ 0 < a)$
and *le-minus-divide-eq*: $a \leq - (b \ / \ c) \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ a * c \leq - b\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ - b \leq a * c\ \mathit{else}\ a \leq 0)$
and *minus-divide-le-eq*: $- (b \ / \ c) \leq a \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ - b \leq a * c\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ a * c \leq - b\ \mathit{else}\ 0 \leq a)$
and *less-minus-divide-eq*: $a < - (b \ / \ c) \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ a * c < - b\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ - b < a * c\ \mathit{else}\ a < 0)$
and *minus-divide-less-eq*: $- (b \ / \ c) < a \longleftrightarrow (\mathit{if}\ 0 < c\ \mathit{then}\ - b < a * c\ \mathit{else}\ \mathit{if}\ c < 0\ \mathit{then}\ a * c < - b\ \mathit{else}\ 0 < a)$
by (*auto simp*: *field-simps not-less dest*: *order.antisym*)

Division and Signs

lemma
shows *zero-less-divide-iff*: $0 < a \ / \ b \longleftrightarrow 0 < a \wedge 0 < b \vee a < 0 \wedge b < 0$
and *divide-less-0-iff*: $a \ / \ b < 0 \longleftrightarrow 0 < a \wedge b < 0 \vee a < 0 \wedge 0 < b$
and *zero-le-divide-iff*: $0 \leq a \ / \ b \longleftrightarrow 0 \leq a \wedge 0 \leq b \vee a \leq 0 \wedge b \leq 0$
and *divide-le-0-iff*: $a \ / \ b \leq 0 \longleftrightarrow 0 \leq a \wedge b \leq 0 \vee a \leq 0 \wedge 0 \leq b$
by (*auto simp add*: *field-split-simps*)

Division and the Number One

Simplify expressions equated with 1

lemma *zero-eq-1-divide-iff* [*simp*]: $0 = 1 \ / \ a \longleftrightarrow a = 0$
by (*cases* $a = 0$) (*auto simp*: *field-simps*)

lemma *one-divide-eq-0-iff* [*simp*]: $1 \ / \ a = 0 \longleftrightarrow a = 0$
using *zero-eq-1-divide-iff*[*of a*] **by** *simp*

Simplify expressions such as $0 < 1/x$ to $0 < x$

lemma *zero-le-divide-1-iff* [*simp*]:
$0 \leq 1 \ / \ a \longleftrightarrow 0 \leq a$
by (*simp add*: *zero-le-divide-iff*)

lemma *zero-less-divide-1-iff* [*simp*]:
$0 < 1 \ / \ a \longleftrightarrow 0 < a$
by (*simp add*: *zero-less-divide-iff*)

lemma *divide-le-0-1-iff* [*simp*]:
$1 \ / \ a \leq 0 \longleftrightarrow a \leq 0$
by (*simp add*: *divide-le-0-iff*)

lemma *divide-less-0-1-iff* [*simp*]:
$1 \;/\; a < 0 \longleftrightarrow a < 0$
by (*simp add*: *divide-less-0-iff*)

lemma *divide-right-mono*:
$[\![a \le b;\ 0 \le c]\!] \Longrightarrow a/c \le b/c$
by (*force simp add*: *divide-strict-right-mono le-less*)

lemma *divide-right-mono-neg*: $a \le b \Longrightarrow c \le 0 \Longrightarrow b \;/\; c \le a \;/\; c$
by (*auto dest*: *divide-right-mono* [*of* - - $-$ *c*])

lemma *divide-left-mono-neg*: $a \le b \Longrightarrow c \le 0 \Longrightarrow 0 < a * b \Longrightarrow c \;/\; a \le c \;/\; b$
by (*auto simp add*: *mult.commute dest*: *divide-left-mono* [*of* - - $-$ *c*])

lemma *inverse-le-iff*: $inverse\ a \le inverse\ b \longleftrightarrow (0 < a * b \longrightarrow b \le a) \wedge (a * b \le 0 \longrightarrow a \le b)$
by (*cases a 0 b 0 rule*: *linorder-cases*[*case-product linorder-cases*])
(*auto simp add*: *field-simps zero-less-mult-iff mult-le-0-iff*)

lemma *inverse-less-iff*: $inverse\ a < inverse\ b \longleftrightarrow (0 < a * b \longrightarrow b < a) \wedge (a * b \le 0 \longrightarrow a < b)$
by (*subst less-le*) (*auto simp*: *inverse-le-iff*)

lemma *divide-le-cancel*: $a \;/\; c \le b \;/\; c \longleftrightarrow (0 < c \longrightarrow a \le b) \wedge (c < 0 \longrightarrow b \le a)$
by (*simp add*: *divide-inverse mult-le-cancel-right*)

lemma *divide-less-cancel*: $a \;/\; c < b \;/\; c \longleftrightarrow (0 < c \longrightarrow a < b) \wedge (c < 0 \longrightarrow b < a) \wedge c \neq 0$
by (*auto simp add*: *divide-inverse mult-less-cancel-right*)

Simplify quotients that are compared with the value 1.

lemma *le-divide-eq-1*:
$(1 \le b \;/\; a) = ((0 < a \wedge a \le b) \vee (a < 0 \wedge b \le a))$
by (*auto simp add*: *le-divide-eq*)

lemma *divide-le-eq-1*:
$(b \;/\; a \le 1) = ((0 < a \wedge b \le a) \vee (a < 0 \wedge a \le b) \vee a{=}0)$
by (*auto simp add*: *divide-le-eq*)

lemma *less-divide-eq-1*:
$(1 < b \;/\; a) = ((0 < a \wedge a < b) \vee (a < 0 \wedge b < a))$
by (*auto simp add*: *less-divide-eq*)

lemma *divide-less-eq-1*:
$(b \;/\; a < 1) = ((0 < a \wedge b < a) \vee (a < 0 \wedge a < b) \vee a{=}0)$
by (*auto simp add*: *divide-less-eq*)

lemma *divide-nonneg-nonneg* [*simp*]:
$0 \leq x \Longrightarrow 0 \leq y \Longrightarrow 0 \leq x / y$
by (*auto simp add*: *field-split-simps*)

lemma *divide-nonpos-nonpos*:
$x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow 0 \leq x / y$
by (*auto simp add*: *field-split-simps*)

lemma *divide-nonneg-nonpos*:
$0 \leq x \Longrightarrow y \leq 0 \Longrightarrow x / y \leq 0$
by (*auto simp add*: *field-split-simps*)

lemma *divide-nonpos-nonneg*:
$x \leq 0 \Longrightarrow 0 \leq y \Longrightarrow x / y \leq 0$
by (*auto simp add*: *field-split-simps*)

Conditional Simplification Rules: No Case Splits

lemma *le-divide-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (1 \leq b/a) = (a \leq b)$
by (*auto simp add*: *le-divide-eq*)

lemma *le-divide-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow (1 \leq b/a) = (b \leq a)$
by (*auto simp add*: *le-divide-eq*)

lemma *divide-le-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (b/a \leq 1) = (b \leq a)$
by (*auto simp add*: *divide-le-eq*)

lemma *divide-le-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow (b/a \leq 1) = (a \leq b)$
by (*auto simp add*: *divide-le-eq*)

lemma *less-divide-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (1 < b/a) = (a < b)$
by (*auto simp add*: *less-divide-eq*)

lemma *less-divide-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow (1 < b/a) = (b < a)$
by (*auto simp add*: *less-divide-eq*)

lemma *divide-less-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (b/a < 1) = (b < a)$
by (*auto simp add*: *divide-less-eq*)

lemma *divide-less-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow b/a < 1 \longleftrightarrow a < b$
by (*auto simp add*: *divide-less-eq*)

lemma *eq-divide-eq-1* [*simp*]:
  $(1 = b/a) = ((a \neq 0 \wedge a = b))$
  by (*auto simp add*: *eq-divide-eq*)

lemma *divide-eq-eq-1* [*simp*]:
  $(b/a = 1) = ((a \neq 0 \wedge a = b))$
  by (*auto simp add*: *divide-eq-eq*)

lemma *abs-div-pos*: $0 < y \Longrightarrow |x| / y = |x / y|$
  by (*simp add*: *order-less-imp-le*)

lemma *zero-le-divide-abs-iff* [*simp*]: $(0 \leq a / |b|) = (0 \leq a \vee b = 0)$
  by (*auto simp*: *zero-le-divide-iff*)

lemma *divide-le-0-abs-iff* [*simp*]: $(a / |b| \leq 0) = (a \leq 0 \vee b = 0)$
  by (*auto simp*: *divide-le-0-iff*)

lemma *field-le-mult-one-interval*:
  assumes ∗: $\bigwedge z.\ [\![0 < z ; z < 1]\!] \Longrightarrow z * x \leq y$
  shows $x \leq y$
proof (*cases* $0 < x$)
  assume $0 < x$
  thus *?thesis*
    using *dense-le-bounded*[*of 0 1 y/x*] ∗
    unfolding *le-divide-eq if-P*[*OF* ‹$0 < x$›] **by** *simp*
next
  assume $\neg 0 < x$ **hence** $x \leq 0$ **by** *simp*
  obtain *s*::*'a* **where** *s*: $0 < s$ $s < 1$ **using** *dense*[*of 0 1*::*'a*] **by** *auto*
  hence $x \leq s * x$ **using** *mult-le-cancel-right*[*of 1 x s*] ‹$x \leq 0$› **by** *auto*
  also note ∗[*OF s*]
  finally show *?thesis* **.**
qed

For creating values between *u* and *v*.

lemma *scaling-mono*:
  assumes $u \leq v$ $0 \leq r$ $r \leq s$
  shows $u + r * (v - u) / s \leq v$
proof −
  have $r/s \leq 1$ **using** *assms*
    using *divide-le-eq-1* **by** *fastforce*
  moreover have $0 \leq v - u$
    using *assms* **by** *simp*
  ultimately have $(r/s) * (v - u) \leq 1 * (v - u)$
    by (*rule mult-right-mono*)
  then show *?thesis*
    by (*simp add*: *field-simps*)
qed

end

Min/max Simplification Rules

lemma *min-mult-distrib-left*:
fixes *x*::*'a*::*linordered-idom*
shows $p * min\ x\ y = (if\ 0 \leq p\ then\ min\ (p*x)\ (p*y)\ else\ max\ (p*x)\ (p*y))$
by (*auto simp add*: *min-def max-def mult-le-cancel-left*)

lemma *min-mult-distrib-right*:
fixes *x*::*'a*::*linordered-idom*
shows $min\ x\ y * p = (if\ 0 \leq p\ then\ min\ (x*p)\ (y*p)\ else\ max\ (x*p)\ (y*p))$
by (*auto simp add*: *min-def max-def mult-le-cancel-right*)

lemma *min-divide-distrib-right*:
fixes *x*::*'a*::*linordered-field*
shows $min\ x\ y\ /\ p = (if\ 0 \leq p\ then\ min\ (x/p)\ (y/p)\ else\ max\ (x/p)\ (y/p))$
by (*simp add*: *min-mult-distrib-right divide-inverse*)

lemma *max-mult-distrib-left*:
fixes *x*::*'a*::*linordered-idom*
shows $p * max\ x\ y = (if\ 0 \leq p\ then\ max\ (p*x)\ (p*y)\ else\ min\ (p*x)\ (p*y))$
by (*auto simp add*: *min-def max-def mult-le-cancel-left*)

lemma *max-mult-distrib-right*:
fixes *x*::*'a*::*linordered-idom*
shows $max\ x\ y * p = (if\ 0 \leq p\ then\ max\ (x*p)\ (y*p)\ else\ min\ (x*p)\ (y*p))$
by (*auto simp add*: *min-def max-def mult-le-cancel-right*)

lemma *max-divide-distrib-right*:
fixes *x*::*'a*::*linordered-field*
shows $max\ x\ y\ /\ p = (if\ 0 \leq p\ then\ max\ (x/p)\ (y/p)\ else\ min\ (x/p)\ (y/p))$
by (*simp add*: *max-mult-distrib-right divide-inverse*)

hide-fact (**open**) *field-inverse field-divide-inverse field-inverse-zero*

code-identifier
code-module *Fields* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

19 Relations – as sets of pairs, and binary predicates

theory *Relation*
imports *Product-Type Sum-Type Fields*
begin

A preliminary: classical rules for reasoning on predicates

declare *predicate1I* [*Pure.intro*!, *intro*!]
declare *predicate1D* [*Pure.dest*, *dest*]

declare *predicate2I* [*Pure.intro*!, *intro*!]
declare *predicate2D* [*Pure.dest*, *dest*]
declare *bot1E* [*elim*!]
declare *bot2E* [*elim*!]
declare *top1I* [*intro*!]
declare *top2I* [*intro*!]
declare *inf1I* [*intro*!]
declare *inf2I* [*intro*!]
declare *inf1E* [*elim*!]
declare *inf2E* [*elim*!]
declare *sup1I1* [*intro?*]
declare *sup2I1* [*intro?*]
declare *sup1I2* [*intro?*]
declare *sup2I2* [*intro?*]
declare *sup1E* [*elim*!]
declare *sup2E* [*elim*!]
declare *sup1CI* [*intro*!]
declare *sup2CI* [*intro*!]
declare *Inf1-I* [*intro*!]
declare *INF1-I* [*intro*!]
declare *Inf2-I* [*intro*!]
declare *INF2-I* [*intro*!]
declare *Inf1-D* [*elim*]
declare *INF1-D* [*elim*]
declare *Inf2-D* [*elim*]
declare *INF2-D* [*elim*]
declare *Inf1-E* [*elim*]
declare *INF1-E* [*elim*]
declare *Inf2-E* [*elim*]
declare *INF2-E* [*elim*]
declare *Sup1-I* [*intro*]
declare *SUP1-I* [*intro*]
declare *Sup2-I* [*intro*]
declare *SUP2-I* [*intro*]
declare *Sup1-E* [*elim*!]
declare *SUP1-E* [*elim*!]
declare *Sup2-E* [*elim*!]
declare *SUP2-E* [*elim*!]

19.1 Fundamental

19.1.1 Relations as sets of pairs

type-synonym $'a\ rel = ('a \times 'a)\ set$

lemma *subrelI*: $(\bigwedge x\ y.\ (x,\ y) \in r \Longrightarrow (x,\ y) \in s) \Longrightarrow r \subseteq s$
— Version of *subsetI* for binary relations
by *auto*

lemma *lfp-induct2*:

$(a, b) \in lfp\ f \Longrightarrow mono\ f \Longrightarrow$
$(\bigwedge a\ b.\ (a, b) \in f\ (lfp\ f \cap \{(x, y).\ P\ x\ y\}) \Longrightarrow P\ a\ b) \Longrightarrow P\ a\ b$
— Version of *lfp-induct* for binary relations
using *lfp-induct-set* [*of* (*a*, *b*) *f case-prod P*] **by** *auto*

19.1.2 Conversions between set and predicate relations

lemma *pred-equals-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) = (\lambda x.\ x \in S) \longleftrightarrow R = S$
by (*simp add*: *set-eq-iff fun-eq-iff*)

lemma *pred-equals-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x, y) \in R) = (\lambda x\ y.\ (x, y) \in S) \longleftrightarrow R = S$
by (*simp add*: *set-eq-iff fun-eq-iff*)

lemma *pred-subset-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) \le (\lambda x.\ x \in S) \longleftrightarrow R \subseteq S$
by (*simp add*: *subset-iff le-fun-def*)

lemma *pred-subset-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x, y) \in R) \le (\lambda x\ y.\ (x, y) \in S) \longleftrightarrow R \subseteq S$
by (*simp add*: *subset-iff le-fun-def*)

lemma *bot-empty-eq* [*pred-set-conv*]: $\bot = (\lambda x.\ x \in \{\})$
by (*auto simp add*: *fun-eq-iff*)

lemma *bot-empty-eq2* [*pred-set-conv*]: $\bot = (\lambda x\ y.\ (x, y) \in \{\})$
by (*auto simp add*: *fun-eq-iff*)

lemma *top-empty-eq*: $\top = (\lambda x.\ x \in UNIV)$
by (*auto simp add*: *fun-eq-iff*)

lemma *top-empty-eq2*: $\top = (\lambda x\ y.\ (x, y) \in UNIV)$
by (*auto simp add*: *fun-eq-iff*)

lemma *inf-Int-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) \sqcap (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cap S)$
by (*simp add*: *inf-fun-def*)

lemma *inf-Int-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x, y) \in R) \sqcap (\lambda x\ y.\ (x, y) \in S) = (\lambda x\ y.\ (x, y) \in R \cap S)$
by (*simp add*: *inf-fun-def*)

lemma *sup-Un-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) \sqcup (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cup S)$
by (*simp add*: *sup-fun-def*)

lemma *sup-Un-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x, y) \in R) \sqcup (\lambda x\ y.\ (x, y) \in S) = (\lambda x\ y.\ (x, y) \in R \cup S)$
by (*simp add*: *sup-fun-def*)

lemma *INF-INT-eq* [*pred-set-conv*]: $(\bigsqcap i \in S.\ (\lambda x.\ x \in r\ i)) = (\lambda x.\ x \in (\bigcap i \in S.\ r\ i))$

by (*simp add*: *fun-eq-iff*)

lemma *INF-INT-eq2* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in r\ i)) = (\lambda x\ y.\ (x,\ y) \in (\bigcap i{\in}S.\ r\ i))$
by (*simp add*: *fun-eq-iff*)

lemma *SUP-UN-eq* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x.\ x \in r\ i)) = (\lambda x.\ x \in (\bigcup i{\in}S.\ r\ i))$
by (*simp add*: *fun-eq-iff*)

lemma *SUP-UN-eq2* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in r\ i)) = (\lambda x\ y.\ (x,\ y) \in (\bigcup i{\in}S.\ r\ i))$
by (*simp add*: *fun-eq-iff*)

lemma *Inf-INT-eq* [*pred-set-conv*]: $\bigsqcap S = (\lambda x.\ x \in (\bigcap (Collect\ `\ S)))$
by (*simp add*: *fun-eq-iff*)

lemma *INF-Int-eq* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x.\ x \in i)) = (\lambda x.\ x \in \bigcap S)$
by (*simp add*: *fun-eq-iff*)

lemma *Inf-INT-eq2* [*pred-set-conv*]: $\bigsqcap S = (\lambda x\ y.\ (x,\ y) \in (\bigcap (Collect\ `\ case\text{-}prod\ `\ S)))$
by (*simp add*: *fun-eq-iff*)

lemma *INF-Int-eq2* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in i)) = (\lambda x\ y.\ (x,\ y) \in \bigcap S)$
by (*simp add*: *fun-eq-iff*)

lemma *Sup-SUP-eq* [*pred-set-conv*]: $\bigsqcup S = (\lambda x.\ x \in \bigcup (Collect\ `\ S))$
by (*simp add*: *fun-eq-iff*)

lemma *SUP-Sup-eq* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x.\ x \in i)) = (\lambda x.\ x \in \bigcup S)$
by (*simp add*: *fun-eq-iff*)

lemma *Sup-SUP-eq2* [*pred-set-conv*]: $\bigsqcup S = (\lambda x\ y.\ (x,\ y) \in (\bigcup (Collect\ `\ case\text{-}prod\ `\ S)))$
by (*simp add*: *fun-eq-iff*)

lemma *SUP-Sup-eq2* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in i)) = (\lambda x\ y.\ (x,\ y) \in \bigcup S)$
by (*simp add*: *fun-eq-iff*)

19.2 Properties of relations

19.2.1 Reflexivity

definition *refl-on* :: $'a\ set \Rightarrow\ 'a\ rel \Rightarrow bool$
where $refl\text{-}on\ A\ r \longleftrightarrow r \subseteq A \times A \wedge (\forall x{\in}A.\ (x,\ x) \in r)$

abbreviation *refl* :: $'a\ rel \Rightarrow bool$ — reflexivity over a type

where *refl* ≡ *refl-on UNIV*

definition *reflp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
where *reflp-on A R* ⟷ (∀ *x*∈*A*. *R x x*)

abbreviation *reflp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
where *reflp* ≡ *reflp-on UNIV*

lemma *reflp-def*[*no-atp*]: *reflp R* ⟷ (∀ *x*. *R x x*)
by (*simp add*: *reflp-on-def*)

reflp-def is for backward compatibility.

lemma *reflp-refl-eq* [*pred-set-conv*]: *reflp* (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *refl r*
by (*simp add*: *refl-on-def reflp-def*)

lemma *refl-onI* [*intro?*]: *r* ⊆ *A* × *A* ⟹ (⋀*x*. *x* ∈ *A* ⟹ (*x*, *x*) ∈ *r*) ⟹ *refl-on A r*
unfolding *refl-on-def* **by** (*iprover intro*!: *ballI*)

lemma *reflI*: (⋀*x*. (*x*, *x*) ∈ *r*) ⟹ *refl r*
by (*auto intro*: *refl-onI*)

lemma *reflp-onI*:
(⋀*x*. *x* ∈ *A* ⟹ *R x x*) ⟹ *reflp-on A R*
by (*simp add*: *reflp-on-def*)

lemma *reflpI*[*intro?*]: (⋀*x*. *R x x*) ⟹ *reflp R*
by (*rule reflp-onI*)

lemma *refl-onD*: *refl-on A r* ⟹ *a* ∈ *A* ⟹ (*a*, *a*) ∈ *r*
unfolding *refl-on-def* **by** *blast*

lemma *refl-onD1*: *refl-on A r* ⟹ (*x*, *y*) ∈ *r* ⟹ *x* ∈ *A*
unfolding *refl-on-def* **by** *blast*

lemma *refl-onD2*: *refl-on A r* ⟹ (*x*, *y*) ∈ *r* ⟹ *y* ∈ *A*
unfolding *refl-on-def* **by** *blast*

lemma *reflD*: *refl r* ⟹ (*a*, *a*) ∈ *r*
unfolding *refl-on-def* **by** *blast*

lemma *reflp-onD*:
reflp-on A R ⟹ *x* ∈ *A* ⟹ *R x x*
by (*simp add*: *reflp-on-def*)

lemma *reflpD*[*dest?*]: *reflp R* ⟹ *R x x*
by (*simp add*: *reflp-onD*)

lemma *reflpE*:

assumes *reflp r*
obtains *r x x*
using *assms* **by** (*auto dest*: *refl-onD simp add*: *reflp-def*)

lemma *reflp-on-subset*: *reflp-on A R* $\Longrightarrow$ *B* $\subseteq$ *A* $\Longrightarrow$ *reflp-on B R*
by (*auto intro*: *reflp-onI dest*: *reflp-onD*)

lemma *reflp-on-image*: *reflp-on* (*f* ' *A*) *R* $\longleftrightarrow$ *reflp-on A* (λa *b*. *R* (*f a*) (*f b*))
by (*simp add*: *reflp-on-def*)

lemma *refl-on-Int*: *refl-on A r* $\Longrightarrow$ *refl-on B s* $\Longrightarrow$ *refl-on* ($A \cap B$) ($r \cap s$)
unfolding *refl-on-def* **by** *blast*

lemma *reflp-on-inf*: *reflp-on A R* $\Longrightarrow$ *reflp-on B S* $\Longrightarrow$ *reflp-on* ($A \cap B$) ($R \sqcap S$)
by (*auto intro*: *reflp-onI dest*: *reflp-onD*)

lemma *reflp-inf*: *reflp r* $\Longrightarrow$ *reflp s* $\Longrightarrow$ *reflp* ($r \sqcap s$)
by (*rule reflp-on-inf*[*of UNIV - UNIV*, *unfolded Int-absorb*])

lemma *refl-on-Un*: *refl-on A r* $\Longrightarrow$ *refl-on B s* $\Longrightarrow$ *refl-on* ($A \cup B$) ($r \cup s$)
unfolding *refl-on-def* **by** *blast*

lemma *reflp-on-sup*: *reflp-on A R* $\Longrightarrow$ *reflp-on B S* $\Longrightarrow$ *reflp-on* ($A \cup B$) ($R \sqcup S$)
by (*auto intro*: *reflp-onI dest*: *reflp-onD*)

lemma *reflp-sup*: *reflp r* $\Longrightarrow$ *reflp s* $\Longrightarrow$ *reflp* ($r \sqcup s$)
by (*rule reflp-on-sup*[*of UNIV - UNIV*, *unfolded Un-absorb*])

lemma *refl-on-INTER*: $\forall x \in S$. *refl-on* (*A x*) (*r x*) $\Longrightarrow$ *refl-on* ($\bigcap$(*A* ' *S*)) ($\bigcap$(*r* ' *S*))
unfolding *refl-on-def* **by** *fast*

lemma *reflp-on-Inf*: $\forall x \in S$. *reflp-on* (*A x*) (*R x*) $\Longrightarrow$ *reflp-on* ($\bigcap$(*A* ' *S*)) ($\bigsqcap$(*R* ' *S*))
by (*auto intro*: *reflp-onI dest*: *reflp-onD*)

lemma *refl-on-UNION*: $\forall x \in S$. *refl-on* (*A x*) (*r x*) $\Longrightarrow$ *refl-on* ($\bigcup$(*A* ' *S*)) ($\bigcup$(*r* ' *S*))
unfolding *refl-on-def* **by** *blast*

lemma *reflp-on-Sup*: $\forall x \in S$. *reflp-on* (*A x*) (*R x*) $\Longrightarrow$ *reflp-on* ($\bigcup$(*A* ' *S*)) ($\bigsqcup$(*R* ' *S*))
by (*auto intro*: *reflp-onI dest*: *reflp-onD*)

lemma *refl-on-empty* [*simp*]: *refl-on* {} {}
by (*simp add*: *refl-on-def*)

lemma *reflp-on-empty* [*simp*]: *reflp-on* {} *R*
by (*auto intro*: *reflp-onI*)

lemma *refl-on-singleton* [*simp*]: *refl-on* $\{x\}$ $\{(x, x)\}$
by (*blast intro*: *refl-onI*)

lemma *refl-on-def'* [*nitpick-unfold*, *code*]:
refl-on A $r \longleftrightarrow (\forall (x, y) \in r.\ x \in A \wedge y \in A) \wedge (\forall x \in A.\ (x, x) \in r)$
by (*auto intro*: *refl-onI dest*: *refl-onD refl-onD1 refl-onD2*)

lemma *reflp-on-equality* [*simp*]: *reflp-on* A $(=)$
by (*simp add*: *reflp-on-def*)

lemma *reflp-on-mono*:
reflp-on A $R \Longrightarrow (\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow Q\ x\ y) \Longrightarrow$ *reflp-on* A Q
by (*auto intro*: *reflp-onI dest*: *reflp-onD*)

lemma *reflp-mono*: *reflp* $R \Longrightarrow (\bigwedge x\ y.\ R\ x\ y \Longrightarrow Q\ x\ y) \Longrightarrow$ *reflp* Q
by (*rule reflp-on-mono*[*of UNIV R Q*]) *simp-all*

lemma (**in** *preorder*) *reflp-on-le*[*simp*]: *reflp-on* A $(\le)$
by (*simp add*: *reflp-onI*)

lemma (**in** *preorder*) *reflp-on-ge*[*simp*]: *reflp-on* A $(\ge)$
by (*simp add*: *reflp-onI*)

19.2.2 Irreflexivity

definition *irrefl-on* :: *'a set* $\Rightarrow$ *'a rel* $\Rightarrow$ *bool* **where**
irrefl-on A $r \longleftrightarrow (\forall a \in A.\ (a, a) \notin r)$

abbreviation *irrefl* :: *'a rel* $\Rightarrow$ *bool* **where**
irrefl $\equiv$ *irrefl-on UNIV*

definition *irreflp-on* :: *'a set* $\Rightarrow$ (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
irreflp-on A $R \longleftrightarrow (\forall a \in A.\ \neg\ R\ a\ a)$

abbreviation *irreflp* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
irreflp $\equiv$ *irreflp-on UNIV*

lemma *irrefl-def*[*no-atp*]: *irrefl* $r \longleftrightarrow (\forall a.\ (a, a) \notin r)$
by (*simp add*: *irrefl-on-def*)

lemma *irreflp-def*[*no-atp*]: *irreflp* $R \longleftrightarrow (\forall a.\ \neg\ R\ a\ a)$
by (*simp add*: *irreflp-on-def*)

irrefl-def and *irreflp-def* are for backward compatibility.

lemma *irreflp-on-irrefl-on-eq* [*pred-set-conv*]: *irreflp-on* A $(\lambda a\ b.\ (a, b) \in r) \longleftrightarrow$ *irrefl-on* A r
by (*simp add*: *irrefl-on-def irreflp-on-def*)

lemmas *irreflp-irrefl-eq* = *irreflp-on-irrefl-on-eq*[*of UNIV*]

lemma *irrefl-onI*: ($\bigwedge a.\ a \in A \Longrightarrow (a,\ a) \notin r$) $\Longrightarrow$ *irrefl-on A r*
by (*simp add*: *irrefl-on-def*)

lemma *irreflI*[*intro?*]: ($\bigwedge a.\ (a,\ a) \notin r$) $\Longrightarrow$ *irrefl r*
by (*rule irrefl-onI*[*of UNIV*, *simplified*])

lemma *irreflp-onI*: ($\bigwedge a.\ a \in A \Longrightarrow \neg\ R\ a\ a$) $\Longrightarrow$ *irreflp-on A R*
by (*rule irrefl-onI*[*to-pred*])

lemma *irreflpI*[*intro?*]: ($\bigwedge a.\ \neg\ R\ a\ a$) $\Longrightarrow$ *irreflp R*
by (*rule irreflI*[*to-pred*])

lemma *irrefl-onD*: *irrefl-on A r* $\Longrightarrow a \in A \Longrightarrow (a,\ a) \notin r$
by (*simp add*: *irrefl-on-def*)

lemma *irreflD*: *irrefl r* $\Longrightarrow (x,\ x) \notin r$
by (*rule irrefl-onD*[*of UNIV*, *simplified*])

lemma *irreflp-onD*: *irreflp-on A R* $\Longrightarrow a \in A \Longrightarrow \neg\ R\ a\ a$
by (*rule irrefl-onD*[*to-pred*])

lemma *irreflpD*: *irreflp R* $\Longrightarrow \neg\ R\ x\ x$
by (*rule irreflD*[*to-pred*])

lemma *irrefl-on-distinct* [*code*]: *irrefl-on A r* $\longleftrightarrow (\forall (a,\ b) \in r.\ a \in A \longrightarrow b \in A \longrightarrow a \neq b)$
by (*auto simp add*: *irrefl-on-def*)

lemmas *irrefl-distinct* = *irrefl-on-distinct* — For backward compatibility

lemma *irrefl-on-subset*: *irrefl-on A r* $\Longrightarrow B \subseteq A \Longrightarrow$ *irrefl-on B r*
by (*auto simp*: *irrefl-on-def*)

lemma *irreflp-on-subset*: *irreflp-on A R* $\Longrightarrow B \subseteq A \Longrightarrow$ *irreflp-on B R*
by (*auto simp*: *irreflp-on-def*)

lemma *irreflp-on-image*: *irreflp-on* (f ‘ A) $R \longleftrightarrow$ *irreflp-on A* ($\lambda a\ b.\ R\ (f\ a)\ (f\ b)$)
by (*simp add*: *irreflp-on-def*)

lemma (**in** *preorder*) *irreflp-on-less*[*simp*]: *irreflp-on A* ($<$)
by (*simp add*: *irreflp-onI*)

lemma (**in** *preorder*) *irreflp-on-greater*[*simp*]: *irreflp-on A* ($>$)
by (*simp add*: *irreflp-onI*)

19.2.3 Asymmetry

definition *asym-on* :: $'a$ *set* $\Rightarrow$ $'a$ *rel* $\Rightarrow$ *bool* **where**
asym-on A r $\longleftrightarrow$ $(\forall x \in A.\ \forall y \in A.\ (x, y) \in r \longrightarrow (y, x) \notin r)$

abbreviation *asym* :: $'a$ *rel* $\Rightarrow$ *bool* **where**
asym $\equiv$ *asym-on UNIV*

definition *asymp-on* :: $'a$ *set* $\Rightarrow$ $('a \Rightarrow 'a \Rightarrow$ *bool*$)$ $\Rightarrow$ *bool* **where**
asymp-on A R $\longleftrightarrow$ $(\forall x \in A.\ \forall y \in A.\ R\ x\ y \longrightarrow \neg\ R\ y\ x)$

abbreviation *asymp* :: $('a \Rightarrow 'a \Rightarrow$ *bool*$)$ $\Rightarrow$ *bool* **where**
asymp $\equiv$ *asymp-on UNIV*

lemma *asymp-on-asym-on-eq*[*pred-set-conv*]: *asymp-on A* $(\lambda x\ y.\ (x, y) \in r)$ $\longleftrightarrow$ *asym-on A r*
by (*simp add*: *asymp-on-def asym-on-def*)

lemmas *asymp-asym-eq* = *asymp-on-asym-on-eq*[*of UNIV*] — For backward compatibility

lemma *asym-onI*[*intro*]:
$(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \notin r) \Longrightarrow$ *asym-on A r*
by (*simp add*: *asym-on-def*)

lemma *asymI*[*intro*]: $(\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (y, x) \notin r) \Longrightarrow$ *asym r*
by (*simp add*: *asym-onI*)

lemma *asymp-onI*[*intro*]:
$(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow \neg\ R\ y\ x) \Longrightarrow$ *asymp-on A R*
by (*rule asym-onI*[*to-pred*])

lemma *asympI*[*intro*]: $(\bigwedge x\ y.\ R\ x\ y \Longrightarrow \neg\ R\ y\ x) \Longrightarrow$ *asymp R*
by (*rule asymI*[*to-pred*])

lemma *asym-onD*: *asym-on A r* $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \notin r$
by (*simp add*: *asym-on-def*)

lemma *asymD*: *asym r* $\Longrightarrow (x, y) \in r \Longrightarrow (y, x) \notin r$
by (*simp add*: *asym-onD*)

lemma *asymp-onD*: *asymp-on A R* $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow \neg\ R\ y\ x$
by (*rule asym-onD*[*to-pred*])

lemma *asympD*: *asymp R* $\Longrightarrow R\ x\ y \Longrightarrow \neg\ R\ y\ x$
by (*rule asymD*[*to-pred*])

lemma *asym-iff*: *asym r* $\longleftrightarrow (\forall x\ y.\ (x,y) \in r \longrightarrow (y,x) \notin r)$
by (*blast dest*: *asymD*)

lemma *asym-on-subset*: *asym-on A r* $\Longrightarrow$ *B* $\subseteq$ *A* $\Longrightarrow$ *asym-on B r*
by (*auto simp*: *asym-on-def*)

lemma *asymp-on-subset*: *asymp-on A R* $\Longrightarrow$ *B* $\subseteq$ *A* $\Longrightarrow$ *asymp-on B R*
by (*auto simp*: *asymp-on-def*)

lemma *asymp-on-image*: *asymp-on* (*f* ‘ *A*) *R* $\longleftrightarrow$ *asymp-on A* (λ*a b*. *R* (*f a*) (*f b*))
by (*simp add*: *asymp-on-def*)

lemma *irrefl-on-if-asym-on*[*simp*]: *asym-on A r* $\Longrightarrow$ *irrefl-on A r*
by (*auto intro*: *irrefl-onI dest*: *asym-onD*)

lemma *irreflp-on-if-asymp-on*[*simp*]: *asymp-on A r* $\Longrightarrow$ *irreflp-on A r*
by (*rule irrefl-on-if-asym-on*[*to-pred*])

lemma (**in** *preorder*) *asymp-on-less*[*simp*]: *asymp-on A* (<)
by (*auto intro*: *dual-order.asym*)

lemma (**in** *preorder*) *asymp-on-greater*[*simp*]: *asymp-on A* (>)
by (*auto intro*: *dual-order.asym*)

19.2.4 Symmetry

definition *sym-on* :: *'a set* $\Rightarrow$ *'a rel* $\Rightarrow$ *bool* **where**
sym-on A r $\longleftrightarrow$ ($\forall x \in A$. $\forall y \in A$. $(x, y) \in r \longrightarrow (y, x) \in r$)

abbreviation *sym* :: *'a rel* $\Rightarrow$ *bool* **where**
sym $\equiv$ *sym-on UNIV*

definition *symp-on* :: *'a set* $\Rightarrow$ (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
symp-on A R $\longleftrightarrow$ ($\forall x \in A$. $\forall y \in A$. *R x y* $\longrightarrow$ *R y x*)

abbreviation *symp* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
symp $\equiv$ *symp-on UNIV*

lemma *sym-def*[*no-atp*]: *sym r* $\longleftrightarrow$ ($\forall x\ y$. $(x, y) \in r \longrightarrow (y, x) \in r$)
by (*simp add*: *sym-on-def*)

lemma *symp-def*[*no-atp*]: *symp R* $\longleftrightarrow$ ($\forall x\ y$. *R x y* $\longrightarrow$ *R y x*)
by (*simp add*: *symp-on-def*)

sym-def and *symp-def* are for backward compatibility.

lemma *symp-on-sym-on-eq*[*pred-set-conv*]: *symp-on A* (λ*x y*. $(x, y) \in r$) $\longleftrightarrow$ *sym-on A r*
by (*simp add*: *sym-on-def symp-on-def*)

lemmas *symp-sym-eq* = *symp-on-sym-on-eq*[*of UNIV*] — For backward compatibility

lemma *sym-on-subset*: *sym-on A r* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *sym-on B r*
by (*auto simp*: *sym-on-def*)

lemma *symp-on-subset*: *symp-on A R* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *symp-on B R*
by (*auto simp*: *symp-on-def*)

lemma *symp-on-image*: *symp-on* (*f* ' *A*) *R* $\longleftrightarrow$ *symp-on A* ($\lambda a\ b.\ R\ (f\ a)\ (f\ b)$)
by (*simp add*: *symp-on-def*)

lemma *sym-onI*: ($\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \in r$) $\Longrightarrow$ *sym-on A r*
by (*simp add*: *sym-on-def*)

lemma *symI* [*intro?*]: ($\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (y, x) \in r$) $\Longrightarrow$ *sym r*
by (*simp add*: *sym-onI*)

lemma *symp-onI*: ($\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x$) $\Longrightarrow$ *symp-on A R*
by (*rule sym-onI*[*to-pred*])

lemma *sympI* [*intro?*]: ($\bigwedge x\ y.\ R\ x\ y \Longrightarrow R\ y\ x$) $\Longrightarrow$ *symp R*
by (*rule symI*[*to-pred*])

lemma *symE*:
assumes *sym r* **and** $(b, a) \in r$
obtains $(a, b) \in r$
using *assms* **by** (*simp add*: *sym-def*)

lemma *sympE*:
assumes *symp r* **and** *r b a*
obtains *r a b*
using *assms* **by** (*rule symE* [*to-pred*])

lemma *sym-onD*: *sym-on A r* $\Longrightarrow$ $x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \in r$
by (*simp add*: *sym-on-def*)

lemma *symD* [*dest?*]: *sym r* $\Longrightarrow$ $(x, y) \in r \Longrightarrow (y, x) \in r$
by (*simp add*: *sym-onD*)

lemma *symp-onD*: *symp-on A R* $\Longrightarrow$ $x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x$
by (*rule sym-onD*[*to-pred*])

lemma *sympD* [*dest?*]: *symp R* $\Longrightarrow$ $R\ x\ y \Longrightarrow R\ y\ x$
by (*rule symD*[*to-pred*])

lemma *sym-Int*: *sym r* $\Longrightarrow$ *sym s* $\Longrightarrow$ *sym* $(r \cap s)$
by (*fast intro*: *symI elim*: *symE*)

lemma *symp-inf*: *symp r* $\Longrightarrow$ *symp s* $\Longrightarrow$ *symp* $(r \sqcap s)$
by (*fact sym-Int* [*to-pred*])

lemma *sym-Un*: *sym r* $\Longrightarrow$ *sym s* $\Longrightarrow$ *sym* $(r \cup s)$
by (*fast intro*: *symI elim*: *symE*)

lemma *symp-sup*: *symp r* $\Longrightarrow$ *symp s* $\Longrightarrow$ *symp* $(r \sqcup s)$
by (*fact sym-Un* [*to-pred*])

lemma *sym-INTER*: $\forall x \in S$. *sym* $(r\ x) \Longrightarrow$ *sym* $(\bigcap (r\ `\ S))$
by (*fast intro*: *symI elim*: *symE*)

lemma *symp-INF*: $\forall x \in S$. *symp* $(r\ x) \Longrightarrow$ *symp* $(\bigsqcap (r\ `\ S))$
by (*fact sym-INTER* [*to-pred*])

lemma *sym-UNION*: $\forall x \in S$. *sym* $(r\ x) \Longrightarrow$ *sym* $(\bigcup (r\ `\ S))$
by (*fast intro*: *symI elim*: *symE*)

lemma *symp-SUP*: $\forall x \in S$. *symp* $(r\ x) \Longrightarrow$ *symp* $(\bigsqcup (r\ `\ S))$
by (*fact sym-UNION* [*to-pred*])

19.2.5 Antisymmetry

definition *antisym-on* :: *'a set* $\Rightarrow$ *'a rel* $\Rightarrow$ *bool* **where**
antisym-on A r $\longleftrightarrow (\forall x \in A.\ \forall y \in A.\ (x, y) \in r \longrightarrow (y, x) \in r \longrightarrow x = y)$

abbreviation *antisym* :: *'a rel* $\Rightarrow$ *bool* **where**
antisym $\equiv$ *antisym-on UNIV*

definition *antisymp-on* :: *'a set* $\Rightarrow$ *('a* $\Rightarrow$ *'a* $\Rightarrow$ *bool)* $\Rightarrow$ *bool* **where**
antisymp-on A R $\longleftrightarrow (\forall x \in A.\ \forall y \in A.\ R\ x\ y \longrightarrow R\ y\ x \longrightarrow x = y)$

abbreviation *antisymp* :: *('a* $\Rightarrow$ *'a* $\Rightarrow$ *bool)* $\Rightarrow$ *bool* **where**
antisymp $\equiv$ *antisymp-on UNIV*

lemma *antisym-def*[*no-atp*]: *antisym r* $\longleftrightarrow (\forall x\ y.\ (x, y) \in r \longrightarrow (y, x) \in r \longrightarrow x = y)$
by (*simp add*: *antisym-on-def*)

lemma *antisymp-def*[*no-atp*]: *antisymp R* $\longleftrightarrow (\forall x\ y.\ R\ x\ y \longrightarrow R\ y\ x \longrightarrow x = y)$
by (*simp add*: *antisymp-on-def*)

antisym-def and *antisymp-def* are for backward compatibility.

lemma *antisymp-on-antisym-on-eq*[*pred-set-conv*]:
antisymp-on A $(\lambda x\ y.\ (x, y) \in r) \longleftrightarrow$ *antisym-on A r*
by (*simp add*: *antisym-on-def antisymp-on-def*)

lemmas *antisymp-antisym-eq* = *antisymp-on-antisym-on-eq*[*of UNIV*] — For backward compatibility

lemma *antisym-on-subset*: *antisym-on A r* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *antisym-on B r*
by (*auto simp*: *antisym-on-def*)

lemma *antisymp-on-subset*: *antisymp-on A R* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *antisymp-on B R*
by (*auto simp*: *antisymp-on-def*)

lemma *antisymp-on-image*:
assumes *inj-on f A*
shows *antisymp-on* (*f ‘ A*) *R* $\longleftrightarrow$ *antisymp-on A* ($\lambda a\ b.\ R\ (f\ a)\ (f\ b)$)
using *assms* **by** (*auto simp*: *antisymp-on-def inj-on-def*)

lemma *antisym-onI*:
$(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \in r \Longrightarrow x = y) \Longrightarrow$ *antisym-on A r*
unfolding *antisym-on-def* **by** *simp*

lemma *antisymI* [*intro?*]:
$(\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (y, x) \in r \Longrightarrow x = y) \Longrightarrow$ *antisym r*
by (*simp add*: *antisym-onI*)

lemma *antisymp-onI*:
$(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y) \Longrightarrow$ *antisymp-on A R*
by (*rule antisym-onI*[*to-pred*])

lemma *antisympI* [*intro?*]:
$(\bigwedge x\ y.\ R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y) \Longrightarrow$ *antisymp R*
by (*rule antisymI*[*to-pred*])

lemma *antisym-onD*:
antisym-on A r $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \in r \Longrightarrow x = y$
by (*simp add*: *antisym-on-def*)

lemma *antisymD* [*dest?*]:
antisym r $\Longrightarrow (x, y) \in r \Longrightarrow (y, x) \in r \Longrightarrow x = y$
by (*simp add*: *antisym-onD*)

lemma *antisymp-onD*:
antisymp-on A R $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y$
by (*rule antisym-onD*[*to-pred*])

lemma *antisympD* [*dest?*]:
antisymp R $\Longrightarrow R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y$
by (*rule antisymD*[*to-pred*])

lemma *antisym-subset*:
$r \subseteq s \Longrightarrow$ *antisym s* $\Longrightarrow$ *antisym r*

unfolding *antisym-def* **by** *blast*

lemma *antisymp-less-eq*:
r ≤ *s* ⟹ *antisymp s* ⟹ *antisymp r*
by (*fact antisym-subset* [*to-pred*])

lemma *antisym-empty* [*simp*]:
antisym {}
unfolding *antisym-def* **by** *blast*

lemma *antisym-bot* [*simp*]:
antisymp ⊥
by (*fact antisym-empty* [*to-pred*])

lemma *antisymp-equality* [*simp*]:
antisymp HOL.eq
by (*auto intro*: *antisympI*)

lemma *antisym-singleton* [*simp*]:
antisym {*x*}
by (*blast intro*: *antisymI*)

lemma *antisym-on-if-asym-on*: *asym-on A r* ⟹ *antisym-on A r*
by (*auto intro*: *antisym-onI dest*: *asym-onD*)

lemma *antisymp-on-if-asymp-on*: *asymp-on A R* ⟹ *antisymp-on A R*
by (*rule antisym-on-if-asym-on*[*to-pred*])

lemma (**in** *preorder*) *antisymp-on-less*[*simp*]: *antisymp-on A* (<)
by (*rule antisymp-on-if-asymp-on*[*OF asymp-on-less*])

lemma (**in** *preorder*) *antisymp-on-greater*[*simp*]: *antisymp-on A* (>)
by (*rule antisymp-on-if-asymp-on*[*OF asymp-on-greater*])

lemma (**in** *order*) *antisymp-on-le*[*simp*]: *antisymp-on A* (≤)
by (*simp add*: *antisymp-onI*)

lemma (**in** *order*) *antisymp-on-ge*[*simp*]: *antisymp-on A* (≥)
by (*simp add*: *antisymp-onI*)

19.2.6 Transitivity

definition *trans-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
trans-on A r ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. ∀ *z* ∈ *A*. (*x*, *y*) ∈ *r* ⟶ (*y*, *z*) ∈ *r* ⟶ (*x*, *z*) ∈ *r*)

abbreviation *trans* :: *'a rel* ⇒ *bool* **where**
trans ≡ *trans-on UNIV*

definition *transp-on* :: $'a\ set \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ **where**
transp-on $A\ R \longleftrightarrow (\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ R\ x\ y \longrightarrow R\ y\ z \longrightarrow R\ x\ z)$

abbreviation *transp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ **where**
transp $\equiv$ *transp-on UNIV*

lemma *trans-def*[*no-atp*]: *trans* $r \longleftrightarrow (\forall x\ y\ z.\ (x, y) \in r \longrightarrow (y, z) \in r \longrightarrow (x, z) \in r)$
by (*simp add*: *trans-on-def*)

lemma *transp-def*: *transp* $R \longleftrightarrow (\forall x\ y\ z.\ R\ x\ y \longrightarrow R\ y\ z \longrightarrow R\ x\ z)$
by (*simp add*: *transp-on-def*)

trans-def and *transp-def* are for backward compatibility.

lemma *transp-on-trans-on-eq*[*pred-set-conv*]: *transp-on* $A\ (\lambda x\ y.\ (x, y) \in r) \longleftrightarrow$ *trans-on* $A\ r$
by (*simp add*: *trans-on-def transp-on-def*)

lemmas *transp-trans-eq* = *transp-on-trans-on-eq*[*of UNIV*] — For backward compatibility

lemma *trans-onI*:
$(\bigwedge x\ y\ z.\ x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, z) \in r \Longrightarrow (x, z) \in r) \Longrightarrow$
trans-on $A\ r$
unfolding *trans-on-def*
by (*intro ballI*) *iprover*

lemma *transI* [*intro?*]: $(\bigwedge x\ y\ z.\ (x, y) \in r \Longrightarrow (y, z) \in r \Longrightarrow (x, z) \in r) \Longrightarrow$ *trans* r
by (*rule trans-onI*)

lemma *transp-onI*:
$(\bigwedge x\ y\ z.\ x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z) \Longrightarrow$ *transp-on* $A\ R$
by (*rule trans-onI*[*to-pred*])

lemma *transpI* [*intro?*]: $(\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z) \Longrightarrow$ *transp* R
by (*rule transI*[*to-pred*])

lemma *transE*:
assumes *trans* r **and** $(x, y) \in r$ **and** $(y, z) \in r$
obtains $(x, z) \in r$
using *assms* **by** (*unfold trans-def*) *iprover*

lemma *transpE*:
assumes *transp* r **and** $r\ x\ y$ **and** $r\ y\ z$
obtains $r\ x\ z$
using *assms* **by** (*rule transE* [*to-pred*])

lemma *trans-onD*:
trans-on A $r \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, z) \in r \Longrightarrow (x, z) \in r$
unfolding *trans-on-def*
by (*elim ballE*) *iprover+*

lemma *transD*[*dest?*]: *trans* $r \Longrightarrow (x, y) \in r \Longrightarrow (y, z) \in r \Longrightarrow (x, z) \in r$
by (*simp add*: *trans-onD*[*of UNIV r x y z*])

lemma *transp-onD*: *transp-on* A $R \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
by (*rule trans-onD*[*to-pred*])

lemma *transpD*[*dest?*]: *transp* $R \Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
by (*rule transD*[*to-pred*])

lemma *trans-on-subset*: *trans-on* A $r \Longrightarrow B \subseteq A \Longrightarrow$ *trans-on* B r
by (*auto simp*: *trans-on-def*)

lemma *transp-on-subset*: *transp-on* A $R \Longrightarrow B \subseteq A \Longrightarrow$ *transp-on* B R
by (*auto simp*: *transp-on-def*)

lemma *transp-on-image*: *transp-on* (f ‘ A) $R \longleftrightarrow$ *transp-on* A ($\lambda a\ b.\ R\ (f\ a)\ (f\ b)$)
by (*simp add*: *transp-on-def*)

lemma *trans-Int*: *trans* $r \Longrightarrow$ *trans* $s \Longrightarrow$ *trans* ($r \cap s$)
by (*fast intro*: *transI elim*: *transE*)

lemma *transp-inf*: *transp* $r \Longrightarrow$ *transp* $s \Longrightarrow$ *transp* ($r \sqcap s$)
by (*fact trans-Int* [*to-pred*])

lemma *trans-INTER*: $\forall x \in S$. *trans* ($r\ x$) $\Longrightarrow$ *trans* ($\bigcap (r$ ‘ $S)$)
by (*fast intro*: *transI elim*: *transD*)

lemma *transp-INF*: $\forall x \in S$. *transp* ($r\ x$) $\Longrightarrow$ *transp* ($\bigsqcap (r$ ‘ $S)$)
by (*fact trans-INTER* [*to-pred*])

lemma *trans-on-join* [*code*]:
trans-on A $r \longleftrightarrow (\forall (x, y1) \in r.\ x \in A \longrightarrow y1 \in A \longrightarrow$
$(\forall (y2, z) \in r.\ y1 = y2 \longrightarrow z \in A \longrightarrow (x, z) \in r))$
by (*auto simp*: *trans-on-def*)

lemma *trans-join*: *trans* $r \longleftrightarrow (\forall (x, y1) \in r.\ \forall (y2, z) \in r.\ y1 = y2 \longrightarrow (x, z) \in r)$
by (*auto simp add*: *trans-def*)

lemma *transp-trans*: *transp* $r \longleftrightarrow$ *trans* $\{(x, y).\ r\ x\ y\}$

by (*simp add*: *trans-def transp-def*)

lemma *transp-equality* [*simp*]: *transp* (=)
by (*auto intro*: *transpI*)

lemma *trans-empty* [*simp*]: *trans* {}
by (*blast intro*: *transI*)

lemma *transp-empty* [*simp*]: *transp* ($\lambda x\ y.$ *False*)
using *trans-empty*[*to-pred*] **by** (*simp add*: *bot-fun-def*)

lemma *trans-singleton* [*simp*]: *trans* {(*a*, *a*)}
by (*blast intro*: *transI*)

lemma *transp-singleton* [*simp*]: *transp* ($\lambda x\ y.\ x = a \wedge y = a$)
by (*simp add*: *transp-def*)

lemma *asym-on-iff-irrefl-on-if-trans-on*: *trans-on A r* $\Longrightarrow$ *asym-on A r* $\longleftrightarrow$ *irrefl-on A r*
by (*auto intro*: *irrefl-on-if-asym-on dest*: *trans-onD irrefl-onD*)

lemma *asymp-on-iff-irreflp-on-if-transp-on*: *transp-on A R* $\Longrightarrow$ *asymp-on A R* $\longleftrightarrow$ *irreflp-on A R*
by (*rule asym-on-iff-irrefl-on-if-trans-on*[*to-pred*])

lemma (**in** *preorder*) *transp-on-le*[*simp*]: *transp-on A* ($\leq$)
by (*auto intro*: *transp-onI order-trans*)

lemma (**in** *preorder*) *transp-on-less*[*simp*]: *transp-on A* (<)
by (*auto intro*: *transp-onI less-trans*)

lemma (**in** *preorder*) *transp-on-ge*[*simp*]: *transp-on A* ($\geq$)
by (*auto intro*: *transp-onI order-trans*)

lemma (**in** *preorder*) *transp-on-greater*[*simp*]: *transp-on A* (>)
by (*auto intro*: *transp-onI less-trans*)

19.2.7 Totality

definition *total-on* :: *'a set* $\Rightarrow$ *'a rel* $\Rightarrow$ *bool* **where**
total-on A r $\longleftrightarrow$ ($\forall x \in A.\ \forall y \in A.\ x \neq y \longrightarrow (x, y) \in r \vee (y, x) \in r$)

abbreviation *total* :: *'a rel* $\Rightarrow$ *bool* **where**
total $\equiv$ *total-on UNIV*

definition *totalp-on* :: *'a set* $\Rightarrow$ (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
totalp-on A R $\longleftrightarrow$ ($\forall x \in A.\ \forall y \in A.\ x \neq y \longrightarrow R\ x\ y \vee R\ y\ x$)

abbreviation *totalp* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**

totalp ≡ *totalp-on UNIV*

lemma *totalp-on-total-on-eq*[*pred-set-conv*]: *totalp-on A* (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *total-on A r*
by (*simp add*: *totalp-on-def total-on-def*)

lemma *total-onI* [*intro?*]:
(⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≠ *y* ⟹ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*) ⟹ *total-on A r*
unfolding *total-on-def* **by** *blast*

lemma *totalI*: (⋀*x y*. *x* ≠ *y* ⟹ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*) ⟹ *total r*
by (*rule total-onI*)

lemma *totalp-onI*: (⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≠ *y* ⟹ *R x y* ∨ *R y x*) ⟹ *totalp-on A R*
by (*rule total-onI*[*to-pred*])

lemma *totalpI*: (⋀*x y*. *x* ≠ *y* ⟹ *R x y* ∨ *R y x*) ⟹ *totalp R*
by (*rule totalI*[*to-pred*])

lemma *totalp-onD*:
totalp-on A R ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≠ *y* ⟹ *R x y* ∨ *R y x*
by (*simp add*: *totalp-on-def*)

lemma *totalpD*: *totalp R* ⟹ *x* ≠ *y* ⟹ *R x y* ∨ *R y x*
by (*simp add*: *totalp-onD*)

lemma *total-on-subset*: *total-on A r* ⟹ *B* ⊆ *A* ⟹ *total-on B r*
by (*auto simp*: *total-on-def*)

lemma *totalp-on-subset*: *totalp-on A R* ⟹ *B* ⊆ *A* ⟹ *totalp-on B R*
by (*auto intro*: *totalp-onI dest*: *totalp-onD*)

lemma *totalp-on-image*:
assumes *inj-on f A*
shows *totalp-on* (*f* ' *A*) *R* ⟷ *totalp-on A* (λ*a b*. *R* (*f a*) (*f b*))
using *assms* **by** (*auto simp*: *totalp-on-def inj-on-def*)

lemma *total-on-empty* [*simp*]: *total-on* {} *r*
by (*simp add*: *total-on-def*)

lemma *totalp-on-empty* [*simp*]: *totalp-on* {} *R*
by (*simp add*: *totalp-on-def*)

lemma *total-on-singleton* [*simp*]: *total-on* {*x*} *r*
by (*simp add*: *total-on-def*)

lemma *totalp-on-singleton* [*simp*]: *totalp-on* {*x*} *R*
by (*simp add*: *totalp-on-def*)

lemma (**in** *linorder*) *totalp-on-less*[*simp*]: *totalp-on A* (<)
by (*auto intro*: *totalp-onI*)

lemma (**in** *linorder*) *totalp-on-greater*[*simp*]: *totalp-on A* (>)
by (*auto intro*: *totalp-onI*)

lemma (**in** *linorder*) *totalp-on-le*[*simp*]: *totalp-on A* (≤)
by (*rule totalp-onI*, *rule linear*)

lemma (**in** *linorder*) *totalp-on-ge*[*simp*]: *totalp-on A* (≥)
by (*rule totalp-onI*, *rule linear*)

19.2.8 Single valued relations

definition *single-valued* :: ('*a* × '*b*) *set* ⇒ *bool*
where *single-valued r* ⟷ (∀ *x y*. (*x*, *y*) ∈ *r* ⟶ (∀ *z*. (*x*, *z*) ∈ *r* ⟶ *y* = *z*))

definition *single-valuedp* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ *bool*
where *single-valuedp r* ⟷ (∀ *x y*. *r x y* ⟶ (∀ *z*. *r x z* ⟶ *y* = *z*))

lemma *single-valuedp-single-valued-eq* [*pred-set-conv*]:
single-valuedp (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *single-valued r*
by (*simp add*: *single-valued-def single-valuedp-def*)

lemma *single-valuedp-iff-Uniq*:
single-valuedp r ⟷ (∀ *x*. $\exists_{\leq 1}$*y*. *r x y*)
unfolding *Uniq-def single-valuedp-def* **by** *auto*

lemma *single-valuedI*:
(⋀*x y*. (*x*, *y*) ∈ *r* ⟹ (⋀*z*. (*x*, *z*) ∈ *r* ⟹ *y* = *z*)) ⟹ *single-valued r*
unfolding *single-valued-def* **by** *blast*

lemma *single-valuedpI*:
(⋀*x y*. *r x y* ⟹ (⋀*z*. *r x z* ⟹ *y* = *z*)) ⟹ *single-valuedp r*
by (*fact single-valuedI* [*to-pred*])

lemma *single-valuedD*:
single-valued r ⟹ (*x*, *y*) ∈ *r* ⟹ (*x*, *z*) ∈ *r* ⟹ *y* = *z*
by (*simp add*: *single-valued-def*)

lemma *single-valuedpD*:
single-valuedp r ⟹ *r x y* ⟹ *r x z* ⟹ *y* = *z*
by (*fact single-valuedD* [*to-pred*])

lemma *single-valued-empty* [*simp*]:
single-valued {}
by (*simp add*: *single-valued-def*)

lemma *single-valuedp-bot* [*simp*]:
single-valuedp $\bot$
by (*fact single-valued-empty* [*to-pred*])

lemma *single-valued-subset*:
$r \subseteq s \Longrightarrow$ *single-valued* $s \Longrightarrow$ *single-valued* r
unfolding *single-valued-def* **by** *blast*

lemma *single-valuedp-less-eq*:
$r \leq s \Longrightarrow$ *single-valuedp* $s \Longrightarrow$ *single-valuedp* r
by (*fact single-valued-subset* [*to-pred*])

19.3 Relation operations

19.3.1 The identity relation

definition *Id* :: *'a rel*
where $Id = \{p.\ \exists x.\ p = (x, x)\}$

lemma *IdI* [*intro*]: $(a, a) \in Id$
by (*simp add*: *Id-def*)

lemma *IdE* [*elim*!]: $p \in Id \Longrightarrow (\bigwedge x.\ p = (x, x) \Longrightarrow P) \Longrightarrow P$
unfolding *Id-def* **by** (*iprover elim*: *CollectE*)

lemma *pair-in-Id-conv* [*iff*]: $(a, b) \in Id \longleftrightarrow a = b$
unfolding *Id-def* **by** *blast*

lemma *refl-Id*: *refl Id*
by (*simp add*: *refl-on-def*)

lemma *antisym-Id*: *antisym Id*
— A strange result, since *Id* is also symmetric.
by (*simp add*: *antisym-def*)

lemma *sym-Id*: *sym Id*
by (*simp add*: *sym-def*)

lemma *trans-Id*: *trans Id*
by (*simp add*: *trans-def*)

lemma *single-valued-Id* [*simp*]: *single-valued Id*
by (*unfold single-valued-def*) *blast*

lemma *irrefl-diff-Id* [*simp*]: *irrefl* $(r - Id)$
by (*simp add*: *irrefl-def*)

lemma *trans-diff-Id*: *trans* $r \Longrightarrow$ *antisym* $r \Longrightarrow$ *trans* $(r - Id)$
unfolding *antisym-def trans-def* **by** *blast*

lemma *total-on-diff-Id* [*simp*]: *total-on A* (*r* − *Id*) = *total-on A r*
by (*simp add*: *total-on-def*)

lemma *Id-fstsnd-eq*: *Id* = {*x*. *fst x* = *snd x*}
by *force*

19.3.2 Diagonal: identity over a set

definition *Id-on* :: ′*a set* ⇒ ′*a rel*
where *Id-on A* = (⋃*x*∈*A*. {(*x*, *x*)})

lemma *Id-on-empty* [*simp*]: *Id-on* {} = {}
by (*simp add*: *Id-on-def*)

lemma *Id-on-eqI*: *a* = *b* ⟹ *a* ∈ *A* ⟹ (*a*, *b*) ∈ *Id-on A*
by (*simp add*: *Id-on-def*)

lemma *Id-onI* [*intro*!]: *a* ∈ *A* ⟹ (*a*, *a*) ∈ *Id-on A*
by (*rule Id-on-eqI*) (*rule refl*)

lemma *Id-onE* [*elim*!]: *c* ∈ *Id-on A* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *c* = (*x*, *x*) ⟹ *P*) ⟹ *P*
— The general elimination rule.
unfolding *Id-on-def* **by** (*iprover elim*!: *UN-E singletonE*)

lemma *Id-on-iff*: (*x*, *y*) ∈ *Id-on A* ⟷ *x* = *y* ∧ *x* ∈ *A*
by *blast*

lemma *Id-on-def′* [*nitpick-unfold*]: *Id-on* {*x*. *A x*} = *Collect* (λ(*x*, *y*). *x* = *y* ∧ *A x*)
by *auto*

lemma *Id-on-subset-Times*: *Id-on A* ⊆ *A* × *A*
by *blast*

lemma *refl-on-Id-on*: *refl-on A* (*Id-on A*)
by (*rule refl-onI* [*OF Id-on-subset-Times Id-onI*])

lemma *antisym-Id-on* [*simp*]: *antisym* (*Id-on A*)
unfolding *antisym-def* **by** *blast*

lemma *sym-Id-on* [*simp*]: *sym* (*Id-on A*)
by (*rule symI*) *clarify*

lemma *trans-Id-on* [*simp*]: *trans* (*Id-on A*)
by (*fast intro*: *transI elim*: *transD*)

lemma *single-valued-Id-on* [*simp*]: *single-valued* (*Id-on A*)
unfolding *single-valued-def* **by** *blast*

19.3.3 Composition

inductive-set *relcomp* :: ('a × 'b) *set* ⇒ ('b × 'c) *set* ⇒ ('a × 'c) *set*
for *r* :: ('a × 'b) *set* **and** *s* :: ('b × 'c) *set*
where *relcompI* [*intro*]: (*a*, *b*) ∈ *r* ⟹ (*b*, *c*) ∈ *s* ⟹ (*a*, *c*) ∈ *relcomp r s*

open-bundle *relcomp-syntax*
begin
notation *relcomp* (**infixr** ‹*O*› *75*) **and** *relcompp* (**infixr** ‹*OO*› *75*)
end

lemmas *relcomppI* = *relcompp.intros*

For historic reasons, the elimination rules are not wholly corresponding. Feel free to consolidate this.

inductive-cases *relcompEpair*: (*a*, *c*) ∈ *r O s*
inductive-cases *relcomppE* [*elim*!]: (*r OO s*) *a c*

lemma *relcompE* [*elim*!]: *xz* ∈ *r O s* ⟹
(⋀*x y z*. *xz* = (*x*, *z*) ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *z*) ∈ *s* ⟹ *P*) ⟹ *P*
apply (*cases xz*)
apply *simp*
apply (*erule relcompEpair*)
apply *iprover*
done

lemma *R-O-Id* [*simp*]: *R O Id* = *R*
by *fast*

lemma *Id-O-R* [*simp*]: *Id O R* = *R*
by *fast*

lemma *relcomp-empty1* [*simp*]: {} *O R* = {}
by *blast*

lemma *relcompp-bot1* [*simp*]: ⊥ *OO R* = ⊥
by (*fact relcomp-empty1* [*to-pred*])

lemma *relcomp-empty2* [*simp*]: *R O* {} = {}
by *blast*

lemma *relcompp-bot2* [*simp*]: *R OO* ⊥ = ⊥
by (*fact relcomp-empty2* [*to-pred*])

lemma *O-assoc*: (*R O S*) *O T* = *R O* (*S O T*)
by *blast*

lemma *relcompp-assoc*: (*r OO s*) *OO t* = *r OO* (*s OO t*)
by (*fact O-assoc* [*to-pred*])

lemma *trans-O-subset*: $trans\ r \Longrightarrow r\ O\ r \subseteq r$
by (*unfold trans-def*) *blast*

lemma *transp-relcompp-less-eq*: $transp\ r \Longrightarrow r\ OO\ r \leq r$
by (*fact trans-O-subset* [*to-pred*])

lemma *relcomp-mono*: $r' \subseteq r \Longrightarrow s' \subseteq s \Longrightarrow r'\ O\ s' \subseteq r\ O\ s$
by *blast*

lemma *relcompp-mono*: $r' \leq r \Longrightarrow s' \leq s \Longrightarrow r'\ OO\ s' \leq r\ OO\ s$
by (*fact relcomp-mono* [*to-pred*])

lemma *relcomp-subset-Sigma*: $r \subseteq A \times B \Longrightarrow s \subseteq B \times C \Longrightarrow r\ O\ s \subseteq A \times C$
by *blast*

lemma *relcomp-distrib* [*simp*]: $R\ O\ (S \cup T) = (R\ O\ S) \cup (R\ O\ T)$
by *auto*

lemma *relcompp-distrib* [*simp*]: $R\ OO\ (S \sqcup T) = R\ OO\ S \sqcup R\ OO\ T$
by (*fact relcomp-distrib* [*to-pred*])

lemma *relcomp-distrib2* [*simp*]: $(S \cup T)\ O\ R = (S\ O\ R) \cup (T\ O\ R)$
by *auto*

lemma *relcompp-distrib2* [*simp*]: $(S \sqcup T)\ OO\ R = S\ OO\ R \sqcup T\ OO\ R$
by (*fact relcomp-distrib2* [*to-pred*])

lemma *relcomp-UNION-distrib*: $s\ O\ \bigcup(r\ `\ I) = (\bigcup i{\in}I.\ s\ O\ r\ i)$
by *auto*

lemma *relcompp-SUP-distrib*: $s\ OO\ \bigsqcup(r\ `\ I) = (\bigsqcup i{\in}I.\ s\ OO\ r\ i)$
by (*fact relcomp-UNION-distrib* [*to-pred*])

lemma *relcomp-UNION-distrib2*: $\bigcup(r\ `\ I)\ O\ s = (\bigcup i{\in}I.\ r\ i\ O\ s)$
by *auto*

lemma *relcompp-SUP-distrib2*: $\bigsqcup(r\ `\ I)\ OO\ s = (\bigsqcup i{\in}I.\ r\ i\ OO\ s)$
by (*fact relcomp-UNION-distrib2* [*to-pred*])

lemma *single-valued-relcomp*: *single-valued* $r \Longrightarrow$ *single-valued* $s \Longrightarrow$ *single-valued* $(r\ O\ s)$
unfolding *single-valued-def* **by** *blast*

lemma *relcomp-unfold*: $r\ O\ s = \{(x, z).\ \exists\, y.\ (x, y) \in r \land (y, z) \in s\}$
by (*auto simp add*: *set-eq-iff*)

lemma *relcompp-apply*: $(R\ OO\ S)\ a\ c \longleftrightarrow (\exists\, b.\ R\ a\ b \land S\ b\ c)$
unfolding *relcomp-unfold* [*to-pred*] **..**

lemma *eq-OO*: (=) *OO* $R = R$
  by *blast*

lemma *OO-eq*: R *OO* (=) $= R$
  by *blast*

19.3.4 Converse

inductive-set *converse* :: $('a \times 'b)$ *set* $\Rightarrow$ $('b \times 'a)$ *set*
  for r :: $('a \times 'b)$ *set*
  where $(a, b) \in r \Longrightarrow (b, a) \in$ *converse* r

open-bundle *converse-syntax*
begin
notation
  converse (‹(‹notation=‹postfix −1››-$^{-1}$)› [1000] 999) **and**
  conversep (‹(‹notation=‹postfix −1−1››-$^{-1-1}$)› [1000] 1000)
notation (*ASCII*)
  converse (‹(‹notation=‹postfix −1››-^−1)› [1000] 999) **and**
  conversep (‹(‹notation=‹postfix −1−1››-^−−1)› [1000] 1000)
end

lemma *converseI* [*sym*]: $(a, b) \in r \Longrightarrow (b, a) \in r^{-1}$
  by (*fact converse.intros*)

lemma *conversepI* : $r\ a\ b \Longrightarrow r^{-1-1}\ b\ a$
  by (*fact conversep.intros*)

lemma *converseD* [*sym*]: $(a, b) \in r^{-1} \Longrightarrow (b, a) \in r$
  by (*erule converse.cases*) *iprover*

lemma *conversepD* : $r^{-1-1}\ b\ a \Longrightarrow r\ a\ b$
  by (*fact converseD* [*to-pred*])

lemma *converseE* [*elim*!]: $yx \in r^{-1} \Longrightarrow (\bigwedge x\ y.\ yx = (y, x) \Longrightarrow (x, y) \in r \Longrightarrow P) \Longrightarrow P$
  — More general than *converseD*, as it "splits" the member of the relation.
  apply (*cases yx*)
  apply *simp*
  apply (*erule converse.cases*)
  apply *iprover*
  done

lemmas *conversepE* [*elim*!] = *conversep.cases*

lemma *converse-iff* [*iff*]: $(a, b) \in r^{-1} \longleftrightarrow (b, a) \in r$
  by (*auto intro*: *converseI*)

lemma *conversep-iff* [*iff*]: $r^{-1-1}\ a\ b = r\ b\ a$

by (*fact converse-iff* [*to-pred*])

lemma *converse-converse* [*simp*]: $(r^{-1})^{-1} = r$
by (*simp add*: *set-eq-iff*)

lemma *conversep-conversep* [*simp*]: $(r^{-1-1})^{-1-1} = r$
by (*fact converse-converse* [*to-pred*])

lemma *converse-empty*[*simp*]: $\{\}^{-1} = \{\}$
by *auto*

lemma *converse-UNIV*[*simp*]: $UNIV^{-1} = UNIV$
by *auto*

lemma *converse-relcomp*: $(r\ O\ s)^{-1} = s^{-1}\ O\ r^{-1}$
by *blast*

lemma *converse-relcompp*: $(r\ OO\ s)^{-1-1} = s^{-1-1}\ OO\ r^{-1-1}$
by (*iprover intro*: *order-antisym conversepI relcomppI elim*: *relcomppE dest*: *conversepD*)

lemma *converse-Int*: $(r \cap s)^{-1} = r^{-1} \cap s^{-1}$
by *blast*

lemma *converse-meet*: $(r \sqcap s)^{-1-1} = r^{-1-1} \sqcap s^{-1-1}$
by (*simp add*: *inf-fun-def*) (*iprover intro*: *conversepI ext dest*: *conversepD*)

lemma *converse-Un*: $(r \cup s)^{-1} = r^{-1} \cup s^{-1}$
by *blast*

lemma *converse-join*: $(r \sqcup s)^{-1-1} = r^{-1-1} \sqcup s^{-1-1}$
by (*simp add*: *sup-fun-def*) (*iprover intro*: *conversepI ext dest*: *conversepD*)

lemma *converse-INTER*: $(\bigcap (r\ `\ S))^{-1} = (\bigcap x \in S.\ (r\ x)^{-1})$
by *fast*

lemma *converse-UNION*: $(\bigcup (r\ `\ S))^{-1} = (\bigcup x \in S.\ (r\ x)^{-1})$
by *blast*

lemma *converse-mono*[*simp*]: $r^{-1} \subseteq s\ ^{-1} \longleftrightarrow r \subseteq s$
by *auto*

lemma *conversep-mono*[*simp*]: $r^{-1-1} \leq s\ ^{-1-1} \longleftrightarrow r \leq s$
by (*fact converse-mono*[*to-pred*])

lemma *converse-inject*[*simp*]: $r^{-1} = s\ ^{-1} \longleftrightarrow r = s$
by *auto*

lemma *conversep-inject*[*simp*]: $r^{-1-1} = s\ ^{-1-1} \longleftrightarrow r = s$

by (*fact converse-inject*[*to-pred*])

lemma *converse-subset-swap*: $r \subseteq s^{-1} \longleftrightarrow r^{-1} \subseteq s$
by *auto*

lemma *conversep-le-swap*: $r \leq s^{-1-1} \longleftrightarrow r^{-1-1} \leq s$
by (*fact converse-subset-swap*[*to-pred*])

lemma *converse-Id* [*simp*]: $Id^{-1} = Id$
by *blast*

lemma *converse-Id-on* [*simp*]: $(Id\text{-}on\ A)^{-1} = Id\text{-}on\ A$
by *blast*

lemma *refl-on-converse* [*simp*]: *refl-on A* (r^{-1}) = *refl-on A r*
by (*auto simp*: *refl-on-def*)

lemma *reflp-on-conversp* [*simp*]: *reflp-on A* $R^{-1-1} \longleftrightarrow$ *reflp-on A R*
by (*auto simp*: *reflp-on-def*)

lemma *irrefl-on-converse* [*simp*]: *irrefl-on A* (r^{-1}) = *irrefl-on A r*
by (*simp add*: *irrefl-on-def*)

lemma *irreflp-on-converse* [*simp*]: *irreflp-on A* (r^{-1-1}) = *irreflp-on A r*
by (*rule irrefl-on-converse*[*to-pred*])

lemma *sym-on-converse* [*simp*]: *sym-on A* (r^{-1}) = *sym-on A r*
by (*auto intro*: *sym-onI dest*: *sym-onD*)

lemma *symp-on-conversep* [*simp*]: *symp-on A* R^{-1-1} = *symp-on A R*
by (*rule sym-on-converse*[*to-pred*])

lemma *asym-on-converse* [*simp*]: *asym-on A* (r^{-1}) = *asym-on A r*
by (*auto dest*: *asym-onD*)

lemma *asymp-on-conversep* [*simp*]: *asymp-on A* R^{-1-1} = *asymp-on A R*
by (*rule asym-on-converse*[*to-pred*])

lemma *antisym-on-converse* [*simp*]: *antisym-on A* (r^{-1}) = *antisym-on A r*
by (*auto intro*: *antisym-onI dest*: *antisym-onD*)

lemma *antisymp-on-conversep* [*simp*]: *antisymp-on A* R^{-1-1} = *antisymp-on A R*
by (*rule antisym-on-converse*[*to-pred*])

lemma *trans-on-converse* [*simp*]: *trans-on A* (r^{-1}) = *trans-on A r*
by (*auto intro*: *trans-onI dest*: *trans-onD*)

lemma *transp-on-conversep* [*simp*]: *transp-on A* R^{-1-1} = *transp-on A R*
by (*rule trans-on-converse*[*to-pred*])

lemma *sym-conv-converse-eq*: *sym r* $\longleftrightarrow$ $r^{-1} = r$
unfolding *sym-def* **by** *fast*

lemma *sym-Un-converse*: *sym* $(r \cup r^{-1})$
unfolding *sym-def* **by** *blast*

lemma *sym-Int-converse*: *sym* $(r \cap r^{-1})$
unfolding *sym-def* **by** *blast*

lemma *total-on-converse* [*simp*]: *total-on A* (r^{-1}) = *total-on A r*
by (*auto simp*: *total-on-def*)

lemma *totalp-on-converse* [*simp*]: *totalp-on A* R^{-1-1} = *totalp-on A R*
by (*rule total-on-converse*[*to-pred*])

lemma *conversep-noteq* [*simp*]: $(\neq)^{-1-1} = (\neq)$
by (*auto simp add*: *fun-eq-iff*)

lemma *conversep-eq* [*simp*]: $(=)^{-1-1} = (=)$
by (*auto simp add*: *fun-eq-iff*)

lemma *converse-unfold* [*code*]: $r^{-1} = \{(y, x). (x, y) \in r\}$
by (*simp add*: *set-eq-iff*)

19.3.5 Domain, range and field

inductive-set *Domain* :: $('a \times 'b)$ *set* $\Rightarrow$ *'a set* **for** *r* :: $('a \times 'b)$ *set*
where *DomainI* [*intro*]: $(a, b) \in r \Longrightarrow a \in$ *Domain r*

lemmas *DomainPI* = *Domainp.DomainI*

inductive-cases *DomainE* [*elim!*]: $a \in$ *Domain r*
inductive-cases *DomainpE* [*elim!*]: *Domainp r a*

inductive-set *Range* :: $('a \times 'b)$ *set* $\Rightarrow$ *'b set* **for** *r* :: $('a \times 'b)$ *set*
where *RangeI* [*intro*]: $(a, b) \in r \Longrightarrow b \in$ *Range r*

lemmas *RangePI* = *Rangep.RangeI*

inductive-cases *RangeE* [*elim!*]: $b \in$ *Range r*
inductive-cases *RangepE* [*elim!*]: *Rangep r b*

definition *Field* :: *'a rel* $\Rightarrow$ *'a set*
where *Field r* = *Domain r* $\cup$ *Range r*

lemma *Field-iff*: $x \in$ *Field r* $\longleftrightarrow$ $(\exists y. (x,y) \in r \vee (y,x) \in r)$
by (*auto simp*: *Field-def*)

lemma *FieldI1*: $(i, j) \in R \Longrightarrow i \in Field\ R$
unfolding *Field-def* **by** *blast*

lemma *FieldI2*: $(i, j) \in R \Longrightarrow j \in Field\ R$
unfolding *Field-def* **by** *auto*

lemma *Domain-fst* [*code*]: *Domain r = fst ' r*
by *force*

lemma *Range-snd* [*code*]: *Range r = snd ' r*
by *force*

lemma *fst-eq-Domain*: *fst ' R = Domain R*
by *force*

lemma *snd-eq-Range*: *snd ' R = Range R*
by *force*

lemma *range-fst* [*simp*]: *range fst = UNIV*
by (*auto simp*: *fst-eq-Domain*)

lemma *range-snd* [*simp*]: *range snd = UNIV*
by (*auto simp*: *snd-eq-Range*)

lemma *Domain-empty* [*simp*]: *Domain {} = {}*
by *auto*

lemma *Range-empty* [*simp*]: *Range {} = {}*
by *auto*

lemma *Field-empty* [*simp*]: *Field {} = {}*
by (*simp add*: *Field-def*)

lemma *Domain-empty-iff*: $Domain\ r = \{\} \longleftrightarrow r = \{\}$
by *auto*

lemma *Range-empty-iff*: $Range\ r = \{\} \longleftrightarrow r = \{\}$
by *auto*

lemma *Domain-insert* [*simp*]: *Domain (insert (a, b) r) = insert a (Domain r)*
by *blast*

lemma *Range-insert* [*simp*]: *Range (insert (a, b) r) = insert b (Range r)*
by *blast*

lemma *Field-insert* [*simp*]: $Field\ (insert\ (a, b)\ r) = \{a, b\} \cup Field\ r$
by (*auto simp add*: *Field-def*)

lemma *Domain-iff*: $a \in Domain\ r \longleftrightarrow (\exists y.\ (a, y) \in r)$

by *blast*

lemma *Range-iff*: $a \in Range\ r \longleftrightarrow (\exists y.\ (y,\ a) \in r)$
by *blast*

lemma *Domain-Id* [*simp*]: $Domain\ Id = UNIV$
by *blast*

lemma *Range-Id* [*simp*]: $Range\ Id = UNIV$
by *blast*

lemma *Domain-Id-on* [*simp*]: $Domain\ (Id\text{-}on\ A) = A$
by *blast*

lemma *Range-Id-on* [*simp*]: $Range\ (Id\text{-}on\ A) = A$
by *blast*

lemma *Domain-Un-eq*: $Domain\ (A \cup B) = Domain\ A \cup Domain\ B$
by *blast*

lemma *Range-Un-eq*: $Range\ (A \cup B) = Range\ A \cup Range\ B$
by *blast*

lemma *Field-Un* [*simp*]: $Field\ (r \cup s) = Field\ r \cup Field\ s$
by (*auto simp*: *Field-def*)

lemma *Domain-Int-subset*: $Domain\ (A \cap B) \subseteq Domain\ A \cap Domain\ B$
by *blast*

lemma *Range-Int-subset*: $Range\ (A \cap B) \subseteq Range\ A \cap Range\ B$
by *blast*

lemma *Domain-Diff-subset*: $Domain\ A - Domain\ B \subseteq Domain\ (A - B)$
by *blast*

lemma *Range-Diff-subset*: $Range\ A - Range\ B \subseteq Range\ (A - B)$
by *blast*

lemma *Domain-Union*: $Domain\ (\bigcup S) = (\bigcup A{\in}S.\ Domain\ A)$
by *blast*

lemma *Range-Union*: $Range\ (\bigcup S) = (\bigcup A{\in}S.\ Range\ A)$
by *blast*

lemma *Field-Union* [*simp*]: $Field\ (\bigcup R) = \bigcup (Field\ `\ R)$
by (*auto simp*: *Field-def*)

lemma *Domain-converse* [*simp*]: $Domain\ (r^{-1}) = Range\ r$
by *auto*

lemma *Range-converse* [*simp*]: *Range* (r^{-1}) = *Domain r*
by *blast*

lemma *Field-converse* [*simp*]: *Field* (r^{-1}) = *Field r*
by (*auto simp*: *Field-def*)

lemma *Domain-Collect-case-prod* [*simp*]: *Domain* $\{(x, y).\ P\ x\ y\} = \{x.\ \exists\, y.\ P\ x\ y\}$
by *auto*

lemma *Range-Collect-case-prod* [*simp*]: *Range* $\{(x, y).\ P\ x\ y\} = \{y.\ \exists\, x.\ P\ x\ y\}$
by *auto*

lemma *Domain-mono*: $r \subseteq s \Longrightarrow$ *Domain r* $\subseteq$ *Domain s*
by *blast*

lemma *Range-mono*: $r \subseteq s \Longrightarrow$ *Range r* $\subseteq$ *Range s*
by *blast*

lemma *mono-Field*: $r \subseteq s \Longrightarrow$ *Field r* $\subseteq$ *Field s*
by (*auto simp*: *Field-def Domain-def Range-def*)

lemma *Domain-unfold*: *Domain r* = $\{x.\ \exists\, y.\ (x, y) \in r\}$
by *blast*

lemma *Field-square* [*simp*]: *Field* $(x \times x) = x$
unfolding *Field-def* **by** *blast*

19.3.6 Image of a set under a relation

definition *Image* :: $('a \times 'b)$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'b$ *set* (**infixr** ‹‘‘› *90*)
where r ‘‘ $s = \{y.\ \exists\, x{\in}s.\ (x, y) \in r\}$

lemma *Image-iff*: $b \in r$‘‘$A \longleftrightarrow (\exists\, x{\in}A.\ (x, b) \in r)$
by (*simp add*: *Image-def*)

lemma *Image-singleton*: r‘‘$\{a\} = \{b.\ (a, b) \in r\}$
by (*simp add*: *Image-def*)

lemma *Image-singleton-iff* [*iff*]: $b \in r$‘‘$\{a\} \longleftrightarrow (a, b) \in r$
by (*rule Image-iff* [*THEN trans*]) *simp*

lemma *ImageI* [*intro*]: $(a, b) \in r \Longrightarrow a \in A \Longrightarrow b \in r$‘‘$A$
unfolding *Image-def* **by** *blast*

lemma *ImageE* [*elim!*]: $b \in r$ ‘‘ $A \Longrightarrow (\bigwedge x.\ (x, b) \in r \Longrightarrow x \in A \Longrightarrow P) \Longrightarrow P$
unfolding *Image-def* **by** (*iprover elim!*: *CollectE bexE*)

lemma *rev-ImageI*: $a \in A \Longrightarrow (a,\ b) \in r \Longrightarrow b \in r \text{ `` } A$
— This version's more effective when we already have the required a
by *blast*

lemma *Image-empty1* [*simp*]: $\{\} \text{ `` } X = \{\}$
by *auto*

lemma *Image-empty2* [*simp*]: $R\text{``}\{\} = \{\}$
by *blast*

lemma *Image-Id* [*simp*]: $Id \text{ `` } A = A$
by *blast*

lemma *Image-Id-on* [*simp*]: $Id\text{-}on\ A \text{ `` } B = A \cap B$
by *blast*

lemma *Image-Int-subset*: $R \text{ `` } (A \cap B) \subseteq R \text{ `` } A \cap R \text{ `` } B$
by *blast*

lemma *Image-Int-eq*: $single\text{-}valued\ (converse\ R) \Longrightarrow R \text{ `` } (A \cap B) = R \text{ `` } A \cap R \text{ `` } B$
by (*auto simp*: *single-valued-def*)

lemma *Image-Un*: $R \text{ `` } (A \cup B) = R \text{ `` } A \cup R \text{ `` } B$
by *blast*

lemma *Un-Image*: $(R \cup S) \text{ `` } A = R \text{ `` } A \cup S \text{ `` } A$
by *blast*

lemma *Image-subset*: $r \subseteq A \times B \Longrightarrow r\text{``}C \subseteq B$
by (*iprover intro*!: *subsetI elim*!: *ImageE dest*!: *subsetD SigmaD2*)

lemma *Image-eq-UN*: $r\text{``}B = (\bigcup y\in B.\ r\text{``}\{y\})$
— NOT suitable for rewriting
by *blast*

lemma *Image-mono*: $r' \subseteq r \Longrightarrow A' \subseteq A \Longrightarrow (r' \text{ `` } A') \subseteq (r \text{ `` } A)$
by *blast*

lemma *Image-UN*: $r \text{ `` } (\bigcup (B \text{ ` } A)) = (\bigcup x\in A.\ r \text{ `` } (B\ x))$
by *blast*

lemma *UN-Image*: $(\bigcup i\in I.\ X\ i) \text{ `` } S = (\bigcup i\in I.\ X\ i \text{ `` } S)$
by *auto*

lemma *Image-INT-subset*: $(r \text{ `` } (\bigcap (B \text{ ` } A))) \subseteq (\bigcap x\in A.\ r \text{ `` } (B\ x))$
by *blast*

Converse inclusion requires some assumptions

lemma *Image-INT-eq*:
  assumes *single-valued* (r^{-1})
    and $A \neq \{\}$
  shows r ‘‘ $(\bigcap (B$ ‘ $A)) = (\bigcap x{\in}A.\ r$ ‘‘ $B\ x)$
proof(*rule equalityI*, *rule Image-INT-subset*)
  show $(\bigcap x{\in}A.\ r$ ‘‘ $B\ x) \subseteq r$ ‘‘ $\bigcap\ (B$ ‘ $A)$
  proof
    fix x
    assume $x \in (\bigcap x{\in}A.\ r$ ‘‘ $B\ x)$
    then show $x \in r$ ‘‘ $\bigcap\ (B$ ‘ $A)$
      using *assms* **unfolding** *single-valued-def* **by** *simp blast*
  qed
qed

lemma *Image-subset-eq*: r‘‘$A \subseteq B \longleftrightarrow A \subseteq -\ ((r^{-1})$ ‘‘ $(-\ B))$
  by *blast*

lemma *Image-Collect-case-prod* [*simp*]: $\{(x,\ y).\ P\ x\ y\}$ ‘‘ $A = \{y.\ \exists\, x{\in}A.\ P\ x\ y\}$
  by *auto*

lemma *Sigma-Image*: $(SIGMA\ x{:}A.\ B\ x)$ ‘‘ $X = (\bigcup x{\in}X \cap A.\ B\ x)$
  by *auto*

lemma *relcomp-Image*: $(X\ O\ Y)$ ‘‘ $Z = Y$ ‘‘ $(X$ ‘‘ $Z)$
  by *auto*

19.3.7 Inverse image

definition *inv-image* :: $'b\ rel \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ rel$
  where $inv\text{-}image\ r\ f = \{(x,\ y).\ (f\ x,\ f\ y) \in r\}$

definition *inv-imagep* :: $('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
  where $inv\text{-}imagep\ r\ f = (\lambda x\ y.\ r\ (f\ x)\ (f\ y))$

lemma [*pred-set-conv*]: $inv\text{-}imagep\ (\lambda x\ y.\ (x,\ y) \in r)\ f = (\lambda x\ y.\ (x,\ y) \in inv\text{-}image\ r\ f)$
  by (*simp add*: *inv-image-def inv-imagep-def*)

lemma *sym-inv-image*: $sym\ r \Longrightarrow sym\ (inv\text{-}image\ r\ f)$
  unfolding *sym-def inv-image-def* **by** *blast*

lemma *trans-inv-image*: $trans\ r \Longrightarrow trans\ (inv\text{-}image\ r\ f)$
  unfolding *trans-def inv-image-def*
  by (*simp* (*no-asm*)) *blast*

lemma *total-inv-image*: $[\![inj\ f;\ total\ r]\!] \Longrightarrow total\ (inv\text{-}image\ r\ f)$
  unfolding *inv-image-def total-on-def* **by** (*auto simp*: *inj-eq*)

lemma *asym-inv-image*: $asym\ R \Longrightarrow asym\ (inv\text{-}image\ R\ f)$

by (*simp add*: *inv-image-def asym-iff*)

lemma *in-inv-image*[*simp*]: $(x, y) \in$ *inv-image r f* $\longleftrightarrow (f\ x, f\ y) \in r$
by (*auto simp*: *inv-image-def*)

lemma *converse-inv-image*[*simp*]: $(\textit{inv-image } R\ f)^{-1} = \textit{inv-image } (R^{-1})\ f$
unfolding *inv-image-def converse-unfold* **by** *auto*

lemma *in-inv-imagep* [*simp*]: *inv-imagep r f x y* = *r* (*f x*) (*f y*)
by (*simp add*: *inv-imagep-def*)

19.3.8 Powerset

definition *Powp* :: $('a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
where *Powp A* = $(\lambda B.\ \forall x \in B.\ A\ x)$

lemma *Powp-Pow-eq* [*pred-set-conv*]: *Powp* $(\lambda x.\ x \in A) = (\lambda x.\ x \in \textit{Pow } A)$
by (*auto simp add*: *Powp-def fun-eq-iff*)

lemmas *Powp-mono* [*mono*] = *Pow-mono* [*to-pred*]

end

20 Finite sets

theory *Finite-Set*
imports *Product-Type Sum-Type Fields Relation*
begin

20.1 Predicate for finite sets

context notes [[*inductive-internals*]]
begin

inductive *finite* :: $'a\ set \Rightarrow bool$
where
emptyI [*simp*, *intro*!]: *finite* {}
| *insertI* [*simp*, *intro*!]: *finite A* $\Longrightarrow$ *finite* (*insert a A*)

end

simproc-setup *finite-Collect* (*finite* (*Collect P*)) = ‹*K Set-Comprehension-Pointfree.proc*›

declare [[*simproc del*: *finite-Collect*]]

lemma *finite-induct* [*case-names empty insert*, *induct set*: *finite*]:
— Discharging $x \notin F$ entails extra work.
assumes *finite F*
assumes *P* {}

and *insert*: $\bigwedge x\ F.\ \mathit{finite}\ F \Longrightarrow x \notin F \Longrightarrow P\ F \Longrightarrow P\ (\mathit{insert}\ x\ F)$
shows $P\ F$
using ‹*finite F*›
proof *induct*
show $P\ \{\}$ **by** *fact*
next
fix $x\ F$
assume *F*: *finite F* **and** *P*: $P\ F$
show $P\ (\mathit{insert}\ x\ F)$
proof *cases*
assume $x \in F$
then have $\mathit{insert}\ x\ F = F$ **by** (*rule insert-absorb*)
with *P* **show** *?thesis* **by** (*simp only*:)
next
assume $x \notin F$
from *F this P* **show** *?thesis* **by** (*rule insert*)
qed
qed

lemma *infinite-finite-induct* [*case-names infinite empty insert*]:
assumes *infinite*: $\bigwedge A.\ \neg\ \mathit{finite}\ A \Longrightarrow P\ A$
and *empty*: $P\ \{\}$
and *insert*: $\bigwedge x\ F.\ \mathit{finite}\ F \Longrightarrow x \notin F \Longrightarrow P\ F \Longrightarrow P\ (\mathit{insert}\ x\ F)$
shows $P\ A$
proof (*cases finite A*)
case *False*
with *infinite* **show** *?thesis* **.**
next
case *True*
then show *?thesis* **by** (*induct A*) (*fact empty insert*)+
qed

20.1.1 Choice principles

lemma *ex-new-if-finite*: — does not depend on def of finite at all
assumes $\neg\ \mathit{finite}\ (\mathit{UNIV} :: 'a\ set)$ **and** *finite A*
shows $\exists\, a::'a.\ a \notin A$
proof −
from *assms* **have** $A \neq \mathit{UNIV}$ **by** *blast*
then show *?thesis* **by** *blast*
qed

A finite choice principle. Does not need the SOME choice operator.

lemma *finite-set-choice*: $\mathit{finite}\ A \Longrightarrow \forall\, x{\in}A.\ \exists\, y.\ P\ x\ y \Longrightarrow \exists f.\ \forall\, x{\in}A.\ P\ x\ (f\ x)$
proof (*induct rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert a A*)

then obtain *f b* **where** *f*: $\forall x \in A.\ P\ x\ (f\ x)$ **and** *ab*: *P a b*
  by *auto*
  show *?case* (**is** $\exists f.$ *?P f*)
  proof
    show *?P* ($\lambda x.$ *if* $x = a$ *then b else f x*)
      using *f ab* **by** *auto*
  qed
qed

20.1.2 Finite sets are the images of initial segments of natural numbers

lemma *finite-imp-nat-seg-image-inj-on*:
  assumes *finite A*
  shows $\exists (n::nat)\ f.\ A = f\ `\ \{i.\ i < n\} \wedge$ *inj-on f* $\{i.\ i < n\}$
  using *assms*
proof *induct*
  case *empty*
  show *?case*
  proof
    show $\exists f.\ \{\} = f\ `\ \{i::nat.\ i < 0\} \wedge$ *inj-on f* $\{i.\ i < 0\}$
      by *simp*
  qed
next
  case (*insert a A*)
  have *notinA*: $a \notin A$ **by** *fact*
  from *insert.hyps* **obtain** *n f* **where** $A = f\ `\ \{i::nat.\ i < n\}$ *inj-on f* $\{i.\ i < n\}$
    by *blast*
  then have *insert a A* $= f(n:=a)\ `\ \{i.\ i < Suc\ n\}$ **and** *inj-on* $(f(n:=a))\ \{i.\ i <$
$Suc\ n\}$
    using *notinA* **by** (*auto simp add*: *image-def Ball-def inj-on-def less-Suc-eq*)
  then show *?case* **by** *blast*
qed

lemma *nat-seg-image-imp-finite*: $A = f\ `\ \{i::nat.\ i < n\} \Longrightarrow$ *finite A*
proof (*induct n arbitrary*: *A*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  let $?B = f\ `\ \{i.\ i < n\}$
  have *finB*: *finite ?B* **by** (*rule Suc.hyps*[*OF refl*])
  show *?case*
  proof (*cases* $\exists k<n.\ f\ n = f\ k$)
    case *True*
    then have $A = ?B$
      using *Suc.prems* **by** (*auto simp*:*less-Suc-eq*)
    then show *?thesis*
      using *finB* **by** *simp*

```
  next
    case False
    then have A = insert (f n) ?B
      using Suc.prems by (auto simp:less-Suc-eq)
    then show ?thesis using finB by simp
  qed
qed

lemma finite-conv-nat-seg-image: finite A ⟷ (∃ n f. A = f ` {i::nat. i < n})
  by (blast intro: nat-seg-image-imp-finite dest: finite-imp-nat-seg-image-inj-on)

lemma finite-imp-inj-to-nat-seg:
  assumes finite A
  shows ∃f n. f ` A = {i::nat. i < n} ∧ inj-on f A
proof −
  from finite-imp-nat-seg-image-inj-on [OF ‹finite A›]
  obtain f and n :: nat where bij: bij-betw f {i. i<n} A
    by (auto simp: bij-betw-def)
  let ?f = the-inv-into {i. i<n} f
  have inj-on ?f A ∧ ?f ` A = {i. i<n}
    by (fold bij-betw-def) (rule bij-betw-the-inv-into[OF bij])
  then show ?thesis by blast
qed

lemma finite-Collect-less-nat [iff]: finite {n::nat. n < k}
  by (fastforce simp: finite-conv-nat-seg-image)

lemma finite-Collect-le-nat [iff]: finite {n::nat. n ≤ k}
  by (simp add: le-eq-less-or-eq Collect-disj-eq)
```

20.2 Finiteness and common set operations

```
lemma rev-finite-subset: finite B ⟹ A ⊆ B ⟹ finite A
proof (induct arbitrary: A rule: finite-induct)
  case empty
  then show ?case by simp
next
  case (insert x F A)
  have A: A ⊆ insert x F and r: A − {x} ⊆ F ⟹ finite (A − {x})
    by fact+
  show finite A
  proof cases
    assume x: x ∈ A
    with A have A − {x} ⊆ F by (simp add: subset-insert-iff)
    with r have finite (A − {x}) .
    then have finite (insert x (A − {x})) ..
    also have insert x (A − {x}) = A
      using x by (rule insert-Diff)
    finally show ?thesis .
```

  next
    show *?thesis* **when** $A \subseteq F$
      using *that* **by** *fact*
    assume $x \notin A$
    with A **show** $A \subseteq F$
      by (*simp add*: *subset-insert-iff*)
  qed
qed

lemma *finite-subset*: $A \subseteq B \Longrightarrow$ *finite* $B \Longrightarrow$ *finite* A
  by (*rule rev-finite-subset*)

simproc-setup *finite* (*finite* A) = ‹
let
  val finite-subset = @{*thm finite-subset*}
  val Eq-TrueI = @{*thm Eq-TrueI*}

  fun is-subset A th = *case Thm.prop-of th of*
      (- $ ***Const-*** ‹*less-eq* ***Type*** ‹*set* -› *for A′ B*›)
      => *if A aconv A′ then SOME*(*B*,*th*) *else NONE*
    | - => *NONE*;

  fun is-finite th = *case Thm.prop-of th of*
      (- $ ***Const-*** ‹*finite* - *for A*›) => *SOME*(*A*,*th*)
    | - => *NONE*;

  fun comb (*A*,*sub-th*) (*A′*,*fin-th*) *ths* = *if A aconv A′ then* (*sub-th*,*fin-th*) :: *ths else ths*

  fun proc ctxt ct =
    (*let*
        val - $ *A* = *Thm.term-of ct*
        val prems = *Simplifier.prems-of ctxt*
        val fins = *map-filter is-finite prems*
        val subsets = *map-filter* (*is-subset A*) *prems*
      in case fold-product comb subsets fins [] *of*
          (*sub-th*,*fin-th*) :: - => *SOME*((*fin-th RS* (*sub-th RS finite-subset*)) *RS Eq-TrueI*)
        | - => *NONE*
      end)
in K proc end
›

declare [[*simproc del*: *finite*]]

lemma *finite-UnI*:
  assumes *finite F* **and** *finite G*
  shows *finite* ($F \cup G$)

using *assms* **by** *induct simp-all*

lemma *finite-Un* [*iff*]: *finite* ($F \cup G$) $\longleftrightarrow$ *finite* $F \wedge$ *finite* G
by (*blast intro*: *finite-UnI finite-subset* [*of* - $F \cup G$])

lemma *finite-insert* [*simp*]: *finite* (*insert a A*) $\longleftrightarrow$ *finite A*
proof −
have *finite* $\{a\} \wedge$ *finite* $A \longleftrightarrow$ *finite A* **by** *simp*
then have *finite* ($\{a\} \cup A$) $\longleftrightarrow$ *finite A* **by** (*simp only*: *finite-Un*)
then show *?thesis* **by** *simp*
qed

lemma *finite-Int* [*simp*, *intro*]: *finite* $F \vee$ *finite* $G \Longrightarrow$ *finite* ($F \cap G$)
by (*blast intro*: *finite-subset*)

lemma *finite-Collect-conjI* [*simp*, *intro*]:
finite $\{x.\ P\ x\} \vee$ *finite* $\{x.\ Q\ x\} \Longrightarrow$ *finite* $\{x.\ P\ x \wedge Q\ x\}$
by (*simp add*: *Collect-conj-eq*)

lemma *finite-Collect-disjI* [*simp*]:
finite $\{x.\ P\ x \vee Q\ x\} \longleftrightarrow$ *finite* $\{x.\ P\ x\} \wedge$ *finite* $\{x.\ Q\ x\}$
by (*simp add*: *Collect-disj-eq*)

lemma *finite-Diff* [*simp*, *intro*]: *finite* $A \Longrightarrow$ *finite* ($A - B$)
by (*rule finite-subset*, *rule Diff-subset*)

lemma *finite-Diff2* [*simp*]:
assumes *finite B*
shows *finite* ($A - B$) $\longleftrightarrow$ *finite A*
proof −
have *finite* $A \longleftrightarrow$ *finite* $((A - B) \cup (A \cap B))$
by (*simp add*: *Un-Diff-Int*)
also have ... $\longleftrightarrow$ *finite* ($A - B$)
using ‹*finite B*› **by** *simp*
finally show *?thesis* **..**
qed

lemma *finite-Diff-insert* [*iff*]: *finite* ($A -$ *insert a B*) $\longleftrightarrow$ *finite* ($A - B$)
proof −
have *finite* ($A - B$) $\longleftrightarrow$ *finite* ($A - B - \{a\}$) **by** *simp*
moreover have $A -$ *insert a B* $= A - B - \{a\}$ **by** *auto*
ultimately show *?thesis* **by** *simp*
qed

lemma *finite-compl* [*simp*]:
finite (A :: *'a set*) $\Longrightarrow$ *finite* ($- A$) $\longleftrightarrow$ *finite* (*UNIV* :: *'a set*)
by (*simp add*: *Compl-eq-Diff-UNIV*)

lemma *finite-Collect-not* [*simp*]:

finite $\{x :: 'a.\ P\ x\} \Longrightarrow$ *finite* $\{x.\ \neg\ P\ x\} \longleftrightarrow$ *finite* $(UNIV :: 'a\ set)$
by (*simp add*: *Collect-neg-eq*)

lemma *finite-Union* [*simp*, *intro*]:
finite $A \Longrightarrow (\bigwedge M.\ M \in A \Longrightarrow$ *finite* $M) \Longrightarrow$ *finite* $(\bigcup A)$
by (*induct rule*: *finite-induct*) *simp-all*

lemma *finite-UN-I* [*intro*]:
finite $A \Longrightarrow (\bigwedge a.\ a \in A \Longrightarrow$ *finite* $(B\ a)) \Longrightarrow$ *finite* $(\bigcup a{\in}A.\ B\ a)$
by (*induct rule*: *finite-induct*) *simp-all*

lemma *finite-UN* [*simp*]: *finite* $A \Longrightarrow$ *finite* $(\bigcup (B\ `\ A)) \longleftrightarrow (\forall x{\in}A.$ *finite* $(B\ x))$
by (*blast intro*: *finite-subset*)

lemma *finite-Inter* [*intro*]: $\exists A{\in}M.$ *finite* $A \Longrightarrow$ *finite* $(\bigcap M)$
by (*blast intro*: *Inter-lower finite-subset*)

lemma *finite-INT* [*intro*]: $\exists x{\in}I.$ *finite* $(A\ x) \Longrightarrow$ *finite* $(\bigcap x{\in}I.\ A\ x)$
by (*blast intro*: *INT-lower finite-subset*)

lemma *finite-imageI* [*simp*, *intro*]: *finite* $F \Longrightarrow$ *finite* $(h\ `\ F)$
by (*induct rule*: *finite-induct*) *simp-all*

lemma *finite-image-set* [*simp*]: *finite* $\{x.\ P\ x\} \Longrightarrow$ *finite* $\{f\ x\ |x.\ P\ x\}$
by (*simp add*: *image-Collect* [*symmetric*])

lemma *finite-image-set2*:
finite $\{x.\ P\ x\} \Longrightarrow$ *finite* $\{y.\ Q\ y\} \Longrightarrow$ *finite* $\{f\ x\ y\ |x\ y.\ P\ x \wedge Q\ y\}$
by (*rule finite-subset* [**where** $B = \bigcup x \in \{x.\ P\ x\}.\ \bigcup y \in \{y.\ Q\ y\}.\ \{f\ x\ y\}$]) *auto*

lemma *finite-imageD*:
assumes *finite* $(f\ `\ A)$ **and** *inj-on* $f\ A$
shows *finite* A
using *assms*
proof (*induct* $f\ `\ A$ *arbitrary*: A)
case *empty*
then show *?case* **by** *simp*
next
case (*insert* $x\ B$)
then have *B-A*: *insert* $x\ B = f\ `\ A$
by *simp*
then obtain y **where** $x = f\ y$ **and** $y \in A$
by *blast*
from *B-A* ‹$x \notin B$› **have** $B = f\ `\ A - \{x\}$
by *blast*
with *B-A* ‹$x \notin B$› ‹$x = f\ y$› ‹*inj-on* $f\ A$› ‹$y \in A$› **have** $B = f\ `\ (A - \{y\})$
by (*simp add*: *inj-on-image-set-diff*)
moreover from ‹*inj-on* $f\ A$› **have** *inj-on* $f\ (A - \{y\})$
by (*rule inj-on-diff*)

ultimately have *finite* $(A - \{y\})$
by (*rule insert.hyps*)
then show *finite A*
by *simp*
qed

lemma *finite-image-iff*: *inj-on f A* $\Longrightarrow$ *finite* $(f$ ‘ $A)$ $\longleftrightarrow$ *finite A*
using *finite-imageD* **by** *blast*

lemma *finite-surj*: *finite A* $\Longrightarrow$ $B \subseteq f$ ‘ A $\Longrightarrow$ *finite B*
by (*erule finite-subset*) (*rule finite-imageI*)

lemma *finite-range-imageI*: *finite* (*range g*) $\Longrightarrow$ *finite* (*range* ($\lambda x.\ f\ (g\ x)$))
by (*drule finite-imageI*) (*simp add*: *range-composition*)

lemma *finite-subset-image*:
assumes *finite B*
shows $B \subseteq f$ ‘ $A \Longrightarrow \exists\, C \subseteq A.$ *finite C* $\wedge\ B = f$ ‘ C
using *assms*
proof *induct*
case *empty*
then show *?case* **by** *simp*
next
case *insert*
then show *?case*
by (*clarsimp simp del*: *image-insert simp add*: *image-insert* [*symmetric*]) *blast*
qed

lemma *all-subset-image*: $(\forall B.\ B \subseteq f$ ‘ $A \longrightarrow P\ B) \longleftrightarrow (\forall B.\ B \subseteq A \longrightarrow P(f$ ‘ $B))$
by (*safe elim*!: *subset-imageE*) (*use image-mono* **in** ‹*blast*+›)

lemma *all-finite-subset-image*:
$(\forall B.$ *finite B* $\wedge\ B \subseteq f$ ‘ $A \longrightarrow P\ B) \longleftrightarrow (\forall B.$ *finite B* $\wedge\ B \subseteq A \longrightarrow P\ (f$ ‘ $B))$
proof *safe*
fix *B* :: *'a set*
assume *B*: *finite B* $B \subseteq f$ ‘ A **and** *P*: $\forall B.$ *finite B* $\wedge\ B \subseteq A \longrightarrow P\ (f$ ‘ $B)$
show *P B*
using *finite-subset-image* [*OF B*] *P* **by** *blast*
qed *blast*

lemma *ex-finite-subset-image*:
$(\exists B.$ *finite B* $\wedge\ B \subseteq f$ ‘ $A \wedge P\ B) \longleftrightarrow (\exists B.$ *finite B* $\wedge\ B \subseteq A \wedge P\ (f$ ‘ $B))$
proof *safe*
fix *B* :: *'a set*
assume *B*: *finite B* $B \subseteq f$ ‘ A **and** *P B*
show $\exists B.$ *finite B* $\wedge\ B \subseteq A \wedge P\ (f$ ‘ $B)$
using *finite-subset-image* [*OF B*] ‹*P B*› **by** *blast*
qed *blast*

lemma *finite-vimage-IntI*: *finite* $F \Longrightarrow$ *inj-on* h $A \Longrightarrow$ *finite* $(h -` F \cap A)$
proof (*induct rule*: *finite-induct*)
  case (*insert* x F)
  then show *?case*
    by (*simp add*: *vimage-insert* [*of h x F*] *finite-subset* [*OF inj-on-vimage-singleton*] *Int-Un-distrib2*)
qed *simp*

lemma *finite-finite-vimage-IntI*:
  assumes *finite* F
    and $\bigwedge y.\ y \in F \Longrightarrow$ *finite* $((h -` \{y\}) \cap A)$
  shows *finite* $(h -` F \cap A)$
proof −
  have $*$: $h -` F \cap A = (\bigcup\ y{\in}F.\ (h -` \{y\}) \cap A)$
    by *blast*
  show *?thesis*
    by (*simp only*: $*$ *assms finite-UN-I*)
qed

lemma *finite-vimageI*: *finite* $F \Longrightarrow$ *inj* $h \Longrightarrow$ *finite* $(h -` F)$
  using *finite-vimage-IntI*[*of F h UNIV*] **by** *auto*

lemma *finite-vimageD'*: *finite* $(f -` A) \Longrightarrow A \subseteq$ *range* $f \Longrightarrow$ *finite* A
  by (*auto simp add*: *subset-image-iff intro*: *finite-subset*[*rotated*])

lemma *finite-vimageD*: *finite* $(h -` F) \Longrightarrow$ *surj* $h \Longrightarrow$ *finite* F
  by (*auto dest*: *finite-vimageD'*)

lemma *finite-vimage-iff*: *bij* $h \Longrightarrow$ *finite* $(h -` F) \longleftrightarrow$ *finite* F
  unfolding *bij-def* **by** (*auto elim*: *finite-vimageD finite-vimageI*)

lemma *finite-inverse-image-gen*:
  assumes *finite* A *inj-on* f D
  shows *finite* $\{j{\in}D.\ f\ j \in A\}$
  using *finite-vimage-IntI* [*OF assms*]
  by (*simp add*: *Collect-conj-eq inf-commute vimage-def*)

lemma *finite-inverse-image*:
  assumes *finite* A *inj* f
  shows *finite* $\{j.\ f\ j \in A\}$
  using *finite-inverse-image-gen* [*OF assms*] **by** *simp*

lemma *finite-Collect-bex* [*simp*]:
  assumes *finite* A
  shows *finite* $\{x.\ \exists\, y{\in}A.\ Q\ x\ y\} \longleftrightarrow (\forall\, y{\in}A.$ *finite* $\{x.\ Q\ x\ y\})$
proof −
  have $\{x.\ \exists\, y{\in}A.\ Q\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ Q\ x\ y\})$ **by** *auto*
  with *assms* **show** *?thesis* **by** *simp*

```
qed

lemma finite-Collect-bounded-ex [simp]:
  assumes finite {y. P y}
  shows finite {x. ∃ y. P y ∧ Q x y} ⟷ (∀ y. P y ⟶ finite {x. Q x y})
proof −
  have {x. ∃ y. P y ∧ Q x y} = (⋃ y∈{y. P y}. {x. Q x y})
    by auto
  with assms show ?thesis
    by simp
qed

lemma finite-Plus: finite A ⟹ finite B ⟹ finite (A <+> B)
  by (simp add: Plus-def)

lemma finite-PlusD:
  fixes A :: 'a set and B :: 'b set
  assumes fin: finite (A <+> B)
  shows finite A finite B
proof −
  have Inl ‘ A ⊆ A <+> B
    by auto
  then have finite (Inl ‘ A :: ('a + 'b) set)
    using fin by (rule finite-subset)
  then show finite A
    by (rule finite-imageD) (auto intro: inj-onI)
next
  have Inr ‘ B ⊆ A <+> B
    by auto
  then have finite (Inr ‘ B :: ('a + 'b) set)
    using fin by (rule finite-subset)
  then show finite B
    by (rule finite-imageD) (auto intro: inj-onI)
qed

lemma finite-Plus-iff [simp]: finite (A <+> B) ⟷ finite A ∧ finite B
  by (auto intro: finite-PlusD finite-Plus)

lemma finite-Plus-UNIV-iff [simp]:
  finite (UNIV :: ('a + 'b) set) ⟷ finite (UNIV :: 'a set) ∧ finite (UNIV :: 'b
set)
  by (subst UNIV-Plus-UNIV [symmetric]) (rule finite-Plus-iff)

lemma finite-SigmaI [simp, intro]:
  finite A ⟹ (⋀a. a∈A ⟹ finite (B a)) ⟹ finite (SIGMA a:A. B a)
  unfolding Sigma-def by blast

lemma finite-SigmaI2:
  assumes finite {x∈A. B x ≠ {}}
```

and $\bigwedge a.\ a \in A \Longrightarrow$ *finite* $(B\ a)$
shows *finite* (*Sigma* $A\ B$)
proof −
from *assms* **have** *finite* (*Sigma* $\{x{\in}A.\ B\ x \neq \{\}\}\ B$)
by *auto*
also have *Sigma* $\{x{:}A.\ B\ x \neq \{\}\}\ B =$ *Sigma* $A\ B$
by *auto*
finally show *?thesis* **.**
qed

lemma *finite-cartesian-product*: *finite* $A \Longrightarrow$ *finite* $B \Longrightarrow$ *finite* $(A \times B)$
by (*rule finite-SigmaI*)

lemma *finite-Prod-UNIV*:
finite (*UNIV* :: *'a set*) $\Longrightarrow$ *finite* (*UNIV* :: *'b set*) $\Longrightarrow$ *finite* (*UNIV* :: (*'a* × *'b*) *set*)
by (*simp only*: *UNIV-Times-UNIV* [*symmetric*] *finite-cartesian-product*)

lemma *finite-cartesian-productD1*:
assumes *finite* $(A \times B)$ **and** $B \neq \{\}$
shows *finite* A
proof −
from *assms* **obtain** $n\ f$ **where** $A \times B = f$ ‘ $\{i{::}nat.\ i < n\}$
by (*auto simp add*: *finite-conv-nat-seg-image*)
then have *fst* ‘ $(A \times B) =$ *fst* ‘ f ‘ $\{i{::}nat.\ i < n\}$
by *simp*
with ‹$B \neq \{\}$› **have** $A = ($*fst* $\circ f)$ ‘ $\{i{::}nat.\ i < n\}$
by (*simp add*: *image-comp*)
then have $\exists\, n\ f.\ A = f$ ‘ $\{i{::}nat.\ i < n\}$
by *blast*
then show *?thesis*
by (*auto simp add*: *finite-conv-nat-seg-image*)
qed

lemma *finite-cartesian-productD2*:
assumes *finite* $(A \times B)$ **and** $A \neq \{\}$
shows *finite* B
proof −
from *assms* **obtain** $n\ f$ **where** $A \times B = f$ ‘ $\{i{::}nat.\ i < n\}$
by (*auto simp add*: *finite-conv-nat-seg-image*)
then have *snd* ‘ $(A \times B) =$ *snd* ‘ f ‘ $\{i{::}nat.\ i < n\}$
by *simp*
with ‹$A \neq \{\}$› **have** $B = ($*snd* $\circ f)$ ‘ $\{i{::}nat.\ i < n\}$
by (*simp add*: *image-comp*)
then have $\exists\, n\ f.\ B = f$ ‘ $\{i{::}nat.\ i < n\}$
by *blast*
then show *?thesis*
by (*auto simp add*: *finite-conv-nat-seg-image*)
qed

lemma *finite-cartesian-product-iff*:
finite $(A \times B) \longleftrightarrow (A = \{\} \vee B = \{\} \vee (finite\ A \wedge finite\ B))$
by (*auto dest*: *finite-cartesian-productD1 finite-cartesian-productD2 finite-cartesian-product*)

lemma *finite-prod*:
finite $(UNIV :: ('a \times 'b)\ set) \longleftrightarrow finite\ (UNIV :: 'a\ set) \wedge finite\ (UNIV :: 'b\ set)$
using *finite-cartesian-product-iff*[*of UNIV UNIV*] **by** *simp*

lemma *finite-Pow-iff* [*iff*]: *finite* (*Pow A*) $\longleftrightarrow$ *finite A*
proof
assume *finite* (*Pow A*)
then have *finite* $((\lambda x.\ \{x\})$ ' $A)$
by (*blast intro*: *finite-subset*)
then show *finite A*
by (*rule finite-imageD* [*unfolded inj-on-def*]) *simp*
next
assume *finite A*
then show *finite* (*Pow A*)
by *induct* (*simp-all add*: *Pow-insert*)
qed

corollary *finite-Collect-subsets* [*simp*, *intro*]: *finite A* $\Longrightarrow$ *finite* $\{B.\ B \subseteq A\}$
by (*simp add*: *Pow-def* [*symmetric*])

lemma *finite-set*: *finite* $(UNIV :: 'a\ set\ set) \longleftrightarrow finite\ (UNIV :: 'a\ set)$
by (*simp only*: *finite-Pow-iff Pow-UNIV*[*symmetric*])

lemma *finite-UnionD*: *finite* $(\bigcup A) \Longrightarrow$ *finite A*
by (*blast intro*: *finite-subset* [*OF subset-Pow-Union*])

lemma *finite-bind*:
assumes *finite S*
assumes $\forall x \in S.\ finite\ (f\ x)$
shows *finite* (*Set.bind S f*)
using *assms* **by** (*simp add*: *bind-UNION*)

lemma *finite-filter* [*simp*]: *finite S* $\Longrightarrow$ *finite* (*Set.filter P S*)
unfolding *Set.filter-def* **by** *simp*

lemma *finite-set-of-finite-funs*:
assumes *finite A finite B*
shows *finite* $\{f.\ \forall x.\ (x \in A \longrightarrow f\ x \in B) \wedge (x \notin A \longrightarrow f\ x = d)\}$ (**is** *finite ?S*)
proof −
let $?F = \lambda f.\ \{(a,b).\ a \in A \wedge b = f\ a\}$
have $?F$ ' $?S \subseteq Pow(A \times B)$
by *auto*
from *finite-subset*[*OF this*] *assms* **have** *1*: *finite* $(?F$ ' $?S)$

    by *simp*
  have *2*: *inj-on ?F ?S*
    by (*fastforce simp add*: *inj-on-def set-eq-iff fun-eq-iff*)
  show *?thesis*
    by (*rule finite-imageD* [*OF 1 2*])
qed

lemma *not-finite-existsD*:
  assumes ¬ *finite* {*a. P a*}
  shows ∃ *a. P a*
proof (*rule classical*)
  assume ¬ *?thesis*
  with *assms* **show** *?thesis* **by** *auto*
qed

lemma *finite-converse* [*iff*]: *finite* (r^{-1}) ⟷ *finite r*
  unfolding *converse-def conversep-iff*
  using [[*simproc add*: *finite-Collect*]]
  by (*auto elim*: *finite-imageD simp*: *inj-on-def*)

lemma *finite-Domain*: *finite r* ⟹ *finite* (*Domain r*)
  by (*induct set*: *finite*) *auto*

lemma *finite-Range*: *finite r* ⟹ *finite* (*Range r*)
  by (*induct set*: *finite*) *auto*

lemma *finite-Field*: *finite r* ⟹ *finite* (*Field r*)
  by (*simp add*: *Field-def finite-Domain finite-Range*)

lemma *finite-Image*[*simp*]: *finite R* ⟹ *finite* (*R* `` *A*)
  by(*rule finite-subset*[*OF - finite-Range*]) *auto*

20.3 Further induction rules on finite sets

lemma *finite-ne-induct* [*case-names singleton insert*, *consumes 2*]:
  assumes *finite F* **and** *F* ≠ {}
  assumes ⋀*x. P* {*x*}
    and ⋀*x F. finite F* ⟹ *F* ≠ {} ⟹ *x* ∉ *F* ⟹ *P F* ⟹ *P* (*insert x F*)
  shows *P F*
  using *assms*
proof *induct*
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert x F*)
  then show *?case* **by** *cases auto*
qed

lemma *finite-subset-induct* [*consumes 2*, *case-names empty insert*]:

```
  assumes finite F and F ⊆ A
    and empty: P {}
    and insert: ⋀a F. finite F ⟹ a ∈ A ⟹ a ∉ F ⟹ P F ⟹ P (insert a F)
  shows P F
  using ‹finite F› ‹F ⊆ A›
proof induct
  show P {} by fact
next
  fix x F
  assume finite F and x ∉ F and P: F ⊆ A ⟹ P F and i: insert x F ⊆ A
  show P (insert x F)
  proof (rule insert)
    from i show x ∈ A by blast
    from i have F ⊆ A by blast
    with P show P F .
    show finite F by fact
    show x ∉ F by fact
  qed
qed

lemma finite-empty-induct:
  assumes finite A
    and P A
    and remove: ⋀a A. finite A ⟹ a ∈ A ⟹ P A ⟹ P (A − {a})
  shows P {}
proof −
  have P (A − B) if B ⊆ A for B :: 'a set
  proof −
    from ‹finite A› that have finite B
      by (rule rev-finite-subset)
    from this ‹B ⊆ A› show P (A − B)
    proof induct
      case empty
      from ‹P A› show ?case by simp
    next
      case (insert b B)
      have P (A − B − {b})
      proof (rule remove)
        from ‹finite A› show finite (A − B)
          by induct auto
        from insert show b ∈ A − B
          by simp
        from insert show P (A − B)
          by simp
      qed
      also have A − B − {b} = A − insert b B
        by (rule Diff-insert [symmetric])
      finally show ?case .
    qed
```

qed
then have P $(A - A)$ **by** *blast*
then show *?thesis* **by** *simp*
qed

lemma *finite-update-induct* [*consumes 1*, *case-names const update*]:
assumes *finite*: *finite* $\{a.\ f\ a \neq c\}$
and *const*: P $(\lambda a.\ c)$
and *update*: $\bigwedge a\ b\ f.$ *finite* $\{a.\ f\ a \neq c\} \Longrightarrow f\ a = c \Longrightarrow b \neq c \Longrightarrow P\ f \Longrightarrow P$ $(f(a := b))$
shows $P\ f$
using *finite*
proof (*induct* $\{a.\ f\ a \neq c\}$ *arbitrary*: f)
case *empty*
with *const* **show** *?case* **by** *simp*
next
case (*insert a A*)
then have $A = \{a'.\ (f(a := c))\ a' \neq c\}$ **and** $f\ a \neq c$
by *auto*
with ‹*finite A*› **have** *finite* $\{a'.\ (f(a := c))\ a' \neq c\}$
by *simp*
have $(f(a := c))\ a = c$
by *simp*
from *insert* ‹$A = \{a'.\ (f(a := c))\ a' \neq c\}$› **have** P $(f(a := c))$
by *simp*
with ‹*finite* $\{a'.\ (f(a := c))\ a' \neq c\}$› ‹$(f(a := c))\ a = c$› ‹$f\ a \neq c$›
have P $((f(a := c))(a := f\ a))$
by (*rule update*)
then show *?case* **by** *simp*
qed

lemma *finite-subset-induct′* [*consumes 2*, *case-names empty insert*]:
assumes *finite* F **and** $F \subseteq A$
and *empty*: P $\{\}$
and *insert*: $\bigwedge a\ F.$ $[\![$*finite* F; $a \in A$; $F \subseteq A$; $a \notin F$; $P\ F$ $]\!] \Longrightarrow P$ (*insert a F*)
shows $P\ F$
using *assms*(*1*,*2*)
proof *induct*
show P $\{\}$ **by** *fact*
next
fix $x\ F$
assume *finite* F **and** $x \notin F$ **and**
P: $F \subseteq A \Longrightarrow P\ F$ **and** i: *insert* $x\ F \subseteq A$
show P (*insert x F*)
proof (*rule insert*)
from i **show** $x \in A$ **by** *blast*
from i **have** $F \subseteq A$ **by** *blast*
with P **show** $P\ F$ **.**
show *finite* F **by** *fact*

```
    show x ∉ F by fact
    show F ⊆ A by fact
  qed
qed
```

20.4 Class *finite*

```
class finite =
  assumes finite-UNIV: finite (UNIV :: 'a set)
begin

lemma finite [simp]: finite (A :: 'a set)
  by (rule subset-UNIV finite-UNIV finite-subset)+

lemma finite-code [code]: finite (A :: 'a set) ⟷ True
  by simp

end

instance prod :: (finite, finite) finite
  by standard (simp only: UNIV-Times-UNIV [symmetric] finite-cartesian-product
finite)

lemma inj-graph: inj (λf. {(x, y). y = f x})
  by (rule inj-onI) (auto simp add: set-eq-iff fun-eq-iff)

instance fun :: (finite, finite) finite
proof
  show finite (UNIV :: ('a ⇒ 'b) set)
  proof (rule finite-imageD)
    let ?graph = λf::'a ⇒ 'b. {(x, y). y = f x}
    have range ?graph ⊆ Pow UNIV
      by simp
    moreover have finite (Pow (UNIV :: ('a * 'b) set))
      by (simp only: finite-Pow-iff finite)
    ultimately show finite (range ?graph)
      by (rule finite-subset)
    show inj ?graph
      by (rule inj-graph)
  qed
qed

instance bool :: finite
  by standard (simp add: UNIV-bool)

instance set :: (finite) finite
  by standard (simp only: Pow-UNIV [symmetric] finite-Pow-iff finite)

instance unit :: finite
```

by *standard* (*simp add*: *UNIV-unit*)

instance *sum* :: (*finite*, *finite*) *finite*
by *standard* (*simp only*: *UNIV-Plus-UNIV* [*symmetric*] *finite-Plus finite*)

20.5 A basic fold functional for finite sets

The intended behaviour is *fold* f z $\{x_1, \ldots, x_n\} = f\ x_1\ (\ldots\ (f\ x_n\ z)\ldots)$ if f is "left-commutative". The commutativity requirement is relativised to the carrier set S:

locale *comp-fun-commute-on* =
fixes S :: $'a$ *set*
fixes f :: $'a \Rightarrow 'b \Rightarrow 'b$
assumes *comp-fun-commute-on*: $x \in S \Longrightarrow y \in S \Longrightarrow f\ y \circ f\ x = f\ x \circ f\ y$
begin

lemma *fun-left-comm*: $x \in S \Longrightarrow y \in S \Longrightarrow f\ y\ (f\ x\ z) = f\ x\ (f\ y\ z)$
using *comp-fun-commute-on* **by** (*simp add*: *fun-eq-iff*)

lemma *commute-left-comp*: $x \in S \Longrightarrow y \in S \Longrightarrow f\ y \circ (f\ x \circ g) = f\ x \circ (f\ y \circ g)$
by (*simp add*: *o-assoc comp-fun-commute-on*)

end

inductive *fold-graph* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b \Rightarrow bool$
for f :: $'a \Rightarrow 'b \Rightarrow 'b$ **and** z :: $'b$
where
emptyI [*intro*]: *fold-graph* f z $\{\}$ z
| *insertI* [*intro*]: $x \notin A \Longrightarrow$ *fold-graph* f z A $y \Longrightarrow$ *fold-graph* f z (*insert* x A) (f x y)

inductive-cases *empty-fold-graphE* [*elim!*]: *fold-graph* f z $\{\}$ x

lemma *fold-graph-closed-lemma*:
fold-graph f z A $x \wedge x \in B$
if *fold-graph* g z A x
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow f\ a\ b = g\ a\ b$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow g\ a\ b \in B$
$z \in B$
using *that*(*1*−*3*)
proof (*induction rule*: *fold-graph.induct*)
case (*insertI* x A y)
have *fold-graph* f z A y $y \in B$
unfolding *atomize-conj*
by (*rule insertI.IH*) (*auto intro*: *insertI.prems*)
then have $g\ x\ y \in B$ **and** *f-eq*: $f\ x\ y = g\ x\ y$
by (*auto simp*: *insertI.prems*)
moreover have *fold-graph* f z (*insert* x A) (f x y)
by (*rule fold-graph.insertI*; *fact*)

ultimately
show *?case*
by (*simp add*: *f-eq*)
qed (*auto intro*!: *that*)

lemma *fold-graph-closed-eq*:
fold-graph f z A = fold-graph g z A
if $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow f\ a\ b = g\ a\ b$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow g\ a\ b \in B$
$z \in B$
using *fold-graph-closed-lemma*[*of f z A - B g*] *fold-graph-closed-lemma*[*of g z A - B f*] *that*
by *auto*

definition *fold* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b$
where *fold f z A* = (*if finite A then* (*THE y. fold-graph f z A y*) *else z*)

lemma *fold-closed-eq*: *fold f z A = fold g z A*
if $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow f\ a\ b = g\ a\ b$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow g\ a\ b \in B$
$z \in B$
unfolding *Finite-Set.fold-def*
by (*subst fold-graph-closed-eq*[**where** *B=B* **and** *g=g*]) (*auto simp*: *that*)

A tempting alternative for the definition is *if finite A then THE y. fold-graph f z A y else e*. It allows the removal of finiteness assumptions from the theorems *fold-comm*, *fold-reindex* and *fold-distrib*. The proofs become ugly. It is not worth the effort. (???)

lemma *finite-imp-fold-graph*: *finite A* $\Longrightarrow \exists x.$ *fold-graph f z A x*
by (*induct rule*: *finite-induct*) *auto*

20.5.1 From *fold-graph* to *fold*

context *comp-fun-commute-on*
begin

lemma *fold-graph-finite*:
assumes *fold-graph f z A y*
shows *finite A*
using *assms* **by** *induct simp-all*

lemma *fold-graph-insertE-aux*:
assumes $A \subseteq S$
assumes *fold-graph f z A y* $a \in A$
shows $\exists y'.\ y = f\ a\ y' \wedge$ *fold-graph f z* $(A - \{a\})\ y'$
using *assms*(*2*−,*1*)
proof (*induct set*: *fold-graph*)
case *emptyI*
then show *?case* **by** *simp*

next
  case (*insertI x A y*)
  show *?case*
  proof (*cases x = a*)
    case *True*
    with *insertI* **show** *?thesis* **by** *auto*
  next
    case *False*
    then obtain y' **where** *y*: $y = f\ a\ y'$ **and** y': *fold-graph f z* $(A - \{a\})\ y'$
      using *insertI* **by** *auto*
    from *insertI* **have** $x \in S\ a \in S$ **by** *auto*
    then have $f\ x\ y = f\ a\ (f\ x\ y')$
      unfolding *y* **by** (*intro fun-left-comm*; *simp*)
    moreover have *fold-graph f z* (*insert x A* $- \{a\}$) $(f\ x\ y')$
      using y' **and** ‹$x \neq a$› **and** ‹$x \notin A$›
      by (*simp add*: *insert-Diff-if fold-graph.insertI*)
    ultimately show *?thesis*
      by *fast*
  qed
qed

lemma *fold-graph-insertE*:
  assumes *insert x A* $\subseteq S$
  assumes *fold-graph f z* (*insert x A*) *v* **and** $x \notin A$
  obtains *y* **where** $v = f\ x\ y$ **and** *fold-graph f z A y*
  using *assms* **by** (*auto dest*: *fold-graph-insertE-aux*[*OF* ‹*insert x A* $\subseteq S$› - *insertI1*])

lemma *fold-graph-determ*:
  assumes $A \subseteq S$
  assumes *fold-graph f z A x fold-graph f z A y*
  shows $y = x$
  using *assms*(*2*−,*1*)
proof (*induct arbitrary*: *y set*: *fold-graph*)
  case *emptyI*
  then show *?case* **by** *fast*
next
  case (*insertI x A y v*)
  from ‹*insert x A* $\subseteq S$› **and** ‹*fold-graph f z* (*insert x A*) *v*› **and** ‹$x \notin A$›
  obtain y' **where** $v = f\ x\ y'$ **and** *fold-graph f z A* y'
    by (*rule fold-graph-insertE*)
  from ‹*fold-graph f z A* y'› *insertI* **have** $y' = y$
    by *simp*
  with ‹$v = f\ x\ y'$› **show** $v = f\ x\ y$
    by *simp*
qed

lemma *fold-equality*: $A \subseteq S \Longrightarrow$ *fold-graph f z A y* $\Longrightarrow$ *fold f z A* $= y$
 by (*cases finite A*) (*auto simp add*: *fold-def intro*: *fold-graph-determ dest*: *fold-graph-finite*)

lemma *fold-graph-fold*:
  assumes $A \subseteq S$
  assumes *finite A*
  shows *fold-graph f z A* (*fold f z A*)
proof –
  from ‹*finite A*› **have** $\exists x$. *fold-graph f z A x*
    by (*rule finite-imp-fold-graph*)
  moreover note *fold-graph-determ*[*OF* ‹$A \subseteq S$›]
  ultimately have $\exists!x$. *fold-graph f z A x*
    by (*rule ex-ex1I*)
  then have *fold-graph f z A* (*The* (*fold-graph f z A*))
    by (*rule theI'*)
  with *assms* **show** *?thesis*
    by (*simp add*: *fold-def*)
qed

The base case for *fold*:

lemma (**in** –) *fold-infinite* [*simp*]: ¬ *finite A* $\Longrightarrow$ *fold f z A* = *z*
  by (*auto simp*: *fold-def*)

lemma (**in** –) *fold-empty* [*simp*]: *fold f z* {} = *z*
  by (*auto simp*: *fold-def*)

The various recursion equations for *fold*:

lemma *fold-insert* [*simp*]:
  assumes *insert x A* $\subseteq S$
  assumes *finite A* **and** $x \notin A$
  shows *fold f z* (*insert x A*) = *f x* (*fold f z A*)
proof (*rule fold-equality*[*OF* ‹*insert x A* $\subseteq S$›])
  fix *z*
  from ‹*insert x A* $\subseteq S$› ‹*finite A*› **have** *fold-graph f z A* (*fold f z A*)
    by (*blast intro*: *fold-graph-fold*)
  with ‹$x \notin A$› **have** *fold-graph f z* (*insert x A*) (*f x* (*fold f z A*))
    by (*rule fold-graph.insertI*)
  then show *fold-graph f z* (*insert x A*) (*f x* (*fold f z A*))
    by *simp*
qed

declare (**in** –) *empty-fold-graphE* [*rule del*] *fold-graph.intros* [*rule del*]
  — No more proofs involve these.

lemma *fold-fun-left-comm*:
  assumes *insert x A* $\subseteq S$ *finite A*
  shows *f x* (*fold f z A*) = *fold f* (*f x z*) *A*
  using *assms*(*2*,*1*)
proof (*induct rule*: *finite-induct*)
  case *empty*
  then show *?case* **by** *simp*

next
  case (*insert y F*)
  then have *fold f* (*f x z*) (*insert y F*) = *f y* (*fold f* (*f x z*) *F*)
    by *simp*
  also have ... = *f x* (*f y* (*fold f z F*))
    using *insert* **by** (*simp add*: *fun-left-comm*[**where** *?y=x*])
  also have ... = *f x* (*fold f z* (*insert y F*))
  proof −
    from *insert* **have** *insert y F* ⊆ *S* **by** *simp*
    from *fold-insert*[*OF this*] *insert* **show** *?thesis* **by** *simp*
  qed
  finally show *?case* **..**
qed

lemma *fold-insert2*:
  insert x A ⊆ *S* ⟹ *finite A* ⟹ *x* ∉ *A* ⟹ *fold f z* (*insert x A*) = *fold f* (*f x z*)
A
  by (*simp add*: *fold-fun-left-comm*)

lemma *fold-rec*:
  assumes *A* ⊆ *S*
  assumes *finite A* **and** *x* ∈ *A*
  shows *fold f z A* = *f x* (*fold f z* (*A* − {*x*}))
proof −
  have *A*: *A* = *insert x* (*A* − {*x*})
    using ‹*x* ∈ *A*› **by** *blast*
  then have *fold f z A* = *fold f z* (*insert x* (*A* − {*x*}))
    by *simp*
  also have ... = *f x* (*fold f z* (*A* − {*x*}))
    by (*rule fold-insert*) (*use assms* **in** ‹*auto*›)
  finally show *?thesis* **.**
qed

lemma *fold-insert-remove*:
  assumes *insert x A* ⊆ *S*
  assumes *finite A*
  shows *fold f z* (*insert x A*) = *f x* (*fold f z* (*A* − {*x*}))
proof −
  from ‹*finite A*› **have** *finite* (*insert x A*)
    by *auto*
  moreover have *x* ∈ *insert x A*
    by *auto*
  ultimately have *fold f z* (*insert x A*) = *f x* (*fold f z* (*insert x A* − {*x*}))
    using ‹*insert x A* ⊆ *S*› **by** (*blast intro*: *fold-rec*)
  then show *?thesis*
    by *simp*
qed

lemma *fold-set-union-disj*:

```
  assumes A ⊆ S B ⊆ S
  assumes finite A finite B A ∩ B = {}
  shows Finite-Set.fold f z (A ∪ B) = Finite-Set.fold f (Finite-Set.fold f z A) B
  using ‹finite B› assms(1,2,3,5)
proof induct
  case (insert x F)
  have fold f z (A ∪ insert x F) = f x (fold f (fold f z A) F)
    using insert by auto
  also have ... = fold f (fold f z A) (insert x F)
    using insert by (blast intro: fold-insert[symmetric])
  finally show ?case .
qed simp


end
```

Other properties of *fold*:

```
lemma finite-set-fold-single [simp]: Finite-Set.fold f z {x} = f x z
proof –
  have fold-graph f z {x} (f x z)
    by (auto intro: fold-graph.intros)
  moreover
  {
    fix X y
    have fold-graph f z X y ⟹ (X = {} ⟶ y = z) ∧ (X = {x} ⟶ y = f x z)
      by (induct rule: fold-graph.induct) auto
  }
  ultimately have (THE y. fold-graph f z {x} y) = f x z
    by blast
  thus ?thesis
    by (simp add: Finite-Set.fold-def)
qed

lemma fold-graph-image:
  assumes inj-on g A
  shows fold-graph f z (g ` A) = fold-graph (f ∘ g) z A
proof
  fix w
  show fold-graph f z (g ` A) w = fold-graph (f o g) z A w
  proof
    assume fold-graph f z (g ` A) w
    then show fold-graph (f ∘ g) z A w
      using assms
    proof (induct g ` A w arbitrary: A)
      case emptyI
      then show ?case by (auto intro: fold-graph.emptyI)
    next
      case (insertI x A r B)
      from ‹inj-on g B› ‹x ∉ A› ‹insert x A = image g B› obtain x' A'
```

where $x' \notin A'$ **and** [*simp*]: $B = insert\ x'\ A'\ x = g\ x'\ A = g\ `\ A'$
by (*rule inj-img-insertE*)
from *insertI.prems* **have** *fold-graph* $(f \circ g)\ z\ A'\ r$
by (*auto intro*: *insertI.hyps*)
with ‹$x' \notin A'$› **have** *fold-graph* $(f \circ g)\ z\ (insert\ x'\ A')\ ((f \circ g)\ x'\ r)$
by (*rule fold-graph.insertI*)
then show *?case*
by *simp*
qed
next
assume *fold-graph* $(f \circ g)\ z\ A\ w$
then show *fold-graph* $f\ z\ (g\ `\ A)\ w$
using *assms*
proof *induct*
case *emptyI*
then show *?case*
by (*auto intro*: *fold-graph.emptyI*)
next
case (*insertI* $x\ A\ r$)
from ‹$x \notin A$› *insertI.prems* **have** $g\ x \notin g\ `\ A$
by *auto*
moreover from *insertI* **have** *fold-graph* $f\ z\ (g\ `\ A)\ r$
by *simp*
ultimately have *fold-graph* $f\ z\ (insert\ (g\ x)\ (g\ `\ A))\ (f\ (g\ x)\ r)$
by (*rule fold-graph.insertI*)
then show *?case*
by *simp*
qed
qed
qed

lemma *fold-image*:
assumes *inj-on* $g\ A$
shows *fold* $f\ z\ (g\ `\ A) =$ *fold* $(f \circ g)\ z\ A$
proof (*cases finite* A)
case *False*
with *assms* **show** *?thesis*
by (*auto dest*: *finite-imageD simp add*: *fold-def*)
next
case *True*
then show *?thesis*
by (*auto simp add*: *fold-def fold-graph-image*[*OF assms*])
qed

lemma *fold-cong*:
assumes *comp-fun-commute-on* $S\ f$ *comp-fun-commute-on* $S\ g$
and $A \subseteq S$ *finite* A
and *cong*: $\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x$
and $s = t$ **and** $A = B$

shows *fold f s A* = *fold g t B*
proof −
have *fold f s A* = *fold g s A*
using ‹*finite A*› ‹$A \subseteq S$› *cong*
proof (*induct A*)
case *empty*
then show *?case* **by** *simp*
next
case *insert*
interpret *f*: *comp-fun-commute-on S f* **by** (*fact* ‹*comp-fun-commute-on S f*›)
interpret *g*: *comp-fun-commute-on S g* **by** (*fact* ‹*comp-fun-commute-on S g*›)
from *insert* **show** *?case* **by** *simp*
qed
with *assms* **show** *?thesis* **by** *simp*
qed

A simplified version for idempotent functions:

locale *comp-fun-idem-on* = *comp-fun-commute-on* +
assumes *comp-fun-idem-on*: $x \in S \Longrightarrow f\ x \circ f\ x = f\ x$
begin

lemma *fun-left-idem*: $x \in S \Longrightarrow f\ x\ (f\ x\ z) = f\ x\ z$
using *comp-fun-idem-on* **by** (*simp add*: *fun-eq-iff*)

lemma *fold-insert-idem*:
assumes *insert x A* $\subseteq S$
assumes *fin*: *finite A*
shows *fold f z* (*insert x A*) = *f x* (*fold f z A*)
proof *cases*
assume $x \in A$
then obtain *B* **where** *A* = *insert x B* **and** $x \notin B$
by (*rule set-insert*)
then show *?thesis*
using *assms* **by** (*simp add*: *comp-fun-idem-on fun-left-idem*)
next
assume $x \notin A$
then show *?thesis*
using *assms* **by** *auto*
qed

declare *fold-insert* [*simp del*] *fold-insert-idem* [*simp*]

lemma *fold-insert-idem2*: *insert x A* $\subseteq S \Longrightarrow$ *finite A* $\Longrightarrow$ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
by (*simp add*: *fold-fun-left-comm*)

end

20.5.2 Liftings to *comp-fun-commute-on* etc.

lemma (**in** *comp-fun-commute-on*) *comp-comp-fun-commute-on*:
range g ⊆ *S* ⟹ *comp-fun-commute-on R* (*f* ∘ *g*)
by *standard* (*force intro*: *comp-fun-commute-on*)

lemma (**in** *comp-fun-idem-on*) *comp-comp-fun-idem-on*:
assumes *range g* ⊆ *S*
shows *comp-fun-idem-on R* (*f* ∘ *g*)
proof
interpret *f-g*: *comp-fun-commute-on R f o g*
by (*fact comp-comp-fun-commute-on*[*OF* ‹*range g* ⊆ *S*›])
show *x* ∈ *R* ⟹ *y* ∈ *R* ⟹ (*f* ∘ *g*) *y* ∘ (*f* ∘ *g*) *x* = (*f* ∘ *g*) *x* ∘ (*f* ∘ *g*) *y* **for** *x y*
by (*fact f-g.comp-fun-commute-on*)
qed (*use* ‹*range g* ⊆ *S*› **in** ‹*force intro*: *comp-fun-idem-on*›)

lemma (**in** *comp-fun-commute-on*) *comp-fun-commute-on-funpow*:
comp-fun-commute-on S (λ*x*. *f x* ^^ *g x*)
proof
fix *x y* **assume** *x* ∈ *S y* ∈ *S*
show *f y* ^^ *g y* ∘ *f x* ^^ *g x* = *f x* ^^ *g x* ∘ *f y* ^^ *g y*
proof (*cases x* = *y*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof (*induct g x arbitrary*: *g*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n g*)
have *hyp1*: *f y* ^^ *g y* ∘ *f x* = *f x* ∘ *f y* ^^ *g y*
proof (*induct g y arbitrary*: *g*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n g*)
define *h* **where** *h z* = *g z* − *1* **for** *z*
with *Suc* **have** *n* = *h y*
by *simp*
with *Suc* **have** *hyp*: *f y* ^^ *h y* ∘ *f x* = *f x* ∘ *f y* ^^ *h y*
by *auto*
from *Suc h-def* **have** *g y* = *Suc* (*h y*)
by *simp*
with ‹*x* ∈ *S*› ‹*y* ∈ *S*› **show** *?case*
by (*simp add*: *comp-assoc hyp*) (*simp add*: *o-assoc comp-fun-commute-on*)
qed
define *h* **where** *h z* = (*if z* = *x then g x* − *1 else g z*) **for** *z*
with *Suc* **have** *n* = *h x*

by *simp*
with *Suc* **have** $f\ y \ \hat{}\hat{}\ h\ y \circ f\ x \ \hat{}\hat{}\ h\ x = f\ x \ \hat{}\hat{}\ h\ x \circ f\ y \ \hat{}\hat{}\ h\ y$
by *auto*
with *False h-def* **have** *hyp2*: $f\ y \ \hat{}\hat{}\ g\ y \circ f\ x \ \hat{}\hat{}\ h\ x = f\ x \ \hat{}\hat{}\ h\ x \circ f\ y \ \hat{}\hat{}\ g\ y$
by *simp*
from *Suc h-def* **have** $g\ x = Suc\ (h\ x)$
by *simp*
then show *?case*
by (*simp del*: *funpow.simps add*: *funpow-Suc-right o-assoc hyp2*) (*simp add*: *comp-assoc hyp1*)
qed
qed
qed

20.5.3 *UNIV* as carrier set

locale *comp-fun-commute* =
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'b$
assumes *comp-fun-commute*: $f\ y \circ f\ x = f\ x \circ f\ y$
begin

lemma (**in** −) *comp-fun-commute-def'*: *comp-fun-commute f* = *comp-fun-commute-on UNIV f*
unfolding *comp-fun-commute-def comp-fun-commute-on-def* **by** *blast*

We abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *comp-fun-commute-on UNIV f*
rewrites $\bigwedge X.\ (X \subseteq UNIV) \equiv True$
and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
proof −
show *comp-fun-commute-on UNIV f*
by *standard* (*simp add*: *comp-fun-commute*)
qed *simp-all*

end

lemma (**in** *comp-fun-commute*) *comp-comp-fun-commute*: *comp-fun-commute* (*f o g*)
unfolding *comp-fun-commute-def'* **by** (*fact comp-comp-fun-commute-on*)

lemma (**in** *comp-fun-commute*) *comp-fun-commute-funpow*: *comp-fun-commute* $(\lambda x.\ f\ x \ \hat{}\hat{}\ g\ x)$
unfolding *comp-fun-commute-def'* **by** (*fact comp-fun-commute-on-funpow*)

locale *comp-fun-idem* = *comp-fun-commute* +

assumes *comp-fun-idem*: *f x o f x = f x*
begin

lemma (**in** −) *comp-fun-idem-def′*: *comp-fun-idem f = comp-fun-idem-on UNIV f*
unfolding *comp-fun-idem-on-def comp-fun-idem-def comp-fun-commute-def′*
unfolding *comp-fun-idem-axioms-def comp-fun-idem-on-axioms-def*
by *blast*

Again, we abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *comp-fun-idem-on UNIV f*
rewrites $\bigwedge X.\ (X \subseteq UNIV) \equiv True$
and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
proof −
show *comp-fun-idem-on UNIV f*
by *standard* (*simp-all add*: *comp-fun-idem comp-fun-commute*)
qed *simp-all*

end

lemma (**in** *comp-fun-idem*) *comp-comp-fun-idem*: *comp-fun-idem* (*f o g*)
unfolding *comp-fun-idem-def′* **by** (*fact comp-comp-fun-idem-on*)

20.5.4 Expressing set operations via *fold*

lemma *comp-fun-commute-const*: *comp-fun-commute* (λ-. *f*)
by *standard* (*rule refl*)

lemma *comp-fun-idem-insert*: *comp-fun-idem insert*
by *standard auto*

lemma *comp-fun-idem-remove*: *comp-fun-idem Set.remove*
by *standard auto*

lemma (**in** *semilattice-inf*) *comp-fun-idem-inf*: *comp-fun-idem inf*
by *standard* (*auto simp add*: *inf-left-commute*)

lemma (**in** *semilattice-sup*) *comp-fun-idem-sup*: *comp-fun-idem sup*
by *standard* (*auto simp add*: *sup-left-commute*)

lemma *union-fold-insert*:
assumes *finite A*
shows $A \cup B = fold\ insert\ B\ A$
proof −
interpret *comp-fun-idem insert*

by (*fact comp-fun-idem-insert*)
from ‹*finite A*› **show** *?thesis*
by (*induct A arbitrary*: *B*) *simp-all*
qed

lemma *minus-fold-remove*:
assumes *finite A*
shows $B - A$ = *fold Set.remove B A*
proof –
interpret *comp-fun-idem Set.remove*
by (*fact comp-fun-idem-remove*)
from ‹*finite A*› **have** *fold Set.remove B A* = $B - A$
by (*induct A arbitrary*: *B*) *auto*
then show *?thesis* **..**
qed

lemma *comp-fun-commute-filter-fold*:
comp-fun-commute ($\lambda x\ A'$. *if P x then Set.insert x A' else A'*)
proof –
interpret *comp-fun-idem Set.insert* **by** (*fact comp-fun-idem-insert*)
show *?thesis* **by** *standard* (*auto simp*: *fun-eq-iff*)
qed

lemma *Set-filter-fold*:
assumes *finite A*
shows *Set.filter P A* = *fold* ($\lambda x\ A'$. *if P x then Set.insert x A' else A'*) {} *A*
using *assms*
proof –
interpret *commute-insert*: *comp-fun-commute* ($\lambda x\ A'$. *if P x then Set.insert x A'*
else A')
by (*fact comp-fun-commute-filter-fold*)
from ‹*finite A*› **show** *?thesis*
by *induct* (*auto simp add*: *Set.filter-def*)
qed

lemma *inter-Set-filter*:
assumes *finite B*
shows $A \cap B$ = *Set.filter* ($\lambda x.\ x \in A$) *B*
using *assms*
by *induct* (*auto simp*: *Set.filter-def*)

lemma *image-fold-insert*:
assumes *finite A*
shows *image f A* = *fold* ($\lambda k\ A$. *Set.insert* (*f k*) *A*) {} *A*
proof –
interpret *comp-fun-commute* $\lambda k\ A$. *Set.insert* (*f k*) *A*
by *standard auto*
show *?thesis*
using *assms* **by** (*induct A*) *auto*

qed

lemma *Ball-fold*:
  assumes *finite A*
  shows *Ball A P* = *fold* (λ*k s*. *s* ∧ *P k*) *True A*
proof −
  interpret *comp-fun-commute* λ*k s*. *s* ∧ *P k*
    by *standard auto*
  show *?thesis*
    using *assms* **by** (*induct A*) *auto*
qed

lemma *Bex-fold*:
  assumes *finite A*
  shows *Bex A P* = *fold* (λ*k s*. *s* ∨ *P k*) *False A*
proof −
  interpret *comp-fun-commute* λ*k s*. *s* ∨ *P k*
    by *standard auto*
  show *?thesis*
    using *assms* **by** (*induct A*) *auto*
qed

lemma *comp-fun-commute-Pow-fold*: *comp-fun-commute* (λ*x A*. *A* ∪ *Set.insert x* ‘ *A*)
  by (*clarsimp simp*: *fun-eq-iff comp-fun-commute-def*) *blast*

lemma *Pow-fold*:
  assumes *finite A*
  shows *Pow A* = *fold* (λ*x A*. *A* ∪ *Set.insert x* ‘ *A*) {{}} *A*
proof −
  interpret *comp-fun-commute* λ*x A*. *A* ∪ *Set.insert x* ‘ *A*
    by (*rule comp-fun-commute-Pow-fold*)
  show *?thesis*
    using *assms* **by** (*induct A*) (*auto simp*: *Pow-insert*)
qed

lemma *fold-union-pair*:
  assumes *finite B*
  shows (⋃*y*∈*B*. {(*x*, *y*)}) ∪ *A* = *fold* (λ*y*. *Set.insert* (*x*, *y*)) *A B*
proof −
  interpret *comp-fun-commute* λ*y*. *Set.insert* (*x*, *y*)
    by *standard auto*
  show *?thesis*
    using *assms* **by** (*induct arbitrary*: *A*) *simp-all*
qed

lemma *comp-fun-commute-product-fold*:
  finite B ⟹ *comp-fun-commute* (λ*x z*. *fold* (λ*y*. *Set.insert* (*x*, *y*)) *z B*)
  by *standard* (*auto simp*: *fold-union-pair* [*symmetric*])

lemma *product-fold*:
assumes *finite A finite B*
shows *A* × *B* = *fold* (λ*x z*. *fold* (λ*y*. *Set.insert* (*x*, *y*)) *z B*) {} *A*
proof −
interpret *commute-product*: *comp-fun-commute* (λ*x z*. *fold* (λ*y*. *Set.insert* (*x*, *y*)) *z B*)
by (*fact comp-fun-commute-product-fold*[*OF* ‹*finite B*›])
from *assms* **show** *?thesis* **unfolding** *Sigma-def*
by (*induct A*) (*simp-all add*: *fold-union-pair*)
qed

context *complete-lattice*
begin

lemma *inf-Inf-fold-inf*:
assumes *finite A*
shows *inf* (*Inf A*) *B* = *fold inf B A*
proof −
interpret *comp-fun-idem inf*
by (*fact comp-fun-idem-inf*)
from ‹*finite A*› *fold-fun-left-comm* **show** *?thesis*
by (*induct A arbitrary*: *B*) (*simp-all add*: *inf-commute fun-eq-iff*)
qed

lemma *sup-Sup-fold-sup*:
assumes *finite A*
shows *sup* (*Sup A*) *B* = *fold sup B A*
proof −
interpret *comp-fun-idem sup*
by (*fact comp-fun-idem-sup*)
from ‹*finite A*› *fold-fun-left-comm* **show** *?thesis*
by (*induct A arbitrary*: *B*) (*simp-all add*: *sup-commute fun-eq-iff*)
qed

lemma *Inf-fold-inf*: *finite A* ⟹ *Inf A* = *fold inf top A*
using *inf-Inf-fold-inf* [*of A top*] **by** (*simp add*: *inf-absorb2*)

lemma *Sup-fold-sup*: *finite A* ⟹ *Sup A* = *fold sup bot A*
using *sup-Sup-fold-sup* [*of A bot*] **by** (*simp add*: *sup-absorb2*)

lemma *inf-INF-fold-inf*:
assumes *finite A*
shows *inf B* (⨅(*f* ‘ *A*)) = *fold* (*inf* ∘ *f*) *B A* (**is** *?inf* = *?fold*)
proof −
interpret *comp-fun-idem inf* **by** (*fact comp-fun-idem-inf*)
interpret *comp-fun-idem inf* ∘ *f* **by** (*fact comp-comp-fun-idem*)
from ‹*finite A*› **have** *?fold* = *?inf*
by (*induct A arbitrary*: *B*) (*simp-all add*: *inf-left-commute*)

then show *?thesis* **..**
qed

lemma *sup-SUP-fold-sup*:
assumes *finite A*
shows *sup B* ($\bigsqcup$(*f* ' *A*)) = *fold* (*sup* ∘ *f*) *B A* (**is** *?sup* = *?fold*)
proof −
interpret *comp-fun-idem sup* **by** (*fact comp-fun-idem-sup*)
interpret *comp-fun-idem sup* ∘ *f* **by** (*fact comp-comp-fun-idem*)
from ‹*finite A*› **have** *?fold* = *?sup*
by (*induct A arbitrary*: *B*) (*simp-all add*: *sup-left-commute*)
then show *?thesis* **..**
qed

lemma *INF-fold-inf*: *finite A* ⟹ $\bigsqcap$(*f* ' *A*) = *fold* (*inf* ∘ *f*) *top A*
using *inf-INF-fold-inf* [*of A top*] **by** *simp*

lemma *SUP-fold-sup*: *finite A* ⟹ $\bigsqcup$(*f* ' *A*) = *fold* (*sup* ∘ *f*) *bot A*
using *sup-SUP-fold-sup* [*of A bot*] **by** *simp*

lemma *finite-Inf-in*:
assumes *finite A A*≠{} **and** *inf*: ⋀*x y*. ⟦*x* ∈ *A*; *y* ∈ *A*⟧ ⟹ *inf x y* ∈ *A*
shows *Inf A* ∈ *A*
proof −
have *Inf B* ∈ *A* **if** *B* ≤ *A B*≠{} **for** *B*
using *finite-subset* [*OF* ‹*B* ⊆ *A*› ‹*finite A*›] *that*
by (*induction B*) (*use inf* **in** ‹*force*+›)
then show *?thesis*
by (*simp add*: *assms*)
qed

lemma *finite-Sup-in*:
assumes *finite A A*≠{} **and** *sup*: ⋀*x y*. ⟦*x* ∈ *A*; *y* ∈ *A*⟧ ⟹ *sup x y* ∈ *A*
shows *Sup A* ∈ *A*
proof −
have *Sup B* ∈ *A* **if** *B* ≤ *A B*≠{} **for** *B*
using *finite-subset* [*OF* ‹*B* ⊆ *A*› ‹*finite A*›] *that*
by (*induction B*) (*use sup* **in** ‹*force*+›)
then show *?thesis*
by (*simp add*: *assms*)
qed

end

20.5.5 Expressing relation operations via *fold*

lemma *Id-on-fold*:
assumes *finite A*
shows *Id-on A* = *Finite-Set.fold* (λ*x*. *Set.insert* (*Pair x x*)) {} *A*

proof −
  interpret *comp-fun-commute* $\lambda x.$ *Set.insert* (*Pair* x x)
    by *standard auto*
  from *assms* **show** *?thesis*
    unfolding *Id-on-def* **by** (*induct* A) *simp-all*
qed

lemma *comp-fun-commute-Image-fold*:
  comp-fun-commute (λ(x,y) A. *if* $x \in S$ *then* *Set.insert* y A *else* A)
proof −
  interpret *comp-fun-idem Set.insert*
    by (*fact comp-fun-idem-insert*)
  show *?thesis*
    by *standard* (*auto simp*: *fun-eq-iff comp-fun-commute split*: *prod.split*)
qed

lemma *Image-fold*:
  assumes *finite* R
  shows R `` S = *Finite-Set.fold* (λ(x,y) A. *if* $x \in S$ *then* *Set.insert* y A *else* A) {} R
proof −
  interpret *comp-fun-commute* (λ(x,y) A. *if* $x \in S$ *then* *Set.insert* y A *else* A)
    by (*rule comp-fun-commute-Image-fold*)
  have ∗: $\bigwedge x$ F. *Set.insert* x F `` $S$ = (*if fst* $x \in S$ *then* *Set.insert* (*snd* $x$) ($F$ `` S) *else* (F `` S))
    by (*force intro*: *rev-ImageI*)
  show *?thesis*
    using *assms* **by** (*induct* R) (*auto simp*: ∗)
qed

lemma *insert-relcomp-union-fold*:
  assumes *finite* S
  shows {x} O $S \cup X$ = *Finite-Set.fold* (λ(w,z) A'. *if snd* $x = w$ *then* *Set.insert* (*fst* x,z) A' *else* A') X S
proof −
  interpret *comp-fun-commute* λ(w,z) A'. *if snd* $x = w$ *then* *Set.insert* (*fst* x,z) A' *else* A'
  proof −
    interpret *comp-fun-idem Set.insert*
      by (*fact comp-fun-idem-insert*)
    show *comp-fun-commute* (λ(w,z) A'. *if snd* $x = w$ *then* *Set.insert* (*fst* x,z) A' *else* A')
      by *standard* (*auto simp add*: *fun-eq-iff split*: *prod.split*)
  qed
  have ∗: {x} O S = {(x', z). x' = *fst* x ∧ (*snd* x, z) ∈ S}
    by (*auto simp*: *relcomp-unfold intro*!: *exI*)
  show *?thesis*
    unfolding ∗ **using** ‹*finite* S› **by** (*induct* S) (*auto split*: *prod.split*)
qed

lemma *insert-relcomp-fold*:
  assumes *finite S*
  shows *Set.insert x R O S* =
    Finite-Set.fold (λ(*w*,*z*) *A'*. *if snd x* = *w then Set.insert* (*fst x*,*z*) *A' else A'*) (*R O S*) *S*
proof −
  have *Set.insert x R O S* = ({*x*} *O S*) ∪ (*R O S*)
    by *auto*
  then show *?thesis*
    by (*auto simp*: *insert-relcomp-union-fold* [*OF assms*])
qed

lemma *comp-fun-commute-relcomp-fold*:
  assumes *finite S*
  shows *comp-fun-commute* (λ(*x*,*y*) *A*.
    Finite-Set.fold (λ(*w*,*z*) *A'*. *if y* = *w then Set.insert* (*x*,*z*) *A' else A'*) *A S*)
proof −
  have ∗: ⋀*a b A*.
    Finite-Set.fold (λ(*w*, *z*) *A'*. *if b* = *w then Set.insert* (*a*, *z*) *A' else A'*) *A S* = {(*a*,*b*)} *O S* ∪ *A*
    by (*auto simp*: *insert-relcomp-union-fold*[*OF assms*] *cong*: *if-cong*)
  show *?thesis*
    by *standard* (*auto simp*: ∗)
qed

lemma *relcomp-fold*:
  assumes *finite R finite S*
  shows *R O S* = *Finite-Set.fold*
    (λ(*x*,*y*) *A*. *Finite-Set.fold* (λ(*w*,*z*) *A'*. *if y* = *w then Set.insert* (*x*,*z*) *A' else A'*) *A S*) {} *R*
proof −
  interpret *commute-relcomp-fold*: *comp-fun-commute*
    (λ(*x*, *y*) *A*. *Finite-Set.fold* (λ(*w*, *z*) *A'*. *if y* = *w then insert* (*x*, *z*) *A' else A'*) *A S*)
    by (*fact comp-fun-commute-relcomp-fold*[*OF* ‹*finite S*›])
  from *assms* **show** *?thesis*
    by (*induct R*) (*auto simp*: *comp-fun-commute-relcomp-fold insert-relcomp-fold cong*: *if-cong*)
qed

20.6 Locales as mini-packages for fold operations

20.6.1 The natural case

locale *folding-on* =
  fixes *S* :: *'a set*
  fixes *f* :: *'a* ⇒ *'b* ⇒ *'b* **and** *z* :: *'b*
  assumes *comp-fun-commute-on*: *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *f y o f x* = *f x o f y*
begin

interpretation *fold?*: *comp-fun-commute-on S f*
  by *standard* (*simp add*: *comp-fun-commute-on*)

definition F :: *'a set* ⇒ *'b*
  where *eq-fold*: $F\ A = \mathit{Finite\text{-}Set.fold}\ f\ z\ A$

lemma *empty* [*simp*]: $F\ \{\} = z$
  by (*simp add*: *eq-fold*)

lemma *infinite* [*simp*]: $\neg\ \mathit{finite}\ A \Longrightarrow F\ A = z$
  by (*simp add*: *eq-fold*)

lemma *insert* [*simp*]:
  assumes $\mathit{insert}\ x\ A \subseteq S$ **and** *finite A* **and** $x \notin A$
  shows $F\ (\mathit{insert}\ x\ A) = f\ x\ (F\ A)$
proof −
  from *fold-insert assms*
  have $\mathit{Finite\text{-}Set.fold}\ f\ z\ (\mathit{insert}\ x\ A)$
      $= f\ x\ (\mathit{Finite\text{-}Set.fold}\ f\ z\ A)$
    by *simp*
  with ‹*finite A*› **show** *?thesis* **by** (*simp add*: *eq-fold fun-eq-iff*)
qed

lemma *remove*:
  assumes $A \subseteq S$ **and** *finite A* **and** $x \in A$
  shows $F\ A = f\ x\ (F\ (A - \{x\}))$
proof −
  from ‹$x \in A$› **obtain** B **where** A: $A = \mathit{insert}\ x\ B$ **and** $x \notin B$
    by (*auto dest*: *mk-disjoint-insert*)
  moreover from ‹*finite A*› A **have** *finite B* **by** *simp*
  ultimately show *?thesis*
    using ‹$A \subseteq S$› **by** *auto*
qed

lemma *insert-remove*:
  assumes $\mathit{insert}\ x\ A \subseteq S$ **and** *finite A*
  shows $F\ (\mathit{insert}\ x\ A) = f\ x\ (F\ (A - \{x\}))$
  using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *remove insert-absorb*)

end

20.6.2 With idempotency

locale *folding-idem-on* = *folding-on* +
  assumes *comp-fun-idem-on*: $x \in S \Longrightarrow y \in S \Longrightarrow f\ x \circ f\ x = f\ x$
begin

declare *insert* [*simp del*]

interpretation *fold?*: *comp-fun-idem-on S f*
  by *standard* (*simp-all add*: *comp-fun-commute-on comp-fun-idem-on*)

lemma *insert-idem* [*simp*]:
  assumes *insert x A* $\subseteq$ *S* **and** *finite A*
  shows *F* (*insert x A*) = *f x* (*F A*)
proof −
  from *fold-insert-idem assms*
  have *fold f z* (*insert x A*) = *f x* (*fold f z A*) **by** *simp*
  with ‹*finite A*› **show** *?thesis* **by** (*simp add*: *eq-fold fun-eq-iff*)
qed

end

20.6.3 *UNIV* as the carrier set

locale *folding* =
  fixes *f* :: *'a* ⇒ *'b* ⇒ *'b* **and** *z* :: *'b*
  assumes *comp-fun-commute*: *f y* ∘ *f x* = *f x* ∘ *f y*
begin

lemma (**in** −) *folding-def'*: *folding f* = *folding-on UNIV f*
  unfolding *folding-def folding-on-def* **by** *blast*

Again, we abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *folding-on UNIV f*
  rewrites ⋀*X*. (*X* ⊆ *UNIV*) ≡ *True*
    and ⋀*x*. *x* ∈ *UNIV* ≡ *True*
    and ⋀*P*. (*True* ⟹ *P*) ≡ *Trueprop P*
    and ⋀*P Q*. (*True* ⟹ *PROP P* ⟹ *PROP Q*) ≡ (*PROP P* ⟹ *True* ⟹ *PROP Q*)
proof −
  show *folding-on UNIV f*
    by *standard* (*simp add*: *comp-fun-commute*)
qed *simp-all*

end

locale *folding-idem* = *folding* +
  assumes *comp-fun-idem*: *f x* ∘ *f x* = *f x*
begin

lemma (**in** −) *folding-idem-def'*: *folding-idem f* = *folding-idem-on UNIV f*
  unfolding *folding-idem-def folding-def' folding-idem-on-def*
  unfolding *folding-idem-axioms-def folding-idem-on-axioms-def*
  by *blast*

Again, we abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *folding-idem-on UNIV f*
  rewrites $\bigwedge X.\ (X \subseteq UNIV) \equiv True$
    and $\bigwedge x.\ x \in UNIV \equiv True$
    and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
    and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
proof −
  show *folding-idem-on UNIV f*
    by *standard* (*simp add*: *comp-fun-idem*)
qed *simp-all*

end

20.7 Finite cardinality

The traditional definition $card\ A \equiv LEAST\ n.\ \exists f.\ A = \{f\ i\ |i.\ i < n\}$ is ugly to work with. But now that we have *fold* things are easy:

global-interpretation *card*: *folding* λ-. *Suc 0*
  defines *card* = *folding-on.F* (λ-. *Suc*) *0*
  by *standard* (*rule refl*)

lemma *card-insert-disjoint*: $finite\ A \Longrightarrow x \notin A \Longrightarrow card\ (insert\ x\ A) = Suc\ (card\ A)$
  by (*fact card.insert*)

lemma *card-insert-if*: $finite\ A \Longrightarrow card\ (insert\ x\ A) = (if\ x \in A\ then\ card\ A\ else\ Suc\ (card\ A))$
  by *auto* (*simp add*: *card.insert-remove card.remove*)

lemma *card-ge-0-finite*: $card\ A > 0 \Longrightarrow finite\ A$
  by (*rule ccontr*) *simp*

lemma *card-0-eq* [*simp*]: $finite\ A \Longrightarrow card\ A = 0 \longleftrightarrow A = \{\}$
  by (*auto dest*: *mk-disjoint-insert*)

lemma *finite-UNIV-card-ge-0*: $finite\ (UNIV :: 'a\ set) \Longrightarrow card\ (UNIV :: 'a\ set) > 0$
  by (*rule ccontr*) *simp*

lemma *card-eq-0-iff*: $card\ A = 0 \longleftrightarrow A = \{\} \lor \neg\ finite\ A$
  by *auto*

lemma *card-range-greater-zero*: $finite\ (range\ f) \Longrightarrow card\ (range\ f) > 0$
  by (*rule ccontr*) (*simp add*: *card-eq-0-iff*)

lemma *card-gt-0-iff*: $0 < card\ A \longleftrightarrow A \neq \{\} \land finite\ A$
  by (*simp add*: *neq0-conv* [*symmetric*] *card-eq-0-iff*)

lemma *card-Suc-Diff1*:
  assumes *finite A x* ∈ *A* **shows** *Suc* (*card* (*A* − {*x*})) = *card A*
proof −
  have *Suc* (*card* (*A* − {*x*})) = *card* (*insert x* (*A* − {*x*}))
    using *assms* **by** (*simp add*: *card.insert-remove*)
  also have ... = *card A*
    using *assms* **by** (*simp add*: *card-insert-if*)
  finally show *?thesis* **.**
qed

lemma *card-insert-le-m1*:
  assumes *n* > *0 card y* ≤ *n* − *1* **shows** *card* (*insert x y*) ≤ *n*
  using *assms*
  by (*cases finite y*) (*auto simp*: *card-insert-if*)

lemma *card-Diff-singleton*:
  assumes *x* ∈ *A* **shows** *card* (*A* − {*x*}) = *card A* − *1*
proof (*cases finite A*)
  case *True*
  with *assms* **show** *?thesis*
    by (*simp add*: *card-Suc-Diff1* [*symmetric*])
qed *auto*

lemma *card-Diff-singleton-if*:
  card (*A* − {*x*}) = (*if x* ∈ *A then card A* − *1 else card A*)
  by (*simp add*: *card-Diff-singleton*)

lemma *card-Diff-insert*[*simp*]:
  assumes *a* ∈ *A* **and** *a* ∉ *B*
  shows *card* (*A* − *insert a B*) = *card* (*A* − *B*) − *1*
proof −
  have *A* − *insert a B* = (*A* − *B*) − {*a*}
    using *assms* **by** *blast*
  then show *?thesis*
    using *assms* **by** (*simp add*: *card-Diff-singleton*)
qed

lemma *card-insert-le*: *card A* ≤ *card* (*insert x A*)
proof (*cases finite A*)
  case *True*
  then show *?thesis* **by** (*simp add*: *card-insert-if*)
qed *auto*

lemma *card-Collect-less-nat*[*simp*]: *card* {*i*::*nat*. *i* < *n*} = *n*
  by (*induct n*) (*simp-all add*:*less-Suc-eq Collect-disj-eq*)

lemma *card-Collect-le-nat*[*simp*]: *card* {*i*::*nat*. *i* ≤ *n*} = *Suc n*
  using *card-Collect-less-nat*[*of Suc n*] **by** (*simp add*: *less-Suc-eq-le*)

lemma *card-mono*:
  assumes *finite B* **and** $A \subseteq B$
  shows *card A* $\leq$ *card B*
proof –
  from *assms* **have** *finite A*
    by (*auto intro*: *finite-subset*)
  then show *?thesis*
    using *assms*
  proof (*induct A arbitrary*: *B*)
    case *empty*
    then show *?case* **by** *simp*
  next
    case (*insert x A*)
    then have $x \in B$
      by *simp*
    from *insert* **have** $A \subseteq B - \{x\}$ **and** *finite* $(B - \{x\})$
      by *auto*
    with *insert.hyps* **have** *card A* $\leq$ *card* $(B - \{x\})$
      by *auto*
    with ‹*finite A*› ‹$x \notin A$› ‹*finite B*› ‹$x \in B$› **show** *?case*
      by *simp* (*simp only*: *card.remove*)
  qed
qed

lemma *card-seteq*:
  assumes *finite B* **and** *A*: $A \subseteq B$ *card B* $\leq$ *card A*
  shows $A = B$
  using *assms*
proof (*induction arbitrary*: *A rule*: *finite-induct*)
  case (*insert b B*)
  then have *A*: *finite A* $A - \{b\} \subseteq B$
    by *force+*
  then have *card B* $\leq$ *card* $(A - \{b\})$
    using *insert* **by** (*auto simp add*: *card-Diff-singleton-if*)
  then have $A - \{b\} = B$
    using *A insert.IH* **by** *auto*
  then show *?case*
    using *insert.hyps insert.prems* **by** *auto*
qed *auto*

lemma *psubset-card-mono*: *finite B* $\Longrightarrow A < B \Longrightarrow$ *card A* $<$ *card B*
  using *card-seteq* [*of B A*] **by** (*auto simp add*: *psubset-eq*)

lemma *card-Un-Int*:
  assumes *finite A finite B*
  shows *card A* + *card B* = *card* $(A \cup B)$ + *card* $(A \cap B)$
  using *assms*
proof (*induct A*)

case *empty*
then show *?case* **by** *simp*
next
case *insert*
then show *?case*
by (*auto simp add*: *insert-absorb Int-insert-left*)
qed

lemma *card-Un-disjoint*: *finite A* $\Longrightarrow$ *finite B* $\Longrightarrow$ $A \cap B = \{\}$ $\Longrightarrow$ *card* $(A \cup B)$ $=$ *card A* $+$ *card B*
using *card-Un-Int* [*of A B*] **by** *simp*

lemma *card-Un-disjnt*: $[\![$*finite A*; *finite B*; *disjnt A B*$]\!]$ $\Longrightarrow$ *card* $(A \cup B)$ $=$ *card A* $+$ *card B*
by (*simp add*: *card-Un-disjoint disjnt-def*)

lemma *card-Un-le*: *card* $(A \cup B) \leq$ *card A* $+$ *card B*
proof (*cases finite A* $\wedge$ *finite B*)
case *True*
then show *?thesis*
using *le-iff-add card-Un-Int* [*of A B*] **by** *auto*
qed *auto*

lemma *card-Diff-subset*:
assumes *finite B*
and $B \subseteq A$
shows *card* $(A - B)$ $=$ *card A* $-$ *card B*
using *assms*
proof (*cases finite A*)
case *False*
with *assms* **show** *?thesis*
by *simp*
next
case *True*
with *assms* **show** *?thesis*
by (*induct B arbitrary*: *A*) *simp-all*
qed

lemma *card-Diff-subset-Int*:
assumes *finite* $(A \cap B)$
shows *card* $(A - B)$ $=$ *card A* $-$ *card* $(A \cap B)$
proof $-$
have $A - B = A - A \cap B$ **by** *auto*
with *assms* **show** *?thesis*
by (*simp add*: *card-Diff-subset*)
qed

lemma *card-Int-Diff*:
assumes *finite A*

shows *card* A = *card* $(A \cap B)$ + *card* $(A - B)$
by (*simp add*: *assms card-Diff-subset-Int card-mono*)

lemma *diff-card-le-card-Diff*:
assumes *finite* B
shows *card* A − *card* $B \leq$ *card* $(A - B)$
proof −
have *card* A − *card* $B \leq$ *card* A − *card* $(A \cap B)$
using *card-mono*[*OF assms Int-lower2*, *of A*] **by** *arith*
also have ... = *card* $(A - B)$
using *assms* **by** (*simp add*: *card-Diff-subset-Int*)
finally show *?thesis* **.**
qed

lemma *card-le-sym-Diff*:
assumes *finite* A *finite* B *card* $A \leq$ *card* B
shows *card*$(A - B) \leq$ *card*$(B - A)$
proof −
have *card*$(A - B)$ = *card* A − *card* $(A \cap B)$ **using** *assms*(*1*,*2*) **by**(*simp add*: *card-Diff-subset-Int*)
also have ... $\leq$ *card* B − *card* $(A \cap B)$ **using** *assms*(*3*) **by** *linarith*
also have ... = *card*$(B - A)$ **using** *assms*(*1*,*2*) **by**(*simp add*: *card-Diff-subset-Int Int-commute*)
finally show *?thesis* **.**
qed

lemma *card-less-sym-Diff*:
assumes *finite* A *finite* B *card* $A <$ *card* B
shows *card*$(A - B) <$ *card*$(B - A)$
proof −
have *card*$(A - B)$ = *card* A − *card* $(A \cap B)$ **using** *assms*(*1*,*2*) **by**(*simp add*: *card-Diff-subset-Int*)
also have ... $<$ *card* B − *card* $(A \cap B)$ **using** *assms*(*1*,*3*) **by** (*simp add*: *card-mono diff-less-mono*)
also have ... = *card*$(B - A)$ **using** *assms*(*1*,*2*) **by**(*simp add*: *card-Diff-subset-Int Int-commute*)
finally show *?thesis* **.**
qed

lemma *card-Diff1-less-iff*: *card* $(A - \{x\}) <$ *card* $A \longleftrightarrow$ *finite* $A \wedge x \in A$
proof (*cases finite* $A \wedge x \in A$)
case *True*
then show *?thesis*
by (*auto simp*: *card-gt-0-iff intro*: *diff-less*)
qed *auto*

lemma *card-Diff1-less*: *finite* $A \Longrightarrow x \in A \Longrightarrow$ *card* $(A - \{x\}) <$ *card* A
unfolding *card-Diff1-less-iff* **by** *auto*

lemma *card-Diff2-less*:
  assumes *finite A* $x \in A$ $y \in A$ **shows** *card* $(A - \{x\} - \{y\}) <$ *card A*
proof (*cases* $x = y$)
  case *True*
  with *assms* **show** *?thesis*
    by (*simp add*: *card-Diff1-less del*: *card-Diff-insert*)
next
  case *False*
  then have *card* $(A - \{x\} - \{y\}) <$ *card* $(A - \{x\})$ *card* $(A - \{x\}) <$ *card A*
    using *assms* **by** (*intro card-Diff1-less*; *simp*)+
  then show *?thesis*
    by (*blast intro*: *less-trans*)
qed

lemma *card-Diff1-le*: *card* $(A - \{x\}) \leq$ *card A*
proof (*cases finite A*)
  case *True*
  then show *?thesis*
    by (*cases* $x \in A$) (*simp-all add*: *card-Diff1-less less-imp-le*)
qed *auto*

lemma *card-psubset*: *finite B* $\Longrightarrow A \subseteq B \Longrightarrow$ *card A* $<$ *card B* $\Longrightarrow A < B$
  by (*erule psubsetI*) *blast*

lemma *card-le-inj*:
  assumes *fA*: *finite A*
    and *fB*: *finite B*
    and *c*: *card A* $\leq$ *card B*
  shows $\exists f.\ f$ ‘ $A \subseteq B \wedge$ *inj-on f A*
  using *fA fB c*
proof (*induct arbitrary*: *B rule*: *finite-induct*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert x s t*)
  then show *?case*
  proof (*induct rule*: *finite-induct* [*OF insert.prems*(*1*)])
    case *1*
    then show *?case* **by** *simp*
  next
    case (*2 y t*)
    from *2.prems*(*1*,*2*,*5*) *2.hyps*(*1*,*2*) **have** *cst*: *card s* $\leq$ *card t*
      by *simp*
    from *2.prems*(*3*) [*OF 2.hyps*(*1*) *cst*]
    obtain *f* **where** $*$: *f* ‘ $s \subseteq t$ *inj-on f s*
      by *blast*
    let *?g* = ($\lambda a.$ *if* $a = x$ *then y else f a*)
    have *?g* ‘ *insert x s* $\subseteq$ *insert y t* $\wedge$ *inj-on ?g* (*insert x s*)
      using $*$ *2.prems*(*2*) *2.hyps*(*2*) **unfolding** *inj-on-def* **by** *auto*

then show *?case* **by** (*rule exI*[**where** *?x*=*?g*])
qed
qed

lemma *card-subset-eq*:
assumes *fB*: *finite B*
and *AB*: $A \subseteq B$
and *c*: *card A* = *card B*
shows $A = B$
proof −
from *fB AB* **have** *fA*: *finite A*
by (*auto intro*: *finite-subset*)
from *fA fB* **have** *fBA*: *finite* $(B - A)$
by *auto*
have *e*: $A \cap (B - A) = \{\}$
by *blast*
have *eq*: $A \cup (B - A) = B$
using *AB* **by** *blast*
from *card-Un-disjoint*[*OF fA fBA e*, *unfolded eq c*] **have** *card* $(B - A) = 0$
by *arith*
then have $B - A = \{\}$
unfolding *card-eq-0-iff* **using** *fA fB* **by** *simp*
with *AB* **show** $A = B$
by *blast*
qed

lemma *insert-partition*:
$x \notin F \Longrightarrow \forall c1 \in insert\ x\ F.\ \forall c2 \in insert\ x\ F.\ c1 \neq c2 \longrightarrow c1 \cap c2 = \{\} \Longrightarrow$
$x \cap \bigcup F = \{\}$
by *auto*

lemma *finite-psubset-induct* [*consumes 1*, *case-names psubset*]:
assumes *finite*: *finite A*
and *major*: $\bigwedge A.\ finite\ A \Longrightarrow (\bigwedge B.\ B \subset A \Longrightarrow P\ B) \Longrightarrow P\ A$
shows *P A*
using *finite*
proof (*induct A taking*: *card rule*: *measure-induct-rule*)
case (*less A*)
have *fin*: *finite A* **by** *fact*
have *ih*: *card B* < *card A* $\Longrightarrow$ *finite B* $\Longrightarrow$ *P B* **for** *B* **by** *fact*
have *P B* **if** $B \subset A$ **for** *B*
proof −
from *that* **have** *card B* < *card A*
using *psubset-card-mono fin* **by** *blast*
moreover
from *that* **have** $B \subseteq A$
by *auto*
then have *finite B*
using *fin finite-subset* **by** *blast*

ultimately show *?thesis* **using** *ih* **by** *simp*
qed
with *fin* **show** *P A* **using** *major* **by** *blast*
qed

lemma *finite-induct-select* [*consumes 1*, *case-names empty select*]:
assumes *finite S*
and *P* {}
and *select*: $\bigwedge T.\ T \subset S \Longrightarrow P\ T \Longrightarrow \exists s{\in}S - T.\ P\ (insert\ s\ T)$
shows *P S*
proof −
have $0 \leq card\ S$ **by** *simp*
then have $\exists\ T \subseteq S.\ card\ T = card\ S \wedge P\ T$
proof (*induct rule*: *dec-induct*)
case *base* **with** ‹*P* {}›
show *?case*
by (*intro exI*[*of* - {}]) *auto*
next
case (*step n*)
then obtain *T* **where** *T*: $T \subseteq S$ $card\ T = n$ *P T*
by *auto*
with ‹$n < card\ S$› **have** $T \subset S$ *P T*
by *auto*
with *select*[*of T*] **obtain** *s* **where** $s \in S$ $s \notin T$ *P* (*insert s T*)
by *auto*
with *step*(*2*) *T* ‹*finite S*› **show** *?case*
by (*intro exI*[*of* - *insert s T*]) (*auto dest*: *finite-subset*)
qed
with ‹*finite S*› **show** *P S*
by (*auto dest*: *card-subset-eq*)
qed

lemma *remove-induct* [*case-names empty infinite remove*]:
assumes *empty*: *P* ({} :: *'a set*)
and *infinite*: $\neg$ *finite B* $\Longrightarrow$ *P B*
and *remove*: $\bigwedge A.\ finite\ A \Longrightarrow A \neq \{\} \Longrightarrow A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow P\ (A - \{x\})) \Longrightarrow P\ A$
shows *P B*
proof (*cases finite B*)
case *False*
then show *?thesis* **by** (*rule infinite*)
next
case *True*
define *A* **where** *A* = *B*
with *True* **have** *finite A* $A \subseteq B$
by *simp-all*
then show *P A*
proof (*induct card A arbitrary*: *A*)
case *0*

then have $A = \{\}$ **by** *auto*
with *empty* **show** *?case* **by** *simp*
next
case (*Suc n A*)
from ‹$A \subseteq B$› **and** ‹*finite B*› **have** *finite A*
by (*rule finite-subset*)
moreover from *Suc.hyps* **have** $A \neq \{\}$ **by** *auto*
moreover note ‹$A \subseteq B$›
moreover have $P\ (A - \{x\})$ **if** x: $x \in A$ **for** x
using *x Suc.prems* ‹*Suc n = card A*› **by** (*intro Suc*) *auto*
ultimately show *?case* **by** (*rule remove*)
qed
qed

lemma *finite-remove-induct* [*consumes 1*, *case-names empty remove*]:
fixes P :: *'a set ⇒ bool*
assumes *finite B*
and $P\ \{\}$
and $\bigwedge A.\ finite\ A \Longrightarrow A \neq \{\} \Longrightarrow A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow P\ (A - \{x\})) \Longrightarrow P\ A$
defines $B' \equiv B$
shows $P\ B'$
by (*induct B' rule*: *remove-induct*) (*simp-all add*: *assms*)

Main cardinality theorem.

lemma *card-partition* [*rule-format*]:
$finite\ C \Longrightarrow finite\ (\bigcup C) \Longrightarrow (\forall c{\in}C.\ card\ c = k) \Longrightarrow$
$(\forall c1 \in C.\ \forall c2 \in C.\ c1 \neq c2 \longrightarrow c1 \cap c2 = \{\}) \Longrightarrow$
$k * card\ C = card\ (\bigcup C)$
proof (*induct rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
then show *?case*
by (*simp add*: *card-Un-disjoint insert-partition finite-subset* [*of* - $\bigcup$ (*insert* - -)])
qed

lemma *card-eq-UNIV-imp-eq-UNIV*:
assumes *fin*: *finite* (*UNIV* :: *'a set*)
and *card*: *card A* = *card* (*UNIV* :: *'a set*)
shows *A* = (*UNIV* :: *'a set*)
proof
show $A \subseteq UNIV$ **by** *simp*
show $UNIV \subseteq A$
proof
show $x \in A$ **for** x
proof (*rule ccontr*)
assume $x \notin A$

then have *A* ⊂ *UNIV* **by** *auto*
with *fin* **have** *card A* < *card* (*UNIV* :: *'a set*)
by (*fact psubset-card-mono*)
with *card* **show** *False* **by** *simp*
qed
qed
qed

The form of a finite set of given cardinality

lemma *card-eq-SucD*:
assumes *card A* = *Suc k*
shows ∃ *b B*. *A* = *insert b B* ∧ *b* ∉ *B* ∧ *card B* = *k* ∧ (*k* = *0* ⟶ *B* = {})
proof −
have *fin*: *finite A*
using *assms* **by** (*auto intro*: *ccontr*)
moreover have *card A* ≠ *0*
using *assms* **by** *auto*
ultimately obtain *b* **where** *b*: *b* ∈ *A*
by *auto*
show *?thesis*
proof (*intro exI conjI*)
show *A* = *insert b* (*A* − {*b*})
using *b* **by** *blast*
show *b* ∉ *A* − {*b*}
by *blast*
show *card* (*A* − {*b*}) = *k* **and** *k* = *0* ⟶ *A* − {*b*} = {}
using *assms b fin* **by** (*fastforce dest*: *mk-disjoint-insert*)+
qed
qed

lemma *card-Suc-eq*:
card A = *Suc k* ⟷
(∃ *b B*. *A* = *insert b B* ∧ *b* ∉ *B* ∧ *card B* = *k* ∧ (*k* = *0* ⟶ *B* = {}))
by (*auto simp*: *card-insert-if card-gt-0-iff elim*!: *card-eq-SucD*)

lemma *card-Suc-eq-finite*:
card A = *Suc k* ⟷ (∃ *b B*. *A* = *insert b B* ∧ *b* ∉ *B* ∧ *card B* = *k* ∧ *finite B*)
unfolding *card-Suc-eq* **using** *card-gt-0-iff* **by** *fastforce*

lemma *card-1-singletonE*:
assumes *card A* = *1*
obtains *x* **where** *A* = {*x*}
using *assms* **by** (*auto simp*: *card-Suc-eq*)

lemma *is-singleton-altdef*: *is-singleton A* ⟷ *card A* = *1*
unfolding *is-singleton-def*
by (*auto elim*!: *card-1-singletonE is-singletonE simp del*: *One-nat-def*)

lemma *card-1-singleton-iff*: *card A* = *Suc 0* ⟷ (∃ *x*. *A* = {*x*})

by (*simp add*: *card-Suc-eq*)

lemma *card-le-Suc0-iff-eq*:
assumes *finite A*
shows *card A* ≤ *Suc 0* ⟷ (∀ *a1* ∈ *A*. ∀ *a2* ∈ *A*. *a1* = *a2*) (**is** *?C* = *?A*)
proof
assume *?C* **thus** *?A* **using** *assms* **by** (*auto simp*: *le-Suc-eq dest*: *card-eq-SucD*)
next
assume *?A*
show *?C*
proof *cases*
assume *A* = {} **thus** *?C* **using** ‹*?A*› **by** *simp*
next
assume *A* ≠ {}
then obtain *a* **where** *A* = {*a*} **using** ‹*?A*› **by** *blast*
thus *?C* **by** *simp*
qed
qed

lemma *card-le-Suc-iff*:
Suc n ≤ *card A* = (∃ *a B*. *A* = *insert a B* ∧ *a* ∉ *B* ∧ *n* ≤ *card B* ∧ *finite B*)
proof (*cases finite A*)
case *True*
then show *?thesis*
by (*fastforce simp*: *card-Suc-eq less-eq-nat.simps split*: *nat.splits*)
qed *auto*

lemma *finite-fun-UNIVD2*:
assumes *fin*: *finite* (*UNIV* :: (*'a* ⇒ *'b*) *set*)
shows *finite* (*UNIV* :: *'b set*)
proof −
from *fin* **have** *finite* (*range* (λ*f* :: *'a* ⇒ *'b*. *f arbitrary*)) **for** *arbitrary*
by (*rule finite-imageI*)
moreover have *UNIV* = *range* (λ*f* :: *'a* ⇒ *'b*. *f arbitrary*) **for** *arbitrary*
by (*rule UNIV-eq-I*) *auto*
ultimately show *finite* (*UNIV* :: *'b set*)
by *simp*
qed

lemma *card-UNIV-unit* [*simp*]: *card* (*UNIV* :: *unit set*) = *1*
unfolding *UNIV-unit* **by** *simp*

lemma *infinite-arbitrarily-large*:
assumes ¬ *finite A*
shows ∃ *B*. *finite B* ∧ *card B* = *n* ∧ *B* ⊆ *A*
proof (*induction n*)
case *0*
show *?case* **by** (*intro exI*[*of* - {}]) *auto*
next

case (*Suc n*)
then obtain *B* **where** *B*: *finite B* ∧ *card B* = *n* ∧ *B* ⊆ *A* **..**
with ‹¬ *finite A*› **have** *A* ≠ *B* **by** *auto*
with *B* **have** *B* ⊂ *A* **by** *auto*
then have ∃ *x*. *x* ∈ *A* − *B*
by (*elim psubset-imp-ex-mem*)
then obtain *x* **where** *x*: *x* ∈ *A* − *B* **..**
with *B* **have** *finite* (*insert x B*) ∧ *card* (*insert x B*) = *Suc n* ∧ *insert x B* ⊆ *A*
by *auto*
then show ∃ *B*. *finite B* ∧ *card B* = *Suc n* ∧ *B* ⊆ *A* **..**
qed

Sometimes, to prove that a set is finite, it is convenient to work with finite subsets and to show that their cardinalities are uniformly bounded. This possibility is formalized in the next criterion.

lemma *finite-if-finite-subsets-card-bdd*:
assumes ⋀*G*. *G* ⊆ *F* ⟹ *finite G* ⟹ *card G* ≤ *C*
shows *finite F* ∧ *card F* ≤ *C*
proof (*cases finite F*)
case *False*
obtain *n*::*nat* **where** *n*: *n* > *max C 0* **by** *auto*
obtain *G* **where** *G*: *G* ⊆ *F card G* = *n* **using** *infinite-arbitrarily-large*[*OF False*]
by *auto*
hence *finite G* **using** ‹*n* > *max C 0*› **using** *card.infinite gr-implies-not0* **by** *blast*
hence *False* **using** *assms G n not-less* **by** *auto*
thus *?thesis* **..**
next
case *True* **thus** *?thesis* **using** *assms*[*of F*] **by** *auto*
qed

lemma *obtain-subset-with-card-n*:
assumes *n* ≤ *card S*
obtains *T* **where** *T* ⊆ *S card T* = *n finite T*
proof −
obtain *n'* **where** *card S* = *n* + *n'*
using *le-Suc-ex*[*OF assms*] **by** *blast*
with *that* **show** *thesis*
proof (*induct n' arbitrary*: *S*)
case *0*
thus *?case* **by** (*cases finite S*) *auto*
next
case *Suc*
thus *?case* **by** (*auto simp add*: *card-Suc-eq*)
qed
qed

lemma *exists-subset-between*:
assumes

card A ≤ *n*
n ≤ *card C*
A ⊆ *C*
finite C
shows ∃*B*. *A* ⊆ *B* ∧ *B* ⊆ *C* ∧ *card B* = *n*
using *assms*
proof (*induct n arbitrary*: *A C*)
case *0*
thus *?case* **using** *finite-subset*[*of A C*] **by** (*intro exI*[*of* - {}], *auto*)
next
case (*Suc n A C*)
show *?case*
proof (*cases A* = {})
case *True*
from *obtain-subset-with-card-n*[*OF Suc*(*3*)]
obtain *B* **where** *B* ⊆ *C card B* = *Suc n* **by** *blast*
thus *?thesis* **unfolding** *True* **by** *blast*
next
case *False*
then obtain *a* **where** *a*: *a* ∈ *A* **by** *auto*
let *?A* = *A* − {*a*}
let *?C* = *C* − {*a*}
have *1*: *card ?A* ≤ *n* **using** *Suc*(*2*−) *a*
using *finite-subset* **by** *fastforce*
have *2*: *card ?C* ≥ *n* **using** *Suc*(*2*−) *a* **by** *auto*
from *Suc*(*1*)[*OF 1 2* - *finite-subset*[*OF* - *Suc*(*5*)]] *Suc*(*2*−)
obtain *B* **where** *?A* ⊆ *B B* ⊆ *?C card B* = *n* **by** *blast*
thus *?thesis* **using** *a Suc*(*2*−)
by (*intro exI*[*of* - *insert a B*], *auto intro*!: *card-insert-disjoint finite-subset*[*of B C*])
qed
qed

20.7.1 Cardinality of image

lemma *card-image-le*: *finite A* ⟹ *card* (*f* ' *A*) ≤ *card A*
by (*induct rule*: *finite-induct*) (*simp-all add*: *le-SucI card-insert-if*)

lemma *card-image*: *inj-on f A* ⟹ *card* (*f* ' *A*) = *card A*
proof (*induct A rule*: *infinite-finite-induct*)
case (*infinite A*)
then have ¬ *finite* (*f* ' *A*) **by** (*auto dest*: *finite-imageD*)
with *infinite* **show** *?case* **by** *simp*
qed *simp-all*

lemma *bij-betw-same-card*: *bij-betw f A B* ⟹ *card A* = *card B*
by (*auto simp*: *card-image bij-betw-def*)

lemma *endo-inj-surj*: *finite A* ⟹ *f* ' *A* ⊆ *A* ⟹ *inj-on f A* ⟹ *f* ' *A* = *A*

by (*simp add*: *card-seteq card-image*)

lemma *eq-card-imp-inj-on*:
assumes *finite A card*$(f$ ‘ $A) =$ *card A*
shows *inj-on f A*
using *assms*
proof (*induct rule*:*finite-induct*)
case *empty*
show *?case* **by** *simp*
next
case (*insert x A*)
then show *?case*
using *card-image-le* [*of A f*] **by** (*simp add*: *card-insert-if split*: *if-splits*)
qed

lemma *inj-on-iff-eq-card*: *finite A* $\Longrightarrow$ *inj-on f A* $\longleftrightarrow$ *card* $(f$ ‘ $A) =$ *card A*
by (*blast intro*: *card-image eq-card-imp-inj-on*)

lemma *card-inj-on-le*:
assumes *inj-on f A f* ‘ $A \subseteq B$ *finite B*
shows *card A* $\leq$ *card B*
proof −
have *finite A*
using *assms* **by** (*blast intro*: *finite-imageD dest*: *finite-subset*)
then show *?thesis*
using *assms* **by** (*force intro*: *card-mono simp*: *card-image* [*symmetric*])
qed

lemma *inj-on-iff-card-le*:
⟦ *finite A*; *finite B* ⟧ $\Longrightarrow$ $(\exists f.$ *inj-on f A* $\wedge$ *f* ‘ $A \leq B) = ($*card A* $\leq$ *card B*$)$
using *card-inj-on-le*[*of - A B*] *card-le-inj*[*of A B*] **by** *blast*

lemma *surj-card-le*: *finite A* $\Longrightarrow B \subseteq f$ ‘ $A \Longrightarrow$ *card B* $\leq$ *card A*
by (*blast intro*: *card-image-le card-mono le-trans*)

lemma *card-bij-eq*:
inj-on f A $\Longrightarrow$ *f* ‘ $A \subseteq B \Longrightarrow$ *inj-on g B* $\Longrightarrow$ *g* ‘ $B \subseteq A \Longrightarrow$ *finite A* $\Longrightarrow$ *finite B*
$\Longrightarrow$ *card A* $=$ *card B*
by (*auto intro*: *le-antisym card-inj-on-le*)

lemma *bij-betw-finite*: *bij-betw f A B* $\Longrightarrow$ *finite A* $\longleftrightarrow$ *finite B*
unfolding *bij-betw-def* **using** *finite-imageD* [*of f A*] **by** *auto*

lemma *inj-on-finite*: *inj-on f A* $\Longrightarrow$ *f* ‘ $A \leq B \Longrightarrow$ *finite B* $\Longrightarrow$ *finite A*
using *finite-imageD finite-subset* **by** *blast*

lemma *card-vimage-inj-on-le*:
assumes *inj-on f D finite A*
shows *card* $(f-$‘$A \cap D) \leq$ *card A*

proof (*rule card-inj-on-le*)
  show *inj-on f* (*f* −' *A* ∩ *D*)
    by (*blast intro*: *assms inj-on-subset*)
qed (*use assms* **in** *auto*)

lemma *card-vimage-inj*: *inj f* ⟹ *A* ⊆ *range f* ⟹ *card* (*f* −' *A*) = *card A*
  by (*auto 4 3 simp*: *subset-image-iff inj-vimage-image-eq*
      intro: *card-image*[*symmetric*, *OF subset-inj-on*])

lemma *card-inverse*[*simp*]: *card* (R^{-1}) = *card R*
proof −
  have ∗: ⋀*R*. *prod.swap* ' *R* = R^{-1} **by** *auto*
  {
    assume ¬*finite R*
    hence *?thesis*
      by *auto*
  } moreover {
    assume *finite R*
    with *card-image-le*[*of R prod.swap*] *card-image-le*[*of* R^{-1} *prod.swap*]
    have *?thesis* **by** (*auto simp*: ∗)
  } ultimately show *?thesis* **by** *blast*
qed

20.7.2 Pigeonhole Principles

lemma *pigeonhole*: *card A* > *card* (*f* ' *A*) ⟹ ¬ *inj-on f A*
  by (*auto dest*: *card-image less-irrefl-nat*)

lemma *pigeonhole-infinite*:
  assumes ¬ *finite A* **and** *finite* (*f*'*A*)
  shows ∃ *a0*∈*A*. ¬ *finite* {*a*∈*A*. *f a* = *f a0*}
  using *assms*(*2*,*1*)
proof (*induct f*'*A arbitrary*: *A rule*: *finite-induct*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert b F*)
  show *?case*
  proof (*cases finite* {*a*∈*A*. *f a* = *b*})
    case *True*
    with ‹¬ *finite A*› **have** ¬ *finite* (*A* − {*a*∈*A*. *f a* = *b*})
      by *simp*
    also have *A* − {*a*∈*A*. *f a* = *b*} = {*a*∈*A*. *f a* ≠ *b*}
      by *blast*
    finally have ¬ *finite* {*a*∈*A*. *f a* ≠ *b*} **.**
    from *insert*(*3*)[*OF* - *this*] *insert*(*2*,*4*) **show** *?thesis*
      by *simp* (*blast intro*: *rev-finite-subset*)
  next
    case *False*

```
    then have {a ∈ A. f a = b} ≠ {} by force
    with False show ?thesis by blast
  qed
qed
```

```
lemma pigeonhole-infinite-rel:
  assumes ¬ finite A
    and finite B
    and ∀ a∈A. ∃ b∈B. R a b
  shows ∃ b∈B. ¬ finite {a:A. R a b}
proof −
  let ?F = λa. {b∈B. R a b}
  from finite-Pow-iff [THEN iffD2, OF ‹finite B›] have finite (?F ‘ A)
    by (blast intro: rev-finite-subset)
  from pigeonhole-infinite [where f = ?F, OF assms(1) this]
  obtain a0 where a0 ∈ A and infinite: ¬ finite {a∈A. ?F a = ?F a0} ..
  obtain b0 where b0 ∈ B and R a0 b0
    using ‹a0 ∈ A› assms(3) by blast
  have finite {a∈A. ?F a = ?F a0} if finite {a∈A. R a b0}
    using ‹b0 ∈ B› ‹R a0 b0› that by (blast intro: rev-finite-subset)
  with infinite ‹b0 ∈ B› show ?thesis
    by blast
qed
```

20.7.3 Cardinality of sums

```
lemma card-Plus:
  assumes finite A finite B
  shows card (A <+> B) = card A + card B
proof −
  have Inl‘A ∩ Inr‘B = {} by fast
  with assms show ?thesis
    by (simp add: Plus-def card-Un-disjoint card-image)
qed
```

```
lemma card-Plus-conv-if:
  card (A <+> B) = (if finite A ∧ finite B then card A + card B else 0)
  by (auto simp add: card-Plus)
```

Relates to equivalence classes. Based on a theorem of F. Kammüller.

```
lemma dvd-partition:
  assumes f: finite (⋃ C)
    and ∀ c∈C. k dvd card c ∀ c1∈C. ∀ c2∈C. c1 ≠ c2 ⟶ c1 ∩ c2 = {}
  shows k dvd card (⋃ C)
proof −
  have finite C
    by (rule finite-UnionD [OF f])
  then show ?thesis
    using assms
```

```
  proof (induct rule: finite-induct)
    case empty
    show ?case by simp
  next
    case (insert c C)
    then have c ∩ ⋃ C = {}
      by auto
    with insert show ?case
      by (simp add: card-Un-disjoint)
  qed
qed
```

20.8 Minimal and maximal elements of finite sets

```
context begin

qualified lemma
  assumes finite A and asymp-on A R and transp-on A R and ∃ x ∈ A. P x
  shows
    bex-min-element-with-property: ∃ x ∈ A. P x ∧ (∀ y ∈ A. R y x ⟶ ¬ P y) and
    bex-max-element-with-property: ∃ x ∈ A. P x ∧ (∀ y ∈ A. R x y ⟶ ¬ P y)
  unfolding atomize-conj
  using assms
proof (induction A rule: finite-induct)
  case empty
  hence False
    by simp-all
  thus ?case ..
next
  case (insert x F)

  from insert.prems have asymp-on F R
    using asymp-on-subset by blast

  from insert.prems have transp-on F R
    using transp-on-subset by blast

  show ?case
  proof (cases P x)
    case True
    show ?thesis
    proof (cases ∃ a∈F. P a)
      case True
      with insert.IH obtain min max where
        min ∈ F and P min and ∀ z ∈ F. R z min ⟶ ¬ P z
        max ∈ F and P max and ∀ z ∈ F. R max z ⟶ ¬ P z
        using ‹asymp-on F R› ‹transp-on F R› by auto

      show ?thesis
```

proof (*rule conjI*)
show $\exists\, y \in$ *insert x F*. *P y* $\wedge$ ($\forall\, z \in$ *insert x F*. *R y z* $\longrightarrow \neg$ *P z*)
proof (*cases R max x*)
case *True*
show *?thesis*
proof (*intro bexI conjI ballI impI*)
show *x* $\in$ *insert x F*
by *simp*
next
show *P x*
using ‹*P x*› **by** *simp*
next
fix *z* **assume** *z* $\in$ *insert x F* **and** *R x z*
hence $z = x \vee z \in F$
by *simp*
thus $\neg$ *P z*
proof (*rule disjE*)
assume $z = x$
hence *R x x*
using ‹*R x z*› **by** *simp*
moreover have $\neg$ *R x x*
using ‹*asymp-on* (*insert x F*) *R*›[*THEN irreflp-on-if-asymp-on*, *THEN irreflp-onD*]
by *simp*
ultimately have *False*
by *simp*
thus *?thesis* **..**
next
assume $z \in F$
moreover have *R max z*
using ‹*R max x*› ‹*R x z*›
using ‹*transp-on* (*insert x F*) *R*›[*THEN transp-onD*, *of max x z*]
using ‹*max* $\in$ *F*› ‹$z \in F$› **by** *simp*
ultimately show *?thesis*
using ‹$\forall\, z \in F$. *R max z* $\longrightarrow \neg$ *P z*› **by** *simp*
qed
qed
next
case *False*
show *?thesis*
proof (*intro bexI conjI ballI impI*)
show *max* $\in$ *insert x F*
using ‹*max* $\in$ *F*› **by** *simp*
next
show *P max*
using ‹*P max*› **by** *simp*
next
fix *z* **assume** *z* $\in$ *insert x F* **and** *R max z*
hence $z = x \vee z \in F$

```
          by simp
        thus ¬ P z
        proof (rule disjE)
          assume z = x
          hence False
            using ‹¬ R max x› ‹R max z› by simp
          thus ?thesis ..
        next
          assume z ∈ F
          thus ?thesis
            using ‹R max z› ‹∀ z∈F. R max z ⟶ ¬ P z› by simp
        qed
      qed
    qed
  next
    show ∃ y ∈ insert x F. P y ∧ (∀ z ∈ insert x F. R z y ⟶ ¬ P z)
    proof (cases R x min)
      case True
      show ?thesis
      proof (intro bexI conjI ballI impI)
        show x ∈ insert x F
          by simp
      next
        show P x
          using ‹P x› by simp
      next
        fix z assume z ∈ insert x F and R z x
        hence z = x ∨ z ∈ F
          by simp
        thus ¬ P z
        proof (rule disjE)
          assume z = x
          hence R x x
            using ‹R z x› by simp
          moreover have ¬ R x x
          using ‹asymp-on (insert x F) R›[THEN irreflp-on-if-asymp-on, THEN
irreflp-onD]
            by simp
          ultimately have False
            by simp
          thus ?thesis ..
        next
          assume z ∈ F
          moreover have R z min
            using ‹R z x› ‹R x min›
            using ‹transp-on (insert x F) R›[THEN transp-onD, of z x min]
            using ‹min ∈ F› ‹z ∈ F› by simp
          ultimately show ?thesis
            using ‹∀ z ∈ F. R z min ⟶ ¬ P z› by simp
```

```
          qed
        qed
      next
        case False
        show ?thesis
        proof (intro bexI conjI ballI impI)
          show min ∈ insert x F
            using ‹min ∈ F› by simp
        next
          show P min
            using ‹P min› by simp
        next
          fix z assume z ∈ insert x F and R z min
          hence z = x ∨ z ∈ F
            by simp
          thus ¬ P z
          proof (rule disjE)
            assume z = x
            hence False
              using ‹¬ R x min› ‹R z min› by simp
            thus ?thesis ..
          next
            assume z ∈ F
            thus ?thesis
              using ‹R z min› ‹∀ z∈F. R z min ⟶ ¬ P z› by simp
          qed
        qed
      qed
    qed
  next
    case False
    then show ?thesis
      using ‹∃ a∈insert x F. P a›
      using ‹asymp-on (insert x F) R›[THEN asymp-onD, of x] insert-iff[of - x
F]
      by blast
  qed
next
  case False
  with insert.prems have ∃ x ∈ F. P x
    by simp
  with insert.IH have
    ∃ y ∈ F. P y ∧ (∀ z∈F. R z y ⟶ ¬ P z)
    ∃ y ∈ F. P y ∧ (∀ z∈F. R y z ⟶ ¬ P z)
    using ‹asymp-on F R› ‹transp-on F R› by auto
  thus ?thesis
    using False by auto
qed
qed
```

qualified lemma
  assumes *finite A* **and** *asymp-on A R* **and** *transp-on A R* **and** $A \neq \{\}$
  shows
    bex-min-element: $\exists\, m \in A.\ \forall\, x \in A.\ x \neq m \longrightarrow \neg\ R\ x\ m$ **and**
    bex-max-element: $\exists\, m \in A.\ \forall\, x \in A.\ x \neq m \longrightarrow \neg\ R\ m\ x$
  using ‹$A \neq \{\}$›
    bex-min-element-with-property[*OF assms*(*1*,*2*,*3*), *of* λ-. *True*, *simplified*]
    bex-max-element-with-property[*OF assms*(*1*,*2*,*3*), *of* λ-. *True*, *simplified*]
  by *blast+*

end

The following alternative form might sometimes be easier to work with.

lemma *is-min-element-in-set-iff*:
  asymp-on A R $\Longrightarrow (\forall\, y \in A.\ y \neq x \longrightarrow \neg\ R\ y\ x) \longleftrightarrow (\forall\, y.\ R\ y\ x \longrightarrow y \notin A)$
  by (*auto dest*: *asymp-onD*)

lemma *is-max-element-in-set-iff*:
  asymp-on A R $\Longrightarrow (\forall\, y \in A.\ y \neq x \longrightarrow \neg\ R\ x\ y) \longleftrightarrow (\forall\, y.\ R\ x\ y \longrightarrow y \notin A)$
  by (*auto dest*: *asymp-onD*)

context begin

qualified lemma
  assumes *finite A* **and** $A \neq \{\}$ **and** *transp-on A R* **and** *totalp-on A R*
  shows
    bex-least-element: $\exists\, l \in A.\ \forall\, x \in A.\ x \neq l \longrightarrow R\ l\ x$ **and**
    bex-greatest-element: $\exists\, g \in A.\ \forall\, x \in A.\ x \neq g \longrightarrow R\ x\ g$
  unfolding *atomize-conj*
  using *assms*
proof (*induction A rule*: *finite-induct*)
  case *empty*
  hence *False* **by** *simp*
  thus *?case* **..**
next
  case (*insert a A′*)

  from *insert.prems*(*2*) **have** *transp-on-A′*: *transp-on A′ R*
    by (*auto intro*: *transp-onI dest*: *transp-onD*)

  from *insert.prems*(*3*) **have**
    totalp-on-a-A′-raw: $\forall\, y \in A'.\ a \neq y \longrightarrow R\ a\ y \vee R\ y\ a$ **and**
    totalp-on-A′: *totalp-on A′ R*
    by (*simp-all add*: *totalp-on-def*)

  show *?case*
  proof (*cases* $A' = \{\}$)
    case *True*

```
    thus ?thesis by simp
  next
    case False
    then obtain least greatest where
      least ∈ A′ and least-of-A′: ∀x∈A′. x ≠ least ⟶ R least x and
      greatest ∈ A′ and greatest-of-A′: ∀x∈A′. x ≠ greatest ⟶ R x greatest
      using insert.IH[OF - transp-on-A′ totalp-on-A′] by auto

    show ?thesis
    proof (rule conjI)
      show ∃l∈insert a A′. ∀x∈insert a A′. x ≠ l ⟶ R l x
      proof (cases R a least)
        case True
        show ?thesis
        proof (intro bexI ballI impI)
          show a ∈ insert a A′
            by simp
        next
          fix x
          show ⋀x. x ∈ insert a A′ ⟹ x ≠ a ⟹ R a x
            using True ‹least ∈ A′› least-of-A′
            using insert.prems(2)[THEN transp-onD, of a least]
            by auto
        qed
      next
        case False
        show ?thesis
        proof (intro bexI ballI impI)
          show least ∈ insert a A′
            using ‹least ∈ A′› by simp
        next
          fix x
          show x ∈ insert a A′ ⟹ x ≠ least ⟹ R least x
            using False ‹least ∈ A′› least-of-A′ totalp-on-a-A′-raw
            by (cases x = a) auto
        qed
      qed
    next
      show ∃g ∈ insert a A′. ∀x ∈ insert a A′. x ≠ g ⟶ R x g
      proof (cases R greatest a)
        case True
        show ?thesis
        proof (intro bexI ballI impI)
          show a ∈ insert a A′
            by simp
        next
          fix x
          show ⋀x. x ∈ insert a A′ ⟹ x ≠ a ⟹ R x a
            using True ‹greatest ∈ A′› greatest-of-A′
```

```
          using insert.prems(2)[THEN transp-onD, of - greatest a]
          by auto
        qed
      next
        case False
        show ?thesis
        proof (intro bexI ballI impI)
          show greatest ∈ insert a A′
            using ‹greatest ∈ A′› by simp
        next
          fix x
          show x ∈ insert a A′ ⟹ x ≠ greatest ⟹ R x greatest
            using False ‹greatest ∈ A′› greatest-of-A′ totalp-on-a-A′-raw
            by (cases x = a) auto
        qed
      qed
    qed
  qed
qed

end
```

20.8.1 Finite orders

```
context order
begin

lemma finite-has-maximal:
  assumes finite A and A ≠ {}
  shows ∃ m ∈ A. ∀ b ∈ A. m ≤ b ⟶ m = b
proof −
  obtain m where m ∈ A and m-is-max: ∀x∈A. x ≠ m ⟶ ¬ m < x
    using Finite-Set.bex-max-element[OF ‹finite A› - - ‹A ≠ {}›, of (<)] by auto
  moreover have ∀b ∈ A. m ≤ b ⟶ m = b
    using m-is-max by (auto simp: le-less)
  ultimately show ?thesis
    by auto
qed

lemma finite-has-maximal2:
  ⟦ finite A; a ∈ A ⟧ ⟹ ∃ m ∈ A. a ≤ m ∧ (∀ b ∈ A. m ≤ b ⟶ m = b)
using finite-has-maximal[of {b ∈ A. a ≤ b}] by fastforce

lemma finite-has-minimal:
  assumes finite A and A ≠ {}
  shows ∃ m ∈ A. ∀ b ∈ A. b ≤ m ⟶ m = b
proof −
  obtain m where m ∈ A and m-is-min: ∀x∈A. x ≠ m ⟶ ¬ x < m
    using Finite-Set.bex-min-element[OF ‹finite A› - - ‹A ≠ {}›, of (<)] by auto
```

moreover have $\forall\, b \in A.\ b \leq m \longrightarrow m = b$
using *m-is-min* **by** (*auto simp*: *le-less*)
ultimately show *?thesis*
by *auto*
qed

lemma *finite-has-minimal2*:
$\llbracket$ *finite A*; $a \in A$ $\rrbracket \Longrightarrow \exists\ m \in A.\ m \leq a \wedge (\forall\ b \in A.\ b \leq m \longrightarrow m = b)$
using *finite-has-minimal*[*of* $\{b \in A.\ b \leq a\}$] **by** *fastforce*

end

20.8.2 Relating injectivity and surjectivity

lemma *finite-surj-inj*:
assumes *finite A* $A \subseteq f$ ' A
shows *inj-on f A*
proof −
have f ' $A = A$
by (*rule card-seteq* [*THEN sym*]) (*auto simp add*: *assms card-image-le*)
then show *?thesis* **using** *assms*
by (*simp add*: *eq-card-imp-inj-on*)
qed

lemma *finite-UNIV-surj-inj*: *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *surj f* $\Longrightarrow$ *inj f*
for f :: $'a \Rightarrow 'a$
by (*blast intro*: *finite-surj-inj subset-UNIV*)

lemma *finite-UNIV-inj-surj*: *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *inj f* $\Longrightarrow$ *surj f*
for f :: $'a \Rightarrow 'a$
by (*fastforce simp*:*surj-def dest*!: *endo-inj-surj*)

lemma *surjective-iff-injective-gen*:
assumes *fS*: *finite S*
and *fT*: *finite T*
and *c*: *card S* = *card T*
and *ST*: f ' $S \subseteq T$
shows $(\forall\, y \in T.\ \exists\, x \in S.\ f\, x = y) \longleftrightarrow$ *inj-on f S*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *h*: *?lhs*
{
fix *x y*
assume *x*: $x \in S$
assume *y*: $y \in S$
assume *f*: $f\, x = f\, y$
from *x fS* **have** *S0*: *card S* $\neq$ *0*
by *auto*
have $x = y$

```
    proof (rule ccontr)
      assume xy: ¬ ?thesis
      have th: card S ≤ card (f ‘ (S − {y}))
        unfolding c
      proof (rule card-mono)
        show finite (f ‘ (S − {y}))
          by (simp add: fS)
        have ⟦x ≠ y; x ∈ S; z ∈ S; f x = f y⟧
         ⟹ ∃ x ∈ S. x ≠ y ∧ f z = f x for z
          by (cases z = y ⟶ z = x) auto
        then show T ⊆ f ‘ (S − {y})
          using h xy x y f by fastforce
      qed
      also have … ≤ card (S − {y})
        by (simp add: card-image-le fS)
      also have … ≤ card S − 1 using y fS by simp
      finally show False using S0 by arith
    qed
  }
  then show ?rhs
    unfolding inj-on-def by blast
next
  assume h: ?rhs
  have f ‘ S = T
    by (simp add: ST c card-image card-subset-eq fT h)
  then show ?lhs by blast
qed

hide-const (open) Finite-Set.fold
```

20.9 Infinite Sets

Some elementary facts about infinite sets, mostly by Stephan Merz. Beware! Because "infinite" merely abbreviates a negation, these lemmas may not work well with *blast*.

```
abbreviation infinite :: 'a set ⇒ bool
  where infinite S ≡ ¬ finite S
```

Infinite sets are non-empty, and if we remove some elements from an infinite set, the result is still infinite.

```
lemma infinite-UNIV-nat [iff]: infinite (UNIV :: nat set)
proof
  assume finite (UNIV :: nat set)
  with finite-UNIV-inj-surj [of Suc] show False
    by simp (blast dest: Suc-neq-Zero surjD)
qed

lemma infinite-UNIV-char-0: infinite (UNIV :: 'a::semiring-char-0 set)
```

proof
  assume *finite* (*UNIV* :: *'a set*)
  with *subset-UNIV* **have** *finite* (*range of-nat* :: *'a set*)
    by (*rule finite-subset*)
  moreover have *inj* (*of-nat* :: *nat* ⇒ *'a*)
    by (*simp add*: *inj-on-def*)
  ultimately have *finite* (*UNIV* :: *nat set*)
    by (*rule finite-imageD*)
  then show *False*
    by *simp*
qed

lemma *infinite-imp-nonempty*: *infinite S* ⟹ *S* ≠ {}
  by *auto*

lemma *infinite-remove*: *infinite S* ⟹ *infinite* (*S* − {*a*})
  by *simp*

lemma *Diff-infinite-finite*:
  assumes *finite T infinite S*
  shows *infinite* (*S* − *T*)
  using ‹*finite T*›
proof *induct*
  from ‹*infinite S*› **show** *infinite* (*S* − {})
    by *auto*
next
  fix *T x*
  assume *ih*: *infinite* (*S* − *T*)
  have *S* − (*insert x T*) = (*S* − *T*) − {*x*}
    by (*rule Diff-insert*)
  with *ih* **show** *infinite* (*S* − (*insert x T*))
    by (*simp add*: *infinite-remove*)
qed

lemma *Un-infinite*: *infinite S* ⟹ *infinite* (*S* ∪ *T*)
  by *simp*

lemma *infinite-Un*: *infinite* (*S* ∪ *T*) ⟷ *infinite S* ∨ *infinite T*
  by *simp*

lemma *infinite-super*:
  assumes *S* ⊆ *T*
    and *infinite S*
  shows *infinite T*
proof
  assume *finite T*
  with ‹*S* ⊆ *T*› **have** *finite S* **by** (*simp add*: *finite-subset*)
  with ‹*infinite S*› **show** *False* **by** *simp*
qed

proposition *infinite-coinduct* [*consumes 1*, *case-names infinite*]:
  assumes *X A*
    and *step*: $\bigwedge A.\ X\ A \Longrightarrow \exists x \in A.\ X\ (A - \{x\}) \vee \textit{infinite}\ (A - \{x\})$
  shows *infinite A*
proof
  assume *finite A*
  then show *False*
    using ‹*X A*›
  proof (*induction rule*: *finite-psubset-induct*)
    case (*psubset A*)
    then obtain *x* **where** $x \in A\ X\ (A - \{x\}) \vee \textit{infinite}\ (A - \{x\})$
      using *local.step psubset.prems* **by** *blast*
    then have $X\ (A - \{x\})$
      using *psubset.hyps* **by** *blast*
    show *False*
    proof (*rule psubset.IH* [**where** $B = A - \{x\}$])
      show $A - \{x\} \subset A$
        using ‹$x \in A$› **by** *blast*
    qed *fact*
  qed
qed

For any function with infinite domain and finite range there is some element that is the image of infinitely many domain elements. In particular, any infinite sequence of elements from a finite set contains some element that occurs infinitely often.

lemma *inf-img-fin-dom′*:
  assumes *img*: *finite* (*f* ‘ *A*)
    and *dom*: *infinite A*
  shows $\exists y \in f\ `\ A.\ \textit{infinite}\ (f\ -`\ \{y\} \cap A)$
proof (*rule ccontr*)
  have $A \subseteq (\bigcup y \in f\ `\ A.\ f\ -`\ \{y\} \cap A)$ **by** *auto*
  moreover assume ¬ *?thesis*
  with *img* **have** *finite* $(\bigcup y \in f\ `\ A.\ f\ -`\ \{y\} \cap A)$ **by** *blast*
  ultimately have *finite A* **by** (*rule finite-subset*)
  with *dom* **show** *False* **by** *contradiction*
qed

lemma *inf-img-fin-domE′*:
  assumes *finite* (*f* ‘ *A*) **and** *infinite A*
  obtains *y* **where** $y \in f`A$ **and** *infinite* $(f\ -`\ \{y\} \cap A)$
  using *assms* **by** (*blast dest*: *inf-img-fin-dom′*)

lemma *inf-img-fin-dom*:
  assumes *img*: *finite* (*f*‘*A*) **and** *dom*: *infinite A*
  shows $\exists y \in f`A.\ \textit{infinite}\ (f\ -`\ \{y\})$
  using *inf-img-fin-dom′*[*OF assms*] **by** *auto*

lemma *inf-img-fin-domE*:
  assumes *finite* (*f*'*A*) **and** *infinite A*
  obtains *y* **where** *y* ∈ *f*'*A* **and** *infinite* (*f* −' {*y*})
  using *assms* **by** (*blast dest*: *inf-img-fin-dom*)

proposition *finite-image-absD*: *finite* (*abs* ' *S*) ⟹ *finite S*
  for *S* :: '*a*::*linordered-ring set*
  by (*rule ccontr*) (*auto simp*: *abs-eq-iff vimage-def dest*: *inf-img-fin-dom*)

20.10 The finite powerset operator

definition *Fpow* :: '*a set* ⇒ '*a set set*
where *Fpow A* ≡ {*X*. *X* ⊆ *A* ∧ *finite X*}

lemma *Fpow-mono*: *A* ⊆ *B* ⟹ *Fpow A* ⊆ *Fpow B*
unfolding *Fpow-def* **by** *auto*

lemma *empty-in-Fpow*: {} ∈ *Fpow A*
unfolding *Fpow-def* **by** *auto*

lemma *Fpow-not-empty*: *Fpow A* ≠ {}
using *empty-in-Fpow* **by** *blast*

lemma *Fpow-subset-Pow*: *Fpow A* ⊆ *Pow A*
unfolding *Fpow-def* **by** *auto*

lemma *Fpow-Pow-finite*: *Fpow A* = *Pow A Int* {*A*. *finite A*}
unfolding *Fpow-def Pow-def* **by** *blast*

lemma *inj-on-image-Fpow*:
  assumes *inj-on f A*
  shows *inj-on* (*image f*) (*Fpow A*)
  using *assms Fpow-subset-Pow*[*of A*] *subset-inj-on*[*of image f Pow A*]
    inj-on-image-Pow **by** *blast*

lemma *image-Fpow-mono*:
  assumes *f* ' *A* ⊆ *B*
  shows (*image f*) ' (*Fpow A*) ⊆ *Fpow B*
  using *assms* **by**(*unfold Fpow-def*, *auto*)

end

21 Reflexive and Transitive closure of a relation

theory *Transitive-Closure*
  imports *Finite-Set*
  abbrevs ^* = * **
    and ^+ = $^+$ $^{++}$
    and ^= = $^=$ $^{==}$

begin

ML-file ‹~~/src/Provers/trancl.ML›

rtrancl is reflexive/transitive closure, *trancl* is transitive closure, *reflcl* is reflexive closure.

These postfix operators have *maximum priority*, forcing their operands to be atomic.

context notes [[*inductive-internals*]]
begin

inductive-set *rtrancl* :: ('a × 'a) set ⇒ ('a × 'a) set (‹(‹notation=‹postfix ∗››-*)› [1000] 999)
  for r :: ('a × 'a) set
  where
    rtrancl-refl [*intro*!, *Pure.intro*!, *simp*]: (a, a) ∈ r*
  | *rtrancl-into-rtrancl* [*Pure.intro*]: (a, b) ∈ r* ⟹ (b, c) ∈ r ⟹ (a, c) ∈ r*

inductive-set *trancl* :: ('a × 'a) set ⇒ ('a × 'a) set (‹(‹notation=‹postfix +››-+)› [1000] 999)
  for r :: ('a × 'a) set
  where
    r-into-trancl [*intro*, *Pure.intro*]: (a, b) ∈ r ⟹ (a, b) ∈ r+
  | *trancl-into-trancl* [*Pure.intro*]: (a, b) ∈ r+ ⟹ (b, c) ∈ r ⟹ (a, c) ∈ r+

notation
  rtranclp (‹(‹notation=‹postfix ∗∗››-**)› [1000] 1000) **and**
  tranclp (‹(‹notation=‹postfix ++››-++)› [1000] 1000)

declare
  rtrancl-def [*nitpick-unfold del*]
  rtranclp-def [*nitpick-unfold del*]
  trancl-def [*nitpick-unfold del*]
  tranclp-def [*nitpick-unfold del*]

end

abbreviation *reflcl* :: ('a × 'a) set ⇒ ('a × 'a) set (‹(‹notation=‹postfix =››-=)› [1000] 999)
  where r= ≡ r ∪ Id

abbreviation *reflclp* :: ('a ⇒ 'a ⇒ bool) ⇒ 'a ⇒ 'a ⇒ bool (‹(‹notation=‹postfix ==››-==)› [1000] 1000)
  where r== ≡ sup r (=)

notation (*ASCII*)
  rtrancl (‹(‹notation=‹postfix ∗››-^*)› [1000] 999) **and**
  trancl (‹(‹notation=‹postfix +››-^+)› [1000] 999) **and**
  reflcl (‹(‹notation=‹postfix =››-^=)› [1000] 999) **and**

rtranclp (‹(‹notation=‹postfix ∗∗››-^∗∗)› [1000] 1000) **and**
tranclp (‹(‹notation=‹postfix ++››-^++)› [1000] 1000) **and**
reflclp (‹(‹notation=‹postfix ==››-^==)› [1000] 1000)

bundle *rtrancl-syntax*
begin
notation
rtrancl (‹(‹notation=‹postfix ∗››-*)› [1000] 999) **and**
rtranclp (‹(‹notation=‹postfix ∗∗››-**)› [1000] 1000)
notation (*ASCII*)
rtrancl (‹(‹notation=‹postfix ∗››-^∗)› [1000] 999) **and**
rtranclp (‹(‹notation=‹postfix ∗∗››-^∗∗)› [1000] 1000)
end

bundle *trancl-syntax*
begin
notation
trancl (‹(‹notation=‹postfix +››-$^+$)› [1000] 999) **and**
tranclp (‹(‹notation=‹postfix ++››-$^{++}$)› [1000] 1000)
notation (*ASCII*)
trancl (‹(‹notation=‹postfix +››-^+)› [1000] 999) **and**
tranclp (‹(‹notation=‹postfix ++››-^++)› [1000] 1000)
end

bundle *reflcl-syntax*
begin
notation
reflcl (‹(‹notation=‹postfix =››-$^=$)› [1000] 999) **and**
reflclp (‹(‹notation=‹postfix ==››-$^{==}$)› [1000] 1000)
notation (*ASCII*)
reflcl (‹(‹notation=‹postfix =››-^=)› [1000] 999) **and**
reflclp (‹(‹notation=‹postfix ==››-^==)› [1000] 1000)
end

21.1 Reflexive closure

lemma *reflcl-set-eq* [*pred-set-conv*]: (*sup* (λ*x y*. (*x*, *y*) ∈ *r*) (=)) = (λ*x y*. (*x*, *y*) ∈ *r* ∪ *Id*)
by (*auto simp*: *fun-eq-iff*)

lemma *refl-reflcl*[*simp*]: *refl* ($r^=$)
by (*simp add*: *refl-on-def*)

lemma *reflp-on-reflclp*[*simp*]: *reflp-on A* $R^{==}$
by (*simp add*: *reflp-on-def*)

lemma *antisym-on-reflcl*[*simp*]: *antisym-on A* ($r^=$) ⟷ *antisym-on A r*
by (*simp add*: *antisym-on-def*)

lemma *antisymp-on-reflclp*[*simp*]: *antisymp-on A* $R^{==}$ $\longleftrightarrow$ *antisymp-on A R*
by (*rule antisym-on-reflcl*[*to-pred*])

lemma *trans-on-reflcl*[*simp*]: *trans-on A r* $\Longrightarrow$ *trans-on A* ($r^{=}$)
by (*auto intro*: *trans-onI dest*: *trans-onD*)

lemma *transp-on-reflclp*[*simp*]: *transp-on A R* $\Longrightarrow$ *transp-on A* $R^{==}$
by (*rule trans-on-reflcl*[*to-pred*])

lemma *antisymp-on-reflclp-if-asymp-on*:
assumes *asymp-on A R*
shows *antisymp-on A* $R^{==}$
unfolding *antisymp-on-reflclp*
using *antisymp-on-if-asymp-on*[*OF* ‹*asymp-on A R*›] **.**

lemma *antisym-on-reflcl-if-asym-on*: *asym-on A R* $\Longrightarrow$ *antisym-on A* ($R^{=}$)
using *antisymp-on-reflclp-if-asymp-on*[*to-set*] **.**

lemma *reflclp-idemp* [*simp*]: $(P^{==})^{==} = P^{==}$
by *blast*

lemma *reflclp-ident-if-reflp*[*simp*]: *reflp R* $\Longrightarrow$ $R^{==} = R$
by (*auto dest*: *reflpD*)

The following are special cases of *reflclp-ident-if-reflp*, but they appear duplicated in multiple, independent theories, which causes name clashes.

lemma (**in** *preorder*) *reflclp-less-eq*[*simp*]: $(\leq)^{==} = (\leq)$
using *reflp-on-le* **by** (*simp only*: *reflclp-ident-if-reflp*)

lemma (**in** *preorder*) *reflclp-greater-eq*[*simp*]: $(\geq)^{==} = (\geq)$
using *reflp-on-ge* **by** (*simp only*: *reflclp-ident-if-reflp*)

lemma *order-reflclp-if-transp-and-asymp*:
assumes *transp R* **and** *asymp R*
shows *class.order* $R^{==}$ *R*
proof *unfold-locales*
show $\bigwedge x\ y.\ R\ x\ y = (R^{==}\ x\ y \wedge \neg\ R^{==}\ y\ x)$
using ‹*asymp R*› *asympD* **by** *fastforce*
next
show $\bigwedge x.\ R^{==}\ x\ x$
by *simp*
next
show $\bigwedge x\ y\ z.\ R^{==}\ x\ y \Longrightarrow R^{==}\ y\ z \Longrightarrow R^{==}\ x\ z$
using *transp-on-reflclp*[*OF* ‹*transp R*›, *THEN transpD*] **.**
next
show $\bigwedge x\ y.\ R^{==}\ x\ y \Longrightarrow R^{==}\ y\ x \Longrightarrow x = y$
using *antisymp-on-reflclp-if-asymp-on*[*OF* ‹*asymp R*›, *THEN antisympD*] **.**
qed

21.2 Reflexive-transitive closure

lemma *r-into-rtrancl* [*intro*]: $\bigwedge p.\ p \in r \Longrightarrow p \in r^*$
— *rtrancl* of r contains r
by (*simp add*: *split-tupled-all rtrancl-refl* [*THEN rtrancl-into-rtrancl*])

lemma *r-into-rtranclp* [*intro*]: $r\ x\ y \Longrightarrow r^{**}\ x\ y$
— *rtrancl* of r contains r
by (*erule rtranclp.rtrancl-refl* [*THEN rtranclp.rtrancl-into-rtrancl*])

lemma *rtranclp-mono*: $r \leq s \Longrightarrow r^{**} \leq s^{**}$
— monotonicity of *rtrancl*
proof (*rule predicate2I*)
show $s^{**}\ x\ y$ **if** $r \leq s\ r^{**}\ x\ y$ **for** $x\ y$
using ‹$r^{**}\ x\ y$› ‹$r \leq s$›
by (*induction rule*: *rtranclp.induct*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+
qed

lemma *mono-rtranclp*[*mono*]: $(\bigwedge a\ b.\ x\ a\ b \longrightarrow y\ a\ b) \Longrightarrow x^{**}\ a\ b \longrightarrow y^{**}\ a\ b$
using *rtranclp-mono*[*of x y*] **by** *auto*

lemmas *rtrancl-mono* = *rtranclp-mono* [*to-set*]

theorem *rtranclp-induct* [*consumes 1*, *case-names base step*, *induct set*: *rtranclp*]:
assumes *a*: $r^{**}\ a\ b$
and *cases*: $P\ a\ \bigwedge y\ z.\ r^{**}\ a\ y \Longrightarrow r\ y\ z \Longrightarrow P\ y \Longrightarrow P\ z$
shows $P\ b$
using *a* **by** (*induct x≡a b*) (*rule cases*)+

lemmas *rtrancl-induct* [*induct set*: *rtrancl*] = *rtranclp-induct* [*to-set*]

lemmas *rtranclp-induct2* =
rtranclp-induct[*of - (ax,ay) (bx,by)*, *split-rule*, *consumes 1*, *case-names refl step*]

lemmas *rtrancl-induct2* =
rtrancl-induct[*of (ax,ay) (bx,by)*, *split-format (complete)*, *consumes 1*, *case-names refl step*]

lemma *refl-rtrancl*: *refl* (r^*)
unfolding *refl-on-def* **by** *fast*

Transitivity of transitive closure.

lemma *trans-rtrancl*: *trans* (r^*)
proof (*rule transI*)
fix $x\ y\ z$
assume $(x, y) \in r^*$
assume $(y, z) \in r^*$
then show $(x, z) \in r^*$
proof *induct*
case *base*

    show $(x, y) \in r^*$ **by** *fact*
  next
    case (*step u v*)
    from ‹$(x, u) \in r^*$› **and** ‹$(u, v) \in r$›
    show $(x, v) \in r^*$ **..**
  qed
qed

lemmas *rtrancl-trans* = *trans-rtrancl* [*THEN transD*]

lemma *rtranclp-trans*:
  assumes r^{**} *x y*
    and r^{**} *y z*
  shows r^{**} *x z*
  using *assms*(*2*,*1*) **by** *induct iprover+*

lemma *rtranclE* [*cases set*: *rtrancl*]:
  fixes *a b* :: *'a*
  assumes *major*: $(a, b) \in r^*$
  obtains
    (*base*) $a = b$
  | (*step*) *y* **where** $(a, y) \in r^*$ **and** $(y, b) \in r$
  — elimination of *rtrancl* – by induction on a special formula
proof −
  have $a = b \vee (\exists y. (a, y) \in r^* \wedge (y, b) \in r)$
    by (*rule major* [*THEN rtrancl-induct*]) *blast+*
  then show *?thesis*
    by (*auto intro*: *base step*)
qed

lemma *rtrancl-Int-subset*: $Id \subseteq s \Longrightarrow (r^* \cap s)\ O\ r \subseteq s \Longrightarrow r^* \subseteq s$
  by (*fastforce elim*: *rtrancl-induct*)

lemma *converse-rtranclp-into-rtranclp*: $r\ a\ b \Longrightarrow r^{**}\ b\ c \Longrightarrow r^{**}\ a\ c$
  by (*rule rtranclp-trans*) *iprover+*

lemmas *converse-rtrancl-into-rtrancl* = *converse-rtranclp-into-rtranclp* [*to-set*]

More r^* equations and inclusions.

lemma *rtranclp-idemp* [*simp*]: $(r^{**})^{**} = r^{**}$
proof −
  have $r^{****}\ x\ y \Longrightarrow r^{**}\ x\ y$ **for** *x y*
    by (*induction rule*: *rtranclp-induct*) (*blast intro*: *rtranclp-trans*)+
  then show *?thesis*
    by (*auto intro*!: *order-antisym*)
qed

lemmas *rtrancl-idemp* [*simp*] = *rtranclp-idemp* [*to-set*]

lemma *rtrancl-idemp-self-comp* [*simp*]: $R^* \ O \ R^* = R^*$
by (*force intro*: *rtrancl-trans*)

lemma *rtrancl-subset-rtrancl*: $r \subseteq s^* \Longrightarrow r^* \subseteq s^*$
by (*drule rtrancl-mono*, *simp*)

lemma *rtranclp-subset*: $R \leq S \Longrightarrow S \leq R^{**} \Longrightarrow S^{**} = R^{**}$
by (*fastforce dest*: *rtranclp-mono*)

lemmas *rtrancl-subset* = *rtranclp-subset* [*to-set*]

lemma *rtranclp-sup-rtranclp*: $(sup\ (R^{**})\ (S^{**}))^{**} = (sup\ R\ S)^{**}$
by (*blast intro*!: *rtranclp-subset intro*: *rtranclp-mono* [*THEN predicate2D*])

lemmas *rtrancl-Un-rtrancl* = *rtranclp-sup-rtranclp* [*to-set*]

lemma *rtranclp-reflclp* [*simp*]: $(R^{==})^{**} = R^{**}$
by (*blast intro*!: *rtranclp-subset*)

lemmas *rtrancl-reflcl* [*simp*] = *rtranclp-reflclp* [*to-set*]

lemma *rtrancl-r-diff-Id*: $(r - Id)^* = r^*$
by (*rule rtrancl-subset* [*symmetric*]) *auto*

lemma *rtranclp-r-diff-Id*: $(inf\ r\ (\neq))^{**} = r^{**}$
by (*rule rtranclp-subset* [*symmetric*]) *auto*

theorem *rtranclp-converseD*:
assumes $(r^{-1-1})^{**}\ x\ y$
shows $r^{**}\ y\ x$
using *assms* **by** *induct* (*iprover intro*: *rtranclp-trans dest*!: *conversepD*)+

lemmas *rtrancl-converseD* = *rtranclp-converseD* [*to-set*]

theorem *rtranclp-converseI*:
assumes $r^{**}\ y\ x$
shows $(r^{-1-1})^{**}\ x\ y$
using *assms* **by** *induct* (*iprover intro*: *rtranclp-trans conversepI*)+

lemmas *rtrancl-converseI* = *rtranclp-converseI* [*to-set*]

lemma *rtrancl-converse*: $(r^{-1})^* = (r^*)^{-1}$
by (*fast dest*!: *rtrancl-converseD intro*!: *rtrancl-converseI*)

lemma *sym-rtrancl*: $sym\ r \Longrightarrow sym\ (r^*)$
by (*simp only*: *sym-conv-converse-eq rtrancl-converse* [*symmetric*])

theorem *converse-rtranclp-induct* [*consumes 1*, *case-names base step*]:
assumes *major*: $r^{**}\ a\ b$

and *cases*: $P\ b\ \bigwedge y\ z.\ r\ y\ z \Longrightarrow r^{**}\ z\ b \Longrightarrow P\ z \Longrightarrow P\ y$
shows $P\ a$
using *rtranclp-converseI* [*OF major*]
by *induct* (*iprover intro*: *cases dest*!: *conversepD rtranclp-converseD*)+

lemmas *converse-rtrancl-induct* = *converse-rtranclp-induct* [*to-set*]

lemmas *converse-rtranclp-induct2* =
converse-rtranclp-induct [*of* - (*ax*, *ay*) (*bx*, *by*), *split-rule*, *consumes 1*, *case-names refl step*]

lemmas *converse-rtrancl-induct2* =
converse-rtrancl-induct [*of* (*ax*, *ay*) (*bx*, *by*), *split-format* (*complete*),
consumes 1, *case-names refl step*]

lemma *converse-rtranclpE* [*consumes 1*, *case-names base step*]:
assumes *major*: $r^{**}\ x\ z$
and *cases*: $x = z \Longrightarrow P\ \bigwedge y.\ r\ x\ y \Longrightarrow r^{**}\ y\ z \Longrightarrow P$
shows P
proof −
have $x = z \lor (\exists\, y.\ r\ x\ y \land r^{**}\ y\ z)$
by (*rule major* [*THEN converse-rtranclp-induct*]) *iprover*+
then show *?thesis*
by (*auto intro*: *cases*)
qed

lemmas *converse-rtranclE* = *converse-rtranclpE* [*to-set*]

lemmas *converse-rtranclpE2* = *converse-rtranclpE* [*of* - (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemmas *converse-rtranclE2* = *converse-rtranclE* [*of* (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemma *r-comp-rtrancl-eq*: $r\ O\ r^* = r^*\ O\ r$
by (*blast elim*: *rtranclE converse-rtranclE*
intro: *rtrancl-into-rtrancl converse-rtrancl-into-rtrancl*)

lemma *rtrancl-unfold*: $r^* = Id \cup r^*\ O\ r$
by (*auto intro*: *rtrancl-into-rtrancl elim*: *rtranclE*)

lemma *rtrancl-Un-separatorE*:
$(a,\ b) \in (P \cup Q)^* \Longrightarrow \forall\, x\ y.\ (a,\ x) \in P^* \longrightarrow (x,\ y) \in Q \longrightarrow x = y \Longrightarrow (a,\ b) \in P^*$
proof (*induct rule*: *rtrancl.induct*)
case *rtrancl-refl*
then show *?case* **by** *blast*
next
case *rtrancl-into-rtrancl*
then show *?case* **by** (*blast intro*: *rtrancl-trans*)
qed

lemma *rtrancl-Un-separator-converseE*:
  $(a, b) \in (P \cup Q)^* \Longrightarrow \forall x\, y.\ (x, b) \in P^* \longrightarrow (y, x) \in Q \longrightarrow y = x \Longrightarrow (a, b) \in P^*$
proof (*induct rule*: *converse-rtrancl-induct*)
  case *base*
  then show *?case* **by** *blast*
next
  case *step*
  then show *?case* **by** (*blast intro*: *rtrancl-trans*)
qed

lemma *Image-closed-trancl*:
  assumes r `` $X \subseteq X$
  shows r^* `` $X = X$
proof −
  from *assms* **have** ∗∗: $\{y.\ \exists x{\in}X.\ (x, y) \in r\} \subseteq X$
    by *auto*
  have $x \in X$ **if** *1*: $(y, x) \in r^*$ **and** *2*: $y \in X$ **for** *x y*
  proof −
    from *1* **show** $x \in X$
    proof *induct*
      case *base*
      show *?case* **by** (*fact 2*)
    next
      case *step*
      with ∗∗ **show** *?case* **by** *auto*
    qed
  qed
  then show *?thesis* **by** *auto*
qed

lemma *rtranclp-ident-if-reflp-and-transp*:
  assumes *reflp R* **and** *transp R*
  shows $R^{**} = R$
proof (*intro ext iffI*)
  fix *x y*
  show $R^{**}\ x\ y \Longrightarrow R\ x\ y$
  proof (*induction y rule*: *rtranclp-induct*)
    case *base*
    show *?case*
      using ‹*reflp R*›[*THEN reflpD*] **.**
  next
    case (*step y z*)
    thus *?case*
      using ‹*transp R*›[*THEN transpD*, *of x y z*] **by** *simp*
  qed
next
  fix *x y*

show $R\ x\ y \Longrightarrow R^{**}\ x\ y$
  using *r-into-rtranclp* **.**
qed

The following are special cases of *rtranclp-ident-if-reflp-and-transp*, but they appear duplicated in multiple, independent theories, which causes name clashes.

lemma (**in** *preorder*) *rtranclp-less-eq*[*simp*]: $(\leq)^{**} = (\leq)$
  using *reflp-on-le transp-on-le* **by** (*simp only*: *rtranclp-ident-if-reflp-and-transp*)

lemma (**in** *preorder*) *rtranclp-greater-eq*[*simp*]: $(\geq)^{**} = (\geq)$
  using *reflp-on-ge transp-on-ge* **by** (*simp only*: *rtranclp-ident-if-reflp-and-transp*)

21.3 Transitive closure

lemma *totalp-on-tranclp*: *totalp-on* $A\ R \Longrightarrow$ *totalp-on* A (*tranclp* R)
  by (*auto intro*: *totalp-onI dest*: *totalp-onD*)

lemma *total-on-trancl*: *total-on* $A\ r \Longrightarrow$ *total-on* A (*trancl* r)
  by (*rule totalp-on-tranclp*[*to-set*])

lemma *trancl-mono*:
  assumes $p \in r^+\ r \subseteq s$
  shows $p \in s^+$
proof −
  have $[\![(a,\ b) \in r^+;\ r \subseteq s]\!] \Longrightarrow (a,\ b) \in s^+$ **for** $a\ b$
    by (*induction rule*: *trancl.induct*) (*iprover dest*: *subsetD*)+
  with *assms* **show** *?thesis*
    by (*cases p*) *force*
qed

lemma *r-into-trancl'*: $\bigwedge p.\ p \in r \Longrightarrow p \in r^+$
  by (*simp only*: *split-tupled-all*) (*erule r-into-trancl*)

Conversions between *trancl* and *rtrancl*.

lemma *tranclp-into-rtranclp*: $r^{++}\ a\ b \Longrightarrow r^{**}\ a\ b$
  by (*erule tranclp.induct*) *iprover*+

lemmas *trancl-into-rtrancl* = *tranclp-into-rtranclp* [*to-set*]

lemma *rtranclp-into-tranclp1*:
  assumes $r^{**}\ a\ b$
  shows $r\ b\ c \Longrightarrow r^{++}\ a\ c$
  using *assms* **by** (*induct arbitrary*: *c*) *iprover*+

lemmas *rtrancl-into-trancl1* = *rtranclp-into-tranclp1* [*to-set*]

lemma *rtranclp-into-tranclp2*:

assumes $r\ a\ b\ r^{**}\ b\ c$ **shows** $r^{++}\ a\ c$
— intro rule from r and *rtrancl*
using ‹$r^{**}\ b\ c$›
proof (*cases rule*: *rtranclp.cases*)
case *rtrancl-refl*
with *assms* **show** *?thesis*
by *iprover*
next
case *rtrancl-into-rtrancl*
with *assms* **show** *?thesis*
by (*auto intro*: *rtranclp-trans* [*THEN rtranclp-into-tranclp1*])
qed

lemmas *rtrancl-into-trancl2* = *rtranclp-into-tranclp2* [*to-set*]

Nice induction rule for *trancl*

lemma *tranclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *tranclp*]:
assumes *a*: $r^{++}\ a\ b$
and *cases*: $\bigwedge y.\ r\ a\ y \Longrightarrow P\ y$ $\bigwedge y\ z.\ r^{++}\ a\ y \Longrightarrow r\ y\ z \Longrightarrow P\ y \Longrightarrow P\ z$
shows $P\ b$
using *a* **by** (*induct x≡a b*) (*iprover intro*: *cases*)+

lemmas *trancl-induct* [*induct set*: *trancl*] = *tranclp-induct* [*to-set*]

lemmas *tranclp-induct2* =
tranclp-induct [*of - (ax, ay) (bx, by)*, *split-rule*, *consumes 1*, *case-names base step*]

lemmas *trancl-induct2* =
trancl-induct [*of (ax, ay) (bx, by)*, *split-format (complete)*,
consumes 1, *case-names base step*]

lemma *tranclp-trans-induct*:
assumes *major*: $r^{++}\ x\ y$
and *cases*: $\bigwedge x\ y.\ r\ x\ y \Longrightarrow P\ x\ y$ $\bigwedge x\ y\ z.\ r^{++}\ x\ y \Longrightarrow P\ x\ y \Longrightarrow r^{++}\ y\ z \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$
shows $P\ x\ y$
— Another induction rule for trancl, incorporating transitivity
by (*iprover intro*: *major* [*THEN tranclp-induct*] *cases*)

lemmas *trancl-trans-induct* = *tranclp-trans-induct* [*to-set*]

lemma *tranclE* [*cases set*: *trancl*]:
assumes $(a, b) \in r^{+}$
obtains
(*base*) $(a, b) \in r$
| (*step*) c **where** $(a, c) \in r^{+}$ **and** $(c, b) \in r$
using *assms* **by** *cases simp-all*

lemma *trancl-Int-subset*: $r \subseteq s \Longrightarrow (r^+ \cap s)\ O\ r \subseteq s \Longrightarrow r^+ \subseteq s$
  by (*fastforce simp add*: *elim*: *trancl-induct*)

lemma *trancl-unfold*: $r^+ = r \cup r^+\ O\ r$
  by (*auto intro*: *trancl-into-trancl elim*: *tranclE*)

Transitivity of r^+

lemma *trans-trancl* [*simp*]: *trans* (r^+)
proof (*rule transI*)
  fix *x y z*
  assume $(x, y) \in r^+$
  assume $(y, z) \in r^+$
  then show $(x, z) \in r^+$
  proof *induct*
    case (*base u*)
    from ‹$(x, y) \in r^+$› **and** ‹$(y, u) \in r$›
    show $(x, u) \in r^+$ **..**
  next
    case (*step u v*)
    from ‹$(x, u) \in r^+$› **and** ‹$(u, v) \in r$›
    show $(x, v) \in r^+$ **..**
  qed
qed

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*]

lemma *tranclp-trans*:
  assumes $r^{++}\ x\ y$
    and $r^{++}\ y\ z$
  shows $r^{++}\ x\ z$
  using *assms*(*2*,*1*) **by** *induct iprover*+

lemma *trancl-id* [*simp*]: *trans* $r \Longrightarrow r^+ = r$
  unfolding *trans-def* **by** (*fastforce simp add*: *elim*: *trancl-induct*)

lemma *rtranclp-tranclp-tranclp*:
  assumes $r^{**}\ x\ y$
  shows $\bigwedge z.\ r^{++}\ y\ z \Longrightarrow r^{++}\ x\ z$
  using *assms* **by** *induct* (*iprover intro*: *tranclp-trans*)+

lemmas *rtrancl-trancl-trancl* = *rtranclp-tranclp-tranclp* [*to-set*]

lemma *tranclp-into-tranclp2*: $r\ a\ b \Longrightarrow r^{++}\ b\ c \Longrightarrow r^{++}\ a\ c$
  by (*erule tranclp-trans* [*OF tranclp.r-into-trancl*])

lemmas *trancl-into-trancl2* = *tranclp-into-tranclp2* [*to-set*]

lemma *tranclp-converseI*:
  assumes $(r^{++})^{-1-1}\ x\ y$ **shows** $(r^{-1-1})^{++}\ x\ y$

using *conversepD* [*OF assms*]
proof (*induction rule*: *tranclp-induct*)
case (*base y*)
then show *?case*
by (*iprover intro*: *conversepI*)
next
case (*step y z*)
then show *?case*
by (*iprover intro*: *conversepI tranclp-trans*)
qed

lemmas *trancl-converseI* = *tranclp-converseI* [*to-set*]

lemma *tranclp-converseD*:
assumes $(r^{-1-1})^{++}$ x y **shows** $(r^{++})^{-1-1}$ x y
proof −
have r^{++} y x
using *assms*
by (*induction rule*: *tranclp-induct*) (*iprover dest*: *conversepD intro*: *tranclp-trans*)+
then show *?thesis*
by (*rule conversepI*)
qed

lemmas *trancl-converseD* = *tranclp-converseD* [*to-set*]

lemma *tranclp-converse*: $(r^{-1-1})^{++} = (r^{++})^{-1-1}$
by (*fastforce simp add*: *fun-eq-iff intro*!: *tranclp-converseI dest*!: *tranclp-converseD*)

lemmas *trancl-converse* = *tranclp-converse* [*to-set*]

lemma *sym-trancl*: *sym r* $\Longrightarrow$ *sym* (r^{+})
by (*simp only*: *sym-conv-converse-eq trancl-converse* [*symmetric*])

lemma *converse-tranclp-induct* [*consumes 1*, *case-names base step*]:
assumes *major*: r^{++} a b
and *cases*: $\bigwedge y.\ r\ y\ b \Longrightarrow P\ y$ $\bigwedge y\ z.\ r\ y\ z \Longrightarrow r^{++}\ z\ b \Longrightarrow P\ z \Longrightarrow P\ y$
shows P a
proof −
have r^{-1-1++} b a
by (*intro tranclp-converseI conversepI major*)
then show *?thesis*
by (*induction rule*: *tranclp-induct*) (*blast intro*: *cases dest*: *tranclp-converseD*)+
qed

lemmas *converse-trancl-induct* = *converse-tranclp-induct* [*to-set*]

lemma *tranclpD*: $R^{++}\ x\ y \Longrightarrow \exists z.\ R\ x\ z \wedge R^{**}\ z\ y$
proof (*induction rule*: *converse-tranclp-induct*)
case (*step u v*)

then show *?case*
by (*blast intro*: *rtranclp-trans*)
qed *auto*

lemmas *tranclD* = *tranclpD* [*to-set*]

lemma *converse-tranclpE*:
assumes *major*: *tranclp r x z*
and *base*: $r\ x\ z \Longrightarrow P$
and *step*: $\bigwedge y.\ r\ x\ y \Longrightarrow tranclp\ r\ y\ z \Longrightarrow P$
shows *P*
proof –
from *tranclpD* [*OF major*] **obtain** *y* **where** *r x y* **and** *rtranclp r y z*
by *iprover*
from *this*(*2*) **show** *P*
proof (*cases rule*: *rtranclp.cases*)
case *rtrancl-refl*
with ‹*r x y*› *base* **show** *P*
by *iprover*
next
case *rtrancl-into-rtrancl*
then have *tranclp r y z*
by (*iprover intro*: *rtranclp-into-tranclp1*)
with ‹*r x y*› *step* **show** *P*
by *iprover*
qed
qed

lemmas *converse-tranclE* = *converse-tranclpE* [*to-set*]

lemma *tranclD2*: $(x, y) \in R^+ \Longrightarrow \exists z.\ (x, z) \in R^* \wedge (z, y) \in R$
by (*blast elim*: *tranclE intro*: *trancl-into-rtrancl*)

lemma *irrefl-tranclI*: $r^{-1} \cap r^* = \{\} \Longrightarrow (x, x) \notin r^+$
by (*blast elim*: *tranclE dest*: *trancl-into-rtrancl*)

lemma *irrefl-trancl-rD*: $\forall x.\ (x, x) \notin r^+ \Longrightarrow (x, y) \in r \Longrightarrow x \neq y$
by (*blast dest*: *r-into-trancl*)

lemma *trancl-subset-Sigma-aux*: $(a, b) \in r^* \Longrightarrow r \subseteq A \times A \Longrightarrow a = b \vee a \in A$
by (*induct rule*: *rtrancl-induct*) *auto*

lemma *trancl-subset-Sigma*:
assumes $r \subseteq A \times A$ **shows** $r^+ \subseteq A \times A$
proof (*rule trancl-Int-subset* [*OF assms*])
show $(r^+ \cap A \times A)\ O\ r \subseteq A \times A$
using *assms* **by** *auto*
qed

lemma *reflclp-tranclp* [*simp*]: $(r^{++})^{==} = r^{**}$
  by (*fast elim*: *rtranclp.cases tranclp-into-rtranclp dest*: *rtranclp-into-tranclp1*)

lemmas *reflcl-trancl* [*simp*] = *reflclp-tranclp* [*to-set*]

lemma *trancl-reflcl* [*simp*]: $(r^{=})^{+} = r^{*}$
proof −
  have $(a, b) \in (r^{=})^{+} \Longrightarrow (a, b) \in r^{*}$ **for** *a b*
    by (*force dest*: *trancl-into-rtrancl*)
  moreover have $(a, b) \in (r^{=})^{+}$ **if** $(a, b) \in r^{*}$ **for** *a b*
    using *that*
  proof (*cases a b rule*: *rtranclE*)
    case *step*
    show *?thesis*
      by (*rule rtrancl-into-trancl1*) (*use step* **in** *auto*)
  qed *auto*
  ultimately show *?thesis*
    by *auto*
qed

lemma *rtrancl-trancl-reflcl* [*code*]: $r^{*} = (r^{+})^{=}$
  by *simp*

lemma *trancl-empty* [*simp*]: $\{\}^{+} = \{\}$
  by (*auto elim*: *trancl-induct*)

lemma *rtrancl-empty* [*simp*]: $\{\}^{*} = Id$
  by (*rule subst* [*OF reflcl-trancl*]) *simp*

lemma *rtrancl--Id*[*simp*]: $Id^{*} = Id$
using *rtrancl-empty rtrancl-idemp*[*of* {}] **by** (*simp*)

lemma *rtranclpD*: $R^{**}\ a\ b \Longrightarrow a = b \vee a \neq b \wedge R^{++}\ a\ b$
  by (*force simp*: *reflclp-tranclp* [*symmetric*] *simp del*: *reflclp-tranclp*)

lemmas *rtranclD* = *rtranclpD* [*to-set*]

lemma *rtrancl-eq-or-trancl*: $(x,y) \in R^{*} \longleftrightarrow x = y \vee x \neq y \wedge (x, y) \in R^{+}$
  by (*fast elim*: *trancl-into-rtrancl dest*: *rtranclD*)

lemma *trancl-unfold-right*: $r^{+} = r^{*}\ O\ r$
  by (*auto dest*: *tranclD2 intro*: *rtrancl-into-trancl1*)

lemma *trancl-unfold-left*: $r^{+} = r\ O\ r^{*}$
  by (*auto dest*: *tranclD intro*: *rtrancl-into-trancl2*)

lemma *tranclp-unfold-left*: *r^++ = r OO r^***
by (*auto intro*!: *ext dest*: *tranclpD intro*: *rtranclp-into-tranclp2*)

lemma *trancl-insert*: $(insert\ (y,\ x)\ r)^+ = r^+ \cup \{(a,\ b).\ (a,\ y) \in r^* \wedge (x,\ b) \in r^*\}$
— primitive recursion for *trancl* over finite relations
proof –
have $\bigwedge a\ b.\ (a,\ b) \in (insert\ (y,\ x)\ r)^+ \Longrightarrow$
$(a,\ b) \in r^+ \cup \{(a,\ b).\ (a,\ y) \in r^* \wedge (x,\ b) \in r^*\}$
by (*erule trancl-induct*) (*blast intro*: *rtrancl-into-trancl1 trancl-into-rtrancl trancl-trans*)+
moreover have $r^+ \cup \{(a,\ b).\ (a,\ y) \in r^* \wedge (x,\ b) \in r^*\} \subseteq (insert\ (y,\ x)\ r)^+$
by (*blast intro*: *trancl-mono rtrancl-mono* [*THEN* [*2*] *rev-subsetD*]
rtrancl-trancl-trancl rtrancl-into-trancl2)
ultimately show *?thesis*
by *auto*
qed

lemma *trancl-insert2*:
$(insert\ (a,\ b)\ r)^+ = r^+ \cup \{(x,\ y).\ ((x,\ a) \in r^+ \vee x = a) \wedge ((b,\ y) \in r^+ \vee y = b)\}$
by (*auto simp*: *trancl-insert rtrancl-eq-or-trancl*)

lemma *rtrancl-insert*: $(insert\ (a,b)\ r)^* = r^* \cup \{(x,\ y).\ (x,\ a) \in r^* \wedge (b,\ y) \in r^*\}$
using *trancl-insert*[*of a b r*]
by (*simp add*: *rtrancl-trancl-reflcl del*: *reflcl-trancl*) *blast*

Simplifying nested closures

lemma *rtrancl-trancl-absorb*[*simp*]: $(R^*)^+ = R^*$
by (*simp add*: *trans-rtrancl*)

lemma *trancl-rtrancl-absorb*[*simp*]: $(R^+)^* = R^*$
by (*subst reflcl-trancl*[*symmetric*]) *simp*

lemma *rtrancl-reflcl-absorb*[*simp*]: $(R^*)^= = R^*$
by *auto*

Domain and *Range*

lemma *Domain-rtrancl* [*simp*]: $Domain\ (R^*) = UNIV$
by *blast*

lemma *Range-rtrancl* [*simp*]: $Range\ (R^*) = UNIV$
by *blast*

lemma *rtrancl-Un-subset*: $(R^* \cup S^*) \subseteq (R \cup S)^*$
by (*rule rtrancl-Un-rtrancl* [*THEN subst*]) *fast*

lemma *in-rtrancl-UnI*: $x \in R^* \vee x \in S^* \Longrightarrow x \in (R \cup S)^*$
by (*blast intro*: *subsetD* [*OF rtrancl-Un-subset*])

lemma *trancl-domain* [*simp*]: $Domain\ (r^+) = Domain\ r$
by (*unfold Domain-unfold*) (*blast dest*: *tranclD*)

lemma *trancl-range* [*simp*]: *Range* (r^+) = *Range* *r*
unfolding *Domain-converse* [*symmetric*] **by** (*simp add*: *trancl-converse* [*symmetric*])

lemma *Not-Domain-rtrancl*:
assumes $x \notin$ *Domain R* **shows** $(x, y) \in R^* \longleftrightarrow x = y$
proof −
have $(x, y) \in R^* \Longrightarrow x = y$
by (*erule rtrancl-induct*) (*use assms* **in** *auto*)
then show *?thesis*
by *auto*
qed

lemma *trancl-subset-Field2*: $r^+ \subseteq$ *Field r* × *Field r*
by (*rule trancl-Int-subset*) (*auto simp*: *Field-def*)

lemma *finite-trancl*[*simp*]: *finite* (r^+) = *finite r*
proof
show *finite* $(r^+) \Longrightarrow$ *finite r*
by (*blast intro*: *r-into-trancl′ finite-subset*)
show *finite r* ⟹ *finite* (r^+)
by (*auto simp*: *finite-Field trancl-subset-Field2* [*THEN finite-subset*])
qed

lemma *finite-rtrancl-Image*[*simp*]: **assumes** *finite R finite A* **shows** *finite* $(R^*$ ‘‘ $A)$
proof (*rule ccontr*)
assume *infinite* $(R^*$ ‘‘ $A)$
with *assms* **show** *False*
by(*simp add*: *rtrancl-trancl-reflcl Un-Image del*: *reflcl-trancl*)
qed

More about converse *rtrancl* and *trancl*, should be merged with main body.

lemma *single-valued-confluent*:
assumes *single-valued r* **and** *xy*: $(x, y) \in r^*$ **and** *xz*: $(x, z) \in r^*$
shows $(y, z) \in r^* \lor (z, y) \in r^*$
using *xy*
proof (*induction rule*: *rtrancl-induct*)
case *base*
show *?case*
by (*simp add*: *assms*)
next
case (*step y z*)
with *xz* ‹*single-valued r*› **show** *?case*
by (*auto elim*: *converse-rtranclE dest*: *single-valuedD intro*: *rtrancl-trans*)
qed

lemma *r-r-into-trancl*: $(a, b) \in R \Longrightarrow (b, c) \in R \Longrightarrow (a, c) \in R^+$
by (*fast intro*: *trancl-trans*)

lemma *trancl-into-trancl*: $(a, b) \in r^+ \Longrightarrow (b, c) \in r \Longrightarrow (a, c) \in r^+$
  by (*induct rule*: *trancl-induct*) (*fast intro*: *r-r-into-trancl trancl-trans*)+

lemma *tranclp-rtranclp-tranclp*:
  assumes $r^{++}\ a\ b\ r^{**}\ b\ c$ **shows** $r^{++}\ a\ c$
proof −
  obtain z **where** $r\ a\ z\ r^{**}\ z\ c$
    using *assms* **by** (*iprover dest*: *tranclpD rtranclp-trans*)
  then show *?thesis*
    by (*blast dest*: *rtranclp-into-tranclp2*)
qed

lemma *rtranclp-conversep*: $r^{-1-1**} = r^{**-1-1}$
  by(*auto simp add*: *fun-eq-iff intro*: *rtranclp-converseI rtranclp-converseD*)

lemmas *symp-rtranclp* = *sym-rtrancl*[*to-pred*]

lemmas *symp-conv-conversep-eq* = *sym-conv-converse-eq*[*to-pred*]

lemmas *rtranclp-tranclp-absorb* [*simp*] = *rtrancl-trancl-absorb*[*to-pred*]
lemmas *tranclp-rtranclp-absorb* [*simp*] = *trancl-rtrancl-absorb*[*to-pred*]
lemmas *rtranclp-reflclp-absorb* [*simp*] = *rtrancl-reflcl-absorb*[*to-pred*]

lemmas *trancl-rtrancl-trancl* = *tranclp-rtranclp-tranclp* [*to-set*]

lemmas *transitive-closure-trans* [*trans*] =
  r-r-into-trancl trancl-trans rtrancl-trans
  trancl.trancl-into-trancl trancl-into-trancl2
  rtrancl.rtrancl-into-rtrancl converse-rtrancl-into-rtrancl
  rtrancl-trancl-trancl trancl-rtrancl-trancl

lemmas *transitive-closurep-trans'* [*trans*] =
  tranclp-trans rtranclp-trans
  tranclp.trancl-into-trancl tranclp-into-tranclp2
  rtranclp.rtrancl-into-rtrancl converse-rtranclp-into-rtranclp
  rtranclp-tranclp-tranclp tranclp-rtranclp-tranclp

declare *trancl-into-rtrancl* [*elim*]

lemma *tranclp-ident-if-transp*:
  assumes *transp R*
  shows $R^{++} = R$
proof (*intro ext iffI*)
  fix $x\ y$
  show $R^{++}\ x\ y \Longrightarrow R\ x\ y$
  proof (*induction y rule*: *tranclp-induct*)
    case (*base y*)
    thus *?case*
      by *simp*

next
case (*step y z*)
thus *?case*
using ‹*transp R*›[*THEN transpD*, *of x y z*] **by** *simp*
qed
next
fix *x y*
show $R\ x\ y \Longrightarrow R^{++}\ x\ y$
using *tranclp.r-into-trancl* **.**
qed

The following are special cases of *tranclp-ident-if-transp*, but they appear duplicated in multiple, independent theories, which causes name clashes.

lemma (**in** *preorder*) *tranclp-less*[*simp*]: $(<)^{++} = (<)$
using *transp-on-less* **by** (*simp only*: *tranclp-ident-if-transp*)

lemma (**in** *preorder*) *tranclp-less-eq*[*simp*]: $(\leq)^{++} = (\leq)$
using *transp-on-le* **by** (*simp only*: *tranclp-ident-if-transp*)

lemma (**in** *preorder*) *tranclp-greater*[*simp*]: $(>)^{++} = (>)$
using *transp-on-greater* **by** (*simp only*: *tranclp-ident-if-transp*)

lemma (**in** *preorder*) *tranclp-greater-eq*[*simp*]: $(\geq)^{++} = (\geq)$
using *transp-on-ge* **by** (*simp only*: *tranclp-ident-if-transp*)

21.4 Symmetric closure

definition *symclp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
where $symclp\ r\ x\ y \longleftrightarrow r\ x\ y \lor r\ y\ x$

lemma *symclpI* [*simp*, *intro?*]:
shows *symclpI1*: $r\ x\ y \Longrightarrow symclp\ r\ x\ y$
and *symclpI2*: $r\ y\ x \Longrightarrow symclp\ r\ x\ y$
by(*simp-all add*: *symclp-def*)

lemma *symclpE* [*consumes 1*, *cases pred*]:
assumes *symclp r x y*
obtains (*base*) *r x y* | (*sym*) *r y x*
using *assms* **by**(*auto simp add*: *symclp-def*)

lemma *symclp-pointfree*: $symclp\ r = sup\ r\ r^{-1-1}$
by(*auto simp add*: *symclp-def fun-eq-iff*)

lemma *symclp-greater*: $r \leq symclp\ r$
by(*simp add*: *symclp-pointfree*)

lemma *symclp-conversep* [*simp*]: $symclp\ r^{-1-1} = symclp\ r$
by(*simp add*: *symclp-pointfree sup.commute*)

lemma *symp-on-symclp* [*simp*]: *symp-on A* (*symclp R*)
by(*auto simp add*: *symp-on-def elim*: *symclpE intro*: *symclpI*)

lemma *symp-symclp-eq*: *symp r* ⟹ *symclp r* = *r*
by(*simp add*: *symclp-pointfree symp-conv-conversep-eq*)

lemma *symp-rtranclp-symclp* [*simp*]: *symp* $(symclp\ r)^{**}$
by(*simp add*: *symp-rtranclp*)

lemma *rtranclp-symclp-sym* [*sym*]: $(symclp\ r)^{**}$ *x y* ⟹ $(symclp\ r)^{**}$ *y x*
by(*rule sympD*[*OF symp-rtranclp-symclp*])

lemma *symclp-idem* [*simp*]: *symclp* (*symclp r*) = *symclp r*
by(*simp add*: *symclp-pointfree sup-commute converse-join*)

lemma *reflp-on-rtranclp* [*simp*]: *reflp-on A* R^{**}
by (*simp add*: *reflp-on-def*)

21.5 The power operation on relations

R ^^ *n* = *R O* ... *O R*, the n-fold composition of *R*

overloading
relpow ≡ *compow* :: *nat* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set*
relpowp ≡ *compow* :: *nat* ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*)
begin

primrec *relpow* :: *nat* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set*
where
relpow 0 R = *Id*
| *relpow* (*Suc n*) *R* = (*R* ^^ *n*) *O R*

primrec *relpowp* :: *nat* ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*)
where
relpowp 0 R = *HOL.eq*
| *relpowp* (*Suc n*) *R* = (*R* ^^ *n*) *OO R*

end

lemmas *relpowp-Suc-right* = *relpowp.simps*(*2*)

lemma *relpowp-relpow-eq* [*pred-set-conv*]:
(λ*x y*. (*x*, *y*) ∈ *R*) ^^ *n* = (λ*x y*. (*x*, *y*) ∈ *R* ^^ *n*) **for** *R* :: '*a rel*
by (*induct n*) (*simp-all add*: *relcompp-relcomp-eq*)

For code generation:

definition *relpow* :: *nat* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set*
where *relpow-code-def* [*code-abbrev*]: *relpow* = *compow*

definition *relpowp* :: *nat* ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*)

where *relpowp-code-def* [*code-abbrev*]: *relpowp* = *compow*

lemma [*code*]:
relpow (*Suc n*) *R* = (*relpow n R*) *O R*
relpow 0 R = *Id*
by (*simp-all add*: *relpow-code-def*)

lemma [*code*]:
relpowp (*Suc n*) *R* = (*R* ^^ *n*) *OO R*
relpowp 0 R = *HOL.eq*
by (*simp-all add*: *relpowp-code-def*)

hide-const (**open**) *relpow*
hide-const (**open**) *relpowp*

lemma *relpow-1* [*simp*]: *R* ^^ *1* = *R*
for *R* :: ('*a* × '*a*) *set*
by *simp*

lemma *relpowp-1* [*simp*]: *P* ^^ *1* = *P*
for *P* :: '*a* ⇒ '*a* ⇒ *bool*
by (*fact relpow-1* [*to-pred*])

lemma *relpowp-Suc-0* [*simp*]: *P* ^^ (*Suc 0*) = *P*
for *P* :: '*a* ⇒ '*a* ⇒ *bool*
by (*auto*)

lemma *relpow-0-I*: (*x*, *x*) ∈ *R* ^^ *0*
by *simp*

lemma *relpowp-0-I*: (*P* ^^ *0*) *x x*
by (*fact relpow-0-I* [*to-pred*])

lemma *relpow-Suc-I*: (*x*, *y*) ∈ *R* ^^ *n* ⟹ (*y*, *z*) ∈ *R* ⟹ (*x*, *z*) ∈ *R* ^^ *Suc n*
by *auto*

lemma *relpowp-Suc-I*[*trans*]: (*P* ^^ *n*) *x y* ⟹ *P y z* ⟹ (*P* ^^ *Suc n*) *x z*
by (*fact relpow-Suc-I* [*to-pred*])

lemma *relpow-Suc-I2*: (*x*, *y*) ∈ *R* ⟹ (*y*, *z*) ∈ *R* ^^ *n* ⟹ (*x*, *z*) ∈ *R* ^^ *Suc n*
by (*induct n arbitrary*: *z*) (*simp*, *fastforce*)

lemma *relpowp-Suc-I2*[*trans*]: *P x y* ⟹ (*P* ^^ *n*) *y z* ⟹ (*P* ^^ *Suc n*) *x z*
by (*fact relpow-Suc-I2* [*to-pred*])

lemma *relpow-0-E*: (*x*, *y*) ∈ *R* ^^ *0* ⟹ (*x* = *y* ⟹ *P*) ⟹ *P*
by *simp*

lemma *relpowp-0-E*: (*P* ^^ *0*) *x y* ⟹ (*x* = *y* ⟹ *Q*) ⟹ *Q*

by (*fact relpow-0-E* [*to-pred*])

lemma *relpow-Suc-E*: $(x, z) \in R \frown\frown Suc\ n \Longrightarrow (\bigwedge y.\ (x, y) \in R \frown\frown n \Longrightarrow (y, z) \in R \Longrightarrow P) \Longrightarrow P$
by *auto*

lemma *relpowp-Suc-E*: $(P \frown\frown Suc\ n)\ x\ z \Longrightarrow (\bigwedge y.\ (P \frown\frown n)\ x\ y \Longrightarrow P\ y\ z \Longrightarrow Q) \Longrightarrow Q$
by (*fact relpow-Suc-E* [*to-pred*])

lemma *relpow-E*:
$(x, z) \in R \frown\frown n \Longrightarrow$
$(n = 0 \Longrightarrow x = z \Longrightarrow P) \Longrightarrow$
$(\bigwedge y\ m.\ n = Suc\ m \Longrightarrow (x, y) \in R \frown\frown m \Longrightarrow (y, z) \in R \Longrightarrow P) \Longrightarrow P$
by (*cases n*) *auto*

lemma *relpowp-E*:
$(P \frown\frown n)\ x\ z \Longrightarrow$
$(n = 0 \Longrightarrow x = z \Longrightarrow Q) \Longrightarrow$
$(\bigwedge y\ m.\ n = Suc\ m \Longrightarrow (P \frown\frown m)\ x\ y \Longrightarrow P\ y\ z \Longrightarrow Q) \Longrightarrow Q$
by (*fact relpow-E* [*to-pred*])

lemma *relpow-Suc-D2*: $(x, z) \in R \frown\frown Suc\ n \Longrightarrow (\exists y.\ (x, y) \in R \wedge (y, z) \in R \frown\frown n)$
by (*induct n arbitrary*: *x z*)
(*blast intro*: *relpow-0-I relpow-Suc-I elim*: *relpow-0-E relpow-Suc-E*)+

lemma *relpowp-Suc-D2*: $(P \frown\frown Suc\ n)\ x\ z \Longrightarrow \exists y.\ P\ x\ y \wedge (P \frown\frown n)\ y\ z$
by (*fact relpow-Suc-D2* [*to-pred*])

lemma *relpow-Suc-E2*: $(x, z) \in R \frown\frown Suc\ n \Longrightarrow (\bigwedge y.\ (x, y) \in R \Longrightarrow (y, z) \in R \frown\frown n \Longrightarrow P) \Longrightarrow P$
by (*blast dest*: *relpow-Suc-D2*)

lemma *relpowp-Suc-E2*: $(P \frown\frown Suc\ n)\ x\ z \Longrightarrow (\bigwedge y.\ P\ x\ y \Longrightarrow (P \frown\frown n)\ y\ z \Longrightarrow Q) \Longrightarrow Q$
by (*fact relpow-Suc-E2* [*to-pred*])

lemma *relpow-Suc-D2'*: $\forall x\ y\ z.\ (x, y) \in R \frown\frown n \wedge (y, z) \in R \longrightarrow (\exists w.\ (x, w) \in R \wedge (w, z) \in R \frown\frown n)$
by (*induct n*) (*simp-all*, *blast*)

lemma *relpowp-Suc-D2'*: $\forall x\ y\ z.\ (P \frown\frown n)\ x\ y \wedge P\ y\ z \longrightarrow (\exists w.\ P\ x\ w \wedge (P \frown\frown n)\ w\ z)$
by (*fact relpow-Suc-D2'* [*to-pred*])

lemma *relpow-E2*:
assumes $(x, z) \in R \frown\frown n\ n = 0 \Longrightarrow x = z \Longrightarrow P$
$\bigwedge y\ m.\ n = Suc\ m \Longrightarrow (x, y) \in R \Longrightarrow (y, z) \in R \frown\frown m \Longrightarrow P$

```
    shows P
proof (cases n)
  case 0
  with assms show ?thesis
    by simp
next
  case (Suc m)
  with assms relpow-Suc-D2′ [of m R] show ?thesis
    by force
qed

lemma relpowp-E2:
  (P ^^ n) x z ⟹
    (n = 0 ⟹ x = z ⟹ Q) ⟹
    (⋀y m. n = Suc m ⟹ P x y ⟹ (P ^^ m) y z ⟹ Q) ⟹ Q
  by (fact relpow-E2 [to-pred])

lemma relpowp-trans[trans]: (R ^^ i) x y ⟹ (R ^^ j) y z ⟹ (R ^^ (i + j)) x z
proof (induction i arbitrary: x)
  case 0
  thus ?case by simp
next
  case (Suc i)
  obtain x′ where R x x′ and (R ^^ i) x′ y
    using ‹(R ^^ Suc i) x y›[THEN relpowp-Suc-D2] by auto

  show (R ^^ (Suc i + j)) x z
    unfolding add-Suc
  proof (rule relpowp-Suc-I2)
    show R x x′
      using ‹R x x′› .
  next
    show (R ^^ (i + j)) x′ z
      using Suc.IH[OF ‹(R ^^ i) x′ y› ‹(R ^^ j) y z›] .
  qed
qed

lemma relpowp-mono:
  fixes x y :: ′a
  shows (⋀x y. R x y ⟹ S x y) ⟹ (R ^^ n) x y ⟹ (S ^^ n) x y
by (induction n arbitrary: y) auto

lemma relpow-trans[trans]: (x, y) ∈ R ^^ i ⟹ (y, z) ∈ R ^^ j ⟹ (x, z) ∈ R
^^ (i + j)
  using relpowp-trans[to-set] .

lemma relpowp-left-unique:
  fixes R :: ′a ⇒ ′a ⇒ bool and n :: nat and x y z :: ′a
  assumes lunique: ⋀x y z. R x z ⟹ R y z ⟹ x = y
```

shows $(R \frown\frown n)\ x\ z \Longrightarrow (R \frown\frown n)\ y\ z \Longrightarrow x = y$
proof (*induction n arbitrary*: $x\ y\ z$)
case *0*
thus *?case*
by *simp*
next
case (*Suc* n')
then obtain $x'\ y' :: 'a$ **where**
$(R \frown\frown n')\ x\ x'$ **and** $R\ x'\ z$ **and**
$(R \frown\frown n')\ y\ y'$ **and** $R\ y'\ z$
by *auto*

have $x' = y'$
using *lunique*[*OF* ‹$R\ x'\ z$› ‹$R\ y'\ z$›] **.**

show $x = y$
proof (*rule Suc.IH*)
show $(R \frown\frown n')\ x\ x'$
using ‹$(R \frown\frown n')\ x\ x'$› **.**
next
show $(R \frown\frown n')\ y\ x'$
using ‹$(R \frown\frown n')\ y\ y'$›
unfolding ‹$x' = y'$› **.**
qed
qed

lemma *relpow-left-unique*:
fixes $R :: ('a \times 'a)$ *set* **and** $n :: nat$ **and** $x\ y\ z :: 'a$
shows $(\bigwedge x\ y\ z.\ (x, z) \in R \Longrightarrow (y, z) \in R \Longrightarrow x = y) \Longrightarrow$
$(x, z) \in R \frown\frown n \Longrightarrow (y, z) \in R \frown\frown n \Longrightarrow x = y$
using *relpowp-left-unique*[*to-set*] **.**

lemma *relpowp-right-unique*:
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $n :: nat$ **and** $x\ y\ z :: 'a$
assumes *runique*: $\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ x\ z \Longrightarrow y = z$
shows $(R \frown\frown n)\ x\ y \Longrightarrow (R \frown\frown n)\ x\ z \Longrightarrow y = z$
proof (*induction n arbitrary*: $x\ y\ z$)
case *0*
thus *?case*
by *simp*
next
case (*Suc* n')
then obtain $x' :: 'a$ **where**
$(R \frown\frown n')\ x\ x'$ **and** $R\ x'\ y$ **and** $R\ x'\ z$
by *auto*
thus $y = z$
using *runique* **by** *simp*
qed

lemma *relpow-right-unique*:
  fixes R :: $('a \times 'a)$ *set* **and** n :: *nat* **and** $x\ y\ z$ :: $'a$
  shows $(\bigwedge x\ y\ z.\ (x, y) \in R \Longrightarrow (x, z) \in R \Longrightarrow y = z) \Longrightarrow$
    $(x, y) \in (R\ \widehat{}\widehat{}\ n) \Longrightarrow (x, z) \in (R\ \widehat{}\widehat{}\ n) \Longrightarrow y = z$
  using *relpowp-right-unique*[*to-set*] **.**

lemma *relpow-add*: $R\ \widehat{}\widehat{}\ (m + n) = R\widehat{}\widehat{}m\ O\ R\widehat{}\widehat{}n$
  by (*induct n*) *auto*

lemma *relpowp-add*: $P\ \widehat{}\widehat{}\ (m + n) = P\ \widehat{}\widehat{}\ m\ OO\ P\ \widehat{}\widehat{}\ n$
  by (*fact relpow-add* [*to-pred*])

lemma *relpow-commute*: $R\ O\ R\ \widehat{}\widehat{}\ n = R\ \widehat{}\widehat{}\ n\ O\ R$
  by (*induct n*) (*simp-all add*: *O-assoc* [*symmetric*])

lemma *relpowp-commute*: $P\ OO\ P\ \widehat{}\widehat{}\ n = P\ \widehat{}\widehat{}\ n\ OO\ P$
  by (*fact relpow-commute* [*to-pred*])

lemma *relpowp-Suc-left*: $R\ \widehat{}\widehat{}\ Suc\ n = R\ OO\ (R\ \widehat{}\widehat{}\ n)$
by (*simp add*: *relpowp-commute*)

lemma *relpow-empty*: $0 < n \Longrightarrow (\{\} :: ('a \times 'a)\ set)\ \widehat{}\widehat{}\ n = \{\}$
  by (*cases n*) *auto*

lemma *relpowp-bot*: $0 < n \Longrightarrow (\bot :: 'a \Rightarrow 'a \Rightarrow bool)\ \widehat{}\widehat{}\ n = \bot$
  by (*fact relpow-empty* [*to-pred*])

lemma *rtrancl-imp-UN-relpow*:
  assumes $p \in R^*$
  shows $p \in (\bigcup n.\ R\ \widehat{}\widehat{}\ n)$
proof (*cases p*)
  case (*Pair x y*)
  with *assms* **have** $(x, y) \in R^*$ **by** *simp*
  then have $(x, y) \in (\bigcup n.\ R\ \widehat{}\widehat{}\ n)$
  proof *induct*
    case *base*
    show *?case* **by** (*blast intro*: *relpow-0-I*)
  next
    case *step*
    then show *?case* **by** (*blast intro*: *relpow-Suc-I*)
  qed
  with *Pair* **show** *?thesis* **by** *simp*
qed

lemma *rtranclp-imp-Sup-relpowp*:
  assumes $(P^{**})\ x\ y$
  shows $(\bigsqcup n.\ P\ \widehat{}\widehat{}\ n)\ x\ y$
  using *assms* **and** *rtrancl-imp-UN-relpow* [*of* (x, y), *to-pred*] **by** *simp*

lemma *relpow-imp-rtrancl*:
  assumes $p \in R$ ^^ n
  shows $p \in R^*$
proof (*cases p*)
  case (*Pair x y*)
  with *assms* **have** $(x, y) \in R$ ^^ n **by** *simp*
  then have $(x, y) \in R^*$
  proof (*induct n arbitrary*: *x y*)
    case *0*
    then show *?case* **by** *simp*
  next
    case *Suc*
    then show *?case*
      by (*blast elim*: *relpow-Suc-E intro*: *rtrancl-into-rtrancl*)
  qed
  with *Pair* **show** *?thesis* **by** *simp*
qed

lemma *relpowp-imp-rtranclp*: $(P$ ^^ $n)\ x\ y \Longrightarrow (P^{**})\ x\ y$
  using *relpow-imp-rtrancl* [*of* (x, y), *to-pred*] **by** *simp*

lemma *rtrancl-is-UN-relpow*: $R^* = (\bigcup n.\ R$ ^^ $n)$
  by (*blast intro*: *rtrancl-imp-UN-relpow relpow-imp-rtrancl*)

lemma *rtranclp-is-Sup-relpowp*: $P^{**} = (\bigsqcup n.\ P$ ^^ $n)$
  using *rtrancl-is-UN-relpow* [*to-pred*, *of P*] **by** *auto*

lemma *rtrancl-power*: $p \in R^* \longleftrightarrow (\exists n.\ p \in R$ ^^ $n)$
  by (*simp add*: *rtrancl-is-UN-relpow*)

lemma *rtranclp-power*: $(P^{**})\ x\ y \longleftrightarrow (\exists n.\ (P$ ^^ $n)\ x\ y)$
  by (*simp add*: *rtranclp-is-Sup-relpowp*)

lemma *trancl-power*: $p \in R^+ \longleftrightarrow (\exists n > 0.\ p \in R$ ^^ $n)$
proof −
  have $(a, b) \in R^+ \longleftrightarrow (\exists n>0.\ (a, b) \in R$ ^^ $n)$ **for** *a b*
  proof *safe*
    show $(a, b) \in R^+ \Longrightarrow \exists n>0.\ (a, b) \in R$ ^^ n
      by (*fastforce simp*: *rtrancl-is-UN-relpow relcomp-unfold dest*: *tranclD2*)
    show $(a, b) \in R^+$ **if** $n > 0$ $(a, b) \in R$ ^^ n **for** *n*
    proof (*cases n*)
      case (*Suc m*)
      with *that* **show** *?thesis*
        by (*auto simp*: *dest*: *relpow-imp-rtrancl rtrancl-into-trancl1*)
    qed (*use that* **in** *auto*)
  qed
  then show *?thesis*
    by (*cases p*) *auto*
qed

lemma *tranclp-power*: (P^{++}) *x y* ⟷ (∃ *n* > *0*. (*P* ^^ *n*) *x y*)
  using *trancl-power* [*to-pred*, *of P* (*x*, *y*)] **by** *simp*

lemma *rtrancl-imp-relpow*: *p* ∈ R^* ⟹ ∃ *n*. *p* ∈ *R* ^^ *n*
  by (*auto dest*: *rtrancl-imp-UN-relpow*)

lemma *rtranclp-imp-relpowp*: (P^{**}) *x y* ⟹ ∃ *n*. (*P* ^^ *n*) *x y*
  by (*auto dest*: *rtranclp-imp-Sup-relpowp*)

By Sternagel/Thiemann:

lemma *relpow-fun-conv*: (*a*, *b*) ∈ *R* ^^ *n* ⟷ (∃ *f*. *f 0* = *a* ∧ *f n* = *b* ∧ (∀ *i*<*n*. (*f i*, *f* (*Suc i*)) ∈ *R*))
proof (*induct n arbitrary*: *b*)
  case *0*
  show *?case* **by** *auto*
next
  case (*Suc n*)
  show *?case*
  proof −
    have (∃ *y*. (∃ *f*. *f 0* = *a* ∧ *f n* = *y* ∧ (∀ *i*<*n*. (*f i*,*f*(*Suc i*)) ∈ *R*)) ∧ (*y*,*b*) ∈ *R*) ⟷
      (∃ *f*. *f 0* = *a* ∧ *f*(*Suc n*) = *b* ∧ (∀ *i*<*Suc n*. (*f i*, *f* (*Suc i*)) ∈ *R*))
      (**is** *?l* ⟷ *?r*)
    proof
      assume *?l*
      then obtain *c f*
        where *1*: *f 0* = *a* *f n* = *c* ⋀*i*. *i* < *n* ⟹ (*f i*, *f* (*Suc i*)) ∈ *R* (*c*,*b*) ∈ *R*
        by *auto*
      let *?g* = λ *m*. *if m* = *Suc n then b else f m*
      show *?r* **by** (*rule exI*[*of* - *?g*]) (*simp add*: *1*)
    next
      assume *?r*
      then obtain *f* **where** *1*: *f 0* = *a* *b* = *f* (*Suc n*) ⋀*i*. *i* < *Suc n* ⟹ (*f i*, *f* (*Suc i*)) ∈ *R*
        by *auto*
      show *?l* **by** (*rule exI*[*of* - *f n*], *rule conjI*, *rule exI*[*of* - *f*], *auto simp add*: *1*)
    qed
    then show *?thesis* **by** (*simp add*: *relcomp-unfold Suc*)
  qed
qed

lemma *relpowp-fun-conv*: (*P* ^^ *n*) *x y* ⟷ (∃ *f*. *f 0* = *x* ∧ *f n* = *y* ∧ (∀ *i*<*n*. *P* (*f i*) (*f* (*Suc i*))))
  by (*fact relpow-fun-conv* [*to-pred*])

lemma *relpow-finite-bounded1*:
  fixes *R* :: (*'a* × *'a*) *set*
  assumes *finite R* **and** *k* > *0*

shows $R^{\frown\frown}k \subseteq (\bigcup n\in\{n.\ 0 < n \wedge n \leq card\ R\}.\ R^{\frown\frown}n)$
(**is** - $\subseteq$ *?r*)
proof –
have $(a, b) \in R^{\frown\frown}(Suc\ k) \Longrightarrow \exists n.\ 0 < n \wedge n \leq card\ R \wedge (a, b) \in R^{\frown\frown}n$ **for** *a b k*
proof (*induct k arbitrary*: *b*)
case *0*
then have $R \neq \{\}$ **by** *auto*
with *card-0-eq*[*OF* ‹*finite R*›] **have** $card\ R \geq Suc\ 0$ **by** *auto*
then show *?case* **using** *0* **by** *force*
next
case (*Suc k*)
then obtain a' **where** $(a, a') \in R^{\frown\frown}(Suc\ k)$ **and** $(a', b) \in R$
by *auto*
from *Suc*(*1*)[*OF* ‹$(a, a') \in R^{\frown\frown}(Suc\ k)$›] **obtain** *n* **where** $n \leq card\ R$ **and** $(a, a') \in R^{\frown\frown}n$
by *auto*
have $(a, b) \in R^{\frown\frown}(Suc\ n)$
using ‹$(a, a') \in R^{\frown\frown}n$› **and** ‹$(a', b) \in R$› **by** *auto*
from ‹$n \leq card\ R$› **consider** $n < card\ R \mid n = card\ R$ **by** *force*
then show *?case*
proof *cases*
case *1*
then show *?thesis*
using ‹$(a, b) \in R^{\frown\frown}(Suc\ n)$› *Suc-leI*[*OF* ‹$n < card\ R$›] **by** *blast*
next
case *2*
from ‹$(a, b) \in R^{\frown\frown}(Suc\ n)$› [*unfolded relpow-fun-conv*]
obtain *f* **where** $f\ 0 = a$ **and** $f\ (Suc\ n) = b$
and *steps*: $\bigwedge i.\ i \leq n \Longrightarrow (f\ i, f\ (Suc\ i)) \in R$ **by** *auto*
let $?p = \lambda i.\ (f\ i, f(Suc\ i))$
let $?N = \{i.\ i \leq n\}$
have $?p\ `\ ?N \subseteq R$
using *steps* **by** *auto*
from *card-mono*[*OF assms*(*1*) *this*] **have** $card\ (?p\ `\ ?N) \leq card\ R$ **.**
also have $\ldots < card\ ?N$
using ‹$n = card\ R$› **by** *simp*
finally have $\neg$ *inj-on ?p ?N*
by (*rule pigeonhole*)
then obtain *i j* **where** *i*: $i \leq n$ **and** *j*: $j \leq n$ **and** *ij*: $i \neq j$ **and** *pij*: $?p\ i = ?p\ j$
by (*auto simp*: *inj-on-def*)
let $?i = min\ i\ j$
let $?j = max\ i\ j$
have *i*: $?i \leq n$ **and** *j*: $?j \leq n$ **and** *pij*: $?p\ ?i = ?p\ ?j$ **and** *ij*: $?i < ?j$
using *i j ij pij* **unfolding** *min-def max-def* **by** *auto*
from *i j pij ij* **obtain** *i j* **where** *i*: $i \leq n$ **and** *j*: $j \leq n$ **and** *ij*: $i < j$
and *pij*: $?p\ i = ?p\ j$
by *blast*

```
      let ?g = λl. if l ≤ i then f l else f (l + (j − i))
      let ?n = Suc (n − (j − i))
      have abl: (a, b) ∈ R ^^ ?n
        unfolding relpow-fun-conv
      proof (rule exI[of - ?g], intro conjI impI allI)
        show ?g ?n = b
          using ‹f(Suc n) = b› j ij by auto
      next
        fix k
        assume k < ?n
        show (?g k, ?g (Suc k)) ∈ R
        proof (cases k < i)
          case True
          with i have k ≤ n
            by auto
          from steps[OF this] show ?thesis
            using True by simp
        next
          case False
          then have i ≤ k by auto
          show ?thesis
          proof (cases k = i)
            case True
            then show ?thesis
              using ij pij steps[OF i] by simp
          next
            case False
            with ‹i ≤ k› have i < k by auto
            then have small: k + (j − i) ≤ n
              using ‹k<?n› by arith
            show ?thesis
              using steps[OF small] ‹i<k› by auto
          qed
        qed
      qed (simp add: ‹f 0 = a›)
      moreover have ?n ≤ n
        using i j ij by arith
      ultimately show ?thesis
        using ‹n = card R› by blast
    qed
  qed
  then show ?thesis
    using gr0-implies-Suc[OF ‹k > 0›] by auto
qed

lemma relpow-finite-bounded:
  fixes R :: ('a × 'a) set
  assumes finite R
  shows R^^k ⊆ (⋃n∈{n. n ≤ card R}. R^^n)
```

proof (*cases k*)
  case (*Suc k'*)
  then show *?thesis*
    using *relpow-finite-bounded1*[*OF assms*, *of k*] **by** *auto*
qed *force*

lemma *rtrancl-finite-eq-relpow*: *finite R* ⟹ $R^* = (\bigcup n\in\{n.\ n \leq card\ R\}.\ R^{\frown\frown}n)$
  by (*fastforce simp*: *rtrancl-power dest*: *relpow-finite-bounded*)

lemma *trancl-finite-eq-relpow*:
  assumes *finite R* **shows** $R^+ = (\bigcup n\in\{n.\ 0 < n \wedge n \leq card\ R\}.\ R^{\frown\frown}n)$
proof −
  have $\bigwedge a\ b\ n.\ [\![0 < n;\ (a,\ b) \in R^{\frown\frown} n]\!] \Longrightarrow \exists x>0.\ x \leq card\ R \wedge (a,\ b) \in R^{\frown\frown}$ x
    using *assms* **by** (*auto dest*: *relpow-finite-bounded1*)
  then show *?thesis*
    by (*auto simp*: *trancl-power*)
qed

lemma *finite-relcomp*[*simp*,*intro*]:
  assumes *finite R* **and** *finite S*
  shows *finite* (*R O S*)
proof−
  have $R\ O\ S = (\bigcup(x,\ y)\in R.\ \bigcup(u,\ v)\in S.\ if\ u = y\ then\ \{(x,\ v)\}\ else\ \{\})$
    by (*force simp*: *split-def image-constant-conv split*: *if-splits*)
  then show *?thesis*
    using *assms* **by** *clarsimp*
qed

lemma *finite-relpow* [*simp*, *intro*]:
  fixes R :: $('a \times 'a)$ *set*
  assumes *finite R*
  shows $n > 0 \Longrightarrow finite\ (R^{\frown\frown}n)$
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  then show *?case* **by** (*cases n*) (*use assms* **in** *simp-all*)
qed

lemma *single-valued-relpow*:
  fixes R :: $('a \times 'a)$ *set*
  shows *single-valued R* ⟹ *single-valued* $(R^{\frown\frown} n)$
proof (*induct n arbitrary*: *R*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)

```
  show ?case
    by (rule single-valuedI)
      (use Suc in ‹fast dest: single-valuedD elim: relpow-Suc-E›)
qed
```

21.6 Bounded transitive closure

```
definition ntrancl :: nat ⇒ ('a × 'a) set ⇒ ('a × 'a) set
  where ntrancl n R = (⋃i∈{i. 0 < i ∧ i ≤ Suc n}. R ^^ i)

lemma ntrancl-Zero [simp, code]: ntrancl 0 R = R
proof
  show R ⊆ ntrancl 0 R
    unfolding ntrancl-def by fastforce
  have 0 < i ∧ i ≤ Suc 0 ⟷ i = 1 for i
    by auto
  then show ntrancl 0 R ≤ R
    unfolding ntrancl-def by auto
qed

lemma ntrancl-Suc [simp]: ntrancl (Suc n) R = ntrancl n R O (Id ∪ R)
proof
  have (a, b) ∈ ntrancl n R O (Id ∪ R) if (a, b) ∈ ntrancl (Suc n) R for a b
  proof −
    from that obtain i where 0 < i i ≤ Suc (Suc n) (a, b) ∈ R ^^ i
      unfolding ntrancl-def by auto
    show ?thesis
    proof (cases i = 1)
      case True
      with ‹(a, b) ∈ R ^^ i› show ?thesis
        by (auto simp: ntrancl-def)
    next
      case False
      with ‹0 < i› obtain j where j: i = Suc j 0 < j
        by (cases i) auto
      with ‹(a, b) ∈ R ^^ i› obtain c where c1: (a, c) ∈ R ^^ j and c2: (c, b)
∈ R
        by auto
      from c1 j ‹i ≤ Suc (Suc n)› have (a, c) ∈ ntrancl n R
        by (fastforce simp: ntrancl-def)
      with c2 show ?thesis by fastforce
    qed
  qed
  then show ntrancl (Suc n) R ⊆ ntrancl n R O (Id ∪ R)
    by auto
  show ntrancl n R O (Id ∪ R) ⊆ ntrancl (Suc n) R
    by (fastforce simp: ntrancl-def)
qed
```

lemma [*code*]: *ntrancl* (*Suc n*) *r* = (*let r′* = *ntrancl n r in r′* ∪ *r′ O r*)
by (*auto simp*: *Let-def*)

lemma *finite-trancl-ntranl*: *finite R* ⟹ *trancl R* = *ntrancl* (*card R* − *1*) *R*
by (*cases card R*) (*auto simp*: *trancl-finite-eq-relpow relpow-empty ntrancl-def*)

21.7 Acyclic relations

definition *acyclic* :: (′*a* × ′*a*) *set* ⇒ *bool*
where *acyclic r* ⟷ (∀ *x*. (*x*,*x*) ∉ r^+)

abbreviation *acyclicP* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*
where *acyclicP r* ≡ *acyclic* {(*x*, *y*). *r x y*}

lemma *acyclic-irrefl* [*code*]: *acyclic r* ⟷ *irrefl* (r^+)
by (*simp add*: *acyclic-def irrefl-def*)

lemma *acyclicI*: ∀ *x*. (*x*, *x*) ∉ r^+ ⟹ *acyclic r*
by (*simp add*: *acyclic-def*)

lemma (**in** *preorder*) *acyclicI-order*:
assumes ∗: ⋀*a b*. (*a*, *b*) ∈ *r* ⟹ *f b* < *f a*
shows *acyclic r*
proof −
have *f b* < *f a* **if** (*a*, *b*) ∈ r^+ **for** *a b*
using *that* **by** *induct* (*auto intro*: ∗ *less-trans*)
then show *?thesis*
by (*auto intro*!: *acyclicI*)
qed

lemma *acyclic-insert* [*iff*]: *acyclic* (*insert* (*y*, *x*) *r*) ⟷ *acyclic r* ∧ (*x*, *y*) ∉ r^*
by (*simp add*: *acyclic-def trancl-insert*) (*blast intro*: *rtrancl-trans*)

lemma *acyclic-converse* [*iff*]: *acyclic* (r^{-1}) ⟷ *acyclic r*
by (*simp add*: *acyclic-def trancl-converse*)

lemmas *acyclicP-converse* [*iff*] = *acyclic-converse* [*to-pred*]

lemma *acyclic-impl-antisym-rtrancl*: *acyclic r* ⟹ *antisym* (r^*)
by (*simp add*: *acyclic-def antisym-def*)
(*blast elim*: *rtranclE intro*: *rtrancl-into-trancl1 rtrancl-trancl-trancl*)

lemma *acyclic-subset*: *acyclic s* ⟹ *r* ⊆ *s* ⟹ *acyclic r*
unfolding *acyclic-def* **by** (*blast intro*: *trancl-mono*)

21.8 Setup of transitivity reasoner

ML ‹

```
structure Trancl-Tac = Trancl-Tac
(
  val r-into-trancl = @{thm trancl.r-into-trancl};
  val trancl-trans  = @{thm trancl-trans};
  val rtrancl-refl = @{thm rtrancl.rtrancl-refl};
  val r-into-rtrancl = @{thm r-into-rtrancl};
  val trancl-into-rtrancl = @{thm trancl-into-rtrancl};
  val rtrancl-trancl-trancl = @{thm rtrancl-trancl-trancl};
  val trancl-rtrancl-trancl = @{thm trancl-rtrancl-trancl};
  val rtrancl-trans = @{thm rtrancl-trans};

  fun decomp Const- ‹Trueprop for t› =
        let
          fun dec Const- ‹Set.member - for Const- ‹Pair - - for a b› rel› =
              let
                fun decr Const- ‹rtrancl - for r› = (r,r*)
                  | decr Const- ‹trancl - for r› = (r,r+)
                  | decr r = (r,r);
                val (rel,r) = decr (Envir.beta-eta-contract rel);
              in SOME (a,b,rel,r) end
          | dec - =  NONE
        in dec t end
    | decomp - = NONE;
);

structure Tranclp-Tac = Trancl-Tac
(
  val r-into-trancl = @{thm tranclp.r-into-trancl};
  val trancl-trans  = @{thm tranclp-trans};
  val rtrancl-refl = @{thm rtranclp.rtrancl-refl};
  val r-into-rtrancl = @{thm r-into-rtranclp};
  val trancl-into-rtrancl = @{thm tranclp-into-rtranclp};
  val rtrancl-trancl-trancl = @{thm rtranclp-tranclp-tranclp};
  val trancl-rtrancl-trancl = @{thm tranclp-rtranclp-tranclp};
  val rtrancl-trans = @{thm rtranclp-trans};

  fun decomp Const- ‹Trueprop for t› =
        let
          fun dec (rel $ a $ b) =
            let
              fun decr Const- ‹rtranclp - for r› = (r,r*)
                | decr Const- ‹tranclp - for r› = (r,r+)
                | decr r = (r,r);
              val (rel,r) = decr rel;
            in SOME (a, b, rel, r) end
          | dec - =  NONE
        in dec t end
    | decomp - = NONE;
);
```

›

setup ‹
map-theory-simpset (*fn ctxt* => *ctxt*
addSolver (*mk-solver Trancl Trancl-Tac.trancl-tac*)
addSolver (*mk-solver Rtrancl Trancl-Tac.rtrancl-tac*)
addSolver (*mk-solver Tranclp Tranclp-Tac.trancl-tac*)
addSolver (*mk-solver Rtranclp Tranclp-Tac.rtrancl-tac*))
›

lemma *transp-rtranclp* [*simp*]: *transp* R^{**}
by(*auto simp add*: *transp-def*)

Optional methods.

method-setup *trancl* =
‹*Scan.succeed* (*SIMPLE-METHOD′ o Trancl-Tac.trancl-tac*)›
‹*simple transitivity reasoner*›
method-setup *rtrancl* =
‹*Scan.succeed* (*SIMPLE-METHOD′ o Trancl-Tac.rtrancl-tac*)›
‹*simple transitivity reasoner*›
method-setup *tranclp* =
‹*Scan.succeed* (*SIMPLE-METHOD′ o Tranclp-Tac.trancl-tac*)›
‹*simple transitivity reasoner* (*predicate version*)›
method-setup *rtranclp* =
‹*Scan.succeed* (*SIMPLE-METHOD′ o Tranclp-Tac.rtrancl-tac*)›
‹*simple transitivity reasoner* (*predicate version*)›

end

22 Well-founded Recursion

theory *Wellfounded*
imports *Transitive-Closure*
begin

22.1 Basic Definitions

definition *wf-on* :: *′a set* ⇒ *′a rel* ⇒ *bool* **where**
wf-on A r ⟷ (∀ *P*. (∀ *x* ∈ *A*. (∀ *y* ∈ *A*. (*y*, *x*) ∈ *r* ⟶ *P y*) ⟶ *P x*) ⟶ (∀ *x* ∈ *A*. *P x*))

abbreviation *wf* :: (*′a* × *′a*) *set* ⇒ *bool* **where**
wf ≡ *wf-on UNIV*

definition *wfp-on* :: *′a set* ⇒ (*′a* ⇒ *′a* ⇒ *bool*) ⇒ *bool* **where**
wfp-on A R ⟷ (∀ *P*. (∀ *x* ∈ *A*. (∀ *y* ∈ *A*. *R y x* ⟶ *P y*) ⟶ *P x*) ⟶ (∀ *x* ∈ *A*. *P x*))

abbreviation *wfP* :: (*′a* ⇒ *′a* ⇒ *bool*) ⇒ *bool* **where**

wfP ≡ *wfp-on UNIV*

alias *wfp* = *wfP*

We keep old name *wfp* for backward compatibility, but offer new name *wfp* to be consistent with similar predicates, e.g., *asymp*, *transp*, *totalp*.

22.2 Equivalence of Definitions

lemma *wfp-on-wf-on-eq*[*pred-set-conv*]: *wfp-on A* ($\lambda x\ y.\ (x, y) \in r$) ⟷ *wf-on A r*
by (*simp add*: *wfp-on-def wf-on-def*)

lemma *wf-def*: *wf r* ⟷ ($\forall P.\ (\forall x.\ (\forall y.\ (y, x) \in r \longrightarrow P\ y) \longrightarrow P\ x) \longrightarrow (\forall x.\ P\ x)$)
unfolding *wf-on-def* **by** *simp*

lemma *wfp-def*: *wfp r* ⟷ *wf* $\{(x, y).\ r\ x\ y\}$
unfolding *wf-def wfp-on-def* **by** *simp*

lemma *wfp-wf-eq*: *wfp* ($\lambda x\ y.\ (x, y) \in r$) = *wf r*
using *wfp-on-wf-on-eq* **.**

22.3 Induction Principles

lemma *wf-on-induct*[*consumes 1*, *case-names in-set less*, *induct set*: *wf-on*]:
assumes *wf-on A r* **and** $x \in A$ **and** $\bigwedge x.\ x \in A \Longrightarrow (\bigwedge y.\ y \in A \Longrightarrow (y, x) \in r \Longrightarrow P\ y) \Longrightarrow P\ x$
shows $P\ x$
using *assms*(*2*,*3*) **by** (*auto intro*: ‹*wf-on A r*›[*unfolded wf-on-def*, *rule-format*])

lemma *wfp-on-induct*[*consumes 1*, *case-names in-set less*, *induct pred*: *wfp-on*]:
assumes *wfp-on A r* **and** $x \in A$ **and** $\bigwedge x.\ x \in A \Longrightarrow (\bigwedge y.\ y \in A \Longrightarrow r\ y\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows $P\ x$
using *assms* **by** (*fact wf-on-induct*[*to-pred*])

lemma *wf-induct*:
assumes *wf r*
and $\bigwedge x.\ \forall y.\ (y, x) \in r \longrightarrow P\ y \Longrightarrow P\ x$
shows $P\ a$
using *assms* **by** (*auto intro*: *wf-on-induct*[*of UNIV*])

lemmas *wfp-induct* = *wf-induct* [*to-pred*]

lemmas *wf-induct-rule* = *wf-induct* [*rule-format*, *consumes 1*, *case-names less*, *induct set*: *wf*]

lemmas *wfp-induct-rule* = *wf-induct-rule* [*to-pred*, *induct set*: *wfp*]

lemma *wf-on-iff-wf*: *wf-on A r* $\longleftrightarrow$ *wf* $\{(x, y) \in r.\ x \in A \wedge y \in A\}$
proof (*rule iffI*)
  assume *wf*: *wf-on A r*
  show *wf* $\{(x, y) \in r.\ x \in A \wedge y \in A\}$
    unfolding *wf-def*
  proof (*intro allI impI ballI*)
    fix *P x*
    assume *IH*: $\forall x.\ (\forall y.\ (y, x) \in \{(x, y).\ (x, y) \in r \wedge x \in A \wedge y \in A\} \longrightarrow P\ y)$
$\longrightarrow P\ x$
    show *P x*
    proof (*cases* $x \in A$)
      case *True*
      show *?thesis*
        using *wf*
      proof (*induction x rule*: *wf-on-induct*)
        case *in-set*
        thus *?case*
          using *True* **.**
      next
        case (*less x*)
        thus *?case*
          by (*auto intro*: *IH*[*rule-format*])
      qed
    next
      case *False*
      then show *?thesis*
        by (*auto intro*: *IH*[*rule-format*])
    qed
  qed
next
  assume *wf*: *wf* $\{(x, y).\ (x, y) \in r \wedge x \in A \wedge y \in A\}$
  show *wf-on A r*
    unfolding *wf-on-def*
  proof (*intro allI impI ballI*)
    fix *P x*
    assume *IH*: $\forall x{\in}A.\ (\forall y{\in}A.\ (y, x) \in r \longrightarrow P\ y) \longrightarrow P\ x$ **and** $x \in A$
    show *P x*
      using *wf* ‹$x \in A$›
    proof (*induction x rule*: *wf-on-induct*)
      case *in-set*
      show *?case*
        by *simp*
    next
      case (*less y*)
      hence $\bigwedge z.\ (z, y) \in r \Longrightarrow z \in A \Longrightarrow P\ z$
        by *simp*
      thus *?case*
        using *IH*[*rule-format*, *OF* ‹$y \in A$›] **by** *simp*

qed
qed
qed

22.4 Introduction Rules

lemma *wfUNIVI*: $(\bigwedge P\ x.\ (\forall x.\ (\forall y.\ (y,\ x) \in r \longrightarrow P\ y) \longrightarrow P\ x) \Longrightarrow P\ x) \Longrightarrow$ *wf r*
unfolding *wf-def* **by** *blast*

lemmas *wfpUNIVI* = *wfUNIVI* [*to-pred*]

Restriction to domain A and range B. If r is well-founded over their intersection, then *wf r*.

lemma *wfI*:
assumes $r \subseteq A \times B$
and $\bigwedge x\ P.\ [\![\forall x.\ (\forall y.\ (y,\ x) \in r \longrightarrow P\ y) \longrightarrow P\ x;\ \ x \in A;\ x \in B]\!] \Longrightarrow P\ x$
shows *wf r*
using *assms* **unfolding** *wf-def* **by** *blast*

22.5 Ordering Properties

lemma *wf-not-sym*: *wf r* $\Longrightarrow (a,\ x) \in r \Longrightarrow (x,\ a) \notin r$
by (*induct a arbitrary*: *x set*: *wf*) *blast*

lemma *wf-asym*:
assumes *wf r* $(a,\ x) \in r$
obtains $(x,\ a) \notin r$
by (*drule wf-not-sym*[*OF assms*])

lemma *wf-imp-asym*: *wf r* $\Longrightarrow$ *asym r*
by (*auto intro*: *asymI elim*: *wf-asym*)

lemma *wfp-imp-asymp*: *wfp r* $\Longrightarrow$ *asymp r*
by (*rule wf-imp-asym*[*to-pred*])

lemma *wf-not-refl* [*simp*]: *wf r* $\Longrightarrow (a,\ a) \notin r$
by (*blast elim*: *wf-asym*)

lemma *wf-irrefl*:
assumes *wf r*
obtains $(a,\ a) \notin r$
by (*drule wf-not-refl*[*OF assms*])

lemma *wf-imp-irrefl*:
assumes *wf r* **shows** *irrefl r*
using *wf-irrefl* [*OF assms*] **by** (*auto simp add*: *irrefl-def*)

lemma *wfp-imp-irreflp*: *wfp r* $\Longrightarrow$ *irreflp r*

```
  by (rule wf-imp-irrefl[to-pred])

lemma wf-wellorderI:
  assumes wf: wf {(x::'a::ord, y). x < y}
    and lin: OFCLASS('a::ord, linorder-class)
  shows OFCLASS('a::ord, wellorder-class)
  apply (rule wellorder-class.intro [OF lin])
  apply (simp add: wellorder-class.intro class.wellorder-axioms.intro wf-induct-rule
[OF wf])
  done

lemma (in wellorder) wf: wf {(x, y). x < y}
  unfolding wf-def by (blast intro: less-induct)

lemma (in wellorder) wfp-on-less[simp]: wfp-on A (<)
  unfolding wfp-on-def
proof (intro allI impI ballI)
  fix P x
  assume hyps: ∀x∈A. (∀y∈A. y < x ⟶ P y) ⟶ P x
  show x ∈ A ⟹ P x
  proof (induction x rule: less-induct)
    case (less x)
    show ?case
    proof (rule hyps[rule-format])
      show x ∈ A
        using ‹x ∈ A› .
    next
      show ⋀y. y ∈ A ⟹ y < x ⟹ P y
        using less.IH .
    qed
  qed
qed
```

22.6 Basic Results

Point-free characterization of well-foundedness

```
lemma wf-onE-pf:
  assumes wf: wf-on A r and B ⊆ A and B ⊆ r `` B
  shows B = {}
proof –
  have x ∉ B if x ∈ A for x
    using wf
  proof (induction x rule: wf-on-induct)
    case in-set
    show ?case
      using that .
  next
    case (less x)
    have x ∉ r `` B
```

```
      using less.IH ‹B ⊆ A› by blast
    thus ?case
      using ‹B ⊆ r `` B› by blast
  qed
  with ‹B ⊆ A› show ?thesis
    by blast
qed

lemma wfE-pf: wf R ⟹ A ⊆ R `` A ⟹ A = {}
  using wf-onE-pf[of UNIV, simplified] .

lemma wf-onI-pf:
  assumes ⋀B. B ⊆ A ⟹ B ⊆ R `` B ⟹ B = {}
  shows wf-on A R
  unfolding wf-on-def
proof (intro allI impI ballI)
  fix P :: 'a ⇒ bool and x :: 'a
  let ?B = {x ∈ A. ¬ P x}
  assume ∀x∈A. (∀y∈A. (y, x) ∈ R ⟶ P y) ⟶ P x
  hence ?B ⊆ R `` ?B by blast
  hence {x ∈ A. ¬ P x} = {}
    using assms(1)[of ?B] by simp
  moreover assume x ∈ A
  ultimately show P x
    by simp
qed

lemma wfI-pf: (⋀A. A ⊆ R `` A ⟹ A = {}) ⟹ wf R
  using wf-onI-pf[of UNIV, simplified] .
```

22.6.1 Minimal-element characterization of well-foundedness

```
lemma wf-on-iff-ex-minimal: wf-on A R ⟷ (∀B ⊆ A. B ≠ {} ⟶ (∃z ∈ B. ∀y.
(y, z) ∈ R ⟶ y ∉ B))
proof (intro iffI allI impI)
  fix B
  assume wf-on A R and B ⊆ A and B ≠ {}
  show ∃z ∈ B. ∀y. (y, z) ∈ R ⟶ y ∉ B
  using wf-onE-pf[OF ‹wf-on A R› ‹B ⊆ A›] ‹B ≠ {}› by blast
next
  assume ex-min: ∀B⊆A. B ≠ {} ⟶ (∃z∈B. ∀y. (y, z) ∈ R ⟶ y ∉ B)
  show wf-on A R
  proof (rule wf-onI-pf)
    fix B
    assume B ⊆ A and B ⊆ R `` B
    have False if B ≠ {}
      using ex-min[rule-format, OF ‹B ⊆ A› ‹B ≠ {}›]
      using ‹B ⊆ R `` B› by blast
    thus B = {}
```

by *blast*
qed
qed

lemma *wf-iff-ex-minimal*: *wf R* $\longleftrightarrow$ $(\forall B.\ B \neq \{\} \longrightarrow (\exists z \in B.\ \forall y.\ (y, z) \in R \longrightarrow y \notin B))$
using *wf-on-iff-ex-minimal*[*of UNIV*, *simplified*] **.**

lemma *wfp-on-iff-ex-minimal*: *wfp-on A R* $\longleftrightarrow$ $(\forall B \subseteq A.\ B \neq \{\} \longrightarrow (\exists z \in B.\ \forall y.\ R\ y\ z \longrightarrow y \notin B))$
using *wf-on-iff-ex-minimal*[*of A*, *to-pred*] **by** *simp*

lemma *wfp-iff-ex-minimal*: *wfp R* $\longleftrightarrow$ $(\forall B.\ B \neq \{\} \longrightarrow (\exists z \in B.\ \forall y.\ R\ y\ z \longrightarrow y \notin B))$
using *wfp-on-iff-ex-minimal*[*of UNIV*, *simplified*] **.**

lemma *wfE-min*:
assumes *wf*: *wf R* **and** *Q*: $x \in Q$
obtains *z* **where** $z \in Q$ $\bigwedge y.\ (y, z) \in R \Longrightarrow y \notin Q$
using *Q wfE-pf*[*OF wf*, *of Q*] **by** *blast*

lemma *wfE-min'*:
wf R $\Longrightarrow Q \neq \{\} \Longrightarrow (\bigwedge z.\ z \in Q \Longrightarrow (\bigwedge y.\ (y, z) \in R \Longrightarrow y \notin Q) \Longrightarrow thesis) \Longrightarrow thesis$
using *wfE-min*[*of R - Q*] **by** *blast*

lemma *wfI-min*:
assumes *a*: $\bigwedge x\ Q.\ x \in Q \Longrightarrow \exists z \in Q.\ \forall y.\ (y, z) \in R \longrightarrow y \notin Q$
shows *wf R*
proof (*rule wfI-pf*)
fix *A*
assume *b*: $A \subseteq R$ `` A
have *False* **if** $x \in A$ **for** *x*
using *a*[*OF that*] *b* **by** *blast*
then show $A = \{\}$ **by** *blast*
qed

lemma *wf-eq-minimal*: *wf r* $\longleftrightarrow$ $(\forall Q\ x.\ x \in Q \longrightarrow (\exists z \in Q.\ \forall y.\ (y, z) \in r \longrightarrow y \notin Q))$
unfolding *wf-iff-ex-minimal* **by** *blast*

lemmas *wfp-eq-minimal* = *wf-eq-minimal* [*to-pred*]

22.6.2 Finite characterization of well-foundedness

lemma *strict-partial-order-wfp-on-finite-set*:
assumes *transp-on* $\mathcal{X}$ *R* **and** *asymp-on* $\mathcal{X}$ *R* **and** *finite* $\mathcal{X}$
shows *wfp-on* $\mathcal{X}$ *R*
unfolding *Wellfounded.wfp-on-iff-ex-minimal*

proof (*intro allI impI*)
fix $\mathcal{W}$
assume $\mathcal{W} \subseteq \mathcal{X}$ **and** $\mathcal{W} \neq \{\}$

have *finite* $\mathcal{W}$
using *finite-subset*[*OF* ‹$\mathcal{W} \subseteq \mathcal{X}$› ‹*finite* $\mathcal{X}$›] **.**

moreover have *asymp-on* $\mathcal{W}$ *R*
using *asymp-on-subset*[*OF* ‹*asymp-on* $\mathcal{X}$ *R*› ‹$\mathcal{W} \subseteq \mathcal{X}$›] **.**

moreover have *transp-on* $\mathcal{W}$ *R*
using *transp-on-subset*[*OF* ‹*transp-on* $\mathcal{X}$ *R*› ‹$\mathcal{W} \subseteq \mathcal{X}$›] **.**

ultimately have $\exists\, m{\in}\mathcal{W}.\ \forall\, x{\in}\mathcal{W}.\ x \neq m \longrightarrow \neg\ R\ x\ m$
using ‹$\mathcal{W} \neq \{\}$› *Finite-Set.bex-min-element*[*of* $\mathcal{W}$ *R*] **by** *iprover*

thus $\exists\, z{\in}\mathcal{W}.\ \forall\, y.\ R\ y\ z \longrightarrow y \notin \mathcal{W}$
using *asymp-onD*[*OF* ‹*asymp-on* $\mathcal{W}$ *R*›] **by** *fast*
qed

22.6.3 Antimonotonicity

lemma *wfp-on-antimono-stronger*:
fixes
$A :: 'a\ set$ **and** $B :: 'b\ set$ **and**
$f :: 'a \Rightarrow 'b$ **and**
$R :: 'b \Rightarrow 'b \Rightarrow bool$ **and** $Q :: 'a \Rightarrow 'a \Rightarrow bool$
assumes
wf: *wfp-on B R* **and**
sub: f ‘ $A \subseteq B$ **and**
mono: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow Q\ x\ y \Longrightarrow R\ (f\ x)\ (f\ y)$
shows *wfp-on A Q*
unfolding *wfp-on-iff-ex-minimal*
proof (*intro allI impI*)
fix $A' :: 'a\ set$
assume $A' \subseteq A$ **and** $A' \neq \{\}$
have f ‘ $A' \subseteq B$
using ‹$A' \subseteq A$› *sub* **by** *blast*
moreover have f ‘ $A' \neq \{\}$
using ‹$A' \neq \{\}$› **by** *blast*
ultimately have $\exists\, z{\in}f$ ‘ $A'.\ \forall\, y.\ R\ y\ z \longrightarrow y \notin f$ ‘ A'
using *wf wfp-on-iff-ex-minimal* **by** *blast*
hence $\exists\, z{\in}A'.\ \forall\, y.\ R\ (f\ y)\ (f\ z) \longrightarrow y \notin A'$
by *blast*
thus $\exists\, z{\in}A'.\ \forall\, y.\ Q\ y\ z \longrightarrow y \notin A'$
using ‹$A' \subseteq A$› *mono* **by** *blast*
qed

lemma *wf-on-antimono-stronger*:

assumes
wf-on B r **and**
f ' $A \subseteq B$ **and**
$(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in q \Longrightarrow (f\ x, f\ y) \in r)$
shows *wf-on A q*
using *assms wfp-on-antimono-stronger*[*to-set*, *of B r f A q*] **by** *blast*

lemma *wf-on-antimono-strong*:
assumes *wf-on B r* **and** $A \subseteq B$ **and** $(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in q \Longrightarrow (x, y) \in r)$
shows *wf-on A q*
using *assms wf-on-antimono-stronger*[*of B r* $\lambda x.\ x$ *A q*] **by** *blast*

lemma *wfp-on-antimono-strong*:
wfp-on B R $\Longrightarrow A \subseteq B \Longrightarrow (\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow Q\ x\ y \Longrightarrow R\ x\ y) \Longrightarrow$ *wfp-on A Q*
using *wf-on-antimono-strong*[*of B - A*, *to-pred*] **.**

lemma *wf-on-antimono*: $A \subseteq B \Longrightarrow q \subseteq r \Longrightarrow$ *wf-on B r* $\leq$ *wf-on A q*
using *wf-on-antimono-strong*[*of B r A q*] **by** *auto*

lemma *wfp-on-antimono*: $A \subseteq B \Longrightarrow Q \leq R \Longrightarrow$ *wfp-on B R* $\leq$ *wfp-on A Q*
using *wfp-on-antimono-strong*[*of B R A Q*] **by** *auto*

lemma *wf-on-subset*: *wf-on B r* $\Longrightarrow A \subseteq B \Longrightarrow$ *wf-on A r*
using *wf-on-antimono-strong* **.**

lemma *wfp-on-subset*: *wfp-on B R* $\Longrightarrow A \subseteq B \Longrightarrow$ *wfp-on A R*
using *wfp-on-antimono-strong* **.**

22.6.4 Well-foundedness of transitive closure

lemma *ex-terminating-rtranclp-strong*:
assumes *wf*: *wfp-on* $\{x'.\ R^{**}\ x\ x'\}\ R^{-1-1}$
shows $\exists y.\ R^{**}\ x\ y \land (\nexists z.\ R\ y\ z)$
proof (*cases* $\exists y.\ R\ x\ y$)
case *True*
with *wf* **show** *?thesis*
proof (*induction rule*: *Wellfounded.wfp-on-induct*)
case *in-set*
thus *?case*
by *simp*
next
case (*less y*)
have $R^{**}\ x\ y$
using *less.hyps mem-Collect-eq*[*of -* R^{**} *x*] **by** *iprover*

moreover obtain *z* **where** *R y z*
using *less.prems* **by** *iprover*

ultimately have R^{**} *x z*
using *rtranclp.rtrancl-into-rtrancl*[*of R x y z*] **by** *iprover*

show *?case*
using ‹*R y z*› ‹R^{**} *x z*› *less.IH*[*of z*] *rtranclp-trans*[*of R y z*] **by** *blast*
qed
next
case *False*
thus *?thesis*
by *blast*
qed

lemma *ex-terminating-rtranclp*:
assumes *wf*: *wfp* R^{-1-1}
shows $\exists\, y.\ R^{**}\ x\ y \wedge (\nexists\, z.\ R\ y\ z)$
using *ex-terminating-rtranclp-strong*[*OF wfp-on-subset*[*OF wf subset-UNIV*]] **.**

lemma *wf-trancl*:
assumes *wf r*
shows *wf* (r^+)
proof −
have *P x* **if** *induct-step*: $\bigwedge x.\ (\bigwedge y.\ (y,\ x) \in r^+ \Longrightarrow P\ y) \Longrightarrow P\ x$ **for** *P x*
proof (*rule induct-step*)
show *P y* **if** $(y,\ x) \in r^+$ **for** *y*
using ‹*wf r*› **and** *that*
proof (*induct x arbitrary*: *y*)
case (*less x*)
note *hyp* = ‹$\bigwedge x'\ y'.\ (x',\ x) \in r \Longrightarrow (y',\ x') \in r^+ \Longrightarrow P\ y'$›
from ‹$(y,\ x) \in r^+$› **show** *P y*
proof *cases*
case *base*
show *P y*
proof (*rule induct-step*)
fix y'
assume $(y',\ y) \in r^+$
with ‹$(y,\ x) \in r$› **show** $P\ y'$
by (*rule hyp* [*of y y'*])
qed
next
case *step*
then obtain x' **where** $(x',\ x) \in r$ **and** $(y,\ x') \in r^+$
by *simp*
then show *P y* **by** (*rule hyp* [*of* x' *y*])
qed
qed
qed
then show *?thesis* **unfolding** *wf-def* **by** *blast*
qed

lemmas *wfp-tranclp* = *wf-trancl* [*to-pred*]

lemma *wf-converse-trancl*: *wf* $(r^{-1}) \Longrightarrow$ *wf* $((r^{+})^{-1})$
  apply (*subst trancl-converse* [*symmetric*])
  apply (*erule wf-trancl*)
  done

Well-foundedness of subsets

lemma *wf-subset*: *wf* $r \Longrightarrow p \subseteq r \Longrightarrow$ *wf* p
  by (*simp add*: *wf-eq-minimal*) *fast*

lemmas *wfp-subset* = *wf-subset* [*to-pred*]

Well-foundedness of the empty relation

lemma *wf-empty* [*iff*]: *wf* {}
  by (*simp add*: *wf-def*)

lemma *wfp-empty* [*iff*]: *wfp* ($\lambda x\ y.$ *False*)
proof −
  have *wfp bot*
    by (*fact wf-empty*[*to-pred bot-empty-eq2*])
  then show *?thesis*
    by (*simp add*: *bot-fun-def*)
qed

lemma *wf-Int1*: *wf* $r \Longrightarrow$ *wf* $(r \cap r')$
  by (*erule wf-subset*) (*rule Int-lower1*)

lemma *wf-Int2*: *wf* $r \Longrightarrow$ *wf* $(r' \cap r)$
  by (*erule wf-subset*) (*rule Int-lower2*)

Exponentiation.

lemma *wf-exp*:
  assumes *wf* $(R\ \widehat{}\widehat{}\ n)$
  shows *wf* R
proof (*rule wfI-pf*)
  fix A **assume** $A \subseteq R$ `` A
  then have $A \subseteq (R\ \widehat{}\widehat{}\ n)$ `` A
    by (*induct n*) *force+*
  with ‹*wf* $(R\ \widehat{}\widehat{}\ n)$› **show** $A = \{\}$
    by (*rule wfE-pf*)
qed

Well-foundedness of *insert*.

lemma *wf-insert* [*iff*]: *wf* (*insert* (y,x) r) $\longleftrightarrow$ *wf* $r \wedge (x,y) \notin r^{*}$ (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs* **then show** *?rhs*

```
    by (blast elim: wf-trancl [THEN wf-irrefl]
        intro: rtrancl-into-trancl1 wf-subset rtrancl-mono [THEN subsetD])
next
  assume R: ?rhs
  then have R′: Q ≠ {} ⟹ (∃z∈Q. ∀y. (y, z) ∈ r ⟶ y ∉ Q) for Q
    by (auto simp: wf-eq-minimal)
  show ?lhs
    unfolding wf-eq-minimal
  proof clarify
    fix Q :: ′a set and q
    assume q ∈ Q
    then obtain a where a ∈ Q and a: ⋀y. (y, a) ∈ r ⟹ y ∉ Q
      using R by (auto simp: wf-eq-minimal)
    show ∃z∈Q. ∀y′. (y′, z) ∈ insert (y, x) r ⟶ y′ ∉ Q
    proof (cases a=x)
      case True
      show ?thesis
      proof (cases y ∈ Q)
        case True
        then obtain z where z ∈ Q (z, y) ∈ r*
                               ⋀z′. (z′, z) ∈ r ⟶ z′ ∈ Q ⟶ (z′, y) ∉ r*
          using R′ [of {z ∈ Q. (z,y) ∈ r*}] by auto
        then have ∀y′. (y′, z) ∈ insert (y, x) r ⟶ y′ ∉ Q
          using R by(blast intro: rtrancl-trans)+
        then show ?thesis
          by (rule bexI) fact
      next
        case False
        then show ?thesis
          using a ‹a ∈ Q› by blast
      qed
    next
      case False
      with a ‹a ∈ Q› show ?thesis
        by blast
    qed
  qed
qed
```

22.6.5 Well-foundedness of image

```
lemma wf-map-prod-image-Dom-Ran:
  fixes r:: (′a × ′a) set
    and f:: ′a ⇒ ′b
  assumes wf-r: wf r
    and inj: ⋀ a a′. a ∈ Domain r ⟹ a′ ∈ Range r ⟹ f a = f a′ ⟹ a = a′
  shows wf (map-prod f f ‘ r)
proof (unfold wf-eq-minimal, clarify)
  fix B :: ′b set and b::′b
```

```
  assume b ∈ B
  define A where A = f −‘ B ∩ Domain r
  show ∃z∈B. ∀y. (y, z) ∈ map-prod f f ‘ r ⟶ y ∉ B
  proof (cases A = {})
    case False
    then obtain a0 where a0 ∈ A and ∀a. (a, a0) ∈ r ⟶ a ∉ A
      using wfE-min[OF wf-r] by auto
    thus ?thesis
      using inj unfolding A-def
      by (intro bexI[of - f a0]) auto
  qed (use ‹b ∈ B› in ‹unfold A-def, auto›)
qed

lemma wf-map-prod-image: wf r ⟹ inj f ⟹ wf (map-prod f f ‘ r)
by(rule wf-map-prod-image-Dom-Ran) (auto dest: inj-onD)

lemma wfp-on-image: wfp-on (f ‘ A) R ⟷ wfp-on A (λa b. R (f a) (f b))
proof (rule iffI)
  assume hyp: wfp-on (f ‘ A) R
  show wfp-on A (λa b. R (f a) (f b))
    unfolding wfp-on-iff-ex-minimal
  proof (intro allI impI)
    fix B
    assume B ⊆ A and B ≠ {}
    hence f ‘ B ⊆ f ‘ A and f ‘ B ≠ {}
      unfolding atomize-conj image-is-empty
      using image-mono by iprover
    hence ∃z∈f ‘ B. ∀y. R y z ⟶ y ∉ f ‘ B
      using hyp[unfolded wfp-on-iff-ex-minimal, rule-format] by iprover
    then obtain fz where fz ∈ f ‘ B and fz-max: ∀y. R y fz ⟶ y ∉ f ‘ B ..

    obtain z where z ∈ B and fz = f z
      using ‹fz ∈ f ‘ B› unfolding image-iff ..

    show ∃z∈B. ∀y. R (f y) (f z) ⟶ y ∉ B
    proof (intro bexI allI impI)
      show z ∈ B
        using ‹z ∈ B› .
    next
      fix y assume R (f y) (f z)
      hence f y ∉ f ‘ B
        using fz-max ‹fz = f z› by iprover
      thus y ∉ B
        by (rule contrapos-nn) (rule imageI)
    qed
  qed
next
  assume hyp: wfp-on A (λa b. R (f a) (f b))
  show wfp-on (f ‘ A) R
```

unfolding *wfp-on-iff-ex-minimal*
proof (*intro allI impI*)
fix *fA*
assume $fA \subseteq f \text{ ` } A$ **and** $fA \neq \{\}$
then obtain A' **where** $A' \subseteq A$ **and** $A' \neq \{\}$ **and** $fA = f \text{ ` } A'$
by (*auto simp only*: *subset-image-iff*)

obtain z **where** $z \in A'$ **and** *z-max*: $\forall y.\ R\ (f\ y)\ (f\ z) \longrightarrow y \notin A'$
using *hyp*[*unfolded wfp-on-iff-ex-minimal*, *rule-format*, *OF* ‹$A' \subseteq A$› ‹$A' \neq$ $\{\}$›] **by** *blast*

show $\exists z{\in}fA.\ \forall y.\ R\ y\ z \longrightarrow y \notin fA$
proof (*intro bexI allI impI*)
show $f\ z \in fA$
unfolding ‹$fA = f \text{ ` } A'$›
using *imageI*[*OF* ‹$z \in A'$›] **.**
next
show $\bigwedge y.\ R\ y\ (f\ z) \Longrightarrow y \notin fA$
unfolding ‹$fA = f \text{ ` } A'$›
using *z-max* **by** *auto*
qed
qed
qed

22.7 Well-Foundedness Results for Unions

lemma *wf-union-compatible*:
assumes *wf R wf S*
assumes $R\ O\ S \subseteq R$
shows *wf* $(R \cup S)$
proof (*rule wfI-min*)
fix $x :: 'a$ **and** Q
let $?Q' = \{x \in Q.\ \forall y.\ (y,\ x) \in R \longrightarrow y \notin Q\}$
assume $x \in Q$
obtain a **where** $a \in\ ?Q'$
by (*rule wfE-min* [*OF* ‹*wf R*› ‹$x \in Q$›]) *blast*
with ‹*wf S*› **obtain** z **where** $z \in\ ?Q'$ **and** *zmin*: $\bigwedge y.\ (y,\ z) \in S \Longrightarrow y \notin\ ?Q'$
by (*erule wfE-min*)
have $y \notin Q$ **if** $(y,\ z) \in S$ **for** y
proof
from *that* **have** $y \notin\ ?Q'$ **by** (*rule zmin*)
assume $y \in Q$
with ‹$y \notin\ ?Q'$› **obtain** w **where** $(w,\ y) \in R$ **and** $w \in Q$ **by** *auto*
from ‹$(w,\ y) \in R$› ‹$(y,\ z) \in S$› **have** $(w,\ z) \in R\ O\ S$ **by** (*rule relcompI*)
with ‹$R\ O\ S \subseteq R$› **have** $(w,\ z) \in R$ **..**
with ‹$z \in\ ?Q'$› **have** $w \notin Q$ **by** *blast*
with ‹$w \in Q$› **show** *False* **by** *contradiction*
qed
with ‹$z \in\ ?Q'$› **show** $\exists z{\in}Q.\ \forall y.\ (y,\ z) \in R \cup S \longrightarrow y \notin Q$ **by** *blast*

qed

Well-foundedness of indexed union with disjoint domains and ranges.

lemma *wf-UN*:
  assumes *r*: $\bigwedge i.\ i \in I \Longrightarrow wf\ (r\ i)$
    and *disj*: $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ r\ i \neq r\ j]\!] \Longrightarrow Domain\ (r\ i) \cap Range\ (r\ j) = \{\}$
  shows $wf\ (\bigcup i{\in}I.\ r\ i)$
  unfolding *wf-eq-minimal*
proof *clarify*
  fix *A* **and** *a* :: *'b*
  assume $a \in A$
  show $\exists z{\in}A.\ \forall y.\ (y,\ z) \in \bigcup(r\ `\ I) \longrightarrow y \notin A$
  proof ($cases\ \exists i{\in}I.\ \exists a{\in}A.\ \exists b{\in}A.\ (b,\ a) \in r\ i$)
    case *True*
    then obtain *i b c* **where** *ibc*: $i \in I\ b \in A\ c \in A\ (c,b) \in r\ i$
      by *blast*
    have *ri*: $\bigwedge Q.\ Q \neq \{\} \Longrightarrow \exists z{\in}Q.\ \forall y.\ (y,\ z) \in r\ i \longrightarrow y \notin Q$
      using *r* [*OF* ‹$i \in I$›] **unfolding** *wf-eq-minimal* **by** *auto*
    show *?thesis*
      using *ri* [*of* $\{a.\ a \in A \wedge (\exists b{\in}A.\ (b,\ a) \in r\ i)\ \}$] *ibc disj*
      by *blast*
  next
    case *False*
    with ‹$a \in A$› **show** *?thesis*
      by *blast*
  qed
qed

lemma *wfp-SUP*:
  $\forall i.\ wfp\ (r\ i) \Longrightarrow \forall i\ j.\ r\ i \neq r\ j \longrightarrow inf\ (Domainp\ (r\ i))\ (Rangep\ (r\ j)) = bot \Longrightarrow$
    $wfp\ (\bigsqcup(range\ r))$
  by (*rule wf-UN*[*to-pred*]) *simp-all*

lemma *wf-Union*:
  assumes $\forall r{\in}R.\ wf\ r$
    and $\forall r{\in}R.\ \forall s{\in}R.\ r \neq s \longrightarrow Domain\ r \cap Range\ s = \{\}$
  shows $wf\ (\bigcup R)$
  using *assms wf-UN*[*of R* $\lambda i.\ i$] **by** *simp*

Intuition: We find an $R \cup S$-min element of a nonempty subset A by case distinction.

1. There is a step $a\ {-R\rightarrow}\ b$ with $a,\ b \in A$. Pick an R-min element z of the (nonempty) set $\{a{\in}A \mid \exists b{\in}A.\ a\ {-R\rightarrow}\ b\}$. By definition, there is $z' \in A$ s.t. $z\ {-R\rightarrow}\ z'$. Because z is R-min in the subset, z' must be R-min in A. Because z' has an R-predecessor, it cannot have an S-successor and is thus S-min in A as well.

2. There is no such step. Pick an S-min element of A. In this case it must be an R-min element of A as well.

```
lemma wf-Un: wf r ⟹ wf s ⟹ Domain r ∩ Range s = {} ⟹ wf (r ∪ s)
  using wf-union-compatible[of s r]
  by (auto simp: Un-ac)

lemma wf-union-merge: wf (R ∪ S) = wf (R O R ∪ S O R ∪ S)
  (is wf ?A = wf ?B)
proof
  assume wf ?A
  with wf-trancl have wfT: wf (?A⁺) .
  moreover have ?B ⊆ ?A⁺
    by (subst trancl-unfold, subst trancl-unfold) blast
  ultimately show wf ?B by (rule wf-subset)
next
  assume wf ?B
  show wf ?A
  proof (rule wfI-min)
    fix Q :: 'a set and x
    assume x ∈ Q
    with ‹wf ?B› obtain z where z ∈ Q and ⋀y. (y, z) ∈ ?B ⟹ y ∉ Q
      by (erule wfE-min)
    then have 1: ⋀y. (y, z) ∈ R O R ⟹ y ∉ Q
      and 2: ⋀y. (y, z) ∈ S O R ⟹ y ∉ Q
      and 3: ⋀y. (y, z) ∈ S ⟹ y ∉ Q
      by auto
    show ∃z∈Q. ∀y. (y, z) ∈ ?A ⟶ y ∉ Q
    proof (cases ∀y. (y, z) ∈ R ⟶ y ∉ Q)
      case True
      with ‹z ∈ Q› 3 show ?thesis by blast
    next
      case False
      then obtain z' where z'∈Q (z', z) ∈ R by blast
      have ∀y. (y, z') ∈ ?A ⟶ y ∉ Q
      proof (intro allI impI)
        fix y assume (y, z') ∈ ?A
        then show y ∉ Q
        proof
          assume (y, z') ∈ R
          then have (y, z) ∈ R O R using ‹(z', z) ∈ R› ..
          with 1 show y ∉ Q .
        next
          assume (y, z') ∈ S
          then have (y, z) ∈ S O R using ‹(z', z) ∈ R› ..
          with 2 show y ∉ Q .
        qed
      qed
      with ‹z' ∈ Q› show ?thesis ..
```

qed
qed
qed

lemma *wf-comp-self*: *wf R* $\longleftrightarrow$ *wf* $(R\ O\ R)$ — special case
by (*rule wf-union-merge* [**where** $S = \{\}$, *simplified*])

22.8 Well-Foundedness of Composition

Bachmair and Dershowitz 1986, Lemma 2. [Provided by Tjark Weber]

lemma *qc-wf-relto-iff*:
assumes $R\ O\ S \subseteq (R \cup S)^*\ O\ R$ — R quasi-commutes over S
shows *wf* $(S^*\ O\ R\ O\ S^*) \longleftrightarrow$ *wf R*
(**is** *wf ?S* $\longleftrightarrow$ -)
proof
show *wf R* **if** *wf ?S*
proof −
have $R \subseteq$ *?S* **by** *auto*
with *wf-subset* [*of ?S*] *that* **show** *wf R*
by *auto*
qed
next
show *wf ?S* **if** *wf R*
proof (*rule wfI-pf*)
fix *A*
assume *A*: $A \subseteq$ *?S* `` *A*
let *?X* = $(R \cup S)^*$ `` *A*
have ∗: $R\ O\ (R \cup S)^* \subseteq (R \cup S)^*\ O\ R$
proof −
have $(x, z) \in (R \cup S)^*\ O\ R$ **if** $(y, z) \in (R \cup S)^*$ **and** $(x, y) \in R$ **for** *x y z*
using *that*
proof (*induct y z*)
case *rtrancl-refl*
then show *?case* **by** *auto*
next
case (*rtrancl-into-rtrancl a b c*)
then have $(x, c) \in ((R \cup S)^*\ O\ (R \cup S)^*)\ O\ R$
using *assms* **by** *blast*
then show *?case* **by** *simp*
qed
then show *?thesis* **by** *auto*
qed
then have $R\ O\ S^* \subseteq (R \cup S)^*\ O\ R$
using *rtrancl-Un-subset* **by** *blast*
then have *?S* $\subseteq (R \cup S)^*\ O\ (R \cup S)^*\ O\ R$
by (*simp add*: *relcomp-mono rtrancl-mono*)
also have ... $= (R \cup S)^*\ O\ R$
by (*simp add*: *O-assoc*[*symmetric*])
finally have *?S* $O\ (R \cup S)^* \subseteq (R \cup S)^*\ O\ R\ O\ (R \cup S)^*$

by (*simp add*: *O-assoc*[*symmetric*] *relcomp-mono*)
also have ... $\subseteq (R \cup S)^* \; O \; (R \cup S)^* \; O \; R$
using $*$ **by** (*simp add*: *relcomp-mono*)
finally have $?S \; O \; (R \cup S)^* \subseteq (R \cup S)^* \; O \; R$
by (*simp add*: *O-assoc*[*symmetric*])
then have $(?S \; O \; (R \cup S)^*) \; `` \; A \subseteq ((R \cup S)^* \; O \; R) \; `` \; A$
by (*simp add*: *Image-mono*)
moreover have $?X \subseteq (?S \; O \; (R \cup S)^*) \; `` \; A$
using *A* **by** (*auto simp*: *relcomp-Image*)
ultimately have $?X \subseteq R \; `` \; ?X$
by (*auto simp*: *relcomp-Image*)
then have $?X = \{\}$
using ‹*wf R*› **by** (*simp add*: *wfE-pf*)
moreover have $A \subseteq ?X$ **by** *auto*
ultimately show $A = \{\}$ **by** *simp*
qed
qed

corollary *wf-relcomp-compatible*:
assumes *wf R* **and** $R \; O \; S \subseteq S \; O \; R$
shows *wf* $(S \; O \; R)$
proof −
have $R \; O \; S \subseteq (R \cup S)^* \; O \; R$
using *assms* **by** *blast*
then have *wf* $(S^* \; O \; R \; O \; S^*)$
by (*simp add*: *assms qc-wf-relto-iff*)
then show *?thesis*
by (*rule Wellfounded.wf-subset*) *blast*
qed

22.9 Acyclic relations

lemma *wf-acyclic*: *wf r* $\Longrightarrow$ *acyclic r*
by (*simp add*: *acyclic-def*) (*blast elim*: *wf-trancl* [*THEN wf-irrefl*])

lemmas *wfp-acyclicP* = *wf-acyclic* [*to-pred*]

22.9.1 Wellfoundedness of finite acyclic relations

lemma *finite-acyclic-wf*:
assumes *finite r acyclic r* **shows** *wf r*
using *assms*
proof (*induction r rule*: *finite-induct*)
case (*insert x r*)
then show *?case*
by (*cases x*) *simp*
qed *simp*

lemma *finite-acyclic-wf-converse*: *finite r* $\Longrightarrow$ *acyclic r* $\Longrightarrow$ *wf* (r^{-1})
apply (*erule finite-converse* [*THEN iffD2*, *THEN finite-acyclic-wf*])

apply (*erule acyclic-converse* [*THEN iffD2*])
done

Observe that the converse of an irreflexive, transitive, and finite relation is again well-founded. Thus, we may employ it for well-founded induction.

lemma *wf-converse*:
assumes *irrefl r* **and** *trans r* **and** *finite r*
shows *wf* (r^{-1})
proof −
have *acyclic r*
using ‹*irrefl r*› **and** ‹*trans r*›
by (*simp add*: *irrefl-def acyclic-irrefl*)
with ‹*finite r*› **show** *?thesis*
by (*rule finite-acyclic-wf-converse*)
qed

lemma *wf-iff-acyclic-if-finite*: *finite r* $\Longrightarrow$ *wf r* = *acyclic r*
by (*blast intro*: *finite-acyclic-wf wf-acyclic*)

22.10 *nat* is well-founded

lemma *less-nat-rel*: $(<) = (\lambda m\ n.\ n = Suc\ m)^{++}$
proof (*rule ext*, *rule ext*, *rule iffI*)
fix *n m* :: *nat*
show $(\lambda m\ n.\ n = Suc\ m)^{++}\ m\ n$ **if** $m < n$
using *that*
proof (*induct n*)
case *0*
then show *?case* **by** *auto*
next
case (*Suc n*)
then show *?case*
by (*auto simp add*: *less-Suc-eq-le le-less intro*: *tranclp.trancl-into-trancl*)
qed
show $m < n$ **if** $(\lambda m\ n.\ n = Suc\ m)^{++}\ m\ n$
using *that* **by** (*induct n*) (*simp-all add*: *less-Suc-eq-le reflexive le-less*)
qed

definition *pred-nat* :: (*nat* × *nat*) *set*
where *pred-nat* = {(*m*, *n*). *n* = *Suc m*}

definition *less-than* :: (*nat* × *nat*) *set*
where *less-than* = $pred\text{-}nat^{+}$

lemma *less-eq*: $(m, n) \in pred\text{-}nat^{+} \longleftrightarrow m < n$
unfolding *less-nat-rel pred-nat-def trancl-def* **by** *simp*

lemma *pred-nat-trancl-eq-le*: $(m, n) \in pred\text{-}nat^{*} \longleftrightarrow m \le n$
unfolding *less-eq rtrancl-eq-or-trancl* **by** *auto*

lemma *wf-pred-nat*: *wf pred-nat*
  unfolding *wf-def*
proof *clarify*
  fix *P x*
  assume $\forall x'. (\forall y. (y, x') \in$ *pred-nat* $\longrightarrow P\ y) \longrightarrow P\ x'$
  then show *P x*
    unfolding *pred-nat-def* **by** (*induction x*) *blast+*
qed

lemma *wf-less-than* [*iff*]: *wf less-than*
  by (*simp add*: *less-than-def wf-pred-nat* [*THEN wf-trancl*])

lemma *trans-less-than* [*iff*]: *trans less-than*
  by (*simp add*: *less-than-def*)

lemma *less-than-iff* [*iff*]: $((x,y) \in$ *less-than*$) = (x<y)$
  by (*simp add*: *less-than-def less-eq*)

lemma *irrefl-less-than*: *irrefl less-than*
  using *irrefl-def* **by** *blast*

lemma *asym-less-than*: *asym less-than*
  by (*rule asymI*) *simp*

lemma *total-less-than*: *total less-than* **and** *total-on-less-than* [*simp*]: *total-on A less-than*
  using *total-on-def* **by** *force+*

lemma *wf-less*: *wf* $\{(x, y{::}nat).\ x < y\}$
  by (*rule Wellfounded.wellorder-class.wf*)

22.11 Accessible Part

Inductive definition of the accessible part *acc r* of a relation; see also [6].

inductive-set *acc* :: $('a \times 'a)$ *set* $\Rightarrow$ *'a set* **for** *r* :: $('a \times 'a)$ *set*
  where *accI*: $(\bigwedge y. (y, x) \in r \Longrightarrow y \in acc\ r) \Longrightarrow x \in acc\ r$

abbreviation *termip* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$
  where *termip r* $\equiv$ *accp* (r^{-1-1})

abbreviation *termi* :: $('a \times 'a)$ *set* $\Rightarrow$ *'a set*
  where *termi r* $\equiv$ *acc* (r^{-1})

lemmas *accpI* = *accp.accI*

lemma *accp-eq-acc* [*code*]: *accp r* = $(\lambda x. x \in$ *Wellfounded.acc* $\{(x, y). r\ x\ y\})$
  by (*simp add*: *acc-def*)

Induction rules

theorem *accp-induct*:
assumes *major*: *accp r a*
assumes *hyp*: $\bigwedge x.\ accp\ r\ x \Longrightarrow \forall y.\ r\ y\ x \longrightarrow P\ y \Longrightarrow P\ x$
shows *P a*
apply (*rule major* [*THEN accp.induct*])
apply (*rule hyp*)
apply (*rule accp.accI*)
apply *auto*
done

lemmas *accp-induct-rule* = *accp-induct* [*rule-format*, *induct set*: *accp*]

theorem *accp-downward*: $accp\ r\ b \Longrightarrow r\ a\ b \Longrightarrow accp\ r\ a$
by (*cases rule*: *accp.cases*)

lemma *not-accp-down*:
assumes *na*: $\neg$ *accp R x*
obtains *z* **where** *R z x* **and** $\neg$ *accp R z*
proof –
assume *a*: $\bigwedge z.\ R\ z\ x \Longrightarrow \neg\ accp\ R\ z \Longrightarrow thesis$
show *thesis*
proof (*cases* $\forall z.\ R\ z\ x \longrightarrow accp\ R\ z$)
case *True*
then have $\bigwedge z.\ R\ z\ x \Longrightarrow accp\ R\ z$ **by** *auto*
then have *accp R x* **by** (*rule accp.accI*)
with *na* **show** *thesis* **..**
next
case *False* **then obtain** *z* **where** *R z x* **and** $\neg$ *accp R z*
by *auto*
with *a* **show** *thesis* **.**
qed
qed

lemma *accp-downwards-aux*: $r^{**}\ b\ a \Longrightarrow accp\ r\ a \longrightarrow accp\ r\ b$
by (*erule rtranclp-induct*) (*blast dest*: *accp-downward*)+

theorem *accp-downwards*: $accp\ r\ a \Longrightarrow r^{**}\ b\ a \Longrightarrow accp\ r\ b$
by (*blast dest*: *accp-downwards-aux*)

theorem *accp-wfpI*: $\forall x.\ accp\ r\ x \Longrightarrow wfp\ r$
proof (*rule wfpUNIVI*)
fix *P x*
assume $\forall x.\ accp\ r\ x\ \forall x.\ (\forall y.\ r\ y\ x \longrightarrow P\ y) \longrightarrow P\ x$
then show *P x*
using *accp-induct*[**where** *P* = *P*] **by** *blast*
qed

theorem *accp-wfpD*: $wfp\ r \Longrightarrow accp\ r\ x$

apply (*erule wfp-induct-rule*)
apply (*rule accp.accI*)
apply *blast*
done

theorem *wfp-iff-accp*: $wfp\ r = (\forall x.\ accp\ r\ x)$
by (*blast intro*: *accp-wfpI dest*: *accp-wfpD*)

Smaller relations have bigger accessible parts:

lemma *accp-subset*:
assumes $R1 \leq R2$
shows $accp\ R2 \leq accp\ R1$
proof (*rule predicate1I*)
fix x
assume $accp\ R2\ x$
then show $accp\ R1\ x$
proof (*induct x*)
fix x
assume $\bigwedge y.\ R2\ y\ x \Longrightarrow accp\ R1\ y$
with *assms* **show** $accp\ R1\ x$
by (*blast intro*: *accp.accI*)
qed
qed

This is a generalized induction theorem that works on subsets of the accessible part.

lemma *accp-subset-induct*:
assumes *subset*: $D \leq accp\ R$
and *dcl*: $\bigwedge x\ z.\ D\ x \Longrightarrow R\ z\ x \Longrightarrow D\ z$
and $D\ x$
and *istep*: $\bigwedge x.\ D\ x \Longrightarrow (\bigwedge z.\ R\ z\ x \Longrightarrow P\ z) \Longrightarrow P\ x$
shows $P\ x$
proof −
from *subset* **and** ‹$D\ x$›
have $accp\ R\ x$ **..**
then show $P\ x$ **using** ‹$D\ x$›
proof (*induct x*)
fix x
assume $D\ x$ **and** $\bigwedge y.\ R\ y\ x \Longrightarrow D\ y \Longrightarrow P\ y$
with *dcl* **and** *istep* **show** $P\ x$ **by** *blast*
qed
qed

Set versions of the above theorems

lemmas *acc-induct* = *accp-induct* [*to-set*]
lemmas *acc-induct-rule* = *acc-induct* [*rule-format*, *induct set*: *acc*]
lemmas *acc-downward* = *accp-downward* [*to-set*]
lemmas *not-acc-down* = *not-accp-down* [*to-set*]
lemmas *acc-downwards-aux* = *accp-downwards-aux* [*to-set*]

lemmas *acc-downwards* = *accp-downwards* [*to-set*]
lemmas *acc-wfI* = *accp-wfpI* [*to-set*]
lemmas *acc-wfD* = *accp-wfpD* [*to-set*]
lemmas *wf-iff-acc* = *wfp-iff-accp* [*to-set*]
lemmas *acc-subset* = *accp-subset* [*to-set*]
lemmas *acc-subset-induct* = *accp-subset-induct* [*to-set*]

22.12 Tools for building wellfounded relations

Inverse Image

```
lemma wf-inv-image [simp,intro!]:
  fixes f :: 'a ⇒ 'b
  assumes wf r
  shows wf (inv-image r f)
proof −
  have ⋀x P. x ∈ P ⟹ ∃z∈P. ∀y. (f y, f z) ∈ r ⟶ y ∉ P
  proof −
    fix P and x::'a
    assume x ∈ P
    then obtain w where w: w ∈ {w. ∃x::'a. x ∈ P ∧ f x = w}
      by auto
    have ∗: ⋀Q u. u ∈ Q ⟹ ∃z∈Q. ∀y. (y, z) ∈ r ⟶ y ∉ Q
      using assms by (auto simp add: wf-eq-minimal)
    show ∃z∈P. ∀y. (f y, f z) ∈ r ⟶ y ∉ P
      using ∗ [OF w] by auto
  qed
  then show ?thesis
    by (clarsimp simp: inv-image-def wf-eq-minimal)
qed

lemma wfp-on-inv-imagep:
  assumes wf: wfp-on (f ‘ A) R
  shows wfp-on A (inv-imagep R f)
  unfolding wfp-on-iff-ex-minimal
proof (intro allI impI)
  fix B assume B ⊆ A and B ≠ {}
  hence ∃z∈f ‘ B. ∀y. R y z ⟶ y ∉ f ‘ B
    using wf[unfolded wfp-on-iff-ex-minimal, rule-format, of f ‘ B] by blast
  thus ∃z∈B. ∀y. inv-imagep R f y z ⟶ y ∉ B
    unfolding inv-imagep-def
    by auto
qed
```

22.12.1 Conversion to a known well-founded relation

```
lemma wfp-on-if-convertible-to-wfp-on:
  assumes
    wf: wfp-on (f ‘ A) Q and
    convertible: (⋀x y. x ∈ A ⟹ y ∈ A ⟹ R x y ⟹ Q (f x) (f y))
```

shows *wfp-on A R*
unfolding *wfp-on-iff-ex-minimal*
proof (*intro allI impI*)
fix B **assume** $B \subseteq A$ **and** $B \neq \{\}$
moreover from *wf* **have** *wfp-on A* (*inv-imagep Q f*)
by (*rule wfp-on-inv-imagep*)
ultimately obtain y **where** $y \in B$ **and** $\bigwedge z.\ Q\ (f\ z)\ (f\ y) \Longrightarrow z \notin B$
unfolding *wfp-on-iff-ex-minimal in-inv-imagep*
by *blast*
thus $\exists z \in B.\ \forall y.\ R\ y\ z \longrightarrow y \notin B$
using ‹$B \subseteq A$› *convertible* **by** *blast*
qed

lemma *wf-on-if-convertible-to-wf-on*: *wf-on* (f ‘ A) $Q \Longrightarrow (\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in R \Longrightarrow (f\ x, f\ y) \in Q) \Longrightarrow$ *wf-on* A R
using *wfp-on-if-convertible-to-wfp-on*[*to-set*] **.**

lemma *wf-if-convertible-to-wf*:
fixes r :: *'a rel* **and** s :: *'b rel* **and** f :: $'a \Rightarrow 'b$
assumes *wf s* **and** *convertible*: $\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (f\ x, f\ y) \in s$
shows *wf r*
proof (*rule wf-on-if-convertible-to-wf-on*)
show *wf-on* (*range f*) *s*
using *wf-on-subset*[*OF* ‹*wf s*› *subset-UNIV*] **.**
next
show $\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (f\ x, f\ y) \in s$
using *convertible* **.**
qed

lemma *wfp-if-convertible-to-wfp*: *wfp* $S \Longrightarrow (\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ (f\ x)\ (f\ y)) \Longrightarrow$ *wfp R*
using *wf-if-convertible-to-wf*[*to-pred*, *of S R f*] **by** *simp*

Converting to *nat* is a very common special case that might be found more easily by Sledgehammer.

lemma *wfp-if-convertible-to-nat*:
fixes f :: - $\Rightarrow$ *nat*
shows $(\bigwedge x\ y.\ R\ x\ y \Longrightarrow f\ x < f\ y) \Longrightarrow$ *wfp R*
by (*rule wfp-if-convertible-to-wfp*[*of* (<) :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *bool*, *simplified*])

22.12.2 Measure functions into *nat*

definition *measure* :: $('a \Rightarrow nat) \Rightarrow ('a \times 'a)$ *set*
where *measure* = *inv-image less-than*

lemma *in-measure*[*simp*, *code-unfold*]: $(x, y) \in$ *measure* $f \longleftrightarrow f\ x < f\ y$
by (*simp add*:*measure-def*)

lemma *wf-measure* [*iff*]: *wf* (*measure f*)

unfolding *measure-def* **by** (*rule wf-less-than* [*THEN wf-inv-image*])

lemma *wf-if-measure*: (⋀*x*. *P x* ⟹ *f*(*g x*) < *f x*) ⟹ *wf* {(*y*,*x*). *P x* ∧ *y* = *g x*}
for *f* :: ′*a* ⇒ *nat*
using *wf-measure*[*of f*] **unfolding** *measure-def inv-image-def less-than-def less-eq*
by (*rule wf-subset*) *auto*

22.12.3 Lexicographic combinations

definition *lex-prod* :: (′*a* ×′*a*) *set* ⇒ (′*b* × ′*b*) *set* ⇒ ((′*a* × ′*b*) × (′*a* × ′*b*)) *set*
(**infixr** ‹<∗*lex*∗>› *80*)
where *ra* <∗*lex*∗> *rb* = {((*a*, *b*), (*a*′, *b*′)). (*a*, *a*′) ∈ *ra* ∨ *a* = *a*′ ∧ (*b*, *b*′) ∈ *rb*}

lemma *in-lex-prod*[*simp*]: ((*a*, *b*), (*a*′, *b*′)) ∈ *r* <∗*lex*∗> *s* ⟷ (*a*, *a*′) ∈ *r* ∨ *a* = *a*′ ∧ (*b*, *b*′) ∈ *s*
by (*auto simp*:*lex-prod-def*)

lemma *wf-lex-prod* [*intro*!]:
assumes *wf ra wf rb*
shows *wf* (*ra* <∗*lex*∗> *rb*)
proof (*rule wfI*)
fix *z* :: ′*a* × ′*b* **and** *P*
assume ∗ [*rule-format*]: ∀ *u*. (∀ *v*. (*v*, *u*) ∈ *ra* <∗*lex*∗> *rb* ⟶ *P v*) ⟶ *P u*
obtain *x y* **where** *zeq*: *z* = (*x*,*y*)
by *fastforce*
have *P*(*x*,*y*) **using** ‹*wf ra*›
proof (*induction x arbitrary*: *y rule*: *wf-induct-rule*)
case (*less x*)
note *lessx* = *less*
show *?case* **using** ‹*wf rb*› *less*
proof (*induction y rule*: *wf-induct-rule*)
case (*less y*)
show *?case*
by (*force intro*: ∗ *less.IH lessx*)
qed
qed
then show *P z*
by (*simp add*: *zeq*)
qed *auto*

lemma *refl-lex-prod*[*simp*]: *refl* r_B ⟹ *refl* (r_A <∗*lex*∗> r_B)
by (*auto intro*!: *reflI dest*: *refl-onD*)

lemma *irrefl-on-lex-prod*[*simp*]:
irrefl-on A r_A ⟹ *irrefl-on B* r_B ⟹ *irrefl-on* (*A* × *B*) (r_A <∗*lex*∗> r_B)
by (*auto intro*!: *irrefl-onI dest*: *irrefl-onD*)

lemma *irrefl-lex-prod*[*simp*]: *irrefl* r_A ⟹ *irrefl* r_B ⟹ *irrefl* (r_A <∗*lex*∗> r_B)
by (*rule irrefl-on-lex-prod*[*of UNIV - UNIV*, *unfolded UNIV-Times-UNIV*])

lemma *sym-on-lex-prod*[*simp*]:
sym-on A r_A ⟹ *sym-on* B r_B ⟹ *sym-on* (A × B) (r_A <*lex*> r_B)
by (*auto intro*!: *sym-onI dest*: *sym-onD*)

lemma *sym-lex-prod*[*simp*]:
sym r_A ⟹ *sym* r_B ⟹ *sym* (r_A <*lex*> r_B)
by (*rule sym-on-lex-prod*[*of UNIV - UNIV*, *unfolded UNIV-Times-UNIV*])

lemma *asym-on-lex-prod*[*simp*]:
asym-on A r_A ⟹ *asym-on* B r_B ⟹ *asym-on* (A × B) (r_A <*lex*> r_B)
by (*auto intro*!: *asym-onI dest*: *asym-onD*)

lemma *asym-lex-prod*[*simp*]:
asym r_A ⟹ *asym* r_B ⟹ *asym* (r_A <*lex*> r_B)
by (*rule asym-on-lex-prod*[*of UNIV - UNIV*, *unfolded UNIV-Times-UNIV*])

lemma *trans-on-lex-prod*[*simp*]:
assumes *trans-on* A r_A **and** *trans-on* B r_B
shows *trans-on* (A × B) (r_A <*lex*> r_B)
proof (*rule trans-onI*)
fix x y z
show $x \in A \times B \Longrightarrow y \in A \times B \Longrightarrow z \in A \times B \Longrightarrow$
(x, y) ∈ r_A <*lex*> r_B ⟹ (y, z) ∈ r_A <*lex*> r_B ⟹ (x, z) ∈ r_A <*lex*> r_B
using *trans-onD*[*OF* ‹*trans-on A* r_A›, *of fst x fst y fst z*]
using *trans-onD*[*OF* ‹*trans-on B* r_B›, *of snd x snd y snd z*]
by *auto*
qed

lemma *trans-lex-prod* [*simp*,*intro*!]: *trans* r_A ⟹ *trans* r_B ⟹ *trans* (r_A <*lex*> r_B)
by (*rule trans-on-lex-prod*[*of UNIV - UNIV*, *unfolded UNIV-Times-UNIV*])

lemma *total-on-lex-prod*[*simp*]:
total-on A r_A ⟹ *total-on* B r_B ⟹ *total-on* (A × B) (r_A <*lex*> r_B)
by (*auto simp*: *total-on-def*)

lemma *total-lex-prod*[*simp*]: *total* r_A ⟹ *total* r_B ⟹ *total* (r_A <*lex*> r_B)
by (*rule total-on-lex-prod*[*of UNIV - UNIV*, *unfolded UNIV-Times-UNIV*])

lexicographic combinations with measure functions

definition *mlex-prod* :: ('*a* ⇒ *nat*) ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* (**infixr** ‹<*mlex*>› *80*)
where *f* <*mlex*> *R* = *inv-image* (*less-than* <*lex*> *R*) (λ*x*. (*f x*, *x*))

lemma
wf-mlex: *wf R* ⟹ *wf* (*f* <*mlex*> *R*) **and**
mlex-less: *f x* < *f y* ⟹ (*x*, *y*) ∈ *f* <*mlex*> *R* **and**

mlex-leq: *f x* ≤ *f y* ⟹ (*x*, *y*) ∈ *R* ⟹ (*x*, *y*) ∈ *f* <*mlex*> *R* **and**
mlex-iff: (*x*, *y*) ∈ *f* <*mlex*> *R* ⟷ *f x* < *f y* ∨ *f x* = *f y* ∧ (*x*, *y*) ∈ *R*
by (*auto simp*: *mlex-prod-def*)

Proper subset relation on finite sets.

definition *finite-psubset* :: (′*a set* × ′*a set*) *set*
where *finite-psubset* = {(*A*, *B*). *A* ⊂ *B* ∧ *finite B*}

lemma *wf-finite-psubset*[*simp*]: *wf finite-psubset*
apply (*unfold finite-psubset-def*)
apply (*rule wf-measure* [*THEN wf-subset*])
apply (*simp add*: *measure-def inv-image-def less-than-def less-eq*)
apply (*fast elim*!: *psubset-card-mono*)
done

lemma *trans-finite-psubset*: *trans finite-psubset*
by (*auto simp*: *finite-psubset-def less-le trans-def*)

lemma *in-finite-psubset*[*simp*]: (*A*, *B*) ∈ *finite-psubset* ⟷ *A* ⊂ *B* ∧ *finite B*
unfolding *finite-psubset-def* **by** *auto*

max- and min-extension of order to finite sets

inductive-set *max-ext* :: (′*a* × ′*a*) *set* ⇒ (′*a set* × ′*a set*) *set*
for *R* :: (′*a* × ′*a*) *set*
where *max-extI*[*intro*]:
finite X ⟹ *finite Y* ⟹ *Y* ≠ {} ⟹ (⋀*x*. *x* ∈ *X* ⟹ ∃ *y*∈*Y*. (*x*, *y*) ∈ *R*) ⟹ (*X*, *Y*) ∈ *max-ext R*

lemma *max-ext-wf*:
assumes *wf*: *wf r*
shows *wf* (*max-ext r*)
proof (*rule acc-wfI*, *intro allI*)
show *M* ∈ *acc* (*max-ext r*) (**is** - ∈ *?W*) **for** *M*
proof (*induct M rule*: *infinite-finite-induct*)
case *empty*
show *?case*
by (*rule accI*) (*auto elim*: *max-ext.cases*)
next
case (*insert a M*)
from *wf* ‹*M* ∈ *?W*› ‹*finite M*› **show** *insert a M* ∈ *?W*
proof (*induct arbitrary*: *M*)
fix *M a*
assume *M* ∈ *?W*
assume [*intro*]: *finite M*
assume *hyp*: ⋀*b M*. (*b*, *a*) ∈ *r* ⟹ *M* ∈ *?W* ⟹ *finite M* ⟹ *insert b M* ∈ *?W*
have *add-less*: *M* ∈ *?W* ⟹ (⋀*y*. *y* ∈ *N* ⟹ (*y*, *a*) ∈ *r*) ⟹ *N* ∪ *M* ∈ *?W*
if *finite N finite M* **for** *N M* :: ′*a set*
using *that* **by** (*induct N arbitrary*: *M*) (*auto simp*: *hyp*)

```
      show insert a M ∈ ?W
      proof (rule accI)
        fix N
        assume Nless: (N, insert a M) ∈ max-ext r
        then have *: ⋀x. x ∈ N ⟹ (x, a) ∈ r ∨ (∃y ∈ M. (x, y) ∈ r)
          by (auto elim!: max-ext.cases)

        let ?N1 = {n ∈ N. (n, a) ∈ r}
        let ?N2 = {n ∈ N. (n, a) ∉ r}
        have N: ?N1 ∪ ?N2 = N by (rule set-eqI) auto
        from Nless have finite N by (auto elim: max-ext.cases)
        then have finites: finite ?N1 finite ?N2 by auto

        have ?N2 ∈ ?W
        proof (cases M = {})
          case [simp]: True
          have Mw: {} ∈ ?W by (rule accI) (auto elim: max-ext.cases)
          from * have ?N2 = {} by auto
          with Mw show ?N2 ∈ ?W by (simp only:)
        next
          case False
          from * finites have N2: (?N2, M) ∈ max-ext r
            using max-extI[OF - - ‹M ≠ {}›, where ?X = ?N2] by auto
          with ‹M ∈ ?W› show ?N2 ∈ ?W by (rule acc-downward)
        qed
        with finites have ?N1 ∪ ?N2 ∈ ?W
          by (rule add-less) simp
        then show N ∈ ?W by (simp only: N)
      qed
    qed
  next
    case infinite
    show ?case
      by (rule accI) (auto elim: max-ext.cases simp: infinite)
  qed
qed

lemma max-ext-additive: (A, B) ∈ max-ext R ⟹ (C, D) ∈ max-ext R ⟹ (A ∪
C, B ∪ D) ∈ max-ext R
  by (force elim!: max-ext.cases)

definition min-ext :: ('a × 'a) set ⇒ ('a set × 'a set) set
  where min-ext r = {(X, Y) | X Y. X ≠ {} ∧ (∀y ∈ Y. (∃x ∈ X. (x, y) ∈ r))}

lemma min-ext-wf:
  assumes wf r
  shows wf (min-ext r)
proof (rule wfI-min)
  show ∃m ∈ Q. (∀n. (n, m) ∈ min-ext r ⟶ n ∉ Q) if nonempty: x ∈ Q
```

for Q :: $'a$ *set set* **and** x
proof (*cases* $Q = \{\{\}\}$)
case *True*
then show *?thesis* **by** (*simp add*: *min-ext-def*)
next
case *False*
with *nonempty* **obtain** e x **where** $x \in Q$ $e \in x$ **by** *force*
then have eU: $e \in \bigcup Q$ **by** *auto*
with ‹*wf r*›
obtain z **where** z: $z \in \bigcup Q$ $\bigwedge y.\ (y,\ z) \in r \Longrightarrow y \notin \bigcup Q$
by (*erule wfE-min*)
from z **obtain** m **where** $m \in Q$ $z \in m$ **by** *auto*
from ‹$m \in Q$› **show** *?thesis*
proof (*intro rev-bexI allI impI*)
fix n
assume *smaller*: $(n,\ m) \in$ *min-ext r*
with ‹$z \in m$› **obtain** y **where** $y \in n$ $(y,\ z) \in r$
by (*auto simp*: *min-ext-def*)
with $z(2)$ **show** $n \notin Q$ **by** *auto*
qed
qed
qed

22.12.4 Bounded increase must terminate

lemma *wf-bounded-measure*:
fixes *ub* :: $'a \Rightarrow nat$
and f :: $'a \Rightarrow nat$
assumes $\bigwedge a\ b.\ (b,\ a) \in r \Longrightarrow ub\ b \leq ub\ a \wedge ub\ a \geq f\ b \wedge f\ b > f\ a$
shows *wf r*
by (*rule wf-subset*[*OF wf-measure*[*of* $\lambda a.\ ub\ a - f\ a$]]) (*auto dest*: *assms*)

lemma *wf-bounded-set*:
fixes *ub* :: $'a \Rightarrow 'b$ *set*
and f :: $'a \Rightarrow 'b$ *set*
assumes $\bigwedge a\ b.\ (b,a) \in r \Longrightarrow$ *finite* $(ub\ a) \wedge ub\ b \subseteq ub\ a \wedge ub\ a \supseteq f\ b \wedge f\ b \supset f$
a
shows *wf r*
apply (*rule wf-bounded-measure*[*of r* $\lambda a.$ *card* $(ub\ a)$ $\lambda a.$ *card* $(f\ a)$])
apply (*drule assms*)
apply (*blast intro*: *card-mono finite-subset psubset-card-mono dest*: *psubset-eq*[*THEN*
iffD2])
done

lemma *finite-subset-wf*:
assumes *finite A*
shows *wf* $\{(X,\ Y).\ X \subset Y \wedge Y \subseteq A\}$
by (*rule wf-subset*[*OF wf-finite-psubset*[*unfolded finite-psubset-def*]])
(*auto intro*: *finite-subset*[*OF - assms*])

hide-const (**open**) *acc accp*

22.13 Code Generation Setup

Code equations with *wf* or *wfp* on the left-hand side are not supported by the code generation module because of the *UNIV* hidden behind the abbreviations. To sidestep this problem, we provide the following wrapper definitions and use *code-abbrev* to register the definitions with the pre- and post-processors of the code generator.

definition *wf-code* :: $('a \times 'a)$ *set* $\Rightarrow$ *bool* **where**
[*code-abbrev*]: *wf-code r* $\longleftrightarrow$ *wf r*

definition *wfp-code* :: $('a \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow$ *bool* **where**
[*code-abbrev*]: *wfp-code R* $\longleftrightarrow$ *wfp R*

end

23 Well-Founded Recursion Combinator

theory *Wfrec*
imports *Wellfounded*
begin

inductive *wfrec-rel* :: $('a \times 'a)$ *set* $\Rightarrow (('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)) \Rightarrow 'a \Rightarrow 'b \Rightarrow$ *bool*
for *R F*
where *wfrecI*: $(\bigwedge z.\ (z, x) \in R \Longrightarrow$ *wfrec-rel R F z* $(g\ z)) \Longrightarrow$ *wfrec-rel R F x* $(F\ g\ x)$

definition *cut* :: $('a \Rightarrow 'b) \Rightarrow ('a \times 'a)$ *set* $\Rightarrow 'a \Rightarrow 'a \Rightarrow 'b$
where *cut f R x* $= (\lambda y.$ *if* $(y, x) \in R$ *then f y else undefined*$)$

definition *adm-wf* :: $('a \times 'a)$ *set* $\Rightarrow (('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)) \Rightarrow$ *bool*
where *adm-wf R F* $\longleftrightarrow (\forall f\ g\ x.\ (\forall z.\ (z, x) \in R \longrightarrow f\ z = g\ z) \longrightarrow F\ f\ x = F\ g\ x)$

definition *wfrec* :: $('a \times 'a)$ *set* $\Rightarrow (('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)) \Rightarrow ('a \Rightarrow 'b)$
where *wfrec R F* $= (\lambda x.$ *THE y. wfrec-rel R* $(\lambda f\ x.\ F$ $($*cut f R x*$)\ x)\ x\ y)$

lemma *cuts-eq*: $($*cut f R x* $=$ *cut g R x*$) \longleftrightarrow (\forall y.\ (y, x) \in R \longrightarrow f\ y = g\ y)$
by (*simp add*: *fun-eq-iff cut-def*)

lemma *cut-apply*: $(x, a) \in R \Longrightarrow$ *cut f R a x* $= f\ x$
by (*simp add*: *cut-def*)

Inductive characterization of *wfrec* combinator; for details see: John Harrison, "Inductive definitions: automation and application".

lemma *theI-unique*: $\exists!x.\ P\ x \Longrightarrow P\ x \longleftrightarrow x = The\ P$
by (*auto intro*: *the-equality*[*symmetric*] *theI*)

lemma *wfrec-unique*:
assumes *adm-wf R F wf R*
shows $\exists!y.$ *wfrec-rel R F x y*
using ‹*wf R*›
proof *induct*
define *f* **where** *f y* = (*THE z. wfrec-rel R F y z*) **for** *y*
case (*less x*)
then have $\bigwedge y\ z.\ (y, x) \in R \Longrightarrow$ *wfrec-rel R F y z* $\longleftrightarrow z = f\ y$
unfolding *f-def* **by** (*rule theI-unique*)
with ‹*adm-wf R F*› **show** *?case*
by (*subst wfrec-rel.simps*) (*auto simp*: *adm-wf-def*)
qed

lemma *adm-lemma*: *adm-wf R* ($\lambda f\ x.\ F$ (*cut f R x*) *x*)
by (*auto simp*: *adm-wf-def intro*!: *arg-cong*[**where** $f=\lambda x.\ F\ x\ y$ **for** *y*] *cuts-eq*[*THEN iffD2*])

lemma *wfrec*: *wf R* $\Longrightarrow$ *wfrec R F a* = *F* (*cut* (*wfrec R F*) *R a*) *a*
apply (*simp add*: *wfrec-def*)
apply (*rule adm-lemma* [*THEN wfrec-unique*, *THEN the1-equality*])
apply *assumption*
apply (*rule wfrec-rel.wfrecI*)
apply (*erule adm-lemma* [*THEN wfrec-unique*, *THEN theI'*])
done

This form avoids giant explosions in proofs. NOTE USE OF $\equiv$.

lemma *def-wfrec*: $f \equiv$ *wfrec R F* $\Longrightarrow$ *wf R* $\Longrightarrow f\ a = F$ (*cut f R a*) *a*
by (*auto intro*: *wfrec*)

23.0.1 Well-founded recursion via genuine fixpoints

lemma *wfrec-fixpoint*:
assumes *wf*: *wf R*
and *adm*: *adm-wf R F*
shows *wfrec R F* = *F* (*wfrec R F*)
proof (*rule ext*)
fix *x*
have *wfrec R F x* = *F* (*cut* (*wfrec R F*) *R x*) *x*
using *wfrec*[*of R F*] *wf* **by** *simp*
also
have $\bigwedge y.\ (y, x) \in R \Longrightarrow$ *cut* (*wfrec R F*) *R x y* = *wfrec R F y*
by (*auto simp add*: *cut-apply*)
then have *F* (*cut* (*wfrec R F*) *R x*) *x* = *F* (*wfrec R F*) *x*
using *adm adm-wf-def*[*of R F*] **by** *auto*
finally show *wfrec R F x* = *F* (*wfrec R F*) *x* **.**
qed

lemma *wfrec-def-adm*: $f \equiv wfrec\ R\ F \Longrightarrow wf\ R \Longrightarrow adm\text{-}wf\ R\ F \Longrightarrow f = F\ f$
using *wfrec-fixpoint* **by** *simp*

23.1 Wellfoundedness of *same-fst*

definition *same-fst* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow ('b \times 'b)\ set) \Rightarrow (('a \times 'b) \times ('a \times 'b))\ set$
where $same\text{-}fst\ P\ R = \{((x', y'), (x, y)) . x' = x \wedge P\ x \wedge (y',y) \in R\ x\}$
— For *wfrec* declarations where the first n parameters stay unchanged in the recursive call.

lemma *same-fstI* [*intro*!]: $P\ x \Longrightarrow (y', y) \in R\ x \Longrightarrow ((x, y'), (x, y)) \in same\text{-}fst\ P\ R$
by (*simp add*: *same-fst-def*)

lemma *wf-same-fst*:
assumes $\bigwedge x.\ P\ x \Longrightarrow wf\ (R\ x)$
shows $wf\ (same\text{-}fst\ P\ R)$
proof −
have $\bigwedge a\ b\ Q.\ \forall a\ b.\ (\forall x.\ P\ a \wedge (x, b) \in R\ a \longrightarrow Q\ (a, x)) \longrightarrow Q\ (a, b) \Longrightarrow Q\ (a, b)$
proof −
fix $Q\ a\ b$
assume $*$: $\forall a\ b.\ (\forall x.\ P\ a \wedge (x,b) \in R\ a \longrightarrow Q\ (a,x)) \longrightarrow Q\ (a,b)$
show $Q(a,b)$
proof (*cases wf* (*R a*))
case *True*
then show *?thesis*
by (*induction b rule*: *wf-induct-rule*) (*use* $*$ **in** *blast*)
qed (*use* $*$ *assms* **in** *blast*)
qed
then show *?thesis*
by (*clarsimp simp add*: *wf-def same-fst-def*)
qed

end

24 Orders as Relations

theory *Order-Relation*
imports *Wfrec*
begin

24.1 Orders on a set

definition $preorder\text{-}on\ A\ r \equiv refl\text{-}on\ A\ r \wedge trans\ r$

definition $partial\text{-}order\text{-}on\ A\ r \equiv preorder\text{-}on\ A\ r \wedge antisym\ r$

definition *linear-order-on* $A\ r \equiv$ *partial-order-on* $A\ r \wedge$ *total-on* $A\ r$

definition *strict-linear-order-on* $A\ r \equiv$ *trans* $r \wedge$ *irrefl* $r \wedge$ *total-on* $A\ r$

definition *well-order-on* $A\ r \equiv$ *linear-order-on* $A\ r \wedge$ *wf*$(r - Id)$

lemmas *order-on-defs* =
preorder-on-def partial-order-on-def linear-order-on-def
strict-linear-order-on-def well-order-on-def

lemma *partial-order-onD*:
assumes *partial-order-on* $A\ r$ **shows** *refl-on* $A\ r$ **and** *trans* r **and** *antisym* r
using *assms* **unfolding** *partial-order-on-def preorder-on-def* **by** *auto*

lemma *preorder-on-empty*[*simp*]: *preorder-on* {} {}
by (*simp add*: *preorder-on-def trans-def*)

lemma *partial-order-on-empty*[*simp*]: *partial-order-on* {} {}
by (*simp add*: *partial-order-on-def*)

lemma *lnear-order-on-empty*[*simp*]: *linear-order-on* {} {}
by (*simp add*: *linear-order-on-def*)

lemma *well-order-on-empty*[*simp*]: *well-order-on* {} {}
by (*simp add*: *well-order-on-def*)

lemma *preorder-on-converse*[*simp*]: *preorder-on* $A\ (r^{-1})$ = *preorder-on* $A\ r$
by (*simp add*: *preorder-on-def*)

lemma *partial-order-on-converse*[*simp*]: *partial-order-on* $A\ (r^{-1})$ = *partial-order-on* $A\ r$
by (*simp add*: *partial-order-on-def*)

lemma *linear-order-on-converse*[*simp*]: *linear-order-on* $A\ (r^{-1})$ = *linear-order-on* $A\ r$
by (*simp add*: *linear-order-on-def*)

lemma *partial-order-on-acyclic*:
partial-order-on $A\ r \Longrightarrow$ *acyclic* $(r - Id)$
by (*simp add*: *acyclic-irrefl partial-order-on-def preorder-on-def trans-diff-Id*)

lemma *partial-order-on-well-order-on*:
finite $r \Longrightarrow$ *partial-order-on* $A\ r \Longrightarrow$ *wf* $(r - Id)$
by (*simp add*: *finite-acyclic-wf partial-order-on-acyclic*)

lemma *strict-linear-order-on-diff-Id*: *linear-order-on* $A\ r \Longrightarrow$ *strict-linear-order-on*

A (*r* − *Id*)
  by (*simp add*: *order-on-defs trans-diff-Id*)

lemma *linear-order-on-singleton* [*simp*]: *linear-order-on* {*x*} {(*x*, *x*)}
  by (*simp add*: *order-on-defs*)

lemma *linear-order-on-acyclic*:
  assumes *linear-order-on A r*
  shows *acyclic* (*r* − *Id*)
  using *strict-linear-order-on-diff-Id*[*OF assms*]
  by (*auto simp add*: *acyclic-irrefl strict-linear-order-on-def*)

lemma *linear-order-on-well-order-on*:
  assumes *finite r*
  shows *linear-order-on A r* ⟷ *well-order-on A r*
  unfolding *well-order-on-def*
  using *assms finite-acyclic-wf*[*OF - linear-order-on-acyclic*, *of r*] **by** *blast*

24.2 Orders on the field

abbreviation *Refl r* ≡ *refl-on* (*Field r*) *r*

abbreviation *Preorder r* ≡ *preorder-on* (*Field r*) *r*

abbreviation *Partial-order r* ≡ *partial-order-on* (*Field r*) *r*

abbreviation *Total r* ≡ *total-on* (*Field r*) *r*

abbreviation *Linear-order r* ≡ *linear-order-on* (*Field r*) *r*

abbreviation *Well-order r* ≡ *well-order-on* (*Field r*) *r*

lemma *subset-Image-Image-iff*:
  Preorder r ⟹ *A* ⊆ *Field r* ⟹ *B* ⊆ *Field r* ⟹
    r '' *A* ⊆ *r* '' *B* ⟷ (∀ *a*∈*A*.∃ *b*∈*B*. (*b*, *a*) ∈ *r*)
  apply (*simp add*: *preorder-on-def refl-on-def Image-def subset-eq*)
  apply (*simp only*: *trans-def*)
  apply *fast*
  done

lemma *subset-Image1-Image1-iff*:
  Preorder r ⟹ *a* ∈ *Field r* ⟹ *b* ∈ *Field r* ⟹ *r* '' {*a*} ⊆ *r* '' {*b*} ⟷ (*b*, *a*) ∈
r
  by (*simp add*: *subset-Image-Image-iff*)

lemma *Refl-antisym-eq-Image1-Image1-iff*:
  assumes *Refl r*
    and *as*: *antisym r*

and *abf*: $a \in Field\ r\ b \in Field\ r$
shows $r `` \{a\} = r `` \{b\} \longleftrightarrow a = b$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then have $*$: $\bigwedge x.\ (a, x) \in r \longleftrightarrow (b, x) \in r$
by (*simp add*: *set-eq-iff*)
have $(a, a) \in r\ (b, b) \in r$ **using** ‹*Refl r*› *abf* **by** (*simp-all add*: *refl-on-def*)
then have $(a, b) \in r\ (b, a) \in r$ **using** $*$[*of a*] $*$[*of b*] **by** *simp-all*
then show *?rhs*
using ‹*antisym r*›[*unfolded antisym-def*] **by** *blast*
next
assume *?rhs*
then show *?lhs* **by** *fast*
qed

lemma *Partial-order-eq-Image1-Image1-iff*:
Partial-order r $\Longrightarrow a \in Field\ r \Longrightarrow b \in Field\ r \Longrightarrow r `` \{a\} = r `` \{b\} \longleftrightarrow a = b$
by (*auto simp*: *order-on-defs Refl-antisym-eq-Image1-Image1-iff*)

lemma *Total-Id-Field*:
assumes *Total r*
and *not-Id*: $\neg\ r \subseteq Id$
shows $Field\ r = Field\ (r - Id)$
proof −
have $Field\ r \subseteq Field\ (r - Id)$
proof (*rule subsetI*)
fix *a* **assume** $*$: $a \in Field\ r$
from *not-Id* **have** $r \neq \{\}$ **by** *fast*
with *not-Id* **obtain** *b* **and** *c* **where** $b \neq c \wedge (b,c) \in r$ **by** *auto*
then have $b \neq c \wedge \{b, c\} \subseteq Field\ r$ **by** (*auto simp*: *Field-def*)
with $*$ **obtain** *d* **where** $d \in Field\ r\ d \neq a$ **by** *auto*
with $*$ ‹*Total r*› **have** $(a, d) \in r \vee (d, a) \in r$ **by** (*simp add*: *total-on-def*)
with ‹$d \neq a$› **show** $a \in Field\ (r - Id)$ **unfolding** *Field-def* **by** *blast*
qed
then show *?thesis*
using *mono-Field*[*of r* − *Id r*] *Diff-subset*[*of r Id*] **by** *auto*
qed

24.3 Relations given by a predicate and the field

definition *relation-of* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set$
where $relation\text{-}of\ P\ A \equiv \{\ (a, b) \in A \times A.\ P\ a\ b\ \}$

lemma *Field-relation-of*:
assumes *refl-on A* (*relation-of P A*) **shows** *Field* (*relation-of P A*) = *A*
using *assms* **unfolding** *refl-on-def Field-def* **by** *auto*

lemma *partial-order-on-relation-ofI*:
assumes *refl*: $\bigwedge a.\ a \in A \Longrightarrow P\ a\ a$
and *trans*: $\bigwedge a\ b\ c.\ [\![\ a \in A;\ b \in A;\ c \in A\ ]\!] \Longrightarrow P\ a\ b \Longrightarrow P\ b\ c \Longrightarrow P\ a\ c$
and *antisym*: $\bigwedge a\ b.\ [\![\ a \in A;\ b \in A\ ]\!] \Longrightarrow P\ a\ b \Longrightarrow P\ b\ a \Longrightarrow a = b$
shows *partial-order-on A* (*relation-of P A*)
proof −
from *refl* **have** *refl-on A* (*relation-of P A*)
unfolding *refl-on-def relation-of-def* **by** *auto*
moreover have *trans* (*relation-of P A*) **and** *antisym* (*relation-of P A*)
unfolding *relation-of-def*
by (*auto intro*: *transI dest*: *trans*, *auto intro*: *antisymI dest*: *antisym*)
ultimately show *?thesis*
unfolding *partial-order-on-def preorder-on-def* **by** *simp*
qed

lemma *Partial-order-relation-ofI*:
assumes *partial-order-on A* (*relation-of P A*) **shows** *Partial-order* (*relation-of P A*)
using *Field-relation-of assms partial-order-on-def preorder-on-def* **by** *fastforce*

24.4 Orders on a type

abbreviation *strict-linear-order* ≡ *strict-linear-order-on UNIV*

abbreviation *linear-order* ≡ *linear-order-on UNIV*

abbreviation *well-order* ≡ *well-order-on UNIV*

24.5 Order-like relations

In this subsection, we develop basic concepts and results pertaining to order-like relations, i.e., to reflexive and/or transitive and/or symmetric and/or total relations. We also further define upper and lower bounds operators.

24.5.1 Auxiliaries

lemma *refl-on-domain*: *refl-on A r* $\Longrightarrow (a, b) \in r \Longrightarrow a \in A \wedge b \in A$
by (*auto simp add*: *refl-on-def*)

corollary *well-order-on-domain*: *well-order-on A r* $\Longrightarrow (a, b) \in r \Longrightarrow a \in A \wedge b \in A$
by (*auto simp add*: *refl-on-domain order-on-defs*)

lemma *well-order-on-Field*: *well-order-on A r* $\Longrightarrow A =$ *Field r*
by (*auto simp add*: *refl-on-def Field-def order-on-defs*)

lemma *well-order-on-Well-order*: *well-order-on A r* $\Longrightarrow A =$ *Field r* $\wedge$ *Well-order r*
using *well-order-on-Field* [*of A*] **by** *auto*

lemma *Total-subset-Id*:
  assumes *Total r*
    and $r \subseteq Id$
  shows $r = \{\} \vee (\exists a.\ r = \{(a, a)\})$
proof −
  have $\exists a.\ r = \{(a, a)\}$ **if** $r \neq \{\}$
  proof −
    from *that* **obtain** *a b* **where** *ab*: $(a, b) \in r$ **by** *fast*
    with ‹$r \subseteq Id$› **have** $a = b$ **by** *blast*
    with *ab* **have** *aa*: $(a, a) \in r$ **by** *simp*
    have $a = c \wedge a = d$ **if** $(c, d) \in r$ **for** *c d*
    proof −
      from *that* **have** $\{a, c, d\} \subseteq Field\ r$
        using *ab* **unfolding** *Field-def* **by** *blast*
      then have $((a, c) \in r \vee (c, a) \in r \vee a = c) \wedge ((a, d) \in r \vee (d, a) \in r \vee a = d)$
        using ‹*Total r*› **unfolding** *total-on-def* **by** *blast*
      with ‹$r \subseteq Id$› **show** *?thesis* **by** *blast*
    qed
    then have $r \subseteq \{(a, a)\}$ **by** *auto*
    with *aa* **show** *?thesis* **by** *blast*
  qed
  then show *?thesis* **by** *blast*
qed

lemma *Linear-order-in-diff-Id*:
  assumes *Linear-order r*
    and $a \in Field\ r$
    and $b \in Field\ r$
  shows $(a, b) \in r \longleftrightarrow (b, a) \notin r - Id$
  using *assms* **unfolding** *order-on-defs total-on-def antisym-def Id-def refl-on-def* **by** *force*

24.5.2 The upper and lower bounds operators

Here we define upper ("above") and lower ("below") bounds operators. We think of r as a *non-strict* relation. The suffix S at the names of some operators indicates that the bounds are strict – e.g., *underS a* is the set of all strict lower bounds of a (w.r.t. r). Capitalization of the first letter in the name reminds that the operator acts on sets, rather than on individual elements.

definition *under* :: $'a\ rel \Rightarrow 'a \Rightarrow 'a\ set$
  where $under\ r\ a \equiv \{b.\ (b, a) \in r\}$

definition *underS* :: $'a\ rel \Rightarrow 'a \Rightarrow 'a\ set$
  where $underS\ r\ a \equiv \{b.\ b \neq a \wedge (b, a) \in r\}$

definition *Under* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow 'a\ set$
where $Under\ r\ A \equiv \{b \in Field\ r.\ \forall a \in A.\ (b,\ a) \in r\}$

definition *UnderS* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow 'a\ set$
where $UnderS\ r\ A \equiv \{b \in Field\ r.\ \forall a \in A.\ b \neq a \wedge (b,\ a) \in r\}$

definition *above* :: $'a\ rel \Rightarrow 'a \Rightarrow 'a\ set$
where $above\ r\ a \equiv \{b.\ (a,\ b) \in r\}$

definition *aboveS* :: $'a\ rel \Rightarrow 'a \Rightarrow 'a\ set$
where $aboveS\ r\ a \equiv \{b.\ b \neq a \wedge (a,\ b) \in r\}$

definition *Above* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow 'a\ set$
where $Above\ r\ A \equiv \{b \in Field\ r.\ \forall a \in A.\ (a,\ b) \in r\}$

definition *AboveS* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow 'a\ set$
where $AboveS\ r\ A \equiv \{b \in Field\ r.\ \forall a \in A.\ b \neq a \wedge (a,\ b) \in r\}$

definition *ofilter* :: $'a\ rel \Rightarrow 'a\ set \Rightarrow bool$
where $ofilter\ r\ A \equiv A \subseteq Field\ r \wedge (\forall a \in A.\ under\ r\ a \subseteq A)$

Note: In the definitions of *Above*[*S*] and *Under*[*S*], we bounded comprehension by *Field r* in order to properly cover the case of *A* being empty.

lemma *underS-subset-under*: $underS\ r\ a \subseteq under\ r\ a$
by (*auto simp add*: *underS-def under-def*)

lemma *underS-notIn*: $a \notin underS\ r\ a$
by (*simp add*: *underS-def*)

lemma *Refl-under-in*: $Refl\ r \Longrightarrow a \in Field\ r \Longrightarrow a \in under\ r\ a$
by (*simp add*: *refl-on-def under-def*)

lemma *AboveS-disjoint*: $A \cap (AboveS\ r\ A) = \{\}$
by (*auto simp add*: *AboveS-def*)

lemma *in-AboveS-underS*: $a \in Field\ r \Longrightarrow a \in AboveS\ r\ (underS\ r\ a)$
by (*auto simp add*: *AboveS-def underS-def*)

lemma *Refl-under-underS*: $Refl\ r \Longrightarrow a \in Field\ r \Longrightarrow under\ r\ a = underS\ r\ a \cup \{a\}$
unfolding *under-def underS-def*
using *refl-on-def*[*of - r*] **by** *fastforce*

lemma *underS-empty*: $a \notin Field\ r \Longrightarrow underS\ r\ a = \{\}$
by (*auto simp*: *Field-def underS-def*)

lemma *under-Field*: $under\ r\ a \subseteq Field\ r$
by (*auto simp*: *under-def Field-def*)

lemma *underS-Field*: *underS r a* ⊆ *Field r*
 by (*auto simp*: *underS-def Field-def*)

lemma *underS-Field2*: *a* ∈ *Field r* ⟹ *underS r a* ⊂ *Field r*
 using *underS-notIn underS-Field* **by** *fast*

lemma *underS-Field3*: *Field r* ≠ {} ⟹ *underS r a* ⊂ *Field r*
 by (*cases a* ∈ *Field r*) (*auto simp*: *underS-Field2 underS-empty*)

lemma *AboveS-Field*: *AboveS r A* ⊆ *Field r*
 by (*auto simp*: *AboveS-def Field-def*)

lemma *under-incr*:
 assumes *trans r*
  and (*a*, *b*) ∈ *r*
 shows *under r a* ⊆ *under r b*
 unfolding *under-def*
proof *safe*
 fix *x* **assume** (*x*, *a*) ∈ *r*
 with *assms trans-def*[*of r*] **show** (*x*, *b*) ∈ *r* **by** *blast*
qed

lemma *underS-incr*:
 assumes *trans r*
  and *antisym r*
  and *ab*: (*a*, *b*) ∈ *r*
 shows *underS r a* ⊆ *underS r b*
 unfolding *underS-def*
proof *safe*
 assume ∗: *b* ≠ *a* **and** ∗∗: (*b*, *a*) ∈ *r*
 with ‹*antisym r*› *antisym-def*[*of r*] *ab* **show** *False*
  by *blast*
next
 fix *x* **assume** *x* ≠ *a* (*x*, *a*) ∈ *r*
 with *ab* ‹*trans r*› *trans-def*[*of r*] **show** (*x*, *b*) ∈ *r*
  by *blast*
qed

lemma *underS-incl-iff*:
 assumes *LO*: *Linear-order r*
  and *INa*: *a* ∈ *Field r*
  and *INb*: *b* ∈ *Field r*
 shows *underS r a* ⊆ *underS r b* ⟷ (*a*, *b*) ∈ *r*
  (**is** *?lhs* ⟷ *?rhs*)
proof
 assume *?rhs*
 with ‹*Linear-order r*› **show** *?lhs*
  by (*simp add*: *order-on-defs underS-incr*)
next

assume ∗: *?lhs*
have $(a, b) \in r$ **if** $a = b$
using *assms that* **by** (*simp add*: *order-on-defs refl-on-def*)
moreover have *False* **if** $a \neq b$ $(b, a) \in r$
proof −
from *that* **have** $b \in$ *underS r a* **unfolding** *underS-def* **by** *blast*
with ∗ **have** $b \in$ *underS r b* **by** *blast*
then show *?thesis* **by** (*simp add*: *underS-notIn*)
qed
ultimately show $(a,b) \in r$
using *assms order-on-defs*[*of Field r r*] *total-on-def*[*of Field r r*] **by** *blast*
qed

lemma *finite-Partial-order-induct*[*consumes 3*, *case-names step*]:
assumes *Partial-order r*
and $x \in$ *Field r*
and *finite r*
and *step*: $\bigwedge x.\ x \in Field\ r \Longrightarrow (\bigwedge y.\ y \in aboveS\ r\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows *P x*
using *assms*(*2*)
proof (*induct rule*: *wf-induct*[*of* $r^{-1} - Id$])
case *1*
from *assms*(*1*,*3*) **show** *wf* $(r^{-1} - Id)$
using *partial-order-on-well-order-on partial-order-on-converse* **by** *blast*
next
case *prems*: (*2 x*)
show *?case*
by (*rule step*) (*use prems* **in** ‹*auto simp*: *aboveS-def intro*: *FieldI2*›)
qed

lemma *finite-Linear-order-induct*[*consumes 3*, *case-names step*]:
assumes *Linear-order r*
and $x \in$ *Field r*
and *finite r*
and *step*: $\bigwedge x.\ x \in Field\ r \Longrightarrow (\bigwedge y.\ y \in aboveS\ r\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows *P x*
using *assms*(*2*)
proof (*induct rule*: *wf-induct*[*of* $r^{-1} - Id$])
case *1*
from *assms*(*1*,*3*) **show** *wf* $(r^{-1} - Id)$
using *linear-order-on-well-order-on linear-order-on-converse*
unfolding *well-order-on-def* **by** *blast*
next
case *prems*: (*2 x*)
show *?case*
by (*rule step*) (*use prems* **in** ‹*auto simp*: *aboveS-def intro*: *FieldI2*›)
qed

24.6 Variations on Well-Founded Relations

This subsection contains some variations of the results from *HOL.Wellfounded*:

- means for slightly more direct definitions by well-founded recursion;
- variations of well-founded induction;
- means for proving a linear order to be a well-order.

24.6.1 Characterizations of well-foundedness

A transitive relation is well-founded iff it is "locally" well-founded, i.e., iff its restriction to the lower bounds of of any element is well-founded.

lemma *trans-wf-iff*:
  assumes *trans r*
  shows $wf\ r \longleftrightarrow (\forall a.\ wf\ (r \cap (r^{-1} `` \{a\} \times r^{-1} `` \{a\})))$
proof −
  define R **where** $R\ a = r \cap (r^{-1} `` \{a\} \times r^{-1} `` \{a\})$ **for** a
  have $wf\ (R\ a)$ **if** $wf\ r$ **for** a
    using *that R-def wf-subset*[*of r R a*] **by** *auto*
  moreover
  have $wf\ r$ **if** $*$: $\forall a.\ wf(R\ a)$
    unfolding *wf-def*
  proof *clarify*
    fix *phi a*
    assume $**$: $\forall a.\ (\forall b.\ (b, a) \in r \longrightarrow phi\ b) \longrightarrow phi\ a$
    define *chi* **where** $chi\ b \longleftrightarrow (b, a) \in r \longrightarrow phi\ b$ **for** b
    with $*$ **have** $wf\ (R\ a)$ **by** *auto*
    then have $(\forall b.\ (\forall c.\ (c, b) \in R\ a \longrightarrow chi\ c) \longrightarrow chi\ b) \longrightarrow (\forall b.\ chi\ b)$
      unfolding *wf-def* **by** *blast*
    also have $\forall b.\ (\forall c.\ (c, b) \in R\ a \longrightarrow chi\ c) \longrightarrow chi\ b$
    proof *safe*
      fix b
      assume $\forall c.\ (c, b) \in R\ a \longrightarrow chi\ c$
      moreover have $(b, a) \in r \Longrightarrow \forall c.\ (c, b) \in r \wedge (c, a) \in r \longrightarrow phi\ c \Longrightarrow phi$
b
      proof −
        assume $(b, a) \in r$ **and** $\forall c.\ (c, b) \in r \wedge (c, a) \in r \longrightarrow phi\ c$
        then have $\forall c.\ (c, b) \in r \longrightarrow phi\ c$
          using *assms trans-def*[*of r*] **by** *blast*
        with $**$ **show** $phi\ b$ **by** *blast*
      qed
      ultimately show $chi\ b$
        by (*auto simp add*: *chi-def R-def*)
    qed
    finally have $\forall b.\ chi\ b$ **.**

with ∗∗ *chi-def* **show** *phi a* **by** *blast*
qed
ultimately show *?thesis* **unfolding** *R-def* **by** *blast*
qed

A transitive relation is well-founded if all initial segments are finite.

corollary *wf-finite-segments*:
assumes *irrefl r* **and** *trans r* **and** $\bigwedge x.$ *finite* $\{y.\ (y, x) \in r\}$
shows *wf r*
proof −
have $\bigwedge a.$ *acyclic* $(r \cap \{x.\ (x, a) \in r\} \times \{x.\ (x, a) \in r\})$
proof −
fix *a*
have *trans* $(r \cap (\{x.\ (x, a) \in r\} \times \{x.\ (x, a) \in r\}))$
using *assms* **unfolding** *trans-def Field-def* **by** *blast*
then show *acyclic* $(r \cap \{x.\ (x, a) \in r\} \times \{x.\ (x, a) \in r\})$
using *assms acyclic-def assms irrefl-def* **by** *fastforce*
qed
then show *?thesis*
by (*clarsimp simp*: *trans-wf-iff wf-iff-acyclic-if-finite converse-def assms*)
qed

The next lemma is a variation of *wf-eq-minimal* from Wellfounded, allowing one to assume the set included in the field.

lemma *wf-eq-minimal2*: *wf r* $\longleftrightarrow$ $(\forall A.\ A \subseteq$ *Field r* $\wedge A \neq \{\} \longrightarrow (\exists a \in A.\ \forall a' \in A.\ (a', a) \notin r))$
proof−
let *?phi* = $\lambda A.\ A \neq \{\} \longrightarrow (\exists a \in A.\ \forall a' \in A.\ (a',a) \notin r)$
have *wf r* $\longleftrightarrow$ $(\forall A.$ *?phi A*$)$
proof
assume *wf r*
show $\forall A.$ *?phi A*
proof *clarify*
fix *A*:: *'a set*
assume $A \neq \{\}$
then obtain *x* **where** $x \in A$
by *auto*
show $\exists a \in A.\ \forall a' \in A.\ (a', a) \notin r$
apply (*rule wfE-min*[*of r x A*])
apply *fact*+
by *blast*
qed
next
assume ∗: $\forall A.$ *?phi A*
then show *wf r*
apply (*clarsimp simp*: *ex-in-conv* [*THEN sym*])
apply (*rule wfI-min*)
by *fast*
qed

also have ($\forall$ *A*. *?phi A*) $\longleftrightarrow$ ($\forall$ *B* $\subseteq$ *Field r*. *?phi B*)
proof
assume $\forall$ *A*. *?phi A*
then show $\forall$ *B* $\subseteq$ *Field r*. *?phi B* **by** *simp*
next
assume $*$: $\forall$ *B* $\subseteq$ *Field r*. *?phi B*
show $\forall$ *A*. *?phi A*
proof *clarify*
fix *A* :: *'a set*
assume $**$: *A* $\neq$ {}
define *B* **where** *B* = *A* $\cap$ *Field r*
show $\exists$ *a* $\in$ *A*. $\forall$ *a'* $\in$ *A*. (*a'*, *a*) $\notin$ *r*
proof (*cases B* = {})
case *True*
with $**$ **obtain** *a* **where** *a*: *a* $\in$ *A* *a* $\notin$ *Field r*
unfolding *B-def* **by** *blast*
with *a* **have** $\forall$ *a'* $\in$ *A*. (*a'*,*a*) $\notin$ *r*
unfolding *Field-def* **by** *blast*
with *a* **show** *?thesis* **by** *blast*
next
case *False*
have *B* $\subseteq$ *Field r* **unfolding** *B-def* **by** *blast*
with *False* $*$ **obtain** *a* **where** *a*: *a* $\in$ *B* $\forall$ *a'* $\in$ *B*. (*a'*, *a*) $\notin$ *r*
by *blast*
have (*a'*, *a*) $\notin$ *r* **if** *a'* $\in$ *A* **for** *a'*
proof
assume *a'a*: (*a'*, *a*) $\in$ *r*
with *that* **have** *a'* $\in$ *B* **unfolding** *B-def Field-def* **by** *blast*
with *a a'a* **show** *False* **by** *blast*
qed
with *a* **show** *?thesis* **unfolding** *B-def* **by** *blast*
qed
qed
qed
finally show *?thesis* **by** *blast*
qed

24.6.2 Characterizations of well-foundedness

The next lemma and its corollary enable one to prove that a linear order is a well-order in a way which is more standard than via well-foundedness of the strict version of the relation.

lemma *Linear-order-wf-diff-Id*:
assumes *Linear-order r*
shows *wf* (*r* − *Id*) $\longleftrightarrow$ ($\forall$ *A* $\subseteq$ *Field r*. *A* $\neq$ {} $\longrightarrow$ ($\exists$ *a* $\in$ *A*. $\forall$ *a'* $\in$ *A*. (*a*, *a'*) $\in$ *r*))
proof (*cases r* $\subseteq$ *Id*)
case *True*
then have $*$: *r* − *Id* = {} **by** *blast*

have *wf* $(r - Id)$ **by** (*simp add*: $*$)
moreover have $\exists\, a \in A.\ \forall\, a' \in A.\ (a,\ a') \in r$
if $*$: $A \subseteq$ *Field* r **and** $**$: $A \neq \{\}$ **for** A
proof −
from ‹*Linear-order r*› *True*
obtain a **where** a: $r = \{\} \vee r = \{(a,\ a)\}$
unfolding *order-on-defs* **using** *Total-subset-Id* [*of r*] **by** *blast*
with $*$ $**$ **have** $A = \{a\} \wedge r = \{(a,\ a)\}$
unfolding *Field-def* **by** *blast*
with a **show** *?thesis* **by** *blast*
qed
ultimately show *?thesis* **by** *blast*
next
case *False*
with ‹*Linear-order r*› **have** *Field*: *Field* r = *Field* $(r - Id)$
unfolding *order-on-defs* **using** *Total-Id-Field* [*of r*] **by** *blast*
show *?thesis*
proof
assume $*$: *wf* $(r - Id)$
show $\forall\, A \subseteq$ *Field* $r.\ A \neq \{\} \longrightarrow (\exists\, a \in A.\ \forall\, a' \in A.\ (a,\ a') \in r)$
proof *clarify*
fix A
assume $**$: $A \subseteq$ *Field* r **and** $***$: $A \neq \{\}$
then have $\exists\, a \in A.\ \forall\, a' \in A.\ (a',a) \notin r - Id$
using *Field* $*$ **unfolding** *wf-eq-minimal2* **by** *simp*
moreover have $\forall\, a \in A.\ \forall\, a' \in A.\ (a,\ a') \in r \longleftrightarrow (a',\ a) \notin r - Id$
using *Linear-order-in-diff-Id* [*OF* ‹*Linear-order r*›] $**$ **by** *blast*
ultimately show $\exists\, a \in A.\ \forall\, a' \in A.\ (a,\ a') \in r$ **by** *blast*
qed
next
assume $*$: $\forall\, A \subseteq$ *Field* $r.\ A \neq \{\} \longrightarrow (\exists\, a \in A.\ \forall\, a' \in A.\ (a,\ a') \in r)$
show *wf* $(r - Id)$
unfolding *wf-eq-minimal2*
proof *clarify*
fix A
assume $**$: $A \subseteq$ *Field*$(r - Id)$ **and** $***$: $A \neq \{\}$
then have $\exists\, a \in A.\ \forall\, a' \in A.\ (a,a') \in r$
using *Field* $*$ **by** *simp*
moreover have $\forall\, a \in A.\ \forall\, a' \in A.\ (a,\ a') \in r \longleftrightarrow (a',\ a) \notin r - Id$
using *Linear-order-in-diff-Id* [*OF* ‹*Linear-order r*›] $**$ *mono-Field*[*of* $r - Id$ r] **by** *blast*
ultimately show $\exists\, a \in A.\ \forall\, a' \in A.\ (a',a) \notin r - Id$
by *blast*
qed
qed
qed

corollary *Linear-order-Well-order-iff*:
Linear-order $r \Longrightarrow$

Well-order r $\longleftrightarrow$ ($\forall$ *A* $\subseteq$ *Field r*. *A* $\neq$ {} $\longrightarrow$ ($\exists$ *a* $\in$ *A*. $\forall$ *a'* $\in$ *A*. (*a*, *a'*) $\in$ *r*))
unfolding *well-order-on-def* **using** *Linear-order-wf-diff-Id*[*of r*] **by** *blast*

end

25 Hilbert's Epsilon-Operator and the Axiom of Choice

theory *Hilbert-Choice*
imports *Wellfounded*
keywords *specification* :: *thy-goal-defn*
begin

25.1 Hilbert's epsilon

axiomatization *Eps* :: ('*a* ⇒ *bool*) ⇒ '*a*
where *someI*: *P x* ⟹ *P* (*Eps P*)

syntax (*epsilon*)
-Eps :: *pttrn* ⇒ *bool* ⇒ '*a* (‹(‹*indent=3 notation=*‹*binder* ϵ ››ϵ -./ -)› [*0*, *10*] *10*)
syntax (*input*)
-Eps :: *pttrn* ⇒ *bool* ⇒ '*a* (‹(‹*indent=3 notation=*‹*binder* @››@ -./ -)› [*0*, *10*] *10*)
syntax
-Eps :: *pttrn* ⇒ *bool* ⇒ '*a* (‹(‹*indent=3 notation=*‹*binder SOME*››*SOME* -./ -)› [*0*, *10*] *10*)

syntax-consts *-Eps* ⇌ *Eps*

translations
SOME x. *P* ⇌ *CONST Eps* (λ*x*. *P*)

print-translation ‹
[(***const-syntax*** ‹*Eps*›, *fn ctxt* => *fn* [*Abs abs*] =>
let val (*x*, *t*) = *Syntax-Trans.atomic-abs-tr' ctxt abs*
in Syntax.const ***syntax-const*** ‹*-Eps*› $ *x* $ *t end*)]
› — to avoid eta-contraction of body

definition *inv-into* :: '*a set* ⇒ ('*a* ⇒ '*b*) ⇒ ('*b* ⇒ '*a*) **where**
inv-into A f = (λ*x*. *SOME y*. *y* ∈ *A* ∧ *f y* = *x*)

lemma *inv-into-def2*: *inv-into A f x* = (*SOME y*. *y* ∈ *A* ∧ *f y* = *x*)
by(*simp add*: *inv-into-def*)

abbreviation *inv* :: ('*a* ⇒ '*b*) ⇒ ('*b* ⇒ '*a*) **where**
inv ≡ *inv-into UNIV*

25.2 Hilbert's Epsilon-operator

lemma *Eps-cong*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $Eps\ P = Eps\ Q$
using *ext*[*of P Q*, *OF assms*] **by** *simp*

Easier to use than *someI* if the witness comes from an existential formula.

lemma *someI-ex* [*elim?*]: $\exists x.\ P\ x \Longrightarrow P\ (SOME\ x.\ P\ x)$
by (*elim exE someI*)

lemma *some-eq-imp*:
assumes $Eps\ P = a\ P\ b$ **shows** $P\ a$
using *assms someI-ex* **by** *force*

Easier to use than *someI* because the conclusion has only one occurrence of P.

lemma *someI2*: $P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (SOME\ x.\ P\ x)$
by (*blast intro*: *someI*)

Easier to use than *someI2* if the witness comes from an existential formula.

lemma *someI2-ex*: $\exists a.\ P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (SOME\ x.\ P\ x)$
by (*blast intro*: *someI2*)

lemma *someI2-bex*: $\exists a\in A.\ P\ a \Longrightarrow (\bigwedge x.\ x \in A \wedge P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (SOME\ x.\ x \in A \wedge P\ x)$
by (*blast intro*: *someI2*)

lemma *some-equality* [*intro*]: $P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow x = a) \Longrightarrow (SOME\ x.\ P\ x) = a$
by (*blast intro*: *someI2*)

lemma *some1-equality*: $\exists!x.\ P\ x \Longrightarrow P\ a \Longrightarrow (SOME\ x.\ P\ x) = a$
by *blast*

lemma *some-eq-ex*: $P\ (SOME\ x.\ P\ x) \longleftrightarrow (\exists x.\ P\ x)$
by (*blast intro*: *someI*)

lemma *some-in-eq*: $(SOME\ x.\ x \in A) \in A \longleftrightarrow A \neq \{\}$
unfolding *ex-in-conv*[*symmetric*] **by** (*rule some-eq-ex*)

lemma *some-eq-trivial* [*simp*]: $(SOME\ y.\ y = x) = x$
by (*rule some-equality*) (*rule refl*)

lemma *some-sym-eq-trivial* [*simp*]: $(SOME\ y.\ x = y) = x$
by (*iprover intro*: *some-equality*)

25.3 Axiom of Choice, Proved Using the Description Operator

lemma *choice*: $\forall x. \exists y. Q\ x\ y \Longrightarrow \exists f. \forall x. Q\ x\ (f\ x)$
  by (*fast elim*: *someI*)

lemma *bchoice*: $\forall x{\in}S. \exists y. Q\ x\ y \Longrightarrow \exists f. \forall x{\in}S. Q\ x\ (f\ x)$
  by (*fast elim*: *someI*)

lemma *choice-iff*: $(\forall x. \exists y. Q\ x\ y) \longleftrightarrow (\exists f. \forall x. Q\ x\ (f\ x))$
  by (*fast elim*: *someI*)

lemma *choice-iff'*: $(\forall x. P\ x \longrightarrow (\exists y. Q\ x\ y)) \longleftrightarrow (\exists f. \forall x. P\ x \longrightarrow Q\ x\ (f\ x))$
  by (*fast elim*: *someI*)

lemma *bchoice-iff*: $(\forall x{\in}S. \exists y. Q\ x\ y) \longleftrightarrow (\exists f. \forall x{\in}S. Q\ x\ (f\ x))$
  by (*fast elim*: *someI*)

lemma *bchoice-iff'*: $(\forall x{\in}S. P\ x \longrightarrow (\exists y. Q\ x\ y)) \longleftrightarrow (\exists f. \forall x{\in}S. P\ x \longrightarrow Q\ x$
$(f\ x))$
  by (*fast elim*: *someI*)

lemma *dependent-nat-choice*:
  assumes *1*: $\exists x. P\ 0\ x$
    and *2*: $\bigwedge x\ n. P\ n\ x \Longrightarrow \exists y. P\ (Suc\ n)\ y \wedge Q\ n\ x\ y$
  shows $\exists f. \forall n. P\ n\ (f\ n) \wedge Q\ n\ (f\ n)\ (f\ (Suc\ n))$
proof (*intro exI allI conjI*)
  fix *n*
  define *f* **where** $f = rec\text{-}nat\ (SOME\ x. P\ 0\ x)\ (\lambda n\ x. SOME\ y. P\ (Suc\ n)\ y \wedge$
$Q\ n\ x\ y)$
  then have $P\ 0\ (f\ 0)\ \bigwedge n. P\ n\ (f\ n) \Longrightarrow P\ (Suc\ n)\ (f\ (Suc\ n)) \wedge Q\ n\ (f\ n)\ (f$
$(Suc\ n))$
    using *someI-ex*[*OF 1*] *someI-ex*[*OF 2*] **by** *simp-all*
  then show $P\ n\ (f\ n)\ Q\ n\ (f\ n)\ (f\ (Suc\ n))$
    by (*induct n*) *auto*
qed

lemma *finite-subset-Union*:
  assumes *finite* $A\ A \subseteq \bigcup\mathcal{B}$
  obtains $\mathcal{F}$ **where** *finite* $\mathcal{F}\ \mathcal{F} \subseteq \mathcal{B}\ A \subseteq \bigcup\mathcal{F}$
proof −
  have $\forall x{\in}A. \exists B{\in}\mathcal{B}. x{\in}B$
    using *assms* **by** *blast*
  then obtain *f* **where** *f*: $\bigwedge x. x \in A \Longrightarrow f\ x \in \mathcal{B} \wedge x \in f\ x$
    by (*auto simp add*: *bchoice-iff Bex-def*)
  show *thesis*
  proof
    show *finite* $(f\ `\ A)$
      using *assms* **by** *auto*
  qed (*use f* **in** *auto*)

qed

25.4 Getting an element of a nonempty set

definition *some-elem* :: $'a$ *set* $\Rightarrow$ $'a$
  where *some-elem* $A = (SOME\ x.\ x \in A)$

lemma *some-elem-eq* [*simp*]: *some-elem* $\{x\} = x$
  by (*simp add*: *some-elem-def*)

lemma *some-elem-nonempty*: $A \neq \{\} \Longrightarrow$ *some-elem* $A \in A$
  unfolding *some-elem-def* **by** (*auto intro*: *someI*)

lemma *is-singleton-some-elem*: *is-singleton* $A \longleftrightarrow A = \{$*some-elem* $A\}$
  by (*auto simp*: *is-singleton-def*)

lemma *some-elem-image-unique*:
  assumes $A \neq \{\}$
    and $*$: $\bigwedge y.\ y \in A \Longrightarrow f\ y = a$
  shows *some-elem* $(f\ `\ A) = a$
  unfolding *some-elem-def*
proof (*rule some1-equality*)
  from ‹$A \neq \{\}$› **obtain** y **where** $y \in A$ **by** *auto*
  with $*$ ‹$y \in A$› **have** $a \in f\ `\ A$ **by** *blast*
  then show $a \in f\ `\ A$ **by** *auto*
  with $*$ **show** $\exists!x.\ x \in f\ `\ A$
    by *auto*
qed

25.5 Function Inverse

lemma *inv-def*: *inv* $f = (\lambda y.\ SOME\ x.\ f\ x = y)$
  by (*simp add*: *inv-into-def*)

lemma *inv-into-into*: $x \in f\ `\ A \Longrightarrow$ *inv-into* $A\ f\ x \in A$
  by (*simp add*: *inv-into-def*) (*fast intro*: *someI2*)

lemma *inv-identity* [*simp*]: *inv* $(\lambda a.\ a) = (\lambda a.\ a)$
  by (*simp add*: *inv-def*)

lemma *inv-id* [*simp*]: *inv id* = *id*
  by (*simp add*: *id-def*)

lemma *inv-into-f-f* [*simp*]: *inj-on* $f\ A \Longrightarrow x \in A \Longrightarrow$ *inv-into* $A\ f\ (f\ x) = x$
  by (*simp add*: *inv-into-def inj-on-def*) (*blast intro*: *someI2*)

lemma *inv-f-f*: *inj* $f \Longrightarrow$ *inv* $f\ (f\ x) = x$
  by *simp*

lemma *f-inv-into-f*: $y \in f`A \Longrightarrow f$ (*inv-into* $A\ f\ y) = y$

by (*simp add*: *inv-into-def*) (*fast intro*: *someI2*)

lemma *inv-into-f-eq*: *inj-on f A* $\Longrightarrow$ $x \in A$ $\Longrightarrow$ $f\ x = y$ $\Longrightarrow$ *inv-into A f y* $= x$
by (*erule subst*) (*fast intro*: *inv-into-f-f*)

lemma *inv-f-eq*: *inj f* $\Longrightarrow$ $f\ x = y$ $\Longrightarrow$ *inv f y* $= x$
by (*simp add*:*inv-into-f-eq*)

lemma *inj-imp-inv-eq*: *inj f* $\Longrightarrow$ $\forall x.\ f\ (g\ x) = x$ $\Longrightarrow$ *inv f* $= g$
by (*blast intro*: *inv-into-f-eq*)

But is it useful?

lemma *inj-transfer*:
assumes *inj*: *inj f*
and *minor*: $\bigwedge y.\ y \in$ *range f* $\Longrightarrow$ P (*inv f y*)
shows $P\ x$
proof −
have $f\ x \in$ *range f* **by** *auto*
then have P(*inv f* $(f\ x)$) **by** (*rule minor*)
then show $P\ x$ **by** (*simp add*: *inv-into-f-f* [*OF inj*])
qed

lemma *inj-iff*: *inj f* $\longleftrightarrow$ *inv f* $\circ f = id$
by (*simp add*: *o-def fun-eq-iff*) (*blast intro*: *inj-on-inverseI inv-into-f-f*)

lemma *inv-o-cancel*[*simp*]: *inj f* $\Longrightarrow$ *inv f* $\circ f = id$
by (*simp add*: *inj-iff*)

lemma *o-inv-o-cancel*[*simp*]: *inj f* $\Longrightarrow$ $g \circ$ *inv f* $\circ f = g$
by (*simp add*: *comp-assoc*)

lemma *inv-into-image-cancel*[*simp*]: *inj-on f A* $\Longrightarrow$ $S \subseteq A$ $\Longrightarrow$ *inv-into A f* ' f ' S $= S$
by (*fastforce simp*: *image-def*)

lemma *inj-imp-surj-inv*: *inj f* $\Longrightarrow$ *surj* (*inv f*)
by (*blast intro*!: *surjI inv-into-f-f*)

lemma *surj-f-inv-f*: *surj f* $\Longrightarrow$ f (*inv f y*) $= y$
by (*simp add*: *f-inv-into-f*)

lemma *bij-inv-eq-iff*: *bij p* $\Longrightarrow$ $x =$ *inv p y* $\longleftrightarrow$ $p\ x = y$
using *surj-f-inv-f*[*of p*] **by** (*auto simp add*: *bij-def*)

lemma *inv-into-injective*:
assumes *eq*: *inv-into A f x* $=$ *inv-into A f y*
and x: $x \in f$'A
and y: $y \in f$'A
shows $x = y$

proof –
  from *eq* **have** f (*inv-into* A f x) = f (*inv-into* A f y)
    by *simp*
  with x y **show** *?thesis*
    by (*simp add*: *f-inv-into-f*)
qed

lemma *inj-on-inv-into*: $B \subseteq f`A \Longrightarrow$ *inj-on* (*inv-into* A f) B
  by (*blast intro*: *inj-onI dest*: *inv-into-injective injD*)

lemma *inj-imp-bij-betw-inv*: *inj* $f \Longrightarrow$ *bij-betw* (*inv* f) (f ‘ M) M
  by (*simp add*: *bij-betw-def image-subsetI inj-on-inv-into*)

lemma *bij-betw-inv-into*: *bij-betw* f A $B \Longrightarrow$ *bij-betw* (*inv-into* A f) B A
  by (*auto simp add*: *bij-betw-def inj-on-inv-into*)

lemma *surj-imp-inj-inv*: *surj* $f \Longrightarrow$ *inj* (*inv* f)
  by (*simp add*: *inj-on-inv-into*)

lemma *surj-iff*: *surj* $f \longleftrightarrow f \circ$ *inv* f = *id*
  by (*auto intro*!: *surjI simp*: *surj-f-inv-f fun-eq-iff*[**where** $'b='a$])

lemma *surj-iff-all*: *surj* $f \longleftrightarrow (\forall x.\ f$ (*inv* f x) = x)
  by (*simp add*: *o-def surj-iff fun-eq-iff*)

lemma *surj-imp-inv-eq*:
  assumes *surj* f **and** *gf*: $\bigwedge x.\ g\ (f\ x) = x$
  shows *inv* $f = g$
proof (*rule ext*)
  fix x
  have g (f (*inv* f x)) = *inv* f x
    by (*rule gf*)
  then show *inv* f x = g x
    by (*simp add*: *surj-f-inv-f* ‹*surj* f›)
qed

lemma *bij-imp-bij-inv*: *bij* $f \Longrightarrow$ *bij* (*inv* f)
  by (*simp add*: *bij-def inj-imp-surj-inv surj-imp-inj-inv*)

lemma *inv-equality*: $(\bigwedge x.\ g\ (f\ x) = x) \Longrightarrow (\bigwedge y.\ f\ (g\ y) = y) \Longrightarrow$ *inv* $f = g$
  by (*rule ext*) (*auto simp add*: *inv-into-def*)

lemma *inv-inv-eq*: *bij* $f \Longrightarrow$ *inv* (*inv* f) = f
  by (*rule inv-equality*) (*auto simp add*: *bij-def surj-f-inv-f*)

bij (*inv* f) implies little about f. Consider f :: *bool* ⇒ *bool* such that f *True* = f *False* = *True*. Then it ia consistent with axiom *someI* that *inv* f could be any function at all, including the identity function. If *inv* f = *id* then *inv* f is a bijection, but *inj* f, *surj* f and *inv* (*inv* f) = f all fail.

lemma *inv-into-comp*:
inj-on f (*g* ' *A*) ⟹ *inj-on g A* ⟹ *x* ∈ *f* ' *g* ' *A* ⟹
inv-into A (*f* ∘ *g*) *x* = (*inv-into A g* ∘ *inv-into* (*g* ' *A*) *f*) *x*
by (*auto simp*: *f-inv-into-f inv-into-into intro*: *inv-into-f-eq comp-inj-on*)

lemma *o-inv-distrib*: *bij f* ⟹ *bij g* ⟹ *inv* (*f* ∘ *g*) = *inv g* ∘ *inv f*
by (*rule inv-equality*) (*auto simp add*: *bij-def surj-f-inv-f*)

lemma *image-f-inv-f*: *surj f* ⟹ *f* ' (*inv f* ' *A*) = *A*
by (*simp add*: *surj-f-inv-f image-comp comp-def*)

lemma *image-inv-f-f*: *inj f* ⟹ *inv f* ' (*f* ' *A*) = *A*
by *simp*

lemma *bij-image-Collect-eq*:
assumes *bij f*
shows *f* ' *Collect P* = {*y*. *P* (*inv f y*)}
proof
show *f* ' *Collect P* ⊆ {*y*. *P* (*inv f y*)}
using *assms* **by** (*force simp add*: *bij-is-inj*)
show {*y*. *P* (*inv f y*)} ⊆ *f* ' *Collect P*
using *assms* **by** (*blast intro*: *bij-is-surj* [*THEN surj-f-inv-f*, *symmetric*])
qed

lemma *bij-vimage-eq-inv-image*:
assumes *bij f*
shows *f* −' *A* = *inv f* ' *A*
proof
show *f* −' *A* ⊆ *inv f* ' *A*
using *assms* **by** (*blast intro*: *bij-is-inj* [*THEN inv-into-f-f*, *symmetric*])
show *inv f* ' *A* ⊆ *f* −' *A*
using *assms* **by** (*auto simp add*: *bij-is-surj* [*THEN surj-f-inv-f*])
qed

lemma *inv-fn-o-fn-is-id*:
fixes *f*::'*a* ⇒ '*a*
assumes *bij f*
shows ((*inv f*)^^*n*) *o* (*f*^^*n*) = (λ*x*. *x*)
proof −
have ((*inv f*)^^*n*)((*f*^^*n*) *x*) = *x* **for** *x n*
proof (*induction n*)
case (*Suc n*)
have ∗: (*inv f*) (*f y*) = *y* **for** *y*
by (*simp add*: *assms bij-is-inj*)
have (*inv f* ^^ *Suc n*) ((*f* ^^ *Suc n*) *x*) = (*inv f*^^*n*) (*inv f* (*f* ((*f*^^*n*) *x*)))
by (*simp add*: *funpow-swap1*)
also have ... = (*inv f*^^*n*) ((*f*^^*n*) *x*)
using ∗ **by** *auto*
also have ... = *x* **using** *Suc.IH* **by** *auto*

```
    finally show ?case by simp
  qed (auto)
  then show ?thesis unfolding o-def by blast
qed

lemma fn-o-inv-fn-is-id:
  fixes f::'a ⇒ 'a
  assumes bij f
  shows (f^^n) o ((inv f)^^n) = (λx. x)
proof –
  have (f^^n) (((inv f)^^n) x) = x for x n
  proof (induction n)
    case (Suc n)
    have *: f(inv f y) = y for y
      using bij-inv-eq-iff[OF assms] by auto
    have (f ^^ Suc n) ((inv f ^^ Suc n) x) = (f^^n) (f (inv f ((inv f^^n) x)))
      by (simp add: funpow-swap1)
    also have ... = (f^^n) ((inv f^^n) x)
      using * by auto
    also have ... = x using Suc.IH by auto
    finally show ?case by simp
  qed (auto)
  then show ?thesis unfolding o-def by blast
qed

lemma inv-fn:
  fixes f::'a ⇒ 'a
  assumes bij f
  shows inv (f^^n) = ((inv f)^^n)
proof –
  have inv (f^^n) x = ((inv f)^^n) x for x
  proof (rule inv-into-f-eq)
    show inj (f ^^ n)
      by (simp add: inj-fn[OF bij-is-inj [OF assms]])
    show (f ^^ n) ((inv f ^^ n) x) = x
      using fn-o-inv-fn-is-id[OF assms, THEN fun-cong] by force
  qed auto
  then show ?thesis by auto
qed

lemma funpow-inj-finite:
  assumes ‹inj p› ‹finite {y. ∃ n. y = (p ^^ n) x}›
  obtains n where ‹n > 0› ‹(p ^^ n) x = x›
proof –
  have ‹infinite (UNIV :: nat set)›
    by simp
  moreover have ‹{y. ∃ n. y = (p ^^ n) x} = (λn. (p ^^ n) x) ` UNIV›
    by auto
  with assms have ‹finite . . .›
```

by *simp*
ultimately have $\exists\, n \in UNIV.\ \neg\ finite\ \{m \in UNIV.\ (p\ \widehat{}\widehat{}\ m)\ x = (p\ \widehat{}\widehat{}\ n)\ x\}$
by (*rule pigeonhole-infinite*)
then obtain n **where** $infinite\ \{m.\ (p\ \widehat{}\widehat{}\ m)\ x = (p\ \widehat{}\widehat{}\ n)\ x\}$ **by** *auto*
then have $infinite\ (\{m.\ (p\ \widehat{}\widehat{}\ m)\ x = (p\ \widehat{}\widehat{}\ n)\ x\} - \{n\})$ **by** *auto*
then have $(\{m.\ (p\ \widehat{}\widehat{}\ m)\ x = (p\ \widehat{}\widehat{}\ n)\ x\} - \{n\}) \neq \{\}$
by (*auto simp add*: *subset-singleton-iff*)
then obtain m **where** m: $(p\ \widehat{}\widehat{}\ m)\ x = (p\ \widehat{}\widehat{}\ n)\ x\ m \neq n$ **by** *auto*

{ fix $m\ n$ **assume** $(p\ \widehat{}\widehat{}\ n)\ x = (p\ \widehat{}\widehat{}\ m)\ x\ m < n$
have $(p\ \widehat{}\widehat{}\ (n - m))\ x = inv\ (p\ \widehat{}\widehat{}\ m)\ ((p\ \widehat{}\widehat{}\ m)\ ((p\ \widehat{}\widehat{}\ (n - m))\ x))$
using ‹*inj p*› **by** (*simp add*: *inv-f-f*)
also have $((p\ \widehat{}\widehat{}\ m)\ ((p\ \widehat{}\widehat{}\ (n - m))\ x)) = (p\ \widehat{}\widehat{}\ n)\ x$
using ‹$m < n$› *funpow-add* [*of m* ‹$n - m$› *p*] **by** *simp*
also have $inv\ (p\ \widehat{}\widehat{}\ m)\ \ldots = x$
using ‹*inj p*› **by** (*simp add*: ‹$(p\ \widehat{}\widehat{}\ n)\ x = -$›)
finally have $(p\ \widehat{}\widehat{}\ (n - m))\ x = x\ 0 < n - m$
using ‹$m < n$› **by** *auto* **}**
note *general* = *this*

show *thesis*
proof (*cases m n rule*: *linorder-cases*)
case *less*
then have ‹$n - m > 0$› ‹$(p\ \widehat{}\widehat{}\ (n - m))\ x = x$›
using *general* [*of n m*] *m* **by** *simp-all*
then show *thesis* **by** (*blast intro*: *that*)
next
case *equal*
then show *thesis* **using** *m* **by** *simp*
next
case *greater*
then have ‹$m - n > 0$› ‹$(p\ \widehat{}\widehat{}\ (m - n))\ x = x$›
using *general* [*of m n*] *m* **by** *simp-all*
then show *thesis* **by** (*blast intro*: *that*)
qed
qed

lemma *mono-inv*:
fixes f::′*a*::*linorder* ⇒ ′*b*::*linorder*
assumes *mono f bij f*
shows *mono* (*inv f*)
proof
fix $x\ y$::′*b* **assume** $x \leq y$
from ‹*bij f*› **obtain** $a\ b$ **where** x: $x = f\ a$ **and** y: $y = f\ b$ **by**(*fastforce simp*: *bij-def surj-def*)
show $inv\ f\ x \leq inv\ f\ y$
proof (*rule le-cases*)
assume $a \leq b$

    thus *?thesis* **using** ‹*bij f*› *x y* **by**(*simp add*: *bij-def inv-f-f*)
  next
    assume $b \leq a$
    hence $f\ b \leq f\ a$ **by**(*rule monoD*[*OF* ‹*mono f*›])
    hence $y \leq x$ **using** *x y* **by** *simp*
    hence $x = y$ **using** ‹$x \leq y$› **by** *auto*
    thus *?thesis* **by** *simp*
  qed
qed

lemma *strict-mono-inv-on-range*:
  fixes *f* :: *'a*::*linorder* ⇒ *'b*::*order*
  assumes *strict-mono f*
  shows *strict-mono-on* (*range f*) (*inv f*)
proof (*clarsimp simp*: *strict-mono-on-def*)
  fix *x y*
  assume $f\ x < f\ y$
  then show *inv f* (*f x*) < *inv f* (*f y*)
    using *assms strict-mono-imp-inj-on strict-mono-less* **by** *fastforce*
qed

lemma *mono-bij-Inf*:
  fixes *f* :: *'a*::*complete-linorder* ⇒ *'b*::*complete-linorder*
  assumes *mono f bij f*
  shows *f* (*Inf A*) = *Inf* (*f'A*)
proof −
  have *surj f* **using** ‹*bij f*› **by** (*auto simp*: *bij-betw-def*)
  have ∗: (*inv f*) (*Inf* (*f'A*)) ≤ *Inf* ((*inv f*)'(*f'A*))
    using *mono-Inf*[*OF mono-inv*[*OF assms*], *of f'A*] **by** *simp*
  have *Inf* (*f'A*) ≤ *f* (*Inf* ((*inv f*)'(*f'A*)))
    using *monoD*[*OF* ‹*mono f*› ∗] **by**(*simp add*: *surj-f-inv-f*[*OF* ‹*surj f*›])
  also have ... = *f*(*Inf A*)
    using *assms* **by** (*simp add*: *bij-is-inj*)
  finally show *?thesis* **using** *mono-Inf*[*OF assms*(*1*), *of A*] **by** *auto*
qed

lemma *finite-fun-UNIVD1*:
  assumes *fin*: *finite* (*UNIV* :: (*'a* ⇒ *'b*) *set*)
    and *card*: *card* (*UNIV* :: *'b set*) ≠ *Suc 0*
  shows *finite* (*UNIV* :: *'a set*)
proof −
  let *?UNIV-b* = *UNIV* :: *'b set*
  from *fin* **have** *finite ?UNIV-b*
    by (*rule finite-fun-UNIVD2*)
  with *card* **have** *card ?UNIV-b* ≥ *Suc* (*Suc 0*)
    by (*cases card ?UNIV-b*) (*auto simp*: *card-eq-0-iff*)
  then have *card ?UNIV-b* = *Suc* (*Suc* (*card ?UNIV-b* − *Suc* (*Suc 0*)))
    by *simp*
  then obtain *b1 b2* :: *'b* **where** *b1b2*: *b1* ≠ *b2*

by (*auto simp*: *card-Suc-eq*)
from *fin* **have** *fin'*: *finite* (*range* (λf :: $'a \Rightarrow 'b$. *inv f b1*))
by (*rule finite-imageI*)
have *UNIV* = *range* (λf :: $'a \Rightarrow 'b$. *inv f b1*)
proof (*rule UNIV-eq-I*)
fix x :: $'a$
from *b1b2* **have** x = *inv* (λy. *if* $y = x$ *then b1 else b2*) *b1*
by (*simp add*: *inv-into-def*)
then show $x \in$ *range* (λf::$'a \Rightarrow 'b$. *inv f b1*)
by *blast*
qed
with *fin'* **show** *?thesis*
by *simp*
qed

Every infinite set contains a countable subset. More precisely we show that a set S is infinite if and only if there exists an injective function from the naturals into S.

The "only if" direction is harder because it requires the construction of a sequence of pairwise different elements of an infinite set S. The idea is to construct a sequence of non-empty and infinite subsets of S obtained by successively removing elements of S.

lemma *infinite-countable-subset*:
assumes *inf*: $\neg$ *finite S*
shows $\exists f$::*nat* $\Rightarrow 'a$. *inj f* $\wedge$ *range f* $\subseteq S$
— Courtesy of Stephan Merz
proof −
define *Sseq* **where** *Sseq* = *rec-nat S* ($\lambda n\ T.\ T - \{$*SOME e*. $e \in T\}$)
define *pick* **where** *pick n* = (*SOME e*. $e \in$ *Sseq n*) **for** n
have $*$: *Sseq n* $\subseteq S$ $\neg$ *finite* (*Sseq n*) **for** n
by (*induct n*) (*auto simp*: *Sseq-def inf*)
then have $**$: $\bigwedge n$. *pick n* $\in$ *Sseq n*
unfolding *pick-def* **by** (*subst* (*asm*) *finite.simps*) (*auto simp add*: *ex-in-conv intro*: *someI-ex*)
with $*$ **have** *range pick* $\subseteq S$ **by** *auto*
moreover have *pick n* $\neq$ *pick* (n + *Suc m*) **for** $m\ n$
proof −
have *pick n* $\notin$ *Sseq* (n + *Suc m*)
by (*induct m*) (*auto simp add*: *Sseq-def pick-def*)
with $**$ **show** *?thesis* **by** *auto*
qed
then have *inj pick*
by (*intro linorder-injI*) (*auto simp add*: *less-iff-Suc-add*)
ultimately show *?thesis* **by** *blast*
qed

lemma *infinite-iff-countable-subset*: $\neg$ *finite S* $\longleftrightarrow$ ($\exists f$::*nat* $\Rightarrow 'a$. *inj f* $\wedge$ *range f* $\subseteq S$)

— Courtesy of Stephan Merz
using *finite-imageD finite-subset infinite-UNIV-char-0 infinite-countable-subset*
by *auto*

lemma *image-inv-into-cancel*:
assumes *surj*: $f`A = A'$
and *sub*: $B' \subseteq A'$
shows $f `((inv\text{-}into\ A\ f)`B') = B'$
using *assms*
proof (*auto simp*: *f-inv-into-f*)
let $?f' = inv\text{-}into\ A\ f$
fix a'
assume $*$: $a' \in B'$
with *sub* **have** $a' \in A'$ **by** *auto*
with *surj* **have** $a' = f\ (?f'\ a')$
by (*auto simp*: *f-inv-into-f*)
with $*$ **show** $a' \in f\ `\ (?f'\ `\ B')$ **by** *blast*
qed

lemma *inv-into-inv-into-eq*:
assumes *bij-betw f A A'*
and *a*: $a \in A$
shows $inv\text{-}into\ A'\ (inv\text{-}into\ A\ f)\ a = f\ a$
proof −
let $?f' = inv\text{-}into\ A\ f$
let $?f'' = inv\text{-}into\ A'\ ?f'$
from *assms* **have** $*$: $bij\text{-}betw\ ?f'\ A'\ A$
by (*auto simp*: *bij-betw-inv-into*)
with *a* **obtain** a' **where** a': $a' \in A'\ ?f'\ a' = a$
unfolding *bij-betw-def* **by** *force*
with *a* $*$ **have** $?f''\ a = a'$
by (*auto simp*: *f-inv-into-f bij-betw-def*)
moreover from *assms* a' **have** $f\ a = a'$
by (*auto simp*: *bij-betw-def*)
ultimately show $?f''\ a = f\ a$ **by** *simp*
qed

lemma *inj-on-iff-surj*:
assumes $A \neq \{\}$
shows $(\exists f.\ inj\text{-}on\ f\ A \wedge f\ `\ A \subseteq A') \longleftrightarrow (\exists g.\ g\ `\ A' = A)$
proof *safe*
fix f
assume *inj*: *inj-on f A* **and** *incl*: $f\ `\ A \subseteq A'$
let $?phi = \lambda a'\ a.\ a \in A \wedge f\ a = a'$
let $?csi = \lambda a.\ a \in A$
let $?g = \lambda a'.\ if\ a' \in f\ `\ A\ then\ (SOME\ a.\ ?phi\ a'\ a)\ else\ (SOME\ a.\ ?csi\ a)$
have $?g\ `\ A' = A$
proof
show $?g\ `\ A' \subseteq A$

```
    proof clarify
      fix a'
      assume *: a' ∈ A'
      show ?g a' ∈ A
      proof (cases a' ∈ f ` A)
        case True
        then obtain a where ?phi a' a by blast
        then have ?phi a' (SOME a. ?phi a' a)
          using someI[of ?phi a' a] by blast
        with True show ?thesis by auto
      next
        case False
        with assms have ?csi (SOME a. ?csi a)
          using someI-ex[of ?csi] by blast
        with False show ?thesis by auto
      qed
    qed
  next
    show A ⊆ ?g ` A'
    proof −
      have ?g (f a) = a ∧ f a ∈ A' if a: a ∈ A for a
      proof −
        let ?b = SOME aa. ?phi (f a) aa
        from a have ?phi (f a) a by auto
        then have *: ?phi (f a) ?b
          using someI[of ?phi(f a) a] by blast
        then have ?g (f a) = ?b using a by auto
        moreover from inj * a have a = ?b
          by (auto simp add: inj-on-def)
        ultimately have ?g(f a) = a by simp
        with incl a show ?thesis by auto
      qed
      then show ?thesis by force
    qed
  qed
  then show ∃ g. g ` A' = A by blast
next
  fix g
  let ?f = inv-into A' g
  have inj-on ?f (g ` A')
    by (auto simp: inj-on-inv-into)
  moreover have ?f (g a') ∈ A' if a': a' ∈ A' for a'
  proof −
    let ?phi = λ b'. b' ∈ A' ∧ g b' = g a'
    from a' have ?phi a' by auto
    then have ?phi (SOME b'. ?phi b')
      using someI[of ?phi] by blast
    then show ?thesis by (auto simp: inv-into-def)
  qed
```

ultimately show $\exists f.\ inj\text{-}on\ f\ (g\ `\ A') \wedge f\ `\ g\ `\ A' \subseteq A'$
  by *auto*
qed

lemma *Ex-inj-on-UNION-Sigma*:
$\exists f.\ (inj\text{-}on\ f\ (\bigcup i \in I.\ A\ i) \wedge f\ `\ (\bigcup i \in I.\ A\ i) \subseteq (SIGMA\ i : I.\ A\ i))$
proof
  let $?phi = \lambda a\ i.\ i \in I \wedge a \in A\ i$
  let $?sm = \lambda a.\ SOME\ i.\ ?phi\ a\ i$
  let $?f = \lambda a.\ (?sm\ a,\ a)$
  have $inj\text{-}on\ ?f\ (\bigcup i \in I.\ A\ i)$
    by (*auto simp*: *inj-on-def*)
  moreover
  have $?sm\ a \in I \wedge a \in A(?sm\ a)$ **if** $i \in I$ **and** $a \in A\ i$ **for** $i\ a$
    using *that someI*[*of ?phi a i*] **by** *auto*
  then have $?f\ `\ (\bigcup i \in I.\ A\ i) \subseteq (SIGMA\ i : I.\ A\ i)$
    by *auto*
  ultimately show $inj\text{-}on\ ?f\ (\bigcup i \in I.\ A\ i) \wedge ?f\ `\ (\bigcup i \in I.\ A\ i) \subseteq (SIGMA\ i :$
$I.\ A\ i)$
    by *auto*
qed

lemma *inv-unique-comp*:
  assumes *fg*: $f \circ g = id$
    and *gf*: $g \circ f = id$
  shows $inv\ f = g$
  using *fg gf inv-equality*[*of g f*] **by** (*auto simp add*: *fun-eq-iff*)

lemma *subset-image-inj*:
  $S \subseteq f\ `\ T \longleftrightarrow (\exists\ U.\ U \subseteq T \wedge inj\text{-}on\ f\ U \wedge S = f\ `\ U)$
proof *safe*
  show $\exists\ U \subseteq T.\ inj\text{-}on\ f\ U \wedge S = f\ `\ U$
    if $S \subseteq f\ `\ T$
  proof −
    from *that* [*unfolded subset-image-iff subset-iff*]
    obtain g **where** g: $\bigwedge x.\ x \in S \Longrightarrow g\ x \in T \wedge x = f\ (g\ x)$
      by (*auto simp add*: *image-iff Bex-def choice-iff'*)
    show *?thesis*
    proof (*intro exI conjI*)
      show $g\ `\ S \subseteq T$
        by (*simp add*: *g image-subsetI*)
      show $inj\text{-}on\ f\ (g\ `\ S)$
        using *g* **by** (*auto simp*: *inj-on-def*)
      show $S = f\ `\ (g\ `\ S)$
        using *g image-subset-iff* **by** *auto*
    qed
  qed
qed *blast*

25.6 Other Consequences of Hilbert's Epsilon

Hilbert's Epsilon and the *split* Operator

Looping simprule!

lemma *split-paired-Eps*: (*SOME x. P x*) = (*SOME* (*a*, *b*). *P* (*a*, *b*))
by *simp*

lemma *Eps-case-prod*: *Eps* (*case-prod P*) = (*SOME xy. P* (*fst xy*) (*snd xy*))
by (*simp add*: *split-def*)

lemma *Eps-case-prod-eq* [*simp*]: (*SOME* (*x′*, *y′*). *x* = *x′* ∧ *y* = *y′*) = (*x*, *y*)
by *blast*

A relation is wellfounded iff it has no infinite descending chain.

lemma *wf-iff-no-infinite-down-chain*: *wf r* ⟷ (∄*f*. ∀*i*. (*f* (*Suc i*), *f i*) ∈ *r*)
(**is** - ⟷ ¬ *?ex*)
proof
assume *wf r*
show ¬ *?ex*
proof
assume *?ex*
then obtain *f* **where** *f*: (*f* (*Suc i*), *f i*) ∈ *r* **for** *i*
by *blast*
from ‹*wf r*› **have** *minimal*: *x* ∈ *Q* ⟹ ∃*z*∈*Q*. ∀*y*. (*y*, *z*) ∈ *r* ⟶ *y* ∉ *Q* **for** *x Q*
by (*auto simp*: *wf-eq-minimal*)
let *?Q* = {*w*. ∃*i*. *w* = *f i*}
fix *n*
have *f n* ∈ *?Q* **by** *blast*
from *minimal* [*OF this*] **obtain** *j* **where** (*y*, *f j*) ∈ *r* ⟹ *y* ∉ *?Q* **for** *y* **by** *blast*
with *this* [*OF* ‹(*f* (*Suc j*), *f j*) ∈ *r*›] **have** *f* (*Suc j*) ∉ *?Q* **by** *simp*
then show *False* **by** *blast*
qed
next
assume ¬ *?ex*
then show *wf r*
proof (*rule contrapos-np*)
assume ¬ *wf r*
then obtain *Q x* **where** *x*: *x* ∈ *Q* **and** *rec*: *z* ∈ *Q* ⟹ ∃*y*. (*y*, *z*) ∈ *r* ∧ *y* ∈ *Q* **for** *z*
by (*auto simp add*: *wf-eq-minimal*)
obtain *descend* :: *nat* ⇒ *′a*
where *descend-0*: *descend 0* = *x*
and *descend-Suc*: *descend* (*Suc n*) = (*SOME y*. *y* ∈ *Q* ∧ (*y*, *descend n*) ∈ *r*) **for** *n*
by (*rule that* [*of rec-nat x* (λ- *rec*. (*SOME y*. *y* ∈ *Q* ∧ (*y*, *rec*) ∈ *r*))]) *simp-all*
have *descend-Q*: *descend n* ∈ *Q* **for** *n*

proof (*induct n*)
case *0*
with *x* **show** *?case* **by** (*simp only*: *descend-0*)
next
case *Suc*
then show *?case* **by** (*simp only*: *descend-Suc*) (*rule someI2-ex*; *use rec* **in** *blast*)
qed
have $(descend\ (Suc\ i),\ descend\ i) \in r$ **for** *i*
by (*simp only*: *descend-Suc*) (*rule someI2-ex*; *use descend-Q rec* **in** *blast*)
then show $\exists f.\ \forall i.\ (f\ (Suc\ i),\ f\ i) \in r$ **by** *blast*
qed
qed

lemma *wf-no-infinite-down-chainE*:
assumes *wf r*
obtains *k* **where** $(f\ (Suc\ k),\ f\ k) \notin r$
using *assms wf-iff-no-infinite-down-chain*[*of r*] **by** *blast*

A dynamically-scoped fact for TFL

lemma *tfl-some*: $\forall P\ x.\ P\ x \longrightarrow P\ (Eps\ P)$
by (*blast intro*: *someI*)

25.7 An aside: bounded accessible part

Finite monotone eventually stable sequences

lemma *finite-mono-remains-stable-implies-strict-prefix*:
fixes $f :: nat \Rightarrow 'a::order$
assumes *S*: *finite* (*range f*) *mono f*
and *eq*: $\forall n.\ f\ n = f\ (Suc\ n) \longrightarrow f\ (Suc\ n) = f\ (Suc\ (Suc\ n))$
shows $\exists N.\ (\forall n{\le}N.\ \forall m{\le}N.\ m < n \longrightarrow f\ m < f\ n) \wedge (\forall n{\ge}N.\ f\ N = f\ n)$
using *assms*
proof −
have $\exists n.\ f\ n = f\ (Suc\ n)$
proof (*rule ccontr*)
assume ¬ *?thesis*
then have $\bigwedge n.\ f\ n \ne f\ (Suc\ n)$ **by** *auto*
with ‹*mono f*› **have** $\bigwedge n.\ f\ n < f\ (Suc\ n)$
by (*auto simp*: *le-less mono-iff-le-Suc*)
with *lift-Suc-mono-less-iff*[*of f*] **have** ∗: $\bigwedge n\ m.\ n < m \Longrightarrow f\ n < f\ m$
by *auto*
have *inj f*
proof (*intro injI*)
fix *x y*
assume $f\ x = f\ y$
then show $x = y$
by (*cases x y rule*: *linorder-cases*) (*auto dest*: ∗)
qed
with ‹*finite* (*range f*)› **have** *finite* (*UNIV*::*nat set*)

```
      by (rule finite-imageD)
    then show False by simp
  qed
  then obtain n where n: f n = f (Suc n) ..
  define N where N = (LEAST n. f n = f (Suc n))
  have N: f N = f (Suc N)
    unfolding N-def using n by (rule LeastI)
  show ?thesis
  proof (intro exI[of - N] conjI allI impI)
    fix n
    assume N ≤ n
    then have ⋀m. N ≤ m ⟹ m ≤ n ⟹ f m = f N
    proof (induct rule: dec-induct)
      case base
      then show ?case by simp
    next
      case (step n)
      then show ?case
        using eq [rule-format, of n − 1] N
        by (cases n) (auto simp add: le-Suc-eq)
    qed
    from this[of n] ‹N ≤ n› show f N = f n by auto
  next
    fix n m :: nat
    assume m < n n ≤ N
    then show f m < f n
    proof (induct rule: less-Suc-induct)
      case (1 i)
      then have i < N by simp
      then have f i ≠ f (Suc i)
        unfolding N-def by (rule not-less-Least)
      with ‹mono f› show ?case by (simp add: mono-iff-le-Suc less-le)
    next
      case 2
      then show ?case by simp
    qed
  qed
qed

lemma finite-mono-strict-prefix-implies-finite-fixpoint:
  fixes f :: nat ⇒ 'a set
  assumes S: ⋀i. f i ⊆ S finite S
    and ex: ∃N. (∀n≤N. ∀m≤N. m < n ⟶ f m ⊂ f n) ∧ (∀n≥N. f N = f n)
  shows f (card S) = (⋃n. f n)
proof −
  from ex obtain N where inj: ⋀n m. n ≤ N ⟹ m ≤ N ⟹ m < n ⟹ f m
⊂ f n
    and eq: ∀n≥N. f N = f n
    by atomize auto
```

have $i \leq N \Longrightarrow i \leq card\ (f\ i)$ **for** i
proof (*induct i*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc i*)
with *inj* [*of Suc i i*] **have** $(f\ i) \subset (f\ (Suc\ i))$ **by** *auto*
moreover have *finite* $(f\ (Suc\ i))$ **using** *S* **by** (*rule finite-subset*)
ultimately have $card\ (f\ i) < card\ (f\ (Suc\ i))$ **by** (*intro psubset-card-mono*)
with *Suc inj* **show** *?case* **by** *auto*
qed
then have $N \leq card\ (f\ N)$ **by** *simp*
also have $\ldots \leq card\ S$ **using** *S* **by** (*intro card-mono*)
finally have §: $f\ (card\ S) = f\ N$ **using** *eq* **by** *auto*
moreover have $\bigcup\ (range\ f) \subseteq f\ N$
proof *clarify*
fix $x\ n$
assume $x \in f\ n$
with *eq inj* [*of N*] **show** $x \in f\ N$
by (*cases* $n < N$) (*auto simp*: *not-less*)
qed
ultimately show *?thesis*
by *auto*
qed

25.8 More on injections, bijections, and inverses

locale *bijection* =
fixes $f :: 'a \Rightarrow 'a$
assumes *bij*: *bij f*
begin

lemma *bij-inv*: *bij* (*inv f*)
using *bij* **by** (*rule bij-imp-bij-inv*)

lemma *surj* [*simp*]: *surj f*
using *bij* **by** (*rule bij-is-surj*)

lemma *inj*: *inj f*
using *bij* **by** (*rule bij-is-inj*)

lemma *surj-inv* [*simp*]: *surj* (*inv f*)
using *inj* **by** (*rule inj-imp-surj-inv*)

lemma *inj-inv*: *inj* (*inv f*)
using *surj* **by** (*rule surj-imp-inj-inv*)

lemma *eqI*: $f\ a = f\ b \Longrightarrow a = b$
using *inj* **by** (*rule injD*)

lemma *eq-iff* [*simp*]: $f\ a = f\ b \longleftrightarrow a = b$
  by (*auto intro*: *eqI*)

lemma *eq-invI*: $inv\ f\ a = inv\ f\ b \Longrightarrow a = b$
  using *inj-inv* **by** (*rule injD*)

lemma *eq-inv-iff* [*simp*]: $inv\ f\ a = inv\ f\ b \longleftrightarrow a = b$
  by (*auto intro*: *eq-invI*)

lemma *inv-left* [*simp*]: $inv\ f\ (f\ a) = a$
  using *inj* **by** (*simp add*: *inv-f-eq*)

lemma *inv-comp-left* [*simp*]: $inv\ f \circ f = id$
  by (*simp add*: *fun-eq-iff*)

lemma *inv-right* [*simp*]: $f\ (inv\ f\ a) = a$
  using *surj* **by** (*simp add*: *surj-f-inv-f*)

lemma *inv-comp-right* [*simp*]: $f \circ inv\ f = id$
  by (*simp add*: *fun-eq-iff*)

lemma *inv-left-eq-iff* [*simp*]: $inv\ f\ a = b \longleftrightarrow f\ b = a$
  by *auto*

lemma *inv-right-eq-iff* [*simp*]: $b = inv\ f\ a \longleftrightarrow f\ b = a$
  by *auto*

end

lemma *infinite-imp-bij-betw*:
  assumes *infinite*: $\neg$ *finite A*
  shows $\exists h.$ *bij-betw h A* $(A - \{a\})$
proof (*cases* $a \in A$)
  case *False*
  then have $A - \{a\} = A$ **by** *blast*
  then show *?thesis*
    using *bij-betw-id*[*of A*] **by** *auto*
next
  case *True*
  with *infinite* **have** $\neg$ *finite* $(A - \{a\})$ **by** *auto*
  with *infinite-iff-countable-subset*[*of* $A - \{a\}$]
  obtain f :: *nat* $\Rightarrow$ *'a* **where** *inj f* **and** f: f ` *UNIV* $\subseteq A - \{a\}$ **by** *blast*
  define g **where** $g\ n = ($*if* $n = 0$ *then a else f* $(Suc\ n))$ **for** n
  define A' **where** $A' = g$ ` *UNIV*
  have $*$: $\forall y.\ f\ y \neq a$ **using** f **by** *blast*
  have *3*: *inj-on g UNIV* $\wedge$ g ` *UNIV* $\subseteq A \wedge a \in g$ ` *UNIV*
    using ‹*inj f*› f $*$ **unfolding** *inj-on-def g-def*
    by (*auto simp add*: *True image-subset-iff*)

then have *4*: *bij-betw g UNIV A′* ∧ *a* ∈ *A′* ∧ *A′* ⊆ *A*
  using *inj-on-imp-bij-betw*[*of g*] **by** (*auto simp*: *A′-def*)
then have *5*: *bij-betw* (*inv g*) *A′ UNIV*
  by (*auto simp add*: *bij-betw-inv-into*)
from *3* **obtain** *n* **where** *n*: *g n* = *a* **by** *auto*
have *6*: *bij-betw g* (*UNIV* − {*n*}) (*A′* − {*a*})
  by (*rule bij-betw-subset*) (*use 3 4 n* **in** ‹*auto simp*: *image-set-diff A′-def*›)
define *v* **where** *v m* = (*if m* < *n then m else Suc m*) **for** *m*
have *m* < *n* ∨ *m* = *n* **if** ⋀*k*. *k* < *n* ∨ *m* ≠ *Suc k* **for** *m*
  using *that* [*of m*−*1*] **by** *auto*
then have *7*: *bij-betw v UNIV* (*UNIV* − {*n*})
  unfolding *bij-betw-def inj-on-def v-def* **by** *auto*
define *h′* **where** *h′* = *g* ∘ *v* ∘ (*inv g*)
with *5 6 7* **have** *8*: *bij-betw h′ A′* (*A′* − {*a*})
  by (*auto simp add*: *bij-betw-trans*)
define *h* **where** *h b* = (*if b* ∈ *A′ then h′ b else b*) **for** *b*
with *8* **have** *bij-betw h A′* (*A′* − {*a*})
  using *bij-betw-cong*[*of A′ h*] **by** *auto*
moreover
have ∀ *b* ∈ *A* − *A′*. *h b* = *b* **by** (*auto simp*: *h-def*)
then have *bij-betw h* (*A* − *A′*) (*A* − *A′*)
  using *bij-betw-cong*[*of A* − *A′ h id*] *bij-betw-id*[*of A* − *A′*] **by** *auto*
moreover
from *4* **have** (*A′* ∩ (*A* − *A′*) = {} ∧ *A′* ∪ (*A* − *A′*) = *A*) ∧
  ((*A′* − {*a*}) ∩ (*A* − *A′*) = {} ∧ (*A′* − {*a*}) ∪ (*A* − *A′*) = *A* − {*a*})
  by *blast*
ultimately have *bij-betw h A* (*A* − {*a*})
  using *bij-betw-combine*[*of h A′ A′* − {*a*} *A* − *A′ A* − *A′*] **by** *simp*
then show *?thesis* **by** *blast*
qed

lemma *infinite-imp-bij-betw2*:
assumes ¬ *finite A*
shows ∃ *h*. *bij-betw h A* (*A* ∪ {*a*})
proof (*cases a* ∈ *A*)
case *True*
then have *A* ∪ {*a*} = *A* **by** *blast*
then show *?thesis* **using** *bij-betw-id*[*of A*] **by** *auto*
next
case *False*
let *?A′* = *A* ∪ {*a*}
from *False* **have** *A* = *?A′* − {*a*} **by** *blast*
moreover from *assms* **have** ¬ *finite ?A′* **by** *auto*
ultimately obtain *f* **where** *bij-betw f ?A′ A*
  using *infinite-imp-bij-betw*[*of ?A′ a*] **by** *auto*
then have *bij-betw* (*inv-into ?A′ f*) *A ?A′* **by** (*rule bij-betw-inv-into*)
then show *?thesis* **by** *auto*
qed

lemma *bij-betw-inv-into-left*: *bij-betw f A A'* $\Longrightarrow a \in A \Longrightarrow$ *inv-into A f* $(f\ a) = a$
  unfolding *bij-betw-def* **by** *clarify* (*rule inv-into-f-f*)

lemma *bij-betw-inv-into-right*: *bij-betw f A A'* $\Longrightarrow a' \in A' \Longrightarrow f$ (*inv-into A f a'*) $= a'$
  unfolding *bij-betw-def* **using** *f-inv-into-f* **by** *force*

lemma *bij-betw-inv-into-subset*:
  bij-betw f A A' $\Longrightarrow B \subseteq A \Longrightarrow f$ ' $B = B' \Longrightarrow$ *bij-betw* (*inv-into A f*) B' B
  by (*auto simp*: *bij-betw-def intro*: *inj-on-inv-into*)

25.9 Specification package – Hilbertized version

lemma *exE-some*: *Ex P* $\Longrightarrow c \equiv$ *Eps P* $\Longrightarrow P\ c$
  by (*simp only*: *someI-ex*)

ML-file ‹*Tools/choice-specification.ML*›

25.10 Complete Distributive Lattices – Properties depending on Hilbert Choice

context *complete-distrib-lattice*
begin

lemma *Sup-Inf*: $\bigsqcup$ (*Inf* ' *A*) $= \bigsqcap$ (*Sup* ' $\{f$ ' $A\ |f.\ \forall B{\in}A.\ f\ B \in B\}$)
proof (*rule order.antisym*)
  show $\bigsqcup$ (*Inf* ' *A*) $\le \bigsqcap$ (*Sup* ' $\{f$ ' $A\ |f.\ \forall B{\in}A.\ f\ B \in B\}$)
    using *Inf-lower2 Sup-upper*
    by (*fastforce simp add*: *intro*: *Sup-least INF-greatest*)
next
  show $\bigsqcap$ (*Sup* ' $\{f$ ' $A\ |f.\ \forall B{\in}A.\ f\ B \in B\}$) $\le \bigsqcup$ (*Inf* ' *A*)
  proof (*simp add*: *Inf-Sup*, *rule SUP-least*, *simp*, *safe*)
    fix *f*
    assume $\forall Y.\ (\exists f.\ Y = f$ ' $A \wedge (\forall Y{\in}A.\ f\ Y \in Y)) \longrightarrow f\ Y \in Y$
    then have *B*: $\bigwedge F\ .\ (\forall\ Y \in A\ .\ F\ Y \in Y) \Longrightarrow \exists\ Z \in A\ .\ f\ (F$ ' $A) = F\ Z$
      by *auto*
    show $\bigsqcap(f$ ' $\{f$ ' $A\ |f.\ \forall Y{\in}A.\ f\ Y \in Y\}) \le \bigsqcup$(*Inf* ' *A*)
    proof (*cases* $\exists\ Z \in A\ .\ \bigsqcap(f$ ' $\{f$ ' $A\ |f.\ \forall Y{\in}A.\ f\ Y \in Y\}) \le$ *Inf Z*)
      case *True*
      from *this* **obtain** *Z* **where** [*simp*]: $Z \in A$ **and** *A*: $\bigsqcap(f$ ' $\{f$ ' $A\ |f.\ \forall Y{\in}A.\ f$
$Y \in Y\}) \le$ *Inf Z*
        by *blast*
      have *B*: ... $\le \bigsqcup$(*Inf* ' *A*)
        by (*simp add*: *SUP-upper*)
      from *A* **and** *B* **show** *?thesis*
        by *simp*
    next
      case *False*
      then have *X*: $\bigwedge\ Z\ .\ Z \in A \Longrightarrow \exists\ x\ .\ x \in Z \wedge \neg \bigsqcap(f$ ' $\{f$ ' $A\ |f.\ \forall Y{\in}A.\ f$

$Y \in Y\}) \leq x$
using *Inf-greatest* **by** *blast*
define F **where** $F = (\lambda\ Z\ .\ SOME\ x\ .\ x \in Z \wedge \neg \bigsqcap(f\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y \in Y\}) \leq x)$
have C: $\bigwedge Y.\ Y \in A \Longrightarrow F\ Y \in Y$
using X **by** (*simp add*: *F-def*, *rule someI2-ex*, *auto*)
have E: $\bigwedge Y.\ Y \in A \Longrightarrow \neg \bigsqcap(f\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y \in Y\}) \leq F\ Y$
using X **by** (*simp add*: *F-def*, *rule someI2-ex*, *auto*)
from C **and** B **obtain** Z **where** D: $Z \in A$ **and** Y: $f\ (F\ `\ A) = F\ Z$
by *blast*
from E **and** D **have** W: $\neg \bigsqcap(f\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y \in Y\}) \leq F\ Z$
by *simp*
have $\bigsqcap(f\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y \in Y\}) \leq f\ (F\ `\ A)$
using C **by** (*blast intro*: *INF-lower*)
with $W\ Y$ **show** *?thesis*
by *simp*
qed
qed
qed

lemma *dual-complete-distrib-lattice*:
class.complete-distrib-lattice Sup Inf sup $(\geq)$ $(>)$ *inf* $\top$ $\bot$
by (*simp add*: *class.complete-distrib-lattice.intro* [*OF dual-complete-lattice*]
class.complete-distrib-lattice-axioms-def Sup-Inf)

lemma *sup-Inf*: $a \sqcup \bigsqcap B = \bigsqcap((\sqcup)\ a\ `\ B)$
proof (*rule order.antisym*)
show $a \sqcup \bigsqcap B \leq \bigsqcap((\sqcup)\ a\ `\ B)$
using *Inf-lower sup.mono* **by** (*fastforce intro*: *INF-greatest*)
next
have $\bigsqcap((\sqcup)\ a\ `\ B) \leq \bigsqcap(Sup\ `\ \{\{f\ \{a\},\ f\ B\}\ |f.\ f\ \{a\} = a \wedge f\ B \in B\})$
by (*rule INF-greatest*, *auto simp add*: *INF-lower*)
also have ... $= \bigsqcup(Inf\ `\ \{\{a\},\ B\})$
by (*unfold Sup-Inf*, *simp*)
finally show $\bigsqcap((\sqcup)\ a\ `\ B) \leq a \sqcup \bigsqcap B$
by *simp*
qed

lemma *inf-Sup*: $a \sqcap \bigsqcup B = \bigsqcup((\sqcap)\ a\ `\ B)$
using *dual-complete-distrib-lattice*
by (*rule complete-distrib-lattice.sup-Inf*)

lemma *INF-SUP*: $(\bigsqcap y.\ \bigsqcup x.\ P\ x\ y) = (\bigsqcup f.\ \bigsqcap x.\ P\ (f\ x)\ x)$
proof (*rule order.antisym*)
show $(SUP\ x.\ INF\ y.\ P\ (x\ y)\ y) \leq (INF\ y.\ SUP\ x.\ P\ x\ y)$
by (*rule SUP-least*, *rule INF-greatest*, *rule SUP-upper2*, *simp-all*, *rule INF-lower2*, *simp*, *blast*)
next
have $(INF\ y.\ SUP\ x.\ ((P\ x\ y))) \leq Inf\ (Sup\ `\ \{\{P\ x\ y\ |\ x\ .\ True\}\ |\ y\ .\ True\ \})$

(**is** *?A* ≤ *?B*)
proof (*rule INF-greatest*, *clarsimp*)
fix *y*
have *?A* ≤ (*SUP x*. *P x y*)
by (*rule INF-lower*, *simp*)
also have ... ≤ *Sup* {*uu*. ∃ *x*. *uu* = *P x y*}
by (*simp add*: *full-SetCompr-eq*)
finally show *?A* ≤ *Sup* {*uu*. ∃ *x*. *uu* = *P x y*}
by *simp*
qed
also have ... ≤ (*SUP x*. *INF y*. *P* (*x y*) *y*)
proof (*subst Inf-Sup*, *rule SUP-least*, *clarsimp*)
fix *f*
assume *A*: ∀ *Y*. (∃ *y*. *Y* = {*uu*. ∃ *x*. *uu* = *P x y*}) ⟶ *f Y* ∈ *Y*

have ⨅(*f* ‘ {*uu*. ∃ *y*. *uu* = {*uu*. ∃ *x*. *uu* = *P x y*}}) ≤
(⨅ *y*. *P* (*SOME x*. *f* {*P x y* |*x*. *True*} = *P x y*) *y*)
proof (*rule INF-greatest*, *clarsimp*)
fix *y*
have (*INF x*∈{*uu*. ∃ *y*. *uu* = {*uu*. ∃ *x*. *uu* = *P x y*}}. *f x*) ≤ *f* {*uu*. ∃ *x*. *uu* = *P x y*}
by (*rule INF-lower*, *blast*)
also have ... ≤ *P* (*SOME x*. *f* {*uu* . ∃ *x*. *uu* = *P x y*} = *P x y*) *y*
by (*rule someI2-ex*) (*use A* **in** *auto*)
finally show ⨅(*f* ‘ {*uu*. ∃ *y*. *uu* = {*uu*. ∃ *x*. *uu* = *P x y*}}) ≤
P (*SOME x*. *f* {*uu*. ∃ *x*. *uu* = *P x y*} = *P x y*) *y*
by *simp*
qed
also have ... ≤ (*SUP x*. *INF y*. *P* (*x y*) *y*)
by (*rule SUP-upper*, *simp*)
finally show ⨅(*f* ‘ {*uu*. ∃ *y*. *uu* = {*uu*. ∃ *x*. *uu* = *P x y*}}) ≤ (⨆ *x*. ⨅ *y*. *P* (*x y*) *y*)
by *simp*
qed
finally show (*INF y*. *SUP x*. *P x y*) ≤ (*SUP x*. *INF y*. *P* (*x y*) *y*)
by *simp*
qed

lemma *INF-SUP-set*: (⨅ *B*∈*A*. ⨆(*g* ‘ *B*)) = (⨆ *B*∈{*f* ‘ *A* |*f*. ∀ *C*∈*A*. *f C* ∈ *C*}. ⨅(*g* ‘ *B*))
(**is** - = (⨆ *B*∈*?F*. -))
proof (*rule order.antisym*)
have ⨅ ((*g* ∘ *f*) ‘ *A*) ≤ ⨆ (*g* ‘ *B*) **if** ⋀*B*. *B* ∈ *A* ⟹ *f B* ∈ *B B* ∈ *A* **for** *f B*
using *that* **by** (*auto intro*: *SUP-upper2 INF-lower2*)
then show (⨆ *x*∈*?F*. ⨅ *a*∈*x*. *g a*) ≤ (⨅ *x*∈*A*. ⨆ *a*∈*x*. *g a*)
by (*auto intro*!: *SUP-least INF-greatest simp add*: *image-comp*)
next
show (⨅ *x*∈*A*. ⨆ *a*∈*x*. *g a*) ≤ (⨆ *x*∈*?F*. ⨅ *a*∈*x*. *g a*)
proof (*cases* {} ∈ *A*)

case *True*
then show *?thesis*
by (*rule INF-lower2*) *simp-all*
next
case *False*
{**fix** *x*
have ($\bigsqcap x \in A. \bigsqcup x \in x.\ g\ x$) $\leq$ ($\bigsqcup u.$ *if* $x \in A$ *then if* $u \in x$ *then* $g\ u$ *else* $\bot$ *else* $\top$)
proof (*cases* $x \in A$)
case *True*
then show *?thesis*
by (*intro INF-lower2 SUP-least SUP-upper2*) *auto*
qed *auto*
}
then have ($\bigsqcap Y \in A. \bigsqcup a \in Y.\ g\ a$) $\leq$ ($\bigsqcap Y. \bigsqcup y.$ *if* $Y \in A$ *then if* $y \in Y$ *then* $g\ y$ *else* $\bot$ *else* $\top$)
by (*rule INF-greatest*)
also have ... = ($\bigsqcup x. \bigsqcap Y.$ *if* $Y \in A$ *then if* $x\ Y \in Y$ *then* $g\ (x\ Y)$ *else* $\bot$ *else* $\top$)
by (*simp only*: *INF-SUP*)
also have ... $\leq$ ($\bigsqcup x \in ?F. \bigsqcap a \in x.\ g\ a$)
proof (*rule SUP-least*)
show ($\bigsqcap B.$ *if* $B \in A$ *then if* $x\ B \in B$ *then* $g\ (x\ B)$ *else* $\bot$ *else* $\top$)
$\leq$ ($\bigsqcup x \in ?F. \bigsqcap x \in x.\ g\ x$) **for** *x*
proof −
define *G* **where** $G \equiv \lambda Y.$ *if* $x\ Y \in Y$ *then* $x\ Y$ *else* (*SOME* $x.\ x \in Y$)
have $\forall Y \in A.\ G\ Y \in Y$
using *False some-in-eq G-def* **by** *auto*
then have *A*: G ‘ $A \in ?F$
by *blast*
show ($\bigsqcap Y.$ *if* $Y \in A$ *then if* $x\ Y \in Y$ *then* $g\ (x\ Y)$ *else* $\bot$ *else* $\top$) $\leq$ ($\bigsqcup x \in ?F. \bigsqcap x \in x.\ g\ x$)
by (*fastforce simp*: *G-def intro*: *SUP-upper2* [*OF A*] *INF-greatest INF-lower2*)
qed
qed
finally show *?thesis* **by** *simp*
qed
qed

lemma *SUP-INF*: ($\bigsqcup y. \bigsqcap x.\ P\ x\ y$) = ($\bigsqcap x. \bigsqcup y.\ P\ (x\ y)\ y$)
using *dual-complete-distrib-lattice*
by (*rule complete-distrib-lattice.INF-SUP*)

lemma *SUP-INF-set*: ($\bigsqcup x \in A. \bigsqcap\ (g$ ‘ $x)$) = ($\bigsqcap x \in \{f$ ‘ $A\ |f.\ \forall Y \in A.\ f\ Y \in Y\}. \bigsqcup\ (g$ ‘ $x)$)
using *dual-complete-distrib-lattice*
by (*rule complete-distrib-lattice.INF-SUP-set*)

end

context *complete-distrib-lattice*
begin

lemma *sup-INF*: $a \sqcup (\bigsqcap b{\in}B.\ f\ b) = (\bigsqcap b{\in}B.\ a \sqcup f\ b)$
by (*simp add*: *sup-Inf image-comp*)

lemma *inf-SUP*: $a \sqcap (\bigsqcup b{\in}B.\ f\ b) = (\bigsqcup b{\in}B.\ a \sqcap f\ b)$
by (*simp add*: *inf-Sup image-comp*)

lemma *Inf-sup*: $\bigsqcap B \sqcup a = (\bigsqcap b{\in}B.\ b \sqcup a)$
by (*simp add*: *sup-Inf sup-commute*)

lemma *Sup-inf*: $\bigsqcup B \sqcap a = (\bigsqcup b{\in}B.\ b \sqcap a)$
by (*simp add*: *inf-Sup inf-commute*)

lemma *INF-sup*: $(\bigsqcap b{\in}B.\ f\ b) \sqcup a = (\bigsqcap b{\in}B.\ f\ b \sqcup a)$
by (*simp add*: *sup-INF sup-commute*)

lemma *SUP-inf*: $(\bigsqcup b{\in}B.\ f\ b) \sqcap a = (\bigsqcup b{\in}B.\ f\ b \sqcap a)$
by (*simp add*: *inf-SUP inf-commute*)

lemma *Inf-sup-eq-top-iff*: $(\bigsqcap B \sqcup a = \top) \longleftrightarrow (\forall b{\in}B.\ b \sqcup a = \top)$
by (*simp only*: *Inf-sup INF-top-conv*)

lemma *Sup-inf-eq-bot-iff*: $(\bigsqcup B \sqcap a = \bot) \longleftrightarrow (\forall b{\in}B.\ b \sqcap a = \bot)$
by (*simp only*: *Sup-inf SUP-bot-conv*)

lemma *INF-sup-distrib2*: $(\bigsqcap a{\in}A.\ f\ a) \sqcup (\bigsqcap b{\in}B.\ g\ b) = (\bigsqcap a{\in}A.\ \bigsqcap b{\in}B.\ f\ a \sqcup g\ b)$
by (*subst INF-commute*) (*simp add*: *sup-INF INF-sup*)

lemma *SUP-inf-distrib2*: $(\bigsqcup a{\in}A.\ f\ a) \sqcap (\bigsqcup b{\in}B.\ g\ b) = (\bigsqcup a{\in}A.\ \bigsqcup b{\in}B.\ f\ a \sqcap g\ b)$
by (*subst SUP-commute*) (*simp add*: *inf-SUP SUP-inf*)

end

instantiation *set* :: (*type*) *complete-distrib-lattice*
begin
instance proof (*standard*, *clarsimp*)
fix A :: (('a *set*) *set*) *set*
fix x::'a
assume A: $\forall \mathcal{S}{\in}A.\ \exists X{\in}\mathcal{S}.\ x \in X$
define F **where** $F \equiv \lambda Y.\ SOME\ X.\ Y \in A \wedge X \in Y \wedge x \in X$
have $(\forall S \in F\ `\ A.\ x \in S)$
using A **unfolding** *F-def* **by** (*fastforce intro*: *someI2-ex*)
moreover have $\forall Y{\in}A.\ F\ Y \in Y$

```
    using A unfolding F-def by (fastforce intro: someI2-ex)
  then have ∃f. F ‘ A  = f ‘ A ∧ (∀ Y∈A. f Y ∈ Y)
    by blast
  ultimately show ∃X. (∃f. X = f ‘ A ∧ (∀ Y∈A. f Y ∈ Y)) ∧ (∀S∈X. x ∈ S)
    by auto
qed
end

instance set :: (type) complete-boolean-algebra ..

instantiation fun :: (type, complete-distrib-lattice) complete-distrib-lattice
begin
instance by standard (simp add: le-fun-def INF-SUP-set image-comp)
end

instance fun :: (type, complete-boolean-algebra) complete-boolean-algebra ..

context complete-linorder
begin

subclass complete-distrib-lattice
proof (standard, rule ccontr)
  fix A :: 'a set set
  let ?F = {f ‘ A |f. ∀ Y∈A. f Y ∈ Y}
  assume ¬ ⨅(Sup ‘ A) ≤ ⨆(Inf ‘ ?F)
  then have C: ⨅(Sup ‘ A) > ⨆(Inf ‘ ?F)
    by (simp add: not-le)
  show False
  proof (cases ∃ z . ⨅(Sup ‘ A) > z ∧ z > ⨆(Inf ‘ ?F))
    case True
    then obtain z where A: z < ⨅(Sup ‘ A) and X: z > ⨆(Inf ‘ ?F)
      by blast
    then have B: ⋀Y. Y ∈ A ⟹ ∃k ∈Y . z < k
      using local.less-Sup-iff by(force dest: less-INF-D)

    define G where G ≡ λY. SOME k . k ∈ Y ∧ z < k
    have E: ⋀Y. Y ∈ A ⟹ G Y ∈ Y
      using B unfolding G-def by (fastforce intro: someI2-ex)
    have z ≤ Inf (G ‘ A)
    proof (rule INF-greatest)
      show  ⋀Y. Y ∈ A ⟹ z ≤ G Y
        using B unfolding G-def by (fastforce intro: someI2-ex)
    qed
    also have ... ≤ ⨆(Inf ‘ ?F)
      by (rule SUP-upper) (use E in blast)
    finally have z ≤ ⨆(Inf ‘ ?F)
      by simp

    with X show ?thesis
```

```
    using local.not-less by blast
  next
   case False
   have B: ⋀Y. Y ∈ A ⟹ ∃ k ∈Y . ⨆(Inf ‘ ?F) < k
    using C local.less-Sup-iff by(force dest: less-INF-D)
   define G where G ≡ λ Y . SOME k . k ∈ Y ∧ ⨆(Inf ‘ ?F) < k
   have E: ⋀Y. Y ∈ A ⟹ G Y ∈ Y
    using B unfolding G-def by (fastforce intro: someI2-ex)
   have ⋀Y. Y ∈ A ⟹ ⨅(Sup ‘ A) ≤ G Y
    using B False local.leI unfolding G-def by (fastforce intro: someI2-ex)
   then have ⨅(Sup ‘ A) ≤ Inf (G ‘ A)
    by (simp add: local.INF-greatest)
   also have Inf (G ‘ A) ≤ ⨆(Inf ‘ ?F)
    by (rule SUP-upper) (use E in blast)
   finally have ⨅(Sup ‘ A) ≤ ⨆(Inf ‘ ?F)
    by simp
   with C show ?thesis
    using not-less by blast
  qed
qed
end

end
```

26 Zorn's Lemma and the Well-ordering Theorem

```
theory Zorn
  imports Order-Relation Hilbert-Choice
begin
```

26.1 Zorn's Lemma for the Subset Relation

26.1.1 Results that do not require an order

Let P be a binary predicate on the set A.

```
locale pred-on =
  fixes A :: 'a set
   and P :: 'a ⇒ 'a ⇒ bool (infix ‹⊏› 50)
begin

abbreviation Peq :: 'a ⇒ 'a ⇒ bool (infix ‹⊑› 50)
  where x ⊑ y ≡ P== x y
```

A chain is a totally ordered subset of A.

```
definition chain :: 'a set ⇒ bool
  where chain C ⟷ C ⊆ A ∧ (∀x∈C. ∀y∈C. x ⊑ y ∨ y ⊑ x)
```

We call a chain that is a proper superset of some set X, but not necessarily a chain itself, a superchain of X.

abbreviation *superchain* :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$ (**infix** ‹$<c$› *50*)
  where $X <c\ C \equiv chain\ C \wedge X \subset C$

A maximal chain is a chain that does not have a superchain.

definition *maxchain* :: $'a\ set \Rightarrow bool$
  where $maxchain\ C \longleftrightarrow chain\ C \wedge (\nexists S.\ C <c\ S)$

We define the successor of a set to be an arbitrary superchain, if such exists, or the set itself, otherwise.

definition *suc* :: $'a\ set \Rightarrow 'a\ set$
  where $suc\ C = (if\ \neg\ chain\ C \vee maxchain\ C\ then\ C\ else\ (SOME\ D.\ C <c\ D))$

lemma *chainI* [*Pure.intro?*]: $C \subseteq A \Longrightarrow (\bigwedge x\ y.\ x \in C \Longrightarrow y \in C \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x) \Longrightarrow chain\ C$
  unfolding *chain-def* **by** *blast*

lemma *chain-total*: $chain\ C \Longrightarrow x \in C \Longrightarrow y \in C \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x$
  by (*simp add*: *chain-def*)

lemma *not-chain-suc* [*simp*]: $\neg\ chain\ X \Longrightarrow suc\ X = X$
  by (*simp add*: *suc-def*)

lemma *maxchain-suc* [*simp*]: $maxchain\ X \Longrightarrow suc\ X = X$
  by (*simp add*: *suc-def*)

lemma *suc-subset*: $X \subseteq suc\ X$
  by (*auto simp*: *suc-def maxchain-def intro*: *someI2*)

lemma *chain-empty* [*simp*]: $chain\ \{\}$
  by (*auto simp*: *chain-def*)

lemma *not-maxchain-Some*: $chain\ C \Longrightarrow \neg\ maxchain\ C \Longrightarrow C <c\ (SOME\ D.\ C <c\ D)$
  by (*rule someI-ex*) (*auto simp*: *maxchain-def*)

lemma *suc-not-equals*: $chain\ C \Longrightarrow \neg\ maxchain\ C \Longrightarrow suc\ C \neq C$
  using *not-maxchain-Some* **by** (*auto simp*: *suc-def*)

lemma *subset-suc*:
  assumes $X \subseteq Y$
  shows $X \subseteq suc\ Y$
  using *assms* **by** (*rule subset-trans*) (*rule suc-subset*)

We build a set $\mathcal{C}$ that is closed under applications of *suc* and contains the union of all its subsets.

inductive-set *suc-Union-closed* (‹$\mathcal{C}$›)
  where
    suc: $X \in \mathcal{C} \Longrightarrow suc\ X \in \mathcal{C}$

| *Union* [*unfolded Pow-iff*]: $X \in Pow\ \mathcal{C} \Longrightarrow \bigcup X \in \mathcal{C}$

Since the empty set as well as the set itself is a subset of every set, $\mathcal{C}$ contains at least $\{\} \in \mathcal{C}$ and $\bigcup\ \mathcal{C} \in \mathcal{C}$.

lemma *suc-Union-closed-empty*: $\{\} \in \mathcal{C}$
  and *suc-Union-closed-Union*: $\bigcup \mathcal{C} \in \mathcal{C}$
  using *Union* [*of* {}] **and** *Union* [*of* $\mathcal{C}$] **by** *simp-all*

Thus closure under *suc* will hit a maximal chain eventually, as is shown below.

lemma *suc-Union-closed-induct* [*consumes 1*, *case-names suc Union*, *induct pred*: *suc-Union-closed*]:
  assumes $X \in \mathcal{C}$
    and $\bigwedge X.\ X \in \mathcal{C} \Longrightarrow Q\ X \Longrightarrow Q\ (suc\ X)$
    and $\bigwedge X.\ X \subseteq \mathcal{C} \Longrightarrow \forall x{\in}X.\ Q\ x \Longrightarrow Q\ (\bigcup X)$
  shows $Q\ X$
  using *assms* **by** *induct blast+*

lemma *suc-Union-closed-cases* [*consumes 1*, *case-names suc Union*, *cases pred*: *suc-Union-closed*]:
  assumes $X \in \mathcal{C}$
    and $\bigwedge Y.\ X = suc\ Y \Longrightarrow Y \in \mathcal{C} \Longrightarrow Q$
    and $\bigwedge Y.\ X = \bigcup Y \Longrightarrow Y \subseteq \mathcal{C} \Longrightarrow Q$
  shows Q
  using *assms* **by** *cases simp-all*

On chains, *suc* yields a chain.

lemma *chain-suc*:
  assumes *chain X*
  shows *chain* (*suc X*)
  using *assms*
 by (*cases* ¬ *chain X* ∨ *maxchain X*) (*force simp*: *suc-def dest*: *not-maxchain-Some*)+

lemma *chain-sucD*:
  assumes *chain X*
  shows $suc\ X \subseteq A \wedge chain\ (suc\ X)$
proof −
  from ‹*chain X*› **have** ∗: *chain* (*suc X*)
    by (*rule chain-suc*)
  then have $suc\ X \subseteq A$
    unfolding *chain-def* **by** *blast*
  with ∗ **show** *?thesis* **by** *blast*
qed

lemma *suc-Union-closed-total'*:
  assumes $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$
    and ∗: $\bigwedge Z.\ Z \in \mathcal{C} \Longrightarrow Z \subseteq Y \Longrightarrow Z = Y \vee suc\ Z \subseteq Y$
  shows $X \subseteq Y \vee suc\ Y \subseteq X$

```
  using ‹X ∈ 𝒞›
proof induct
  case (suc X)
  with ∗ show ?case by (blast del: subsetI intro: subset-suc)
next
  case Union
  then show ?case by blast
qed

lemma suc-Union-closed-subsetD:
  assumes Y ⊆ X and X ∈ 𝒞 and Y ∈ 𝒞
  shows X = Y ∨ suc Y ⊆ X
  using assms(2,3,1)
proof (induct arbitrary: Y)
  case (suc X)
  note ∗ = ‹⋀Y. Y ∈ 𝒞 ⟹ Y ⊆ X ⟹ X = Y ∨ suc Y ⊆ X›
  with suc-Union-closed-total' [OF ‹Y ∈ 𝒞› ‹X ∈ 𝒞›]
  have Y ⊆ X ∨ suc X ⊆ Y by blast
  then show ?case
  proof
    assume Y ⊆ X
    with ∗ and ‹Y ∈ 𝒞› subset-suc show ?thesis
      by fastforce
  next
    assume suc X ⊆ Y
    with ‹Y ⊆ suc X› show ?thesis by blast
  qed
next
  case (Union X)
  show ?case
  proof (rule ccontr)
    assume ¬ ?thesis
    with ‹Y ⊆ ⋃X› obtain x y z
      where ¬ suc Y ⊆ ⋃X
        and x ∈ X and y ∈ x and y ∉ Y
        and z ∈ suc Y and ∀x∈X. z ∉ x by blast
    with ‹X ⊆ 𝒞› have x ∈ 𝒞 by blast
    from Union and ‹x ∈ X› have ∗: ⋀y. y ∈ 𝒞 ⟹ y ⊆ x ⟹ x = y ∨ suc y ⊆
x
      by blast
    with suc-Union-closed-total' [OF ‹Y ∈ 𝒞› ‹x ∈ 𝒞›] have Y ⊆ x ∨ suc x ⊆ Y
      by blast
    then show False
    proof
      assume Y ⊆ x
      with ∗ [OF ‹Y ∈ 𝒞›] ‹y ∈ x› ‹y ∉ Y› ‹x ∈ X› ‹¬ suc Y ⊆ ⋃X› show False
        by blast
    next
      assume suc x ⊆ Y
```

with ‹$y \notin Y$› *suc-subset* ‹$y \in x$› **show** *False* **by** *blast*
qed
qed
qed

The elements of $\mathcal{C}$ are totally ordered by the subset relation.

lemma *suc-Union-closed-total*:
assumes $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$
shows $X \subseteq Y \vee Y \subseteq X$
proof (*cases* $\forall Z{\in}\mathcal{C}.\ Z \subseteq Y \longrightarrow Z = Y \vee suc\ Z \subseteq Y$)
case *True*
with *suc-Union-closed-total'* [*OF assms*]
have $X \subseteq Y \vee suc\ Y \subseteq X$ **by** *blast*
with *suc-subset* [*of Y*] **show** *?thesis* **by** *blast*
next
case *False*
then obtain Z **where** $Z \in \mathcal{C}$ **and** $Z \subseteq Y$ **and** $Z \neq Y$ **and** $\neg\ suc\ Z \subseteq Y$
by *blast*
with *suc-Union-closed-subsetD* **and** ‹$Y \in \mathcal{C}$› **show** *?thesis*
by *blast*
qed

Once we hit a fixed point w.r.t. *suc*, all other elements of $\mathcal{C}$ are subsets of this fixed point.

lemma *suc-Union-closed-suc*:
assumes $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$ **and** $suc\ Y = Y$
shows $X \subseteq Y$
using ‹$X \in \mathcal{C}$›
proof *induct*
case (*suc X*)
with ‹$Y \in \mathcal{C}$› **and** *suc-Union-closed-subsetD* **have** $X = Y \vee suc\ X \subseteq Y$
by *blast*
then show *?case*
by (*auto simp*: ‹$suc\ Y = Y$›)
next
case *Union*
then show *?case* **by** *blast*
qed

lemma *eq-suc-Union*:
assumes $X \in \mathcal{C}$
shows $suc\ X = X \longleftrightarrow X = \bigcup\mathcal{C}$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then have $\bigcup\mathcal{C} \subseteq X$
by (*rule suc-Union-closed-suc* [*OF suc-Union-closed-Union* ‹$X \in \mathcal{C}$›])
with ‹$X \in \mathcal{C}$› **show** *?rhs*
by *blast*

next
  from ‹$X \in \mathcal{C}$› **have** $suc\ X \in \mathcal{C}$ **by** (*rule suc*)
  then have $suc\ X \subseteq \bigcup\mathcal{C}$ **by** *blast*
  moreover assume *?rhs*
  ultimately have $suc\ X \subseteq X$ **by** *simp*
  moreover have $X \subseteq suc\ X$ **by** (*rule suc-subset*)
  ultimately show *?lhs* **..**
qed

lemma *suc-in-carrier*:
  assumes $X \subseteq A$
  shows $suc\ X \subseteq A$
  using *assms*
  by (*cases* $\neg$ *chain X* $\vee$ *maxchain X*) (*auto dest*: *chain-sucD*)

lemma *suc-Union-closed-in-carrier*:
  assumes $X \in \mathcal{C}$
  shows $X \subseteq A$
  using *assms*
  by *induct* (*auto dest*: *suc-in-carrier*)

All elements of $\mathcal{C}$ are chains.

lemma *suc-Union-closed-chain*:
  assumes $X \in \mathcal{C}$
  shows *chain X*
  using *assms*
proof *induct*
  case (*suc X*)
  then show *?case*
    using *not-maxchain-Some* **by** (*simp add*: *suc-def*)
next
  case (*Union X*)
  then have $\bigcup X \subseteq A$
    by (*auto dest*: *suc-Union-closed-in-carrier*)
  moreover have $\forall x\in\bigcup X.\ \forall y\in\bigcup X.\ x \sqsubseteq y \vee y \sqsubseteq x$
  proof (*intro ballI*)
    fix *x y*
    assume $x \in \bigcup X$ **and** $y \in \bigcup X$
    then obtain *u v* **where** $x \in u$ **and** $u \in X$ **and** $y \in v$ **and** $v \in X$
      by *blast*
    with *Union* **have** $u \in \mathcal{C}$ **and** $v \in \mathcal{C}$ **and** *chain u* **and** *chain v*
      by *blast+*
    with *suc-Union-closed-total* **have** $u \subseteq v \vee v \subseteq u$
      by *blast*
    then show $x \sqsubseteq y \vee y \sqsubseteq x$
    proof
      assume $u \subseteq v$
      from ‹*chain v*› **show** *?thesis*
      proof (*rule chain-total*)

```
      show y ∈ v by fact
      show x ∈ v using ‹u ⊆ v› and ‹x ∈ u› by blast
    qed
  next
    assume v ⊆ u
    from ‹chain u› show ?thesis
    proof (rule chain-total)
      show x ∈ u by fact
      show y ∈ u using ‹v ⊆ u› and ‹y ∈ v› by blast
    qed
  qed
 qed
 ultimately show ?case unfolding chain-def ..
qed
```

26.1.2 Hausdorff's Maximum Principle

There exists a maximal totally ordered subset of A. (Note that we do not require A to be partially ordered.)

```
theorem Hausdorff: ∃ C. maxchain C
proof –
 let ?M = ⋃𝒞
 have maxchain ?M
 proof (rule ccontr)
  assume ¬ ?thesis
  then have suc ?M ≠ ?M
   using suc-not-equals and suc-Union-closed-chain [OF suc-Union-closed-Union]
by simp
  moreover have suc ?M = ?M
   using eq-suc-Union [OF suc-Union-closed-Union] by simp
  ultimately show False by contradiction
 qed
 then show ?thesis by blast
qed
```

Make notation $\mathcal{C}$ available again.

```
no-notation suc-Union-closed (‹𝒞›)

lemma chain-extend: chain C ⟹ z ∈ A ⟹ ∀ x∈C. x ⊑ z ⟹ chain ({z} ∪ C)
 unfolding chain-def by blast

lemma maxchain-imp-chain: maxchain C ⟹ chain C
 by (simp add: maxchain-def)

end
```

Hide constant *pred-on.suc-Union-closed*, which was just needed for the proof of Hausforff's maximum principle.

hide-const *pred-on.suc-Union-closed*

lemma *chain-mono*:
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
and *pred-on.chain A P C*
shows *pred-on.chain A Q C*
using *assms* **unfolding** *pred-on.chain-def* **by** *blast*

26.1.3 Results for the proper subset relation

interpretation *subset*: *pred-on A* $(\subset)$ **for** *A* **.**

lemma *subset-maxchain-max*:
assumes *subset.maxchain A C*
and $X \in A$
and $\bigcup C \subseteq X$
shows $\bigcup C = X$
proof (*rule ccontr*)
let $?C = \{X\} \cup C$
from ‹*subset.maxchain A C*› **have** *subset.chain A C*
and $*$: $\bigwedge S.$ *subset.chain A S* $\Longrightarrow \neg\ C \subset S$
by (*auto simp*: *subset.maxchain-def*)
moreover have $\forall x \in C.\ x \subseteq X$ **using** ‹$\bigcup C \subseteq X$› **by** *auto*
ultimately have *subset.chain A ?C*
using *subset.chain-extend* [*of A C X*] **and** ‹$X \in A$› **by** *auto*
moreover assume $**$: $\bigcup C \neq X$
moreover from $**$ **have** $C \subset ?C$ **using** ‹$\bigcup C \subseteq X$› **by** *auto*
ultimately show *False* **using** $*$ **by** *blast*
qed

lemma *subset-chain-def*: $\bigwedge \mathcal{A}.$ *subset.chain* $\mathcal{A}\ \mathcal{C} = (\mathcal{C} \subseteq \mathcal{A} \wedge (\forall X \in \mathcal{C}.\ \forall Y \in \mathcal{C}.\ X \subseteq Y \vee Y \subseteq X))$
by (*auto simp*: *subset.chain-def*)

lemma *subset-chain-insert*:
subset.chain $\mathcal{A}$ (*insert B* $\mathcal{B}$) $\longleftrightarrow B \in \mathcal{A} \wedge (\forall X \in \mathcal{B}.\ X \subseteq B \vee B \subseteq X) \wedge$ *subset.chain* $\mathcal{A}\ \mathcal{B}$
by (*fastforce simp add*: *subset-chain-def*)

26.1.4 Zorn's lemma

If every chain has an upper bound, then there is a maximal set.

theorem *subset-Zorn*:
assumes $\bigwedge C.$ *subset.chain A C* $\Longrightarrow \exists U \in A.\ \forall X \in C.\ X \subseteq U$
shows $\exists M \in A.\ \forall X \in A.\ M \subseteq X \longrightarrow X = M$
proof −
from *subset.Hausdorff* [*of A*] **obtain** *M* **where** *subset.maxchain A M* **..**
then have *subset.chain A M*
by (*rule subset.maxchain-imp-chain*)

with *assms* **obtain** Y **where** $Y \in A$ **and** $\forall X \in M.\ X \subseteq Y$
by *blast*
moreover have $\forall X \in A.\ Y \subseteq X \longrightarrow Y = X$
proof (*intro ballI impI*)
fix X
assume $X \in A$ **and** $Y \subseteq X$
show $Y = X$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
with ‹$Y \subseteq X$› **have** $\neg X \subseteq Y$ **by** *blast*
from *subset.chain-extend* [*OF* ‹*subset.chain A M*› ‹$X \in A$›] **and** ‹$\forall X \in M.\ X \subseteq Y$›
have *subset.chain A* $(\{X\} \cup M)$
using ‹$Y \subseteq X$› **by** *auto*
moreover have $M \subset \{X\} \cup M$
using ‹$\forall X \in M.\ X \subseteq Y$› **and** ‹$\neg X \subseteq Y$› **by** *auto*
ultimately show *False*
using ‹*subset.maxchain A M*› **by** (*auto simp*: *subset.maxchain-def*)
qed
qed
ultimately show *?thesis* **by** *blast*
qed

Alternative version of Zorn's lemma for the subset relation.

lemma *subset-Zorn'*:
assumes $\bigwedge C.$ *subset.chain A C* $\Longrightarrow \bigcup C \in A$
shows $\exists M \in A.\ \forall X \in A.\ M \subseteq X \longrightarrow X = M$
proof –
from *subset.Hausdorff* [*of A*] **obtain** M **where** *subset.maxchain A M* **..**
then have *subset.chain A M*
by (*rule subset.maxchain-imp-chain*)
with *assms* **have** $\bigcup M \in A$ **.**
moreover have $\forall Z \in A.\ \bigcup M \subseteq Z \longrightarrow \bigcup M = Z$
proof (*intro ballI impI*)
fix Z
assume $Z \in A$ **and** $\bigcup M \subseteq Z$
with *subset-maxchain-max* [*OF* ‹*subset.maxchain A M*›]
show $\bigcup M = Z$ **.**
qed
ultimately show *?thesis* **by** *blast*
qed

26.2 Zorn's Lemma for Partial Orders

Relate old to new definitions.

definition *chain-subset* :: *'a set set* $\Rightarrow$ *bool* (‹$chain_\subseteq$›)
where $chain_\subseteq\ C \longleftrightarrow (\forall A \in C.\ \forall B \in C.\ A \subseteq B \vee B \subseteq A)$

definition *chains* :: *'a set set* $\Rightarrow$ *'a set set set*

where *chains* $A = \{C.\ C \subseteq A \wedge chain_{\subseteq}\ C\}$

definition *Chains* :: $('a \times 'a)\ set \Rightarrow 'a\ set\ set$
where *Chains* $r = \{C.\ \forall a{\in}C.\ \forall b{\in}C.\ (a,\ b) \in r \vee (b,\ a) \in r\}$

lemma *chains-extend*: $c \in chains\ S \Longrightarrow z \in S \Longrightarrow \forall x \in c.\ x \subseteq z \Longrightarrow \{z\} \cup c \in chains\ S$
for z :: $'a\ set$
unfolding *chains-def chain-subset-def* **by** *blast*

lemma *mono-Chains*: $r \subseteq s \Longrightarrow Chains\ r \subseteq Chains\ s$
unfolding *Chains-def* **by** *blast*

lemma *chain-subset-alt-def*: $chain_{\subseteq}\ C = subset.chain\ UNIV\ C$
unfolding *chain-subset-def subset.chain-def* **by** *fast*

lemma *chains-alt-def*: $chains\ A = \{C.\ subset.chain\ A\ C\}$
by (*simp add*: *chains-def chain-subset-alt-def subset.chain-def*)

lemma *Chains-subset*: $Chains\ r \subseteq \{C.\ pred\text{-}on.chain\ UNIV\ (\lambda x\ y.\ (x,\ y) \in r)\ C\}$
by (*force simp add*: *Chains-def pred-on.chain-def*)

lemma *Chains-subset'*:
assumes *refl r*
shows $\{C.\ pred\text{-}on.chain\ UNIV\ (\lambda x\ y.\ (x,\ y) \in r)\ C\} \subseteq Chains\ r$
using *assms*
by (*auto simp add*: *Chains-def pred-on.chain-def refl-on-def*)

lemma *Chains-alt-def*:
assumes *refl r*
shows $Chains\ r = \{C.\ pred\text{-}on.chain\ UNIV\ (\lambda x\ y.\ (x,\ y) \in r)\ C\}$
using *assms Chains-subset Chains-subset'* **by** *blast*

lemma *Chains-relation-of*:
assumes $C \in Chains\ (relation\text{-}of\ P\ A)$ **shows** $C \subseteq A$
using *assms* **unfolding** *Chains-def relation-of-def* **by** *auto*

lemma *pairwise-chain-Union*:
assumes P: $\bigwedge S.\ S \in \mathcal{C} \Longrightarrow pairwise\ R\ S$ **and** $chain_{\subseteq}\ \mathcal{C}$
shows *pairwise* $R\ (\bigcup \mathcal{C})$
using ‹$chain_{\subseteq}\ \mathcal{C}$› **unfolding** *pairwise-def chain-subset-def*
by (*blast intro*: P [*unfolded pairwise-def*, *rule-format*])

lemma *Zorn-Lemma*: $\forall C{\in}chains\ A.\ \bigcup C \in A \Longrightarrow \exists M{\in}A.\ \forall X{\in}A.\ M \subseteq X \longrightarrow X = M$
using *subset-Zorn'* [*of A*] **by** (*force simp*: *chains-alt-def*)

lemma *Zorn-Lemma2*: $\forall C{\in}chains\ A.\ \exists U{\in}A.\ \forall X{\in}C.\ X \subseteq U \Longrightarrow \exists M{\in}A.\ \forall X{\in}A.\ M \subseteq X \longrightarrow X = M$

using *subset-Zorn* [*of A*] **by** (*auto simp*: *chains-alt-def*)

26.3 Other variants of Zorn's Lemma

lemma *chainsD*: $c \in chains\ S \Longrightarrow x \in c \Longrightarrow y \in c \Longrightarrow x \subseteq y \vee y \subseteq x$
unfolding *chains-def chain-subset-def* **by** *blast*

lemma *chainsD2*: $c \in chains\ S \Longrightarrow c \subseteq S$
unfolding *chains-def* **by** *blast*

lemma *Zorns-po-lemma*:
assumes *po*: *Partial-order r*
and *u*: $\bigwedge C.\ C \in Chains\ r \Longrightarrow \exists u{\in}Field\ r.\ \forall a{\in}C.\ (a,\ u) \in r$
shows $\exists m{\in}Field\ r.\ \forall a{\in}Field\ r.\ (m,\ a) \in r \longrightarrow a = m$
proof −
have *Preorder r*
using *po* **by** (*simp add*: *partial-order-on-def*)

Mirror r in the set of subsets below (wrt r) elements of A.

let $?B = \lambda x.\ r^{-1}$ `` $\{x\}$
let $?S = ?B$ ` *Field r*
have $\exists u{\in}Field\ r.\ \forall A{\in}C.\ A \subseteq r^{-1}$ `` $\{u\}$ (**is** $\exists u{\in}Field\ r.\ ?P\ u$)
if *1*: $C \subseteq ?S$ **and** *2*: $\forall A{\in}C.\ \forall B{\in}C.\ A \subseteq B \vee B \subseteq A$ **for** C
proof −
let $?A = \{x{\in}Field\ r.\ \exists M{\in}C.\ M = ?B\ x\}$
from *1* **have** $C = ?B$ ` $?A$ **by** (*auto simp*: *image-def*)
have $?A \in Chains\ r$
proof (*simp add*: *Chains-def*, *intro allI impI*, *elim conjE*)
fix *a b*
assume $a \in Field\ r$ **and** $?B\ a \in C$ **and** $b \in Field\ r$ **and** $?B\ b \in C$
with *2* **have** $?B\ a \subseteq ?B\ b \vee ?B\ b \subseteq ?B\ a$ **by** *auto*
then show $(a,\ b) \in r \vee (b,\ a) \in r$
using ‹*Preorder r*› **and** ‹$a \in Field\ r$› **and** ‹$b \in Field\ r$›
by (*simp add*:*subset-Image1-Image1-iff*)
qed
then obtain *u* **where** *uA*: $u \in Field\ r\ \forall a{\in}?A.\ (a,\ u) \in r$
by (*auto simp*: *dest*: *u*)
have *?P u*
proof *auto*
fix *a B* **assume** *aB*: $B \in C\ a \in B$
with *1* **obtain** *x* **where** $x \in Field\ r$ **and** $B = r^{-1}$ `` $\{x\}$ **by** *auto*
then show $(a,\ u) \in r$
using *uA* **and** *aB* **and** ‹*Preorder r*›
unfolding *preorder-on-def refl-on-def* **by** *simp* (*fast dest*: *transD*)
qed
then show *?thesis*
using ‹$u \in Field\ r$› **by** *blast*
qed
then have $\forall C{\in}chains\ ?S.\ \exists U{\in}?S.\ \forall A{\in}C.\ A \subseteq U$
by (*auto simp*: *chains-def chain-subset-def*)

from *Zorn-Lemma2* [*OF this*] **obtain** *m B*
where $m \in$ *Field r*
and $B = r^{-1}$ " $\{m\}$
and $\forall x \in$*Field r*. $B \subseteq r^{-1}$ " $\{x\} \longrightarrow r^{-1}$ " $\{x\} = B$
by *auto*
then have $\forall a \in$*Field r*. $(m, a) \in r \longrightarrow a = m$
using *po* **and** ‹*Preorder r*› **and** ‹$m \in$ *Field r*›
by (*auto simp*: *subset-Image1-Image1-iff Partial-order-eq-Image1-Image1-iff*)
then show *?thesis*
using ‹$m \in$ *Field r*› **by** *blast*
qed

lemma *predicate-Zorn*:
assumes *po*: *partial-order-on A* (*relation-of P A*)
and *ch*: $\bigwedge C$. $C \in$ *Chains* (*relation-of P A*) $\Longrightarrow \exists u \in A. \forall a \in C. P\ a\ u$
shows $\exists m \in A. \forall a \in A. P\ m\ a \longrightarrow a = m$
proof −
have $a \in A$ **if** $C \in$ *Chains* (*relation-of P A*) **and** $a \in C$ **for** *C a*
using *that* **unfolding** *Chains-def relation-of-def* **by** *auto*
moreover have $(a, u) \in$ *relation-of P A* **if** $a \in A$ **and** $u \in A$ **and** *P a u* **for** *a u*
unfolding *relation-of-def* **using** *that* **by** *auto*
ultimately have $\exists m \in A. \forall a \in A. (m, a) \in$ *relation-of P A* $\longrightarrow a = m$
using *Zorns-po-lemma*[*OF Partial-order-relation-ofI*[*OF po*], *rule-format*] *ch*
unfolding *Field-relation-of*[*OF partial-order-onD*(*1*)[*OF po*]] **by** *blast*
then show *?thesis*
by (*auto simp*: *relation-of-def*)
qed

lemma *Union-in-chain*: ⟦*finite* $\mathcal{B}$; $\mathcal{B} \neq \{\}$; *subset.chain* $\mathcal{A}$ $\mathcal{B}$⟧ $\Longrightarrow \bigcup\mathcal{B} \in \mathcal{B}$
proof (*induction* $\mathcal{B}$ *rule*: *finite-induct*)
case (*insert B* $\mathcal{B}$)
show *?case*
proof (*cases* $\mathcal{B} = \{\}$)
case *False*
then show *?thesis*
using *insert sup.absorb2* **by** (*auto simp*: *subset-chain-insert dest*!: *bspec* [**where** $x=\bigcup\mathcal{B}$])
qed *auto*
qed *simp*

lemma *Inter-in-chain*: ⟦*finite* $\mathcal{B}$; $\mathcal{B} \neq \{\}$; *subset.chain* $\mathcal{A}$ $\mathcal{B}$⟧ $\Longrightarrow \bigcap\mathcal{B} \in \mathcal{B}$
proof (*induction* $\mathcal{B}$ *rule*: *finite-induct*)
case (*insert B* $\mathcal{B}$)
show *?case*
proof (*cases* $\mathcal{B} = \{\}$)
case *False*
then show *?thesis*
using *insert inf.absorb2* **by** (*auto simp*: *subset-chain-insert dest*!: *bspec* [**where** $x=\bigcap\mathcal{B}$])

qed *auto*
qed *simp*

lemma *finite-subset-Union-chain*:
assumes *finite* A $A \subseteq \bigcup \mathcal{B}$ $\mathcal{B} \neq \{\}$ **and** *sub*: *subset.chain* $\mathcal{A}$ $\mathcal{B}$
obtains B **where** $B \in \mathcal{B}$ $A \subseteq B$
proof –
obtain $\mathcal{F}$ **where** $\mathcal{F}$: *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{B}$ $A \subseteq \bigcup \mathcal{F}$
using *assms* **by** (*auto intro*: *finite-subset-Union*)
show *thesis*
proof (*cases* $\mathcal{F} = \{\}$)
case *True*
then show *?thesis*
using ‹$A \subseteq \bigcup \mathcal{F}$› ‹$\mathcal{B} \neq \{\}$› *that* **by** *fastforce*
next
case *False*
show *?thesis*
proof
show $\bigcup \mathcal{F} \in \mathcal{B}$
using *sub* ‹$\mathcal{F} \subseteq \mathcal{B}$› ‹*finite* $\mathcal{F}$›
by (*simp add*: *Union-in-chain False subset.chain-def subset-iff*)
show $A \subseteq \bigcup \mathcal{F}$
using ‹$A \subseteq \bigcup \mathcal{F}$› **by** *blast*
qed
qed
qed

lemma *subset-Zorn-nonempty*:
assumes $\mathcal{A} \neq \{\}$ **and** *ch*: $\bigwedge \mathcal{C}$. $[\![\mathcal{C} \neq \{\}$; *subset.chain* $\mathcal{A}$ $\mathcal{C}]\!] \Longrightarrow \bigcup \mathcal{C} \in \mathcal{A}$
shows $\exists M \in \mathcal{A}. \forall X \in \mathcal{A}. M \subseteq X \longrightarrow X = M$
proof (*rule subset-Zorn*)
show $\exists U \in \mathcal{A}. \forall X \in \mathcal{C}. X \subseteq U$ **if** *subset.chain* $\mathcal{A}$ $\mathcal{C}$ **for** $\mathcal{C}$
proof (*cases* $\mathcal{C} = \{\}$)
case *True*
then show *?thesis*
using ‹$\mathcal{A} \neq \{\}$› **by** *blast*
next
case *False*
show *?thesis*
by (*blast intro*!: *ch False that Union-upper*)
qed
qed

26.4 The Well Ordering Theorem

definition *init-seg-of* :: $(('a \times 'a)\ set \times ('a \times 'a)\ set)\ set$
where *init-seg-of* $= \{(r, s).\ r \subseteq s \wedge (\forall a\ b\ c.\ (a, b) \in s \wedge (b, c) \in r \longrightarrow (a, b) \in r)\}$

abbreviation *initial-segment-of-syntax* :: $('a \times 'a)$ *set* ⇒ $('a \times 'a)$ *set* ⇒ *bool*
  (**infix** ‹*initial′-segment′-of*› *55*)
 where *r initial-segment-of s* ≡ (*r*, *s*) ∈ *init-seg-of*

lemma *refl-on-init-seg-of* [*simp*]: *r initial-segment-of r*
 by (*simp add*: *init-seg-of-def*)

lemma *trans-init-seg-of*:
 r initial-segment-of s ⟹ *s initial-segment-of t* ⟹ *r initial-segment-of t*
 by (*simp* (*no-asm-use*) *add*: *init-seg-of-def*) *blast*

lemma *antisym-init-seg-of*: *r initial-segment-of s* ⟹ *s initial-segment-of r* ⟹ *r*
= *s*
 unfolding *init-seg-of-def* **by** *safe*

lemma *Chains-init-seg-of-Union*: *R* ∈ *Chains init-seg-of* ⟹ *r*∈*R* ⟹ *r initial-segment-of*
⋃*R*
 by (*auto simp*: *init-seg-of-def Ball-def Chains-def*) *blast*

lemma *chain-subset-trans-Union*:
 assumes $chain_{\subseteq}$ *R* ∀ *r*∈*R*. *trans r*
 shows *trans* (⋃*R*)
proof (*intro transI*, *elim UnionE*)
 fix *S1 S2* :: *'a rel* **and** *x y z* :: *'a*
 assume *S1* ∈ *R S2* ∈ *R*
 with *assms*(*1*) **have** *S1* ⊆ *S2* ∨ *S2* ⊆ *S1*
  unfolding *chain-subset-def* **by** *blast*
 moreover assume (*x*, *y*) ∈ *S1* (*y*, *z*) ∈ *S2*
 ultimately have ((*x*, *y*) ∈ *S1* ∧ (*y*, *z*) ∈ *S1*) ∨ ((*x*, *y*) ∈ *S2* ∧ (*y*, *z*) ∈ *S2*)
  by *blast*
 with ‹*S1* ∈ *R*› ‹*S2* ∈ *R*› *assms*(*2*) **show** (*x*, *z*) ∈ ⋃*R*
  by (*auto elim*: *transE*)
qed

lemma *chain-subset-antisym-Union*:
 assumes $chain_{\subseteq}$ *R* ∀ *r*∈*R*. *antisym r*
 shows *antisym* (⋃*R*)
proof (*intro antisymI*, *elim UnionE*)
 fix *S1 S2* :: *'a rel* **and** *x y* :: *'a*
 assume *S1* ∈ *R S2* ∈ *R*
 with *assms*(*1*) **have** *S1* ⊆ *S2* ∨ *S2* ⊆ *S1*
  unfolding *chain-subset-def* **by** *blast*
 moreover assume (*x*, *y*) ∈ *S1* (*y*, *x*) ∈ *S2*
 ultimately have ((*x*, *y*) ∈ *S1* ∧ (*y*, *x*) ∈ *S1*) ∨ ((*x*, *y*) ∈ *S2* ∧ (*y*, *x*) ∈ *S2*)
  by *blast*
 with ‹*S1* ∈ *R*› ‹*S2* ∈ *R*› *assms*(*2*) **show** *x* = *y*
  unfolding *antisym-def* **by** *auto*
qed

```
lemma chain-subset-Total-Union:
  assumes chain⊆ R and ∀ r∈R. Total r
  shows Total (⋃ R)
proof (simp add: total-on-def Ball-def, auto del: disjCI)
  fix r s a b
  assume A: r ∈ R s ∈ R a ∈ Field r b ∈ Field s a ≠ b
  from ‹chain⊆ R› and ‹r ∈ R› and ‹s ∈ R› have r ⊆ s ∨ s ⊆ r
    by (auto simp add: chain-subset-def)
  then show (∃ r∈R. (a, b) ∈ r) ∨ (∃ r∈R. (b, a) ∈ r)
  proof
    assume r ⊆ s
    then have (a, b) ∈ s ∨ (b, a) ∈ s
      using assms(2) A mono-Field[of r s]
      by (auto simp add: total-on-def)
    then show ?thesis
      using ‹s ∈ R› by blast
  next
    assume s ⊆ r
    then have (a, b) ∈ r ∨ (b, a) ∈ r
      using assms(2) A mono-Field[of s r]
      by (fastforce simp add: total-on-def)
    then show ?thesis
      using ‹r ∈ R› by blast
  qed
qed

lemma wf-Union-wf-init-segs:
  assumes R ∈ Chains init-seg-of
    and ∀ r∈R. wf r
  shows wf (⋃ R)
proof (simp add: wf-iff-no-infinite-down-chain, rule ccontr, auto)
  fix f
  assume 1: ∀ i. ∃ r∈R. (f (Suc i), f i) ∈ r
  then obtain r where r ∈ R and (f (Suc 0), f 0) ∈ r by auto
  have (f (Suc i), f i) ∈ r for i
  proof (induct i)
    case 0
    show ?case by fact
  next
    case (Suc i)
    then obtain s where s: s ∈ R (f (Suc (Suc i)), f(Suc i)) ∈ s
      using 1 by auto
    then have s initial-segment-of r ∨ r initial-segment-of s
      using assms(1) ‹r ∈ R› by (simp add: Chains-def)
    with Suc s show ?case by (simp add: init-seg-of-def) blast
  qed
  then show False
    using assms(2) and ‹r ∈ R›
    by (simp add: wf-iff-no-infinite-down-chain) blast
```

qed

lemma *initial-segment-of-Diff*: *p initial-segment-of q* ⟹ *p* − *s initial-segment-of q* − *s*
unfolding *init-seg-of-def* **by** *blast*

lemma *Chains-inits-DiffI*: *R* ∈ *Chains init-seg-of* ⟹ {*r* − *s* |*r*. *r* ∈ *R*} ∈ *Chains init-seg-of*
unfolding *Chains-def* **by** (*blast intro*: *initial-segment-of-Diff*)

theorem *well-ordering*: ∃ *r*::'*a rel*. *Well-order r* ∧ *Field r* = *UNIV*
proof −
— The initial segment relation on well-orders:
let *?WO* = {*r*::'*a rel*. *Well-order r*}
define *I* **where** *I* = *init-seg-of* ∩ *?WO* × *?WO*
then have *I-init*: *I* ⊆ *init-seg-of* **by** *simp*
then have *subch*: ⋀*R*. *R* ∈ *Chains I* ⟹ $chain_{\subseteq}$ *R*
unfolding *init-seg-of-def chain-subset-def Chains-def* **by** *blast*
have *Chains-wo*: ⋀*R r*. *R* ∈ *Chains I* ⟹ *r* ∈ *R* ⟹ *Well-order r*
by (*simp add*: *Chains-def I-def*) *blast*
have *FI*: *Field I* = *?WO*
by (*auto simp add*: *I-def init-seg-of-def Field-def*)
then have *0*: *Partial-order I*
by (*auto simp*: *partial-order-on-def preorder-on-def antisym-def antisym-init-seg-of refl-on-def*
trans-def I-def elim!: *trans-init-seg-of*)
— *I*-chains have upper bounds in *?WO* wrt *I*: their Union
have ⋃ *R* ∈ *?WO* ∧ (∀ *r*∈*R*. (*r*, ⋃ *R*) ∈ *I*) **if** *R* ∈ *Chains I* **for** *R*
proof −
from *that* **have** *Ris*: *R* ∈ *Chains init-seg-of*
using *mono-Chains* [*OF I-init*] **by** *blast*
have *subch*: $chain_{\subseteq}$ *R*
using ‹*R* ∈ *Chains I*› *I-init* **by** (*auto simp*: *init-seg-of-def chain-subset-def Chains-def*)
have ∀ *r*∈*R*. *Refl r* **and** ∀ *r*∈*R*. *trans r* **and** ∀ *r*∈*R*. *antisym r*
and ∀ *r*∈*R*. *Total r* **and** ∀ *r*∈*R*. *wf* (*r* − *Id*)
using *Chains-wo* [*OF* ‹*R* ∈ *Chains I*›] **by** (*simp-all add*: *order-on-defs*)
have *Refl* (⋃ *R*)
using ‹∀ *r*∈*R*. *Refl r*› **unfolding** *refl-on-def* **by** *fastforce*
moreover have *trans* (⋃ *R*)
by (*rule chain-subset-trans-Union* [*OF subch* ‹∀ *r*∈*R*. *trans r*›])
moreover have *antisym* (⋃ *R*)
by (*rule chain-subset-antisym-Union* [*OF subch* ‹∀ *r*∈*R*. *antisym r*›])
moreover have *Total* (⋃ *R*)
by (*rule chain-subset-Total-Union* [*OF subch* ‹∀ *r*∈*R*. *Total r*›])
moreover have *wf* ((⋃ *R*) − *Id*)
proof −
have (⋃ *R*) − *Id* = ⋃ {*r* − *Id* | *r*. *r* ∈ *R*} **by** *blast*
with ‹∀ *r*∈*R*. *wf* (*r* − *Id*)› **and** *wf-Union-wf-init-segs* [*OF Chains-inits-DiffI*

[*OF Ris*]]
show *?thesis* **by** *fastforce*
qed
ultimately have *Well-order* ($\bigcup R$)
by (*simp add*:*order-on-defs*)
moreover have $\forall r \in R.$ *r initial-segment-of* $\bigcup R$
using *Ris* **by** (*simp add*: *Chains-init-seg-of-Union*)
ultimately show *?thesis*
using *mono-Chains* [*OF I-init*] *Chains-wo*[*of R*] **and** ‹*R* ∈ *Chains I*›
unfolding *I-def* **by** *blast*
qed
then have *1*: $\exists u \in$*Field I*. $\forall r \in R.$ $(r, u) \in I$ **if** $R \in$ *Chains I* **for** *R*
using *that* **by** (*subst FI*) *blast*
— Zorn's Lemma yields a maximal well-order *m*:
then obtain *m* :: *'a rel*
where *Well-order m*
and *max*: $\forall r.$ *Well-order r* $\wedge$ $(m, r) \in I \longrightarrow r = m$
using *Zorns-po-lemma*[*OF 0 1*] **unfolding** *FI* **by** *fastforce*
— Now show by contradiction that *m* covers the whole type:
have *False* **if** $x \notin$ *Field m* **for** *x* :: *'a*
proof −
— Assuming that *x* is not covered and extend *m* at the top with *x*
have $m \neq \{\}$
proof
assume $m = \{\}$
moreover have *Well-order* $\{(x, x)\}$
by (*simp add*: *order-on-defs refl-on-def trans-def antisym-def total-on-def Field-def*)
ultimately show *False* **using** *max*
by (*auto simp*: *I-def init-seg-of-def simp del*: *Field-insert*)
qed
then have *Field m* $\neq \{\}$ **by** (*auto simp*: *Field-def*)
moreover have *wf* $(m - Id)$
using ‹*Well-order m*› **by** (*simp add*: *well-order-on-def*)
— The extension of *m* by *x*:
let $?s = \{(a, x) \mid a.\ a \in$ *Field m*$\}$
let $?m =$ *insert* (x, x) $m \cup ?s$
have *Fm*: *Field ?m* = *insert x* (*Field m*)
by (*auto simp*: *Field-def*)
have *Refl m* **and** *trans m* **and** *antisym m* **and** *Total m* **and** *wf* $(m - Id)$
using ‹*Well-order m*› **by** (*simp-all add*: *order-on-defs*)
— We show that the extension is a well-order
have *Refl ?m*
using ‹*Refl m*› *Fm* **unfolding** *refl-on-def* **by** *blast*
moreover have *trans ?m* **using** ‹*trans m*› **and** ‹$x \notin$ *Field m*›
unfolding *trans-def Field-def* **by** *blast*
moreover have *antisym ?m*
using ‹*antisym m*› **and** ‹$x \notin$ *Field m*› **unfolding** *antisym-def Field-def* **by** *blast*

moreover have *Total ?m*
using ‹*Total m*› **and** *Fm* **by** (*auto simp*: *total-on-def*)
moreover have *wf* (*?m* − *Id*)
proof −
have *wf ?s*
using ‹*x* ∉ *Field m*› **by** (*auto simp*: *wf-eq-minimal Field-def Bex-def*)
then show *?thesis*
using ‹*wf* (*m* − *Id*)› **and** ‹*x* ∉ *Field m*› *wf-subset* [*OF* ‹*wf ?s*› *Diff-subset*]
by (*auto simp*: *Un-Diff Field-def intro*: *wf-Un*)
qed
ultimately have *Well-order ?m*
by (*simp add*: *order-on-defs*)
— We show that the extension is above *m*
moreover have (*m*, *?m*) ∈ *I*
using ‹*Well-order ?m*› **and** ‹*Well-order m*› **and** ‹*x* ∉ *Field m*›
by (*fastforce simp*: *I-def init-seg-of-def Field-def*)
ultimately
— This contradicts maximality of *m*:
show *False*
using *max* **and** ‹*x* ∉ *Field m*› **unfolding** *Field-def* **by** *blast*
qed
then have *Field m* = *UNIV* **by** *auto*
with ‹*Well-order m*› **show** *?thesis* **by** *blast*
qed

corollary *well-order-on*: ∃ *r*::′*a rel*. *well-order-on A r*
proof −
obtain *r* :: ′*a rel* **where** *wo*: *Well-order r* **and** *univ*: *Field r* = *UNIV*
using *well-ordering* [**where** ′*a* = ′*a*] **by** *blast*
let *?r* = {(*x*, *y*). *x* ∈ *A* ∧ *y* ∈ *A* ∧ (*x*, *y*) ∈ *r*}
have *1*: *Field ?r* = *A*
using *wo univ* **by** (*fastforce simp*: *Field-def order-on-defs refl-on-def*)
from ‹*Well-order r*› **have** *Refl r trans r antisym r Total r wf* (*r* − *Id*)
by (*simp-all add*: *order-on-defs*)
from ‹*Refl r*› **have** *Refl ?r*
by (*auto simp*: *refl-on-def 1 univ*)
moreover from ‹*trans r*› **have** *trans ?r*
unfolding *trans-def* **by** *blast*
moreover from ‹*antisym r*› **have** *antisym ?r*
unfolding *antisym-def* **by** *blast*
moreover from ‹*Total r*› **have** *Total ?r*
by (*simp add*:*total-on-def 1 univ*)
moreover have *wf* (*?r* − *Id*)
by (*rule wf-subset* [*OF* ‹*wf* (*r* − *Id*)›]) *blast*
ultimately have *Well-order ?r*
by (*simp add*: *order-on-defs*)
with *1* **show** *?thesis* **by** *auto*
qed

lemma *dependent-wf-choice*:
  fixes $P :: ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
  assumes *wf R*
    and *adm*: $\bigwedge f\ g\ x\ r.\ (\bigwedge z.\ (z, x) \in R \Longrightarrow f\ z = g\ z) \Longrightarrow P\ f\ x\ r = P\ g\ x\ r$
    and *P*: $\bigwedge x\ f.\ (\bigwedge y.\ (y, x) \in R \Longrightarrow P\ f\ y\ (f\ y)) \Longrightarrow \exists r.\ P\ f\ x\ r$
  shows $\exists f.\ \forall x.\ P\ f\ x\ (f\ x)$
proof (*intro exI allI*)
  fix *x*
  define *f* **where** $f \equiv wfrec\ R\ (\lambda f\ x.\ SOME\ r.\ P\ f\ x\ r)$
  from ‹*wf R*› **show** $P\ f\ x\ (f\ x)$
  proof (*induct x*)
    case (*less x*)
    show $P\ f\ x\ (f\ x)$
    proof (*subst* (*2*) *wfrec-def-adm*[*OF f-def* ‹*wf R*›])
      show *adm-wf R* $(\lambda f\ x.\ SOME\ r.\ P\ f\ x\ r)$
        by (*auto simp*: *adm-wf-def intro*!: *arg-cong*[**where** *f*=*Eps*] *adm*)
      show $P\ f\ x\ (Eps\ (P\ f\ x))$
        using *P* **by** (*rule someI-ex*) *fact*
    qed
  qed
qed

lemma (**in** *wellorder*) *dependent-wellorder-choice*:
  assumes $\bigwedge r\ f\ g\ x.\ (\bigwedge y.\ y < x \Longrightarrow f\ y = g\ y) \Longrightarrow P\ f\ x\ r = P\ g\ x\ r$
    and *P*: $\bigwedge x\ f.\ (\bigwedge y.\ y < x \Longrightarrow P\ f\ y\ (f\ y)) \Longrightarrow \exists r.\ P\ f\ x\ r$
  shows $\exists f.\ \forall x.\ P\ f\ x\ (f\ x)$
  using *wf* **by** (*rule dependent-wf-choice*) (*auto intro*!: *assms*)

end

27 Well-Order Relations as Needed by Bounded Natural Functors

theory *BNF-Wellorder-Relation*
  imports *Order-Relation*
begin

In this section, we develop basic concepts and results pertaining to well-order relations. Note that we consider well-order relations as *non-strict relations*, i.e., as containing the diagonals of their fields.

locale *wo-rel* =
  fixes $r :: 'a\ rel$
  assumes *WELL*: *Well-order r*
begin

The following context encompasses all this section. In other words, for the whole section, we consider a fixed well-order relation *r*.

abbreviation *under* **where** *under* ≡ *Order-Relation.under r*

abbreviation *underS* **where** *underS* ≡ *Order-Relation.underS r*
abbreviation *Under* **where** *Under* ≡ *Order-Relation.Under r*
abbreviation *UnderS* **where** *UnderS* ≡ *Order-Relation.UnderS r*
abbreviation *above* **where** *above* ≡ *Order-Relation.above r*
abbreviation *aboveS* **where** *aboveS* ≡ *Order-Relation.aboveS r*
abbreviation *Above* **where** *Above* ≡ *Order-Relation.Above r*
abbreviation *AboveS* **where** *AboveS* ≡ *Order-Relation.AboveS r*
abbreviation *ofilter* **where** *ofilter* ≡ *Order-Relation.ofilter r*
lemmas *ofilter-def* = *Order-Relation.ofilter-def*[*of r*]

27.1 Auxiliaries

lemma *REFL*: *Refl r*
using *WELL order-on-defs*[*of - r*] **by** *auto*

lemma *TRANS*: *trans r*
using *WELL order-on-defs*[*of - r*] **by** *auto*

lemma *ANTISYM*: *antisym r*
using *WELL order-on-defs*[*of - r*] **by** *auto*

lemma *TOTAL*: *Total r*
using *WELL order-on-defs*[*of - r*] **by** *auto*

lemma *TOTALS*: ∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. (*a*,*b*) ∈ *r* ∨ (*b*,*a*) ∈ *r*
using *REFL TOTAL refl-on-def*[*of - r*] *total-on-def*[*of - r*] **by** *force*

lemma *LIN*: *Linear-order r*
using *WELL well-order-on-def*[*of - r*] **by** *auto*

lemma *WF*: *wf* (*r* − *Id*)
using *WELL well-order-on-def*[*of - r*] **by** *auto*

lemma *cases-Total*:
⋀ *phi a b*. ⟦{*a*,*b*} <= *Field r*; ((*a*,*b*) ∈ *r* ⟹ *phi a b*); ((*b*,*a*) ∈ *r* ⟹ *phi a b*)⟧
⟹ *phi a b*
using *TOTALS* **by** *auto*

lemma *cases-Total3*:
⋀ *phi a b*. ⟦{*a*,*b*} ≤ *Field r*; ((*a*,*b*) ∈ *r* − *Id* ∨ (*b*,*a*) ∈ *r* − *Id* ⟹ *phi a b*);
(*a* = *b* ⟹ *phi a b*)⟧ ⟹ *phi a b*
using *TOTALS* **by** *auto*

27.2 Well-founded induction and recursion adapted to non-strict well-order relations

Here we provide induction and recursion principles specific to *non-strict* well-order relations. Although minor variations of those for well-founded relations, they will be useful for doing away with the tediousness of having to take out the diagonal each time in order to switch to a well-founded relation.

lemma *well-order-induct*:
assumes *IND*: $\bigwedge x.\ \forall y.\ y \neq x \wedge (y, x) \in r \longrightarrow P\ y \Longrightarrow P\ x$
shows *P a*
proof−
have $\bigwedge x.\ \forall y.\ (y, x) \in r - Id \longrightarrow P\ y \Longrightarrow P\ x$
using *IND* **by** *blast*
thus *P a* **using** *WF wf-induct*[*of r* − *Id P a*] **by** *blast*
qed

definition
worec :: $(('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
where
worec F ≡ *wfrec* (*r* − *Id*) *F*

definition
adm-wo :: $(('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b) \Rightarrow bool$
where
adm-wo H ≡ ∀ *f g x*. (∀ *y* ∈ *underS x*. *f y* = *g y*) ⟶ *H f x* = *H g x*

lemma *worec-fixpoint*:
assumes *ADM*: *adm-wo H*
shows *worec H* = *H* (*worec H*)
proof−
let *?rS* = *r* − *Id*
have *adm-wf* (*r* − *Id*) *H*
unfolding *adm-wf-def*
using *ADM adm-wo-def*[*of H*] *underS-def*[*of r*] **by** *auto*
hence *wfrec ?rS H* = *H* (*wfrec ?rS H*)
using *WF wfrec-fixpoint*[*of ?rS H*] **by** *simp*
thus *?thesis* **unfolding** *worec-def* **.**
qed

27.3 The notions of maximum, minimum, supremum, successor and order filter

We define the successor *of a set*, and not of an element (the latter is of course a particular case). Also, we define the maximum *of two elements*, *max2*, and the minimum *of a set*, *minim* – we chose these variants since we consider them the most useful for well-orders. The minimum is defined in terms of the auxiliary relational operator *isMinim*. Then, supremum and successor are defined in terms of minimum as expected. The minimum is only meaningful for non-empty sets, and the successor is only meaningful for sets for which strict upper bounds exist. Order filters for well-orders are also known as "initial segments".

definition *max2* :: $'a \Rightarrow 'a \Rightarrow 'a$

where *max2 a b* ≡ *if* (*a*,*b*) ∈ *r then b else a*

definition *isMinim* :: ′*a set* ⇒ ′*a* ⇒ *bool*
where *isMinim A b* ≡ *b* ∈ *A* ∧ (∀ *a* ∈ *A*. (*b*,*a*) ∈ *r*)

definition *minim* :: ′*a set* ⇒ ′*a*
where *minim A* ≡ *THE b*. *isMinim A b*

definition *supr* :: ′*a set* ⇒ ′*a*
where *supr A* ≡ *minim* (*Above A*)

definition *suc* :: ′*a set* ⇒ ′*a*
where *suc A* ≡ *minim* (*AboveS A*)

27.3.1 Properties of max2

lemma *max2-greater-among*:
assumes *a* ∈ *Field r* **and** *b* ∈ *Field r*
shows (*a*, *max2 a b*) ∈ *r* ∧ (*b*, *max2 a b*) ∈ *r* ∧ *max2 a b* ∈ {*a*,*b*}
proof−
{**assume** (*a*,*b*) ∈ *r*
hence *?thesis* **using** *max2-def assms REFL refl-on-def*
by (*auto simp add*: *refl-on-def*)
}
moreover
{**assume** *a* = *b*
hence (*a*,*b*) ∈ *r* **using** *REFL assms*
by (*auto simp add*: *refl-on-def*)
}
moreover
{**assume** ∗: *a* ≠ *b* ∧ (*b*,*a*) ∈ *r*
hence (*a*,*b*) ∉ *r* **using** *ANTISYM*
by (*auto simp add*: *antisym-def*)
hence *?thesis* **using** ∗ *max2-def assms REFL refl-on-def*
by (*auto simp add*: *refl-on-def*)
}
ultimately show *?thesis* **using** *assms TOTAL*
total-on-def[*of Field r r*] **by** *blast*
qed

lemma *max2-greater*:
assumes *a* ∈ *Field r* **and** *b* ∈ *Field r*
shows (*a*, *max2 a b*) ∈ *r* ∧ (*b*, *max2 a b*) ∈ *r*
using *assms* **by** (*auto simp add*: *max2-greater-among*)

lemma *max2-among*:
assumes *a* ∈ *Field r* **and** *b* ∈ *Field r*
shows *max2 a b* ∈ {*a*, *b*}
using *assms max2-greater-among*[*of a b*] **by** *simp*

lemma *max2-equals1*:
  assumes $a \in Field\ r$ **and** $b \in Field\ r$
  shows $(max2\ a\ b = a) = ((b,a) \in r)$
  using *assms ANTISYM* **unfolding** *antisym-def* **using** *TOTALS*
  by(*auto simp add*: *max2-def max2-among*)

lemma *max2-equals2*:
  assumes $a \in Field\ r$ **and** $b \in Field\ r$
  shows $(max2\ a\ b = b) = ((a,b) \in r)$
  using *assms ANTISYM* **unfolding** *antisym-def* **using** *TOTALS*
  unfolding *max2-def* **by** *auto*

lemma *in-notinI*:
  assumes $(j,i) \notin r \vee j = i$ **and** $i \in Field\ r$ **and** $j \in Field\ r$
  shows $(i,j) \in r$ **using** *assms max2-def max2-greater-among* **by** *fastforce*

27.3.2 Existence and uniqueness for isMinim and well-definedness of minim

lemma *isMinim-unique*:
  assumes *isMinim B a isMinim B a′*
  shows $a = a'$
  using *assms ANTISYM antisym-def*[*of r*] **by** (*auto simp*: *isMinim-def*)

lemma *Well-order-isMinim-exists*:
  assumes *SUB*: $B \leq Field\ r$ **and** *NE*: $B \neq \{\}$
  shows $\exists\, b.\ isMinim\ B\ b$
proof−
  from *spec*[*OF WF*[*unfolded wf-eq-minimal*[*of r − Id*]], *of B*] *NE* **obtain** *b* **where**
    $*$: $b \in B \wedge (\forall\, b'.\ b' \neq b \wedge (b',b) \in r \longrightarrow b' \notin B)$ **by** *auto*
  have $\forall\, b'.\ b' \in B \longrightarrow (b,\ b') \in r$
  proof
    fix b'
    show $b' \in B \longrightarrow (b,\ b') \in r$
    proof
      assume *As*: $b' \in B$
      hence $**$: $b \in Field\ r \wedge b' \in Field\ r$ **using** *As SUB* $*$ **by** *auto*
      from *As* $*$ **have** $b' = b \vee (b',b) \notin r$ **by** *auto*
      moreover have $b' = b \Longrightarrow (b,\ b') \in r$
        using $**$ *REFL* **by** (*auto simp add*: *refl-on-def*)
      moreover have $b' \neq b \wedge (b',b) \notin r \Longrightarrow (b,b') \in r$
        using $**$ *TOTAL* **by** (*auto simp add*: *total-on-def*)
      ultimately show $(b,b') \in r$ **by** *blast*
    qed
  qed
  then show *?thesis*
    unfolding *isMinim-def* **using** $*$ **by** *auto*
qed

lemma *minim-isMinim*:
  assumes *SUB*: $B \leq$ *Field r* **and** *NE*: $B \neq \{\}$
  shows *isMinim B* (*minim B*)
proof−
  let *?phi* = (λ *b*. *isMinim B b*)
  from *assms Well-order-isMinim-exists*
  obtain *b* **where** $*$: *?phi b* **by** *blast*
  moreover
  have $\bigwedge b'$. *?phi* $b' \Longrightarrow b' = b$
    using *isMinim-unique* $*$ **by** *auto*
  ultimately show *?thesis*
    unfolding *minim-def* **using** *theI*[*of ?phi b*] **by** *blast*
qed

27.3.3 Properties of minim

lemma *minim-in*:
  assumes $B \leq$ *Field r* **and** $B \neq \{\}$
  shows *minim* $B \in B$
  using *assms minim-isMinim*[*of B*] **by** (*auto simp*: *isMinim-def*)

lemma *minim-inField*:
  assumes $B \leq$ *Field r* **and** $B \neq \{\}$
  shows *minim* $B \in$ *Field r*
proof−
  have *minim* $B \in B$ **using** *assms* **by** (*simp add*: *minim-in*)
  thus *?thesis* **using** *assms* **by** *blast*
qed

lemma *minim-least*:
  assumes *SUB*: $B \leq$ *Field r* **and** *IN*: $b \in B$
  shows (*minim B*, *b*) $\in r$
proof−
  from *minim-isMinim*[*of B*] *assms*
  have *isMinim B* (*minim B*) **by** *auto*
  thus *?thesis* **by** (*auto simp add*: *isMinim-def IN*)
qed

lemma *equals-minim*:
  assumes *SUB*: $B \leq$ *Field r* **and** *IN*: $a \in B$ **and**
    LEAST: $\bigwedge b.\ b \in B \Longrightarrow (a,b) \in r$
  shows $a =$ *minim B*
proof−
  from *minim-isMinim*[*of B*] *assms*
  have *isMinim B* (*minim B*) **by** *auto*
  moreover have *isMinim B a* **using** *IN LEAST isMinim-def* **by** *auto*
  ultimately show *?thesis*
    using *isMinim-unique* **by** *auto*

qed

27.3.4 Properties of successor

lemma *suc-AboveS*:
  assumes *SUB*: $B \leq$ *Field r* **and** *ABOVES*: *AboveS B* $\neq \{\}$
  shows *suc B* $\in$ *AboveS B*
proof(*unfold suc-def*)
  have *AboveS B* $\leq$ *Field r*
    using *AboveS-Field*[*of r*] **by** *auto*
  thus *minim* (*AboveS B*) $\in$ *AboveS B*
    using *assms* **by** (*simp add*: *minim-in*)
qed

lemma *suc-greater*:
  assumes *SUB*: $B \leq$ *Field r* **and** *ABOVES*: *AboveS B* $\neq \{\}$ **and** *IN*: $b \in B$
  shows *suc B* $\neq b \wedge (b,$*suc B*$) \in r$
  using *IN AboveS-def*[*of r*] *assms suc-AboveS* **by** *auto*

lemma *suc-least-AboveS*:
  assumes *ABOVES*: $a \in$ *AboveS B*
  shows (*suc B*,a) $\in r$
  using *assms minim-least AboveS-Field*[*of r*] **by** (*auto simp*: *suc-def*)

lemma *suc-inField*:
  assumes $B \leq$ *Field r* **and** *AboveS B* $\neq \{\}$
  shows *suc B* $\in$ *Field r*
  using *suc-AboveS assms AboveS-Field*[*of r*] **by** *auto*

lemma *equals-suc-AboveS*:
  assumes $B \leq$ *Field r* **and** $a \in$ *AboveS B* **and** $\bigwedge a'.\ a' \in$ *AboveS B* $\Longrightarrow (a,a')$
$\in r$
  shows $a =$ *suc B*
  using *assms equals-minim AboveS-Field*[*of r B*] **by** (*auto simp*: *suc-def*)

lemma *suc-underS*:
  assumes *IN*: $a \in$ *Field r*
  shows $a =$ *suc* (*underS a*)
proof−
  have *underS a* $\leq$ *Field r*
    using *underS-Field*[*of r*] **by** *auto*
  moreover
  have $a \in$ *AboveS* (*underS a*)
    using *in-AboveS-underS IN* **by** *fast*
  moreover
  have $\forall a' \in$ *AboveS* (*underS a*). $(a,a') \in r$
  proof(*clarify*)
    fix a'
    assume ∗: $a' \in$ *AboveS* (*underS a*)

```
    hence **: a' ∈ Field r
      using AboveS-Field by fast
    {assume (a,a') ∉ r
      hence a' = a ∨ (a',a) ∈ r
        using TOTAL IN ** by (auto simp add: total-on-def)
      moreover
      {assume a' = a
        hence (a,a') ∈ r
          using REFL IN ** by (auto simp add: refl-on-def)
      }
      moreover
      {assume a' ≠ a ∧ (a',a) ∈ r
        hence a' ∈ underS a
          unfolding underS-def by simp
        hence a' ∉ AboveS (underS a)
          using AboveS-disjoint by fast
        with * have False by simp
      }
      ultimately have (a,a') ∈ r by blast
    }
    thus (a, a') ∈ r by blast
  qed
  ultimately show ?thesis
    using equals-suc-AboveS by auto
qed
```

27.3.5 Properties of order filters

```
lemma under-ofilter: ofilter (under a)
  using TRANS by (auto simp: ofilter-def under-def Field-iff trans-def)

lemma underS-ofilter: ofilter (underS a)
  unfolding ofilter-def underS-def under-def
proof safe
  fix b assume (a, b) ∈ r (b, a) ∈ r and DIFF: b ≠ a
  thus False
    using ANTISYM antisym-def[of r] by blast
next
  fix b x
  assume (b,a) ∈ r b ≠ a (x,b) ∈ r
  thus (x,a) ∈ r
    using TRANS trans-def[of r] by blast
next
  fix x
  assume x ≠ a and (x, a) ∈ r
  then show x ∈ Field r
    unfolding Field-def
    by auto
qed
```

lemma *Field-ofilter*:
  ofilter (*Field r*)
  by(*unfold ofilter-def under-def*, *auto simp add*: *Field-def*)

lemma *ofilter-underS-Field*:
  ofilter A = ((∃ *a* ∈ *Field r*. *A* = *underS a*) ∨ (*A* = *Field r*))
proof
  assume (∃ *a*∈*Field r*. *A* = *underS a*) ∨ *A* = *Field r*
  thus *ofilter A*
    by (*auto simp*: *underS-ofilter Field-ofilter*)
next
  assume ∗: *ofilter A*
  let *?One* = (∃ *a*∈*Field r*. *A* = *underS a*)
  let *?Two* = (*A* = *Field r*)
  show *?One* ∨ *?Two*
  proof(*cases ?Two*)
    let *?B* = (*Field r*) − *A*
    let *?a* = *minim ?B*
    assume *A* ≠ *Field r*
    moreover have *A* ≤ *Field r* **using** ∗ *ofilter-def* **by** *simp*
    ultimately have *1*: *?B* ≠ {} **by** *blast*
    hence *2*: *?a* ∈ *Field r* **using** *minim-inField*[*of ?B*] **by** *blast*
    have *3*: *?a* ∈ *?B* **using** *minim-in*[*of ?B*] *1* **by** *blast*
    hence *4*: *?a* ∉ *A* **by** *blast*
    have *5*: *A* ≤ *Field r* **using** ∗ *ofilter-def* **by** *auto*

    moreover
    have *A* = *underS ?a*
    proof
      show *A* ≤ *underS ?a*
      proof
        fix *x* **assume** ∗∗: *x* ∈ *A*
        hence *11*: *x* ∈ *Field r* **using** *5* **by** *auto*
        have *12*: *x* ≠ *?a* **using** *4* ∗∗ **by** *auto*
        have *13*: *under x* ≤ *A* **using** ∗ *ofilter-def* ∗∗ **by** *auto*
        {**assume** (*x*,*?a*) ∉ *r*
          hence (*?a*,*x*) ∈ *r*
            using *TOTAL total-on-def*[*of Field r r*]
              2 4 11 12 **by** *auto*
          hence *?a* ∈ *under x* **using** *under-def*[*of r*] **by** *auto*
          hence *?a* ∈ *A* **using** ∗∗ *13* **by** *blast*
          with *4* **have** *False* **by** *simp*
        }
        then have (*x*,*?a*) ∈ *r* **by** *blast*
        thus *x* ∈ *underS ?a*
          unfolding *underS-def* **by** (*auto simp add*: *12*)
      qed
    next

```
      show underS ?a ≤ A
      proof
        fix x
        assume **: x ∈ underS ?a
        hence 11: x ∈ Field r
          using Field-def unfolding underS-def by fastforce
        {assume x ∉ A
          hence x ∈ ?B using 11 by auto
          hence (?a,x) ∈ r using 3 minim-least[of ?B x] by blast
          hence False
            using ANTISYM antisym-def[of r] ** unfolding underS-def by auto
        }
        thus x ∈ A by blast
      qed
    qed
    ultimately have ?One using 2 by blast
    thus ?thesis by simp
  next
    assume A = Field r
    then show ?thesis
      by simp
  qed
qed

lemma ofilter-UNION:
  (⋀ i. i ∈ I ⟹ ofilter(A i)) ⟹ ofilter (⋃ i ∈ I. A i)
  unfolding ofilter-def by blast

lemma ofilter-under-UNION:
  assumes ofilter A
  shows A = (⋃ a ∈ A. under a)
proof
  have ∀ a ∈ A. under a ≤ A
    using assms ofilter-def by auto
  thus (⋃ a ∈ A. under a) ≤ A by blast
next
  have ∀ a ∈ A. a ∈ under a
    using REFL Refl-under-in[of r] assms ofilter-def[of A] by blast
  thus A ≤ (⋃ a ∈ A. under a) by blast
qed
```

27.3.6 Other properties

```
lemma ofilter-linord:
  assumes OF1: ofilter A and OF2: ofilter B
  shows A ≤ B ∨ B ≤ A
proof(cases A = Field r)
  assume Case1: A = Field r
  hence B ≤ A using OF2 ofilter-def by auto
```

```
  thus ?thesis by simp
next
  assume Case2: A ≠ Field r
  with ofilter-underS-Field OF1 obtain a where
    1: a ∈ Field r ∧ A = underS a by auto
  show ?thesis
  proof(cases B = Field r)
    assume Case21: B = Field r
    hence A ≤ B using OF1 ofilter-def by auto
    thus ?thesis by simp
  next
    assume Case22: B ≠ Field r
    with ofilter-underS-Field OF2 obtain b where
      2: b ∈ Field r ∧ B = underS b by auto
    have a = b ∨ (a,b) ∈ r ∨ (b,a) ∈ r
      using 1 2 TOTAL total-on-def[of - r] by auto
    moreover
    {assume a = b with 1 2 have ?thesis by auto
    }
    moreover
    {assume (a,b) ∈ r
      with underS-incr[of r] TRANS ANTISYM 1 2
      have A ≤ B by auto
      hence ?thesis by auto
    }
    moreover
    {assume (b,a) ∈ r
      with underS-incr[of r] TRANS ANTISYM 1 2
      have B ≤ A by auto
      hence ?thesis by auto
    }
    ultimately show ?thesis by blast
  qed
qed

lemma ofilter-AboveS-Field:
  assumes ofilter A
  shows A ∪ (AboveS A) = Field r
proof
  show A ∪ (AboveS A) ≤ Field r
    using assms ofilter-def AboveS-Field[of r] by auto
next
  {fix x assume *: x ∈ Field r and **: x ∉ A
    {fix y assume ***: y ∈ A
      with ** have 1: y ≠ x by auto
      {assume (y,x) ∉ r
        moreover
        have y ∈ Field r using assms ofilter-def *** by auto
        ultimately have (x,y) ∈ r
```

```
     using 1 ∗ TOTAL total-on-def[of - r] by auto
    with ∗∗∗ assms ofilter-def under-def[of r] have x ∈ A by auto
    with ∗∗ have False by contradiction
   }
   hence (y,x) ∈ r by blast
   with 1 have y ≠ x ∧ (y,x) ∈ r by auto
  }
  with ∗ have x ∈ AboveS A unfolding AboveS-def by auto
 }
 thus Field r ≤ A ∪ (AboveS A) by blast
qed

lemma suc-ofilter-in:
 assumes OF: ofilter A and ABOVE-NE: AboveS A ≠ {} and
  REL: (b,suc A) ∈ r and DIFF: b ≠ suc A
 shows b ∈ A
proof−
 have ∗: suc A ∈ Field r ∧ b ∈ Field r
  using WELL REL well-order-on-domain[of Field r] by auto
 {assume ∗∗: b ∉ A
  hence b ∈ AboveS A
   using OF ∗ ofilter-AboveS-Field by auto
  hence (suc A, b) ∈ r
   using suc-least-AboveS by auto
  hence False using REL DIFF ANTISYM ∗
   by (auto simp add: antisym-def)
 }
 thus ?thesis by blast
qed

end

end
```

28 Well-Order Embeddings as Needed by Bounded Natural Functors

```
theory BNF-Wellorder-Embedding
 imports Hilbert-Choice BNF-Wellorder-Relation
begin
```

In this section, we introduce well-order *embeddings* and *isomorphisms* and prove their basic properties. The notion of embedding is considered from the point of view of the theory of ordinals, and therefore requires the source to be injected as an *initial segment* (i.e., *order filter*) of the target. A main result of this section is the existence of embeddings (in one direction or another) between any two well-orders, having as a consequence the fact that, given any two sets on any two types, one is smaller than (i.e., can be injected into) the other.

28.1 Auxiliaries

lemma *UNION-inj-on-ofilter*:
assumes *WELL*: *Well-order r* **and**
OF: $\bigwedge$ *i. i* $\in$ *I* $\Longrightarrow$ *wo-rel.ofilter r* (*A i*) **and**
INJ: $\bigwedge$ *i. i* $\in$ *I* $\Longrightarrow$ *inj-on f* (*A i*)
shows *inj-on f* ($\bigcup$ *i* $\in$ *I. A i*)
proof−
have *wo-rel r* **using** *WELL* **by** (*simp add*: *wo-rel-def*)
hence $\bigwedge$ *i j.* ⟦*i* $\in$ *I*; *j* $\in$ *I*⟧ $\Longrightarrow$ *A i* <= *A j* $\vee$ *A j* <= *A i*
using *wo-rel.ofilter-linord*[*of r*] *OF* **by** *blast*
with *WELL INJ* **show** *?thesis*
by (*auto simp add*: *inj-on-UNION-chain*)
qed

lemma *under-underS-bij-betw*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′* **and**
IN: *a* $\in$ *Field r* **and** *IN′*: *f a* $\in$ *Field r′* **and**
BIJ: *bij-betw f* (*underS r a*) (*underS r′* (*f a*))
shows *bij-betw f* (*under r a*) (*under r′* (*f a*))
proof−
have *a* $\notin$ *underS r a* $\wedge$ *f a* $\notin$ *underS r′* (*f a*)
unfolding *underS-def* **by** *auto*
moreover
{**have** *Refl r* $\wedge$ *Refl r′* **using** *WELL WELL′*
by (*auto simp add*: *order-on-defs*)
hence *under r a* = *underS r a* $\cup$ {*a*} $\wedge$
under r′ (*f a*) = *underS r′* (*f a*) $\cup$ {*f a*}
using *IN IN′* **by**(*auto simp add*: *Refl-under-underS*)
}
ultimately show *?thesis*
using *BIJ notIn-Un-bij-betw*[*of a underS r a f underS r′* (*f a*)] **by** *auto*
qed

28.2 (Well-order) embeddings, strict embeddings, isomorphisms and order-compatible functions

Standardly, a function is an embedding of a well-order in another if it injectively and order-compatibly maps the former into an order filter of the latter. Here we opt for a more succinct definition (operator *embed*), asking that, for any element in the source, the function should be a bijection between the set of strict lower bounds of that element and the set of strict lower bounds of its image. (Later we prove equivalence with the standard definition – lemma *embed-iff-compat-inj-on-ofilter*.) A *strict embedding* (operator *embedS*) is a non-bijective embedding and an isomorphism (operator *iso*) is a bijective embedding.

definition *embed* :: $'a$ *rel* ⇒ $'a'$ *rel* ⇒ ($'a$ ⇒ $'a'$) ⇒ *bool*
where
embed r r' f ≡ ∀ *a* ∈ *Field r*. *bij-betw f* (*under r a*) (*under r'* (*f a*))

lemmas *embed-defs* = *embed-def embed-def*[*abs-def*]

Strict embeddings:

definition *embedS* :: $'a$ *rel* ⇒ $'a'$ *rel* ⇒ ($'a$ ⇒ $'a'$) ⇒ *bool*
where
embedS r r' f ≡ *embed r r' f* ∧ ¬ *bij-betw f* (*Field r*) (*Field r'*)

lemmas *embedS-defs* = *embedS-def embedS-def*[*abs-def*]

definition *iso* :: $'a$ *rel* ⇒ $'a'$ *rel* ⇒ ($'a$ ⇒ $'a'$) ⇒ *bool*
where
iso r r' f ≡ *embed r r' f* ∧ *bij-betw f* (*Field r*) (*Field r'*)

lemmas *iso-defs* = *iso-def iso-def*[*abs-def*]

definition *compat* :: $'a$ *rel* ⇒ $'a'$ *rel* ⇒ ($'a$ ⇒ $'a'$) ⇒ *bool*
where
compat r r' f ≡ ∀ *a b*. (*a*,*b*) ∈ *r* ⟶ (*f a*, *f b*) ∈ *r'*

lemma *compat-wf*:
assumes *CMP*: *compat r r' f* **and** *WF*: *wf r'*
shows *wf r*
proof−
have *r* ≤ *inv-image r' f*
unfolding *inv-image-def* **using** *CMP*
by (*auto simp add*: *compat-def*)
with *WF* **show** *?thesis*
using *wf-inv-image*[*of r' f*] *wf-subset*[*of inv-image r' f*] **by** *auto*
qed

lemma *id-embed*: *embed r r id*
by(*auto simp add*: *id-def embed-def bij-betw-def*)

lemma *id-iso*: *iso r r id*
by(*auto simp add*: *id-def embed-def iso-def bij-betw-def*)

lemma *embed-compat*:
assumes *EMB*: *embed r r' f*
shows *compat r r' f*
unfolding *compat-def*
proof *clarify*
fix *a b*
assume ∗: (*a*,*b*) ∈ *r*
hence *1*: *b* ∈ *Field r* **using** *Field-def*[*of r*] **by** *blast*
have *a* ∈ *under r b*

using $*$ *under-def*[*of r*] **by** *simp*
hence $f\ a \in under\ r'\ (f\ b)$
using *EMB embed-def*[*of r r' f*]
bij-betw-def[*of f under r b under r'* (*f b*)]
image-def[*of f under r b*] *1* **by** *auto*
thus $(f\ a, f\ b) \in r'$
by (*auto simp add*: *under-def*)
qed

lemma *embed-in-Field*:
assumes *EMB*: *embed r r' f* **and** *IN*: $a \in Field\ r$
shows $f\ a \in Field\ r'$
proof −
have $a \in Domain\ r \vee a \in Range\ r$
using *IN* **unfolding** *Field-def* **by** *blast*
then show *?thesis*
using *embed-compat* [*OF EMB*]
unfolding *Field-def compat-def* **by** *force*
qed

lemma *comp-embed*:
assumes *EMB*: *embed r r' f* **and** *EMB'*: *embed r' r'' f'*
shows *embed r r''* $(f' \circ f)$
proof(*unfold embed-def*, *auto*)
fix *a* **assume** $*$: $a \in Field\ r$
hence *bij-betw f* (*under r a*) (*under r'* (*f a*))
using *embed-def*[*of r*] *EMB* **by** *auto*
moreover
{**have** $f\ a \in Field\ r'$
using *EMB* $*$ **by** (*auto simp add*: *embed-in-Field*)
hence *bij-betw f'* (*under r'* (*f a*)) (*under r''* (*f'* (*f a*)))
using *embed-def*[*of r'*] *EMB'* **by** *auto*
}
ultimately
show *bij-betw* $(f' \circ f)$ (*under r a*) (*under r''* (*f'*(*f a*)))
by(*auto simp add*: *bij-betw-trans*)
qed

lemma *comp-iso*:
assumes *EMB*: *iso r r' f* **and** *EMB'*: *iso r' r'' f'*
shows *iso r r''* $(f' \circ f)$
using *assms* **unfolding** *iso-def*
by (*auto simp add*: *comp-embed bij-betw-trans*)

That *embedS* is also preserved by function composition shall be proved only later.

lemma *embed-Field*: *embed r r' f* $\Longrightarrow$ *f*'(*Field r*) $\leq$ *Field r'*
by (*auto simp add*: *embed-in-Field*)

lemma *embed-preserves-ofilter*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *OF*: *wo-rel.ofilter r A*
shows *wo-rel.ofilter r' (f'A)*
proof−

from *WELL* **have** *Well*: *wo-rel r* **unfolding** *wo-rel-def* **.**
from *WELL'* **have** *Well'*: *wo-rel r'* **unfolding** *wo-rel-def* **.**
from *OF* **have** *0*: $A \leq$ *Field r* **by**(*auto simp add*: *Well wo-rel.ofilter-def*)

show *?thesis* **using** *Well' WELL EMB 0 embed-Field*[*of r r' f*]
proof(*unfold wo-rel.ofilter-def*, *auto simp add*: *image-def*)
fix *a b'*
assume ∗: $a \in A$ **and** ∗∗: $b' \in$ *under r'* (*f a*)
hence $a \in$ *Field r* **using** *0* **by** *auto*
hence *bij-betw f* (*under r a*) (*under r'* (*f a*))
using ∗ *EMB* **by** (*auto simp add*: *embed-def*)
hence *f'*(*under r a*) = *under r'* (*f a*)
by (*simp add*: *bij-betw-def*)
with ∗∗ *image-def*[*of f under r a*] **obtain** *b* **where**
1: $b \in$ *under r a* $\wedge$ $b' = f\ b$ **by** *blast*
hence $b \in A$ **using** *Well* ∗ *OF*
by (*auto simp add*: *wo-rel.ofilter-def*)
with *1* **show** $\exists\, b \in A.\ b' = f\ b$ **by** *blast*
qed
qed

lemma *embed-Field-ofilter*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f*
shows *wo-rel.ofilter r'* (*f'*(*Field r*))
proof−
have *wo-rel.ofilter r* (*Field r*)
using *WELL* **by** (*auto simp add*: *wo-rel-def wo-rel.Field-ofilter*)
with *WELL WELL' EMB*
show *?thesis* **by** (*auto simp add*: *embed-preserves-ofilter*)
qed

lemma *embed-inj-on*:
assumes *WELL*: *Well-order r* **and** *EMB*: *embed r r' f*
shows *inj-on f* (*Field r*)
proof(*unfold inj-on-def*, *clarify*)

from *WELL* **have** *Well*: *wo-rel r* **unfolding** *wo-rel-def* **.**
with *wo-rel.TOTAL*[*of r*]
have *Total*: *Total r* **by** *simp*
from *Well wo-rel.REFL*[*of r*]
have *Refl*: *Refl r* **by** *simp*

fix *a b*
assume ∗: *a* ∈ *Field r* **and** ∗∗: *b* ∈ *Field r* **and**
∗∗∗: *f a* = *f b*
hence *1*: *a* ∈ *Field r* ∧ *b* ∈ *Field r*
unfolding *Field-def* **by** *auto*
{**assume** (*a*,*b*) ∈ *r*
hence *a* ∈ *under r b* ∧ *b* ∈ *under r b*
using *Refl* **by**(*auto simp add*: *under-def refl-on-def*)
hence *a* = *b*
using *EMB 1* ∗∗∗
by (*auto simp add*: *embed-def bij-betw-def inj-on-def*)
}
moreover
{**assume** (*b*,*a*) ∈ *r*
hence *a* ∈ *under r a* ∧ *b* ∈ *under r a*
using *Refl* **by**(*auto simp add*: *under-def refl-on-def*)
hence *a* = *b*
using *EMB 1* ∗∗∗
by (*auto simp add*: *embed-def bij-betw-def inj-on-def*)
}
ultimately
show *a* = *b* **using** *Total 1*
by (*auto simp add*: *total-on-def*)
qed

lemma *embed-underS*:
assumes *WELL*: *Well-order r* **and**
EMB: *embed r r′ f* **and** *IN*: *a* ∈ *Field r*
shows *bij-betw f* (*underS r a*) (*underS r′* (*f a*))
proof−
have *f a* ∈ *Field r′* **using** *assms embed-Field*[*of r r′ f*] **by** *auto*
then have *0*: *under r a* = *underS r a* ∪ {*a*}
by (*simp add*: *IN Refl-under-underS WELL wo-rel.REFL wo-rel.intro*)
moreover have *1*: *bij-betw f* (*under r a*) (*under r′* (*f a*))
using *assms* **by** (*auto simp add*: *embed-def*)
moreover have *under r′* (*f a*) = *underS r′* (*f a*) ∪ {*f a*}
proof
show *under r′* (*f a*) ⊆ *underS r′* (*f a*) ∪ {*f a*}
using *underS-def under-def* **by** *fastforce*
show *underS r′* (*f a*) ∪ {*f a*} ⊆ *under r′* (*f a*)
using *bij-betwE 0 1 underS-subset-under* **by** *fastforce*
qed
moreover have *a* ∉ *underS r a* ∧ *f a* ∉ *underS r′* (*f a*)
unfolding *underS-def* **by** *blast*
ultimately show *?thesis*
by (*auto simp add*: *notIn-Un-bij-betw3*)
qed

lemma *embed-iff-compat-inj-on-ofilter*:

assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows *embed r r' f* = (*compat r r' f* ∧ *inj-on f* (*Field r*) ∧ *wo-rel.ofilter r'* (*f'*(*Field r*)))
using *assms*
proof(*auto simp add*: *embed-compat embed-inj-on embed-Field-ofilter*,
unfold embed-def, *auto*)
fix *a*
assume ∗: *inj-on f* (*Field r*) **and**
∗∗: *compat r r' f* **and**
∗∗∗: *wo-rel.ofilter r'* (*f'*(*Field r*)) **and**
∗∗∗∗: *a* ∈ *Field r*

have *Well*: *wo-rel r*
using *WELL wo-rel-def*[*of r*] **by** *simp*
hence *Refl*: *Refl r*
using *wo-rel.REFL*[*of r*] **by** *simp*
have *Total*: *Total r*
using *Well wo-rel.TOTAL*[*of r*] **by** *simp*
have *Well'*: *wo-rel r'*
using *WELL' wo-rel-def*[*of r'*] **by** *simp*
hence *Antisym'*: *antisym r'*
using *wo-rel.ANTISYM*[*of r'*] **by** *simp*
have (*a*,*a*) ∈ *r*
using ∗∗∗∗ *Well wo-rel.REFL*[*of r*]
refl-on-def[*of - r*] **by** *auto*
hence (*f a*, *f a*) ∈ *r'*
using ∗∗ **by**(*auto simp add*: *compat-def*)
hence *0*: *f a* ∈ *Field r'*
unfolding *Field-def* **by** *auto*
have *f a* ∈ *f'*(*Field r*)
using ∗∗∗∗ **by** *auto*
hence *2*: *under r'* (*f a*) ≤ *f'*(*Field r*)
using *Well'* ∗∗∗ *wo-rel.ofilter-def*[*of r' f'*(*Field r*)] **by** *fastforce*

show *bij-betw f* (*under r a*) (*under r'* (*f a*))
proof(*unfold bij-betw-def*, *auto*)
show *inj-on f* (*under r a*) **by** (*rule subset-inj-on*[*OF* ∗ *under-Field*])
next
fix *b* **assume** *b* ∈ *under r a*
thus *f b* ∈ *under r'* (*f a*)
unfolding *under-def* **using** ∗∗
by (*auto simp add*: *compat-def*)
next
fix *b'* **assume** ∗∗∗∗∗: *b'* ∈ *under r'* (*f a*)
hence *b'* ∈ *f'*(*Field r*)
using *2* **by** *auto*
with *Field-def*[*of r*] **obtain** *b* **where**
3: *b* ∈ *Field r* **and** *4*: *b'* = *f b* **by** *auto*
have (*b*,*a*) ∈ *r*

```
    proof−
      {assume (a,b) ∈ r
        with ∗∗ 4 have (f a, b′) ∈ r′
          by (auto simp add: compat-def)
        with ∗∗∗∗∗ Antisym′ have f a = b′
          by(auto simp add: under-def antisym-def)
        with 3 ∗∗∗∗ 4 ∗ have a = b
          by(auto simp add: inj-on-def)
      }
      moreover
      {assume a = b
        hence (b,a) ∈ r using Refl ∗∗∗∗ 3
          by (auto simp add: refl-on-def)
      }
      ultimately
      show ?thesis using Total ∗∗∗∗ 3 by (fastforce simp add: total-on-def)
    qed
    with 4 show  b′ ∈ f‘(under r a)
      unfolding under-def by auto
  qed
qed

lemma inv-into-ofilter-embed:
  assumes WELL: Well-order r and OF: wo-rel.ofilter r A and
    BIJ: ∀ b ∈ A. bij-betw f (under r b) (under r′ (f b)) and
    IMAGE: f ‘ A = Field r′
  shows embed r′ r (inv-into A f)
proof−

  have Well: wo-rel r
    using WELL wo-rel-def[of r] by simp
  have Refl: Refl r
    using Well wo-rel.REFL[of r] by simp
  have Total: Total r
    using Well wo-rel.TOTAL[of r] by simp

  have 1: bij-betw f A (Field r′)
  proof(unfold bij-betw-def inj-on-def, auto simp add: IMAGE)
    fix b1 b2
    assume ∗: b1 ∈ A and ∗∗: b2 ∈ A and
      ∗∗∗: f b1 = f b2
    have 11: b1 ∈ Field r ∧ b2 ∈ Field r
      using ∗ ∗∗ Well OF by (auto simp add: wo-rel.ofilter-def)
    moreover
    {assume (b1,b2) ∈ r
      hence b1 ∈ under r b2 ∧ b2 ∈ under r b2
        unfolding under-def using 11 Refl
        by (auto simp add: refl-on-def)
      hence b1 = b2 using BIJ ∗ ∗∗ ∗∗∗
```

```
      by (simp add: bij-betw-def inj-on-def)
    }
    moreover
    {assume (b2,b1) ∈ r
      hence b1 ∈ under r b1 ∧ b2 ∈ under r b1
        unfolding under-def using 11 Refl
        by (auto simp add: refl-on-def)
      hence b1 = b2 using BIJ * ** ***
        by (simp add: bij-betw-def inj-on-def)
    }
    ultimately
    show b1 = b2
      using Total by (auto simp add: total-on-def)
  qed

  let ?f' = (inv-into A f)

  have 2: ∀ b ∈ A. bij-betw ?f' (under r' (f b)) (under r b)
  proof(clarify)
    fix b assume *: b ∈ A
    hence under r b ≤ A
      using Well OF by(auto simp add: wo-rel.ofilter-def)
    moreover
    have f ` (under r b) = under r' (f b)
      using * BIJ by (auto simp add: bij-betw-def)
    ultimately
    show bij-betw ?f' (under r' (f b)) (under r b)
      using 1 by (auto simp add: bij-betw-inv-into-subset)
  qed

  have 3: ∀ b' ∈ Field r'. bij-betw ?f' (under r' b') (under r (?f' b'))
  proof(clarify)
    fix b' assume *: b' ∈ Field r'
    have b' = f (?f' b') using * 1
      by (auto simp add: bij-betw-inv-into-right)
    moreover
    {obtain b where 31: b ∈ A and f b = b' using IMAGE * by force
      hence ?f' b' = b using 1 by (auto simp add: bij-betw-inv-into-left)
      with 31 have ?f' b' ∈ A by auto
    }
    ultimately
    show  bij-betw ?f' (under r' b') (under r (?f' b'))
      using 2 by auto
  qed

  thus ?thesis unfolding embed-def .
qed

lemma inv-into-underS-embed:
```

assumes *WELL*: *Well-order r* **and**
BIJ: ∀ *b* ∈ *underS r a*. *bij-betw f* (*under r b*) (*under r′* (*f b*)) **and**
IN: *a* ∈ *Field r* **and**
IMAGE: *f* ‘ (*underS r a*) = *Field r′*
shows *embed r′ r* (*inv-into* (*underS r a*) *f*)
using *assms*
by(*auto simp add*: *wo-rel-def wo-rel.underS-ofilter inv-into-ofilter-embed*)

lemma *inv-into-Field-embed*:
assumes *WELL*: *Well-order r* **and** *EMB*: *embed r r′ f* **and**
IMAGE: *Field r′* ≤ *f* ‘ (*Field r*)
shows *embed r′ r* (*inv-into* (*Field r*) *f*)
proof−
have (∀ *b* ∈ *Field r*. *bij-betw f* (*under r b*) (*under r′* (*f b*)))
using *EMB* **by** (*auto simp add*: *embed-def*)
moreover
have *f* ‘ (*Field r*) ≤ *Field r′*
using *EMB WELL* **by** (*auto simp add*: *embed-Field*)
ultimately
show *?thesis* **using** *assms*
by(*auto simp add*: *wo-rel-def wo-rel.Field-ofilter inv-into-ofilter-embed*)
qed

lemma *inv-into-Field-embed-bij-betw*:
assumes *EMB*: *embed r r′ f* **and** *BIJ*: *bij-betw f* (*Field r*) (*Field r′*)
shows *embed r′ r* (*inv-into* (*Field r*) *f*)
proof−
have *Field r′* ≤ *f* ‘ (*Field r*)
using *BIJ* **by** (*auto simp add*: *bij-betw-def*)
then have *iso*: *iso r r′ f*
by (*simp add*: *BIJ EMB iso-def*)
have ∗: ∀ *a*. *a* ∈ *Field r* ⟶ *bij-betw f* (*under r a*) (*under r′* (*f a*))
using *EMB embed-def* **by** *fastforce*
show *?thesis*
proof (*clarsimp simp add*: *embed-def*)
fix *a*
assume *a*: *a* ∈ *Field r′*
then have *ar*: *a* ∈ *f* ‘ *Field r*
using *BIJ bij-betw-imp-surj-on* **by** *blast*
have [*simp*]: *f* (*inv-into* (*Field r*) *f a*) = *a*
by (*simp add*: *ar f-inv-into-f*)
show *bij-betw* (*inv-into* (*Field r*) *f*) (*under r′ a*) (*under r* (*inv-into* (*Field r*) *f*
a))
proof (*rule bij-betw-inv-into-subset* [*OF BIJ*])
show *under r* (*inv-into* (*Field r*) *f a*) ⊆ *Field r*
by (*simp add*: *under-Field*)
have *inv-into* (*Field r*) *f a* ∈ *Field r*
by (*simp add*: *ar inv-into-into*)
then show *f* ‘ *under r* (*inv-into* (*Field r*) *f a*) = *under r′ a*

```
      using bij-betw-imp-surj-on * by fastforce
    qed
  qed
qed
```

28.3 Given any two well-orders, one can be embedded in the other

Here is an overview of the proof of of this fact, stated in theorem *wellorders-totally-ordered*:

Fix the well-orders $r::'a$ *rel* and $r'::'a'$ *rel*. Attempt to define an embedding $f::'a \Rightarrow 'a'$ from r to r' in the natural way by well-order recursion ("hoping" that *Field* r turns out to be smaller than *Field* r'), but also record, at the recursive step, in a function $g::'a \Rightarrow$ *bool*, the extra information of whether *Field* r' gets exhausted or not.

If *Field* r' does not get exhausted, then *Field* r is indeed smaller and f is the desired embedding from r to r' (lemma *wellorders-totally-ordered-aux*).

Otherwise, it means that *Field* r' is the smaller one, and the inverse of (the "good" segment of) f is the desired embedding from r' to r (lemma *wellorders-totally-ordered-aux2*).

```
lemma wellorders-totally-ordered-aux:
  fixes r ::'a rel  and r'::'a' rel and
    f :: 'a ⇒ 'a' and a::'a
  assumes WELL: Well-order r and WELL': Well-order r' and IN: a ∈ Field r
and
    IH: ∀ b ∈ underS r a. bij-betw f (under r b) (under r' (f b)) and
    NOT: f ` (underS r a) ≠ Field r' and SUC: f a = wo-rel.suc r' (f`(underS r
a))
  shows bij-betw f (under r a) (under r' (f a))
proof−

  have Well: wo-rel r using WELL unfolding wo-rel-def .
  hence Refl: Refl r using wo-rel.REFL[of r] by auto
  have Trans: trans r using Well wo-rel.TRANS[of r] by auto
  have Well': wo-rel r' using WELL' unfolding wo-rel-def .
  have OF: wo-rel.ofilter r (underS r a)
    by (auto simp add: Well wo-rel.underS-ofilter)
  hence UN: underS r a = (⋃ b ∈ underS r a. under r b)
    using Well wo-rel.ofilter-under-UNION[of r underS r a] by blast

  {fix b assume *: b ∈ underS r a
    hence t0: (b,a) ∈ r ∧ b ≠ a unfolding underS-def by auto
    have t1: b ∈ Field r
      using * underS-Field[of r a] by auto
    have t2: f`(under r b) = under r' (f b)
      using IH * by (auto simp add: bij-betw-def)
    hence t3: wo-rel.ofilter r' (f`(under r b))
      using Well' by (auto simp add: wo-rel.under-ofilter)
```

```
    have f`(under r b) ≤ Field r'
      using t2 by (auto simp add: under-Field)
    moreover
    have b ∈ under r b
      using t1 by(auto simp add: Refl Refl-under-in)
    ultimately
    have t4:  f b ∈ Field r' by auto
    have f`(under r b) = under r' (f b) ∧
         wo-rel.ofilter r' (f`(under r b)) ∧
         f b ∈ Field r'
      using t2 t3 t4 by auto
  }
  hence bFact:
    ∀ b ∈ underS r a. f`(under r b) = under r' (f b) ∧
                      wo-rel.ofilter r' (f`(under r b)) ∧
                      f b ∈ Field r' by blast

  have subField: f`(underS r a) ≤ Field r'
    using bFact by blast

  have OF': wo-rel.ofilter r' (f`(underS r a))
  proof-
    have f`(underS r a) = f`(⋃ b ∈ underS r a. under r b)
      using UN by auto
    also have ... = (⋃ b ∈ underS r a. f`(under r b)) by blast
    also have ... = (⋃ b ∈ underS r a. (under r' (f b)))
      using bFact by auto
    finally
    have f`(underS r a) = (⋃ b ∈ underS r a. (under r' (f b))) .
    thus ?thesis
      using Well' bFact
        wo-rel.ofilter-UNION[of r' underS r a λ b. under r' (f b)] by fastforce
  qed

  have f`(underS r a) ∪ AboveS r' (f`(underS r a)) = Field r'
    using Well' OF' by (auto simp add: wo-rel.ofilter-AboveS-Field)
  hence NE: AboveS r' (f`(underS r a)) ≠ {}
    using subField NOT by blast

  have INCL1: f`(underS r a) ≤ underS r' (f a)
  proof(auto)
    fix b assume *: b ∈ underS r a
    have f b ≠ f a ∧ (f b, f a) ∈ r'
      using subField Well' SUC NE *
        wo-rel.suc-greater[of r' f`(underS r a) f b] by force
    thus f b ∈ underS r' (f a)
      unfolding underS-def by simp
  qed
```

```
  have INCL2: underS r' (f a) ≤ f`(underS r a)
  proof
    fix b' assume b' ∈ underS r' (f a)
    hence b' ≠ f a ∧ (b', f a) ∈ r'
      unfolding underS-def by simp
    thus b' ∈ f`(underS r a)
      using Well' SUC NE OF'
        wo-rel.suc-ofilter-in[of r' f ` underS r a b'] by auto
  qed

  have INJ: inj-on f (underS r a)
  proof−
    have ∀ b ∈ underS r a. inj-on f (under r b)
      using IH by (auto simp add: bij-betw-def)
    moreover
    have ∀ b. wo-rel.ofilter r (under r b)
      using Well by (auto simp add: wo-rel.under-ofilter)
    ultimately show ?thesis
      using WELL bFact UN
        UNION-inj-on-ofilter[of r underS r a λb. under r b f]
      by auto
  qed

  have BIJ: bij-betw f (underS r a) (underS r' (f a))
    unfolding bij-betw-def
    using INJ INCL1 INCL2 by auto

  have f a ∈ Field r'
    using Well' subField NE SUC
    by (auto simp add: wo-rel.suc-inField)
  thus ?thesis
    using WELL WELL' IN BIJ under-underS-bij-betw[of r r' a f] by auto
qed

lemma wellorders-totally-ordered-aux2:
  fixes r ::'a rel  and r'::'a' rel and
    f :: 'a ⇒ 'a' and g :: 'a ⇒ bool  and a::'a
  assumes WELL: Well-order r and WELL': Well-order r' and
    MAIN1:
    ⋀ a. (False ∉ g`(underS r a) ∧ f`(underS r a) ≠ Field r'
          ⟶ f a = wo-rel.suc r' (f`(underS r a)) ∧ g a = True)
         ∧
         (¬(False ∉ (g`(underS r a)) ∧ f`(underS r a) ≠ Field r')
          ⟶ g a = False) and
    MAIN2: ⋀ a. a ∈ Field r ∧ False ∉ g`(under r a) ⟶
             bij-betw f (under r a) (under r' (f a)) and
    Case: a ∈ Field r ∧ False ∈ g`(under r a)
  shows ∃f'. embed r' r f'
proof−
```

have *Well*: *wo-rel r* **using** *WELL* **unfolding** *wo-rel-def* **.**
hence *Refl*: *Refl r* **using** *wo-rel.REFL[of r]* **by** *auto*
have *Trans*: *trans r* **using** *Well wo-rel.TRANS[of r]* **by** *auto*
have *Antisym*: *antisym r* **using** *Well wo-rel.ANTISYM[of r]* **by** *auto*
have *Well'*: *wo-rel r'* **using** *WELL'* **unfolding** *wo-rel-def* **.**

have *0*: *under r a* = *underS r a* ∪ {*a*}
  using *Refl Case* **by**(*auto simp add*: *Refl-under-underS*)

have *1*: *g a* = *False*
proof−
  {**assume** *g a* ≠ *False*
    with *0 Case* **have** *False* ∈ *g*'(*underS r a*) **by** *blast*
    with *MAIN1* **have** *g a* = *False* **by** *blast*}
  thus *?thesis* **by** *blast*
qed
let *?A* = {*a* ∈ *Field r*. *g a* = *False*}
let *?a* = (*wo-rel.minim r ?A*)

have *2*: *?A* ≠ {} ∧ *?A* ≤ *Field r* **using** *Case 1* **by** *blast*

have *3*: *False* ∉ *g*'(*underS r ?a*)
proof
  assume *False* ∈ *g*'(*underS r ?a*)
  then obtain *b* **where** *b* ∈ *underS r ?a* **and** *31*: *g b* = *False* **by** *auto*
  hence *32*: (*b*,*?a*) ∈ *r* ∧ *b* ≠ *?a*
    by (*auto simp add*: *underS-def*)
  hence *b* ∈ *Field r* **unfolding** *Field-def* **by** *auto*
  with *31* **have** *b* ∈ *?A* **by** *auto*
  hence (*?a*,*b*) ∈ *r* **using** *wo-rel.minim-least 2 Well* **by** *fastforce*

  with *32 Antisym* **show** *False*
    by (*auto simp add*: *antisym-def*)
qed
have *temp*: *?a* ∈ *?A*
  using *Well 2 wo-rel.minim-in[of r ?A]* **by** *auto*
hence *4*: *?a* ∈ *Field r* **by** *auto*

have *5*: *g ?a* = *False* **using** *temp* **by** *blast*

have *6*: *f*'(*underS r ?a*) = *Field r'*
  using *MAIN1[of ?a] 3 5* **by** *blast*

have *7*: ∀ *b* ∈ *underS r ?a*. *bij-betw f* (*under r b*) (*under r'* (*f b*))
proof
  fix *b* **assume** *as*: *b* ∈ *underS r ?a*
  moreover
  have *wo-rel.ofilter r* (*underS r ?a*)
    using *Well* **by** (*auto simp add*: *wo-rel.underS-ofilter*)

ultimately
have *False* $\notin$ *g*'(*under r b*) **using** *3 Well* **by** (*subst* (*asm*) *wo-rel.ofilter-def*) *fast+*
moreover have *b* $\in$ *Field r*
unfolding *Field-def* **using** *as* **by** (*auto simp add*: *underS-def*)
ultimately
show *bij-betw f* (*under r b*) (*under r'* (*f b*))
using *MAIN2* **by** *auto*
qed

have *embed r' r* (*inv-into* (*underS r ?a*) *f*)
using *WELL WELL' 7 4 6 inv-into-underS-embed*[*of r ?a f r'*] **by** *auto*
thus *?thesis*
unfolding *embed-def* **by** *blast*
qed

theorem *wellorders-totally-ordered*:
fixes *r* ::*'a rel* **and** *r'*::*'a' rel*
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows ($\exists f$. *embed r r' f*) $\vee$ ($\exists f'$. *embed r' r f'*)
proof−

have *Well*: *wo-rel r* **using** *WELL* **unfolding** *wo-rel-def* **.**
hence *Refl*: *Refl r* **using** *wo-rel.REFL*[*of r*] **by** *auto*
have *Trans*: *trans r* **using** *Well wo-rel.TRANS*[*of r*] **by** *auto*
have *Well'*: *wo-rel r'* **using** *WELL'* **unfolding** *wo-rel-def* **.**

obtain *H* **where** *H-def*: *H* =
(λ*h a*. *if False* $\notin$ (*snd* $\circ$ *h*)'(*underS r a*) $\wedge$ (*fst* $\circ$ *h*)'(*underS r a*) $\neq$ *Field r'*
then (*wo-rel.suc r'* ((*fst* $\circ$ *h*)'(*underS r a*)), *True*)
else (*undefined*, *False*)) **by** *blast*
have *Adm*: *wo-rel.adm-wo r H*
using *Well*
proof(*unfold wo-rel.adm-wo-def*, *clarify*)
fix *h1*::*'a* $\Rightarrow$ *'a'* $*$ *bool* **and** *h2*::*'a* $\Rightarrow$ *'a'* $*$ *bool* **and** *x*
assume $\forall$ *y*$\in$*underS r x*. *h1 y* = *h2 y*
hence $\forall$ *y*$\in$*underS r x*. (*fst* $\circ$ *h1*) *y* = (*fst* $\circ$ *h2*) *y* $\wedge$
(*snd* $\circ$ *h1*) *y* = (*snd* $\circ$ *h2*) *y* **by** *auto*
hence (*fst* $\circ$ *h1*)'(*underS r x*) = (*fst* $\circ$ *h2*)'(*underS r x*) $\wedge$
(*snd* $\circ$ *h1*)'(*underS r x*) = (*snd* $\circ$ *h2*)'(*underS r x*)
by (*auto simp add*: *image-def*)
thus *H h1 x* = *H h2 x* **by** (*simp add*: *H-def del*: *not-False-in-image-Ball*)
qed

obtain *h*::*'a* $\Rightarrow$ *'a'* $*$ *bool* **and** *f*::*'a* $\Rightarrow$ *'a'* **and** *g*::*'a* $\Rightarrow$ *bool*
where *h-def*: *h* = *wo-rel.worec r H* **and**
f-def: *f* = *fst* $\circ$ *h* **and** *g-def*: *g* = *snd* $\circ$ *h* **by** *blast*
obtain *test* **where** *test-def*:
test = (λ *a*. *False* $\notin$ (*g*'(*underS r a*)) $\wedge$ *f*'(*underS r a*) $\neq$ *Field r'*) **by** *blast*

have $*$: $\bigwedge$ *a. h a* $=$ *H h a*
using *Adm Well wo-rel.worec-fixpoint*[*of r H*] **by** (*simp add*: *h-def*)
have *Main1*:
$\bigwedge$ *a.* (*test a* $\longrightarrow$ *f a* $=$ *wo-rel.suc r'* (*f'*(*underS r a*)) $\wedge$ *g a* $=$ *True*) $\wedge$
(¬(*test a*) $\longrightarrow$ *g a* $=$ *False*)
proof−
fix *a* **show** (*test a* $\longrightarrow$ *f a* $=$ *wo-rel.suc r'* (*f'*(*underS r a*)) $\wedge$ *g a* $=$ *True*) $\wedge$
(¬(*test a*) $\longrightarrow$ *g a* $=$ *False*)
using $*$[*of a*] *test-def f-def g-def H-def* **by** *auto*
qed

let *?phi* $=$ λ *a. a* $\in$ *Field r* $\wedge$ *False* $\notin$ *g'*(*under r a*) $\longrightarrow$
bij-betw f (*under r a*) (*under r'* (*f a*))
have *Main2*: $\bigwedge$ *a. ?phi a*
proof−
fix *a* **show** *?phi a*
proof(*rule wo-rel.well-order-induct*[*of r ?phi*],
simp only: *Well*, *clarify*)
fix *a*
assume *IH*: $\forall$ *b. b* $\neq$ *a* $\wedge$ (*b*,*a*) $\in$ *r* $\longrightarrow$ *?phi b* **and**
$*$: *a* $\in$ *Field r* **and**
$**$: *False* $\notin$ *g'*(*under r a*)
have *1*: $\forall$ *b* $\in$ *underS r a. bij-betw f* (*under r b*) (*under r'* (*f b*))
proof(*clarify*)
fix *b* **assume** $***$: *b* $\in$ *underS r a*
hence *0*: (*b*,*a*) $\in$ *r* $\wedge$ *b* $\neq$ *a* **unfolding** *underS-def* **by** *auto*
moreover have *b* $\in$ *Field r*
using $***$ *underS-Field*[*of r a*] **by** *auto*
moreover have *False* $\notin$ *g'*(*under r b*)
using *0* $**$ *Trans under-incr*[*of r b a*] **by** *auto*
ultimately show *bij-betw f* (*under r b*) (*under r'* (*f b*))
using *IH* **by** *auto*
qed

have *21*: *False* $\notin$ *g'*(*underS r a*)
using $**$ *underS-subset-under*[*of r a*] **by** *auto*
have *22*: *g'*(*under r a*) $\leq$ {*True*} **using** $**$ **by** *auto*
moreover have *23*: *a* $\in$ *under r a*
using *Refl* $*$ **by** (*auto simp add*: *Refl-under-in*)
ultimately have *24*: *g a* $=$ *True* **by** *blast*
have *2*: *f'*(*underS r a*) $\neq$ *Field r'*
proof
assume *f'*(*underS r a*) $=$ *Field r'*
hence *g a* $=$ *False* **using** *Main1 test-def* **by** *blast*
with *24* **show** *False* **using** $**$ **by** *blast*
qed

have *3*: *f a* $=$ *wo-rel.suc r'* (*f'*(*underS r a*))

```
    using 21 2 Main1 test-def by blast

   show bij-betw f (under r a) (under r' (f a))
    using WELL  WELL' 1 2 3 *
     wellorders-totally-ordered-aux[of r r' a f] by auto
  qed
 qed

 let ?chi = (λ a. a ∈ Field r ∧ False ∈ g`(under r a))
 show ?thesis
 proof(cases ∃ a. ?chi a)
  assume ¬ (∃ a. ?chi a)
  hence ∀ a ∈ Field r.  bij-betw f (under r a) (under r' (f a))
   using Main2 by blast
  thus ?thesis unfolding embed-def by blast
 next
  assume ∃ a. ?chi a
  then obtain a where ?chi a by blast
  hence ∃ f'. embed r' r f'
   using wellorders-totally-ordered-aux2[of r r' g f a]
    WELL WELL' Main1 Main2 test-def by fast
  thus ?thesis by blast
 qed
qed
```

28.4 Uniqueness of embeddings

Here we show a fact complementary to the one from the previous subsection – namely, that between any two well-orders there is *at most* one embedding, and is the one definable by the expected well-order recursive equation. As a consequence, any two embeddings of opposite directions are mutually inverse.

```
lemma embed-determined:
 assumes WELL: Well-order r and WELL': Well-order r' and
  EMB: embed r r' f and IN: a ∈ Field r
 shows f a = wo-rel.suc r' (f`(underS r a))
proof−
 have bij-betw f (underS r a) (underS r' (f a))
  using assms by (auto simp add: embed-underS)
 hence f`(underS r a) = underS r' (f a)
  by (auto simp add: bij-betw-def)
 moreover
 {have f a ∈ Field r' using IN
   using EMB WELL embed-Field[of r r' f] by auto
  hence f a = wo-rel.suc r' (underS r' (f a))
   using WELL' by (auto simp add: wo-rel-def wo-rel.suc-underS)
 }
 ultimately show ?thesis by simp
```

qed

lemma *embed-unique*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMBf: *embed r r' f* **and** *EMBg*: *embed r r' g*
shows $a \in$ *Field r* $\longrightarrow f\ a = g\ a$
proof(*rule wo-rel.well-order-induct*[*of r*], *auto simp add*: *WELL wo-rel-def*)
fix *a*
assume *IH*: $\forall b.\ b \neq a \wedge (b,a) \in r \longrightarrow b \in$ *Field r* $\longrightarrow f\ b = g\ b$ **and**
$*$: $a \in$ *Field r*
hence $\forall b \in$ *underS r a*. $f\ b = g\ b$
unfolding *underS-def* **by** (*auto simp add*: *Field-def*)
hence *f'*(*underS r a*) = *g'*(*underS r a*) **by** *force*
thus $f\ a = g\ a$
using *assms* $*$ *embed-determined*[*of r r' f a*] *embed-determined*[*of r r' g a*] **by** *auto*
qed

lemma *embed-bothWays-inverse*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *EMB'*: *embed r' r f'*
shows ($\forall a \in$ *Field r*. $f'(f\ a) = a$) $\wedge$ ($\forall a' \in$ *Field r'*. $f(f'\ a') = a'$)
proof−
have *embed r r* $(f' \circ f)$ **using** *assms*
by(*auto simp add*: *comp-embed*)
moreover have *embed r r id* **using** *assms*
by (*auto simp add*: *id-embed*)
ultimately have $\forall a \in$ *Field r*. $f'(f\ a) = a$
using *assms embed-unique*[*of r r f'* $\circ$ *f id*] *id-def* **by** *auto*
moreover
{**have** *embed r' r'* $(f \circ f')$ **using** *assms*
by(*auto simp add*: *comp-embed*)
moreover have *embed r' r' id* **using** *assms*
by (*auto simp add*: *id-embed*)
ultimately have $\forall a' \in$ *Field r'*. $f(f'\ a') = a'$
using *assms embed-unique*[*of r' r' f* $\circ$ *f' id*] *id-def* **by** *auto*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *embed-bothWays-bij-betw*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *EMB'*: *embed r' r g*
shows *bij-betw f* (*Field r*) (*Field r'*)
proof−
let *?A* = *Field r* **let** *?A'* = *Field r'*
have *embed r r* $(g \circ f)$ $\wedge$ *embed r' r'* $(f \circ g)$
using *assms* **by** (*auto simp add*: *comp-embed*)
hence *1*: ($\forall a \in$ *?A*. $g(f\ a) = a$) $\wedge$ ($\forall a' \in$ *?A'*. $f(g\ a') = a'$)

using *WELL id-embed*[*of r*] *embed-unique*[*of r r g* ∘ *f id*]
WELL′ id-embed[*of r′*] *embed-unique*[*of r′ r′ f* ∘ *g id*]
id-def **by** *auto*
have *2*: (∀ *a* ∈ *?A*. *f a* ∈ *?A′*) ∧ (∀ *a′* ∈ *?A′*. *g a′* ∈ *?A*)
using *assms embed-Field*[*of r r′ f*] *embed-Field*[*of r′ r g*] **by** *blast*

show *?thesis*
proof(*unfold bij-betw-def inj-on-def*, *auto simp add*: *2*)
fix *a b* **assume** ∗: *a* ∈ *?A b* ∈ *?A* **and** ∗∗: *f a* = *f b*
have *a* = *g*(*f a*) ∧ *b* = *g*(*f b*) **using** ∗ *1* **by** *auto*
with ∗∗ **show** *a* = *b* **by** *auto*
next
fix *a′* **assume** ∗: *a′* ∈ *?A′*
hence *g a′* ∈ *?A* ∧ *f*(*g a′*) = *a′* **using** *1 2* **by** *auto*
thus *a′* ∈ *f* ‘ *?A* **by** *force*
qed
qed

lemma *embed-bothWays-iso*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′* **and**
EMB: *embed r r′ f* **and** *EMB′*: *embed r′ r g*
shows *iso r r′ f*
unfolding *iso-def* **using** *assms* **by** (*auto simp add*: *embed-bothWays-bij-betw*)

28.5 More properties of embeddings, strict embeddings and isomorphisms

lemma *embed-bothWays-Field-bij-betw*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′* **and**
EMB: *embed r r′ f* **and** *EMB′*: *embed r′ r f′*
shows *bij-betw f* (*Field r*) (*Field r′*)
proof−
have (∀ *a* ∈ *Field r*. *f′*(*f a*) = *a*) ∧ (∀ *a′* ∈ *Field r′*. *f*(*f′ a′*) = *a′*)
using *assms* **by** (*auto simp add*: *embed-bothWays-inverse*)
moreover
have *f*‘(*Field r*) ≤ *Field r′* ∧ *f′* ‘ (*Field r′*) ≤ *Field r*
using *assms* **by** (*auto simp add*: *embed-Field*)
ultimately
show *?thesis* **using** *bij-betw-byWitness*[*of Field r f′ f Field r′*] **by** *auto*
qed

lemma *embedS-comp-embed*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′*
and *EMB*: *embedS r r′ f* **and** *EMB′*: *embed r′ r′′ f′*
shows *embedS r r′′* (*f′* ∘ *f*)
proof−
let *?g* = (*f′* ∘ *f*) **let** *?h* = *inv-into* (*Field r*) *?g*
have *1*: *embed r r′ f* ∧ ¬ (*bij-betw f* (*Field r*) (*Field r′*))
using *EMB* **by** (*auto simp add*: *embedS-def*)

hence *2*: *embed r r″ ?g*
using *EMB′ comp-embed*[*of r r′ f r″ f′*] **by** *auto*
moreover
{**assume** *bij-betw ?g* (*Field r*) (*Field r″*)
hence *embed r″ r ?h* **using** *2*
by (*auto simp add*: *inv-into-Field-embed-bij-betw*)
hence *embed r′ r* (*?h* ∘ *f′*) **using** *EMB′*
by (*auto simp add*: *comp-embed*)
hence *bij-betw f* (*Field r*) (*Field r′*) **using** *WELL WELL′ 1*
by (*auto simp add*: *embed-bothWays-Field-bij-betw*)
with *1* **have** *False* **by** *blast*
}
ultimately show *?thesis* **unfolding** *embedS-def* **by** *auto*
qed

lemma *embed-comp-embedS*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′*
and *EMB*: *embed r r′ f* **and** *EMB′*: *embedS r′ r″ f′*
shows *embedS r r″* (*f′* ∘ *f*)
proof−
let *?g* = (*f′* ∘ *f*) **let** *?h* = *inv-into* (*Field r*) *?g*
have *1*: *embed r′ r″ f′* ∧ ¬ (*bij-betw f′* (*Field r′*) (*Field r″*))
using *EMB′* **by** (*auto simp add*: *embedS-def*)
hence *2*: *embed r r″ ?g*
using *WELL EMB comp-embed*[*of r r′ f r″ f′*] **by** *auto*
moreover have §: *f′ ‘ Field r′* ⊆ *Field r″*
by (*simp add*: *1 embed-Field*)
{**assume** §: *bij-betw ?g* (*Field r*) (*Field r″*)
hence *embed r″ r ?h* **using** *2 WELL*
by (*auto simp add*: *inv-into-Field-embed-bij-betw*)
hence *embed r′ r* (*inv-into* (*Field r*) *?g* ∘ *f′*)
using *1 BNF-Wellorder-Embedding.comp-embed WELL′* **by** *blast*
then have *bij-betw f′* (*Field r′*) (*Field r″*)
using *EMB WELL WELL′* § *bij-betw-comp-iff* **by** (*blast intro*: *embed-bothWays-Field-bij-betw*)
with *1* **have** *False* **by** *blast*
}
ultimately show *?thesis* **unfolding** *embedS-def* **by** *auto*
qed

lemma *embed-comp-iso*:
assumes *EMB*: *embed r r′ f* **and** *EMB′*: *iso r′ r″ f′*
shows *embed r r″* (*f′* ∘ *f*) **using** *assms* **unfolding** *iso-def*
by (*auto simp add*: *comp-embed*)

lemma *iso-comp-embed*:
assumes *EMB*: *iso r r′ f* **and** *EMB′*: *embed r′ r″ f′*
shows *embed r r″* (*f′* ∘ *f*)
using *assms* **unfolding** *iso-def* **by** (*auto simp add*: *comp-embed*)

lemma *embedS-comp-iso*:
  assumes *EMB*: *embedS r r' f* **and** *EMB'*: *iso r' r'' f'*
  shows *embedS r r'' (f' ∘ f)*
proof −
  have §: *embed r r' f* ∧ ¬ *bij-betw f (Field r) (Field r')*
    using *EMB embedS-def* **by** *blast*
  then have *embed r r'' (f' ∘ f)*
    using *embed-comp-iso EMB'* **by** *blast*
  then have *f ‘ Field r ⊆ Field r'*
    by (*simp add*: *embed-Field* §)
  then have ¬ *bij-betw (f' ∘ f) (Field r) (Field r'')*
    using § *EMB'* **by** (*auto simp*: *bij-betw-comp-iff2 iso-def*)
  then show *?thesis*
    by (*simp add*: ‹*embed r r'' (f' ∘ f)*› *embedS-def*)
qed

lemma *iso-comp-embedS*:
  assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
    and *EMB*: *iso r r' f* **and** *EMB'*: *embedS r' r'' f'*
  shows *embedS r r'' (f' ∘ f)*
  using *assms* **unfolding** *iso-def* **by** (*auto simp add*: *embed-comp-embedS*)

lemma *embedS-Field*:
  assumes *WELL*: *Well-order r* **and** *EMB*: *embedS r r' f*
  shows *f ‘ (Field r) < Field r'*
proof−
  have *f‘(Field r) ≤ Field r'* **using** *assms*
    by (*auto simp add*: *embed-Field embedS-def*)
  moreover
  {**have** *inj-on f (Field r)* **using** *assms*
      by (*auto simp add*: *embedS-def embed-inj-on*)
    hence *f‘(Field r) ≠ Field r'* **using** *EMB*
      by (*auto simp add*: *embedS-def bij-betw-def*)
  }
  ultimately show *?thesis* **by** *blast*
qed

lemma *embedS-iff*:
  assumes *WELL*: *Well-order r* **and** *ISO*: *embed r r' f*
  shows *embedS r r' f = (f ‘ (Field r) < Field r')*
proof
  assume *embedS r r' f*
  thus *f ‘ Field r ⊂ Field r'*
    using *WELL* **by** (*auto simp add*: *embedS-Field*)
next
  assume *f ‘ Field r ⊂ Field r'*
  hence ¬ *bij-betw f (Field r) (Field r')*
    unfolding *bij-betw-def* **by** *blast*
  thus *embedS r r' f* **unfolding** *embedS-def*

```
    using ISO by auto
qed

lemma iso-Field: iso r r' f ⟹ f ` (Field r) = Field r'
  by (auto simp add: iso-def bij-betw-def)

lemma iso-iff:
  assumes Well-order r
  shows iso r r' f = (embed r r' f ∧ f ` (Field r) = Field r')
proof
  assume iso r r' f
  thus embed r r' f ∧ f ` (Field r) = Field r'
    by (auto simp add: iso-Field iso-def)
next
  assume *: embed r r' f ∧ f ` Field r = Field r'
  hence inj-on f (Field r) using assms by (auto simp add: embed-inj-on)
  with * have bij-betw f (Field r) (Field r')
    unfolding bij-betw-def by simp
  with * show iso r r' f unfolding iso-def by auto
qed

lemma iso-iff2: iso r r' f ⟷
                bij-betw f (Field r) (Field r') ∧
                (∀ a ∈ Field r. ∀ b ∈ Field r. (a, b) ∈ r ⟷ (f a, f b) ∈ r')
  (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  then have bij-betw f (Field r) (Field r') and emb: embed r r' f
    by (auto simp: bij-betw-def iso-def)
  then obtain g where g: ⋀x. x ∈ Field r ⟹ g (f x) = x
    by (auto simp: bij-betw-iff-bijections)
  moreover
  have (a, b) ∈ r if a ∈ Field r b ∈ Field r (f a, f b) ∈ r' for a b
    using that emb g g [OF FieldI1] — yes it's weird
    by (force simp add: embed-def under-def bij-betw-iff-bijections)
  ultimately show ?rhs
    using L by (auto simp: compat-def iso-def dest: embed-compat)
next
  assume R: ?rhs
  then show ?lhs
    apply (clarsimp simp add: iso-def embed-def under-def bij-betw-iff-bijections)
    apply (rule-tac x=g in exI)
    apply (fastforce simp add: intro: FieldI1)+
    done
qed

lemma iso-iff3:
  assumes WELL: Well-order r and WELL': Well-order r'
  shows iso r r' f = (bij-betw f (Field r) (Field r') ∧ compat r r' f)
```

proof
  assume *iso r r′ f*
  thus *bij-betw f* (*Field r*) (*Field r′*) ∧ *compat r r′ f*
    unfolding *compat-def* **using** *WELL* **by** (*auto simp add*: *iso-iff2 Field-def*)
next
  have *Well*: *wo-rel r* ∧ *wo-rel r′* **using** *WELL WELL′*
    by (*auto simp add*: *wo-rel-def*)
  assume ∗: *bij-betw f* (*Field r*) (*Field r′*) ∧ *compat r r′ f*
  thus *iso r r′ f*
    unfolding *compat-def* **using** *assms*
  proof(*auto simp add*: *iso-iff2*)
    fix *a b* **assume** ∗∗: *a* ∈ *Field r b* ∈ *Field r* **and**
      ∗∗∗: (*f a*, *f b*) ∈ *r′*
    {**assume** (*b*,*a*) ∈ *r* ∨ *b* = *a*
      hence (*b*,*a*) ∈ *r***using** *Well* ∗∗ *wo-rel.REFL*[*of r*] *refl-on-def*[*of - r*] **by** *blast*
      hence (*f b*, *f a*) ∈ *r′* **using** ∗ **unfolding** *compat-def* **by** *auto*
      hence *f a* = *f b*
        using *Well* ∗∗∗ *wo-rel.ANTISYM*[*of r′*] *antisym-def*[*of r′*] **by** *blast*
      hence *a* = *b* **using** ∗ ∗∗ **unfolding** *bij-betw-def inj-on-def* **by** *auto*
      hence (*a*,*b*) ∈ *r* **using** *Well* ∗∗ *wo-rel.REFL*[*of r*] *refl-on-def*[*of - r*] **by** *blast*
    }
    thus (*a*,*b*) ∈ *r*
      using *Well* ∗∗ *wo-rel.TOTAL*[*of r*] *total-on-def*[*of - r*] **by** *blast*
  qed
qed

lemma *iso-imp-inj-on*:
  assumes *iso r r′ f* **shows** *inj-on f* (*Field r*)
  using *assms* **unfolding** *iso-iff2 bij-betw-def* **by** *blast*

lemma *iso-backward-Field*:
  assumes *x* ∈ *Field r′ iso r r′ f*
  shows *inv-into* (*Field r*) *f x* ∈ *Field r*
  using *assms iso-Field* **by** (*blast intro*!: *inv-into-into*)

lemma *iso-backward*:
  assumes (*x*,*y*) ∈ *r′* **and** *iso*: *iso r r′ f*
  shows (*inv-into* (*Field r*) *f x*, *inv-into* (*Field r*) *f y*) ∈ *r*
proof −
  have §: ⋀*x*. (∃ *xa*∈*Field r*. *x* = *f xa*) = (*x* ∈ *Field r′*)
    using *assms iso-Field* [*OF iso*] **by** (*force simp add*:)
  have *x* ∈ *Field r′ y* ∈ *Field r′*
    using *assms* **by** (*auto simp*: *Field-def*)
  with § [*of x*] § [*of y*] *assms* **show** *?thesis*
    by (*auto simp add*: *iso-iff2 bij-betw-inv-into-left*)
qed

lemma *iso-forward*:
  assumes (*x*,*y*) ∈ *r iso r r′ f* **shows** (*f x*,*f y*) ∈ *r′*

using *assms* **by** (*auto simp*: *Field-def iso-iff2*)

lemma *iso-trans*:
assumes *trans r* **and** *iso*: *iso r r′ f* **shows** *trans r′*
unfolding *trans-def*
proof *clarify*
fix *x y z*
assume *xyz*: (*x*, *y*) ∈ *r′* (*y*, *z*) ∈ *r′*
then have ∗: (*inv-into* (*Field r*) *f x*, *inv-into* (*Field r*) *f y*) ∈ *r*
(*inv-into* (*Field r*) *f y*, *inv-into* (*Field r*) *f z*) ∈ *r*
using *iso-backward* [*OF* - *iso*] **by** *blast*+
then have *inv-into* (*Field r*) *f x* ∈ *Field r inv-into* (*Field r*) *f y* ∈ *Field r*
by (*auto simp*: *Field-def*)
with ∗ **have** (*inv-into* (*Field r*) *f x*, *inv-into* (*Field r*) *f z*) ∈ *r*
using *assms*(*1*) **by** (*blast intro*: *transD*)
then have (*f* (*inv-into* (*Field r*) *f x*), *f* (*inv-into* (*Field r*) *f z*)) ∈ *r′*
by (*blast intro*: *iso iso-forward*)
moreover have *x* ∈ *f* ‘ *Field r z* ∈ *f* ‘ *Field r*
using *xyz iso iso-Field* **by** (*blast intro*: *FieldI1 FieldI2*)+
ultimately show (*x*, *z*) ∈ *r′*
by (*simp add*: *f-inv-into-f*)
qed

lemma *iso-Total*:
assumes *Total r* **and** *iso*: *iso r r′ f* **shows** *Total r′*
unfolding *total-on-def*
proof *clarify*
fix *x y*
assume *xy*: *x* ∈ *Field r′ y* ∈ *Field r′ x* ≠ *y* (*y*,*x*) ∉ *r′*
then have §: *inv-into* (*Field r*) *f x* ∈ *Field r inv-into* (*Field r*) *f y* ∈ *Field r*
using *iso-backward-Field* [*OF* - *iso*] **by** *auto*
moreover have *x* ∈ *f* ‘ *Field r y* ∈ *f* ‘ *Field r*
using *xy iso iso-Field* **by** (*blast intro*: *FieldI1 FieldI2*)+
ultimately have *False* **if** *inv-into* (*Field r*) *f x* = *inv-into* (*Field r*) *f y*
using *inv-into-injective* [*OF that*] ‹*x* ≠ *y*› **by** *simp*
then have (*inv-into* (*Field r*) *f x*, *inv-into* (*Field r*) *f y*) ∈ *r* ∨ (*inv-into* (*Field r*) *f y*, *inv-into* (*Field r*) *f x*) ∈ *r*
using *assms* § **by** (*auto simp*: *total-on-def*)
then show (*x*, *y*) ∈ *r′*
using *assms xy* **by** (*auto simp*: *iso-Field f-inv-into-f dest*!: *iso-forward* [*OF* - *iso*])
qed

lemma *iso-wf*:
assumes *wf r* **and** *iso*: *iso r r′ f* **shows** *wf r′*
proof −
have *bij-betw f* (*Field r*) (*Field r′*)
and *iff*: (∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. (*a*, *b*) ∈ *r* ⟷ (*f a*, *f b*) ∈ *r′*)
using *assms* **by** (*auto simp*: *iso-iff2*)

```
  show ?thesis
  proof (rule wfI-min)
    fix x::'b and Q
    assume x ∈ Q
    let ?g = inv-into (Field r) f
    obtain z0 where z0 ∈ ?g ‘ Q
      using ‹x ∈ Q› by blast
    then obtain z where z: z ∈ ?g ‘ Q and ⋀x y. ⟦(y, z) ∈ r; x ∈ Q⟧ ⟹ y ≠
?g x
      by (rule wfE-min [OF ‹wf r›]) auto
    then have ∀ y. (y, inv-into Q ?g z) ∈ r' ⟶ y ∉ Q
      by (clarsimp simp: f-inv-into-f[OF z] z dest!: iso-backward [OF - iso]) blast
    moreover have inv-into Q ?g z ∈ Q
      by (simp add: inv-into-into z)
    ultimately show ∃ z∈Q. ∀ y. (y, z) ∈ r' ⟶ y ∉ Q ..
  qed
qed

end
```

29 Constructions on Wellorders as Needed by Bounded Natural Functors

```
theory BNF-Wellorder-Constructions
  imports BNF-Wellorder-Embedding
begin
```

In this section, we study basic constructions on well-orders, such as restriction to a set/order filter, copy via direct images, ordinal-like sum of disjoint well-orders, and bounded square. We also define between well-orders the relations *ordLeq*, of being embedded (abbreviated $\leq o$), *ordLess*, of being strictly embedded (abbreviated $<o$), and *ordIso*, of being isomorphic (abbreviated $=o$). We study the connections between these relations, order filters, and the aforementioned constructions. A main result of this section is that $<o$ is well-founded.

29.1 Restriction to a set

```
abbreviation Restr :: 'a rel ⇒ 'a set ⇒ 'a rel
  where Restr r A ≡ r Int (A × A)

lemma Restr-subset:
  A ≤ B ⟹ Restr (Restr r B) A = Restr r A
  by blast

lemma Restr-Field: Restr r (Field r) = r
  unfolding Field-def by auto
```

lemma *Refl-Restr*: *Refl r* ⟹ *Refl*(*Restr r A*)
unfolding *refl-on-def Field-def* **by** *auto*

lemma *linear-order-on-Restr*:
linear-order-on A r ⟹ *linear-order-on* (*A* ∩ *above r x*) (*Restr r* (*above r x*))
by(*simp add*: *order-on-defs refl-on-def trans-def antisym-def total-on-def*)(*safe*; *blast*)

lemma *antisym-Restr*:
antisym r ⟹ *antisym*(*Restr r A*)
unfolding *antisym-def Field-def* **by** *auto*

lemma *Total-Restr*:
Total r ⟹ *Total*(*Restr r A*)
unfolding *total-on-def Field-def* **by** *auto*

lemma *total-on-imp-Total-Restr*: *total-on A r* ⟹ *Total* (*Restr r A*)
by (*auto simp*: *Field-def total-on-def*)

lemma *trans-Restr*:
trans r ⟹ *trans*(*Restr r A*)
unfolding *trans-def Field-def* **by** *blast*

lemma *Preorder-Restr*:
Preorder r ⟹ *Preorder*(*Restr r A*)
unfolding *preorder-on-def* **by** (*simp add*: *Refl-Restr trans-Restr*)

lemma *Partial-order-Restr*:
Partial-order r ⟹ *Partial-order*(*Restr r A*)
unfolding *partial-order-on-def* **by** (*simp add*: *Preorder-Restr antisym-Restr*)

lemma *Linear-order-Restr*:
Linear-order r ⟹ *Linear-order*(*Restr r A*)
unfolding *linear-order-on-def* **by** (*simp add*: *Partial-order-Restr Total-Restr*)

lemma *Well-order-Restr*:
assumes *Well-order r*
shows *Well-order*(*Restr r A*)
using *assms*
by (*auto simp*: *well-order-on-def Linear-order-Restr elim*: *wf-subset*)

lemma *Field-Restr-subset*: *Field*(*Restr r A*) ≤ *A*
by (*auto simp add*: *Field-def*)

lemma *Refl-Field-Restr*:
Refl r ⟹ *Field*(*Restr r A*) = (*Field r*) *Int A*
unfolding *refl-on-def Field-def* **by** *blast*

lemma *Refl-Field-Restr2*:
  $[\![$*Refl r*; $A \leq$ *Field r*$]\!] \Longrightarrow$ *Field*(*Restr r A*) = *A*
  by (*auto simp add*: *Refl-Field-Restr*)

lemma *well-order-on-Restr*:
  assumes *WELL*: *Well-order r* **and** *SUB*: $A \leq$ *Field r*
  shows *well-order-on A* (*Restr r A*)
  using *assms*
  using *Well-order-Restr*[*of r A*] *Refl-Field-Restr2*[*of r A*]
    order-on-defs[*of Field r r*] **by** *auto*

29.2 Order filters versus restrictions and embeddings

lemma *Field-Restr-ofilter*:
  $[\![$*Well-order r*; *wo-rel.ofilter r A*$]\!] \Longrightarrow$ *Field*(*Restr r A*) = *A*
  by (*auto simp add*: *wo-rel-def wo-rel.ofilter-def wo-rel.REFL Refl-Field-Restr2*)

lemma *ofilter-Restr-under*:
  assumes *WELL*: *Well-order r* **and** *OF*: *wo-rel.ofilter r A* **and** *IN*: $a \in A$
  shows *under* (*Restr r A*) *a* = *under r a*
  unfolding *wo-rel.ofilter-def under-def*
proof
  show $\{b.\ (b,\ a) \in$ *Restr r A*$\} \subseteq \{b.\ (b,\ a) \in r\}$
    by *auto*
next
  have *under r a* $\subseteq A$
  proof
    fix *x*
    assume $*$: $x \in$ *under r a*
    then have $a \in$ *Field r*
      unfolding *under-def* **using** *Field-def* **by** *fastforce*
    then show $x \in A$ **using** *IN assms* $*$
      by (*auto simp add*: *wo-rel-def wo-rel.ofilter-def*)
  qed
  then show $\{b.\ (b,\ a) \in r\} \subseteq \{b.\ (b,\ a) \in$ *Restr r A*$\}$
    unfolding *under-def* **using** *assms* **by** *auto*
qed

lemma *ofilter-embed*:
  assumes *Well-order r*
  shows *wo-rel.ofilter r A* = ($A \leq$ *Field r* $\wedge$ *embed* (*Restr r A*) *r id*)
proof
  assume $*$: *wo-rel.ofilter r A*
  show $A \leq$ *Field r* $\wedge$ *embed* (*Restr r A*) *r id*
    unfolding *embed-def*
  proof *safe*
    fix *a* **assume** $a \in A$ **thus** $a \in$ *Field r* **using** *assms* $*$
      by (*auto simp add*: *wo-rel-def wo-rel.ofilter-def*)
  next

```
    fix a assume a ∈ Field (Restr r A)
    thus bij-betw id (under (Restr r A) a) (under r (id a)) using assms *
      by (simp add: ofilter-Restr-under Field-Restr-ofilter)
  qed
next
  assume *: A ≤ Field r ∧ embed (Restr r A) r id
  hence Field(Restr r A) ≤ Field r
    using assms  embed-Field[of Restr r A r id] id-def
      Well-order-Restr[of r] by auto
  {fix a assume a ∈ A
    hence a ∈ Field(Restr r A) using * assms
      by (simp add: order-on-defs Refl-Field-Restr2)
    hence bij-betw id (under (Restr r A) a) (under r a)
      using * unfolding embed-def by auto
    hence under r a ≤ under (Restr r A) a
      unfolding bij-betw-def by auto
    also have ... ≤ Field(Restr r A) by (simp add: under-Field)
    also have ... ≤ A by (simp add: Field-Restr-subset)
    finally have under r a ≤ A .
  }
  thus wo-rel.ofilter r A using assms * by (simp add: wo-rel-def wo-rel.ofilter-def)
qed

lemma ofilter-Restr-Int:
  assumes WELL: Well-order r and OFA: wo-rel.ofilter r A
  shows wo-rel.ofilter (Restr r B) (A Int B)
proof−
  let ?rB = Restr r B
  have Well: wo-rel r unfolding wo-rel-def using WELL .
  hence Refl: Refl r by (simp add: wo-rel.REFL)
  hence Field: Field ?rB = Field r Int B
    using Refl-Field-Restr by blast
  have WellB: wo-rel ?rB ∧ Well-order ?rB using WELL
    by (simp add: Well-order-Restr wo-rel-def)

  show ?thesis using WellB assms
    unfolding wo-rel.ofilter-def under-def ofilter-def
  proof safe
    fix a assume a ∈ A and *: a ∈ B
    hence a ∈ Field r using OFA Well by (auto simp add: wo-rel.ofilter-def)
    with * show a ∈ Field ?rB using Field by auto
  next
    fix a b assume a ∈ A and (b,a) ∈ r
    thus b ∈ A using Well OFA by (auto simp add: wo-rel.ofilter-def under-def)
  qed
qed

lemma ofilter-Restr-subset:
  assumes WELL: Well-order r and OFA: wo-rel.ofilter r A and SUB: A ≤ B
```

```
  shows wo-rel.ofilter (Restr r B) A
proof−
  have A Int B = A using SUB by blast
  thus ?thesis using assms ofilter-Restr-Int[of r A B] by auto
qed

lemma ofilter-subset-embed:
  assumes WELL: Well-order r and
    OFA: wo-rel.ofilter r A and OFB: wo-rel.ofilter r B
  shows (A ≤ B) = (embed (Restr r A) (Restr r B) id)
proof−
  let ?rA = Restr r A  let ?rB = Restr r B
  have Well: wo-rel r unfolding wo-rel-def using WELL .
  hence Refl: Refl r by (simp add: wo-rel.REFL)
  hence FieldA: Field ?rA = Field r Int A
    using Refl-Field-Restr by blast
  have FieldB: Field ?rB = Field r Int B
    using Refl Refl-Field-Restr by blast
  have WellA: wo-rel ?rA ∧ Well-order ?rA using WELL
    by (simp add: Well-order-Restr wo-rel-def)
  have WellB: wo-rel ?rB ∧ Well-order ?rB using WELL
    by (simp add: Well-order-Restr wo-rel-def)

  show ?thesis
  proof
    assume ∗: A ≤ B
    hence wo-rel.ofilter (Restr r B) A using assms
      by (simp add: ofilter-Restr-subset)
    hence embed (Restr ?rB A) (Restr r B) id
      using WellB ofilter-embed[of ?rB A] by auto
    thus embed (Restr r A) (Restr r B) id
      using ∗ by (simp add: Restr-subset)
  next
    assume ∗: embed (Restr r A) (Restr r B) id
    {fix a assume ∗∗: a ∈ A
      hence a ∈ Field r using Well OFA by (auto simp add: wo-rel.ofilter-def)
      with ∗∗ FieldA have a ∈ Field ?rA by auto
      hence a ∈ Field ?rB using ∗ WellA embed-Field[of ?rA ?rB id] by auto
      hence a ∈ B using FieldB by auto
    }
    thus A ≤ B by blast
  qed
qed

lemma ofilter-subset-embedS-iso:
  assumes WELL: Well-order r and
    OFA: wo-rel.ofilter r A and OFB: wo-rel.ofilter r B
  shows ((A < B) = (embedS (Restr r A) (Restr r B) id)) ∧
      ((A = B) = (iso (Restr r A) (Restr r B) id))
```

```
proof−
  let ?rA = Restr r A  let ?rB = Restr r B
  have Well: wo-rel r unfolding wo-rel-def using WELL .
  hence Refl: Refl r by (simp add: wo-rel.REFL)
  hence Field ?rA = Field r Int A
    using Refl-Field-Restr by blast
  hence FieldA: Field ?rA = A using OFA Well
    by (auto simp add: wo-rel.ofilter-def)
  have Field ?rB = Field r Int B
    using Refl Refl-Field-Restr by blast
  hence FieldB: Field ?rB = B using OFB Well
    by (auto simp add: wo-rel.ofilter-def)

  show ?thesis unfolding embedS-def iso-def
    using assms ofilter-subset-embed[of r A B]
      FieldA FieldB bij-betw-id-iff[of A B] by auto
qed

lemma ofilter-subset-embedS:
  assumes WELL: Well-order r and
    OFA: wo-rel.ofilter r A and OFB: wo-rel.ofilter r B
  shows (A < B) = embedS (Restr r A) (Restr r B) id
  using assms by (simp add: ofilter-subset-embedS-iso)

lemma embed-implies-iso-Restr:
  assumes WELL: Well-order r and WELL′: Well-order r′ and
    EMB: embed r′ r f
  shows iso r′ (Restr r (f ‘ (Field r′))) f
proof−
  let ?A′ = Field r′
  let ?r′′ = Restr r (f ‘ ?A′)
  have 0: Well-order ?r′′ using WELL Well-order-Restr by blast
  have 1: wo-rel.ofilter r (f ‘ ?A′) using assms embed-Field-ofilter  by blast
  hence Field ?r′′ = f ‘ (Field r′) using WELL Field-Restr-ofilter by blast
  hence bij-betw f ?A′ (Field ?r′′)
    using EMB embed-inj-on WELL′ unfolding bij-betw-def by blast
  moreover
  {have ∀ a b. (a,b) ∈ r′ ⟶ a ∈ Field r′ ∧ b ∈ Field r′
      unfolding Field-def by auto
    hence compat r′ ?r′′ f
      using assms embed-iff-compat-inj-on-ofilter
      unfolding compat-def by blast
  }
  ultimately show ?thesis using WELL′ 0 iso-iff3 by blast
qed
```

29.3 The strict inclusion on proper ofilters is well-founded

```
definition ofilterIncl :: ′a rel ⇒ ′a set rel
```

```
  where
    ofilterIncl r ≡ {(A,B). wo-rel.ofilter r A ∧ A ≠ Field r ∧
                     wo-rel.ofilter r B ∧ B ≠ Field r ∧ A < B}

lemma wf-ofilterIncl:
  assumes WELL: Well-order r
  shows wf(ofilterIncl r)
proof−
  have Well: wo-rel r using WELL by (simp add: wo-rel-def)
  hence Lo: Linear-order r by (simp add: wo-rel.LIN)
  let ?h = (λ A. wo-rel.suc r A)
  let ?rS = r − Id
  have wf ?rS using WELL by (simp add: order-on-defs)
  moreover
  have compat (ofilterIncl r) ?rS ?h
  proof(unfold compat-def ofilterIncl-def,
      intro allI impI, simp, elim conjE)
    fix A B
    assume ∗: wo-rel.ofilter r A A ≠ Field r and
      ∗∗: wo-rel.ofilter r B B ≠ Field r and ∗∗∗: A < B
    then obtain a and b where 0: a ∈ Field r ∧ b ∈ Field r and
      1: A = underS r a ∧ B = underS r b
      using Well by (auto simp add: wo-rel.ofilter-underS-Field)
    hence a ≠ b using ∗∗∗ by auto
    moreover
    have (a,b) ∈ r using 0 1 Lo ∗∗∗
      by (auto simp add: underS-incl-iff)
    moreover
    have a = wo-rel.suc r A ∧ b = wo-rel.suc r B
      using Well 0 1 by (simp add: wo-rel.suc-underS)
    ultimately
    show (wo-rel.suc r A, wo-rel.suc r B) ∈ r ∧ wo-rel.suc r A ≠ wo-rel.suc r B
      by simp
  qed
  ultimately show wf (ofilterIncl r) by (simp add: compat-wf)
qed
```

29.4 Ordering the well-orders by existence of embeddings

We define three relations between well-orders:

- *ordLeq*, of being embedded (abbreviated $\leq o$);
- *ordLess*, of being strictly embedded (abbreviated $<o$);
- *ordIso*, of being isomorphic (abbreviated $=o$).

The prefix "ord" and the index "o" in these names stand for "ordinal-like". These relations shall be proved to be inter-connected in a similar fashion as the trio $\leq$, $<$, $=$ associated to a total order on a set.

definition *ordLeq* :: ('*a rel* ∗ '*a*' *rel*) *set*
where
ordLeq = {(*r*,*r*'). *Well-order r* ∧ *Well-order r*' ∧ (∃*f*. *embed r r*' *f*)}

abbreviation *ordLeq2* :: '*a rel* ⇒ '*a*' *rel* ⇒ *bool* (**infix** ‹<=*o*› *50*)
where *r* <=*o r*' ≡ (*r*,*r*') ∈ *ordLeq*

abbreviation *ordLeq3* :: '*a rel* ⇒ '*a*' *rel* ⇒ *bool* (**infix** ‹≤*o*› *50*)
where *r* ≤*o r*' ≡ *r* <=*o r*'

definition *ordLess* :: ('*a rel* ∗ '*a*' *rel*) *set*
where
ordLess = {(*r*,*r*'). *Well-order r* ∧ *Well-order r*' ∧ (∃*f*. *embedS r r*' *f*)}

abbreviation *ordLess2* :: '*a rel* ⇒ '*a*' *rel* ⇒ *bool* (**infix** ‹<*o*› *50*)
where *r* <*o r*' ≡ (*r*,*r*') ∈ *ordLess*

definition *ordIso* :: ('*a rel* ∗ '*a*' *rel*) *set*
where
ordIso = {(*r*,*r*'). *Well-order r* ∧ *Well-order r*' ∧ (∃*f*. *iso r r*' *f*)}

abbreviation *ordIso2* :: '*a rel* ⇒ '*a*' *rel* ⇒ *bool* (**infix** ‹=*o*› *50*)
where *r* =*o r*' ≡ (*r*,*r*') ∈ *ordIso*

lemmas *ordRels-def* = *ordLeq-def ordLess-def ordIso-def*

lemma *ordLeq-Well-order-simp*:
assumes *r* ≤*o r*'
shows *Well-order r* ∧ *Well-order r*'
using *assms* **unfolding** *ordLeq-def* **by** *simp*

Notice that the relations ≤*o*, <*o*, =*o* connect well-orders on potentially *distinct* types. However, some of the lemmas below, including the next one, restrict implicitly the type of these relations to (('*a rel*) ∗ ('*a rel*)) *set* , i.e., to '*a rel rel*.

lemma *ordLeq-reflexive*:
Well-order r ⟹ *r* ≤*o r*
unfolding *ordLeq-def* **using** *id-embed*[*of r*] **by** *blast*

lemma *ordLeq-transitive*[*trans*]:
assumes *r* ≤*o r*' **and** *r*' ≤*o r*''
shows *r* ≤*o r*''
using *assms* **by** (*auto simp*: *ordLeq-def intro*: *comp-embed*)

lemma *ordLeq-total*:
⟦*Well-order r*; *Well-order r*'⟧ ⟹ *r* ≤*o r*' ∨ *r*' ≤*o r*
unfolding *ordLeq-def* **using** *wellorders-totally-ordered* **by** *blast*

lemma *ordIso-reflexive*:

Well-order r ⟹ *r* =*o r*
unfolding *ordIso-def* **using** *id-iso*[*of r*] **by** *blast*

lemma *ordIso-transitive*[*trans*]:
assumes ∗: *r* =*o r*′ **and** ∗∗: *r*′ =*o r*″
shows *r* =*o r*″
using *assms* **by** (*auto simp*: *ordIso-def intro*: *comp-iso*)

lemma *ordIso-symmetric*:
assumes ∗: *r* =*o r*′
shows *r*′ =*o r*
proof−
obtain *f* **where** *1*: *Well-order r* ∧ *Well-order r*′ **and**
2: *embed r r*′ *f* ∧ *bij-betw f* (*Field r*) (*Field r*′)
using ∗ **by** (*auto simp add*: *ordIso-def iso-def*)
let *?f*′ = *inv-into* (*Field r*) *f*
have *embed r*′ *r ?f*′ ∧ *bij-betw ?f*′ (*Field r*′) (*Field r*)
using *1 2* **by** (*simp add*: *bij-betw-inv-into inv-into-Field-embed-bij-betw*)
thus *r*′ =*o r* **unfolding** *ordIso-def* **using** *1* **by** (*auto simp add*: *iso-def*)
qed

lemma *ordLeq-ordLess-trans*[*trans*]:
assumes *r* ≤*o r*′ **and** *r*′ <*o r*″
shows *r* <*o r*″
proof−
have *Well-order r* ∧ *Well-order r*″
using *assms* **unfolding** *ordLeq-def ordLess-def* **by** *auto*
thus *?thesis* **using** *assms* **unfolding** *ordLeq-def ordLess-def*
using *embed-comp-embedS* **by** *blast*
qed

lemma *ordLess-ordLeq-trans*[*trans*]:
assumes *r* <*o r*′ **and** *r*′ ≤*o r*″
shows *r* <*o r*″
using *embedS-comp-embed assms* **by** (*force simp*: *ordLeq-def ordLess-def*)

lemma *ordLeq-ordIso-trans*[*trans*]:
assumes *r* ≤*o r*′ **and** *r*′ =*o r*″
shows *r* ≤*o r*″
using *embed-comp-iso assms* **by** (*force simp*: *ordLeq-def ordIso-def*)

lemma *ordIso-ordLeq-trans*[*trans*]:
assumes *r* =*o r*′ **and** *r*′ ≤*o r*″
shows *r* ≤*o r*″
using *iso-comp-embed assms* **by** (*force simp*: *ordLeq-def ordIso-def*)

lemma *ordLess-ordIso-trans*[*trans*]:
assumes *r* <*o r*′ **and** *r*′ =*o r*″
shows *r* <*o r*″

using *embedS-comp-iso assms* **by** (*force simp*: *ordLess-def ordIso-def*)

lemma *ordIso-ordLess-trans*[*trans*]:
assumes $r =o\ r'$ **and** $r' <o\ r''$
shows $r <o\ r''$
using *iso-comp-embedS assms* **by** (*force simp*: *ordLess-def ordIso-def*)

lemma *ordLess-not-embed*:
assumes $r <o\ r'$
shows ¬(∃*f′*. *embed r′ r f′*)
proof−
obtain *f* **where** *1*: *Well-order r* ∧ *Well-order r′* **and** *2*: *embed r r′ f* **and**
3: ¬ *bij-betw f* (*Field r*) (*Field r′*)
using *assms* **unfolding** *ordLess-def* **by** (*auto simp add*: *embedS-def*)
{**fix** *f′* **assume** ∗: *embed r′ r f′*
hence *bij-betw f* (*Field r*) (*Field r′*) **using** *1 2*
by (*simp add*: *embed-bothWays-Field-bij-betw*)
with *3* **have** *False* **by** *contradiction*
}
thus *?thesis* **by** *blast*
qed

lemma *ordLess-Field*:
assumes *OL*: *r1* <*o r2* **and** *EMB*: *embed r1 r2 f*
shows ¬ (*f*‘(*Field r1*) = *Field r2*)
proof−
let *?A1* = *Field r1* **let** *?A2* = *Field r2*
obtain *g* **where**
0: *Well-order r1* ∧ *Well-order r2* **and**
1: *embed r1 r2 g* ∧ ¬(*bij-betw g ?A1 ?A2*)
using *OL* **unfolding** *ordLess-def* **by** (*auto simp add*: *embedS-def*)
hence ∀ *a* ∈ *?A1*. *f a* = *g a*
using *0 EMB embed-unique*[*of r1*] **by** *auto*
hence ¬(*bij-betw f ?A1 ?A2*)
using *1 bij-betw-cong*[*of ?A1*] **by** *blast*
moreover
have *inj-on f ?A1* **using** *EMB 0* **by** (*simp add*: *embed-inj-on*)
ultimately show *?thesis* **by** (*simp add*: *bij-betw-def*)
qed

lemma *ordLess-iff*:
$r <o\ r'$ = (*Well-order r* ∧ *Well-order r′* ∧ ¬(∃*f′*. *embed r′ r f′*))
proof
assume ∗: $r <o\ r'$
hence ¬(∃*f′*. *embed r′ r f′*) **using** *ordLess-not-embed*[*of r r′*] **by** *simp*
with ∗ **show** *Well-order r* ∧ *Well-order r′* ∧ ¬ (∃*f′*. *embed r′ r f′*)
unfolding *ordLess-def* **by** *auto*
next
assume ∗: *Well-order r* ∧ *Well-order r′* ∧ ¬ (∃*f′*. *embed r′ r f′*)

then obtain *f* **where** *1*: *embed r r′ f*
using *wellorders-totally-ordered*[*of r r′*] **by** *blast*
moreover
{**assume** *bij-betw f* (*Field r*) (*Field r′*)
with ∗ *1* **have** *embed r′ r* (*inv-into* (*Field r*) *f*)
using *inv-into-Field-embed-bij-betw*[*of r r′ f*] **by** *auto*
with ∗ **have** *False* **by** *blast*
}
ultimately show (*r*,*r′*) ∈ *ordLess*
unfolding *ordLess-def* **using** ∗ **by** (*fastforce simp add*: *embedS-def*)
qed

lemma *ordLess-irreflexive*: ¬ *r* <*o r*
using *id-embed*[*of r*] **by** (*auto simp*: *ordLess-iff*)

lemma *ordLeq-iff-ordLess-or-ordIso*:
$r \leq o\ r' = (r <o\ r' \vee r =o\ r')$
unfolding *ordRels-def embedS-defs iso-defs* **by** *blast*

lemma *ordIso-iff-ordLeq*:
$(r =o\ r') = (r \leq o\ r' \wedge r' \leq o\ r)$
proof
assume *r* =*o r′*
then obtain *f* **where** *1*: *Well-order r* ∧ *Well-order r′* ∧
embed r r′ f ∧ *bij-betw f* (*Field r*) (*Field r′*)
unfolding *ordIso-def iso-defs* **by** *auto*
hence *embed r r′ f* ∧ *embed r′ r* (*inv-into* (*Field r*) *f*)
by (*simp add*: *inv-into-Field-embed-bij-betw*)
thus $r \leq o\ r' \wedge r' \leq o\ r$
unfolding *ordLeq-def* **using** *1* **by** *auto*
next
assume $r \leq o\ r' \wedge r' \leq o\ r$
then obtain *f* **and** *g* **where** *1*: *Well-order r* ∧ *Well-order r′* ∧
embed r r′ f ∧ *embed r′ r g*
unfolding *ordLeq-def* **by** *auto*
hence *iso r r′ f* **by** (*auto simp add*: *embed-bothWays-iso*)
thus *r* =*o r′* **unfolding** *ordIso-def* **using** *1* **by** *auto*
qed

lemma *not-ordLess-ordLeq*:
$r <o\ r' \Longrightarrow \neg\ r' \leq o\ r$
using *ordLess-ordLeq-trans ordLess-irreflexive* **by** *blast*

lemma *not-ordLeq-ordLess*:
$r \leq o\ r' \Longrightarrow \neg\ r' <o\ r$
using *not-ordLess-ordLeq* **by** *blast*

lemma *ordLess-or-ordLeq*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′*

shows $r <o\ r' \lor r' \leq o\ r$
proof−
have $r \leq o\ r' \lor r' \leq o\ r$
using *assms* **by** (*simp add*: *ordLeq-total*)
moreover
{**assume** $\neg\ r <o\ r' \land r \leq o\ r'$
hence $r =o\ r'$ **using** *ordLeq-iff-ordLess-or-ordIso* **by** *blast*
hence $r' \leq o\ r$ **using** *ordIso-symmetric ordIso-iff-ordLeq* **by** *blast*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *not-ordLess-ordIso*:
$r <o\ r' \Longrightarrow \neg\ r =o\ r'$
using *ordLess-ordIso-trans ordIso-symmetric ordLess-irreflexive* **by** *blast*

lemma *not-ordLeq-iff-ordLess*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows $(\neg\ r' \leq o\ r) = (r <o\ r')$
using *assms not-ordLess-ordLeq ordLess-or-ordLeq* **by** *blast*

lemma *not-ordLess-iff-ordLeq*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows $(\neg\ r' <o\ r) = (r \leq o\ r')$
using *assms not-ordLess-ordLeq ordLess-or-ordLeq* **by** *blast*

lemma *ordLess-transitive*[*trans*]:
$[\![r <o\ r'; r' <o\ r'']\!] \Longrightarrow r <o\ r''$
using *ordLess-ordLeq-trans ordLeq-iff-ordLess-or-ordIso* **by** *blast*

corollary *ordLess-trans*: *trans ordLess*
unfolding *trans-def* **using** *ordLess-transitive* **by** *blast*

lemmas *ordIso-equivalence* = *ordIso-transitive ordIso-reflexive ordIso-symmetric*

lemma *ordIso-imp-ordLeq*:
$r =o\ r' \Longrightarrow r \leq o\ r'$
using *ordIso-iff-ordLeq* **by** *blast*

lemma *ordLess-imp-ordLeq*:
$r <o\ r' \Longrightarrow r \leq o\ r'$
using *ordLeq-iff-ordLess-or-ordIso* **by** *blast*

lemma *ofilter-subset-ordLeq*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $(A \leq B) = (Restr\ r\ A \leq o\ Restr\ r\ B)$
proof
assume $A \leq B$

thus *Restr r A* $\leq o$ *Restr r B*
unfolding *ordLeq-def* **using** *assms*
Well-order-Restr Well-order-Restr ofilter-subset-embed **by** *blast*
next
assume $*$: *Restr r A* $\leq o$ *Restr r B*
then obtain *f* **where** *embed* (*Restr r A*) (*Restr r B*) *f*
unfolding *ordLeq-def* **by** *blast*
{assume $B < A$
hence *Restr r B* $<o$ *Restr r A*
unfolding *ordLess-def* **using** *assms*
Well-order-Restr Well-order-Restr ofilter-subset-embedS **by** *blast*
hence *False* **using** $*$ *not-ordLess-ordLeq* **by** *blast*
}
thus $A \leq B$ **using** *OFA OFB WELL*
wo-rel-def[*of r*] *wo-rel.ofilter-linord*[*of r A B*] **by** *blast*
qed

lemma *ofilter-subset-ordLess*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $(A < B) =$ (*Restr r A* $<o$ *Restr r B*)
proof−
let *?rA* = *Restr r A* **let** *?rB* = *Restr r B*
have *1*: *Well-order ?rA* $\wedge$ *Well-order ?rB*
using *WELL Well-order-Restr* **by** *blast*
have $(A < B) = (\neg B \leq A)$ **using** *assms*
wo-rel-def wo-rel.ofilter-linord[*of r A B*] **by** *blast*
also have ... = (¬ *Restr r B* $\leq o$ *Restr r A*)
using *assms ofilter-subset-ordLeq* **by** *blast*
also have ... = (*Restr r A* $<o$ *Restr r B*)
using *1 not-ordLeq-iff-ordLess* **by** *blast*
finally show *?thesis* **.**
qed

lemma *ofilter-ordLess*:
⟦*Well-order r*; *wo-rel.ofilter r A*⟧ ⟹ ($A <$ *Field r*) = (*Restr r A* $<o$ *r*)
by (*simp add*: *ofilter-subset-ordLess wo-rel.Field-ofilter*
wo-rel-def Restr-Field)

corollary *underS-Restr-ordLess*:
assumes *Well-order r* **and** *Field r* $\neq \{\}$
shows *Restr r* (*underS r a*) $<o$ *r*
proof−
have *underS r a* $<$ *Field r* **using** *assms*
by (*simp add*: *underS-Field3*)
thus *?thesis* **using** *assms*
by (*simp add*: *ofilter-ordLess wo-rel.underS-ofilter wo-rel-def*)
qed

lemma *embed-ordLess-ofilterIncl*:
  assumes
    OL12: *r1* <*o r2* **and** *OL23*: *r2* <*o r3* **and**
    EMB13: *embed r1 r3 f13* **and** *EMB23*: *embed r2 r3 f23*
  shows (*f13*‘(*Field r1*), *f23*‘(*Field r2*)) ∈ (*ofilterIncl r3*)
proof−
  have *OL13*: *r1* <*o r3*
    using *OL12 OL23* **using** *ordLess-transitive* **by** *auto*
  let *?A1* = *Field r1* **let** *?A2* =*Field r2* **let** *?A3* =*Field r3*
  obtain *f12 g23* **where**
    0: *Well-order r1* ∧ *Well-order r2* ∧ *Well-order r3* **and**
    1: *embed r1 r2 f12* ∧ ¬(*bij-betw f12 ?A1 ?A2*) **and**
    2: *embed r2 r3 g23* ∧ ¬(*bij-betw g23 ?A2 ?A3*)
    using *OL12 OL23* **by** (*auto simp add*: *ordLess-def embedS-def*)
  hence ∀ *a* ∈ *?A2*. *f23 a* = *g23 a*
    using *EMB23 embed-unique*[*of r2 r3*] **by** *blast*
  hence *3*: ¬(*bij-betw f23 ?A2 ?A3*)
    using *2 bij-betw-cong*[*of ?A2 f23 g23*] **by** *blast*

  have *4*: *wo-rel.ofilter r2* (*f12* ‘ *?A1*) ∧ *f12* ‘ *?A1* ≠ *?A2*
    using *0 1 OL12* **by** (*simp add*: *embed-Field-ofilter ordLess-Field*)
  have *5*: *wo-rel.ofilter r3* (*f23* ‘ *?A2*) ∧ *f23* ‘ *?A2* ≠ *?A3*
    using *0 EMB23 OL23* **by** (*simp add*: *embed-Field-ofilter ordLess-Field*)
  have *6*: *wo-rel.ofilter r3* (*f13* ‘ *?A1*) ∧ *f13* ‘ *?A1* ≠ *?A3*
    using *0 EMB13 OL13* **by** (*simp add*: *embed-Field-ofilter ordLess-Field*)

  have *f12* ‘ *?A1* < *?A2*
    using *0 4* **by** (*auto simp add*: *wo-rel-def wo-rel.ofilter-def*)
  moreover have *inj-on f23 ?A2*
    using *EMB23 0* **by** (*simp add*: *wo-rel-def embed-inj-on*)
  ultimately
  have *f23* ‘ (*f12* ‘ *?A1*) < *f23* ‘ *?A2* **by** (*simp add*: *image-strict-mono*)
  moreover
  {**have** *embed r1 r3* (*f23* ∘ *f12*)
      using *1 EMB23 0* **by** (*auto simp add*: *comp-embed*)
    hence ∀ *a* ∈ *?A1*. *f23*(*f12 a*) = *f13 a*
      using *EMB13 0 embed-unique*[*of r1 r3 f23* ∘ *f12 f13*] **by** *auto*
    hence *f23* ‘ (*f12* ‘ *?A1*) = *f13* ‘ *?A1* **by** *force*
  }
  ultimately
  have *f13* ‘ *?A1* < *f23* ‘ *?A2* **by** *simp*

  with *5 6* **show** *?thesis*
    unfolding *ofilterIncl-def* **by** *auto*
qed

lemma *ordLess-iff-ordIso-Restr*:
  assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′*
  shows (*r′* <*o r*) = (∃ *a* ∈ *Field r*. *r′* =*o Restr r* (*underS r a*))

proof *safe*
fix *a* **assume** ∗: $a \in$ *Field r* **and** ∗∗: $r' =o$ *Restr r* (*underS r a*)
hence *Restr r* (*underS r a*) $<o$ *r* **using** *WELL underS-Restr-ordLess*[*of r*] **by** *blast*
thus $r' <o$ *r* **using** ∗∗ *ordIso-ordLess-trans* **by** *blast*
next
assume $r' <o$ *r*
then obtain *f* **where** *1*: *Well-order r* ∧ *Well-order r′* **and**
2: *embed r′ r f* ∧ *f* ‘ (*Field r′*) ≠ *Field r*
unfolding *ordLess-def embedS-def*[*abs-def*] *bij-betw-def* **using** *embed-inj-on* **by** *blast*
hence *wo-rel.ofilter r* (*f* ‘ (*Field r′*)) **using** *embed-Field-ofilter* **by** *blast*
then obtain *a* **where** *3*: $a \in$ *Field r* **and** *4*: *underS r a* = *f* ‘ (*Field r′*)
using *1 2* **by** (*auto simp add*: *wo-rel.ofilter-underS-Field wo-rel-def*)
have *iso r′* (*Restr r* (*f* ‘ (*Field r′*))) *f*
using *embed-implies-iso-Restr 2 assms* **by** *blast*
moreover have *Well-order* (*Restr r* (*f* ‘ (*Field r′*)))
using *WELL Well-order-Restr* **by** *blast*
ultimately have $r' =o$ *Restr r* (*f* ‘ (*Field r′*))
using *WELL′* **unfolding** *ordIso-def* **by** *auto*
hence $r' =o$ *Restr r* (*underS r a*) **using** *4* **by** *auto*
thus ∃ *a* ∈ *Field r*. $r' =o$ *Restr r* (*underS r a*) **using** *3* **by** *auto*
qed

lemma *internalize-ordLess*:
($r' <o$ *r*) = (∃ *p*. *Field p* < *Field r* ∧ $r' =o$ *p* ∧ *p* $<o$ *r*)
proof
assume ∗: $r' <o$ *r*
hence *0*: *Well-order r* ∧ *Well-order r′* **unfolding** *ordLess-def* **by** *auto*
with ∗ **obtain** *a* **where** *1*: $a \in$ *Field r* **and** *2*: $r' =o$ *Restr r* (*underS r a*)
using *ordLess-iff-ordIso-Restr* **by** *blast*
let *?p* = *Restr r* (*underS r a*)
have *wo-rel.ofilter r* (*underS r a*) **using** *0*
by (*simp add*: *wo-rel-def wo-rel.underS-ofilter*)
hence *Field ?p* = *underS r a* **using** *0 Field-Restr-ofilter* **by** *blast*
hence *Field ?p* < *Field r* **using** *underS-Field2 1* **by** *fast*
moreover have *?p* $<o$ *r* **using** *underS-Restr-ordLess*[*of r a*] *0 1* **by** *blast*
ultimately show ∃ *p*. *Field p* < *Field r* ∧ $r' =o$ *p* ∧ *p* $<o$ *r* **using** *2* **by** *blast*
next
assume ∃ *p*. *Field p* < *Field r* ∧ $r' =o$ *p* ∧ *p* $<o$ *r*
thus $r' <o$ *r* **using** *ordIso-ordLess-trans* **by** *blast*
qed

lemma *internalize-ordLeq*:
($r' \leq o$ *r*) = (∃ *p*. *Field p* ≤ *Field r* ∧ $r' =o$ *p* ∧ *p* $\leq o$ *r*)
proof
assume ∗: $r' \leq o$ *r*
moreover
have $r' <o$ *r* ⟹ ∃ *p*. *Field p* ≤ *Field r* ∧ $r' =o$ *p* ∧ *p* $\leq o$ *r*

using *ordLeq-iff-ordLess-or-ordIso internalize-ordLess*[*of* r' r] **by** *blast*
moreover
have $r \leq o\ r$ **using** $*$ *ordLeq-def ordLeq-reflexive* **by** *blast*
ultimately show $\exists\, p.\ \mathit{Field}\ p \leq \mathit{Field}\ r \wedge r' =o\ p \wedge p \leq o\ r$
using *ordLeq-iff-ordLess-or-ordIso* **by** *blast*
next
assume $\exists\, p.\ \mathit{Field}\ p \leq \mathit{Field}\ r \wedge r' =o\ p \wedge p \leq o\ r$
thus $r' \leq o\ r$ **using** *ordIso-ordLeq-trans* **by** *blast*
qed

lemma *ordLeq-iff-ordLess-Restr*:
assumes *WELL*: *Well-order* r **and** *WELL'*: *Well-order* r'
shows $(r \leq o\ r') = (\forall\, a \in \mathit{Field}\ r.\ \mathit{Restr}\ r\ (\mathit{underS}\ r\ a) <o\ r')$
proof *safe*
assume $*$: $r \leq o\ r'$
fix a **assume** $a \in \mathit{Field}\ r$
hence $\mathit{Restr}\ r\ (\mathit{underS}\ r\ a) <o\ r$
using *WELL underS-Restr-ordLess*[*of* r] **by** *blast*
thus $\mathit{Restr}\ r\ (\mathit{underS}\ r\ a) <o\ r'$
using $*$ *ordLess-ordLeq-trans* **by** *blast*
next
assume $*$: $\forall\, a \in \mathit{Field}\ r.\ \mathit{Restr}\ r\ (\mathit{underS}\ r\ a) <o\ r'$
{**assume** $r' <o\ r$
then obtain a **where** $a \in \mathit{Field}\ r \wedge r' =o\ \mathit{Restr}\ r\ (\mathit{underS}\ r\ a)$
using *assms ordLess-iff-ordIso-Restr* **by** *blast*
hence *False* **using** $*$ *not-ordLess-ordIso ordIso-symmetric* **by** *blast*
}
thus $r \leq o\ r'$ **using** *ordLess-or-ordLeq assms* **by** *blast*
qed

lemma *finite-ordLess-infinite*:
assumes *WELL*: *Well-order* r **and** *WELL'*: *Well-order* r' **and**
FIN: *finite*(*Field* r) **and** *INF*: ¬*finite*(*Field* r')
shows $r <o\ r'$
proof−
{**assume** $r' \leq o\ r$
then obtain h **where** $\mathit{inj\text{-}on}\ h\ (\mathit{Field}\ r') \wedge h\ `\ (\mathit{Field}\ r') \leq \mathit{Field}\ r$
unfolding *ordLeq-def* **using** *assms embed-inj-on embed-Field* **by** *blast*
hence *False* **using** *finite-imageD finite-subset FIN INF* **by** *blast*
}
thus *?thesis* **using** *WELL WELL' ordLess-or-ordLeq* **by** *blast*
qed

lemma *finite-well-order-on-ordIso*:
assumes *FIN*: *finite* A **and**
WELL: *well-order-on* A r **and** *WELL'*: *well-order-on* A r'
shows $r =o\ r'$
proof−
have *0*: $\mathit{Well\text{-}order}\ r \wedge \mathit{Well\text{-}order}\ r' \wedge \mathit{Field}\ r = A \wedge \mathit{Field}\ r' = A$

```
    using assms well-order-on-Well-order by blast
  moreover
  have ∀ r r'. well-order-on A r ∧ well-order-on A r' ∧ r ≤o r'
               ⟶ r =o r'
  proof(clarify)
    fix r r' assume ∗: well-order-on A r and ∗∗: well-order-on A r'
    have 2: Well-order r ∧ Well-order r' ∧ Field r = A ∧ Field r' = A
      using ∗ ∗∗ well-order-on-Well-order by blast
    assume r ≤o r'
    then obtain f where 1: embed r r' f and
      inj-on f A ∧ f ‘ A ≤ A
      unfolding ordLeq-def using 2 embed-inj-on embed-Field by blast
    hence bij-betw f A A unfolding bij-betw-def using FIN endo-inj-surj by blast
    thus r =o r' unfolding ordIso-def iso-def[abs-def] using 1 2 by auto
  qed
  ultimately show ?thesis using assms ordLeq-total ordIso-symmetric by blast
qed
```

29.5 <o is well-founded

Of course, it only makes sense to state that the <o is well-founded on the restricted type *'a rel rel*. We prove this by first showing that, for any set of well-orders all embedded in a fixed well-order, the function mapping each well-order in the set to an order filter of the fixed well-order is compatible w.r.t. to <o versus *strict inclusion*; and we already know that strict inclusion of order filters is well-founded.

```
definition ord-to-filter :: 'a rel ⇒ 'a rel ⇒ 'a set
  where ord-to-filter r0 r ≡ (SOME f. embed r r0 f) ‘ (Field r)

lemma ord-to-filter-compat:
  compat (ordLess Int (ordLess⁻¹ ‘‘{r0} × ordLess⁻¹ ‘‘{r0}))
        (ofilterIncl r0)
        (ord-to-filter r0)
proof(unfold compat-def ord-to-filter-def, clarify)
  fix r1::'a rel and r2::'a rel
  let ?A1 = Field r1  let ?A2 =Field r2 let ?A0 =Field r0
  let ?phi10 = λ f10. embed r1 r0 f10 let ?f10 = SOME f. ?phi10 f
  let ?phi20 = λ f20. embed r2 r0 f20 let ?f20 = SOME f. ?phi20 f
  assume ∗: r1 <o r0 r2 <o r0 and ∗∗: r1 <o r2
  hence (∃ f. ?phi10 f) ∧ (∃ f. ?phi20 f)
    by (auto simp add: ordLess-def embedS-def)
  hence ?phi10 ?f10 ∧ ?phi20 ?f20 by (auto simp add: someI-ex)
  thus (?f10 ‘ ?A1, ?f20 ‘ ?A2) ∈ ofilterIncl r0
    using ∗ ∗∗ by (simp add: embed-ordLess-ofilterIncl)
qed

theorem wf-ordLess: wf ordLess
proof−
```

{fix *r0* :: ('*a* × '*a*) *set*

let *?ordLess* = *ordLess*::('*d rel* ∗ '*d rel*) *set*
let *?R* = *?ordLess Int* (*?ordLess*$^{-1}$ ‘‘{*r0*} × *?ordLess*$^{-1}$ ‘‘{*r0*})
{assume *Case1*: *Well-order r0*
hence *wf ?R*
using *wf-ofilterIncl*[*of r0*]
compat-wf[*of ?R ofilterIncl r0 ord-to-filter r0*]
ord-to-filter-compat[*of r0*] **by** *auto*
}
moreover
{assume *Case2*: ¬ *Well-order r0*
hence *?R* = {} **unfolding** *ordLess-def* **by** *auto*
hence *wf ?R* **using** *wf-empty* **by** *simp*
}
ultimately have *wf ?R* **by** *blast*
}
thus *?thesis* **by** (*simp add*: *trans-wf-iff ordLess-trans*)
qed

corollary *exists-minim-Well-order*:
assumes *NE*: *R* ≠ {} **and** *WELL*: ∀ *r* ∈ *R*. *Well-order r*
shows ∃ *r* ∈ *R*. ∀ *r*' ∈ *R*. *r* ≤*o r*'
proof−
obtain *r* **where** *r* ∈ *R* ∧ (∀ *r*' ∈ *R*. ¬ *r*' <*o r*)
using *NE spec*[*OF spec*[*OF subst*[*OF wf-eq-minimal*, *of* %*x*. *x*, *OF wf-ordLess*]], *of* - *R*]
equals0I[*of R*] **by** *blast*
with *not-ordLeq-iff-ordLess WELL* **show** *?thesis* **by** *blast*
qed

29.6 Copy via direct images

The direct image operator is the dual of the inverse image operator *inv-image* from *Relation.thy*. It is useful for transporting a well-order between different types.

definition *dir-image* :: '*a rel* ⇒ ('*a* ⇒ '*a*') ⇒ '*a*' *rel*
where
dir-image r f = {(*f a*, *f b*)| *a b*. (*a*,*b*) ∈ *r*}

lemma *dir-image-Field*:
Field(*dir-image r f*) = *f* ‘ (*Field r*)
unfolding *dir-image-def Field-def Range-def Domain-def* **by** *fast*

lemma *dir-image-minus-Id*:
inj-on f (*Field r*) ⟹ (*dir-image r f*) − *Id* = *dir-image* (*r* − *Id*) *f*
unfolding *inj-on-def Field-def dir-image-def* **by** *auto*

lemma *Refl-dir-image*:

assumes *Refl r*
shows *Refl*(*dir-image r f*)
proof−
{**fix** a' b'
assume (a',b') ∈ *dir-image r f*
then obtain *a b* **where** *1*: $a' = f\ a \wedge b' = f\ b \wedge (a,b) \in r$
unfolding *dir-image-def* **by** *blast*
hence *a* ∈ *Field r* ∧ *b* ∈ *Field r* **using** *Field-def* **by** *fastforce*
hence (*a*,*a*) ∈ *r* ∧ (*b*,*b*) ∈ *r* **using** *assms* **by** (*simp add*: *refl-on-def*)
with *1* **have** (a',a') ∈ *dir-image r f* ∧ (b',b') ∈ *dir-image r f*
unfolding *dir-image-def* **by** *auto*
}
thus *?thesis*
by(*unfold refl-on-def Field-def Domain-def Range-def*, *auto*)
qed

lemma *trans-dir-image*:
assumes *TRANS*: *trans r* **and** *INJ*: *inj-on f* (*Field r*)
shows *trans*(*dir-image r f*)
unfolding *trans-def*
proof *safe*
fix a' b' c'
assume (a',b') ∈ *dir-image r f* (b',c') ∈ *dir-image r f*
then obtain *a b1 b2 c* **where** *1*: $a' = f\ a \wedge b' = f\ b1 \wedge b' = f\ b2 \wedge c' = f\ c$
and
2: (*a*,*b1*) ∈ *r* ∧ (*b2*,*c*) ∈ *r*
unfolding *dir-image-def* **by** *blast*
hence *b1* ∈ *Field r* ∧ *b2* ∈ *Field r*
unfolding *Field-def* **by** *auto*
hence *b1* = *b2* **using** *1 INJ* **unfolding** *inj-on-def* **by** *auto*
hence (*a*,*c*) ∈ *r* **using** *2 TRANS* **unfolding** *trans-def* **by** *blast*
thus (a',c') ∈ *dir-image r f*
unfolding *dir-image-def* **using** *1* **by** *auto*
qed

lemma *Preorder-dir-image*:
⟦*Preorder r*; *inj-on f* (*Field r*)⟧ ⟹ *Preorder* (*dir-image r f*)
by (*simp add*: *preorder-on-def Refl-dir-image trans-dir-image*)

lemma *antisym-dir-image*:
assumes *AN*: *antisym r* **and** *INJ*: *inj-on f* (*Field r*)
shows *antisym*(*dir-image r f*)
unfolding *antisym-def*
proof *safe*
fix a' b'
assume (a',b') ∈ *dir-image r f* (b',a') ∈ *dir-image r f*
then obtain *a1 b1 a2 b2* **where** *1*: $a' = f\ a1 \wedge a' = f\ a2 \wedge b' = f\ b1 \wedge b' = f$
b2 **and**
2: (*a1*,*b1*) ∈ *r* ∧ (*b2*,*a2*) ∈ *r* **and**

3: {*a1*,*a2*,*b1*,*b2*} ≤ *Field r*
unfolding *dir-image-def Field-def* **by** *blast*
hence *a1* = *a2* ∧ *b1* = *b2* **using** *INJ* **unfolding** *inj-on-def* **by** *auto*
hence *a1* = *b2* **using** *2 AN* **unfolding** *antisym-def* **by** *auto*
thus *a′* = *b′* **using** *1* **by** *auto*
qed

lemma *Partial-order-dir-image*:
⟦*Partial-order r*; *inj-on f* (*Field r*)⟧ ⟹ *Partial-order* (*dir-image r f*)
by (*simp add*: *partial-order-on-def Preorder-dir-image antisym-dir-image*)

lemma *Total-dir-image*:
assumes *TOT*: *Total r* **and** *INJ*: *inj-on f* (*Field r*)
shows *Total*(*dir-image r f*)
proof(*unfold total-on-def*, *intro ballI impI*)
fix *a′ b′*
assume *a′* ∈ *Field* (*dir-image r f*) *b′* ∈ *Field* (*dir-image r f*)
then obtain *a* **and** *b* **where** *1*: *a* ∈ *Field r* ∧ *b* ∈ *Field r* ∧ *f a* = *a′* ∧ *f b* = *b′*
unfolding *dir-image-Field*[*of r f*] **by** *blast*
moreover assume *a′* ≠ *b′*
ultimately have *a* ≠ *b* **using** *INJ* **unfolding** *inj-on-def* **by** *auto*
hence (*a*,*b*) ∈ *r* ∨ (*b*,*a*) ∈ *r* **using** *1 TOT* **unfolding** *total-on-def* **by** *auto*
thus (*a′*,*b′*) ∈ *dir-image r f* ∨ (*b′*,*a′*) ∈ *dir-image r f*
using *1* **unfolding** *dir-image-def* **by** *auto*
qed

lemma *Linear-order-dir-image*:
⟦*Linear-order r*; *inj-on f* (*Field r*)⟧ ⟹ *Linear-order* (*dir-image r f*)
by (*simp add*: *linear-order-on-def Partial-order-dir-image Total-dir-image*)

lemma *wf-dir-image*:
assumes *WF*: *wf r* **and** *INJ*: *inj-on f* (*Field r*)
shows *wf*(*dir-image r f*)
proof(*unfold wf-eq-minimal2*, *intro allI impI*, *elim conjE*)
fix *A′*::*′b set*
assume *SUB*: *A′* ≤ *Field*(*dir-image r f*) **and** *NE*: *A′* ≠ {}
obtain *A* **where** *A-def*: *A* = {*a* ∈ *Field r*. *f a* ∈ *A′*} **by** *blast*
have *A* ≠ {} ∧ *A* ≤ *Field r* **using** *A-def SUB NE* **by** (*auto simp*: *dir-image-Field*)
then obtain *a* **where** *1*: *a* ∈ *A* ∧ (∀ *b* ∈ *A*. (*b*,*a*) ∉ *r*)
using *spec*[*OF WF*[*unfolded wf-eq-minimal2*], *of A*] **by** *blast*
have ∀ *b′* ∈ *A′*. (*b′*,*f a*) ∉ *dir-image r f*
proof(*clarify*)
fix *b′* **assume** ∗: *b′* ∈ *A′* **and** ∗∗: (*b′*,*f a*) ∈ *dir-image r f*
obtain *b1 a1* **where** *2*: *b′* = *f b1* ∧ *f a* = *f a1* **and**
3: (*b1*,*a1*) ∈ *r* ∧ {*a1*,*b1*} ≤ *Field r*
using ∗∗ **unfolding** *dir-image-def Field-def* **by** *blast*
hence *a* = *a1* **using** *1 A-def INJ* **unfolding** *inj-on-def* **by** *auto*
hence *b1* ∈ *A* ∧ (*b1*,*a*) ∈ *r* **using** *2 3 A-def* ∗ **by** *auto*
with *1* **show** *False* **by** *auto*

qed
thus $\exists\, a' \in A'.\ \forall\, b' \in A'.\ (b',\ a') \notin$ *dir-image r f*
using *A-def 1* **by** *blast*
qed

lemma *Well-order-dir-image*:
$[\![$*Well-order r*; *inj-on f* (*Field r*)$]\!] \Longrightarrow$ *Well-order* (*dir-image r f*)
unfolding *well-order-on-def*
using *Linear-order-dir-image*[*of r f*] *wf-dir-image*[*of r* − *Id f*]
dir-image-minus-Id[*of f r*]
subset-inj-on[*of f Field r Field*(*r* − *Id*)]
mono-Field[*of r* − *Id r*] **by** *auto*

lemma *dir-image-bij-betw*:
$[\![$*inj-on f* (*Field r*)$]\!] \Longrightarrow$ *bij-betw f* (*Field r*) (*Field* (*dir-image r f*))
unfolding *bij-betw-def* **by** (*simp add*: *dir-image-Field order-on-defs*)

lemma *dir-image-compat*:
compat r (*dir-image r f*) *f*
unfolding *compat-def dir-image-def* **by** *auto*

lemma *dir-image-iso*:
$[\![$*Well-order r*; *inj-on f* (*Field r*)$]\!] \Longrightarrow$ *iso r* (*dir-image r f*) *f*
using *iso-iff3 dir-image-compat dir-image-bij-betw Well-order-dir-image* **by** *blast*

lemma *dir-image-ordIso*:
$[\![$*Well-order r*; *inj-on f* (*Field r*)$]\!] \Longrightarrow$ *r =o dir-image r f*
unfolding *ordIso-def* **using** *dir-image-iso Well-order-dir-image* **by** *blast*

lemma *Well-order-iso-copy*:
assumes *WELL*: *well-order-on A r* **and** *BIJ*: *bij-betw f A A'*
shows $\exists\, r'.$ *well-order-on A' r'* $\wedge$ *r =o r'*
proof−
let *?r'* = *dir-image r f*
have *1*: *A* = *Field r* $\wedge$ *Well-order r*
using *WELL well-order-on-Well-order* **by** *blast*
hence *2*: *iso r ?r' f*
using *dir-image-iso* **using** *BIJ* **unfolding** *bij-betw-def* **by** *auto*
hence *f* ‘ (*Field r*) = *Field ?r'* **using** *1 iso-iff*[*of r ?r'*] **by** *blast*
hence *Field ?r'* = *A'*
using *1 BIJ* **unfolding** *bij-betw-def* **by** *auto*
moreover have *Well-order ?r'*
using *1 Well-order-dir-image BIJ* **unfolding** *bij-betw-def* **by** *blast*
ultimately show *?thesis* **unfolding** *ordIso-def* **using** *1 2* **by** *blast*
qed

29.7 Bounded square

This construction essentially defines, for an order relation *r*, a lexicographic order *bsqr r* on (*Field r*) × (*Field r*), applying the following criteria (in this order):

- compare the maximums;
- compare the first components;
- compare the second components.

The only application of this construction that we are aware of is at proving that the square of an infinite set has the same cardinal as that set. The essential property required there (and which is ensured by this construction) is that any proper order filter of the product order is included in a rectangle, i.e., in a product of proper filters on the original relation (assumed to be a well-order).

```
definition bsqr :: 'a rel => ('a * 'a)rel
  where
    bsqr r = {((a1,a2),(b1,b2)).
          {a1,a2,b1,b2} ≤ Field r ∧
          (a1 = b1 ∧ a2 = b2 ∨
           (wo-rel.max2 r a1 a2, wo-rel.max2 r b1 b2) ∈ r − Id ∨
           wo-rel.max2 r a1 a2 = wo-rel.max2 r b1 b2 ∧ (a1,b1) ∈ r − Id ∨
            wo-rel.max2 r a1 a2 = wo-rel.max2 r b1 b2 ∧ a1 = b1  ∧ (a2,b2) ∈ r
− Id
          )}

lemma Field-bsqr:
  Field (bsqr r) = Field r × Field r
proof
  show Field (bsqr r) ≤ Field r × Field r
  proof−
    {fix a1 a2 assume (a1,a2) ∈ Field (bsqr r)
      moreover
      have ⋀ b1 b2. ((a1,a2),(b1,b2)) ∈ bsqr r ∨ ((b1,b2),(a1,a2)) ∈ bsqr r ⟹
                  a1 ∈ Field r ∧ a2 ∈ Field r unfolding bsqr-def by auto
      ultimately have a1 ∈ Field r ∧ a2 ∈ Field r unfolding Field-def by auto
    }
    thus ?thesis unfolding Field-def by force
  qed
next
  show Field r × Field r ≤ Field (bsqr r)
  proof safe
    fix a1 a2 assume a1 ∈ Field r and a2 ∈ Field r
    hence ((a1,a2),(a1,a2)) ∈ bsqr r unfolding bsqr-def by blast
    thus (a1,a2) ∈ Field (bsqr r) unfolding Field-def by auto
  qed
```

qed

lemma *bsqr-Refl*: *Refl*(*bsqr r*)
  by(*unfold refl-on-def Field-bsqr*, *auto simp add*: *bsqr-def*)

lemma *bsqr-Trans*:
  assumes *Well-order r*
  shows *trans* (*bsqr r*)
  unfolding *trans-def*
proof *safe*

  have *Well*: *wo-rel r* **using** *assms wo-rel-def* **by** *auto*
  hence *Trans*: *trans r* **using** *wo-rel.TRANS* **by** *auto*
  have *Anti*: *antisym r* **using** *wo-rel.ANTISYM Well* **by** *auto*
  hence *TransS*: *trans*(*r* − *Id*) **using** *Trans* **by** (*simp add*: *trans-diff-Id*)

  fix *a1 a2 b1 b2 c1 c2*
  assume ∗: ((*a1*,*a2*),(*b1*,*b2*)) ∈ *bsqr r* **and** ∗∗: ((*b1*,*b2*),(*c1*,*c2*)) ∈ *bsqr r*
  hence *0*: {*a1*,*a2*,*b1*,*b2*,*c1*,*c2*} ≤ *Field r* **unfolding** *bsqr-def* **by** *auto*
  have *1*: *a1* = *b1* ∧ *a2* = *b2* ∨ (*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*) ∈ *r*
− *Id* ∨
        wo-rel.max2 r a1 a2 = *wo-rel.max2 r b1 b2* ∧ (*a1*,*b1*) ∈ *r* − *Id* ∨
        wo-rel.max2 r a1 a2 = *wo-rel.max2 r b1 b2* ∧ *a1* = *b1* ∧ (*a2*,*b2*) ∈ *r* −
Id
    using ∗ **unfolding** *bsqr-def* **by** *auto*
  have *2*: *b1* = *c1* ∧ *b2* = *c2* ∨ (*wo-rel.max2 r b1 b2*, *wo-rel.max2 r c1 c2*) ∈ *r*
− *Id* ∨
        wo-rel.max2 r b1 b2 = *wo-rel.max2 r c1 c2* ∧ (*b1*,*c1*) ∈ *r* − *Id* ∨
        wo-rel.max2 r b1 b2 = *wo-rel.max2 r c1 c2* ∧ *b1* = *c1* ∧ (*b2*,*c2*) ∈ *r* −
Id
    using ∗∗ **unfolding** *bsqr-def* **by** *auto*
  show ((*a1*,*a2*),(*c1*,*c2*)) ∈ *bsqr r*
  proof−
    {**assume** *Case1*: *a1* = *b1* ∧ *a2* = *b2*
      hence *?thesis* **using** ∗∗ **by** *simp*
    }
    moreover
    {**assume** *Case2*: (*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*) ∈ *r* − *Id*
      {**assume** *Case21*: *b1* = *c1* ∧ *b2* = *c2*
        hence *?thesis* **using** ∗ **by** *simp*
      }
      moreover
      {**assume** *Case22*: (*wo-rel.max2 r b1 b2*, *wo-rel.max2 r c1 c2*) ∈ *r* − *Id*
        hence (*wo-rel.max2 r a1 a2*, *wo-rel.max2 r c1 c2*) ∈ *r* − *Id*
          using *Case2 TransS trans-def*[*of r* − *Id*] **by** *blast*
        hence *?thesis* **using** *0* **unfolding** *bsqr-def* **by** *auto*
      }
      moreover
      {**assume** *Case23-4*: *wo-rel.max2 r b1 b2* = *wo-rel.max2 r c1 c2*

```
        hence ?thesis using Case2 0 unfolding bsqr-def by auto
      }
      ultimately have ?thesis using 0 2 by auto
    }
    moreover
    {assume Case3: wo-rel.max2 r a1 a2 = wo-rel.max2 r b1 b2 ∧ (a1,b1) ∈ r −
Id
      {assume Case31: b1 = c1 ∧ b2 = c2
        hence ?thesis using ∗ by simp
      }
      moreover
      {assume Case32: (wo-rel.max2 r b1 b2, wo-rel.max2 r c1 c2) ∈ r − Id
        hence ?thesis using Case3 0 unfolding bsqr-def by auto
      }
      moreover
      {assume Case33: wo-rel.max2 r b1 b2 = wo-rel.max2 r c1 c2 ∧ (b1,c1) ∈ r
− Id
        hence (a1,c1) ∈ r − Id
          using Case3 TransS trans-def[of r − Id] by blast
        hence ?thesis using Case3 Case33 0 unfolding bsqr-def by auto
      }
      moreover
      {assume Case33: wo-rel.max2 r b1 b2 = wo-rel.max2 r c1 c2 ∧ b1 = c1
        hence ?thesis using Case3 0 unfolding bsqr-def by auto
      }
      ultimately have ?thesis using 0 2 by auto
    }
    moreover
    {assume Case4: wo-rel.max2 r a1 a2 = wo-rel.max2 r b1 b2 ∧ a1 = b1 ∧
(a2,b2) ∈ r − Id
      {assume Case41: b1 = c1 ∧ b2 = c2
        hence ?thesis using ∗ by simp
      }
      moreover
      {assume Case42: (wo-rel.max2 r b1 b2, wo-rel.max2 r c1 c2) ∈ r − Id
        hence ?thesis using Case4 0 unfolding bsqr-def by force
      }
      moreover
      {assume Case43: wo-rel.max2 r b1 b2 = wo-rel.max2 r c1 c2 ∧ (b1,c1) ∈ r
− Id
        hence ?thesis using Case4 0 unfolding bsqr-def by auto
      }
      moreover
      {assume Case44: wo-rel.max2 r b1 b2 = wo-rel.max2 r c1 c2 ∧ b1 = c1 ∧
(b2,c2) ∈ r − Id
        hence (a2,c2) ∈ r − Id
          using Case4 TransS trans-def[of r − Id] by blast
        hence ?thesis using Case4 Case44 0 unfolding bsqr-def by auto
      }
```

```
    ultimately have ?thesis using 0 2 by auto
   }
   ultimately show ?thesis using 0 1 by auto
  qed
qed

lemma bsqr-antisym:
  assumes Well-order r
  shows antisym (bsqr r)
proof(unfold antisym-def, clarify)

  have Well: wo-rel r using assms wo-rel-def by auto
  hence Trans: trans r using wo-rel.TRANS by auto
  have Anti: antisym r using wo-rel.ANTISYM Well by auto
  hence TransS: trans(r − Id) using Trans by (simp add: trans-diff-Id)
  hence IrrS: ∀ a b. ¬((a,b) ∈ r − Id ∧ (b,a) ∈ r − Id)
    using Anti trans-def[of r − Id] antisym-def[of r − Id] by blast

  fix a1 a2 b1 b2
  assume *: ((a1,a2),(b1,b2)) ∈ bsqr r and **: ((b1,b2),(a1,a2)) ∈ bsqr r
  hence {a1,a2,b1,b2} ≤ Field r unfolding bsqr-def by auto
  moreover
  have a1 = b1 ∧ a2 = b2 ∨ (wo-rel.max2 r a1 a2, wo-rel.max2 r b1 b2) ∈ r −
Id ∨
         wo-rel.max2 r a1 a2 = wo-rel.max2 r b1 b2 ∧ (a1,b1) ∈ r − Id ∨
         wo-rel.max2 r a1 a2 = wo-rel.max2 r b1 b2 ∧ a1 = b1 ∧ (a2,b2) ∈ r −
Id
    using * unfolding bsqr-def by auto
  moreover
  have b1 = a1 ∧ b2 = a2 ∨ (wo-rel.max2 r b1 b2, wo-rel.max2 r a1 a2) ∈ r −
Id ∨
         wo-rel.max2 r b1 b2 = wo-rel.max2 r a1 a2 ∧ (b1,a1) ∈ r − Id ∨
         wo-rel.max2 r b1 b2 = wo-rel.max2 r a1 a2 ∧ b1 = a1 ∧ (b2,a2) ∈ r −
Id
    using ** unfolding bsqr-def by auto
  ultimately show a1 = b1 ∧ a2 = b2
    using IrrS by auto
qed

lemma bsqr-Total:
  assumes Well-order r
  shows Total(bsqr r)
proof−

  have Well: wo-rel r using assms wo-rel-def by auto
  hence Total: ∀ a ∈ Field r. ∀ b ∈ Field r. (a,b) ∈ r ∨ (b,a) ∈ r
    using wo-rel.TOTALS by auto

  {fix a1 a2 b1 b2 assume {(a1,a2), (b1,b2)} ≤ Field(bsqr r)
```

hence *0*: *a1* ∈ *Field r* ∧ *a2* ∈ *Field r* ∧ *b1* ∈ *Field r* ∧ *b2* ∈ *Field r*
using *Field-bsqr* **by** *blast*
have ((*a1*,*a2*) = (*b1*,*b2*) ∨ ((*a1*,*a2*),(*b1*,*b2*)) ∈ *bsqr r* ∨ ((*b1*,*b2*),(*a1*,*a2*)) ∈ *bsqr r*)
proof(*rule wo-rel.cases-Total*[*of r a1 a2*], *clarsimp simp add*: *Well*, *simp add*: *0*)

assume *Case1*: (*a1*,*a2*) ∈ *r*
hence *1*: *wo-rel.max2 r a1 a2* = *a2*
using *Well 0* **by** (*simp add*: *wo-rel.max2-equals2*)
show *?thesis*
proof(*rule wo-rel.cases-Total*[*of r b1 b2*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case11*: (*b1*,*b2*) ∈ *r*
hence *2*: *wo-rel.max2 r b1 b2* = *b2*
using *Well 0* **by** (*simp add*: *wo-rel.max2-equals2*)
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a2 b2*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case111*: (*a2*,*b2*) ∈ *r* − *Id* ∨ (*b2*,*a2*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 2* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case112*: *a2* = *b2*
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a1 b1*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case1121*: (*a1*,*b1*) ∈ *r* − *Id* ∨ (*b1*,*a1*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 2 Case112* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case1122*: *a1* = *b1*
thus *?thesis* **using** *Case112* **by** *auto*
qed
qed
next
assume *Case12*: (*b2*,*b1*) ∈ *r*
hence *3*: *wo-rel.max2 r b1 b2* = *b1* **using** *Well 0* **by** (*simp add*: *wo-rel.max2-equals1*)
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a2 b1*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case121*: (*a2*,*b1*) ∈ *r* − *Id* ∨ (*b1*,*a2*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 3* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case122*: *a2* = *b1*
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a1 b1*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case1221*: (*a1*,*b1*) ∈ *r* − *Id* ∨ (*b1*,*a1*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 3 Case122* **unfolding** *bsqr-def* **by** *auto*
next

assume *Case1222*: *a1* = *b1*
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a2 b2*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case12221*: (*a2*,*b2*) ∈ *r* − *Id* ∨ (*b2*,*a2*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 3 Case122 Case1222* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case12222*: *a2* = *b2*
thus *?thesis* **using** *Case122 Case1222* **by** *auto*
qed
qed
qed
qed
next
assume *Case2*: (*a2*,*a1*) ∈ *r*
hence *1*: *wo-rel.max2 r a1 a2* = *a1* **using** *Well 0* **by** (*simp add*: *wo-rel.max2-equals1*)
show *?thesis*
proof(*rule wo-rel.cases-Total*[*of r b1 b2*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case21*: (*b1*,*b2*) ∈ *r*
hence *2*: *wo-rel.max2 r b1 b2* = *b2* **using** *Well 0* **by** (*simp add*: *wo-rel.max2-equals2*)
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a1 b2*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case211*: (*a1*,*b2*) ∈ *r* − *Id* ∨ (*b2*,*a1*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 2* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case212*: *a1* = *b2*
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a1 b1*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case2121*: (*a1*,*b1*) ∈ *r* − *Id* ∨ (*b1*,*a1*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 2 Case212* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case2122*: *a1* = *b1*
show *?thesis*
proof(*rule wo-rel.cases-Total3*[*of r a2 b2*], *clarsimp simp add*: *Well*, *simp add*: *0*)
assume *Case21221*: (*a2*,*b2*) ∈ *r* − *Id* ∨ (*b2*,*a2*) ∈ *r* − *Id*
thus *?thesis* **using** *0 1 2 Case212 Case2122* **unfolding** *bsqr-def* **by** *auto*
next
assume *Case21222*: *a2* = *b2*
thus *?thesis* **using** *Case2122 Case212* **by** *auto*
qed
qed
qed
next

```
      assume Case22: (b2,b1) ∈ r
        hence 3: wo-rel.max2 r b1 b2 = b1  using Well 0 by (simp add:
wo-rel.max2-equals1)
      show ?thesis
      proof(rule wo-rel.cases-Total3[of r a1 b1], clarsimp simp add: Well, simp
add: 0)
        assume Case221: (a1,b1) ∈ r − Id ∨ (b1,a1) ∈ r − Id
        thus ?thesis using 0 1 3 unfolding bsqr-def by auto
      next
        assume Case222: a1 = b1
        show ?thesis
        proof(rule wo-rel.cases-Total3[of r a2 b2], clarsimp simp add: Well, simp
add: 0)
          assume Case2221: (a2,b2) ∈ r − Id ∨ (b2,a2) ∈ r − Id
          thus ?thesis using 0 1 3 Case222 unfolding bsqr-def by auto
        next
          assume Case2222: a2 = b2
          thus ?thesis using Case222 by auto
        qed
      qed
    qed
  qed
 }
 thus ?thesis unfolding total-on-def by fast
qed

lemma bsqr-Linear-order:
 assumes Well-order r
 shows Linear-order(bsqr r)
 unfolding order-on-defs
 using assms bsqr-Refl bsqr-Trans bsqr-antisym bsqr-Total by blast

lemma bsqr-Well-order:
 assumes Well-order r
 shows Well-order(bsqr r)
 using assms
proof(simp add: bsqr-Linear-order Linear-order-Well-order-iff, intro allI impI)
 have 0: ∀ A ≤ Field r. A ≠ {} ⟶ (∃ a ∈ A. ∀ a′ ∈ A. (a,a′) ∈ r)
   using assms well-order-on-def Linear-order-Well-order-iff by blast
 fix D assume ∗: D ≤ Field (bsqr r) and ∗∗: D ≠ {}
 hence 1: D ≤ Field r × Field r unfolding Field-bsqr by simp

 obtain M where M-def: M = {wo-rel.max2 r a1 a2| a1 a2. (a1,a2) ∈ D} by
blast
 have M ≠ {} using 1 M-def ∗∗ by auto
 moreover
 have M ≤ Field r unfolding M-def
   using 1 assms wo-rel-def[of r] wo-rel.max2-among[of r] by fastforce
 ultimately obtain m where m-min: m ∈ M ∧ (∀ a ∈ M. (m,a) ∈ r)
```

using *0* **by** *blast*

obtain *A1* **where** *A1-def*: $A1 = \{a1.\ \exists a2.\ (a1,a2) \in D \wedge wo\text{-}rel.max2\ r\ a1\ a2 = m\}$ **by** *blast*
have $A1 \leq Field\ r$ **unfolding** *A1-def* **using** *1* **by** *auto*
moreover have $A1 \neq \{\}$ **unfolding** *A1-def* **using** *m-min* **unfolding** *M-def* **by** *blast*
ultimately obtain *a1* **where** *a1-min*: $a1 \in A1 \wedge (\forall a \in A1.\ (a1,a) \in r)$
using *0* **by** *blast*

obtain *A2* **where** *A2-def*: $A2 = \{a2.\ (a1,a2) \in D \wedge wo\text{-}rel.max2\ r\ a1\ a2 = m\}$ **by** *blast*
have $A2 \leq Field\ r$ **unfolding** *A2-def* **using** *1* **by** *auto*
moreover have $A2 \neq \{\}$ **unfolding** *A2-def*
using *m-min a1-min* **unfolding** *A1-def M-def* **by** *blast*
ultimately obtain *a2* **where** *a2-min*: $a2 \in A2 \wedge (\forall a \in A2.\ (a2,a) \in r)$
using *0* **by** *blast*

have *2*: $wo\text{-}rel.max2\ r\ a1\ a2 = m$
using *a1-min a2-min* **unfolding** *A1-def A2-def* **by** *auto*
have *3*: $(a1,a2) \in D$ **using** *a2-min* **unfolding** *A2-def* **by** *auto*

moreover
{fix *b1 b2* **assume** $***$: $(b1,b2) \in D$
hence *4*: $\{a1,a2,b1,b2\} \leq Field\ r$ **using** *1 3* **by** *blast*
have *5*: $(wo\text{-}rel.max2\ r\ a1\ a2,\ wo\text{-}rel.max2\ r\ b1\ b2) \in r$
using $***$ *a1-min a2-min m-min* **unfolding** *A1-def A2-def M-def* **by** *auto*
have $((a1,a2),(b1,b2)) \in bsqr\ r$
proof(*cases* $wo\text{-}rel.max2\ r\ a1\ a2 = wo\text{-}rel.max2\ r\ b1\ b2$)
assume *Case1*: $wo\text{-}rel.max2\ r\ a1\ a2 \neq wo\text{-}rel.max2\ r\ b1\ b2$
thus *?thesis* **unfolding** *bsqr-def* **using** *4 5* **by** *auto*
next
assume *Case2*: $wo\text{-}rel.max2\ r\ a1\ a2 = wo\text{-}rel.max2\ r\ b1\ b2$
hence $b1 \in A1$ **unfolding** *A1-def* **using** *2* $***$ **by** *auto*
hence *6*: $(a1,b1) \in r$ **using** *a1-min* **by** *auto*
show *?thesis*
proof(*cases* $a1 = b1$)
assume *Case21*: $a1 \neq b1$
thus *?thesis* **unfolding** *bsqr-def* **using** *4 Case2 6* **by** *auto*
next
assume *Case22*: $a1 = b1$
hence $b2 \in A2$ **unfolding** *A2-def* **using** *2* $***$ *Case2* **by** *auto*
hence *7*: $(a2,b2) \in r$ **using** *a2-min* **by** *auto*
thus *?thesis* **unfolding** *bsqr-def* **using** *4 7 Case2 Case22* **by** *auto*
qed
qed
}

ultimately show $\exists d \in D.\ \forall d' \in D.\ (d,d') \in bsqr\ r$ **by** *fastforce*

qed

lemma *bsqr-max2*:
  assumes *WELL*: *Well-order r* **and** *LEQ*: ((*a1*,*a2*),(*b1*,*b2*)) ∈ *bsqr r*
  shows (*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*) ∈ *r*
proof−
  have {(*a1*,*a2*),(*b1*,*b2*)} ≤ *Field*(*bsqr r*)
    using *LEQ* **unfolding** *Field-def* **by** *auto*
  hence {*a1*,*a2*,*b1*,*b2*} ≤ *Field r* **unfolding** *Field-bsqr* **by** *auto*
  hence {*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*} ≤ *Field r*
    using *WELL wo-rel-def*[*of r*] *wo-rel.max2-among*[*of r*] **by** *fastforce*
  moreover have (*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*) ∈ *r* ∨ *wo-rel.max2 r a1 a2* = *wo-rel.max2 r b1 b2*
    using *LEQ* **unfolding** *bsqr-def* **by** *auto*
  ultimately show *?thesis* **using** *WELL* **unfolding** *order-on-defs refl-on-def* **by** *auto*
qed

lemma *bsqr-ofilter*:
  assumes *WELL*: *Well-order r* **and**
    OF: *wo-rel.ofilter* (*bsqr r*) *D* **and** *SUB*: *D* < *Field r* × *Field r* **and**
    NE: ¬ (∃ *a*. *Field r* = *under r a*)
  shows ∃ *A*. *wo-rel.ofilter r A* ∧ *A* < *Field r* ∧ *D* ≤ *A* × *A*
proof−
  let *?r′* = *bsqr r*
  have *Well*: *wo-rel r* **using** *WELL wo-rel-def* **by** *blast*
  hence *Trans*: *trans r* **using** *wo-rel.TRANS* **by** *blast*
  have *Well′*: *Well-order ?r′* ∧ *wo-rel ?r′*
    using *WELL bsqr-Well-order wo-rel-def* **by** *blast*

  have *D* < *Field ?r′* **unfolding** *Field-bsqr* **using** *SUB* **.**
  with *OF* **obtain** *a1* **and** *a2* **where**
    (*a1*,*a2*) ∈ *Field ?r′* **and** *1*: *D* = *underS ?r′* (*a1*,*a2*)
    using *Well′ wo-rel.ofilter-underS-Field*[*of ?r′ D*] **by** *auto*
  hence *2*: {*a1*,*a2*} ≤ *Field r* **unfolding** *Field-bsqr* **by** *auto*
  let *?m* = *wo-rel.max2 r a1 a2*
  have *D* ≤ (*under r ?m*) × (*under r ?m*)
  proof(*unfold 1*)
    {**fix** *b1 b2*
      let *?n* = *wo-rel.max2 r b1 b2*
      assume (*b1*,*b2*) ∈ *underS ?r′* (*a1*,*a2*)
      hence *3*: ((*b1*,*b2*),(*a1*,*a2*)) ∈ *?r′*
        unfolding *underS-def* **by** *blast*
      hence (*?n*,*?m*) ∈ *r* **using** *WELL* **by** (*simp add*: *bsqr-max2*)
      moreover
      {**have** (*b1*,*b2*) ∈ *Field ?r′* **using** *3* **unfolding** *Field-def* **by** *auto*
        hence {*b1*,*b2*} ≤ *Field r* **unfolding** *Field-bsqr* **by** *auto*
        hence (*b1*,*?n*) ∈ *r* ∧ (*b2*,*?n*) ∈ *r*
          using *Well* **by** (*simp add*: *wo-rel.max2-greater*)

```
      }
      ultimately have (b1,?m) ∈ r ∧ (b2,?m) ∈ r
        using Trans trans-def[of r] by blast
       hence (b1,b2) ∈ (under r ?m) × (under r ?m) unfolding under-def by
simp}
    thus underS ?r' (a1,a2) ≤ (under r ?m) × (under r ?m) by auto
  qed
  moreover have wo-rel.ofilter r (under r ?m)
    using Well by (simp add: wo-rel.under-ofilter)
  moreover have under r ?m < Field r
    using NE under-Field[of r ?m] by blast
  ultimately show ?thesis by blast
qed

definition Func where
  Func A B = {f . (∀ a ∈ A. f a ∈ B) ∧ (∀ a. a ∉ A ⟶ f a = undefined)}

lemma Func-empty:
  Func {} B = {λx. undefined}
  unfolding Func-def by auto

lemma Func-elim:
  assumes g ∈ Func A B and a ∈ A
  shows ∃ b. b ∈ B ∧ g a = b
  using assms unfolding Func-def by (cases g a = undefined) auto

definition curr where
  curr A f ≡ λ a. if a ∈ A then λb. f (a,b) else undefined

lemma curr-in:
  assumes f: f ∈ Func (A × B) C
  shows curr A f ∈ Func A (Func B C)
  using assms unfolding curr-def Func-def by auto

lemma curr-inj:
  assumes f1 ∈ Func (A × B) C and f2 ∈ Func (A × B) C
  shows curr A f1 = curr A f2 ⟷ f1 = f2
proof safe
  assume c: curr A f1 = curr A f2
  show f1 = f2
  proof (rule ext, clarify)
    fix a b show f1 (a, b) = f2 (a, b)
    proof (cases (a,b) ∈ A × B)
      case False
      thus ?thesis using assms unfolding Func-def by auto
    next
      case True hence a: a ∈ A and b: b ∈ B by auto
      thus ?thesis
        using c unfolding curr-def fun-eq-iff by(elim allE[of - a]) simp
```

    qed
  qed
qed

lemma *curr-surj*:
  assumes $g \in Func\ A\ (Func\ B\ C)$
  shows $\exists\ f \in Func\ (A \times B)\ C.\ curr\ A\ f = g$
proof
  let $?f = \lambda\ ab.\ if\ fst\ ab \in A \wedge snd\ ab \in B\ then\ g\ (fst\ ab)\ (snd\ ab)\ else\ undefined$
  show $curr\ A\ ?f = g$
  proof (*rule ext*)
    fix *a* **show** $curr\ A\ ?f\ a = g\ a$
    proof ($cases\ a \in A$)
      case *False*
      hence $g\ a = undefined$ **using** *assms* **unfolding** *Func-def* **by** *auto*
      thus *?thesis* **unfolding** *curr-def* **using** *False* **by** *simp*
    next
      case *True*
      obtain *g1* **where** $g1 \in Func\ B\ C$ **and** $g\ a = g1$
        using *assms* **using** *Func-elim*[*OF assms True*] **by** *blast*
      thus *?thesis* **using** *True* **unfolding** *Func-def curr-def* **by** *auto*
    qed
  qed
  show $?f \in Func\ (A \times B)\ C$ **using** *assms* **unfolding** *Func-def mem-Collect-eq*
by *auto*
qed

lemma *bij-betw-curr*:
  $bij\text{-}betw\ (curr\ A)\ (Func\ (A \times B)\ C)\ (Func\ A\ (Func\ B\ C))$
  unfolding *bij-betw-def inj-on-def*
  using *curr-surj curr-in curr-inj* **by** *blast*

definition *Func-map* **where**
  $Func\text{-}map\ B2\ f1\ f2\ g\ b2 \equiv if\ b2 \in B2\ then\ f1\ (g\ (f2\ b2))\ else\ undefined$

lemma *Func-map*:
  assumes g: $g \in Func\ A2\ A1$ **and** $f1$: $f1\ `\ A1 \subseteq B1$ **and** $f2$: $f2\ `\ B2 \subseteq A2$
  shows $Func\text{-}map\ B2\ f1\ f2\ g \in Func\ B2\ B1$
  using *assms* **unfolding** *Func-def Func-map-def mem-Collect-eq* **by** *auto*

lemma *Func-non-emp*:
  assumes $B \neq \{\}$
  shows $Func\ A\ B \neq \{\}$
proof−
  obtain *b* **where** *b*: $b \in B$ **using** *assms* **by** *auto*
  hence $(\lambda\ a.\ if\ a \in A\ then\ b\ else\ undefined) \in Func\ A\ B$ **unfolding** *Func-def* **by**
auto
  thus *?thesis* **by** *blast*
qed

lemma *Func-is-emp*:
  Func A B = {} ⟷ *A* ≠ {} ∧ *B* = {} (**is** *?L* ⟷ *?R*)
proof
  assume *?L*
  then show *?R*
    using *Func-empty Func-non-emp*[*of B A*] **by** *blast*
next
  assume *?R*
  then show *?L*
    using *Func-empty Func-non-emp*[*of B A*] **by** (*auto simp*: *Func-def*)
qed

lemma *Func-map-surj*:
  assumes *B1*: *f1* ' *A1* = *B1* **and** *A2*: *inj-on f2 B2 f2* ' *B2* ⊆ *A2*
    and *B2A2*: *B2* = {} ⟹ *A2* = {}
  shows *Func B2 B1* = *Func-map B2 f1 f2* ' *Func A2 A1*
proof(*cases B2* = {})
  case *True*
  thus *?thesis* **using** *B2A2* **by** (*auto simp*: *Func-empty Func-map-def*)
next
  case *False* **note** *B2* = *False*
  show *?thesis*
  proof *safe*
    fix *h* **assume** *h*: *h* ∈ *Func B2 B1*
    define *j1* **where** *j1* = *inv-into A1 f1*
    have ∀ *a2* ∈ *f2* ' *B2*. ∃ *b2*. *b2* ∈ *B2* ∧ *f2 b2* = *a2* **by** *blast*
    then obtain *k* **where** *k*: ∀ *a2* ∈ *f2* ' *B2*. *k a2* ∈ *B2* ∧ *f2* (*k a2*) = *a2*
      by *atomize-elim* (*rule bchoice*)
    {**fix** *b2* **assume** *b2*: *b2* ∈ *B2*
      hence *f2* (*k* (*f2 b2*)) = *f2 b2* **using** *k A2*(*2*) **by** *auto*
      moreover have *k* (*f2 b2*) ∈ *B2* **using** *b2 A2*(*2*) *k* **by** *auto*
      ultimately have *k* (*f2 b2*) = *b2* **using** *b2 A2*(*1*) **unfolding** *inj-on-def* **by** *blast*
    } **note** *kk* = *this*
    obtain *b22* **where** *b22*: *b22* ∈ *B2* **using** *B2* **by** *auto*
    define *j2* **where** [*abs-def*]: *j2 a2* = (*if a2* ∈ *f2* ' *B2 then k a2 else b22*) **for** *a2*
    have *j2A2*: *j2* ' *A2* ⊆ *B2* **unfolding** *j2-def* **using** *k b22* **by** *auto*
    have *j2*: ⋀ *b2*. *b2* ∈ *B2* ⟹ *j2* (*f2 b2*) = *b2*
      using *kk* **unfolding** *j2-def* **by** *auto*
    define *g* **where** *g* = *Func-map A2 j1 j2 h*
    have *Func-map B2 f1 f2 g* = *h*
    proof (*rule ext*)
      fix *b2* **show** *Func-map B2 f1 f2 g b2* = *h b2*
      proof(*cases b2* ∈ *B2*)
        case *True*
        show *?thesis*
        proof (*cases h b2* = *undefined*)
          case *True*

hence *b1*: *h b2* ∈ *f1* ' *A1* **using** *h* ‹*b2* ∈ *B2*› **unfolding** *B1 Func-def* **by** *auto*
show *?thesis* **using** *A2 f-inv-into-f*[*OF b1*]
unfolding *True g-def Func-map-def j1-def j2*[*OF* ‹*b2* ∈ *B2*›] **by** *auto*
qed(*insert A2 True j2*[*OF True*] *h B1*, *unfold j1-def g-def Func-def Func-map-def*,
auto intro: *f-inv-into-f*)
qed(*insert h*, *unfold Func-def Func-map-def*, *auto*)
qed
moreover have *g* ∈ *Func A2 A1* **unfolding** *g-def* **apply**(*rule Func-map*[*OF h*])
using *j2A2 B1 A2* **unfolding** *j1-def* **by** (*fast intro*: *inv-into-into*)+
ultimately show *h* ∈ *Func-map B2 f1 f2* ' *Func A2 A1*
unfolding *Func-map-def*[*abs-def*] **by** *auto*
qed(*use B1 Func-map*[*OF - - A2*(*2*)] **in** *auto*)
qed

end

30 Cardinal-Order Relations as Needed by Bounded Natural Functors

theory *BNF-Cardinal-Order-Relation*
imports *Zorn BNF-Wellorder-Constructions*
begin

In this section, we define cardinal-order relations to be minim well-orders on their field. Then we define the cardinal of a set to be *some* cardinal-order relation on that set, which will be unique up to order isomorphism. Then we study the connection between cardinals and:

- standard set-theoretic constructions: products, sums, unions, lists, powersets, set-of finite sets operator;
- finiteness and infiniteness (in particular, with the numeric cardinal operator for finite sets, *card*, from the theory *Finite-Sets.thy*).

On the way, we define the canonical ω cardinal and finite cardinals. We also define (again, up to order isomorphism) the successor of a cardinal, and show that any cardinal admits a successor.

Main results of this section are the existence of cardinal relations and the facts that, in the presence of infiniteness, most of the standard set-theoretic constructions (except for the powerset) *do not increase cardinality*. In particular, e.g., the set of words/lists over any infinite set has the same cardinality (hence, is in bijection) with that set.

30.1 Cardinal orders

A cardinal order in our setting shall be a well-order *minim* w.r.t. the order-embedding relation, $\leq o$ (which is the same as being *minimal* w.r.t. the strict order-embedding relation, $<o$), among all the well-orders on its field.

definition *card-order-on* :: *'a set* ⇒ *'a rel* ⇒ *bool*
  where
    card-order-on A r ≡ *well-order-on A r* ∧ (∀ *r'*. *well-order-on A r'* ⟶ *r* ≤*o r'*)

abbreviation *Card-order r* ≡ *card-order-on* (*Field r*) *r*
abbreviation *card-order r* ≡ *card-order-on UNIV r*

lemma *card-order-on-well-order-on*:
  assumes *card-order-on A r*
  shows *well-order-on A r*
  using *assms* **unfolding** *card-order-on-def* **by** *simp*

lemma *card-order-on-Card-order*:
  card-order-on A r ⟹ *A* = *Field r* ∧ *Card-order r*
  unfolding *card-order-on-def* **using** *well-order-on-Field* **by** *blast*

The existence of a cardinal relation on any given set (which will mean that any set has a cardinal) follows from two facts:

- Zermelo's theorem (proved in *Zorn.thy* as theorem *well-order-on*), which states that on any given set there exists a well-order;
- The well-founded-ness of $<o$, ensuring that then there exists a minimal such well-order, i.e., a cardinal order.

theorem *card-order-on*: ∃ *r*. *card-order-on A r*
proof –
  define *R* **where** *R* ≡ {*r*. *well-order-on A r*}
  have *R* ≠ {} ∧ (∀ *r* ∈ *R*. *Well-order r*)
    using *well-order-on*[*of A*] *R-def well-order-on-Well-order* **by** *blast*
  with *exists-minim-Well-order*[*of R*] **show** *?thesis*
    by (*auto simp*: *R-def card-order-on-def*)
qed

lemma *card-order-on-ordIso*:
  assumes *CO*: *card-order-on A r* **and** *CO'*: *card-order-on A r'*
  shows *r* =*o r'*
  using *assms* **unfolding** *card-order-on-def*
  using *ordIso-iff-ordLeq* **by** *blast*

lemma *Card-order-ordIso*:
  assumes *CO*: *Card-order r* **and** *ISO*: *r'* =*o r*
  shows *Card-order r'*
  using *ISO* **unfolding** *ordIso-def*

proof(*unfold card-order-on-def*, *auto*)
  fix *p′* **assume** *well-order-on* (*Field r′*) *p′*
  hence *0*: *Well-order p′* ∧ *Field p′* = *Field r′*
    using *well-order-on-Well-order* **by** *blast*
  obtain *f* **where** *1*: *iso r′ r f* **and** *2*: *Well-order r* ∧ *Well-order r′*
    using *ISO* **unfolding** *ordIso-def* **by** *auto*
  hence *3*: *inj-on f* (*Field r′*) ∧ *f* ‘ (*Field r′*) = *Field r*
    by (*auto simp add*: *iso-iff embed-inj-on*)
  let *?p* = *dir-image p′ f*
  have *4*: *p′* =*o ?p* ∧ *Well-order ?p*
    using *0 2 3* **by** (*auto simp add*: *dir-image-ordIso Well-order-dir-image*)
  moreover have *Field ?p* = *Field r*
    using *0 3* **by** (*auto simp add*: *dir-image-Field*)
  ultimately have *well-order-on* (*Field r*) *?p* **by** *auto*
  hence *r* ≤*o ?p* **using** *CO* **unfolding** *card-order-on-def* **by** *auto*
  thus *r′* ≤*o p′*
   using *ISO 4 ordLeq-ordIso-trans ordIso-ordLeq-trans ordIso-symmetric* **by** *blast*
qed

lemma *Card-order-ordIso2*:
  assumes *CO*: *Card-order r* **and** *ISO*: *r* =*o r′*
  shows *Card-order r′*
  using *assms Card-order-ordIso ordIso-symmetric* **by** *blast*

30.2 Cardinal of a set

We define the cardinal of set to be *some* cardinal order on that set. We shall prove that this notion is unique up to order isomorphism, meaning that order isomorphism shall be the true identity of cardinals.

definition *card-of* :: *′a set* ⇒ *′a rel* (‹(‹*open-block notation*=‹*mixfix card-of*››|-|)›)
  where *card-of A* = (*SOME r*. *card-order-on A r*)

lemma *card-of-card-order-on*: *card-order-on A* |*A*|
  unfolding *card-of-def* **by** (*auto simp add*: *card-order-on someI-ex*)

lemma *card-of-well-order-on*: *well-order-on A* |*A*|
  using *card-of-card-order-on card-order-on-def* **by** *blast*

lemma *Field-card-of*: *Field* |*A*| = *A*
  using *card-of-card-order-on*[*of A*] **unfolding** *card-order-on-def*
  using *well-order-on-Field* **by** *blast*

lemma *card-of-Card-order*: *Card-order* |*A*|
  by (*simp only*: *card-of-card-order-on Field-card-of*)

corollary *ordIso-card-of-imp-Card-order*:
  r =*o* |*A*| ⟹ *Card-order r*
  using *card-of-Card-order Card-order-ordIso* **by** *blast*

lemma *card-of-Well-order*: *Well-order* |*A*|
  using *card-of-Card-order* **unfolding** *card-order-on-def* **by** *auto*

lemma *card-of-refl*: |*A*| =*o* |*A*|
  using *card-of-Well-order ordIso-reflexive* **by** *blast*

lemma *card-of-least*: *well-order-on A r* ⟹ |*A*| ≤*o r*
  using *card-of-card-order-on* **unfolding** *card-order-on-def* **by** *blast*

lemma *card-of-ordIso*:
  (∃*f*. *bij-betw f A B*) = (|*A*| =*o* |*B*|)
proof(*auto*)
  fix *f* **assume** ∗: *bij-betw f A B*
  then obtain *r* **where** *well-order-on B r* ∧ |*A*| =*o r*
    using *Well-order-iso-copy card-of-well-order-on* **by** *blast*
  hence |*B*| ≤*o* |*A*| **using** *card-of-least*
      ordLeq-ordIso-trans ordIso-symmetric **by** *blast*
  moreover
  {let *?g* = *inv-into A f*
    have *bij-betw ?g B A* **using** ∗ *bij-betw-inv-into* **by** *blast*
    then obtain *r* **where** *well-order-on A r* ∧ |*B*| =*o r*
      using *Well-order-iso-copy card-of-well-order-on* **by** *blast*
    hence |*A*| ≤*o* |*B*|
      using *card-of-least ordLeq-ordIso-trans ordIso-symmetric* **by** *blast*
  }
  ultimately show |*A*| =*o* |*B*| **using** *ordIso-iff-ordLeq* **by** *blast*
next
  assume |*A*| =*o* |*B*|
  then obtain *f* **where** *iso* (|*A*|) (|*B*|) *f*
    unfolding *ordIso-def* **by** *auto*
  hence *bij-betw f A B* **unfolding** *iso-def Field-card-of* **by** *simp*
  thus ∃*f*. *bij-betw f A B* **by** *auto*
qed

lemma *card-of-ordLeq*:
  (∃*f*. *inj-on f A* ∧ *f* ' *A* ≤ *B*) = (|*A*| ≤*o* |*B*|)
proof(*auto*)
  fix *f* **assume** ∗: *inj-on f A* **and** ∗∗: *f* ' *A* ≤ *B*
  {assume |*B*| <*o* |*A*|
    hence |*B*| ≤*o* |*A*| **using** *ordLeq-iff-ordLess-or-ordIso* **by** *blast*
    then obtain *g* **where** *embed* (|*B*|) (|*A*|) *g*
      unfolding *ordLeq-def* **by** *auto*
    hence *1*: *inj-on g B* ∧ *g* ' *B* ≤ *A* **using** *embed-inj-on*[*of* |*B*| |*A*| *g*]
        card-of-Well-order[*of B*] *Field-card-of*[*of B*] *Field-card-of*[*of A*]
        embed-Field[*of* |*B*| |*A*| *g*] **by** *auto*
    obtain *h* **where** *bij-betw h A B*
      using ∗ ∗∗ *1 Schroeder-Bernstein*[*of f*] **by** *fastforce*
    hence |*A*| ≤*o* |*B*| **using** *card-of-ordIso ordIso-iff-ordLeq* **by** *auto*
  }

thus $|A| \le o\ |B|$ **using** *ordLess-or-ordLeq*[*of* $|B|$ $|A|$]
by (*auto simp*: *card-of-Well-order*)
next
assume $*$: $|A| \le o\ |B|$
obtain f **where** *embed* $|A|$ $|B|$ f
using $*$ **unfolding** *ordLeq-def* **by** *auto*
hence *inj-on* $f\ A \wedge f$ ' $A \le B$
using *embed-inj-on*[*of* $|A|$ $|B|$] *card-of-Well-order* *embed-Field*[*of* $|A|$ $|B|$]
by (*auto simp*: *Field-card-of*)
thus $\exists f.$ *inj-on* $f\ A \wedge f$ ' $A \le B$ **by** *auto*
qed

lemma *card-of-ordLeq2*:
$A \neq \{\} \Longrightarrow (\exists g.\ g$ ' $B = A) = (\ |A| \le o\ |B|\ )$
using *card-of-ordLeq*[*of* A B] *inj-on-iff-surj*[*of* A B] **by** *auto*

lemma *card-of-ordLess*:
$(\neg(\exists f.$ *inj-on* $f\ A \wedge f$ ' $A \le B)) = (\ |B| <o\ |A|\ )$
proof −
have $(\neg(\exists f.$ *inj-on* $f\ A \wedge f$ ' $A \le B)) = (\neg\ |A| \le o\ |B|\ )$
using *card-of-ordLeq* **by** *blast*
also have ... $= (\ |B| <o\ |A|\ )$
using *not-ordLeq-iff-ordLess* **by** (*auto intro*: *card-of-Well-order*)
finally show *?thesis* **.**
qed

lemma *card-of-ordLess2*:
$B \neq \{\} \Longrightarrow (\neg(\exists f.\ f$ ' $A = B)) = (\ |A| <o\ |B|\ )$
using *card-of-ordLess*[*of* B A] *inj-on-iff-surj*[*of* B A] **by** *auto*

lemma *card-of-ordIsoI*:
assumes *bij-betw* $f\ A\ B$
shows $|A| =o\ |B|$
using *assms* **unfolding** *card-of-ordIso*[*symmetric*] **by** *auto*

lemma *card-of-ordLeqI*:
assumes *inj-on* $f\ A$ **and** $\bigwedge a.\ a \in A \Longrightarrow f\ a \in B$
shows $|A| \le o\ |B|$
using *assms* **unfolding** *card-of-ordLeq*[*symmetric*] **by** *auto*

lemma *card-of-unique*:
card-order-on $A\ r \Longrightarrow r =o\ |A|$
by (*simp only*: *card-order-on-ordIso card-of-card-order-on*)

lemma *card-of-mono1*:
$A \le B \Longrightarrow |A| \le o\ |B|$
using *inj-on-id*[*of* A] *card-of-ordLeq*[*of* A B] **by** *fastforce*

lemma *card-of-mono2*:

assumes $r \le o\ r'$
shows $|Field\ r| \le o\ |Field\ r'|$
proof –
obtain f **where**
1: *well-order-on* (*Field r*) *r* ∧ *well-order-on* (*Field r*) *r* ∧ *embed r r' f*
using *assms* **unfolding** *ordLeq-def*
by (*auto simp add*: *well-order-on-Well-order*)
hence *inj-on f* (*Field r*) ∧ *f* ' (*Field r*) ≤ *Field r'*
by (*auto simp add*: *embed-inj-on embed-Field*)
thus $|Field\ r| \le o\ |Field\ r'|$ **using** *card-of-ordLeq* **by** *blast*
qed

lemma *card-of-cong*: $r =o\ r' \Longrightarrow |Field\ r| =o\ |Field\ r'|$
by (*simp add*: *ordIso-iff-ordLeq card-of-mono2*)

lemma *card-of-Field-ordIso*:
assumes *Card-order r*
shows $|Field\ r| =o\ r$
proof –
have *card-order-on* (*Field r*) *r*
using *assms card-order-on-Card-order* **by** *blast*
moreover have *card-order-on* (*Field r*) $|Field\ r|$
using *card-of-card-order-on* **by** *blast*
ultimately show *?thesis* **using** *card-order-on-ordIso* **by** *blast*
qed

lemma *Card-order-iff-ordIso-card-of*:
Card-order r = ($r =o\ |Field\ r|$)
using *ordIso-card-of-imp-Card-order card-of-Field-ordIso ordIso-symmetric* **by** *blast*

lemma *Card-order-iff-ordLeq-card-of*:
Card-order r = ($r \le o\ |Field\ r|$)
proof –
have *Card-order r* = ($r =o\ |Field\ r|$)
unfolding *Card-order-iff-ordIso-card-of* **by** *simp*
also have ... = ($r \le o\ |Field\ r| \wedge |Field\ r| \le o\ r$)
unfolding *ordIso-iff-ordLeq* **by** *simp*
also have ... = ($r \le o\ |Field\ r|$)
using *card-of-least*
by (*auto simp*: *card-of-least ordLeq-Well-order-simp*)
finally show *?thesis* **.**
qed

lemma *Card-order-iff-Restr-underS*:
assumes *Well-order r*
shows *Card-order r* = (∀ *a* ∈ *Field r*. *Restr r* (*underS r a*) $<o\ |Field\ r|$)
using *assms ordLeq-iff-ordLess-Restr card-of-Well-order*
unfolding *Card-order-iff-ordLeq-card-of* **by** *blast*

lemma *card-of-underS*:
  assumes *r*: *Card-order r* **and** *a*: *a* ∈ *Field r*
  shows |*underS r a*| <*o r*
proof –
  let *?A* = *underS r a* **let** *?r′* = *Restr r ?A*
  have *1*: *Well-order r*
    using *r* **unfolding** *card-order-on-def* **by** *simp*
  have *Well-order ?r′* **using** *1 Well-order-Restr* **by** *auto*
  with *card-of-card-order-on* **have** |*Field ?r′*| ≤*o ?r′*
    unfolding *card-order-on-def* **by** *auto*
  moreover have *Field ?r′* = *?A*
    using *1 wo-rel.underS-ofilter Field-Restr-ofilter*
    unfolding *wo-rel-def* **by** *fastforce*
  ultimately have |*?A*| ≤*o ?r′* **by** *simp*
  also have *?r′* <*o* |*Field r*|
    using *1 a r Card-order-iff-Restr-underS* **by** *blast*
  also have |*Field r*| =*o r*
    using *r ordIso-symmetric* **unfolding** *Card-order-iff-ordIso-card-of* **by** *auto*
  finally show *?thesis* **.**
qed

lemma *ordLess-Field*:
  assumes *r* <*o r′*
  shows |*Field r*| <*o r′*
proof –
  have *well-order-on* (*Field r*) *r* **using** *assms* **unfolding** *ordLess-def*
    by (*auto simp add*: *well-order-on-Well-order*)
  hence |*Field r*| ≤*o r* **using** *card-of-least* **by** *blast*
  thus *?thesis* **using** *assms ordLeq-ordLess-trans* **by** *blast*
qed

lemma *internalize-card-of-ordLeq*:
  (|*A*| ≤*o r*) = (∃ *B* ≤ *Field r*. |*A*| =*o* |*B*| ∧ |*B*| ≤*o r*)
proof
  assume |*A*| ≤*o r*
  then obtain *p* **where** *1*: *Field p* ≤ *Field r* ∧ |*A*| =*o p* ∧ *p* ≤*o r*
    using *internalize-ordLeq*[*of* |*A*| *r*] **by** *blast*
  hence *Card-order p* **using** *card-of-Card-order Card-order-ordIso2* **by** *blast*
  hence |*Field p*| =*o p* **using** *card-of-Field-ordIso* **by** *blast*
  hence |*A*| =*o* |*Field p*| ∧ |*Field p*| ≤*o r*
    using *1 ordIso-equivalence ordIso-ordLeq-trans* **by** *blast*
  thus ∃ *B* ≤ *Field r*. |*A*| =*o* |*B*| ∧ |*B*| ≤*o r* **using** *1* **by** *blast*
next
  assume ∃ *B* ≤ *Field r*. |*A*| =*o* |*B*| ∧ |*B*| ≤*o r*
  thus |*A*| ≤*o r* **using** *ordIso-ordLeq-trans* **by** *blast*
qed

lemma *internalize-card-of-ordLeq2*:

($|A| \leq o\ |C|$) = ($\exists B \leq C.\ |A| =o\ |B| \wedge |B| \leq o\ |C|$)
using *internalize-card-of-ordLeq*[*of A* $|C|$] *Field-card-of*[*of C*] **by** *auto*

30.3 Cardinals versus set operations on arbitrary sets

Here we embark in a long journey of simple results showing that the standard set-theoretic operations are well-behaved w.r.t. the notion of cardinal – essentially, this means that they preserve the "cardinal identity" $=o$ and are monotonic w.r.t. $\leq o$.

lemma *card-of-empty*: $|\{\}| \leq o\ |A|$
using *card-of-ordLeq inj-on-id* **by** *blast*

lemma *card-of-empty1*:
assumes *Well-order r* $\vee$ *Card-order r*
shows $|\{\}| \leq o\ r$
proof –
have *Well-order r* **using** *assms* **unfolding** *card-order-on-def* **by** *auto*
hence $|Field\ r| \leq o\ r$
using *assms card-of-least* **by** *blast*
moreover have $|\{\}| \leq o\ |Field\ r|$ **by** (*simp add*: *card-of-empty*)
ultimately show *?thesis* **using** *ordLeq-transitive* **by** *blast*
qed

corollary *Card-order-empty*:
Card-order r $\Longrightarrow |\{\}| \leq o\ r$ **by** (*simp add*: *card-of-empty1*)

lemma *card-of-empty2*:
assumes $|A| =o\ |\{\}|$
shows $A = \{\}$
using *assms card-of-ordIso*[*of A*] *bij-betw-empty2* **by** *blast*

lemma *card-of-empty3*:
assumes $|A| \leq o\ |\{\}|$
shows $A = \{\}$
using *assms*
by (*simp add*: *ordIso-iff-ordLeq card-of-empty1 card-of-empty2 ordLeq-Well-order-simp*)

lemma *card-of-empty-ordIso*:
$|\{\}::'a\ set| =o\ |\{\}::'b\ set|$
using *card-of-ordIso* **unfolding** *bij-betw-def inj-on-def* **by** *blast*

lemma *card-of-image*:
$|f\ `\ A| \leq o\ |A|$
proof(*cases* $A = \{\}$)
case *False*
hence $f\ `\ A \neq \{\}$ **by** *auto*
thus *?thesis*

  using *card-of-ordLeq2*[*of f ‘ A A*] **by** *auto*
qed (*simp add*: *card-of-empty*)

lemma *surj-imp-ordLeq*:
 assumes $B \subseteq f$ ‘ A
 shows $|B| \leq o\ |A|$
proof −
 have $|B| \leq o\ |f$ ‘ $A|$ **using** *assms card-of-mono1* **by** *auto*
 thus *?thesis* **using** *card-of-image ordLeq-transitive* **by** *blast*
qed

lemma *card-of-singl-ordLeq*:
 assumes $A \neq \{\}$
 shows $|\{b\}| \leq o\ |A|$
proof −
 obtain *a* **where** ∗: $a \in A$ **using** *assms* **by** *auto*
 let *?h* = λ *b*′::′*b*. *if b*′ = *b then a else undefined*
 have *inj-on ?h* {*b*} ∧ *?h* ‘ {*b*} ≤ *A*
  using ∗ **unfolding** *inj-on-def* **by** *auto*
 thus *?thesis* **unfolding** *card-of-ordLeq*[*symmetric*] **by** (*intro exI*)
qed

corollary *Card-order-singl-ordLeq*:
 ⟦*Card-order r*; *Field r* ≠ {}⟧ ⟹ $|\{b\}| \leq o\ r$
 using *card-of-singl-ordLeq*[*of Field r b*]
  card-of-Field-ordIso[*of r*] *ordLeq-ordIso-trans* **by** *blast*

lemma *card-of-Pow*: $|A| < o\ |Pow\ A|$
 using *card-of-ordLess2*[*of Pow A A*] *Cantors-theorem*[*of A*]
  Pow-not-empty[*of A*] **by** *auto*

corollary *Card-order-Pow*:
 Card-order r ⟹ *r* <*o* |*Pow*(*Field r*)|
 using *card-of-Pow card-of-Field-ordIso ordIso-ordLess-trans ordIso-symmetric* **by** *blast*

lemma *card-of-Plus1*: |*A*| ≤*o* |*A* <+> *B*| **and** *card-of-Plus2*: |*B*| ≤*o* |*A* <+> *B*|
 using *card-of-ordLeq* **by** *force+*

corollary *Card-order-Plus1*:
 Card-order r ⟹ *r* ≤*o* |(*Field r*) <+> *B*|
 using *card-of-Plus1 card-of-Field-ordIso ordIso-ordLeq-trans ordIso-symmetric* **by** *blast*

corollary *Card-order-Plus2*:
 Card-order r ⟹ *r* ≤*o* |*A* <+> (*Field r*)|
 using *card-of-Plus2 card-of-Field-ordIso ordIso-ordLeq-trans ordIso-symmetric* **by** *blast*

```
lemma card-of-Plus-empty1: |A| =o |A <+> {}|
proof -
  have bij-betw Inl A (A <+> {}) unfolding bij-betw-def inj-on-def by auto
  thus ?thesis using card-of-ordIso by auto
qed

lemma card-of-Plus-empty2: |A| =o |{} <+> A|
proof -
  have bij-betw Inr A ({} <+> A) unfolding bij-betw-def inj-on-def by auto
  thus ?thesis using card-of-ordIso by auto
qed

lemma card-of-Plus-commute: |A <+> B| =o |B <+> A|
proof -
  let ?f = λc. case c of Inl a ⇒ Inr a | Inr b ⇒ Inl b
  have bij-betw ?f (A <+> B) (B <+> A)
    unfolding bij-betw-def inj-on-def by force
  thus ?thesis using card-of-ordIso by blast
qed

lemma card-of-Plus-assoc:
  fixes A :: 'a set and B :: 'b set and C :: 'c set
  shows |(A <+> B) <+> C| =o |A <+> B <+> C|
proof -
  define f :: ('a + 'b) + 'c ⇒ 'a + 'b + 'c
    where [abs-def]: f k =
      (case k of
        Inl ab ⇒
          (case ab of
            Inl a ⇒ Inl a
          | Inr b ⇒ Inr (Inl b))
      | Inr c ⇒ Inr (Inr c))
    for k
  have A <+> B <+> C ⊆ f ` ((A <+> B) <+> C)
  proof
    fix x assume x: x ∈ A <+> B <+> C
    show x ∈ f ` ((A <+> B) <+> C)
    proof(cases x)
      case (Inl a)
      hence a ∈ A x = f (Inl (Inl a))
        using x unfolding f-def by auto
      thus ?thesis by auto
    next
      case (Inr bc) with x show ?thesis
        by (cases bc) (force simp: f-def)+
    qed
  qed
  hence bij-betw f ((A <+> B) <+> C) (A <+> B <+> C)
    unfolding bij-betw-def inj-on-def f-def by fastforce
```

thus *?thesis* **using** *card-of-ordIso* **by** *blast*
qed

lemma *card-of-Plus-mono1*:
assumes |*A*| ≤*o* |*B*|
shows |*A* <+> *C*| ≤*o* |*B* <+> *C*|
proof −
obtain *f* **where** *f*: *inj-on f A* ∧ *f* ' *A* ≤ *B*
using *assms card-of-ordLeq*[*of A*] **by** *fastforce*
define *g* **where** *g* ≡ λ*d*. *case d of Inl a* ⇒ *Inl*(*f a*) | *Inr* (*c*::'*c*) ⇒ *Inr c*
have *inj-on g* (*A* <+> *C*) ∧ *g* ' (*A* <+> *C*) ≤ (*B* <+> *C*)
using *f* **unfolding** *inj-on-def g-def* **by** *force*
thus *?thesis* **using** *card-of-ordLeq* **by** *blast*
qed

corollary *ordLeq-Plus-mono1*:
assumes *r* ≤*o* *r*'
shows |(*Field r*) <+> *C*| ≤*o* |(*Field r*') <+> *C*|
using *assms card-of-mono2 card-of-Plus-mono1* **by** *blast*

lemma *card-of-Plus-mono2*:
assumes |*A*| ≤*o* |*B*|
shows |*C* <+> *A*| ≤*o* |*C* <+> *B*|
using *card-of-Plus-mono1*[*OF assms*]
by (*blast intro*: *card-of-Plus-commute ordIso-ordLeq-trans ordLeq-ordIso-trans*)

corollary *ordLeq-Plus-mono2*:
assumes *r* ≤*o* *r*'
shows |*A* <+> (*Field r*)| ≤*o* |*A* <+> (*Field r*')|
using *assms card-of-mono2 card-of-Plus-mono2* **by** *blast*

lemma *card-of-Plus-mono*:
assumes |*A*| ≤*o* |*B*| **and** |*C*| ≤*o* |*D*|
shows |*A* <+> *C*| ≤*o* |*B* <+> *D*|
using *assms card-of-Plus-mono1*[*of A B C*] *card-of-Plus-mono2*[*of C D B*]
ordLeq-transitive **by** *blast*

corollary *ordLeq-Plus-mono*:
assumes *r* ≤*o* *r*' **and** *p* ≤*o* *p*'
shows |(*Field r*) <+> (*Field p*)| ≤*o* |(*Field r*') <+> (*Field p*')|
using *assms card-of-mono2*[*of r r*'] *card-of-mono2*[*of p p*'] *card-of-Plus-mono* **by** *blast*

lemma *card-of-Plus-cong1*:
assumes |*A*| =*o* |*B*|
shows |*A* <+> *C*| =*o* |*B* <+> *C*|
using *assms* **by** (*simp add*: *ordIso-iff-ordLeq card-of-Plus-mono1*)

corollary *ordIso-Plus-cong1*:

assumes $r =o\ r'$
shows $|(Field\ r) <+> C| =o\ |(Field\ r') <+> C|$
using *assms card-of-cong card-of-Plus-cong1* **by** *blast*

lemma *card-of-Plus-cong2*:
assumes $|A| =o\ |B|$
shows $|C <+> A| =o\ |C <+> B|$
using *assms* **by** (*simp add*: *ordIso-iff-ordLeq card-of-Plus-mono2*)

corollary *ordIso-Plus-cong2*:
assumes $r =o\ r'$
shows $|A <+> (Field\ r)| =o\ |A <+> (Field\ r')|$
using *assms card-of-cong card-of-Plus-cong2* **by** *blast*

lemma *card-of-Plus-cong*:
assumes $|A| =o\ |B|$ **and** $|C| =o\ |D|$
shows $|A <+> C| =o\ |B <+> D|$
using *assms* **by** (*simp add*: *ordIso-iff-ordLeq card-of-Plus-mono*)

corollary *ordIso-Plus-cong*:
assumes $r =o\ r'$ **and** $p =o\ p'$
shows $|(Field\ r) <+> (Field\ p)| =o\ |(Field\ r') <+> (Field\ p')|$
using *assms card-of-cong*[*of r r'*] *card-of-cong*[*of p p'*] *card-of-Plus-cong* **by** *blast*

lemma *card-of-Un-Plus-ordLeq*:
$|A \cup B| \leq o\ |A <+> B|$
proof −
let $?f = \lambda\ c.\ if\ c \in A\ then\ Inl\ c\ else\ Inr\ c$
have $inj\text{-}on\ ?f\ (A \cup B) \wedge ?f\ `\ (A \cup B) \leq A <+> B$
unfolding *inj-on-def* **by** *auto*
thus *?thesis* **using** *card-of-ordLeq* **by** *blast*
qed

lemma *card-of-Times1*:
assumes $A \neq \{\}$
shows $|B| \leq o\ |B \times A|$
proof(*cases* $B = \{\}$)
case *False*
have $fst\ `(B \times A) = B$ **using** *assms* **by** *auto*
thus *?thesis* **using** *inj-on-iff-surj*[*of B B* $\times$ *A*]
card-of-ordLeq False **by** *blast*
qed (*simp add*: *card-of-empty*)

lemma *card-of-Times-commute*: $|A \times B| =o\ |B \times A|$
proof −
have $bij\text{-}betw\ (\lambda(a,b).\ (b,a))\ (A \times B)\ (B \times A)$
unfolding *bij-betw-def inj-on-def* **by** *auto*
thus *?thesis* **using** *card-of-ordIso* **by** *blast*
qed

lemma *card-of-Times2*:
  assumes $A \neq \{\}$ **shows** $|B| \leq o\ |A \times B|$
  using *assms card-of-Times1*[*of A B*] *card-of-Times-commute*[*of B A*]
    ordLeq-ordIso-trans **by** *blast*

corollary *Card-order-Times1*:
  $[\![Card\text{-}order\ r;\ B \neq \{\}]\!] \Longrightarrow r \leq o\ |(Field\ r) \times B|$
  using *card-of-Times1*[*of B*] *card-of-Field-ordIso*
    ordIso-ordLeq-trans ordIso-symmetric **by** *blast*

corollary *Card-order-Times2*:
  $[\![Card\text{-}order\ r;\ A \neq \{\}]\!] \Longrightarrow r \leq o\ |A \times (Field\ r)|$
  using *card-of-Times2*[*of A*] *card-of-Field-ordIso*
    ordIso-ordLeq-trans ordIso-symmetric **by** *blast*

lemma *card-of-Times3*: $|A| \leq o\ |A \times A|$
  using *card-of-Times1*[*of A*]
  by(*cases* $A = \{\}$, *simp add*: *card-of-empty*)

lemma *card-of-Plus-Times-bool*: $|A <+> A| =o\ |A \times (UNIV::bool\ set)|$
proof −
  let $?f = \lambda c::'a + 'a.\ case\ c\ of\ Inl\ a \Rightarrow (a,True)$
                  $|Inr\ a \Rightarrow (a,False)$
  have *bij-betw* $?f\ (A <+> A)\ (A \times (UNIV::bool\ set))$
  proof −
    have $\bigwedge c1\ c2.\ ?f\ c1 = ?f\ c2 \Longrightarrow c1 = c2$
      by (*force split*: *sum.split-asm*)
    moreover
    have $\bigwedge c.\ c \in A <+> A \Longrightarrow ?f\ c \in A \times (UNIV::bool\ set)$
      by (*force split*: *sum.split-asm*)
    moreover
    {**fix** *a bl* **assume** $(a,bl) \in A \times (UNIV::bool\ set)$
      hence $(a,bl) \in ?f\ `\ (A <+> A)$
      by (*cases bl*) (*force split*: *sum.split-asm*)+
    }
    ultimately show *?thesis* **unfolding** *bij-betw-def inj-on-def* **by** *auto*
  qed
  thus *?thesis* **using** *card-of-ordIso* **by** *blast*
qed

lemma *card-of-Times-mono1*:
  assumes $|A| \leq o\ |B|$
  shows $|A \times C| \leq o\ |B \times C|$
proof −
  obtain *f* **where** *f*: $inj\text{-}on\ f\ A \wedge f\ `\ A \leq B$
    using *assms card-of-ordLeq*[*of A*] **by** *fastforce*
  define *g* **where** $g \equiv (\lambda(a,c::'c).\ (f\ a,c))$
  have $inj\text{-}on\ g\ (A \times C) \wedge g\ `\ (A \times C) \leq (B \times C)$

using *f* **unfolding** *inj-on-def* **using** *g-def* **by** *auto*
thus *?thesis* **using** *card-of-ordLeq* **by** *blast*
qed

corollary *ordLeq-Times-mono1*:
assumes $r \leq o\ r'$
shows $|(Field\ r) \times C| \leq o\ |(Field\ r') \times C|$
using *assms card-of-mono2 card-of-Times-mono1* **by** *blast*

lemma *card-of-Times-mono2*:
assumes $|A| \leq o\ |B|$
shows $|C \times A| \leq o\ |C \times B|$
using *assms card-of-Times-mono1*[*of A B C*]
by (*blast intro*: *card-of-Times-commute ordIso-ordLeq-trans ordLeq-ordIso-trans*)

corollary *ordLeq-Times-mono2*:
assumes $r \leq o\ r'$
shows $|A \times (Field\ r)| \leq o\ |A \times (Field\ r')|$
using *assms card-of-mono2 card-of-Times-mono2* **by** *blast*

lemma *card-of-Sigma-mono1*:
assumes $\forall i \in I.\ |A\ i| \leq o\ |B\ i|$
shows $|SIGMA\ i : I.\ A\ i| \leq o\ |SIGMA\ i : I.\ B\ i|$
proof −
have $\forall i.\ i \in I \longrightarrow (\exists f.\ inj\text{-}on\ f\ (A\ i) \wedge f\ `\ (A\ i) \leq B\ i)$
using *assms* **by** (*auto simp add*: *card-of-ordLeq*)
with *choice*[*of* $\lambda\ i\ f.\ i \in I \longrightarrow inj\text{-}on\ f\ (A\ i) \wedge f\ `\ (A\ i) \leq B\ i$]
obtain *F* **where** *F*: $\forall i \in I.\ inj\text{-}on\ (F\ i)\ (A\ i) \wedge (F\ i)\ `\ (A\ i) \leq B\ i$
by *atomize-elim* (*auto intro*: *bchoice*)
define *g* **where** $g \equiv (\lambda(i,a::'b).\ (i,F\ i\ a))$
have $inj\text{-}on\ g\ (Sigma\ I\ A) \wedge g\ `\ (Sigma\ I\ A) \leq (Sigma\ I\ B)$
using *F* **unfolding** *inj-on-def* **using** *g-def* **by** *force*
thus *?thesis* **using** *card-of-ordLeq* **by** *blast*
qed

lemma *card-of-UNION-Sigma*:
$|\bigcup i \in I.\ A\ i| \leq o\ |SIGMA\ i : I.\ A\ i|$
using *Ex-inj-on-UNION-Sigma* [*of A I*] *card-of-ordLeq* **by** *blast*

lemma *card-of-bool*:
assumes $a1 \neq a2$
shows $|UNIV::bool\ set| =o\ |\{a1,a2\}|$
proof −
let $?f = \lambda\ bl.\ if\ bl\ then\ a1\ else\ a2$
have *bij-betw ?f UNIV* $\{a1,a2\}$
proof −
have $\bigwedge bl1\ bl2.\ ?f\ bl1 = ?f\ bl2 \Longrightarrow bl1 = bl2$
using *assms* **by** (*force split*: *if-split-asm*)
moreover

have $\bigwedge bl.\ ?f\ bl \in \{a1,a2\}$
using *assms* **by** (*force split*: *if-split-asm*)
ultimately show *?thesis* **unfolding** *bij-betw-def inj-on-def* **by** *force*
qed
thus *?thesis* **using** *card-of-ordIso* **by** *blast*
qed

lemma *card-of-Plus-Times-aux*:
assumes $A2$: $a1 \neq a2 \wedge \{a1,a2\} \leq A$ **and**
LEQ: $|A| \leq o\ |B|$
shows $|A <+> B| \leq o\ |A \times B|$
proof −
have 1: $|UNIV::bool\ set| \leq o\ |A|$
using *A2 card-of-mono1*[*of* $\{a1,a2\}$] *card-of-bool*[*of a1 a2*]
by (*blast intro*: *ordIso-ordLeq-trans*)
have $|A <+> B| \leq o\ |B <+> B|$
using *LEQ card-of-Plus-mono1* **by** *blast*
moreover have $|B <+> B| =o\ |B \times (UNIV::bool\ set)|$
using *card-of-Plus-Times-bool* **by** *blast*
moreover have $|B \times (UNIV::bool\ set)| \leq o\ |B \times A|$
using *1* **by** (*simp add*: *card-of-Times-mono2*)
moreover have $|B \times A| =o\ |A \times B|$
using *card-of-Times-commute* **by** *blast*
ultimately show $|A <+> B| \leq o\ |A \times B|$
by (*blast intro*: *ordLeq-transitive dest*: *ordLeq-ordIso-trans*)
qed

lemma *card-of-Plus-Times*:
assumes $A2$: $a1 \neq a2 \wedge \{a1,a2\} \leq A$ **and** $B2$: $b1 \neq b2 \wedge \{b1,b2\} \leq B$
shows $|A <+> B| \leq o\ |A \times B|$
proof −
{assume $|A| \leq o\ |B|$
hence *?thesis* **using** *assms* **by** (*auto simp add*: *card-of-Plus-Times-aux*)
}
moreover
{assume $|B| \leq o\ |A|$
hence $|B <+> A| \leq o\ |B \times A|$
using *assms* **by** (*auto simp add*: *card-of-Plus-Times-aux*)
hence *?thesis*
using *card-of-Plus-commute card-of-Times-commute*
ordIso-ordLeq-trans ordLeq-ordIso-trans **by** *blast*
}
ultimately show *?thesis*
using *card-of-Well-order*[*of A*] *card-of-Well-order*[*of B*]
ordLeq-total[*of* $|A|$] **by** *blast*
qed

lemma *card-of-Times-Plus-distrib*:
$|A \times (B <+> C)| =o\ |A \times B <+> A \times C|$ (**is** $|?RHS| =o\ |?LHS|$)

proof –
let *?f* = λ(*a*, *bc*). *case bc of Inl b* ⇒ *Inl* (*a*, *b*) | *Inr c* ⇒ *Inr* (*a*, *c*)
have *bij-betw ?f ?RHS ?LHS* **unfolding** *bij-betw-def inj-on-def* **by** *force*
thus *?thesis* **using** *card-of-ordIso* **by** *blast*
qed

lemma *card-of-ordLeq-finite*:
assumes |*A*| ≤*o* |*B*| **and** *finite B*
shows *finite A*
using *assms* **unfolding** *ordLeq-def*
using *embed-inj-on*[*of* |*A*| |*B*|] *embed-Field*[*of* |*A*| |*B*|]
Field-card-of[*of A*] *Field-card-of*[*of B*] *inj-on-finite*[*of - A B*] **by** *fastforce*

lemma *card-of-ordLeq-infinite*:
assumes |*A*| ≤*o* |*B*| **and** ¬ *finite A*
shows ¬ *finite B*
using *assms card-of-ordLeq-finite* **by** *auto*

lemma *card-of-ordIso-finite*:
assumes |*A*| =*o* |*B*|
shows *finite A* = *finite B*
using *assms* **unfolding** *ordIso-def iso-def*[*abs-def*]
by (*auto simp*: *bij-betw-finite Field-card-of*)

lemma *card-of-ordIso-finite-Field*:
assumes *Card-order r* **and** *r* =*o* |*A*|
shows *finite*(*Field r*) = *finite A*
using *assms card-of-Field-ordIso card-of-ordIso-finite ordIso-equivalence* **by** *blast*

30.4 Cardinals versus set operations involving infinite sets

Here we show that, for infinite sets, most set-theoretic constructions do not increase the cardinality. The cornerstone for this is theorem *Card-order-Times-same-infinite*, which states that self-product does not increase cardinality – the proof of this fact adapts a standard set-theoretic argument, as presented, e.g., in the proof of theorem 1.5.11 at page 47 in [4]. Then everything else follows fairly easily.

lemma *infinite-iff-card-of-nat*:
¬ *finite A* ⟷ (|*UNIV*::*nat set*| ≤*o* |*A*|)
unfolding *infinite-iff-countable-subset card-of-ordLeq* **..**

The next two results correspond to the ZF fact that all infinite cardinals are limit ordinals:

lemma *Card-order-infinite-not-under*:
assumes *CARD*: *Card-order r* **and** *INF*: ¬*finite* (*Field r*)
shows ¬ (∃ *a*. *Field r* = *under r a*)
proof(*auto*)
have *0*: *Well-order r* ∧ *wo-rel r* ∧ *Refl r*

using *CARD* **unfolding** *wo-rel-def card-order-on-def order-on-defs* **by** *auto*
fix *a* **assume** $*$: *Field r* = *under r a*
show *False*
proof(*cases a* ∈ *Field r*)
assume *Case1*: *a* ∉ *Field r*
hence *under r a* = {} **unfolding** *Field-def under-def* **by** *auto*
thus *False* **using** *INF* $*$ **by** *auto*
next
let *?r′* = *Restr r* (*underS r a*)
assume *Case2*: *a* ∈ *Field r*
hence *1*: *under r a* = *underS r a* ∪ {*a*} ∧ *a* ∉ *underS r a*
using *0 Refl-under-underS*[*of r a*] *underS-notIn*[*of a r*] **by** *blast*
have *2*: *wo-rel.ofilter r* (*underS r a*) ∧ *underS r a* < *Field r*
using *0 wo-rel.underS-ofilter* $*$ *1 Case2* **by** *fast*
hence *?r′* <*o r* **using** *0* **using** *ofilter-ordLess* **by** *blast*
moreover
have *Field ?r′* = *underS r a* ∧ *Well-order ?r′*
using *2 0 Field-Restr-ofilter*[*of r*] *Well-order-Restr*[*of r*] **by** *blast*
ultimately have |*underS r a*| <*o r* **using** *ordLess-Field*[*of ?r′*] **by** *auto*
moreover have |*under r a*| =*o r* **using** $*$ *CARD card-of-Field-ordIso*[*of r*] **by** *auto*
ultimately have |*underS r a*| <*o* |*under r a*|
using *ordIso-symmetric ordLess-ordIso-trans* **by** *blast*
moreover
{**have** ∃*f*. *bij-betw f* (*under r a*) (*underS r a*)
using *infinite-imp-bij-betw*[*of Field r a*] *INF* $*$ *1* **by** *auto*
hence |*under r a*| =*o* |*underS r a*| **using** *card-of-ordIso* **by** *blast*
}
ultimately show *False* **using** *not-ordLess-ordIso ordIso-symmetric* **by** *blast*
qed
qed

lemma *infinite-Card-order-limit*:
assumes *r*: *Card-order r* **and** ¬*finite* (*Field r*)
and *a*: *a* ∈ *Field r*
shows ∃ *b* ∈ *Field r*. *a* ≠ *b* ∧ (*a*,*b*) ∈ *r*
proof −
have *Field r* ≠ *under r a*
using *assms Card-order-infinite-not-under* **by** *blast*
moreover have *under r a* ≤ *Field r*
using *under-Field* **.**
ultimately obtain *b* **where** *b*: *b* ∈ *Field r* ∧ ¬ (*b*,*a*) ∈ *r*
unfolding *under-def* **by** *blast*
moreover have *ba*: *b* ≠ *a*
using *b r* **unfolding** *card-order-on-def well-order-on-def*
linear-order-on-def partial-order-on-def preorder-on-def refl-on-def **by** *auto*
ultimately have (*a*,*b*) ∈ *r*
using *a r* **unfolding** *card-order-on-def well-order-on-def linear-order-on-def*
total-on-def **by** *blast*

thus *?thesis* **using** *b ba* **by** *auto*
qed

theorem *Card-order-Times-same-infinite*:
assumes *CO*: *Card-order r* **and** *INF*: $\neg$*finite*(*Field r*)
shows |*Field r* × *Field r*| ≤*o r*
proof −
define *phi* **where**
phi ≡ λ*r*::'*a rel*. *Card-order r* ∧ ¬*finite*(*Field r*) ∧ ¬ |*Field r* × *Field r*| ≤*o r*
have *temp1*: ∀ *r*. *phi r* ⟶ *Well-order r*
unfolding *phi-def card-order-on-def* **by** *auto*
have *Ft*: ¬(∃ *r*. *phi r*)
proof
assume ∃ *r*. *phi r*
hence {*r*. *phi r*} ≠ {} ∧ {*r*. *phi r*} ≤ {*r*. *Well-order r*}
using *temp1* **by** *auto*
then obtain *r* **where** *1*: *phi r* **and** *2*: ∀ *r'*. *phi r'* ⟶ *r* ≤*o r'* **and**
3: *Card-order r* ∧ *Well-order r*
using *exists-minim-Well-order*[*of* {*r*. *phi r*}] *temp1 phi-def* **by** *blast*
let *?A* = *Field r* **let** *?r'* = *bsqr r*
have *4*: *Well-order ?r'* ∧ *Field ?r'* = *?A* × *?A* ∧ |*?A*| =*o r*
using *3 bsqr-Well-order Field-bsqr card-of-Field-ordIso* **by** *blast*
have *5*: *Card-order* |*?A* × *?A*| ∧ *Well-order* |*?A* × *?A*|
using *card-of-Card-order card-of-Well-order* **by** *blast*

have *r* <*o* |*?A* × *?A*|
using *1 3 5 ordLess-or-ordLeq* **unfolding** *phi-def* **by** *blast*
moreover have |*?A* × *?A*| ≤*o ?r'*
using *card-of-least*[*of ?A* × *?A*] *4* **by** *auto*
ultimately have *r* <*o ?r'* **using** *ordLess-ordLeq-trans* **by** *auto*
then obtain *f* **where** *6*: *embed r ?r' f* **and** *7*: ¬ *bij-betw f ?A* (*?A* × *?A*)
unfolding *ordLess-def embedS-def*[*abs-def*]
by (*auto simp add*: *Field-bsqr*)
let *?B* = *f* ' *?A*
have |*?A*| =*o* |*?B*|
using *3 6 embed-inj-on inj-on-imp-bij-betw card-of-ordIso* **by** *blast*
hence *8*: *r* =*o* |*?B*| **using** *4 ordIso-transitive ordIso-symmetric* **by** *blast*

have *wo-rel.ofilter ?r' ?B*
using *6 embed-Field-ofilter 3 4* **by** *blast*
hence *wo-rel.ofilter ?r' ?B* ∧ *?B* ≠ *?A* × *?A* ∧ *?B* ≠ *Field ?r'*
using *7* **unfolding** *bij-betw-def* **using** *6 3 embed-inj-on 4* **by** *auto*
hence *temp2*: *wo-rel.ofilter ?r' ?B* ∧ *?B* < *?A* × *?A*
using *4 wo-rel-def*[*of ?r'*] *wo-rel.ofilter-def*[*of ?r' ?B*] **by** *blast*
have ¬ (∃ *a*. *Field r* = *under r a*)
using *1* **unfolding** *phi-def* **using** *Card-order-infinite-not-under*[*of r*] **by** *auto*
then obtain *A1* **where** *temp3*: *wo-rel.ofilter r A1* ∧ *A1* < *?A* **and** *9*: *?B* ≤
A1 × *A1*
using *temp2 3 bsqr-ofilter*[*of r ?B*] **by** *blast*

hence |?B| ≤o |A1 × A1| **using** *card-of-mono1* **by** *blast*
hence *10*: r ≤o |A1 × A1| **using** *8 ordIso-ordLeq-trans* **by** *blast*
let ?r1 = Restr r A1
have ?r1 <o r **using** *temp3 ofilter-ordLess 3* **by** *blast*
moreover
{**have** *well-order-on A1 ?r1* **using** *3 temp3 well-order-on-Restr* **by** *blast*
hence |A1| ≤o ?r1 **using** *3 Well-order-Restr card-of-least* **by** *blast*
}
ultimately have *11*: |A1| <o r **using** *ordLeq-ordLess-trans* **by** *blast*

have ¬ *finite* (*Field r*) **using** *1* **unfolding** *phi-def* **by** *simp*
hence ¬ *finite ?B* **using** *8 3 card-of-ordIso-finite-Field*[*of r ?B*] **by** *blast*
hence ¬ *finite A1* **using** *9 finite-cartesian-product finite-subset* **by** *blast*
moreover have *temp4*: *Field* |A1| = A1 ∧ *Well-order* |A1| ∧ *Card-order* |A1|
using *card-of-Card-order*[*of A1*] *card-of-Well-order*[*of A1*]
by (*simp add*: *Field-card-of*)
moreover have ¬ r ≤o | A1 |
using *temp4 11 3* **using** *not-ordLeq-iff-ordLess* **by** *blast*
ultimately have ¬ *finite*(*Field* |A1|) ∧ *Card-order* |A1| ∧ ¬ r ≤o | A1 |
by (*simp add*: *card-of-card-order-on*)
hence |*Field* |A1| × *Field* |A1| | ≤o |A1|
using *2* **unfolding** *phi-def* **by** *blast*
hence |A1 × A1 | ≤o |A1| **using** *temp4* **by** *auto*
hence r ≤o |A1| **using** *10 ordLeq-transitive* **by** *blast*
thus *False* **using** *11 not-ordLess-ordLeq* **by** *auto*
qed
thus *?thesis* **using** *assms* **unfolding** *phi-def* **by** *blast*
qed

corollary *card-of-Times-same-infinite*:
assumes ¬*finite A*
shows |A × A| =o |A|
proof −
let ?r = |A|
have *Field ?r* = A ∧ *Card-order ?r*
using *Field-card-of card-of-Card-order*[*of A*] **by** *fastforce*
hence |A × A| ≤o |A|
using *Card-order-Times-same-infinite*[*of ?r*] *assms* **by** *auto*
thus *?thesis* **using** *card-of-Times3 ordIso-iff-ordLeq* **by** *blast*
qed

lemma *card-of-Times-infinite*:
assumes *INF*: ¬*finite A* **and** *NE*: B ≠ {} **and** *LEQ*: |B| ≤o |A|
shows |A × B| =o |A| ∧ |B × A| =o |A|
proof −
have |A| ≤o |A × B| ∧ |A| ≤o |B × A|
using *assms* **by** (*simp add*: *card-of-Times1 card-of-Times2*)
moreover
{**have** |A × B| ≤o |A × A| ∧ |B × A| ≤o |A × A|

using *LEQ card-of-Times-mono1 card-of-Times-mono2* **by** *blast*
moreover have $|A \times A| =o |A|$ **using** *INF card-of-Times-same-infinite* **by** *blast*
ultimately have $|A \times B| \leq o |A| \wedge |B \times A| \leq o |A|$
using *ordLeq-ordIso-trans*[*of* $|A \times B|$] *ordLeq-ordIso-trans*[*of* $|B \times A|$] **by** *auto*
}
ultimately show *?thesis* **by** (*simp add*: *ordIso-iff-ordLeq*)
qed

corollary *Card-order-Times-infinite*:
assumes *INF*: $\neg$*finite*(*Field r*) **and** *CARD*: *Card-order r* **and**
NE: *Field p* $\neq$ {} **and** *LEQ*: $p \leq o\ r$
shows | (*Field r*) $\times$ (*Field p*) | $=o\ r \wedge$ | (*Field p*) $\times$ (*Field r*) | $=o\ r$
proof –
have |*Field r* $\times$ *Field p*| $=o$ |*Field r*| $\wedge$ |*Field p* $\times$ *Field r*| $=o$ |*Field r*|
using *assms* **by** (*simp add*: *card-of-Times-infinite card-of-mono2*)
thus *?thesis*
using *assms card-of-Field-ordIso* **by** (*blast intro*: *ordIso-transitive*)
qed

lemma *card-of-Sigma-ordLeq-infinite*:
assumes *INF*: $\neg$*finite B* **and**
LEQ-I: $|I| \leq o |B|$ **and** *LEQ*: $\forall i \in I.\ |A\ i| \leq o |B|$
shows |*SIGMA* $i : I.\ A\ i| \leq o |B|$
proof(*cases* $I = \{\}$)
case *False*
have |*SIGMA* $i : I.\ A\ i| \leq o |I \times B|$
using *card-of-Sigma-mono1*[*OF LEQ*] **by** *blast*
moreover have $|I \times B| =o |B|$
using *INF False LEQ-I* **by** (*auto simp add*: *card-of-Times-infinite*)
ultimately show *?thesis* **using** *ordLeq-ordIso-trans* **by** *blast*
qed (*simp add*: *card-of-empty*)

lemma *card-of-Sigma-ordLeq-infinite-Field*:
assumes *INF*: $\neg$*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LEQ-I: $|I| \leq o\ r$ **and** *LEQ*: $\forall i \in I.\ |A\ i| \leq o\ r$
shows |*SIGMA* $i : I.\ A\ i| \leq o\ r$
proof –
let *?B* = *Field r*
have *1*: $r =o$ |*?B*| $\wedge$ |*?B*| $=o\ r$
using *r card-of-Field-ordIso ordIso-symmetric* **by** *blast*
hence $|I| \leq o$ |*?B*| $\forall i \in I.\ |A\ i| \leq o$ |*?B*|
using *LEQ-I LEQ ordLeq-ordIso-trans* **by** *blast+*
hence |*SIGMA* $i : I.\ A\ i| \leq o$ |*?B*| **using** *INF LEQ*
card-of-Sigma-ordLeq-infinite **by** *blast*
thus *?thesis* **using** *1 ordLeq-ordIso-trans* **by** *blast*
qed

```
lemma card-of-Times-ordLeq-infinite-Field:
  ⟦¬finite (Field r); |A| ≤o r; |B| ≤o r; Card-order r⟧ ⟹ |A × B| ≤o r
  by(simp add: card-of-Sigma-ordLeq-infinite-Field)

lemma card-of-Times-infinite-simps:
  ⟦¬finite A; B ≠ {}; |B| ≤o |A|⟧ ⟹ |A × B| =o |A|
  ⟦¬finite A; B ≠ {}; |B| ≤o |A|⟧ ⟹ |A| =o |A × B|
  ⟦¬finite A; B ≠ {}; |B| ≤o |A|⟧ ⟹ |B × A| =o |A|
  ⟦¬finite A; B ≠ {}; |B| ≤o |A|⟧ ⟹ |A| =o |B × A|
  by (auto simp add: card-of-Times-infinite ordIso-symmetric)

lemma card-of-UNION-ordLeq-infinite:
  assumes INF: ¬finite B and LEQ-I: |I| ≤o |B| and LEQ: ∀ i ∈ I. |A i| ≤o |B|
  shows |⋃ i ∈ I. A i| ≤o |B|
proof(cases I = {})
  case False
  have |⋃ i ∈ I. A i| ≤o |SIGMA i : I. A i|
    using card-of-UNION-Sigma by blast
  moreover have |SIGMA i : I. A i| ≤o |B|
    using assms card-of-Sigma-ordLeq-infinite by blast
  ultimately show ?thesis using ordLeq-transitive by blast
qed (simp add: card-of-empty)

corollary card-of-UNION-ordLeq-infinite-Field:
  assumes INF: ¬finite (Field r) and r: Card-order r and
    LEQ-I: |I| ≤o r and LEQ: ∀ i ∈ I. |A i| ≤o r
  shows |⋃ i ∈ I. A i| ≤o r
proof −
  let ?B = Field r
  have 1: r =o |?B| ∧ |?B| =o r
    using r card-of-Field-ordIso ordIso-symmetric by blast
  hence |I| ≤o |?B|  ∀ i ∈ I. |A i| ≤o |?B|
    using LEQ-I LEQ ordLeq-ordIso-trans by blast+
  hence  |⋃ i ∈ I. A i| ≤o |?B| using INF LEQ
      card-of-UNION-ordLeq-infinite by blast
  thus ?thesis using 1 ordLeq-ordIso-trans by blast
qed

lemma card-of-Plus-infinite1:
  assumes INF: ¬finite A and LEQ: |B| ≤o |A|
  shows |A <+> B| =o |A|
proof(cases B = {})
  case True
  then show ?thesis
    by (simp add: card-of-Plus-empty1 card-of-Plus-empty2 ordIso-symmetric)
next
  case False
  let ?Inl = Inl::'a ⇒ 'a + 'b  let ?Inr = Inr::'b ⇒ 'a + 'b
  assume ∗: B ≠ {}
```

then obtain *b1* **where** *1*: *b1* ∈ *B* **by** *blast*
show *?thesis*
proof(*cases B* = {*b1*})
case *True*
have *2*: *bij-betw ?Inl A* ((*?Inl* ‘ *A*))
unfolding *bij-betw-def inj-on-def* **by** *auto*
hence *3*: ¬*finite* (*?Inl* ‘ *A*)
using *INF bij-betw-finite*[*of ?Inl A*] **by** *blast*
let *?A′* = *?Inl* ‘ *A* ∪ {*?Inr b1*}
obtain *g* **where** *bij-betw g* (*?Inl* ‘ *A*) *?A′*
using *3 infinite-imp-bij-betw2*[*of ?Inl* ‘ *A*] **by** *auto*
moreover have *?A′* = *A* <+> *B* **using** *True* **by** *blast*
ultimately have *bij-betw g* (*?Inl* ‘ *A*) (*A* <+> *B*) **by** *simp*
hence *bij-betw* (*g* ∘ *?Inl*) *A* (*A* <+> *B*)
using *2* **by** (*auto simp add*: *bij-betw-trans*)
thus *?thesis* **using** *card-of-ordIso ordIso-symmetric* **by** *blast*
next
case *False*
with ∗ *1* **obtain** *b2* **where** *3*: *b1* ≠ *b2* ∧ {*b1*,*b2*} ≤ *B* **by** *fastforce*
obtain *f* **where** *inj-on f B* ∧ *f* ‘ *B* ≤ *A*
using *LEQ card-of-ordLeq*[*of B*] **by** *fastforce*
with *3* **have** *f b1* ≠ *f b2* ∧ {*f b1*, *f b2*} ≤ *A*
unfolding *inj-on-def* **by** *auto*
with *3* **have** |*A* <+> *B*| ≤*o* |*A* × *B*|
by (*auto simp add*: *card-of-Plus-Times*)
moreover have |*A* × *B*| =*o* |*A*|
using *assms* ∗ **by** (*simp add*: *card-of-Times-infinite-simps*)
ultimately have |*A* <+> *B*| ≤*o* |*A*| **using** *ordLeq-ordIso-trans* **by** *blast*
thus *?thesis* **using** *card-of-Plus1 ordIso-iff-ordLeq* **by** *blast*
qed
qed

lemma *card-of-Plus-infinite2*:
assumes *INF*: ¬*finite A* **and** *LEQ*: |*B*| ≤*o* |*A*|
shows |*B* <+> *A*| =*o* |*A*|
using *assms card-of-Plus-commute card-of-Plus-infinite1*
ordIso-equivalence **by** *blast*

lemma *card-of-Plus-infinite*:
assumes *INF*: ¬*finite A* **and** *LEQ*: |*B*| ≤*o* |*A*|
shows |*A* <+> *B*| =*o* |*A*| ∧ |*B* <+> *A*| =*o* |*A*|
using *assms* **by** (*auto simp*: *card-of-Plus-infinite1 card-of-Plus-infinite2*)

corollary *Card-order-Plus-infinite*:
assumes *INF*: ¬*finite*(*Field r*) **and** *CARD*: *Card-order r* **and**
LEQ: *p* ≤*o* *r*
shows | (*Field r*) <+> (*Field p*) | =*o* *r* ∧ | (*Field p*) <+> (*Field r*) | =*o* *r*
proof −
have | *Field r* <+> *Field p* | =*o* | *Field r* | ∧

    | *Field p* <+> *Field r* | =*o* | *Field r* |
  using *assms* **by** (*simp add*: *card-of-Plus-infinite card-of-mono2*)
 thus *?thesis*
  using *assms card-of-Field-ordIso* **by** (*blast intro*: *ordIso-transitive*)

qed

30.5 The cardinal ω and the finite cardinals

The cardinal ω, of natural numbers, shall be the standard non-strict order relation on *nat*, that we abbreviate by *natLeq*. The finite cardinals shall be the restrictions of these relations to the numbers smaller than fixed numbers *n*, that we abbreviate by *natLeq-on n*.

definition (*natLeq*::(*nat* ∗ *nat*) *set*) ≡ {(*x*,*y*). *x* ≤ *y*}
definition (*natLess*::(*nat* ∗ *nat*) *set*) ≡ {(*x*,*y*). *x* < *y*}

abbreviation *natLeq-on* :: *nat* ⇒ (*nat* ∗ *nat*) *set*
 where *natLeq-on n* ≡ {(*x*,*y*). *x* < *n* ∧ *y* < *n* ∧ *x* ≤ *y*}

lemma *infinite-cartesian-product*:
 assumes ¬*finite A* ¬*finite B*
 shows ¬*finite* (*A* × *B*)
using *assms finite-cartesian-productD2* **by** *auto*

30.5.1 First as well-orders

lemma *Field-natLeq*: *Field natLeq* = (*UNIV*::*nat set*)
 by(*unfold Field-def natLeq-def*, *auto*)

lemma *natLeq-Refl*: *Refl natLeq*
 unfolding *refl-on-def Field-def natLeq-def* **by** *auto*

lemma *natLeq-trans*: *trans natLeq*
 unfolding *trans-def natLeq-def* **by** *auto*

lemma *natLeq-Preorder*: *Preorder natLeq*
 unfolding *preorder-on-def*
 by (*auto simp add*: *natLeq-Refl natLeq-trans*)

lemma *natLeq-antisym*: *antisym natLeq*
 unfolding *antisym-def natLeq-def* **by** *auto*

lemma *natLeq-Partial-order*: *Partial-order natLeq*
 unfolding *partial-order-on-def*
 by (*auto simp add*: *natLeq-Preorder natLeq-antisym*)

lemma *natLeq-Total*: *Total natLeq*
 unfolding *total-on-def natLeq-def* **by** *auto*

lemma *natLeq-Linear-order*: *Linear-order natLeq*
unfolding *linear-order-on-def*
by (*auto simp add*: *natLeq-Partial-order natLeq-Total*)

lemma *natLeq-natLess-Id*: *natLess* = *natLeq* − *Id*
unfolding *natLeq-def natLess-def* **by** *auto*

lemma *natLeq-Well-order*: *Well-order natLeq*
unfolding *well-order-on-def*
using *natLeq-Linear-order wf-less natLeq-natLess-Id natLeq-def natLess-def* **by** *auto*

lemma *Field-natLeq-on*: *Field* (*natLeq-on n*) = {*x*. *x* < *n*}
unfolding *Field-def* **by** *auto*

lemma *natLeq-underS-less*: *underS natLeq n* = {*x*. *x* < *n*}
unfolding *underS-def natLeq-def* **by** *auto*

lemma *Restr-natLeq*: *Restr natLeq* {*x*. *x* < *n*} = *natLeq-on n*
unfolding *natLeq-def* **by** *force*

lemma *Restr-natLeq2*:
Restr natLeq (*underS natLeq n*) = *natLeq-on n*
by (*auto simp add*: *Restr-natLeq natLeq-underS-less*)

lemma *natLeq-on-Well-order*: *Well-order*(*natLeq-on n*)
using *Restr-natLeq*[*of n*] *natLeq-Well-order*
Well-order-Restr[*of natLeq* {*x*. *x* < *n*}] **by** *auto*

corollary *natLeq-on-well-order-on*: *well-order-on* {*x*. *x* < *n*} (*natLeq-on n*)
using *natLeq-on-Well-order Field-natLeq-on* **by** *auto*

lemma *natLeq-on-wo-rel*: *wo-rel*(*natLeq-on n*)
unfolding *wo-rel-def* **using** *natLeq-on-Well-order* **.**

30.5.2 Then as cardinals

lemma *natLeq-Card-order*: *Card-order natLeq*
proof −
have *natLeq-on n* <*o* |*UNIV*::*nat set*| **for** *n*
proof −
have *finite*(*Field* (*natLeq-on n*)) **by** (*auto simp*: *Field-def*)
moreover have ¬*finite*(*UNIV*::*nat set*) **by** *auto*
ultimately show *?thesis*
using *finite-ordLess-infinite*[*of natLeq-on n* |*UNIV*::*nat set*|]
card-of-Well-order[*of UNIV*::*nat set*] *natLeq-on-Well-order*
by (*force simp add*: *Field-card-of*)
qed
then show *?thesis*

apply (*simp add*: *natLeq-Well-order Card-order-iff-Restr-underS Restr-natLeq2*)
apply (*force simp add*: *Field-natLeq*)
done
qed

corollary *card-of-Field-natLeq*:
$|Field\ natLeq| =o\ natLeq$
using *Field-natLeq natLeq-Card-order Card-order-iff-ordIso-card-of*[*of natLeq*]
ordIso-symmetric[*of natLeq*] **by** *blast*

corollary *card-of-nat*:
$|UNIV::nat\ set| =o\ natLeq$
using *Field-natLeq card-of-Field-natLeq* **by** *auto*

corollary *infinite-iff-natLeq-ordLeq*:
$\neg finite\ A = (\ natLeq \leq o\ |A|\ )$
using *infinite-iff-card-of-nat*[*of A*] *card-of-nat*
ordIso-ordLeq-trans ordLeq-ordIso-trans ordIso-symmetric **by** *blast*

corollary *finite-iff-ordLess-natLeq*:
$finite\ A = (\ |A| <o\ natLeq)$
using *infinite-iff-natLeq-ordLeq not-ordLeq-iff-ordLess*
card-of-Well-order natLeq-Well-order **by** *blast*

30.6 The successor of a cardinal

First we define *isCardSuc r r′*, the notion of r' being a successor cardinal of r. Although the definition does not require r to be a cardinal, only this case will be meaningful.

definition *isCardSuc* :: $'a\ rel \Rightarrow 'a\ set\ rel \Rightarrow bool$
where
$isCardSuc\ r\ r' \equiv$
$Card\text{-}order\ r' \wedge r <o\ r' \wedge$
$(\forall (r''::'a\ set\ rel).\ Card\text{-}order\ r'' \wedge r <o\ r'' \longrightarrow r' \leq o\ r'')$

Now we introduce the cardinal-successor operator *cardSuc*, by picking *some* cardinal-order relation fulfilling *isCardSuc*. Again, the picked item shall be proved unique up to order-isomorphism.

definition *cardSuc* :: $'a\ rel \Rightarrow 'a\ set\ rel$
where $cardSuc\ r \equiv SOME\ r'.\ isCardSuc\ r\ r'$

lemma *exists-minim-Card-order*:
$[\![R \neq \{\}; \forall r \in R.\ Card\text{-}order\ r]\!] \Longrightarrow \exists r \in R.\ \forall r' \in R.\ r \leq o\ r'$
unfolding *card-order-on-def* **using** *exists-minim-Well-order* **by** *blast*

lemma *exists-isCardSuc*:
assumes *Card-order r*
shows $\exists r'.\ isCardSuc\ r\ r'$

proof −
  let *?R* = {(*r*′::′*a set rel*). *Card-order r*′ ∧ *r* <*o r*′}
  have |*Pow*(*Field r*)| ∈ *?R* ∧ (∀ *r* ∈ *?R*. *Card-order r*) **using** *assms*
    by (*simp add*: *card-of-Card-order Card-order-Pow*)
  then obtain *r* **where** *r* ∈ *?R* ∧ (∀ *r*′ ∈ *?R*. *r* ≤*o r*′)
    using *exists-minim-Card-order*[*of ?R*] **by** *blast*
  thus *?thesis* **unfolding** *isCardSuc-def* **by** *auto*
qed

lemma *cardSuc-isCardSuc*:
  assumes *Card-order r*
  shows *isCardSuc r* (*cardSuc r*)
  unfolding *cardSuc-def* **using** *assms*
  by (*simp add*: *exists-isCardSuc someI-ex*)

lemma *cardSuc-Card-order*:
  Card-order r ⟹ *Card-order*(*cardSuc r*)
  using *cardSuc-isCardSuc* **unfolding** *isCardSuc-def* **by** *blast*

lemma *cardSuc-greater*:
  Card-order r ⟹ *r* <*o cardSuc r*
  using *cardSuc-isCardSuc* **unfolding** *isCardSuc-def* **by** *blast*

lemma *cardSuc-ordLeq*:
  Card-order r ⟹ *r* ≤*o cardSuc r*
  using *cardSuc-greater ordLeq-iff-ordLess-or-ordIso* **by** *blast*

The minimality property of *cardSuc* originally present in its definition is local to the type ′*a set rel*, i.e., that of *cardSuc r*:

lemma *cardSuc-least-aux*:
  ⟦*Card-order* (*r*::′*a rel*); *Card-order* (*r*′::′*a set rel*); *r* <*o r*′⟧ ⟹ *cardSuc r* ≤*o r*′
  using *cardSuc-isCardSuc* **unfolding** *isCardSuc-def* **by** *blast*

But from this we can infer general minimality:

lemma *cardSuc-least*:
  assumes *CARD*: *Card-order r* **and** *CARD*′: *Card-order r*′ **and** *LESS*: *r* <*o r*′
  shows *cardSuc r* ≤*o r*′
proof −
  let *?p* = *cardSuc r*
  have *0*: *Well-order ?p* ∧ *Well-order r*′
    using *assms cardSuc-Card-order* **unfolding** *card-order-on-def* **by** *blast*
  { **assume** *r*′ <*o ?p*
    then obtain *r*′′ **where** *1*: *Field r*′′ < *Field ?p* **and** *2*: *r*′ =*o r*′′ ∧ *r*′′ <*o ?p*
      using *internalize-ordLess*[*of r*′ *?p*] **by** *blast*

    have *Card-order r*′′ **using** *CARD*′ *Card-order-ordIso2 2* **by** *blast*
    moreover have *r* <*o r*′′ **using** *LESS 2 ordLess-ordIso-trans* **by** *blast*
    ultimately have *?p* ≤*o r*′′ **using** *cardSuc-least-aux CARD* **by** *blast*
    hence *False* **using** *2 not-ordLess-ordLeq* **by** *blast*

}
thus *?thesis* **using** *0 ordLess-or-ordLeq* **by** *blast*
qed

lemma *cardSuc-ordLess-ordLeq*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows $(r <o\ r') = (cardSuc\ r \leq o\ r')$
proof
show $cardSuc\ r \leq o\ r' \Longrightarrow r <o\ r'$
using *assms cardSuc-greater ordLess-ordLeq-trans* **by** *blast*
qed (*auto simp add*: *assms cardSuc-least*)

lemma *cardSuc-ordLeq-ordLess*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows $(r' <o\ cardSuc\ r) = (r' \leq o\ r)$
proof −
have *Well-order r* ∧ *Well-order r'*
using *assms* **unfolding** *card-order-on-def* **by** *auto*
moreover have *Well-order*(*cardSuc r*)
using *assms cardSuc-Card-order card-order-on-def* **by** *blast*
ultimately show *?thesis*
using *assms cardSuc-ordLess-ordLeq* **by** (*blast dest*: *not-ordLeq-iff-ordLess*)
qed

lemma *cardSuc-mono-ordLeq*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows $(cardSuc\ r \leq o\ cardSuc\ r') = (r \leq o\ r')$
using *assms cardSuc-ordLeq-ordLess cardSuc-ordLess-ordLeq cardSuc-Card-order*
by *blast*

lemma *cardSuc-invar-ordIso*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows $(cardSuc\ r =o\ cardSuc\ r') = (r =o\ r')$
proof −
have *0*: *Well-order r* ∧ *Well-order r'* ∧ *Well-order*(*cardSuc r*) ∧ *Well-order*(*cardSuc r'*)
using *assms* **by** (*simp add*: *card-order-on-well-order-on cardSuc-Card-order*)
thus *?thesis*
using *ordIso-iff-ordLeq*[*of r r'*] *ordIso-iff-ordLeq*
using *cardSuc-mono-ordLeq*[*of r r'*] *cardSuc-mono-ordLeq*[*of r' r*] *assms* **by** *blast*
qed

lemma *card-of-cardSuc-finite*:
finite(*Field*(*cardSuc* |*A*|)) = *finite A*
proof
assume ∗: *finite* (*Field* (*cardSuc* |*A*|))
have *0*: |*Field*(*cardSuc* |*A*|)| =*o cardSuc* |*A*|
using *card-of-Card-order cardSuc-Card-order card-of-Field-ordIso* **by** *blast*

hence *|A| ≤o |Field(cardSuc |A|)|*
using *card-of-Card-order[of A] cardSuc-ordLeq[of |A|] ordIso-symmetric ordLeq-ordIso-trans* **by** *blast*
thus *finite A* **using** ∗ *card-of-ordLeq-finite* **by** *blast*
next
assume *finite A*
then have *finite (Field |Pow A|)* **unfolding** *Field-card-of* **by** *simp*
moreover
have *cardSuc |A| ≤o |Pow A|*
by (*rule iffD1*[*OF cardSuc-ordLess-ordLeq card-of-Pow*]) (*simp-all add*: *card-of-Card-order*)
ultimately show *finite* (*Field* (*cardSuc |A|*))
by (*blast intro*: *card-of-ordLeq-finite card-of-mono2*)
qed

lemma *cardSuc-finite*:
assumes *Card-order r*
shows *finite* (*Field* (*cardSuc r*)) = *finite* (*Field r*)
proof −
let *?A = Field r*
have *|?A| =o r* **using** *assms* **by** (*simp add*: *card-of-Field-ordIso*)
hence *cardSuc |?A| =o cardSuc r* **using** *assms*
by (*simp add*: *card-of-Card-order cardSuc-invar-ordIso*)
moreover have *|Field* (*cardSuc |?A|*) *| =o cardSuc |?A|*
by (*simp add*: *card-of-card-order-on Field-card-of card-of-Field-ordIso cardSuc-Card-order*)
moreover
{ have *|Field* (*cardSuc r*) *| =o cardSuc r*
using *assms* **by** (*simp add*: *card-of-Field-ordIso cardSuc-Card-order*)
hence *cardSuc r =o |Field* (*cardSuc r*) *|*
using *ordIso-symmetric* **by** *blast*
}
ultimately have *|Field* (*cardSuc |?A|*) *| =o |Field* (*cardSuc r*) *|*
using *ordIso-transitive* **by** *blast*
hence *finite* (*Field* (*cardSuc |?A|*)) = *finite* (*Field* (*cardSuc r*))
using *card-of-ordIso-finite* **by** *blast*
thus *?thesis* **by** (*simp only*: *card-of-cardSuc-finite*)
qed

lemma *Field-cardSuc-not-empty*:
assumes *Card-order r*
shows *Field* (*cardSuc r*) ≠ {}
proof
assume *Field*(*cardSuc r*) = {}
then have *|Field*(*cardSuc r*)*| ≤o r* **using** *assms Card-order-empty*[*of r*] **by** *auto*
then have *cardSuc r ≤o r* **using** *assms card-of-Field-ordIso cardSuc-Card-order ordIso-symmetric ordIso-ordLeq-trans* **by** *blast*
then show *False* **using** *cardSuc-greater not-ordLess-ordLeq assms* **by** *blast*
qed

typedef *'a suc* = *Field* (*cardSuc* |*UNIV* :: *'a set*|)
using *Field-cardSuc-not-empty card-of-Card-order* **by** *blast*

definition *card-suc* :: *'a rel* ⇒ *'a suc rel* **where**
card-suc ≡ λ-. *map-prod Abs-suc Abs-suc* ' *cardSuc* |*UNIV* :: *'a set*|

lemma *Field-card-suc*: *Field* (*card-suc r*) = *UNIV*
proof −
let *?r* = *cardSuc* |*UNIV*|
let *?ar* = λ*x*. *Abs-suc* (*Rep-suc x*)
have *1*: ⋀*P*. (∀ *x*∈*Field ?r*. *P x*) = (∀ *x*. *P* (*Rep-suc x*)) **using** *Rep-suc-induct Rep-suc* **by** *blast*
have *2*: ⋀*P*. (∃ *x*∈*Field ?r*. *P x*) = (∃ *x*. *P* (*Rep-suc x*)) **using** *Rep-suc-cases Rep-suc* **by** *blast*
have *3*: ⋀*A a b*. (*a*, *b*) ∈ *A* ⟹ (*Abs-suc a*, *Abs-suc b*) ∈ *map-prod Abs-suc Abs-suc* ' *A* **unfolding** *map-prod-def* **by** *auto*
have ∀ *x*∈*Field ?r*. (∃ *b*∈*Field ?r*. (*x*, *b*) ∈ *?r*) ∨ (∃ *a*∈*Field ?r*. (*a*, *x*) ∈ *?r*)
unfolding *Field-def Range.simps Domain.simps Un-iff* **by** *blast*
then have ∀ *x*. (∃ *b*. (*Rep-suc x*, *Rep-suc b*) ∈ *?r*) ∨ (∃ *a*. (*Rep-suc a*, *Rep-suc x*) ∈ *?r*) **unfolding** *1 2* **.**
then have ∀ *x*. (∃ *b*. (*?ar x*, *?ar b*) ∈ *map-prod Abs-suc Abs-suc* ' *?r*) ∨ (∃ *a*. (*?ar a*, *?ar x*) ∈ *map-prod Abs-suc Abs-suc* ' *?r*) **using** *3* **by** *fast*
then have ∀ *x*. (∃ *b*. (*x*, *b*) ∈ *card-suc r*) ∨ (∃ *a*. (*a*, *x*) ∈ *card-suc r*) **unfolding** *card-suc-def Rep-suc-inverse* **.**
then show *?thesis* **unfolding** *Field-def Domain.simps Range.simps set-eq-iff Un-iff eqTrueI*[*OF UNIV-I*] *ex-simps simp-thms* **.**
qed

lemma *card-suc-alt*: *card-suc r* = *dir-image* (*cardSuc* |*UNIV* :: *'a set*|) *Abs-suc*
unfolding *card-suc-def dir-image-def* **by** *auto*

lemma *cardSuc-Well-order*: *Card-order r* ⟹ *Well-order*(*cardSuc r*)
using *cardSuc-Card-order* **unfolding** *card-order-on-def* **by** *blast*

lemma *cardSuc-ordIso-card-suc*:
assumes *card-order r*
shows *cardSuc r* =*o card-suc* (*r* :: *'a rel*)
proof −
have *cardSuc* (*r* :: *'a rel*) =*o cardSuc* |*UNIV* :: *'a set*|
using *cardSuc-invar-ordIso*[*THEN iffD2*, *OF - card-of-Card-order card-of-unique*[*OF assms*]] *assms*
by (*simp add*: *card-order-on-Card-order*)
also have *cardSuc* |*UNIV* :: *'a set*| =*o card-suc* (*r* :: *'a rel*)
unfolding *card-suc-alt*
by (*rule dir-image-ordIso*) (*simp-all add*: *inj-on-def Abs-suc-inject cardSuc-Well-order card-of-Card-order*)
finally show *?thesis* **.**
qed

lemma *Card-order-card-suc*: *card-order r* ⟹ *Card-order* (*card-suc r*)
using *cardSuc-Card-order*[*THEN Card-order-ordIso2*[*OF - cardSuc-ordIso-card-suc*]]
card-order-on-Card-order **by** *blast*

lemma *card-order-card-suc*: *card-order r* ⟹ *card-order* (*card-suc r*)
using *Card-order-card-suc arg-cong2*[*OF Field-card-suc refl*, *of card-order-on*] **by**
blast

lemma *card-suc-greater*: *card-order r* ⟹ *r* <*o card-suc r*
using *ordLess-ordIso-trans*[*OF cardSuc-greater cardSuc-ordIso-card-suc*] *card-order-on-Card-order*
by *blast*

lemma *card-of-Plus-ordLess-infinite*:
assumes *INF*: ¬*finite C* **and** *LESS1*: |*A*| <*o* |*C*| **and** *LESS2*: |*B*| <*o* |*C*|
shows |*A* <+> *B*| <*o* |*C*|
proof(*cases A* = {} ∨ *B* = {})
case *True*
hence |*A*| =*o* |*A* <+> *B*| ∨ |*B*| =*o* |*A* <+> *B*|
using *card-of-Plus-empty1 card-of-Plus-empty2* **by** *blast*
hence |*A* <+> *B*| =*o* |*A*| ∨ |*A* <+> *B*| =*o* |*B*|
using *ordIso-symmetric*[*of* |*A*|] *ordIso-symmetric*[*of* |*B*|] **by** *blast*
thus *?thesis* **using** *LESS1 LESS2*
ordIso-ordLess-trans[*of* |*A* <+> *B*| |*A*|]
ordIso-ordLess-trans[*of* |*A* <+> *B*| |*B*|] **by** *blast*
next
case *False*
have *False* **if** |*C*| ≤*o* |*A* <+> *B*|
proof −
have §: ¬*finite A* ∨ ¬*finite B*
using *that INF card-of-ordLeq-finite finite-Plus* **by** *blast*
consider |*A*| ≤*o* |*B*| | |*B*| ≤*o* |*A*|
using *ordLeq-total* [*OF card-of-Well-order card-of-Well-order*] **by** *blast*
then show *False*
proof *cases*
case *1*
hence ¬*finite B* **using** § *card-of-ordLeq-finite* **by** *blast*
hence |*A* <+> *B*| =*o* |*B*| **using** *False 1*
by (*auto simp add*: *card-of-Plus-infinite*)
thus *False* **using** *LESS2 not-ordLess-ordLeq that ordLeq-ordIso-trans* **by** *blast*
next
case *2*
hence ¬*finite A* **using** § *card-of-ordLeq-finite* **by** *blast*
hence |*A* <+> *B*| =*o* |*A*| **using** *False 2*
by (*auto simp add*: *card-of-Plus-infinite*)
thus *False* **using** *LESS1 not-ordLess-ordLeq that ordLeq-ordIso-trans* **by** *blast*
qed
qed
thus *?thesis*
using *ordLess-or-ordLeq*[*of* |*A* <+> *B*| |*C*|]

card-of-Well-order[of A <+> B] card-of-Well-order[of C] **by** *auto*
qed

lemma *card-of-Plus-ordLess-infinite-Field*:
assumes *INF*: ¬*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LESS1: |*A*| <*o r* **and** *LESS2*: |*B*| <*o r*
shows |*A* <+> *B*| <*o r*
proof −
let *?C* = *Field r*
have *1*: *r* =*o* |*?C*| ∧ |*?C*| =*o r*
using *r card-of-Field-ordIso ordIso-symmetric* **by** *blast*
hence |*A*| <*o* |*?C*| |*B*| <*o* |*?C*|
using *LESS1 LESS2 ordLess-ordIso-trans* **by** *blast+*
hence |*A* <+> *B*| <*o* |*?C*| **using** *INF*
card-of-Plus-ordLess-infinite **by** *blast*
thus *?thesis* **using** *1 ordLess-ordIso-trans* **by** *blast*
qed

lemma *card-of-Plus-ordLeq-infinite-Field*:
assumes *r*: ¬*finite* (*Field r*) **and** *A*: |*A*| ≤*o r* **and** *B*: |*B*| ≤*o r*
and *c*: *Card-order r*
shows |*A* <+> *B*| ≤*o r*
proof −
let *?r'* = *cardSuc r*
have *Card-order ?r'* ∧ ¬*finite* (*Field ?r'*) **using** *assms*
by (*simp add*: *cardSuc-Card-order cardSuc-finite*)
moreover have |*A*| <*o ?r'* **and** |*B*| <*o ?r'* **using** *A B c*
by (*auto simp*: *card-of-card-order-on Field-card-of cardSuc-ordLeq-ordLess*)
ultimately have |*A* <+> *B*| <*o ?r'*
using *card-of-Plus-ordLess-infinite-Field* **by** *blast*
thus *?thesis* **using** *c r*
by (*simp add*: *card-of-card-order-on Field-card-of cardSuc-ordLeq-ordLess*)
qed

lemma *card-of-Un-ordLeq-infinite-Field*:
assumes *C*: ¬*finite* (*Field r*) **and** *A*: |*A*| ≤*o r* **and** *B*: |*B*| ≤*o r*
and *Card-order r*
shows |*A Un B*| ≤*o r*
using *assms card-of-Plus-ordLeq-infinite-Field card-of-Un-Plus-ordLeq*
ordLeq-transitive **by** *fast*

lemma *card-of-Un-ordLess-infinite*:
assumes *INF*: ¬*finite C* **and**
LESS1: |*A*| <*o* |*C*| **and** *LESS2*: |*B*| <*o* |*C*|
shows |*A* ∪ *B*| <*o* |*C*|
using *assms card-of-Plus-ordLess-infinite card-of-Un-Plus-ordLeq*
ordLeq-ordLess-trans **by** *blast*

lemma *card-of-Un-ordLess-infinite-Field*:

assumes *INF*: ¬*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LESS1: |*A*| <*o r* **and** *LESS2*: |*B*| <*o r*
shows |*A Un B*| <*o r*
proof −
let *?C* = *Field r*
have *1*: *r* =*o* |*?C*| ∧ |*?C*| =*o r* **using** *r card-of-Field-ordIso*
ordIso-symmetric **by** *blast*
hence |*A*| <*o* |*?C*| |*B*| <*o* |*?C*|
using *LESS1 LESS2 ordLess-ordIso-trans* **by** *blast+*
hence |*A Un B*| <*o* |*?C*| **using** *INF*
card-of-Un-ordLess-infinite **by** *blast*
thus *?thesis* **using** *1 ordLess-ordIso-trans* **by** *blast*
qed

30.7 Regular cardinals

definition *cofinal* **where**
cofinal A r ≡ ∀ *a* ∈ *Field r*. ∃ *b* ∈ *A*. *a* ≠ *b* ∧ (*a*,*b*) ∈ *r*

definition *regularCard* **where**
regularCard r ≡ ∀ *K*. *K* ≤ *Field r* ∧ *cofinal K r* ⟶ |*K*| =*o r*

definition *relChain* **where**
relChain r As ≡ ∀ *i j*. (*i*,*j*) ∈ *r* ⟶ *As i* ≤ *As j*

lemma *regularCard-UNION*:
assumes *r*: *Card-order r* *regularCard r*
and *As*: *relChain r As*
and *Bsub*: *B* ≤ (⋃ *i* ∈ *Field r*. *As i*)
and *cardB*: |*B*| <*o r*
shows ∃ *i* ∈ *Field r*. *B* ≤ *As i*
proof −
let *?phi* = λ*b j*. *j* ∈ *Field r* ∧ *b* ∈ *As j*
have ∀ *b*∈*B*. ∃ *j*. *?phi b j* **using** *Bsub* **by** *blast*
then obtain *f* **where** *f*: ⋀*b*. *b* ∈ *B* ⟹ *?phi b* (*f b*)
using *bchoice*[*of B ?phi*] **by** *blast*
let *?K* = *f* ‘ *B*
{**assume** *1*: ⋀*i*. *i* ∈ *Field r* ⟹ ¬ *B* ≤ *As i*
have *2*: *cofinal ?K r*
unfolding *cofinal-def*
proof (*intro strip*)
fix *i* **assume** *i*: *i* ∈ *Field r*
with *1* **obtain** *b* **where** *b*: *b* ∈ *B* ∧ *b* ∉ *As i* **by** *blast*
hence *i* ≠ *f b* ∧ ¬ (*f b*,*i*) ∈ *r*
using *As f* **unfolding** *relChain-def* **by** *auto*
hence *i* ≠ *f b* ∧ (*i*, *f b*) ∈ *r* **using** *r*
unfolding *card-order-on-def well-order-on-def linear-order-on-def*
total-on-def **using** *i f b* **by** *auto*
with *b* **show** ∃ *b* ∈ *f*‘*B*. *i* ≠ *b* ∧ (*i*,*b*) ∈ *r* **by** *blast*

```
    qed
    moreover have ?K ≤ Field r using f by blast
    ultimately have |?K| =o r using 2 r unfolding regularCard-def by blast
    moreover
    have |?K| <o r using cardB ordLeq-ordLess-trans card-of-image by blast
    ultimately have False using not-ordLess-ordIso by blast
  }
  thus ?thesis by blast
qed

lemma infinite-cardSuc-regularCard:
  assumes r-inf: ¬finite (Field r) and r-card: Card-order r
  shows regularCard (cardSuc r)
proof −
  let ?r' = cardSuc r
  have r': Card-order ?r' ⋀p. Card-order p ⟶ (p ≤o r) = (p <o ?r')
    using r-card by (auto simp: cardSuc-Card-order cardSuc-ordLeq-ordLess)
  show ?thesis
    unfolding regularCard-def proof auto
    fix K assume 1: K ≤ Field ?r' and 2: cofinal K ?r'
    hence |K| ≤o |Field ?r'| by (simp only: card-of-mono1)
    also have 22: |Field ?r'| =o ?r'
      using r' by (simp add: card-of-Field-ordIso[of ?r'])
    finally have |K| ≤o ?r' .
    moreover
    { let ?L = ⋃ j ∈ K. underS ?r' j
      let ?J = Field r
      have rJ: r =o |?J|
        using r-card card-of-Field-ordIso ordIso-symmetric by blast
      assume |K| <o ?r'
      hence |K| ≤o r using r' card-of-Card-order[of K] by blast
      hence |K| ≤o |?J| using rJ ordLeq-ordIso-trans by blast
      moreover
      {have ∀ j∈K. |underS ?r' j| <o ?r'
          using r' 1 by (auto simp: card-of-underS)
        hence ∀ j∈K. |underS ?r' j| ≤o r
          using r' card-of-Card-order by blast
        hence ∀ j∈K. |underS ?r' j| ≤o |?J|
          using rJ ordLeq-ordIso-trans by blast
      }
      ultimately have |?L| ≤o |?J|
        using r-inf card-of-UNION-ordLeq-infinite by blast
      hence |?L| ≤o r using rJ ordIso-symmetric ordLeq-ordIso-trans by blast
      hence |?L| <o ?r' using r' card-of-Card-order by blast
      moreover
      {
        have Field ?r' ≤ ?L
          using 2 unfolding underS-def cofinal-def by auto
        hence |Field ?r'| ≤o |?L| by (simp add: card-of-mono1)
```

hence *?r'* $\leq$*o* |*?L*|
using *22 ordIso-ordLeq-trans ordIso-symmetric* **by** *blast*
}
ultimately have |*?L*| <*o* |*?L*| **using** *ordLess-ordLeq-trans* **by** *blast*
hence *False* **using** *ordLess-irreflexive* **by** *blast*
}
ultimately show |*K*| =*o* *?r'*
unfolding *ordLeq-iff-ordLess-or-ordIso* **by** *blast*
qed
qed

lemma *cardSuc-UNION*:
assumes *r*: *Card-order r* **and** ¬*finite* (*Field r*)
and *As*: *relChain* (*cardSuc r*) *As*
and *Bsub*: *B* ≤ (⋃ *i* ∈ *Field* (*cardSuc r*). *As i*)
and *cardB*: |*B*| ≤*o* *r*
shows ∃ *i* ∈ *Field* (*cardSuc r*). *B* ≤ *As i*
proof −
let *?r'* = *cardSuc r*
have *Card-order ?r'* ∧ |*B*| <*o* *?r'*
using *r cardB cardSuc-ordLeq-ordLess cardSuc-Card-order*
card-of-Card-order **by** *blast*
moreover have *regularCard ?r'*
using *assms* **by**(*simp add*: *infinite-cardSuc-regularCard*)
ultimately show *?thesis*
using *As Bsub cardB regularCard-UNION* **by** *blast*
qed

30.8 Others

lemma *card-of-Func-Times*:
|*Func* (*A* × *B*) *C*| =*o* |*Func A* (*Func B C*)|
unfolding *card-of-ordIso*[*symmetric*]
using *bij-betw-curr* **by** *blast*

lemma *card-of-Pow-Func*:
|*Pow A*| =*o* |*Func A* (*UNIV*::*bool set*)|
proof −
define *F* **where** [*abs-def*]: *F A' a* ≡
(*if a* ∈ *A then* (*if a* ∈ *A' then True else False*) *else undefined*) **for** *A' a*
have *bij-betw F* (*Pow A*) (*Func A* (*UNIV*::*bool set*))
unfolding *bij-betw-def inj-on-def* **proof** (*intro ballI impI conjI*)
fix *A1 A2* **assume** *A1* ∈ *Pow A A2* ∈ *Pow A F A1* = *F A2*
thus *A1* = *A2* **unfolding** *F-def Pow-def fun-eq-iff* **by** (*auto split*: *if-split-asm*)
next
show *F* ‘ *Pow A* = *Func A UNIV*
proof *safe*
fix *f* **assume** *f*: *f* ∈ *Func A* (*UNIV*::*bool set*)
show *f* ∈ *F* ‘ *Pow A*

```
      unfolding image-iff
    proof
      show f = F {a ∈ A. f a = True}
        using f unfolding Func-def F-def by force
    qed auto
  qed(unfold Func-def F-def, auto)
 qed
 thus ?thesis unfolding card-of-ordIso[symmetric] by blast
qed

lemma card-of-Func-UNIV:
 |Func (UNIV::'a set) (B::'b set)| =o |{f::'a ⇒ 'b. range f ⊆ B}|
proof −
 let ?F = λ f (a::'a). ((f a)::'b)
 have bij-betw ?F {f. range f ⊆ B} (Func UNIV B)
  unfolding bij-betw-def inj-on-def
 proof safe
  fix h :: 'a ⇒ 'b assume h: h ∈ Func UNIV B
  then obtain f where f: ∀ a. h a = f a by blast
  hence range f ⊆ B using h unfolding Func-def by auto
  thus h ∈ (λf a. f a) ‘ {f. range f ⊆ B} using f by auto
 qed(unfold Func-def fun-eq-iff, auto)
 then show ?thesis
  by (blast intro: ordIso-symmetric card-of-ordIsoI)
qed

lemma Func-Times-Range:
 |Func A (B × C)| =o |Func A B × Func A C| (is |?LHS| =o |?RHS|)
proof −
 let ?F = λfg. (λx. if x ∈ A then fst (fg x) else undefined,
          λx. if x ∈ A then snd (fg x) else undefined)
 let ?G = λ(f, g) x. if x ∈ A then (f x, g x) else undefined
 have bij-betw ?F ?LHS ?RHS unfolding bij-betw-def inj-on-def
 proof (intro conjI impI ballI equalityI subsetI)
  fix f g assume ∗: f ∈ Func A (B × C) g ∈ Func A (B × C) ?F f = ?F g
  show f = g
  proof
   fix x from ∗ have fst (f x) = fst (g x) ∧ snd (f x) = snd (g x)
    by (cases x ∈ A) (auto simp: Func-def fun-eq-iff split: if-splits)
   then show f x = g x by (subst (1 2) surjective-pairing) simp
  qed
 next
  fix fg assume fg ∈ Func A B × Func A C
  thus fg ∈ ?F ‘ Func A (B × C)
   by (intro image-eqI[of - - ?G fg]) (auto simp: Func-def)
 qed (auto simp: Func-def fun-eq-iff)
 thus ?thesis using card-of-ordIso by blast
qed
```

30.9 Regular vs. stable cardinals

definition *stable* :: 'a rel ⇒ bool
  where
    stable r ≡ ∀ (A::'a set) (F :: 'a ⇒ 'a set).
        |A| <o r ∧ (∀ a ∈ A. |F a| <o r)
        ⟶ |SIGMA a : A. F a| <o r

lemma *regularCard-stable*:
  assumes *cr*: *Card-order r* **and** *ir*: ¬*finite* (*Field r*) **and** *reg*: *regularCard r*
  shows *stable r*
  unfolding *stable-def* **proof** *safe*
  fix *A* :: 'a set **and** *F* :: 'a ⇒ 'a set **assume** *A*: |A| <o r **and** *F*: ∀ a∈A. |F a| <o r
  {**assume** r ≤o |Sigma A F|
   hence |Field r| ≤o |Sigma A F| **using** *card-of-Field-ordIso*[*OF cr*] *ordIso-ordLeq-trans* **by** *blast*
    moreover have *Fi*: *Field r* ≠ {} **using** *ir* **by** *auto*
    ultimately have ∃f. f ' Sigma A F = Field r **using** *card-of-ordLeq2*[*OF Fi*] **by** *blast*
    then obtain *f* **where** *f*: f ' Sigma A F = Field r **by** *blast*
    have *r*: *wo-rel r* **using** *cr* **unfolding** *card-order-on-def wo-rel-def* **by** *auto*
    {**fix** *a* **assume** *a*: a ∈ A
      define *L* **where** L = {(a,u) | u. u ∈ F a}
      have *fL*: f ' L ⊆ Field r **using** *f a* **unfolding** *L-def* **by** *auto*
      have *bij-betw snd* {(a, u) |u. u ∈ F a} (F a)
        unfolding *bij-betw-def inj-on-def* **by** (*auto simp*: *image-def*)
      then have |L| =o |F a| **unfolding** *L-def card-of-ordIso*[*symmetric*] **by** *blast*
      hence |L| <o r **using** *F a ordIso-ordLess-trans*[*of* |L| |F a|] **unfolding** *L-def* **by** *auto*
      hence |f ' L| <o r **using** *ordLeq-ordLess-trans*[*OF card-of-image*, *of L*] **unfolding** *L-def* **by** *auto*
      hence ¬ *cofinal* (f ' L) r **using** *reg fL* **unfolding** *regularCard-def*
        by (*force simp add*: *dest*: *not-ordLess-ordIso*)
      then obtain *k* **where** *k*: k ∈ Field r **and** ∀ l ∈ L. ¬ (f l ≠ k ∧ (k, f l) ∈ r)
        unfolding *cofinal-def image-def* **by** *auto*
      hence ∃ k ∈ Field r. ∀ l ∈ L. (f l, k) ∈ r
        using *wo-rel.in-notinI*[*OF r - - ‹k ∈ Field r›*] *fL* **unfolding** *image-subset-iff* **by** *fast*
      hence ∃ k ∈ Field r. ∀ u ∈ F a. (f (a,u), k) ∈ r **unfolding** *L-def* **by** *auto*
    }
    then have *x*: ⋀a. a∈A ⟹ ∃k. k ∈ Field r ∧ (∀ u∈F a. (f (a, u), k) ∈ r) **by** *blast*
    obtain *gg* **where** ⋀a. a∈A ⟹ gg a = (SOME k. k ∈ Field r ∧ (∀ u∈F a. (f (a, u), k) ∈ r)) **by** *simp*
    then have *gg*: ∀ a∈A. ∀ u∈F a. (f (a, u), gg a) ∈ r **using** *someI-ex*[*OF x*] **by** *auto*
    obtain *j0* **where** *j0*: j0 ∈ Field r **using** *Fi* **by** *auto*
    define *g* **where** [*abs-def*]: g a = (if F a ≠ {} then gg a else j0) **for** *a*
    have *g*: ∀ a ∈ A. ∀ u ∈ F a. (f (a,u),g a) ∈ r **using** *gg* **unfolding** *g-def* **by**

auto
hence *1*: *Field r* ⊆ (⋃ *a* ∈ *A*. *under r* (*g a*))
using *f*[*symmetric*] **unfolding** *under-def image-def* **by** *auto*
have *gA*: *g* ' *A* ⊆ *Field r* **using** *gg j0* **unfolding** *Field-def g-def* **by** *auto*
moreover have *cofinal* (*g* ' *A*) *r* **unfolding** *cofinal-def*
proof *safe*
fix *i* **assume** *i* ∈ *Field r*
then obtain *j* **where** *ij*: (*i*,*j*) ∈ *r i* ≠ *j* **using** *cr ir infinite-Card-order-limit*
by *fast*
hence *j* ∈ *Field r* **using** *card-order-on-def cr well-order-on-domain* **by** *fast*
then obtain *a* **where** *a*: *a* ∈ *A* **and** *j*: (*j*, *g a*) ∈ *r*
using *1* **unfolding** *under-def* **by** *auto*
hence (*i*, *g a*) ∈ *r* **using** *ij wo-rel.TRANS*[*OF r*] **unfolding** *trans-def* **by**
blast
moreover have *i* ≠ *g a*
using *ij j r* **unfolding** *wo-rel-def* **unfolding** *well-order-on-def linear-order-on-def*
partial-order-on-def antisym-def **by** *auto*
ultimately show ∃ *j*∈*g* ' *A*. *i* ≠ *j* ∧ (*i*, *j*) ∈ *r* **using** *a* **by** *auto*
qed
ultimately have |*g* ' *A*| =*o r* **using** *reg* **unfolding** *regularCard-def* **by** *auto*
moreover have |*g* ' *A*| ≤*o* |*A*| **using** *card-of-image* **by** *blast*
ultimately have *False* **using** *A* **using** *not-ordLess-ordIso ordLeq-ordLess-trans*
by *blast*
}
thus |*Sigma A F*| <*o r*
using *cr not-ordLess-iff-ordLeq* **using** *card-of-Well-order card-order-on-well-order-on*
by *blast*
qed

lemma *stable-regularCard*:
assumes *cr*: *Card-order r* **and** *ir*: ¬*finite* (*Field r*) **and** *st*: *stable r*
shows *regularCard r*
unfolding *regularCard-def* **proof** *safe*
fix *K* **assume** *K*: *K* ⊆ *Field r* **and** *cof*: *cofinal K r*
have |*K*| ≤*o r* **using** *K card-of-Field-ordIso card-of-mono1 cr ordLeq-ordIso-trans*
by *blast*
moreover
{assume *Kr*: |*K*| <*o r*
have *x*: ⋀*a*. *a* ∈ *Field r* ⟹ ∃ *b*. *b* ∈ *K* ∧ *a* ≠ *b* ∧ (*a*, *b*) ∈ *r* **using** *cof*
unfolding *cofinal-def* **by** *blast*
then obtain *f* **where** ⋀*a*. *a* ∈ *Field r* ⟹ *f a* = (*SOME b*. *b* ∈ *K* ∧ *a* ≠ *b* ∧
(*a*, *b*) ∈ *r*) **by** *simp*
then have ∀ *a*∈*Field r*. *f a* ∈ *K* ∧ *a* ≠ *f a* ∧ (*a*, *f a*) ∈ *r* **using** *someI-ex*[*OF*
x] **by** *simp*
hence *Field r* ⊆ (⋃ *a* ∈ *K*. *underS r a*) **unfolding** *underS-def* **by** *auto*
hence *r* ≤*o* |⋃ *a* ∈ *K*. *underS r a*|
using *cr Card-order-iff-ordLeq-card-of card-of-mono1 ordLeq-transitive* **by**
blast
also have |⋃ *a* ∈ *K*. *underS r a*| ≤*o* |*SIGMA a*: *K*. *underS r a*| **by** (*rule*

card-of-UNION-Sigma)
also
{have ∀ *a* ∈ *K*. |*underS r a*| <*o r* **using** *K card-of-underS*[*OF cr*] *subsetD* **by** *auto*
hence |*SIGMA a*: *K*. *underS r a*| <*o r* **using** *st Kr* **unfolding** *stable-def* **by** *auto*
}
finally have *r* <*o r* **.**
hence *False* **using** *ordLess-irreflexive* **by** *blast*
}
ultimately show |*K*| =*o r* **using** *ordLeq-iff-ordLess-or-ordIso* **by** *blast*
qed

lemma *internalize-card-of-ordLess*:
(|*A*| <*o r*) = (∃ *B* < *Field r*. |*A*| =*o* |*B*| ∧ |*B*| <*o r*)
proof
assume |*A*| <*o r*
then obtain *p* **where** *1*: *Field p* < *Field r* ∧ |*A*| =*o p* ∧ *p* <*o r*
using *internalize-ordLess*[*of* |*A*| *r*] **by** *blast*
hence *Card-order p* **using** *card-of-Card-order Card-order-ordIso2* **by** *blast*
hence |*Field p*| =*o p* **using** *card-of-Field-ordIso* **by** *blast*
hence |*A*| =*o* |*Field p*| ∧ |*Field p*| <*o r*
using *1 ordIso-equivalence ordIso-ordLess-trans* **by** *blast*
thus ∃ *B* < *Field r*. |*A*| =*o* |*B*| ∧ |*B*| <*o r* **using** *1* **by** *blast*
next
assume ∃ *B* < *Field r*. |*A*| =*o* |*B*| ∧ |*B*| <*o r*
thus |*A*| <*o r* **using** *ordIso-ordLess-trans* **by** *blast*
qed

lemma *card-of-Sigma-cong1*:
assumes ∀ *i* ∈ *I*. |*A i*| =*o* |*B i*|
shows |*SIGMA i* : *I*. *A i*| =*o* |*SIGMA i* : *I*. *B i*|
using *assms* **by** (*auto simp add*: *card-of-Sigma-mono1 ordIso-iff-ordLeq*)

lemma *card-of-Sigma-cong2*:
assumes *bij-betw f* (*I*::'*i set*) (*J*::'*j set*)
shows |*SIGMA i* : *I*. (*A*::'*j* ⇒ '*a set*) (*f i*)| =*o* |*SIGMA j* : *J*. *A j*|
proof −
let *?LEFT* = *SIGMA i* : *I*. *A* (*f i*)
let *?RIGHT* = *SIGMA j* : *J*. *A j*
define *u* **where** *u* ≡ λ(*i*::'*i*,*a*::'*a*). (*f i*,*a*)
have *bij-betw u ?LEFT ?RIGHT*
using *assms* **unfolding** *u-def bij-betw-def inj-on-def* **by** *auto*
thus *?thesis* **using** *card-of-ordIso* **by** *blast*
qed

lemma *card-of-Sigma-cong*:
assumes *BIJ*: *bij-betw f I J* **and** *ISO*: ∀ *j* ∈ *J*. |*A j*| =*o* |*B j*|
shows |*SIGMA i* : *I*. *A* (*f i*)| =*o* |*SIGMA j* : *J*. *B j*|

proof −
  have ∀ *i* ∈ *I*. |*A*(*f i*)| =*o* |*B*(*f i*)|
    using *ISO BIJ* **unfolding** *bij-betw-def* **by** *blast*
  hence |*SIGMA i* : *I*. *A* (*f i*)| =*o* |*SIGMA i* : *I*. *B* (*f i*)| **by** (*rule card-of-Sigma-cong1*)
  moreover have |*SIGMA i* : *I*. *B* (*f i*)| =*o* |*SIGMA j* : *J*. *B j*|
    using *BIJ card-of-Sigma-cong2* **by** *blast*
  ultimately show *?thesis* **using** *ordIso-transitive* **by** *blast*
qed

lemma *stable-elim*:
  assumes *ST*: *stable r* **and** *A-LESS*: |*A*| <*o r* **and**
    F-LESS: ⋀ *a*. *a* ∈ *A* ⟹ |*F a*| <*o r*
  shows |*SIGMA a* : *A*. *F a*| <*o r*
proof −
  obtain *A′* **where** *1*: *A′* < *Field r* ∧ |*A′*| <*o r* **and** *2*: |*A*| =*o* |*A′*|
    using *internalize-card-of-ordLess*[*of A r*] *A-LESS* **by** *blast*
  then obtain *G* **where** *3*: *bij-betw G A′ A*
    using *card-of-ordIso ordIso-symmetric* **by** *blast*

  {**fix** *a* **assume** *a* ∈ *A*
    hence ∃ *B′*. *B′* ≤ *Field r* ∧ |*F a*| =*o* |*B′*| ∧ |*B′*| <*o r*
      using *internalize-card-of-ordLess*[*of F a r*] *F-LESS* **by** *blast*
  }
  then obtain *F′* **where**
    temp: ∀ *a* ∈ *A*. *F′ a* ≤ *Field r* ∧ |*F a*| =*o* |*F′ a*| ∧ |*F′ a*| <*o r*
    using *bchoice*[*of A* λ *a B′*. *B′* ≤ *Field r* ∧ |*F a*| =*o* |*B′*| ∧ |*B′*| <*o r*] **by** *blast*
  hence *4*: ∀ *a* ∈ *A*. *F′ a* ≤ *Field r* ∧ |*F′ a*| <*o r* **by** *auto*
  have *5*: ∀ *a* ∈ *A*. |*F′ a*| =*o* |*F a*| **using** *temp ordIso-symmetric* **by** *auto*

  have ∀ *a′* ∈ *A′*. *F′*(*G a′*) ≤ *Field r* ∧ |*F′*(*G a′*)| <*o r*
    using *3 4 bij-betw-ball*[*of G A′ A*] **by** *auto*
  hence |*SIGMA a′* : *A′*. *F′*(*G a′*)| <*o r*
    using *ST 1* **unfolding** *stable-def* **by** *auto*
  moreover have |*SIGMA a′* : *A′*. *F′*(*G a′*)| =*o* |*SIGMA a* : *A*. *F a*|
    using *card-of-Sigma-cong*[*of G A′ A F′ F*] *5 3* **by** *blast*
  ultimately show *?thesis* **using** *ordIso-symmetric ordIso-ordLess-trans* **by** *blast*
qed

lemma *stable-natLeq*: *stable natLeq*
proof(*unfold stable-def*, *safe*)
  fix *A* :: ′*a set* **and** *F* :: ′*a* ⇒ ′*a set*
  assume |*A*| <*o natLeq* **and** ∀ *a*∈*A*. |*F a*| <*o natLeq*
  hence *finite A* ∧ (∀ *a* ∈ *A*. *finite*(*F a*))
    by (*auto simp add*: *finite-iff-ordLess-natLeq*)
  hence *finite*(*Sigma A F*) **by** (*simp only*: *finite-SigmaI*[*of A F*])
  thus |*Sigma A F* | <*o natLeq*
    by (*auto simp add*: *finite-iff-ordLess-natLeq*)
qed

corollary *regularCard-natLeq*: *regularCard natLeq*
using *stable-regularCard*[*OF natLeq-Card-order - stable-natLeq*] *Field-natLeq* **by** *simp*

lemma *stable-ordIso1*:
assumes *ST*: *stable r* **and** *ISO*: $r' =o\ r$
shows *stable* r'
proof(*unfold stable-def*, *auto*)
fix A::$'b$ *set* **and** F::$'b \Rightarrow 'b$ *set*
assume $|A| <o\ r'$ **and** $\forall a \in A.\ |F\ a| <o\ r'$
hence $(\ |A| <o\ r) \wedge (\forall a \in A.\ |F\ a| <o\ r)$
using *ISO ordLess-ordIso-trans* **by** *blast*
hence $|SIGMA\ a : A.\ F\ a| <o\ r$ **using** *assms stable-elim* **by** *blast*
thus $|SIGMA\ a : A.\ F\ a| <o\ r'$
using *ISO ordIso-symmetric ordLess-ordIso-trans* **by** *blast*
qed

lemma *stable-UNION*:
assumes *stable r* **and** $|A| <o\ r$ **and** $\bigwedge a.\ a \in A \Longrightarrow |F\ a| <o\ r$
shows $|\bigcup a \in A.\ F\ a| <o\ r$
using *assms card-of-UNION-Sigma stable-elim ordLeq-ordLess-trans* **by** *blast*

corollary *card-of-UNION-ordLess-infinite*:
assumes *stable* $|B|$ **and** $|I| <o\ |B|$ **and** $\forall i \in I.\ |A\ i| <o\ |B|$
shows $|\bigcup i \in I.\ A\ i| <o\ |B|$
using *assms stable-UNION* **by** *blast*

corollary *card-of-UNION-ordLess-infinite-Field*:
assumes *ST*: *stable r* **and** *r*: *Card-order r* **and**
LEQ-I: $|I| <o\ r$ **and** *LEQ*: $\forall i \in I.\ |A\ i| <o\ r$
shows $|\bigcup i \in I.\ A\ i| <o\ r$
proof −
let *?B* = *Field r*
have *1*: $r =o\ |?B| \wedge |?B| =o\ r$ **using** *r card-of-Field-ordIso*
ordIso-symmetric **by** *blast*
hence $|I| <o\ |?B|\ \ \forall i \in I.\ |A\ i| <o\ |?B|$
using *LEQ-I LEQ ordLess-ordIso-trans* **by** *blast+*
moreover have *stable* $|?B|$ **using** *stable-ordIso1 ST 1* **by** *blast*
ultimately have $|\bigcup i \in I.\ A\ i| <o\ |?B|$ **using** *LEQ*
card-of-UNION-ordLess-infinite **by** *blast*
thus *?thesis* **using** *1 ordLess-ordIso-trans* **by** *blast*
qed

end

31 Cardinal Arithmetic as Needed by Bounded Natural Functors

theory *BNF-Cardinal-Arithmetic*
imports *BNF-Cardinal-Order-Relation*
begin

lemma *dir-image*: $[\![\bigwedge x\ y.\ (f\ x = f\ y) = (x = y);\ Card\text{-}order\ r]\!] \Longrightarrow r =o\ dir\text{-}image\ r\ f$
by (*rule dir-image-ordIso*) (*auto simp add*: *inj-on-def card-order-on-def*)

lemma *card-order-dir-image*:
assumes *bij*: *bij f* **and** *co*: *card-order r*
shows *card-order* (*dir-image r f*)
proof −
have *Field* (*dir-image r f*) = *UNIV*
using *assms card-order-on-Card-order*[*of UNIV r*]
unfolding *bij-def dir-image-Field* **by** *auto*
moreover from *bij* **have** $\bigwedge x\ y.\ (f\ x = f\ y) = (x = y)$
unfolding *bij-def inj-on-def* **by** *auto*
with *co* **have** *Card-order* (*dir-image r f*)
using *card-order-on-Card-order Card-order-ordIso2*[*OF - dir-image*] **by** *blast*
ultimately show *?thesis* **by** *auto*
qed

lemma *ordIso-refl*: *Card-order r* ⟹ *r =o r*
by (*rule card-order-on-ordIso*)

lemma *ordLeq-refl*: *Card-order r* ⟹ *r ≤o r*
by (*rule ordIso-imp-ordLeq*, *rule card-order-on-ordIso*)

lemma *card-of-ordIso-subst*: *A* = *B* ⟹ |*A*| =o |*B*|
by (*simp only*: *ordIso-refl card-of-Card-order*)

lemma *Field-card-order*: *card-order r* ⟹ *Field r* = *UNIV*
using *card-order-on-Card-order*[*of UNIV r*] **by** *simp*

31.1 Zero

definition *czero* **where**
czero = *card-of* {}

lemma *czero-ordIso*: *czero =o czero*
using *card-of-empty-ordIso* **by** (*simp add*: *czero-def*)

lemma *card-of-ordIso-czero-iff-empty*:
|*A*| =o (*czero* :: '*b rel*) ⟷ *A* = ({} :: '*a set*)
unfolding *czero-def* **by** (*rule iffI*[*OF card-of-empty2*]) (*auto simp*: *card-of-refl card-of-empty-ordIso*)

abbreviation *Cnotzero* **where**
Cnotzero $(r :: 'a\ rel) \equiv \neg(r =o\ (czero :: 'a\ rel)) \wedge Card\text{-}order\ r$

lemma *Cnotzero-imp-not-empty*: $Cnotzero\ r \Longrightarrow Field\ r \neq \{\}$
unfolding *Card-order-iff-ordIso-card-of czero-def* **by** *force*

lemma *czeroI*:
$[\![Card\text{-}order\ r;\ Field\ r = \{\}]\!] \Longrightarrow r =o\ czero$
using *Cnotzero-imp-not-empty ordIso-transitive*[*OF - czero-ordIso*] **by** *blast*

lemma *czeroE*:
$r =o\ czero \Longrightarrow Field\ r = \{\}$
unfolding *czero-def*
by (*drule card-of-cong*) (*simp only*: *Field-card-of card-of-empty2*)

lemma *Cnotzero-mono*:
$[\![Cnotzero\ r;\ Card\text{-}order\ q;\ r \leq o\ q]\!] \Longrightarrow Cnotzero\ q$
by (*force intro*: *czeroI dest*: *card-of-mono2 card-of-empty3 czeroE*)

31.2 (In)finite cardinals

definition *cinfinite* **where**
$cinfinite\ r \equiv (\neg\ finite\ (Field\ r))$

abbreviation *Cinfinite* **where**
$Cinfinite\ r \equiv cinfinite\ r \wedge Card\text{-}order\ r$

definition *cfinite* **where**
$cfinite\ r = finite\ (Field\ r)$

abbreviation *Cfinite* **where**
$Cfinite\ r \equiv cfinite\ r \wedge Card\text{-}order\ r$

lemma *Cfinite-ordLess-Cinfinite*: $[\![Cfinite\ r;\ Cinfinite\ s]\!] \Longrightarrow r <o\ s$
unfolding *cfinite-def cinfinite-def*
by (*blast intro*: *finite-ordLess-infinite card-order-on-well-order-on*)

lemmas *natLeq-card-order* = *natLeq-Card-order*[*unfolded Field-natLeq*]

lemma *natLeq-cinfinite*: *cinfinite natLeq*
unfolding *cinfinite-def Field-natLeq* **by** (*rule infinite-UNIV-nat*)

lemma *natLeq-Cinfinite*: *Cinfinite natLeq*
using *natLeq-cinfinite natLeq-Card-order* **by** *simp*

lemma *natLeq-ordLeq-cinfinite*:

assumes *inf*: *Cinfinite r*
shows *natLeq* $\leq o$ *r*
proof –
from *inf* **have** *natLeq* $\leq o$ |*Field r*| **unfolding** *cinfinite-def*
using *infinite-iff-natLeq-ordLeq* **by** *blast*
also from *inf* **have** |*Field r*| $=o$ *r* **by** (*simp add*: *card-of-unique ordIso-symmetric*)
finally show *?thesis* **.**
qed

lemma *cinfinite-not-czero*: *cinfinite r* $\Longrightarrow \neg$ (*r* $=o$ (*czero* :: '*a rel*))
unfolding *cinfinite-def* **by** (*cases Field r* = {}) (*auto dest*: *czeroE*)

lemma *Cinfinite-Cnotzero*: *Cinfinite r* $\Longrightarrow$ *Cnotzero r*
using *cinfinite-not-czero* **by** *auto*

lemma *Cinfinite-cong*: ⟦*r1* $=o$ *r2*; *Cinfinite r1*⟧ $\Longrightarrow$ *Cinfinite r2*
using *Card-order-ordIso2*[*of r1 r2*] **unfolding** *cinfinite-def ordIso-iff-ordLeq*
by (*auto dest*: *card-of-ordLeq-infinite*[*OF card-of-mono2*])

lemma *cinfinite-mono*: ⟦*r1* $\leq o$ *r2*; *cinfinite r1*⟧ $\Longrightarrow$ *cinfinite r2*
unfolding *cinfinite-def* **by** (*auto dest*: *card-of-ordLeq-infinite*[*OF card-of-mono2*])

lemma *regularCard-ordIso*:
assumes *k* $=o$ *k'* **and** *Cinfinite k* **and** *regularCard k*
shows *regularCard k'*
proof–
have *stable k* **using** *assms cinfinite-def regularCard-stable* **by** *blast*
hence *stable k'* **using** *assms stable-ordIso1 ordIso-symmetric* **by** *blast*
thus *?thesis* **using** *assms cinfinite-def stable-regularCard Cinfinite-cong* **by** *blast*
qed

corollary *card-of-UNION-ordLess-infinite-Field-regularCard*:
assumes *regularCard r* **and** *Cinfinite r* **and** |*I*| $<o$ *r* **and** $\forall$ *i* $\in$ *I*. |*A i*| $<o$ *r*
shows |$\bigcup$ *i* $\in$ *I*. *A i*| $<o$ *r*
using *card-of-UNION-ordLess-infinite-Field regularCard-stable assms cinfinite-def*
by *blast*

31.3 Binary sum

definition *csum* (**infixr** ‹+*c*› *65*)
where *r1* $+c$ *r2* $\equiv$ |*Field r1* <+> *Field r2*|

lemma *Field-csum*: *Field* (*r* $+c$ *s*) = *Inl* ‘ *Field r* $\cup$ *Inr* ‘ *Field s*
unfolding *csum-def Field-card-of* **by** *auto*

lemma *Card-order-csum*: *Card-order* (*r1* $+c$ *r2*)
unfolding *csum-def* **by** (*simp add*: *card-of-Card-order*)

lemma *csum-Cnotzero1*: *Cnotzero r1* $\Longrightarrow$ *Cnotzero* (*r1* $+c$ *r2*)

using *Cnotzero-imp-not-empty*
by (*auto intro*: *card-of-Card-order simp*: *csum-def card-of-ordIso-czero-iff-empty*)

lemma *card-order-csum*:
assumes *card-order r1 card-order r2*
shows *card-order* (*r1* +*c r2*)
proof −
have *Field r1* = *UNIV Field r2* = *UNIV* **using** *assms card-order-on-Card-order* **by** *auto*
thus *?thesis* **unfolding** *csum-def* **by** (*auto simp*: *card-of-card-order-on*)
qed

lemma *cinfinite-csum*:
cinfinite r1 ∨ *cinfinite r2* ⟹ *cinfinite* (*r1* +*c r2*)
unfolding *cinfinite-def csum-def* **by** (*auto simp*: *Field-card-of*)

lemma *Cinfinite-csum*:
Cinfinite r1 ∨ *Cinfinite r2* ⟹ *Cinfinite* (*r1* +*c r2*)
using *card-of-Card-order*
by (*auto simp*: *cinfinite-def csum-def Field-card-of*)

lemma *Cinfinite-csum1*: *Cinfinite r1* ⟹ *Cinfinite* (*r1* +*c r2*)
by (*blast intro*!: *Cinfinite-csum elim*:)

lemma *Cinfinite-csum-weak*:
⟦*Cinfinite r1*; *Cinfinite r2*⟧ ⟹ *Cinfinite* (*r1* +*c r2*)
by (*erule Cinfinite-csum1*)

lemma *csum-cong*: ⟦*p1* =*o r1*; *p2* =*o r2*⟧ ⟹ *p1* +*c p2* =*o r1* +*c r2*
by (*simp only*: *csum-def ordIso-Plus-cong*)

lemma *csum-cong1*: *p1* =*o r1* ⟹ *p1* +*c q* =*o r1* +*c q*
by (*simp only*: *csum-def ordIso-Plus-cong1*)

lemma *csum-cong2*: *p2* =*o r2* ⟹ *q* +*c p2* =*o q* +*c r2*
by (*simp only*: *csum-def ordIso-Plus-cong2*)

lemma *csum-mono*: ⟦*p1* ≤*o r1*; *p2* ≤*o r2*⟧ ⟹ *p1* +*c p2* ≤*o r1* +*c r2*
by (*simp only*: *csum-def ordLeq-Plus-mono*)

lemma *csum-mono1*: *p1* ≤*o r1* ⟹ *p1* +*c q* ≤*o r1* +*c q*
by (*simp only*: *csum-def ordLeq-Plus-mono1*)

lemma *csum-mono2*: *p2* ≤*o r2* ⟹ *q* +*c p2* ≤*o q* +*c r2*
by (*simp only*: *csum-def ordLeq-Plus-mono2*)

lemma *ordLeq-csum1*: *Card-order p1* ⟹ *p1* ≤*o p1* +*c p2*
by (*simp only*: *csum-def Card-order-Plus1*)

lemma *ordLeq-csum2*: *Card-order p2* ⟹ *p2* ≤*o p1* +*c p2*
by (*simp only*: *csum-def Card-order-Plus2*)

lemma *csum-com*: *p1* +*c p2* =*o p2* +*c p1*
by (*simp only*: *csum-def card-of-Plus-commute*)

lemma *csum-assoc*: (*p1* +*c p2*) +*c p3* =*o p1* +*c p2* +*c p3*
by (*simp only*: *csum-def Field-card-of card-of-Plus-assoc*)

lemma *Cfinite-csum*: ⟦*Cfinite r*; *Cfinite s*⟧ ⟹ *Cfinite* (*r* +*c s*)
unfolding *cfinite-def csum-def Field-card-of* **using** *card-of-card-order-on* **by** *simp*

lemma *csum-csum*: (*r1* +*c r2*) +*c* (*r3* +*c r4*) =*o* (*r1* +*c r3*) +*c* (*r2* +*c r4*)
proof −
have (*r1* +*c r2*) +*c* (*r3* +*c r4*) =*o r1* +*c r2* +*c* (*r3* +*c r4*)
by (*rule csum-assoc*)
also have *r1* +*c r2* +*c* (*r3* +*c r4*) =*o r1* +*c* (*r2* +*c r3*) +*c r4*
by (*intro csum-assoc csum-cong2 ordIso-symmetric*)
also have *r1* +*c* (*r2* +*c r3*) +*c r4* =*o r1* +*c* (*r3* +*c r2*) +*c r4*
by (*intro csum-com csum-cong1 csum-cong2*)
also have *r1* +*c* (*r3* +*c r2*) +*c r4* =*o r1* +*c r3* +*c r2* +*c r4*
by (*intro csum-assoc csum-cong2 ordIso-symmetric*)
also have *r1* +*c r3* +*c r2* +*c r4* =*o* (*r1* +*c r3*) +*c* (*r2* +*c r4*)
by (*intro csum-assoc ordIso-symmetric*)
finally show *?thesis* **.**
qed

lemma *Plus-csum*: |*A* <+> *B*| =*o* |*A*| +*c* |*B*|
by (*simp only*: *csum-def Field-card-of card-of-refl*)

lemma *Un-csum*: |*A* ∪ *B*| ≤*o* |*A*| +*c* |*B*|
using *ordLeq-ordIso-trans*[*OF card-of-Un-Plus-ordLeq Plus-csum*] **by** *blast*

31.4 One

definition *cone* **where**
cone = *card-of* {()}

lemma *Card-order-cone*: *Card-order cone*
unfolding *cone-def* **by** (*rule card-of-Card-order*)

lemma *Cfinite-cone*: *Cfinite cone*
unfolding *cfinite-def* **by** (*simp add*: *Card-order-cone*)

lemma *cone-not-czero*: ¬ (*cone* =*o czero*)
unfolding *czero-def cone-def ordIso-iff-ordLeq*
using *card-of-empty3 empty-not-insert* **by** *blast*

lemma *cone-ordLeq-Cnotzero*: *Cnotzero r* ⟹ *cone* ≤*o r*
unfolding *cone-def* **by** (*rule Card-order-singl-ordLeq*) (*auto intro*: *czeroI*)

31.5 Two

definition *ctwo* **where**
ctwo = |*UNIV* :: *bool set*|

lemma *Card-order-ctwo*: *Card-order ctwo*
unfolding *ctwo-def* **by** (*rule card-of-Card-order*)

lemma *ctwo-not-czero*: ¬ (*ctwo* =*o czero*)
using *card-of-empty3*[*of UNIV* :: *bool set*] *ordIso-iff-ordLeq*
unfolding *czero-def ctwo-def* **using** *UNIV-not-empty* **by** *auto*

lemma *ctwo-Cnotzero*: *Cnotzero ctwo*
by (*simp add*: *ctwo-not-czero Card-order-ctwo*)

31.6 Family sum

definition *Csum* **where**
Csum r rs ≡ |*SIGMA i* : *Field r*. *Field* (*rs i*)|

syntax *-Csum* ::
pttrn => ('*a* ∗ '*a*) *set* => '*b* ∗ '*b set* => (('*a* ∗ '*b*) ∗ ('*a* ∗ '*b*)) *set*
(‹(‹*indent=3 notation=*‹*binder CSUM*››*CSUM* -:-. -)› [*0*, *51*, *10*] *10*)

syntax-consts
-Csum == *Csum*

translations
CSUM i:*r*. *rs* == *CONST Csum r* (%*i*. *rs*)

lemma *SIGMA-CSUM*: |*SIGMA i* : *I*. *As i*| = (*CSUM i* : |*I*|. |*As i*|)
by (*auto simp*: *Csum-def Field-card-of*)

31.7 Product

definition *cprod* (**infixr** ‹∗*c*› *80*) **where**
r1 ∗*c r2* = |*Field r1* × *Field r2*|

lemma *card-order-cprod*:
assumes *card-order r1 card-order r2*
shows *card-order* (*r1* ∗*c r2*)
proof −
have *Field r1* = *UNIV Field r2* = *UNIV*
using *assms card-order-on-Card-order* **by** *auto*
thus *?thesis* **by** (*auto simp*: *cprod-def card-of-card-order-on*)
qed

lemma *Card-order-cprod*: *Card-order* (*r1* ∗*c* *r2*)
by (*simp only*: *cprod-def Field-card-of card-of-card-order-on*)

lemma *cprod-mono1*: *p1* ≤*o* *r1* ⟹ *p1* ∗*c* *q* ≤*o* *r1* ∗*c* *q*
by (*simp only*: *cprod-def ordLeq-Times-mono1*)

lemma *cprod-mono2*: *p2* ≤*o* *r2* ⟹ *q* ∗*c* *p2* ≤*o* *q* ∗*c* *r2*
by (*simp only*: *cprod-def ordLeq-Times-mono2*)

lemma *cprod-mono*: ⟦*p1* ≤*o* *r1*; *p2* ≤*o* *r2*⟧ ⟹ *p1* ∗*c* *p2* ≤*o* *r1* ∗*c* *r2*
by (*rule ordLeq-transitive*[*OF cprod-mono1 cprod-mono2*])

lemma *ordLeq-cprod2*: ⟦*Cnotzero p1*; *Card-order p2*⟧ ⟹ *p2* ≤*o* *p1* ∗*c* *p2*
unfolding *cprod-def* **by** (*rule Card-order-Times2*) (*auto intro*: *czeroI*)

lemma *cinfinite-cprod*: ⟦*cinfinite r1*; *cinfinite r2*⟧ ⟹ *cinfinite* (*r1* ∗*c* *r2*)
by (*simp add*: *cinfinite-def cprod-def Field-card-of infinite-cartesian-product*)

lemma *cinfinite-cprod2*: ⟦*Cnotzero r1*; *Cinfinite r2*⟧ ⟹ *cinfinite* (*r1* ∗*c* *r2*)
by (*rule cinfinite-mono*) (*auto intro*: *ordLeq-cprod2*)

lemma *Cinfinite-cprod2*: ⟦*Cnotzero r1*; *Cinfinite r2*⟧ ⟹ *Cinfinite* (*r1* ∗*c* *r2*)
by (*blast intro*: *cinfinite-cprod2 Card-order-cprod*)

lemma *cprod-cong*: ⟦*p1* =*o* *r1*; *p2* =*o* *r2*⟧ ⟹ *p1* ∗*c* *p2* =*o* *r1* ∗*c* *r2*
unfolding *ordIso-iff-ordLeq* **by** (*blast intro*: *cprod-mono*)

lemma *cprod-cong1*: ⟦*p1* =*o* *r1*⟧ ⟹ *p1* ∗*c* *p2* =*o* *r1* ∗*c* *p2*
unfolding *ordIso-iff-ordLeq* **by** (*blast intro*: *cprod-mono1*)

lemma *cprod-cong2*: *p2* =*o* *r2* ⟹ *q* ∗*c* *p2* =*o* *q* ∗*c* *r2*
unfolding *ordIso-iff-ordLeq* **by** (*blast intro*: *cprod-mono2*)

lemma *cprod-com*: *p1* ∗*c* *p2* =*o* *p2* ∗*c* *p1*
by (*simp only*: *cprod-def card-of-Times-commute*)

lemma *card-of-Csum-Times*:
∀ *i* ∈ *I*. |*A i*| ≤*o* |*B*| ⟹ (*CSUM i* : |*I*|. |*A i*|) ≤*o* |*I*| ∗*c* |*B*|
by (*simp only*: *Csum-def cprod-def Field-card-of card-of-Sigma-mono1*)

lemma *card-of-Csum-Times'*:
assumes *Card-order r* ∀ *i* ∈ *I*. |*A i*| ≤*o* *r*
shows (*CSUM i* : |*I*|. |*A i*|) ≤*o* |*I*| ∗*c* *r*
proof −
from *assms*(*1*) **have** ∗: *r* =*o* |*Field r*| **by** (*simp add*: *card-of-unique*)
with *assms*(*2*) **have** ∀ *i* ∈ *I*. |*A i*| ≤*o* |*Field r*| **by** (*blast intro*: *ordLeq-ordIso-trans*)
hence (*CSUM i* : |*I*|. |*A i*|) ≤*o* |*I*| ∗*c* |*Field r*| **by** (*simp only*: *card-of-Csum-Times*)
also from ∗ **have** |*I*| ∗*c* |*Field r*| ≤*o* |*I*| ∗*c* *r*

by (*simp only*: *Field-card-of card-of-refl cprod-def ordIso-imp-ordLeq*)
finally show *?thesis* **.**
qed

lemma *cprod-csum-distrib1*: *r1* $*c$ *r2* $+c$ *r1* $*c$ *r3* $=o$ *r1* $*c$ (*r2* $+c$ *r3*)
unfolding *csum-def cprod-def* **by** (*simp add*: *Field-card-of card-of-Times-Plus-distrib ordIso-symmetric*)

lemma *csum-absorb2'*: ⟦*Card-order r2*; *r1* $\leq o$ *r2*; *cinfinite r1* ∨ *cinfinite r2*⟧ ⟹ *r1* $+c$ *r2* $=o$ *r2*
unfolding *csum-def*
using *Card-order-Plus-infinite*
by (*fastforce simp*: *cinfinite-def dest*: *cinfinite-mono*)

lemma *csum-absorb1'*:
assumes *card*: *Card-order r2*
and *r12*: *r1* $\leq o$ *r2* **and** *cr12*: *cinfinite r1* ∨ *cinfinite r2*
shows *r2* $+c$ *r1* $=o$ *r2*
proof −
have *r1* $+c$ *r2* $=o$ *r2*
by (*simp add*: *csum-absorb2' assms*)
then show *?thesis*
by (*blast intro*: *ordIso-transitive csum-com*)
qed

lemma *csum-absorb1*: ⟦*Cinfinite r2*; *r1* $\leq o$ *r2*⟧ ⟹ *r2* $+c$ *r1* $=o$ *r2*
by (*rule csum-absorb1'*) *auto*

lemma *csum-absorb2*: ⟦*Cinfinite r2* ; *r1* $\leq o$ *r2*⟧ ⟹ *r1* $+c$ *r2* $=o$ *r2*
using *ordIso-transitive csum-com csum-absorb1* **by** *blast*

lemma *regularCard-csum*:
assumes *Cinfinite r Cinfinite s regularCard r regularCard s*
shows *regularCard* (*r* $+c$ *s*)
proof (*cases r* $\leq o$ *s*)
case *True*
then show *?thesis* **using** *regularCard-ordIso*[*of s*] *csum-absorb2'*[*THEN ordIso-symmetric*] *assms* **by** *auto*
next
case *False*
have *Well-order s Well-order r* **using** *assms card-order-on-well-order-on* **by** *auto*
then have *s* $\leq o$ *r* **using** *not-ordLeq-iff-ordLess False ordLess-imp-ordLeq* **by** *auto*
then show *?thesis* **using** *regularCard-ordIso*[*of r*] *csum-absorb1'*[*THEN ordIso-symmetric*] *assms* **by** *auto*
qed

lemma *csum-mono-strict*:
assumes *Card-order*: *Card-order r Card-order q*

and *Cinfinite*: *Cinfinite r′ Cinfinite q′*
and *less*: *r <o r′ q <o q′*
shows *r +c q <o r′ +c q′*
proof −
have *Well-order*: *Well-order r Well-order q Well-order r′ Well-order q′*
using *card-order-on-well-order-on Card-order Cinfinite* **by** *auto*
show *?thesis*
proof (*cases Cinfinite r*)
case *outer*: *True*
then show *?thesis*
proof (*cases Cinfinite q*)
case *inner*: *True*
then show *?thesis*
proof (*cases r ≤o q*)
case *True*
then have *r +c q =o q* **using** *csum-absorb2 inner* **by** *blast*
then show *?thesis*
using *ordIso-ordLess-trans ordLess-ordLeq-trans less Cinfinite ordLeq-csum2* **by** *blast*
next
case *False*
then have *q ≤o r* **using** *not-ordLeq-iff-ordLess Well-order ordLess-imp-ordLeq* **by** *blast*
then have *r +c q =o r* **using** *csum-absorb1 outer* **by** *blast*
then show *?thesis*
using *ordIso-ordLess-trans ordLess-ordLeq-trans less Cinfinite ordLeq-csum1* **by** *blast*
qed
next
case *False*
then have *Cfinite q* **using** *Card-order cinfinite-def cfinite-def* **by** *blast*
then have *q ≤o r* **using** *finite-ordLess-infinite cfinite-def cinfinite-def outer Well-order ordLess-imp-ordLeq* **by** *blast*
then have *r +c q =o r* **by** (*rule csum-absorb1*[*OF outer*])
then show *?thesis* **using** *ordIso-ordLess-trans ordLess-ordLeq-trans less ordLeq-csum1 Cinfinite* **by** *blast*
qed
next
case *False*
then have *outer*: *Cfinite r* **using** *Card-order cinfinite-def cfinite-def* **by** *blast*
then show *?thesis*
proof (*cases Cinfinite q*)
case *True*
then have *r ≤o q* **using** *finite-ordLess-infinite cinfinite-def cfinite-def outer Well-order ordLess-imp-ordLeq* **by** *blast*
then have *r +c q =o q* **by** (*rule csum-absorb2*[*OF True*])
then show *?thesis* **using** *ordIso-ordLess-trans ordLess-ordLeq-trans less ordLeq-csum2 Cinfinite* **by** *blast*

```
    next
      case False
      then have Cfinite q using Card-order cinfinite-def cfinite-def by blast
      then have Cfinite (r +c q) using Cfinite-csum outer by blast
     moreover have Cinfinite (r′ +c q′) using Cinfinite-csum1 Cinfinite by blast
      ultimately show ?thesis using Cfinite-ordLess-Cinfinite by blast
    qed
  qed
qed
```

31.8 Exponentiation

```
definition cexp (infixr ‹^c› 90) where
  r1 ^c r2 ≡ |Func (Field r2) (Field r1)|

lemma Card-order-cexp: Card-order (r1 ^c r2)
  unfolding cexp-def by (rule card-of-Card-order)

lemma cexp-mono′:
  assumes 1: p1 ≤o r1 and 2: p2 ≤o r2
    and n: Field p2 = {} ⟹ Field r2 = {}
  shows p1 ^c p2 ≤o r1 ^c r2
proof(cases Field p1 = {})
  case True
  hence Field p2 ≠ {} ⟹ Func (Field p2) {} = {} unfolding Func-is-emp by
simp
  with True have |Field |Func (Field p2) (Field p1)|| ≤o cone
    unfolding cone-def Field-card-of
    by (cases Field p2 = {}, auto intro: surj-imp-ordLeq simp: Func-empty)
  hence |Func (Field p2) (Field p1)| ≤o cone by (simp add: Field-card-of cexp-def)
  hence p1 ^c p2 ≤o cone unfolding cexp-def .
  thus ?thesis
  proof (cases Field p2 = {})
    case True
    with n have Field r2 = {} .
    hence cone ≤o r1 ^c r2 unfolding cone-def cexp-def Func-def
      by (auto intro: card-of-ordLeqI[where f=λ- -. undefined])
    thus ?thesis using ‹p1 ^c p2 ≤o cone› ordLeq-transitive by auto
  next
    case False with True have |Field (p1 ^c p2)| =o czero
      unfolding card-of-ordIso-czero-iff-empty cexp-def Field-card-of Func-def by
auto
    thus ?thesis unfolding cexp-def card-of-ordIso-czero-iff-empty Field-card-of
      by (simp add: card-of-empty)
  qed
next
  case False
  have 1: |Field p1| ≤o |Field r1| and 2: |Field p2| ≤o |Field r2|
    using 1 2 by (auto simp: card-of-mono2)
```

obtain *f1* **where** *f1*: *f1 ' Field r1 = Field p1*
using *1* **unfolding** *card-of-ordLeq2*[*OF False*, *symmetric*] **by** *auto*
obtain *f2* **where** *f2*: *inj-on f2* (*Field p2*) *f2 ' Field p2* ⊆ *Field r2*
using *2* **unfolding** *card-of-ordLeq*[*symmetric*] **by** *blast*
have *0*: *Func-map* (*Field p2*) *f1 f2 '* (*Field* (*r1 ^c r2*)) = *Field* (*p1 ^c p2*)
unfolding *cexp-def Field-card-of* **using** *Func-map-surj*[*OF f1 f2 n*, *symmetric*]
.
have *00*: *Field* (*p1 ^c p2*) ≠ {} **unfolding** *cexp-def Field-card-of Func-is-emp*
using *False* **by** *simp*
show *?thesis*
using *0 card-of-ordLeq2*[*OF 00*] **unfolding** *cexp-def Field-card-of* **by** *blast*
qed

lemma *cexp-mono*:
assumes *1*: *p1* ≤*o r1* **and** *2*: *p2* ≤*o r2*
and *n*: *p2* =*o czero* ⟹ *r2* =*o czero* **and** *card*: *Card-order p2*
shows *p1 ^c p2* ≤*o r1 ^c r2*
by (*rule cexp-mono'*[*OF 1 2 czeroE*[*OF n*[*OF czeroI*[*OF card*]]]])

lemma *cexp-mono1*:
assumes *1*: *p1* ≤*o r1* **and** *q*: *Card-order q*
shows *p1 ^c q* ≤*o r1 ^c q*
using *ordLeq-refl*[*OF q*] **by** (*rule cexp-mono*[*OF 1*]) (*auto simp*: *q*)

lemma *cexp-mono2'*:
assumes *2*: *p2* ≤*o r2* **and** *q*: *Card-order q*
and *n*: *Field p2* = {} ⟹ *Field r2* = {}
shows *q ^c p2* ≤*o q ^c r2*
using *ordLeq-refl*[*OF q*] **by** (*rule cexp-mono'*[*OF - 2 n*]) *auto*

lemma *cexp-mono2*:
assumes *2*: *p2* ≤*o r2* **and** *q*: *Card-order q*
and *n*: *p2* =*o czero* ⟹ *r2* =*o czero* **and** *card*: *Card-order p2*
shows *q ^c p2* ≤*o q ^c r2*
using *ordLeq-refl*[*OF q*] **by** (*rule cexp-mono*[*OF - 2 n card*]) *auto*

lemma *cexp-mono2-Cnotzero*:
assumes *p2* ≤*o r2 Card-order q Cnotzero p2*
shows *q ^c p2* ≤*o q ^c r2*
using *assms*(*3*) *czeroI* **by** (*blast intro*: *cexp-mono2'*[*OF assms*(*1*,*2*)])

lemma *cexp-cong*:
assumes *1*: *p1* =*o r1* **and** *2*: *p2* =*o r2*
and *Cr*: *Card-order r2*
and *Cp*: *Card-order p2*
shows *p1 ^c p2* =*o r1 ^c r2*
proof −
obtain *f* **where** *bij-betw f* (*Field p2*) (*Field r2*)
using *2 card-of-ordIso*[*of Field p2 Field r2*] *card-of-cong* **by** *auto*

hence *0*: *Field p2* = {} ⟷ *Field r2* = {} **unfolding** *bij-betw-def* **by** *auto*
have *r*: *p2 =o czero* ⟹ *r2 =o czero*
and *p*: *r2 =o czero* ⟹ *p2 =o czero*
using *0 Cr Cp czeroE czeroI* **by** *auto*
show *?thesis* **using** *0 1 2* **unfolding** *ordIso-iff-ordLeq*
using *r p cexp-mono*[*OF - - - Cp*] *cexp-mono*[*OF - - - Cr*] **by** *blast*
qed

lemma *cexp-cong1*:
assumes *1*: *p1 =o r1* **and** *q*: *Card-order q*
shows *p1 ^c q =o r1 ^c q*
by (*rule cexp-cong*[*OF 1 - q q*]) (*rule ordIso-refl*[*OF q*])

lemma *cexp-cong2*:
assumes *2*: *p2 =o r2* **and** *q*: *Card-order q* **and** *p*: *Card-order p2*
shows *q ^c p2 =o q ^c r2*
by (*rule cexp-cong*[*OF - 2*]) (*auto simp only*: *ordIso-refl Card-order-ordIso2*[*OF p 2*] *q p*)

lemma *cexp-cone*:
assumes *Card-order r*
shows *r ^c cone =o r*
proof –
have *r ^c cone =o* |*Field r*|
unfolding *cexp-def cone-def Field-card-of Func-empty*
card-of-ordIso[*symmetric*] *bij-betw-def Func-def inj-on-def image-def*
by (*rule exI*[*of - λf. f* ()]) *auto*
also have |*Field r*| *=o r* **by** (*rule card-of-Field-ordIso*[*OF assms*])
finally show *?thesis* **.**
qed

lemma *cexp-cprod*:
assumes *r1*: *Card-order r1*
shows (*r1 ^c r2*) *^c r3 =o r1 ^c* (*r2 *c r3*) (**is** *?L =o ?R*)
proof –
have *?L =o r1 ^c* (*r3 *c r2*)
unfolding *cprod-def cexp-def Field-card-of*
using *card-of-Func-Times* **by**(*rule ordIso-symmetric*)
also have *r1 ^c* (*r3 *c r2*) *=o ?R*
using *cprod-com r1* **by** (*intro cexp-cong2*, *auto simp*: *Card-order-cprod*)
finally show *?thesis* **.**
qed

lemma *cprod-infinite1'*: ⟦*Cinfinite r*; *Cnotzero p*; *p ≤o r*⟧ ⟹ *r *c p =o r*
unfolding *cinfinite-def cprod-def*
by (*rule Card-order-Times-infinite*[*THEN conjunct1*]) (*blast intro*: *czeroI*)+

lemma *cprod-infinite*: *Cinfinite r* ⟹ *r *c r =o r*
using *cprod-infinite1' Cinfinite-Cnotzero ordLeq-refl* **by** *blast*

lemma *cexp-cprod-ordLeq*:
assumes *r1*: *Card-order r1* **and** *r2*: *Cinfinite r2*
and *r3*: *Cnotzero r3 r3* $\leq o$ *r2*
shows (*r1* ^*c r2*) ^*c r3* =*o r1* ^*c r2* (**is** *?L* =*o ?R*)
proof−
have *?L* =*o r1* ^*c* (*r2* ∗*c r3*) **using** *cexp-cprod*[*OF r1*] **.**
also have *r1* ^*c* (*r2* ∗*c r3*) =*o ?R*
using *assms* **by** (*fastforce simp*: *Card-order-cprod intro*: *cprod-infinite1' cexp-cong2*)
finally show *?thesis* **.**
qed

lemma *Cnotzero-UNIV*: *Cnotzero* |*UNIV*|
by (*auto simp*: *card-of-Card-order card-of-ordIso-czero-iff-empty*)

lemma *ordLess-ctwo-cexp*:
assumes *Card-order r*
shows *r* <*o ctwo* ^*c r*
proof −
have *r* <*o* |*Pow* (*Field r*)| **using** *assms* **by** (*rule Card-order-Pow*)
also have |*Pow* (*Field r*)| =*o ctwo* ^*c r*
unfolding *ctwo-def cexp-def Field-card-of* **by** (*rule card-of-Pow-Func*)
finally show *?thesis* **.**
qed

lemma *ordLeq-cexp1*:
assumes *Cnotzero r Card-order q*
shows *q* $\leq o$ *q* ^*c r*
proof (*cases q* =*o* (*czero* :: *'a rel*))
case *True* **thus** *?thesis* **by** (*simp only*: *card-of-empty cexp-def czero-def ordIso-ordLeq-trans*)
next
case *False*
have *q* =*o q* ^*c cone*
by (*blast intro*: *assms ordIso-symmetric cexp-cone*)
also have *q* ^*c cone* $\leq o$ *q* ^*c r*
using *assms*
by (*intro cexp-mono2*) (*simp-all add*: *cone-ordLeq-Cnotzero cone-not-czero Card-order-cone*)
finally show *?thesis* **.**
qed

lemma *ordLeq-cexp2*:
assumes *ctwo* $\leq o$ *q Card-order r*
shows *r* $\leq o$ *q* ^*c r*
proof (*cases r* =*o* (*czero* :: *'a rel*))
case *True* **thus** *?thesis* **by** (*simp only*: *card-of-empty cexp-def czero-def ordIso-ordLeq-trans*)
next

case *False*
have *r* $<o$ *ctwo* $\hat{}c$ *r*
by (*blast intro*: *assms ordLess-ctwo-cexp*)
also have *ctwo* $\hat{}c$ *r* $\leq o$ *q* $\hat{}c$ *r*
by (*blast intro*: *assms cexp-mono1*)
finally show *?thesis* **by** (*rule ordLess-imp-ordLeq*)
qed

lemma *cinfinite-cexp*: ⟦*ctwo* $\leq o$ *q*; *Cinfinite r*⟧ $\Longrightarrow$ *cinfinite* (*q* $\hat{}c$ *r*)
by (*rule cinfinite-mono*[*OF ordLeq-cexp2*]) *simp-all*

lemma *Cinfinite-cexp*:
⟦*ctwo* $\leq o$ *q*; *Cinfinite r*⟧ $\Longrightarrow$ *Cinfinite* (*q* $\hat{}c$ *r*)
by (*simp add*: *cinfinite-cexp Card-order-cexp*)

lemma *card-order-cexp*:
assumes *card-order r1 card-order r2*
shows *card-order* (*r1* $\hat{}c$ *r2*)
proof −
have *Field r1* = *UNIV Field r2* = *UNIV* **using** *assms card-order-on-Card-order* **by** *auto*
thus *?thesis* **unfolding** *cexp-def Func-def* **using** *card-of-card-order-on* **by** *simp*
qed

lemma *ctwo-ordLess-natLeq*: *ctwo* $<o$ *natLeq*
unfolding *ctwo-def* **using** *finite-UNIV natLeq-cinfinite natLeq-Card-order*
by (*intro Cfinite-ordLess-Cinfinite*) (*auto simp*: *cfinite-def card-of-Card-order*)

lemma *ctwo-ordLess-Cinfinite*: *Cinfinite r* $\Longrightarrow$ *ctwo* $<o$ *r*
by (*rule ordLess-ordLeq-trans*[*OF ctwo-ordLess-natLeq natLeq-ordLeq-cinfinite*])

lemma *ctwo-ordLeq-Cinfinite*:
assumes *Cinfinite r*
shows *ctwo* $\leq o$ *r*
by (*rule ordLess-imp-ordLeq*[*OF ctwo-ordLess-Cinfinite*[*OF assms*]])

lemma *Un-Cinfinite-bound*: ⟦$|A| \leq o$ *r*; $|B| \leq o$ *r*; *Cinfinite r*⟧ $\Longrightarrow$ $|A \cup B| \leq o$ *r*
by (*auto simp add*: *cinfinite-def card-of-Un-ordLeq-infinite-Field*)

lemma *Un-Cinfinite-bound-strict*: ⟦$|A| <o$ *r*; $|B| <o$ *r*; *Cinfinite r*⟧ $\Longrightarrow$ $|A \cup B| <o$ *r*
by (*auto simp add*: *cinfinite-def card-of-Un-ordLess-infinite-Field*)

lemma *UNION-Cinfinite-bound*: ⟦$|I| \leq o$ *r*; $\forall i \in I.\ |A\ i| \leq o$ *r*; *Cinfinite r*⟧ $\Longrightarrow$ $|\bigcup i \in I.\ A\ i| \leq o$ *r*
by (*auto simp add*: *card-of-UNION-ordLeq-infinite-Field cinfinite-def*)

lemma *csum-cinfinite-bound*:
assumes *p* $\leq o$ *r* *q* $\leq o$ *r* *Card-order p Card-order q Cinfinite r*

shows *p +c q ≤o r*
proof −
have *|Field p| ≤o r |Field q| ≤o r*
using *assms card-of-least ordLeq-transitive* **unfolding** *card-order-on-def* **by** *blast+*
with *assms* **show** *?thesis* **unfolding** *cinfinite-def csum-def*
by (*blast intro*: *card-of-Plus-ordLeq-infinite-Field*)
qed

lemma *cprod-cinfinite-bound*:
assumes *p ≤o r q ≤o r Card-order p Card-order q Cinfinite r*
shows *p ∗c q ≤o r*
proof −
from *assms*(*1−4*) **have** *|Field p| ≤o r |Field q| ≤o r*
unfolding *card-order-on-def* **using** *card-of-least ordLeq-transitive* **by** *blast+*
with *assms* **show** *?thesis* **unfolding** *cinfinite-def cprod-def*
by (*blast intro*: *card-of-Times-ordLeq-infinite-Field*)
qed

lemma *cprod-infinite2'*: ⟦*Cnotzero r1*; *Cinfinite r2*; *r1 ≤o r2*⟧ ⟹ *r1 ∗c r2 =o r2*
unfolding *ordIso-iff-ordLeq*
by (*intro conjI cprod-cinfinite-bound ordLeq-cprod2 ordLeq-refl*)
(*auto dest*!: *ordIso-imp-ordLeq not-ordLeq-ordLess simp*: *czero-def Card-order-empty*)

lemma *regularCard-cprod*:
assumes *Cinfinite r Cinfinite s regularCard r regularCard s*
shows *regularCard* (*r ∗c s*)
proof (*cases r ≤o s*)
case *True*
with *assms Cinfinite-Cnotzero* **show** *?thesis*
by (*force intro*: *regularCard-ordIso ordIso-symmetric*[*OF cprod-infinite2'*])
next
case *False*
have *Well-order r Well-order s* **using** *assms card-order-on-well-order-on* **by** *auto*
then have *s ≤o r* **using** *not-ordLeq-iff-ordLess ordLess-imp-ordLeq False* **by** *blast*
with *assms Cinfinite-Cnotzero* **show** *?thesis*
by (*force intro*: *regularCard-ordIso ordIso-symmetric*[*OF cprod-infinite1'*])
qed

lemma *cprod-csum-cexp*:
r1 ∗c r2 ≤o (*r1 +c r2*) *^c ctwo*
unfolding *cprod-def csum-def cexp-def ctwo-def Field-card-of*
proof −
let *?f* = *λ*(*a*, *b*). *%x*. *if x then Inl a else Inr b*
have *inj-on ?f* (*Field r1* × *Field r2*) (**is** *inj-on - ?LHS*)
by (*auto simp*: *inj-on-def fun-eq-iff split*: *bool.split*)
moreover

have ?f ‘ ?LHS ⊆ Func (UNIV :: bool set) (Field r1 <+> Field r2) (**is** - ⊆ ?RHS)
by (auto simp: Func-def)
ultimately show |?LHS| ≤o |?RHS| **using** card-of-ordLeq **by** blast
qed

lemma Cfinite-cprod-Cinfinite: ⟦Cfinite r; Cinfinite s⟧ ⟹ r *c s ≤o s
by (intro cprod-cinfinite-bound)
(auto intro: ordLeq-refl ordLess-imp-ordLeq[OF Cfinite-ordLess-Cinfinite])

lemma cprod-cexp: (r *c s) ^c t =o r ^c t *c s ^c t
unfolding cprod-def cexp-def Field-card-of **by** (rule Func-Times-Range)

lemma cprod-cexp-csum-cexp-Cinfinite:
assumes t: Cinfinite t
shows (r *c s) ^c t ≤o (r +c s) ^c t
proof −
have (r *c s) ^c t ≤o ((r +c s) ^c ctwo) ^c t
by (rule cexp-mono1[OF cprod-csum-cexp conjunct2[OF t]])
also have ((r +c s) ^c ctwo) ^c t =o (r +c s) ^c (ctwo *c t)
by (rule cexp-cprod[OF Card-order-csum])
also have (r +c s) ^c (ctwo *c t) =o (r +c s) ^c (t *c ctwo)
by (rule cexp-cong2[OF cprod-com Card-order-csum Card-order-cprod])
also have (r +c s) ^c (t *c ctwo) =o ((r +c s) ^c t) ^c ctwo
by (rule ordIso-symmetric[OF cexp-cprod[OF Card-order-csum]])
also have ((r +c s) ^c t) ^c ctwo =o (r +c s) ^c t
by (rule cexp-cprod-ordLeq[OF Card-order-csum t ctwo-Cnotzero ctwo-ordLeq-Cinfinite[OF t]])
finally show ?thesis **.**
qed

lemma Cfinite-cexp-Cinfinite:
assumes s: Cfinite s **and** t: Cinfinite t
shows s ^c t ≤o ctwo ^c t
proof (cases s ≤o ctwo)
case True **thus** ?thesis **using** t **by** (blast intro: cexp-mono1)
next
case False
hence ctwo ≤o s **using** ordLeq-total[of s ctwo] Card-order-ctwo s
by (auto intro: card-order-on-well-order-on)
hence Cnotzero s **using** Cnotzero-mono[OF ctwo-Cnotzero] s **by** blast
hence st: Cnotzero (s *c t) **by** (intro Cinfinite-Cnotzero[OF Cinfinite-cprod2]) (auto simp: t)
have s ^c t ≤o (ctwo ^c s) ^c t
using assms **by** (blast intro: cexp-mono1 ordLess-imp-ordLeq[OF ordLess-ctwo-cexp])
also have (ctwo ^c s) ^c t =o ctwo ^c (s *c t)
by (blast intro: Card-order-ctwo cexp-cprod)
also have ctwo ^c (s *c t) ≤o ctwo ^c t
using assms st **by** (intro cexp-mono2-Cnotzero Cfinite-cprod-Cinfinite Card-order-ctwo)

finally show *?thesis* **.**
qed

lemma *csum-Cfinite-cexp-Cinfinite*:
assumes *r*: *Card-order r* **and** *s*: *Cfinite s* **and** *t*: *Cinfinite t*
shows (*r* +*c s*) ^*c t* ≤*o* (*r* +*c ctwo*) ^*c t*
proof (*cases Cinfinite r*)
case *True*
hence *r* +*c s* =*o r* **by** (*intro csum-absorb1 ordLess-imp-ordLeq*[*OF Cfinite-ordLess-Cinfinite*] *s*)
hence (*r* +*c s*) ^*c t* =*o r* ^*c t* **using** *t* **by** (*blast intro*: *cexp-cong1*)
also have *r* ^*c t* ≤*o* (*r* +*c ctwo*) ^*c t* **using** *t* **by** (*blast intro*: *cexp-mono1 ordLeq-csum1 r*)
finally show *?thesis* **.**
next
case *False*
with *r* **have** *Cfinite r* **unfolding** *cinfinite-def cfinite-def* **by** *auto*
hence *Cfinite* (*r* +*c s*) **by** (*intro Cfinite-csum s*)
hence (*r* +*c s*) ^*c t* ≤*o ctwo* ^*c t* **by** (*intro Cfinite-cexp-Cinfinite t*)
also have *ctwo* ^*c t* ≤*o* (*r* +*c ctwo*) ^*c t* **using** *t*
by (*blast intro*: *cexp-mono1 ordLeq-csum2 Card-order-ctwo*)
finally show *?thesis* **.**
qed

lemma *Cinfinite-cardSuc*: *Cinfinite r* ⟹ *Cinfinite* (*cardSuc r*)
by (*simp add*: *cinfinite-def cardSuc-Card-order cardSuc-finite*)

lemma *cardSuc-UNION-Cinfinite*:
assumes *Cinfinite r relChain* (*cardSuc r*) *As B* ≤ (⋃*i* ∈ *Field* (*cardSuc r*). *As i*) |*B*| <=*o r*
shows ∃ *i* ∈ *Field* (*cardSuc r*). *B* ≤ *As i*
using *cardSuc-UNION assms* **unfolding** *cinfinite-def* **by** *blast*

lemma *Cinfinite-card-suc*: ⟦ *Cinfinite r* ; *card-order r* ⟧ ⟹ *Cinfinite* (*card-suc r*)
using *Cinfinite-cong*[*OF cardSuc-ordIso-card-suc Cinfinite-cardSuc*] **.**

lemma *card-suc-least*: ⟦*card-order r*; *Card-order s*; *r* <*o s*⟧ ⟹ *card-suc r* ≤*o s*
by (*rule ordIso-ordLeq-trans*[*OF ordIso-symmetric*[*OF cardSuc-ordIso-card-suc*]])
(*auto intro*!: *cardSuc-least simp*: *card-order-on-Card-order*)

lemma *regularCard-cardSuc*: *Cinfinite k* ⟹ *regularCard* (*cardSuc k*)
by (*rule infinite-cardSuc-regularCard*) (*auto simp*: *cinfinite-def*)

lemma *regularCard-card-suc*: *card-order r* ⟹ *Cinfinite r* ⟹ *regularCard* (*card-suc r*)
using *cardSuc-ordIso-card-suc Cinfinite-cardSuc regularCard-cardSuc regularCard-ordIso*
by *blast*

end

32 Function Definition Base

theory *Fun-Def-Base*
imports *Ctr-Sugar Set Wellfounded*
begin

ML-file ‹*Tools/Function/function-lib.ML*›
named-theorems *termination-simp simplification rules for termination proofs*
ML-file ‹*Tools/Function/function-common.ML*›
ML-file ‹*Tools/Function/function-context-tree.ML*›

attribute-setup *fundef-cong* =
‹*Attrib.add-del Function-Context-Tree.cong-add Function-Context-Tree.cong-del*›
declaration of congruence rule for function definitions

ML-file ‹*Tools/Function/sum-tree.ML*›

end

33 Definition of Bounded Natural Functors

theory *BNF-Def*
imports *BNF-Cardinal-Arithmetic Fun-Def-Base*
keywords
print-bnfs :: *diag* **and**
bnf :: *thy-goal-defn*
begin

lemma *Collect-case-prodD*: $x \in$ *Collect* (*case-prod A*) $\Longrightarrow$ *A* (*fst x*) (*snd x*)
by *auto*

inductive
rel-sum :: $('a \Rightarrow 'c \Rightarrow bool) \Rightarrow ('b \Rightarrow 'd \Rightarrow bool) \Rightarrow 'a + 'b \Rightarrow 'c + 'd \Rightarrow bool$
for *R1 R2*
where
R1 a c $\Longrightarrow$ *rel-sum R1 R2* (*Inl a*) (*Inl c*)
| *R2 b d* $\Longrightarrow$ *rel-sum R1 R2* (*Inr b*) (*Inr d*)

definition
rel-fun :: $('a \Rightarrow 'c \Rightarrow bool) \Rightarrow ('b \Rightarrow 'd \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('c \Rightarrow 'd) \Rightarrow bool$
where
rel-fun A B = $(\lambda f\ g.\ \forall x\ y.\ A\ x\ y \longrightarrow B\ (f\ x)\ (g\ y))$

lemma *rel-funI* [*intro*]:
assumes $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ (f\ x)\ (g\ y)$

shows *rel-fun A B f g*
using *assms* **by** (*simp add*: *rel-fun-def*)

lemma *rel-funD*:
assumes *rel-fun A B f g* **and** *A x y*
shows *B* (*f x*) (*g y*)
using *assms* **by** (*simp add*: *rel-fun-def*)

lemma *rel-fun-mono*:
$\llbracket$ *rel-fun X A f g*; $\bigwedge x\ y.\ Y\ x\ y \longrightarrow X\ x\ y$; $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y$ $\rrbracket \Longrightarrow$ *rel-fun Y B f g*
by(*simp add*: *rel-fun-def*)

lemma *rel-fun-mono'* [*mono*]:
$\llbracket \bigwedge x\ y.\ Y\ x\ y \longrightarrow X\ x\ y; \bigwedge x\ y.\ A\ x\ y \longrightarrow B\ x\ y \rrbracket \Longrightarrow$ *rel-fun X A f g* $\longrightarrow$ *rel-fun Y B f g*
by(*simp add*: *rel-fun-def*)

definition *rel-set* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow bool$
where *rel-set R* = $(\lambda A\ B.\ (\forall x{\in}A.\ \exists y{\in}B.\ R\ x\ y) \wedge (\forall y{\in}B.\ \exists x{\in}A.\ R\ x\ y))$

lemma *rel-setI*:
assumes $\bigwedge x.\ x \in A \Longrightarrow \exists y{\in}B.\ R\ x\ y$
assumes $\bigwedge y.\ y \in B \Longrightarrow \exists x{\in}A.\ R\ x\ y$
shows *rel-set R A B*
using *assms* **unfolding** *rel-set-def* **by** *simp*

lemma *predicate2-transferD*:
$\llbracket$*rel-fun R1* (*rel-fun R2* (=)) *P Q*; $a \in A$; $b \in B$; $A \subseteq \{(x,\ y).\ R1\ x\ y\}$; $B \subseteq \{(x,\ y).\ R2\ x\ y\}\rrbracket \Longrightarrow$
P (*fst a*) (*fst b*) $\longleftrightarrow$ *Q* (*snd a*) (*snd b*)
unfolding *rel-fun-def* **by** (*blast dest*!: *Collect-case-prodD*)

definition *collect* **where**
collect F x = $(\bigcup f \in F.\ f\ x)$

lemma *fstI*: $x = (y,\ z) \Longrightarrow fst\ x = y$
by *simp*

lemma *sndI*: $x = (y,\ z) \Longrightarrow snd\ x = z$
by *simp*

lemma *bijI'*: $\llbracket \bigwedge x\ y.\ (f\ x = f\ y) = (x = y); \bigwedge y.\ \exists x.\ y = f\ x \rrbracket \Longrightarrow bij\ f$
unfolding *bij-def inj-on-def* **by** *auto blast*

definition *Gr A f* = $\{(a,\ f\ a) \mid a.\ a \in A\}$

definition *Grp A f* = $(\lambda a\ b.\ b = f\ a \wedge a \in A)$

definition *vimage2p* **where**
  $vimage2p\ f\ g\ R = (\lambda x\ y.\ R\ (f\ x)\ (g\ y))$

lemma *collect-comp*: $collect\ F \circ g = collect\ ((\lambda f.\ f \circ g)\ `\ F)$
  by (*rule ext*) (*simp add*: *collect-def*)

definition *convol* (‹(‹indent=1 notation=‹mixfix convol››⟨-,/ -⟩)›) **where**
  $\langle f,\ g\rangle \equiv \lambda a.\ (f\ a,\ g\ a)$

lemma *fst-convol*: $fst \circ \langle f,\ g\rangle = f$
  apply(*rule ext*)
  unfolding *convol-def* **by** *simp*

lemma *snd-convol*: $snd \circ \langle f,\ g\rangle = g$
  apply(*rule ext*)
  unfolding *convol-def* **by** *simp*

lemma *convol-mem-GrpI*:
  $x \in A \Longrightarrow \langle id,\ g\rangle\ x \in (Collect\ (case\text{-}prod\ (Grp\ A\ g)))$
  unfolding *convol-def Grp-def* **by** *auto*

definition *csquare* **where**
  $csquare\ A\ f1\ f2\ p1\ p2 \longleftrightarrow (\forall\ a \in A.\ f1\ (p1\ a) = f2\ (p2\ a))$

lemma *eq-alt*: $(=) = Grp\ UNIV\ id$
  unfolding *Grp-def* **by** *auto*

lemma *leq-conversepI*: $R = (=) \Longrightarrow R \leq R^{-1-1}$
  by *auto*

lemma *leq-OOI*: $R = (=) \Longrightarrow R \leq R\ OO\ R$
  by *auto*

lemma *OO-Grp-alt*: $(Grp\ A\ f)^{-1-1}\ OO\ Grp\ A\ g = (\lambda x\ y.\ \exists z.\ z \in A \wedge f\ z = x \wedge g\ z = y)$
  unfolding *Grp-def* **by** *auto*

lemma *Grp-UNIV-id*: $f = id \Longrightarrow (Grp\ UNIV\ f)^{-1-1}\ OO\ Grp\ UNIV\ f = Grp\ UNIV\ f$
  unfolding *Grp-def* **by** *auto*

lemma *Grp-UNIV-idI*: $x = y \Longrightarrow Grp\ UNIV\ id\ x\ y$
  unfolding *Grp-def* **by** *auto*

lemma *Grp-mono*: $A \leq B \Longrightarrow Grp\ A\ f \leq Grp\ B\ f$
  unfolding *Grp-def* **by** *auto*

lemma *GrpI*: $[\![f\ x = y;\ x \in A]\!] \Longrightarrow Grp\ A\ f\ x\ y$

unfolding *Grp-def* **by** *auto*

lemma *GrpE*: $Grp\ A\ f\ x\ y \Longrightarrow ([\![f\ x = y;\ x \in A]\!] \Longrightarrow R) \Longrightarrow R$
unfolding *Grp-def* **by** *auto*

lemma *Collect-case-prod-Grp-eqD*: $z \in Collect\ (case\text{-}prod\ (Grp\ A\ f)) \Longrightarrow (f \circ fst)\ z = snd\ z$
unfolding *Grp-def comp-def* **by** *auto*

lemma *Collect-case-prod-Grp-in*: $z \in Collect\ (case\text{-}prod\ (Grp\ A\ f)) \Longrightarrow fst\ z \in A$
unfolding *Grp-def comp-def* **by** *auto*

definition *pick-middlep* $P\ Q\ a\ c = (SOME\ b.\ P\ a\ b \wedge Q\ b\ c)$

lemma *pick-middlep*:
$(P\ OO\ Q)\ a\ c \Longrightarrow P\ a\ (pick\text{-}middlep\ P\ Q\ a\ c) \wedge Q\ (pick\text{-}middlep\ P\ Q\ a\ c)\ c$
unfolding *pick-middlep-def* **by** (*rule someI-ex*) *auto*

definition *fstOp* **where**
fstOp P Q ac = (*fst ac*, *pick-middlep P Q* (*fst ac*) (*snd ac*))

definition *sndOp* **where**
sndOp P Q ac = (*pick-middlep P Q* (*fst ac*) (*snd ac*), (*snd ac*))

lemma *fstOp-in*: $ac \in Collect\ (case\text{-}prod\ (P\ OO\ Q)) \Longrightarrow fstOp\ P\ Q\ ac \in Collect\ (case\text{-}prod\ P)$
unfolding *fstOp-def mem-Collect-eq*
by (*subst* (*asm*) *surjective-pairing*, *unfold prod.case*) (*erule pick-middlep*[*THEN conjunct1*])

lemma *fst-fstOp*: $fst\ bc = (fst \circ fstOp\ P\ Q)\ bc$
unfolding *comp-def fstOp-def* **by** *simp*

lemma *snd-sndOp*: $snd\ bc = (snd \circ sndOp\ P\ Q)\ bc$
unfolding *comp-def sndOp-def* **by** *simp*

lemma *sndOp-in*: $ac \in Collect\ (case\text{-}prod\ (P\ OO\ Q)) \Longrightarrow sndOp\ P\ Q\ ac \in Collect\ (case\text{-}prod\ Q)$
unfolding *sndOp-def mem-Collect-eq*
by (*subst* (*asm*) *surjective-pairing*, *unfold prod.case*) (*erule pick-middlep*[*THEN conjunct2*])

lemma *csquare-fstOp-sndOp*:
csquare (*Collect* (*f* (*P OO Q*))) *snd fst* (*fstOp P Q*) (*sndOp P Q*)
unfolding *csquare-def fstOp-def sndOp-def* **using** *pick-middlep* **by** *simp*

lemma *snd-fst-flip*: $snd\ xy = (fst \circ (\%(x, y).\ (y, x)))\ xy$
by (*simp split*: *prod.split*)

lemma *fst-snd-flip*: *fst xy* = (*snd* ∘ (%(*x*, *y*). (*y*, *x*))) *xy*
by (*simp split*: *prod.split*)

lemma *flip-pred*: *A* ⊆ *Collect* (*case-prod* (R^{-1-1})) ⟹ (%(*x*, *y*). (*y*, *x*)) ‘ *A* ⊆ *Collect* (*case-prod R*)
by *auto*

lemma *predicate2-eqD*: *A* = *B* ⟹ *A a b* ⟷ *B a b*
by *simp*

lemma *case-sum-o-inj*: *case-sum f g* ∘ *Inl* = *f case-sum f g* ∘ *Inr* = *g*
by *auto*

lemma *map-sum-o-inj*: *map-sum f g* ∘ *Inl* = *Inl* ∘ *f map-sum f g* ∘ *Inr* = *Inr* ∘ *g*
by *auto*

lemma *card-order-csum-cone-cexp-def*:
card-order r ⟹ (|*A1*| +*c cone*) ^*c r* = |*Func UNIV* (*Inl* ‘ *A1* ∪ {*Inr* ()})|
unfolding *cexp-def cone-def Field-csum Field-card-of* **by** (*auto dest*: *Field-card-order*)

lemma *If-the-inv-into-in-Func*:
⟦*inj-on g C*; *C* ⊆ *B* ∪ {*x*}⟧ ⟹
(λ*i*. *if i* ∈ *g* ‘ *C then the-inv-into C g i else x*) ∈ *Func UNIV* (*B* ∪ {*x*})
unfolding *Func-def* **by** (*auto dest*: *the-inv-into-into*)

lemma *If-the-inv-into-f-f*:
⟦*i* ∈ *C*; *inj-on g C*⟧ ⟹ ((λ*i*. *if i* ∈ *g* ‘ *C then the-inv-into C g i else x*) ∘ *g*) *i* = *id i*
unfolding *Func-def* **by** (*auto elim*: *the-inv-into-f-f*)

lemma *the-inv-f-o-f-id*: *inj f* ⟹ (*the-inv f* ∘ *f*) *z* = *id z*
by (*simp add*: *the-inv-f-f*)

lemma *vimage2pI*: *R* (*f x*) (*g y*) ⟹ *vimage2p f g R x y*
unfolding *vimage2p-def* **.**

lemma *rel-fun-iff-leq-vimage2p*: (*rel-fun R S*) *f g* = (*R* ≤ *vimage2p f g S*)
unfolding *rel-fun-def vimage2p-def* **by** *auto*

lemma *convol-image-vimage2p*: ⟨*f* ∘ *fst*, *g* ∘ *snd*⟩ ‘ *Collect* (*case-prod* (*vimage2p f g R*)) ⊆ *Collect* (*case-prod R*)
unfolding *vimage2p-def convol-def* **by** *auto*

lemma *vimage2p-Grp*: *vimage2p f g P* = *Grp UNIV f OO P OO* (*Grp UNIV* $g)^{-1-1}$
unfolding *vimage2p-def Grp-def* **by** *auto*

lemma *subst-Pair*: *P x y* ⟹ *a* = (*x*, *y*) ⟹ *P* (*fst a*) (*snd a*)
by *simp*

lemma *comp-apply-eq*: $f\ (g\ x) = h\ (k\ x) \Longrightarrow (f \circ g)\ x = (h \circ k)\ x$
unfolding *comp-apply* **by** *assumption*

lemma *refl-ge-eq*: $(\bigwedge x.\ R\ x\ x) \Longrightarrow (=) \leq R$
by *auto*

lemma *ge-eq-refl*: $(=) \leq R \Longrightarrow R\ x\ x$
by *auto*

lemma *reflp-eq*: $reflp\ R = ((=) \leq R)$
by (*auto simp*: *reflp-def fun-eq-iff*)

lemma *transp-relcompp*: $transp\ r \longleftrightarrow r\ OO\ r \leq r$
by (*auto simp*: *transp-def*)

lemma *symp-conversep*: $symp\ R = (R^{-1-1} \leq R)$
by (*auto simp*: *symp-def fun-eq-iff*)

lemma *diag-imp-eq-le*: $(\bigwedge x.\ x \in A \Longrightarrow R\ x\ x) \Longrightarrow \forall\, x\ y.\ x \in A \longrightarrow y \in A \longrightarrow x = y \longrightarrow R\ x\ y$
by *blast*

definition *eq-onp* :: $('a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
where $eq\text{-}onp\ R = (\lambda x\ y.\ R\ x \wedge x = y)$

lemma *eq-onp-Grp*: $eq\text{-}onp\ P = BNF\text{-}Def.Grp\ (Collect\ P)\ id$
unfolding *eq-onp-def Grp-def* **by** *auto*

lemma *eq-onp-to-eq*: $eq\text{-}onp\ P\ x\ y \Longrightarrow x = y$
by (*simp add*: *eq-onp-def*)

lemma *eq-onp-top-eq-eq*: $eq\text{-}onp\ top = (=)$
by (*simp add*: *eq-onp-def*)

lemma *eq-onp-same-args*: $eq\text{-}onp\ P\ x\ x = P\ x$
by (*auto simp add*: *eq-onp-def*)

lemma *eq-onp-eqD*: $eq\text{-}onp\ P = Q \Longrightarrow P\ x = Q\ x\ x$
unfolding *eq-onp-def* **by** *blast*

lemma *Ball-Collect*: $Ball\ A\ P = (A \subseteq (Collect\ P))$
by *auto*

lemma *eq-onp-mono0*: $\forall\, x{\in}A.\ P\ x \longrightarrow Q\ x \Longrightarrow \forall\, x{\in}A.\ \forall\, y{\in}A.\ eq\text{-}onp\ P\ x\ y \longrightarrow eq\text{-}onp\ Q\ x\ y$
unfolding *eq-onp-def* **by** *auto*

lemma *eq-onp-True*: $eq\text{-}onp\ (\lambda\text{-}.\ True) = (=)$

unfolding *eq-onp-def* **by** *simp*

lemma *Ball-image-comp*: *Ball* (*f* ' *A*) *g* = *Ball A* (*g* ∘ *f*)
by *auto*

lemma *rel-fun-Collect-case-prodD*:
rel-fun A B f g ⟹ *X* ⊆ *Collect* (*case-prod A*) ⟹ *x* ∈ *X* ⟹ *B* ((*f* ∘ *fst*) *x*) ((*g* ∘ *snd*) *x*)
unfolding *rel-fun-def* **by** *auto*

lemma *eq-onp-mono-iff*: *eq-onp P* ≤ *eq-onp Q* ⟷ *P* ≤ *Q*
unfolding *eq-onp-def* **by** *auto*

ML-file ‹*Tools/BNF/bnf-util.ML*›
ML-file ‹*Tools/BNF/bnf-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-def-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-def.ML*›

end

34 Composition of Bounded Natural Functors

theory *BNF-Composition*
imports *BNF-Def*
begin

lemma *ssubst-mem*: ⟦*t* = *s*; *s* ∈ *X*⟧ ⟹ *t* ∈ *X*
by *simp*

lemma *empty-natural*: (λ-. {}) ∘ *f* = *image g* ∘ (λ-. {})
by (*rule ext*) *simp*

lemma *Cinfinite-gt-empty*: *Cinfinite r* ⟹ |{}| <*o r*
by (*simp add*: *cinfinite-def finite-ordLess-infinite card-of-ordIso-finite Field-card-of card-of-well-order-on emptyI card-order-on-well-order-on*)

lemma *Union-natural*: *Union* ∘ *image* (*image f*) = *image f* ∘ *Union*
by (*rule ext*) (*auto simp only*: *comp-apply*)

lemma *in-Union-o-assoc*: *x* ∈ (*Union* ∘ *gset* ∘ *gmap*) *A* ⟹ *x* ∈ (*Union* ∘ (*gset* ∘ *gmap*)) *A*
by (*unfold comp-assoc*)

lemma *regularCard-UNION-bound*:
assumes *Cinfinite r regularCard r* **and** |*I*| <*o r* ⋀*i*. *i* ∈ *I* ⟹ |*A i*| <*o r*
shows |⋃*i*∈*I*. *A i*| <*o r*
using *assms cinfinite-def regularCard-stable stable-UNION* **by** *blast*

lemma *comp-single-set-bd-strict*:

assumes *fbd*: *Cinfinite fbd regularCard fbd* **and**
gbd: *Cinfinite gbd regularCard gbd* **and**
fset-bd: $\bigwedge x.$ |*fset x*| <*o fbd* **and**
gset-bd: $\bigwedge x.$ |*gset x*| <*o gbd*
shows |$\bigcup$(*fset* ' *gset x*)| <*o gbd* **c fbd*
proof (*cases fbd* <*o gbd*)
case *True*
then have |*fset x*| <*o gbd* **for** *x* **using** *fset-bd ordLess-transitive* **by** *blast*
then have |$\bigcup$(*fset* ' *gset x*)| <*o gbd* **using** *regularCard-UNION-bound*[*OF gbd gset-bd*] **by** *blast*
then have |$\bigcup$ (*fset* ' *gset x*)| <*o fbd* **c gbd*
using *ordLess-ordLeq-trans ordLeq-cprod2 gbd*(*1*) *fbd*(*1*) *cinfinite-not-czero* **by** *blast*
then show *?thesis* **using** *ordLess-ordIso-trans cprod-com* **by** *blast*
next
case *False*
have *Well-order fbd Well-order gbd* **using** *fbd*(*1*) *gbd*(*1*) *card-order-on-well-order-on* **by** *auto*
then have *gbd* ≤*o fbd* **using** *not-ordLess-iff-ordLeq False* **by** *blast*
then have |*gset x*| <*o fbd* **for** *x* **using** *gset-bd ordLess-ordLeq-trans* **by** *blast*
then have |$\bigcup$(*fset* ' *gset x*)| <*o fbd* **using** *regularCard-UNION-bound*[*OF fbd*] *fset-bd* **by** *blast*
then show *?thesis* **using** *ordLess-ordLeq-trans ordLeq-cprod2 gbd*(*1*) *fbd*(*1*) *cinfinite-not-czero* **by** *blast*
qed

lemma *comp-single-set-bd*:
assumes *fbd-Card-order*: *Card-order fbd* **and**
fset-bd: $\bigwedge x.$ |*fset x*| ≤*o fbd* **and**
gset-bd: $\bigwedge x.$ |*gset x*| ≤*o gbd*
shows |$\bigcup$(*fset* ' *gset x*)| ≤*o gbd* **c fbd*
apply *simp*
apply (*rule ordLeq-transitive*)
apply (*rule card-of-UNION-Sigma*)
apply (*subst SIGMA-CSUM*)
apply (*rule ordLeq-transitive*)
apply (*rule card-of-Csum-Times'*)
apply (*rule fbd-Card-order*)
apply (*rule ballI*)
apply (*rule fset-bd*)
apply (*rule ordLeq-transitive*)
apply (*rule cprod-mono1*)
apply (*rule gset-bd*)
apply (*rule ordIso-imp-ordLeq*)
apply (*rule ordIso-refl*)
apply (*rule Card-order-cprod*)
done

lemma *csum-dup*: *cinfinite r* ⟹ *Card-order r* ⟹ *p* +*c p'* =*o r* +*c r* ⟹ *p* +*c*

$p' =o\ r$
  apply (*erule ordIso-transitive*)
  apply (*frule csum-absorb2'*)
  apply (*erule ordLeq-refl*)
  by *simp*

lemma *cprod-dup*: $cinfinite\ r \Longrightarrow Card\text{-}order\ r \Longrightarrow p *c\ p' =o\ r *c\ r \Longrightarrow p *c\ p' =o\ r$
  apply (*erule ordIso-transitive*)
  apply (*rule cprod-infinite*)
  by *simp*

lemma *Union-image-insert*: $\bigcup(f\ `\ insert\ a\ B) = f\ a \cup \bigcup(f\ `\ B)$
  by *simp*

lemma *Union-image-empty*: $A \cup \bigcup(f\ `\ \{\}) = A$
  by *simp*

lemma *image-o-collect*: $collect\ ((\lambda f.\ image\ g \circ f)\ `\ F) = image\ g \circ collect\ F$
  by (*rule ext*) (*auto simp add*: *collect-def*)

lemma *conj-subset-def*: $A \subseteq \{x.\ P\ x \wedge Q\ x\} = (A \subseteq \{x.\ P\ x\} \wedge A \subseteq \{x.\ Q\ x\})$
  by *blast*

lemma *UN-image-subset*: $\bigcup(f\ `\ g\ x) \subseteq X = (g\ x \subseteq \{x.\ f\ x \subseteq X\})$
  by *blast*

lemma *comp-set-bd-Union-o-collect*: $|\bigcup(\bigcup((\lambda f.\ f\ x)\ `\ X))| \le o\ hbd \Longrightarrow |(Union \circ collect\ X)\ x| \le o\ hbd$
  by (*unfold comp-apply collect-def*) *simp*

lemma *comp-set-bd-Union-o-collect-strict*: $|\bigcup(\bigcup((\lambda f.\ f\ x)\ `\ X))| <o\ hbd \Longrightarrow |(Union \circ collect\ X)\ x| <o\ hbd$
  by (*unfold comp-apply collect-def*) *simp*

lemma *Collect-inj*: $Collect\ P = Collect\ Q \Longrightarrow P = Q$
  by *blast*

lemma *Grp-fst-snd*: $(Grp\ (Collect\ (case\text{-}prod\ R))\ fst)^{-1-1}\ OO\ Grp\ (Collect\ (case\text{-}prod\ R))\ snd = R$
  unfolding *Grp-def fun-eq-iff relcompp.simps* **by** *auto*

lemma *OO-Grp-cong*: $A = B \Longrightarrow (Grp\ A\ f)^{-1-1}\ OO\ Grp\ A\ g = (Grp\ B\ f)^{-1-1}\ OO\ Grp\ B\ g$
  by (*rule arg-cong*)

lemma *vimage2p-relcompp-mono*: $R\ OO\ S \le T \Longrightarrow$
  $vimage2p\ f\ g\ R\ OO\ vimage2p\ g\ h\ S \le vimage2p\ f\ h\ T$
  unfolding *vimage2p-def* **by** *auto*

lemma *type-copy-map-cong0*: $M\ (g\ x) = N\ (h\ x) \Longrightarrow (f \circ M \circ g)\ x = (f \circ N \circ h)\ x$
by *auto*

lemma *type-copy-set-bd*: $(\bigwedge y.\ |S\ y| <o\ bd) \Longrightarrow |(S \circ Rep)\ x| <o\ bd$
by *auto*

lemma *vimage2p-cong*: $R = S \Longrightarrow vimage2p\ f\ g\ R = vimage2p\ f\ g\ S$
by *simp*

lemma *Ball-comp-iff*: $(\lambda x.\ Ball\ (A\ x)\ f) \circ g = (\lambda x.\ Ball\ ((A \circ g)\ x)\ f)$
unfolding *o-def* **by** *auto*

lemma *conj-comp-iff*: $(\lambda x.\ P\ x \wedge Q\ x) \circ g = (\lambda x.\ (P \circ g)\ x \wedge (Q \circ g)\ x)$
unfolding *o-def* **by** *auto*

context
fixes *Rep Abs*
assumes *type-copy*: *type-definition Rep Abs UNIV*
begin

lemma *type-copy-map-id0*: $M = id \Longrightarrow Abs \circ M \circ Rep = id$
using *type-definition.Rep-inverse*[*OF type-copy*] **by** *auto*

lemma *type-copy-map-comp0*: $M = M1 \circ M2 \Longrightarrow f \circ M \circ g = (f \circ M1 \circ Rep) \circ (Abs \circ M2 \circ g)$
using *type-definition.Abs-inverse*[*OF type-copy UNIV-I*] **by** *auto*

lemma *type-copy-set-map0*: $S \circ M = image\ f \circ S' \Longrightarrow (S \circ Rep) \circ (Abs \circ M \circ g) = image\ f \circ (S' \circ g)$
using *type-definition.Abs-inverse*[*OF type-copy UNIV-I*] **by** (*auto simp*: *o-def fun-eq-iff*)

lemma *type-copy-wit*: $x \in (S \circ Rep)\ (Abs\ y) \Longrightarrow x \in S\ y$
using *type-definition.Abs-inverse*[*OF type-copy UNIV-I*] **by** *auto*

lemma *type-copy-vimage2p-Grp-Rep*: $vimage2p\ f\ Rep\ (Grp\ (Collect\ P)\ h) = Grp\ (Collect\ (\lambda x.\ P\ (f\ x)))\ (Abs \circ h \circ f)$
unfolding *vimage2p-def Grp-def fun-eq-iff*
by (*auto simp*: *type-definition.Abs-inverse*[*OF type-copy UNIV-I*]
type-definition.Rep-inverse[*OF type-copy*] *dest*: *sym*)

lemma *type-copy-vimage2p-Grp-Abs*:
$\bigwedge h.\ vimage2p\ g\ Abs\ (Grp\ (Collect\ P)\ h) = Grp\ (Collect\ (\lambda x.\ P\ (g\ x)))\ (Rep \circ h \circ g)$
unfolding *vimage2p-def Grp-def fun-eq-iff*
by (*auto simp*: *type-definition.Abs-inverse*[*OF type-copy UNIV-I*]
type-definition.Rep-inverse[*OF type-copy*] *dest*: *sym*)

lemma *type-copy-ex-RepI*: $(\exists\, b.\ F\ b) = (\exists\, b.\ F\ (Rep\ b))$
proof *safe*
  fix *b* **assume** *F b*
  show $\exists\, b'.\ F\ (Rep\ b')$
  proof (*rule exI*)
   from ‹*F b*› **show** *F* (*Rep* (*Abs b*)) **using** *type-definition.Abs-inverse*[*OF type-copy*]
by *auto*
  qed
qed *blast*

lemma *vimage2p-relcompp-converse*:
  vimage2p f g $(R^{-1-1}$ *OO S*$)$ = (*vimage2p Rep f R*)$^{-1-1}$ *OO vimage2p Rep g S*
  unfolding *vimage2p-def relcompp.simps conversep.simps fun-eq-iff image-def*
  by (*auto simp*: *type-copy-ex-RepI*)

end

bnf *DEADID*: *'a*
  map: *id* :: *'a* ⇒ *'a*
  bd: *natLeq*
  rel: (=) :: *'a* ⇒ *'a* ⇒ *bool*
  by (*auto simp add*: *natLeq-card-order natLeq-cinfinite regularCard-natLeq*)

definition *id-bnf* :: *'a* ⇒ *'a* **where**
  id-bnf ≡ (λ*x*. *x*)

lemma *id-bnf-apply*: *id-bnf x* = *x*
  unfolding *id-bnf-def* **by** *simp*

bnf *ID*: *'a*
  map: *id-bnf* :: (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b*
  sets: λ*x*. {*x*}
  bd: *natLeq*
  rel: *id-bnf* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a* ⇒ *'b* ⇒ *bool*
  pred: *id-bnf* :: (*'a* ⇒ *bool*) ⇒ *'a* ⇒ *bool*
  unfolding *id-bnf-def*
  apply (*auto simp*: *Grp-def fun-eq-iff relcompp.simps natLeq-card-order natLeq-cinfinite regularCard-natLeq*)
  apply (*rule finite-ordLess-infinite*[*OF - natLeq-Well-order*])
  apply (*auto simp add*: *Field-card-of Field-natLeq card-of-well-order-on*)[*3*]
  done

lemma *type-definition-id-bnf-UNIV*: *type-definition id-bnf id-bnf UNIV*
  unfolding *id-bnf-def* **by** *unfold-locales auto*

ML-file ‹*Tools/BNF/bnf-comp-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-comp.ML*›

hide-fact
DEADID.inj-map DEADID.inj-map-strong DEADID.map-comp DEADID.map-cong DEADID.map-cong0
DEADID.map-cong-simp DEADID.map-id DEADID.map-id0 DEADID.map-ident DEADID.map-transfer
DEADID.rel-Grp DEADID.rel-compp DEADID.rel-compp-Grp DEADID.rel-conversep DEADID.rel-eq
DEADID.rel-flip DEADID.rel-map DEADID.rel-mono DEADID.rel-transfer
ID.inj-map ID.inj-map-strong ID.map-comp ID.map-cong ID.map-cong0 ID.map-cong-simp ID.map-id
ID.map-id0 ID.map-ident ID.map-transfer ID.rel-Grp ID.rel-compp ID.rel-compp-Grp ID.rel-conversep
ID.rel-eq ID.rel-flip ID.rel-map ID.rel-mono ID.rel-transfer ID.set-map ID.set-transfer

end

35 Registration of Basic Types as Bounded Natural Functors

theory *Basic-BNFs*
imports *BNF-Def*
begin

inductive-set *setl* :: $'a + 'b \Rightarrow 'a$ *set* **for** *s* :: $'a + 'b$ **where**
$s = Inl\ x \Longrightarrow x \in setl\ s$
inductive-set *setr* :: $'a + 'b \Rightarrow 'b$ *set* **for** *s* :: $'a + 'b$ **where**
$s = Inr\ x \Longrightarrow x \in setr\ s$

lemma *sum-set-defs*[*code*]:
$setl = (\lambda x.\ case\ x\ of\ Inl\ z \Rightarrow \{z\} \mid \text{-} \Rightarrow \{\})$
$setr = (\lambda x.\ case\ x\ of\ Inr\ z \Rightarrow \{z\} \mid \text{-} \Rightarrow \{\})$
by (*auto simp*: *fun-eq-iff intro*: *setl.intros setr.intros elim*: *setl.cases setr.cases split*: *sum.splits*)

lemma *rel-sum-simps*[*code*, *simp*]:
rel-sum R1 R2 (*Inl a1*) (*Inl b1*) = *R1 a1 b1*
rel-sum R1 R2 (*Inl a1*) (*Inr b2*) = *False*
rel-sum R1 R2 (*Inr a2*) (*Inl b1*) = *False*
rel-sum R1 R2 (*Inr a2*) (*Inr b2*) = *R2 a2 b2*
by (*auto intro*: *rel-sum.intros elim*: *rel-sum.cases*)

inductive
pred-sum :: $('a \Rightarrow bool) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'a + 'b \Rightarrow bool$ **for** *P1 P2*
where
$P1\ a \Longrightarrow pred\text{-}sum\ P1\ P2\ (Inl\ a)$
$\mid P2\ b \Longrightarrow pred\text{-}sum\ P1\ P2\ (Inr\ b)$

lemma *pred-sum-inject*[*code*, *simp*]:

```
  pred-sum P1 P2 (Inl a) ⟷ P1 a
  pred-sum P1 P2 (Inr b) ⟷ P2 b
  by (simp add: pred-sum.simps)+

bnf 'a + 'b
  map: map-sum
  sets: setl setr
  bd: natLeq
  wits: Inl Inr
  rel: rel-sum
  pred: pred-sum
proof −
  show map-sum id id = id by (rule map-sum.id)
next
  fix f1 :: 'o ⇒ 's and f2 :: 'p ⇒ 't and g1 :: 's ⇒ 'q and g2 :: 't ⇒ 'r
  show map-sum (g1 ∘ f1) (g2 ∘ f2) = map-sum g1 g2 ∘ map-sum f1 f2
    by (rule map-sum.comp[symmetric])
next
  fix x and f1 :: 'o ⇒ 'q and f2 :: 'p ⇒ 'r and g1 g2
  assume a1: ⋀z. z ∈ setl x ⟹ f1 z = g1 z and
         a2: ⋀z. z ∈ setr x ⟹ f2 z = g2 z
  thus map-sum f1 f2 x = map-sum g1 g2 x
  proof (cases x)
    case Inl thus ?thesis using a1 by (clarsimp simp: sum-set-defs(1))
  next
    case Inr thus ?thesis using a2 by (clarsimp simp: sum-set-defs(2))
  qed
next
  fix f1 :: 'o ⇒ 'q and f2 :: 'p ⇒ 'r
  show setl ∘ map-sum f1 f2 = image f1 ∘ setl
    by (rule ext, unfold o-apply) (simp add: sum-set-defs(1) split: sum.split)
next
  fix f1 :: 'o ⇒ 'q and f2 :: 'p ⇒ 'r
  show setr ∘ map-sum f1 f2 = image f2 ∘ setr
    by (rule ext, unfold o-apply) (simp add: sum-set-defs(2) split: sum.split)
next
  show card-order natLeq by (rule natLeq-card-order)
next
  show cinfinite natLeq by (rule natLeq-cinfinite)
next
  show regularCard natLeq by (rule regularCard-natLeq)
next
  fix x :: 'o + 'p
  show |setl x| <o natLeq
    apply (rule finite-iff-ordLess-natLeq[THEN iffD1])
    by (simp add: sum-set-defs(1) split: sum.split)
next
  fix x :: 'o + 'p
  show |setr x| <o natLeq
```

apply (*rule finite-iff-ordLess-natLeq*[*THEN iffD1*])
by (*simp add*: *sum-set-defs*(*2*) *split*: *sum.split*)
next
fix *R1 R2 S1 S2*
show *rel-sum R1 R2 OO rel-sum S1 S2* $\leq$ *rel-sum* (*R1 OO S1*) (*R2 OO S2*)
by (*force elim*: *rel-sum.cases*)
next
fix *R S*
show *rel-sum R S* = ($\lambda x\ y$.
$\exists z$. (*setl z* $\subseteq$ {(x, y). *R x y*} $\wedge$ *setr z* $\subseteq$ {(x, y). *S x y*}) $\wedge$
map-sum fst fst z = x $\wedge$ *map-sum snd snd z* = y)
unfolding *sum-set-defs relcompp.simps conversep.simps fun-eq-iff*
by (*fastforce elim*: *rel-sum.cases split*: *sum.splits*)
qed (*auto simp*: *sum-set-defs fun-eq-iff pred-sum.simps split*: *sum.splits*)

inductive-set *fsts* :: $'a \times 'b \Rightarrow 'a$ *set* **for** p :: $'a \times 'b$ **where**
fst p $\in$ *fsts p*
inductive-set *snds* :: $'a \times 'b \Rightarrow 'b$ *set* **for** p :: $'a \times 'b$ **where**
snd p $\in$ *snds p*

lemma *prod-set-defs*[*code*]: *fsts* = (λp. {*fst p*}) *snds* = (λp. {*snd p*})
by (*auto intro*: *fsts.intros snds.intros elim*: *fsts.cases snds.cases*)

inductive
rel-prod :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('c \Rightarrow 'd \Rightarrow bool) \Rightarrow 'a \times 'c \Rightarrow 'b \times 'd \Rightarrow bool$
for *R1 R2*
where
$[\![R1\ a\ b;\ R2\ c\ d]\!] \Longrightarrow$ *rel-prod R1 R2* (a, c) (b, d)

inductive
pred-prod :: $('a \Rightarrow bool) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'a \times 'b \Rightarrow bool$ **for** *P1 P2*
where
$[\![P1\ a;\ P2\ b]\!] \Longrightarrow$ *pred-prod P1 P2* (a, b)

lemma *rel-prod-inject* [*code*, *simp*]:
rel-prod R1 R2 (a, b) (c, d) $\longleftrightarrow$ *R1 a c* $\wedge$ *R2 b d*
by (*auto intro*: *rel-prod.intros elim*: *rel-prod.cases*)

lemma *pred-prod-inject* [*code*, *simp*]:
pred-prod P1 P2 (a, b) $\longleftrightarrow$ *P1 a* $\wedge$ *P2 b*
by (*auto intro*: *pred-prod.intros elim*: *pred-prod.cases*)

lemma *rel-prod-conv*:
rel-prod R1 R2 = (λ(a, b) (c, d). *R1 a c* $\wedge$ *R2 b d*)
by *force*

definition
pred-fun :: $('a \Rightarrow bool) \Rightarrow ('b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
where

pred-fun A B = (λf. $\forall x$. *A x* $\longrightarrow$ *B* (*f x*))

lemma *pred-funI*: ($\bigwedge x$. *A x* $\Longrightarrow$ *B* (*f x*)) $\Longrightarrow$ *pred-fun A B f*
unfolding *pred-fun-def* **by** *simp*

bnf *'a* $\times$ *'b*
map: *map-prod*
sets: *fsts snds*
bd: *natLeq*
rel: *rel-prod*
pred: *pred-prod*
proof (*unfold prod-set-defs*)
show *map-prod id id* = *id* **by** (*rule map-prod.id*)
next
fix *f1 f2 g1 g2*
show *map-prod* (*g1* $\circ$ *f1*) (*g2* $\circ$ *f2*) = *map-prod g1 g2* $\circ$ *map-prod f1 f2*
by (*rule map-prod.comp*[*symmetric*])
next
fix *x f1 f2 g1 g2*
assume $\bigwedge z$. *z* $\in$ {*fst x*} $\Longrightarrow$ *f1 z* = *g1 z* $\bigwedge z$. *z* $\in$ {*snd x*} $\Longrightarrow$ *f2 z* = *g2 z*
thus *map-prod f1 f2 x* = *map-prod g1 g2 x* **by** (*cases x*) *simp*
next
fix *f1 f2*
show (λx. {*fst x*}) $\circ$ *map-prod f1 f2* = *image f1* $\circ$ (λx. {*fst x*})
by (*rule ext*, *unfold o-apply*) *simp*
next
fix *f1 f2*
show (λx. {*snd x*}) $\circ$ *map-prod f1 f2* = *image f2* $\circ$ (λx. {*snd x*})
by (*rule ext*, *unfold o-apply*) *simp*
next
show *card-order natLeq* **by** (*rule natLeq-card-order*)
next
show *cinfinite natLeq* **by** (*rule natLeq-cinfinite*)
next
show *regularCard natLeq* **by** (*rule regularCard-natLeq*)
next
fix *x*
show |{*fst x*}| <*o natLeq*
by (*simp add*: *finite-iff-ordLess-natLeq*[*symmetric*])
next
fix *x*
show |{*snd x*}| <*o natLeq*
by (*simp add*: *finite-iff-ordLess-natLeq*[*symmetric*])
next
fix *R1 R2 S1 S2*
show *rel-prod R1 R2 OO rel-prod S1 S2* $\leq$ *rel-prod* (*R1 OO S1*) (*R2 OO S2*)
by *auto*
next
fix *R S*

```
  show rel-prod R S = (λx y.
    ∃z. ({fst z} ⊆ {(x, y). R x y} ∧ {snd z} ⊆ {(x, y). S x y}) ∧
      map-prod fst fst z = x ∧ map-prod snd snd z = y)
  unfolding prod-set-defs rel-prod-inject relcompp.simps conversep.simps fun-eq-iff
  by auto
qed auto

lemma card-order-bd-fun: card-order (natLeq +c card-suc ( |UNIV| ))
 by (auto simp: card-order-csum natLeq-card-order card-order-card-suc card-of-card-order-on)

lemma Cinfinite-bd-fun: Cinfinite (natLeq +c card-suc ( |UNIV| ))
  by (auto simp: Cinfinite-csum natLeq-Cinfinite)

lemma regularCard-bd-fun: regularCard (natLeq +c card-suc ( |UNIV| ))
  (is regularCard (- +c card-suc ?U))
proof (cases Cinfinite ?U)
  case True
  then show ?thesis
    by (intro regularCard-csum natLeq-Cinfinite Cinfinite-card-suc
          card-of-card-order-on regularCard-natLeq regularCard-card-suc)
next
  case False
  then have card-suc ?U ≤o natLeq
    unfolding cinfinite-def Field-card-of
    by (intro card-suc-least;
      simp add: natLeq-Card-order card-of-card-order-on flip: finite-iff-ordLess-natLeq)
  then have natLeq =o natLeq +c card-suc ?U
    using natLeq-Cinfinite csum-absorb1 ordIso-symmetric by blast
  then show ?thesis
    by (intro regularCard-ordIso[OF - natLeq-Cinfinite regularCard-natLeq])
qed

lemma ordLess-bd-fun: |UNIV::'a set| <o natLeq +c card-suc ( |UNIV::'a set| )
  (is - <o (- +c card-suc (?U :: 'a rel)))
proof (cases Cinfinite ?U)
  case True
 have ?U <o card-suc ?U using card-of-card-order-on natLeq-card-order card-suc-greater
by blast
 also have card-suc ?U =o natLeq +c card-suc ?U by (rule csum-absorb2[THEN
ordIso-symmetric])
   (auto simp: True card-of-card-order-on intro!: Cinfinite-card-suc natLeq-ordLeq-cinfinite)
  finally show ?thesis .
next
  case False
  then have ?U <o natLeq
   by (auto simp: cinfinite-def Field-card-of card-of-card-order-on finite-iff-ordLess-natLeq[symmetric])
  then show ?thesis
    by (rule ordLess-ordLeq-trans[OF - ordLeq-csum1[OF natLeq-Card-order]])
qed
```

bnf $'a \Rightarrow 'b$
map: $(\circ)$
sets: *range*
bd: *natLeq* +*c* *card-suc* (|*UNIV*::'*a set*|)
rel: *rel-fun* (=)
pred: *pred-fun* (λ-. *True*)
proof
fix *f* **show** *id* $\circ$ *f* = *id f* **by** *simp*
next
fix *f g* **show** $(\circ)$ $(g \circ f) = (\circ)$ $g \circ (\circ)$ *f*
unfolding *comp-def*[*abs-def*] **..**
next
fix *x f g*
assume $\bigwedge z.\ z \in$ *range* $x \Longrightarrow f\ z = g\ z$
thus $f \circ x = g \circ x$ **by** *auto*
next
fix *f* **show** *range* $\circ$ $(\circ)$ *f* = (') *f* $\circ$ *range*
by (*auto simp add*: *fun-eq-iff*)
next
show *card-order* (*natLeq* +*c* *card-suc* (|*UNIV*|))
by (*rule card-order-bd-fun*)
next
show *cinfinite* (*natLeq* +*c* *card-suc* (|*UNIV*|))
by (*rule Cinfinite-bd-fun*[*THEN conjunct1*])
next
show *regularCard* (*natLeq* +*c* *card-suc* (|*UNIV*|))
by (*rule regularCard-bd-fun*)
next
fix *f* :: $'d \Rightarrow 'a$
show |*range f*| <*o* *natLeq* +*c* *card-suc* |*UNIV* :: '*d set*|
by (*rule ordLeq-ordLess-trans*[*OF card-of-image ordLess-bd-fun*])
next
fix *R S*
show *rel-fun* (=) *R OO rel-fun* (=) $S \leq$ *rel-fun* (=) (*R OO S*) **by** (*auto simp*: *rel-fun-def*)
next
fix *R*
show *rel-fun* (=) *R* = ($\lambda x\ y$.
$\exists z.$ *range* $z \subseteq \{(x, y).\ R\ x\ y\} \wedge$ *fst* $\circ\ z = x \wedge$ *snd* $\circ\ z = y$)
unfolding *rel-fun-def subset-iff* **by** (*force simp*: *fun-eq-iff*[*symmetric*])
qed (*auto simp*: *pred-fun-def*)

end

36 Shared Fixpoint Operations on Bounded Natural Functors

theory *BNF-Fixpoint-Base*
imports *BNF-Composition Basic-BNFs*
begin

lemma *conj-imp-eq-imp-imp*: $(P \wedge Q \Longrightarrow PROP\ R) \equiv (P \Longrightarrow Q \Longrightarrow PROP\ R)$
by *standard simp-all*

lemma *predicate2D-conj*: $P \leq Q \wedge R \Longrightarrow R \wedge (P\ x\ y \longrightarrow Q\ x\ y)$
by *blast*

lemma *eq-sym-Unity-conv*: $(x = (() = ())) = x$
by *blast*

lemma *case-unit-Unity*: $(case\ u\ of\ () \Rightarrow f) = f$
by (*cases u*) (*hypsubst*, *rule unit.case*)

lemma *case-prod-Pair-iden*: $(case\ p\ of\ (x,\ y) \Rightarrow (x,\ y)) = p$
by *simp*

lemma *unit-all-impI*: $(P\ () \Longrightarrow Q\ ()) \Longrightarrow \forall x.\ P\ x \longrightarrow Q\ x$
by *simp*

lemma *pointfree-idE*: $f \circ g = id \Longrightarrow f\ (g\ x) = x$
unfolding *comp-def fun-eq-iff* **by** *simp*

lemma *o-bij*:
assumes *gf*: $g \circ f = id$ **and** *fg*: $f \circ g = id$
shows *bij f*
unfolding *bij-def inj-on-def surj-def* **proof** *safe*
fix *a1 a2* **assume** $f\ a1 = f\ a2$
hence $g\ (f\ a1) = g\ (f\ a2)$ **by** *simp*
thus $a1 = a2$ **using** *gf* **unfolding** *fun-eq-iff* **by** *simp*
next
fix *b*
have $b = f\ (g\ b)$
using *fg* **unfolding** *fun-eq-iff* **by** *simp*
thus $\exists a.\ b = f\ a$ **by** *blast*
qed

lemma *case-sum-step*:
$case\text{-}sum\ (case\text{-}sum\ f'\ g')\ g\ (Inl\ p) = case\text{-}sum\ f'\ g'\ p$
$case\text{-}sum\ f\ (case\text{-}sum\ f'\ g')\ (Inr\ p) = case\text{-}sum\ f'\ g'\ p$
by *auto*

lemma *obj-one-pointE*: $\forall x.\ s = x \longrightarrow P \Longrightarrow P$
by *blast*

lemma *type-copy-obj-one-point-absE*:
assumes *type-definition Rep Abs UNIV* $\forall x.\ s = Abs\ x \longrightarrow P$ **shows** *P*
using *type-definition.Rep-inverse*[*OF assms*(*1*)]
by (*intro mp*[*OF spec*[*OF assms*(*2*), *of Rep s*]]) *simp*

lemma *obj-sumE-f*:
assumes $\forall x.\ s = f\ (Inl\ x) \longrightarrow P$ $\forall x.\ s = f\ (Inr\ x) \longrightarrow P$
shows $\forall x.\ s = f\ x \longrightarrow P$
proof
fix *x* **from** *assms* **show** $s = f\ x \longrightarrow P$ **by** (*cases x*) *auto*
qed

lemma *case-sum-if*:
case-sum f g (*if p then Inl x else Inr y*) = (*if p then f x else g y*)
by *simp*

lemma *prod-set-simps*[*simp*]:
fsts (*x*, *y*) = {*x*}
snds (*x*, *y*) = {*y*}
unfolding *prod-set-defs* **by** *simp+*

lemma *sum-set-simps*[*simp*]:
setl (*Inl x*) = {*x*}
setl (*Inr x*) = {}
setr (*Inl x*) = {}
setr (*Inr x*) = {*x*}
unfolding *sum-set-defs* **by** *simp+*

lemma *Inl-Inr-False*: (*Inl x* = *Inr y*) = *False*
by *simp*

lemma *Inr-Inl-False*: (*Inr x* = *Inl y*) = *False*
by *simp*

lemma *spec2*: $\forall x\ y.\ P\ x\ y \Longrightarrow P\ x\ y$
by *blast*

lemma *rewriteR-comp-comp*: $[\![g \circ h = r]\!] \Longrightarrow f \circ g \circ h = f \circ r$
unfolding *comp-def fun-eq-iff* **by** *auto*

lemma *rewriteR-comp-comp2*: $[\![g \circ h = r1 \circ r2;\ f \circ r1 = l]\!] \Longrightarrow f \circ g \circ h = l \circ r2$
unfolding *comp-def fun-eq-iff* **by** *auto*

lemma *rewriteL-comp-comp*: $[\![f \circ g = l]\!] \Longrightarrow f \circ (g \circ h) = l \circ h$
unfolding *comp-def fun-eq-iff* **by** *auto*

lemma *rewriteL-comp-comp2*: $[\![f \circ g = l1 \circ l2;\ l2 \circ h = r]\!] \Longrightarrow f \circ (g \circ h) = l1$

$\circ$ *r*
unfolding *comp-def fun-eq-iff* **by** *auto*

lemma *convol-o*: $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$
unfolding *convol-def* **by** *auto*

lemma *map-prod-o-convol*: *map-prod h1 h2* $\circ \langle f, g\rangle = \langle h1 \circ f, h2 \circ g\rangle$
unfolding *convol-def* **by** *auto*

lemma *map-prod-o-convol-id*: (*map-prod f id* $\circ \langle id, g\rangle$) $x = \langle id \circ f, g\rangle$ x
unfolding *map-prod-o-convol id-comp comp-id* **..**

lemma *o-case-sum*: $h \circ$ *case-sum f g* = *case-sum* ($h \circ f$) ($h \circ g$)
unfolding *comp-def* **by** (*auto split*: *sum.splits*)

lemma *case-sum-o-map-sum*: *case-sum f g* $\circ$ *map-sum h1 h2* = *case-sum* ($f \circ h1$) ($g \circ h2$)
unfolding *comp-def* **by** (*auto split*: *sum.splits*)

lemma *case-sum-o-map-sum-id*: (*case-sum id g* $\circ$ *map-sum f id*) x = *case-sum* ($f \circ id$) g x
unfolding *case-sum-o-map-sum id-comp comp-id* **..**

lemma *rel-fun-def-butlast*:
rel-fun R (*rel-fun S T*) *f g* = ($\forall x\ y.\ R\ x\ y \longrightarrow$ (*rel-fun S T*) ($f\ x$) ($g\ y$))
unfolding *rel-fun-def* **..**

lemma *subst-eq-imp*: ($\forall a\ b.\ a = b \longrightarrow P\ a\ b$) $\equiv$ ($\forall a.\ P\ a\ a$)
by *auto*

lemma *eq-subset*: $(=) \le (\lambda a\ b.\ P\ a\ b \lor a = b)$
by *auto*

lemma *eq-le-Grp-id-iff*: ($(=) \le$ *Grp* (*Collect R*) *id*) = (*All R*)
unfolding *Grp-def id-apply* **by** *blast*

lemma *Grp-id-mono-subst*: ($\bigwedge x\ y.$ *Grp P id x y* $\Longrightarrow$ *Grp Q id* ($f\ x$) ($f\ y$)) $\equiv$
($\bigwedge x.\ x \in P \Longrightarrow f\ x \in Q$)
unfolding *Grp-def* **by** *rule auto*

lemma *vimage2p-mono*: *vimage2p f g R x y* $\Longrightarrow R \le S \Longrightarrow$ *vimage2p f g S x y*
unfolding *vimage2p-def* **by** *blast*

lemma *vimage2p-refl*: ($\bigwedge x.\ R\ x\ x$) $\Longrightarrow$ *vimage2p f f R x x*
unfolding *vimage2p-def* **by** *auto*

lemma
assumes *type-definition Rep Abs UNIV*
shows *type-copy-Rep-o-Abs*: *Rep* $\circ$ *Abs* = *id* **and** *type-copy-Abs-o-Rep*: *Abs* $\circ$ *Rep*

= *id*
unfolding *fun-eq-iff comp-apply id-apply*
type-definition.Abs-inverse[*OF assms UNIV-I*] *type-definition.Rep-inverse*[*OF assms*] **by** *simp-all*

lemma *type-copy-map-comp0-undo*:
assumes *type-definition Rep Abs UNIV*
type-definition Rep′ Abs′ UNIV
type-definition Rep′′ Abs′′ UNIV
shows $Abs' \circ M \circ Rep'' = (Abs' \circ M1 \circ Rep) \circ (Abs \circ M2 \circ Rep'') \Longrightarrow M1 \circ M2 = M$
by (*rule sym*) (*auto simp*: *fun-eq-iff type-definition.Abs-inject*[*OF assms*(*2*) *UNIV-I UNIV-I*]
type-definition.Abs-inverse[*OF assms*(*1*) *UNIV-I*]
type-definition.Abs-inverse[*OF assms*(*3*) *UNIV-I*] *dest*: *spec*[*of - Abs′′ x* **for** *x*])

lemma *vimage2p-id*: *vimage2p id id R* = *R*
unfolding *vimage2p-def* **by** *auto*

lemma *vimage2p-comp*: *vimage2p* (*f1* ∘ *f2*) (*g1* ∘ *g2*) = *vimage2p f2 g2* ∘ *vimage2p f1 g1*
unfolding *fun-eq-iff vimage2p-def o-apply* **by** *simp*

lemma *vimage2p-rel-fun*: *rel-fun* (*vimage2p f g R*) *R f g*
unfolding *rel-fun-def vimage2p-def* **by** *auto*

lemma *fun-cong-unused-0*: $f = (\lambda x.\ g) \Longrightarrow f\ (\lambda x.\ 0) = g$
by (*erule arg-cong*)

lemma *inj-on-convol-ident*: *inj-on* (λ*x*. (*x*, *f x*)) *X*
unfolding *inj-on-def* **by** *simp*

lemma *map-sum-if-distrib-then*:
⋀*f g e x y*. *map-sum f g* (*if e then Inl x else y*) = (*if e then Inl* (*f x*) *else map-sum f g y*)
⋀*f g e x y*. *map-sum f g* (*if e then Inr x else y*) = (*if e then Inr* (*g x*) *else map-sum f g y*)
by *simp-all*

lemma *map-sum-if-distrib-else*:
⋀*f g e x y*. *map-sum f g* (*if e then x else Inl y*) = (*if e then map-sum f g x else Inl* (*f y*))
⋀*f g e x y*. *map-sum f g* (*if e then x else Inr y*) = (*if e then map-sum f g x else Inr* (*g y*))
by *simp-all*

lemma *case-prod-app*: *case-prod f x y* = *case-prod* (λ*l r*. *f l r y*) *x*
by (*cases x*) *simp*

lemma *case-sum-map-sum*: *case-sum l r* (*map-sum f g x*) = *case-sum* (*l* ∘ *f*) (*r* ∘ *g*) *x*
by (*cases x*) *simp-all*

lemma *case-sum-transfer*:
rel-fun (*rel-fun R T*) (*rel-fun* (*rel-fun S T*) (*rel-fun* (*rel-sum R S*) *T*)) *case-sum case-sum*
unfolding *rel-fun-def* **by** (*auto split*: *sum.splits*)

lemma *case-prod-map-prod*: *case-prod h* (*map-prod f g x*) = *case-prod* (λ*l r*. *h* (*f l*) (*g r*)) *x*
by (*cases x*) *simp-all*

lemma *case-prod-o-map-prod*: *case-prod f* ∘ *map-prod g1 g2* = *case-prod* (λ*l r*. *f* (*g1 l*) (*g2 r*))
unfolding *comp-def* **by** *auto*

lemma *case-prod-transfer*:
(*rel-fun* (*rel-fun A* (*rel-fun B C*)) (*rel-fun* (*rel-prod A B*) *C*)) *case-prod case-prod*
unfolding *rel-fun-def* **by** *simp*

lemma *eq-ifI*: (*P* ⟶ *t* = *u1*) ⟹ (¬ *P* ⟶ *t* = *u2*) ⟹ *t* = (*if P then u1 else u2*)
by *simp*

lemma *comp-transfer*:
rel-fun (*rel-fun B C*) (*rel-fun* (*rel-fun A B*) (*rel-fun A C*)) (∘) (∘)
unfolding *rel-fun-def* **by** *simp*

lemma *If-transfer*: *rel-fun* (=) (*rel-fun A* (*rel-fun A A*)) *If If*
unfolding *rel-fun-def* **by** *simp*

lemma *Abs-transfer*:
assumes *type-copy1*: *type-definition Rep1 Abs1 UNIV*
assumes *type-copy2*: *type-definition Rep2 Abs2 UNIV*
shows *rel-fun R* (*vimage2p Rep1 Rep2 R*) *Abs1 Abs2*
unfolding *vimage2p-def rel-fun-def*
type-definition.Abs-inverse[*OF type-copy1 UNIV-I*]
type-definition.Abs-inverse[*OF type-copy2 UNIV-I*] **by** *simp*

lemma *Inl-transfer*:
rel-fun S (*rel-sum S T*) *Inl Inl*
by *auto*

lemma *Inr-transfer*:
rel-fun T (*rel-sum S T*) *Inr Inr*
by *auto*

lemma *Pair-transfer*: *rel-fun A* (*rel-fun B* (*rel-prod A B*)) *Pair Pair*

unfolding *rel-fun-def* **by** *simp*

lemma *eq-onp-live-step*: $x = y \Longrightarrow$ *eq-onp* P a $a \wedge x \longleftrightarrow P$ $a \wedge y$
by (*simp only*: *eq-onp-same-args*)

lemma *top-conj*: *top* $x \wedge P \longleftrightarrow P$ $P \wedge$ *top* $x \longleftrightarrow P$
by *blast+*

lemma *fst-convol'*: *fst* $(\langle f,\ g\rangle\ x) = f\ x$
using *fst-convol* **unfolding** *convol-def* **by** *simp*

lemma *snd-convol'*: *snd* $(\langle f,\ g\rangle\ x) = g\ x$
using *snd-convol* **unfolding** *convol-def* **by** *simp*

lemma *convol-expand-snd*: *fst* $\circ f = g \Longrightarrow \langle g,$ *snd* $\circ f\rangle = f$
unfolding *convol-def* **by** *auto*

lemma *convol-expand-snd'*:
assumes (*fst* $\circ f = g$)
shows $h =$ *snd* $\circ f \longleftrightarrow \langle g,\ h\rangle = f$
proof −
from *assms* **have** ∗: $\langle g,$ *snd* $\circ f\rangle = f$ **by** (*rule convol-expand-snd*)
then have $h =$ *snd* $\circ f \longleftrightarrow h =$ *snd* $\circ \langle g,$ *snd* $\circ f\rangle$ **by** *simp*
moreover have ... $\longleftrightarrow h =$ *snd* $\circ f$ **by** (*simp add*: *snd-convol*)
moreover have ... $\longleftrightarrow \langle g,\ h\rangle = f$ **by** (*subst* (*2*) ∗[*symmetric*]) (*auto simp*: *convol-def fun-eq-iff*)
ultimately show *?thesis* **by** *simp*
qed

lemma *case-sum-expand-Inr-pointfree*: $f \circ$ *Inl* $= g \Longrightarrow$ *case-sum* g ($f \circ$ *Inr*) $= f$
by (*auto split*: *sum.splits*)

lemma *case-sum-expand-Inr'*: $f \circ$ *Inl* $= g \Longrightarrow h = f \circ$ *Inr* $\longleftrightarrow$ *case-sum* g $h = f$
by (*rule iffI*) (*auto simp add*: *fun-eq-iff split*: *sum.splits*)

lemma *case-sum-expand-Inr*: $f \circ$ *Inl* $= g \Longrightarrow f\ x =$ *case-sum* g ($f \circ$ *Inr*) x
by (*auto split*: *sum.splits*)

lemma *id-transfer*: *rel-fun* A A *id id*
unfolding *rel-fun-def* **by** *simp*

lemma *fst-transfer*: *rel-fun* (*rel-prod* A B) A *fst fst*
unfolding *rel-fun-def* **by** *simp*

lemma *snd-transfer*: *rel-fun* (*rel-prod* A B) B *snd snd*
unfolding *rel-fun-def* **by** *simp*

lemma *convol-transfer*:
rel-fun (*rel-fun* R S) (*rel-fun* (*rel-fun* R T) (*rel-fun* R (*rel-prod* S T))) *BNF-Def.convol*

BNF-Def.convol
  unfolding *rel-fun-def convol-def* **by** *auto*

lemma *Let-const*: *Let x* (λ-. *c*) = *c*
  unfolding *Let-def* **..**

ML-file ‹*Tools/BNF/bnf-fp-util-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-fp-util.ML*›
ML-file ‹*Tools/BNF/bnf-fp-def-sugar-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-fp-def-sugar.ML*›
ML-file ‹*Tools/BNF/bnf-fp-n2m-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-fp-n2m.ML*›
ML-file ‹*Tools/BNF/bnf-fp-n2m-sugar.ML*›

end

37 Least Fixpoint (Datatype) Operation on Bounded Natural Functors

theory *BNF-Least-Fixpoint*
imports *BNF-Fixpoint-Base*
keywords
  datatype :: *thy-defn* **and**
  datatype-compat :: *thy-defn*
begin

lemma *subset-emptyI*: $(\bigwedge x.\ x \in A \Longrightarrow \mathit{False}) \Longrightarrow A \subseteq \{\}$
  by *blast*

lemma *image-Collect-subsetI*: $(\bigwedge x.\ P\ x \Longrightarrow f\ x \in B) \Longrightarrow f\ `\ \{x.\ P\ x\} \subseteq B$
  by *blast*

lemma *Collect-restrict*: $\{x.\ x \in X \wedge P\ x\} \subseteq X$
  by *auto*

lemma *prop-restrict*: $[\![x \in Z;\ Z \subseteq \{x.\ x \in X \wedge P\ x\}]\!] \Longrightarrow P\ x$
  by *auto*

lemma *underS-I*: $[\![i \neq j;\ (i,\ j) \in R]\!] \Longrightarrow i \in \mathit{underS}\ R\ j$
  unfolding *underS-def* **by** *simp*

lemma *underS-E*: $i \in \mathit{underS}\ R\ j \Longrightarrow i \neq j \wedge (i,\ j) \in R$
  unfolding *underS-def* **by** *simp*

lemma *underS-Field*: $i \in \mathit{underS}\ R\ j \Longrightarrow i \in \mathit{Field}\ R$
  unfolding *underS-def Field-def* **by** *auto*

lemma *ex-bij-betw*: $|A| \leq o\ (r :: 'b\ \mathit{rel}) \Longrightarrow \exists f\ B :: 'b\ \mathit{set}.\ \mathit{bij\text{-}betw}\ f\ B\ A$

by (*subst* (*asm*) *internalize-card-of-ordLeq*) (*auto dest*!: *iffD2*[*OF card-of-ordIso ordIso-symmetric*])

lemma *bij-betwI′*:
  ⟦⋀*x y*. ⟦*x* ∈ *X*; *y* ∈ *X*⟧ ⟹ (*f x* = *f y*) = (*x* = *y*);
   ⋀*x*. *x* ∈ *X* ⟹ *f x* ∈ *Y*;
   ⋀*y*. *y* ∈ *Y* ⟹ ∃ *x* ∈ *X*. *y* = *f x*⟧ ⟹ *bij-betw f X Y*
  unfolding *bij-betw-def inj-on-def* **by** *blast*

lemma *surj-fun-eq*:
  assumes *surj-on*: *f* ‘ *X* = *UNIV* **and** *eq-on*: ∀ *x* ∈ *X*. (*g1* ∘ *f*) *x* = (*g2* ∘ *f*) *x*
  shows *g1* = *g2*
proof (*rule ext*)
  fix *y*
  from *surj-on* **obtain** *x* **where** *x* ∈ *X* **and** *y* = *f x* **by** *blast*
  thus *g1 y* = *g2 y* **using** *eq-on* **by** *simp*
qed

lemma *Card-order-wo-rel*: *Card-order r* ⟹ *wo-rel r*
  unfolding *wo-rel-def card-order-on-def* **by** *blast*

lemma *Cinfinite-limit*: ⟦*x* ∈ *Field r*; *Cinfinite r*⟧ ⟹ ∃ *y* ∈ *Field r*. *x* ≠ *y* ∧ (*x*, *y*) ∈ *r*
  unfolding *cinfinite-def* **by** (*auto simp add*: *infinite-Card-order-limit*)

lemma *Card-order-trans*:
  ⟦*Card-order r*; *x* ≠ *y*; (*x*, *y*) ∈ *r*; *y* ≠ *z*; (*y*, *z*) ∈ *r*⟧ ⟹ *x* ≠ *z* ∧ (*x*, *z*) ∈ *r*
  unfolding *card-order-on-def well-order-on-def linear-order-on-def*
    partial-order-on-def preorder-on-def trans-def antisym-def **by** *blast*

lemma *Cinfinite-limit2*:
  assumes *x1*: *x1* ∈ *Field r* **and** *x2*: *x2* ∈ *Field r* **and** *r*: *Cinfinite r*
  shows ∃ *y* ∈ *Field r*. (*x1* ≠ *y* ∧ (*x1*, *y*) ∈ *r*) ∧ (*x2* ≠ *y* ∧ (*x2*, *y*) ∈ *r*)
proof −
  from *r* **have** *trans*: *trans r* **and** *total*: *Total r* **and** *antisym*: *antisym r*
    unfolding *card-order-on-def well-order-on-def linear-order-on-def*
      partial-order-on-def preorder-on-def **by** *auto*
  obtain *y1* **where** *y1*: *y1* ∈ *Field r x1* ≠ *y1* (*x1*, *y1*) ∈ *r*
    using *Cinfinite-limit*[*OF x1 r*] **by** *blast*
  obtain *y2* **where** *y2*: *y2* ∈ *Field r x2* ≠ *y2* (*x2*, *y2*) ∈ *r*
    using *Cinfinite-limit*[*OF x2 r*] **by** *blast*
  show *?thesis*
  proof (*cases y1* = *y2*)
    case *True* **with** *y1 y2* **show** *?thesis* **by** *blast*
  next
    case *False*
    with *y1*(*1*) *y2*(*1*) *total* **have** (*y1*, *y2*) ∈ *r* ∨ (*y2*, *y1*) ∈ *r*
      unfolding *total-on-def* **by** *auto*
    thus *?thesis*

proof
assume ∗: $(y1, y2) \in r$
with *trans y1*(*3*) **have** $(x1, y2) \in r$ **unfolding** *trans-def* **by** *blast*
with *False y1 y2* ∗ *antisym* **show** *?thesis* **by** (*cases* $x1 = y2$) (*auto simp*: *antisym-def*)
next
assume ∗: $(y2, y1) \in r$
with *trans y2*(*3*) **have** $(x2, y1) \in r$ **unfolding** *trans-def* **by** *blast*
with *False y1 y2* ∗ *antisym* **show** *?thesis* **by** (*cases* $x2 = y1$) (*auto simp*: *antisym-def*)
qed
qed
qed

lemma *Cinfinite-limit-finite*:
$[\![$*finite X*; $X \subseteq$ *Field r*; *Cinfinite r*$]\!] \Longrightarrow \exists\, y \in$ *Field r*. $\forall\, x \in X$. $(x \neq y \wedge (x, y) \in r)$
proof (*induct X rule*: *finite-induct*)
case *empty* **thus** *?case* **unfolding** *cinfinite-def* **using** *ex-in-conv*[*of Field r*] *finite.emptyI* **by** *auto*
next
case (*insert x X*)
then obtain *y* **where** *y*: $y \in$ *Field r* $\forall\, x \in X$. $(x \neq y \wedge (x, y) \in r)$ **by** *blast*
then obtain *z* **where** *z*: $z \in$ *Field r* $x \neq z \wedge (x, z) \in r$ $y \neq z \wedge (y, z) \in r$
using *Cinfinite-limit2*[*OF - y*(*1*) *insert*(*5*), *of x*] *insert*(*4*) **by** *blast*
show *?case*
apply (*intro bexI ballI*)
apply (*erule insertE*)
apply *hypsubst*
apply (*rule z*(*2*))
using *Card-order-trans*[*OF insert*(*5*)[*THEN conjunct2*]] *y*(*2*) *z*(*3*)
apply *blast*
apply (*rule z*(*1*))
done
qed

lemma *insert-subsetI*: $[\![x \in A; X \subseteq A]\!] \Longrightarrow$ *insert x X* $\subseteq A$
by *auto*

lemmas *well-order-induct-imp* = *wo-rel.well-order-induct*[*of r* $\lambda x.\ x \in$ *Field r* $\longrightarrow$ *P x* **for** *r P*]

lemma *meta-spec2*:
assumes $(\bigwedge x\ y.$ *PROP P x y*$)$
shows *PROP P x y*
by (*rule assms*)

lemma *nchotomy-relcomppE*:
assumes $\bigwedge y.\ \exists\, x.\ y = f\ x$ $(r\ OO\ s)\ a\ c$ $\bigwedge b.\ r\ a\ (f\ b) \Longrightarrow s\ (f\ b)\ c \Longrightarrow P$

shows P
proof (*rule relcompp.cases*[*OF assms*(*2*)], *hypsubst*)
fix b **assume** $r\ a\ b\ s\ b\ c$
moreover from *assms*(*1*) **obtain** b' **where** $b = f\ b'$ **by** *blast*
ultimately show P **by** (*blast intro*: *assms*(*3*))
qed

lemma *predicate2D-vimage2p*: $[\![R \le \mathit{vimage2p}\ f\ g\ S;\ R\ x\ y]\!] \Longrightarrow S\ (f\ x)\ (g\ y)$
unfolding *vimage2p-def* **by** *auto*

lemma *ssubst-Pair-rhs*: $[\![(r,\ s) \in R;\ s' = s]\!] \Longrightarrow (r,\ s') \in R$
by (*rule ssubst*)

lemma *all-mem-range1*:
$(\bigwedge y.\ y \in \mathit{range}\ f \Longrightarrow P\ y) \equiv (\bigwedge x.\ P\ (f\ x))$
by (*rule equal-intr-rule*) *fast+*

lemma *all-mem-range2*:
$(\bigwedge fa\ y.\ fa \in \mathit{range}\ f \Longrightarrow y \in \mathit{range}\ fa \Longrightarrow P\ y) \equiv (\bigwedge x\ xa.\ P\ (f\ x\ xa))$
by (*rule equal-intr-rule*) *fast+*

lemma *all-mem-range3*:
$(\bigwedge fa\ fb\ y.\ fa \in \mathit{range}\ f \Longrightarrow fb \in \mathit{range}\ fa \Longrightarrow y \in \mathit{range}\ fb \Longrightarrow P\ y) \equiv (\bigwedge x\ xa\ xb.\ P\ (f\ x\ xa\ xb))$
by (*rule equal-intr-rule*) *fast+*

lemma *all-mem-range4*:
$(\bigwedge fa\ fb\ fc\ y.\ fa \in \mathit{range}\ f \Longrightarrow fb \in \mathit{range}\ fa \Longrightarrow fc \in \mathit{range}\ fb \Longrightarrow y \in \mathit{range}\ fc \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc.\ P\ (f\ x\ xa\ xb\ xc))$
by (*rule equal-intr-rule*) *fast+*

lemma *all-mem-range5*:
$(\bigwedge fa\ fb\ fc\ fd\ y.\ fa \in \mathit{range}\ f \Longrightarrow fb \in \mathit{range}\ fa \Longrightarrow fc \in \mathit{range}\ fb \Longrightarrow fd \in \mathit{range}\ fc \Longrightarrow$
$y \in \mathit{range}\ fd \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc\ xd.\ P\ (f\ x\ xa\ xb\ xc\ xd))$
by (*rule equal-intr-rule*) *fast+*

lemma *all-mem-range6*:
$(\bigwedge fa\ fb\ fc\ fd\ fe\ ff\ y.\ fa \in \mathit{range}\ f \Longrightarrow fb \in \mathit{range}\ fa \Longrightarrow fc \in \mathit{range}\ fb \Longrightarrow fd \in \mathit{range}\ fc \Longrightarrow$
$fe \in \mathit{range}\ fd \Longrightarrow ff \in \mathit{range}\ fe \Longrightarrow y \in \mathit{range}\ ff \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc\ xd\ xe\ xf.\ P\ (f\ x\ xa\ xb\ xc\ xd\ xe\ xf))$
by (*rule equal-intr-rule*) (*fastforce*, *fast*)

lemma *all-mem-range7*:
$(\bigwedge fa\ fb\ fc\ fd\ fe\ ff\ fg\ y.\ fa \in \mathit{range}\ f \Longrightarrow fb \in \mathit{range}\ fa \Longrightarrow fc \in \mathit{range}\ fb \Longrightarrow fd \in \mathit{range}\ fc \Longrightarrow$

fe ∈ *range fd* ⟹ *ff* ∈ *range fe* ⟹ *fg* ∈ *range ff* ⟹ *y* ∈ *range fg* ⟹ *P y*) ≡
(⋀*x xa xb xc xd xe xf xg*. *P* (*f x xa xb xc xd xe xf xg*))
by (*rule equal-intr-rule*) (*fastforce*, *fast*)

lemma *all-mem-range8*:
(⋀*fa fb fc fd fe ff fg fh y*. *fa* ∈ *range f* ⟹ *fb* ∈ *range fa* ⟹ *fc* ∈ *range fb* ⟹
fd ∈ *range fc* ⟹
fe ∈ *range fd* ⟹ *ff* ∈ *range fe* ⟹ *fg* ∈ *range ff* ⟹ *fh* ∈ *range fg* ⟹ *y* ∈
range fh ⟹ *P y*) ≡
(⋀*x xa xb xc xd xe xf xg xh*. *P* (*f x xa xb xc xd xe xf xg xh*))
by (*rule equal-intr-rule*) (*fastforce*, *fast*)

lemmas *all-mem-range* = *all-mem-range1 all-mem-range2 all-mem-range3 all-mem-range4*
all-mem-range5
all-mem-range6 all-mem-range7 all-mem-range8

lemma *pred-fun-True-id*: *NO-MATCH id p* ⟹ *pred-fun* (λ*x*. *True*) *p f* = *pred-fun*
(λ*x*. *True*) *id* (*p* ∘ *f*)
unfolding *fun.pred-map* **unfolding** *comp-def id-def* **..**

ML-file ‹*Tools/BNF/bnf-lfp-util.ML*›
ML-file ‹*Tools/BNF/bnf-lfp-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-lfp.ML*›
ML-file ‹*Tools/BNF/bnf-lfp-compat.ML*›
ML-file ‹*Tools/BNF/bnf-lfp-rec-sugar-more.ML*›
ML-file ‹*Tools/BNF/bnf-lfp-size.ML*›

ML-file ‹*Tools/datatype-simprocs.ML*›

simproc-setup *datatype-no-proper-subterm*
((*x* :: *'a* :: *size*) = *y*) = ‹*K Datatype-Simprocs.no-proper-subterm-proc*›

end

38 Equivalence Relations in Higher-Order Set Theory

theory *Equiv-Relations*
imports *BNF-Least-Fixpoint*
begin

38.1 Equivalence relations – set version

definition *equiv* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *bool*
where *equiv A r* ⟷ *refl-on A r* ∧ *sym r* ∧ *trans r*

lemma *equivI*: *refl-on A r* ⟹ *sym r* ⟹ *trans r* ⟹ *equiv A r*
by (*simp add*: *equiv-def*)

lemma *equivE*:
  assumes *equiv A r*
  obtains *refl-on A r* **and** *sym r* **and** *trans r*
  using *assms* **by** (*simp add*: *equiv-def*)

Suppes, Theorem 70: r is an equiv relation iff $r^{-1}\ O\ r = r$.

First half: $equiv\ A\ r \Longrightarrow r^{-1}\ O\ r = r$.

lemma *sym-trans-comp-subset*: $sym\ r \Longrightarrow trans\ r \Longrightarrow r^{-1}\ O\ r \subseteq r$
  unfolding *trans-def sym-def converse-unfold* **by** *blast*

lemma *refl-on-comp-subset*: $refl\text{-}on\ A\ r \Longrightarrow r \subseteq r^{-1}\ O\ r$
  unfolding *refl-on-def* **by** *blast*

lemma *equiv-comp-eq*: $equiv\ A\ r \Longrightarrow r^{-1}\ O\ r = r$
  unfolding *equiv-def*
  by (*iprover intro*: *sym-trans-comp-subset refl-on-comp-subset equalityI*)

Second half.

lemma *comp-equivI*:
  assumes $r^{-1}\ O\ r = r$ $Domain\ r = A$
  shows *equiv A r*
proof −
  have $*$: $\bigwedge x\ y.\ (x,\ y) \in r \Longrightarrow (y,\ x) \in r$
    using *assms* **by** *blast*
  show *?thesis*
    unfolding *equiv-def refl-on-def sym-def trans-def*
    using *assms* **by** (*auto intro*: $*$)
qed

38.2 Equivalence classes

lemma *equiv-class-subset*: $equiv\ A\ r \Longrightarrow (a,\ b) \in r \Longrightarrow r``\{a\} \subseteq r``\{b\}$
  — lemma for the next result
  unfolding *equiv-def trans-def sym-def* **by** *blast*

theorem *equiv-class-eq*: $equiv\ A\ r \Longrightarrow (a,\ b) \in r \Longrightarrow r``\{a\} = r``\{b\}$
  by (*intro equalityI equiv-class-subset*; *force simp add*: *equiv-def sym-def*)

lemma *equiv-class-self*: $equiv\ A\ r \Longrightarrow a \in A \Longrightarrow a \in r``\{a\}$
  unfolding *equiv-def refl-on-def* **by** *blast*

lemma *subset-equiv-class*: $equiv\ A\ r \Longrightarrow r``\{b\} \subseteq r``\{a\} \Longrightarrow b \in A \Longrightarrow (a,\ b) \in r$
  — lemma for the next result
  unfolding *equiv-def refl-on-def* **by** *blast*

lemma *eq-equiv-class*: $r``\{a\} = r``\{b\} \Longrightarrow equiv\ A\ r \Longrightarrow b \in A \Longrightarrow (a,\ b) \in r$
  by (*iprover intro*: *equalityD2 subset-equiv-class*)

lemma *equiv-class-nondisjoint*: *equiv* $A\ r \Longrightarrow x \in (r``\{a\} \cap r``\{b\}) \Longrightarrow (a,\ b) \in r$
unfolding *equiv-def trans-def sym-def* **by** *blast*

lemma *equiv-type*: *equiv* $A\ r \Longrightarrow r \subseteq A \times A$
unfolding *equiv-def refl-on-def* **by** *blast*

lemma *equiv-class-eq-iff*: *equiv* $A\ r \Longrightarrow (x,\ y) \in r \longleftrightarrow r``\{x\} = r``\{y\} \wedge x \in A \wedge y \in A$
by (*blast intro*!: *equiv-class-eq dest*: *eq-equiv-class equiv-type*)

lemma *eq-equiv-class-iff*: *equiv* $A\ r \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow r``\{x\} = r``\{y\} \longleftrightarrow (x,\ y) \in r$
by (*blast intro*!: *equiv-class-eq dest*: *eq-equiv-class equiv-type*)

lemma *disjnt-equiv-class*: *equiv* $A\ r \Longrightarrow$ *disjnt* $(r``\{a\})\ (r``\{b\}) \longleftrightarrow (a,\ b) \notin r$
by (*auto dest*: *equiv-class-self simp*: *equiv-class-eq-iff disjnt-def*)

38.3 Quotients

definition *quotient* :: $'a$ *set* $\Rightarrow ('a \times 'a)$ *set* $\Rightarrow 'a$ *set set* (**infixl** ‹'/'/› *90*)
where $A//r = (\bigcup x \in A.\ \{r``\{x\}\})$ — set of equiv classes

lemma *quotientI*: $x \in A \Longrightarrow r``\{x\} \in A//r$
unfolding *quotient-def* **by** *blast*

lemma *quotientE*: $X \in A//r \Longrightarrow (\bigwedge x.\ X = r``\{x\} \Longrightarrow x \in A \Longrightarrow P) \Longrightarrow P$
unfolding *quotient-def* **by** *blast*

lemma *Union-quotient*: *equiv* $A\ r \Longrightarrow \bigcup (A//r) = A$
unfolding *equiv-def refl-on-def quotient-def* **by** *blast*

lemma *quotient-disj*: *equiv* $A\ r \Longrightarrow X \in A//r \Longrightarrow Y \in A//r \Longrightarrow X = Y \vee X \cap Y = \{\}$
unfolding *quotient-def equiv-def trans-def sym-def* **by** *blast*

lemma *quotient-eqI*:
assumes *equiv* $A\ r\ X \in A//r\ Y \in A//r$ **and** *xy*: $x \in X\ y \in Y\ (x,\ y) \in r$
shows $X = Y$
proof −
obtain $a\ b$ **where** $a \in A$ **and** a: $X = r\ ``\ \{a\}$ **and** $b \in A$ **and** $b$: $Y = r\ ``\ \{b\}$
using *assms* **by** (*auto elim*!: *quotientE*)
then have $(a,b) \in r$
using *xy* ‹*equiv A r*› **unfolding** *equiv-def sym-def trans-def* **by** *blast*
then show *?thesis*
unfolding *a b* **by** (*rule equiv-class-eq* [*OF* ‹*equiv A r*›])
qed

lemma *quotient-eq-iff*:
assumes *equiv* $A\ r\ X \in A//r\ Y \in A//r$ **and** *xy*: $x \in X\ y \in Y$

shows $X = Y \longleftrightarrow (x, y) \in r$
proof
assume L: $X = Y$
with *assms* **show** $(x, y) \in r$
unfolding *equiv-def sym-def trans-def* **by** (*blast elim*!: *quotientE*)
next
assume §: $(x, y) \in r$ **show** $X = Y$
by (*rule quotient-eqI*) (*use* § *assms* **in** ‹*blast*+›)
qed

lemma *eq-equiv-class-iff2*: $equiv\ A\ r \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow \{x\}//r = \{y\}//r \longleftrightarrow (x, y) \in r$
by (*simp add*: *quotient-def eq-equiv-class-iff*)

lemma *quotient-empty* [*simp*]: $\{\}//r = \{\}$
by (*simp add*: *quotient-def*)

lemma *quotient-is-empty* [*iff*]: $A//r = \{\} \longleftrightarrow A = \{\}$
by (*simp add*: *quotient-def*)

lemma *quotient-is-empty2* [*iff*]: $\{\} = A//r \longleftrightarrow A = \{\}$
by (*simp add*: *quotient-def*)

lemma *singleton-quotient*: $\{x\}//r = \{r \text{ `` } \{x\}\}$
by (*simp add*: *quotient-def*)

lemma *quotient-diff1*: $inj\text{-}on\ (\lambda a.\ \{a\}//r)\ A \Longrightarrow a \in A \Longrightarrow (A - \{a\})//r = A//r - \{a\}//r$
unfolding *quotient-def inj-on-def* **by** *blast*

38.4 Refinement of one equivalence relation WRT another

lemma *refines-equiv-class-eq*: $R \subseteq S \Longrightarrow equiv\ A\ R \Longrightarrow equiv\ A\ S \Longrightarrow R``(S``\{a\}) = S``\{a\}$
by (*auto simp*: *equiv-class-eq-iff*)

lemma *refines-equiv-class-eq2*: $R \subseteq S \Longrightarrow equiv\ A\ R \Longrightarrow equiv\ A\ S \Longrightarrow S``(R``\{a\}) = S``\{a\}$
by (*auto simp*: *equiv-class-eq-iff*)

lemma *refines-equiv-image-eq*: $R \subseteq S \Longrightarrow equiv\ A\ R \Longrightarrow equiv\ A\ S \Longrightarrow (\lambda X.\ S``X) \text{ ` } (A//R) = A//S$
by (*auto simp*: *quotient-def image-UN refines-equiv-class-eq2*)

lemma *finite-refines-finite*:
$finite\ (A//R) \Longrightarrow R \subseteq S \Longrightarrow equiv\ A\ R \Longrightarrow equiv\ A\ S \Longrightarrow finite\ (A//S)$
by (*erule finite-surj* [**where** $f = \lambda X.\ S``X$]) (*simp add*: *refines-equiv-image-eq*)

lemma *finite-refines-card-le*:

finite $(A//R) \Longrightarrow R \subseteq S \Longrightarrow$ *equiv* A $R \Longrightarrow$ *equiv* A $S \Longrightarrow$ *card* $(A//S) \leq$ *card* $(A//R)$
by (*subst refines-equiv-image-eq* [*of* R S A, *symmetric*])
(*auto simp*: *card-image-le* [**where** $f = \lambda X.\ S``X$])

38.5 Defining unary operations upon equivalence classes

A congruence-preserving function.

definition *congruent* :: $('a \times 'a)$ *set* $\Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool*
where *congruent* r $f \longleftrightarrow (\forall (y, z) \in r.\ f\ y = f\ z)$

lemma *congruentI*: $(\bigwedge y\ z.\ (y, z) \in r \Longrightarrow f\ y = f\ z) \Longrightarrow$ *congruent* r f
by (*auto simp add*: *congruent-def*)

lemma *congruentD*: *congruent* r $f \Longrightarrow (y, z) \in r \Longrightarrow f\ y = f\ z$
by (*auto simp add*: *congruent-def*)

abbreviation *RESPECTS* :: $('a \Rightarrow 'b) \Rightarrow ('a \times 'a)$ *set* $\Rightarrow$ *bool* (**infixr** ‹*respects*› *80*)
where f *respects* $r \equiv$ *congruent* r f

lemma *UN-constant-eq*: $a \in A \Longrightarrow \forall y \in A.\ f\ y = c \Longrightarrow (\bigcup y \in A.\ f\ y) = c$
— lemma required to prove *UN-equiv-class*
by *auto*

lemma *UN-equiv-class*:
assumes *equiv* A r f *respects* r $a \in A$
shows $(\bigcup x \in r``\{a\}.\ f\ x) = f\ a$
— Conversion rule
proof –
have §: $\forall x{\in}r `` \{a\}.\ f\ x = f\ a$
using *assms* **unfolding** *equiv-def congruent-def sym-def* **by** *blast*
show *?thesis*
by (*iprover intro*: *assms UN-constant-eq* [*OF equiv-class-self* §])
qed

lemma *UN-equiv-class-type*:
assumes r: *equiv* A r f *respects* r **and** X: $X \in A//r$ **and** AB: $\bigwedge x.\ x \in A \Longrightarrow f\ x \in B$
shows $(\bigcup x \in X.\ f\ x) \in B$
using *assms* **unfolding** *quotient-def*
by (*auto simp*: *UN-equiv-class* [*OF* r])

Sufficient conditions for injectiveness. Could weaken premises! major premise could be an inclusion; *bcong* could be $\bigwedge y.\ y \in A \Longrightarrow f\ y \in B$.

lemma *UN-equiv-class-inject*:
assumes *equiv* A r f *respects* r

and *eq*: $(\bigcup x \in X.\ f\ x) = (\bigcup y \in Y.\ f\ y)$
and *X*: $X \in A//r$ **and** *Y*: $Y \in A//r$
and *fr*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \Longrightarrow (x, y) \in r$
shows $X = Y$
proof –
obtain *a b* **where** $a \in A$ **and** *a*: $X = r$ `` $\{a\}$ **and** $b \in A$ **and** *b*: $Y = r$ `` $\{b\}$
using *assms* **by** (*auto elim*!: *quotientE*)
then have $\bigcup (f$ ` r `` $\{a\}) = f\ a$ $\bigcup (f$ ` $r$ `` $\{b\}) = f\ b$
by (*iprover intro*: *UN-equiv-class* [*OF* ‹*equiv A r*›] *assms*)+
then have $f\ a = f\ b$
using *eq* **unfolding** *a b* **by** (*iprover intro*: *trans sym*)
then have $(a,b) \in r$
using *fr* ‹$a \in A$› ‹$b \in A$› **by** *blast*
then show *?thesis*
unfolding *a b* **by** (*rule equiv-class-eq* [*OF* ‹*equiv A r*›])
qed

38.6 Defining binary operations upon equivalence classes

A congruence-preserving function of two arguments.

definition *congruent2* :: $('a \times 'a)$ *set* $\Rightarrow$ $('b \times 'b)$ *set* $\Rightarrow$ $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow$ *bool*
where *congruent2 r1 r2 f* $\longleftrightarrow$ $(\forall (y1, z1) \in r1.\ \forall (y2, z2) \in r2.\ f\ y1\ y2 = f\ z1\ z2)$

lemma *congruent2I'*:
assumes $\bigwedge y1\ z1\ y2\ z2.\ (y1, z1) \in r1 \Longrightarrow (y2, z2) \in r2 \Longrightarrow f\ y1\ y2 = f\ z1\ z2$
shows *congruent2 r1 r2 f*
using *assms* **by** (*auto simp add*: *congruent2-def*)

lemma *congruent2D*: *congruent2 r1 r2 f* $\Longrightarrow (y1, z1) \in r1 \Longrightarrow (y2, z2) \in r2 \Longrightarrow f\ y1\ y2 = f\ z1\ z2$
by (*auto simp add*: *congruent2-def*)

Abbreviation for the common case where the relations are identical.

abbreviation *RESPECTS2*:: $('a \Rightarrow 'a \Rightarrow 'b) \Rightarrow ('a \times 'a)$ *set* $\Rightarrow$ *bool* (**infixr** ‹*respects2*› *80*)
where *f respects2 r* $\equiv$ *congruent2 r r f*

lemma *congruent2-implies-congruent*:
equiv A r1 $\Longrightarrow$ *congruent2 r1 r2 f* $\Longrightarrow a \in A \Longrightarrow$ *congruent r2* $(f\ a)$
unfolding *congruent-def congruent2-def equiv-def refl-on-def* **by** *blast*

lemma *congruent2-implies-congruent-UN*:
assumes *equiv A1 r1 equiv A2 r2 congruent2 r1 r2 f* $a \in A2$
shows *congruent r1* $(\lambda x1.\ \bigcup x2 \in r2``\{a\}.\ f\ x1\ x2)$
unfolding *congruent-def*
proof *clarify*
fix *c d*

assume *cd*: $(c,d) \in r1$
then have $c \in A1$ $d \in A1$
using ‹*equiv A1 r1*› **by** (*auto elim*!: *equiv-type* [*THEN subsetD*, *THEN SigmaE2*])
moreover have $f\ c\ a = f\ d\ a$
using *assms cd* **unfolding** *congruent2-def equiv-def refl-on-def* **by** *blast*
ultimately show $\bigcup\ (f\ c\ \text{`}\ r2\ \text{``}\ \{a\}) = \bigcup\ (f\ d\ \text{`}\ r2\ \text{``}\ \{a\})$
using *assms* **by** (*simp add*: *UN-equiv-class congruent2-implies-congruent*)
qed

lemma *UN-equiv-class2*:
$equiv\ A1\ r1 \Longrightarrow equiv\ A2\ r2 \Longrightarrow congruent2\ r1\ r2\ f \Longrightarrow a1 \in A1 \Longrightarrow a2 \in A2 \Longrightarrow$
$(\bigcup x1 \in r1\text{``}\{a1\}.\ \bigcup x2 \in r2\text{``}\{a2\}.\ f\ x1\ x2) = f\ a1\ a2$
by (*simp add*: *UN-equiv-class congruent2-implies-congruent congruent2-implies-congruent-UN*)

lemma *UN-equiv-class-type2*:
$equiv\ A1\ r1 \Longrightarrow equiv\ A2\ r2 \Longrightarrow congruent2\ r1\ r2\ f$
$\Longrightarrow X1 \in A1//r1 \Longrightarrow X2 \in A2//r2$
$\Longrightarrow (\bigwedge x1\ x2.\ x1 \in A1 \Longrightarrow x2 \in A2 \Longrightarrow f\ x1\ x2 \in B)$
$\Longrightarrow (\bigcup x1 \in X1.\ \bigcup x2 \in X2.\ f\ x1\ x2) \in B$
unfolding *quotient-def*
by (*blast intro*: *UN-equiv-class-type congruent2-implies-congruent-UN*
congruent2-implies-congruent quotientI)

lemma *UN-UN-split-split-eq*:
$(\bigcup (x1,\ x2) \in X.\ \bigcup (y1,\ y2) \in Y.\ A\ x1\ x2\ y1\ y2) =$
$(\bigcup x \in X.\ \bigcup y \in Y.\ (\lambda(x1,\ x2).\ (\lambda(y1,\ y2).\ A\ x1\ x2\ y1\ y2)\ y)\ x)$
— Allows a natural expression of binary operators,
— without explicit calls to *split*
by *auto*

lemma *congruent2I*:
$equiv\ A1\ r1 \Longrightarrow equiv\ A2\ r2$
$\Longrightarrow (\bigwedge y\ z\ w.\ w \in A2 \Longrightarrow (y,z) \in r1 \Longrightarrow f\ y\ w = f\ z\ w)$
$\Longrightarrow (\bigwedge y\ z\ w.\ w \in A1 \Longrightarrow (y,z) \in r2 \Longrightarrow f\ w\ y = f\ w\ z)$
$\Longrightarrow congruent2\ r1\ r2\ f$
— Suggested by John Harrison – the two subproofs may be
— *much* simpler than the direct proof.
unfolding *congruent2-def equiv-def refl-on-def*
by (*blast intro*: *trans*)

lemma *congruent2-commuteI*:
assumes *equivA*: *equiv A r*
and *commute*: $\bigwedge y\ z.\ y \in A \Longrightarrow z \in A \Longrightarrow f\ y\ z = f\ z\ y$
and *congt*: $\bigwedge y\ z\ w.\ w \in A \Longrightarrow (y,z) \in r \Longrightarrow f\ w\ y = f\ w\ z$
shows *f respects2 r*
proof (*rule congruent2I* [*OF equivA equivA*])

note *eqv* = *equivA* [*THEN equiv-type*, *THEN subsetD*, *THEN SigmaE2*]
show $\bigwedge y\ z\ w.\ [\![w \in A;\ (y,\ z) \in r]\!] \Longrightarrow f\ y\ w = f\ z\ w$
by (*iprover intro*: *commute* [*THEN trans*] *sym congt elim*: *eqv*)
show $\bigwedge y\ z\ w.\ [\![w \in A;\ (y,\ z) \in r]\!] \Longrightarrow f\ w\ y = f\ w\ z$
by (*iprover intro*: *congt elim*: *eqv*)
qed

38.7 Quotients and finiteness

Suggested by Florian Kammüller

lemma *finite-quotient*:
assumes *finite A* $r \subseteq A \times A$
shows *finite* $(A//r)$
— recall *equiv* $?A\ ?r \Longrightarrow ?r \subseteq ?A \times ?A$
proof –
have $A//r \subseteq Pow\ A$
using *assms* **unfolding** *quotient-def* **by** *blast*
moreover have *finite* (*Pow A*)
using *assms* **by** *simp*
ultimately show *?thesis*
by (*iprover intro*: *finite-subset*)
qed

lemma *finite-equiv-class*: *finite* $A \Longrightarrow r \subseteq A \times A \Longrightarrow X \in A//r \Longrightarrow$ *finite X*
unfolding *quotient-def*
by (*erule rev-finite-subset*) *blast*

lemma *equiv-imp-dvd-card*:
assumes *finite A* *equiv A r* $\bigwedge X.\ X \in A//r \Longrightarrow k\ dvd\ card\ X$
shows *k dvd card A*
proof (*rule Union-quotient* [*THEN subst*])
show *k dvd card* $(\bigcup\ (A\ //\ r))$
apply (*rule dvd-partition*)
using *assms*
by (*auto simp*: *Union-quotient dest*: *quotient-disj*)
qed (*use assms* **in** *blast*)

38.8 Projection

definition *proj* :: $('b \times 'a)\ set \Rightarrow 'b \Rightarrow 'a\ set$
where $proj\ r\ x = r\ ``\ \{x\}$

lemma *proj-preserves*: $x \in A \Longrightarrow proj\ r\ x \in A//r$
unfolding *proj-def* **by** (*rule quotientI*)

lemma *proj-in-iff*:
assumes *equiv A r*
shows $proj\ r\ x \in A//r \longleftrightarrow x \in A$
(**is** $?lhs \longleftrightarrow ?rhs$)

proof
  assume *?rhs*
  then show *?lhs* **by** (*simp add*: *proj-preserves*)
next
  assume *?lhs*
  then show *?rhs*
    unfolding *proj-def quotient-def*
  proof *safe*
    fix y
    assume y: $y \in A$ **and** r `` $\{x\} = r$ `` $\{y\}$
    moreover have $y \in r$ `` $\{y\}$
      using *assms y* **unfolding** *equiv-def refl-on-def* **by** *blast*
    ultimately have $(x, y) \in r$ **by** *blast*
    then show $x \in A$
      using *assms* **unfolding** *equiv-def refl-on-def* **by** *blast*
  qed
qed

lemma *proj-iff*: *equiv* A $r \Longrightarrow \{x, y\} \subseteq A \Longrightarrow$ *proj* r x = *proj* r $y \longleftrightarrow (x, y) \in r$
  by (*simp add*: *proj-def eq-equiv-class-iff*)

lemma *proj-image*: *proj* r ` $A = A//r$
  unfolding *proj-def*[*abs-def*] *quotient-def* **by** *blast*

lemma *in-quotient-imp-non-empty*: *equiv* A $r \Longrightarrow X \in A//r \Longrightarrow X \neq \{\}$
  unfolding *quotient-def* **using** *equiv-class-self* **by** *fast*

lemma *in-quotient-imp-in-rel*: *equiv* A $r \Longrightarrow X \in A//r \Longrightarrow \{x, y\} \subseteq X \Longrightarrow (x, y) \in r$
  using *quotient-eq-iff*[*THEN iffD1*] **by** *fastforce*

lemma *in-quotient-imp-closed*: *equiv* A $r \Longrightarrow X \in A//r \Longrightarrow x \in X \Longrightarrow (x, y) \in r \Longrightarrow y \in X$
  unfolding *quotient-def equiv-def trans-def* **by** *blast*

lemma *in-quotient-imp-subset*: *equiv* A $r \Longrightarrow X \in A//r \Longrightarrow X \subseteq A$
  using *in-quotient-imp-in-rel equiv-type* **by** *fastforce*

38.9 Equivalence relations – predicate version

Partial equivalences.

definition *part-equivp* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ *bool*
  where *part-equivp* $R \longleftrightarrow (\exists x.\ R\ x\ x) \wedge (\forall x\ y.\ R\ x\ y \longleftrightarrow R\ x\ x \wedge R\ y\ y \wedge R\ x = R\ y)$
    — John-Harrison-style characterization

lemma *part-equivpI*: $\exists x.\ R\ x\ x \Longrightarrow$ *symp* $R \Longrightarrow$ *transp* $R \Longrightarrow$ *part-equivp* R

by (*auto simp add*: *part-equivp-def*) (*auto elim*: *sympE transpE*)

lemma *part-equivpE*:
assumes *part-equivp R*
obtains *x* **where** *R x x* **and** *symp R* **and** *transp R*
proof −
from *assms* **have** *1*: $\exists x.\ R\ x\ x$
and *2*: $\bigwedge x\ y.\ R\ x\ y \longleftrightarrow R\ x\ x \wedge R\ y\ y \wedge R\ x = R\ y$
unfolding *part-equivp-def* **by** *blast+*
from *1* **obtain** *x* **where** *R x x* **..**
moreover have *symp R*
proof (*rule sympI*)
fix *x y*
assume *R x y*
with *2* [*of x y*] **show** *R y x* **by** *auto*
qed
moreover have *transp R*
proof (*rule transpI*)
fix *x y z*
assume *R x y* **and** *R y z*
with *2* [*of x y*] *2* [*of y z*] **show** *R x z* **by** *auto*
qed
ultimately show *thesis* **by** (*rule that*)
qed

lemma *part-equivp-refl-symp-transp*: *part-equivp R* $\longleftrightarrow$ ($\exists x.\ R\ x\ x$) $\wedge$ *symp R* $\wedge$ *transp R*
by (*auto intro*: *part-equivpI elim*: *part-equivpE*)

lemma *part-equivp-symp*: *part-equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y x*
by (*erule part-equivpE*, *erule sympE*)

lemma *part-equivp-transp*: *part-equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y z* $\Longrightarrow$ *R x z*
by (*erule part-equivpE*, *erule transpE*)

lemma *part-equivp-typedef*: *part-equivp R* $\Longrightarrow \exists d.\ d \in \{c.\ \exists x.\ R\ x\ x \wedge c = Collect\ (R\ x)\}$
by (*auto elim*: *part-equivpE*)

Total equivalences.

definition *equivp* :: ($'a \Rightarrow\ 'a \Rightarrow bool$) $\Rightarrow$ *bool*
where *equivp R* $\longleftrightarrow$ ($\forall x\ y.\ R\ x\ y = (R\ x = R\ y)$) — John-Harrison-style characterization

lemma *equivpI*: *reflp R* $\Longrightarrow$ *symp R* $\Longrightarrow$ *transp R* $\Longrightarrow$ *equivp R*
by (*auto elim*: *reflpE sympE transpE simp add*: *equivp-def*)

lemma *equivpE*:
assumes *equivp R*

obtains *reflp R* **and** *symp R* **and** *transp R*
using *assms* **by** (*auto intro*!: *that reflpI sympI transpI simp add*: *equivp-def*)

lemma *equivp-implies-part-equivp*: *equivp R* $\Longrightarrow$ *part-equivp R*
by (*auto intro*: *part-equivpI elim*: *equivpE reflpE*)

lemma *equivp-equiv*: *equiv UNIV A* $\longleftrightarrow$ *equivp* ($\lambda x\ y.\ (x, y) \in A$)
by (*auto intro*!: *equivI equivpI* [*to-set*] *elim*!: *equivE equivpE* [*to-set*])

lemma *equivp-reflp-symp-transp*: *equivp R* $\longleftrightarrow$ *reflp R* $\wedge$ *symp R* $\wedge$ *transp R*
by (*auto intro*: *equivpI elim*: *equivpE*)

lemma *identity-equivp*: *equivp* (=)
by (*auto intro*: *equivpI reflpI sympI transpI*)

lemma *equivp-reflp*: *equivp R* $\Longrightarrow$ *R x x*
by (*erule equivpE*, *erule reflpE*)

lemma *equivp-symp*: *equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y x*
by (*erule equivpE*, *erule sympE*)

lemma *equivp-transp*: *equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y z* $\Longrightarrow$ *R x z*
by (*erule equivpE*, *erule transpE*)

lemma *equivp-rtranclp*: *symp r* $\Longrightarrow$ *equivp* r^{**}
by(*intro equivpI reflpI sympI transpI*)(*auto dest*: *sympD*[*OF symp-rtranclp*])

lemmas *equivp-rtranclp-symclp* [*simp*] = *equivp-rtranclp*[*OF symp-on-symclp*]

lemma *equivp-vimage2p*: *equivp R* $\Longrightarrow$ *equivp* (*vimage2p f f R*)
by(*auto simp add*: *equivp-def vimage2p-def dest*: *fun-cong*)

lemma *equivp-imp-transp*: *equivp R* $\Longrightarrow$ *transp R*
by(*simp add*: *equivp-reflp-symp-transp*)

38.10 Equivalence closure

definition *equivclp* :: ($'a \Rightarrow 'a \Rightarrow bool$) $\Rightarrow$ $'a \Rightarrow 'a \Rightarrow bool$ **where**
equivclp r = (*symclp r*)**

lemma *transp-equivclp* [*simp*]: *transp* (*equivclp r*)
by(*simp add*: *equivclp-def*)

lemma *reflp-equivclp* [*simp*]: *reflp* (*equivclp r*)
by(*simp add*: *equivclp-def*)

lemma *symp-equivclp* [*simp*]: *symp* (*equivclp r*)
by(*simp add*: *equivclp-def*)

lemma *equivp-evquivclp* [*simp*]: *equivp* (*equivclp r*)
by(*simp add*: *equivpI*)

lemma *tranclp-equivclp* [*simp*]: $(equivclp\ r)^{++} = equivclp\ r$
by(*simp add*: *equivclp-def*)

lemma *rtranclp-equivclp* [*simp*]: $(equivclp\ r)^{**} = equivclp\ r$
by(*simp add*: *equivclp-def*)

lemma *symclp-equivclp* [*simp*]: *symclp* (*equivclp r*) = *equivclp r*
by(*simp add*: *equivclp-def symp-symclp-eq*)

lemma *equivclp-symclp* [*simp*]: *equivclp* (*symclp r*) = *equivclp r*
by(*simp add*: *equivclp-def*)

lemma *equivclp-conversep* [*simp*]: *equivclp* (*conversep r*) = *equivclp r*
by(*simp add*: *equivclp-def*)

lemma *equivclp-sym* [*sym*]: *equivclp r x y* $\Longrightarrow$ *equivclp r y x*
by(*rule sympD*[*OF symp-equivclp*])

lemma *equivclp-OO-equivclp-le-equivclp*: *equivclp r OO equivclp r* $\leq$ *equivclp r*
by(*rule transp-relcompp-less-eq transp-equivclp*)+

lemma *rtranlcp-le-equivclp*: $r^{**} \leq$ *equivclp r*
unfolding *equivclp-def* **by**(*rule rtranclp-mono*)(*simp add*: *symclp-pointfree*)

lemma *rtranclp-conversep-le-equivclp*: $r^{-1-1**} \leq$ *equivclp r*
unfolding *equivclp-def* **by**(*rule rtranclp-mono*)(*simp add*: *symclp-pointfree*)

lemma *symclp-rtranclp-le-equivclp*: *symclp* $r^{**} \leq$ *equivclp r*
unfolding *symclp-pointfree*
by(*rule le-supI*)(*simp-all add*: *rtranclp-conversep*[*symmetric*] *rtranlcp-le-equivclp rtranclp-conversep-le-equivclp*)

lemma *r-OO-conversep-into-equivclp*:
r^{**} *OO* $r^{-1-1**} \leq$ *equivclp r*
by(*blast intro*: *order-trans*[*OF - equivclp-OO-equivclp-le-equivclp*] *relcompp-mono rtranlcp-le-equivclp rtranclp-conversep-le-equivclp del*: *predicate2I*)

lemma *equivclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *equivclp*]:
assumes *a*: *equivclp r a b*
and *cases*: *P a* $\bigwedge y\ z.$ *equivclp r a y* $\Longrightarrow$ *r y z* $\vee$ *r z y* $\Longrightarrow$ *P y* $\Longrightarrow$ *P z*
shows *P b*
using *a* **unfolding** *equivclp-def*
by(*induction rule*: *rtranclp-induct*; *fold equivclp-def*; *blast intro*: *cases elim*: *symclpE*)

lemma *converse-equivclp-induct* [*consumes 1*, *case-names base step*]:

assumes *major*: *equivclp r a b*
and *cases*: *P b* $\bigwedge$*y z. r y z* ∨ *r z y* ⟹ *equivclp r z b* ⟹ *P z* ⟹ *P y*
shows *P a*
using *major* **unfolding** *equivclp-def*
by(*induction rule*: *converse-rtranclp-induct*; *fold equivclp-def*; *blast intro*: *cases elim*: *symclpE*)

lemma *equivclp-refl* [*simp*]: *equivclp r x x*
by(*rule reflpD*[*OF reflp-equivclp*])

lemma *r-into-equivclp* [*intro*]: *r x y* ⟹ *equivclp r x y*
unfolding *equivclp-def* **by**(*blast intro*: *symclpI*)

lemma *converse-r-into-equivclp* [*intro*]: *r y x* ⟹ *equivclp r x y*
unfolding *equivclp-def* **by**(*blast intro*: *symclpI*)

lemma *rtranclp-into-equivclp*: r^{**} *x y* ⟹ *equivclp r x y*
using *rtranlcp-le-equivclp*[*of r*] **by** *blast*

lemma *converse-rtranclp-into-equivclp*: r^{**} *y x* ⟹ *equivclp r x y*
by(*blast intro*: *equivclp-sym rtranclp-into-equivclp*)

lemma *equivclp-into-equivclp*: ⟦ *equivclp r a b*; *r b c* ∨ *r c b* ⟧ ⟹ *equivclp r a c*
unfolding *equivclp-def* **by**(*erule rtranclp.rtrancl-into-rtrancl*)(*auto intro*: *symclpI*)

lemma *equivclp-trans* [*trans*]: ⟦ *equivclp r a b*; *equivclp r b c* ⟧ ⟹ *equivclp r a c*
using *equivclp-OO-equivclp-le-equivclp*[*of r*] **by** *blast*

hide-const (**open**) *proj*

end

theory *Basic-BNF-LFPs*
imports *BNF-Least-Fixpoint*
begin

definition *xtor* :: *'a* ⇒ *'a* **where**
xtor x = *x*

lemma *xtor-map*: *f* (*xtor x*) = *xtor* (*f x*)
unfolding *xtor-def* **by** (*rule refl*)

lemma *xtor-map-unique*: *u* ∘ *xtor* = *xtor* ∘ *f* ⟹ *u* = *f*
unfolding *o-def xtor-def* **.**

lemma *xtor-set*: *f* (*xtor x*) = *f x*
unfolding *xtor-def* **by** (*rule refl*)

lemma *xtor-rel*: $R\ (xtor\ x)\ (xtor\ y) = R\ x\ y$
unfolding *xtor-def* **by** (*rule refl*)

lemma *xtor-induct*: $(\bigwedge x.\ P\ (xtor\ x)) \Longrightarrow P\ z$
unfolding *xtor-def* **by** *assumption*

lemma *xtor-xtor*: $xtor\ (xtor\ x) = x$
unfolding *xtor-def* **by** (*rule refl*)

lemmas *xtor-inject* = *xtor-rel*[*of* (=)]

lemma *xtor-rel-induct*: $(\bigwedge x\ y.\ vimage2p\ id\text{-}bnf\ id\text{-}bnf\ R\ x\ y \Longrightarrow IR\ (xtor\ x)\ (xtor\ y)) \Longrightarrow R \leq IR$
unfolding *xtor-def vimage2p-def id-bnf-def* **..**

lemma *xtor-iff-xtor*: $u = xtor\ w \longleftrightarrow xtor\ u = w$
unfolding *xtor-def* **..**

lemma *Inl-def-alt*: $Inl \equiv (\lambda a.\ xtor\ (id\text{-}bnf\ (Inl\ a)))$
unfolding *xtor-def id-bnf-def* **by** (*rule reflexive*)

lemma *Inr-def-alt*: $Inr \equiv (\lambda a.\ xtor\ (id\text{-}bnf\ (Inr\ a)))$
unfolding *xtor-def id-bnf-def* **by** (*rule reflexive*)

lemma *Pair-def-alt*: $Pair \equiv (\lambda a\ b.\ xtor\ (id\text{-}bnf\ (a,\ b)))$
unfolding *xtor-def id-bnf-def* **by** (*rule reflexive*)

definition *ctor-rec* :: $'a \Rightarrow\ 'a$ **where**
ctor-rec $x = x$

lemma *ctor-rec*: $g = id \Longrightarrow ctor\text{-}rec\ f\ (xtor\ x) = f\ ((id\text{-}bnf \circ g \circ id\text{-}bnf)\ x)$
unfolding *ctor-rec-def id-bnf-def xtor-def comp-def id-def* **by** *hypsubst* (*rule refl*)

lemma *ctor-rec-unique*: $g = id \Longrightarrow f \circ xtor = s \circ (id\text{-}bnf \circ g \circ id\text{-}bnf) \Longrightarrow f = ctor\text{-}rec\ s$
unfolding *ctor-rec-def id-bnf-def xtor-def comp-def id-def* **by** *hypsubst* (*rule refl*)

lemma *ctor-rec-def-alt*: $f = ctor\text{-}rec\ (f \circ id\text{-}bnf)$
unfolding *ctor-rec-def id-bnf-def comp-def* **by** (*rule refl*)

lemma *ctor-rec-o-map*: $ctor\text{-}rec\ f \circ g = ctor\text{-}rec\ (f \circ (id\text{-}bnf \circ g \circ id\text{-}bnf))$
unfolding *ctor-rec-def id-bnf-def comp-def* **by** (*rule refl*)

lemma *ctor-rec-transfer*: $rel\text{-}fun\ (rel\text{-}fun\ (vimage2p\ id\text{-}bnf\ id\text{-}bnf\ R)\ S)\ (rel\text{-}fun\ R\ S)\ ctor\text{-}rec\ ctor\text{-}rec$
unfolding *rel-fun-def vimage2p-def id-bnf-def ctor-rec-def* **by** *simp*

lemma *eq-fst-iff*: $a = fst\ p \longleftrightarrow (\exists\, b.\ p = (a,\ b))$

by (*cases p*) *auto*

lemma *eq-snd-iff*: $b = snd\ p \longleftrightarrow (\exists a.\ p = (a, b))$
by (*cases p*) *auto*

lemma *ex-neg-all-pos*: $((\exists x.\ P\ x) \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
by *standard blast+*

lemma *hypsubst-in-prems*: $(\bigwedge x.\ y = x \Longrightarrow z = f\ x \Longrightarrow P) \equiv (z = f\ y \Longrightarrow P)$
by *standard blast+*

lemma *isl-map-sum*:
isl (*map-sum f g s*) = *isl s*
by (*cases s*) *simp-all*

lemma *map-sum-sel*:
isl s ⟹ *projl* (*map-sum f g s*) = *f* (*projl s*)
¬ *isl s* ⟹ *projr* (*map-sum f g s*) = *g* (*projr s*)
by (*cases s*; *simp*)+

lemma *set-sum-sel*:
isl s ⟹ *projl s* ∈ *setl s*
¬ *isl s* ⟹ *projr s* ∈ *setr s*
by (*cases s*; *auto intro*: *setl.intros setr.intros*)+

lemma *rel-sum-sel*: *rel-sum R1 R2 a b* = (*isl a* = *isl b* ∧
(*isl a* ⟶ *isl b* ⟶ *R1* (*projl a*) (*projl b*)) ∧
(¬ *isl a* ⟶ ¬ *isl b* ⟶ *R2* (*projr a*) (*projr b*)))
by (*cases a b rule*: *sum.exhaust*[*case-product sum.exhaust*]) *simp-all*

lemma *isl-transfer*: *rel-fun* (*rel-sum A B*) (=) *isl isl*
unfolding *rel-fun-def rel-sum-sel* **by** *simp*

lemma *rel-prod-sel*: *rel-prod R1 R2 p q* = (*R1* (*fst p*) (*fst q*) ∧ *R2* (*snd p*) (*snd q*))
by (*force simp*: *rel-prod.simps elim*: *rel-prod.cases*)

ML-file ‹*Tools/BNF/bnf-lfp-basic-sugar.ML*›

declare *prod.size* [*no-atp*]

hide-const (**open**) *xtor ctor-rec*

hide-fact (**open**)
xtor-def xtor-map xtor-set xtor-rel xtor-induct xtor-xtor xtor-inject ctor-rec-def ctor-rec
ctor-rec-def-alt ctor-rec-o-map xtor-rel-induct Inl-def-alt Inr-def-alt Pair-def-alt

end

39 MESON Proof Method

theory *Meson*
imports *Nat*
begin

39.1 Negation Normal Form

de Morgan laws

lemma *not-conjD*: $\neg(P \wedge Q) \Longrightarrow \neg P \vee \neg Q$
and *not-disjD*: $\neg(P \vee Q) \Longrightarrow \neg P \wedge \neg Q$
and *not-notD*: $\neg\neg P \Longrightarrow P$
and *not-allD*: $\bigwedge P.\ \neg(\forall x.\ P(x)) \Longrightarrow \exists x.\ \neg P(x)$
and *not-exD*: $\bigwedge P.\ \neg(\exists x.\ P(x)) \Longrightarrow \forall x.\ \neg P(x)$
by *fast+*

Removal of $\longrightarrow$ and $\longleftrightarrow$ (positive and negative occurrences)

lemma *imp-to-disjD*: $P \longrightarrow Q \Longrightarrow \neg P \vee Q$
and *not-impD*: $\neg(P \longrightarrow Q) \Longrightarrow P \wedge \neg Q$
and *iff-to-disjD*: $P = Q \Longrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$
and *not-iffD*: $\neg(P = Q) \Longrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$
— Much more efficient than $P \wedge \neg\ Q \vee Q \wedge \neg\ P$ for computing CNF
and *not-refl-disj-D*: $x \neq x \vee P \Longrightarrow P$
by *fast+*

39.2 Pulling out the existential quantifiers

Conjunction

lemma *conj-exD1*: $\bigwedge P\ Q.\ (\exists x.\ P(x)) \wedge Q \Longrightarrow \exists x.\ P(x) \wedge Q$
and *conj-exD2*: $\bigwedge P\ Q.\ P \wedge (\exists x.\ Q(x)) \Longrightarrow \exists x.\ P \wedge Q(x)$
by *fast+*

Disjunction

lemma *disj-exD*: $\bigwedge P\ Q.\ (\exists x.\ P(x)) \vee (\exists x.\ Q(x)) \Longrightarrow \exists x.\ P(x) \vee Q(x)$
— DO NOT USE with forall-Skolemization: makes fewer schematic variables!!
— With ex-Skolemization, makes fewer Skolem constants
and *disj-exD1*: $\bigwedge P\ Q.\ (\exists x.\ P(x)) \vee Q \Longrightarrow \exists x.\ P(x) \vee Q$
and *disj-exD2*: $\bigwedge P\ Q.\ P \vee (\exists x.\ Q(x)) \Longrightarrow \exists x.\ P \vee Q(x)$
by *fast+*

lemma *disj-assoc*: $(P \vee Q) \vee R \Longrightarrow P \vee (Q \vee R)$
and *disj-comm*: $P \vee Q \Longrightarrow Q \vee P$
and *disj-FalseD1*: $\mathit{False} \vee P \Longrightarrow P$
and *disj-FalseD2*: $P \vee \mathit{False} \Longrightarrow P$
by *fast+*

Generation of contrapositives

Inserts negated disjunct after removing the negation; P is a literal. Model elimination requires assuming the negation of every attempted subgoal, hence the negated disjuncts.

lemma *make-neg-rule*: $\neg P \lor Q \Longrightarrow ((\neg P \Longrightarrow P) \Longrightarrow Q)$
by *blast*

Version for Plaisted's "Postive refinement" of the Meson procedure

lemma *make-refined-neg-rule*: $\neg P \lor Q \Longrightarrow (P \Longrightarrow Q)$
by *blast*

P should be a literal

lemma *make-pos-rule*: $P \lor Q \Longrightarrow ((P \Longrightarrow \neg P) \Longrightarrow Q)$
by *blast*

Versions of *make-neg-rule* and *make-pos-rule* that don't insert new assumptions, for ordinary resolution.

lemmas *make-neg-rule'* = *make-refined-neg-rule*

lemma *make-pos-rule'*: $[\![P \lor Q; \neg P]\!] \Longrightarrow Q$
by *blast*

Generation of a goal clause – put away the final literal

lemma *make-neg-goal*: $\neg P \Longrightarrow ((\neg P \Longrightarrow P) \Longrightarrow False)$
by *blast*

lemma *make-pos-goal*: $P \Longrightarrow ((P \Longrightarrow \neg P) \Longrightarrow False)$
by *blast*

39.3 Lemmas for Forward Proof

There is a similarity to congruence rules. They are also useful in ordinary proofs.

lemma *conj-forward*: $[\![P' \land Q'; \ P' \Longrightarrow P; \ Q' \Longrightarrow Q \ ]\!] \Longrightarrow P \land Q$
by *blast*

lemma *disj-forward*: $[\![P' \lor Q'; \ P' \Longrightarrow P; \ Q' \Longrightarrow Q \ ]\!] \Longrightarrow P \lor Q$
by *blast*

lemma *imp-forward*: $[\![P' \longrightarrow Q'; \ P \Longrightarrow P'; \ Q' \Longrightarrow Q \ ]\!] \Longrightarrow P \longrightarrow Q$
by *blast*

lemma *imp-forward2*: $[\![P' \longrightarrow Q'; \ P \Longrightarrow P'; \ P' \Longrightarrow Q' \Longrightarrow Q \ ]\!] \Longrightarrow P \longrightarrow Q$
by *blast*

lemma *disj-forward2*: $[\![\ P' \lor Q'; \ P' \Longrightarrow P; \ [\![Q'; P \Longrightarrow False]\!] \Longrightarrow Q]\!] \Longrightarrow P \lor Q$
apply *blast*

done

lemma *all-forward*: $[|\ \forall x.\ P'(x);\ \ !!x.\ P'(x) ==> P(x)\ |] ==> \forall x.\ P(x)$
by *blast*

lemma *ex-forward*: $[|\ \exists x.\ P'(x);\ \ !!x.\ P'(x) ==> P(x)\ |] ==> \exists x.\ P(x)$
by *blast*

39.4 Clausification helper

lemma *TruepropI*: $P \equiv Q \Longrightarrow \mathit{Trueprop}\ P \equiv \mathit{Trueprop}\ Q$
by *simp*

lemma *ext-cong-neq*: $F\ g \neq F\ h \Longrightarrow F\ g \neq F\ h \wedge (\exists x.\ g\ x \neq h\ x)$
apply (*erule contrapos-np*)
apply *clarsimp*
apply (*rule cong*[**where** $f = F$])
by *auto*

Combinator translation helpers

definition *COMBI* :: $'a \Rightarrow 'a$ **where**
$\mathit{COMBI}\ P = P$

definition *COMBK* :: $'a \Rightarrow 'b \Rightarrow 'a$ **where**
$\mathit{COMBK}\ P\ Q = P$

definition *COMBB* :: $('b => 'c) \Rightarrow ('a => 'b) \Rightarrow 'a \Rightarrow 'c$ **where**
$\mathit{COMBB}\ P\ Q\ R = P\ (Q\ R)$

definition *COMBC* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'b \Rightarrow 'a \Rightarrow 'c$ **where**
$\mathit{COMBC}\ P\ Q\ R = P\ R\ Q$

definition *COMBS* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ **where**
$\mathit{COMBS}\ P\ Q\ R = P\ R\ (Q\ R)$

lemma *abs-S*: $\lambda x.\ (f\ x)\ (g\ x) \equiv \mathit{COMBS}\ f\ g$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBS-def*)
done

lemma *abs-I*: $\lambda x.\ x \equiv \mathit{COMBI}$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBI-def*)
done

lemma *abs-K*: $\lambda x.\ y \equiv \mathit{COMBK}\ y$
apply (*rule eq-reflection*)

apply (*rule ext*)
apply (*simp add*: *COMBK-def*)
done

lemma *abs-B*: $\lambda x.\ a\ (g\ x) \equiv COMBB\ a\ g$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBB-def*)
done

lemma *abs-C*: $\lambda x.\ (f\ x)\ b \equiv COMBC\ f\ b$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBC-def*)
done

39.5 Skolemization helpers

definition *skolem* :: $'a \Rightarrow 'a$ **where**
$skolem = (\lambda x.\ x)$

lemma *skolem-COMBK-iff*: $P \longleftrightarrow skolem\ (COMBK\ P\ (i{::}nat))$
unfolding *skolem-def COMBK-def* **by** (*rule refl*)

lemmas *skolem-COMBK-I* = *iffD1* [*OF skolem-COMBK-iff*]

39.6 Meson package

ML-file ‹*Tools/Meson/meson.ML*›
ML-file ‹*Tools/Meson/meson-clausify.ML*›
ML-file ‹*Tools/Meson/meson-tactic.ML*›

hide-const (**open**) *COMBI COMBK COMBB COMBC COMBS skolem*
hide-fact (**open**) *not-conjD not-disjD not-notD not-allD not-exD imp-to-disjD*
not-impD iff-to-disjD not-iffD not-refl-disj-D conj-exD1 conj-exD2 disj-exD
disj-exD1 disj-exD2 disj-assoc disj-comm disj-FalseD1 disj-FalseD2 TruepropI
ext-cong-neq COMBI-def COMBK-def COMBB-def COMBC-def COMBS-def
abs-I abs-K
abs-B abs-C abs-S skolem-def skolem-COMBK-iff skolem-COMBK-I
end

40 Automatic Theorem Provers (ATPs)

theory *ATP*
imports *Meson Hilbert-Choice*
begin

40.1 ATP problems and proofs

ML-file ‹*Tools/ATP/atp-util.ML*›
ML-file ‹*Tools/ATP/atp-problem.ML*›
ML-file ‹*Tools/ATP/atp-proof.ML*›
ML-file ‹*Tools/ATP/atp-proof-redirect.ML*›

40.2 Higher-order reasoning helpers

definition *fFalse* :: *bool* **where**
fFalse ⟷ *False*

definition *fTrue* :: *bool* **where**
fTrue ⟷ *True*

definition *fNot* :: *bool* ⇒ *bool* **where**
fNot P ⟷ ¬ *P*

definition *fComp* :: ('*a* ⇒ *bool*) ⇒ '*a* ⇒ *bool* **where**
fComp P = (λ*x*. ¬ *P x*)

definition *fconj* :: *bool* ⇒ *bool* ⇒ *bool* **where**
fconj P Q ⟷ *P* ∧ *Q*

definition *fdisj* :: *bool* ⇒ *bool* ⇒ *bool* **where**
fdisj P Q ⟷ *P* ∨ *Q*

definition *fimplies* :: *bool* ⇒ *bool* ⇒ *bool* **where**
fimplies P Q ⟷ (*P* ⟶ *Q*)

definition *fAll* :: ('*a* ⇒ *bool*) ⇒ *bool* **where**
fAll P ⟷ *All P*

definition *fEx* :: ('*a* ⇒ *bool*) ⇒ *bool* **where**
fEx P ⟷ *Ex P*

definition *fequal* :: '*a* ⇒ '*a* ⇒ *bool* **where**
fequal x y ⟷ (*x* = *y*)

definition *fChoice* :: ('*a* ⇒ *bool*) ⇒ '*a* **where**
  fChoice ≡ *Hilbert-Choice.Eps*

lemma *fTrue-ne-fFalse*: *fFalse* ≠ *fTrue*
unfolding *fFalse-def fTrue-def* **by** *simp*

lemma *fNot-table*:
fNot fFalse = *fTrue*
fNot fTrue = *fFalse*
unfolding *fFalse-def fTrue-def fNot-def* **by** *auto*

lemma *fconj-table*:
fconj fFalse P = *fFalse*
fconj P fFalse = *fFalse*
fconj fTrue fTrue = *fTrue*
unfolding *fFalse-def fTrue-def fconj-def* **by** *auto*

lemma *fdisj-table*:
fdisj fTrue P = *fTrue*
fdisj P fTrue = *fTrue*
fdisj fFalse fFalse = *fFalse*
unfolding *fFalse-def fTrue-def fdisj-def* **by** *auto*

lemma *fimplies-table*:
fimplies P fTrue = *fTrue*
fimplies fFalse P = *fTrue*
fimplies fTrue fFalse = *fFalse*
unfolding *fFalse-def fTrue-def fimplies-def* **by** *auto*

lemma *fAll-table*:
Ex (*fComp P*) ∨ *fAll P* = *fTrue*
All P ∨ *fAll P* = *fFalse*
unfolding *fFalse-def fTrue-def fComp-def fAll-def* **by** *auto*

lemma *fEx-table*:
All (*fComp P*) ∨ *fEx P* = *fTrue*
Ex P ∨ *fEx P* = *fFalse*
unfolding *fFalse-def fTrue-def fComp-def fEx-def* **by** *auto*

lemma *fequal-table*:
fequal x x = *fTrue*
x = *y* ∨ *fequal x y* = *fFalse*
unfolding *fFalse-def fTrue-def fequal-def* **by** *auto*

lemma *fNot-law*:
fNot P ≠ *P*
unfolding *fNot-def* **by** *auto*

lemma *fComp-law*:
fComp P x ⟷ ¬ *P x*
unfolding *fComp-def* **..**

lemma *fconj-laws*:
fconj P P ⟷ *P*
fconj P Q ⟷ *fconj Q P*
fNot (*fconj P Q*) ⟷ *fdisj* (*fNot P*) (*fNot Q*)
unfolding *fNot-def fconj-def fdisj-def* **by** *auto*

lemma *fdisj-laws*:
fdisj P P ⟷ *P*

fdisj P Q ⟷ *fdisj Q P*
fNot (*fdisj P Q*) ⟷ *fconj* (*fNot P*) (*fNot Q*)
unfolding *fNot-def fconj-def fdisj-def* **by** *auto*

lemma *fimplies-laws*:
fimplies P Q ⟷ *fdisj* (¬ *P*) *Q*
fNot (*fimplies P Q*) ⟷ *fconj P* (*fNot Q*)
unfolding *fNot-def fconj-def fdisj-def fimplies-def* **by** *auto*

lemma *fAll-law*:
fNot (*fAll R*) ⟷ *fEx* (*fComp R*)
unfolding *fNot-def fComp-def fAll-def fEx-def* **by** *auto*

lemma *fEx-law*:
fNot (*fEx R*) ⟷ *fAll* (*fComp R*)
unfolding *fNot-def fComp-def fAll-def fEx-def* **by** *auto*

lemma *fequal-laws*:
fequal x y = *fequal y x*
fequal x y = *fFalse* ∨ *fequal y z* = *fFalse* ∨ *fequal x z* = *fTrue*
fequal x y = *fFalse* ∨ *fequal* (*f x*) (*f y*) = *fTrue*
unfolding *fFalse-def fTrue-def fequal-def* **by** *auto*

lemma *fChoice-iff-Ex*: *P* (*fChoice P*) ⟷ *HOL.Ex P*
  unfolding *fChoice-def*
  by (*fact some-eq-ex*)

We use the *Ex* constant on the right-hand side of *fChoice-iff-Ex* because we want to use the TPTP-native version if fChoice is introduced in a logic that supports FOOL. In logics that don't support it, it gets replaced by fEx during processing. Notice that we cannot use $\exists x.\ P\ x$, as existentials are not skolimized by the metis proof method but *Ex P* is eta-expanded if FOOL is supported.

40.3 Basic connection between ATPs and HOL

ML-file ‹*Tools/lambda-lifting.ML*›
ML-file ‹*Tools/monomorph.ML*›
ML-file ‹*Tools/ATP/atp-problem-generate.ML*›
ML-file ‹*Tools/ATP/atp-proof-reconstruct.ML*›

end

41 Metis Proof Method

theory *Metis*
imports *ATP*
begin

context notes [[*ML-catch-all*]]
begin
ML-file ‹~~/*src*/*Tools*/*Metis*/*metis.ML*›
end

41.1 Literal selection and lambda-lifting helpers

definition *select* :: *'a* ⇒ *'a* **where**
select = (λ*x*. *x*)

lemma *not-atomize*: (¬ *A* ⟹ *False*) ≡ *Trueprop A*
by (*cut-tac atomize-not* [*of* ¬ *A*]) *simp*

lemma *atomize-not-select*: (*A* ⟹ *select False*) ≡ *Trueprop* (¬ *A*)
unfolding *select-def* **by** (*rule atomize-not*)

lemma *not-atomize-select*: (¬ *A* ⟹ *select False*) ≡ *Trueprop A*
unfolding *select-def* **by** (*rule not-atomize*)

lemma *select-FalseI*: *False* ⟹ *select False*
by *simp*

definition *lambda* :: *'a* ⇒ *'a* **where**
lambda = (λ*x*. *x*)

lemma *eq-lambdaI*: *x* ≡ *y* ⟹ *x* ≡ *lambda y*
unfolding *lambda-def* **by** *assumption*

41.2 Metis package

ML-file ‹*Tools*/*Metis*/*metis-generate.ML*›
ML-file ‹*Tools*/*Metis*/*metis-reconstruct.ML*›
ML-file ‹*Tools*/*Metis*/*metis-instantiations.ML*›
ML-file ‹*Tools*/*Metis*/*metis-tactic.ML*›

hide-const (**open**) *select fFalse fTrue fNot fComp fconj fdisj fimplies fAll fEx fequal lambda*
hide-fact (**open**) *select-def not-atomize atomize-not-select not-atomize-select select-FalseI*
fFalse-def fTrue-def fNot-def fconj-def fdisj-def fimplies-def fAll-def fEx-def fequal-def
fTrue-ne-fFalse fNot-table fconj-table fdisj-table fimplies-table fAll-table fEx-table
fequal-table fAll-table fEx-table fNot-law fComp-law fconj-laws fdisj-laws fimplies-laws
fequal-laws fAll-law fEx-law lambda-def eq-lambdaI

end

42 Generic theorem transfer using relations

theory *Transfer*
imports *Basic-BNF-LFPs Hilbert-Choice Metis*
begin

42.1 Relator for function space

bundle *lifting-syntax*
begin
notation *rel-fun* (**infixr** ‹===>› *55*)
notation *map-fun* (**infixr** ‹--->› *55*)
end

context includes *lifting-syntax*
begin

lemma *rel-funD2*:
assumes *rel-fun A B f g* **and** *A x x*
shows *B* (*f x*) (*g x*)
using *assms* **by** (*rule rel-funD*)

lemma *rel-funE*:
assumes *rel-fun A B f g* **and** *A x y*
obtains *B* (*f x*) (*g y*)
using *assms* **by** (*simp add*: *rel-fun-def*)

lemmas *rel-fun-eq* = *fun.rel-eq*

lemma *rel-fun-eq-rel*:
shows *rel-fun* (=) $R = (\lambda f\ g.\ \forall x.\ R\ (f\ x)\ (g\ x))$
by (*simp add*: *rel-fun-def*)

42.2 Transfer method

Explicit tag for relation membership allows for backward proof methods.

definition *Rel* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
where *Rel r* ≡ *r*

Handling of equality relations

definition *is-equality* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where *is-equality* $R \longleftrightarrow R = (=)$

lemma *is-equality-eq*: *is-equality* (=)
unfolding *is-equality-def* **by** *simp*

Reverse implication for monotonicity rules

definition *rev-implies* **where**
rev-implies $x\ y \longleftrightarrow (y \longrightarrow x)$

Handling of meta-logic connectives

definition *transfer-forall* **where**
transfer-forall ≡ *All*

definition *transfer-implies* **where**
transfer-implies ≡ (⟶)

definition *transfer-bforall* :: (′*a* ⇒ *bool*) ⇒ (′*a* ⇒ *bool*) ⇒ *bool*
where *transfer-bforall* ≡ (λ*P Q*. ∀ *x*. *P x* ⟶ *Q x*)

lemma *transfer-forall-eq*: (⋀*x*. *P x*) ≡ *Trueprop* (*transfer-forall* (λ*x*. *P x*))
unfolding *atomize-all transfer-forall-def* **..**

lemma *transfer-implies-eq*: (*A* ⟹ *B*) ≡ *Trueprop* (*transfer-implies A B*)
unfolding *atomize-imp transfer-implies-def* **..**

lemma *transfer-bforall-unfold*:
Trueprop (*transfer-bforall P* (λ*x*. *Q x*)) ≡ (⋀*x*. *P x* ⟹ *Q x*)
unfolding *transfer-bforall-def atomize-imp atomize-all* **..**

lemma *transfer-start*: ⟦*P*; *Rel* (=) *P Q*⟧ ⟹ *Q*
unfolding *Rel-def* **by** *simp*

lemma *transfer-start′*: ⟦*P*; *Rel* (⟶) *P Q*⟧ ⟹ *Q*
unfolding *Rel-def* **by** *simp*

lemma *transfer-prover-start*: ⟦*x* = *x′*; *Rel R x′ y*⟧ ⟹ *Rel R x y*
by *simp*

lemma *untransfer-start*: ⟦*Q*; *Rel* (=) *P Q*⟧ ⟹ *P*
unfolding *Rel-def* **by** *simp*

lemma *Rel-eq-refl*: *Rel* (=) *x x*
unfolding *Rel-def* **..**

lemma *Rel-app*:
assumes *Rel* (*A* ===> *B*) *f g* **and** *Rel A x y*
shows *Rel B* (*f x*) (*g y*)
using *assms* **unfolding** *Rel-def rel-fun-def* **by** *fast*

lemma *Rel-abs*:
assumes ⋀*x y*. *Rel A x y* ⟹ *Rel B* (*f x*) (*g y*)
shows *Rel* (*A* ===> *B*) (λ*x*. *f x*) (λ*y*. *g y*)
using *assms* **unfolding** *Rel-def rel-fun-def* **by** *fast*

42.3 Predicates on relations, i.e. "class constraints"

definition *left-total* :: (′*a* ⇒ ′*b* ⇒ *bool*) ⇒ *bool*
where *left-total R* ⟷ (∀ *x*. ∃ *y*. *R x y*)

definition *left-unique* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where *left-unique* $R \longleftrightarrow (\forall x\ y\ z.\ R\ x\ z \longrightarrow R\ y\ z \longrightarrow x = y)$

definition *right-total* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where *right-total* $R \longleftrightarrow (\forall y.\ \exists x.\ R\ x\ y)$

definition *right-unique* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where *right-unique* $R \longleftrightarrow (\forall x\ y\ z.\ R\ x\ y \longrightarrow R\ x\ z \longrightarrow y = z)$

definition *bi-total* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where *bi-total* $R \longleftrightarrow (\forall x.\ \exists y.\ R\ x\ y) \wedge (\forall y.\ \exists x.\ R\ x\ y)$

definition *bi-unique* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where *bi-unique* $R \longleftrightarrow$
$(\forall x\ y\ z.\ R\ x\ y \longrightarrow R\ x\ z \longrightarrow y = z) \wedge$
$(\forall x\ y\ z.\ R\ x\ z \longrightarrow R\ y\ z \longrightarrow x = y)$

lemma *left-unique-iff*: *left-unique* $R \longleftrightarrow (\forall z.\ \exists_{\leq 1} x.\ R\ x\ z)$
unfolding *Uniq-def left-unique-def* **by** *force*

lemma *left-uniqueI*: $(\bigwedge x\ y\ z.\ [\![\ A\ x\ z;\ A\ y\ z\ ]\!] \Longrightarrow x = y) \Longrightarrow$ *left-unique* A
unfolding *left-unique-def* **by** *blast*

lemma *left-uniqueD*: $[\![$ *left-unique* $A;\ A\ x\ z;\ A\ y\ z\ ]\!] \Longrightarrow x = y$
unfolding *left-unique-def* **by** *blast*

lemma *left-totalI*:
$(\bigwedge x.\ \exists y.\ R\ x\ y) \Longrightarrow$ *left-total* R
unfolding *left-total-def* **by** *blast*

lemma *left-totalE*:
assumes *left-total* R
obtains $(\bigwedge x.\ \exists y.\ R\ x\ y)$
using *assms* **unfolding** *left-total-def* **by** *blast*

lemma *bi-uniqueDr*: $[\![$ *bi-unique* $A;\ A\ x\ y;\ A\ x\ z\ ]\!] \Longrightarrow y = z$
by(*simp add*: *bi-unique-def*)

lemma *bi-uniqueDl*: $[\![$ *bi-unique* $A;\ A\ x\ y;\ A\ z\ y\ ]\!] \Longrightarrow x = z$
by(*simp add*: *bi-unique-def*)

lemma *bi-unique-iff*: *bi-unique* $R \longleftrightarrow (\forall z.\ \exists_{\leq 1} x.\ R\ x\ z) \wedge (\forall z.\ \exists_{\leq 1} x.\ R\ z\ x)$
unfolding *Uniq-def bi-unique-def* **by** *force*

lemma *right-unique-iff*: *right-unique* $R \longleftrightarrow (\forall z.\ \exists_{\leq 1} x.\ R\ z\ x)$
unfolding *Uniq-def right-unique-def* **by** *force*

lemma *right-uniqueI*: $(\bigwedge x\ y\ z.\ [\![\ A\ x\ y;\ A\ x\ z\ ]\!] \Longrightarrow y = z) \Longrightarrow$ *right-unique* A

unfolding *right-unique-def* **by** *fast*

lemma *right-uniqueD*: $[\![$ *right-unique A*; *A x y*; *A x z* $]\!] \Longrightarrow y = z$
unfolding *right-unique-def* **by** *fast*

lemma *right-totalI*: $(\bigwedge y.\ \exists x.\ A\ x\ y) \Longrightarrow$ *right-total A*
by(*simp add*: *right-total-def*)

lemma *right-totalE*:
assumes *right-total A*
obtains *x* **where** *A x y*
using *assms* **by**(*auto simp add*: *right-total-def*)

lemma *right-total-alt-def2*:
right-total R $\longleftrightarrow$ ((*R* ===> ($\longrightarrow$)) ===> ($\longrightarrow$)) *All All* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
unfolding *right-total-def rel-fun-def* **by** *blast*
next
assume §: *?rhs*
show *?lhs*
using § [*unfolded rel-fun-def*, *rule-format*, *of* $\lambda x.$ *True* $\lambda y.\ \exists x.\ R\ x\ y$]
unfolding *right-total-def* **by** *blast*
qed

lemma *right-unique-alt-def2*:
right-unique R $\longleftrightarrow$ (*R* ===> *R* ===> ($\longrightarrow$)) (=) (=)
unfolding *right-unique-def rel-fun-def* **by** *auto*

lemma *bi-total-alt-def2*:
bi-total R $\longleftrightarrow$ ((*R* ===> (=)) ===> (=)) *All All* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
unfolding *bi-total-def rel-fun-def* **by** *blast*
next
assume §: *?rhs*
show *?lhs*
using § [*unfolded rel-fun-def*, *rule-format*, *of* $\lambda x.\ \exists y.\ R\ x\ y$ $\lambda y.$ *True*]
using § [*unfolded rel-fun-def*, *rule-format*, *of* $\lambda x.$ *True* $\lambda y.\ \exists x.\ R\ x\ y$]
by (*auto simp*: *bi-total-def*)
qed

lemma *bi-unique-alt-def2*:
bi-unique R $\longleftrightarrow$ (*R* ===> *R* ===> (=)) (=) (=)
unfolding *bi-unique-def rel-fun-def* **by** *auto*

lemma [*simp*]:
shows *left-unique-conversep*: *left-unique* A^{-1-1} $\longleftrightarrow$ *right-unique A*
and *right-unique-conversep*: *right-unique* A^{-1-1} $\longleftrightarrow$ *left-unique A*

by(*auto simp add*: *left-unique-def right-unique-def*)

lemma [*simp*]:
shows *left-total-conversep*: *left-total* A^{-1-1} $\longleftrightarrow$ *right-total A*
and *right-total-conversep*: *right-total* A^{-1-1} $\longleftrightarrow$ *left-total A*
by(*simp-all add*: *left-total-def right-total-def*)

lemma *bi-unique-conversep* [*simp*]: *bi-unique* R^{-1-1} = *bi-unique R*
by(*auto simp add*: *bi-unique-def*)

lemma *bi-total-conversep* [*simp*]: *bi-total* R^{-1-1} = *bi-total R*
by(*auto simp add*: *bi-total-def*)

lemma *right-unique-alt-def*: *right-unique R* = (*conversep R OO R* $\leq$ (=)) **unfolding** *right-unique-def* **by** *blast*
lemma *left-unique-alt-def*: *left-unique R* = (*R OO* (*conversep R*) $\leq$ (=)) **unfolding** *left-unique-def* **by** *blast*

lemma *right-total-alt-def*: *right-total R* = (*conversep R OO R* $\geq$ (=)) **unfolding** *right-total-def* **by** *blast*
lemma *left-total-alt-def*: *left-total R* = (*R OO conversep R* $\geq$ (=)) **unfolding** *left-total-def* **by** *blast*

lemma *bi-total-alt-def*: *bi-total A* = (*left-total A* $\wedge$ *right-total A*)
unfolding *left-total-def right-total-def bi-total-def* **by** *blast*

lemma *bi-unique-alt-def*: *bi-unique A* = (*left-unique A* $\wedge$ *right-unique A*)
unfolding *left-unique-def right-unique-def bi-unique-def* **by** *blast*

lemma *bi-totalI*: *left-total R* $\Longrightarrow$ *right-total R* $\Longrightarrow$ *bi-total R*
unfolding *bi-total-alt-def* **..**

lemma *bi-uniqueI*: *left-unique R* $\Longrightarrow$ *right-unique R* $\Longrightarrow$ *bi-unique R*
unfolding *bi-unique-alt-def* **..**

end

lemma *is-equality-lemma*: ($\bigwedge$*R*. *is-equality R* $\Longrightarrow$ *PROP* (*P R*)) $\equiv$ *PROP* (*P* (=))
unfolding *is-equality-def*
proof (*rule equal-intr-rule*)
show ($\bigwedge$*R*. *R* = (=) $\Longrightarrow$ *PROP P R*) $\Longrightarrow$ *PROP P* (=)
apply (*drule meta-spec*)
apply (*erule meta-mp* [*OF* - *refl*])
done
qed *simp*

lemma *Domainp-lemma*: ($\bigwedge$*R*. *Domainp T* = *R* $\Longrightarrow$ *PROP* (*P R*)) $\equiv$ *PROP* (*P* (*Domainp T*))

proof (*rule equal-intr-rule*)
  show ($\bigwedge R$. *Domainp T* = *R* $\Longrightarrow$ *PROP P R*) $\Longrightarrow$ *PROP P* (*Domainp T*)
    apply (*drule meta-spec*)
    apply (*erule meta-mp* [*OF* - *refl*])
    done
qed *simp*

ML-file ‹*Tools/Transfer/transfer.ML*›
declare *refl* [*transfer-rule*]

hide-const (**open**) *Rel*

context includes *lifting-syntax*
begin

Handling of domains

lemma *Domainp-iff*: *Domainp T x* $\longleftrightarrow$ ($\exists$ *y*. *T x y*)
  by *auto*

lemma *Domainp-refl*[*transfer-domain-rule*]:
  Domainp T = *Domainp T* **..**

lemma *Domain-eq-top*[*transfer-domain-rule*]: *Domainp* (=) = *top* **by** *auto*

lemma *Domainp-pred-fun-eq*[*relator-domain*]:
  assumes *left-unique T*
  shows *Domainp* (*T* ===> *S*) = *pred-fun* (*Domainp T*) (*Domainp S*)   (**is** *?lhs* = *?rhs*)
proof (*intro ext iffI*)
  fix *x*
  assume *?lhs x*
  then show *?rhs x*
    using *assms* **unfolding** *rel-fun-def pred-fun-def* **by** *blast*
next
  fix *x*
  assume *?rhs x*
  then have $\exists$ *g*. $\forall$ *y xa*. *T xa y* $\longrightarrow$ *S* (*x xa*) (*g y*)
    using *assms* **unfolding** *Domainp-iff left-unique-def pred-fun-def*
    by (*intro choice*) *blast*
  then show *?lhs x*
    by *blast*
qed

Properties are preserved by relation composition.

lemma *OO-def*: *R OO S* = (λ*x z*. $\exists$ *y*. *R x y* $\wedge$ *S y z*)
  by *auto*

lemma *bi-total-OO*: ⟦*bi-total A*; *bi-total B*⟧ $\Longrightarrow$ *bi-total* (*A OO B*)
  unfolding *bi-total-def OO-def* **by** *fast*

lemma *bi-unique-OO*: $[\![$*bi-unique A*; *bi-unique B*$]\!] \Longrightarrow$ *bi-unique* (*A OO B*)
unfolding *bi-unique-def OO-def* **by** *blast*

lemma *right-total-OO*:
$[\![$*right-total A*; *right-total B*$]\!] \Longrightarrow$ *right-total* (*A OO B*)
unfolding *right-total-def OO-def* **by** *fast*

lemma *right-unique-OO*:
$[\![$*right-unique A*; *right-unique B*$]\!] \Longrightarrow$ *right-unique* (*A OO B*)
unfolding *right-unique-def OO-def* **by** *fast*

lemma *left-total-OO*: *left-total R* $\Longrightarrow$ *left-total S* $\Longrightarrow$ *left-total* (*R OO S*)
unfolding *left-total-def OO-def* **by** *fast*

lemma *left-unique-OO*: *left-unique R* $\Longrightarrow$ *left-unique S* $\Longrightarrow$ *left-unique* (*R OO S*)
unfolding *left-unique-def OO-def* **by** *blast*

42.4 Properties of relators

lemma *left-total-eq*[*transfer-rule*]: *left-total* (=)
unfolding *left-total-def* **by** *blast*

lemma *left-unique-eq*[*transfer-rule*]: *left-unique* (=)
unfolding *left-unique-def* **by** *blast*

lemma *right-total-eq* [*transfer-rule*]: *right-total* (=)
unfolding *right-total-def* **by** *simp*

lemma *right-unique-eq* [*transfer-rule*]: *right-unique* (=)
unfolding *right-unique-def* **by** *simp*

lemma *bi-total-eq*[*transfer-rule*]: *bi-total* (=)
unfolding *bi-total-def* **by** *simp*

lemma *bi-unique-eq*[*transfer-rule*]: *bi-unique* (=)
unfolding *bi-unique-def* **by** *simp*

lemma *left-total-fun*[*transfer-rule*]:
assumes *left-unique A left-total B*
shows *left-total* (*A* ===> *B*)
unfolding *left-total-def*
proof
fix *f*
show *Ex* ((*A* ===> *B*) *f*)
unfolding *rel-fun-def*
proof (*intro exI strip*)
fix *x y*
assume *A*: *A x y*

    have (*THE x. A x y*) = *x*
      using *A assms* **by** (*simp add*: *left-unique-def the-equality*)
    then show *B* (*f x*) (*SOME z. B* (*f* (*THE x. A x y*)) *z*)
      using *assms* **by** (*force simp*: *left-total-def intro*: *someI-ex*)
  qed
qed

lemma *left-unique-fun*[*transfer-rule*]:
  ⟦*left-total A*; *left-unique B*⟧ ⟹ *left-unique* (*A* ===> *B*)
  unfolding *left-total-def left-unique-def rel-fun-def*
  by (*clarify*, *rule ext*, *fast*)

lemma *right-total-fun* [*transfer-rule*]:
  assumes *right-unique A right-total B*
  shows *right-total* (*A* ===> *B*)
  unfolding *right-total-def*
proof
  fix *g*
  show ∃ *x*. (*A* ===> *B*) *x g*
    unfolding *rel-fun-def*
  proof (*intro exI strip*)
    fix *x y*
    assume *A*: *A x y*
    have (*THE y. A x y*) = *y*
      using *A assms* **by** (*simp add*: *right-unique-def the-equality*)
    then show *B* (*SOME z. B z* (*g* (*THE y. A x y*))) (*g y*)
      using *assms* **by** (*force simp*: *right-total-def intro*: *someI-ex*)
  qed
qed

lemma *right-unique-fun* [*transfer-rule*]:
  ⟦*right-total A*; *right-unique B*⟧ ⟹ *right-unique* (*A* ===> *B*)
  unfolding *right-total-def right-unique-def rel-fun-def*
  by (*clarify*, *rule ext*, *fast*)

lemma *bi-total-fun*[*transfer-rule*]:
  ⟦*bi-unique A*; *bi-total B*⟧ ⟹ *bi-total* (*A* ===> *B*)
  unfolding *bi-unique-alt-def bi-total-alt-def*
  by (*blast intro*: *right-total-fun left-total-fun*)

lemma *bi-unique-fun*[*transfer-rule*]:
  ⟦*bi-total A*; *bi-unique B*⟧ ⟹ *bi-unique* (*A* ===> *B*)
  unfolding *bi-unique-alt-def bi-total-alt-def*
  by (*blast intro*: *right-unique-fun left-unique-fun*)

end

lemma *if-conn*:
  (*if P* ∧ *Q then t else e*) = (*if P then if Q then t else e else e*)

(*if* P ∨ Q *then* t *else* e) = (*if* P *then* t *else if* Q *then* t *else* e)
(*if* P ⟶ Q *then* t *else* e) = (*if* P *then if* Q *then* t *else* e *else* t)
(*if* ¬ P *then* t *else* e) = (*if* P *then* e *else* t)
by *auto*

ML-file ‹*Tools/Transfer/transfer-bnf.ML*›
ML-file ‹*Tools/BNF/bnf-fp-rec-sugar-transfer.ML*›

declare *pred-fun-def* [*simp*]
declare *rel-fun-eq* [*relator-eq*]

declare *fun.Domainp-rel*[*relator-domain del*]

42.5 Transfer rules

context includes *lifting-syntax*
begin

lemma *Domainp-forall-transfer* [*transfer-rule*]:
assumes *right-total A*
shows ((A ===> (=)) ===> (=))
(*transfer-bforall* (*Domainp A*)) *transfer-forall*
using *assms* **unfolding** *right-total-def*
unfolding *transfer-forall-def transfer-bforall-def rel-fun-def Domainp-iff*
by *fast*

Transfer rules using implication instead of equality on booleans.

lemma *transfer-forall-transfer* [*transfer-rule*]:
bi-total A ⟹ ((A ===> (=)) ===> (=)) *transfer-forall transfer-forall*
right-total A ⟹ ((A ===> (=)) ===> *implies*) *transfer-forall transfer-forall*
right-total A ⟹ ((A ===> *implies*) ===> *implies*) *transfer-forall transfer-forall*
bi-total A ⟹ ((A ===> (=)) ===> *rev-implies*) *transfer-forall transfer-forall*
bi-total A ⟹ ((A ===> *rev-implies*) ===> *rev-implies*) *transfer-forall transfer-forall*
unfolding *transfer-forall-def rev-implies-def rel-fun-def right-total-def bi-total-def*
by *fast+*

lemma *transfer-implies-transfer* [*transfer-rule*]:
((=) ===> (=) ===> (=)) *transfer-implies transfer-implies*
(*rev-implies* ===> *implies* ===> *implies*) *transfer-implies transfer-implies*
(*rev-implies* ===> (=) ===> *implies*) *transfer-implies transfer-implies*
((=) ===> *implies* ===> *implies*) *transfer-implies transfer-implies*
((=) ===> (=) ===> *implies*) *transfer-implies transfer-implies*
(*implies* ===> *rev-implies* ===> *rev-implies*) *transfer-implies transfer-implies*
(*implies* ===> (=) ===> *rev-implies*) *transfer-implies transfer-implies*
((=) ===> *rev-implies* ===> *rev-implies*) *transfer-implies transfer-implies*
((=) ===> (=) ===> *rev-implies*) *transfer-implies transfer-implies*

unfolding *transfer-implies-def rev-implies-def rel-fun-def* **by** *auto*

lemma *eq-imp-transfer* [*transfer-rule*]:
right-unique A ⟹ (*A* ===> *A* ===> (⟶)) (=) (=)
unfolding *right-unique-alt-def2* **.**

Transfer rules using equality.

lemma *left-unique-transfer* [*transfer-rule*]:
assumes *right-total A*
assumes *right-total B*
assumes *bi-unique A*
shows ((*A* ===> *B* ===> (=)) ===> *implies*) *left-unique left-unique*
using *assms* **unfolding** *left-unique-def right-total-def bi-unique-def rel-fun-def*
by *metis*

lemma *eq-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*A* ===> *A* ===> (=)) (=) (=)
using *assms* **unfolding** *bi-unique-def rel-fun-def* **by** *auto*

lemma *right-total-Ex-transfer*[*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> (=)) (*Bex* (*Collect* (*Domainp A*))) *Ex*
using *assms* **unfolding** *right-total-def Bex-def rel-fun-def Domainp-iff*
by *fast*

lemma *right-total-All-transfer*[*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> (=)) (*Ball* (*Collect* (*Domainp A*))) *All*
using *assms* **unfolding** *right-total-def Ball-def rel-fun-def Domainp-iff*
by *fast*

context
includes *lifting-syntax*
begin

lemma *right-total-fun-eq-transfer*:
assumes [*transfer-rule*]: *right-total A bi-unique B*
shows ((*A* ===> *B*) ===> (*A* ===> *B*) ===> (=)) (λ*f g*. ∀ *x*∈*Collect*(*Domainp A*). *f x* = *g x*) (=)
unfolding *fun-eq-iff*
by *transfer-prover*

end

lemma *All-transfer* [*transfer-rule*]:
assumes *bi-total A*
shows ((*A* ===> (=)) ===> (=)) *All All*
using *assms* **unfolding** *bi-total-def rel-fun-def* **by** *fast*

lemma *Ex-transfer* [*transfer-rule*]:
  assumes *bi-total A*
  shows ((*A* ===> (=)) ===> (=)) *Ex Ex*
  using *assms* **unfolding** *bi-total-def rel-fun-def* **by** *fast*

lemma *Ex1-parametric* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-unique A bi-total A*
  shows ((*A* ===> (=)) ===> (=)) *Ex1 Ex1*
unfolding *Ex1-def* **by** *transfer-prover*

declare *If-transfer* [*transfer-rule*]

lemma *Let-transfer* [*transfer-rule*]: (*A* ===> (*A* ===> *B*) ===> *B*) *Let Let*
  unfolding *rel-fun-def* **by** *simp*

declare *id-transfer* [*transfer-rule*]

declare *comp-transfer* [*transfer-rule*]

lemma *curry-transfer* [*transfer-rule*]:
  ((*rel-prod A B* ===> *C*) ===> *A* ===> *B* ===> *C*) *curry curry*
  unfolding *curry-def* **by** *transfer-prover*

lemma *fun-upd-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-unique A*
  shows ((*A* ===> *B*) ===> *A* ===> *B* ===> *A* ===> *B*) *fun-upd fun-upd*
  unfolding *fun-upd-def* **by** *transfer-prover*

lemma *case-nat-transfer* [*transfer-rule*]:
  (*A* ===> ((=) ===> *A*) ===> (=) ===> *A*) *case-nat case-nat*
  unfolding *rel-fun-def* **by** (*simp split*: *nat.split*)

lemma *rec-nat-transfer* [*transfer-rule*]:
  (*A* ===> ((=) ===> *A* ===> *A*) ===> (=) ===> *A*) *rec-nat rec-nat*
  unfolding *rel-fun-def*
  apply *safe*
  subgoal for - - - - - *n*
    by (*induction n*) *simp-all*
  done

lemma *funpow-transfer* [*transfer-rule*]:
  ((=) ===> (*A* ===> *A*) ===> (*A* ===> *A*)) *compow compow*
  unfolding *funpow-def* **by** *transfer-prover*

lemma *mono-transfer*[*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-total A*
  assumes [*transfer-rule*]: (*A* ===> *A* ===> (=)) (≤) (≤)

assumes [*transfer-rule*]: (*B* ===> *B* ===> (=)) (≤) (≤)
shows ((*A* ===> *B*) ===> (=)) *mono mono*
unfolding *mono-def* **by** *transfer-prover*

lemma *right-total-relcompp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *right-total B*
shows ((*A* ===> *B* ===> (=)) ===> (*B* ===> *C* ===> (=)) ===> *A* ===> *C* ===> (=))
(λ*R S x z*. ∃ *y*∈*Collect* (*Domainp B*). *R x y* ∧ *S y z*) (*OO*)
unfolding *OO-def* **by** *transfer-prover*

lemma *relcompp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (*B* ===> *C* ===> (=)) ===> *A* ===> *C* ===> (=)) (*OO*) (*OO*)
unfolding *OO-def* **by** *transfer-prover*

lemma *right-total-Domainp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *right-total B*
shows ((*A* ===> *B* ===> (=)) ===> *A* ===> (=)) (λ*T x*. ∃ *y*∈*Collect*(*Domainp B*). *T x y*) *Domainp*
apply(*subst*(*2*) *Domainp-iff*[*abs-def*]) **by** *transfer-prover*

lemma *Domainp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> *A* ===> (=)) *Domainp Domainp*
unfolding *Domainp-iff* **by** *transfer-prover*

lemma *reflp-transfer*[*transfer-rule*]:
bi-total A ⟹ ((*A* ===> *A* ===> (=)) ===> (=)) *reflp reflp*
right-total A ⟹ ((*A* ===> *A* ===> *implies*) ===> *implies*) *reflp reflp*
right-total A ⟹ ((*A* ===> *A* ===> (=)) ===> *implies*) *reflp reflp*
bi-total A ⟹ ((*A* ===> *A* ===> *rev-implies*) ===> *rev-implies*) *reflp reflp*
bi-total A ⟹ ((*A* ===> *A* ===> (=)) ===> *rev-implies*) *reflp reflp*
unfolding *reflp-def rev-implies-def bi-total-def right-total-def rel-fun-def*
by *fast+*

lemma *right-unique-transfer* [*transfer-rule*]:
⟦ *right-total A*; *right-total B*; *bi-unique B* ⟧
⟹ ((*A* ===> *B* ===> (=)) ===> *implies*) *right-unique right-unique*
unfolding *right-unique-def right-total-def bi-unique-def rel-fun-def*
by *metis*

lemma *left-total-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *left-total left-total*
unfolding *left-total-def* **by** *transfer-prover*

lemma *right-total-parametric* [*transfer-rule*]:

assumes [*transfer-rule*]: *bi-total A bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *right-total right-total*
unfolding *right-total-def* **by** *transfer-prover*

lemma *left-unique-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A bi-total A bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *left-unique left-unique*
unfolding *left-unique-def* **by** *transfer-prover*

lemma *prod-pred-parametric* [*transfer-rule*]:
((*A* ===> (=)) ===> (*B* ===> (=)) ===> *rel-prod A B* ===> (=)) *pred-prod pred-prod*
unfolding *prod.pred-set Basic-BNFs.fsts-def Basic-BNFs.snds-def fstsp.simps sndsp.simps*

by *simp transfer-prover*

lemma *apfst-parametric* [*transfer-rule*]:
((*A* ===> *B*) ===> *rel-prod A C* ===> *rel-prod B C*) *apfst apfst*
unfolding *apfst-def* **by** *transfer-prover*

lemma *rel-fun-eq-eq-onp*: ((=) ===> *eq-onp P*) = *eq-onp* ($\lambda f.\ \forall x.\ P(f\ x)$)
unfolding *eq-onp-def rel-fun-def* **by** *auto*

lemma *rel-fun-eq-onp-rel*:
shows ((*eq-onp R*) ===> *S*) = ($\lambda f\ g.\ \forall x.\ R\ x \longrightarrow S\ (f\ x)\ (g\ x)$)
by (*auto simp add*: *eq-onp-def rel-fun-def*)

lemma *eq-onp-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows ((*A* ===> (=)) ===> *A* ===> *A* ===> (=)) *eq-onp eq-onp*
unfolding *eq-onp-def* **by** *transfer-prover*

lemma *rtranclp-parametric* [*transfer-rule*]:
assumes *bi-unique A bi-total A*
shows ((*A* ===> *A* ===> (=)) ===> *A* ===> *A* ===> (=)) *rtranclp rtranclp*
proof(*rule rel-funI iffI*)+
fix *R* :: $'a \Rightarrow 'a \Rightarrow$ *bool* **and** $R'\ x\ y\ x'\ y'$
assume *R*: (*A* ===> *A* ===> (=)) $R\ R'$ **and** $A\ x\ x'$
{
assume $R^{**}\ x\ y\ A\ y\ y'$
thus $R'^{**}\ x'\ y'$
proof(*induction arbitrary*: y')
case *base*
with ‹*bi-unique A*› ‹$A\ x\ x'$› **have** $x' = y'$ **by**(*rule bi-uniqueDr*)
thus *?case* **by** *simp*
next
case (*step* $y\ z\ z'$)
from ‹*bi-total A*› **obtain** y' **where** $A\ y\ y'$ **unfolding** *bi-total-def* **by** *blast*

hence R'^{**} *x' y'* **by**(*rule step.IH*)
moreover from *R* ‹*A y y'*› ‹*A z z'*› ‹*R y z*›
have *R' y' z'* **by**(*auto dest*: *rel-funD*)
ultimately show *?case* **..**
qed
next
assume R'^{**} *x' y' A y y'*
thus R^{**} *x y*
proof(*induction arbitrary*: *y*)
case *base*
with ‹*bi-unique A*› ‹*A x x'*› **have** *x* = *y* **by**(*rule bi-uniqueDl*)
thus *?case* **by** *simp*
next
case (*step y' z' z*)
from ‹*bi-total A*› **obtain** *y* **where** *A y y'* **unfolding** *bi-total-def* **by** *blast*
hence R^{**} *x y* **by**(*rule step.IH*)
moreover from *R* ‹*A y y'*› ‹*A z z'*› ‹*R' y' z'*›
have *R y z* **by**(*auto dest*: *rel-funD*)
ultimately show *?case* **..**
qed
}
qed

lemma *right-unique-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A bi-unique B bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *right-unique right-unique*
unfolding *right-unique-def* **by** *transfer-prover*

lemma *map-fun-parametric* [*transfer-rule*]:
((*A* ===> *B*) ===> (*C* ===> *D*) ===> (*B* ===> *C*) ===> *A* ===> *D*) *map-fun map-fun*
unfolding *map-fun-def* **by** *transfer-prover*

end

42.6 *of-bool* and *of-nat*

context
includes *lifting-syntax*
begin

lemma *transfer-rule-of-bool*:
‹((⟷) ===> (≅)) *of-bool of-bool*›
if [*transfer-rule*]: ‹*0* ≅ *0*› ‹*1* ≅ *1*›
for *R* :: ‹*'a*::*zero-neq-one* ⇒ *'b*::*zero-neq-one* ⇒ *bool*› (**infix** ‹≅› *50*)
unfolding *of-bool-def* **by** *transfer-prover*

lemma *transfer-rule-of-nat*:
((=) ===> (≅)) *of-nat of-nat*

if [*transfer-rule*]: ‹*0* ≅ *0*› ‹*1* ≅ *1*›
‹((≅) ===> (≅) ===> (≅)) (+) (+)›
for *R* :: ‹′*a*::*semiring-1* ⇒ ′*b*::*semiring-1* ⇒ *bool*› (**infix** ‹≅› *50*)
unfolding *of-nat-def* **by** *transfer-prover*

end

end

43 Lifting package

theory *Lifting*
imports *Equiv-Relations Transfer*
keywords
parametric **and**
print-quot-maps print-quotients :: *diag* **and**
lift-definition :: *thy-goal-defn* **and**
setup-lifting lifting-forget lifting-update :: *thy-decl*
begin

43.1 Function map

context includes *lifting-syntax*
begin

lemma *map-fun-id*:
(*id* −−−> *id*) = *id*
by (*simp add*: *fun-eq-iff*)

43.2 Quotient Predicate

definition *Quotient* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ (′*a* ⇒ ′*b*) ⇒ (′*b* ⇒ ′*a*) ⇒ (′*a* ⇒ ′*b* ⇒ *bool*) ⇒ *bool*
where
Quotient R Abs Rep T ⟷
(∀ *a*. *Abs* (*Rep a*) = *a*) ∧
(∀ *a*. *R* (*Rep a*) (*Rep a*)) ∧
(∀ *r s*. *R r s* ⟷ *R r r* ∧ *R s s* ∧ *Abs r* = *Abs s*) ∧
T = (λ*x y*. *R x x* ∧ *Abs x* = *y*)

lemma *QuotientI*:
assumes ⋀*a*. *Abs* (*Rep a*) = *a*
and ⋀*a*. *R* (*Rep a*) (*Rep a*)
and ⋀*r s*. *R r s* ⟷ *R r r* ∧ *R s s* ∧ *Abs r* = *Abs s*
and *T* = (λ*x y*. *R x x* ∧ *Abs x* = *y*)
shows *Quotient R Abs Rep T*
using *assms* **unfolding** *Quotient-def* **by** *blast*

context

fixes *R Abs Rep T*
assumes *a*: *Quotient R Abs Rep T*
begin

lemma *Quotient-abs-rep*: $Abs\ (Rep\ a) = a$
using *a* **unfolding** *Quotient-def*
by *simp*

lemma *Quotient-rep-reflp*: $R\ (Rep\ a)\ (Rep\ a)$
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-rel*:
$R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s \longleftrightarrow R\ r\ s$ — orientation does not loop on rewriting
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-cr-rel*: $T = (\lambda x\ y.\ R\ x\ x \wedge Abs\ x = y)$
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-refl1*: $R\ r\ s \Longrightarrow R\ r\ r$
using *a* **unfolding** *Quotient-def*
by *fast*

lemma *Quotient-refl2*: $R\ r\ s \Longrightarrow R\ s\ s$
using *a* **unfolding** *Quotient-def*
by *fast*

lemma *Quotient-rel-rep*: $R\ (Rep\ a)\ (Rep\ b) \longleftrightarrow a = b$
using *a* **unfolding** *Quotient-def*
by *metis*

lemma *Quotient-rep-abs*: $R\ r\ r \Longrightarrow R\ (Rep\ (Abs\ r))\ r$
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-rep-abs-eq*: $R\ t\ t \Longrightarrow R \le (=) \Longrightarrow Rep\ (Abs\ t) = t$
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-rep-abs-fold-unmap*:
assumes $x' \equiv Abs\ x$ **and** $R\ x\ x$ **and** $Rep\ x' \equiv Rep'\ x'$
shows $R\ (Rep'\ x')\ x$
proof −
have $R\ (Rep\ x')\ x$ **using** *assms*(*1*−*2*) *Quotient-rep-abs* **by** *auto*
then show *?thesis* **using** *assms*(*3*) **by** *simp*
qed

lemma *Quotient-Rep-eq*:
  assumes $x' \equiv Abs\ x$
  shows $Rep\ x' \equiv Rep\ x'$
by *simp*

lemma *Quotient-rel-abs*: $R\ r\ s \Longrightarrow Abs\ r = Abs\ s$
  using *a* **unfolding** *Quotient-def*
  by *blast*

lemma *Quotient-rel-abs2*:
  assumes $R\ (Rep\ x)\ y$
  shows $x = Abs\ y$
proof −
  from *assms* **have** $Abs\ (Rep\ x) = Abs\ y$ **by** (*auto intro*: *Quotient-rel-abs*)
  then show *?thesis* **using** *assms*(*1*) **by** (*simp add*: *Quotient-abs-rep*)
qed

lemma *Quotient-symp*: *symp R*
  using *a* **unfolding** *Quotient-def* **using** *sympI* **by** (*metis* (*full-types*))

lemma *Quotient-transp*: *transp R*
  using *a* **unfolding** *Quotient-def* **using** *transpI* **by** (*metis* (*full-types*))

lemma *Quotient-part-equivp*: *part-equivp R*
by (*metis Quotient-rep-reflp Quotient-symp Quotient-transp part-equivpI*)

end

lemma *identity-quotient*: *Quotient* (=) *id id* (=)
unfolding *Quotient-def* **by** *simp*

TODO: Use one of these alternatives as the real definition.

lemma *Quotient-alt-def*:
  $Quotient\ R\ Abs\ Rep\ T \longleftrightarrow$
    $(\forall\, a\ b.\ T\ a\ b \longrightarrow Abs\ a = b) \wedge$
    $(\forall\, b.\ T\ (Rep\ b)\ b) \wedge$
    $(\forall\, x\ y.\ R\ x\ y \longleftrightarrow T\ x\ (Abs\ x) \wedge T\ y\ (Abs\ y) \wedge Abs\ x = Abs\ y)$
apply *safe*
apply (*simp* (*no-asm-use*) *only*: *Quotient-def*, *fast*)
apply (*simp* (*no-asm-use*) *only*: *Quotient-def*, *fast*)
apply (*simp* (*no-asm-use*) *only*: *Quotient-def*, *fast*)
apply (*simp* (*no-asm-use*) *only*: *Quotient-def*, *fast*)
apply (*simp* (*no-asm-use*) *only*: *Quotient-def*, *fast*)
apply (*simp* (*no-asm-use*) *only*: *Quotient-def*, *fast*)
apply (*rule QuotientI*)
apply *simp*
apply *metis*
apply *simp*
apply (*rule ext*, *rule ext*, *metis*)

done

lemma *Quotient-alt-def2*:
Quotient R Abs Rep T $\longleftrightarrow$
($\forall$ *a b*. *T a b* $\longrightarrow$ *Abs a* = *b*) $\wedge$
($\forall$ *b*. *T* (*Rep b*) *b*) $\wedge$
($\forall$ *x y*. *R x y* $\longleftrightarrow$ *T x* (*Abs y*) $\wedge$ *T y* (*Abs x*))
unfolding *Quotient-alt-def* **by** (*safe*, *metis*+)

lemma *Quotient-alt-def3*:
Quotient R Abs Rep T $\longleftrightarrow$
($\forall$ *a b*. *T a b* $\longrightarrow$ *Abs a* = *b*) $\wedge$ ($\forall$ *b*. *T* (*Rep b*) *b*) $\wedge$
($\forall$ *x y*. *R x y* $\longleftrightarrow$ ($\exists$ *z*. *T x z* $\wedge$ *T y z*))
unfolding *Quotient-alt-def2* **by** (*safe*, *metis*+)

lemma *Quotient-alt-def4*:
Quotient R Abs Rep T $\longleftrightarrow$
($\forall$ *a b*. *T a b* $\longrightarrow$ *Abs a* = *b*) $\wedge$ ($\forall$ *b*. *T* (*Rep b*) *b*) $\wedge$ *R* = *T OO conversep T*
unfolding *Quotient-alt-def3 fun-eq-iff* **by** *auto*

lemma *Quotient-alt-def5*:
Quotient R Abs Rep T $\longleftrightarrow$
T $\leq$ *BNF-Def.Grp UNIV Abs* $\wedge$ *BNF-Def.Grp UNIV Rep* $\leq$ T^{-1-1} $\wedge$ *R* = *T* *OO* T^{-1-1}
unfolding *Quotient-alt-def4 Grp-def* **by** *blast*

lemma *fun-quotient*:
assumes *1*: *Quotient R1 abs1 rep1 T1*
assumes *2*: *Quotient R2 abs2 rep2 T2*
shows *Quotient* (*R1* ===> *R2*) (*rep1* ---> *abs2*) (*abs1* ---> *rep2*) (*T1* ===> *T2*)
using *assms* **unfolding** *Quotient-alt-def2*
unfolding *rel-fun-def fun-eq-iff map-fun-apply*
by (*safe*, *metis*+)

lemma *apply-rsp*:
fixes *f g*::'*a* $\Rightarrow$ '*c*
assumes *q*: *Quotient R1 Abs1 Rep1 T1*
and *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
using *a* **by** (*auto elim*: *rel-funE*)

lemma *apply-rsp'*:
assumes *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
using *a* **by** (*auto elim*: *rel-funE*)

lemma *apply-rsp''*:
assumes *Quotient R Abs Rep T*

and (R ===> S) f f
shows S (f (*Rep* x)) (f (*Rep* x))
proof –
from *assms*(*1*) **have** R (*Rep* x) (*Rep* x) **by** (*rule Quotient-rep-reflp*)
then show *?thesis* **using** *assms*(*2*) **by** (*auto intro*: *apply-rsp*′)
qed

43.3 Quotient composition

lemma *Quotient-compose*:
assumes *1*: *Quotient R1 Abs1 Rep1 T1*
assumes *2*: *Quotient R2 Abs2 Rep2 T2*
shows *Quotient* (*T1 OO R2 OO conversep T1*) (*Abs2* ∘ *Abs1*) (*Rep1* ∘ *Rep2*) (*T1 OO T2*)
using *assms* **unfolding** *Quotient-alt-def4* **by** *fastforce*

lemma *equivp-reflp2*:
equivp R ⟹ *reflp R*
by (*erule equivpE*)

43.4 Respects predicate

definition *Respects* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ ′*a set*
where *Respects R* = {*x*. *R x x*}

lemma *in-respects*: *x* ∈ *Respects R* ⟷ *R x x*
unfolding *Respects-def* **by** *simp*

lemma *UNIV-typedef-to-Quotient*:
assumes *type-definition Rep Abs UNIV*
and *T-def*: *T* ≡ (λ*x y*. *x* = *Rep y*)
shows *Quotient* (=) *Abs Rep T*
proof –
interpret *type-definition Rep Abs UNIV* **by** *fact*
from *Abs-inject Rep-inverse Abs-inverse T-def* **show** *?thesis*
by (*fastforce intro*!: *QuotientI fun-eq-iff*)
qed

lemma *UNIV-typedef-to-equivp*:
fixes *Abs* :: ′*a* ⇒ ′*b*
and *Rep* :: ′*b* ⇒ ′*a*
assumes *type-definition Rep Abs* (*UNIV*::′*a set*)
shows *equivp* ((=) ::′*a*⇒′*a*⇒*bool*)
by (*rule identity-equivp*)

lemma *typedef-to-Quotient*:
assumes *type-definition Rep Abs S*
and *T-def*: *T* ≡ (λ*x y*. *x* = *Rep y*)
shows *Quotient* (*eq-onp* (λ*x*. *x* ∈ *S*)) *Abs Rep T*
proof –

interpret *type-definition Rep Abs S* **by** *fact*
from *Rep Abs-inject Rep-inverse Abs-inverse T-def* **show** *?thesis*
by (*auto intro*!: *QuotientI simp*: *eq-onp-def fun-eq-iff*)
qed

lemma *typedef-to-part-equivp*:
assumes *type-definition Rep Abs S*
shows *part-equivp* (*eq-onp* ($\lambda x.\ x \in S$))
proof (*intro part-equivpI*)
interpret *type-definition Rep Abs S* **by** *fact*
show $\exists x.$ *eq-onp* ($\lambda x.\ x \in S$) *x x* **using** *Rep* **by** (*auto simp*: *eq-onp-def*)
next
show *symp* (*eq-onp* ($\lambda x.\ x \in S$)) **by** (*auto intro*: *sympI simp*: *eq-onp-def*)
next
show *transp* (*eq-onp* ($\lambda x.\ x \in S$)) **by** (*auto intro*: *transpI simp*: *eq-onp-def*)
qed

lemma *open-typedef-to-Quotient*:
assumes *type-definition Rep Abs* $\{x.\ P\ x\}$
and *T-def*: $T \equiv (\lambda x\ y.\ x = Rep\ y)$
shows *Quotient* (*eq-onp P*) *Abs Rep T*
using *typedef-to-Quotient* [*OF assms*] **by** *simp*

lemma *open-typedef-to-part-equivp*:
assumes *type-definition Rep Abs* $\{x.\ P\ x\}$
shows *part-equivp* (*eq-onp P*)
using *typedef-to-part-equivp* [*OF assms*] **by** *simp*

lemma *type-definition-Quotient-not-empty*: *Quotient* (*eq-onp P*) *Abs Rep T* $\Longrightarrow$ $\exists x.\ P\ x$
unfolding *eq-onp-def* **by** (*drule Quotient-rep-reflp*) *blast*

lemma *type-definition-Quotient-not-empty-witness*: *Quotient* (*eq-onp P*) *Abs Rep* $T \Longrightarrow P$ (*Rep undefined*)
unfolding *eq-onp-def* **by** (*drule Quotient-rep-reflp*) *blast*

Generating transfer rules for quotients.

context
fixes *R Abs Rep T*
assumes *1*: *Quotient R Abs Rep T*
begin

lemma *Quotient-right-unique*: *right-unique T*
using *1* **unfolding** *Quotient-alt-def right-unique-def* **by** *metis*

lemma *Quotient-right-total*: *right-total T*
using *1* **unfolding** *Quotient-alt-def right-total-def* **by** *metis*

lemma *Quotient-rel-eq-transfer*: (T ===> T ===> (=)) R (=)

using *1* **unfolding** *Quotient-alt-def rel-fun-def* **by** *simp*

lemma *Quotient-abs-induct*:
assumes $\bigwedge y.\ R\ y\ y \Longrightarrow P\ (Abs\ y)$ **shows** $P\ x$
using *1 assms* **unfolding** *Quotient-def* **by** *metis*

end

Generating transfer rules for total quotients.

context
fixes *R Abs Rep T*
assumes *1*: *Quotient R Abs Rep T* **and** *2*: *reflp R*
begin

lemma *Quotient-left-total*: *left-total T*
using *1 2* **unfolding** *Quotient-alt-def left-total-def reflp-def* **by** *auto*

lemma *Quotient-bi-total*: *bi-total T*
using *1 2* **unfolding** *Quotient-alt-def bi-total-def reflp-def* **by** *auto*

lemma *Quotient-id-abs-transfer*: $((=)\ ===>\ T)\ (\lambda x.\ x)\ Abs$
using *1 2* **unfolding** *Quotient-alt-def reflp-def rel-fun-def* **by** *simp*

lemma *Quotient-total-abs-induct*: $(\bigwedge y.\ P\ (Abs\ y)) \Longrightarrow P\ x$
using *1 2* **unfolding** *Quotient-alt-def reflp-def* **by** *metis*

lemma *Quotient-total-abs-eq-iff*: $Abs\ x = Abs\ y \longleftrightarrow R\ x\ y$
using *Quotient-rel* [*OF 1*] *2* **unfolding** *reflp-def* **by** *simp*

end

Generating transfer rules for a type defined with *typedef*.

context
fixes *Rep Abs A T*
assumes *type*: *type-definition Rep Abs A*
assumes *T-def*: $T \equiv (\lambda(x::'a)\ (y::'b).\ x = Rep\ y)$
begin

lemma *typedef-left-unique*: *left-unique T*
unfolding *left-unique-def T-def*
by (*simp add*: *type-definition.Rep-inject* [*OF type*])

lemma *typedef-bi-unique*: *bi-unique T*
unfolding *bi-unique-def T-def*
by (*simp add*: *type-definition.Rep-inject* [*OF type*])

lemma *typedef-right-unique*: *right-unique T*

using *T-def type Quotient-right-unique typedef-to-Quotient*
by *blast*

lemma *typedef-right-total*: *right-total T*
using *T-def type Quotient-right-total typedef-to-Quotient*
by *blast*

lemma *typedef-rep-transfer*: $(T ===> (=))\ (\lambda x.\ x)\ Rep$
unfolding *rel-fun-def T-def* **by** *simp*

end

Generating the correspondence rule for a constant defined with *lift-definition*.

lemma *Quotient-to-transfer*:
assumes *Quotient R Abs Rep T* **and** *R c c* **and** $c' \equiv Abs\ c$
shows $T\ c\ c'$
using *assms* **by** (*auto dest*: *Quotient-cr-rel*)

Proving reflexivity

lemma *Quotient-to-left-total*:
assumes *q*: *Quotient R Abs Rep T*
and *r-R*: *reflp R*
shows *left-total T*
using *r-R Quotient-cr-rel*[*OF q*] **unfolding** *left-total-def* **by** (*auto elim*: *reflpE*)

lemma *Quotient-composition-ge-eq*:
assumes *left-total T*
assumes $R \geq (=)$
shows $(T\ OO\ R\ OO\ T^{-1-1}) \geq (=)$
using *assms* **unfolding** *left-total-def* **by** *fast*

lemma *Quotient-composition-le-eq*:
assumes *left-unique T*
assumes $R \leq (=)$
shows $(T\ OO\ R\ OO\ T^{-1-1}) \leq (=)$
using *assms* **unfolding** *left-unique-def* **by** *blast*

lemma *eq-onp-le-eq*:
eq-onp $P \leq (=)$ **unfolding** *eq-onp-def* **by** *blast*

lemma *reflp-ge-eq*:
reflp $R \Longrightarrow R \geq (=)$ **unfolding** *reflp-def* **by** *blast*

Proving a parametrized correspondence relation

definition $POS :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$ **where**
$POS\ A\ B \equiv A \leq B$

definition $NEG :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$ **where**
$NEG\ A\ B \equiv B \leq A$

lemma *pos-OO-eq*:
  shows *POS* (*A OO* (=)) *A*
unfolding *POS-def OO-def* **by** *blast*

lemma *pos-eq-OO*:
  shows *POS* ((=) *OO A*) *A*
unfolding *POS-def OO-def* **by** *blast*

lemma *neg-OO-eq*:
  shows *NEG* (*A OO* (=)) *A*
unfolding *NEG-def OO-def* **by** *auto*

lemma *neg-eq-OO*:
  shows *NEG* ((=) *OO A*) *A*
unfolding *NEG-def OO-def* **by** *blast*

lemma *POS-trans*:
  assumes *POS A B*
  assumes *POS B C*
  shows *POS A C*
using *assms* **unfolding** *POS-def* **by** *auto*

lemma *NEG-trans*:
  assumes *NEG A B*
  assumes *NEG B C*
  shows *NEG A C*
using *assms* **unfolding** *NEG-def* **by** *auto*

lemma *POS-NEG*:
  POS A B ≡ *NEG B A*
  unfolding *POS-def NEG-def* **by** *auto*

lemma *NEG-POS*:
  NEG A B ≡ *POS B A*
  unfolding *POS-def NEG-def* **by** *auto*

lemma *POS-pcr-rule*:
  assumes *POS* (*A OO B*) *C*
  shows *POS* (*A OO B OO X*) (*C OO X*)
using *assms* **unfolding** *POS-def OO-def* **by** *blast*

lemma *NEG-pcr-rule*:
  assumes *NEG* (*A OO B*) *C*
  shows *NEG* (*A OO B OO X*) (*C OO X*)
using *assms* **unfolding** *NEG-def OO-def* **by** *blast*

lemma *POS-apply*:
  assumes *POS R R'*

assumes $R\ f\ g$
shows $R'\ f\ g$
using *assms* **unfolding** *POS-def* **by** *auto*

Proving a parametrized correspondence relation

lemma *fun-mono*:
assumes $A \geq C$
assumes $B \leq D$
shows $(A\ {=}{=}{=}{>}\ B) \leq (C\ {=}{=}{=}{>}\ D)$
using *assms* **unfolding** *rel-fun-def* **by** *blast*

lemma *pos-fun-distr*: $((R\ {=}{=}{=}{>}\ S)\ OO\ (R'\ {=}{=}{=}{>}\ S')) \leq ((R\ OO\ R')\ {=}{=}{=}{>}\ (S\ OO\ S'))$
unfolding *OO-def rel-fun-def* **by** *blast*

lemma *functional-relation*: *right-unique* $R \Longrightarrow$ *left-total* $R \Longrightarrow \forall x.\ \exists!y.\ R\ x\ y$
unfolding *right-unique-def left-total-def* **by** *blast*

lemma *functional-converse-relation*: *left-unique* $R \Longrightarrow$ *right-total* $R \Longrightarrow \forall y.\ \exists!x.\ R\ x\ y$
unfolding *left-unique-def right-total-def* **by** *blast*

lemma *neg-fun-distr1*:
assumes *1*: *left-unique* R *right-total* R
assumes *2*: *right-unique* R' *left-total* R'
shows $(R\ OO\ R'\ {=}{=}{=}{>}\ S\ OO\ S') \leq ((R\ {=}{=}{=}{>}\ S)\ OO\ (R'\ {=}{=}{=}{>}\ S'))$
using *functional-relation*[*OF 2*] *functional-converse-relation*[*OF 1*]
unfolding *rel-fun-def OO-def*
apply *clarify*
apply (*subst all-comm*)
apply (*subst all-conj-distrib*[*symmetric*])
apply (*intro choice*)
by *metis*

lemma *neg-fun-distr2*:
assumes *1*: *right-unique* R' *left-total* R'
assumes *2*: *left-unique* S' *right-total* S'
shows $(R\ OO\ R'\ {=}{=}{=}{>}\ S\ OO\ S') \leq ((R\ {=}{=}{=}{>}\ S)\ OO\ (R'\ {=}{=}{=}{>}\ S'))$
using *functional-converse-relation*[*OF 2*] *functional-relation*[*OF 1*]
unfolding *rel-fun-def OO-def*
apply *clarify*
apply (*subst all-comm*)
apply (*subst all-conj-distrib*[*symmetric*])
apply (*intro choice*)
by *metis*

43.5 Domains

lemma *composed-equiv-rel-eq-onp*:

assumes *left-unique R*
assumes $(R ===> (=))\ P\ P'$
assumes *Domainp* $R = P''$
shows $(R\ OO\ eq\text{-}onp\ P'\ OO\ R^{-1-1}) = eq\text{-}onp\ (inf\ P''\ P)$
using *assms* **unfolding** *OO-def conversep-iff Domainp-iff*[*abs-def*] *left-unique-def rel-fun-def eq-onp-def*
fun-eq-iff **by** *blast*

lemma *composed-equiv-rel-eq-eq-onp*:
assumes *left-unique R*
assumes *Domainp* $R = P$
shows $(R\ OO\ (=)\ OO\ R^{-1-1}) = eq\text{-}onp\ P$
using *assms* **unfolding** *OO-def conversep-iff Domainp-iff*[*abs-def*] *left-unique-def eq-onp-def*
fun-eq-iff is-equality-def **by** *metis*

lemma *pcr-Domainp-par-left-total*:
assumes *Domainp* $B = P$
assumes *left-total A*
assumes $(A ===> (=))\ P'\ P$
shows *Domainp* $(A\ OO\ B) = P'$
using *assms*
unfolding *Domainp-iff*[*abs-def*] *OO-def bi-unique-def left-total-def rel-fun-def*
by (*fast intro*: *fun-eq-iff*)

lemma *pcr-Domainp-par*:
assumes *Domainp* $B = P2$
assumes *Domainp* $A = P1$
assumes $(A ===> (=))\ P2'\ P2$
shows *Domainp* $(A\ OO\ B) = (inf\ P1\ P2')$
using *assms* **unfolding** *rel-fun-def Domainp-iff*[*abs-def*] *OO-def*
by (*fast intro*: *fun-eq-iff*)

definition *rel-pred-comp* :: $('a => 'b => bool) => ('b => bool) => 'a => bool$
where *rel-pred-comp* $R\ P \equiv \lambda x.\ \exists\, y.\ R\ x\ y \wedge P\ y$

lemma *pcr-Domainp*:
assumes *Domainp* $B = P$
shows *Domainp* $(A\ OO\ B) = (\lambda x.\ \exists\, y.\ A\ x\ y \wedge P\ y)$
using *assms* **by** *blast*

lemma *pcr-Domainp-total*:
assumes *left-total B*
assumes *Domainp* $A = P$
shows *Domainp* $(A\ OO\ B) = P$
using *assms* **unfolding** *left-total-def*
by *fast*

lemma *Quotient-to-Domainp*:

assumes *Quotient R Abs Rep T*
shows *Domainp T* = (*λx. R x x*)
by (*simp add*: *Domainp-iff*[*abs-def*] *Quotient-cr-rel*[*OF assms*])

lemma *eq-onp-to-Domainp*:
assumes *Quotient* (*eq-onp P*) *Abs Rep T*
shows *Domainp T* = *P*
by (*simp add*: *eq-onp-def Domainp-iff*[*abs-def*] *Quotient-cr-rel*[*OF assms*])

end

lemma *right-total-UNIV-transfer*:
assumes *right-total A*
shows (*rel-set A*) (*Collect* (*Domainp A*)) *UNIV*
using *assms* **unfolding** *right-total-def rel-set-def Domainp-iff* **by** *blast*

43.6 ML setup

ML-file ‹*Tools/Lifting/lifting-util.ML*›

named-theorems *relator-eq-onp*
theorems that a relator of an eq-onp is an eq-onp of the corresponding predicate
ML-file ‹*Tools/Lifting/lifting-info.ML*›

declare *fun-quotient*[*quot-map*]
declare *fun-mono*[*relator-mono*]
lemmas [*relator-distr*] = *pos-fun-distr neg-fun-distr1 neg-fun-distr2*

ML-file ‹*Tools/Lifting/lifting-bnf.ML*›
ML-file ‹*Tools/Lifting/lifting-term.ML*›
ML-file ‹*Tools/Lifting/lifting-def.ML*›
ML-file ‹*Tools/Lifting/lifting-setup.ML*›
ML-file ‹*Tools/Lifting/lifting-def-code-dt.ML*›

lemma *pred-prod-beta*: *pred-prod P Q xy* ⟷ *P* (*fst xy*) ∧ *Q* (*snd xy*)
by(*cases xy*) *simp*

lemma *pred-prod-split*: *P* (*pred-prod Q R xy*) ⟷ (∀ *x y*. *xy* = (*x*, *y*) ⟶ *P* (*Q x* ∧ *R y*))
by(*cases xy*) *simp*

hide-const (**open**) *POS NEG*

end

44 Definition of Quotient Types

theory *Quotient*
imports *Lifting*
keywords
print-quotmapsQ3 print-quotientsQ3 print-quotconsts :: *diag* **and**
quotient-type :: *thy-goal-defn* **and** / **and**
quotient-definition :: *thy-goal-defn* **and**
copy-bnf :: *thy-defn* **and**
lift-bnf :: *thy-goal-defn*
begin

Basic definition for equivalence relations that are represented by predicates.

Composition of Relations

abbreviation
rel-conj :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('b \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$ (**infixr** ‹*OOO*› *75*)
where
$r1\ OOO\ r2 \equiv r1\ OO\ r2\ OO\ r1$

lemma *eq-comp-r*:
shows $((=)\ OOO\ R) = R$
by (*auto simp add*: *fun-eq-iff*)

context includes *lifting-syntax*
begin

44.1 Quotient Predicate

definition
$Quotient3\ R\ Abs\ Rep \longleftrightarrow$
$(\forall a.\ Abs\ (Rep\ a) = a) \wedge (\forall a.\ R\ (Rep\ a)\ (Rep\ a)) \wedge$
$(\forall r\ s.\ R\ r\ s \longleftrightarrow R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s)$

lemma *Quotient3I*:
assumes $\bigwedge a.\ Abs\ (Rep\ a) = a$
and $\bigwedge a.\ R\ (Rep\ a)\ (Rep\ a)$
and $\bigwedge r\ s.\ R\ r\ s \longleftrightarrow R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s$
shows *Quotient3 R Abs Rep*
using *assms* **unfolding** *Quotient3-def* **by** *blast*

context
fixes *R Abs Rep*
assumes *a*: *Quotient3 R Abs Rep*
begin

lemma *Quotient3-abs-rep*:
$Abs\ (Rep\ a) = a$
using *a*

unfolding *Quotient3-def*
by *simp*

lemma *Quotient3-rep-reflp*:
R (*Rep a*) (*Rep a*)
using *a*
unfolding *Quotient3-def*
by *blast*

lemma *Quotient3-rel*:
$R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s \longleftrightarrow R\ r\ s$ — orientation does not loop on rewriting
using *a*
unfolding *Quotient3-def*
by *blast*

lemma *Quotient3-refl1*:
$R\ r\ s \Longrightarrow R\ r\ r$
using *a* **unfolding** *Quotient3-def*
by *fast*

lemma *Quotient3-refl2*:
$R\ r\ s \Longrightarrow R\ s\ s$
using *a* **unfolding** *Quotient3-def*
by *fast*

lemma *Quotient3-rel-rep*:
$R\ (Rep\ a)\ (Rep\ b) \longleftrightarrow a = b$
using *a*
unfolding *Quotient3-def*
by *metis*

lemma *Quotient3-rep-abs*:
$R\ r\ r \Longrightarrow R\ (Rep\ (Abs\ r))\ r$
using *a* **unfolding** *Quotient3-def*
by *blast*

lemma *Quotient3-rel-abs*:
$R\ r\ s \Longrightarrow Abs\ r = Abs\ s$
using *a* **unfolding** *Quotient3-def*
by *blast*

lemma *Quotient3-symp*:
symp R
using *a* **unfolding** *Quotient3-def* **using** *sympI* **by** *metis*

lemma *Quotient3-transp*:
transp R
using *a* **unfolding** *Quotient3-def* **using** *transpI* **by** (*metis* (*full-types*))

lemma *Quotient3-part-equivp*:
part-equivp R
by (*metis Quotient3-rep-reflp Quotient3-symp Quotient3-transp part-equivpI*)

lemma *abs-o-rep*:
Abs ∘ *Rep* = *id*
unfolding *fun-eq-iff*
by (*simp add*: *Quotient3-abs-rep*)

lemma *equals-rsp*:
assumes *b*: *R xa xb R ya yb*
shows *R xa ya* = *R xb yb*
using *b Quotient3-symp Quotient3-transp*
by (*blast elim*: *sympE transpE*)

lemma *rep-abs-rsp*:
assumes *b*: *R x1 x2*
shows *R x1* (*Rep* (*Abs x2*))
using *b Quotient3-rel Quotient3-abs-rep Quotient3-rep-reflp*
by *metis*

lemma *rep-abs-rsp-left*:
assumes *b*: *R x1 x2*
shows *R* (*Rep* (*Abs x1*)) *x2*
using *b Quotient3-rel Quotient3-abs-rep Quotient3-rep-reflp*
by *metis*

end

lemma *identity-quotient3*:
Quotient3 (=) *id id*
unfolding *Quotient3-def id-def*
by *blast*

lemma *fun-quotient3*:
assumes *q1*: *Quotient3 R1 abs1 rep1*
and *q2*: *Quotient3 R2 abs2 rep2*
shows *Quotient3* (*R1* ===> *R2*) (*rep1* −−−> *abs2*) (*abs1* −−−> *rep2*)
proof −
have (*rep1* −−−> *abs2*) ((*abs1* −−−> *rep2*) *a*) = *a* **for** *a*
using *q1 q2* **by** (*simp add*: *Quotient3-def fun-eq-iff*)
moreover
have (*R1* ===> *R2*) ((*abs1* −−−> *rep2*) *a*) ((*abs1* −−−> *rep2*) *a*) **for** *a*
by (*rule rel-funI*)
(*use q1 q2 Quotient3-rel-abs* [*of R1 abs1 rep1*] *Quotient3-rel-rep* [*of R2 abs2 rep2*]
in ‹*simp* (*no-asm*) *add*: *Quotient3-def*, *simp*›)
moreover
have (*R1* ===> *R2*) *r s* = ((*R1* ===> *R2*) *r r* ∧ (*R1* ===> *R2*) *s s* ∧

(*rep1* ---> *abs2*) *r* = (*rep1* ---> *abs2*) *s*) **for** *r s*
proof –
have (*R1* ===> *R2*) *r s* ⟹ (*R1* ===> *R2*) *r r* **unfolding** *rel-fun-def*
using *Quotient3-part-equivp*[*OF q1*] *Quotient3-part-equivp*[*OF q2*]
by (*metis* (*full-types*) *part-equivp-def*)
moreover have (*R1* ===> *R2*) *r s* ⟹ (*R1* ===> *R2*) *s s* **unfolding** *rel-fun-def*
using *Quotient3-part-equivp*[*OF q1*] *Quotient3-part-equivp*[*OF q2*]
by (*metis* (*full-types*) *part-equivp-def*)
moreover have (*R1* ===> *R2*) *r s* ⟹ (*rep1* ---> *abs2*) *r* = (*rep1* ---> *abs2*) *s*
by (*auto simp add*: *rel-fun-def fun-eq-iff*)
(*use q1 q2* **in** ‹*unfold Quotient3-def*, *metis*›)
moreover have ((*R1* ===> *R2*) *r r* ∧ (*R1* ===> *R2*) *s s* ∧
(*rep1* ---> *abs2*) *r* = (*rep1* ---> *abs2*) *s*) ⟹ (*R1* ===> *R2*) *r s*
by (*auto simp add*: *rel-fun-def fun-eq-iff*)
(*use q1 q2* **in** ‹*unfold Quotient3-def*, *metis map-fun-apply*›)
ultimately show *?thesis* **by** *blast*
qed
ultimately show *?thesis* **by** (*intro Quotient3I*) (*assumption+*)
qed

lemma *lambda-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep1* ---> *Abs2*) (λ*x*. *Rep2* (*f* (*Abs1 x*))) = (λ*x*. *f x*)
unfolding *fun-eq-iff*
using *Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*]
by *simp*

lemma *lambda-prs1*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep1* ---> *Abs2*) (λ*x*. (*Abs1* ---> *Rep2*) *f x*) = (λ*x*. *f x*)
unfolding *fun-eq-iff*
using *Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*]
by *simp*

In the following theorem R1 can be instantiated with anything, but we know some of the types of the Rep and Abs functions; so by solving Quotient assumptions we can get a unique R1 that will be provable; which is why we need to use *apply-rsp* and not the primed version

lemma *apply-rspQ3*:
fixes *f g*::'*a* ⇒ '*c*
assumes *q*: *Quotient3 R1 Abs1 Rep1*
and *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
using *a* **by** (*auto elim*: *rel-funE*)

lemma *apply-rspQ3''*:
assumes *Quotient3 R Abs Rep*
and $(R \ ===> \ S)\ f\ f$
shows $S\ (f\ (Rep\ x))\ (f\ (Rep\ x))$
proof −
from *assms*(*1*) **have** $R\ (Rep\ x)\ (Rep\ x)$ **by** (*rule Quotient3-rep-reflp*)
then show *?thesis* **using** *assms*(*2*) **by** (*auto intro*: *apply-rsp'*)
qed

44.2 lemmas for regularisation of ball and bex

lemma *ball-reg-eqv*:
fixes $P :: 'a \Rightarrow bool$
assumes *a*: *equivp R*
shows $Ball\ (Respects\ R)\ P = (All\ P)$
using *a*
unfolding *equivp-def*
by (*auto simp add*: *in-respects*)

lemma *bex-reg-eqv*:
fixes $P :: 'a \Rightarrow bool$
assumes *a*: *equivp R*
shows $Bex\ (Respects\ R)\ P = (Ex\ P)$
using *a*
unfolding *equivp-def*
by (*auto simp add*: *in-respects*)

lemma *ball-reg-right*:
assumes *a*: $\bigwedge x.\ x \in R \Longrightarrow P\ x \longrightarrow Q\ x$
shows $All\ P \longrightarrow Ball\ R\ Q$
using *a* **by** *fast*

lemma *bex-reg-left*:
assumes *a*: $\bigwedge x.\ x \in R \Longrightarrow Q\ x \longrightarrow P\ x$
shows $Bex\ R\ Q \longrightarrow Ex\ P$
using *a* **by** *fast*

lemma *ball-reg-left*:
assumes *a*: *equivp R*
shows $(\bigwedge x.\ (Q\ x \longrightarrow P\ x)) \Longrightarrow Ball\ (Respects\ R)\ Q \longrightarrow All\ P$
using *a* **by** (*metis equivp-reflp in-respects*)

lemma *bex-reg-right*:
assumes *a*: *equivp R*
shows $(\bigwedge x.\ (Q\ x \longrightarrow P\ x)) \Longrightarrow Ex\ Q \longrightarrow Bex\ (Respects\ R)\ P$
using *a* **by** (*metis equivp-reflp in-respects*)

lemma *ball-reg-eqv-range*:
fixes $P{::}'a \Rightarrow bool$

and *x*::′*a*
assumes *a*: *equivp R2*
shows (*Ball* (*Respects* (*R1* ===> *R2*)) (λ*f*. *P* (*f x*)) = *All* (λ*f*. *P* (*f x*)))
proof (*intro allI iffI*)
fix *f*
assume ∀ *f* ∈ *Respects* (*R1* ===> *R2*). *P* (*f x*)
moreover have (λ*y*. *f x*) ∈ *Respects* (*R1* ===> *R2*)
using *a equivp-reflp-symp-transp*[*of R2*]
by(*auto simp add*: *in-respects rel-fun-def elim*: *equivpE reflpE*)
ultimately show *P* (*f x*)
by *auto*
qed *auto*

lemma *bex-reg-eqv-range*:
assumes *a*: *equivp R2*
shows (*Bex* (*Respects* (*R1* ===> *R2*)) (λ*f*. *P* (*f x*)) = *Ex* (λ*f*. *P* (*f x*)))
proof −
have (λ*y*. *f x*) ∈ *Respects* (*R1* ===> *R2*) **for** *f*
using *a equivp-reflp-symp-transp*[*of R2*]
by (*auto simp add*: *Respects-def in-respects rel-fun-def elim*: *equivpE reflpE*)
then show *?thesis*
by *auto*
qed

lemma *all-reg*:
assumes *a*: ∀ *x* :: ′*a*. (*P x* ⟶ *Q x*)
and *b*: *All P*
shows *All Q*
using *a b* **by** *fast*

lemma *ex-reg*:
assumes *a*: ∀ *x* :: ′*a*. (*P x* ⟶ *Q x*)
and *b*: *Ex P*
shows *Ex Q*
using *a b* **by** *fast*

lemma *ball-reg*:
assumes *a*: ∀ *x* :: ′*a*. (*x* ∈ *R* ⟶ *P x* ⟶ *Q x*)
and *b*: *Ball R P*
shows *Ball R Q*
using *a b* **by** *fast*

lemma *bex-reg*:
assumes *a*: ∀ *x* :: ′*a*. (*x* ∈ *R* ⟶ *P x* ⟶ *Q x*)
and *b*: *Bex R P*
shows *Bex R Q*
using *a b* **by** *fast*

lemma *ball-all-comm*:
  assumes $\bigwedge y.\ (\forall x \in P.\ A\ x\ y) \longrightarrow (\forall x.\ B\ x\ y)$
  shows $(\forall x \in P.\ \forall y.\ A\ x\ y) \longrightarrow (\forall x.\ \forall y.\ B\ x\ y)$
  using *assms* **by** *auto*

lemma *bex-ex-comm*:
  assumes $(\exists y.\ \exists x.\ A\ x\ y) \longrightarrow (\exists y.\ \exists x \in P.\ B\ x\ y)$
  shows $(\exists x.\ \exists y.\ A\ x\ y) \longrightarrow (\exists x \in P.\ \exists y.\ B\ x\ y)$
  using *assms* **by** *auto*

44.3 Bounded abstraction

definition
  Babs :: $'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
where
  $x \in p \Longrightarrow Babs\ p\ m\ x = m\ x$

lemma *babs-rsp*:
  assumes *q*: *Quotient3 R1 Abs1 Rep1*
    and *a*: (*R1* ===> *R2*) *f g*
   shows (*R1* ===> *R2*) (*Babs* (*Respects R1*) *f*) (*Babs* (*Respects R1*) *g*)
proof
  fix *x y*
  assume *R1 x y*
  then have $x \in$ *Respects R1* $\wedge$ $y \in$ *Respects R1*
   unfolding *in-respects rel-fun-def* **using** *Quotient3-rel*[*OF q*]**by** *metis*
  then show *R2* (*Babs* (*Respects R1*) *f x*) (*Babs* (*Respects R1*) *g y*)
  using ‹*R1 x y*› *a* **by** (*simp add*: *Babs-def rel-fun-def*)
qed

lemma *babs-prs*:
  assumes *q1*: *Quotient3 R1 Abs1 Rep1*
  and   *q2*: *Quotient3 R2 Abs2 Rep2*
shows ((*Rep1* ---> *Abs2*) (*Babs* (*Respects R1*) ((*Abs1* ---> *Rep2*) *f*))) = *f*
proof −
  have *Abs2* (*Babs* (*Respects R1*) ((*Abs1* ---> *Rep2*) *f*) (*Rep1 x*)) = *f x* **for** *x*
  proof −
   have *Rep1 x* $\in$ *Respects R1*
    by (*simp add*: *in-respects Quotient3-rel-rep*[*OF q1*])
   then show *?thesis*
    by (*simp add*: *Babs-def Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*])
  qed
  then show *?thesis*
   by *force*
qed

lemma *babs-simp*:
  assumes *q*: *Quotient3 R1 Abs Rep*

shows ((*R1* ===> *R2*) (*Babs* (*Respects R1*) *f*) (*Babs* (*Respects R1*) *g*)) = ((*R1* ===> *R2*) *f g*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *rel-fun-def* **by** (*metis Babs-def in-respects Quotient3-rel*[*OF q*])
qed (*simp add*: *babs-rsp*[*OF q*])

If a user proves that a particular functional relation is an equivalence, this may be useful in regularising

lemma *babs-reg-eqv*:
shows *equivp R* ⟹ *Babs* (*Respects R*) *P* = *P*
by (*simp add*: *fun-eq-iff Babs-def in-respects equivp-reflp*)

lemma *ball-rsp*:
assumes *a*: (*R* ===> (=)) *f g*
shows *Ball* (*Respects R*) *f* = *Ball* (*Respects R*) *g*
using *a* **by** (*auto simp add*: *Ball-def in-respects elim*: *rel-funE*)

lemma *bex-rsp*:
assumes *a*: (*R* ===> (=)) *f g*
shows (*Bex* (*Respects R*) *f* = *Bex* (*Respects R*) *g*)
using *a* **by** (*auto simp add*: *Bex-def in-respects elim*: *rel-funE*)

lemma *bex1-rsp*:
assumes *a*: (*R* ===> (=)) *f g*
shows *Ex1* (λ*x*. *x* ∈ *Respects R* ∧ *f x*) = *Ex1* (λ*x*. *x* ∈ *Respects R* ∧ *g x*)
using *a* **by** (*auto elim*: *rel-funE simp add*: *Ex1-def in-respects*)

Two lemmas needed for cleaning of quantifiers

lemma *all-prs*:
assumes *a*: *Quotient3 R absf repf*
shows *Ball* (*Respects R*) ((*absf* ---> *id*) *f*) = *All f*
using *a* **unfolding** *Quotient3-def Ball-def in-respects id-apply comp-def map-fun-def*
by *metis*

lemma *ex-prs*:
assumes *a*: *Quotient3 R absf repf*
shows *Bex* (*Respects R*) ((*absf* ---> *id*) *f*) = *Ex f*
using *a* **unfolding** *Quotient3-def Bex-def in-respects id-apply comp-def map-fun-def*
by *metis*

44.4 *Bex1-rel* quantifier

definition
Bex1-rel :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ *bool*) ⇒ *bool*

where
Bex1-rel R P $\longleftrightarrow$ ($\exists$ *x* $\in$ *Respects R*. *P x*) $\wedge$ ($\forall$ *x* $\in$ *Respects R*. $\forall$ *y* $\in$ *Respects R*. ((*P x* $\wedge$ *P y*) $\longrightarrow$ (*R x y*)))

lemma *bex1-rel-aux*:
⟦$\forall$ *xa ya*. *R xa ya* $\longrightarrow$ *x xa* = *y ya*; *Bex1-rel R x*⟧ $\Longrightarrow$ *Bex1-rel R y*
unfolding *Bex1-rel-def*
by (*metis in-respects*)

lemma *bex1-rel-aux2*:
⟦$\forall$ *xa ya*. *R xa ya* $\longrightarrow$ *x xa* = *y ya*; *Bex1-rel R y*⟧ $\Longrightarrow$ *Bex1-rel R x*
unfolding *Bex1-rel-def*
by (*metis in-respects*)

lemma *bex1-rel-rsp*:
assumes *a*: *Quotient3 R absf repf*
shows ((*R* ===> (=)) ===> (=)) (*Bex1-rel R*) (*Bex1-rel R*)
unfolding *rel-fun-def* **by** (*metis bex1-rel-aux bex1-rel-aux2*)

lemma *ex1-prs*:
assumes *Quotient3 R absf repf*
shows ((*absf* −−−> *id*) −−−> *id*) (*Bex1-rel R*) *f* = *Ex1 f*
(**is** *?lhs* = *?rhs*)
using *assms*
by (*auto simp add*: *Bex1-rel-def Respects-def*) (*metis* (*full-types*) *Quotient3-def*)+

lemma *bex1-bexeq-reg*:
shows ($\exists$!*x*$\in$*Respects R*. *P x*) $\longrightarrow$ (*Bex1-rel R* (λ*x*. *P x*))
by (*auto simp add*: *Ex1-def Bex1-rel-def Bex-def Ball-def in-respects*)

lemma *bex1-bexeq-reg-eqv*:
assumes *a*: *equivp R*
shows ($\exists$!*x*. *P x*) $\longrightarrow$ *Bex1-rel R P*
using *equivp-reflp*[*OF a*]
by (*metis* (*full-types*) *Bex1-rel-def in-respects*)

44.5 Various respects and preserve lemmas

lemma *quot-rel-rsp*:
assumes *a*: *Quotient3 R Abs Rep*
shows (*R* ===> *R* ===> (=)) *R R*
by (*rule rel-funI*)+ (*meson assms equals-rsp*)

lemma *o-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
and *q3*: *Quotient3 R3 Abs3 Rep3*
shows ((*Abs2* −−−> *Rep3*) −−−> (*Abs1* −−−> *Rep2*) −−−> (*Rep1* −−−> *Abs3*)) ($\circ$) = ($\circ$)

and (*id* ---> (*Abs1* ---> *id*) ---> *Rep1* ---> *id*) (∘) = (∘)
using *Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*] *Quotient3-abs-rep*[*OF q3*]
by (*simp-all add*: *fun-eq-iff*)

lemma *o-rsp*:
((*R2* ===> *R3*) ===> (*R1* ===> *R2*) ===> (*R1* ===> *R3*)) (∘) (∘)
((=) ===> (*R1* ===> (=)) ===> *R1* ===> (=)) (∘) (∘)
by (*force elim*: *rel-funE*)+

lemma *cond-prs*:
assumes *a*: *Quotient3 R absf repf*
shows *absf* (*if a then repf b else repf c*) = (*if a then b else c*)
using *a* **unfolding** *Quotient3-def* **by** *auto*

lemma *if-prs*:
assumes *q*: *Quotient3 R Abs Rep*
shows (*id* ---> *Rep* ---> *Rep* ---> *Abs*) *If* = *If*
using *Quotient3-abs-rep*[*OF q*]
by (*auto simp add*: *fun-eq-iff*)

lemma *if-rsp*:
assumes *q*: *Quotient3 R Abs Rep*
shows ((=) ===> *R* ===> *R* ===> *R*) *If If*
by *force*

lemma *let-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep2* ---> (*Abs2* ---> *Rep1*) ---> *Abs1*) *Let* = *Let*
using *Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*]
by (*auto simp add*: *fun-eq-iff*)

lemma *let-rsp*:
shows (*R1* ===> (*R1* ===> *R2*) ===> *R2*) *Let Let*
by (*force elim*: *rel-funE*)

lemma *id-rsp*:
shows (*R* ===> *R*) *id id*
by *auto*

lemma *id-prs*:
assumes *a*: *Quotient3 R Abs Rep*
shows (*Rep* ---> *Abs*) *id* = *id*
by (*simp add*: *fun-eq-iff Quotient3-abs-rep* [*OF a*])

end

locale *quot-type* =

fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$
and $Abs :: 'a\ set \Rightarrow 'b$
and $Rep :: 'b \Rightarrow 'a\ set$
assumes *equivp*: *part-equivp* R
and *rep-prop*: $\bigwedge y.\ \exists x.\ R\ x\ x \wedge Rep\ y = Collect\ (R\ x)$
and *rep-inverse*: $\bigwedge x.\ Abs\ (Rep\ x) = x$
and *abs-inverse*: $\bigwedge c.\ (\exists x.\ ((R\ x\ x) \wedge (c = Collect\ (R\ x)))) \Longrightarrow (Rep\ (Abs\ c)) = c$
and *rep-inject*: $\bigwedge x\ y.\ (Rep\ x = Rep\ y) = (x = y)$
begin

definition
abs :: $'a \Rightarrow 'b$
where
abs x = *Abs* (*Collect* (*R x*))

definition
rep :: $'b \Rightarrow 'a$
where
rep a = (*SOME x. x* $\in$ *Rep a*)

lemma *some-collect*:
assumes *R r r*
shows *R* (*SOME x. x* $\in$ *Collect* (*R r*)) = *R r*
by *simp* (*metis assms exE-some equivp*[*simplified part-equivp-def*])

lemma *Quotient*: *Quotient3 R abs rep*
unfolding *Quotient3-def abs-def rep-def*
proof (*intro conjI allI*)
fix *a r s*
show *x*: *R* (*SOME x. x* $\in$ *Rep a*) (*SOME x. x* $\in$ *Rep a*) **proof** −
obtain *x* **where** *r*: *R x x* **and** *rep*: *Rep a* = *Collect* (*R x*) **using** *rep-prop*[*of a*] **by** *auto*
have *R* (*SOME x. x* $\in$ *Rep a*) *x* **using** *r rep some-collect* **by** *metis*
then have *R x* (*SOME x. x* $\in$ *Rep a*) **using** *part-equivp-symp*[*OF equivp*] **by** *fast*
then show *R* (*SOME x. x* $\in$ *Rep a*) (*SOME x. x* $\in$ *Rep a*)
using *part-equivp-transp*[*OF equivp*] **by** (*metis* ‹*R* (*SOME x. x* $\in$ *Rep a*) *x*›)
qed
have *Collect* (*R* (*SOME x. x* $\in$ *Rep a*)) = (*Rep a*) **by** (*metis some-collect rep-prop*)
then show *Abs* (*Collect* (*R* (*SOME x. x* $\in$ *Rep a*))) = *a* **using** *rep-inverse* **by** *auto*
have $R\ r\ r \Longrightarrow R\ s\ s \Longrightarrow Abs\ (Collect\ (R\ r)) = Abs\ (Collect\ (R\ s)) \longleftrightarrow R\ r = R\ s$
proof −
assume *R r r* **and** *R s s*
then have $Abs\ (Collect\ (R\ r)) = Abs\ (Collect\ (R\ s)) \longleftrightarrow Collect\ (R\ r) = Collect\ (R\ s)$

by (*metis abs-inverse*)
also have *Collect* (*R r*) = *Collect* (*R s*) ⟷ (λ*A x*. *x* ∈ *A*) (*Collect* (*R r*)) = (λ*A x*. *x* ∈ *A*) (*Collect* (*R s*))
by (*rule iffI*) *simp-all*
finally show *Abs* (*Collect* (*R r*)) = *Abs* (*Collect* (*R s*)) ⟷ *R r* = *R s* **by** *simp*
qed
then show *R r s* ⟷ *R r r* ∧ *R s s* ∧ (*Abs* (*Collect* (*R r*)) = *Abs* (*Collect* (*R s*)))
using *equivp*[*simplified part-equivp-def*] **by** *metis*
qed

end

44.6 Quotient composition

lemma *OOO-quotient3*:
fixes *R1* :: ′*a* ⇒ ′*a* ⇒ *bool*
fixes *Abs1* :: ′*a* ⇒ ′*b* **and** *Rep1* :: ′*b* ⇒ ′*a*
fixes *Abs2* :: ′*b* ⇒ ′*c* **and** *Rep2* :: ′*c* ⇒ ′*b*
fixes *R2*′ :: ′*a* ⇒ ′*a* ⇒ *bool*
fixes *R2* :: ′*b* ⇒ ′*b* ⇒ *bool*
assumes *R1*: *Quotient3 R1 Abs1 Rep1*
assumes *R2*: *Quotient3 R2 Abs2 Rep2*
assumes *Abs1*: ⋀*x y*. *R2*′ *x y* ⟹ *R1 x x* ⟹ *R1 y y* ⟹ *R2* (*Abs1 x*) (*Abs1 y*)
assumes *Rep1*: ⋀*x y*. *R2 x y* ⟹ *R2*′ (*Rep1 x*) (*Rep1 y*)
shows *Quotient3* (*R1 OO R2*′ *OO R1*) (*Abs2* ∘ *Abs1*) (*Rep1* ∘ *Rep2*)
proof −
have ∗: (*R1 OOO R2*′) *r r* ∧ (*R1 OOO R2*′) *s s* ∧ (*Abs2* ∘ *Abs1*) *r* = (*Abs2* ∘ *Abs1*) *s*
⟷ (*R1 OOO R2*′) *r s* **for** *r s*
proof (*intro iffI conjI*; *clarify*)
show (*R1 OOO R2*′) *r s*
if *r*: *R1 r a R2*′ *a b R1 b r* **and** *eq*: (*Abs2* ∘ *Abs1*) *r* = (*Abs2* ∘ *Abs1*) *s*
and *s*: *R1 s c R2*′ *c d R1 d s* **for** *a b c d*
proof −
have *R1 r* (*Rep1* (*Abs1 r*))
using *r Quotient3-refl1 R1 rep-abs-rsp* **by** *fastforce*
moreover have *R2*′ (*Rep1* (*Abs1 r*)) (*Rep1* (*Abs1 s*))
using *that*
by *simp* (*metis* (*full-types*) *Rep1 Abs1 Quotient3-rel R2 Quotient3-refl1* [*OF R1*]
Quotient3-refl2 [*OF R1*] *Quotient3-rel-abs* [*OF R1*])
moreover have *R1* (*Rep1* (*Abs1 s*)) *s*
by (*metis s Quotient3-rel R1 rep-abs-rsp-left*)
ultimately show *?thesis*
by (*metis relcomppI*)
qed
next

fix *x y*
assume *xy*: *R1 r x R2′ x y R1 y s*
then have *R2* (*Abs1 x*) (*Abs1 y*)
by (*iprover dest*: *Abs1 elim*: *Quotient3-refl1* [*OF R1*] *Quotient3-refl2* [*OF R1*])
then have *R2′* (*Rep1* (*Abs1 x*)) (*Rep1* (*Abs1 x*)) *R2′* (*Rep1* (*Abs1 y*)) (*Rep1* (*Abs1 y*))
by (*simp-all add*: *Quotient3-refl1* [*OF R2*] *Quotient3-refl2* [*OF R2*] *Rep1*)
with ‹*R1 r x*› ‹*R1 y s*› **show** (*R1 OOO R2′*) *r r* (*R1 OOO R2′*) *s s*
by (*metis* (*full-types*) *Quotient3-def R1 relcompp.relcompI*)+
show (*Abs2* ∘ *Abs1*) *r* = (*Abs2* ∘ *Abs1*) *s*
using *xy* **by** *simp* (*metis* (*full-types*) *Abs1 Quotient3-rel R1 R2*)
qed
show *?thesis*
apply (*rule Quotient3I*)
using ∗ **apply** (*simp-all add*: *o-def Quotient3-abs-rep* [*OF R2*] *Quotient3-abs-rep* [*OF R1*])
apply (*metis Quotient3-rep-reflp R1 R2 Rep1 relcompp.relcompI*)
done
qed

lemma *OOO-eq-quotient3*:
fixes *R1* :: ′*a* ⇒ ′*a* ⇒ *bool*
fixes *Abs1* :: ′*a* ⇒ ′*b* **and** *Rep1* :: ′*b* ⇒ ′*a*
fixes *Abs2* :: ′*b* ⇒ ′*c* **and** *Rep2* :: ′*c* ⇒ ′*b*
assumes *R1*: *Quotient3 R1 Abs1 Rep1*
assumes *R2*: *Quotient3* (=) *Abs2 Rep2*
shows *Quotient3* (*R1 OOO* (=)) (*Abs2* ∘ *Abs1*) (*Rep1* ∘ *Rep2*)
using *assms*
by (*rule OOO-quotient3*) *auto*

44.7 Quotient3 to Quotient

lemma *Quotient3-to-Quotient*:
assumes *Quotient3 R Abs Rep*
and *T* ≡ λ*x y*. *R x x* ∧ *Abs x* = *y*
shows *Quotient R Abs Rep T*
using *assms* **unfolding** *Quotient3-def* **by** (*intro QuotientI*) *blast*+

lemma *Quotient3-to-Quotient-equivp*:
assumes *q*: *Quotient3 R Abs Rep*
and *T-def*: *T* ≡ λ*x y*. *Abs x* = *y*
and *eR*: *equivp R*
shows *Quotient R Abs Rep T*
proof (*intro QuotientI*)
show *Abs* (*Rep a*) = *a* **for** *a*
using *q* **by**(*rule Quotient3-abs-rep*)
show *R* (*Rep a*) (*Rep a*) **for** *a*
using *q* **by**(*rule Quotient3-rep-reflp*)

show *R r s* = (*R r r* ∧ *R s s* ∧ *Abs r* = *Abs s*) **for** *r s*
using *q* **by**(*rule Quotient3-rel*[*symmetric*])
show *T* = (λ*x y*. *R x x* ∧ *Abs x* = *y*)
using *T-def equivp-reflp*[*OF eR*] **by** *simp*
qed

44.8 ML setup

Auxiliary data for the quotient package

named-theorems *quot-equiv equivalence relation theorems*
and *quot-respect respectfulness theorems*
and *quot-preserve preservation theorems*
and *id-simps identity simp rules for maps*
and *quot-thm quotient theorems*
ML-file ‹*Tools/Quotient/quotient-info.ML*›

declare [[*mapQ3 fun* = (*rel-fun*, *fun-quotient3*)]]

lemmas [*quot-thm*] = *fun-quotient3*
lemmas [*quot-respect*] = *quot-rel-rsp if-rsp o-rsp let-rsp id-rsp*
lemmas [*quot-preserve*] = *if-prs o-prs let-prs id-prs*
lemmas [*quot-equiv*] = *identity-equivp*

Lemmas about simplifying id's.

lemmas [*id-simps*] =
id-def[*symmetric*]
map-fun-id
id-apply
id-o
o-id
eq-comp-r
vimage-id

Translation functions for the lifting process.

ML-file ‹*Tools/Quotient/quotient-term.ML*›

Definitions of the quotient types.

ML-file ‹*Tools/Quotient/quotient-type.ML*›

Definitions for quotient constants.

ML-file ‹*Tools/Quotient/quotient-def.ML*›

An auxiliary constant for recording some information about the lifted theorem in a tactic.

definition
Quot-True :: ′*a* ⇒ *bool*
where

Quot-True x ⟷ *True*

lemma
shows *QT-all*: *Quot-True* (*All P*) ⟹ *Quot-True P*
and *QT-ex*: *Quot-True* (*Ex P*) ⟹ *Quot-True P*
and *QT-ex1*: *Quot-True* (*Ex1 P*) ⟹ *Quot-True P*
and *QT-lam*: *Quot-True* (λ*x*. *P x*) ⟹ (⋀*x*. *Quot-True* (*P x*))
and *QT-ext*: (⋀*x*. *Quot-True* (*a x*) ⟹ *f x* = *g x*) ⟹ (*Quot-True a* ⟹ *f* = *g*)
by (*simp-all add*: *Quot-True-def ext*)

lemma *QT-imp*: *Quot-True a* ≡ *Quot-True b*
by (*simp add*: *Quot-True-def*)

context includes *lifting-syntax*
begin

Tactics for proving the lifted theorems

ML-file ‹*Tools/Quotient/quotient-tacs.ML*›

end

44.9 Methods / Interface

method-setup *lifting* =
‹*Attrib.thms* >> (*fn thms* => *fn ctxt* =>
SIMPLE-METHOD′ (*Quotient-Tacs.lift-tac ctxt* [] *thms*))›
‹*lift theorems to quotient types*›

method-setup *lifting-setup* =
‹*Attrib.thm* >> (*fn thm* => *fn ctxt* =>
SIMPLE-METHOD′ (*Quotient-Tacs.lift-procedure-tac ctxt* [] *thm*))›
‹*set up the three goals for the quotient lifting procedure*›

method-setup *descending* =
‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Quotient-Tacs.descend-tac ctxt* []))›
‹*decend theorems to the raw level*›

method-setup *descending-setup* =
‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Quotient-Tacs.descend-procedure-tac ctxt* []))›
‹*set up the three goals for the decending theorems*›

method-setup *partiality-descending* =
‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Quotient-Tacs.partiality-descend-tac ctxt* []))›
‹*decend theorems to the raw level*›

method-setup *partiality-descending-setup* =
‹*Scan.succeed* (*fn ctxt* =>
SIMPLE-METHOD′ (*Quotient-Tacs.partiality-descend-procedure-tac ctxt* []))›
‹*set up the three goals for the decending theorems*›

method-setup *regularize* =
‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Quotient-Tacs.regularize-tac ctxt*))›
‹*prove the regularization goals from the quotient lifting procedure*›

method-setup *injection* =
‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Quotient-Tacs.all-injection-tac ctxt*))›
‹*prove the rep/abs injection goals from the quotient lifting procedure*›

method-setup *cleaning* =
‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Quotient-Tacs.clean-tac ctxt*))›
‹*prove the cleaning goals from the quotient lifting procedure*›

attribute-setup *quot-lifted* =
‹*Scan.succeed Quotient-Tacs.lifted-attrib*›
‹*lift theorems to quotient types*›

no-notation *rel-conj* (**infixr** ‹*OOO*› *75*)

45 Lifting of BNFs

lemma *sum-insert-Inl-unit*: $x \in A \Longrightarrow (\bigwedge y.\ x = Inr\ y \Longrightarrow Inr\ y \in B) \Longrightarrow x \in insert\ (Inl\ ())\ B$
by (*cases x*) (*simp-all*)

lemma *lift-sum-unit-vimage-commute*:
$insert\ (Inl\ ())\ (Inr\ `\ f\ -`\ A) = map\text{-}sum\ id\ f\ -`\ insert\ (Inl\ ())\ (Inr\ `\ A)$
by (*auto simp*: *map-sum-def split*: *sum.splits*)

lemma *insert-Inl-int-map-sum-unit*: $insert\ (Inl\ ())\ A \cap range\ (map\text{-}sum\ id\ f) \neq \{\}$
by (*auto simp*: *map-sum-def split*: *sum.splits*)

lemma *image-map-sum-unit-subset*:
$A \subseteq insert\ (Inl\ ())\ (Inr\ `\ B) \Longrightarrow map\text{-}sum\ id\ f\ `\ A \subseteq insert\ (Inl\ ())\ (Inr\ `\ f\ `\ B)$
by *auto*

lemma *subset-lift-sum-unitD*: $A \subseteq insert\ (Inl\ ())\ (Inr\ `\ B) \Longrightarrow Inr\ x \in A \Longrightarrow x \in B$
unfolding *insert-def* **by** *auto*

lemma *UNIV-sum-unit-conv*: $insert\ (Inl\ ())\ (range\ Inr) = UNIV$
unfolding *UNIV-sum UNIV-unit image-insert image-empty Un-insert-left sup-bot.left-neutral*..

lemma *subset-vimage-image-subset*: $A \subseteq f -` B \Longrightarrow f ` A \subseteq B$
by *auto*

lemma *relcompp-mem-Grp-neq-bot*:
$A \cap range\ f \neq \{\} \Longrightarrow (\lambda x\ y.\ x \in A \wedge y \in A)\ OO\ (Grp\ UNIV\ f)^{-1-1} \neq bot$
unfolding *Grp-def relcompp-apply fun-eq-iff* **by** *blast*

lemma *comp-projr-Inr*: $projr \circ Inr = id$
by *auto*

lemma *in-rel-sum-in-image-projr*:
$B \subseteq \{(x,y).\ rel\text{-}sum\ ((=) :: unit \Rightarrow unit \Rightarrow bool)\ A\ x\ y\} \Longrightarrow$
$Inr ` C = fst ` B \Longrightarrow snd ` B = Inr ` D \Longrightarrow map\text{-}prod\ projr\ projr ` B \subseteq \{(x,y).\ A\ x\ y\}$
by (*force simp*: *projr-def image-iff dest*!: *spec*[*of* - *Inl* ()] *split*: *sum.splits*)

lemma *subset-rel-sumI*: $B \subseteq \{(x,y).\ A\ x\ y\} \Longrightarrow rel\text{-}sum\ ((=) :: unit => unit => bool)\ A$
(*if* $x \in B$ *then Inr* (*fst x*) *else Inl* ())
(*if* $x \in B$ *then Inr* (*snd x*) *else Inl* ())
by *auto*

lemma *relcompp-eq-Grp-neq-bot*: $(=)\ OO\ (Grp\ UNIV\ f)^{-1-1} \neq bot$
unfolding *Grp-def relcompp-apply fun-eq-iff* **by** *blast*

lemma *rel-fun-rel-OO1*: $(rel\text{-}fun\ Q\ (rel\text{-}fun\ R\ (=)))\ A\ B \Longrightarrow conversep\ Q\ OO\ A\ OO\ R \leq B$
by (*auto simp*: *rel-fun-def*)

lemma *rel-fun-rel-OO2*: $(rel\text{-}fun\ Q\ (rel\text{-}fun\ R\ (=)))\ A\ B \Longrightarrow Q\ OO\ B\ OO\ conversep\ R \leq A$
by (*auto simp*: *rel-fun-def*)

lemma *rel-sum-eq2-nonempty*: $rel\text{-}sum\ (=)\ A\ OO\ rel\text{-}sum\ (=)\ B \neq bot$
by (*auto simp*: *fun-eq-iff relcompp-apply intro*!: *exI*[*of* - *Inl* -])

lemma *rel-sum-eq3-nonempty*: $rel\text{-}sum\ (=)\ A\ OO\ (rel\text{-}sum\ (=)\ B\ OO\ rel\text{-}sum\ (=)\ C) \neq bot$
by (*auto simp*: *fun-eq-iff relcompp-apply intro*!: *exI*[*of* - *Inl* -])

lemma *hypsubst*: $A = B \Longrightarrow x \in B \Longrightarrow (x \in A \Longrightarrow P) \Longrightarrow P$ **by** *simp*

lemma *Quotient-crel-quotient*: $Quotient\ R\ Abs\ Rep\ T \Longrightarrow equivp\ R \Longrightarrow T \equiv (\lambda x\ y.\ Abs\ x = y)$
by (*drule Quotient-cr-rel*) (*auto simp*: *fun-eq-iff equivp-reflp intro*!: *eq-reflection*)

lemma *Quotient-crel-typedef*: $Quotient\ (eq\text{-}onp\ P)\ Abs\ Rep\ T \Longrightarrow T \equiv (\lambda x\ y.\ x = Rep\ y)$

unfolding *Quotient-def*
by (*auto 0 4 simp*: *fun-eq-iff eq-onp-def intro*: *sym intro*!: *eq-reflection*)

lemma *Quotient-crel-typecopy*: *Quotient* (=) *Abs Rep T* $\Longrightarrow$ *T* $\equiv$ ($\lambda x\ y.\ x = Rep\ y$)
by (*subst* (*asm*) *eq-onp-True*[*symmetric*]) (*rule Quotient-crel-typedef*)

lemma *equivp-add-relconj*:
assumes *equiv*: *equivp R equivp R′* **and** *le*: *S OO T OO U* $\le$ *R OO STU OO R′*
shows *R OO S OO T OO U OO R′* $\le$ *R OO STU OO R′*
proof −
have *trans*: *R OO R* $\le$ *R R′ OO R′* $\le$ *R′*
using *equiv* **unfolding** *equivp-reflp-symp-transp transp-relcompp* **by** *blast*+
have *R OO S OO T OO U OO R′* = *R OO* (*S OO T OO U*) *OO R′*
unfolding *relcompp-assoc* **..**
also have . . . $\le$ *R OO* (*R OO STU OO R′*) *OO R′*
by (*intro le relcompp-mono order-refl*)
also have . . . $\le$ (*R OO R*) *OO STU OO* (*R′ OO R′*)
unfolding *relcompp-assoc* **..**
also have . . . $\le$ *R OO STU OO R′*
by (*intro trans relcompp-mono order-refl*)
finally show *?thesis* **.**
qed

lemma *Grp-conversep-eq-onp*: ((*BNF-Def.Grp UNIV f*)$^{-1-1}$ *OO BNF-Def.Grp UNIV f*) = *eq-onp* ($\lambda x.\ x \in range\ f$)
by (*auto simp*: *fun-eq-iff Grp-def eq-onp-def image-iff*)

lemma *Grp-conversep-nonempty*: (*BNF-Def.Grp UNIV f*)$^{-1-1}$ *OO BNF-Def.Grp UNIV f* $\neq$ *bot*
by (*auto simp*: *fun-eq-iff Grp-def*)

lemma *relcomppI2*: *r a b* $\Longrightarrow$ *s b c* $\Longrightarrow$ *t c d* $\Longrightarrow$ (*r OO s OO t*) *a d*
by (*auto*)

lemma *rel-conj-eq-onp*: *equivp R* $\Longrightarrow$ *rel-conj R* (*eq-onp P*) $\le$ *R*
by (*auto simp*: *eq-onp-def transp-def equivp-def*)

lemma *Quotient-Quotient3*: *Quotient R Abs Rep T* $\Longrightarrow$ *Quotient3 R Abs Rep*
unfolding *Quotient-def Quotient3-def* **by** *blast*

lemma *Quotient-reflp-imp-equivp*: *Quotient R Abs Rep T* $\Longrightarrow$ *reflp R* $\Longrightarrow$ *equivp R*
using *Quotient-symp Quotient-transp equivpI* **by** *blast*

lemma *Quotient-eq-onp-typedef*:
Quotient (*eq-onp P*) *Abs Rep cr* $\Longrightarrow$ *type-definition Rep Abs* {*x*. *P x*}
unfolding *Quotient-def eq-onp-def*
by *unfold-locales auto*

lemma *Quotient-eq-onp-type-copy*:
Quotient (=) *Abs Rep cr* ⟹ *type-definition Rep Abs UNIV*
unfolding *Quotient-def eq-onp-def*
by *unfold-locales auto*

ML-file ‹*Tools/BNF/bnf-lift.ML*›

hide-fact
sum-insert-Inl-unit lift-sum-unit-vimage-commute insert-Inl-int-map-sum-unit image-map-sum-unit-subset subset-lift-sum-unitD UNIV-sum-unit-conv subset-vimage-image-subset relcompp-mem-Grp-neq-bot comp-projr-Inr in-rel-sum-in-image-projr subset-rel-sumI relcompp-eq-Grp-neq-bot rel-fun-rel-OO1 rel-fun-rel-OO2 rel-sum-eq2-nonempty rel-sum-eq3-nonempty hypsubst equivp-add-relconj Grp-conversep-eq-onp Grp-conversep-nonempty rel-comppI2 rel-conj-eq-onp Quotient-reflp-imp-equivp Quotient-Quotient3

end

46 Binary Numerals

theory *Num*
imports *BNF-Least-Fixpoint Transfer*
begin

46.1 The *num* type

datatype *num* = *One* | *Bit0 num* | *Bit1 num*

Increment function for type *num*

primrec *inc* :: ‹*num* ⇒ *num*›
where
‹*inc One* = *Bit0 One*›
| ‹*inc* (*Bit0 x*) = *Bit1 x*›
| ‹*inc* (*Bit1 x*) = *Bit0* (*inc x*)›

Converting between type *num* and type *nat*

primrec *nat-of-num* :: ‹*num* ⇒ *nat*›
where
‹*nat-of-num One* = *Suc 0*›
| ‹*nat-of-num* (*Bit0 x*) = *nat-of-num x* + *nat-of-num x*›
| ‹*nat-of-num* (*Bit1 x*) = *Suc* (*nat-of-num x* + *nat-of-num x*)›

primrec *num-of-nat* :: ‹*nat* ⇒ *num*›
where
‹*num-of-nat 0* = *One*›
| ‹*num-of-nat* (*Suc n*) = (*if 0* < *n then inc* (*num-of-nat n*) *else One*)›

lemma *nat-of-num-pos*: ‹*0* < *nat-of-num x*›
  by (*induct x*) *simp-all*

lemma *nat-of-num-neq-0*: ‹ *nat-of-num x* ≠ *0*›
  by (*induct x*) *simp-all*

lemma *nat-of-num-inc*: ‹*nat-of-num* (*inc x*) = *Suc* (*nat-of-num x*)›
  by (*induct x*) *simp-all*

lemma *num-of-nat-double*: ‹*0* < *n* ⟹ *num-of-nat* (*n* + *n*) = *Bit0* (*num-of-nat n*)›
  by (*induct n*) *simp-all*

Type *num* is isomorphic to the strictly positive natural numbers.

lemma *nat-of-num-inverse*: ‹*num-of-nat* (*nat-of-num x*) = *x*›
  by (*induct x*) (*simp-all add*: *num-of-nat-double nat-of-num-pos*)

lemma *num-of-nat-inverse*: ‹*0* < *n* ⟹ *nat-of-num* (*num-of-nat n*) = *n*›
  by (*induct n*) (*simp-all add*: *nat-of-num-inc*)

lemma *num-eq-iff*: ‹*x* = *y* ⟷ *nat-of-num x* = *nat-of-num y*›
  apply *safe*
  apply (*drule arg-cong* [**where** *f*=*num-of-nat*])
  apply (*simp add*: *nat-of-num-inverse*)
  done

lemma *num-induct* [*case-names One inc*]:
  fixes *P* :: ‹*num* ⇒ *bool*›
  assumes *One*: ‹*P One*›
    and *inc*: ‹⋀*x*. *P x* ⟹ *P* (*inc x*)›
  shows ‹*P x*›
proof −
  obtain *n* **where** *n*: ‹*Suc n* = *nat-of-num x*›
    by (*cases* ‹*nat-of-num x*›) (*simp-all add*: *nat-of-num-neq-0*)
  have ‹*P* (*num-of-nat* (*Suc n*))›
  proof (*induct n*)
    case *0*
    from *One* **show** *?case* **by** *simp*
  next
    case (*Suc n*)
    then have ‹*P* (*inc* (*num-of-nat* (*Suc n*)))› **by** (*rule inc*)
    then show ‹*P* (*num-of-nat* (*Suc* (*Suc n*)))› **by** *simp*
  qed
  with *n* **show** ‹*P x*›
    by (*simp add*: *nat-of-num-inverse*)
qed

From now on, there are two possible models for *num*: as positive naturals (rule *num-induct*) and as digit representation (rules *num.induct*, *num.cases*).

46.2 Numeral operations

instantiation *num* :: ‹{*plus*,*times*,*linorder*}›
begin

definition [*code del*]: ‹$m + n = \textit{num-of-nat}\ (\textit{nat-of-num}\ m + \textit{nat-of-num}\ n)$›

definition [*code del*]: ‹$m * n = \textit{num-of-nat}\ (\textit{nat-of-num}\ m * \textit{nat-of-num}\ n)$›

definition [*code del*]: ‹$m \leq n \longleftrightarrow \textit{nat-of-num}\ m \leq \textit{nat-of-num}\ n$›

definition [*code del*]: ‹$m < n \longleftrightarrow \textit{nat-of-num}\ m < \textit{nat-of-num}\ n$›

instance
  by *standard* (*auto simp add*: *less-num-def less-eq-num-def num-eq-iff*)

end

lemma *nat-of-num-add*: ‹$\textit{nat-of-num}\ (x + y) = \textit{nat-of-num}\ x + \textit{nat-of-num}\ y$›
  unfolding *plus-num-def*
  by (*intro num-of-nat-inverse add-pos-pos nat-of-num-pos*)

lemma *nat-of-num-mult*: ‹$\textit{nat-of-num}\ (x * y) = \textit{nat-of-num}\ x * \textit{nat-of-num}\ y$›
  unfolding *times-num-def*
  by (*intro num-of-nat-inverse mult-pos-pos nat-of-num-pos*)

lemma *add-num-simps* [*simp*, *code*]:
  ‹*One* + *One* = *Bit0 One*›
  ‹*One* + *Bit0 n* = *Bit1 n*›
  ‹*One* + *Bit1 n* = *Bit0* (*n* + *One*)›
  ‹*Bit0 m* + *One* = *Bit1 m*›
  ‹*Bit0 m* + *Bit0 n* = *Bit0* (*m* + *n*)›
  ‹*Bit0 m* + *Bit1 n* = *Bit1* (*m* + *n*)›
  ‹*Bit1 m* + *One* = *Bit0* (*m* + *One*)›
  ‹*Bit1 m* + *Bit0 n* = *Bit1* (*m* + *n*)›
  ‹*Bit1 m* + *Bit1 n* = *Bit0* (*m* + *n* + *One*)›
  by (*simp-all add*: *num-eq-iff nat-of-num-add*)

lemma *mult-num-simps* [*simp*, *code*]:
  ‹*m* ∗ *One* = *m*›
  ‹*One* ∗ *n* = *n*›
  ‹*Bit0 m* ∗ *Bit0 n* = *Bit0* (*Bit0* (*m* ∗ *n*))›
  ‹*Bit0 m* ∗ *Bit1 n* = *Bit0* (*m* ∗ *Bit1 n*)›
  ‹*Bit1 m* ∗ *Bit0 n* = *Bit0* (*Bit1 m* ∗ *n*)›
  ‹*Bit1 m* ∗ *Bit1 n* = *Bit1* (*m* + *n* + *Bit0* (*m* ∗ *n*))›
  by (*simp-all add*: *num-eq-iff nat-of-num-add nat-of-num-mult distrib-right distrib-left*)

lemma *eq-num-simps*:
‹*One* = *One* ⟷ *True*›
‹*One* = *Bit0* *n* ⟷ *False*›
‹*One* = *Bit1* *n* ⟷ *False*›
‹*Bit0* *m* = *One* ⟷ *False*›
‹*Bit1* *m* = *One* ⟷ *False*›
‹*Bit0* *m* = *Bit0* *n* ⟷ *m* = *n*›
‹*Bit0* *m* = *Bit1* *n* ⟷ *False*›
‹*Bit1* *m* = *Bit0* *n* ⟷ *False*›
‹*Bit1* *m* = *Bit1* *n* ⟷ *m* = *n*›
by *simp-all*

lemma *le-num-simps* [*simp*, *code*]:
‹*One* ≤ *n* ⟷ *True*›
‹*Bit0* *m* ≤ *One* ⟷ *False*›
‹*Bit1* *m* ≤ *One* ⟷ *False*›
‹*Bit0* *m* ≤ *Bit0* *n* ⟷ *m* ≤ *n*›
‹*Bit0* *m* ≤ *Bit1* *n* ⟷ *m* ≤ *n*›
‹*Bit1* *m* ≤ *Bit1* *n* ⟷ *m* ≤ *n*›
‹*Bit1* *m* ≤ *Bit0* *n* ⟷ *m* < *n*›
using *nat-of-num-pos* [*of n*] *nat-of-num-pos* [*of m*]
by (*auto simp add*: *less-eq-num-def less-num-def*)

lemma *less-num-simps* [*simp*, *code*]:
‹*m* < *One* ⟷ *False*›
‹*One* < *Bit0* *n* ⟷ *True*›
‹*One* < *Bit1* *n* ⟷ *True*›
‹*Bit0* *m* < *Bit0* *n* ⟷ *m* < *n*›
‹*Bit0* *m* < *Bit1* *n* ⟷ *m* ≤ *n*›
‹*Bit1* *m* < *Bit1* *n* ⟷ *m* < *n*›
‹*Bit1* *m* < *Bit0* *n* ⟷ *m* < *n*›
using *nat-of-num-pos* [*of n*] *nat-of-num-pos* [*of m*]
by (*auto simp add*: *less-eq-num-def less-num-def*)

lemma *le-num-One-iff*: ‹*x* ≤ *One* ⟷ *x* = *One*›
by (*simp add*: *antisym-conv*)

Rules using *One* and *inc* as constructors.

lemma *add-One*: ‹*x* + *One* = *inc* *x*›
by (*simp add*: *num-eq-iff nat-of-num-add nat-of-num-inc*)

lemma *add-One-commute*: ‹*One* + *n* = *n* + *One*›
by (*induct n*) *simp-all*

lemma *add-inc*: ‹*x* + *inc* *y* = *inc* (*x* + *y*)›
by (*simp add*: *num-eq-iff nat-of-num-add nat-of-num-inc*)

lemma *mult-inc*: ‹*x* ∗ *inc* *y* = *x* ∗ *y* + *x*›

by (*simp add*: *num-eq-iff nat-of-num-mult nat-of-num-add nat-of-num-inc*)

The *num-of-nat* conversion.

lemma *num-of-nat-One*: ‹$n \leq 1 \Longrightarrow$ *num-of-nat n* = *One*›
by (*cases n*) *simp-all*

lemma *num-of-nat-plus-distrib*:
‹$0 < m \Longrightarrow 0 < n \Longrightarrow$ *num-of-nat* $(m + n)$ = *num-of-nat m* + *num-of-nat n*›
by (*induct n*) (*auto simp add*: *add-One add-One-commute add-inc*)

A double-and-decrement function.

primrec *BitM* :: ‹*num* ⇒ *num*›
where
‹*BitM One* = *One*›
| ‹*BitM* (*Bit0 n*) = *Bit1* (*BitM n*)›
| ‹*BitM* (*Bit1 n*) = *Bit1* (*Bit0 n*)›

lemma *BitM-plus-one*: ‹*BitM n* + *One* = *Bit0 n*›
by (*induct n*) *simp-all*

lemma *one-plus-BitM*: ‹*One* + *BitM n* = *Bit0 n*›
unfolding *add-One-commute BitM-plus-one* **..**

lemma *BitM-inc-eq*:
‹*BitM* (*inc n*) = *Bit1 n*›
by (*induction n*) *simp-all*

lemma *inc-BitM-eq*:
‹*inc* (*BitM n*) = *Bit0 n*›
by (*simp add*: *BitM-plus-one*[*symmetric*] *add-One*)

Squaring and exponentiation.

primrec *sqr* :: ‹*num* ⇒ *num*›
where
‹*sqr One* = *One*›
| ‹*sqr* (*Bit0 n*) = *Bit0* (*Bit0* (*sqr n*))›
| ‹*sqr* (*Bit1 n*) = *Bit1* (*Bit0* (*sqr n* + *n*))›

primrec *pow* :: ‹*num* ⇒ *num* ⇒ *num*›
where
‹*pow x One* = *x*›
| ‹*pow x* (*Bit0 y*) = *sqr* (*pow x y*)›
| ‹*pow x* (*Bit1 y*) = *sqr* (*pow x y*) ∗ *x*›

lemma *nat-of-num-sqr*: ‹*nat-of-num* (*sqr x*) = *nat-of-num x* ∗ *nat-of-num x*›
by (*induct x*) (*simp-all add*: *algebra-simps nat-of-num-add*)

lemma *sqr-conv-mult*: ‹*sqr x* = *x* ∗ *x*›
by (*simp add*: *num-eq-iff nat-of-num-sqr nat-of-num-mult*)

```
lemma num-double [simp]:
  ‹Bit0 num.One * n = Bit0 n›
  by (simp add: num-eq-iff nat-of-num-mult)
```

46.3 Binary numerals

We embed binary representations into a generic algebraic structure using *numeral*.

```
class numeral = one + semigroup-add
begin

primrec numeral :: ‹num ⇒ 'a›
  where
    numeral-One: ‹numeral One = 1›
  | numeral-Bit0: ‹numeral (Bit0 n) = numeral n + numeral n›
  | numeral-Bit1: ‹numeral (Bit1 n) = numeral n + numeral n + 1›

lemma numeral-code [code]:
  ‹numeral One = 1›
  ‹numeral (Bit0 n) = (let m = numeral n in m + m)›
  ‹numeral (Bit1 n) = (let m = numeral n in m + m + 1)›
  by (simp-all add: Let-def)

lemma one-plus-numeral-commute: ‹1 + numeral x = numeral x + 1›
proof (induct x)
  case One
  then show ?case by simp
next
  case Bit0
  then show ?case by (simp add: add.assoc [symmetric]) (simp add: add.assoc)
next
  case Bit1
  then show ?case by (simp add: add.assoc [symmetric]) (simp add: add.assoc)
qed

lemma numeral-inc: ‹numeral (inc x) = numeral x + 1›
proof (induct x)
  case One
  then show ?case by simp
next
  case Bit0
  then show ?case by simp
next
  case (Bit1 x)
  have ‹numeral x + (1 + numeral x) + 1 = numeral x + (numeral x + 1) + 1›
    by (simp only: one-plus-numeral-commute)
  with Bit1 show ?case
    by (simp add: add.assoc)
```

qed

declare *numeral.simps* [*simp del*]

abbreviation ‹*Numeral1* ≡ *numeral One*›

declare *numeral-One* [*code-post*]

end

Numeral syntax.

syntax
  -Numeral :: ‹*num-const* ⇒ *'a*› (‹(‹*open-block notation*=‹*literal number*››-)›)

ML-file ‹*Tools/numeral.ML*›

parse-translation ‹

```
let
  fun numeral-tr [(c as Const (syntax-const ‹-constrain›, -)) $ t $ u] =
        c $ numeral-tr [t] $ u
    | numeral-tr [Const (num, -)] =
        (Numeral.mk-number-syntax o #value o Lexicon.read-num) num
    | numeral-tr ts = raise TERM (numeral-tr, ts);
in [(syntax-const ‹-Numeral›, K numeral-tr)] end
```

›

typed-print-translation ‹

```
let
  fun num-tr' ctxt T [n] =
    let
      val k = Numeral.dest-num-syntax n;
      val t' =
        Syntax.const syntax-const ‹-Numeral› $
          Syntax.free (string-of-int k);
    in
      (case T of
        Type (type-name ‹fun›, [-, T']) =>
          if Printer.type-emphasis ctxt T' then
            Syntax.const syntax-const ‹-constrain› $ t' $
              Syntax-Phases.term-of-typ ctxt T'
          else t'
      | - => if T = dummyT then t' else raise Match)
    end;
in
  [(const-syntax ‹numeral›, num-tr')]
end
```

›

46.4 Class-specific numeral rules

numeral is a morphism.

46.4.1 Structures with addition: class *numeral*

context *numeral*
begin

lemma *numeral-add*: ‹*numeral* (*m* + *n*) = *numeral m* + *numeral n*›
by (*induct n rule*: *num-induct*)
(*simp-all only*: *numeral-One add-One add-inc numeral-inc add.assoc*)

lemma *numeral-plus-numeral*: ‹*numeral m* + *numeral n* = *numeral* (*m* + *n*)›
by (*rule numeral-add* [*symmetric*])

lemma *numeral-plus-one*: ‹*numeral n* + *1* = *numeral* (*n* + *One*)›
using *numeral-add* [*of n One*] **by** (*simp add*: *numeral-One*)

lemma *one-plus-numeral*: ‹*1* + *numeral n* = *numeral* (*One* + *n*)›
using *numeral-add* [*of One n*] **by** (*simp add*: *numeral-One*)

lemma *one-add-one*: ‹*1* + *1* = *2*›
using *numeral-add* [*of One One*] **by** (*simp add*: *numeral-One*)

lemmas *add-numeral-special* =
numeral-plus-one one-plus-numeral one-add-one

end

46.4.2 Structures with negation: class *neg-numeral*

class *neg-numeral* = *numeral* + *group-add*
begin

lemma *uminus-numeral-One*: ‹− *Numeral1* = − *1*›
by (*simp add*: *numeral-One*)

Numerals form an abelian subgroup.

inductive *is-num* :: ‹*'a* ⇒ *bool*›
where
‹*is-num 1*›
| ‹*is-num x* ⟹ *is-num* (− *x*)›
| ‹*is-num x* ⟹ *is-num y* ⟹ *is-num* (*x* + *y*)›

lemma *is-num-numeral*: ‹*is-num* (*numeral k*)›
by (*induct k*) (*simp-all add*: *numeral.simps is-num.intros*)

lemma *is-num-add-commute*: ‹*is-num x* ⟹ *is-num y* ⟹ *x* + *y* = *y* + *x*›

```
proof(induction x rule: is-num.induct)
  case 1
  then show ?case
  proof (induction y rule: is-num.induct)
    case 1
    then show ?case by simp
  next
    case (2 y)
    then have ‹y + (1 + − y) + y = y + (− y + 1) + y›
      by (simp add: add.assoc)
    then have ‹y + (1 + − y) = y + (− y + 1)›
      by simp
    then show ?case
      by (rule add-left-imp-eq[of y])
  next
    case (3 x y)
    then have ‹1 + (x + y) = x + 1 + y›
      by (simp add: add.assoc [symmetric])
    then show ?case using 3
      by (simp add: add.assoc)
  qed
next
  case (2 x)
  then have ‹x + (− x + y) + x = x + (y + − x) + x›
    by (simp add: add.assoc)
  then have ‹x + (− x + y) = x + (y + − x)›
    by simp
  then show ?case
    by (rule add-left-imp-eq[of x])
next
  case (3 x z)
  moreover have ‹x + (y + z) = (x + y) + z›
    by (simp add: add.assoc[symmetric])
  ultimately show ?case
    by (simp add: add.assoc)
qed

lemma is-num-add-left-commute: ‹is-num x ⟹ is-num y ⟹ x + (y + z) = y
+ (x + z)›
  by (simp only: add.assoc [symmetric] is-num-add-commute)

lemmas is-num-normalize =
  add.assoc is-num-add-commute is-num-add-left-commute
  is-num.intros is-num-numeral
  minus-add

definition dbl :: ‹'a ⇒ 'a›
  where ‹dbl x = x + x›
```

definition *dbl-inc* :: ‹′a ⇒ ′a›
where ‹*dbl-inc x* = *x* + *x* + *1*›

definition *dbl-dec* :: ‹′a ⇒ ′a›
where ‹*dbl-dec x* = *x* + *x* − *1*›

definition *sub* :: ‹*num* ⇒ *num* ⇒ ′a›
where ‹*sub k l* = *numeral k* − *numeral l*›

lemma *numeral-BitM*: ‹*numeral* (*BitM n*) = *numeral* (*Bit0 n*) − *1*›
by (*simp only*: *BitM-plus-one* [*symmetric*] *numeral-add numeral-One eq-diff-eq*)

lemma *sub-inc-One-eq*:
‹*sub* (*inc n*) *num.One* = *numeral n*›
by (*simp-all add*: *sub-def diff-eq-eq numeral-inc numeral.numeral-One*)

lemma *dbl-simps* [*simp*]:
‹*dbl* (− *numeral k*) = − *dbl* (*numeral k*)›
‹*dbl 0* = *0*›
‹*dbl 1* = *2*›
‹*dbl* (− *1*) = − *2*›
‹*dbl* (*numeral k*) = *numeral* (*Bit0 k*)›
by (*simp-all add*: *dbl-def numeral.simps minus-add*)

lemma *dbl-inc-simps* [*simp*]:
‹*dbl-inc* (− *numeral k*) = − *dbl-dec* (*numeral k*)›
‹*dbl-inc 0* = *1*›
‹*dbl-inc 1* = *3*›
‹*dbl-inc* (− *1*) = − *1*›
‹*dbl-inc* (*numeral k*) = *numeral* (*Bit1 k*)›
by (*simp-all add*: *dbl-inc-def dbl-dec-def numeral.simps numeral-BitM is-num-normalize algebra-simps*
del: *add-uminus-conv-diff*)

lemma *dbl-dec-simps* [*simp*]:
‹*dbl-dec* (− *numeral k*) = − *dbl-inc* (*numeral k*)›
‹*dbl-dec 0* = − *1*›
‹*dbl-dec 1* = *1*›
‹*dbl-dec* (− *1*) = − *3*›
‹*dbl-dec* (*numeral k*) = *numeral* (*BitM k*)›
by (*simp-all add*: *dbl-dec-def dbl-inc-def numeral.simps numeral-BitM is-num-normalize*)

lemma *sub-num-simps* [*simp*]:
‹*sub One One* = *0*›
‹*sub One* (*Bit0 l*) = − *numeral* (*BitM l*)›
‹*sub One* (*Bit1 l*) = − *numeral* (*Bit0 l*)›
‹*sub* (*Bit0 k*) *One* = *numeral* (*BitM k*)›
‹*sub* (*Bit1 k*) *One* = *numeral* (*Bit0 k*)›
‹*sub* (*Bit0 k*) (*Bit0 l*) = *dbl* (*sub k l*)›

‹*sub* (*Bit0 k*) (*Bit1 l*) = *dbl-dec* (*sub k l*)›
‹*sub* (*Bit1 k*) (*Bit0 l*) = *dbl-inc* (*sub k l*)›
‹*sub* (*Bit1 k*) (*Bit1 l*) = *dbl* (*sub k l*)›
by (*simp-all add*: *dbl-def dbl-dec-def dbl-inc-def sub-def numeral.simps numeral-BitM is-num-normalize del*: *add-uminus-conv-diff add*: *diff-conv-add-uminus*)

lemma *add-neg-numeral-simps*:
‹*numeral m* + − *numeral n* = *sub m n*›
‹− *numeral m* + *numeral n* = *sub n m*›
‹− *numeral m* + − *numeral n* = − (*numeral m* + *numeral n*)›
by (*simp-all add*: *sub-def numeral-add numeral.simps is-num-normalize del*: *add-uminus-conv-diff add*: *diff-conv-add-uminus*)

lemma *add-neg-numeral-special*:
‹*1* + − *numeral m* = *sub One m*›
‹− *numeral m* + *1* = *sub One m*›
‹*numeral m* + − *1* = *sub m One*›
‹− *1* + *numeral n* = *sub n One*›
‹− *1* + − *numeral n* = − *numeral* (*inc n*)›
‹− *numeral m* + − *1* = − *numeral* (*inc m*)›
‹*1* + − *1* = *0*›
‹− *1* + *1* = *0*›
‹− *1* + − *1* = − *2*›
by (*simp-all add*: *sub-def numeral-add numeral.simps is-num-normalize right-minus numeral-inc*
del: *add-uminus-conv-diff add*: *diff-conv-add-uminus*)

lemma *diff-numeral-simps*:
‹*numeral m* − *numeral n* = *sub m n*›
‹*numeral m* − − *numeral n* = *numeral* (*m* + *n*)›
‹− *numeral m* − *numeral n* = − *numeral* (*m* + *n*)›
‹− *numeral m* − − *numeral n* = *sub n m*›
by (*simp-all add*: *sub-def numeral-add numeral.simps is-num-normalize del*: *add-uminus-conv-diff add*: *diff-conv-add-uminus*)

lemma *diff-numeral-special*:
‹*1* − *numeral n* = *sub One n*›
‹*numeral m* − *1* = *sub m One*›
‹*1* − − *numeral n* = *numeral* (*One* + *n*)›
‹− *numeral m* − *1* = − *numeral* (*m* + *One*)›
‹− *1* − *numeral n* = − *numeral* (*inc n*)›
‹*numeral m* − − *1* = *numeral* (*inc m*)›
‹− *1* − − *numeral n* = *sub n One*›
‹− *numeral m* − − *1* = *sub One m*›
‹*1* − *1* = *0*›
‹− *1* − *1* = − *2*›
‹*1* − − *1* = *2*›
‹− *1* − − *1* = *0*›
by (*simp-all add*: *sub-def numeral-add numeral.simps is-num-normalize numeral-inc*

del: *add-uminus-conv-diff* *add*: *diff-conv-add-uminus*)

end

46.4.3 Structures with multiplication: class *semiring-numeral*

class *semiring-numeral* = *semiring* + *monoid-mult*
begin

subclass *numeral* **..**

lemma *numeral-mult*: ‹*numeral* (*m* ∗ *n*) = *numeral* *m* ∗ *numeral* *n*›
by (*induct* *n* *rule*: *num-induct*)
(*simp-all* *add*: *numeral-One* *mult-inc* *numeral-inc* *numeral-add* *distrib-left*)

lemma *numeral-times-numeral*: ‹*numeral* *m* ∗ *numeral* *n* = *numeral* (*m* ∗ *n*)›
by (*rule* *numeral-mult* [*symmetric*])

lemma *mult-2*: ‹*2* ∗ *z* = *z* + *z*›
by (*simp* *add*: *one-add-one* [*symmetric*] *distrib-right*)

lemma *mult-2-right*: ‹*z* ∗ *2* = *z* + *z*›
by (*simp* *add*: *one-add-one* [*symmetric*] *distrib-left*)

lemma *left-add-twice*:
‹*a* + (*a* + *b*) = *2* ∗ *a* + *b*›
by (*simp* *add*: *mult-2* *ac-simps*)

lemma *numeral-Bit0-eq-double*:
‹*numeral* (*Bit0* *n*) = *2* ∗ *numeral* *n*›
by (*simp* *add*: *mult-2*) (*simp* *add*: *numeral-Bit0*)

lemma *numeral-Bit1-eq-inc-double*:
‹*numeral* (*Bit1* *n*) = *2* ∗ *numeral* *n* + *1*›
by (*simp* *add*: *mult-2*) (*simp* *add*: *numeral-Bit1*)

end

46.4.4 Structures with a zero: class *semiring-1*

context *semiring-1*
begin

subclass *semiring-numeral* **..**

lemma *of-nat-numeral* [*simp*]: ‹*of-nat* (*numeral* *n*) = *numeral* *n*›
by (*induct* *n*) (*simp-all* *only*: *numeral.simps* *numeral-class.numeral.simps* *of-nat-add* *of-nat-1*)

end

lemma *nat-of-num-numeral* [*code-abbrev*]: ‹*nat-of-num* = *numeral*›
proof
  fix *n*
  have ‹*numeral n* = *nat-of-num n*›
    by (*induct n*) (*simp-all add*: *numeral.simps*)
  then show ‹*nat-of-num n* = *numeral n*›
    by *simp*
qed

lemma *nat-of-num-code* [*code*]:
  ‹*nat-of-num One* = *1*›
  ‹*nat-of-num* (*Bit0 n*) = (*let m* = *nat-of-num n in m* + *m*)›
  ‹*nat-of-num* (*Bit1 n*) = (*let m* = *nat-of-num n in Suc* (*m* + *m*))›
  by (*simp-all add*: *Let-def*)

46.4.5 Equality: class *semiring-char-0*

context *semiring-char-0*
begin

lemma *numeral-eq-iff*: ‹*numeral m* = *numeral n* ⟷ *m* = *n*›
  by (*simp only*: *of-nat-numeral* [*symmetric*] *nat-of-num-numeral* [*symmetric*]
    of-nat-eq-iff num-eq-iff)

lemma *numeral-eq-one-iff*: ‹*numeral n* = *1* ⟷ *n* = *One*›
  by (*rule numeral-eq-iff* [*of n One*, *unfolded numeral-One*])

lemma *one-eq-numeral-iff*: ‹*1* = *numeral n* ⟷ *One* = *n*›
  by (*rule numeral-eq-iff* [*of One n*, *unfolded numeral-One*])

lemma *numeral-neq-zero*: ‹*numeral n* ≠ *0*›
  by (*simp add*: *of-nat-numeral* [*symmetric*] *nat-of-num-numeral* [*symmetric*] *nat-of-num-pos*)

lemma *zero-neq-numeral*: ‹*0* ≠ *numeral n*›
  unfolding *eq-commute* [*of 0*] **by** (*rule numeral-neq-zero*)

lemmas *eq-numeral-simps* [*simp*] =
  numeral-eq-iff
  numeral-eq-one-iff
  one-eq-numeral-iff
  numeral-neq-zero
  zero-neq-numeral

end

46.4.6 Comparisons: class *linordered-nonzero-semiring*

context *linordered-nonzero-semiring*
begin

lemma *numeral-le-iff*: ‹*numeral m* ≤ *numeral n* ⟷ *m* ≤ *n*›
proof −
  have ‹*of-nat* (*numeral m*) ≤ *of-nat* (*numeral n*) ⟷ *m* ≤ *n*›
    by (*simp only*: *less-eq-num-def nat-of-num-numeral of-nat-le-iff*)
  then show *?thesis* **by** *simp*
qed

lemma *one-le-numeral*: ‹*1* ≤ *numeral n*›
  using *numeral-le-iff* [*of One n*] **by** (*simp add*: *numeral-One*)

lemma *numeral-le-one-iff*: ‹*numeral n* ≤ *1* ⟷ *n* ≤ *One*›
  using *numeral-le-iff* [*of n One*] **by** (*simp add*: *numeral-One*)

lemma *numeral-less-iff*: ‹*numeral m* < *numeral n* ⟷ *m* < *n*›
proof −
  have ‹*of-nat* (*numeral m*) < *of-nat* (*numeral n*) ⟷ *m* < *n*›
    unfolding *less-num-def nat-of-num-numeral of-nat-less-iff* **..**
  then show *?thesis* **by** *simp*
qed

lemma *not-numeral-less-one*: ‹¬ *numeral n* < *1*›
  using *numeral-less-iff* [*of n One*] **by** (*simp add*: *numeral-One*)

lemma *one-less-numeral-iff*: ‹*1* < *numeral n* ⟷ *One* < *n*›
  using *numeral-less-iff* [*of One n*] **by** (*simp add*: *numeral-One*)

lemma *zero-le-numeral*: ‹*0* ≤ *numeral n*›
  using *dual-order.trans one-le-numeral zero-le-one* **by** *blast*

lemma *zero-less-numeral*: ‹*0* < *numeral n*›
  using *less-linear not-numeral-less-one order.strict-trans zero-less-one* **by** *blast*

lemma *not-numeral-le-zero*: ‹¬ *numeral n* ≤ *0*›
  by (*simp add*: *not-le zero-less-numeral*)

lemma *not-numeral-less-zero*: ‹¬ *numeral n* < *0*›
  by (*simp add*: *not-less zero-le-numeral*)

lemma *one-of-nat-le-iff* [*simp*]: ‹*1* ≤ *of-nat k* ⟷ *1* ≤ *k*›
  using *of-nat-le-iff* [*of 1*] **by** *simp*

lemma *numeral-nat-le-iff* [*simp*]: ‹*numeral n* ≤ *of-nat k* ⟷ *numeral n* ≤ *k*›
  using *of-nat-le-iff* [*of* ‹*numeral n*›] **by** *simp*

lemma *of-nat-le-1-iff* [*simp*]: ‹*of-nat k* ≤ *1* ⟷ *k* ≤ *1*›
  using *of-nat-le-iff* [*of - 1*] **by** *simp*

lemma *of-nat-le-numeral-iff* [*simp*]: ‹*of-nat k* ≤ *numeral n* ⟷ *k* ≤ *numeral n*›

using *of-nat-le-iff* [*of* - ‹*numeral n*›] **by** *simp*

lemma *one-of-nat-less-iff* [*simp*]: ‹*1* < *of-nat k* ⟷ *1* < *k*›
using *of-nat-less-iff* [*of 1*] **by** *simp*

lemma *numeral-nat-less-iff* [*simp*]: ‹*numeral n* < *of-nat k* ⟷ *numeral n* < *k*›
using *of-nat-less-iff* [*of* ‹*numeral n*›] **by** *simp*

lemma *of-nat-less-1-iff* [*simp*]: ‹*of-nat k* < *1* ⟷ *k* < *1*›
using *of-nat-less-iff* [*of* - *1*] **by** *simp*

lemma *of-nat-less-numeral-iff* [*simp*]: ‹*of-nat k* < *numeral n* ⟷ *k* < *numeral n*›
using *of-nat-less-iff* [*of* - ‹*numeral n*›] **by** *simp*

lemma *of-nat-eq-numeral-iff* [*simp*]: ‹*of-nat k* = *numeral n* ⟷ *k* = *numeral n*›
using *of-nat-eq-iff* [*of* - ‹*numeral n*›] **by** *simp*

lemmas *le-numeral-extra* =
zero-le-one not-one-le-zero
order-refl [*of 0*] *order-refl* [*of 1*]

lemmas *less-numeral-extra* =
zero-less-one not-one-less-zero
less-irrefl [*of 0*] *less-irrefl* [*of 1*]

lemmas *le-numeral-simps* [*simp*] =
numeral-le-iff
one-le-numeral
numeral-le-one-iff
zero-le-numeral
not-numeral-le-zero

lemmas *less-numeral-simps* [*simp*] =
numeral-less-iff
one-less-numeral-iff
not-numeral-less-one
zero-less-numeral
not-numeral-less-zero

lemma *min-0-1* [*simp*]:
fixes *min'* :: ‹*'a* ⇒ *'a* ⇒ *'a*›
defines ‹*min'* ≡ *min*›
shows
‹*min' 0 1* = *0*›
‹*min' 1 0* = *0*›
‹*min' 0* (*numeral x*) = *0*›
‹*min'* (*numeral x*) *0* = *0*›
‹*min' 1* (*numeral x*) = *1*›
‹*min'* (*numeral x*) *1* = *1*›

by (*simp-all add*: *min′-def min-def le-num-One-iff*)

lemma *max-0-1* [*simp*]:
fixes *max′* :: ‹′*a* ⇒ ′*a* ⇒ ′*a*›
defines ‹*max′* ≡ *max*›
shows
‹*max′* *0* *1* = *1*›
‹*max′* *1* *0* = *1*›
‹*max′* *0* (*numeral x*) = *numeral x*›
‹*max′* (*numeral x*) *0* = *numeral x*›
‹*max′* *1* (*numeral x*) = *numeral x*›
‹*max′* (*numeral x*) *1* = *numeral x*›
by (*simp-all add*: *max′-def max-def le-num-One-iff*)

end

Unfold *min* and *max* on numerals.

lemmas *max-number-of* [*simp*] =
max-def [*of* ‹*numeral u*› ‹*numeral v*›]
max-def [*of* ‹*numeral u*› ‹− *numeral v*›]
max-def [*of* ‹− *numeral u*› ‹*numeral v*›]
max-def [*of* ‹− *numeral u*› ‹− *numeral v*›] **for** *u v*

lemmas *min-number-of* [*simp*] =
min-def [*of* ‹*numeral u*› ‹*numeral v*›]
min-def [*of* ‹*numeral u*› ‹− *numeral v*›]
min-def [*of* ‹− *numeral u*› ‹*numeral v*›]
min-def [*of* ‹− *numeral u*› ‹− *numeral v*›] **for** *u v*

46.4.7 Multiplication and negation: class *ring-1*

context *ring-1*
begin

subclass *neg-numeral* **..**

lemma *mult-neg-numeral-simps*:
‹− *numeral m* ∗ − *numeral n* = *numeral* (*m* ∗ *n*)›
‹− *numeral m* ∗ *numeral n* = − *numeral* (*m* ∗ *n*)›
‹*numeral m* ∗ − *numeral n* = − *numeral* (*m* ∗ *n*)›
by (*simp-all only*: *mult-minus-left mult-minus-right minus-minus numeral-mult*)

lemma *mult-minus1* [*simp*]: ‹− *1* ∗ *z* = − *z*›
by (*simp add*: *numeral.simps*)

lemma *mult-minus1-right* [*simp*]: ‹*z* ∗ − *1* = − *z*›
by (*simp add*: *numeral.simps*)

lemma *minus-sub-one-diff-one* [*simp*]:

‹− *sub m One* − *1* = − *numeral m*›
proof −
have ‹*sub m One* + *1* = *numeral m*›
by (*simp flip*: *eq-diff-eq add*: *diff-numeral-special*)
then have ‹− (*sub m One* + *1*) = − *numeral m*›
by *simp*
then show *?thesis*
by *simp*
qed

end

46.4.8 Equality using *iszero* for rings with non-zero characteristic

context *ring-1*
begin

definition *iszero* :: ‹′*a* ⇒ *bool*›
where ‹*iszero z* ⟷ *z* = *0*›

lemma *iszero-0* [*simp*]: ‹*iszero 0*›
by (*simp add*: *iszero-def*)

lemma *not-iszero-1* [*simp*]: ‹¬ *iszero 1*›
by (*simp add*: *iszero-def*)

lemma *not-iszero-Numeral1*: ‹¬ *iszero Numeral1*›
by (*simp add*: *numeral-One*)

lemma *not-iszero-neg-1* [*simp*]: ‹¬ *iszero* (− *1*)›
by (*simp add*: *iszero-def*)

lemma *not-iszero-neg-Numeral1*: ‹¬ *iszero* (− *Numeral1*)›
by (*simp add*: *numeral-One*)

lemma *iszero-neg-numeral* [*simp*]: ‹*iszero* (− *numeral w*) ⟷ *iszero* (*numeral w*)›
unfolding *iszero-def* **by** (*rule neg-equal-0-iff-equal*)

lemma *eq-iff-iszero-diff*: ‹*x* = *y* ⟷ *iszero* (*x* − *y*)›
unfolding *iszero-def* **by** (*rule eq-iff-diff-eq-0*)

The *eq-numeral-iff-iszero* lemmas are not declared [*simp*] by default, because for rings of characteristic zero, better simp rules are possible. For a type like integers mod *n*, type-instantiated versions of these rules should be added to the simplifier, along with a type-specific rule for deciding propositions of the form *iszero* (*numeral w*).

bh: Maybe it would not be so bad to just declare these as simp rules anyway? I should test whether these rules take precedence over the *ring-char-0* rules in the simplifier.

lemma *eq-numeral-iff-iszero*:
‹*numeral x = numeral y* ⟷ *iszero* (*sub x y*)›
‹*numeral x* = − *numeral y* ⟷ *iszero* (*numeral* (*x* + *y*))›
‹− *numeral x* = *numeral y* ⟷ *iszero* (*numeral* (*x* + *y*))›
‹− *numeral x* = − *numeral y* ⟷ *iszero* (*sub y x*)›
‹*numeral x* = *1* ⟷ *iszero* (*sub x One*)›
‹*1* = *numeral y* ⟷ *iszero* (*sub One y*)›
‹− *numeral x* = *1* ⟷ *iszero* (*numeral* (*x* + *One*))›
‹*1* = − *numeral y* ⟷ *iszero* (*numeral* (*One* + *y*))›
‹*numeral x* = *0* ⟷ *iszero* (*numeral x*)›
‹*0* = *numeral y* ⟷ *iszero* (*numeral y*)›
‹− *numeral x* = *0* ⟷ *iszero* (*numeral x*)›
‹*0* = − *numeral y* ⟷ *iszero* (*numeral y*)›
unfolding *eq-iff-iszero-diff diff-numeral-simps diff-numeral-special*
by *simp-all*

end

46.4.9 Equality and negation: class *ring-char-0*

context *ring-char-0*
begin

lemma *not-iszero-numeral* [*simp*]: ‹¬ *iszero* (*numeral w*)›
by (*simp add*: *iszero-def*)

lemma *neg-numeral-eq-iff*: ‹− *numeral m* = − *numeral n* ⟷ *m* = *n*›
by *simp*

lemma *numeral-neq-neg-numeral*: ‹*numeral m* ≠ − *numeral n*›
by (*simp add*: *eq-neg-iff-add-eq-0 numeral-plus-numeral*)

lemma *neg-numeral-neq-numeral*: ‹− *numeral m* ≠ *numeral n*›
by (*rule numeral-neq-neg-numeral* [*symmetric*])

lemma *zero-neq-neg-numeral*: ‹*0* ≠ − *numeral n*›
by *simp*

lemma *neg-numeral-neq-zero*: ‹− *numeral n* ≠ *0*›
by *simp*

lemma *one-neq-neg-numeral*: ‹*1* ≠ − *numeral n*›
using *numeral-neq-neg-numeral* [*of One n*] **by** (*simp add*: *numeral-One*)

lemma *neg-numeral-neq-one*: ‹− *numeral n* ≠ *1*›
using *neg-numeral-neq-numeral* [*of n One*] **by** (*simp add*: *numeral-One*)

lemma *neg-one-neq-numeral*: ‹− *1* ≠ *numeral n*›
using *neg-numeral-neq-numeral* [*of One n*] **by** (*simp add*: *numeral-One*)

lemma *numeral-neq-neg-one*: ‹*numeral n* ≠ − *1*›
using *numeral-neq-neg-numeral* [*of n One*] **by** (*simp add*: *numeral-One*)

lemma *neg-one-eq-numeral-iff*: ‹− *1* = − *numeral n* ⟷ *n* = *One*›
using *neg-numeral-eq-iff* [*of One n*] **by** (*auto simp add*: *numeral-One*)

lemma *numeral-eq-neg-one-iff*: ‹− *numeral n* = − *1* ⟷ *n* = *One*›
using *neg-numeral-eq-iff* [*of n One*] **by** (*auto simp add*: *numeral-One*)

lemma *neg-one-neq-zero*: ‹− *1* ≠ *0*›
by *simp*

lemma *zero-neq-neg-one*: ‹*0* ≠ − *1*›
by *simp*

lemma *neg-one-neq-one*: ‹− *1* ≠ *1*›
using *neg-numeral-neq-numeral* [*of One One*] **by** (*simp only*: *numeral-One not-False-eq-True*)

lemma *one-neq-neg-one*: ‹*1* ≠ − *1*›
using *numeral-neq-neg-numeral* [*of One One*] **by** (*simp only*: *numeral-One not-False-eq-True*)

lemmas *eq-neg-numeral-simps* [*simp*] =
neg-numeral-eq-iff
numeral-neq-neg-numeral neg-numeral-neq-numeral
one-neq-neg-numeral neg-numeral-neq-one
zero-neq-neg-numeral neg-numeral-neq-zero
neg-one-neq-numeral numeral-neq-neg-one
neg-one-eq-numeral-iff numeral-eq-neg-one-iff
neg-one-neq-zero zero-neq-neg-one
neg-one-neq-one one-neq-neg-one

end

46.4.10 Structures with negation and order: class *linordered-idom*

context *linordered-idom*
begin

subclass *ring-char-0* **..**

lemma *neg-numeral-le-iff*: ‹− *numeral m* ≤ − *numeral n* ⟷ *n* ≤ *m*›
by (*simp only*: *neg-le-iff-le numeral-le-iff*)

lemma *neg-numeral-less-iff*: ‹− *numeral m* < − *numeral n* ⟷ *n* < *m*›
by (*simp only*: *neg-less-iff-less numeral-less-iff*)

lemma *neg-numeral-less-zero*: ‹− *numeral n* < *0*›
by (*simp only*: *neg-less-0-iff-less zero-less-numeral*)

lemma *neg-numeral-le-zero*: ‹− *numeral n* ≤ *0*›
by (*simp only*: *neg-le-0-iff-le zero-le-numeral*)

lemma *not-zero-less-neg-numeral*: ‹¬ *0* < − *numeral n*›
by (*simp only*: *not-less neg-numeral-le-zero*)

lemma *not-zero-le-neg-numeral*: ‹¬ *0* ≤ − *numeral n*›
by (*simp only*: *not-le neg-numeral-less-zero*)

lemma *neg-numeral-less-numeral*: ‹− *numeral m* < *numeral n*›
using *neg-numeral-less-zero zero-less-numeral* **by** (*rule less-trans*)

lemma *neg-numeral-le-numeral*: ‹− *numeral m* ≤ *numeral n*›
by (*simp only*: *less-imp-le neg-numeral-less-numeral*)

lemma *not-numeral-less-neg-numeral*: ‹¬ *numeral m* < − *numeral n*›
by (*simp only*: *not-less neg-numeral-le-numeral*)

lemma *not-numeral-le-neg-numeral*: ‹¬ *numeral m* ≤ − *numeral n*›
by (*simp only*: *not-le neg-numeral-less-numeral*)

lemma *neg-numeral-less-one*: ‹− *numeral m* < *1*›
by (*rule neg-numeral-less-numeral* [*of m One*, *unfolded numeral-One*])

lemma *neg-numeral-le-one*: ‹− *numeral m* ≤ *1*›
by (*rule neg-numeral-le-numeral* [*of m One*, *unfolded numeral-One*])

lemma *not-one-less-neg-numeral*: ‹¬ *1* < − *numeral m*›
by (*simp only*: *not-less neg-numeral-le-one*)

lemma *not-one-le-neg-numeral*: ‹¬ *1* ≤ − *numeral m*›
by (*simp only*: *not-le neg-numeral-less-one*)

lemma *not-numeral-less-neg-one*: ‹¬ *numeral m* < − *1*›
using *not-numeral-less-neg-numeral* [*of m One*] **by** (*simp add*: *numeral-One*)

lemma *not-numeral-le-neg-one*: ‹¬ *numeral m* ≤ − *1*›
using *not-numeral-le-neg-numeral* [*of m One*] **by** (*simp add*: *numeral-One*)

lemma *neg-one-less-numeral*: ‹− *1* < *numeral m*›
using *neg-numeral-less-numeral* [*of One m*] **by** (*simp add*: *numeral-One*)

lemma *neg-one-le-numeral*: ‹− *1* ≤ *numeral m*›
using *neg-numeral-le-numeral* [*of One m*] **by** (*simp add*: *numeral-One*)

lemma *neg-numeral-less-neg-one-iff*: ‹− *numeral m* < − *1* ⟷ *m* ≠ *One*›
by (*cases m*) *simp-all*

lemma *neg-numeral-le-neg-one*: ‹$- \ numeral \ m \leq - 1$›
by *simp*

lemma *not-neg-one-less-neg-numeral*: ‹$\neg - 1 < - numeral \ m$›
by *simp*

lemma *not-neg-one-le-neg-numeral-iff*: ‹$\neg - 1 \leq - numeral \ m \longleftrightarrow m \neq One$›
by (*cases m*) *simp-all*

lemma *sub-non-negative*: ‹$sub \ n \ m \geq 0 \longleftrightarrow n \geq m$›
by (*simp only*: *sub-def le-diff-eq*) *simp*

lemma *sub-positive*: ‹$sub \ n \ m > 0 \longleftrightarrow n > m$›
by (*simp only*: *sub-def less-diff-eq*) *simp*

lemma *sub-non-positive*: ‹$sub \ n \ m \leq 0 \longleftrightarrow n \leq m$›
by (*simp only*: *sub-def diff-le-eq*) *simp*

lemma *sub-negative*: ‹$sub \ n \ m < 0 \longleftrightarrow n < m$›
by (*simp only*: *sub-def diff-less-eq*) *simp*

lemmas *le-neg-numeral-simps* [*simp*] =
neg-numeral-le-iff
neg-numeral-le-numeral not-numeral-le-neg-numeral
neg-numeral-le-zero not-zero-le-neg-numeral
neg-numeral-le-one not-one-le-neg-numeral
neg-one-le-numeral not-numeral-le-neg-one
neg-numeral-le-neg-one not-neg-one-le-neg-numeral-iff

lemma *le-minus-one-simps* [*simp*]:
‹$- 1 \leq 0$›
‹$- 1 \leq 1$›
‹$\neg 0 \leq - 1$›
‹$\neg 1 \leq - 1$›
by *simp-all*

lemmas *less-neg-numeral-simps* [*simp*] =
neg-numeral-less-iff
neg-numeral-less-numeral not-numeral-less-neg-numeral
neg-numeral-less-zero not-zero-less-neg-numeral
neg-numeral-less-one not-one-less-neg-numeral
neg-one-less-numeral not-numeral-less-neg-one
neg-numeral-less-neg-one-iff not-neg-one-less-neg-numeral

lemma *less-minus-one-simps* [*simp*]:
‹$- 1 < 0$›
‹$- 1 < 1$›
‹$\neg 0 < - 1$›
‹$\neg 1 < - 1$›

by (*simp-all add*: *less-le*)

lemma *abs-numeral* [*simp*]: ‹|*numeral n*| = *numeral n*›
by *simp*

lemma *abs-neg-numeral* [*simp*]: ‹|− *numeral n*| = *numeral n*›
by (*simp only*: *abs-minus-cancel abs-numeral*)

lemma *abs-neg-one* [*simp*]: ‹|− *1*| = *1*›
by *simp*

end

46.4.11 Natural numbers

lemma *numeral-num-of-nat*:
‹*numeral* (*num-of-nat n*) = *n*› **if** ‹*n* > *0*›
using *that nat-of-num-numeral num-of-nat-inverse* **by** *simp*

lemma *Suc-1* [*simp*]: ‹*Suc 1* = *2*›
unfolding *Suc-eq-plus1* **by** (*rule one-add-one*)

lemma *Suc-numeral* [*simp*]: ‹*Suc* (*numeral n*) = *numeral* (*n* + *One*)›
unfolding *Suc-eq-plus1* **by** (*rule numeral-plus-one*)

definition *pred-numeral* :: ‹*num* ⇒ *nat*›
where ‹*pred-numeral k* = *numeral k* − *1*›

declare [[*code drop*: *pred-numeral*]]

lemma *numeral-eq-Suc*: ‹*numeral k* = *Suc* (*pred-numeral k*)›
by (*simp add*: *pred-numeral-def*)

lemma *eval-nat-numeral*:
‹*numeral One* = *Suc 0*›
‹*numeral* (*Bit0 n*) = *Suc* (*numeral* (*BitM n*))›
‹*numeral* (*Bit1 n*) = *Suc* (*numeral* (*Bit0 n*))›
by (*simp-all add*: *numeral.simps BitM-plus-one*)

lemma *pred-numeral-simps* [*simp*]:
‹*pred-numeral One* = *0*›
‹*pred-numeral* (*Bit0 k*) = *numeral* (*BitM k*)›
‹*pred-numeral* (*Bit1 k*) = *numeral* (*Bit0 k*)›
by (*simp-all only*: *pred-numeral-def eval-nat-numeral diff-Suc-Suc diff-0*)

lemma *pred-numeral-inc* [*simp*]:
‹*pred-numeral* (*inc k*) = *numeral k*›
by (*simp only*: *pred-numeral-def numeral-inc diff-add-inverse2*)

lemma *numeral-2-eq-2*: ‹*2* = *Suc* (*Suc 0*)›
 by (*simp add*: *eval-nat-numeral*)

lemma *numeral-3-eq-3*: ‹*3* = *Suc* (*Suc* (*Suc 0*))›
 by (*simp add*: *eval-nat-numeral*)

lemma *numeral-1-eq-Suc-0*: ‹*Numeral1* = *Suc 0*›
 by (*simp only*: *numeral-One One-nat-def*)

lemma *Suc-nat-number-of-add*: ‹*Suc* (*numeral v* + *n*) = *numeral* (*v* + *One*) + *n*›
 by *simp*

lemma *numerals*: ‹*Numeral1* = (*1*::*nat*)› ‹*2* = *Suc* (*Suc 0*)›
 by (*rule numeral-One*) (*rule numeral-2-eq-2*)

lemmas *numeral-nat* = *eval-nat-numeral BitM.simps One-nat-def*

Comparisons involving *Suc*.

lemma *eq-numeral-Suc* [*simp*]: ‹*numeral k* = *Suc n* ⟷ *pred-numeral k* = *n*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *Suc-eq-numeral* [*simp*]: ‹*Suc n* = *numeral k* ⟷ *n* = *pred-numeral k*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *less-numeral-Suc* [*simp*]: ‹*numeral k* < *Suc n* ⟷ *pred-numeral k* < *n*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *less-Suc-numeral* [*simp*]: ‹*Suc n* < *numeral k* ⟷ *n* < *pred-numeral k*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *le-numeral-Suc* [*simp*]: ‹*numeral k* ≤ *Suc n* ⟷ *pred-numeral k* ≤ *n*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *le-Suc-numeral* [*simp*]: ‹*Suc n* ≤ *numeral k* ⟷ *n* ≤ *pred-numeral k*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *diff-Suc-numeral* [*simp*]: ‹*Suc n* − *numeral k* = *n* − *pred-numeral k*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *diff-numeral-Suc* [*simp*]: ‹*numeral k* − *Suc n* = *pred-numeral k* − *n*›
 by (*simp add*: *numeral-eq-Suc*)

lemma *max-Suc-numeral* [*simp*]: ‹*max* (*Suc n*) (*numeral k*) = *Suc* (*max n* (*pred-numeral k*))›
 by (*simp add*: *numeral-eq-Suc*)

lemma *max-numeral-Suc* [*simp*]: ‹*max* (*numeral k*) (*Suc n*) = *Suc* (*max* (*pred-numeral k*) *n*)›
 by (*simp add*: *numeral-eq-Suc*)

lemma *min-Suc-numeral* [*simp*]: ‹*min* (*Suc n*) (*numeral k*) = *Suc* (*min n* (*pred-numeral k*))›
by (*simp add*: *numeral-eq-Suc*)

lemma *min-numeral-Suc* [*simp*]: ‹*min* (*numeral k*) (*Suc n*) = *Suc* (*min* (*pred-numeral k*) *n*)›
by (*simp add*: *numeral-eq-Suc*)

For *case-nat* and *rec-nat*.

lemma *case-nat-numeral* [*simp*]: ‹*case-nat a f* (*numeral v*) = (*let pv* = *pred-numeral v in f pv*)›
by (*simp add*: *numeral-eq-Suc*)

lemma *case-nat-add-eq-if* [*simp*]:
‹*case-nat a f* ((*numeral v*) + *n*) = (*let pv* = *pred-numeral v in f* (*pv* + *n*))›
by (*simp add*: *numeral-eq-Suc*)

lemma *rec-nat-numeral* [*simp*]:
‹*rec-nat a f* (*numeral v*) = (*let pv* = *pred-numeral v in f pv* (*rec-nat a f pv*))›
by (*simp add*: *numeral-eq-Suc Let-def*)

lemma *rec-nat-add-eq-if* [*simp*]:
‹*rec-nat a f* (*numeral v* + *n*) = (*let pv* = *pred-numeral v in f* (*pv* + *n*) (*rec-nat a f* (*pv* + *n*)))›
by (*simp add*: *numeral-eq-Suc Let-def*)

Case analysis on $n < (2::'a)$.

lemma *less-2-cases*: ‹$n < 2 \Longrightarrow n = 0 \lor n = Suc\ 0$›
by (*auto simp add*: *numeral-2-eq-2*)

lemma *less-2-cases-iff*: ‹$n < 2 \longleftrightarrow n = 0 \lor n = Suc\ 0$›
by (*auto simp add*: *numeral-2-eq-2*)

Removal of Small Numerals: 0, 1 and (in additive positions) 2.

bh: Are these rules really a good idea? LCP: well, it already happens for 0 and 1!

lemma *add-2-eq-Suc* [*simp*]: ‹$2 + n = Suc\ (Suc\ n)$›
by *simp*

lemma *add-2-eq-Suc'* [*simp*]: ‹$n + 2 = Suc\ (Suc\ n)$›
by *simp*

Can be used to eliminate long strings of Sucs, but not by default.

lemma *Suc3-eq-add-3*: ‹$Suc\ (Suc\ (Suc\ n)) = 3 + n$›
by *simp*

lemmas *nat-1-add-1* = *one-add-one* [**where** ′*a*=*nat*]

context *semiring-numeral*
begin

lemma *numeral-add-unfold-funpow*:
‹*numeral k* + *a* = ((+) *1* ^^ *numeral k*) *a*›
proof (*rule sym*, *induction k arbitrary*: *a*)
case *One*
then show *?case*
by (*simp add*: *Num.numeral-One numeral-One*)
next
case (*Bit0 k*)
then show *?case*
by (*simp add*: *Num.numeral-Bit0 numeral-Bit0 ac-simps funpow-add*)
next
case (*Bit1 k*)
then show *?case*
by (*simp add*: *Num.numeral-Bit1 numeral-Bit1 ac-simps funpow-add*)
qed

end

context *semiring-1*
begin

lemma *numeral-unfold-funpow*:
‹*numeral k* = ((+) *1* ^^ *numeral k*) *0*›
using *numeral-add-unfold-funpow* [*of k 0*] **by** *simp*

end

context
includes *lifting-syntax*
begin

lemma *transfer-rule-numeral*:
‹((=) ===> *R*) *numeral numeral*›
if [*transfer-rule*]: ‹*R 0 0*› ‹*R 1 1*›
‹(*R* ===> *R* ===> *R*) (+) (+)›
for *R* :: ‹′*a*::{*semiring-numeral*,*monoid-add*} ⇒ ′*b*::{*semiring-numeral*,*monoid-add*} ⇒ *bool*›
proof −
have ‹((=) ===> *R*) (λ*k*. ((+) *1* ^^ *numeral k*) *0*) (λ*k*. ((+) *1* ^^ *numeral k*) *0*)›
by *transfer-prover*
moreover have ‹*numeral* = (λ*k*. ((+) (*1*::′*a*) ^^ *numeral k*) *0*)›
using *numeral-add-unfold-funpow* [**where** *?′a* = ′*a*, *of* - *0*]
by (*simp add*: *fun-eq-iff*)

moreover have ‹*numeral* = (λ*k*. ((+) (*1*::′*b*) ^^ *numeral k*) *0*)›
  using *numeral-add-unfold-funpow* [**where** *?′a* = ′*b*, *of* - *0*]
  by (*simp add*: *fun-eq-iff*)
ultimately show *?thesis*
  by *simp*
qed

end

46.5 Particular lemmas concerning *2*::′*a*

context *linordered-field*
begin

subclass *field-char-0* **..**

lemma *half-gt-zero-iff*: ‹*0* < *a* / *2* ⟷ *0* < *a*›
  by (*auto simp add*: *field-simps*)

lemma *half-gt-zero* [*simp*]: ‹*0* < *a* ⟹ *0* < *a* / *2*›
  by (*simp add*: *half-gt-zero-iff*)

end

46.6 Numeral equations as default simplification rules

declare (**in** *numeral*) *numeral-One* [*simp*]
declare (**in** *numeral*) *numeral-plus-numeral* [*simp*]
declare (**in** *numeral*) *add-numeral-special* [*simp*]
declare (**in** *neg-numeral*) *add-neg-numeral-simps* [*simp*]
declare (**in** *neg-numeral*) *add-neg-numeral-special* [*simp*]
declare (**in** *neg-numeral*) *diff-numeral-simps* [*simp*]
declare (**in** *neg-numeral*) *diff-numeral-special* [*simp*]
declare (**in** *semiring-numeral*) *numeral-times-numeral* [*simp*]
declare (**in** *ring-1*) *mult-neg-numeral-simps* [*simp*]

46.6.1 Special Simplification for Constants

These distributive laws move literals inside sums and differences.

lemmas *distrib-right-numeral* [*simp*] = *distrib-right* [*of* - - ‹*numeral v*›] **for** *v*
lemmas *distrib-left-numeral* [*simp*] = *distrib-left* [*of* ‹*numeral v*›] **for** *v*
lemmas *left-diff-distrib-numeral* [*simp*] = *left-diff-distrib* [*of* - - ‹*numeral v*›] **for** *v*
lemmas *right-diff-distrib-numeral* [*simp*] = *right-diff-distrib* [*of* ‹*numeral v*›] **for** *v*

These are actually for fields, like real

lemmas *zero-less-divide-iff-numeral* [*simp*, *no-atp*] = *zero-less-divide-iff* [*of* ‹*numeral w*›] **for** *w*

lemmas *divide-less-0-iff-numeral* [*simp*, *no-atp*] = *divide-less-0-iff* [*of* ‹*numeral w*›] **for** *w*
lemmas *zero-le-divide-iff-numeral* [*simp*, *no-atp*] = *zero-le-divide-iff* [*of* ‹*numeral w*›] **for** *w*
lemmas *divide-le-0-iff-numeral* [*simp*, *no-atp*] = *divide-le-0-iff* [*of* ‹*numeral w*›] **for** *w*

Replaces *inverse #nn* by *1/#nn*. It looks strange, but then other simprocs simplify the quotient.

lemmas *inverse-eq-divide-numeral* [*simp*] =
inverse-eq-divide [*of* ‹*numeral w*›] **for** *w*

lemmas *inverse-eq-divide-neg-numeral* [*simp*] =
inverse-eq-divide [*of* ‹− *numeral w*›] **for** *w*

These laws simplify inequalities, moving unary minus from a term into the literal.

lemmas *equation-minus-iff-numeral* [*no-atp*] =
equation-minus-iff [*of* ‹*numeral v*›] **for** *v*

lemmas *minus-equation-iff-numeral* [*no-atp*] =
minus-equation-iff [*of* - ‹*numeral v*›] **for** *v*

lemmas *le-minus-iff-numeral* [*no-atp*] =
le-minus-iff [*of* ‹*numeral v*›] **for** *v*

lemmas *minus-le-iff-numeral* [*no-atp*] =
minus-le-iff [*of* - ‹*numeral v*›] **for** *v*

lemmas *less-minus-iff-numeral* [*no-atp*] =
less-minus-iff [*of* ‹*numeral v*›] **for** *v*

lemmas *minus-less-iff-numeral* [*no-atp*] =
minus-less-iff [*of* - ‹*numeral v*›] **for** *v*

Cancellation of constant factors in comparisons ($<$ and $\leq$)

lemmas *mult-less-cancel-left-numeral* [*simp*, *no-atp*] = *mult-less-cancel-left* [*of* ‹*numeral v*›] **for** *v*
lemmas *mult-less-cancel-right-numeral* [*simp*, *no-atp*] = *mult-less-cancel-right* [*of* - ‹*numeral v*›] **for** *v*
lemmas *mult-le-cancel-left-numeral* [*simp*, *no-atp*] = *mult-le-cancel-left* [*of* ‹*numeral v*›] **for** *v*
lemmas *mult-le-cancel-right-numeral* [*simp*, *no-atp*] = *mult-le-cancel-right* [*of* - ‹*numeral v*›] **for** *v*

Multiplying out constant divisors in comparisons ($<$, $\leq$ and $=$)

named-theorems *divide-const-simps* ‹*simplification rules to simplify comparisons involving constant divisors*›

lemmas *le-divide-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-le-divide-eq [*of* ‹*numeral w*›, *OF zero-less-numeral*]
neg-le-divide-eq [*of* ‹− *numeral w*›, *OF neg-numeral-less-zero*] **for** *w*

lemmas *divide-le-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-divide-le-eq [*of* ‹*numeral w*›, *OF zero-less-numeral*]
neg-divide-le-eq [*of* ‹− *numeral w*›, *OF neg-numeral-less-zero*] **for** *w*

lemmas *less-divide-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-less-divide-eq [*of* ‹*numeral w*›, *OF zero-less-numeral*]
neg-less-divide-eq [*of* ‹− *numeral w*›, *OF neg-numeral-less-zero*] **for** *w*

lemmas *divide-less-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-divide-less-eq [*of* ‹*numeral w*›, *OF zero-less-numeral*]
neg-divide-less-eq [*of* ‹− *numeral w*›, *OF neg-numeral-less-zero*] **for** *w*

lemmas *eq-divide-eq-numeral1* [*simp*,*divide-const-simps*] =
eq-divide-eq [*of* - - ‹*numeral w*›]
eq-divide-eq [*of* - - ‹− *numeral w*›] **for** *w*

lemmas *divide-eq-eq-numeral1* [*simp*,*divide-const-simps*] =
divide-eq-eq [*of* - ‹*numeral w*›]
divide-eq-eq [*of* - ‹− *numeral w*›] **for** *w*

46.6.2 Optional Simplification Rules Involving Constants

Simplify quotients that are compared with a literal constant.

lemmas *le-divide-eq-numeral* [*divide-const-simps*] =
le-divide-eq [*of* ‹*numeral w*›]
le-divide-eq [*of* ‹− *numeral w*›] **for** *w*

lemmas *divide-le-eq-numeral* [*divide-const-simps*] =
divide-le-eq [*of* - - ‹*numeral w*›]
divide-le-eq [*of* - - ‹− *numeral w*›] **for** *w*

lemmas *less-divide-eq-numeral* [*divide-const-simps*] =
less-divide-eq [*of* ‹*numeral w*›]
less-divide-eq [*of* ‹− *numeral w*›] **for** *w*

lemmas *divide-less-eq-numeral* [*divide-const-simps*] =
divide-less-eq [*of* - - ‹*numeral w*›]
divide-less-eq [*of* - - ‹− *numeral w*›] **for** *w*

lemmas *eq-divide-eq-numeral* [*divide-const-simps*] =
eq-divide-eq [*of* ‹*numeral w*›]
eq-divide-eq [*of* ‹− *numeral w*›] **for** *w*

lemmas *divide-eq-eq-numeral* [*divide-const-simps*] =

divide-eq-eq [*of* - - ‹*numeral w*›]
divide-eq-eq [*of* - - ‹− *numeral w*›] **for** *w*

Not good as automatic simprules because they cause case splits.

lemmas [*divide-const-simps*] =
le-divide-eq-1 divide-le-eq-1 less-divide-eq-1 divide-less-eq-1

46.7 Setting up simprocs

lemma *mult-numeral-1*: ‹*Numeral1* ∗ *a* = *a*›
for *a* :: ‹′*a*::*semiring-numeral*›
by *simp*

lemma *mult-numeral-1-right*: ‹*a* ∗ *Numeral1* = *a*›
for *a* :: ‹′*a*::*semiring-numeral*›
by *simp*

lemma *divide-numeral-1*: ‹*a* / *Numeral1* = *a*›
for *a* :: ‹′*a*::*field*›
by *simp*

lemma *inverse-numeral-1*: ‹*inverse Numeral1* = (*Numeral1*::′*a*::*division-ring*)›
by *simp*

Theorem lists for the cancellation simprocs. The use of a binary numeral for 1 reduces the number of special cases.

lemma *mult-1s-semiring-numeral*:
‹*Numeral1* ∗ *a* = *a*›
‹*a* ∗ *Numeral1* = *a*›
for *a* :: ‹′*a*::*semiring-numeral*›
by *simp-all*

lemma *mult-1s-ring-1*:
‹− *Numeral1* ∗ *b* = − *b*›
‹*b* ∗ − *Numeral1* = − *b*›
for *b* :: ‹′*a*::*ring-1*›
by *simp-all*

lemmas *mult-1s* = *mult-1s-semiring-numeral mult-1s-ring-1*

setup ‹
Reorient-Proc.add
(*fn Const* (***const-name*** ‹*numeral*›, -) $ - => *true*
| *Const* (***const-name*** ‹*uminus*›, -) $ (*Const* (***const-name*** ‹*numeral*›, -) $ -)
=> *true*
| - => *false*)
›

simproc-setup *reorient-numeral* (‹*numeral w* = *x*› | ‹− *numeral w* = *y*›) =

‹K Reorient-Proc.proc›

46.7.1 Simplification of arithmetic operations on integer constants

lemmas *arith-special* =
add-numeral-special add-neg-numeral-special
diff-numeral-special

lemmas *arith-extra-simps* =
numeral-plus-numeral add-neg-numeral-simps add-0-left add-0-right
minus-zero
diff-numeral-simps diff-0 diff-0-right
numeral-times-numeral mult-neg-numeral-simps
mult-zero-left mult-zero-right
abs-numeral abs-neg-numeral

For making a minimal simpset, one must include these default simprules. Also include *simp-thms*.

lemmas *arith-simps* =
add-num-simps mult-num-simps sub-num-simps
BitM.simps dbl-simps dbl-inc-simps dbl-dec-simps
abs-zero abs-one arith-extra-simps

lemmas *more-arith-simps* =
neg-le-iff-le
minus-zero left-minus right-minus
mult-1-left mult-1-right
mult-minus-left mult-minus-right
minus-add-distrib minus-minus mult.assoc

lemmas *of-nat-simps* =
of-nat-0 of-nat-1 of-nat-Suc of-nat-add of-nat-mult

Simplification of relational operations.

lemmas *eq-numeral-extra* =
zero-neq-one one-neq-zero

lemmas *rel-simps* =
le-num-simps less-num-simps eq-num-simps
le-numeral-simps le-neg-numeral-simps le-minus-one-simps le-numeral-extra
less-numeral-simps less-neg-numeral-simps less-minus-one-simps less-numeral-extra
eq-numeral-simps eq-neg-numeral-simps eq-numeral-extra

lemma *Let-numeral* [*simp*]: ‹*Let* (*numeral v*) *f* = *f* (*numeral v*)›
— Unfold all *let*s involving constants
unfolding *Let-def* **..**

lemma *Let-neg-numeral* [*simp*]: ‹*Let* (− *numeral v*) *f* = *f* (− *numeral v*)›

— Unfold all *let*s involving constants
unfolding *Let-def* **..**

declaration ‹
let
fun number-of ctxt T n =
if not (*Sign.of-sort* (*Proof-Context.theory-of ctxt*) (*T*, ***sort*** ‹*numeral*›))
then raise CTERM (*number-of*, [])
else Numeral.mk-cnumber (*Thm.ctyp-of ctxt T*) *n*;
in
K (
Lin-Arith.set-number-of number-of
#> *Lin-Arith.add-simps*
@{*thms arith-simps more-arith-simps rel-simps pred-numeral-simps*
arith-special numeral-One of-nat-simps uminus-numeral-One
Suc-numeral Let-numeral Let-neg-numeral Let-0 Let-1
le-Suc-numeral le-numeral-Suc less-Suc-numeral less-numeral-Suc
Suc-eq-numeral eq-numeral-Suc mult-Suc mult-Suc-right of-nat-numeral})
end
›

46.7.2 Simplification of arithmetic when nested to the right

lemma *add-numeral-left* [*simp*]: ‹*numeral v* + (*numeral w* + *z*) = (*numeral*(*v* + *w*) + *z*)›
by (*simp-all add*: *add.assoc* [*symmetric*])

lemma *add-neg-numeral-left* [*simp*]:
‹*numeral v* + (− *numeral w* + *y*) = (*sub v w* + *y*)›
‹− *numeral v* + (*numeral w* + *y*) = (*sub w v* + *y*)›
‹− *numeral v* + (− *numeral w* + *y*) = (− *numeral*(*v* + *w*) + *y*)›
by (*simp-all add*: *add.assoc* [*symmetric*])

lemma *mult-numeral-left-semiring-numeral*:
‹*numeral v* ∗ (*numeral w* ∗ *z*) = (*numeral*(*v* ∗ *w*) ∗ *z* :: ′*a*::*semiring-numeral*)›
by (*simp add*: *mult.assoc* [*symmetric*])

lemma *mult-numeral-left-ring-1*:
‹− *numeral v* ∗ (*numeral w* ∗ *y*) = (− *numeral*(*v* ∗ *w*) ∗ *y* :: ′*a*::*ring-1*)›
‹*numeral v* ∗ (− *numeral w* ∗ *y*) = (− *numeral*(*v* ∗ *w*) ∗ *y* :: ′*a*::*ring-1*)›
‹− *numeral v* ∗ (− *numeral w* ∗ *y*) = (*numeral*(*v* ∗ *w*) ∗ *y* :: ′*a*::*ring-1*)›
by (*simp-all add*: *mult.assoc* [*symmetric*])

lemmas *mult-numeral-left* [*simp*] =
mult-numeral-left-semiring-numeral
mult-numeral-left-ring-1

46.8 Code module namespace

code-identifier

code-module *Num* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

46.9 Printing of evaluated natural numbers as numerals

lemma [*code-post*]:
‹*Suc 0 = 1*›
‹*Suc 1 = 2*›
‹*Suc* (*numeral n*) = *numeral* (*inc n*)›
by (*simp-all add*: *numeral-inc*)

lemmas [*code-post*] = *inc.simps*

46.10 More on auxiliary conversion

context *semiring-1*
begin

lemma *num-of-nat-numeral-eq* [*simp*]:
‹*num-of-nat* (*numeral q*) = *q*›
by (*simp flip*: *nat-of-num-numeral add*: *nat-of-num-inverse*)

lemma *numeral-num-of-nat-unfold*:
‹*numeral* (*num-of-nat n*) = (*if n = 0 then 1 else of-nat n*)›
apply (*simp only*: *of-nat-numeral* [*symmetric*, *of* ‹*num-of-nat n*›] *flip*: *nat-of-num-numeral*)
apply (*auto simp add*: *num-of-nat-inverse*)
done

end

hide-const (**open**) *One Bit0 Bit1 BitM inc pow sqr sub dbl dbl-inc dbl-dec*

end

47 Exponentiation

theory *Power*
imports *Num*
begin

47.1 Powers for Arbitrary Monoids

class *power* = *one* + *times*
begin

primrec *power* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ‹^› *80*)
where
power-0: *a* ^ *0* = *1*
| *power-Suc*: *a* ^ *Suc n* = *a* ∗ *a* ^ *n*

notation (*latex* **output**)
power (‹(-̂)› [*1000*] *1000*)

Special syntax for squares.

abbreviation *power2* :: ′*a* ⇒ ′*a* (‹(‹*notation*=‹*postfix 2*››-^{2})› [*1000*] *999*)
where $x^2 \equiv x$ ^ *2*

end

context
includes *lifting-syntax*
begin

lemma *power-transfer* [*transfer-rule*]:
‹(*R* ===> (=) ===> *R*) (^) (^)›
if [*transfer-rule*]: ‹*R 1 1*›
‹(*R* ===> *R* ===> *R*) (∗) (∗)›
for *R* :: ‹′*a*::*power* ⇒ ′*b*::*power* ⇒ *bool*›
by (*simp only*: *power-def* [*abs-def*]) *transfer-prover*

end

context *monoid-mult*
begin

subclass *power* **.**

lemma *power-one* [*simp*]: *1* ^ *n* = *1*
by (*induct n*) *simp-all*

lemma *power-one-right* [*simp*]: *a* ^ *1* = *a*
by *simp*

lemma *power-Suc0-right* [*simp*]: *a* ^ *Suc 0* = *a*
by *simp*

lemma *power-commutes*: *a* ^ *n* ∗ *a* = *a* ∗ *a* ^ *n*
by (*induct n*) (*simp-all add*: *mult.assoc*)

lemma *power-Suc2*: *a* ^ *Suc n* = *a* ^ *n* ∗ *a*
by (*simp add*: *power-commutes*)

lemma *power-add*: *a* ^ (*m* + *n*) = *a* ^ *m* ∗ *a* ^ *n*
by (*induct m*) (*simp-all add*: *algebra-simps*)

lemma *power-mult*: *a* ^ (*m* ∗ *n*) = (*a* ^ *m*) ^ *n*
by (*induct n*) (*simp-all add*: *power-add*)

lemma *power-even-eq*: $a \,\hat{}\, (2 * n) = (a \,\hat{}\, n)^2$
  by (*subst mult.commute*) (*simp add*: *power-mult*)

lemma *power-odd-eq*: $a \,\hat{}\, Suc\ (2{*}n) = a * (a \,\hat{}\, n)^2$
  by (*simp add*: *power-even-eq*)

lemma *power-numeral-even*: $z \,\hat{}\, numeral\ (Num.Bit0\ w) = (let\ w = z \,\hat{}\, (numeral\ w)\ in\ w * w)$
  by (*simp only*: *numeral-Bit0 power-add Let-def*)

lemma *power-numeral-odd*: $z \,\hat{}\, numeral\ (Num.Bit1\ w) = (let\ w = z \,\hat{}\, (numeral\ w)\ in\ z * w * w)$
  by (*simp only*: *numeral-Bit1 One-nat-def add-Suc-right add-0-right*
    power-Suc power-add Let-def mult.assoc)

lemma *power2-eq-square*: $a^2 = a * a$
  by (*simp add*: *numeral-2-eq-2*)

lemma *power3-eq-cube*: $a \,\hat{}\, 3 = a * a * a$
  by (*simp add*: *numeral-3-eq-3 mult.assoc*)

lemma *power4-eq-xxxx*: $x\hat{}4 = x * x * x * x$
  by (*simp add*: *mult.assoc power-numeral-even*)

lemma *power-numeral-reduce*: $x \,\hat{}\, numeral\ n = x * x \,\hat{}\, pred\text{-}numeral\ n$
  by (*simp add*: *numeral-eq-Suc*)

lemma *funpow-times-power*: $(times\ x\ \hat{}\hat{}\ f\ x) = times\ (x \,\hat{}\, f\ x)$
proof (*induct f x arbitrary*: *f*)
  case *0*
  then show *?case* **by** (*simp add*: *fun-eq-iff*)
next
  case (*Suc n*)
  define *g* **where** $g\ x = f\ x - 1$ **for** *x*
  with *Suc* **have** $n = g\ x$ **by** *simp*
  with *Suc* **have** $times\ x\ \hat{}\hat{}\ g\ x = times\ (x \,\hat{}\, g\ x)$ **by** *simp*
  moreover from *Suc g-def* **have** $f\ x = g\ x + 1$ **by** *simp*
  ultimately show *?case*
    by (*simp add*: *power-add funpow-add fun-eq-iff mult.assoc*)
qed

lemma *power-commuting-commutes*:
  assumes $x * y = y * x$
  shows $x \,\hat{}\, n * y = y * x \,\hat{}\, n$
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)

have *x* ^ *Suc n* ∗ *y* = *x* ^ *n* ∗ *y* ∗ *x*
by (*subst power-Suc2*) (*simp add*: *assms ac-simps*)
also have ... = *y* ∗ *x* ^ *Suc n*
by (*simp only*: *Suc power-Suc2*) (*simp add*: *ac-simps*)
finally show *?case* **.**
qed

lemma *power-minus-mult*: *0* < *n* ⟹ *a* ^ (*n* − *1*) ∗ *a* = *a* ^ *n*
by (*simp add*: *power-commutes split*: *nat-diff-split*)

lemma *left-right-inverse-power*:
assumes *x* ∗ *y* = *1*
shows *x* ^ *n* ∗ *y* ^ *n* = *1*
proof (*induct n*)
case (*Suc n*)
moreover have *x* ^ *Suc n* ∗ *y* ^ *Suc n* = *x*^*n* ∗ (*x* ∗ *y*) ∗ *y*^*n*
by (*simp add*: *power-Suc2*[*symmetric*] *mult.assoc*[*symmetric*])
ultimately show *?case* **by** (*simp add*: *assms*)
qed *simp*

end

context *comm-monoid-mult*
begin

lemma *power-mult-distrib* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*, *divide-simps*]:
(*a* ∗ *b*) ^ *n* = (*a* ^ *n*) ∗ (*b* ^ *n*)
by (*induction n*) (*simp-all add*: *ac-simps*)

end

Extract constant factors from powers.

declare *power-mult-distrib* [**where** *a* = *numeral w* **for** *w*, *simp*]
declare *power-mult-distrib* [**where** *b* = *numeral w* **for** *w*, *simp*]

lemma *power-add-numeral* [*simp*]: *a*^*numeral m* ∗ *a*^*numeral n* = *a*^*numeral* (*m* + *n*)
for *a* :: *'a*::*monoid-mult*
by (*simp add*: *power-add* [*symmetric*])

lemma *power-add-numeral2* [*simp*]: *a*^*numeral m* ∗ (*a*^*numeral n* ∗ *b*) = *a*^*numeral* (*m* + *n*) ∗ *b*
for *a* :: *'a*::*monoid-mult*
by (*simp add*: *mult.assoc* [*symmetric*])

lemma *power-mult-numeral* [*simp*]: (*a*^*numeral m*)^*numeral n* = *a*^*numeral* (*m* ∗ *n*)
for *a* :: *'a*::*monoid-mult*

by (*simp only*: *numeral-mult power-mult*)

context *semiring-numeral*
begin

lemma *numeral-sqr*: *numeral* (*Num.sqr k*) = *numeral k* ∗ *numeral k*
by (*simp only*: *sqr-conv-mult numeral-mult*)

lemma *numeral-pow*: *numeral* (*Num.pow k l*) = *numeral k* ^ *numeral l*
by (*induct l*)
(*simp-all only*: *numeral-class.numeral.simps pow.simps*
numeral-sqr numeral-mult power-add power-one-right)

lemma *power-numeral* [*simp*]: *numeral k* ^ *numeral l* = *numeral* (*Num.pow k l*)
by (*rule numeral-pow* [*symmetric*])

end

context *semiring-1*
begin

lemma *of-nat-power* [*simp*]: *of-nat* (*m* ^ *n*) = *of-nat m* ^ *n*
by (*induct n*) *simp-all*

lemma *zero-power*: *0* < *n* ⟹ *0* ^ *n* = *0*
by (*cases n*) *simp-all*

lemma *power-zero-numeral* [*simp*]: *0* ^ *numeral k* = *0*
by (*simp add*: *numeral-eq-Suc*)

lemma *zero-power2*: $0^2 = 0$
by (*rule power-zero-numeral*)

lemma *one-power2*: $1^2 = 1$
by (*rule power-one*)

lemma *power-0-Suc* [*simp*]: *0* ^ *Suc n* = *0*
by *simp*

It looks plausible as a simprule, but its effect can be strange.

lemma *power-0-left*: *0* ^ *n* = (*if n* = *0 then 1 else 0*)
by (*cases n*) *simp-all*

end

context *semiring-char-0* **begin**

lemma *numeral-power-eq-of-nat-cancel-iff* [*simp*]:
numeral x ^ *n* = *of-nat y* ⟷ *numeral x* ^ *n* = *y*

using *of-nat-eq-iff* **by** *fastforce*

lemma *real-of-nat-eq-numeral-power-cancel-iff* [*simp*]:
of-nat y = *numeral x* ^ *n* ⟷ *y* = *numeral x* ^ *n*
using *numeral-power-eq-of-nat-cancel-iff* [*of x n y*] **by** (*metis* (*mono-tags*))

lemma *of-nat-eq-of-nat-power-cancel-iff*[*simp*]: (*of-nat b*) ^ *w* = *of-nat x* ⟷ *b* ^ *w* = *x*
by (*metis of-nat-power of-nat-eq-iff*)

lemma *of-nat-power-eq-of-nat-cancel-iff*[*simp*]: *of-nat x* = (*of-nat b*) ^ *w* ⟷ *x* = *b* ^ *w*
by (*metis of-nat-eq-of-nat-power-cancel-iff*)

end

context *comm-semiring-1*
begin

The divides relation.

lemma *le-imp-power-dvd*:
assumes *m* ≤ *n*
shows *a* ^ *m dvd a* ^ *n*
proof
from *assms* **have** *a* ^ *n* = *a* ^(*m* + (*n* − *m*)) **by** *simp*
also have ... = *a* ^ *m* ∗ *a* ^(*n* − *m*) **by** (*rule power-add*)
finally show *a* ^ *n* = *a* ^ *m* ∗ *a* ^(*n* − *m*) **.**
qed

lemma *power-le-dvd*: *a* ^ *n dvd b* ⟹ *m* ≤ *n* ⟹ *a* ^ *m dvd b*
by (*rule dvd-trans* [*OF le-imp-power-dvd*])

lemma *dvd-power-same*: *x dvd y* ⟹ *x* ^ *n dvd y* ^ *n*
by (*induct n*) (*auto simp add*: *mult-dvd-mono*)

lemma *dvd-power-le*: *x dvd y* ⟹ *m* ≥ *n* ⟹ *x* ^ *n dvd y* ^ *m*
by (*rule power-le-dvd* [*OF dvd-power-same*])

lemma *dvd-power* [*simp*]:
fixes *n* :: *nat*
assumes *n* > *0* ∨ *x* = *1*
shows *x dvd* (*x* ^ *n*)
using *assms*
proof
assume *0* < *n*
then have *x* ^ *n* = *x* ^ *Suc* (*n* − *1*) **by** *simp*
then show *x dvd* (*x* ^ *n*) **by** *simp*
next
assume *x* = *1*

then show x *dvd* $(x \,\widehat{}\, n)$ **by** *simp*
qed

end

context *semiring-1-no-zero-divisors*
begin

subclass *power* **.**

lemma *power-eq-0-iff* [*simp*]: $a \,\widehat{}\, n = 0 \longleftrightarrow a = 0 \wedge n > 0$
by (*induct n*) *auto*

lemma *power-not-zero*: $a \neq 0 \Longrightarrow a \,\widehat{}\, n \neq 0$
by (*induct n*) *auto*

lemma *zero-eq-power2* [*simp*]: $a^2 = 0 \longleftrightarrow a = 0$
unfolding *power2-eq-square* **by** *simp*

end

context *ring-1*
begin

lemma *power-minus*: $(-\ a) \,\widehat{}\, n = (-\ 1) \,\widehat{}\, n * a \,\widehat{}\, n$
proof (*induct n*)
case *0*
show *?case* **by** *simp*
next
case (*Suc n*)
then show *?case*
by (*simp del*: *power-Suc add*: *power-Suc2 mult.assoc*)
qed

lemma *power-minus'*: *NO-MATCH 1 x* $\Longrightarrow (-x) \,\widehat{}\, n = (-1)\widehat{}n * x \,\widehat{}\, n$
by (*rule power-minus*)

lemma *power-minus-Bit0*: $(-\ x) \,\widehat{}\,$*numeral* (*Num.Bit0 k*) $= x \,\widehat{}\,$*numeral* (*Num.Bit0 k*)
by (*induct k*, *simp-all only*: *numeral-class.numeral.simps power-add power-one-right mult-minus-left mult-minus-right minus-minus*)

lemma *power-minus-Bit1*: $(-\ x) \,\widehat{}\,$ *numeral* (*Num.Bit1 k*) $= -\ (x \,\widehat{}\,$ *numeral* (*Num.Bit1 k*))
by (*simp only*: *eval-nat-numeral*(*3*) *power-Suc power-minus-Bit0 mult-minus-left*)

lemma *power2-minus* [*simp*]: $(-\ a)^2 = a^2$
by (*fact power-minus-Bit0*)

lemma *power-minus1-even* [*simp*]: (− *1*) ^ (*2*∗*n*) = *1*
proof (*induct n*)
  case *0*
  show *?case* **by** *simp*
next
  case (*Suc n*)
  then show *?case* **by** (*simp add*: *power-add power2-eq-square*)
qed

lemma *power-minus1-odd*: (− *1*) ^ *Suc* (*2*∗*n*) = −*1*
  by *simp*

lemma *power-minus-even* [*simp*]: (−*a*) ^ (*2*∗*n*) = *a* ^ (*2*∗*n*)
  by (*simp add*: *power-minus* [*of a*])

end

context *ring-1-no-zero-divisors*
begin

lemma *power2-eq-1-iff*: a^2 = *1* ⟷ *a* = *1* ∨ *a* = − *1*
  using *square-eq-1-iff* [*of a*] **by** (*simp add*: *power2-eq-square*)

end

context *idom*
begin

lemma *power2-eq-iff*: x^2 = y^2 ⟷ *x* = *y* ∨ *x* = − *y*
  unfolding *power2-eq-square* **by** (*rule square-eq-iff*)

end

context *semidom-divide*
begin

lemma *power-diff*:
  a ^ (*m* − *n*) = (*a* ^ *m*) *div* (*a* ^ *n*) **if** *a* ≠ *0* **and** *n* ≤ *m*
proof −
  define *q* **where** *q* = *m* − *n*
  with ‹*n* ≤ *m*› **have** *m* = *q* + *n* **by** *simp*
  with ‹*a* ≠ *0*› *q-def* **show** *?thesis*
    by (*simp add*: *power-add*)
qed

lemma *power-diff-if*:
  a ^ (*m* − *n*) = (*if n* ≤ *m then* (*a* ^ *m*) *div* (*a* ^ *n*) *else 1*) **if** *a* ≠ *0*
  by (*simp add*: *power-diff that*)

end

context *algebraic-semidom*
begin

lemma *div-power*: $b\ dvd\ a \Longrightarrow (a\ div\ b)\ \widehat{}\ n = a\ \widehat{}\ n\ div\ b\ \widehat{}\ n$
by (*induct n*) (*simp-all add*: *div-mult-div-if-dvd dvd-power-same*)

lemma *is-unit-power-iff*: *is-unit* $(a\ \widehat{}\ n) \longleftrightarrow$ *is-unit* $a \vee n = 0$
by (*induct n*) (*auto simp add*: *is-unit-mult-iff*)

lemma *dvd-power-iff*:
assumes $x \neq 0$
shows $x\ \widehat{}\ m\ dvd\ x\ \widehat{}\ n \longleftrightarrow$ *is-unit* $x \vee m \leq n$
proof
assume $*$: $x\ \widehat{}\ m\ dvd\ x\ \widehat{}\ n$
{
assume $m > n$
note $*$
also have $x\ \widehat{}\ n = x\ \widehat{}\ n * 1$ **by** *simp*
also from ‹$m > n$› **have** $m = n + (m - n)$ **by** *simp*
also have $x\ \widehat{}\ \ldots = x\ \widehat{}\ n * x\ \widehat{}\ (m - n)$ **by** (*rule power-add*)
finally have $x\ \widehat{}\ (m - n)\ dvd\ 1$
using *assms* **by** (*subst* (*asm*) *dvd-times-left-cancel-iff*) *simp-all*
with ‹$m > n$› **have** *is-unit* x **by** (*simp add*: *is-unit-power-iff*)
}
thus *is-unit* $x \vee m \leq n$ **by** *force*
qed (*auto intro*: *unit-imp-dvd simp*: *is-unit-power-iff le-imp-power-dvd*)

end

context *normalization-semidom-multiplicative*
begin

lemma *normalize-power*: *normalize* $(a\ \widehat{}\ n) =$ *normalize* $a\ \widehat{}\ n$
by (*induct n*) (*simp-all add*: *normalize-mult*)

lemma *unit-factor-power*: *unit-factor* $(a\ \widehat{}\ n) =$ *unit-factor* $a\ \widehat{}\ n$
by (*induct n*) (*simp-all add*: *unit-factor-mult*)

end

context *division-ring*
begin

Perhaps these should be simprules.

lemma *power-inverse* [*field-simps*, *field-split-simps*, *divide-simps*]: *inverse* $a\ \widehat{}\ n =$ *inverse* $(a\ \widehat{}\ n)$

proof (*cases* $a = 0$)
  case *True*
  then show *?thesis* **by** (*simp add*: *power-0-left*)
next
  case *False*
  then have *inverse* $(a \,\hat{}\, n) =$ *inverse* $a \,\hat{}\, n$
    by (*induct n*) (*simp-all add*: *nonzero-inverse-mult-distrib power-commutes*)
  then show *?thesis* **by** *simp*
qed

lemma *power-one-over* [*field-simps*, *field-split-simps*, *divide-simps*]: $(1 \,/\, a) \,\hat{}\, n = 1 \,/\, a \,\hat{}\, n$
  using *power-inverse* [*of a*] **by** (*simp add*: *divide-inverse*)

end

context *field*
begin

lemma *power-divide* [*field-simps*, *field-split-simps*, *divide-simps*]: $(a \,/\, b) \,\hat{}\, n = a \,\hat{}\, n \,/\, b \,\hat{}\, n$
  by (*induct n*) *simp-all*

end

47.2 Exponentiation on ordered types

context *linordered-semidom*
begin

lemma *zero-less-power* [*simp*]: $0 < a \Longrightarrow 0 < a \,\hat{}\, n$
  by (*induct n*) *simp-all*

lemma *zero-le-power* [*simp*]: $0 \le a \Longrightarrow 0 \le a \,\hat{}\, n$
  by (*induct n*) *simp-all*

lemma *power-mono*: $a \le b \Longrightarrow 0 \le a \Longrightarrow a \,\hat{}\, n \le b \,\hat{}\, n$
  by (*induct n*) (*auto intro*: *mult-mono order-trans* [*of 0 a b*])

lemma *one-le-power* [*simp*]: $1 \le a \Longrightarrow 1 \le a \,\hat{}\, n$
  using *power-mono* [*of 1 a n*] **by** *simp*

lemma *power-le-one*: $0 \le a \Longrightarrow a \le 1 \Longrightarrow a \,\hat{}\, n \le 1$
  using *power-mono* [*of a 1 n*] **by** *simp*

lemma *power-gt1-lemma*:
  assumes *gt1*: $1 < a$
  shows $1 < a * a \,\hat{}\, n$
proof −

from *gt1* **have** $0 \le a$
by (*fact order-trans* [*OF zero-le-one less-imp-le*])
from *gt1* **have** $1 * 1 < a * 1$ **by** *simp*
also from *gt1* **have** $\ldots \le a * a \,\hat{}\, n$
by (*simp only*: *mult-mono* ‹$0 \le a$› *one-le-power order-less-imp-le zero-le-one order-refl*)
finally show *?thesis* **by** *simp*
qed

lemma *power-gt1*: $1 < a \Longrightarrow 1 < a \,\hat{}\, Suc\ n$
by (*simp add*: *power-gt1-lemma*)

lemma *one-less-power* [*simp*]: $1 < a \Longrightarrow 0 < n \Longrightarrow 1 < a \,\hat{}\, n$
by (*cases n*) (*simp-all add*: *power-gt1-lemma*)

lemma *power-le-imp-le-exp*:
assumes *gt1*: $1 < a$
shows $a \,\hat{}\, m \le a \,\hat{}\, n \Longrightarrow m \le n$
proof (*induct m arbitrary*: *n*)
case *0*
show *?case* **by** *simp*
next
case (*Suc m*)
show *?case*
proof (*cases n*)
case *0*
with *Suc* **have** $a * a \,\hat{}\, m \le 1$ **by** *simp*
with *gt1* **show** *?thesis*
by (*force simp only*: *power-gt1-lemma not-less* [*symmetric*])
next
case (*Suc n*)
with *Suc.prems Suc.hyps* **show** *?thesis*
by (*force dest*: *mult-left-le-imp-le simp add*: *less-trans* [*OF zero-less-one gt1*])
qed
qed

lemma *of-nat-zero-less-power-iff* [*simp*]: $of\text{-}nat\ x \,\hat{}\, n > 0 \longleftrightarrow x > 0 \lor n = 0$
by (*induct n*) *auto*

Surely we can strengthen this? It holds for $0<a<1$ too.

lemma *power-inject-exp* [*simp*]:
‹$a \,\hat{}\, m = a \,\hat{}\, n \longleftrightarrow m = n$› **if** ‹$1 < a$›
using *that* **by** (*force simp add*: *order-class.order.antisym power-le-imp-le-exp*)

Can relax the first premise to $0 < a$ in the case of the natural numbers.

lemma *power-less-imp-less-exp*: $1 < a \Longrightarrow a \,\hat{}\, m < a \,\hat{}\, n \Longrightarrow m < n$
by (*simp add*: *order-less-le* [*of m n*] *less-le* [*of a^m a^n*] *power-le-imp-le-exp*)

lemma *power-strict-mono*: $a < b \Longrightarrow 0 \le a \Longrightarrow 0 < n \Longrightarrow a \,\hat{}\, n < b \,\hat{}\, n$

proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  then show *?case*
    by (*cases* $n = 0$) (*auto simp*: *mult-strict-mono le-less-trans* [*of 0 a b*])
qed

lemma *power-mono-iff* [*simp*]:
  shows $[\![a \geq 0;\ b \geq 0;\ n>0]\!] \Longrightarrow a\ \hat{}\ n \leq b\ \hat{}\ n \longleftrightarrow a \leq b$
  using *power-mono* [*of a b*] *power-strict-mono* [*of b a*] *not-le* **by** *auto*

Lemma for *power-strict-decreasing*

lemma *power-Suc-less*: $0 < a \Longrightarrow a < 1 \Longrightarrow a * a\ \hat{}\ n < a\ \hat{}\ n$
  by (*induct n*) (*auto simp*: *mult-strict-left-mono*)

lemma *power-strict-decreasing*: $n < N \Longrightarrow 0 < a \Longrightarrow a < 1 \Longrightarrow a\ \hat{}\ N < a\ \hat{}\ n$
proof (*induction N*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc N*)
  then show *?case*
    using *mult-strict-mono*[*of a 1 a* ^ *N a* ^ *n*]
    by (*auto simp add*: *power-Suc-less less-Suc-eq*)
qed

Proof resembles that of *power-strict-decreasing*.

lemma *power-decreasing*: $n \leq N \Longrightarrow 0 \leq a \Longrightarrow a \leq 1 \Longrightarrow a\ \hat{}\ N \leq a\ \hat{}\ n$
proof (*induction N*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc N*)
  then show *?case*
    using *mult-mono*[*of a 1 a*^*N a* ^ *n*]
    by (*auto simp add*: *le-Suc-eq*)
qed

lemma *power-decreasing-iff* [*simp*]: $[\![0 < b;\ b < 1]\!] \Longrightarrow b\ \hat{}\ m \leq b\ \hat{}\ n \longleftrightarrow n \leq m$
  using *power-strict-decreasing* [*of m n b*]
  by (*auto intro*: *power-decreasing ccontr*)

lemma *power-strict-decreasing-iff* [*simp*]: $[\![0 < b;\ b < 1]\!] \Longrightarrow b\ \hat{}\ m < b\ \hat{}\ n \longleftrightarrow n < m$
  using *power-decreasing-iff* [*of b m n*] **unfolding** *le-less*
  by (*auto dest*: *power-strict-decreasing le-neq-implies-less*)

lemma *power-Suc-less-one*: $0 < a \Longrightarrow a < 1 \Longrightarrow a \mathbin{\hat{}} Suc\ n < 1$
  using *power-strict-decreasing* [*of 0 Suc n a*] **by** *simp*

Proof again resembles that of *power-strict-decreasing*.

lemma *power-increasing*: $n \le N \Longrightarrow 1 \le a \Longrightarrow a \mathbin{\hat{}} n \le a \mathbin{\hat{}} N$
proof (*induct N*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc N*)
  then show *?case*
    using *mult-mono*[*of 1 a a ^ n a ^ N*]
    by (*auto simp add*: *le-Suc-eq order-trans* [*OF zero-le-one*])
qed

Lemma for *power-strict-increasing*.

lemma *power-less-power-Suc*: $1 < a \Longrightarrow a \mathbin{\hat{}} n < a * a \mathbin{\hat{}} n$
  by (*induct n*) (*auto simp*: *mult-strict-left-mono less-trans* [*OF zero-less-one*])

lemma *power-strict-increasing*: $n < N \Longrightarrow 1 < a \Longrightarrow a \mathbin{\hat{}} n < a \mathbin{\hat{}} N$
proof (*induct N*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc N*)
  then show *?case*
    using *mult-strict-mono*[*of 1 a a^n a^N*]
   by (*auto simp add*: *power-less-power-Suc less-Suc-eq less-trans* [*OF zero-less-one*] *less-imp-le*)
qed

lemma *power-increasing-iff* [*simp*]: $1 < b \Longrightarrow b \mathbin{\hat{}} x \le b \mathbin{\hat{}} y \longleftrightarrow x \le y$
  by (*blast intro*: *power-le-imp-le-exp power-increasing less-imp-le*)

lemma *power-strict-increasing-iff* [*simp*]: $1 < b \Longrightarrow b \mathbin{\hat{}} x < b \mathbin{\hat{}} y \longleftrightarrow x < y$
  by (*blast intro*: *power-less-imp-less-exp power-strict-increasing*)

lemma *power-le-imp-le-base*:
  assumes *le*: $a \mathbin{\hat{}} Suc\ n \le b \mathbin{\hat{}} Suc\ n$
    and $0 \le b$
  shows $a \le b$
proof (*rule ccontr*)
  assume ¬ *?thesis*
  then have $b < a$ **by** (*simp only*: *linorder-not-le*)
  then have $b \mathbin{\hat{}} Suc\ n < a \mathbin{\hat{}} Suc\ n$
    by (*simp only*: *assms*(*2*) *power-strict-mono*)
  with *le* **show** *False*
    by (*simp add*: *linorder-not-less* [*symmetric*])

qed

lemma *power-less-imp-less-base*:
  assumes *less*: $a \,\hat{}\, n < b \,\hat{}\, n$
  assumes *nonneg*: $0 \leq b$
  shows $a < b$
proof (*rule contrapos-pp* [*OF less*])
  assume $\neg$ *?thesis*
  then have $b \leq a$ **by** (*simp only*: *linorder-not-less*)
  from *this nonneg* **have** $b \,\hat{}\, n \leq a \,\hat{}\, n$ **by** (*rule power-mono*)
  then show $\neg\ a \,\hat{}\, n < b \,\hat{}\, n$ **by** (*simp only*: *linorder-not-less*)
qed

lemma *power-inject-base*: $a \,\hat{}\, Suc\ n = b \,\hat{}\, Suc\ n \Longrightarrow 0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a = b$
  by (*blast intro*: *power-le-imp-le-base order.antisym eq-refl sym*)

lemma *power-eq-imp-eq-base*: $a \,\hat{}\, n = b \,\hat{}\, n \Longrightarrow 0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 < n \Longrightarrow a = b$
  by (*cases n*) (*simp-all del*: *power-Suc*, *rule power-inject-base*)

lemma *power-eq-iff-eq-base*: $0 < n \Longrightarrow 0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a \,\hat{}\, n = b \,\hat{}\, n \longleftrightarrow a = b$
  using *power-eq-imp-eq-base* [*of a n b*] **by** *auto*

lemma *power2-le-imp-le*: $x^2 \leq y^2 \Longrightarrow 0 \leq y \Longrightarrow x \leq y$
  unfolding *numeral-2-eq-2* **by** (*rule power-le-imp-le-base*)

lemma *power2-less-imp-less*: $x^2 < y^2 \Longrightarrow 0 \leq y \Longrightarrow x < y$
  by (*rule power-less-imp-less-base*)

lemma *power2-eq-imp-eq*: $x^2 = y^2 \Longrightarrow 0 \leq x \Longrightarrow 0 \leq y \Longrightarrow x = y$
  unfolding *numeral-2-eq-2* **by** (*erule* (*2*) *power-eq-imp-eq-base*) *simp*

lemma *power-Suc-le-self*: $0 \leq a \Longrightarrow a \leq 1 \Longrightarrow a \,\hat{}\, Suc\ n \leq a$
  using *power-decreasing* [*of 1 Suc n a*] **by** *simp*

lemma *power2-eq-iff-nonneg* [*simp*]:
  assumes $0 \leq x\ 0 \leq y$
  shows $(x \,\hat{}\, 2 = y \,\hat{}\, 2) \longleftrightarrow x = y$
using *assms power2-eq-imp-eq* **by** *blast*

lemma *of-nat-less-numeral-power-cancel-iff* [*simp*]:
  of-nat $x <$ *numeral* $i \,\hat{}\, n \longleftrightarrow x <$ *numeral* $i \,\hat{}\, n$
  using *of-nat-less-iff* [*of x numeral i* $\hat{}$ *n*, *unfolded of-nat-numeral of-nat-power*] **.**

lemma *of-nat-le-numeral-power-cancel-iff* [*simp*]:
  of-nat $x \leq$ *numeral* $i \,\hat{}\, n \longleftrightarrow x \leq$ *numeral* $i \,\hat{}\, n$
  using *of-nat-le-iff* [*of x numeral i* $\hat{}$ *n*, *unfolded of-nat-numeral of-nat-power*] **.**

lemma *numeral-power-less-of-nat-cancel-iff* [*simp*]:
numeral i ^ *n* < *of-nat x* ⟷ *numeral i* ^ *n* < *x*
using *of-nat-less-iff* [*of numeral i* ^ *n x*, *unfolded of-nat-numeral of-nat-power*] **.**

lemma *numeral-power-le-of-nat-cancel-iff* [*simp*]:
numeral i ^ *n* ≤ *of-nat x* ⟷ *numeral i* ^ *n* ≤ *x*
using *of-nat-le-iff* [*of numeral i* ^ *n x*, *unfolded of-nat-numeral of-nat-power*] **.**

lemma *of-nat-le-of-nat-power-cancel-iff* [*simp*]: (*of-nat b*) ^ *w* ≤ *of-nat x* ⟷ *b* ^ *w* ≤ *x*
by (*metis of-nat-le-iff of-nat-power*)

lemma *of-nat-power-le-of-nat-cancel-iff* [*simp*]: *of-nat x* ≤ (*of-nat b*) ^ *w* ⟷ *x* ≤ *b* ^ *w*
by (*metis of-nat-le-iff of-nat-power*)

lemma *of-nat-less-of-nat-power-cancel-iff* [*simp*]: (*of-nat b*) ^ *w* < *of-nat x* ⟷ *b* ^ *w* < *x*
by (*metis of-nat-less-iff of-nat-power*)

lemma *of-nat-power-less-of-nat-cancel-iff* [*simp*]: *of-nat x* < (*of-nat b*) ^ *w* ⟷ *x* < *b* ^ *w*
by (*metis of-nat-less-iff of-nat-power*)

lemma *power2-nonneg-ge-1-iff*:
assumes *x* ≥ *0*
shows *x* ^ *2* ≥ *1* ⟷ *x* ≥ *1*
using *assms* **by** (*auto intro*: *power2-le-imp-le*)

lemma *power2-nonneg-gt-1-iff*:
assumes *x* ≥ *0*
shows *x* ^ *2* > *1* ⟷ *x* > *1*
using *assms* **by** (*auto intro*: *power-less-imp-less-base*)

end

Some *nat*-specific lemmas:

lemma *mono-ge2-power-minus-self*:
assumes *k* ≥ *2* **shows** *mono* (λ*m*. *k* ^ *m* − *m*)
unfolding *mono-iff-le-Suc*
proof
fix *n*
have *k* ^ *n* < *k* ^ *Suc n* **using** *power-strict-increasing-iff* [*of k n Suc n*] *assms* **by** *linarith*
thus *k* ^ *n* − *n* ≤ *k* ^ *Suc n* − *Suc n* **by** *linarith*
qed

lemma *self-le-ge2-pow* [*simp*]:
assumes *k* ≥ *2* **shows** *m* ≤ *k* ^ *m*

proof (*induction m*)
  case *0* **show** *?case* **by** *simp*
next
  case (*Suc m*)
  hence *Suc m* $\leq$ *Suc* (*k* ^ *m*) **by** *simp*
  also have ... $\leq$ *k^m* + *k^m* **using** *one-le-power*[*of k m*] *assms* **by** *linarith*
  also have ... $\leq$ *k* ∗ *k^m* **by** (*metis mult-2 mult-le-mono1*[*OF assms*])
  finally show *?case* **by** *simp*
qed

lemma *diff-le-diff-pow*[*simp*]:
  assumes *k* $\geq$ *2* **shows** *m* − *n* $\leq$ *k* ^ *m* − *k* ^ *n*
proof (*cases n* $\leq$ *m*)
  case *True*
  thus *?thesis*
   using *monoD*[*OF mono-ge2-power-minus-self*[*OF assms*] *True*] *self-le-ge2-pow*[*OF assms*, *of m*]
    by (*simp add*: *le-diff-conv le-diff-conv2*)
qed *auto*

context *linordered-ring-strict*
begin

lemma *sum-squares-eq-zero-iff*: *x* ∗ *x* + *y* ∗ *y* = *0* ⟷ *x* = *0* ∧ *y* = *0*
  by (*simp add*: *add-nonneg-eq-0-iff*)

lemma *sum-squares-le-zero-iff*: *x* ∗ *x* + *y* ∗ *y* $\leq$ *0* ⟷ *x* = *0* ∧ *y* = *0*
  by (*simp add*: *le-less not-sum-squares-lt-zero sum-squares-eq-zero-iff*)

lemma *sum-squares-gt-zero-iff*: *0* < *x* ∗ *x* + *y* ∗ *y* ⟷ *x* ≠ *0* ∨ *y* ≠ *0*
  by (*simp add*: *not-le* [*symmetric*] *sum-squares-le-zero-iff*)

end

context *linordered-idom*
begin

lemma *zero-le-power2* [*simp*]: *0* $\leq$ a^2
  by (*simp add*: *power2-eq-square*)

lemma *zero-less-power2* [*simp*]: *0* < a^2 ⟷ *a* ≠ *0*
  by (*force simp add*: *power2-eq-square zero-less-mult-iff linorder-neq-iff*)

lemma *power2-less-0* [*simp*]: ¬ a^2 < *0*
  by (*force simp add*: *power2-eq-square mult-less-0-iff*)

lemma *power-abs*: |*a* ^ *n*| = |*a*| ^ *n* — FIXME simp?
  by (*induct n*) (*simp-all add*: *abs-mult*)

lemma *power-sgn* [*simp*]: *sgn* (*a* ^ *n*) = *sgn a* ^ *n*
  by (*induct n*) (*simp-all add*: *sgn-mult*)

lemma *abs-power-minus* [*simp*]: |(− *a*) ^ *n*| = |*a* ^ *n*|
  by (*simp add*: *power-abs*)

lemma *zero-less-power-abs-iff* [*simp*]: $0 < |a| ^ n \longleftrightarrow a \neq 0 \vee n = 0$
proof (*induct n*)
  case *0*
  show *?case* **by** *simp*
next
  case *Suc*
  then show *?case* **by** (*auto simp*: *zero-less-mult-iff*)
qed

lemma *zero-le-power-abs* [*simp*]: $0 \leq |a| ^ n$
  by (*rule zero-le-power* [*OF abs-ge-zero*])

lemma *power2-less-eq-zero-iff* [*simp*]: $a^2 \leq 0 \longleftrightarrow a = 0$
  by (*simp add*: *le-less*)

lemma *abs-power2* [*simp*]: $|a^2| = a^2$
  by (*simp add*: *power2-eq-square*)

lemma *power2-abs* [*simp*]: $|a|^2 = a^2$
  by (*simp add*: *power2-eq-square*)

lemma *odd-power-less-zero*: $a < 0 \Longrightarrow a$ ^ *Suc* (*2* ∗ *n*) < *0*
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  have *a* ^ *Suc* (*2* ∗ *Suc n*) = (*a*∗*a*) ∗ *a* ^ *Suc*(*2*∗*n*)
    by (*simp add*: *ac-simps power-add power2-eq-square*)
  then show *?case*
    by (*simp del*: *power-Suc add*: *Suc mult-less-0-iff mult-neg-neg*)
qed

lemma *odd-0-le-power-imp-0-le*: *0* ≤ *a* ^ *Suc* (*2* ∗ *n*) ⟹ *0* ≤ *a*
  using *odd-power-less-zero* [*of a n*]
  by (*force simp add*: *linorder-not-less* [*symmetric*])

lemma *zero-le-even-power'*[*simp*]: *0* ≤ *a* ^ (*2* ∗ *n*)
proof (*induct n*)
  case *0*
  show *?case* **by** *simp*
next

case (*Suc n*)
have $a \widehat{\ } (2 * Suc\ n) = (a*a) * a \widehat{\ } (2*n)$
by (*simp add*: *ac-simps power-add power2-eq-square*)
then show *?case*
by (*simp add*: *Suc zero-le-mult-iff*)
qed

lemma *sum-power2-ge-zero*: $0 \le x^2 + y^2$
by (*intro add-nonneg-nonneg zero-le-power2*)

lemma *not-sum-power2-lt-zero*: $\neg\ x^2 + y^2 < 0$
unfolding *not-less* **by** (*rule sum-power2-ge-zero*)

lemma *sum-power2-eq-zero-iff*: $x^2 + y^2 = 0 \longleftrightarrow x = 0 \wedge y = 0$
unfolding *power2-eq-square* **by** (*simp add*: *add-nonneg-eq-0-iff*)

lemma *sum-power2-le-zero-iff*: $x^2 + y^2 \le 0 \longleftrightarrow x = 0 \wedge y = 0$
by (*simp add*: *le-less sum-power2-eq-zero-iff not-sum-power2-lt-zero*)

lemma *sum-power2-gt-zero-iff*: $0 < x^2 + y^2 \longleftrightarrow x \neq 0 \vee y \neq 0$
unfolding *not-le* [*symmetric*] **by** (*simp add*: *sum-power2-le-zero-iff*)

lemma *abs-le-square-iff*: $|x| \le |y| \longleftrightarrow x^2 \le y^2$
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
then have $|x|^2 \le |y|^2$ **by** (*rule power-mono*) *simp*
then show *?rhs* **by** *simp*
next
assume *?rhs*
then show *?lhs*
by (*auto intro*!: *power2-le-imp-le* [*OF - abs-ge-zero*])
qed

lemma *power2-le-iff-abs-le*:
$y \ge 0 \Longrightarrow x^2 \le y^2 \longleftrightarrow |x| \le y$
by (*metis abs-le-square-iff abs-of-nonneg*)

lemma *abs-square-le-1*:$x^2 \le 1 \longleftrightarrow |x| \le 1$
using *abs-le-square-iff* [*of x 1*] **by** *simp*

lemma *abs-square-eq-1*: $x^2 = 1 \longleftrightarrow |x| = 1$
by (*auto simp add*: *abs-if power2-eq-1-iff*)

lemma *abs-square-less-1*: $x^2 < 1 \longleftrightarrow |x| < 1$
using *abs-square-eq-1* [*of x*] *abs-square-le-1* [*of x*] **by** (*auto simp add*: *le-less*)

lemma *square-le-1*:
assumes $-\ 1 \le x\ x \le 1$

shows $x^2 \leq 1$
using *assms*
by (*metis add.inverse-inverse linear mult-le-one neg-equal-0-iff-equal neg-le-iff-le power2-eq-square power-minus-Bit0*)

end

47.3 Miscellaneous rules

context *linordered-semidom*
begin

lemma *self-le-power*: $1 \leq a \Longrightarrow 0 < n \Longrightarrow a \leq a \,\hat{}\, n$
using *power-increasing* [*of 1 n a*] *power-one-right* [*of a*] **by** *auto*

lemma *power-le-one-iff*: $0 \leq a \Longrightarrow a \,\hat{}\, n \leq 1 \longleftrightarrow (n = 0 \vee a \leq 1)$
by (*metis* (*mono-tags*) *gr0I nle-le one-le-power power-le-one self-le-power power-0*)

lemma *power-less1-D*: $a\hat{}n < 1 \Longrightarrow a < 1$
using *not-le one-le-power* **by** *blast*

lemma *power-less-one-iff*: $0 \leq a \Longrightarrow a \,\hat{}\, n < 1 \longleftrightarrow (n > 0 \wedge a < 1)$
by (*metis* (*mono-tags*) *power-one power-strict-mono power-less1-D less-le-not-le neq0-conv power-0*)

end

lemma *power2-ge-1-iff*: $x \,\hat{}\, 2 \geq 1 \longleftrightarrow x \geq 1 \vee x \leq (-1 :: 'a :: linordered\text{-}idom)$
using *abs-le-square-iff* [*of 1 x*] **by** (*auto simp*: *abs-if split*: *if-splits*)

lemma (**in** *power*) *power-eq-if*: $p \,\hat{}\, m = (if\ m{=}0\ then\ 1\ else\ p * (p \,\hat{}\, (m - 1)))$
unfolding *One-nat-def* **by** (*cases m*) *simp-all*

lemma (**in** *comm-semiring-1*) *power2-sum*: $(x + y)^2 = x^2 + y^2 + 2 * x * y$
by (*simp add*: *algebra-simps power2-eq-square mult-2-right*)

context *comm-ring-1*
begin

lemma *power2-diff*: $(x - y)^2 = x^2 + y^2 - 2 * x * y$
by (*simp add*: *algebra-simps power2-eq-square mult-2-right*)

lemma *power2-commute*: $(x - y)^2 = (y - x)^2$
by (*simp add*: *algebra-simps power2-eq-square*)

lemma *minus-power-mult-self*: $(- a) \,\hat{}\, n * (- a) \,\hat{}\, n = a \,\hat{}\, (2 * n)$
by (*simp add*: *power-mult-distrib* [*symmetric*])
(*simp add*: *power2-eq-square* [*symmetric*] *power-mult* [*symmetric*])

lemma *minus-one-mult-self* [*simp*]: $(- 1) \hat{~} n * (- 1) \hat{~} n = 1$
  using *minus-power-mult-self* [*of 1 n*] **by** *simp*

lemma *left-minus-one-mult-self* [*simp*]: $(- 1) \hat{~} n * ((- 1) \hat{~} n * a) = a$
  by (*simp add*: *mult.assoc* [*symmetric*])

end

Simprules for comparisons where common factors can be cancelled.

lemmas *zero-compare-simps* =
  add-strict-increasing add-strict-increasing2 add-increasing
  zero-le-mult-iff zero-le-divide-iff
  zero-less-mult-iff zero-less-divide-iff
  mult-le-0-iff divide-le-0-iff
  mult-less-0-iff divide-less-0-iff
  zero-le-power2 power2-less-0

47.4 Exponentiation for the Natural Numbers

lemma *nat-one-le-power* [*simp*]: $Suc\ 0 \leq i \Longrightarrow Suc\ 0 \leq i \hat{~} n$
  by (*rule one-le-power* [*of i n*, *unfolded One-nat-def*])

lemma *nat-zero-less-power-iff* [*simp*]: $x \hat{~} n > 0 \longleftrightarrow x > 0 \lor n = 0$
  for x :: *nat*
  by (*induct n*) *auto*

lemma *nat-power-eq-Suc-0-iff* [*simp*]: $x \hat{~} m = Suc\ 0 \longleftrightarrow m = 0 \lor x = Suc\ 0$
  by (*induct m*) *auto*

lemma *power-Suc-0* [*simp*]: $Suc\ 0 \hat{~} n = Suc\ 0$
  by *simp*

Valid for the naturals, but what if $0 < i < 1$? Premises cannot be weakened: consider the case where $i = 0$, $m = 1$ and $n = 0$.

lemma *nat-power-less-imp-less*:
  fixes i :: *nat*
  assumes *nonneg*: $0 < i$
  assumes *less*: $i \hat{~} m < i \hat{~} n$
  shows $m < n$
proof (*cases* $i = 1$)
  case *True*
  with *less power-one* [**where** $'a = nat$] **show** *?thesis* **by** *simp*
next
  case *False*
  with *nonneg* **have** $1 < i$ **by** *auto*
  from *power-strict-increasing-iff* [*OF this*] *less* **show** *?thesis* ..
qed

lemma *power-gt-expt*: $n > Suc\ 0 \Longrightarrow n\hat{~}k > k$

by (*induction k*) (*auto simp*: *less-trans-Suc n-less-m-mult-n*)

lemma *less-exp* [*simp*]:
‹$n < 2 \hat{} n$›
by (*simp add*: *power-gt-expt*)

lemma *power-dvd-imp-le*:
fixes i :: *nat*
assumes $i \hat{} m$ *dvd* $i \hat{} n$ $1 < i$
shows $m \leq n$
using *assms* **by** (*auto intro*: *power-le-imp-le-exp* [*OF* ‹$1 < i$› *dvd-imp-le*])

lemma *dvd-power-iff-le*:
fixes *k*::*nat*
shows $2 \leq k \Longrightarrow ((k \hat{} m)$ *dvd* $(k \hat{} n) \longleftrightarrow m \leq n)$
using *le-imp-power-dvd power-dvd-imp-le* **by** *force*

lemma *power2-nat-le-eq-le*: $m^2 \leq n^2 \longleftrightarrow m \leq n$
for $m\ n$ :: *nat*
by (*auto intro*: *power2-le-imp-le power-mono*)

lemma *power2-nat-le-imp-le*:
fixes $m\ n$ :: *nat*
assumes $m^2 \leq n$
shows $m \leq n$
proof (*cases m*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc k*)
show *?thesis*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have $n < m$ **by** *simp*
with *assms Suc* **show** *False*
by (*simp add*: *power2-eq-square*)
qed
qed

lemma *ex-power-ivl1*: **fixes** $b\ k$ :: *nat* **assumes** $b \geq 2$
shows $k \geq 1 \Longrightarrow \exists n.\ b\hat{}n \leq k \wedge k < b\hat{}(n+1)$ (**is** - $\Longrightarrow \exists n.$ *?P k n*)
proof(*induction k*)
case *0* **thus** *?case* **by** *simp*
next
case (*Suc k*)
show *?case*
proof *cases*
assume *k=0*
hence *?P* (*Suc k*) *0* **using** *assms* **by** *simp*

```
    thus ?case ..
  next
    assume k≠0
    with Suc obtain n where IH: ?P k n by auto
    show ?case
    proof (cases k = b^(n+1) − 1)
      case True
      hence ?P (Suc k) (n+1) using assms
        by (simp add: power-less-power-Suc)
      thus ?thesis ..
    next
      case False
      hence ?P (Suc k) n using IH by auto
      thus ?thesis ..
    qed
  qed
qed

lemma ex-power-ivl2: fixes b k :: nat assumes b ≥ 2 k ≥ 2
  shows ∃ n. b^n < k ∧ k ≤ b^(n+1)
proof −
  have 1 ≤ k − 1 using assms(2) by arith
  from ex-power-ivl1[OF assms(1) this]
  obtain n where b ^ n ≤ k − 1 ∧ k − 1 < b ^ (n + 1) ..
  hence b^n < k ∧ k ≤ b^(n+1) using assms by auto
  thus ?thesis ..
qed
```

47.4.1 Cardinality of the Powerset

```
lemma card-UNIV-bool [simp]: card (UNIV :: bool set) = 2
  unfolding UNIV-bool by simp

lemma card-Pow: finite A ⟹ card (Pow A) = 2 ^ card A
proof (induct rule: finite-induct)
  case empty
  show ?case by simp
next
  case (insert x A)
  from ‹x ∉ A› have disjoint: Pow A ∩ insert x ‘ Pow A = {} by blast
  from ‹x ∉ A› have inj-on: inj-on (insert x) (Pow A)
    unfolding inj-on-def by auto

  have card (Pow (insert x A)) = card (Pow A ∪ insert x ‘ Pow A)
    by (simp only: Pow-insert)
  also have ... = card (Pow A) + card (insert x ‘ Pow A)
    by (rule card-Un-disjoint) (use ‹finite A› disjoint in simp-all)
  also from inj-on have card (insert x ‘ Pow A) = card (Pow A)
    by (rule card-image)
```

also have ... + ... = *2* ∗ ... **by** (*simp add*: *mult-2*)
also from *insert*(*3*) **have** ... = *2* ^ *Suc* (*card A*) **by** *simp*
also from *insert*(*1*,*2*) **have** *Suc* (*card A*) = *card* (*insert x A*)
by (*rule card-insert-disjoint* [*symmetric*])
finally show *?case* **.**
qed

47.5 Code generator tweak

code-identifier
code-module *Power* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

48 Big sum and product over finite (non-empty) sets

theory *Groups-Big*
imports *Power Equiv-Relations*
begin

48.1 Generic monoid operation over a set

locale *comm-monoid-set* = *comm-monoid*
begin

48.1.1 Standard sum or product indexed by a finite set

interpretation *comp-fun-commute f*
by *standard* (*simp add*: *fun-eq-iff left-commute*)

interpretation *comp?*: *comp-fun-commute f* ∘ *g*
by (*fact comp-comp-fun-commute*)

definition *F* :: ('*b* ⇒ '*a*) ⇒ '*b set* ⇒ '*a*
where *eq-fold*: *F g A* = *Finite-Set.fold* (*f* ∘ *g*) **1** *A*

lemma *infinite* [*simp*]: ¬ *finite A* ⟹ *F g A* = **1**
by (*simp add*: *eq-fold*)

lemma *empty* [*simp*]: *F g* {} = **1**
by (*simp add*: *eq-fold*)

lemma *insert* [*simp*]: *finite A* ⟹ *x* ∉ *A* ⟹ *F g* (*insert x A*) = *g x* ∗ *F g A*
by (*simp add*: *eq-fold*)

lemma *remove*:
assumes *finite A* **and** *x* ∈ *A*
shows *F g A* = *g x* ∗ *F g* (*A* − {*x*})

proof −
  from ‹$x \in A$› **obtain** B **where** B: $A = insert\ x\ B$ **and** $x \notin B$
    by (*auto dest*: *mk-disjoint-insert*)
  moreover from ‹*finite* A› B **have** *finite* B **by** *simp*
  ultimately show *?thesis* **by** *simp*
qed

lemma *insert-remove*: *finite* $A \Longrightarrow F\ g\ (insert\ x\ A) = g\ x * F\ g\ (A - \{x\})$
  by (*cases* $x \in A$) (*simp-all add*: *remove insert-absorb*)

lemma *insert-if*: *finite* $A \Longrightarrow F\ g\ (insert\ x\ A) = (if\ x \in A\ then\ F\ g\ A\ else\ g\ x *$
$F\ g\ A)$
  by (*cases* $x \in A$) (*simp-all add*: *insert-absorb*)

lemma *neutral*: $\forall x{\in}A.\ g\ x = \mathbf{1} \Longrightarrow F\ g\ A = \mathbf{1}$
  by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *neutral-const* [*simp*]: $F\ (\lambda\text{-}.\ \mathbf{1})\ A = \mathbf{1}$
  by (*simp add*: *neutral*)

lemma *union-inter*:
  assumes *finite* A **and** *finite* B
  shows $F\ g\ (A \cup B) * F\ g\ (A \cap B) = F\ g\ A * F\ g\ B$
  — The reversed orientation looks more natural, but LOOPS as a simprule!
  using *assms*
proof (*induct A*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert x A*)
  then show *?case*
   by (*auto simp*: *insert-absorb Int-insert-left commute* [*of - g x*] *assoc left-commute*)
qed

corollary *union-inter-neutral*:
  assumes *finite* A **and** *finite* B
    and $\forall x \in A \cap B.\ g\ x = \mathbf{1}$
  shows $F\ g\ (A \cup B) = F\ g\ A * F\ g\ B$
  using *assms* **by** (*simp add*: *union-inter* [*symmetric*] *neutral*)

corollary *union-disjoint*:
  assumes *finite* A **and** *finite* B
  assumes $A \cap B = \{\}$
  shows $F\ g\ (A \cup B) = F\ g\ A * F\ g\ B$
  using *assms* **by** (*simp add*: *union-inter-neutral*)

lemma *union-diff2*:
  assumes *finite* A **and** *finite* B
  shows $F\ g\ (A \cup B) = F\ g\ (A - B) * F\ g\ (B - A) * F\ g\ (A \cap B)$

proof −
  have $A \cup B = A - B \cup (B - A) \cup A \cap B$
    by *auto*
  with *assms* **show** *?thesis*
    by *simp* (*subst union-disjoint*, *auto*)+
qed

lemma *subset-diff*:
  assumes $B \subseteq A$ **and** *finite* A
  shows $F\ g\ A = F\ g\ (A - B) * F\ g\ B$
proof −
  from *assms* **have** *finite* $(A - B)$ **by** *auto*
  moreover from *assms* **have** *finite* B **by** (*rule finite-subset*)
  moreover from *assms* **have** $(A - B) \cap B = \{\}$ **by** *auto*
  ultimately have $F\ g\ (A - B \cup B) = F\ g\ (A - B) * F\ g\ B$ **by** (*rule union-disjoint*)
  moreover from *assms* **have** $A \cup B = A$ **by** *auto*
  ultimately show *?thesis* **by** *simp*
qed

lemma *Int-Diff*:
  assumes *finite* A
  shows $F\ g\ A = F\ g\ (A \cap B) * F\ g\ (A - B)$
  by (*subst subset-diff* [**where** $B = A - B$]) (*auto simp*: *Diff-Diff-Int assms*)

lemma *setdiff-irrelevant*:
  assumes *finite* A
  shows $F\ g\ (A - \{x.\ g\ x = z\}) = F\ g\ A$
  using *assms* **by** (*induct A*) (*simp-all add*: *insert-Diff-if*)

lemma *not-neutral-contains-not-neutral*:
  assumes $F\ g\ A \neq \mathbf{1}$
  obtains a **where** $a \in A$ **and** $g\ a \neq \mathbf{1}$
proof −
  from *assms* **have** $\exists a \in A.\ g\ a \neq \mathbf{1}$
  proof (*induct A rule*: *infinite-finite-induct*)
    case *infinite*
    then show *?case* **by** *simp*
  next
    case *empty*
    then show *?case* **by** *simp*
  next
    case (*insert a A*)
    then show *?case* **by** *fastforce*
  qed
  with *that* **show** *thesis* **by** *blast*
qed

lemma *reindex*:
  assumes *inj-on h A*

shows $F\ g\ (h\ `\ A) = F\ (g \circ h)\ A$
proof (*cases finite A*)
case *True*
with *assms* **show** *?thesis*
by (*simp add*: *eq-fold fold-image comp-assoc*)
next
case *False*
with *assms* **have** $\neg$ *finite* $(h\ `\ A)$ **by** (*blast dest*: *finite-imageD*)
with *False* **show** *?thesis* **by** *simp*
qed

lemma *cong* [*fundef-cong*]:
assumes $A = B$
assumes *g-h*: $\bigwedge x.\ x \in B \Longrightarrow g\ x = h\ x$
shows $F\ g\ A = F\ h\ B$
using *g-h* **unfolding** ‹$A = B$›
by (*induct B rule*: *infinite-finite-induct*) *auto*

lemma *cong-simp* [*cong*]:
$[\![\ A = B;\ \bigwedge x.\ x \in B =simp=> g\ x = h\ x\ ]\!] \Longrightarrow F\ (\lambda x.\ g\ x)\ A = F\ (\lambda x.\ h\ x)\ B$
by (*rule cong*) (*simp-all add*: *simp-implies-def*)

lemma *reindex-cong*:
assumes *inj-on l B*
assumes $A = l\ `\ B$
assumes $\bigwedge x.\ x \in B \Longrightarrow g\ (l\ x) = h\ x$
shows $F\ g\ A = F\ h\ B$
using *assms* **by** (*simp add*: *reindex*)

lemma *image-eq*:
assumes *inj-on g A*
shows $F\ (\lambda x.\ x)\ (g\ `\ A) = F\ g\ A$
using *assms reindex-cong* **by** *fastforce*

lemma *UNION-disjoint*:
assumes *finite I* **and** $\forall i{\in}I.$ *finite* $(A\ i)$
and $\forall i{\in}I.\ \forall j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$
shows $F\ g\ (\bigcup (A\ `\ I)) = F\ (\lambda x.\ F\ g\ (A\ x))\ I$
using *assms*
proof (*induction rule*: *finite-induct*)
case (*insert i I*)
then have $\forall j{\in}I.\ j \neq i$
by *blast*
with *insert.prems* **have** $A\ i \cap \bigcup (A\ `\ I) = \{\}$
by *blast*
with *insert* **show** *?case*
by (*simp add*: *union-disjoint*)
qed *auto*

```
lemma Union-disjoint:
  assumes ∀ A∈C. finite A ∀ A∈C. ∀ B∈C. A ≠ B ⟶ A ∩ B = {}
  shows F g (⋃ C) = (F ∘ F) g C
proof (cases finite C)
  case True
  from UNION-disjoint [OF this assms] show ?thesis by simp
next
  case False
  then show ?thesis by (auto dest: finite-UnionD intro: infinite)
qed

lemma distrib: F (λx. g x ∗ h x) A = F g A ∗ F h A
 by (induct A rule: infinite-finite-induct) (simp-all add: assoc commute left-commute)

lemma Sigma:
  assumes finite A ∀ x∈A. finite (B x)
  shows F (λx. F (g x) (B x)) A = F (case-prod g) (SIGMA x:A. B x)
  unfolding Sigma-def
proof (subst UNION-disjoint)
  show F (λx. F (g x) (B x)) A = F (λx. F (λ(x, y). g x y) (⋃ y∈B x. {(x, y)}))
A
  proof (rule cong [OF refl])
    show F (g x) (B x) = F (λ(x, y). g x y) (⋃ y∈B x. {(x, y)})
      if x ∈ A for x
      using that assms by (simp add: UNION-disjoint)
  qed
qed (use assms in auto)

lemma related:
  assumes Re: R 1 1
    and Rop: ∀ x1 y1 x2 y2. R x1 x2 ∧ R y1 y2 ⟶ R (x1 ∗ y1) (x2 ∗ y2)
    and fin: finite S
    and R-h-g: ∀ x∈S. R (h x) (g x)
  shows R (F h S) (F g S)
  using fin by (rule finite-subset-induct) (use assms in auto)

lemma mono-neutral-cong-left:
  assumes finite T
    and S ⊆ T
    and ∀ i ∈ T − S. h i = 1
    and ⋀x. x ∈ S ⟹ g x = h x
  shows F g S = F h T
proof−
  have eq: T = S ∪ (T − S) using ‹S ⊆ T› by blast
  have d: S ∩ (T − S) = {} using ‹S ⊆ T› by blast
  from ‹finite T› ‹S ⊆ T› have f: finite S finite (T − S)
    by (auto intro: finite-subset)
  show ?thesis using assms(4)
     by (simp add: union-disjoint [OF f d, unfolded eq [symmetric]] neutral [OF
```

assms(*3*)])
qed

lemma *mono-neutral-cong-right*:
  finite $T \Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x)$
$\Longrightarrow$
    $F\ g\ T = F\ h\ S$
  by (*auto intro*!: *mono-neutral-cong-left* [*symmetric*])

lemma *mono-neutral-left*: *finite* $T \Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow F\ g$
$S = F\ g\ T$
  by (*blast intro*: *mono-neutral-cong-left*)

lemma *mono-neutral-right*: *finite* $T \Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow F$
$g\ T = F\ g\ S$
  by (*blast intro*!: *mono-neutral-left* [*symmetric*])

lemma *mono-neutral-cong*:
  assumes [*simp*]: *finite T finite S*
    and $*$: $\bigwedge i.\ i \in T - S \Longrightarrow h\ i = \mathbf{1}$ $\bigwedge i.\ i \in S - T \Longrightarrow g\ i = \mathbf{1}$
    and *gh*: $\bigwedge x.\ x \in S \cap T \Longrightarrow g\ x = h\ x$
 shows $F\ g\ S = F\ h\ T$
proof−
  have $F\ g\ S = F\ g\ (S \cap T)$
    by(*rule mono-neutral-right*)(*auto intro*: $*$)
  also have $\ldots = F\ h\ (S \cap T)$ **using** *refl gh* **by**(*rule cong*)
  also have $\ldots = F\ h\ T$
    by(*rule mono-neutral-left*)(*auto intro*: $*$)
  finally show *?thesis* **.**
qed

lemma *reindex-bij-betw*: *bij-betw* $h\ S\ T \Longrightarrow F\ (\lambda x.\ g\ (h\ x))\ S = F\ g\ T$
  by (*auto simp*: *bij-betw-def reindex*)

lemma *reindex-bij-witness*:
  assumes *witness*:
    $\bigwedge a.\ a \in S \Longrightarrow i\ (j\ a) = a$
    $\bigwedge a.\ a \in S \Longrightarrow j\ a \in T$
    $\bigwedge b.\ b \in T \Longrightarrow j\ (i\ b) = b$
    $\bigwedge b.\ b \in T \Longrightarrow i\ b \in S$
  assumes *eq*:
    $\bigwedge a.\ a \in S \Longrightarrow h\ (j\ a) = g\ a$
  shows $F\ g\ S = F\ h\ T$
proof −
  have *bij-betw* $j\ S\ T$
    using *bij-betw-byWitness*[**where** $A{=}S$ **and** $f{=}j$ **and** $f'{=}i$ **and** $A'{=}T$] *witness*
by *auto*
  moreover have $F\ g\ S = F\ (\lambda x.\ h\ (j\ x))\ S$
    by (*intro cong*) (*auto simp*: *eq*)

ultimately show *?thesis*
by (*simp add*: *reindex-bij-betw*)
qed

lemma *reindex-bij-betw-not-neutral*:
assumes *fin*: *finite* S' *finite* T'
assumes *bij*: *bij-betw* h $(S - S')$ $(T - T')$
assumes *nn*:
$\bigwedge a.\ a \in S' \Longrightarrow g\ (h\ a) = z$
$\bigwedge b.\ b \in T' \Longrightarrow g\ b = z$
shows $F\ (\lambda x.\ g\ (h\ x))\ S = F\ g\ T$
proof –
have [*simp*]: *finite* $S \longleftrightarrow$ *finite* T
using *bij-betw-finite*[*OF bij*] *fin* **by** *auto*
show *?thesis*
proof (*cases finite* S)
case *True*
with *nn* **have** $F\ (\lambda x.\ g\ (h\ x))\ S = F\ (\lambda x.\ g\ (h\ x))\ (S - S')$
by (*intro mono-neutral-cong-right*) *auto*
also have $\ldots = F\ g\ (T - T')$
using *bij* **by** (*rule reindex-bij-betw*)
also have $\ldots = F\ g\ T$
using *nn* ‹*finite* S› **by** (*intro mono-neutral-cong-left*) *auto*
finally show *?thesis* **.**
next
case *False*
then show *?thesis* **by** *simp*
qed
qed

lemma *reindex-nontrivial*:
assumes *finite* A
and *nz*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow h\ x = h\ y \Longrightarrow g\ (h\ x) = \mathbf{1}$
shows $F\ g\ (h\ `\ A) = F\ (g \circ h)\ A$
proof (*subst reindex-bij-betw-not-neutral* [*symmetric*])
show *bij-betw* $h\ (A - \{x \in A.\ (g \circ h)\ x = \mathbf{1}\})\ (h\ `\ A - h\ `\ \{x \in A.\ (g \circ h)\ x = \mathbf{1}\})$
using *nz* **by** (*auto intro*!: *inj-onI simp*: *bij-betw-def*)
qed (*use* ‹*finite* A› **in** *auto*)

lemma *reindex-bij-witness-not-neutral*:
assumes *fin*: *finite* S' *finite* T'
assumes *witness*:
$\bigwedge a.\ a \in S - S' \Longrightarrow i\ (j\ a) = a$
$\bigwedge a.\ a \in S - S' \Longrightarrow j\ a \in T - T'$
$\bigwedge b.\ b \in T - T' \Longrightarrow j\ (i\ b) = b$
$\bigwedge b.\ b \in T - T' \Longrightarrow i\ b \in S - S'$
assumes *nn*:
$\bigwedge a.\ a \in S' \Longrightarrow g\ a = z$

  $\bigwedge b.\ b \in T' \Longrightarrow h\ b = z$
  assumes *eq*:
  $\bigwedge a.\ a \in S \Longrightarrow h\ (j\ a) = g\ a$
  shows $F\ g\ S = F\ h\ T$
proof −
  have *bij*: *bij-betw j* $(S - (S' \cap S))\ (T - (T' \cap T))$
    using *witness* **by** (*intro bij-betw-byWitness*[**where** f'=*i*]) *auto*
  have *F-eq*: $F\ g\ S = F\ (\lambda x.\ h\ (j\ x))\ S$
    by (*intro cong*) (*auto simp*: *eq*)
  show *?thesis*
    unfolding *F-eq* **using** *fin nn eq*
    by (*intro reindex-bij-betw-not-neutral*[*OF - - bij*]) *auto*
qed

lemma *delta-remove*:
  assumes *fS*: *finite S*
  shows $F\ (\lambda k.$ *if* $k = a$ *then* $b\ k$ *else* $c\ k)\ S = ($*if* $a \in S$ *then* $b\ a * F\ c\ (S-\{a\})$
else $F\ c\ (S-\{a\}))$
proof −
  let *?f* = ($\lambda k.$ *if* $k = a$ *then* $b\ k$ *else* $c\ k$)
  show *?thesis*
  proof (*cases* $a \in S$)
    case *False*
    then have $\forall k \in S.$ *?f* $k = c\ k$ **by** *simp*
    with *False* **show** *?thesis* **by** *simp*
  next
    case *True*
    let *?A* = $S - \{a\}$
    let *?B* = $\{a\}$
    from *True* **have** *eq*: S = *?A* ∪ *?B* **by** *blast*
    have *dj*: *?A* ∩ *?B* = {} **by** *simp*
    from *fS* **have** *fAB*: *finite ?A finite ?B* **by** *auto*
    have *F ?f S = F ?f ?A ∗ F ?f ?B*
      using *union-disjoint* [*OF fAB dj*, *of ?f*, *unfolded eq* [*symmetric*]] **by** *simp*
    with *True* **show** *?thesis*
      using *comm-monoid-set.remove comm-monoid-set-axioms fS* **by** *fastforce*
  qed
qed

lemma *delta* [*simp*]:
  assumes *fS*: *finite S*
  shows $F\ (\lambda k.$ *if* $k = a$ *then* $b\ k$ *else* $\mathbf{1})\ S = ($*if* $a \in S$ *then* $b\ a$ *else* $\mathbf{1})$
  by (*simp add*: *delta-remove* [*OF assms*])

lemma *delta'* [*simp*]:
  assumes *fin*: *finite S*
  shows $F\ (\lambda k.$ *if* $a = k$ *then* $b\ k$ *else* $\mathbf{1})\ S = ($*if* $a \in S$ *then* $b\ a$ *else* $\mathbf{1})$
  using *delta* [*OF fin*, *of a b*, *symmetric*] **by** (*auto intro*: *cong*)

lemma *If-cases*:
fixes P :: $'b \Rightarrow$ *bool* **and** $g\ h$ :: $'b \Rightarrow 'a$
assumes *fin*: *finite A*
shows $F\ (\lambda x.\ \textit{if}\ P\ x\ \textit{then}\ h\ x\ \textit{else}\ g\ x)\ A = F\ h\ (A \cap \{x.\ P\ x\}) * F\ g\ (A \cap -\{x.\ P\ x\})$
proof −
have *a*: $A = A \cap \{x.\ P\ x\} \cup A \cap -\{x.\ P\ x\}\ (A \cap \{x.\ P\ x\}) \cap (A \cap -\{x.\ P\ x\}) = \{\}$
by *blast+*
from *fin* **have** *f*: *finite* $(A \cap \{x.\ P\ x\})$ *finite* $(A \cap -\{x.\ P\ x\})$ **by** *auto*
let *?g* = $\lambda x.$ *if P x then h x else g x*
from *union-disjoint* [*OF f a*(*2*), *of ?g*] *a*(*1*) **show** *?thesis*
by (*subst* (*1 2*) *cong*) *simp-all*
qed

lemma *cartesian-product*: $F\ (\lambda x.\ F\ (g\ x)\ B)\ A = F\ (\textit{case-prod}\ g)\ (A \times B)$
proof (*cases* $A = \{\} \vee B = \{\}$)
case *True*
then show *?thesis*
by *auto*
next
case *False*
then have $A \neq \{\}\ B \neq \{\}$ **by** *auto*
show *?thesis*
proof (*cases finite A* $\wedge$ *finite B*)
case *True*
then show *?thesis*
by (*simp add*: *Sigma*)
next
case *False*
then consider *infinite A* | *infinite B* **by** *auto*
then have *infinite* $(A \times B)$
by *cases* (*use* ‹$A \neq \{\}$› ‹$B \neq \{\}$› **in** ‹*auto dest*: *finite-cartesian-productD1 finite-cartesian-productD2*›)
then show *?thesis*
using *False* **by** *auto*
qed
qed

lemma *cartesian-product'*:
$F\ g\ (A \times B) = F\ (\lambda x.\ F\ (\lambda y.\ g\ (x,y))\ B)\ A$
unfolding *cartesian-product* **by** *simp*

lemma *inter-restrict*:
assumes *finite A*
shows $F\ g\ (A \cap B) = F\ (\lambda x.\ \textit{if}\ x \in B\ \textit{then}\ g\ x\ \textit{else}\ \mathbf{1})\ A$
proof −
let *?g* = $\lambda x.$ *if* $x \in A \cap B$ *then g x else* $\mathbf{1}$

have $\forall i{\in}A - A \cap B.$ (*if* $i \in A \cap B$ *then* $g\ i$ *else* $\mathbf{1}$) $= \mathbf{1}$ **by** *simp*
moreover have $A \cap B \subseteq A$ **by** *blast*
ultimately have $F\ ?g\ (A \cap B) = F\ ?g\ A$
using ‹*finite A*› **by** (*intro mono-neutral-left*) *auto*
then show *?thesis* **by** *simp*
qed

lemma *inter-filter*:
finite $A \Longrightarrow F\ g\ \{x \in A.\ P\ x\} = F\ (\lambda x.$ *if* $P\ x$ *then* $g\ x$ *else* $\mathbf{1})\ A$
by (*simp add*: *inter-restrict* [*symmetric*, *of* A $\{x.\ P\ x\}$ g, *simplified mem-Collect-eq*] *Int-def*)

lemma *Union-comp*:
assumes $\forall A \in B.$ *finite* A
and $\bigwedge A1\ A2\ x.\ A1 \in B \Longrightarrow A2 \in B \Longrightarrow A1 \neq A2 \Longrightarrow x \in A1 \Longrightarrow x \in A2 \Longrightarrow g\ x = \mathbf{1}$
shows $F\ g\ (\bigcup B) = (F \circ F)\ g\ B$
using *assms*
proof (*induct B rule*: *infinite-finite-induct*)
case (*infinite A*)
then have $\neg$ *finite* $(\bigcup A)$ **by** (*blast dest*: *finite-UnionD*)
with *infinite* **show** *?case* **by** *simp*
next
case *empty*
then show *?case* **by** *simp*
next
case (*insert A B*)
then have *finite* A *finite* B *finite* $(\bigcup B)$ $A \notin B$
and $\forall x{\in}A \cap \bigcup B.\ g\ x = \mathbf{1}$
and H: $F\ g\ (\bigcup B) = (F \circ F)\ g\ B$ **by** *auto*
then have $F\ g\ (A \cup \bigcup B) = F\ g\ A * F\ g\ (\bigcup B)$
by (*simp add*: *union-inter-neutral*)
with ‹*finite B*› ‹$A \notin B$› **show** *?case*
by (*simp add*: H)
qed

lemma *swap*: $F\ (\lambda i.\ F\ (g\ i)\ B)\ A = F\ (\lambda j.\ F\ (\lambda i.\ g\ i\ j)\ A)\ B$
unfolding *cartesian-product*
by (*rule reindex-bij-witness* [**where** $i = \lambda(i, j).\ (j, i)$ **and** $j = \lambda(i, j).\ (j, i)$]) *auto*

lemma *swap-restrict*:
finite $A \Longrightarrow$ *finite* $B \Longrightarrow$
$F\ (\lambda x.\ F\ (g\ x)\ \{y.\ y \in B \wedge R\ x\ y\})\ A = F\ (\lambda y.\ F\ (\lambda x.\ g\ x\ y)\ \{x.\ x \in A \wedge R\ x\ y\})\ B$
by (*simp add*: *inter-filter*) (*rule swap*)

lemma *image-gen*:
assumes *fin*: *finite S*

shows $F\ h\ S = F\ (\lambda y.\ F\ h\ \{x.\ x \in S \wedge g\ x = y\})\ (g\ `\ S)$
proof −
have $\{y.\ y\in g`S \wedge g\ x = y\} = \{g\ x\}$ **if** $x \in S$ **for** x
using *that* **by** *auto*
then have $F\ h\ S = F\ (\lambda x.\ F\ (\lambda y.\ h\ x)\ \{y.\ y\in g`S \wedge g\ x = y\})\ S$
by *simp*
also have $\ldots = F\ (\lambda y.\ F\ h\ \{x.\ x \in S \wedge g\ x = y\})\ (g\ `\ S)$
by (*rule swap-restrict* [*OF fin finite-imageI* [*OF fin*]])
finally show *?thesis* **.**
qed

lemma *group*:
assumes *fS*: *finite S* **and** *fT*: *finite T* **and** *fST*: $g\ `\ S \subseteq T$
shows $F\ (\lambda y.\ F\ h\ \{x.\ x \in S \wedge g\ x = y\})\ T = F\ h\ S$
unfolding *image-gen*[*OF fS*, *of h g*]
by (*auto intro*: *neutral mono-neutral-right*[*OF fT fST*])

lemma *Plus*:
fixes A :: *'b set* **and** B :: *'c set*
assumes *fin*: *finite A finite B*
shows $F\ g\ (A <+> B) = F\ (g \circ Inl)\ A * F\ (g \circ Inr)\ B$
proof −
have $A <+> B = Inl\ `\ A \cup Inr\ `\ B$ **by** *auto*
moreover from *fin* **have** *finite* (*Inl* ‘ *A*) *finite* (*Inr* ‘ *B*) **by** *auto*
moreover have $Inl\ `\ A \cap Inr\ `\ B = \{\}$ **by** *auto*
moreover have *inj-on Inl A inj-on Inr B* **by** (*auto intro*: *inj-onI*)
ultimately show *?thesis*
using *fin* **by** (*simp add*: *union-disjoint reindex*)
qed

lemma *same-carrier*:
assumes *finite C*
assumes *subset*: $A \subseteq C\ B \subseteq C$
assumes *trivial*: $\bigwedge a.\ a \in C - A \Longrightarrow g\ a = \mathbf{1}\ \bigwedge b.\ b \in C - B \Longrightarrow h\ b = \mathbf{1}$
shows $F\ g\ A = F\ h\ B \longleftrightarrow F\ g\ C = F\ h\ C$
proof −
have *finite A* **and** *finite B* **and** *finite* $(C - A)$ **and** *finite* $(C - B)$
using ‹*finite C*› *subset* **by** (*auto elim*: *finite-subset*)
from *subset* **have** [*simp*]: $A - (C - A) = A$ **by** *auto*
from *subset* **have** [*simp*]: $B - (C - B) = B$ **by** *auto*
from *subset* **have** $C = A \cup (C - A)$ **by** *auto*
then have $F\ g\ C = F\ g\ (A \cup (C - A))$ **by** *simp*
also have $\ldots = F\ g\ (A - (C - A)) * F\ g\ (C - A - A) * F\ g\ (A \cap (C - A))$
using ‹*finite A*› ‹*finite* $(C - A)$› **by** (*simp only*: *union-diff2*)
finally have ∗: $F\ g\ C = F\ g\ A$ **using** *trivial* **by** *simp*
from *subset* **have** $C = B \cup (C - B)$ **by** *auto*
then have $F\ h\ C = F\ h\ (B \cup (C - B))$ **by** *simp*
also have $\ldots = F\ h\ (B - (C - B)) * F\ h\ (C - B - B) * F\ h\ (B \cap (C - B))$
using ‹*finite B*› ‹*finite* $(C - B)$› **by** (*simp only*: *union-diff2*)

finally have $F\ h\ C = F\ h\ B$
  using *trivial* **by** *simp*
with $*$ **show** *?thesis* **by** *simp*
qed

lemma *same-carrierI*:
  assumes *finite* C
  assumes *subset*: $A \subseteq C\ B \subseteq C$
  assumes *trivial*: $\bigwedge a.\ a \in C - A \Longrightarrow g\ a = \mathbf{1}\ \bigwedge b.\ b \in C - B \Longrightarrow h\ b = \mathbf{1}$
  assumes $F\ g\ C = F\ h\ C$
  shows $F\ g\ A = F\ h\ B$
  using *assms same-carrier* [*of* $C\ A\ B$] **by** *simp*

lemma *eq-general*:
  assumes B: $\bigwedge y.\ y \in B \Longrightarrow \exists!x.\ x \in A \wedge h\ x = y$ **and** A: $\bigwedge x.\ x \in A \Longrightarrow h\ x \in B \wedge \gamma(h\ x) = \varphi\ x$
  shows $F\ \varphi\ A = F\ \gamma\ B$
proof $-$
  have *eq*: $B = h\ \text{`}\ A$
    by (*auto dest*: *assms*)
  have h: *inj-on* $h\ A$
    using *assms* **by** (*blast intro*: *inj-onI*)
  have $F\ \varphi\ A = F\ (\gamma \circ h)\ A$
    using A **by** *auto*
  also have $\ldots = F\ \gamma\ B$
    by (*simp add*: *eq reindex h*)
  finally show *?thesis* **.**
qed

lemma *eq-general-inverses*:
  assumes B: $\bigwedge y.\ y \in B \Longrightarrow k\ y \in A \wedge h(k\ y) = y$ **and** A: $\bigwedge x.\ x \in A \Longrightarrow h\ x \in B \wedge k(h\ x) = x \wedge \gamma(h\ x) = \varphi\ x$
  shows $F\ \varphi\ A = F\ \gamma\ B$
  by (*rule eq-general* [**where** *h*=*h*]) (*force intro*: *dest*: $A\ B$)+

48.1.2 HOL Light variant: sum/product indexed by the non-neutral subset

NB only a subset of the properties above are proved

definition G :: $['b \Rightarrow 'a,'b\ set] \Rightarrow 'a$
  where $G\ p\ I \equiv$ *if finite* $\{x \in I.\ p\ x \neq \mathbf{1}\}$ *then* $F\ p\ \{x \in I.\ p\ x \neq \mathbf{1}\}$ *else* $\mathbf{1}$

lemma *finite-Collect-op*:
  shows $[\![$*finite* $\{i \in I.\ x\ i \neq \mathbf{1}\}$; *finite* $\{i \in I.\ y\ i \neq \mathbf{1}\}]\!] \Longrightarrow$ *finite* $\{i \in I.\ x\ i * y\ i \neq \mathbf{1}\}$
  apply (*rule finite-subset* [**where** $B = \{i \in I.\ x\ i \neq \mathbf{1}\} \cup \{i \in I.\ y\ i \neq \mathbf{1}\}$])
  using *left-neutral* **by** *force*+

lemma *empty'* [*simp*]: $G\ p\ \{\} = \mathbf{1}$

by (*auto simp*: *G-def*)

lemma *eq-sum* [*simp*]: *finite* $I \Longrightarrow G\ p\ I = F\ p\ I$
by (*auto simp*: *G-def intro*: *mono-neutral-cong-left*)

lemma *insert'* [*simp*]:
assumes *finite* $\{x \in I.\ p\ x \neq \mathbf{1}\}$
shows $G\ p$ (*insert* $i\ I$) = (*if* $i \in I$ *then* $G\ p\ I$ *else* $p\ i * G\ p\ I$)
proof –
have $\{x.\ x = i \wedge p\ x \neq \mathbf{1} \vee x \in I \wedge p\ x \neq \mathbf{1}\}$ = (*if* $p\ i = \mathbf{1}$ *then* $\{x \in I.\ p\ x \neq \mathbf{1}\}$ *else insert* i $\{x \in I.\ p\ x \neq \mathbf{1}\}$)
by *auto*
then show *?thesis*
using *assms* **by** (*simp add*: *G-def conj-disj-distribR insert-absorb*)
qed

lemma *distrib-triv'*:
assumes *finite I*
shows $G\ (\lambda i.\ g\ i * h\ i)\ I = G\ g\ I * G\ h\ I$
by (*simp add*: *assms local.distrib*)

lemma *non-neutral'*: $G\ g\ \{x \in I.\ g\ x \neq \mathbf{1}\} = G\ g\ I$
by (*simp add*: *G-def*)

lemma *distrib'*:
assumes *finite* $\{x \in I.\ g\ x \neq \mathbf{1}\}$ *finite* $\{x \in I.\ h\ x \neq \mathbf{1}\}$
shows $G\ (\lambda i.\ g\ i * h\ i)\ I = G\ g\ I * G\ h\ I$
proof –
have $a * a \neq a \Longrightarrow a \neq \mathbf{1}$ **for** a
by *auto*
then have $G\ (\lambda i.\ g\ i * h\ i)\ I = G\ (\lambda i.\ g\ i * h\ i)\ (\{i \in I.\ g\ i \neq \mathbf{1}\} \cup \{i \in I.\ h\ i \neq \mathbf{1}\})$
using *assms* **by** (*force simp*: *G-def finite-Collect-op intro*!: *mono-neutral-cong*)
also have $\ldots = G\ g\ I * G\ h\ I$
proof –
have $F\ g\ (\{i \in I.\ g\ i \neq \mathbf{1}\} \cup \{i \in I.\ h\ i \neq \mathbf{1}\}) = G\ g\ I$
$F\ h\ (\{i \in I.\ g\ i \neq \mathbf{1}\} \cup \{i \in I.\ h\ i \neq \mathbf{1}\}) = G\ h\ I$
by (*auto simp*: *G-def assms intro*: *mono-neutral-right*)
then show *?thesis*
using *assms* **by** (*simp add*: *distrib*)
qed
finally show *?thesis* **.**
qed

lemma *cong'*:
assumes $A = B$
assumes *g-h*: $\bigwedge x.\ x \in B \Longrightarrow g\ x = h\ x$
shows $G\ g\ A = G\ h\ B$
using *assms* **by** (*auto simp*: *G-def cong*: *conj-cong intro*: *cong*)

lemma *mono-neutral-cong-left'*:
  assumes $S \subseteq T$
    and $\bigwedge i.\ i \in T - S \Longrightarrow h\ i = \mathbf{1}$
    and $\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x$
  shows $G\ g\ S = G\ h\ T$
proof –
  have $*$: $\{x \in S.\ g\ x \neq \mathbf{1}\} = \{x \in T.\ h\ x \neq \mathbf{1}\}$
    using *assms* **by** (*metis DiffI subset-eq*)
  then have $finite\ \{x \in S.\ g\ x \neq \mathbf{1}\} = finite\ \{x \in T.\ h\ x \neq \mathbf{1}\}$
    by *simp*
  then show *?thesis*
    using *assms* **by** (*auto simp add*: *G-def* $*$ *intro*: *cong*)
qed

lemma *mono-neutral-cong-right'*:
  $S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x) \Longrightarrow$
    $G\ g\ T = G\ h\ S$
  by (*auto intro*!: *mono-neutral-cong-left'* [*symmetric*])

lemma *mono-neutral-left'*: $S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow G\ g\ S = G\ g\ T$
  by (*blast intro*: *mono-neutral-cong-left'*)

lemma *mono-neutral-right'*: $S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow G\ g\ T = G\ g\ S$
  by (*blast intro*!: *mono-neutral-left'* [*symmetric*])

end

48.2 Generalized summation over a set

context *comm-monoid-add*
begin

sublocale *sum*: *comm-monoid-set plus 0*
  defines $sum = sum.F$ **and** $sum' = sum.G$ **..**

abbreviation *Sum* (‹$\sum$›)
  where $\sum \equiv sum\ (\lambda x.\ x)$

end

Now: lots of fancy syntax. First, $sum\ (\lambda x.\ e)\ A$ is written $\sum x{\in}A.\ e$.

syntax (*ASCII*)
  -sum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b::comm-monoid-add* (‹(‹*indent=3 notation=*‹*binder SUM*››*SUM* (-/:-)./ -)› [*0*, *51*, *10*] *10*)
syntax
  -sum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b::comm-monoid-add* (‹(‹*indent=2 notation=*‹*binder* $\sum$››$\sum$ (-/∈-)./ -)› [*0*, *51*, *10*] *10*)

syntax-consts
-sum ⇌ *sum*

translations — Beware of argument permutation!
$\sum i{\in}A.\ b$ ⇌ *CONST sum* ($\lambda i.\ b$) *A*

Instead of $\sum x{\in}\{x.\ P\}.\ e$ we introduce the shorter $\sum x|P.\ e$.

syntax (*ASCII*)
-qsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* (‹(‹indent=3 notation=‹binder SUM Collect››SUM - |/ -./ -)› [0, 0, 10] 10)
syntax
-qsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* (‹(‹indent=2 notation=‹binder ∑ Collect››∑ - | (-)./ -)› [0, 0, 10] 10)

syntax-consts
-qsum == *sum*
translations
$\sum x|P.\ t$ => *CONST sum* ($\lambda x.\ t$) $\{x.\ P\}$
print-translation ‹
[(***const-syntax*** ‹sum›, *K* (*Collect-binder-tr'* ***syntax-const*** ‹-qsum›))]
›

48.2.1 Properties in more restricted classes of structures

lemma *sum-Un*:
finite A ⟹ *finite B* ⟹ *sum f* (*A* ∪ *B*) = *sum f A* + *sum f B* − *sum f* (*A* ∩ *B*)
for *f* :: *'b* ⇒ *'a*::*ab-group-add*
by (*subst sum.union-inter* [*symmetric*]) (*auto simp add*: *algebra-simps*)

lemma *sum-Un2*:
assumes *finite* (*A* ∪ *B*)
shows *sum f* (*A* ∪ *B*) = *sum f* (*A* − *B*) + *sum f* (*B* − *A*) + *sum f* (*A* ∩ *B*)
proof −
have *A* ∪ *B* = *A* − *B* ∪ (*B* − *A*) ∪ *A* ∩ *B*
by *auto*
with *assms* **show** *?thesis*
by *simp* (*subst sum.union-disjoint*, *auto*)+
qed

lemma *sum-diff*:
fixes *f* :: *'b* ⇒ *'a*::*ab-group-add*
assumes *finite A B* ⊆ *A*
shows *sum f* (*A* − *B*) = *sum f A* − *sum f B*
using *sum.subset-diff* [*of B A f*] *assms* **by** *simp*

lemma *sum-diff1*:
fixes *f* :: *'b* ⇒ *'a*::*ab-group-add*

assumes *finite A*
shows *sum f* $(A - \{a\}) = ($*if* $a \in A$ *then sum f A* $-$ *f a else sum f A*$)$
using *assms* **by** (*simp add*: *sum-diff*)

lemma *sum-diff1'-aux*:
fixes $f :: 'a \Rightarrow 'b$::*ab-group-add*
assumes *finite F* $\{i \in I.\ f\ i \neq 0\} \subseteq F$
shows $sum'\ f\ (I - \{i\}) = ($*if* $i \in I$ *then* $sum'\ f\ I - f\ i$ *else* $sum'\ f\ I)$
using *assms*
proof *induct*
case (*insert x F*)
have *1*: *finite* $\{x \in I.\ f\ x \neq 0\} \Longrightarrow$ *finite* $\{x \in I.\ x \neq i \wedge f\ x \neq 0\}$
by (*erule rev-finite-subset*) *auto*
have *2*: *finite* $\{x \in I.\ x \neq i \wedge f\ x \neq 0\} \Longrightarrow$ *finite* $\{x \in I.\ f\ x \neq 0\}$
apply (*drule finite-insert* [*THEN iffD2*])
by (*erule rev-finite-subset*) *auto*
have *3*: *finite* $\{i \in I.\ f\ i \neq 0\}$
using *finite-subset insert* **by** *blast*
show *?case*
using *insert sum-diff1* [*of* $\{i \in I.\ f\ i \neq 0\}$ *f i*]
by (*auto simp*: *sum.G-def 1 2 3 set-diff-eq conj-ac*)
qed (*simp add*: *sum.G-def*)

lemma *sum-diff1'*:
fixes $f :: 'a \Rightarrow 'b$::*ab-group-add*
assumes *finite* $\{i \in I.\ f\ i \neq 0\}$
shows $sum'\ f\ (I - \{i\}) = ($*if* $i \in I$ *then* $sum'\ f\ I - f\ i$ *else* $sum'\ f\ I)$
by (*rule sum-diff1'-aux* [*OF assms order-refl*])

lemma (**in** *ordered-comm-monoid-add*) *sum-mono*:
$(\bigwedge i.\ i \in K \Longrightarrow f\ i \leq g\ i) \Longrightarrow (\sum i \in K.\ f\ i) \leq (\sum i \in K.\ g\ i)$
by (*induct K rule*: *infinite-finite-induct*) (*use add-mono* **in** *auto*)

lemma (**in** *ordered-cancel-comm-monoid-add*) *sum-strict-mono-strong*:
assumes *finite A* $a \in A$ $f\ a < g\ a$
and $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
shows *sum f A* $<$ *sum g A*
proof –
have *sum f A* $= f\ a +$ *sum f* $(A - \{a\})$
by (*simp add*: *assms sum.remove*)
also have ... $\leq f\ a +$ *sum g* $(A - \{a\})$
using *assms* **by** (*meson DiffD1 add-left-mono sum-mono*)
also have ... $< g\ a +$ *sum g* $(A - \{a\})$
using *assms add-less-le-mono* **by** *blast*
also have ... $=$ *sum g A*
using *assms* **by** (*intro sum.remove* [*symmetric*])
finally show *?thesis* **.**
qed

```
lemma (in strict-ordered-comm-monoid-add) sum-strict-mono:
  assumes finite A A ≠ {}
    and ⋀x. x ∈ A ⟹ f x < g x
  shows sum f A < sum g A
  using assms
proof (induct rule: finite-ne-induct)
  case singleton
  then show ?case by simp
next
  case insert
  then show ?case by (auto simp: add-strict-mono)
qed

lemma sum-strict-mono-ex1:
  fixes f g :: 'i ⇒ 'a::ordered-cancel-comm-monoid-add
  assumes finite A
    and ∀ x∈A. f x ≤ g x
    and ∃ a∈A. f a < g a
  shows sum f A < sum g A
proof−
  from assms(3) obtain a where a: a ∈ A f a < g a by blast
  have sum f A = sum f ((A − {a}) ∪ {a})
    by(simp add: insert-absorb[OF ‹a ∈ A›])
  also have ... = sum f (A − {a}) + sum f {a}
    using ‹finite A› by(subst sum.union-disjoint) auto
  also have sum f (A − {a}) ≤ sum g (A − {a})
    by (rule sum-mono) (simp add: assms(2))
  also from a have sum f {a} < sum g {a} by simp
  also have sum g (A − {a}) + sum g {a} = sum g((A − {a}) ∪ {a})
    using ‹finite A› by (subst sum.union-disjoint[symmetric]) auto
  also have ... = sum g A by (simp add: insert-absorb[OF ‹a ∈ A›])
  finally show ?thesis
    by (auto simp add: add-right-mono add-strict-left-mono)
qed

lemma sum-mono-inv:
  fixes f g :: 'i ⇒ 'a :: ordered-cancel-comm-monoid-add
  assumes eq: sum f I = sum g I
  assumes le: ⋀i. i ∈ I ⟹ f i ≤ g i
  assumes i: i ∈ I
  assumes I: finite I
  shows f i = g i
proof (rule ccontr)
  assume ¬ ?thesis
  with le[OF i] have f i < g i by simp
  with i have ∃ i∈I. f i < g i ..
  from sum-strict-mono-ex1[OF I - this] le have sum f I < sum g I
    by blast
  with eq show False by simp
```

qed

lemma *member-le-sum*:
fixes f :: - ⇒ ′b::{*semiring-1*, *ordered-comm-monoid-add*}
assumes $i \in A$
and *le*: $\bigwedge x.\ x \in A - \{i\} \Longrightarrow 0 \leq f\ x$
and *finite A*
shows $f\ i \leq sum\ f\ A$
proof −
have $f\ i \leq sum\ f\ (A \cap \{i\})$
by (*simp add*: *assms*)
also have ... $= (\sum x{\in}A.\ if\ x \in \{i\}\ then\ f\ x\ else\ 0)$
using *assms sum.inter-restrict* **by** *blast*
also have ... $\leq sum\ f\ A$
apply (*rule sum-mono*)
apply (*auto simp*: *le*)
done
finally show *?thesis* **.**
qed

lemma *sum-negf*: $(\sum x{\in}A.\ - f\ x) = - (\sum x{\in}A.\ f\ x)$
for f :: ′b ⇒ ′a::*ab-group-add*
by (*induct A rule*: *infinite-finite-induct*) *auto*

lemma *sum-subtractf*: $(\sum x{\in}A.\ f\ x - g\ x) = (\sum x{\in}A.\ f\ x) - (\sum x{\in}A.\ g\ x)$
for $f\ g$ :: ′b ⇒′a::*ab-group-add*
using *sum.distrib* [*of f* − *g A*] **by** (*simp add*: *sum-negf*)

lemma *sum-subtractf-nat*:
$(\bigwedge x.\ x \in A \Longrightarrow g\ x \leq f\ x) \Longrightarrow (\sum x{\in}A.\ f\ x - g\ x) = (\sum x{\in}A.\ f\ x) - (\sum x{\in}A.\ g\ x)$
for $f\ g$ :: ′a ⇒ *nat*
by (*induct A rule*: *infinite-finite-induct*) (*auto simp*: *sum-mono*)

context *ordered-comm-monoid-add*
begin

lemma *sum-nonneg*: $(\bigwedge x.\ x \in A \Longrightarrow 0 \leq f\ x) \Longrightarrow 0 \leq sum\ f\ A$
proof (*induct A rule*: *infinite-finite-induct*)
case *infinite*
then show *?case* **by** *simp*
next
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
then have $0 + 0 \leq f\ x + sum\ f\ F$ **by** (*blast intro*: *add-mono*)
with *insert* **show** *?case* **by** *simp*
qed

lemma *sum-nonpos*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \leq 0) \Longrightarrow sum\ f\ A \leq 0$
proof (*induct A rule*: *infinite-finite-induct*)
  case *infinite*
  then show *?case* **by** *simp*
next
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert x F*)
  then have $f\ x + sum\ f\ F \leq 0 + 0$ **by** (*blast intro*: *add-mono*)
  with *insert* **show** *?case* **by** *simp*
qed

lemma *sum-nonneg-eq-0-iff*:
  $finite\ A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow 0 \leq f\ x) \Longrightarrow sum\ f\ A = 0 \longleftrightarrow (\forall x \in A.\ f\ x = 0)$
  by (*induct set*: *finite*) (*simp-all add*: *add-nonneg-eq-0-iff sum-nonneg*)

lemma *sum-nonneg-0*:
  $finite\ s \Longrightarrow (\bigwedge i.\ i \in s \Longrightarrow f\ i \geq 0) \Longrightarrow (\sum\ i \in s.\ f\ i) = 0 \Longrightarrow i \in s \Longrightarrow f\ i = 0$
  by (*simp add*: *sum-nonneg-eq-0-iff*)

lemma *sum-nonneg-leq-bound*:
  assumes $finite\ s\ \bigwedge i.\ i \in s \Longrightarrow f\ i \geq 0\ (\sum i \in s.\ f\ i) = B\ i \in s$
  shows $f\ i \leq B$
proof −
  from *assms* **have** $f\ i \leq f\ i + (\sum i \in s - \{i\}.\ f\ i)$
    by (*intro add-increasing2 sum-nonneg*) *auto*
  also have $\ldots = B$
    using *sum.remove*[*of s i f*] *assms* **by** *simp*
  finally show *?thesis* **by** *auto*
qed

lemma *sum-mono2*:
  assumes *fin*: *finite B*
    and *sub*: $A \subseteq B$
    and *nn*: $\bigwedge b.\ b \in B{-}A \Longrightarrow 0 \leq f\ b$
  shows $sum\ f\ A \leq sum\ f\ B$
proof −
  have $sum\ f\ A \leq sum\ f\ A + sum\ f\ (B{-}A)$
    by (*auto intro*: *add-increasing2* [*OF sum-nonneg*] *nn*)
  also from *fin finite-subset*[*OF sub fin*] **have** $\ldots = sum\ f\ (A \cup (B{-}A))$
    by (*simp add*: *sum.union-disjoint del*: *Un-Diff-cancel*)
  also from *sub* **have** $A \cup (B{-}A) = B$ **by** *blast*
  finally show *?thesis* **.**
qed

lemma *sum-le-included*:
  assumes *finite s finite t*

and ∀ *y*∈*t*. *0* ≤ *g y* (∀ *x*∈*s*. ∃ *y*∈*t*. *i y* = *x* ∧ *f x* ≤ *g y*)
shows *sum f s* ≤ *sum g t*
proof −
have *sum f s* ≤ *sum* (λ*y*. *sum g* {*x*. *x*∈*t* ∧ *i x* = *y*}) *s*
proof (*rule sum-mono*)
fix *y*
assume *y* ∈ *s*
with *assms* **obtain** *z* **where** *z*: *z* ∈ *t y* = *i z f y* ≤ *g z* **by** *auto*
with *assms* **show** *f y* ≤ *sum g* {*x* ∈ *t*. *i x* = *y*} (**is** *?A y* ≤ *?B y*)
using *order-trans*[*of ?A* (*i z*) *sum g* {*z*} *?B* (*i z*), *intro*]
by (*auto intro*!: *sum-mono2*)
qed
also have ... ≤ *sum* (λ*y*. *sum g* {*x*. *x*∈*t* ∧ *i x* = *y*}) (*i* ‘ *t*)
using *assms*(*2*−*4*) **by** (*auto intro*!: *sum-mono2 sum-nonneg*)
also have ... ≤ *sum g t*
using *assms* **by** (*auto simp*: *sum.image-gen*[*symmetric*])
finally show *?thesis* **.**
qed

end

lemma (**in** *canonically-ordered-monoid-add*) *sum-eq-0-iff* [*simp*]:
finite F ⟹ (*sum f F* = *0*) = (∀ *a*∈*F*. *f a* = *0*)
by (*intro ballI sum-nonneg-eq-0-iff zero-le*)

context *semiring-0*
begin

lemma *sum-distrib-left*: *r* ∗ *sum f A* = (∑ *n*∈*A*. *r* ∗ *f n*)
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *algebra-simps*)

lemma *sum-distrib-right*: *sum f A* ∗ *r* = (∑ *n*∈*A*. *f n* ∗ *r*)
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *algebra-simps*)

end

lemma *sum-divide-distrib*: *sum f A* / *r* = (∑ *n*∈*A*. *f n* / *r*)
for *r* :: ′*a*::*field*
proof (*induct A rule*: *infinite-finite-induct*)
case *infinite*
then show *?case* **by** *simp*
next
case *empty*
then show *?case* **by** *simp*
next
case *insert*
then show *?case* **by** (*simp add*: *add-divide-distrib*)
qed

lemma *sum-abs*[*iff*]: $|sum\ f\ A| \leq sum\ (\lambda i.\ |f\ i|)\ A$
  for f :: $'a \Rightarrow\ 'b$::*ordered-ab-group-add-abs*
proof (*induct A rule*: *infinite-finite-induct*)
  case *infinite*
  then show *?case* **by** *simp*
next
  case *empty*
  then show *?case* **by** *simp*
next
  case *insert*
  then show *?case* **by** (*auto intro*: *abs-triangle-ineq order-trans*)
qed

lemma *sum-abs-ge-zero*[*iff*]: $0 \leq sum\ (\lambda i.\ |f\ i|)\ A$
  for f :: $'a \Rightarrow\ 'b$::*ordered-ab-group-add-abs*
  by (*simp add*: *sum-nonneg*)

lemma *abs-sum-abs*[*simp*]: $|\sum a\in A.\ |f\ a|| = (\sum a\in A.\ |f\ a|)$
  for f :: $'a \Rightarrow\ 'b$::*ordered-ab-group-add-abs*
proof (*induct A rule*: *infinite-finite-induct*)
  case *infinite*
  then show *?case* **by** *simp*
next
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert a A*)
  then have $|\sum a\in insert\ a\ A.\ |f\ a|| = ||f\ a| + (\sum a\in A.\ |f\ a|)|$ **by** *simp*
  also from *insert* **have** $\ldots = ||f\ a| + |\sum a\in A.\ |f\ a|||$ **by** *simp*
  also have $\ldots = |f\ a| + |\sum a\in A.\ |f\ a||$ **by** (*simp del*: *abs-of-nonneg*)
  also from *insert* **have** $\ldots = (\sum a\in insert\ a\ A.\ |f\ a|)$ **by** *simp*
  finally show *?case* **.**
qed

lemma *sum-product*:
  fixes f :: $'a \Rightarrow\ 'b$::*semiring-0*
  shows $sum\ f\ A * sum\ g\ B = (\sum i\in A.\ \sum j\in B.\ f\ i * g\ j)$
  by (*simp add*: *sum-distrib-left sum-distrib-right*) (*rule sum.swap*)

lemma *sum-mult-sum-if-inj*:
  fixes f :: $'a \Rightarrow\ 'b$::*semiring-0*
  shows $inj\text{-}on\ (\lambda(a,\ b).\ f\ a * g\ b)\ (A \times B) \Longrightarrow$
    $sum\ f\ A * sum\ g\ B = sum\ id\ \{f\ a * g\ b\ |a\ b.\ a \in A \wedge b \in B\}$
  by(*auto simp*: *sum-product sum.cartesian-product intro*!: *sum.reindex-cong*[*symmetric*])

lemma *sum-SucD*: $sum\ f\ A = Suc\ n \Longrightarrow \exists a\in A.\ 0 < f\ a$
  by (*induct A rule*: *infinite-finite-induct*) *auto*

lemma *sum-eq-Suc0-iff*:

finite A $\Longrightarrow$ *sum f A* = *Suc 0* $\longleftrightarrow$ ($\exists$ *a*$\in$*A*. *f a* = *Suc 0* $\wedge$ ($\forall$ *b*$\in$*A*. *a* $\neq$ *b* $\longrightarrow$ *f b* = *0*))
by (*induct A rule*: *finite-induct*) (*auto simp add*: *add-is-1*)

lemmas *sum-eq-1-iff* = *sum-eq-Suc0-iff*[*simplified One-nat-def*[*symmetric*]]

lemma *sum-Un-nat*:
finite A $\Longrightarrow$ *finite B* $\Longrightarrow$ *sum f* (*A* $\cup$ *B*) = *sum f A* + *sum f B* − *sum f* (*A* $\cap$ *B*)
for *f* :: *'a* $\Rightarrow$ *nat*
— For the natural numbers, we have subtraction.
by (*subst sum.union-inter* [*symmetric*]) (*auto simp*: *algebra-simps*)

lemma *sum-diff1-nat*: *sum f* (*A* − {*a*}) = (*if a* $\in$ *A then sum f A* − *f a else sum f A*)
for *f* :: *'a* $\Rightarrow$ *nat*
proof (*induct A rule*: *infinite-finite-induct*)
case *infinite*
then show *?case* **by** *simp*
next
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
then show *?case*
proof (*cases a* $\in$ *F*)
case *True*
then have $\exists$ *B*. *F* = *insert a B* $\wedge$ *a* $\notin$ *B*
by (*auto simp*: *mk-disjoint-insert*)
then show *?thesis* **using** *insert*
by (*auto simp*: *insert-Diff-if*)
qed (*auto*)
qed

lemma *sum-diff-nat*:
fixes *f* :: *'a* $\Rightarrow$ *nat*
assumes *finite B* **and** *B* $\subseteq$ *A*
shows *sum f* (*A* − *B*) = *sum f A* − *sum f B*
using *assms*
proof *induct*
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
note *IH* = ‹*F* $\subseteq$ *A* $\Longrightarrow$ *sum f* (*A* − *F*) = *sum f A* − *sum f F*›
from ‹*x* $\notin$ *F*› ‹*insert x F* $\subseteq$ *A*› **have** *x* $\in$ *A* − *F* **by** *simp*
then have *A*: *sum f* ((*A* − *F*) − {*x*}) = *sum f* (*A* − *F*) − *f x*
by (*simp add*: *sum-diff1-nat*)
from ‹*insert x F* $\subseteq$ *A*› **have** *F* $\subseteq$ *A* **by** *simp*
with *IH* **have** *sum f* (*A* − *F*) = *sum f A* − *sum f F* **by** *simp*

with A **have** B: *sum f* $((A - F) - \{x\}) =$ *sum f A* $-$ *sum f F* $-$ *f x*
  by *simp*
from ‹$x \notin F$› **have** $A -$ *insert x F* $= (A - F) - \{x\}$ **by** *auto*
with B **have** C: *sum f* $(A -$ *insert x F*$) =$ *sum f A* $-$ *sum f F* $-$ *f x*
  by *simp*
from ‹*finite F*› ‹$x \notin F$› **have** *sum f* (*insert x F*) $=$ *sum f F* $+$ *f x*
  by *simp*
with C **have** *sum f* $(A -$ *insert x F*$) =$ *sum f A* $-$ *sum f* (*insert x F*)
  by *simp*
then show *?case* **by** *simp*
qed

lemma *sum-comp-morphism*:
$h\ 0 = 0 \Longrightarrow (\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y) \Longrightarrow sum\ (h \circ g)\ A = h\ (sum\ g\ A)$
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma (**in** *comm-semiring-1*) *dvd-sum*: $(\bigwedge a.\ a \in A \Longrightarrow d\ dvd\ f\ a) \Longrightarrow d\ dvd\ sum$
$f\ A$
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma (**in** *ordered-comm-monoid-add*) *sum-pos*:
finite $I \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow 0 < f\ i) \Longrightarrow 0 < sum\ f\ I$
by (*induct I rule*: *finite-ne-induct*) (*auto intro*: *add-pos-pos*)

lemma (**in** *ordered-comm-monoid-add*) *sum-pos2*:
assumes I: *finite* $I\ i \in I\ 0 < f\ i\ \bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i$
shows $0 < sum\ f\ I$
proof $-$
have $0 < f\ i + sum\ f\ (I - \{i\})$
  using *assms* **by** (*intro add-pos-nonneg sum-nonneg*) *auto*
also have $\ldots = sum\ f\ I$
  using *assms* **by** (*simp add*: *sum.remove*)
finally show *?thesis* **.**
qed

lemma *sum-strict-mono2*:
fixes $f :: 'a \Rightarrow 'b$::*ordered-cancel-comm-monoid-add*
assumes *finite* $B\ A \subseteq B\ b \in B{-}A\ f\ b > 0$ **and** $\bigwedge x.\ x \in B \Longrightarrow f\ x \geq 0$
shows $sum\ f\ A < sum\ f\ B$
proof $-$
have $B - A \neq \{\}$
  using *assms*(*3*) **by** *blast*
have $sum\ f\ (B{-}A) > 0$
  by (*rule sum-pos2*) (*use assms* **in** *auto*)
moreover have $sum\ f\ B = sum\ f\ (B{-}A) + sum\ f\ A$
  by (*rule sum.subset-diff*) (*use assms* **in** *auto*)
ultimately show *?thesis*
  using *add-strict-increasing* **by** *auto*
qed

lemma *sum-cong-Suc*:
  assumes *0* $\notin$ *A* $\bigwedge$*x*. *Suc x* $\in$ *A* $\Longrightarrow$ *f* (*Suc x*) = *g* (*Suc x*)
  shows *sum f A* = *sum g A*
proof (*rule sum.cong*)
  fix *x*
  assume *x* $\in$ *A*
  with *assms*(*1*) **show** *f x* = *g x*
    by (*cases x*) (*auto intro*!: *assms*(*2*))
qed *simp-all*

48.2.2 Cardinality as special case of *sum*

lemma *card-eq-sum*: *card A* = *sum* (λ*x*. *1*) *A*
proof −
  have *plus* ∘ (λ-. *Suc 0*) = (λ-. *Suc*)
    by (*simp add*: *fun-eq-iff*)
  then have *Finite-Set.fold* (*plus* ∘ (λ-. *Suc 0*)) = *Finite-Set.fold* (λ-. *Suc*)
    by (*rule arg-cong*)
  then have *Finite-Set.fold* (*plus* ∘ (λ-. *Suc 0*)) *0 A* = *Finite-Set.fold* (λ-. *Suc*) *0*
A
    by (*blast intro*: *fun-cong*)
  then show *?thesis*
    by (*simp add*: *card.eq-fold sum.eq-fold*)
qed

context *semiring-1*
begin

lemma *sum-constant* [*simp*]:
  ($\sum$*x* $\in$ *A*. *y*) = *of-nat* (*card A*) ∗ *y*
  by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *algebra-simps*)

context
  fixes *A*
  assumes ‹*finite A*›
begin

lemma *sum-of-bool-eq* [*simp*]:
  ‹($\sum$*x* $\in$ *A*. *of-bool* (*P x*)) = *of-nat* (*card* (*A* ∩ {*x*. *P x*}))› **if** ‹*finite A*›
  using ‹*finite A*› **by** *induction simp-all*

lemma *sum-mult-of-bool-eq* [*simp*]:
  ‹($\sum$*x* $\in$ *A*. *f x* ∗ *of-bool* (*P x*)) = ($\sum$*x* $\in$ (*A* ∩ {*x*. *P x*}). *f x*)›
  by (*rule sum.mono-neutral-cong*) (*use* ‹*finite A*› **in** *auto*)

lemma *sum-of-bool-mult-eq* [*simp*]:
  ‹($\sum$*x* $\in$ *A*. *of-bool* (*P x*) ∗ *f x*) = ($\sum$*x* $\in$ (*A* ∩ {*x*. *P x*}). *f x*)›
  by (*rule sum.mono-neutral-cong*) (*use* ‹*finite A*› **in** *auto*)

```
end

end

lemma sum-Suc: sum (λx. Suc(f x)) A = sum f A + card A
  using sum.distrib[of f λ-. 1 A] by simp

lemma sum-bounded-above:
  fixes K :: 'a::{semiring-1,ordered-comm-monoid-add}
  assumes le: ⋀i. i∈A ⟹ f i ≤ K
  shows sum f A ≤ of-nat (card A) ∗ K
proof (cases finite A)
  case True
  then show ?thesis
    using le sum-mono[where K=A and g = λx. K] by simp
next
  case False
  then show ?thesis by simp
qed

lemma sum-bounded-above-divide:
  fixes K :: 'a::linordered-field
  assumes le: ⋀i. i∈A ⟹ f i ≤ K / of-nat (card A) and fin: finite A A ≠ {}
  shows sum f A ≤ K
  using sum-bounded-above [of A f K / of-nat (card A), OF le] fin by simp

lemma sum-bounded-above-strict:
  fixes K :: 'a::{ordered-cancel-comm-monoid-add,semiring-1}
  assumes ⋀i. i∈A ⟹ f i < K card A > 0
  shows sum f A < of-nat (card A) ∗ K
  using assms sum-strict-mono[where A=A and g = λx. K]
  by (simp add: card-gt-0-iff)

lemma sum-bounded-below:
  fixes K :: 'a::{semiring-1,ordered-comm-monoid-add}
  assumes le: ⋀i. i∈A ⟹ K ≤ f i
  shows of-nat (card A) ∗ K ≤ sum f A
proof (cases finite A)
  case True
  then show ?thesis
    using le sum-mono[where K=A and f = λx. K] by simp
next
  case False
  then show ?thesis by simp
qed

lemma convex-sum-bound-le:
  fixes x :: 'a ⇒ 'b::linordered-idom
```

assumes *0*: $\bigwedge i.\ i \in I \Longrightarrow 0 \leq x\ i$ **and** *1*: *sum x I* = *1*
and *δ*: $\bigwedge i.\ i \in I \Longrightarrow |a\ i - b| \leq \delta$
shows $|(\sum i{\in}I.\ a\ i * x\ i) - b| \leq \delta$
proof −
have [*simp*]: $(\sum i{\in}I.\ c * x\ i) = c$ **for** *c*
by (*simp flip*: *sum-distrib-left 1*)
then have $|(\sum i{\in}I.\ a\ i * x\ i) - b| = |\sum i{\in}I.\ (a\ i - b) * x\ i|$
by (*simp add*: *sum-subtractf left-diff-distrib*)
also have ... $\leq (\sum i{\in}I.\ |(a\ i - b) * x\ i|)$
using *abs-abs abs-of-nonneg* **by** *blast*
also have ... $\leq (\sum i{\in}I.\ |(a\ i - b)| * x\ i)$
by (*simp add*: *abs-mult 0*)
also have ... $\leq (\sum i{\in}I.\ \delta * x\ i)$
by (*rule sum-mono*) (*use δ 0 mult-right-mono* **in** *blast*)
also have ... $= \delta$
by *simp*
finally show *?thesis* **.**
qed

lemma *card-UN-disjoint*:
assumes *finite I* **and** $\forall i{\in}I.$ *finite* $(A\ i)$
and $\forall i{\in}I.\ \forall j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$
shows *card* $(\bigcup (A\ `\ I)) = (\sum i{\in}I.\ card(A\ i))$
proof −
have $(\sum i{\in}I.\ card\ (A\ i)) = (\sum i{\in}I.\ \sum x{\in}A\ i.\ 1)$
by *simp*
with *assms* **show** *?thesis*
by (*simp add*: *card-eq-sum sum.UNION-disjoint del*: *sum-constant*)
qed

lemma *card-Union-disjoint*:
assumes *pairwise disjnt C* **and** *fin*: $\bigwedge A.\ A \in C \Longrightarrow$ *finite A*
shows *card* $(\bigcup C)$ = *sum card C*
proof (*cases finite C*)
case *True*
then show *?thesis*
using *card-UN-disjoint* [*OF True*, *of* λ*x*. *x*] *assms*
by (*simp add*: *disjnt-def fin pairwise-def*)
next
case *False*
then show *?thesis*
using *assms card-eq-0-iff finite-UnionD* **by** *fastforce*
qed

lemma *card-Union-le-sum-card-weak*:
fixes *U* :: *'a set set*
assumes $\forall u \in U.$ *finite u*
shows *card* $(\bigcup U) \leq$ *sum card U*
proof (*cases finite U*)

case *False*
then show *card* ($\bigcup$ *U*) ≤ *sum card U*
using *card-eq-0-iff finite-UnionD* **by** *auto*
next
case *True*
then show *card* ($\bigcup$ *U*) ≤ *sum card U*
proof (*induct U rule*: *finite-induct*)
case *empty*
then show *?case* **by** *auto*
next
case (*insert x F*)
then have *card*($\bigcup$(*insert x F*)) ≤ *card*(*x*) + *card* ($\bigcup$ *F*) **using** *card-Un-le* **by** *auto*
also have ... ≤ *card*(*x*) + *sum card F* **using** *insert.hyps* **by** *auto*
also have ... = *sum card* (*insert x F*) **using** *sum.insert-if* **and** *insert.hyps* **by** *auto*
finally show *?case* **.**
qed
qed

lemma *card-Union-le-sum-card*:
fixes *U* :: *'a set set*
shows *card* ($\bigcup$ *U*) ≤ *sum card U*
by (*metis Union-upper card.infinite card-Union-le-sum-card-weak finite-subset zero-le*)

lemma *card-UN-le*:
assumes *finite I*
shows *card*($\bigcup$ *i*∈*I*. *A i*) ≤ ($\sum$ *i*∈*I*. *card*(*A i*))
using *assms*
proof *induction*
case (*insert i I*)
then show *?case*
using *card-Un-le nat-add-left-cancel-le* **by** (*force intro*: *order-trans*)
qed *auto*

lemma *card-quotient-disjoint*:
assumes *finite A inj-on* (λ*x*. {*x*} // *r*) *A*
shows *card* (*A*//*r*) = *card A*
proof −
have ∀ *i*∈*A*. ∀ *j*∈*A*. *i* ≠ *j* ⟶ *r* `` {*j*} ≠ *r* `` {*i*}
using *assms* **by** (*fastforce simp add*: *quotient-def inj-on-def*)
with *assms* **show** *?thesis*
by (*simp add*: *quotient-def card-UN-disjoint*)
qed

lemma *sum-multicount-gen*:
assumes *finite s finite t* ∀ *j*∈*t*. (*card* {*i*∈*s*. *R i j*} = *k j*)
shows *sum* (λ*i*. (*card* {*j*∈*t*. *R i j*})) *s* = *sum k t*

```
  (is ?l = ?r)
proof−
  have ?l = sum (λi. sum (λx.1) {j∈t. R i j}) s
    by auto
  also have ... = ?r
    unfolding sum.swap-restrict [OF assms(1−2)]
    using assms(3) by auto
  finally show ?thesis .
qed

lemma sum-multicount:
  assumes finite S finite T ∀j∈T. (card {i∈S. R i j} = k)
  shows sum (λi. card {j∈T. R i j}) S = k ∗ card T (is ?l = ?r)
proof−
  have ?l = sum (λi. k) T
    by (rule sum-multicount-gen) (auto simp: assms)
  also have ... = ?r by (simp add: mult.commute)
  finally show ?thesis by auto
qed

lemma sum-card-image:
  assumes finite A
  assumes pairwise (λs t. disjnt (f s) (f t)) A
  shows sum card (f ‘ A) = sum (λa. card (f a)) A
using assms
proof (induct A)
  case (insert a A)
  show ?case
  proof cases
    assume f a = {}
    with insert show ?case
     by (subst sum.mono-neutral-right[where S=f ‘ A]) (auto simp: pairwise-insert)
  next
    assume f a ≠ {}
    then have sum card (insert (f a) (f ‘ A)) = card (f a) + sum card (f ‘ A)
      using insert
      by (subst sum.insert) (auto simp: pairwise-insert)
    with insert show ?case by (simp add: pairwise-insert)
  qed
qed simp
```

By Jakub Kdzioka:

```
lemma sum-fun-comp:
  assumes finite S finite R g ‘ S ⊆ R
  shows (∑x ∈ S. f (g x)) = (∑y ∈ R. of-nat (card {x ∈ S. g x = y}) ∗ f y)
proof −
  let ?r = relation-of (λp q. g p = g q) S
  have eqv: equiv S ?r
    unfolding relation-of-def by (auto intro: comp-equivI)
```

have *finite*: $C \in S//?r \Longrightarrow$ *finite* C **for** C
  by (*fact finite-equiv-class*[*OF* ‹*finite S*› *equiv-type*[*OF* ‹*equiv S ?r*›]])
have *disjoint*: $A \in S//?r \Longrightarrow B \in S//?r \Longrightarrow A \neq B \Longrightarrow A \cap B = \{\}$ **for** A B
  using *eqv quotient-disj* **by** *blast*

let *?cls* = $\lambda y.\ \{x \in S.\ y = g\ x\}$
have *quot-as-img*: $S//?r$ = *?cls* ‘ g ‘ S
  by (*auto simp add*: *relation-of-def quotient-def*)
have *cls-inj*: *inj-on ?cls* (g ‘ S)
  by (*auto intro*: *inj-onI*)

have *rest-0*: ($\sum y \in R - g$ ‘ S. *of-nat* (*card* (*?cls* y)) $*$ $f\ y$) = *0*
proof −
  have *of-nat* (*card* (*?cls* y)) $*$ $f\ y$ = *0* **if** *asm*: $y \in R - g$ ‘ S **for** y
  proof −
    from *asm* **have** $*$: *?cls* $y = \{\}$ **by** *auto*
    show *?thesis* **unfolding** $*$ **by** *simp*
  qed
  thus *?thesis* **by** *simp*
qed

have ($\sum x \in S.\ f\ (g\ x)$) = ($\sum C \in S//?r.\ \sum x \in C.\ f\ (g\ x)$)
  using *eqv finite disjoint*
  by (*simp flip*: *sum.Union-disjoint*[*simplified*] *add*: *Union-quotient*)
also have ... = ($\sum y \in g$ ‘ S. $\sum x \in$ *?cls* y. $f\ (g\ x)$)
  unfolding *quot-as-img* **by** (*simp add*: *sum.reindex*[*OF cls-inj*])
also have ... = ($\sum y \in g$ ‘ S. $\sum x \in$ *?cls* y. $f\ y$)
  by *auto*
also have ... = ($\sum y \in g$ ‘ S. *of-nat* (*card* (*?cls* y)) $*$ $f\ y$)
  by (*simp flip*: *sum-constant*)
also have ... = ($\sum y \in R$. *of-nat* (*card* (*?cls* y)) $*$ $f\ y$)
  using *rest-0* **by** (*simp add*: *sum.subset-diff*[*OF* ‹g ‘ $S \subseteq R$› ‹*finite R*›])
finally show *?thesis*
  by (*simp add*: *eq-commute*)
qed

48.2.3 Cardinality of products

lemma *card-SigmaI* [*simp*]:
  finite $A \Longrightarrow \forall a \in A$. *finite* ($B\ a$) $\Longrightarrow$ *card* (*SIGMA* x: A. $B\ x$) = ($\sum a \in A$. *card* ($B\ a$))
  by (*simp add*: *card-eq-sum sum.Sigma del*: *sum-constant*)

lemma *card-cartesian-product*: *card* ($A \times B$) = *card* A $*$ *card* B
  by (*cases finite* $A \wedge$ *finite* B)
  (*auto simp add*: *card-eq-0-iff dest*: *finite-cartesian-productD1 finite-cartesian-productD2*)

lemma *card-cartesian-product-singleton*: *card* ({*x*} × *A*) = *card A*
by (*simp add*: *card-cartesian-product*)

48.3 Generalized product over a set

context *comm-monoid-mult*
begin

sublocale *prod*: *comm-monoid-set times 1*
defines *prod* = *prod.F* **and** *prod′* = *prod.G* **..**

abbreviation *Prod* (‹∏›)
where ∏ ≡ *prod* (λ*x*. *x*)

end

syntax (*ASCII*)
-prod :: *pttrn* => *′a set* => *′b* => *′b*::*comm-monoid-mult* (‹(‹*indent=4 notation=*‹*binder PROD*››*PROD* (-/:-)./ -)› [*0*, *51*, *10*] *10*)
syntax
-prod :: *pttrn* => *′a set* => *′b* => *′b*::*comm-monoid-mult* (‹(‹*indent=2 notation=*‹*binder* ∏››∏ (-/∈-)./ -)› [*0*, *51*, *10*] *10*)

syntax-consts
-prod == *prod*

translations — Beware of argument permutation!
∏*i*∈*A*. *b* == *CONST prod* (λ*i*. *b*) *A*

Instead of ∏*x*∈{*x*. *P*}. *e* we introduce the shorter ∏*x*|*P*. *e*.

syntax (*ASCII*)
-qprod :: *pttrn* ⇒ *bool* ⇒ *′a* ⇒ *′a* (‹(‹*indent=4 notation=*‹*binder PROD Collect*››*PROD* - |/ -./ -)› [*0*, *0*, *10*] *10*)
syntax
-qprod :: *pttrn* ⇒ *bool* ⇒ *′a* ⇒ *′a* (‹(‹*indent=2 notation=*‹*binder* ∏ *Collect*››∏ - | (-)./ -)› [*0*, *0*, *10*] *10*)

syntax-consts
-qprod == *prod*
translations
∏*x*|*P*. *t* => *CONST prod* (λ*x*. *t*) {*x*. *P*}
print-translation ‹
[(***const-syntax***‹*prod*›, *K* (*Collect-binder-tr′* ***syntax-const***‹*-qprod*›))]
›

context *comm-monoid-mult*
begin

lemma *prod-dvd-prod*: (⋀*a*. *a* ∈ *A* ⟹ *f a dvd g a*) ⟹ *prod f A dvd prod g A*

proof (*induct A rule*: *infinite-finite-induct*)
  case *infinite*
  then show *?case* **by** (*auto intro*: *dvdI*)
next
  case *empty*
  then show *?case* **by** (*auto intro*: *dvdI*)
next
  case (*insert a A*)
  then have *f a dvd g a* **and** *prod f A dvd prod g A*
    by *simp-all*
  then obtain *r s* **where** $g\ a = f\ a * r$ **and** $prod\ g\ A = prod\ f\ A * s$
    by (*auto elim*!: *dvdE*)
  then have $g\ a * prod\ g\ A = f\ a * prod\ f\ A * (r * s)$
    by (*simp add*: *ac-simps*)
  with *insert.hyps* **show** *?case*
    by (*auto intro*: *dvdI*)
qed

lemma *prod-dvd-prod-subset*: *finite* $B \Longrightarrow A \subseteq B \Longrightarrow$ *prod f A dvd prod f B*
  by (*auto simp add*: *prod.subset-diff ac-simps intro*: *dvdI*)

end

48.3.1 Properties in more restricted classes of structures

context *linordered-nonzero-semiring*
begin

lemma *prod-ge-1*: $(\bigwedge x.\ x \in A \Longrightarrow 1 \le f\ x) \Longrightarrow 1 \le prod\ f\ A$
proof (*induct A rule*: *infinite-finite-induct*)
  case *infinite*
  then show *?case* **by** *simp*
next
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert x F*)
  have $1 * 1 \le f\ x * prod\ f\ F$
    by (*rule mult-mono'*) (*use insert* **in** *auto*)
  with *insert* **show** *?case* **by** *simp*
qed

lemma *prod-le-1*:
  fixes $f :: 'b \Rightarrow 'a$
  assumes $\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x \wedge f\ x \le 1$
  shows $prod\ f\ A \le 1$
    using *assms*
proof (*induct A rule*: *infinite-finite-induct*)
  case *infinite*

then show *?case* **by** *simp*
next
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
then show *?case* **by** (*force simp*: *mult.commute intro*: *dest*: *mult-le-one*)
qed

end

context *comm-semiring-1*
begin

lemma *dvd-prod-eqI* [*intro*]:
assumes $finite\ A$ **and** $a \in A$ **and** $b = f\ a$
shows $b\ dvd\ prod\ f\ A$
proof −
from ‹$finite\ A$› **have** $prod\ f\ (insert\ a\ (A - \{a\})) = f\ a * prod\ f\ (A - \{a\})$
by (*intro prod.insert*) *auto*
also from ‹$a \in A$› **have** $insert\ a\ (A - \{a\}) = A$
by *blast*
finally have $prod\ f\ A = f\ a * prod\ f\ (A - \{a\})$ **.**
with ‹$b = f\ a$› **show** *?thesis*
by *simp*
qed

lemma *dvd-prodI* [*intro*]: $finite\ A \Longrightarrow a \in A \Longrightarrow f\ a\ dvd\ prod\ f\ A$
by *auto*

lemma *prod-zero*:
assumes $finite\ A$ **and** $\exists\, a \in A.\ f\ a = 0$
shows $prod\ f\ A = 0$
using *assms*
proof (*induct A*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert a A*)
then have $f\ a = 0 \vee (\exists\, a \in A.\ f\ a = 0)$ **by** *simp*
then have $f\ a * prod\ f\ A = 0$ **by** (*rule disjE*) (*simp-all add*: *insert*)
with *insert* **show** *?case* **by** *simp*
qed

lemma *prod-dvd-prod-subset2*:
assumes $finite\ B$ **and** $A \subseteq B$ **and** $\bigwedge a.\ a \in A \Longrightarrow f\ a\ dvd\ g\ a$
shows $prod\ f\ A\ dvd\ prod\ g\ B$
proof −
from *assms* **have** $prod\ f\ A\ dvd\ prod\ g\ A$

```
    by (auto intro: prod-dvd-prod)
  moreover from assms have prod g A dvd prod g B
    by (auto intro: prod-dvd-prod-subset)
  ultimately show ?thesis by (rule dvd-trans)
qed

end

lemma (in semidom) prod-zero-iff [simp]:
  fixes f :: 'b ⇒ 'a
  assumes finite A
  shows prod f A = 0 ⟷ (∃ a∈A. f a = 0)
  using assms by (induct A) (auto simp: no-zero-divisors)

lemma (in semidom-divide) prod-diff1:
  assumes finite A and f a ≠ 0
  shows prod f (A − {a}) = (if a ∈ A then prod f A div f a else prod f A)
proof (cases a ∉ A)
  case True
  then show ?thesis by simp
next
  case False
  with assms show ?thesis
  proof induct
    case empty
    then show ?case by simp
  next
    case (insert b B)
    then show ?case
    proof (cases a = b)
      case True
      with insert show ?thesis by simp
    next
      case False
      with insert have a ∈ B by simp
      define C where C = B − {a}
      with ‹finite B› ‹a ∈ B› have B = insert a C finite C a ∉ C
        by auto
      with insert show ?thesis
        by (auto simp add: insert-commute ac-simps)
    qed
  qed
qed

lemma sum-zero-power [simp]: (∑ i∈A. c i ∗ 0^i) = (if finite A ∧ 0 ∈ A then c
0 else 0)
  for c :: nat ⇒ 'a::division-ring
  by (induct A rule: infinite-finite-induct) auto
```

lemma *sum-zero-power'* [*simp*]:
  ($\sum i{\in}A.\ c\ i * 0\hat{}i\ /\ d\ i$) = (*if finite* $A \wedge 0 \in A$ *then* $c\ 0\ /\ d\ 0$ *else* 0)
  for c :: *nat* ⇒ *'a::field*
  using *sum-zero-power* [*of* $\lambda i.\ c\ i\ /\ d\ i\ A$] **by** *auto*

lemma (**in** *field*) *prod-inversef*: *prod* (*inverse* ∘ *f*) *A* = *inverse* (*prod f A*)
 proof (*cases finite A*)
  case *True*
  then show *?thesis*
   by (*induct A rule*: *finite-induct*) *simp-all*
 next
  case *False*
  then show *?thesis*
   by *auto*
 qed

lemma (**in** *field*) *prod-dividef*: ($\prod x{\in}A.\ f\ x\ /\ g\ x$) = *prod f A* / *prod g A*
  using *prod-inversef* [*of g A*] **by** (*simp add*: *divide-inverse prod.distrib*)

lemma *prod-Un*:
  fixes *f* :: *'b* ⇒ *'a* :: *field*
  assumes *finite A* **and** *finite B*
   and $\forall x{\in}A \cap B.\ f\ x \neq 0$
  shows *prod f* ($A \cup B$) = *prod f A* ∗ *prod f B* / *prod f* ($A \cap B$)
proof −
  from *assms* **have** *prod f A* ∗ *prod f B* = *prod f* ($A \cup B$) ∗ *prod f* ($A \cap B$)
   by (*simp add*: *prod.union-inter* [*symmetric*, *of A B*])
  with *assms* **show** *?thesis*
   by *simp*
qed

context *linordered-semidom*
begin

lemma *prod-nonneg*: ($\bigwedge a.\ a{\in}A \Longrightarrow 0 \le f\ a$) $\Longrightarrow 0 \le$ *prod f A*
  by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *prod-pos*: ($\bigwedge a.\ a{\in}A \Longrightarrow 0 < f\ a$) $\Longrightarrow 0 <$ *prod f A*
  by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *prod-mono*:
  ($\bigwedge i.\ i \in A \Longrightarrow 0 \le f\ i \wedge f\ i \le g\ i$) $\Longrightarrow$ *prod f A* ≤ *prod g A*
  by (*induct A rule*: *infinite-finite-induct*) (*force intro*!: *prod-nonneg mult-mono*)+

Only one needs to be strict

lemma *prod-mono-strict*:
  assumes $i \in A\ f\ i < g\ i$
  assumes *finite A*
  assumes $\bigwedge i.\ i \in A \Longrightarrow 0 \le f\ i \wedge f\ i \le g\ i$

assumes $\bigwedge i.\ i \in A \Longrightarrow 0 < g\ i$
shows *prod f A* $<$ *prod g A*
proof –
have *prod f A* $= f\ i * $ *prod f* $(A - \{i\})$
using *assms* **by** (*intro prod.remove*)
also have ... $\leq f\ i *$ *prod g* $(A - \{i\})$
using *assms* **by** (*intro mult-left-mono prod-mono*) *auto*
also have ... $< g\ i *$ *prod g* $(A - \{i\})$
using *assms* **by** (*intro mult-strict-right-mono prod-pos*) *auto*
also have ... = *prod g A*
using *assms* **by** (*intro prod.remove* [*symmetric*])
finally show *?thesis* **.**
qed

lemma *prod-le-power*:
assumes *A*: $\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i \wedge f\ i \leq n$ *card* $A \leq k$ **and** $n \geq 1$
shows *prod f A* $\leq n \mathbin{\char`^} k$
using *A*
proof (*induction A arbitrary*: *k rule*: *infinite-finite-induct*)
case (*insert i A*)
then obtain k' **where** k': *card* $A \leq k'$ $k =$ *Suc* k'
using *Suc-le-D* **by** *force*
have $f\ i *$ *prod f A* $\leq n * n \mathbin{\char`^} k'$
using *insert* ‹$n \geq 1$› k' **by** (*intro prod-nonneg mult-mono*; *force*)
then show *?case*
by (*auto simp*: ‹$k =$ *Suc* k'› *insert.hyps*)
qed (*use* ‹$n \geq 1$› **in** *auto*)

end

lemma *prod-mono2*:
fixes f :: $'a \Rightarrow 'b$ :: *linordered-idom*
assumes *fin*: *finite B*
and *sub*: $A \subseteq B$
and *nn*: $\bigwedge b.\ b \in B{-}A \Longrightarrow 1 \leq f\ b$
and *A*: $\bigwedge a.\ a \in A \Longrightarrow 0 \leq f\ a$
shows *prod f A* $\leq$ *prod f B*
proof –
have *prod f A* $\leq$ *prod f A* $*$ *prod f* $(B{-}A)$
by (*metis prod-ge-1 A mult-le-cancel-left1 nn not-less prod-nonneg*)
also from *fin finite-subset*[*OF sub fin*] **have** ... = *prod f* $(A \cup (B{-}A))$
by (*simp add*: *prod.union-disjoint del*: *Un-Diff-cancel*)
also from *sub* **have** $A \cup (B{-}A) = B$ **by** *blast*
finally show *?thesis* **.**
qed

lemma *less-1-prod*:
fixes f :: $'a \Rightarrow 'b$::*linordered-idom*
shows *finite I* $\Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow 1 < f\ i) \Longrightarrow 1 <$ *prod f I*

by (*induct I rule*: *finite-ne-induct*) (*auto intro*: *less-1-mult*)

lemma *less-1-prod2*:
fixes *f* :: *'a* ⇒ *'b*::*linordered-idom*
assumes *I*: *finite I i* ∈ *I 1* < *f i* ⋀*i*. *i* ∈ *I* ⟹ *1* ≤ *f i*
shows *1* < *prod f I*
proof −
have *1* < *f i* ∗ *prod f* (*I* − {*i*})
using *assms*
by (*meson DiffD1 leI less-1-mult less-le-trans mult-le-cancel-left1 prod-ge-1*)
also have ... = *prod f I*
using *assms* **by** (*simp add*: *prod.remove*)
finally show *?thesis* **.**
qed

lemma (**in** *linordered-field*) *abs-prod*: |*prod f A*| = (∏ *x*∈*A*. |*f x*|)
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *abs-mult*)

lemma *prod-eq-1-iff* [*simp*]: *finite A* ⟹ *prod f A* = *1* ⟷ (∀ *a*∈*A*. *f a* = *1*)
for *f* :: *'a* ⇒ *nat*
by (*induct A rule*: *finite-induct*) *simp-all*

lemma *prod-pos-nat-iff* [*simp*]: *finite A* ⟹ *prod f A* > *0* ⟷ (∀ *a*∈*A*. *f a* > *0*)
for *f* :: *'a* ⇒ *nat*
using *prod-zero-iff* **by** (*simp del*: *neq0-conv add*: *zero-less-iff-neq-zero*)

lemma *prod-constant* [*simp*]: (∏ *x*∈ *A*. *y*) = *y* ^ *card A*
for *y* :: *'a*::*comm-monoid-mult*
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *prod-power-distrib*: *prod f A* ^ *n* = *prod* (λ*x*. (*f x*) ^ *n*) *A*
for *f* :: *'a* ⇒ *'b*::*comm-semiring-1*
by (*induct A rule*: *infinite-finite-induct*) (*auto simp add*: *power-mult-distrib*)

lemma *power-sum*: *c* ^ (∑ *a*∈*A*. *f a*) = (∏ *a*∈*A*. *c* ^ *f a*)
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *power-add*)

lemma *prod-gen-delta*:
fixes *b* :: *'b* ⇒ *'a*::*comm-monoid-mult*
assumes *fin*: *finite S*
shows *prod* (λ*k*. *if k* = *a then b k else c*) *S* =
(*if a* ∈ *S then b a* ∗ *c* ^ (*card S* − *1*) *else c* ^ *card S*)
proof −
let *?f* = (λ*k*. *if k*=*a then b k else c*)
show *?thesis*
proof (*cases a* ∈ *S*)
case *False*
then have ∀ *k*∈ *S*. *?f k* = *c* **by** *simp*
with *False* **show** *?thesis* **by** (*simp add*: *prod-constant*)

```
  next
    case True
    let ?A = S − {a}
    let ?B = {a}
    from True have eq: S = ?A ∪ ?B by blast
    have disjoint: ?A ∩ ?B = {} by simp
    from fin have fin': finite ?A finite ?B by auto
    have f-A0: prod ?f ?A = prod (λi. c) ?A
      by (rule prod.cong) auto
    from fin True have card-A: card ?A = card S − 1 by auto
    have f-A1: prod ?f ?A = c ^ card ?A
      unfolding f-A0 by (rule prod-constant)
    have prod ?f ?A ∗ prod ?f ?B = prod ?f S
      using prod.union-disjoint[OF fin' disjoint, of ?f, unfolded eq[symmetric]]
      by simp
    with True card-A show ?thesis
      by (simp add: f-A1 field-simps cong add: prod.cong cong del: if-weak-cong)
  qed
qed

lemma sum-image-le:
  fixes g :: 'a ⇒ 'b::ordered-comm-monoid-add
  assumes finite I ⋀i. i ∈ I =⇒ 0 ≤ g(f i)
    shows sum g (f ‘ I) ≤ sum (g ◦ f) I
  using assms
proof induction
  case empty
  then show ?case by auto
next
  case (insert i I)
  hence ∗: sum g (f ‘ I) ≤ g (f i) + sum g (f ‘ I)
          sum g (f ‘ I) ≤ sum (g ◦ f) I using add-increasing by blast+
  have sum g (f ‘ insert i I) = sum g (insert (f i) (f ‘ I)) by simp
  also have . . . ≤ g (f i) + sum g (f ‘ I) by (simp add: ∗ insert sum.insert-if)
  also from ∗ have . . . ≤ g (f i) + sum (g ◦ f) I by (intro add-left-mono)
  also from insert have . . . = sum (g ◦ f) (insert i I) by (simp add: sum.insert-if)
  finally show ?case .
qed

end
```

49 Chain-complete partial orders and their fixpoints

```
theory Complete-Partial-Order
  imports Product-Type
begin
```

49.1 Chains

A chain is a totally-ordered set. Chains are parameterized over the order for maximal flexibility, since type classes are not enough.

definition *chain* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
where $chain\ ord\ S \longleftrightarrow (\forall x \in S.\ \forall y \in S.\ ord\ x\ y \lor ord\ y\ x)$

lemma *chainI*:
assumes $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow ord\ x\ y \lor ord\ y\ x$
shows *chain ord S*
using *assms* **unfolding** *chain-def* **by** *fast*

lemma *chainD*:
assumes *chain ord S* **and** $x \in S$ **and** $y \in S$
shows $ord\ x\ y \lor ord\ y\ x$
using *assms* **unfolding** *chain-def* **by** *fast*

lemma *chainE*:
assumes *chain ord S* **and** $x \in S$ **and** $y \in S$
obtains $ord\ x\ y \mid ord\ y\ x$
using *assms* **unfolding** *chain-def* **by** *fast*

lemma *chain-empty*: *chain ord* {}
by (*simp add*: *chain-def*)

lemma *chain-equality*: $chain\ (=)\ A \longleftrightarrow (\forall x \in A.\ \forall y \in A.\ x = y)$
by (*auto simp add*: *chain-def*)

lemma *chain-subset*: $chain\ ord\ A \Longrightarrow B \subseteq A \Longrightarrow chain\ ord\ B$
by (*rule chainI*) (*blast dest*: *chainD*)

lemma *chain-imageI*:
assumes *chain*: *chain le-a Y*
and *mono*: $\bigwedge x\ y.\ x \in Y \Longrightarrow y \in Y \Longrightarrow le\text{-}a\ x\ y \Longrightarrow le\text{-}b\ (f\ x)\ (f\ y)$
shows *chain le-b* (*f* ‘ *Y*)
by (*blast intro*: *chainI dest*: *chainD*[*OF chain*] *mono*)

49.2 Chain-complete partial orders

A *ccpo* has a least upper bound for any chain. In particular, the empty set is a chain, so every *ccpo* must have a bottom element.

class *ccpo* = *order* + *Sup* +
assumes *ccpo-Sup-upper*: $chain\ (\le)\ A \Longrightarrow x \in A \Longrightarrow x \le Sup\ A$
assumes *ccpo-Sup-least*: $chain\ (\le)\ A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow x \le z) \Longrightarrow Sup\ A \le z$
begin

lemma *chain-singleton*: *Complete-Partial-Order.chain* $(\le)\ \{x\}$

```
  by (rule chainI) simp

lemma ccpo-Sup-singleton [simp]: ⨆{x} = x
  by (rule order.antisym) (auto intro: ccpo-Sup-least ccpo-Sup-upper simp add:
chain-singleton)
```

49.3 Transfinite iteration of a function

```
context notes [[inductive-internals]]
begin

inductive-set iterates :: ('a ⇒ 'a) ⇒ 'a set
  for f :: 'a ⇒ 'a
  where
    step: x ∈ iterates f ⟹ f x ∈ iterates f
  | Sup: chain (≤) M ⟹ ∀ x∈M. x ∈ iterates f ⟹ Sup M ∈ iterates f

end

lemma iterates-le-f: x ∈ iterates f ⟹ monotone (≤) (≤) f ⟹ x ≤ f x
  by (induct x rule: iterates.induct)
    (force dest: monotoneD intro!: ccpo-Sup-upper ccpo-Sup-least)+

lemma chain-iterates:
  assumes f: monotone (≤) (≤) f
  shows chain (≤) (iterates f) (is chain - ?C)
proof (rule chainI)
  fix x y
  assume x ∈ ?C y ∈ ?C
  then show x ≤ y ∨ y ≤ x
  proof (induct x arbitrary: y rule: iterates.induct)
    fix x y
    assume y: y ∈ ?C
      and IH: ⋀z. z ∈ ?C ⟹ x ≤ z ∨ z ≤ x
    from y show f x ≤ y ∨ y ≤ f x
    proof (induct y rule: iterates.induct)
      case (step y)
      with IH f show ?case by (auto dest: monotoneD)
    next
      case (Sup M)
      then have chM: chain (≤) M
        and IH': ⋀z. z ∈ M ⟹ f x ≤ z ∨ z ≤ f x by auto
      show f x ≤ Sup M ∨ Sup M ≤ f x
      proof (cases ∃ z∈M. f x ≤ z)
        case True
        then have f x ≤ Sup M
          by (blast intro: ccpo-Sup-upper[OF chM] order-trans)
        then show ?thesis ..
      next
```

```
      case False
      with IH′ show ?thesis
        by (auto intro: ccpo-Sup-least[OF chM])
    qed
  qed
next
  case (Sup M y)
  show ?case
  proof (cases ∃x∈M. y ≤ x)
    case True
    then have y ≤ Sup M
      by (blast intro: ccpo-Sup-upper[OF Sup(1)] order-trans)
    then show ?thesis ..
  next
    case False with Sup
    show ?thesis by (auto intro: ccpo-Sup-least)
  qed
qed
qed

lemma bot-in-iterates: Sup {} ∈ iterates f
  by (auto intro: iterates.Sup simp add: chain-empty)
```

49.4 Fixpoint combinator

```
definition fixp :: ('a ⇒ 'a) ⇒ 'a
  where fixp f = Sup (iterates f)

lemma iterates-fixp:
  assumes f: monotone (≤) (≤) f
  shows fixp f ∈ iterates f
  unfolding fixp-def
  by (simp add: iterates.Sup chain-iterates f)

lemma fixp-unfold:
  assumes f: monotone (≤) (≤) f
  shows fixp f = f (fixp f)
proof (rule order.antisym)
  show fixp f ≤ f (fixp f)
    by (intro iterates-le-f iterates-fixp f)
  have f (fixp f) ≤ Sup (iterates f)
    by (intro ccpo-Sup-upper chain-iterates f iterates.step iterates-fixp)
  then show f (fixp f) ≤ fixp f
    by (simp only: fixp-def)
qed

lemma fixp-lowerbound:
  assumes f: monotone (≤) (≤) f
    and z: f z ≤ z
```

```
  shows fixp f ≤ z
  unfolding fixp-def
proof (rule ccpo-Sup-least[OF chain-iterates[OF f]])
  fix x
  assume x ∈ iterates f
  then show x ≤ z
  proof (induct x rule: iterates.induct)
    case (step x)
    from f ‹x ≤ z› have f x ≤ f z by (rule monotoneD)
    also note z
    finally show f x ≤ z .
  next
    case (Sup M)
    then show ?case
      by (auto intro: ccpo-Sup-least)
  qed
qed

end
```

49.5 Fixpoint induction

```
setup ‹Sign.map-naming (Name-Space.mandatory-path ccpo)›

definition admissible :: ('a set ⇒ 'a) ⇒ ('a ⇒ 'a ⇒ bool) ⇒ ('a ⇒ bool) ⇒ bool
  where admissible lub ord P ⟷ (∀ A. chain ord A ⟶ A ≠ {} ⟶ (∀ x∈A. P
x) ⟶ P (lub A))

lemma admissibleI:
  assumes ⋀A. chain ord A ⟹ A ≠ {} ⟹ ∀ x∈A. P x ⟹ P (lub A)
  shows ccpo.admissible lub ord P
  using assms unfolding ccpo.admissible-def by fast

lemma admissibleD:
  assumes ccpo.admissible lub ord P
  assumes chain ord A
  assumes A ≠ {}
  assumes ⋀x. x ∈ A ⟹ P x
  shows P (lub A)
  using assms by (auto simp: ccpo.admissible-def)

setup ‹Sign.map-naming Name-Space.parent-path›

lemma (in ccpo) fixp-induct:
  assumes adm: ccpo.admissible Sup (≤) P
  assumes mono: monotone (≤) (≤) f
  assumes bot: P (Sup {})
  assumes step: ⋀x. P x ⟹ P (f x)
  shows P (fixp f)
```

unfolding *fixp-def*
using *adm chain-iterates*[*OF mono*]
proof (*rule ccpo.admissibleD*)
show *iterates f* $\neq$ {}
using *bot-in-iterates* **by** *auto*
next
fix *x*
assume $x \in$ *iterates f*
then show *P x*
proof (*induct rule*: *iterates.induct*)
case *prems*: (*step x*)
from *this*(*2*) **show** *?case* **by** (*rule step*)
next
case (*Sup M*)
then show *?case* **by** (*cases M* = {}) (*auto intro*: *step bot ccpo.admissibleD adm*)
qed
qed

lemma *admissible-True*: *ccpo.admissible lub ord* (λx. *True*)
unfolding *ccpo.admissible-def* **by** *simp*

lemma *admissible-const*: *ccpo.admissible lub ord* (λx. *t*)
by (*auto intro*: *ccpo.admissibleI*)

lemma *admissible-conj*:
assumes *ccpo.admissible lub ord* (λx. *P x*)
assumes *ccpo.admissible lub ord* (λx. *Q x*)
shows *ccpo.admissible lub ord* (λx. *P x* $\wedge$ *Q x*)
using *assms* **unfolding** *ccpo.admissible-def* **by** *simp*

lemma *admissible-all*:
assumes $\bigwedge y$. *ccpo.admissible lub ord* (λx. *P x y*)
shows *ccpo.admissible lub ord* (λx. $\forall y$. *P x y*)
using *assms* **unfolding** *ccpo.admissible-def* **by** *fast*

lemma *admissible-ball*:
assumes $\bigwedge y$. $y \in A \Longrightarrow$ *ccpo.admissible lub ord* (λx. *P x y*)
shows *ccpo.admissible lub ord* (λx. $\forall y \in A$. *P x y*)
using *assms* **unfolding** *ccpo.admissible-def* **by** *fast*

lemma *chain-compr*: *chain ord A* $\Longrightarrow$ *chain ord* $\{x \in A.\ P\ x\}$
unfolding *chain-def* **by** *fast*

context *ccpo*
begin

lemma *admissible-disj*:

fixes $P\ Q :: 'a \Rightarrow bool$
assumes P: $ccpo.admissible\ Sup\ (\leq)\ (\lambda x.\ P\ x)$
assumes Q: $ccpo.admissible\ Sup\ (\leq)\ (\lambda x.\ Q\ x)$
shows $ccpo.admissible\ Sup\ (\leq)\ (\lambda x.\ P\ x \vee Q\ x)$
proof (*rule ccpo.admissibleI*)
fix $A :: 'a\ set$
assume *chain*: $chain\ (\leq)\ A$
assume A: $A \neq \{\}$ **and** $P\text{-}Q$: $\forall x{\in}A.\ P\ x \vee Q\ x$
have $(\exists x{\in}A.\ P\ x) \wedge (\forall x{\in}A.\ \exists y{\in}A.\ x \leq y \wedge P\ y) \vee (\exists x{\in}A.\ Q\ x) \wedge (\forall x{\in}A.\ \exists y{\in}A.\ x \leq y \wedge Q\ y)$
(**is** $?P \vee ?Q$ **is** $?P1 \wedge ?P2 \vee$ -)
proof (*rule disjCI*)
assume $\neg\ ?Q$
then consider $\forall x{\in}A.\ \neg\ Q\ x$ | a **where** $a \in A\ \forall y{\in}A.\ a \leq y \longrightarrow \neg\ Q\ y$
by *blast*
then show $?P$
proof *cases*
case *1*
with $P\text{-}Q$ **have** $\forall x{\in}A.\ P\ x$ **by** *blast*
with A **show** $?P$ **by** *blast*
next
case *2*
note $a = \langle a \in A\rangle$
show $?P$
proof
from $P\text{-}Q$ *2* **have** $*$: $\forall y{\in}A.\ a \leq y \longrightarrow P\ y$ **by** *blast*
with a **have** $P\ a$ **by** *blast*
with a **show** $?P1$ **by** *blast*
show $?P2$
proof
fix x
assume x: $x \in A$
with *chain* a **show** $\exists y{\in}A.\ x \leq y \wedge P\ y$
proof (*rule chainE*)
assume *le*: $a \leq x$
with $*\ a\ x$ **have** $P\ x$ **by** *blast*
with x *le* **show** *?thesis* **by** *blast*
next
assume $a \geq x$
with a $\langle P\ a\rangle$ **show** *?thesis* **by** *blast*
qed
qed
qed
qed
qed
moreover
have $Sup\ A = Sup\ \{x \in A.\ P\ x\}$ **if** $\bigwedge x.\ x{\in}A \Longrightarrow \exists y{\in}A.\ x \leq y \wedge P\ y$ **for** P
proof (*rule order.antisym*)
have *chain-P*: $chain\ (\leq)\ \{x \in A.\ P\ x\}$

```
    by (rule chain-compr [OF chain])
  show Sup A ≤ Sup {x ∈ A. P x}
  proof (rule ccpo-Sup-least [OF chain])
    show ⋀x. x ∈ A ⟹ x ≤ ⨆ {x ∈ A. P x}
        by (blast intro: ccpo-Sup-upper[OF chain-P] order-trans dest: that)
    qed
  show Sup {x ∈ A. P x} ≤ Sup A
    apply (rule ccpo-Sup-least [OF chain-P])
    apply (simp add: ccpo-Sup-upper [OF chain])
    done
 qed
 ultimately
 consider ∃x. x ∈ A ∧ P x Sup A = Sup {x ∈ A. P x}
   | ∃x. x ∈ A ∧ Q x Sup A = Sup {x ∈ A. Q x}
   by blast
 then show P (Sup A) ∨ Q (Sup A)
 proof cases
   case 1
   then show ?thesis
     using ccpo.admissibleD [OF P chain-compr [OF chain]] by force
 next
   case 2
   then show ?thesis
     using ccpo.admissibleD [OF Q chain-compr [OF chain]] by force
 qed
qed

end

instance complete-lattice ⊆ ccpo
 by standard (fast intro: Sup-upper Sup-least)+

lemma lfp-eq-fixp:
 assumes mono: mono f
 shows lfp f = fixp f
proof (rule order.antisym)
 from mono have f': monotone (≤) (≤) f
   unfolding mono-def monotone-def .
 show lfp f ≤ fixp f
   by (rule lfp-lowerbound, subst fixp-unfold [OF f'], rule order-refl)
 show fixp f ≤ lfp f
   by (rule fixp-lowerbound [OF f']) (simp add: lfp-fixpoint [OF mono])
qed

hide-const (open) iterates fixp

end
```

50 Datatype option

theory *Option*
  imports *Lifting*
begin

datatype *′a option* =
    None
  | *Some* (*the*: *′a*)

datatype-compat *option*

lemma [*case-names None Some*, *cases type*: *option*]:
  — for backward compatibility – names of variables differ
  $(y = None \Longrightarrow P) \Longrightarrow (\bigwedge a.\ y = Some\ a \Longrightarrow P) \Longrightarrow P$
  by (*rule option.exhaust*)

lemma [*case-names None Some*, *induct type*: *option*]:
  — for backward compatibility – names of variables differ
  $P\ None \Longrightarrow (\bigwedge option.\ P\ (Some\ option)) \Longrightarrow P\ option$
  by (*rule option.induct*)

Compatibility:

setup ‹*Sign.mandatory-path option*›
lemmas *inducts* = *option.induct*
lemmas *cases* = *option.case*
setup ‹*Sign.parent-path*›

lemma *not-None-eq* [*iff*]: $x \neq None \longleftrightarrow (\exists\, y.\ x = Some\ y)$
  by (*induct x*) *auto*

lemma *not-Some-eq* [*iff*]: $(\forall\, y.\ x \neq Some\ y) \longleftrightarrow x = None$
  by (*induct x*) *auto*

lemma *comp-the-Some*[*simp*]: *the o Some* = *id*
by *auto*

Although it may appear that both of these equalities are helpful only when applied to assumptions, in practice it seems better to give them the uniform iff attribute.

lemma *inj-Some* [*simp*]: *inj-on Some A*
  by (*rule inj-onI*) *simp*

lemma *case-optionE*:
  assumes *c*: (*case x of None* ⇒ *P* | *Some y* ⇒ *Q y*)
  obtains
    (*None*) *x* = *None* **and** *P*
  | (*Some*) *y* **where** *x* = *Some y* **and** *Q y*
  using *c* **by** (*cases x*) *simp-all*

lemma *split-option-all*: $(\forall x.\ P\ x) \longleftrightarrow P\ None \wedge (\forall x.\ P\ (Some\ x))$
  by (*auto intro*: *option.induct*)

lemma *split-option-ex*: $(\exists x.\ P\ x) \longleftrightarrow P\ None \vee (\exists x.\ P\ (Some\ x))$
  using *split-option-all*[*of* $\lambda x.\ \neg\ P\ x$] **by** *blast*

lemma *UNIV-option-conv*: $UNIV = insert\ None\ (range\ Some)$
  by (*auto intro*: *classical*)

lemma *rel-option-None1* [*simp*]: $rel\text{-}option\ P\ None\ x \longleftrightarrow x = None$
  by (*cases x*) *simp-all*

lemma *rel-option-None2* [*simp*]: $rel\text{-}option\ P\ x\ None \longleftrightarrow x = None$
  by (*cases x*) *simp-all*

lemma *option-rel-Some1*: $rel\text{-}option\ A\ (Some\ x)\ y \longleftrightarrow (\exists y'.\ y = Some\ y' \wedge A\ x\ y')$
by(*cases y*) *simp-all*

lemma *option-rel-Some2*: $rel\text{-}option\ A\ x\ (Some\ y) \longleftrightarrow (\exists x'.\ x = Some\ x' \wedge A\ x'\ y)$
by(*cases x*) *simp-all*

lemma *rel-option-inf*: $inf\ (rel\text{-}option\ A)\ (rel\text{-}option\ B) = rel\text{-}option\ (inf\ A\ B)$
  (**is** $?lhs = ?rhs$)
proof (*rule antisym*)
  show $?lhs \leq ?rhs$ **by** (*auto elim*: *option.rel-cases*)
  show $?rhs \leq ?lhs$ **by** (*auto elim*: *option.rel-mono-strong*)
qed

lemma *rel-option-reflI*:
  $(\bigwedge x.\ x \in set\text{-}option\ y \Longrightarrow P\ x\ x) \Longrightarrow rel\text{-}option\ P\ y\ y$
  by (*cases y*) *auto*

50.0.1 Operations

lemma *ospec* [*dest*]: $(\forall x \in set\text{-}option\ A.\ P\ x) \Longrightarrow A = Some\ x \Longrightarrow P\ x$
  by *simp*

setup ‹*map-theory-claset* (*fn ctxt* => *ctxt addSD2* (*ospec*, @{*thm ospec*}))›

lemma *elem-set* [*iff*]: $(x \in set\text{-}option\ xo) = (xo = Some\ x)$
  by (*cases xo*) *auto*

lemma *set-empty-eq* [*simp*]: $(set\text{-}option\ xo = \{\}) = (xo = None)$
  by (*cases xo*) *auto*

lemma *map-option-case*: $map\text{-}option\ f\ y = (case\ y\ of\ None \Rightarrow None \mid Some\ x \Rightarrow$

Some (*f x*))
by (*auto split*: *option.split*)

lemma *map-option-is-None* [*iff*]: (*map-option f opt* = *None*) = (*opt* = *None*)
by (*simp add*: *map-option-case split*: *option.split*)

lemma *None-eq-map-option-iff* [*iff*]: *None* = *map-option f x* ⟷ *x* = *None*
by(*cases x*) *simp-all*

lemma *map-option-eq-Some* [*iff*]: (*map-option f xo* = *Some y*) = (∃ *z*. *xo* = *Some z* ∧ *f z* = *y*)
by (*simp add*: *map-option-case split*: *option.split*)

lemma *map-option-o-case-sum* [*simp*]:
map-option f ∘ *case-sum g h* = *case-sum* (*map-option f* ∘ *g*) (*map-option f* ∘ *h*)
by (*rule o-case-sum*)

lemma *map-option-cong*: *x* = *y* ⟹ (⋀*a*. *y* = *Some a* ⟹ *f a* = *g a*) ⟹ *map-option f x* = *map-option g y*
by (*cases x*) *auto*

lemma *map-option-idI*: (⋀*y*. *y* ∈ *set-option x* ⟹ *f y* = *y*) ⟹ *map-option f x* = *x*
by(*cases x*)(*simp-all*)

functor *map-option*: *map-option*
by (*simp-all add*: *option.map-comp fun-eq-iff option.map-id*)

lemma *case-map-option* [*simp*]: *case-option g h* (*map-option f x*) = *case-option g* (*h* ∘ *f*) *x*
by (*cases x*) *simp-all*

lemma *None-notin-image-Some* [*simp*]: *None* ∉ *Some* ‘ *A*
by *auto*

lemma *notin-range-Some*: *x* ∉ *range Some* ⟷ *x* = *None*
by(*cases x*) *auto*

lemma *rel-option-iff*:
rel-option R x y = (*case* (*x*, *y*) *of* (*None*, *None*) ⇒ *True*
| (*Some x*, *Some y*) ⇒ *R x y*
| - ⇒ *False*)
by (*auto split*: *prod.split option.split*)

definition *combine-options* :: ('*a* ⇒ '*a* ⇒ '*a*) ⇒ '*a option* ⇒ '*a option* ⇒ '*a option*
where *combine-options f x y* =
(*case x of None* ⇒ *y* | *Some x* ⇒ (*case y of None* ⇒ *Some x* | *Some y* ⇒

Some (*f x y*)))

lemma *combine-options-simps* [*simp*]:
combine-options f None y = *y*
combine-options f x None = *x*
combine-options f (*Some a*) (*Some b*) = *Some* (*f a b*)
by (*simp-all add*: *combine-options-def split*: *option.splits*)

lemma *combine-options-cases* [*case-names None1 None2 Some*]:
$(x = None \Longrightarrow P\ x\ y) \Longrightarrow (y = None \Longrightarrow P\ x\ y) \Longrightarrow$
$(\bigwedge a\ b.\ x = Some\ a \Longrightarrow y = Some\ b \Longrightarrow P\ x\ y) \Longrightarrow P\ x\ y$
by (*cases x*; *cases y*) *simp-all*

lemma *combine-options-commute*:
$(\bigwedge x\ y.\ f\ x\ y = f\ y\ x) \Longrightarrow$ *combine-options f x y* = *combine-options f y x*
using *combine-options-cases*[*of x*]
by (*induction x y rule*: *combine-options-cases*) *simp-all*

lemma *combine-options-assoc*:
$(\bigwedge x\ y\ z.\ f\ (f\ x\ y)\ z = f\ x\ (f\ y\ z)) \Longrightarrow$
combine-options f (*combine-options f x y*) *z* =
combine-options f x (*combine-options f y z*)
by (*auto simp*: *combine-options-def split*: *option.splits*)

lemma *combine-options-left-commute*:
$(\bigwedge x\ y.\ f\ x\ y = f\ y\ x) \Longrightarrow (\bigwedge x\ y\ z.\ f\ (f\ x\ y)\ z = f\ x\ (f\ y\ z)) \Longrightarrow$
combine-options f y (*combine-options f x z*) =
combine-options f x (*combine-options f y z*)
by (*auto simp*: *combine-options-def split*: *option.splits*)

lemmas *combine-options-ac* =
combine-options-commute combine-options-assoc combine-options-left-commute

context
begin

qualified definition *is-none* :: *'a option* ⇒ *bool*
where [*code-post*]: *is-none x* ⟷ *x* = *None*

lemma *is-none-simps* [*simp*]:
is-none None
¬ *is-none* (*Some x*)
by (*simp-all add*: *is-none-def*)

lemma *is-none-code* [*code*]:
is-none None = *True*
is-none (*Some x*) = *False*
by *simp-all*

lemma *rel-option-unfold*:
$rel\text{-}option\ R\ x\ y \longleftrightarrow$
$(is\text{-}none\ x \longleftrightarrow is\text{-}none\ y) \wedge (\neg\ is\text{-}none\ x \longrightarrow \neg\ is\text{-}none\ y \longrightarrow R\ (the\ x)\ (the\ y))$
by (*simp add*: *rel-option-iff split*: *option.split*)

lemma *rel-optionI*:
$[\![\ is\text{-}none\ x \longleftrightarrow is\text{-}none\ y;\ [\![\ \neg\ is\text{-}none\ x;\ \neg\ is\text{-}none\ y\ ]\!] \Longrightarrow P\ (the\ x)\ (the\ y)\ ]\!] \Longrightarrow rel\text{-}option\ P\ x\ y$
by (*simp add*: *rel-option-unfold*)

lemma *is-none-map-option* [*simp*]: $is\text{-}none\ (map\text{-}option\ f\ x) \longleftrightarrow is\text{-}none\ x$
by (*simp add*: *is-none-def*)

lemma *the-map-option*: $\neg\ is\text{-}none\ x \Longrightarrow the\ (map\text{-}option\ f\ x) = f\ (the\ x)$
by (*auto simp add*: *is-none-def*)

qualified primrec $bind :: 'a\ option \Rightarrow ('a \Rightarrow 'b\ option) \Rightarrow 'b\ option$
where
bind-lzero: $bind\ None\ f = None$
| *bind-lunit*: $bind\ (Some\ x)\ f = f\ x$

lemma *is-none-bind*: $is\text{-}none\ (bind\ f\ g) \longleftrightarrow is\text{-}none\ f \vee is\text{-}none\ (g\ (the\ f))$
by (*cases f*) *simp-all*

lemma *bind-runit*[*simp*]: $bind\ x\ Some = x$
by (*cases x*) *auto*

lemma *bind-assoc*[*simp*]: $bind\ (bind\ x\ f)\ g = bind\ x\ (\lambda y.\ bind\ (f\ y)\ g)$
by (*cases x*) *auto*

lemma *bind-rzero*[*simp*]: $bind\ x\ (\lambda x.\ None) = None$
by (*cases x*) *auto*

qualified lemma *bind-cong*: $x = y \Longrightarrow (\bigwedge a.\ y = Some\ a \Longrightarrow f\ a = g\ a) \Longrightarrow bind\ x\ f = bind\ y\ g$
by (*cases x*) *auto*

lemma *bind-split*: $P\ (bind\ m\ f) \longleftrightarrow (m = None \longrightarrow P\ None) \wedge (\forall v.\ m = Some\ v \longrightarrow P\ (f\ v))$
by (*cases m*) *auto*

lemma *bind-split-asm*: $P\ (bind\ m\ f) \longleftrightarrow \neg\ (m = None \wedge \neg\ P\ None \vee (\exists x.\ m = Some\ x \wedge \neg\ P\ (f\ x)))$
by (*cases m*) *auto*

lemmas *bind-splits* = *bind-split bind-split-asm*

lemma *bind-eq-Some-conv*: *bind f g* = *Some x* ⟷ (∃ *y*. *f* = *Some y* ∧ *g y* = *Some x*)
  by (*cases f*) *simp-all*

lemma *bind-eq-None-conv*: *Option.bind a f* = *None* ⟷ *a* = *None* ∨ *f* (*the a*) = *None*
by(*cases a*) *simp-all*

lemma *map-option-bind*: *map-option f* (*bind x g*) = *bind x* (*map-option f* ∘ *g*)
  by (*cases x*) *simp-all*

lemma *bind-option-cong*:
  ⟦ *x* = *y*; ⋀*z*. *z* ∈ *set-option y* ⟹ *f z* = *g z* ⟧ ⟹ *bind x f* = *bind y g*
  by (*cases y*) *simp-all*

lemma *bind-option-cong-simp*:
  ⟦ *x* = *y*; ⋀*z*. *z* ∈ *set-option y* =*simp*=> *f z* = *g z* ⟧ ⟹ *bind x f* = *bind y g*
  unfolding *simp-implies-def* **by** (*rule bind-option-cong*)

lemma *bind-option-cong-code*: *x* = *y* ⟹ *bind x f* = *bind y f*
  by *simp*

lemma *bind-map-option*: *bind* (*map-option f x*) *g* = *bind x* (*g* ∘ *f*)
by(*cases x*) *simp-all*

lemma *set-bind-option* [*simp*]: *set-option* (*bind x f*) = (⋃ ((*set-option* ∘ *f*) ‘ *set-option x*))
by(*cases x*) *auto*

lemma *map-conv-bind-option*: *map-option f x* = *Option.bind x* (*Some* ∘ *f*)
by(*cases x*) *simp-all*

end

setup ‹*Code-Simp.map-ss* (*Simplifier.add-cong* @{*thm bind-option-cong-code*})›

context
begin

qualified definition *these* :: ′*a option set* ⇒ ′*a set*
  where *these A* = *the* ‘ {*x* ∈ *A*. *x* ≠ *None*}

lemma *these-empty* [*simp*]: *these* {} = {}
  by (*simp add*: *these-def*)

lemma *these-insert-None* [*simp*]: *these* (*insert None A*) = *these A*
  by (*auto simp add*: *these-def*)

lemma *these-insert-Some* [*simp*]: *these* (*insert* (*Some x*) *A*) = *insert x* (*these A*)
proof −
  have {*y* ∈ *insert* (*Some x*) *A*. *y* ≠ *None*} = *insert* (*Some x*) {*y* ∈ *A*. *y* ≠ *None*}
    by *auto*
  then show *?thesis* **by** (*simp add*: *these-def*)
qed

lemma *in-these-eq*: *x* ∈ *these A* ⟷ *Some x* ∈ *A*
proof
  assume *Some x* ∈ *A*
  then obtain *B* **where** *A* = *insert* (*Some x*) *B* **by** *auto*
  then show *x* ∈ *these A* **by** (*auto simp add*: *these-def intro*!: *image-eqI*)
next
  assume *x* ∈ *these A*
  then show *Some x* ∈ *A* **by** (*auto simp add*: *these-def*)
qed

lemma *these-image-Some-eq* [*simp*]: *these* (*Some* ` *A*) = *A*
  by (*auto simp add*: *these-def intro*!: *image-eqI*)

lemma *Some-image-these-eq*: *Some* ` *these A* = {*x*∈*A*. *x* ≠ *None*}
  by (*auto simp add*: *these-def image-image intro*!: *image-eqI*)

lemma *these-empty-eq*: *these B* = {} ⟷ *B* = {} ∨ *B* = {*None*}
  by (*auto simp add*: *these-def*)

lemma *these-not-empty-eq*: *these B* ≠ {} ⟷ *B* ≠ {} ∧ *B* ≠ {*None*}
  by (*auto simp add*: *these-empty-eq*)

end

lemma *finite-range-Some*: *finite* (*range* (*Some* :: ′*a* ⇒ ′*a option*)) = *finite* (*UNIV* :: ′*a set*)
  by (*auto dest*: *finite-imageD intro*: *inj-Some*)

50.1 Transfer rules for the Transfer package

context includes *lifting-syntax*
begin

lemma *option-bind-transfer* [*transfer-rule*]:
  (*rel-option A* ===> (*A* ===> *rel-option B*) ===> *rel-option B*)
    Option.bind Option.bind
  unfolding *rel-fun-def split-option-all* **by** *simp*

lemma *pred-option-parametric* [*transfer-rule*]:
  ((*A* ===> (=)) ===> *rel-option A* ===> (=)) *pred-option pred-option*
  by (*rule rel-funI*)+ (*auto simp add*: *rel-option-unfold Option.is-none-def dest*:

rel-funD)

end

50.1.1 Interaction with finite sets

lemma *finite-option-UNIV* [*simp*]:
finite (*UNIV* :: *'a option set*) = *finite* (*UNIV* :: *'a set*)
by (*auto simp add*: *UNIV-option-conv elim*: *finite-imageD intro*: *inj-Some*)

instance *option* :: (*finite*) *finite*
by *standard* (*simp add*: *UNIV-option-conv*)

50.1.2 Code generator setup

lemma *equal-None-code-unfold* [*code-unfold*]:
HOL.equal x None ⟷ *Option.is-none x*
HOL.equal None = *Option.is-none*
by (*auto simp add*: *equal Option.is-none-def*)

code-printing
type-constructor *option* ⇀
(*SML*) - *option*
and (*OCaml*) - *option*
and (*Haskell*) *Maybe* -
and (*Scala*) !*Option*[(-)]
| **constant** *None* ⇀
(*SML*) *NONE*
and (*OCaml*) *None*
and (*Haskell*) *Nothing*
and (*Scala*) !*None*
| **constant** *Some* ⇀
(*SML*) *SOME*
and (*OCaml*) *Some* -
and (*Haskell*) *Just*
and (*Scala*) *Some*
| **class-instance** *option* :: *equal* ⇀
(*Haskell*) −
| **constant** *HOL.equal* :: *'a option* ⇒ *'a option* ⇒ *bool* ⇀
(*Haskell*) **infix** *4* ==

code-reserved
(*SML*) *option NONE SOME*
and (*OCaml*) *option None Some*
and (*Scala*) *Option None Some*

end

51 Partial Function Definitions

theory *Partial-Function*
  imports *Complete-Partial-Order Option*
  keywords *partial-function* :: *thy-defn*
begin

named-theorems *partial-function-mono monotonicity rules for partial function definitions*
ML-file ‹*Tools/Function/partial-function.ML*›

lemma (**in** *ccpo*) *in-chain-finite*:
  assumes *Complete-Partial-Order.chain* (≤) *A finite A A* ≠ {}
  shows ⨆*A* ∈ *A*
using *assms*(*2*,*1*,*3*)
proof *induction*
  case *empty* **thus** *?case* **by** *simp*
next
  case (*insert x A*)
  note *chain* = ‹*Complete-Partial-Order.chain* (≤) (*insert x A*)›
  show *?case*
  proof(*cases A* = {})
    case *True* **thus** *?thesis* **by** *simp*
  next
    case *False*
    from *chain* **have** *chain'*: *Complete-Partial-Order.chain* (≤) *A*
      by(*rule chain-subset*) *blast*
    hence ⨆*A* ∈ *A* **using** *False* **by**(*rule insert.IH*)
    show *?thesis*
    proof(*cases x* ≤ ⨆*A*)
      case *True*
      have ⨆(*insert x A*) ≤ ⨆*A* **using** *chain*
          by(*rule ccpo-Sup-least*)(*auto simp add*: *True intro*: *ccpo-Sup-upper*[*OF chain'*])
      hence ⨆(*insert x A*) = ⨆*A*
      by(*rule order.antisym*)(*blast intro*: *ccpo-Sup-upper*[*OF chain*] *ccpo-Sup-least*[*OF chain'*])
      with ‹⨆*A* ∈ *A*› **show** *?thesis* **by** *simp*
    next
      case *False*
      with *chainD*[*OF chain*, *of x* ⨆*A*] ‹⨆*A* ∈ *A*›
      have ⨆(*insert x A*) = *x*
          by(*auto intro*: *order.antisym ccpo-Sup-least*[*OF chain*] *order-trans*[*OF ccpo-Sup-upper*[*OF chain'*]] *ccpo-Sup-upper*[*OF chain*])
      thus *?thesis* **by** *simp*
    qed
  qed
qed

lemma (**in** *ccpo*) *admissible-chfin*:
($\forall$ *S*. *Complete-Partial-Order.chain* ($\leq$) *S* $\longrightarrow$ *finite S*)
$\Longrightarrow$ *ccpo.admissible Sup* ($\leq$) *P*
using *in-chain-finite* **by** (*blast intro*: *ccpo.admissibleI*)

51.1 Axiomatic setup

This techical locale constains the requirements for function definitions with ccpo fixed points.

definition *fun-ord ord f g* $\longleftrightarrow$ ($\forall$ *x*. *ord* (*f x*) (*g x*))
definition *fun-lub L A* = (λ*x*. *L* {*y*. $\exists$*f*$\in$*A*. *y* = *f x*})
definition *img-ord f ord* = (λ*x y*. *ord* (*f x*) (*f y*))
definition *img-lub f g Lub* = (λ*A*. *g* (*Lub* (*f* ‘ *A*)))

lemma *chain-fun*:
assumes *A*: *chain* (*fun-ord ord*) *A*
shows *chain ord* {*y*. $\exists$*f*$\in$*A*. *y* = *f a*} (**is** *chain ord ?C*)
proof (*rule chainI*)
fix *x y* **assume** *x* $\in$ *?C y* $\in$ *?C*
then obtain *f g* **where** *fg*: *f* $\in$ *A g* $\in$ *A*
and [*simp*]: *x* = *f a y* = *g a* **by** *blast*
from *chainD*[*OF A fg*]
show *ord x y* $\vee$ *ord y x* **unfolding** *fun-ord-def* **by** *auto*
qed

lemma *call-mono*[*partial-function-mono*]: *monotone* (*fun-ord ord*) *ord* (λ*f*. *f t*)
by (*rule monotoneI*) (*auto simp*: *fun-ord-def*)

lemma *let-mono*[*partial-function-mono*]:
($\bigwedge$*x*. *monotone orda ordb* (λ*f*. *b f x*))
$\Longrightarrow$ *monotone orda ordb* (λ*f*. *Let t* (*b f*))
by (*simp add*: *Let-def*)

lemma *if-mono*[*partial-function-mono*]: *monotone orda ordb F*
$\Longrightarrow$ *monotone orda ordb G*
$\Longrightarrow$ *monotone orda ordb* (λ*f*. *if c then F f else G f*)
unfolding *monotone-def* **by** *simp*

definition *mk-less R* = (λ*x y*. *R x y* $\wedge$ $\neg$ *R y x*)

locale *partial-function-definitions* =
fixes *leq* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*
fixes *lub* :: *'a set* $\Rightarrow$ *'a*
assumes *leq-refl*: *leq x x*
assumes *leq-trans*: *leq x y* $\Longrightarrow$ *leq y z* $\Longrightarrow$ *leq x z*
assumes *leq-antisym*: *leq x y* $\Longrightarrow$ *leq y x* $\Longrightarrow$ *x* = *y*
assumes *lub-upper*: *chain leq A* $\Longrightarrow$ *x* $\in$ *A* $\Longrightarrow$ *leq x* (*lub A*)
assumes *lub-least*: *chain leq A* $\Longrightarrow$ ($\bigwedge$*x*. *x* $\in$ *A* $\Longrightarrow$ *leq x z*) $\Longrightarrow$ *leq* (*lub A*) *z*

lemma *partial-function-lift*:
  assumes *partial-function-definitions ord lb*
  shows *partial-function-definitions* (*fun-ord ord*) (*fun-lub lb*) (**is** *partial-function-definitions ?ordf ?lubf*)
proof −
  interpret *partial-function-definitions ord lb* **by** *fact*

  show *?thesis*
  proof
    fix *x* **show** *?ordf x x*
      unfolding *fun-ord-def* **by** (*auto simp*: *leq-refl*)
  next
    fix *x y z* **assume** *?ordf x y ?ordf y z*
    thus *?ordf x z* **unfolding** *fun-ord-def*
      by (*force dest*: *leq-trans*)
  next
    fix *x y* **assume** *?ordf x y ?ordf y x*
    thus $x = y$ **unfolding** *fun-ord-def*
      by (*force intro*!: *dest*: *leq-antisym*)
  next
    fix *A f* **assume** *f*: $f \in A$ **and** *A*: *chain ?ordf A*
    thus *?ordf f* (*?lubf A*)
      unfolding *fun-lub-def fun-ord-def*
      by (*blast intro*: *lub-upper chain-fun*[*OF A*] *f*)
  next
    fix *A* :: ('*b* ⇒ '*a*) *set* **and** *g* :: '*b* ⇒ '*a*
    assume *A*: *chain ?ordf A* **and** *g*: $\bigwedge f.\ f \in A \Longrightarrow$ *?ordf f g*
    show *?ordf* (*?lubf A*) *g* **unfolding** *fun-lub-def fun-ord-def*
      by (*blast intro*: *lub-least chain-fun*[*OF A*] *dest*: *g*[*unfolded fun-ord-def*])
   qed
qed

lemma *ccpo*: **assumes** *partial-function-definitions ord lb*
  shows *class.ccpo lb ord* (*mk-less ord*)
using *assms* **unfolding** *partial-function-definitions-def mk-less-def*
by *unfold-locales blast+*

lemma *partial-function-image*:
  assumes *partial-function-definitions ord Lub*
  assumes *inj*: $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
  assumes *inv*: $\bigwedge x.\ f\ (g\ x) = x$
  shows *partial-function-definitions* (*img-ord f ord*) (*img-lub f g Lub*)
proof −
  let *?iord* = *img-ord f ord*
  let *?ilub* = *img-lub f g Lub*

  interpret *partial-function-definitions ord Lub* **by** *fact*
  show *?thesis*
  proof

fix *A x* **assume** *chain ?iord A x* ∈ *A*
then have *chain ord* (*f* ' *A*) *f x* ∈ *f* ' *A*
by (*auto simp*: *img-ord-def intro*: *chainI dest*: *chainD*)
thus *?iord x* (*?ilub A*)
unfolding *inv img-lub-def img-ord-def* **by** (*rule lub-upper*)
next
fix *A x* **assume** *chain ?iord A*
and *1*: $\bigwedge z.\ z \in A \Longrightarrow$ *?iord z x*
then have *chain ord* (*f* ' *A*)
by (*auto simp*: *img-ord-def intro*: *chainI dest*: *chainD*)
thus *?iord* (*?ilub A*) *x*
unfolding *inv img-lub-def img-ord-def*
by (*rule lub-least*) (*auto dest*: *1*[*unfolded img-ord-def*])
qed (*auto simp*: *img-ord-def intro*: *leq-refl dest*: *leq-trans leq-antisym inj*)
qed

context *partial-function-definitions*
begin

abbreviation *le-fun* ≡ *fun-ord leq*
abbreviation *lub-fun* ≡ *fun-lub lub*
abbreviation *fixp-fun* ≡ *ccpo.fixp lub-fun le-fun*
abbreviation *mono-body* ≡ *monotone le-fun leq*
abbreviation *admissible* ≡ *ccpo.admissible lub-fun le-fun*

Interpret manually, to avoid flooding everything with facts about orders

lemma *ccpo*: *class.ccpo lub-fun le-fun* (*mk-less le-fun*)
apply (*rule ccpo*)
apply (*rule partial-function-lift*)
apply (*rule partial-function-definitions-axioms*)
done

The crucial fixed-point theorem

lemma *mono-body-fixp*:
($\bigwedge x.$ *mono-body* ($\lambda f.$ *F f x*)) $\Longrightarrow$ *fixp-fun F* = *F* (*fixp-fun F*)
by (*rule ccpo.fixp-unfold*[*OF ccpo*]) (*auto simp*: *monotone-def fun-ord-def*)

Version with curry/uncurry combinators, to be used by package

lemma *fixp-rule-uc*:
fixes *F* :: *'c* ⇒ *'c* **and**
U :: *'c* ⇒ *'b* ⇒ *'a* **and**
C :: (*'b* ⇒ *'a*) ⇒ *'c*
assumes *mono*: $\bigwedge x.$ *mono-body* ($\lambda f.$ *U* (*F* (*C f*)) *x*)
assumes *eq*: *f* ≡ *C* (*fixp-fun* ($\lambda f.$ *U* (*F* (*C f*))))
assumes *inverse*: $\bigwedge f.$ *C* (*U f*) = *f*
shows *f* = *F f*
proof −
have *f* = *C* (*fixp-fun* ($\lambda f.$ *U* (*F* (*C f*)))) **by** (*simp add*: *eq*)
also have ... = *C* (*U* (*F* (*C* (*fixp-fun* ($\lambda f.$ *U* (*F* (*C f*)))))))

by (*subst mono-body-fixp*[*of* %*f*. *U* (*F* (*C f*)), *OF mono*]) (*rule refl*)
also have ... = *F* (*C* (*fixp-fun* (λ*f*. *U* (*F* (*C f*))))) **by** (*rule inverse*)
also have ... = *F f* **by** (*simp add*: *eq*)
finally show *f* = *F f* **.**
qed

Fixpoint induction rule

lemma *fixp-induct-uc*:
fixes *F* :: ′*c* ⇒ ′*c*
and *U* :: ′*c* ⇒ ′*b* ⇒ ′*a*
and *C* :: (′*b* ⇒ ′*a*) ⇒ ′*c*
and *P* :: (′*b* ⇒ ′*a*) ⇒ *bool*
assumes *mono*: ⋀*x*. *mono-body* (λ*f*. *U* (*F* (*C f*)) *x*)
and *eq*: *f* ≡ *C* (*fixp-fun* (λ*f*. *U* (*F* (*C f*))))
and *inverse*: ⋀*f*. *U* (*C f*) = *f*
and *adm*: *ccpo.admissible lub-fun le-fun P*
and *bot*: *P* (λ-. *lub* {})
and *step*: ⋀*f*. *P* (*U f*) ⟹ *P* (*U* (*F f*))
shows *P* (*U f*)
unfolding *eq inverse*
proof (*rule ccpo.fixp-induct*[*OF ccpo adm*])
show *monotone le-fun le-fun* (λ*f*. *U* (*F* (*C f*)))
using *mono* **by** (*auto simp*: *monotone-def fun-ord-def*)
next
show *P* (*lub-fun* {})
by (*auto simp*: *bot fun-lub-def*)
next
fix *x*
assume *P x*
then show *P* (*U* (*F* (*C x*)))
using *step*[*of C x*] **by** (*simp add*: *inverse*)
qed

Rules for *monotone le-fun leq*:

lemma *const-mono*[*partial-function-mono*]: *monotone ord leq* (λ*f*. *c*)
by (*rule monotoneI*) (*rule leq-refl*)

end

51.2 Flat interpretation: tailrec and option

definition
flat-ord b x y ⟷ *x* = *b* ∨ *x* = *y*

definition
flat-lub b A = (*if A* ⊆ {*b*} *then b else* (*THE x*. *x* ∈ *A* − {*b*}))

lemma *flat-interpretation*:
partial-function-definitions (*flat-ord b*) (*flat-lub b*)

proof
  fix *A x* **assume** *1*: *chain* (*flat-ord b*) *A x* ∈ *A*
  show *flat-ord b x* (*flat-lub b A*)
  proof *cases*
    assume *x* = *b*
    thus *?thesis* **by** (*simp add*: *flat-ord-def*)
  next
    assume *x* ≠ *b*
    with *1* **have** *A* − {*b*} = {*x*}
      by (*auto elim*: *chainE simp*: *flat-ord-def*)
    then have *flat-lub b A* = *x*
      by (*auto simp*: *flat-lub-def*)
    thus *?thesis* **by** (*auto simp*: *flat-ord-def*)
  qed
next
  fix *A z* **assume** *A*: *chain* (*flat-ord b*) *A*
    and *z*: ⋀*x*. *x* ∈ *A* ⟹ *flat-ord b x z*
  show *flat-ord b* (*flat-lub b A*) *z*
  proof *cases*
    assume *A* ⊆ {*b*}
    thus *?thesis*
      by (*auto simp*: *flat-lub-def flat-ord-def*)
  next
    assume *nb*: ¬ *A* ⊆ {*b*}
    then obtain *y* **where** *y*: *y* ∈ *A y* ≠ *b* **by** *auto*
    with *A* **have** *A* − {*b*} = {*y*}
      by (*auto elim*: *chainE simp*: *flat-ord-def*)
    with *nb* **have** *flat-lub b A* = *y*
      by (*auto simp*: *flat-lub-def*)
    with *z y* **show** *?thesis* **by** *auto*
  qed
qed (*auto simp*: *flat-ord-def*)

lemma *flat-ordI*: (*x* ≠ *a* ⟹ *x* = *y*) ⟹ *flat-ord a x y*
by(*auto simp add*: *flat-ord-def*)

lemma *flat-ord-antisym*: ⟦ *flat-ord a x y*; *flat-ord a y x* ⟧ ⟹ *x* = *y*
by(*auto simp add*: *flat-ord-def*)

lemma *antisymp-flat-ord*: *antisymp* (*flat-ord a*)
by(*rule antisympI*)(*auto dest*: *flat-ord-antisym*)

interpretation *tailrec*:
  partial-function-definitions flat-ord undefined flat-lub undefined
  rewrites *flat-lub undefined* {} ≡ *undefined*
by (*rule flat-interpretation*)(*simp add*: *flat-lub-def*)

interpretation *option*:
  partial-function-definitions flat-ord None flat-lub None

rewrites *flat-lub None* {} ≡ *None*
by (*rule flat-interpretation*)(*simp add*: *flat-lub-def*)

abbreviation *tailrec-ord* ≡ *flat-ord undefined*
abbreviation *mono-tailrec* ≡ *monotone* (*fun-ord tailrec-ord*) *tailrec-ord*

lemma *tailrec-admissible*:
ccpo.admissible (*fun-lub* (*flat-lub c*)) (*fun-ord* (*flat-ord c*))
($\lambda a.\ \forall x.\ a\ x \neq c \longrightarrow P\ x\ (a\ x)$)
proof(*intro ccpo.admissibleI strip*)
fix *A x*
assume *chain*: *Complete-Partial-Order.chain* (*fun-ord* (*flat-ord c*)) *A*
and *P* [*rule-format*]: $\forall f{\in}A.\ \forall x.\ f\ x \neq c \longrightarrow P\ x\ (f\ x)$
and *defined*: *fun-lub* (*flat-lub c*) *A x* ≠ *c*
from *defined* **obtain** *f* **where** *f*: *f* ∈ *A f x* ≠ *c*
by(*auto simp add*: *fun-lub-def flat-lub-def split*: *if-split-asm*)
hence *P x* (*f x*) **by**(*rule P*)
moreover from *chain f* **have** $\forall f' \in A.\ f'\ x = c \lor f'\ x = f\ x$
by(*auto 4 4 simp add*: *Complete-Partial-Order.chain-def flat-ord-def fun-ord-def*)
hence *fun-lub* (*flat-lub c*) *A x* = *f x*
using *f* **by**(*auto simp add*: *fun-lub-def flat-lub-def*)
ultimately show *P x* (*fun-lub* (*flat-lub c*) *A x*) **by** *simp*
qed

lemma *fixp-induct-tailrec*:
fixes *F* :: $'c \Rightarrow 'c$ **and**
U :: $'c \Rightarrow 'b \Rightarrow 'a$ **and**
C :: $('b \Rightarrow 'a) \Rightarrow 'c$ **and**
P :: $'b \Rightarrow 'a \Rightarrow$ *bool* **and**
x :: $'b$
assumes *mono*: $\bigwedge x.$ *monotone* (*fun-ord* (*flat-ord c*)) (*flat-ord c*) ($\lambda f.$ *U* (*F* (*C f*)) *x*)
assumes *eq*: *f* ≡ *C* (*ccpo.fixp* (*fun-lub* (*flat-lub c*)) (*fun-ord* (*flat-ord c*)) ($\lambda f.$ *U* (*F* (*C f*))))
assumes *inverse2*: $\bigwedge f.\ U\ (C\ f) = f$
assumes *step*: $\bigwedge f\ x\ y.\ (\bigwedge x\ y.\ U\ f\ x = y \Longrightarrow y \neq c \Longrightarrow P\ x\ y) \Longrightarrow U\ (F\ f)\ x = y \Longrightarrow y \neq c \Longrightarrow P\ x\ y$
assumes *result*: $U\ f\ x = y$
assumes *defined*: $y \neq c$
shows *P x y*
proof −
have $\forall x\ y.\ U\ f\ x = y \longrightarrow y \neq c \longrightarrow P\ x\ y$
by(*rule partial-function-definitions.fixp-induct-uc*[*OF flat-interpretation*, *of* - *U F C*, *OF mono eq inverse2*])
(*auto intro*: *step tailrec-admissible simp add*: *fun-lub-def flat-lub-def*)
thus *?thesis* **using** *result defined* **by** *blast*
qed

lemma *admissible-image*:
assumes *pfun*: *partial-function-definitions le lub*
assumes *adm*: *ccpo.admissible lub le* $(P \circ g)$
assumes *inj*: $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
assumes *inv*: $\bigwedge x.\ f\ (g\ x) = x$
shows *ccpo.admissible* (*img-lub f g lub*) (*img-ord f le*) *P*
proof (*rule ccpo.admissibleI*)
fix *A* **assume** *chain* (*img-ord f le*) *A*
then have *ch′*: *chain le* (*f* ‘ *A*)
by (*auto simp*: *img-ord-def intro*: *chainI dest*: *chainD*)
assume $A \neq \{\}$
assume *P-A*: $\forall x \in A.\ P\ x$
have $(P \circ g)$ (*lub* (*f* ‘ *A*)) **using** *adm ch′*
proof (*rule ccpo.admissibleD*)
fix *x* **assume** $x \in f$ ‘ *A*
with *P-A* **show** $(P \circ g)\ x$ **by** (*auto simp*: *inj*[*OF inv*])
qed(*simp add*: ‹$A \neq \{\}$›)
thus *P* (*img-lub f g lub A*) **unfolding** *img-lub-def* **by** *simp*
qed

lemma *admissible-fun*:
assumes *pfun*: *partial-function-definitions le lub*
assumes *adm*: $\bigwedge x.$ *ccpo.admissible lub le* (*Q x*)
shows *ccpo.admissible* (*fun-lub lub*) (*fun-ord le*) $(\lambda f.\ \forall x.\ Q\ x\ (f\ x))$
proof (*rule ccpo.admissibleI*)
fix *A* :: $('b \Rightarrow 'a)$ *set*
assume *Q*: $\forall f \in A.\ \forall x.\ Q\ x\ (f\ x)$
assume *ch*: *chain* (*fun-ord le*) *A*
assume $A \neq \{\}$
hence *non-empty*: $\bigwedge a.\ \{y.\ \exists f \in A.\ y = f\ a\} \neq \{\}$ **by** *auto*
show $\forall x.\ Q\ x$ (*fun-lub lub A x*)
unfolding *fun-lub-def*
by (*rule allI*, *rule ccpo.admissibleD*[*OF adm chain-fun*[*OF ch*] *non-empty*])
(*auto simp*: *Q*)
qed

abbreviation *option-ord* ≡ *flat-ord None*
abbreviation *mono-option* ≡ *monotone* (*fun-ord option-ord*) *option-ord*

lemma *bind-mono*[*partial-function-mono*]:
assumes *mf*: *mono-option B* **and** *mg*: $\bigwedge y.$ *mono-option* $(\lambda f.\ C\ y\ f)$
shows *mono-option* $(\lambda f.$ *Option.bind* $(B\ f)\ (\lambda y.\ C\ y\ f))$
proof (*rule monotoneI*)
fix *f g* :: $'a \Rightarrow 'b$ *option* **assume** *fg*: *fun-ord option-ord f g*
with *mf*
have *option-ord* $(B\ f)\ (B\ g)$ **by** (*rule monotoneD*[*of - - - f g*])
then have *option-ord* (*Option.bind* $(B\ f)\ (\lambda y.\ C\ y\ f)$) (*Option.bind* $(B\ g)\ (\lambda y.$
$C\ y\ f)$)

unfolding *flat-ord-def* **by** *auto*
also from *mg*
have $\bigwedge y'$. *option-ord* (*C y' f*) (*C y' g*)
by (*rule monotoneD*) (*rule fg*)
then have *option-ord* (*Option.bind* (*B g*) ($\lambda y'$. *C y' f*)) (*Option.bind* (*B g*) ($\lambda y'$. *C y' g*))
unfolding *flat-ord-def* **by** (*cases B g*) *auto*
finally (*option.leq-trans*)
show *option-ord* (*Option.bind* (*B f*) (λy. *C y f*)) (*Option.bind* (*B g*) ($\lambda y'$. *C y' g*)) **.**
qed

lemma *flat-lub-in-chain*:
assumes *ch*: *chain* (*flat-ord b*) *A*
assumes *lub*: *flat-lub b A* = *a*
shows $a = b \vee a \in A$
proof (*cases* $A \subseteq \{b\}$)
case *True*
then have *flat-lub b A* = *b* **unfolding** *flat-lub-def* **by** *simp*
with *lub* **show** *?thesis* **by** *simp*
next
case *False*
then obtain *c* **where** $c \in A$ **and** $c \neq b$ **by** *auto*
{ fix *z* **assume** $z \in A$
from *chainD*[*OF ch* ‹$c \in A$› *this*] **have** $z = c \vee z = b$
unfolding *flat-ord-def* **using** ‹$c \neq b$› **by** *auto* **}**
with *False* **have** $A - \{b\} = \{c\}$ **by** *auto*
with *False* **have** *flat-lub b A* = *c* **by** (*auto simp*: *flat-lub-def*)
with ‹$c \in A$› *lub* **show** *?thesis* **by** *simp*
qed

lemma *option-admissible*: *option.admissible* (%(*f*::'*a* ⇒ '*b option*).
($\forall x\ y.\ f\ x$ = *Some y* ⟶ *P x y*))
proof (*rule ccpo.admissibleI*)
fix *A* :: ('*a* ⇒ '*b option*) *set*
assume *ch*: *chain option.le-fun A*
and *IH*: $\forall f{\in}A.\ \forall x\ y.\ f\ x$ = *Some y* ⟶ *P x y*
from *ch* **have** *ch'*: $\bigwedge x$. *chain option-ord* {*y*. $\exists f{\in}A.\ y = f\ x$} **by** (*rule chain-fun*)
show $\forall x\ y$. *option.lub-fun A x* = *Some y* ⟶ *P x y*
proof (*intro allI impI*)
fix *x y* **assume** *option.lub-fun A x* = *Some y*
from *flat-lub-in-chain*[*OF ch' this*[*unfolded fun-lub-def*]]
have *Some y* ∈ {*y*. $\exists f{\in}A.\ y = f\ x$} **by** *simp*
then have $\exists f{\in}A.\ f\ x$ = *Some y* **by** *auto*
with *IH* **show** *P x y* **by** *auto*
qed
qed

lemma *fixp-induct-option*:

fixes F :: $'c \Rightarrow 'c$ **and**
U :: $'c \Rightarrow 'b \Rightarrow 'a$ *option* **and**
C :: $('b \Rightarrow 'a$ *option*$) \Rightarrow 'c$ **and**
P :: $'b \Rightarrow 'a \Rightarrow$ *bool*
assumes *mono*: $\bigwedge x.$ *mono-option* $(\lambda f.\ U\ (F\ (C\ f))\ x)$
assumes *eq*: $f \equiv C$ (*ccpo.fixp* (*fun-lub* (*flat-lub None*)) (*fun-ord option-ord*) $(\lambda f.\ U\ (F\ (C\ f))))$
assumes *inverse2*: $\bigwedge f.\ U\ (C\ f) = f$
assumes *step*: $\bigwedge f\ x\ y.\ (\bigwedge x\ y.\ U\ f\ x = Some\ y \Longrightarrow P\ x\ y) \Longrightarrow U\ (F\ f)\ x = Some\ y \Longrightarrow P\ x\ y$
assumes *defined*: $U\ f\ x = Some\ y$
shows $P\ x\ y$
using *step defined option.fixp-induct-uc*[*of U F C*, *OF mono eq inverse2 option-admissible*]
unfolding *fun-lub-def flat-lub-def* **by**(*auto 9 2*)

declaration ‹*Partial-Function.init tailrec* ***term*** ‹*tailrec.fixp-fun*›
term ‹*tailrec.mono-body*› @{*thm tailrec.fixp-rule-uc*} @{*thm tailrec.fixp-induct-uc*}
(*SOME* @{*thm fixp-induct-tailrec*[*where c = undefined*]})›

declaration ‹*Partial-Function.init option* ***term*** ‹*option.fixp-fun*›
term ‹*option.mono-body*› @{*thm option.fixp-rule-uc*} @{*thm option.fixp-induct-uc*}
(*SOME* @{*thm fixp-induct-option*})›

hide-const (**open**) *chain*

end

theory *Argo*
imports *HOL*
begin

ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-expr.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-term.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-lit.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-proof.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-rewr.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-cls.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-common.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-cc.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-simplex.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-thy.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-heap.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-cdcl.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-core.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-clausify.ML*›
ML-file ‹~~/*src*/*Tools*/*Argo*/*argo-solver.ML*›

ML-file ‹*Tools/Argo/argo-tactic.ML*›

end

52 Reconstructing external resolution proofs for propositional logic

theory *SAT*
imports *Argo*
begin

ML-file ‹*Tools/prop-logic.ML*›
ML-file ‹*Tools/sat-solver.ML*›
ML-file ‹*Tools/sat.ML*›
ML-file ‹*Tools/Argo/argo-sat-solver.ML*›

method-setup *sat* = ‹*Scan.succeed* (*SIMPLE-METHOD′ o SAT.sat-tac*)›
SAT solver

method-setup *satx* = ‹*Scan.succeed* (*SIMPLE-METHOD′ o SAT.satx-tac*)›
SAT solver (*with definitional CNF*)

end

53 Function Definitions and Termination Proofs

theory *Fun-Def*
imports *Basic-BNF-LFPs Partial-Function SAT*
keywords
function termination :: *thy-goal-defn* **and**
fun fun-cases :: *thy-defn*
begin

53.1 Definitions with default value

definition *THE-default* :: $'a \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a$
where *THE-default d P* = (*if* ($\exists!x.\ P\ x$) *then* (*THE x. P x*) *else d*)

lemma *THE-defaultI′*: $\exists!x.\ P\ x \Longrightarrow P$ (*THE-default d P*)
by (*simp add*: *theI′ THE-default-def*)

lemma *THE-default1-equality*: $\exists!x.\ P\ x \Longrightarrow P\ a \Longrightarrow$ *THE-default d P* = *a*
by (*simp add*: *the1-equality THE-default-def*)

lemma *THE-default-none*: $\neg (\exists!x.\ P\ x) \Longrightarrow$ *THE-default d P* = *d*
by (*simp add*: *THE-default-def*)

lemma *fundef-ex1-existence*:
assumes *f-def*: $f \equiv (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $G\ x\ (f\ x)$
apply (*simp only*: *f-def*)
apply (*rule THE-defaultI'*)
apply (*rule ex1*)
done

lemma *fundef-ex1-uniqueness*:
assumes *f-def*: $f \equiv (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
assumes *elm*: $G\ x\ (h\ x)$
shows $h\ x = f\ x$
by (*auto simp add*: *f-def ex1 elm THE-default1-equality*[*symmetric*])

lemma *fundef-ex1-iff*:
assumes *f-def*: $f \equiv (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $(G\ x\ y) = (f\ x = y)$
by (*auto simp add*: *ex1 f-def THE-default1-equality THE-defaultI'*)

lemma *fundef-default-value*:
assumes *f-def*: $f \equiv (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *graph*: $\bigwedge x\ y.\ G\ x\ y \Longrightarrow D\ x$
assumes $\neg\ D\ x$
shows $f\ x = d\ x$
proof −
have $\neg(\exists\, y.\ G\ x\ y)$
proof
assume $\exists\, y.\ G\ x\ y$
then have $D\ x$ **using** *graph* **..**
with ‹$\neg\ D\ x$› **show** *False* **..**
qed
then have $\neg(\exists!y.\ G\ x\ y)$ **by** *blast*
then show *?thesis*
unfolding *f-def* **by** (*rule THE-default-none*)
qed

definition *in-rel-def*[*simp*]: *in-rel* $R\ x\ y \equiv (x,\ y) \in R$

lemma *wf-in-rel*: *wf* $R \Longrightarrow$ *wfp* (*in-rel* R)
by (*simp add*: *wfp-def*)

ML-file ‹*Tools/Function/function-core.ML*›
ML-file ‹*Tools/Function/mutual.ML*›
ML-file ‹*Tools/Function/pattern-split.ML*›
ML-file ‹*Tools/Function/relation.ML*›

ML-file ‹*Tools/Function/function-elims.ML*›

method-setup *relation* = ‹
Args.term >> (fn t => fn ctxt => SIMPLE-METHOD′ (Function-Relation.relation-infer-tac ctxt t))
› *prove termination using a user−specified wellfounded relation*

ML-file ‹*Tools/Function/function.ML*›
ML-file ‹*Tools/Function/pat-completeness.ML*›

method-setup *pat-completeness* = ‹
Scan.succeed (SIMPLE-METHOD′ o Pat-Completeness.pat-completeness-tac)
› *prove completeness of (co)datatype patterns*

ML-file ‹*Tools/Function/fun.ML*›
ML-file ‹*Tools/Function/induction-schema.ML*›

method-setup *induction-schema* = ‹
Scan.succeed (CONTEXT-TACTIC oo Induction-Schema.induction-schema-tac)
› *prove an induction principle*

53.2 Measure functions

inductive *is-measure* :: ($'a \Rightarrow nat$) $\Rightarrow$ *bool*
where *is-measure-trivial*: *is-measure f*

named-theorems *measure-function rules that guide the heuristic generation of measure functions*
ML-file ‹*Tools/Function/measure-functions.ML*›

lemma *measure-size*[*measure-function*]: *is-measure size*
by (*rule is-measure-trivial*)

lemma *measure-fst*[*measure-function*]: *is-measure f* $\Longrightarrow$ *is-measure* ($\lambda p.$ *f* (*fst p*))
by (*rule is-measure-trivial*)

lemma *measure-snd*[*measure-function*]: *is-measure f* $\Longrightarrow$ *is-measure* ($\lambda p.$ *f* (*snd p*))
by (*rule is-measure-trivial*)

ML-file ‹*Tools/Function/lexicographic-order.ML*›

method-setup *lexicographic-order* = ‹
Method.sections clasimp-modifiers >>
(K (SIMPLE-METHOD o Lexicographic-Order.lexicographic-order-tac false))
› *termination prover for lexicographic orderings*

53.3 Congruence rules

lemma *let-cong* [*fundef-cong*]: $M = N \Longrightarrow (\bigwedge x.\ x = N \Longrightarrow f\ x = g\ x) \Longrightarrow Let\ M\ f = Let\ N\ g$
unfolding *Let-def* **by** *blast*

lemmas [*fundef-cong*] =
if-cong image-cong
bex-cong ball-cong imp-cong map-option-cong Option.bind-cong

lemma *split-cong* [*fundef-cong*]:
$(\bigwedge x\ y.\ (x, y) = q \Longrightarrow f\ x\ y = g\ x\ y) \Longrightarrow p = q \Longrightarrow case\text{-}prod\ f\ p = case\text{-}prod\ g\ q$
by (*auto simp*: *split-def*)

lemma *comp-cong* [*fundef-cong*]: $f\ (g\ x) = f'\ (g'\ x') \Longrightarrow (f \circ g)\ x = (f' \circ g')\ x'$
by (*simp only*: *o-apply*)

53.4 Simp rules for termination proofs

declare
trans-less-add1 [*termination-simp*]
trans-less-add2 [*termination-simp*]
trans-le-add1 [*termination-simp*]
trans-le-add2 [*termination-simp*]
less-imp-le-nat[*termination-simp*]
le-imp-less-Suc[*termination-simp*]

lemma *size-prod-simp*[*termination-simp*]: $size\text{-}prod\ f\ g\ p = f\ (fst\ p) + g\ (snd\ p) + Suc\ 0$
by (*induct p*) *auto*

53.5 Decomposition

lemma *less-by-empty*: $A = \{\} \Longrightarrow A \subseteq B$
and *union-comp-emptyL*: $A\ O\ C = \{\} \Longrightarrow B\ O\ C = \{\} \Longrightarrow (A \cup B)\ O\ C = \{\}$
and *union-comp-emptyR*: $A\ O\ B = \{\} \Longrightarrow A\ O\ C = \{\} \Longrightarrow A\ O\ (B \cup C) = \{\}$
and *wf-no-loop*: $R\ O\ R = \{\} \Longrightarrow wf\ R$
by (*auto simp add*: *wf-comp-self* [*of R*])

53.6 Reduction pairs

definition $reduction\text{-}pair\ P \longleftrightarrow wf\ (fst\ P) \wedge fst\ P\ O\ snd\ P \subseteq fst\ P$

lemma *reduction-pairI*[*intro*]: $wf\ R \Longrightarrow R\ O\ S \subseteq R \Longrightarrow reduction\text{-}pair\ (R, S)$
by (*auto simp*: *reduction-pair-def*)

lemma *reduction-pair-lemma*:
assumes *rp*: *reduction-pair P*
assumes $R \subseteq fst\ P$
assumes $S \subseteq snd\ P$

assumes *wf S*
shows *wf* (*R* ∪ *S*)
proof −
from *rp* ‹*S* ⊆ *snd P*› **have** *wf* (*fst P*) *fst P O S* ⊆ *fst P*
unfolding *reduction-pair-def* **by** *auto*
with ‹*wf S*› **have** *wf* (*fst P* ∪ *S*)
by (*auto intro*: *wf-union-compatible*)
moreover from ‹*R* ⊆ *fst P*› **have** *R* ∪ *S* ⊆ *fst P* ∪ *S* **by** *auto*
ultimately show *?thesis* **by** (*rule wf-subset*)
qed

definition *rp-inv-image* = (λ(*R*,*S*) *f*. (*inv-image R f*, *inv-image S f*))

lemma *rp-inv-image-rp*: *reduction-pair P* ⟹ *reduction-pair* (*rp-inv-image P f*)
unfolding *reduction-pair-def rp-inv-image-def split-def* **by** *force*

53.7 Concrete orders for SCNP termination proofs

definition *pair-less* = *less-than* <∗*lex*∗> *less-than*
definition *pair-leq* = $pair\text{-}less^=$
definition *max-strict* = *max-ext pair-less*
definition *max-weak* = *max-ext pair-leq* ∪ {({}, {})}
definition *min-strict* = *min-ext pair-less*
definition *min-weak* = *min-ext pair-leq* ∪ {({}, {})}

lemma *wf-pair-less*[*simp*]: *wf pair-less*
by (*auto simp*: *pair-less-def*)

lemma *total-pair-less* [*iff*]: *total-on A pair-less* **and** *trans-pair-less* [*iff*]: *trans pair-less*
by (*auto simp*: *total-on-def pair-less-def*)

Introduction rules for *pair-less*/*pair-leq*

lemma *pair-leqI1*: *a* < *b* ⟹ ((*a*, *s*), (*b*, *t*)) ∈ *pair-leq*
and *pair-leqI2*: *a* ≤ *b* ⟹ *s* ≤ *t* ⟹ ((*a*, *s*), (*b*, *t*)) ∈ *pair-leq*
and *pair-lessI1*: *a* < *b* ⟹ ((*a*, *s*), (*b*, *t*)) ∈ *pair-less*
and *pair-lessI2*: *a* ≤ *b* ⟹ *s* < *t* ⟹ ((*a*, *s*), (*b*, *t*)) ∈ *pair-less*
by (*auto simp*: *pair-leq-def pair-less-def*)

lemma *pair-less-iff1* [*simp*]: ((*x*,*y*), (*x*,*z*)) ∈ *pair-less* ⟷ *y*<*z*
by (*simp add*: *pair-less-def*)

Introduction rules for max

lemma *smax-emptyI*: *finite Y* ⟹ *Y* ≠ {} ⟹ ({}, *Y*) ∈ *max-strict*
and *smax-insertI*:
y ∈ *Y* ⟹ (*x*, *y*) ∈ *pair-less* ⟹ (*X*, *Y*) ∈ *max-strict* ⟹ (*insert x X*, *Y*) ∈ *max-strict*
and *wmax-emptyI*: *finite X* ⟹ ({}, *X*) ∈ *max-weak*
and *wmax-insertI*:

$y \in YS \Longrightarrow (x, y) \in$ *pair-leq* $\Longrightarrow (XS, YS) \in$ *max-weak* $\Longrightarrow$ (*insert x XS*, YS) $\in$ *max-weak*
by (*auto simp*: *max-strict-def max-weak-def elim*!: *max-ext.cases*)

Introduction rules for min

lemma *smin-emptyI*: $X \neq \{\} \Longrightarrow (X, \{\}) \in$ *min-strict*
and *smin-insertI*:
$x \in XS \Longrightarrow (x, y) \in$ *pair-less* $\Longrightarrow (XS, YS) \in$ *min-strict* $\Longrightarrow$ (XS, *insert y* YS) $\in$ *min-strict*
and *wmin-emptyI*: $(X, \{\}) \in$ *min-weak*
and *wmin-insertI*:
$x \in XS \Longrightarrow (x, y) \in$ *pair-leq* $\Longrightarrow (XS, YS) \in$ *min-weak* $\Longrightarrow$ (XS, *insert y YS*) $\in$ *min-weak*
by (*auto simp*: *min-strict-def min-weak-def min-ext-def*)

Reduction Pairs.

lemma *max-ext-compat*:
assumes $R\ O\ S \subseteq R$
shows *max-ext R O* (*max-ext S* $\cup \{(\{\}, \{\})\}) \subseteq$ *max-ext R*
proof −
have $\bigwedge X\ Y\ Z.\ (X, Y) \in$ *max-ext R* $\Longrightarrow (Y, Z) \in$ *max-ext S* $\Longrightarrow (X, Z) \in$ *max-ext R*
proof −
fix $X\ Y\ Z$
assume $(X,Y)\in$*max-ext R*
$(Y, Z)\in$*max-ext S*
then have $*$: *finite X finite Y finite Z* $Y\neq\{\}$ $Z\neq\{\}$
$(\bigwedge x.\ x\in X \Longrightarrow \exists y\in Y.\ (x, y)\in R)$
$(\bigwedge y.\ y\in Y \Longrightarrow \exists z\in Z.\ (y, z)\in S)$
by (*auto elim*: *max-ext.cases*)
moreover have $\bigwedge x.\ x\in X \Longrightarrow \exists z\in Z.\ (x, z)\in R$
proof −
fix x
assume $x\in X$
then obtain y **where** *1*: $y\in Y$ $(x, y)\in R$
using $*$ **by** *auto*
then obtain z **where** $z\in Z$ $(y, z)\in S$
using $*$ **by** *auto*
then show $\exists z\in Z.\ (x, z)\in R$
using *assms 1* **by** (*auto elim*: *max-ext.cases*)
qed
ultimately show $(X,Z)\in$*max-ext R*
by *auto*
qed
then show *max-ext R O* (*max-ext S* $\cup \{(\{\}, \{\})\}) \subseteq$ *max-ext R*
by *auto*
qed

lemma *max-rpair-set*: *reduction-pair* (*max-strict*, *max-weak*)

unfolding *max-strict-def max-weak-def*
apply (*intro reduction-pairI max-ext-wf*)
apply *simp*
apply (*rule max-ext-compat*)
apply (*auto simp*: *pair-less-def pair-leq-def*)
done

lemma *min-ext-compat*:
assumes $R\ O\ S \subseteq R$
shows *min-ext* R O (*min-ext* $S \cup \{(\{\},\{\})\}) \subseteq$ *min-ext* R
proof −
have $\bigwedge X\ Y\ Z\ z.\ \forall y \in Y.\ \exists x \in X.\ (x,\ y) \in R \Longrightarrow \forall z \in Z.\ \exists y \in Y.\ (y,\ z) \in S$
$\Longrightarrow z \in Z \Longrightarrow \exists x \in X.\ (x,\ z) \in R$
proof −
fix $X\ Y\ Z\ z$
assume $*$: $\forall y \in Y.\ \exists x \in X.\ (x,\ y) \in R$
$\forall z \in Z.\ \exists y \in Y.\ (y,\ z) \in S$
$z \in Z$
then obtain y' **where** *1*: $y' \in Y\ (y',\ z) \in S$
by *auto*
then obtain x' **where** *2*: $x' \in X\ (x',\ y') \in R$
using $*$ **by** *auto*
show $\exists x \in X.\ (x,\ z) \in R$
using *1 2 assms* **by** *auto*
qed
then show *?thesis*
using *assms* **by** (*auto simp*: *min-ext-def*)
qed

lemma *min-rpair-set*: *reduction-pair* (*min-strict*, *min-weak*)
unfolding *min-strict-def min-weak-def*
apply (*intro reduction-pairI min-ext-wf*)
apply *simp*
apply (*rule min-ext-compat*)
apply (*auto simp*: *pair-less-def pair-leq-def*)
done

53.8 Yet more induction principles on the natural numbers

lemma *nat-descend-induct* [*case-names base descend*]:
fixes $P :: nat \Rightarrow bool$
assumes *H1*: $\bigwedge k.\ k > n \Longrightarrow P\ k$
assumes *H2*: $\bigwedge k.\ k \leq n \Longrightarrow (\bigwedge i.\ i > k \Longrightarrow P\ i) \Longrightarrow P\ k$
shows $P\ m$
using *assms* **by** *induction-schema* (*force intro*!: *wf-measure* [*of* $\lambda k.\ Suc\ n - k$])+

lemma *induct-nat-012*[*case-names 0 1 ge2*]:
$P\ 0 \Longrightarrow P\ (Suc\ 0) \Longrightarrow (\bigwedge n.\ P\ n \Longrightarrow P\ (Suc\ n) \Longrightarrow P\ (Suc\ (Suc\ n))) \Longrightarrow P\ n$
proof (*induction-schema*, *pat-completeness*)

show *wf* (*Wellfounded.measure id*)
by *simp*
qed *auto*

53.9 Tool setup

ML-file ‹*Tools/Function/termination.ML*›
ML-file ‹*Tools/Function/scnp-solve.ML*›
ML-file ‹*Tools/Function/scnp-reconstruct.ML*›
ML-file ‹*Tools/Function/fun-cases.ML*›

ML-val — setup inactive
‹
Context.theory-map (*Function-Common.set-termination-prover*
(*K* (*ScnpReconstruct.decomp-scnp-tac* [*ScnpSolve.MAX*, *ScnpSolve.MIN*, *ScnpSolve.MS*])))
›

end

54 The Integers as Equivalence Classes over Pairs of Natural Numbers

theory *Int*
imports *Quotient Groups-Big Fun-Def*
begin

54.1 Definition of integers as a quotient type

definition *intrel* :: (*nat* × *nat*) ⇒ (*nat* × *nat*) ⇒ *bool*
where *intrel* = (λ(*x*, *y*) (*u*, *v*). *x* + *v* = *u* + *y*)

lemma *intrel-iff* [*simp*]: *intrel* (*x*, *y*) (*u*, *v*) ⟷ *x* + *v* = *u* + *y*
by (*simp add*: *intrel-def*)

quotient-type *int* = *nat* × *nat* / *intrel*
morphisms *Rep-Integ Abs-Integ*
proof (*rule equivpI*)
show *reflp intrel* **by** (*auto simp*: *reflp-def*)
show *symp intrel* **by** (*auto simp*: *symp-def*)
show *transp intrel* **by** (*auto simp*: *transp-def*)
qed

54.2 Integers form a commutative ring

instantiation *int* :: *comm-ring-1*
begin

lift-definition *zero-int* :: *int* **is** (*0*, *0*) **.**

lift-definition *one-int* :: *int* **is** (*1*, *0*) **.**

lift-definition *plus-int* :: *int* ⇒ *int* ⇒ *int*
 is λ(*x*, *y*) (*u*, *v*). (*x* + *u*, *y* + *v*)
 by *clarsimp*

lift-definition *uminus-int* :: *int* ⇒ *int*
 is λ(*x*, *y*). (*y*, *x*)
 by *clarsimp*

lift-definition *minus-int* :: *int* ⇒ *int* ⇒ *int*
 is λ(*x*, *y*) (*u*, *v*). (*x* + *v*, *y* + *u*)
 by *clarsimp*

lift-definition *times-int* :: *int* ⇒ *int* ⇒ *int*
 is λ(*x*, *y*) (*u*, *v*). (*x*∗*u* + *y*∗*v*, *x*∗*v* + *y*∗*u*)
proof (*unfold intrel-def*, *clarify*)
 fix *s t u v w x y z* :: *nat*
 assume *s* + *v* = *u* + *t* **and** *w* + *z* = *y* + *x*
 then have (*s* + *v*) ∗ *w* + (*u* + *t*) ∗ *x* + *u* ∗ (*w* + *z*) + *v* ∗ (*y* + *x*) =
  (*u* + *t*) ∗ *w* + (*s* + *v*) ∗ *x* + *u* ∗ (*y* + *x*) + *v* ∗ (*w* + *z*)
  by *simp*
 then show (*s* ∗ *w* + *t* ∗ *x*) + (*u* ∗ *z* + *v* ∗ *y*) = (*u* ∗ *y* + *v* ∗ *z*) + (*s* ∗ *x* + *t* ∗ *w*)
  by (*simp add*: *algebra-simps*)
qed

instance
 by *standard* (*transfer*; *clarsimp simp*: *algebra-simps*)+

end

abbreviation *int* :: *nat* ⇒ *int*
 where *int* ≡ *of-nat*

lemma *int-def*: *int n* = *Abs-Integ* (*n*, *0*)
 by (*induct n*) (*simp add*: *zero-int.abs-eq*, *simp add*: *one-int.abs-eq plus-int.abs-eq*)

lemma *int-transfer* [*transfer-rule*]:
 includes *lifting-syntax*
 shows *rel-fun* (=) *pcr-int* (λ*n*. (*n*, *0*)) *int*
 by (*simp add*: *rel-fun-def int.pcr-cr-eq cr-int-def int-def*)

lemma *int-diff-cases*: **obtains** (*diff*) *m n* **where** *z* = *int m* − *int n*
 by *transfer clarsimp*

54.3 Integers are totally ordered

instantiation *int* :: *linorder*
begin

lift-definition *less-eq-int* :: *int* ⇒ *int* ⇒ *bool*
  is $\lambda(x, y)\ (u, v).\ x + v \leq u + y$
  by *auto*

lift-definition *less-int* :: *int* ⇒ *int* ⇒ *bool*
  is $\lambda(x, y)\ (u, v).\ x + v < u + y$
  by *auto*

instance
  by *standard* (*transfer*, *force*)+

end

instantiation *int* :: *distrib-lattice*
begin

definition (*inf* :: *int* ⇒ *int* ⇒ *int*) = *min*

definition (*sup* :: *int* ⇒ *int* ⇒ *int*) = *max*

instance
  by *standard* (*auto simp add*: *inf-int-def sup-int-def max-min-distrib2*)

end

54.4 Ordering properties of arithmetic operations

instance *int* :: *ordered-cancel-ab-semigroup-add*
proof
  fix *i j k* :: *int*
  show $i \leq j \Longrightarrow k + i \leq k + j$
    by *transfer clarsimp*
qed

Strict Monotonicity of Multiplication.

Strict, in 1st argument; proof is by induction on $k > 0$.

lemma *zmult-zless-mono2-lemma*: $i < j \Longrightarrow 0 < k \Longrightarrow int\ k * i < int\ k * j$
  for *i j* :: *int*
proof (*induct k*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc k*)
  then show *?case*

```
  by (cases k = 0) (simp-all add: distrib-right add-strict-mono)
qed

lemma zero-le-imp-eq-int:
 assumes k ≥ (0::int) shows ∃ n. k = int n
proof −
 have b ≤ a ⟹ ∃ n::nat. a = n + b for a b
  using exI[of - a − b] by simp
 with assms show ?thesis
  by transfer auto
qed

lemma zero-less-imp-eq-int:
 assumes k > (0::int) shows ∃ n>0. k = int n
proof −
 have b < a ⟹ ∃ n::nat. n>0 ∧ a = n + b for a b
  using exI[of - a − b] by simp
 with assms show ?thesis
  by transfer auto
qed

lemma zmult-zless-mono2: i < j ⟹ 0 < k ⟹ k ∗ i < k ∗ j
 for i j k :: int
 by (drule zero-less-imp-eq-int) (auto simp add: zmult-zless-mono2-lemma)
```

The integers form an ordered integral domain.

```
instantiation int :: linordered-idom
begin

definition zabs-def: |i::int| = (if i < 0 then − i else i)

definition zsgn-def: sgn (i::int) = (if i = 0 then 0 else if 0 < i then 1 else − 1)

instance
proof
 fix i j k :: int
 show i < j ⟹ 0 < k ⟹ k ∗ i < k ∗ j
  by (rule zmult-zless-mono2)
 show |i| = (if i < 0 then −i else i)
  by (simp only: zabs-def)
 show sgn (i::int) = (if i=0 then 0 else if 0<i then 1 else − 1)
  by (simp only: zsgn-def)
qed

end

instance int :: discrete-linordered-semidom
proof
 fix k l :: int
```

show ‹$k < l \longleftrightarrow k + 1 \leq l$› (**is** ‹$?P \longleftrightarrow ?Q$›)
proof
assume *?Q*
then show *?P*
by *simp*
next
assume *?P*
then have ‹$l - k > 0$›
by *simp*
with *zero-less-imp-eq-int* **obtain** *n* **where** ‹$l - k = int\ n$›
by *blast*
then have ‹$n > 0$›
using ‹$l - k > 0$› **by** *simp*
then have ‹$n \geq 1$›
by *simp*
then have ‹$int\ n \geq int\ 1$›
by (*rule of-nat-mono*)
with ‹$l - k = int\ n$› **show** *?Q*
by (*simp add*: *algebra-simps*)
qed
qed

lemma *zless-imp-add1-zle*: $w < z \Longrightarrow w + 1 \leq z$
for *w z* :: *int*
by *transfer clarsimp*

lemma *zless-iff-Suc-zadd*: $w < z \longleftrightarrow (\exists n.\ z = w + int\ (Suc\ n))$
for *w z* :: *int*
proof −
have $\bigwedge a\ b\ c\ d.\ a + d < c + b \Longrightarrow \exists n.\ c + b = Suc\ (a + n + d)$
proof −
fix *a b c d* :: *nat*
assume $a + d < c + b$
then have $c + b = Suc\ (a + (c + b - Suc\ (a + d)) + d)$
by *arith*
then show $\exists n.\ c + b = Suc\ (a + n + d)$
by (*rule exI*)
qed
then show *?thesis*
by *transfer auto*
qed

lemma *zabs-less-one-iff* [*simp*]: $|z| < 1 \longleftrightarrow z = 0$ (**is** $?lhs \longleftrightarrow ?rhs$)
for *z* :: *int*
proof
assume *?rhs*
then show *?lhs* **by** *simp*
next
assume *?lhs*

with *zless-imp-add1-zle* [*of* |*z*| *1*] **have** |*z*| + *1* ≤ *1* **by** *simp*
then have |*z*| ≤ *0* **by** *simp*
then show *?rhs* **by** *simp*
qed

54.5 Embedding of the Integers into any *ring-1*: *of-int*

context *ring-1*
begin

lift-definition *of-int* :: *int* ⇒ *'a*
is λ(*i*, *j*). *of-nat i* − *of-nat j*
by (*clarsimp simp add*: *diff-eq-eq eq-diff-eq diff-add-eq*
of-nat-add [*symmetric*] *simp del*: *of-nat-add*)

lemma *of-int-0* [*simp*]: *of-int 0* = *0*
by *transfer simp*

lemma *of-int-1* [*simp*]: *of-int 1* = *1*
by *transfer simp*

lemma *of-int-add* [*simp*]: *of-int* (*w* + *z*) = *of-int w* + *of-int z*
by *transfer* (*clarsimp simp add*: *algebra-simps*)

lemma *of-int-minus* [*simp*]: *of-int* (− *z*) = − (*of-int z*)
by (*transfer fixing*: *uminus*) *clarsimp*

lemma *of-int-diff* [*simp*]: *of-int* (*w* − *z*) = *of-int w* − *of-int z*
using *of-int-add* [*of w* − *z*] **by** *simp*

lemma *of-int-mult* [*simp*]: *of-int* (*w*∗*z*) = *of-int w* ∗ *of-int z*
by (*transfer fixing*: *times*) (*clarsimp simp add*: *algebra-simps*)

lemma *mult-of-int-commute*: *of-int x* ∗ *y* = *y* ∗ *of-int x*
by (*transfer fixing*: *times*) (*auto simp*: *algebra-simps mult-of-nat-commute*)

Collapse nested embeddings.

lemma *of-int-of-nat-eq* [*simp*]: *of-int* (*int n*) = *of-nat n*
by (*induct n*) *auto*

lemma *of-int-numeral* [*simp*, *code-post*]: *of-int* (*numeral k*) = *numeral k*
by (*simp add*: *of-nat-numeral* [*symmetric*] *of-int-of-nat-eq* [*symmetric*])

lemma *of-int-neg-numeral* [*code-post*]: *of-int* (− *numeral k*) = − *numeral k*
by *simp*

lemma *of-int-power* [*simp*]: *of-int* (*z* ^ *n*) = *of-int z* ^ *n*
by (*induct n*) *simp-all*

lemma *of-int-of-bool* [*simp*]:
of-int (*of-bool P*) = *of-bool P*
by *auto*

end

context *ring-char-0*
begin

lemma *of-int-eq-iff* [*simp*]: *of-int w* = *of-int z* ⟷ *w* = *z*
by *transfer* (*clarsimp simp add*: *algebra-simps of-nat-add* [*symmetric*] *simp del*: *of-nat-add*)

Special cases where either operand is zero.

lemma *of-int-eq-0-iff* [*simp*]: *of-int z* = *0* ⟷ *z* = *0*
using *of-int-eq-iff* [*of z 0*] **by** *simp*

lemma *of-int-0-eq-iff* [*simp*]: *0* = *of-int z* ⟷ *z* = *0*
using *of-int-eq-iff* [*of 0 z*] **by** *simp*

lemma *of-int-eq-1-iff* [*iff*]: *of-int z* = *1* ⟷ *z* = *1*
using *of-int-eq-iff* [*of z 1*] **by** *simp*

lemma *numeral-power-eq-of-int-cancel-iff* [*simp*]:
numeral x ^ *n* = *of-int y* ⟷ *numeral x* ^ *n* = *y*
using *of-int-eq-iff*[*of numeral x* ^ *n y*, *unfolded of-int-numeral of-int-power*] **.**

lemma *of-int-eq-numeral-power-cancel-iff* [*simp*]:
of-int y = *numeral x* ^ *n* ⟷ *y* = *numeral x* ^ *n*
using *numeral-power-eq-of-int-cancel-iff* [*of x n y*] **by** (*metis* (*mono-tags*))

lemma *neg-numeral-power-eq-of-int-cancel-iff* [*simp*]:
(− *numeral x*) ^ *n* = *of-int y* ⟷ (− *numeral x*) ^ *n* = *y*
using *of-int-eq-iff*[*of* (− *numeral x*) ^ *n y*]
by *simp*

lemma *of-int-eq-neg-numeral-power-cancel-iff* [*simp*]:
of-int y = (− *numeral x*) ^ *n* ⟷ *y* = (− *numeral x*) ^ *n*
using *neg-numeral-power-eq-of-int-cancel-iff*[*of x n y*] **by** (*metis* (*mono-tags*))

lemma *of-int-eq-of-int-power-cancel-iff*[*simp*]: (*of-int b*) ^ *w* = *of-int x* ⟷ *b* ^ *w* = *x*
by (*metis of-int-power of-int-eq-iff*)

lemma *of-int-power-eq-of-int-cancel-iff*[*simp*]: *of-int x* = (*of-int b*) ^ *w* ⟷ *x* = *b* ^ *w*
by (*metis of-int-eq-of-int-power-cancel-iff*)

end

context *linordered-idom*
begin

Every *linordered-idom* has characteristic zero.

subclass *ring-char-0* **..**

lemma *of-int-le-iff* [*simp*]: *of-int* $w \leq$ *of-int* $z \longleftrightarrow w \leq z$
by (*transfer fixing*: *less-eq*)
(*clarsimp simp add*: *algebra-simps of-nat-add* [*symmetric*] *simp del*: *of-nat-add*)

lemma *of-int-less-iff* [*simp*]: *of-int* $w <$ *of-int* $z \longleftrightarrow w < z$
by (*simp add*: *less-le order-less-le*)

lemma *of-int-0-le-iff* [*simp*]: $0 \leq$ *of-int* $z \longleftrightarrow 0 \leq z$
using *of-int-le-iff* [*of 0 z*] **by** *simp*

lemma *of-int-le-0-iff* [*simp*]: *of-int* $z \leq 0 \longleftrightarrow z \leq 0$
using *of-int-le-iff* [*of z 0*] **by** *simp*

lemma *of-int-0-less-iff* [*simp*]: $0 <$ *of-int* $z \longleftrightarrow 0 < z$
using *of-int-less-iff* [*of 0 z*] **by** *simp*

lemma *of-int-less-0-iff* [*simp*]: *of-int* $z < 0 \longleftrightarrow z < 0$
using *of-int-less-iff* [*of z 0*] **by** *simp*

lemma *of-int-1-le-iff* [*simp*]: $1 \leq$ *of-int* $z \longleftrightarrow 1 \leq z$
using *of-int-le-iff* [*of 1 z*] **by** *simp*

lemma *of-int-le-1-iff* [*simp*]: *of-int* $z \leq 1 \longleftrightarrow z \leq 1$
using *of-int-le-iff* [*of z 1*] **by** *simp*

lemma *of-int-1-less-iff* [*simp*]: $1 <$ *of-int* $z \longleftrightarrow 1 < z$
using *of-int-less-iff* [*of 1 z*] **by** *simp*

lemma *of-int-less-1-iff* [*simp*]: *of-int* $z < 1 \longleftrightarrow z < 1$
using *of-int-less-iff* [*of z 1*] **by** *simp*

lemma *of-int-pos*: $z > 0 \Longrightarrow$ *of-int* $z > 0$
by *simp*

lemma *of-int-nonneg*: $z \geq 0 \Longrightarrow$ *of-int* $z \geq 0$
by *simp*

lemma *of-int-abs* [*simp*]: *of-int* $|x| = |$*of-int* $x|$
by (*auto simp add*: *abs-if*)

lemma *of-int-lessD*:
assumes $|$*of-int* $n| < x$

shows $n = 0 \lor x > 1$
proof (*cases* $n = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have $|n| \neq 0$ **by** *simp*
then have $|n| > 0$ **by** *simp*
then have $|n| \geq 1$
using *zless-imp-add1-zle* [*of 0* $|n|$] **by** *simp*
then have $|of\text{-}int\ n| \geq 1$
unfolding *of-int-1-le-iff* [*of* $|n|$, *symmetric*] **by** *simp*
then have $1 < x$ **using** *assms* **by** (*rule le-less-trans*)
then show *?thesis* **..**
qed

lemma *of-int-leD*:
assumes $|of\text{-}int\ n| \leq x$
shows $n = 0 \lor 1 \leq x$
proof (*cases* $n = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have $|n| \neq 0$ **by** *simp*
then have $|n| > 0$ **by** *simp*
then have $|n| \geq 1$
using *zless-imp-add1-zle* [*of 0* $|n|$] **by** *simp*
then have $|of\text{-}int\ n| \geq 1$
unfolding *of-int-1-le-iff* [*of* $|n|$, *symmetric*] **by** *simp*
then have $1 \leq x$ **using** *assms* **by** (*rule order-trans*)
then show *?thesis* **..**
qed

lemma *numeral-power-le-of-int-cancel-iff* [*simp*]:
numeral x ^ $n \leq$ *of-int* $a \longleftrightarrow$ *numeral* x ^ $n \leq a$
by (*metis* (*mono-tags*) *local.of-int-eq-numeral-power-cancel-iff of-int-le-iff*)

lemma *of-int-le-numeral-power-cancel-iff* [*simp*]:
of-int $a \leq$ *numeral* x ^ $n \longleftrightarrow a \leq$ *numeral* x ^ n
by (*metis* (*mono-tags*) *local.numeral-power-eq-of-int-cancel-iff of-int-le-iff*)

lemma *numeral-power-less-of-int-cancel-iff* [*simp*]:
numeral x ^ $n <$ *of-int* $a \longleftrightarrow$ *numeral* x ^ $n < a$
by (*metis* (*mono-tags*) *local.of-int-eq-numeral-power-cancel-iff of-int-less-iff*)

lemma *of-int-less-numeral-power-cancel-iff* [*simp*]:
of-int $a <$ *numeral* x ^ $n \longleftrightarrow a <$ *numeral* x ^ n
by (*metis* (*mono-tags*) *local.of-int-eq-numeral-power-cancel-iff of-int-less-iff*)

lemma *neg-numeral-power-le-of-int-cancel-iff* [*simp*]:
(− *numeral x*) ^ *n* ≤ *of-int a* ⟷ (− *numeral x*) ^ *n* ≤ *a*
by (*metis* (*mono-tags*) *of-int-le-iff of-int-neg-numeral of-int-power*)

lemma *of-int-le-neg-numeral-power-cancel-iff* [*simp*]:
of-int a ≤ (− *numeral x*) ^ *n* ⟷ *a* ≤ (− *numeral x*) ^ *n*
by (*metis* (*mono-tags*) *of-int-le-iff of-int-neg-numeral of-int-power*)

lemma *neg-numeral-power-less-of-int-cancel-iff* [*simp*]:
(− *numeral x*) ^ *n* < *of-int a* ⟷ (− *numeral x*) ^ *n* < *a*
using *of-int-less-iff*[*of* (− *numeral x*) ^ *n a*]
by *simp*

lemma *of-int-less-neg-numeral-power-cancel-iff* [*simp*]:
of-int a < (− *numeral x*) ^ *n* ⟷ *a* < (− *numeral x*::*int*) ^ *n*
using *of-int-less-iff*[*of a* (− *numeral x*) ^ *n*]
by *simp*

lemma *of-int-le-of-int-power-cancel-iff*[*simp*]: (*of-int b*) ^ *w* ≤ *of-int x* ⟷ *b* ^ *w* ≤ *x*
by (*metis* (*mono-tags*) *of-int-le-iff of-int-power*)

lemma *of-int-power-le-of-int-cancel-iff*[*simp*]: *of-int x* ≤ (*of-int b*) ^ *w*⟷ *x* ≤ *b* ^ *w*
by (*metis* (*mono-tags*) *of-int-le-iff of-int-power*)

lemma *of-int-less-of-int-power-cancel-iff*[*simp*]: (*of-int b*) ^ *w* < *of-int x* ⟷ *b* ^ *w* < *x*
by (*metis* (*mono-tags*) *of-int-less-iff of-int-power*)

lemma *of-int-power-less-of-int-cancel-iff*[*simp*]: *of-int x* < (*of-int b*) ^ *w*⟷ *x* < *b* ^ *w*
by (*metis* (*mono-tags*) *of-int-less-iff of-int-power*)

lemma *of-int-max*: *of-int* (*max x y*) = *max* (*of-int x*) (*of-int y*)
by (*auto simp*: *max-def*)

lemma *of-int-min*: *of-int* (*min x y*) = *min* (*of-int x*) (*of-int y*)
by (*auto simp*: *min-def*)

end

context *division-ring*
begin

lemmas *mult-inverse-of-int-commute* =
mult-commute-imp-mult-inverse-commute[*OF mult-of-int-commute*]

end

Comparisons involving *of-int*.

lemma *of-int-eq-numeral-iff* [*iff*]: *of-int z* = (*numeral n* :: *'a*::*ring-char-0*) ⟷ *z* = *numeral n*
  using *of-int-eq-iff* **by** *fastforce*

lemma *of-int-le-numeral-iff* [*simp*]:
  of-int z ≤ (*numeral n* :: *'a*::*linordered-idom*) ⟷ *z* ≤ *numeral n*
  using *of-int-le-iff* [*of z numeral n*] **by** *simp*

lemma *of-int-numeral-le-iff* [*simp*]:
  (*numeral n* :: *'a*::*linordered-idom*) ≤ *of-int z* ⟷ *numeral n* ≤ *z*
  using *of-int-le-iff* [*of numeral n*] **by** *simp*

lemma *of-int-less-numeral-iff* [*simp*]:
  of-int z < (*numeral n* :: *'a*::*linordered-idom*) ⟷ *z* < *numeral n*
  using *of-int-less-iff* [*of z numeral n*] **by** *simp*

lemma *of-int-numeral-less-iff* [*simp*]:
  (*numeral n* :: *'a*::*linordered-idom*) < *of-int z* ⟷ *numeral n* < *z*
  using *of-int-less-iff* [*of numeral n z*] **by** *simp*

lemma *of-nat-less-of-int-iff*: (*of-nat n*::*'a*::*linordered-idom*) < *of-int x* ⟷ *int n* < *x*
  by (*metis of-int-of-nat-eq of-int-less-iff*)

lemma *of-int-eq-id* [*simp*]: *of-int* = *id*
proof
  show *of-int z* = *id z* **for** *z*
    by (*cases z rule*: *int-diff-cases*) *simp*
qed

instance *int* :: *no-top*
proof
  fix *x*::*int*
  have *x* < *x* + *1*
    by *simp*
  then show ∃ *y*. *x* < *y*
    by (*rule exI*)
qed

instance *int* :: *no-bot*
proof
  fix *x*::*int*
  have *x* − *1*< *x*
    by *simp*
  then show ∃ *y*. *y* < *x*
    by (*rule exI*)

qed

54.6 Magnitude of an Integer, as a Natural Number: *nat*

lift-definition *nat* :: *int* ⇒ *nat* **is** $\lambda(x, y).\ x - y$
  by *auto*

lemma *nat-int* [*simp*]: *nat* (*int* *n*) = *n*
  by *transfer simp*

lemma *int-nat-eq* [*simp*]: *int* (*nat* *z*) = (*if* *0* ≤ *z* *then* *z* *else* *0*)
  by *transfer clarsimp*

lemma *nat-0-le*: *0* ≤ *z* ⟹ *int* (*nat* *z*) = *z*
  by *simp*

lemma *nat-le-0* [*simp*]: *z* ≤ *0* ⟹ *nat* *z* = *0*
  by *transfer clarsimp*

lemma *nat-le-eq-zle*: *0* < *w* ∨ *0* ≤ *z* ⟹ *nat* *w* ≤ *nat* *z* ⟷ *w* ≤ *z*
  by *transfer* (*clarsimp*, *arith*)

An alternative condition is *0* ≤ *w*.

lemma *nat-mono-iff*: *0* < *z* ⟹ *nat* *w* < *nat* *z* ⟷ *w* < *z*
  by (*simp add*: *nat-le-eq-zle linorder-not-le* [*symmetric*])

lemma *nat-less-eq-zless*: *0* ≤ *w* ⟹ *nat* *w* < *nat* *z* ⟷ *w* < *z*
  by (*simp add*: *nat-le-eq-zle linorder-not-le* [*symmetric*])

lemma *zless-nat-conj* [*simp*]: *nat* *w* < *nat* *z* ⟷ *0* < *z* ∧ *w* < *z*
  by *transfer* (*clarsimp*, *arith*)

lemma *nonneg-int-cases*:
  assumes *0* ≤ *k*
  obtains *n* **where** *k* = *int* *n*
proof −
  from *assms* **have** *k* = *int* (*nat* *k*)
    by *simp*
  then show *thesis*
    by (*rule that*)
qed

lemma *pos-int-cases*:
  assumes *0* < *k*
  obtains *n* **where** *k* = *int* *n* **and** *n* > *0*
proof −
  from *assms* **have** *0* ≤ *k*
    by *simp*
  then obtain *n* **where** *k* = *int* *n*

by (*rule nonneg-int-cases*)
moreover have $n > 0$
using ‹$k = int\ n$› *assms* **by** *simp*
ultimately show *thesis*
by (*rule that*)
qed

lemma *nonpos-int-cases*:
assumes $k \leq 0$
obtains n **where** $k = -\ int\ n$
proof −
from *assms* **have** $-\ k \geq 0$
by *simp*
then obtain n **where** $-\ k = int\ n$
by (*rule nonneg-int-cases*)
then have $k = -\ int\ n$
by *simp*
then show *thesis*
by (*rule that*)
qed

lemma *neg-int-cases*:
assumes $k < 0$
obtains n **where** $k = -\ int\ n$ **and** $n > 0$
proof −
from *assms* **have** $-\ k > 0$
by *simp*
then obtain n **where** $-\ k = int\ n$ **and** $-\ k > 0$
by (*blast elim*: *pos-int-cases*)
then have $k = -\ int\ n$ **and** $n > 0$
by *simp-all*
then show *thesis*
by (*rule that*)
qed

lemma *nat-eq-iff*: $nat\ w = m \longleftrightarrow (if\ 0 \leq w\ then\ w = int\ m\ else\ m = 0)$
by *transfer* (*clarsimp simp add*: *le-imp-diff-is-add*)

lemma *nat-eq-iff2*: $m = nat\ w \longleftrightarrow (if\ 0 \leq w\ then\ w = int\ m\ else\ m = 0)$
using *nat-eq-iff* [*of w m*] **by** *auto*

lemma *nat-0* [*simp*]: $nat\ 0 = 0$
by (*simp add*: *nat-eq-iff*)

lemma *nat-1* [*simp*]: $nat\ 1 = Suc\ 0$
by (*simp add*: *nat-eq-iff*)

lemma *nat-numeral* [*simp*]: $nat\ (numeral\ k) = numeral\ k$
by (*simp add*: *nat-eq-iff*)

lemma *nat-neg-numeral* [*simp*]: *nat* (− *numeral k*) = *0*
by *simp*

lemma *nat-2*: *nat 2* = *Suc* (*Suc 0*)
by *simp*

lemma *nat-less-iff*: *0* ≤ *w* ⟹ *nat w* < *m* ⟷ *w* < *of-nat m*
by *transfer* (*clarsimp*, *arith*)

lemma *nat-le-iff*: *nat x* ≤ *n* ⟷ *x* ≤ *int n*
by *transfer* (*clarsimp simp add*: *le-diff-conv*)

lemma *nat-mono*: *x* ≤ *y* ⟹ *nat x* ≤ *nat y*
by *transfer auto*

lemma *nat-0-iff*[*simp*]: *nat i* = *0* ⟷ *i* ≤ *0*
for *i* :: *int*
by *transfer clarsimp*

lemma *int-eq-iff*: *of-nat m* = *z* ⟷ *m* = *nat z* ∧ *0* ≤ *z*
by (*auto simp add*: *nat-eq-iff2*)

lemma *zero-less-nat-eq* [*simp*]: *0* < *nat z* ⟷ *0* < *z*
using *zless-nat-conj* [*of 0*] **by** *auto*

lemma *nat-add-distrib*: *0* ≤ *z* ⟹ *0* ≤ *z′* ⟹ *nat* (*z* + *z′*) = *nat z* + *nat z′*
by *transfer clarsimp*

lemma *nat-diff-distrib′*: *0* ≤ *x* ⟹ *0* ≤ *y* ⟹ *nat* (*x* − *y*) = *nat x* − *nat y*
by *transfer clarsimp*

lemma *nat-diff-distrib*: *0* ≤ *z′* ⟹ *z′* ≤ *z* ⟹ *nat* (*z* − *z′*) = *nat z* − *nat z′*
by (*rule nat-diff-distrib′*) *auto*

lemma *nat-zminus-int* [*simp*]: *nat* (− *int n*) = *0*
by *transfer simp*

lemma *le-nat-iff*: *k* ≥ *0* ⟹ *n* ≤ *nat k* ⟷ *int n* ≤ *k*
by *transfer auto*

lemma *zless-nat-eq-int-zless*: *m* < *nat z* ⟷ *int m* < *z*
by *transfer* (*clarsimp simp add*: *less-diff-conv*)

lemma (**in** *ring-1*) *of-nat-nat* [*simp*]: *0* ≤ *z* ⟹ *of-nat* (*nat z*) = *of-int z*
by *transfer* (*clarsimp simp add*: *of-nat-diff*)

lemma *diff-nat-numeral* [*simp*]: (*numeral v* :: *nat*) − *numeral v′* = *nat* (*numeral v* − *numeral v′*)

by (*simp only*: *nat-diff-distrib′ zero-le-numeral nat-numeral*)

lemma *nat-abs-triangle-ineq*:
nat |*k* + *l*| ≤ *nat* |*k*| + *nat* |*l*|
by (*simp add*: *nat-add-distrib* [*symmetric*] *nat-le-eq-zle abs-triangle-ineq*)

lemma *nat-of-bool* [*simp*]:
nat (*of-bool P*) = *of-bool P*
by *auto*

lemma *split-nat* [*linarith-split*]: *P* (*nat i*) ⟷ ((∀ *n*. *i* = *int n* ⟶ *P n*) ∧ (*i* < *0* ⟶ *P 0*))
(**is** *?P* = (*?L* ∧ *?R*))
for *i* :: *int*
proof (*cases i* < *0*)
case *True*
then show *?thesis*
by *auto*
next
case *False*
have *?P* = *?L*
proof
assume *?P*
then show *?L* **using** *False* **by** *auto*
next
assume *?L*
moreover from *False* **have** *int* (*nat i*) = *i*
by (*simp add*: *not-less*)
ultimately show *?P*
by *simp*
qed
with *False* **show** *?thesis* **by** *simp*
qed

lemma *all-nat*: (∀ *x*. *P x*) ⟷ (∀ *x*≥*0*. *P* (*nat x*))
by (*auto split*: *split-nat*)

lemma *ex-nat*: (∃ *x*. *P x*) ⟷ (∃ *x*. *0* ≤ *x* ∧ *P* (*nat x*))
proof
assume ∃ *x*. *P x*
then obtain *x* **where** *P x* **..**
then have *int x* ≥ *0* ∧ *P* (*nat* (*int x*)) **by** *simp*
then show ∃ *x*≥*0*. *P* (*nat x*) **..**
next
assume ∃ *x*≥*0*. *P* (*nat x*)
then show ∃ *x*. *P x* **by** *auto*
qed

For termination proofs:

lemma *measure-function-int*[*measure-function*]: *is-measure* (*nat* ∘ *abs*) **..**

54.7 Lemmas about the Function *of-nat* and Orderings

lemma *negative-zless-0*: $-$ (*int* (*Suc n*)) $<$ (*0* :: *int*)
  by (*simp add*: *order-less-le del*: *of-nat-Suc*)

lemma *negative-zless* [*iff*]: $-$ (*int* (*Suc n*)) $<$ *int m*
  by (*rule negative-zless-0* [*THEN order-less-le-trans*], *simp*)

lemma *negative-zle-0*: $-$ *int n* $\leq$ *0*
  by (*simp add*: *minus-le-iff*)

lemma *negative-zle* [*iff*]: $-$ *int n* $\leq$ *int m*
  by (*rule order-trans* [*OF negative-zle-0 of-nat-0-le-iff*])

lemma *not-zle-0-negative* [*simp*]: ¬ *0* $\leq$ $-$ *int* (*Suc n*)
  by (*subst le-minus-iff*) (*simp del*: *of-nat-Suc*)

lemma *int-zle-neg*: *int n* $\leq$ $-$ *int m* $\longleftrightarrow$ *n* = *0* ∧ *m* = *0*
  by *transfer simp*

lemma *not-int-zless-negative* [*simp*]: ¬ *int n* $<$ $-$ *int m*
  by (*simp add*: *linorder-not-less*)

lemma *negative-eq-positive* [*simp*]: $-$ *int n* = *of-nat m* $\longleftrightarrow$ *n* = *0* ∧ *m* = *0*
  by (*force simp add*: *order-eq-iff* [*of* $-$ *of-nat n*] *int-zle-neg*)

lemma *zle-iff-zadd*: *w* $\leq$ *z* $\longleftrightarrow$ (∃ *n*. *z* = *w* + *int n*)
  (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
  assume *?rhs*
  then show *?lhs* **by** *auto*
next
  assume *?lhs*
  then have *0* $\leq$ *z* $-$ *w* **by** *simp*
  then obtain *n* **where** *z* $-$ *w* = *int n*
    using *zero-le-imp-eq-int* [*of z* $-$ *w*] **by** *blast*
  then have *z* = *w* + *int n* **by** *simp*
  then show *?rhs* **..**
qed

lemma *zadd-int-left*: *int m* + (*int n* + *z*) = *int* (*m* + *n*) + *z*
  by *simp*

lemma *negD*:
  assumes *x* $<$ *0* **shows** ∃ *n*. *x* = $-$ (*int* (*Suc n*))
proof $-$
  have $\bigwedge$*a b*. *a* $<$ *b* $\Longrightarrow$ ∃ *n*. *Suc* (*a* + *n*) = *b*

proof −
fix *a b*:: *nat*
assume $a < b$
then have $Suc\ (a + (b - Suc\ a)) = b$
by *arith*
then show $\exists n.\ Suc\ (a + n) = b$
by (*rule exI*)
qed
with *assms* **show** *?thesis*
by *transfer auto*
qed

54.8 Cases and induction

Now we replace the case analysis rule by a more conventional one: whether an integer is negative or not.

This version is symmetric in the two subgoals.

lemma *int-cases2* [*case-names nonneg nonpos*, *cases type*: *int*]:
$(\bigwedge n.\ z = int\ n \Longrightarrow P) \Longrightarrow (\bigwedge n.\ z = -\ (int\ n) \Longrightarrow P) \Longrightarrow P$
by (*cases* $z < 0$) (*auto simp add*: *linorder-not-less dest*!: *negD nat-0-le* [*THEN sym*])

This is the default, with a negative case.

lemma *int-cases* [*case-names nonneg neg*, *cases type*: *int*]:
assumes *pos*: $\bigwedge n.\ z = int\ n \Longrightarrow P$ **and** *neg*: $\bigwedge n.\ z = -\ (int\ (Suc\ n)) \Longrightarrow P$
shows *P*
proof (*cases* $z < 0$)
case *True*
with *neg* **show** *?thesis*
by (*blast dest*!: *negD*)
next
case *False*
with *pos* **show** *?thesis*
by (*force simp add*: *linorder-not-less dest*: *nat-0-le* [*THEN sym*])
qed

lemma *int-cases3* [*case-names zero pos neg*]:
fixes *k* :: *int*
assumes $k = 0 \Longrightarrow P$ **and** $\bigwedge n.\ k = int\ n \Longrightarrow n > 0 \Longrightarrow P$
and $\bigwedge n.\ k = -\ int\ n \Longrightarrow n > 0 \Longrightarrow P$
shows *P*
proof (*cases k 0*::*int rule*: *linorder-cases*)
case *equal*
with *assms*(*1*) **show** *P* **by** *simp*
next
case *greater*
then have ∗: $nat\ k > 0$ **by** *simp*
moreover from ∗ **have** $k = int\ (nat\ k)$ **by** *auto*

ultimately show *P* **using** *assms*(*2*) **by** *blast*
next
case *less*
then have ∗: *nat* (− *k*) > *0* **by** *simp*
moreover from ∗ **have** *k* = − *int* (*nat* (− *k*)) **by** *auto*
ultimately show *P* **using** *assms*(*3*) **by** *blast*
qed

lemma *int-of-nat-induct* [*case-names nonneg neg*, *induct type*: *int*]:
$(\bigwedge n.\ P\ (int\ n)) \Longrightarrow (\bigwedge n.\ P\ (-\ (int\ (Suc\ n)))) \Longrightarrow P\ z$
by (*cases z*) *auto*

lemma *sgn-mult-dvd-iff* [*simp*]:
$sgn\ r * l\ dvd\ k \longleftrightarrow l\ dvd\ k \wedge (r = 0 \longrightarrow k = 0)$ **for** *k l r* :: *int*
by (*cases r rule*: *int-cases3*) *auto*

lemma *mult-sgn-dvd-iff* [*simp*]:
$l * sgn\ r\ dvd\ k \longleftrightarrow l\ dvd\ k \wedge (r = 0 \longrightarrow k = 0)$ **for** *k l r* :: *int*
using *sgn-mult-dvd-iff* [*of r l k*] **by** (*simp add*: *ac-simps*)

lemma *dvd-sgn-mult-iff* [*simp*]:
$l\ dvd\ sgn\ r * k \longleftrightarrow l\ dvd\ k \vee r = 0$ **for** *k l r* :: *int*
by (*cases r rule*: *int-cases3*) *simp-all*

lemma *dvd-mult-sgn-iff* [*simp*]:
$l\ dvd\ k * sgn\ r \longleftrightarrow l\ dvd\ k \vee r = 0$ **for** *k l r* :: *int*
using *dvd-sgn-mult-iff* [*of l r k*] **by** (*simp add*: *ac-simps*)

lemma *int-sgnE*:
fixes *k* :: *int*
obtains *n* **and** *l* **where** $k = sgn\ l * int\ n$
proof −
have $k = sgn\ k * int\ (nat\ |k|)$
by (*simp add*: *sgn-mult-abs*)
then show *?thesis* **..**
qed

54.8.1 Binary comparisons

Preliminaries

lemma *le-imp-0-less*:
fixes *z* :: *int*
assumes *le*: $0 \leq z$
shows $0 < 1 + z$
proof −
have $0 \leq z$ **by** *fact*
also have $\ldots < z + 1$ **by** (*rule less-add-one*)
also have $\ldots = 1 + z$ **by** (*simp add*: *ac-simps*)
finally show $0 < 1 + z$ **.**

qed

lemma *odd-less-0-iff*: $1 + z + z < 0 \longleftrightarrow z < 0$
  for z :: *int*
proof (*cases z*)
  case (*nonneg n*)
  then show *?thesis*
    by (*simp add*: *linorder-not-less add.assoc add-increasing le-imp-0-less* [*THEN order-less-imp-le*])
next
  case (*neg n*)
  then show *?thesis*
    by (*simp del*: *of-nat-Suc of-nat-add of-nat-1*
      add: *algebra-simps of-nat-1* [**where** *'a=int*, *symmetric*] *of-nat-add* [*symmetric*])
qed

54.8.2 Comparisons, for Ordered Rings

lemma *odd-nonzero*: $1 + z + z \neq 0$
  for z :: *int*
proof (*cases z*)
  case (*nonneg n*)
  have *le*: $0 \leq z + z$
    by (*simp add*: *nonneg add-increasing*)
  then show *?thesis*
    using *le-imp-0-less* [*OF le*] **by** (*auto simp*: *ac-simps*)
next
  case (*neg n*)
  show *?thesis*
  proof
    assume *eq*: $1 + z + z = 0$
    have $0 < 1 + (int\ n + int\ n)$
      by (*simp add*: *le-imp-0-less add-increasing*)
    also have $\ldots = - (1 + z + z)$
      by (*simp add*: *neg add.assoc* [*symmetric*])
    also have $\ldots = 0$ **by** (*simp add*: *eq*)
    finally have $0<0$ **..**
    then show *False* **by** *blast*
  qed
qed

54.9 The Set of Integers

context *ring-1*
begin

definition *Ints* :: *'a set* (‹$\mathbb{Z}$›)
  where $\mathbb{Z}$ = *range of-int*

lemma *Ints-of-int* [*simp*]: *of-int* $z \in \mathbb{Z}$

by (*simp add*: *Ints-def*)

lemma *Ints-of-nat* [*simp*]: *of-nat* $n \in \mathbb{Z}$
using *Ints-of-int* [*of of-nat n*] **by** *simp*

lemma *Ints-0* [*simp*]: $0 \in \mathbb{Z}$
using *Ints-of-int* [*of 0*] **by** *simp*

lemma *Ints-1* [*simp*]: $1 \in \mathbb{Z}$
using *Ints-of-int* [*of 1*] **by** *simp*

lemma *Ints-numeral* [*simp*]: *numeral* $n \in \mathbb{Z}$
by (*subst of-nat-numeral* [*symmetric*], *rule Ints-of-nat*)

lemma *Ints-add* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a + b \in \mathbb{Z}$
by (*force simp add*: *Ints-def simp flip*: *of-int-add intro*: *range-eqI*)

lemma *Ints-minus* [*simp*]: $a \in \mathbb{Z} \Longrightarrow -a \in \mathbb{Z}$
by (*force simp add*: *Ints-def simp flip*: *of-int-minus intro*: *range-eqI*)

lemma *minus-in-Ints-iff*: $-x \in \mathbb{Z} \longleftrightarrow x \in \mathbb{Z}$
using *Ints-minus*[*of x*] *Ints-minus*[*of* $-x$] **by** *auto*

lemma *Ints-diff* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a - b \in \mathbb{Z}$
by (*force simp add*: *Ints-def simp flip*: *of-int-diff intro*: *range-eqI*)

lemma *Ints-mult* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a * b \in \mathbb{Z}$
by (*force simp add*: *Ints-def simp flip*: *of-int-mult intro*: *range-eqI*)

lemma *Ints-power* [*simp*]: $a \in \mathbb{Z} \Longrightarrow a \mathbin{\widehat{}} n \in \mathbb{Z}$
by (*induct n*) *simp-all*

lemma *Ints-cases* [*cases set*: *Ints*]:
assumes $q \in \mathbb{Z}$
obtains (*of-int*) z **where** $q =$ *of-int* z
unfolding *Ints-def*
proof −
from ‹$q \in \mathbb{Z}$› **have** $q \in$ *range of-int* **unfolding** *Ints-def* **.**
then obtain z **where** $q =$ *of-int* z **..**
then show *thesis* **..**
qed

lemma *Ints-induct* [*case-names of-int*, *induct set*: *Ints*]:
$q \in \mathbb{Z} \Longrightarrow (\bigwedge z.\ P\ (\textit{of-int}\ z)) \Longrightarrow P\ q$
by (*rule Ints-cases*) *auto*

lemma *Nats-subset-Ints*: $\mathbb{N} \subseteq \mathbb{Z}$
unfolding *Nats-def Ints-def*
by (*rule subsetI*, *elim imageE*, *hypsubst*, *subst of-int-of-nat-eq*[*symmetric*], *rule*

imageI) *simp-all*

lemma *Nats-altdef1*: $\mathbb{N} = \{$*of-int n* $|n.\ n \geq 0\}$
proof (*intro subsetI equalityI*)
  fix x :: $'a$
  assume $x \in \{$*of-int n* $|n.\ n \geq 0\}$
  then obtain n **where** $x =$ *of-int n* $n \geq 0$
    by (*auto elim*!: *Ints-cases*)
  then have $x =$ *of-nat* (*nat n*)
    by (*subst of-nat-nat*) *simp-all*
  then show $x \in \mathbb{N}$
    by *simp*
next
  fix x :: $'a$
  assume $x \in \mathbb{N}$
  then obtain n **where** $x =$ *of-nat n*
    by (*auto elim*!: *Nats-cases*)
  then have $x =$ *of-int* (*int n*) **by** *simp*
  also have *int n* ≥ 0 **by** *simp*
  then have *of-int* (*int n*) $\in \{$*of-int n* $|n.\ n \geq 0\}$ **by** *blast*
  finally show $x \in \{$*of-int n* $|n.\ n \geq 0\}$ **.**
qed

end

lemma *Ints-sum* [*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{Z}) \Longrightarrow$ *sum f A* $\in \mathbb{Z}$
  by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *Ints-prod* [*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{Z}) \Longrightarrow$ *prod f A* $\in \mathbb{Z}$
  by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma (**in** *linordered-idom*) *Ints-abs* [*simp*]:
  shows $a \in \mathbb{Z} \Longrightarrow$ *abs a* $\in \mathbb{Z}$
  by (*auto simp*: *abs-if*)

lemma (**in** *linordered-idom*) *Nats-altdef2*: $\mathbb{N} = \{n \in \mathbb{Z}.\ n \geq 0\}$
proof (*intro subsetI equalityI*)
  fix x :: $'a$
  assume $x \in \{n \in \mathbb{Z}.\ n \geq 0\}$
  then obtain n **where** $x =$ *of-int n* $n \geq 0$
    by (*auto elim*!: *Ints-cases*)
  then have $x =$ *of-nat* (*nat n*)
    by (*subst of-nat-nat*) *simp-all*
  then show $x \in \mathbb{N}$
    by *simp*
qed (*auto elim*!: *Nats-cases*)

lemma (**in** *idom-divide*) *of-int-divide-in-Ints*:
  of-int a div of-int b $\in \mathbb{Z}$ **if** *b dvd a*

proof −
  from *that* **obtain** c **where** $a = b * c$ **..**
  then show *?thesis*
    by (*cases of-int b = 0*) *simp-all*
qed

The premise involving $\mathbb{Z}$ prevents $a = 1 / (2::'a)$.

lemma *Ints-double-eq-0-iff*:
  fixes a :: *'a::ring-char-0*
  assumes *in-Ints*: $a \in \mathbb{Z}$
  shows $a + a = 0 \longleftrightarrow a = 0$
    (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
  from *in-Ints* **have** $a \in$ *range of-int*
    unfolding *Ints-def* [*symmetric*] **.**
  then obtain z **where** a: a = *of-int* z **..**
  show *?thesis*
  proof
    assume *?rhs*
    then show *?lhs* **by** *simp*
  next
    assume *?lhs*
    with a **have** *of-int* $(z + z)$ = (*of-int 0* :: *'a*) **by** *simp*
    then have $z + z = 0$ **by** (*simp only*: *of-int-eq-iff*)
    then have $z = 0$ **by** (*simp only*: *double-zero*)
    with a **show** *?rhs* **by** *simp*
  qed
qed

lemma *Ints-odd-nonzero*:
  fixes a :: *'a::ring-char-0*
  assumes *in-Ints*: $a \in \mathbb{Z}$
  shows $1 + a + a \neq 0$
proof −
  from *in-Ints* **have** $a \in$ *range of-int*
    unfolding *Ints-def* [*symmetric*] **.**
  then obtain z **where** a: a = *of-int* z **..**
  show *?thesis*
  proof
    assume $1 + a + a = 0$
    with a **have** *of-int* $(1 + z + z)$ = (*of-int 0* :: *'a*) **by** *simp*
    then have $1 + z + z = 0$ **by** (*simp only*: *of-int-eq-iff*)
    with *odd-nonzero* **show** *False* **by** *blast*
  qed
qed

lemma *Nats-numeral* [*simp*]: *numeral* $w \in \mathbb{N}$
  using *of-nat-in-Nats* [*of numeral w*] **by** *simp*

lemma *Ints-odd-less-0*:
  fixes *a* :: *'a::linordered-idom*
  assumes *in-Ints*: $a \in \mathbb{Z}$
  shows $1 + a + a < 0 \longleftrightarrow a < 0$
proof −
  from *in-Ints* **have** $a \in$ *range of-int*
    unfolding *Ints-def* [*symmetric*] **.**
  then obtain *z* **where** *a*: *a* = *of-int z* **..**
  with *a* **have** $1 + a + a < 0 \longleftrightarrow$ *of-int* (*1* + *z* + *z*) < (*of-int 0* :: *'a*)
    by *simp*
  also have ... $\longleftrightarrow z < 0$
    by (*simp only*: *of-int-less-iff odd-less-0-iff*)
  also have ... $\longleftrightarrow a < 0$
    by (*simp add*: *a*)
  finally show *?thesis* **.**
qed

54.10 *sum* and *prod*

context *semiring-1*
begin

lemma *of-nat-sum* [*simp*]:
  of-nat (*sum f A*) = ($\sum$ *x*∈*A*. *of-nat* (*f x*))
  by (*induction A rule*: *infinite-finite-induct*) *auto*

end

context *ring-1*
begin

lemma *of-int-sum* [*simp*]:
  of-int (*sum f A*) = ($\sum$ *x*∈*A*. *of-int* (*f x*))
  by (*induction A rule*: *infinite-finite-induct*) *auto*

end

context *comm-semiring-1*
begin

lemma *of-nat-prod* [*simp*]:
  of-nat (*prod f A*) = ($\prod$ *x*∈*A*. *of-nat* (*f x*))
  by (*induction A rule*: *infinite-finite-induct*) *auto*

end

context *comm-ring-1*
begin

lemma *of-int-prod* [*simp*]:
of-int (*prod f A*) = ($\prod$ *x*∈*A*. *of-int* (*f x*))
by (*induction A rule*: *infinite-finite-induct*) *auto*

end

54.11 Setting up simplification procedures

ML-file ‹*Tools*/*int-arith.ML*›

declaration ‹*K* (
Lin-Arith.add-discrete-type ***type-name*** ‹*Int.int*›
#> *Lin-Arith.add-lessD* @{*thm zless-imp-add1-zle*}
#> *Lin-Arith.add-inj-thms* @{*thms of-nat-le-iff* [*THEN iffD2*] *of-nat-eq-iff* [*THEN iffD2*]}
#> *Lin-Arith.add-inj-const* (***const-name*** ‹*of-nat*›, ***typ*** ‹*nat* ⇒ *int*›)
#> *Lin-Arith.add-simps*
@{*thms of-int-0 of-int-1 of-int-add of-int-mult of-int-numeral of-int-neg-numeral nat-0 nat-1 diff-nat-numeral nat-numeral*
neg-less-iff-less
True-implies-equals
distrib-left [*where a* = *numeral v for v*]
distrib-left [*where a* = − *numeral v for v*]
div-by-1 div-0
times-divide-eq-right times-divide-eq-left
minus-divide-left [*THEN sym*] *minus-divide-right* [*THEN sym*]
add-divide-distrib diff-divide-distrib
of-int-minus of-int-diff
of-int-of-nat-eq}
#> *Lin-Arith.add-simprocs* [*Int-Arith.zero-one-idom-simproc*]
)›

simproc-setup *fast-arith*
((*m*::′*a*::*linordered-idom*) < *n* |
(*m*::′*a*::*linordered-idom*) ≤ *n* |
(*m*::′*a*::*linordered-idom*) = *n*) =
‹*K Lin-Arith.simproc*›

54.12 More Inequality Reasoning

lemma *zless-add1-eq*: *w* < *z* + *1* ⟷ *w* < *z* ∨ *w* = *z*
for *w z* :: *int*
by *arith*

lemma *add1-zle-eq*: *w* + *1* ≤ *z* ⟷ *w* < *z*
for *w z* :: *int*
by *arith*

lemma *zle-diff1-eq* [*simp*]: *w* ≤ *z* − *1* ⟷ *w* < *z*
for *w z* :: *int*

by *arith*

lemma *zle-add1-eq-le* [*simp*]: $w < z + 1 \longleftrightarrow w \leq z$
for *w z* :: *int*
by *arith*

lemma *int-one-le-iff-zero-less*: $1 \leq z \longleftrightarrow 0 < z$
for *z* :: *int*
by *arith*

lemma *Ints-nonzero-abs-ge1*:
fixes *x*:: *'a* :: *linordered-idom*
assumes $x \in Ints$ $x \neq 0$
shows $1 \leq abs\ x$
proof (*rule Ints-cases* [*OF* ‹$x \in Ints$›])
fix *z*::*int*
assume $x = of\text{-}int\ z$
with ‹$x \neq 0$›
show $1 \leq |x|$
apply (*auto simp*: *abs-if*)
by (*metis diff-0 of-int-1 of-int-le-iff of-int-minus zle-diff1-eq*)
qed

lemma *Ints-nonzero-abs-less1*:
fixes *x*:: *'a* :: *linordered-idom*
shows $[\![x \in Ints; abs\ x < 1]\!] \Longrightarrow x = 0$
using *Ints-nonzero-abs-ge1* [*of x*] **by** *auto*

lemma *Ints-eq-abs-less1*:
fixes *x*:: *'a* :: *linordered-idom*
shows $[\![x \in Ints; y \in Ints]\!] \Longrightarrow x = y \longleftrightarrow abs\ (x-y) < 1$
using *eq-iff-diff-eq-0* **by** (*fastforce intro*: *Ints-nonzero-abs-less1*)

54.13 The functions *nat* and *int*

Simplify the term $w + - z$.

lemma *one-less-nat-eq* [*simp*]: $Suc\ 0 < nat\ z \longleftrightarrow 1 < z$
using *zless-nat-conj* [*of 1 z*] **by** *auto*

lemma *int-eq-iff-numeral* [*simp*]:
$int\ m = numeral\ v \longleftrightarrow m = numeral\ v$
by (*simp add*: *int-eq-iff*)

lemma *nat-abs-int-diff*:
$nat\ |int\ a - int\ b| = (if\ a \leq b\ then\ b - a\ else\ a - b)$
by *auto*

lemma *nat-int-add*: $nat\ (int\ a + int\ b) = a + b$
by *auto*

context *ring-1*
begin

lemma *of-int-of-nat* [*nitpick-simp*]:
of-int k = (*if k* < *0 then* − *of-nat* (*nat* (− *k*)) *else of-nat* (*nat k*))
proof (*cases k* < *0*)
case *True*
then have $0 \le - k$ **by** *simp*
then have *of-nat* (*nat* (− *k*)) = *of-int* (− *k*) **by** (*rule of-nat-nat*)
with *True* **show** *?thesis* **by** *simp*
next
case *False*
then show *?thesis* **by** (*simp add*: *not-less*)
qed

end

lemma *transfer-rule-of-int*:
includes *lifting-syntax*
fixes *R* :: *'a*::*ring-1* ⇒ *'b*::*ring-1* ⇒ *bool*
assumes [*transfer-rule*]: *R 0 0 R 1 1*
(*R* ===> *R* ===> *R*) (+) (+)
(*R* ===> *R*) *uminus uminus*
shows ((=) ===> *R*) *of-int of-int*
proof −
note *assms*
note *transfer-rule-of-nat* [*transfer-rule*]
have [*transfer-rule*]: ((=) ===> *R*) *of-nat of-nat*
by *transfer-prover*
show *?thesis*
by (*unfold of-int-of-nat* [*abs-def*]) *transfer-prover*
qed

lemma *nat-mult-distrib*:
fixes *z z'* :: *int*
assumes $0 \le z$
shows *nat* (*z* ∗ *z'*) = *nat z* ∗ *nat z'*
proof (*cases* $0 \le z'$)
case *False*
with *assms* **have** $z * z' \le 0$
by (*simp add*: *not-le mult-le-0-iff*)
then have *nat* (*z* ∗ *z'*) = *0* **by** *simp*
moreover from *False* **have** *nat z'* = *0* **by** *simp*
ultimately show *?thesis* **by** *simp*
next
case *True*
with *assms* **have** *ge-0*: $z * z' \ge 0$ **by** (*simp add*: *zero-le-mult-iff*)
show *?thesis*

by (*rule injD* [*of of-nat* :: *nat* ⇒ *int*, *OF inj-of-nat*])
(*simp only*: *of-nat-mult of-nat-nat* [*OF True*]
of-nat-nat [*OF assms*] *of-nat-nat* [*OF ge-0*], *simp*)
qed

lemma *nat-mult-distrib-neg*:
assumes $z \leq (0::int)$ **shows** $nat\ (z * z') = nat\ (-\ z) * nat\ (-\ z')$ (**is** *?L* = *?R*)
proof −
have $?L = nat\ (-\ z * -\ z')$
using *assms* **by** *auto*
also have ... = *?R*
by (*rule nat-mult-distrib*) (*use assms* **in** *auto*)
finally show *?thesis* **.**
qed

lemma *nat-abs-mult-distrib*: $nat\ |w * z| = nat\ |w| * nat\ |z|$
by (*cases* $z = 0 \lor w = 0$)
(*auto simp add*: *abs-if nat-mult-distrib* [*symmetric*]
nat-mult-distrib-neg [*symmetric*] *mult-less-0-iff*)

lemma *int-in-range-abs* [*simp*]: *int n* ∈ *range abs*
proof (*rule range-eqI*)
show $int\ n = |int\ n|$ **by** *simp*
qed

lemma *range-abs-Nats* [*simp*]: *range abs* = (ℕ :: *int set*)
proof −
have $|k| \in \mathbb{N}$ **for** *k* :: *int*
by (*cases k*) *simp-all*
moreover have *k* ∈ *range abs* **if** *k* ∈ ℕ **for** *k* :: *int*
using *that* **by** *induct simp*
ultimately show *?thesis* **by** *blast*
qed

lemma *Suc-nat-eq-nat-zadd1*: $0 \leq z \Longrightarrow Suc\ (nat\ z) = nat\ (1 + z)$
for *z* :: *int*
by (*rule sym*) (*simp add*: *nat-eq-iff*)

lemma *diff-nat-eq-if*:
$nat\ z - nat\ z' =$
(*if* $z' < 0$ *then nat z*
else
let $d = z - z'$
in if $d < 0$ *then* 0 *else nat d*)
by (*simp add*: *Let-def nat-diff-distrib* [*symmetric*])

lemma *nat-numeral-diff-1* [*simp*]: $numeral\ v - (1::nat) = nat\ (numeral\ v - 1)$
using *diff-nat-numeral* [*of v Num.One*] **by** *simp*

54.14 Induction principles for int

Well-founded segments of the integers.

definition *int-ge-less-than* :: *int* ⇒ (*int* × *int*) *set*
  where *int-ge-less-than* $d = \{(z', z).\ d \leq z' \wedge z' < z\}$

lemma *wf-int-ge-less-than*: *wf* (*int-ge-less-than d*)
proof −
  have *int-ge-less-than d* ⊆ *measure* ($\lambda z.$ *nat* $(z - d)$)
    by (*auto simp add*: *int-ge-less-than-def*)
  then show *?thesis*
    by (*rule wf-subset* [*OF wf-measure*])
qed

This variant looks odd, but is typical of the relations suggested by Rank-Finder.

definition *int-ge-less-than2* :: *int* ⇒ (*int* × *int*) *set*
  where *int-ge-less-than2* $d = \{(z',z).\ d \leq z \wedge z' < z\}$

lemma *wf-int-ge-less-than2*: *wf* (*int-ge-less-than2 d*)
proof −
  have *int-ge-less-than2 d* ⊆ *measure* ($\lambda z.$ *nat* $(1 + z - d)$)
    by (*auto simp add*: *int-ge-less-than2-def*)
  then show *?thesis*
    by (*rule wf-subset* [*OF wf-measure*])
qed

theorem *int-ge-induct* [*case-names base step*, *induct set*: *int*]:
  fixes *i* :: *int*
  assumes *ge*: $k \leq i$
    and *base*: $P\ k$
    and *step*: $\bigwedge i.\ k \leq i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
  shows $P\ i$
proof −
  have $\bigwedge i{::}int.\ n = nat\ (i - k) \Longrightarrow k \leq i \Longrightarrow P\ i$ **for** *n*
  proof (*induct n*)
    case *0*
    then have $i = k$ **by** *arith*
    with *base* **show** $P\ i$ **by** *simp*
  next
    case (*Suc n*)
    then have $n = nat\ ((i - 1) - k)$ **by** *arith*
    moreover have *k*: $k \leq i - 1$ **using** *Suc.prems* **by** *arith*
    ultimately have $P\ (i - 1)$ **by** (*rule Suc.hyps*)
    from *step* [*OF k this*] **show** *?case* **by** *simp*
  qed
  with *ge* **show** *?thesis* **by** *fast*
qed

theorem *int-gr-induct* [*case-names base step*, *induct set*: *int*]:
  fixes *i k* :: *int*
  assumes $k < i$ P $(k + 1)$ $\bigwedge i.\ k < i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
  shows *P i*
proof −
  have $k+1 \leq i$
    using *assms* **by** *auto*
  then show *?thesis*
    by (*induction i rule*: *int-ge-induct*) (*auto simp*: *assms*)
qed

theorem *int-le-induct* [*consumes 1*, *case-names base step*]:
  fixes *i k* :: *int*
  assumes *le*: $i \leq k$
    and *base*: *P k*
    and *step*: $\bigwedge i.\ i \leq k \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
  shows *P i*
proof −
  have $\bigwedge i::int.\ n = nat(k-i) \Longrightarrow i \leq k \Longrightarrow P\ i$ **for** *n*
  proof (*induct n*)
    case *0*
    then have $i = k$ **by** *arith*
    with *base* **show** *P i* **by** *simp*
  next
    case (*Suc n*)
    then have $n = nat\ (k - (i + 1))$ **by** *arith*
    moreover have *k*: $i + 1 \leq k$ **using** *Suc.prems* **by** *arith*
    ultimately have $P\ (i + 1)$ **by** (*rule Suc.hyps*)
    from *step*[*OF k this*] **show** *?case* **by** *simp*
  qed
  with *le* **show** *?thesis* **by** *fast*
qed

theorem *int-less-induct* [*consumes 1*, *case-names base step*]:
  fixes *i k* :: *int*
  assumes $i < k$ P $(k - 1)$ $\bigwedge i.\ i < k \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
  shows *P i*
proof −
  have $i \leq k-1$
    using *assms* **by** *auto*
  then show *?thesis*
    by (*induction i rule*: *int-le-induct*) (*auto simp*: *assms*)
qed

theorem *int-induct* [*case-names base step1 step2*]:
  fixes *k* :: *int*
  assumes *base*: *P k*

and *step1*: $\bigwedge i.\ k \le i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
and *step2*: $\bigwedge i.\ k \ge i \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
shows $P\ i$
proof –
have $i \le k \vee i \ge k$ **by** *arith*
then show *?thesis*
proof
assume $i \ge k$
then show *?thesis*
using *base* **by** (*rule int-ge-induct*) (*fact step1*)
next
assume $i \le k$
then show *?thesis*
using *base* **by** (*rule int-le-induct*) (*fact step2*)
qed
qed

54.15 Intermediate value theorems

lemma *nat-ivt-aux*:
$[\![\forall i{<}n.\ |f\ (Suc\ i) - f\ i| \le 1;\ f\ 0 \le k;\ k \le f\ n]\!] \Longrightarrow \exists\, i \le n.\ f\ i = k$
for $m\ n :: nat$ **and** $k :: int$
proof (*induct n*)
case (*Suc n*)
show *?case*
proof (*cases* $k = f\ (Suc\ n)$)
case *False*
with *Suc* **have** $k \le f\ n$
by *auto*
with *Suc* **show** *?thesis*
by (*auto simp add*: *abs-if split*: *if-split-asm intro*: *le-SucI*)
qed (*use Suc* **in** *auto*)
qed *auto*

lemma *nat-intermed-int-val*:
fixes $m\ n :: nat$ **and** $k :: int$
assumes $\forall i.\ m \le i \wedge i < n \longrightarrow |f\ (Suc\ i) - f\ i| \le 1\ \ m \le n\ \ f\ m \le k\ \ k \le f\ n$
shows $\exists i.\ m \le i \wedge i \le n \wedge f\ i = k$
proof –
obtain *i* **where** $i \le n - m\ \ k = f\ (m + i)$
using *nat-ivt-aux* [*of* $n - m\ f \circ plus\ m\ k$] *assms* **by** *auto*
with *assms* **show** *?thesis*
using *exI*[*of* - $m + i$] **by** *auto*
qed

lemma *nat0-intermed-int-val*:
$\exists\, i{\le}n.\ f\ i = k$
if $\forall\, i{<}n.\ |f\ (i + 1) - f\ i| \le 1\ \ f\ 0 \le k\ \ k \le f\ n$
for $n :: nat$ **and** $k :: int$

using *nat-intermed-int-val* [*of 0 n f k*] *that* **by** *auto*

54.16 Products and 1, by T. M. Rasmussen

lemma *abs-zmult-eq-1*:
fixes *m n* :: *int*
assumes *mn*: $|m * n| = 1$
shows $|m| = 1$
proof −
from *mn* **have** *0*: $m \neq 0$ $n \neq 0$ **by** *auto*
have $\neg\ 2 \le |m|$
proof
assume $2 \le |m|$
then have $2 * |n| \le |m| * |n|$ **by** (*simp add*: *mult-mono 0*)
also have $\ldots = |m * n|$ **by** (*simp add*: *abs-mult*)
also from *mn* **have** $\ldots = 1$ **by** *simp*
finally have $2 * |n| \le 1$ **.**
with *0* **show** *False* **by** *arith*
qed
with *0* **show** *?thesis* **by** *auto*
qed

lemma *pos-zmult-eq-1-iff-lemma*: $m * n = 1 \Longrightarrow m = 1 \vee m = -\ 1$
for *m n* :: *int*
using *abs-zmult-eq-1* [*of m n*] **by** *arith*

lemma *pos-zmult-eq-1-iff*:
fixes *m n* :: *int*
assumes $0 < m$
shows $m * n = 1 \longleftrightarrow m = 1 \wedge n = 1$
proof −
from *assms* **have** $m * n = 1 \Longrightarrow m = 1$
by (*auto dest*: *pos-zmult-eq-1-iff-lemma*)
then show *?thesis*
by (*auto dest*: *pos-zmult-eq-1-iff-lemma*)
qed

lemma *zmult-eq-1-iff*: $m * n = 1 \longleftrightarrow (m = 1 \wedge n = 1) \vee (m = -\ 1 \wedge n = -\ 1)$ (**is** *?L = ?R*)
for *m n* :: *int*
proof
assume *L*: *?L* **show** *?R*
using *pos-zmult-eq-1-iff-lemma* [*OF L*] *L* **by** *force*
qed *auto*

lemma *zmult-eq-neg1-iff*: $a * b = (-1 :: int) \longleftrightarrow a = 1 \wedge b = -1 \vee a = -1 \wedge b = 1$
using *zmult-eq-1-iff*[*of a* $-b$] **by** *auto*

lemma *infinite-UNIV-int* [*simp*]: ¬ *finite* (*UNIV*::*int set*)
proof
  assume *finite* (*UNIV*::*int set*)
  moreover have *inj* (λi::*int*. $2 * i$)
    by (*rule injI*) *simp*
  ultimately have *surj* (λi::*int*. $2 * i$)
    by (*rule finite-UNIV-inj-surj*)
  then obtain *i* :: *int* **where** $1 = 2 * i$ **by** (*rule surjE*)
  then show *False* **by** (*simp add*: *pos-zmult-eq-1-iff*)
qed

54.17 The divides relation

lemma *zdvd-antisym-nonneg*: $0 \leq m \Longrightarrow 0 \leq n \Longrightarrow m$ *dvd* $n \Longrightarrow n$ *dvd* $m \Longrightarrow m = n$
  for *m n* :: *int*
  by (*auto simp add*: *dvd-def mult.assoc zero-le-mult-iff zmult-eq-1-iff*)

lemma *zdvd-antisym-abs*:
  fixes *a b* :: *int*
  assumes *a dvd b* **and** *b dvd a*
  shows $|a| = |b|$
proof (*cases* $a = 0$)
  case *True*
  with *assms* **show** *?thesis* **by** *simp*
next
  case *False*
  from ‹*a dvd b*› **obtain** *k* **where** *k*: $b = a * k$
    unfolding *dvd-def* **by** *blast*
  from ‹*b dvd a*› **obtain** k' **where** k': $a = b * k'$
    unfolding *dvd-def* **by** *blast*
  from k k' **have** $a = a * k * k'$ **by** *simp*
  with *mult-cancel-left1*[**where** *c*=*a* **and** *b*=*k***k*′] **have** kk': $k * k' = 1$
    using ‹$a \neq 0$› **by** (*simp add*: *mult.assoc*)
  then have $k = 1 \wedge k' = 1 \vee k = -1 \wedge k' = -1$
    by (*simp add*: *zmult-eq-1-iff*)
  with k k' **show** *?thesis* **by** *auto*
qed

lemma *zdvd-zdiffD*: k *dvd* $m - n \Longrightarrow k$ *dvd* $n \Longrightarrow k$ *dvd* m
  for *k m n* :: *int*
  using *dvd-add-right-iff* [*of* $k - n$ *m*] **by** *simp*

lemma *zdvd-reduce*: k *dvd* $n + k * m \longleftrightarrow k$ *dvd* n
  for *k m n* :: *int*
  using *dvd-add-times-triv-right-iff* [*of k n m*] **by** (*simp add*: *ac-simps*)

lemma *dvd-imp-le-int*:
  fixes *d i* :: *int*

assumes $i \neq 0$ **and** *d dvd i*
shows $|d| \leq |i|$
proof –
from ‹*d dvd i*› **obtain** *k* **where** $i = d * k$ **..**
with ‹$i \neq 0$› **have** $k \neq 0$ **by** *auto*
then have $1 \leq |k|$ **and** $0 \leq |d|$ **by** *auto*
then have $|d| * 1 \leq |d| * |k|$ **by** (*rule mult-left-mono*)
with ‹$i = d * k$› **show** *?thesis* **by** (*simp add*: *abs-mult*)
qed

lemma *zdvd-not-zless*:
fixes *m n* :: *int*
assumes $0 < m$ **and** $m < n$
shows ¬ *n dvd m*
proof
from *assms* **have** $0 < n$ **by** *auto*
assume *n dvd m* **then obtain** *k* **where** *k*: $m = n * k$ **..**
with ‹$0 < m$› **have** $0 < n * k$ **by** *auto*
with ‹$0 < n$› **have** $0 < k$ **by** (*simp add*: *zero-less-mult-iff*)
with *k* ‹$0 < n$› ‹$m < n$› **have** $n * k < n * 1$ **by** *simp*
with ‹$0 < n$› ‹$0 < k$› **show** *False* **unfolding** *mult-less-cancel-left* **by** *auto*
qed

lemma *zdvd-mult-cancel*:
fixes *k m n* :: *int*
assumes *d*: $k * m$ *dvd* $k * n$
and $k \neq 0$
shows *m dvd n*
proof –
from *d* **obtain** *h* **where** *h*: $k * n = k * m * h$
unfolding *dvd-def* **by** *blast*
have $n = m * h$
proof (*rule ccontr*)
assume ¬ *?thesis*
with ‹$k \neq 0$› **have** $k * n \neq k * (m * h)$ **by** *simp*
with *h* **show** *False*
by (*simp add*: *mult.assoc*)
qed
then show *?thesis* **by** *simp*
qed

lemma *int-dvd-int-iff* [*simp*]:
int m dvd int n ⟷ *m dvd n*
proof –
have *m dvd n* **if** $int\ n = int\ m * k$ **for** *k*
proof (*cases k*)
case (*nonneg q*)
with *that* **have** $n = m * q$
by (*simp del*: *of-nat-mult add*: *of-nat-mult* [*symmetric*])

```
    then show ?thesis ..
  next
    case (neg q)
    with that have int n = int m * (− int (Suc q))
      by simp
    also have ... = − (int m * int (Suc q))
      by (simp only: mult-minus-right)
    also have ... = − int (m * Suc q)
      by (simp only: of-nat-mult [symmetric])
    finally have − int (m * Suc q) = int n ..
    then show ?thesis
      by (simp only: negative-eq-positive) auto
  qed
  then show ?thesis by (auto simp add: dvd-def)
qed

lemma dvd-nat-abs-iff [simp]:
  n dvd nat |k| ⟷ int n dvd k
proof −
  have n dvd nat |k| ⟷ int n dvd int (nat |k|)
    by (simp only: int-dvd-int-iff)
  then show ?thesis
    by simp
qed

lemma nat-abs-dvd-iff [simp]:
  nat |k| dvd n ⟷ k dvd int n
proof −
  have nat |k| dvd n ⟷ int (nat |k|) dvd int n
    by (simp only: int-dvd-int-iff)
  then show ?thesis
    by simp
qed

lemma zdvd1-eq [simp]: x dvd 1 ⟷ |x| = 1 (is ?lhs ⟷ ?rhs)
  for x :: int
proof
  assume ?lhs
  then have nat |x| dvd nat |1|
    by (simp only: nat-abs-dvd-iff) simp
  then have nat |x| = 1
    by simp
  then show ?rhs
    by (cases x < 0) simp-all
next
  assume ?rhs
  then have x = 1 ∨ x = − 1
    by auto
  then show ?lhs
```

```
    by (auto intro: dvdI)
qed

lemma zdvd-mult-cancel1:
  fixes m :: int
  assumes mp: m ≠ 0
  shows m * n dvd m ⟷ |n| = 1
    (is ?lhs ⟷ ?rhs)
proof
  assume ?rhs
  then show ?lhs
    by (cases n > 0) (auto simp add: minus-equation-iff)
next
  assume ?lhs
  then have m * n dvd m * 1 by simp
  from zdvd-mult-cancel[OF this mp] show ?rhs
    by (simp only: zdvd1-eq)
qed

lemma nat-dvd-iff: nat z dvd m ⟷ (if 0 ≤ z then z dvd int m else m = 0)
  using nat-abs-dvd-iff [of z m] by (cases z ≥ 0) auto

lemma eq-nat-nat-iff: 0 ≤ z ⟹ 0 ≤ z' ⟹ nat z = nat z' ⟷ z = z'
  by (auto elim: nonneg-int-cases)

lemma nat-power-eq: 0 ≤ z ⟹ nat (z ^ n) = nat z ^ n
  by (induct n) (simp-all add: nat-mult-distrib)

lemma numeral-power-eq-nat-cancel-iff [simp]:
  numeral x ^ n = nat y ⟷ numeral x ^ n = y
  using nat-eq-iff2 by auto

lemma nat-eq-numeral-power-cancel-iff [simp]:
  nat y = numeral x ^ n ⟷ y = numeral x ^ n
  using numeral-power-eq-nat-cancel-iff[of x n y]
  by (metis (mono-tags))

lemma numeral-power-le-nat-cancel-iff [simp]:
  numeral x ^ n ≤ nat a ⟷ numeral x ^ n ≤ a
  using nat-le-eq-zle[of numeral x ^ n a]
  by (auto simp: nat-power-eq)

lemma nat-le-numeral-power-cancel-iff [simp]:
  nat a ≤ numeral x ^ n ⟷ a ≤ numeral x ^ n
  by (simp add: nat-le-iff)

lemma numeral-power-less-nat-cancel-iff [simp]:
  numeral x ^ n < nat a ⟷ numeral x ^ n < a
  using nat-less-eq-zless[of numeral x ^ n a]
```

by (*auto simp*: *nat-power-eq*)

lemma *nat-less-numeral-power-cancel-iff* [*simp*]:
nat a < numeral x ^ n ⟷ *a < numeral x ^ n*
using *nat-less-eq-zless*[*of a numeral x ^ n*]
by (*cases a < 0*) (*auto simp*: *nat-power-eq less-le-trans*[**where** *y=0*])

lemma *zdvd-imp-le*: *z ≤ n* **if** *z dvd n 0 < n* **for** *n z* :: *int*
proof (*cases n*)
case (*nonneg n*)
show *?thesis*
by (*cases z*) (*use nonneg dvd-imp-le that* **in** *auto*)
qed (*use that* **in** *auto*)

lemma *zdvd-period*:
fixes *a d* :: *int*
assumes *a dvd d*
shows *a dvd* (*x* + *t*) ⟷ *a dvd* ((*x* + *c* ∗ *d*) + *t*)
(**is** *?lhs* ⟷ *?rhs*)
proof −
from *assms* **have** *a dvd* (*x* + *t*) ⟷ *a dvd* ((*x* + *t*) + *c* ∗ *d*)
by (*simp add*: *dvd-add-left-iff*)
then show *?thesis*
by (*simp add*: *ac-simps*)
qed

54.18 Powers with integer exponents

The following allows writing powers with an integer exponent. While the type signature is very generic, most theorems will assume that the underlying type is a division ring or a field.

The notation 'powi' is inspired by the 'powr' notation for real/complex exponentiation.

definition *power-int* :: *'a* :: {*inverse*, *power*} ⇒ *int* ⇒ *'a* (**infixr** ‹*powi*› *80*) **where**
power-int x n = (*if n ≥ 0 then x ^ nat n else inverse x ^* (*nat* (−*n*)))

lemma *power-int-0-right* [*simp*]: *power-int x 0* = *1*
and *power-int-1-right* [*simp*]:
power-int (*y* :: *'a* :: {*power*, *inverse*, *monoid-mult*}) *1* = *y*
and *power-int-minus1-right* [*simp*]:
power-int (*y* :: *'a* :: {*power*, *inverse*, *monoid-mult*}) (−*1*) = *inverse y*
by (*simp-all add*: *power-int-def*)

lemma *power-int-of-nat* [*simp*]: *power-int x* (*int n*) = *x ^ n*
by (*simp add*: *power-int-def*)

lemma *power-int-numeral* [*simp*]: *power-int x* (*numeral n*) = *x ^ numeral n*
by (*simp add*: *power-int-def*)

lemma *powi-numeral-reduce*: x *powi numeral* $n = x * x$ *powi int* (*pred-numeral* n)
by (*simp add*: *numeral-eq-Suc*)

lemma *powi-minus-numeral-reduce*: x *powi* $-$ (*numeral* n) = *inverse* $x * x$ *powi* $-$ *int*(*pred-numeral* n)
by (*simp add*: *numeral-eq-Suc power-int-def*)

lemma *int-cases4* [*case-names nonneg neg*]:
fixes m :: *int*
obtains n **where** m = *int* n | n **where** $n > 0$ $m = -$*int* n
proof (*cases* $m \geq 0$)
case *True*
thus *?thesis* **using** *that*(*1*)[*of nat m*] **by** *auto*
next
case *False*
thus *?thesis* **using** *that*(*2*)[*of nat* ($-m$)] **by** *auto*
qed

context
assumes *SORT-CONSTRAINT*('*a*::*division-ring*)
begin

lemma *power-int-minus*: *power-int* (x::'*a*) ($-n$) = *inverse* (*power-int* x n)
by (*auto simp*: *power-int-def power-inverse*)

lemma *power-int-minus-divide*: *power-int* (x::'*a*) ($-n$) = *1* / (*power-int* x n)
by (*simp add*: *divide-inverse power-int-minus*)

lemma *power-int-eq-0-iff* [*simp*]: *power-int* (x::'*a*) $n = 0 \longleftrightarrow x = 0 \wedge n \neq 0$
by (*auto simp*: *power-int-def*)

lemma *power-int-0-left-if*: *power-int* (*0* :: '*a*) m = (*if* $m = 0$ *then 1 else 0*)
by (*auto simp*: *power-int-def*)

lemma *power-int-0-left* [*simp*]: $m \neq 0 \Longrightarrow$ *power-int* (*0* :: '*a*) $m = 0$
by (*simp add*: *power-int-0-left-if*)

lemma *power-int-1-left* [*simp*]: *power-int 1* n = (*1* :: '*a* :: *division-ring*)
by (*auto simp*: *power-int-def*)

lemma *power-diff-conv-inverse*: $x \neq 0 \Longrightarrow m \leq n \Longrightarrow (x$ :: '*a*) ^ $(n - m) = x$ ^ $n *$ *inverse* x ^ m
by (*simp add*: *field-simps flip*: *power-add*)

lemma *power-mult-inverse-distrib*: x ^ $m *$ *inverse* (x :: '*a*) = *inverse* $x * x$ ^ m
proof (*cases* $x = 0$)
case [*simp*]: *False*

show *?thesis*
proof (*cases m*)
case (*Suc m'*)
have $x \,\hat{}\, Suc\ m' * inverse\ x = x \,\hat{}\, m'$
by (*subst power-Suc2*) (*auto simp*: *mult.assoc*)
also have $\ldots = inverse\ x * x \,\hat{}\, Suc\ m'$
by (*subst power-Suc*) (*auto simp*: *mult.assoc* [*symmetric*])
finally show *?thesis* **using** *Suc* **by** *simp*
qed *auto*
qed *auto*

lemma *power-mult-power-inverse-commute*:
$x \,\hat{}\, m * inverse\ (x :: 'a) \,\hat{}\, n = inverse\ x \,\hat{}\, n * x \,\hat{}\, m$
proof (*induction n*)
case (*Suc n*)
have $x \,\hat{}\, m * inverse\ x \,\hat{}\, Suc\ n = (x \,\hat{}\, m * inverse\ x \,\hat{}\, n) * inverse\ x$
by (*simp only*: *power-Suc2 mult.assoc*)
also have $x \,\hat{}\, m * inverse\ x \,\hat{}\, n = inverse\ x \,\hat{}\, n * x \,\hat{}\, m$
by (*rule Suc*)
also have $\ldots * inverse\ x = (inverse\ x \,\hat{}\, n * inverse\ x) * x \,\hat{}\, m$
by (*simp add*: *mult.assoc power-mult-inverse-distrib*)
also have $\ldots = inverse\ x \,\hat{}\, (Suc\ n) * x \,\hat{}\, m$
by (*simp only*: *power-Suc2*)
finally show *?case* **.**
qed *auto*

lemma *power-int-add*:
assumes $x \neq 0 \vee m + n \neq 0$
shows *power-int* $(x::'a)$ $(m + n) = $ *power-int* $x\ m$ $*$ *power-int* $x\ n$
proof (*cases* $x = 0$)
case *True*
thus *?thesis* **using** *assms* **by** (*auto simp*: *power-int-0-left-if*)
next
case [*simp*]: *False*
show *?thesis*
proof (*cases m n rule*: *int-cases4* [*case-product int-cases4*])
case (*nonneg-nonneg a b*)
thus *?thesis*
by (*auto simp*: *power-int-def nat-add-distrib power-add*)
next
case (*nonneg-neg a b*)
thus *?thesis*
by (*auto simp*: *power-int-def nat-diff-distrib not-le power-diff-conv-inverse*
power-mult-power-inverse-commute)
next
case (*neg-nonneg a b*)
thus *?thesis*
by (*auto simp*: *power-int-def nat-diff-distrib not-le power-diff-conv-inverse*
power-mult-power-inverse-commute)

next
case (*neg-neg a b*)
thus *?thesis*
by (*auto simp*: *power-int-def nat-add-distrib add.commute simp flip*: *power-add*)
qed
qed

lemma *power-int-add-1*:
assumes $x \neq 0 \lor m \neq -1$
shows *power-int* $(x::'a)$ $(m + 1)$ = *power-int* x m $*$ x
using *assms* **by** (*subst power-int-add*) *auto*

lemma *power-int-add-1'*:
assumes $x \neq 0 \lor m \neq -1$
shows *power-int* $(x::'a)$ $(m + 1)$ = x $*$ *power-int* x m
using *assms* **by** (*subst add.commute*, *subst power-int-add*) *auto*

lemma *power-int-commutes*: *power-int* $(x :: 'a)$ $n * x = x *$ *power-int* x n
by (*cases* $x = 0$) (*auto simp flip*: *power-int-add-1 power-int-add-1'*)

lemma *power-int-inverse* [*field-simps*, *field-split-simps*, *divide-simps*]:
power-int (*inverse* $(x :: 'a)$) n = *inverse* (*power-int* x n)
by (*auto simp*: *power-int-def power-inverse*)

lemma *power-int-mult*: *power-int* $(x :: 'a)$ $(m * n)$ = *power-int* (*power-int* x m) n
by (*auto simp*: *power-int-def zero-le-mult-iff simp flip*: *power-mult power-inverse nat-mult-distrib*)

end

context
assumes *SORT-CONSTRAINT*('*a*::*field*)
begin

lemma *power-int-diff*:
assumes $x \neq 0 \lor m \neq n$
shows *power-int* $(x::'a)$ $(m - n)$ = *power-int* x m / *power-int* x n
using *power-int-add*[*of* x m $-n$] *assms* **by** (*auto simp*: *field-simps power-int-minus*)

lemma *power-int-minus-mult*: $x \neq 0 \lor n \neq 0 \Longrightarrow$ *power-int* $(x :: 'a)$ $(n - 1) *$ x = *power-int* x n
by (*auto simp flip*: *power-int-add-1*)

lemma *power-int-mult-distrib*: *power-int* $(x * y :: 'a)$ m = *power-int* x m $*$ *power-int* y m
by (*auto simp*: *power-int-def power-mult-distrib*)

lemmas *power-int-mult-distrib-numeral1* = *power-int-mult-distrib* [**where** x = *numeral* w **for** w, *simp*]

lemmas *power-int-mult-distrib-numeral2* = *power-int-mult-distrib* [**where** *y* = *numeral w* **for** *w*, *simp*]

lemma *power-int-divide-distrib*: *power-int* (*x* / *y* :: *'a*) *m* = *power-int x m* / *power-int y m*
  using *power-int-mult-distrib*[*of x inverse y m*] **unfolding** *power-int-inverse*
  by (*simp add*: *field-simps*)

end

lemma *power-int-add-numeral* [*simp*]:
  power-int x (*numeral m*) ∗ *power-int x* (*numeral n*) = *power-int x* (*numeral* (*m* + *n*))
  for *x* :: *'a* :: *division-ring*
  by (*simp add*: *power-int-add* [*symmetric*])

lemma *power-int-add-numeral2* [*simp*]:
  power-int x (*numeral m*) ∗ (*power-int x* (*numeral n*) ∗ *b*) = *power-int x* (*numeral* (*m* + *n*)) ∗ *b*
  for *x* :: *'a* :: *division-ring*
  by (*simp add*: *mult.assoc* [*symmetric*])

lemma *power-int-mult-numeral* [*simp*]:
  power-int (*power-int x* (*numeral m*)) (*numeral n*) = *power-int x* (*numeral* (*m* ∗ *n*))
  for *x* :: *'a* :: *division-ring*
  by (*simp only*: *numeral-mult power-int-mult*)

lemma *power-int-not-zero*: (*x* :: *'a* :: *division-ring*) ≠ *0* ∨ *n* = *0* ⟹ *power-int x n* ≠ *0*
  by (*subst power-int-eq-0-iff*) *auto*

lemma *power-int-one-over* [*field-simps*, *field-split-simps*, *divide-simps*]:
  power-int (*1* / *x* :: *'a* :: *division-ring*) *n* = *1* / *power-int x n*
  using *power-int-inverse*[*of x*] **by** (*simp add*: *divide-inverse*)

context
  assumes *SORT-CONSTRAINT*(*'a* :: *linordered-field*)
begin

lemma *power-int-numeral-neg-numeral* [*simp*]:
  power-int (*numeral m*) (−*numeral n*) = (*inverse* (*numeral* (*Num.pow m n*)) :: *'a*)
  by (*simp add*: *power-int-minus*)

lemma *zero-less-power-int* [*simp*]: *0* < (*x* :: *'a*) ⟹ *0* < *power-int x n*
  by (*auto simp*: *power-int-def*)

lemma *zero-le-power-int* [*simp*]: $0 \leq (x :: 'a) \Longrightarrow 0 \leq$ *power-int x n*
by (*auto simp*: *power-int-def*)

lemma *power-int-mono*: $(x :: 'a) \leq y \Longrightarrow n \geq 0 \Longrightarrow 0 \leq x \Longrightarrow$ *power-int x n* $\leq$ *power-int y n*
by (*cases n rule*: *int-cases4*) (*auto intro*: *power-mono*)

lemma *one-le-power-int* [*simp*]: $1 \leq (x :: 'a) \Longrightarrow n \geq 0 \Longrightarrow 1 \leq$ *power-int x n*
using *power-int-mono* [*of 1 x n*] **by** *simp*

lemma *power-int-le-one*: $0 \leq (x :: 'a) \Longrightarrow n \geq 0 \Longrightarrow x \leq 1 \Longrightarrow$ *power-int x n* $\leq$ *1*
using *power-int-mono* [*of x 1 n*] **by** *simp*

lemma *power-int-le-imp-le-exp*:
assumes *gt1*: $1 < (x :: 'a :: linordered\text{-}field)$
assumes *power-int x m* $\leq$ *power-int x n* $n \geq 0$
shows $m \leq n$
proof (*cases* $m < 0$)
case *True*
with ‹$n \geq 0$› **show** *?thesis* **by** *simp*
next
case *False*
with *assms* **have** $x \mathbin{\hat{}} nat\ m \leq x \mathbin{\hat{}} nat\ n$
by (*simp add*: *power-int-def*)
from *gt1* **and** *this* **show** *?thesis*
using *False* ‹$n \geq 0$› **by** *auto*
qed

lemma *power-int-le-imp-less-exp*:
assumes *gt1*: $1 < (x :: 'a :: linordered\text{-}field)$
assumes *power-int x m* $<$ *power-int x n* $n \geq 0$
shows $m < n$
proof (*cases* $m < 0$)
case *True*
with ‹$n \geq 0$› **show** *?thesis* **by** *simp*
next
case *False*
with *assms* **have** $x \mathbin{\hat{}} nat\ m < x \mathbin{\hat{}} nat\ n$
by (*simp add*: *power-int-def*)
from *gt1* **and** *this* **show** *?thesis*
using *False* ‹$n \geq 0$› **by** *auto*
qed

lemma *power-int-strict-mono*:
$(a :: 'a :: linordered\text{-}field) < b \Longrightarrow 0 \leq a \Longrightarrow 0 < n \Longrightarrow$ *power-int a n* $<$ *power-int b n*
by (*auto simp*: *power-int-def intro*!: *power-strict-mono*)

lemma *power-int-mono-iff* [*simp*]:
fixes *a b* :: *'a* :: *linordered-field*
shows $[\![a \geq 0;\ b \geq 0;\ n > 0]\!] \Longrightarrow$ *power-int a n* $\leq$ *power-int b n* $\longleftrightarrow a \leq b$
by (*auto simp*: *power-int-def intro*!: *power-strict-mono*)

lemma *power-int-strict-increasing*:
fixes *a* :: *'a* :: *linordered-field*
assumes $n < N$ $1 < a$
shows *power-int a N* > *power-int a n*
proof −
have ∗: *a* ^ *nat* (*N* − *n*) > *a* ^ *0*
using *assms* **by** (*intro power-strict-increasing*) *auto*
have *power-int a N* = *power-int a n* ∗ *power-int a* (*N* − *n*)
using *assms* **by** (*simp flip*: *power-int-add*)
also have . . . > *power-int a n* ∗ *1*
using *assms* ∗
by (*intro mult-strict-left-mono zero-less-power-int*) (*auto simp*: *power-int-def*)
finally show *?thesis* **by** *simp*
qed

lemma *power-int-increasing*:
fixes *a* :: *'a* :: *linordered-field*
assumes $n \leq N$ $a \geq 1$
shows *power-int a N* ≥ *power-int a n*
proof −
have ∗: *a* ^ *nat* (*N* − *n*) ≥ *a* ^ *0*
using *assms* **by** (*intro power-increasing*) *auto*
have *power-int a N* = *power-int a n* ∗ *power-int a* (*N* − *n*)
using *assms* **by** (*simp flip*: *power-int-add*)
also have . . . ≥ *power-int a n* ∗ *1*
using *assms* ∗ **by** (*intro mult-left-mono*) (*auto simp*: *power-int-def*)
finally show *?thesis* **by** *simp*
qed

lemma *power-int-strict-decreasing*:
fixes *a* :: *'a* :: *linordered-field*
assumes $n < N$ $0 < a$ $a < 1$
shows *power-int a N* < *power-int a n*
proof −
have ∗: *a* ^ *nat* (*N* − *n*) < *a* ^ *0*
using *assms* **by** (*intro power-strict-decreasing*) *auto*
have *power-int a N* = *power-int a n* ∗ *power-int a* (*N* − *n*)
using *assms* **by** (*simp flip*: *power-int-add*)
also have . . . < *power-int a n* ∗ *1*
using *assms* ∗
by (*intro mult-strict-left-mono zero-less-power-int*) (*auto simp*: *power-int-def*)
finally show *?thesis* **by** *simp*
qed

lemma *power-int-decreasing*:
fixes $a :: 'a :: linordered\text{-}field$
assumes $n \leq N$ $0 \leq a$ $a \leq 1$ $a \neq 0 \vee N \neq 0 \vee n = 0$
shows *power-int a N* $\leq$ *power-int a n*
proof (*cases a = 0*)
case *False*
have $*$: a ^ *nat* $(N - n) \leq a$ ^ *0*
using *assms* **by** (*intro power-decreasing*) *auto*
have *power-int a N* = *power-int a n* $*$ *power-int a* $(N - n)$
using *assms False* **by** (*simp flip*: *power-int-add*)
also have ... $\leq$ *power-int a n* $*$ *1*
using *assms* $*$ **by** (*intro mult-left-mono*) (*auto simp*: *power-int-def*)
finally show *?thesis* **by** *simp*
qed (*use assms* **in** ‹*auto simp*: *power-int-0-left-if*›)

lemma *one-less-power-int*: $1 < (a :: 'a) \Longrightarrow 0 < n \Longrightarrow 1 <$ *power-int a n*
using *power-int-strict-increasing*[*of 0 n a*] **by** *simp*

lemma *power-int-abs*: |*power-int a n* :: *'a*| = *power-int* |*a*| *n*
by (*auto simp*: *power-int-def power-abs*)

lemma *power-int-sgn* [*simp*]: *sgn* (*power-int a n* :: *'a*) = *power-int* (*sgn a*) *n*
by (*auto simp*: *power-int-def*)

lemma *abs-power-int-minus* [*simp*]: |*power-int* (− *a*) *n* :: *'a*| = |*power-int a n*|
by (*simp add*: *power-int-abs*)

lemma *power-int-strict-antimono*:
assumes $(a :: 'a :: linordered\text{-}field) < b$ $0 < a$ $n < 0$
shows *power-int a n* > *power-int b n*
proof −
have *inverse* (*power-int a* (−*n*)) > *inverse* (*power-int b* (−*n*))
using *assms* **by** (*intro less-imp-inverse-less power-int-strict-mono zero-less-power-int*) *auto*
thus *?thesis* **by** (*simp add*: *power-int-minus*)
qed

lemma *power-int-antimono*:
assumes $(a :: 'a :: linordered\text{-}field) \leq b$ $0 < a$ $n < 0$
shows *power-int a n* $\geq$ *power-int b n*
using *power-int-strict-antimono*[*of a b n*] *assms* **by** (*cases a = b*) *auto*

end

54.19 Finiteness of intervals

lemma *finite-interval-int1* [*iff*]: *finite* $\{i :: int.\ a \leq i \wedge i \leq b\}$
proof (*cases* $a \leq b$)

case *True*
then show *?thesis*
proof (*induct b rule*: *int-ge-induct*)
case *base*
have $\{i.\ a \le i \wedge i \le a\} = \{a\}$ **by** *auto*
then show *?case* **by** *simp*
next
case (*step b*)
then have $\{i.\ a \le i \wedge i \le b + 1\} = \{i.\ a \le i \wedge i \le b\} \cup \{b + 1\}$ **by** *auto*
with *step* **show** *?case* **by** *simp*
qed
next
case *False*
then show *?thesis*
by (*metis* (*lifting*, *no-types*) *Collect-empty-eq finite.emptyI order-trans*)
qed

lemma *finite-interval-int2* [*iff*]: *finite* $\{i :: int.\ a \le i \wedge i < b\}$
by (*rule rev-finite-subset*[*OF finite-interval-int1*[*of a b*]]) *auto*

lemma *finite-interval-int3* [*iff*]: *finite* $\{i :: int.\ a < i \wedge i \le b\}$
by (*rule rev-finite-subset*[*OF finite-interval-int1*[*of a b*]]) *auto*

lemma *finite-interval-int4* [*iff*]: *finite* $\{i :: int.\ a < i \wedge i < b\}$
by (*rule rev-finite-subset*[*OF finite-interval-int1*[*of a b*]]) *auto*

54.20 Configuration of the code generator

Constructors

definition *Pos* :: *num* ⇒ *int*
where [*simp*, *code-abbrev*]: *Pos* = *numeral*

definition *Neg* :: *num* ⇒ *int*
where [*simp*, *code-abbrev*]: *Neg n* = − (*Pos n*)

code-datatype *0*::*int Pos Neg*

Auxiliary operations.

definition *dup* :: *int* ⇒ *int*
where [*simp*]: *dup k* = *k* + *k*

lemma *dup-code* [*code*]:
dup 0 = *0*
dup (*Pos n*) = *Pos* (*Num.Bit0 n*)
dup (*Neg n*) = *Neg* (*Num.Bit0 n*)
by (*simp-all add*: *numeral-Bit0*)

definition *sub* :: *num* ⇒ *num* ⇒ *int*
where [*simp*]: *sub m n* = *numeral m* − *numeral n*

lemma *sub-code* [*code*]:
sub Num.One Num.One = 0
sub (*Num.Bit0 m*) *Num.One = Pos* (*Num.BitM m*)
sub (*Num.Bit1 m*) *Num.One = Pos* (*Num.Bit0 m*)
sub Num.One (*Num.Bit0 n*) = *Neg* (*Num.BitM n*)
sub Num.One (*Num.Bit1 n*) = *Neg* (*Num.Bit0 n*)
sub (*Num.Bit0 m*) (*Num.Bit0 n*) = *dup* (*sub m n*)
sub (*Num.Bit1 m*) (*Num.Bit1 n*) = *dup* (*sub m n*)
sub (*Num.Bit1 m*) (*Num.Bit0 n*) = *dup* (*sub m n*) + *1*
sub (*Num.Bit0 m*) (*Num.Bit1 n*) = *dup* (*sub m n*) − *1*
by (*simp-all only*: *sub-def dup-def numeral.simps Pos-def Neg-def numeral-BitM*)

lemma *sub-BitM-One-eq*:
‹(*Num.sub* (*Num.BitM n*) *num.One*) = *2* ∗ (*Num.sub n Num.One* :: *int*)›
by (*cases n*) *simp-all*

Implementations.

lemma *one-int-code* [*code*]: *1* = *Pos Num.One*
by *simp*

lemma *plus-int-code* [*code*]:
k + *0* = *k*
0 + *l* = *l*
Pos m + *Pos n* = *Pos* (*m* + *n*)
Pos m + *Neg n* = *sub m n*
Neg m + *Pos n* = *sub n m*
Neg m + *Neg n* = *Neg* (*m* + *n*)
for *k l* :: *int*
by *simp-all*

lemma *uminus-int-code* [*code*]:
uminus 0 = (*0*::*int*)
uminus (*Pos m*) = *Neg m*
uminus (*Neg m*) = *Pos m*
by *simp-all*

lemma *minus-int-code* [*code*]:
k − *0* = *k*
0 − *l* = *uminus l*
Pos m − *Pos n* = *sub m n*
Pos m − *Neg n* = *Pos* (*m* + *n*)
Neg m − *Pos n* = *Neg* (*m* + *n*)
Neg m − *Neg n* = *sub n m*
for *k l* :: *int*
by *simp-all*

lemma *times-int-code* [*code*]:
k ∗ *0* = *0*

0 $*$ *l* = *0*
Pos m $*$ *Pos n* = *Pos* (*m* $*$ *n*)
Pos m $*$ *Neg n* = *Neg* (*m* $*$ *n*)
Neg m $*$ *Pos n* = *Neg* (*m* $*$ *n*)
Neg m $*$ *Neg n* = *Pos* (*m* $*$ *n*)
for *k l* :: *int*
by *simp-all*

instantiation *int* :: *equal*
begin

definition *HOL.equal k l* ⟷ *k* = (*l*::*int*)

instance
by *standard* (*rule equal-int-def*)

end

lemma *equal-int-code* [*code*]:
HOL.equal 0 (*0*::*int*) ⟷ *True*
HOL.equal 0 (*Pos l*) ⟷ *False*
HOL.equal 0 (*Neg l*) ⟷ *False*
HOL.equal (*Pos k*) *0* ⟷ *False*
HOL.equal (*Pos k*) (*Pos l*) ⟷ *HOL.equal k l*
HOL.equal (*Pos k*) (*Neg l*) ⟷ *False*
HOL.equal (*Neg k*) *0* ⟷ *False*
HOL.equal (*Neg k*) (*Pos l*) ⟷ *False*
HOL.equal (*Neg k*) (*Neg l*) ⟷ *HOL.equal k l*
by (*auto simp add*: *equal*)

lemma *equal-int-refl* [*code nbe*]: *HOL.equal k k* ⟷ *True*
for *k* :: *int*
by (*fact equal-refl*)

lemma *less-eq-int-code* [*code*]:
0 ≤ (*0*::*int*) ⟷ *True*
0 ≤ *Pos l* ⟷ *True*
0 ≤ *Neg l* ⟷ *False*
Pos k ≤ *0* ⟷ *False*
Pos k ≤ *Pos l* ⟷ *k* ≤ *l*
Pos k ≤ *Neg l* ⟷ *False*
Neg k ≤ *0* ⟷ *True*
Neg k ≤ *Pos l* ⟷ *True*
Neg k ≤ *Neg l* ⟷ *l* ≤ *k*
by *simp-all*

lemma *less-int-code* [*code*]:
0 < (*0*::*int*) ⟷ *False*
0 < *Pos l* ⟷ *True*

0 < *Neg l* ⟷ *False*
Pos k < *0* ⟷ *False*
Pos k < *Pos l* ⟷ *k* < *l*
Pos k < *Neg l* ⟷ *False*
Neg k < *0* ⟷ *True*
Neg k < *Pos l* ⟷ *True*
Neg k < *Neg l* ⟷ *l* < *k*
by *simp-all*

lemma *nat-code* [*code*]:
nat (*Int.Neg k*) = *0*
nat 0 = *0*
nat (*Int.Pos k*) = *nat-of-num k*
by (*simp-all add*: *nat-of-num-numeral*)

lemma (**in** *ring-1*) *of-int-code* [*code*]:
of-int (*Int.Neg k*) = − *numeral k*
of-int 0 = *0*
of-int (*Int.Pos k*) = *numeral k*
by *simp-all*

Serializer setup.

code-identifier
code-module *Int* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

quickcheck-params [*default-type* = *int*]

hide-const (**open**) *Pos Neg sub dup*

De-register *int* as a quotient type:

lifting-update *int.lifting*
lifting-forget *int.lifting*

54.21 Duplicates

lemmas *int-sum* = *of-nat-sum* [**where** *'a*=*int*]
lemmas *int-prod* = *of-nat-prod* [**where** *'a*=*int*]
lemmas *zle-int* = *of-nat-le-iff* [**where** *'a*=*int*]
lemmas *int-int-eq* = *of-nat-eq-iff* [**where** *'a*=*int*]
lemmas *nonneg-eq-int* = *nonneg-int-cases*
lemmas *double-eq-0-iff* = *double-zero*

lemmas *int-distrib* =
distrib-right [*of z1 z2 w*]
distrib-left [*of w z1 z2*]
left-diff-distrib [*of z1 z2 w*]
right-diff-distrib [*of w z1 z2*]
for *z1 z2 w* :: *int*

end

55 Big infimum (minimum) and supremum (maximum) over finite (non-empty) sets

theory *Lattices-Big*
imports *Groups-Big Option*
begin

55.1 Generic lattice operations over a set

55.1.1 Without neutral element

locale *semilattice-set* = *semilattice*
begin

interpretation *comp-fun-idem f*
by *standard* (*simp-all add*: *fun-eq-iff left-commute*)

definition F :: $'a$ *set* $\Rightarrow$ $'a$
where
eq-fold': $F\ A$ = *the* (*Finite-Set.fold* ($\lambda x\ y$. *Some* (*case* y *of None* $\Rightarrow x$ | *Some* z $\Rightarrow f\ x\ z$)) *None* A)

lemma *eq-fold*:
assumes *finite* A
shows F (*insert* x A) = *Finite-Set.fold* $f\ x\ A$
proof (*rule sym*)
let *?f* = $\lambda x\ y$. *Some* (*case* y *of None* $\Rightarrow x$ | *Some* $z \Rightarrow f\ x\ z$)
interpret *comp-fun-idem ?f*
by *standard* (*simp-all add*: *fun-eq-iff commute left-commute split*: *option.split*)
from *assms* **show** *Finite-Set.fold* $f\ x\ A$ = F (*insert* x A)
proof *induct*
case *empty* **then show** *?case* **by** (*simp add*: *eq-fold'*)
next
case (*insert* y B) **then show** *?case* **by** (*simp add*: *insert-commute* [*of* x] *eq-fold'*)
qed
qed

lemma *singleton* [*simp*]:
F $\{x\}$ = x
by (*simp add*: *eq-fold*)

lemma *insert-not-elem*:
assumes *finite* A **and** $x \notin A$ **and** $A \neq \{\}$
shows F (*insert* x A) = $x * F\ A$
proof −
from ‹$A \neq \{\}$› **obtain** b **where** $b \in A$ **by** *blast*
then obtain B **where** $*$: A = *insert* b B $b \notin B$ **by** (*blast dest*: *mk-disjoint-insert*)

with ‹*finite A*› **and** ‹$x \notin A$›
have *finite* (*insert x B*) **and** $b \notin$ *insert x B* **by** *auto*
then have *F* (*insert b* (*insert x B*)) = $x * F$ (*insert b B*)
by (*simp add*: *eq-fold*)
then show *?thesis* **by** (*simp add*: ∗ *insert-commute*)
qed

lemma *in-idem*:
assumes *finite A* **and** $x \in A$
shows $x * F\ A = F\ A$
proof −
from *assms* **have** $A \neq \{\}$ **by** *auto*
with ‹*finite A*› **show** *?thesis* **using** ‹$x \in A$›
by (*induct A rule*: *finite-ne-induct*) (*auto simp add*: *ac-simps insert-not-elem*)
qed

lemma *insert* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows *F* (*insert x A*) = $x * F\ A$
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *insert-absorb in-idem insert-not-elem*)

lemma *union*:
assumes *finite A* $A \neq \{\}$ **and** *finite B* $B \neq \{\}$
shows $F\ (A \cup B) = F\ A * F\ B$
using *assms* **by** (*induct A rule*: *finite-ne-induct*) (*simp-all add*: *ac-simps*)

lemma *remove*:
assumes *finite A* **and** $x \in A$
shows $F\ A =$ (*if* $A - \{x\} = \{\}$ *then x else* $x * F\ (A - \{x\})$)
proof −
from *assms* **obtain** *B* **where** $A =$ *insert x B* **and** $x \notin B$ **by** (*blast dest*: *mk-disjoint-insert*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *insert-remove*:
assumes *finite A*
shows *F* (*insert x A*) = (*if* $A - \{x\} = \{\}$ *then x else* $x * F\ (A - \{x\})$)
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *insert-absorb remove*)

lemma *subset*:
assumes *finite A* $B \neq \{\}$ **and** $B \subseteq A$
shows $F\ B * F\ A = F\ A$
proof −
from *assms* **have** $A \neq \{\}$ **and** *finite B* **by** (*auto dest*: *finite-subset*)
with *assms* **show** *?thesis* **by** (*simp add*: *union* [*symmetric*] *Un-absorb1*)
qed

lemma *closed*:

assumes *finite A* $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x, y\}$
shows $F\ A \in A$
using ‹*finite A*› ‹$A \neq \{\}$› **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **then show** *?case* **by** *simp*
next
case *insert* **with** *elem* **show** *?case* **by** *force*
qed

lemma *hom-commute*:
assumes *hom*: $\bigwedge x\ y.\ h\ (x * y) = h\ x * h\ y$
and *N*: *finite N* $N \neq \{\}$
shows $h\ (F\ N) = F\ (h\ `\ N)$
using *N* **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
case (*insert n N*)
then have $h\ (F\ (insert\ n\ N)) = h\ (n * F\ N)$ **by** *simp*
also have $\ldots = h\ n * h\ (F\ N)$ **by** (*rule hom*)
also have $h\ (F\ N) = F\ (h\ `\ N)$ **by** (*rule insert*)
also have $h\ n * \ldots = F\ (insert\ (h\ n)\ (h\ `\ N))$
using *insert* **by** *simp*
also have $insert\ (h\ n)\ (h\ `\ N) = h\ `\ insert\ n\ N$ **by** *simp*
finally show *?case* **.**
qed

lemma *infinite*: $\neg$ *finite A* $\Longrightarrow F\ A =$ *the None*
unfolding *eq-fold′* **by** (*cases finite* (*UNIV*::′*a set*)) (*auto intro*: *finite-subset fold-infinite*)

end

locale *semilattice-order-set* = *binary?*: *semilattice-order* + *semilattice-set*
begin

lemma *bounded-iff*:
assumes *finite A* **and** $A \neq \{\}$
shows $x \leq F\ A \longleftrightarrow (\forall a{\in}A.\ x \leq a)$
using *assms* **by** (*induct rule*: *finite-ne-induct*) *simp-all*

lemma *boundedI*:
assumes *finite A*
assumes $A \neq \{\}$
assumes $\bigwedge a.\ a \in A \Longrightarrow x \leq a$
shows $x \leq F\ A$
using *assms* **by** (*simp add*: *bounded-iff*)

lemma *boundedE*:
assumes *finite A* **and** $A \neq \{\}$ **and** $x \leq F\ A$
obtains $\bigwedge a.\ a \in A \Longrightarrow x \leq a$

using *assms* **by** (*simp add*: *bounded-iff*)

lemma *coboundedI*:
assumes *finite A*
and $a \in A$
shows $F\ A \leq a$
proof −
from *assms* **have** $A \neq \{\}$ **by** *auto*
from ‹*finite A*› ‹$A \neq \{\}$› ‹$a \in A$› **show** *?thesis*
proof (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** (*simp add*: *refl*)
next
case (*insert x B*)
from *insert* **have** $a = x \vee a \in B$ **by** *simp*
then show *?case* **using** *insert* **by** (*auto intro*: *coboundedI2*)
qed
qed

lemma *subset-imp*:
assumes $A \subseteq B$ **and** $A \neq \{\}$ **and** *finite B*
shows $F\ B \leq F\ A$
proof (*cases* $A = B$)
case *True* **then show** *?thesis* **by** (*simp add*: *refl*)
next
case *False*
have *B*: $B = A \cup (B - A)$ **using** ‹$A \subseteq B$› **by** *blast*
then have $F\ B = F\ (A \cup (B - A))$ **by** *simp*
also have $\ldots = F\ A * F\ (B - A)$ **using** *False assms* **by** (*subst union*) (*auto intro*: *finite-subset*)
also have $\ldots \leq F\ A$ **by** *simp*
finally show *?thesis* **.**
qed

end

55.1.2 With neutral element

locale *semilattice-neutr-set* = *semilattice-neutr*
begin

interpretation *comp-fun-idem f*
by *standard* (*simp-all add*: *fun-eq-iff left-commute*)

definition F :: *'a set* ⇒ *'a*
where
eq-fold: $F\ A =$ *Finite-Set.fold f* **1** A

lemma *infinite* [*simp*]:
¬ *finite A* ⟹ $F\ A =$ **1**

by (*simp add*: *eq-fold*)

lemma *empty* [*simp*]:
F {} = **1**
by (*simp add*: *eq-fold*)

lemma *insert* [*simp*]:
assumes *finite A*
shows F (*insert x A*) $= x * F\ A$
using *assms* **by** (*simp add*: *eq-fold*)

lemma *in-idem*:
assumes *finite A* **and** $x \in A$
shows $x * F\ A = F\ A$
proof −
from *assms* **have** $A \neq \{\}$ **by** *auto*
with ‹*finite A*› **show** *?thesis* **using** ‹$x \in A$›
by (*induct A rule*: *finite-ne-induct*) (*auto simp add*: *ac-simps*)
qed

lemma *union*:
assumes *finite A* **and** *finite B*
shows $F\ (A \cup B) = F\ A * F\ B$
using *assms* **by** (*induct A*) (*simp-all add*: *ac-simps*)

lemma *remove*:
assumes *finite A* **and** $x \in A$
shows $F\ A = x * F\ (A - \{x\})$
proof −
from *assms* **obtain** B **where** $A =$ *insert x B* **and** $x \notin B$ **by** (*blast dest*: *mk-disjoint-insert*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *insert-remove*:
assumes *finite A*
shows F (*insert x A*) $= x * F\ (A - \{x\})$
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *insert-absorb remove*)

lemma *subset*:
assumes *finite A* **and** $B \subseteq A$
shows $F\ B * F\ A = F\ A$
proof −
from *assms* **have** *finite B* **by** (*auto dest*: *finite-subset*)
with *assms* **show** *?thesis* **by** (*simp add*: *union* [*symmetric*] *Un-absorb1*)
qed

lemma *closed*:
assumes *finite A* $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x, y\}$

shows *F A* ∈ *A*
using ‹*finite A*› ‹*A* ≠ {}› **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **then show** *?case* **by** *simp*
next
case *insert* **with** *elem* **show** *?case* **by** *force*
qed

end

locale *semilattice-order-neutr-set* = *binary?*: *semilattice-neutr-order* + *semilattice-neutr-set*
begin

lemma *bounded-iff*:
assumes *finite A*
shows $x \le F\ A \longleftrightarrow (\forall a{\in}A.\ x \le a)$
using *assms* **by** (*induct A*) *simp-all*

lemma *boundedI*:
assumes *finite A*
assumes $\bigwedge a.\ a \in A \Longrightarrow x \le a$
shows $x \le F\ A$
using *assms* **by** (*simp add*: *bounded-iff*)

lemma *boundedE*:
assumes *finite A* **and** $x \le F\ A$
obtains $\bigwedge a.\ a \in A \Longrightarrow x \le a$
using *assms* **by** (*simp add*: *bounded-iff*)

lemma *coboundedI*:
assumes *finite A*
and $a \in A$
shows $F\ A \le a$
proof −
from *assms* **have** *A* ≠ {} **by** *auto*
from ‹*finite A*› ‹*A* ≠ {}› ‹*a* ∈ *A*› **show** *?thesis*
proof (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** (*simp add*: *refl*)
next
case (*insert x B*)
from *insert* **have** $a = x \vee a \in B$ **by** *simp*
then show *?case* **using** *insert* **by** (*auto intro*: *coboundedI2*)
qed
qed

lemma *subset-imp*:
assumes $A \subseteq B$ **and** *finite B*
shows $F\ B \le F\ A$
proof (*cases A* = *B*)

case *True* **then show** *?thesis* **by** (*simp add*: *refl*)
next
case *False*
have *B*: $B = A \cup (B - A)$ **using** ‹$A \subseteq B$› **by** *blast*
then have $F\ B = F\ (A \cup (B - A))$ **by** *simp*
also have $\ldots = F\ A * F\ (B - A)$ **using** *False assms* **by** (*subst union*) (*auto intro*: *finite-subset*)
also have $\ldots \leq F\ A$ **by** *simp*
finally show *?thesis* **.**
qed

end

55.2 Lattice operations on finite sets

context *semilattice-inf*
begin

sublocale *Inf-fin*: *semilattice-order-set inf less-eq less*
defines
Inf-fin (‹$\bigsqcap_{fin}$ -› [*900*] *900*) = *Inf-fin.F* **..**

end

context *semilattice-sup*
begin

sublocale *Sup-fin*: *semilattice-order-set sup greater-eq greater*
defines
Sup-fin (‹$\bigsqcup_{fin}$ -› [*900*] *900*) = *Sup-fin.F* **..**

end

55.3 Infimum and Supremum over non-empty sets

context *lattice*
begin

lemma *Inf-fin-le-Sup-fin* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $\bigsqcap_{fin} A \leq \bigsqcup_{fin} A$
proof −
from ‹$A \neq \{\}$› **obtain** *a* **where** $a \in A$ **by** *blast*
with ‹*finite A*› **have** $\bigsqcap_{fin} A \leq a$ **by** (*rule Inf-fin.coboundedI*)
moreover from ‹*finite A*› ‹$a \in A$› **have** $a \leq \bigsqcup_{fin} A$ **by** (*rule Sup-fin.coboundedI*)
ultimately show *?thesis* **by** (*rule order-trans*)
qed

lemma *sup-Inf-absorb* [*simp*]:
finite A $\Longrightarrow a \in A \Longrightarrow \bigsqcap_{fin} A \sqcup a = a$

by (*rule sup-absorb2*) (*rule Inf-fin.coboundedI*)

lemma *inf-Sup-absorb* [*simp*]:
finite A $\Longrightarrow$ $a \in A$ $\Longrightarrow$ $a \sqcap \bigsqcup_{fin} A = a$
by (*rule inf-absorb1*) (*rule Sup-fin.coboundedI*)

end

context *distrib-lattice*
begin

lemma *sup-Inf1-distrib*:
assumes *finite A*
and $A \neq \{\}$
shows $sup\ x\ (\bigsqcap_{fin} A) = \bigsqcap_{fin}\{sup\ x\ a | a.\ a \in A\}$
using *assms* **by** (*simp add*: *image-def Inf-fin.hom-commute* [**where** *h*=*sup x*, *OF sup-inf-distrib1*])
(*rule arg-cong* [**where** *f*=*Inf-fin*], *blast*)

lemma *sup-Inf2-distrib*:
assumes *A*: *finite A* $A \neq \{\}$ **and** *B*: *finite B* $B \neq \{\}$
shows $sup\ (\bigsqcap_{fin} A)\ (\bigsqcap_{fin} B) = \bigsqcap_{fin}\{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
using *A* **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **then show** *?case*
by (*simp add*: *sup-Inf1-distrib* [*OF B*])
next
case (*insert x A*)
have *finB*: *finite* $\{sup\ x\ b\ | b.\ b \in B\}$
by (*rule finite-surj* [**where** *f* = *sup x*, *OF B*(*1*)], *auto*)
have *finAB*: *finite* $\{sup\ a\ b\ | a\ b.\ a \in A \wedge b \in B\}$
proof −
have $\{sup\ a\ b\ | a\ b.\ a \in A \wedge b \in B\} = (\bigcup a \in A.\ \bigcup b \in B.\ \{sup\ a\ b\})$
by *blast*
thus *?thesis* **by**(*simp add*: *insert*(*1*) *B*(*1*))
qed
have *ne*: $\{sup\ a\ b\ | a\ b.\ a \in A \wedge b \in B\} \neq \{\}$ **using** *insert B* **by** *blast*
have $sup\ (\bigsqcap_{fin}(insert\ x\ A))\ (\bigsqcap_{fin} B) = sup\ (inf\ x\ (\bigsqcap_{fin} A))\ (\bigsqcap_{fin} B)$
using *insert* **by** *simp*
also have $\ldots = inf\ (sup\ x\ (\bigsqcap_{fin} B))\ (sup\ (\bigsqcap_{fin} A)\ (\bigsqcap_{fin} B))$ **by**(*rule sup-inf-distrib2*)
also have $\ldots = inf\ (\bigsqcap_{fin}\{sup\ x\ b | b.\ b \in B\})\ (\bigsqcap_{fin}\{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\})$
using *insert* **by**(*simp add*:*sup-Inf1-distrib*[*OF B*])
also have $\ldots = \bigsqcap_{fin}(\{sup\ x\ b\ | b.\ b \in B\} \cup \{sup\ a\ b\ | a\ b.\ a \in A \wedge b \in B\})$
(**is** - = $\bigsqcap_{fin}$*?M*)
using *B insert*
by (*simp add*: *Inf-fin.union* [*OF finB - finAB ne*])
also have *?M* = $\{sup\ a\ b\ | a\ b.\ a \in insert\ x\ A \wedge b \in B\}$
by *blast*
finally show *?case* **.**

qed

lemma *inf-Sup1-distrib*:
  assumes *finite A* **and** $A \neq \{\}$
  shows $inf\ x\ (\bigsqcup_{fin} A) = \bigsqcup_{fin}\{inf\ x\ a|a.\ a \in A\}$
using *assms* **by** (*simp add*: *image-def Sup-fin.hom-commute* [**where** *h=inf x*, *OF inf-sup-distrib1*])
  (*rule arg-cong* [**where** *f=Sup-fin*], *blast*)

lemma *inf-Sup2-distrib*:
  assumes *A*: *finite A* $A \neq \{\}$ **and** *B*: *finite B* $B \neq \{\}$
  shows $inf\ (\bigsqcup_{fin} A)\ (\bigsqcup_{fin} B) = \bigsqcup_{fin}\{inf\ a\ b|a\ b.\ a \in A \wedge b \in B\}$
using *A* **proof** (*induct rule*: *finite-ne-induct*)
  case *singleton* **thus** *?case*
    by(*simp add*: *inf-Sup1-distrib* [*OF B*])
next
  case (*insert x A*)
  have *finB*: *finite* $\{inf\ x\ b\ |b.\ b \in B\}$
    by(*rule finite-surj*[**where** *f* = %*b*. *inf x b*, *OF B*(*1*)], *auto*)
  have *finAB*: *finite* $\{inf\ a\ b\ |a\ b.\ a \in A \wedge b \in B\}$
  proof −
    have $\{inf\ a\ b\ |a\ b.\ a \in A \wedge b \in B\} = (\bigcup a{\in}A.\ \bigcup b{\in}B.\ \{inf\ a\ b\})$
      by *blast*
    thus *?thesis* **by**(*simp add*: *insert*(*1*) *B*(*1*))
  qed
  have *ne*: $\{inf\ a\ b\ |a\ b.\ a \in A \wedge b \in B\} \neq \{\}$ **using** *insert B* **by** *blast*
  have $inf\ (\bigsqcup_{fin}(insert\ x\ A))\ (\bigsqcup_{fin} B) = inf\ (sup\ x\ (\bigsqcup_{fin} A))\ (\bigsqcup_{fin} B)$
    using *insert* **by** *simp*
  also have $\ldots = sup\ (inf\ x\ (\bigsqcup_{fin} B))\ (inf\ (\bigsqcup_{fin} A)\ (\bigsqcup_{fin} B))$ **by**(*rule inf-sup-distrib2*)
  also have $\ldots = sup\ (\bigsqcup_{fin}\{inf\ x\ b|b.\ b \in B\})\ (\bigsqcup_{fin}\{inf\ a\ b|a\ b.\ a \in A \wedge b \in B\})$
    using *insert* **by**(*simp add*:*inf-Sup1-distrib*[*OF B*])
  also have $\ldots = \bigsqcup_{fin}(\{inf\ x\ b\ |b.\ b \in B\} \cup \{inf\ a\ b\ |a\ b.\ a \in A \wedge b \in B\})$
    (**is** - = $\bigsqcup_{fin}$ *?M*)
    using *B insert*
    by (*simp add*: *Sup-fin.union* [*OF finB - finAB ne*])
  also have *?M* = $\{inf\ a\ b\ |a\ b.\ a \in insert\ x\ A \wedge b \in B\}$
    by *blast*
  finally show *?case* **.**
qed

end

context *complete-lattice*
begin

lemma *Inf-fin-Inf*:
  assumes *finite A* **and** $A \neq \{\}$
  shows $\bigsqcap_{fin} A = \bigsqcap A$

proof −
  from *assms* **obtain** *b B* **where** *A* = *insert b B* **and** *finite B* **by** *auto*
  then show *?thesis*
    by (*simp add*: *Inf-fin.eq-fold inf-Inf-fold-inf inf.commute* [*of b*])
qed

lemma *Sup-fin-Sup*:
  assumes *finite A* **and** $A \neq \{\}$
  shows $\bigsqcup_{fin} A = \bigsqcup A$
proof −
  from *assms* **obtain** *b B* **where** *A* = *insert b B* **and** *finite B* **by** *auto*
  then show *?thesis*
    by (*simp add*: *Sup-fin.eq-fold sup-Sup-fold-sup sup.commute* [*of b*])
qed

end

55.4 Minimum and Maximum over non-empty sets

context *linorder*
begin

sublocale *Min*: *semilattice-order-set min less-eq less*
  + *Max*: *semilattice-order-set max greater-eq greater*
defines
  Min = *Min.F* **and** *Max* = *Max.F* **..**

end

syntax
  -MIN1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder MIN››MIN -./ -)› [*0*, *10*] *10*)
  -MIN :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder MIN››MIN -∈-./ -)› [*0*, *0*, *10*] *10*)
  -MAX1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder MAX››MAX -./ -)› [*0*, *10*] *10*)
  -MAX :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder MAX››MAX -∈-./ -)› [*0*, *0*, *10*] *10*)

syntax-consts
  -MIN1 -MIN ⇌ *Min* **and**
  -MAX1 -MAX ⇌ *Max*

translations
  MIN x y. f ⇌ *MIN x. MIN y. f*
  MIN x. f ⇌ *CONST Min* (*CONST range* (λ*x. f*))
  MIN x∈*A. f* ⇌ *CONST Min* ((λ*x. f*) ‘ *A*)
  MAX x y. f ⇌ *MAX x. MAX y. f*
  MAX x. f ⇌ *CONST Max* (*CONST range* (λ*x. f*))

MAX x∈A. f ⇌ *CONST Max* ((λ*x*. *f*) ‘ *A*)

An aside: *Min*/*Max* on linear orders as special case of *Inf-fin*/*Sup-fin*

lemma *Inf-fin-Min*:
Inf-fin = (*Min* :: ′*a*::{*semilattice-inf*, *linorder*} *set* ⇒ ′*a*)
by (*simp add*: *Inf-fin-def Min-def inf-min*)

lemma *Sup-fin-Max*:
Sup-fin = (*Max* :: ′*a*::{*semilattice-sup*, *linorder*} *set* ⇒ ′*a*)
by (*simp add*: *Sup-fin-def Max-def sup-max*)

context *linorder*
begin

lemma *dual-min*:
ord.min greater-eq = *max*
by (*auto simp add*: *ord.min-def max-def fun-eq-iff*)

lemma *dual-max*:
ord.max greater-eq = *min*
by (*auto simp add*: *ord.max-def min-def fun-eq-iff*)

lemma *dual-Min*:
linorder.Min greater-eq = *Max*
proof −
interpret *dual*: *linorder greater-eq greater* **by** (*fact dual-linorder*)
show *?thesis* **by** (*simp add*: *dual.Min-def dual-min Max-def*)
qed

lemma *dual-Max*:
linorder.Max greater-eq = *Min*
proof −
interpret *dual*: *linorder greater-eq greater* **by** (*fact dual-linorder*)
show *?thesis* **by** (*simp add*: *dual.Max-def dual-max Min-def*)
qed

lemmas *Min-singleton* = *Min.singleton*
lemmas *Max-singleton* = *Max.singleton*
lemmas *Min-insert* = *Min.insert*
lemmas *Max-insert* = *Max.insert*
lemmas *Min-Un* = *Min.union*
lemmas *Max-Un* = *Max.union*
lemmas *hom-Min-commute* = *Min.hom-commute*
lemmas *hom-Max-commute* = *Max.hom-commute*

lemma *Min-in* [*simp*]:
assumes *finite A* **and** *A* ≠ {}
shows *Min A* ∈ *A*
using *assms* **by** (*auto simp add*: *min-def Min.closed*)

lemma *Max-in* [*simp*]:
  assumes *finite A* **and** $A \neq \{\}$
  shows *Max* $A \in A$
  using *assms* **by** (*auto simp add*: *max-def Max.closed*)

lemma *Min-insert2*:
  assumes *finite A* **and** *min*: $\bigwedge b.\ b \in A \Longrightarrow a \leq b$
  shows *Min* (*insert a A*) = *a*
proof (*cases* $A = \{\}$)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  with ‹*finite A*› **have** *Min* (*insert a A*) = *min a* (*Min A*)
    by *simp*
  moreover from ‹*finite A*› ‹$A \neq \{\}$› *min* **have** $a \leq$ *Min A* **by** *simp*
  ultimately show *?thesis* **by** (*simp add*: *min.absorb1*)
qed

lemma *Max-insert2*:
  assumes *finite A* **and** *max*: $\bigwedge b.\ b \in A \Longrightarrow b \leq a$
  shows *Max* (*insert a A*) = *a*
proof (*cases* $A = \{\}$)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  with ‹*finite A*› **have** *Max* (*insert a A*) = *max a* (*Max A*)
    by *simp*
  moreover from ‹*finite A*› ‹$A \neq \{\}$› *max* **have** *Max A* $\leq a$ **by** *simp*
  ultimately show *?thesis* **by** (*simp add*: *max.absorb1*)
qed

lemma *Max-const*[*simp*]: ⟦ *finite A*; $A \neq \{\}$ ⟧ $\Longrightarrow$ *Max* ((λ-. *c*) ‘ *A*) = *c*
using *Max-in image-is-empty* **by** *blast*

lemma *Min-const*[*simp*]: ⟦ *finite A*; $A \neq \{\}$ ⟧ $\Longrightarrow$ *Min* ((λ-. *c*) ‘ *A*) = *c*
using *Min-in image-is-empty* **by** *blast*

lemma *Min-le* [*simp*]:
  assumes *finite A* **and** $x \in A$
  shows *Min A* $\leq x$
  using *assms* **by** (*fact Min.coboundedI*)

lemma *Max-ge* [*simp*]:
  assumes *finite A* **and** $x \in A$
  shows $x \leq$ *Max A*
  using *assms* **by** (*fact Max.coboundedI*)

lemma *Min-eqI*:
  assumes *finite A*
  assumes $\bigwedge y.\ y \in A \Longrightarrow y \geq x$
    and $x \in A$
  shows $Min\ A = x$
proof (*rule order.antisym*)
  from ‹$x \in A$› **have** $A \neq \{\}$ **by** *auto*
  with *assms* **show** $Min\ A \geq x$ **by** *simp*
next
  from *assms* **show** $x \geq Min\ A$ **by** *simp*
qed

lemma *Max-eqI*:
  assumes *finite A*
  assumes $\bigwedge y.\ y \in A \Longrightarrow y \leq x$
    and $x \in A$
  shows $Max\ A = x$
proof (*rule order.antisym*)
  from ‹$x \in A$› **have** $A \neq \{\}$ **by** *auto*
  with *assms* **show** $Max\ A \leq x$ **by** *simp*
next
  from *assms* **show** $x \leq Max\ A$ **by** *simp*
qed

lemma *eq-Min-iff*:
  $[\![$ *finite A*; $A \neq \{\}$ $]\!] \Longrightarrow m = Min\ A \longleftrightarrow m \in A \land (\forall a \in A.\ m \leq a)$
by (*meson Min-in Min-le order.antisym*)

lemma *Min-eq-iff*:
  $[\![$ *finite A*; $A \neq \{\}$ $]\!] \Longrightarrow Min\ A = m \longleftrightarrow m \in A \land (\forall a \in A.\ m \leq a)$
by (*meson Min-in Min-le order.antisym*)

lemma *eq-Max-iff*:
  $[\![$ *finite A*; $A \neq \{\}$ $]\!] \Longrightarrow m = Max\ A \longleftrightarrow m \in A \land (\forall a \in A.\ a \leq m)$
by (*meson Max-in Max-ge order.antisym*)

lemma *Max-eq-iff*:
  $[\![$ *finite A*; $A \neq \{\}$ $]\!] \Longrightarrow Max\ A = m \longleftrightarrow m \in A \land (\forall a \in A.\ a \leq m)$
by (*meson Max-in Max-ge order.antisym*)

context
  fixes $A :: 'a\ set$
  assumes *fin-nonempty*: *finite A* $A \neq \{\}$
begin

lemma *Min-ge-iff* [*simp*]:
  $x \leq Min\ A \longleftrightarrow (\forall a{\in}A.\ x \leq a)$
  using *fin-nonempty* **by** (*fact Min.bounded-iff*)

lemma *Max-le-iff* [*simp*]:
Max A $\leq$ *x* $\longleftrightarrow$ ($\forall$ *a*$\in$*A*. *a* $\leq$ *x*)
using *fin-nonempty* **by** (*fact Max.bounded-iff*)

lemma *Min-gr-iff* [*simp*]:
x $<$ *Min A* $\longleftrightarrow$ ($\forall$ *a*$\in$*A*. *x* $<$ *a*)
using *fin-nonempty* **by** (*induct rule*: *finite-ne-induct*) *simp-all*

lemma *Max-less-iff* [*simp*]:
Max A $<$ *x* $\longleftrightarrow$ ($\forall$ *a*$\in$*A*. *a* $<$ *x*)
using *fin-nonempty* **by** (*induct rule*: *finite-ne-induct*) *simp-all*

lemma *Min-le-iff*:
Min A $\leq$ *x* $\longleftrightarrow$ ($\exists$ *a*$\in$*A*. *a* $\leq$ *x*)
using *fin-nonempty* **by** (*induct rule*: *finite-ne-induct*) (*simp-all add*: *min-le-iff-disj*)

lemma *Max-ge-iff*:
x $\leq$ *Max A* $\longleftrightarrow$ ($\exists$ *a*$\in$*A*. *x* $\leq$ *a*)
using *fin-nonempty* **by** (*induct rule*: *finite-ne-induct*) (*simp-all add*: *le-max-iff-disj*)

lemma *Min-less-iff*:
Min A $<$ *x* $\longleftrightarrow$ ($\exists$ *a*$\in$*A*. *a* $<$ *x*)
using *fin-nonempty* **by** (*induct rule*: *finite-ne-induct*) (*simp-all add*: *min-less-iff-disj*)

lemma *Max-gr-iff*:
x $<$ *Max A* $\longleftrightarrow$ ($\exists$ *a*$\in$*A*. *x* $<$ *a*)
using *fin-nonempty* **by** (*induct rule*: *finite-ne-induct*) (*simp-all add*: *less-max-iff-disj*)

end

Handy results about *Max* and *Min* by Chelsea Edmonds

lemma *obtains-Max*:
assumes *finite A* **and** *A* $\neq$ {}
obtains *x* **where** *x* $\in$ *A* **and** *Max A* = *x*
using *assms Max-in* **by** *blast*

lemma *obtains-MAX*:
assumes *finite A* **and** *A* $\neq$ {}
obtains *x* **where** *x* $\in$ *A* **and** *Max* (*f* ‘ *A*) = *f x*
using *obtains-Max*
by (*metis* (*mono-tags*, *opaque-lifting*) *assms*(*1*) *assms*(*2*) *empty-is-image finite-imageI image-iff*)

lemma *obtains-Min*:
assumes *finite A* **and** *A* $\neq$ {}
obtains *x* **where** *x* $\in$ *A* **and** *Min A* = *x*
using *assms Min-in* **by** *blast*

lemma *obtains-MIN*:
assumes *finite A* **and** $A \neq \{\}$
obtains x **where** $x \in A$ **and** *Min* (f ' A) = $f\ x$
using *obtains-Min assms empty-is-image finite-imageI image-iff*
by (*metis* (*mono-tags*, *opaque-lifting*))

lemma *Max-eq-if*:
assumes *finite A* *finite B* $\forall a{\in}A.\ \exists b{\in}B.\ a \leq b$ $\forall b{\in}B.\ \exists a{\in}A.\ b \leq a$
shows *Max A* = *Max B*
proof *cases*
assume $A = \{\}$ **thus** *?thesis* **using** *assms* **by** *simp*
next
assume $A \neq \{\}$ **thus** *?thesis* **using** *assms*
by(*blast intro*: *order.antisym Max-in Max-ge-iff* [*THEN iffD2*])
qed

lemma *Min-antimono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows *Min N* $\leq$ *Min M*
using *assms* **by** (*fact Min.subset-imp*)

lemma *Max-mono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows *Max M* $\leq$ *Max N*
using *assms* **by** (*fact Max.subset-imp*)

lemma *mono-Min-commute*:
assumes *mono f*
assumes *finite A* **and** $A \neq \{\}$
shows f (*Min A*) = *Min* (f ' A)
proof (*rule linorder-class.Min-eqI* [*symmetric*])
from ‹*finite A*› **show** *finite* (f ' A) **by** *simp*
from *assms* **show** f (*Min A*) $\in f$ ' A **by** *simp*
fix x
assume $x \in f$ ' A
then obtain y **where** $y \in A$ **and** $x = f\ y$ **..**
with *assms* **have** *Min A* $\leq y$ **by** *auto*
with ‹*mono f*› **have** f (*Min A*) $\leq f\ y$ **by** (*rule monoE*)
with ‹$x = f\ y$› **show** f (*Min A*) $\leq x$ **by** *simp*
qed

lemma *mono-Max-commute*:
assumes *mono f*
assumes *finite A* **and** $A \neq \{\}$
shows f (*Max A*) = *Max* (f ' A)
proof (*rule linorder-class.Max-eqI* [*symmetric*])
from ‹*finite A*› **show** *finite* (f ' A) **by** *simp*
from *assms* **show** f (*Max A*) $\in f$ ' A **by** *simp*
fix x

assume $x \in f$ ' A
then obtain y **where** $y \in A$ **and** $x = f\ y$ **..**
with *assms* **have** $y \leq Max\ A$ **by** *auto*
with ‹*mono f*› **have** $f\ y \leq f\ (Max\ A)$ **by** (*rule monoE*)
with ‹$x = f\ y$› **show** $x \leq f\ (Max\ A)$ **by** *simp*
qed

lemma *finite-linorder-max-induct* [*consumes 1*, *case-names empty insert*]:
assumes *fin*: *finite A*
and *empty*: $P\ \{\}$
and *insert*: $\bigwedge b\ A.\ finite\ A \Longrightarrow \forall a \in A.\ a < b \Longrightarrow P\ A \Longrightarrow P\ (insert\ b\ A)$
shows $P\ A$
using *fin empty insert*
proof (*induct rule*: *finite-psubset-induct*)
case (*psubset A*)
have *IH*: $\bigwedge B.\ [\![B < A;\ P\ \{\};\ (\bigwedge A\ b.\ [\![finite\ A;\ \forall a \in A.\ a<b;\ P\ A]\!] \Longrightarrow P\ (insert\ b\ A))]\!] \Longrightarrow P\ B$ **by** *fact*
have *fin*: *finite A* **by** *fact*
have *empty*: $P\ \{\}$ **by** *fact*
have *step*: $\bigwedge b\ A.\ [\![finite\ A;\ \forall a \in A.\ a < b;\ P\ A]\!] \Longrightarrow P\ (insert\ b\ A)$ **by** *fact*
show $P\ A$
proof (*cases* $A = \{\}$)
assume $A = \{\}$
then show $P\ A$ **using** ‹$P\ \{\}$› **by** *simp*
next
let $?B = A - \{Max\ A\}$
let $?A = insert\ (Max\ A)\ ?B$
have *finite ?B* **using** ‹*finite A*› **by** *simp*
assume $A \neq \{\}$
with ‹*finite A*› **have** $Max\ A \in A$ **by** *auto*
then have A: $?A = A$ **using** *insert-Diff-single insert-absorb* **by** *auto*
then have $P\ ?B$ **using** ‹$P\ \{\}$› *step IH* [*of ?B*] **by** *blast*
moreover
have $\forall a \in ?B.\ a < Max\ A$ **using** *Max-ge* [*OF* ‹*finite A*›] **by** *fastforce*
ultimately show $P\ A$ **using** A *insert-Diff-single step* [*OF* ‹*finite ?B*›] **by** *fastforce*
qed
qed

lemma *finite-linorder-min-induct* [*consumes 1*, *case-names empty insert*]:
$[\![finite\ A;\ P\ \{\};\ \bigwedge b\ A.\ [\![finite\ A;\ \forall a \in A.\ b < a;\ P\ A]\!] \Longrightarrow P\ (insert\ b\ A)]\!] \Longrightarrow P\ A$
by (*rule linorder.finite-linorder-max-induct* [*OF dual-linorder*])

lemma *finite-ranking-induct*[*consumes 1*, *case-names empty insert*]:
fixes $f :: 'b \Rightarrow 'a$
assumes *finite S*
assumes $P\ \{\}$
assumes $\bigwedge x\ S.\ finite\ S \Longrightarrow (\bigwedge y.\ y \in S \Longrightarrow f\ y \leq f\ x) \Longrightarrow P\ S \Longrightarrow P\ (insert\ x$

S)
  shows *P S*
  using ‹*finite S*›
proof (*induction rule*: *finite-psubset-induct*)
  case (*psubset A*)
  {
    assume $A \neq \{\}$
    hence f ‘ $A \neq \{\}$ **and** *finite* (f ‘ A)
      using *psubset finite-image-iff* **by** *simp+*
    then obtain *a* **where** $f\ a = Max\ (f$ ‘ $A)$ **and** $a \in A$
      by (*metis Max-in*[*of f* ‘ *A*] *imageE*)
    then have $P\ (A - \{a\})$
      using *psubset member-remove* **by** *blast*
    moreover
    have $\bigwedge y.\ y \in A \Longrightarrow f\ y \leq f\ a$
      using ‹$f\ a = Max\ (f$ ‘ $A)$› ‹*finite* (f ‘ A)› **by** *simp*
    ultimately
    have *?case*
      by (*metis* ‹$a \in A$› *DiffD1 insert-Diff assms*(*3*) *finite-Diff psubset.hyps*)
  }
  thus *?case*
    using *assms*(*2*) **by** *blast*
qed

lemma *Least-Min*:
  assumes *finite* $\{a.\ P\ a\}$ **and** $\exists\, a.\ P\ a$
  shows $(LEAST\ a.\ P\ a) = Min\ \{a.\ P\ a\}$
proof −
  { fix A :: $'a\ set$
    assume *A*: *finite A* $A \neq \{\}$
    have $(LEAST\ a.\ a \in A) = Min\ A$
    using *A* **proof** (*induct A rule*: *finite-ne-induct*)
      case *singleton* **show** *?case* **by** (*rule Least-equality*) *simp-all*
    next
      case (*insert a A*)
      have $(LEAST\ b.\ b = a \vee b \in A) = min\ a\ (LEAST\ a.\ a \in A)$
        by (*auto intro*!: *Least-equality simp add*: *min-def not-le Min-le-iff insert.hyps dest*!: *less-imp-le*)
      with *insert* **show** *?case* **by** *simp*
    qed
  } from *this* [*of* $\{a.\ P\ a\}$] *assms* **show** *?thesis* **by** *simp*
qed

lemma *infinite-growing*:
  assumes $X \neq \{\}$
  assumes $*$: $\bigwedge x.\ x \in X \Longrightarrow \exists\, y{\in}X.\ y > x$
  shows $\neg$ *finite X*
proof
  assume *finite X*

  with ‹$X \neq \{\}$› **have** *Max X* $\in$ *X* $\forall x \in X.\ x \leq$ *Max X*
    by *auto*
  with $*$[*of Max X*] **show** *False*
    by *auto*
qed

end

lemma *sum-le-card-Max*: *finite A* $\Longrightarrow$ *sum f A* $\leq$ *card A* $*$ *Max* (*f* ‘ *A*)
using *sum-bounded-above*[*of A f Max* (*f* ‘ *A*)] **by** *simp*

lemma *card-Min-le-sum*: *finite A* $\Longrightarrow$ *card A* $*$ *Min* (*f* ‘ *A*) $\leq$ *sum f A*
using *sum-bounded-below*[*of A Min* (*f* ‘ *A*) *f*] **by** *simp*

context *linordered-ab-semigroup-add*
begin

lemma *Min-add-commute*:
  fixes *k*
  assumes *finite S* **and** $S \neq \{\}$
  shows *Min* (($\lambda x.\ f\ x + k$) ‘ *S*) = *Min*(*f* ‘ *S*) + *k*
proof −
  have *m*: $\bigwedge x\ y.\ min\ x\ y + k = min\ (x{+}k)\ (y{+}k)$
    by (*simp add*: *min-def order.antisym add-right-mono*)
  have ($\lambda x.\ f\ x + k$) ‘ *S* = ($\lambda y.\ y + k$) ‘ (*f* ‘ *S*) **by** *auto*
  also have *Min* ... = *Min* (*f* ‘ *S*) + *k*
    using *assms hom-Min-commute* [*of* $\lambda y.\ y{+}k$ *f* ‘ *S*, *OF m*, *symmetric*] **by** *simp*
  finally show *?thesis* **by** *simp*
qed

lemma *Max-add-commute*:
  fixes *k*
  assumes *finite S* **and** $S \neq \{\}$
  shows *Max* (($\lambda x.\ f\ x + k$) ‘ *S*) = *Max*(*f* ‘ *S*) + *k*
proof −
  have *m*: $\bigwedge x\ y.\ max\ x\ y + k = max\ (x{+}k)\ (y{+}k)$
    by (*simp add*: *max-def order.antisym add-right-mono*)
  have ($\lambda x.\ f\ x + k$) ‘ *S* = ($\lambda y.\ y + k$) ‘ (*f* ‘ *S*) **by** *auto*
  also have *Max* ... = *Max* (*f* ‘ *S*) + *k*
    using *assms hom-Max-commute* [*of* $\lambda y.\ y{+}k$ *f* ‘ *S*, *OF m*, *symmetric*] **by** *simp*
  finally show *?thesis* **by** *simp*
qed

end

context *linordered-ab-group-add*
begin

lemma *minus-Max-eq-Min* [*simp*]:

finite S ⟹ *S* ≠ {} ⟹ − *Max S* = *Min* (*uminus* ' *S*)
by (*induct S rule*: *finite-ne-induct*) (*simp-all add*: *minus-max-eq-min*)

lemma *minus-Min-eq-Max* [*simp*]:
finite S ⟹ *S* ≠ {} ⟹ − *Min S* = *Max* (*uminus* ' *S*)
by (*induct S rule*: *finite-ne-induct*) (*simp-all add*: *minus-min-eq-max*)

end

context *complete-linorder*
begin

lemma *Min-Inf*:
assumes *finite A* **and** *A* ≠ {}
shows *Min A* = *Inf A*
proof −
from *assms* **obtain** *b B* **where** *A* = *insert b B* **and** *finite B* **by** *auto*
then show *?thesis*
by (*simp add*: *Min.eq-fold complete-linorder-inf-min* [*symmetric*] *inf-Inf-fold-inf* *inf.commute* [*of b*])
qed

lemma *Max-Sup*:
assumes *finite A* **and** *A* ≠ {}
shows *Max A* = *Sup A*
proof −
from *assms* **obtain** *b B* **where** *A* = *insert b B* **and** *finite B* **by** *auto*
then show *?thesis*
by (*simp add*: *Max.eq-fold complete-linorder-sup-max* [*symmetric*] *sup-Sup-fold-sup* *sup.commute* [*of b*])
qed

end

lemma *disjnt-ge-max*:
‹*disjnt X Y*› **if** ‹*finite Y*› ‹⋀*x*. *x* ∈ *X* ⟹ *x* > *Max Y*›
using *that* **by** (*auto simp add*: *disjnt-def*) (*use Max-less-iff* **in** *blast*)

55.5 Arg Min

context *ord*
begin

definition *is-arg-min* :: ('*b* ⇒ '*a*) ⇒ ('*b* ⇒ *bool*) ⇒ '*b* ⇒ *bool* **where**
is-arg-min f P x = (*P x* ∧ ¬(∃ *y*. *P y* ∧ *f y* < *f x*))

definition *arg-min* :: ('*b* ⇒ '*a*) ⇒ ('*b* ⇒ *bool*) ⇒ '*b* **where**
arg-min f P = (*SOME x*. *is-arg-min f P x*)

definition *arg-min-on* :: $('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'b$ **where**
arg-min-on f S = *arg-min f* $(\lambda x.\ x \in S)$

end

syntax
 -arg-min :: $('b \Rightarrow 'a) \Rightarrow pttrn \Rightarrow bool \Rightarrow 'b$
  (‹(‹indent=3 notation=‹binder ARG-MIN››ARG′-MIN - -./ -)› [1000, 0, 10]
10)
syntax-consts
 -arg-min ⇌ *arg-min*
translations
 ARG-MIN f x. P ⇌ *CONST arg-min f* $(\lambda x.\ P)$

lemma *is-arg-min-linorder*: **fixes** f :: $'a \Rightarrow 'b$ :: *linorder*
shows *is-arg-min f P x* = $(P\ x \wedge (\forall y.\ P\ y \longrightarrow f\ x \leq f\ y))$
by(*auto simp add*: *is-arg-min-def*)

lemma *is-arg-min-antimono*: **fixes** f :: $'a \Rightarrow ('b::order)$
shows $[\![$ *is-arg-min f P x*; $f\ y \leq f\ x$; $P\ y$ $]\!] \Longrightarrow$ *is-arg-min f P y*
by (*simp add*: *order.order-iff-strict is-arg-min-def*)

lemma *arg-minI*:
 $[\![$ $P\ x$;
  $\bigwedge y.\ P\ y \Longrightarrow \neg\ f\ y < f\ x$;
  $\bigwedge x.\ [\![\ P\ x;\ \forall y.\ P\ y \longrightarrow \neg\ f\ y < f\ x\ ]\!] \Longrightarrow Q\ x$ $]\!]$
 $\Longrightarrow Q$ (*arg-min f P*)
 unfolding *arg-min-def is-arg-min-def*
 by (*blast intro*!: *someI2-ex*)

lemma *arg-min-equality*:
 $[\![\ P\ k;\ \bigwedge x.\ P\ x \Longrightarrow f\ k \leq f\ x\ ]\!] \Longrightarrow f$ (*arg-min f P*) $= f\ k$
 for f :: - ⇒ *'a::order*
 by (*rule arg-minI*; *force simp*: *not-less less-le-not-le*)

lemma *wf-linord-ex-has-least*:
 $[\![\ wf\ r;\ \forall x\ y.\ (x,\ y) \in r^+ \longleftrightarrow (y,\ x) \notin r^*;\ P\ k\ ]\!]$
  $\Longrightarrow \exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow (m\ x,\ m\ y) \in r^*)$
  by (*force dest*!: *wf-trancl* [*THEN wf-eq-minimal* [*THEN iffD1*, *THEN spec*],
where $x = m$ ‘ *Collect P*])

lemma *ex-has-least-nat*: $P\ k \Longrightarrow \exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow m\ x \leq m\ y)$
 for m :: $'a \Rightarrow nat$
 unfolding *pred-nat-trancl-eq-le* [*symmetric*]
 apply (*rule wf-pred-nat* [*THEN wf-linord-ex-has-least*])
  apply (*simp add*: *less-eq linorder-not-le pred-nat-trancl-eq-le*)
 by *assumption*

lemma *arg-min-nat-lemma*:

$P\ k \Longrightarrow P(\textit{arg-min}\ m\ P) \wedge (\forall\, y.\ P\ y \longrightarrow m\ (\textit{arg-min}\ m\ P) \leq m\ y)$
for $m :: 'a \Rightarrow nat$
unfolding *arg-min-def is-arg-min-linorder*
apply (*rule someI-ex*)
apply (*erule ex-has-least-nat*)
done

lemmas *arg-min-natI* = *arg-min-nat-lemma* [*THEN conjunct1*]

lemma *is-arg-min-arg-min-nat*: **fixes** $m :: 'a \Rightarrow nat$
shows $P\ x \Longrightarrow \textit{is-arg-min}\ m\ P\ (\textit{arg-min}\ m\ P)$
by (*metis arg-min-nat-lemma is-arg-min-linorder*)

lemma *arg-min-nat-le*: $P\ x \Longrightarrow m\ (\textit{arg-min}\ m\ P) \leq m\ x$
for $m :: 'a \Rightarrow nat$
by (*rule arg-min-nat-lemma* [*THEN conjunct2*, *THEN spec*, *THEN mp*])

lemma *ex-min-if-finite*:
$[\![\ \textit{finite}\ S;\ S \neq \{\}\ ]\!] \Longrightarrow \exists\, m{\in}S.\ \neg(\exists\, x{\in}S.\ x < (m{::}'a{::}order))$
by(*induction rule*: *finite.induct*) (*auto intro*: *order.strict-trans*)

lemma *ex-is-arg-min-if-finite*: **fixes** $f :: 'a \Rightarrow 'b :: order$
shows $[\![\ \textit{finite}\ S;\ S \neq \{\}\ ]\!] \Longrightarrow \exists\, x.\ \textit{is-arg-min}\ f\ (\lambda x.\ x \in S)\ x$
unfolding *is-arg-min-def*
using *ex-min-if-finite*[*of f ‘ S*]
by *auto*

lemma *arg-min-SOME-Min*:
$\textit{finite}\ S \Longrightarrow \textit{arg-min-on}\ f\ S = (\textit{SOME}\ y.\ y \in S \wedge f\ y = \textit{Min}(f\ `\ S))$
unfolding *arg-min-on-def arg-min-def is-arg-min-linorder*
apply(*rule arg-cong*[**where** *f* = *Eps*])
apply (*auto simp*: *fun-eq-iff intro*: *Min-eqI*[*symmetric*])
done

lemma *arg-min-if-finite*: **fixes** $f :: 'a \Rightarrow 'b :: order$
assumes *finite S* $S \neq \{\}$
shows $\textit{arg-min-on}\ f\ S \in S$ **and** $\neg(\exists\, x{\in}S.\ f\ x < f\ (\textit{arg-min-on}\ f\ S))$
using *ex-is-arg-min-if-finite*[*OF assms*, *of f*]
unfolding *arg-min-on-def arg-min-def is-arg-min-def*
by(*auto dest*!: *someI-ex*)

lemma *arg-min-least*: **fixes** $f :: 'a \Rightarrow 'b :: linorder$
shows $[\![\ \textit{finite}\ S;\ \ S \neq \{\};\ \ y \in S\ ]\!] \Longrightarrow f(\textit{arg-min-on}\ f\ S) \leq f\ y$
by(*simp add*: *arg-min-SOME-Min inv-into-def2*[*symmetric*] *f-inv-into-f*)

lemma *arg-min-inj-eq*: **fixes** $f :: 'a \Rightarrow 'b :: order$
shows $[\![\ \textit{inj-on}\ f\ \{x.\ P\ x\};\ P\ a;\ \forall\, y.\ P\ y \longrightarrow f\ a \leq f\ y\ ]\!] \Longrightarrow \textit{arg-min}\ f\ P = a$
apply(*simp add*: *arg-min-def is-arg-min-def*)
apply(*rule someI2*[*of - a*])

apply (*simp add*: *less-le-not-le*)
by (*metis inj-on-eq-iff less-le mem-Collect-eq*)

55.6 Arg Max

context *ord*
begin

definition *is-arg-max* :: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'b \Rightarrow bool$ **where**
$is\text{-}arg\text{-}max\ f\ P\ x = (P\ x \wedge \neg(\exists\, y.\ P\ y \wedge f\ y > f\ x))$

definition *arg-max* :: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'b$ **where**
$arg\text{-}max\ f\ P = (SOME\ x.\ is\text{-}arg\text{-}max\ f\ P\ x)$

definition *arg-max-on* :: $('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'b$ **where**
$arg\text{-}max\text{-}on\ f\ S = arg\text{-}max\ f\ (\lambda x.\ x \in S)$

end

syntax
-*arg-max* :: $('b \Rightarrow 'a) \Rightarrow pttrn \Rightarrow bool \Rightarrow 'a$
(‹(‹indent=3 notation=‹binder ARG-MAX››ARG′-MAX - -./ -)› [1000, 0, 10] 10)
syntax-consts
-*arg-max* ⇌ *arg-max*
translations
ARG-MAX f x. P ⇌ *CONST arg-max f* (λ*x. P*)

lemma *is-arg-max-linorder*: **fixes** $f :: 'a \Rightarrow 'b :: linorder$
shows $is\text{-}arg\text{-}max\ f\ P\ x = (P\ x \wedge (\forall\, y.\ P\ y \longrightarrow f\ x \geq f\ y))$
by(*auto simp add*: *is-arg-max-def*)

lemma *arg-maxI*:
$P\ x \Longrightarrow$
$(\bigwedge y.\ P\ y \Longrightarrow \neg\ f\ y > f\ x) \Longrightarrow$
$(\bigwedge x.\ P\ x \Longrightarrow \forall\, y.\ P\ y \longrightarrow \neg\ f\ y > f\ x \Longrightarrow Q\ x) \Longrightarrow$
$Q\ (arg\text{-}max\ f\ P)$
unfolding *arg-max-def is-arg-max-def*
by (*blast intro*!: *someI2-ex elim*:)

lemma *arg-max-equality*:
$[\![\ P\ k;\ \bigwedge x.\ P\ x \Longrightarrow f\ x \leq f\ k\ ]\!] \Longrightarrow f\ (arg\text{-}max\ f\ P) = f\ k$
for f :: - ⇒ *'a*::*order*
apply (*rule arg-maxI* [**where** $f = f$])
apply *assumption*
apply (*simp add*: *less-le-not-le*)
by (*metis le-less*)

lemma *ex-has-greatest-nat-lemma*:

$P\ k \Longrightarrow \forall x.\ P\ x \longrightarrow (\exists y.\ P\ y \wedge \neg f\ y \leq f\ x) \Longrightarrow \exists y.\ P\ y \wedge \neg f\ y < f\ k + n$
for $f :: 'a \Rightarrow nat$
by (*induct n*) (*force simp*: *le-Suc-eq*)+

lemma *ex-has-greatest-nat*:
assumes $P\ k$
and $\forall y.\ P\ y \longrightarrow (f{::}\ 'a \Rightarrow nat)\ y < b$
shows $\exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow f\ y \leq f\ x)$
proof (*rule ccontr*)
assume $\nexists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow f\ y \leq f\ x)$
then have $\forall x.\ P\ x \longrightarrow (\exists y.\ P\ y \wedge \neg f\ y \leq f\ x)$
by *auto*
then have $\exists y.\ P\ y \wedge \neg f\ y < f\ k + (b - f\ k)$
using *assms ex-has-greatest-nat-lemma*[*of P k f b* − *f k*]
by *blast*
then show *False*
using *assms* **by** *auto*
qed

lemma *arg-max-nat-lemma*:
$[\![\ P\ k;\ \ \forall y.\ P\ y \longrightarrow f\ y < b\ ]\!]$
$\Longrightarrow P\ (arg\text{-}max\ f\ P) \wedge (\forall y.\ P\ y \longrightarrow f\ y \leq f\ (arg\text{-}max\ f\ P))$
for $f :: 'a \Rightarrow nat$
unfolding *arg-max-def is-arg-max-linorder*
by (*rule someI-ex*) (*metis ex-has-greatest-nat*)

lemmas *arg-max-natI* = *arg-max-nat-lemma* [*THEN conjunct1*]

lemma *arg-max-nat-le*: $P\ x \Longrightarrow \forall y.\ P\ y \longrightarrow f\ y < b \Longrightarrow f\ x \leq f\ (arg\text{-}max\ f\ P)$
for $f :: 'a \Rightarrow nat$
using *arg-max-nat-lemma* **by** *metis*

end

56 Division in euclidean (semi)rings

theory *Euclidean-Rings*
imports *Int Lattices-Big*
begin

56.1 Euclidean (semi)rings with explicit division and remainder

class *euclidean-semiring* = *semidom-modulo* +
fixes *euclidean-size* :: $'a \Rightarrow nat$
assumes *size-0* [*simp*]: *euclidean-size 0* = *0*
assumes *mod-size-less*:
$b \neq 0 \Longrightarrow$ *euclidean-size* (*a mod b*) < *euclidean-size b*
assumes *size-mult-mono*:

  b ≠ *0* ⟹ *euclidean-size a* ≤ *euclidean-size* (*a* ∗ *b*)
begin

lemma *euclidean-size-eq-0-iff* [*simp*]:
 euclidean-size b = *0* ⟷ *b* = *0*
proof
 assume *b* = *0*
 then show *euclidean-size b* = *0*
  by *simp*
next
 assume *euclidean-size b* = *0*
 show *b* = *0*
 proof (*rule ccontr*)
  assume *b* ≠ *0*
  with *mod-size-less* **have** *euclidean-size* (*b mod b*) < *euclidean-size b* **.**
  with ‹*euclidean-size b* = *0*› **show** *False*
   by *simp*
 qed
qed

lemma *euclidean-size-greater-0-iff* [*simp*]:
 euclidean-size b > *0* ⟷ *b* ≠ *0*
 using *euclidean-size-eq-0-iff* [*symmetric*, *of b*] **by** *safe simp*

lemma *size-mult-mono'*: *b* ≠ *0* ⟹ *euclidean-size a* ≤ *euclidean-size* (*b* ∗ *a*)
 by (*subst mult.commute*) (*rule size-mult-mono*)

lemma *dvd-euclidean-size-eq-imp-dvd*:
 assumes *a* ≠ *0* **and** *euclidean-size a* = *euclidean-size b*
  and *b dvd a*
 shows *a dvd b*
proof (*rule ccontr*)
 assume ¬ *a dvd b*
 hence *b mod a* ≠ *0* **using** *mod-0-imp-dvd* [*of b a*] **by** *blast*
 then have *b mod a* ≠ *0* **by** (*simp add*: *mod-eq-0-iff-dvd*)
 from ‹*b dvd a*› **have** *b dvd b mod a* **by** (*simp add*: *dvd-mod-iff*)
 then obtain *c* **where** *b mod a* = *b* ∗ *c* **unfolding** *dvd-def* **by** *blast*
  with ‹*b mod a* ≠ *0*› **have** *c* ≠ *0* **by** *auto*
 with ‹*b mod a* = *b* ∗ *c*› **have** *euclidean-size* (*b mod a*) ≥ *euclidean-size b*
  using *size-mult-mono* **by** *force*
 moreover from ‹¬ *a dvd b*› **and** ‹*a* ≠ *0*›
 have *euclidean-size* (*b mod a*) < *euclidean-size a*
  using *mod-size-less* **by** *blast*
 ultimately show *False* **using** ‹*euclidean-size a* = *euclidean-size b*›
  by *simp*
qed

lemma *euclidean-size-times-unit*:
 assumes *is-unit a*

shows *euclidean-size* $(a * b)$ = *euclidean-size* b
proof (*rule antisym*)
from *assms* **have** [*simp*]: $a \neq 0$ **by** *auto*
thus *euclidean-size* $(a * b) \geq$ *euclidean-size* b **by** (*rule size-mult-mono'*)
from *assms* **have** *is-unit* (*1 div a*) **by** *simp*
hence *1 div a* $\neq 0$ **by** (*intro notI*) *simp-all*
hence *euclidean-size* $(a * b) \leq$ *euclidean-size* ((*1 div a*) $*$ $(a * b)$)
by (*rule size-mult-mono'*)
also from *assms* **have** (*1 div a*) $*$ $(a * b) = b$
by (*simp add*: *algebra-simps unit-div-mult-swap*)
finally show *euclidean-size* $(a * b) \leq$ *euclidean-size* b **.**
qed

lemma *euclidean-size-unit*:
is-unit a $\Longrightarrow$ *euclidean-size a* = *euclidean-size 1*
using *euclidean-size-times-unit* [*of a 1*] **by** *simp*

lemma *unit-iff-euclidean-size*:
is-unit a $\longleftrightarrow$ *euclidean-size a* = *euclidean-size 1* $\wedge$ $a \neq 0$
proof *safe*
assume *A*: $a \neq 0$ **and** *B*: *euclidean-size a* = *euclidean-size 1*
show *is-unit a*
by (*rule dvd-euclidean-size-eq-imp-dvd* [*OF A B*]) *simp-all*
qed (*auto intro*: *euclidean-size-unit*)

lemma *euclidean-size-times-nonunit*:
assumes $a \neq 0$ $b \neq 0$ $\neg$ *is-unit a*
shows *euclidean-size b* < *euclidean-size* $(a * b)$
proof (*rule ccontr*)
assume $\neg$*euclidean-size b* < *euclidean-size* $(a * b)$
with *size-mult-mono'*[*OF assms*(*1*), *of b*]
have *eq*: *euclidean-size* $(a * b)$ = *euclidean-size b* **by** *simp*
have $a * b$ *dvd b*
by (*rule dvd-euclidean-size-eq-imp-dvd* [*OF - eq*])
(*use assms* **in** *simp-all*)
hence $a * b$ *dvd* $1 * b$ **by** *simp*
with ‹$b \neq 0$› **have** *is-unit a* **by** (*subst* (*asm*) *dvd-times-right-cancel-iff*)
with *assms*(*3*) **show** *False* **by** *contradiction*
qed

lemma *dvd-imp-size-le*:
assumes *a dvd b* $b \neq 0$
shows *euclidean-size a* $\leq$ *euclidean-size b*
using *assms* **by** (*auto simp*: *size-mult-mono*)

lemma *dvd-proper-imp-size-less*:
assumes *a dvd b* $\neg$ *b dvd a* $b \neq 0$
shows *euclidean-size a* < *euclidean-size b*
proof –

from *assms*(*1*) **obtain** *c* **where** *b* = *a* ∗ *c* **by** (*erule dvdE*)
hence *z*: *b* = *c* ∗ *a* **by** (*simp add*: *mult.commute*)
from *z assms* **have** ¬*is-unit c* **by** (*auto simp*: *mult.commute mult-unit-dvd-iff*)
with *z assms* **show** *?thesis*
by (*auto intro*!: *euclidean-size-times-nonunit*)
qed

lemma *unit-imp-mod-eq-0*:
a mod b = *0* **if** *is-unit b*
using *that* **by** (*simp add*: *mod-eq-0-iff-dvd unit-imp-dvd*)

lemma *mod-eq-self-iff-div-eq-0*:
a mod b = *a* ⟷ *a div b* = *0* (**is** *?P* ⟷ *?Q*)
proof
assume *?P*
with *div-mult-mod-eq* [*of a b*] **show** *?Q*
by *auto*
next
assume *?Q*
with *div-mult-mod-eq* [*of a b*] **show** *?P*
by *simp*
qed

lemma *coprime-mod-left-iff* [*simp*]:
coprime (*a mod b*) *b* ⟷ *coprime a b* **if** *b* ≠ *0*
by (*rule iffI*; *rule coprimeI*)
(*use that* **in** ‹*auto dest*!: *dvd-mod-imp-dvd coprime-common-divisor simp add*: *dvd-mod-iff*›)

lemma *coprime-mod-right-iff* [*simp*]:
coprime a (*b mod a*) ⟷ *coprime a b* **if** *a* ≠ *0*
using *that coprime-mod-left-iff* [*of a b*] **by** (*simp add*: *ac-simps*)

end

class *euclidean-ring* = *idom-modulo* + *euclidean-semiring*
begin

lemma *dvd-diff-commute* [*ac-simps*]:
a dvd c − *b* ⟷ *a dvd b* − *c*
proof −
have *a dvd c* − *b* ⟷ *a dvd* (*c* − *b*) ∗ − *1*
by (*subst dvd-mult-unit-iff*) *simp-all*
then show *?thesis*
by *simp*
qed

end

56.2 Euclidean (semi)rings with cancel rules

class *euclidean-semiring-cancel* = *euclidean-semiring* +
  assumes *div-mult-self1* [*simp*]: $b \neq 0 \Longrightarrow (a + c * b)$ *div* $b = c + a$ *div* b
  and *div-mult-mult1* [*simp*]: $c \neq 0 \Longrightarrow (c * a)$ *div* $(c * b) = a$ *div* b
begin

lemma *div-mult-self2* [*simp*]:
  assumes $b \neq 0$
  shows $(a + b * c)$ *div* $b = c + a$ *div* b
  using *assms div-mult-self1* [*of b a c*] **by** (*simp add*: *mult.commute*)

lemma *div-mult-self3* [*simp*]:
  assumes $b \neq 0$
  shows $(c * b + a)$ *div* $b = c + a$ *div* b
  using *assms* **by** (*simp add*: *add.commute*)

lemma *div-mult-self4* [*simp*]:
  assumes $b \neq 0$
  shows $(b * c + a)$ *div* $b = c + a$ *div* b
  using *assms* **by** (*simp add*: *add.commute*)

lemma *mod-mult-self1* [*simp*]: $(a + c * b)$ *mod* $b = a$ *mod* b
proof (*cases* $b = 0$)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False*
  have $a + c * b = (a + c * b)$ *div* $b * b + (a + c * b)$ *mod* b
    by (*simp add*: *div-mult-mod-eq*)
  also from *False div-mult-self1* [*of b a c*] **have**
    $\ldots = (c + a$ *div* $b) * b + (a + c * b)$ *mod* b
      by (*simp add*: *algebra-simps*)
  finally have $a = a$ *div* $b * b + (a + c * b)$ *mod* b
    by (*simp add*: *add.commute* [*of a*] *add.assoc distrib-right*)
  then have a *div* $b * b + (a + c * b)$ *mod* $b = a$ *div* $b * b + a$ *mod* b
    by (*simp add*: *div-mult-mod-eq*)
  then show *?thesis* **by** *simp*
qed

lemma *mod-mult-self2* [*simp*]:
  $(a + b * c)$ *mod* $b = a$ *mod* b
  by (*simp add*: *mult.commute* [*of b*])

lemma *mod-mult-self3* [*simp*]:
  $(c * b + a)$ *mod* $b = a$ *mod* b
  by (*simp add*: *add.commute*)

lemma *mod-mult-self4* [*simp*]:
  $(b * c + a)$ *mod* $b = a$ *mod* b
  by (*simp add*: *add.commute*)

lemma *mod-mult-self1-is-0* [*simp*]:
  b ∗ *a mod b* = *0*
  using *mod-mult-self2* [*of 0 b a*] **by** *simp*

lemma *mod-mult-self2-is-0* [*simp*]:
  a ∗ *b mod b* = *0*
  using *mod-mult-self1* [*of 0 a b*] **by** *simp*

lemma *div-add-self1*:
  assumes *b* ≠ *0*
  shows (*b* + *a*) *div b* = *a div b* + *1*
  using *assms div-mult-self1* [*of b a 1*] **by** (*simp add*: *add.commute*)

lemma *div-add-self2*:
  assumes *b* ≠ *0*
  shows (*a* + *b*) *div b* = *a div b* + *1*
  using *assms div-add-self1* [*of b a*] **by** (*simp add*: *add.commute*)

lemma *mod-add-self1* [*simp*]:
  (*b* + *a*) *mod b* = *a mod b*
  using *mod-mult-self1* [*of a 1 b*] **by** (*simp add*: *add.commute*)

lemma *mod-add-self2* [*simp*]:
  (*a* + *b*) *mod b* = *a mod b*
  using *mod-mult-self1* [*of a 1 b*] **by** *simp*

lemma *mod-div-trivial* [*simp*]:
  a mod b div b = *0*
proof (*cases b* = *0*)
  assume *b* = *0*
  thus *?thesis* **by** *simp*
next
  assume *b* ≠ *0*
  hence *a div b* + *a mod b div b* = (*a mod b* + *a div b* ∗ *b*) *div b*
    by (*rule div-mult-self1* [*symmetric*])
  also have ... = *a div b*
    by (*simp only*: *mod-div-mult-eq*)
  also have ... = *a div b* + *0*
    by *simp*
  finally show *?thesis*
    by (*rule add-left-imp-eq*)
qed

lemma *mod-mod-trivial* [*simp*]:
  a mod b mod b = *a mod b*
proof −
  have *a mod b mod b* = (*a mod b* + *a div b* ∗ *b*) *mod b*
    by (*simp only*: *mod-mult-self1*)

also have ... = *a mod b*
  by (*simp only*: *mod-div-mult-eq*)
finally show *?thesis* **.**
qed

lemma *mod-mod-cancel*:
  assumes *c dvd b*
  shows *a mod b mod c* = *a mod c*
proof −
  from ‹*c dvd b*› **obtain** *k* **where** *b* = *c* ∗ *k*
    by (*rule dvdE*)
  have *a mod b mod c* = *a mod* (*c* ∗ *k*) *mod c*
    by (*simp only*: ‹*b* = *c* ∗ *k*›)
  also have ... = (*a mod* (*c* ∗ *k*) + *a div* (*c* ∗ *k*) ∗ *k* ∗ *c*) *mod c*
    by (*simp only*: *mod-mult-self1*)
  also have ... = (*a div* (*c* ∗ *k*) ∗ (*c* ∗ *k*) + *a mod* (*c* ∗ *k*)) *mod c*
    by (*simp only*: *ac-simps*)
  also have ... = *a mod c*
    by (*simp only*: *div-mult-mod-eq*)
  finally show *?thesis* **.**
qed

lemma *div-mult-mult2* [*simp*]:
  c ≠ *0* ⟹ (*a* ∗ *c*) *div* (*b* ∗ *c*) = *a div b*
  by (*drule div-mult-mult1*) (*simp add*: *mult.commute*)

lemma *div-mult-mult1-if* [*simp*]:
  (*c* ∗ *a*) *div* (*c* ∗ *b*) = (*if c* = *0 then 0 else a div b*)
  by *simp-all*

lemma *mod-mult-mult1*:
  (*c* ∗ *a*) *mod* (*c* ∗ *b*) = *c* ∗ (*a mod b*)
proof (*cases c* = *0*)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False*
  from *div-mult-mod-eq*
  have ((*c* ∗ *a*) *div* (*c* ∗ *b*)) ∗ (*c* ∗ *b*) + (*c* ∗ *a*) *mod* (*c* ∗ *b*) = *c* ∗ *a* **.**
  with *False* **have** *c* ∗ ((*a div b*) ∗ *b* + *a mod b*) + (*c* ∗ *a*) *mod* (*c* ∗ *b*)
    = *c* ∗ *a* + *c* ∗ (*a mod b*) **by** (*simp add*: *algebra-simps*)
  with *div-mult-mod-eq* **show** *?thesis* **by** *simp*
qed

lemma *mod-mult-mult2*:
  (*a* ∗ *c*) *mod* (*b* ∗ *c*) = (*a mod b*) ∗ *c*
  using *mod-mult-mult1* [*of c a b*] **by** (*simp add*: *mult.commute*)

lemma *mult-mod-left*: (*a mod b*) ∗ *c* = (*a* ∗ *c*) *mod* (*b* ∗ *c*)
  by (*fact mod-mult-mult2* [*symmetric*])

lemma *mult-mod-right*: $c * (a \ mod \ b) = (c * a) \ mod \ (c * b)$
by (*fact mod-mult-mult1* [*symmetric*])

lemma *dvd-mod*: $k \ dvd \ m \Longrightarrow k \ dvd \ n \Longrightarrow k \ dvd \ (m \ mod \ n)$
unfolding *dvd-def* **by** (*auto simp add*: *mod-mult-mult1*)

lemma *div-plus-div-distrib-dvd-left*:
$c \ dvd \ a \Longrightarrow (a + b) \ div \ c = a \ div \ c + b \ div \ c$
by (*cases* $c = 0$) *auto*

lemma *div-plus-div-distrib-dvd-right*:
$c \ dvd \ b \Longrightarrow (a + b) \ div \ c = a \ div \ c + b \ div \ c$
using *div-plus-div-distrib-dvd-left* [*of c b a*]
by (*simp add*: *ac-simps*)

lemma *sum-div-partition*:
‹$(\sum a \in A. \ f \ a) \ div \ b = (\sum a \in A \cap \{a. \ b \ dvd \ f \ a\}. \ f \ a \ div \ b) + (\sum a \in A \cap \{a. \ \neg \ b \ dvd \ f \ a\}. \ f \ a) \ div \ b$›
if ‹*finite A*›
proof −
have ‹$A = A \cap \{a. \ b \ dvd \ f \ a\} \cup A \cap \{a. \ \neg \ b \ dvd \ f \ a\}$›
by *auto*
then have ‹$(\sum a \in A. \ f \ a) = (\sum a \in A \cap \{a. \ b \ dvd \ f \ a\} \cup A \cap \{a. \ \neg \ b \ dvd \ f \ a\}. \ f \ a)$›
by *simp*
also have ‹$\ldots = (\sum a \in A \cap \{a. \ b \ dvd \ f \ a\}. \ f \ a) + (\sum a \in A \cap \{a. \ \neg \ b \ dvd \ f \ a\}. \ f \ a)$›
using ‹*finite A*› **by** (*auto intro*: *sum.union-inter-neutral*)
finally have $*$: ‹$sum \ f \ A = sum \ f \ (A \cap \{a. \ b \ dvd \ f \ a\}) + sum \ f \ (A \cap \{a. \ \neg \ b \ dvd \ f \ a\})$› **.**
define *B* **where** *B*: ‹$B = A \cap \{a. \ b \ dvd \ f \ a\}$›
with ‹*finite A*› **have** ‹*finite B*› **and** ‹$a \in B \Longrightarrow b \ dvd \ f \ a$› **for** *a*
by *simp-all*
then have ‹$(\sum a \in B. \ f \ a) \ div \ b = (\sum a \in B. \ f \ a \ div \ b)$› **and** ‹$b \ dvd \ (\sum a \in B. \ f \ a)$›
by *induction* (*simp-all add*: *div-plus-div-distrib-dvd-left*)
then show *?thesis* **using** $*$
by (*simp add*: *B div-plus-div-distrib-dvd-left*)
qed

named-theorems *mod-simps*

Addition respects modular equivalence.

lemma *mod-add-left-eq* [*mod-simps*]:
$(a \ mod \ c + b) \ mod \ c = (a + b) \ mod \ c$
proof −
have $(a + b) \ mod \ c = (a \ div \ c * c + a \ mod \ c + b) \ mod \ c$
by (*simp only*: *div-mult-mod-eq*)
also have $\ldots = (a \ mod \ c + b + a \ div \ c * c) \ mod \ c$

```
    by (simp only: ac-simps)
  also have ... = (a mod c + b) mod c
    by (rule mod-mult-self1)
  finally show ?thesis
    by (rule sym)
qed

lemma mod-add-right-eq [mod-simps]:
  (a + b mod c) mod c = (a + b) mod c
  using mod-add-left-eq [of b c a] by (simp add: ac-simps)

lemma mod-add-eq:
  (a mod c + b mod c) mod c = (a + b) mod c
  by (simp add: mod-add-left-eq mod-add-right-eq)

lemma mod-sum-eq [mod-simps]:
  (∑ i∈A. f i mod a) mod a = sum f A mod a
proof (induct A rule: infinite-finite-induct)
  case (insert i A)
  then have (∑ i∈insert i A. f i mod a) mod a
    = (f i mod a + (∑ i∈A. f i mod a)) mod a
    by simp
  also have ... = (f i + (∑ i∈A. f i mod a) mod a) mod a
    by (simp add: mod-simps)
  also have ... = (f i + (∑ i∈A. f i) mod a) mod a
    by (simp add: insert.hyps)
  finally show ?case
    by (simp add: insert.hyps mod-simps)
qed simp-all

lemma mod-add-cong:
  assumes a mod c = a' mod c
  assumes b mod c = b' mod c
  shows (a + b) mod c = (a' + b') mod c
proof –
  have (a mod c + b mod c) mod c = (a' mod c + b' mod c) mod c
    unfolding assms ..
  then show ?thesis
    by (simp add: mod-add-eq)
qed
```

Multiplication respects modular equivalence.

```
lemma mod-mult-left-eq [mod-simps]:
  ((a mod c) * b) mod c = (a * b) mod c
proof –
  have (a * b) mod c = ((a div c * c + a mod c) * b) mod c
    by (simp only: div-mult-mod-eq)
  also have ... = (a mod c * b + a div c * b * c) mod c
    by (simp only: algebra-simps)
```

also have ... = (a *mod* $c * b$) *mod* c
  by (*rule mod-mult-self1*)
finally show *?thesis*
  by (*rule sym*)
qed

lemma *mod-mult-right-eq* [*mod-simps*]:
  (a * (b *mod* c)) *mod* c = (a * b) *mod* c
  using *mod-mult-left-eq* [*of* b c a] **by** (*simp add*: *ac-simps*)

lemma *mod-mult-eq*:
  ((a *mod* c) * (b *mod* c)) *mod* c = (a * b) *mod* c
  by (*simp add*: *mod-mult-left-eq mod-mult-right-eq*)

lemma *mod-prod-eq* [*mod-simps*]:
  ($\prod$ i∈A. f i *mod* a) *mod* a = *prod* f A *mod* a
proof (*induct A rule*: *infinite-finite-induct*)
  case (*insert i A*)
  then have ($\prod$ i∈*insert* i A. f i *mod* a) *mod* a
    = (f i *mod* a * ($\prod$ i∈A. f i *mod* a)) *mod* a
    by *simp*
  also have ... = (f i * (($\prod$ i∈A. f i *mod* a) *mod* a)) *mod* a
    by (*simp add*: *mod-simps*)
  also have ... = (f i * (($\prod$ i∈A. f i) *mod* a)) *mod* a
    by (*simp add*: *insert.hyps*)
  finally show *?case*
    by (*simp add*: *insert.hyps mod-simps*)
qed *simp-all*

lemma *mod-mult-cong*:
  assumes a *mod* c = a' *mod* c
  assumes b *mod* c = b' *mod* c
  shows (a * b) *mod* c = (a' * b') *mod* c
proof −
  have (a *mod* c * (b *mod* c)) *mod* c = (a' *mod* c * (b' *mod* c)) *mod* c
    unfolding *assms* **..**
  then show *?thesis*
    by (*simp add*: *mod-mult-eq*)
qed

Exponentiation respects modular equivalence.

lemma *power-mod* [*mod-simps*]:
  ((a *mod* b) ^ n) *mod* b = (a ^ n) *mod* b
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  have (a *mod* b) ^ *Suc* n *mod* b = (a *mod* b) * ((a *mod* b) ^ n *mod* b) *mod* b

    by (simp add: mod-mult-right-eq)
  with Suc show ?case
    by (simp add: mod-mult-left-eq mod-mult-right-eq)
qed

lemma *power-diff-power-eq*:
  ‹a ^ m div a ^ n = (if n ≤ m then a ^ (m − n) else 1 div a ^ (n − m))›
    if ‹a ≠ 0›
proof (cases ‹n ≤ m›)
  case True
  with that power-diff [symmetric, of a n m] **show** ?thesis **by** simp
next
  case False
  then obtain q **where** n: ‹n = m + Suc q›
    by (auto simp add: not-le dest: less-imp-Suc-add)
  then have ‹a ^ m div a ^ n = (a ^ m * 1) div (a ^ m * a ^ Suc q)›
    by (simp add: power-add ac-simps)
  moreover from that **have** ‹a ^ m ≠ 0›
    by simp
  ultimately have ‹a ^ m div a ^ n = 1 div a ^ Suc q›
    by (subst (asm) div-mult-mult1) simp
  with False n **show** ?thesis
    by simp
qed

end

class *euclidean-ring-cancel* = *euclidean-ring* + *euclidean-semiring-cancel*
begin

subclass *idom-divide* ..

lemma *div-minus-minus* [simp]: (− a) div (− b) = a div b
  using div-mult-mult1 [of − 1 a b] **by** simp

lemma *mod-minus-minus* [simp]: (− a) mod (− b) = − (a mod b)
  using mod-mult-mult1 [of − 1 a b] **by** simp

lemma *div-minus-right*: a div (− b) = (− a) div b
  using div-minus-minus [of − a b] **by** simp

lemma *mod-minus-right*: a mod (− b) = − ((− a) mod b)
  using mod-minus-minus [of − a b] **by** simp

lemma *div-minus1-right* [simp]: a div (− 1) = − a
  using div-minus-right [of a 1] **by** simp

lemma *mod-minus1-right* [simp]: a mod (− 1) = 0

using *mod-minus-right* [*of a 1*] **by** *simp*

Negation respects modular equivalence.

lemma *mod-minus-eq* [*mod-simps*]:
(− (*a mod b*)) *mod b* = (− *a*) *mod b*
proof −
have (− *a*) *mod b* = (− (*a div b* ∗ *b* + *a mod b*)) *mod b*
by (*simp only*: *div-mult-mod-eq*)
also have ... = (− (*a mod b*) + − (*a div b*) ∗ *b*) *mod b*
by (*simp add*: *ac-simps*)
also have ... = (− (*a mod b*)) *mod b*
by (*rule mod-mult-self1*)
finally show *?thesis*
by (*rule sym*)
qed

lemma *mod-minus-cong*:
assumes *a mod b* = *a′ mod b*
shows (− *a*) *mod b* = (− *a′*) *mod b*
proof −
have (− (*a mod b*)) *mod b* = (− (*a′ mod b*)) *mod b*
unfolding *assms* **..**
then show *?thesis*
by (*simp add*: *mod-minus-eq*)
qed

Subtraction respects modular equivalence.

lemma *mod-diff-left-eq* [*mod-simps*]:
(*a mod c* − *b*) *mod c* = (*a* − *b*) *mod c*
using *mod-add-cong* [*of a c a mod c* − *b* − *b*]
by *simp*

lemma *mod-diff-right-eq* [*mod-simps*]:
(*a* − *b mod c*) *mod c* = (*a* − *b*) *mod c*
using *mod-add-cong* [*of a c a* − *b* − (*b mod c*)] *mod-minus-cong* [*of b mod c c b*]
by *simp*

lemma *mod-diff-eq*:
(*a mod c* − *b mod c*) *mod c* = (*a* − *b*) *mod c*
using *mod-add-cong* [*of a c a mod c* − *b* − (*b mod c*)] *mod-minus-cong* [*of b mod c c b*]
by *simp*

lemma *mod-diff-cong*:
assumes *a mod c* = *a′ mod c*
assumes *b mod c* = *b′ mod c*
shows (*a* − *b*) *mod c* = (*a′* − *b′*) *mod c*
using *assms mod-add-cong* [*of a c a′* − *b* − *b′*] *mod-minus-cong* [*of b c b′*]
by *simp*

lemma *minus-mod-self2* [*simp*]:
  (*a* − *b*) *mod b* = *a mod b*
  using *mod-diff-right-eq* [*of a b b*]
  by (*simp add*: *mod-diff-right-eq*)

lemma *minus-mod-self1* [*simp*]:
  (*b* − *a*) *mod b* = − *a mod b*
  using *mod-add-self2* [*of* − *a b*] **by** *simp*

lemma *mod-eq-dvd-iff*:
  a mod c = *b mod c* ⟷ *c dvd a* − *b* (**is** *?P* ⟷ *?Q*)
proof
  assume *?P*
  then have (*a mod c* − *b mod c*) *mod c* = *0*
    by *simp*
  then show *?Q*
    by (*simp add*: *dvd-eq-mod-eq-0 mod-simps*)
next
  assume *?Q*
  then obtain *d* **where** *d*: *a* − *b* = *c* ∗ *d* **..**
  then have *a* = *c* ∗ *d* + *b*
    by (*simp add*: *algebra-simps*)
  then show *?P* **by** *simp*
qed

lemma *mod-eqE*:
  assumes *a mod c* = *b mod c*
  obtains *d* **where** *b* = *a* + *c* ∗ *d*
proof −
  from *assms* **have** *c dvd a* − *b*
    by (*simp add*: *mod-eq-dvd-iff*)
  then obtain *d* **where** *a* − *b* = *c* ∗ *d* **..**
  then have *b* = *a* + *c* ∗ − *d*
    by (*simp add*: *algebra-simps*)
  with *that* **show** *thesis* **.**
qed

lemma *invertible-coprime*:
  coprime a c **if** *a* ∗ *b mod c* = *1*
  by (*rule coprimeI*) (*use that dvd-mod-iff* [*of* - *c a* ∗ *b*] **in** *auto*)

end

56.3 Uniquely determined division

class *unique-euclidean-semiring* = *euclidean-semiring* +
  assumes *euclidean-size-mult*: ‹*euclidean-size* (*a* ∗ *b*) = *euclidean-size a* ∗ *euclidean-size b*›

fixes *division-segment* :: ‹'a ⇒ 'a›
assumes *is-unit-division-segment* [*simp*]: ‹*is-unit* (*division-segment a*)›
and *division-segment-mult*:
‹$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *division-segment* $(a * b) =$ *division-segment* $a *$ *division-segment* b›
and *division-segment-mod*:
‹$b \neq 0 \Longrightarrow \neg\ b$ *dvd* $a \Longrightarrow$ *division-segment* (a *mod* b) = *division-segment* b›
assumes *div-bounded*:
‹$b \neq 0 \Longrightarrow$ *division-segment* r = *division-segment* b
$\Longrightarrow$ *euclidean-size* $r <$ *euclidean-size* b
$\Longrightarrow (q * b + r)$ *div* $b = q$›
begin

lemma *division-segment-not-0* [*simp*]:
‹*division-segment* $a \neq 0$›
using *is-unit-division-segment* [*of a*] *is-unitE* [*of* ‹*division-segment a*›] **by** *blast*

lemma *euclidean-relationI* [*case-names by0 divides euclidean-relation*]:
‹(a *div* b, a *mod* b) = (q, r)›
if *by0*: ‹$b = 0 \Longrightarrow q = 0 \wedge r = a$›
and *divides*: ‹$b \neq 0 \Longrightarrow b$ *dvd* $a \Longrightarrow r = 0 \wedge a = q * b$›
and *euclidean-relation*: ‹$b \neq 0 \Longrightarrow \neg\ b$ *dvd* $a \Longrightarrow$ *division-segment* r = *division-segment* b
$\wedge$ *euclidean-size* $r <$ *euclidean-size* $b \wedge a = q * b + r$›
proof (*cases* ‹$b = 0$›)
case *True*
with *by0* **show** *?thesis*
by *simp*
next
case *False*
show *?thesis*
proof (*cases* ‹b *dvd* a›)
case *True*
with ‹$b \neq 0$› *divides*
show *?thesis*
by *simp*
next
case *False*
with ‹$b \neq 0$› *euclidean-relation*
have ‹*division-segment* r = *division-segment* b›
‹*euclidean-size* $r <$ *euclidean-size* b› ‹$a = q * b + r$›
by *simp-all*
from ‹$b \neq 0$› ‹*division-segment* r = *division-segment* b›
‹*euclidean-size* $r <$ *euclidean-size* b›
have ‹$(q * b + r)$ *div* $b = q$›
by (*rule div-bounded*)
with ‹$a = q * b + r$›
have ‹$q = a$ *div* b›
by *simp*

from ‹*a* = *q* ∗ *b* + *r*›
have ‹*a div b* ∗ *b* + *a mod b* = *q* ∗ *b* + *r*›
by (*simp add*: *div-mult-mod-eq*)
with ‹*q* = *a div b*›
have ‹*q* ∗ *b* + *a mod b* = *q* ∗ *b* + *r*›
by *simp*
then have ‹*r* = *a mod b*›
by *simp*
with ‹*q* = *a div b*›
show *?thesis*
by *simp*
qed
qed

subclass *euclidean-semiring-cancel*
proof
fix *a b c*
assume ‹*b* ≠ *0*›
have ‹((*a* + *c* ∗ *b*) *div b*, (*a* + *c* ∗ *b*) *mod b*) = (*c* + *a div b*, *a mod b*)›
proof (*induction rule*: *euclidean-relationI*)
case *by0*
with ‹*b* ≠ *0*›
show *?case*
by *simp*
next
case *divides*
then show *?case*
by (*simp add*: *algebra-simps dvd-add-left-iff*)
next
case *euclidean-relation*
then have ‹¬ *b dvd a*›
by (*simp add*: *dvd-add-left-iff*)
have ‹*a mod b* + (*b* ∗ *c* + *b* ∗ (*a div b*)) = *b* ∗ *c* + ((*a div b*) ∗ *b* + *a mod b*)›
by (*simp add*: *ac-simps*)
with ‹*b* ≠ *0*› **have** ∗: ‹*a mod b* + (*b* ∗ *c* + *b* ∗ (*a div b*)) = *b* ∗ *c* + *a*›
by (*simp add*: *div-mult-mod-eq*)
from ‹¬ *b dvd a*› *euclidean-relation* **show** *?case*
by (*simp-all add*: *algebra-simps division-segment-mod mod-size-less* ∗)
qed
then show ‹(*a* + *c* ∗ *b*) *div b* = *c* + *a div b*›
by *simp*
next
fix *a b c*
assume ‹*c* ≠ *0*›
have ‹((*c* ∗ *a*) *div* (*c* ∗ *b*), (*c* ∗ *a*) *mod* (*c* ∗ *b*)) = (*a div b*, *c* ∗ (*a mod b*))›
proof (*induction rule*: *euclidean-relationI*)
case *by0*
with ‹*c* ≠ *0*› **show** *?case*
by *simp*

next
case *divides*
then show *?case*
by (*auto simp add*: *algebra-simps*)
next
case *euclidean-relation*
then have ‹$b \neq 0$› ‹a *mod* $b \neq 0$›
by (*simp-all add*: *mod-eq-0-iff-dvd*)
have ‹$c * (a$ *mod* $b) + b * (c * (a$ *div* $b)) = c * ((a$ *div* $b) * b + a$ *mod* $b)$›
by (*simp add*: *algebra-simps*)
with ‹$b \neq 0$› **have** ∗: ‹$c * (a$ *mod* $b) + b * (c * (a$ *div* $b)) = c * a$›
by (*simp add*: *div-mult-mod-eq*)
from ‹$b \neq 0$› ‹$c \neq 0$› **have** ‹*euclidean-size* c ∗ *euclidean-size* (a *mod* b)
< *euclidean-size* c ∗ *euclidean-size* b›
using *mod-size-less* [*of b a*] **by** *simp*
with *euclidean-relation* ‹$b \neq 0$› ‹a *mod* $b \neq 0$› **show** *?case*
by (*simp add*: *algebra-simps division-segment-mult division-segment-mod euclidean-size-mult* ∗)
qed
then show ‹$(c * a)$ *div* $(c * b) = a$ *div* b›
by *simp*
qed

lemma *div-eq-0-iff*:
‹a *div* $b = 0 \longleftrightarrow$ *euclidean-size* $a <$ *euclidean-size* $b \vee b = 0$› (**is** - $\longleftrightarrow$ *?P*)
if ‹*division-segment* a = *division-segment* b›
proof (*cases* ‹$a = 0 \vee b = 0$›)
case *True*
then show *?thesis* **by** *auto*
next
case *False*
then have ‹$a \neq 0$› ‹$b \neq 0$›
by *simp-all*
have ‹a *div* $b = 0 \longleftrightarrow$ *euclidean-size* $a <$ *euclidean-size* b›
proof
assume ‹a *div* $b = 0$›
then have ‹a *mod* $b = a$›
using *div-mult-mod-eq* [*of a b*] **by** *simp*
with ‹$b \neq 0$› *mod-size-less* [*of b a*]
show ‹*euclidean-size* $a <$ *euclidean-size* b›
by *simp*
next
assume ‹*euclidean-size* $a <$ *euclidean-size* b›
have ‹$(a$ *div* b, a *mod* $b) = (0, a)$›
proof (*induction rule*: *euclidean-relationI*)
case *by0*
show *?case*
by *simp*
next

case *divides*
with ‹*euclidean-size a* < *euclidean-size b*› **show** *?case*
using *dvd-imp-size-le* [*of b a*] ‹*a* ≠ *0*› **by** *simp*
next
case *euclidean-relation*
with ‹*euclidean-size a* < *euclidean-size b*› *that*
show *?case*
by *simp*
qed
then show ‹*a div b* = *0*›
by *simp*
qed
with ‹*b* ≠ *0*› **show** *?thesis*
by *simp*
qed

lemma *div-mult1-eq*:
‹(*a* ∗ *b*) *div c* = *a* ∗ (*b div c*) + *a* ∗ (*b mod c*) *div c*›
proof −
have ∗: ‹(*a* ∗ *b*) *mod c* + (*a* ∗ (*c* ∗ (*b div c*)) + *c* ∗ (*a* ∗ (*b mod c*) *div c*)) = *a* ∗ *b*› (**is** ‹*?A* + (*?B* + *?C*) = -›)
proof −
have ‹*?A* = *a* ∗ (*b mod c*) *mod c*›
by (*simp add*: *mod-mult-right-eq*)
then have ‹*?C* + *?A* = *a* ∗ (*b mod c*)›
by (*simp add*: *mult-div-mod-eq*)
then have ‹*?B* + (*?C* + *?A*) = *a* ∗ (*c* ∗ (*b div c*) + (*b mod c*))›
by (*simp add*: *algebra-simps*)
also have ‹. . . = *a* ∗ *b*›
by (*simp add*: *mult-div-mod-eq*)
finally show *?thesis*
by (*simp add*: *algebra-simps*)
qed
have ‹((*a* ∗ *b*) *div c*, (*a* ∗ *b*) *mod c*) = (*a* ∗ (*b div c*) + *a* ∗ (*b mod c*) *div c*, (*a* ∗ *b*) *mod c*)›
proof (*induction rule*: *euclidean-relationI*)
case *by0*
then show *?case* **by** *simp*
next
case *divides*
with ∗ **show** *?case*
by (*simp add*: *algebra-simps*)
next
case *euclidean-relation*
with ∗ **show** *?case*
by (*simp add*: *division-segment-mod mod-size-less algebra-simps*)
qed
then show *?thesis*
by *simp*

qed

lemma *div-add1-eq*:
‹(*a* + *b*) *div c* = *a div c* + *b div c* + (*a mod c* + *b mod c*) *div c*›
proof −
have ∗: ‹(*a* + *b*) *mod c* + (*c* ∗ (*a div c*) + (*c* ∗ (*b div c*) + *c* ∗ ((*a mod c* + *b mod c*) *div c*))) = *a* + *b*›
(**is** ‹*?A* + (*?B* + (*?C* + *?D*)) = -›)
proof −
have ‹*?A* + (*?B* + (*?C* + *?D*)) = *?A* + *?D* + (*?B* + *?C*)›
by (*simp add*: *ac-simps*)
also have ‹*?A* + *?D* = (*a mod c* + *b mod c*) *mod c* + *?D*›
by (*simp add*: *mod-add-eq*)
also have ‹. . . = *a mod c* + *b mod c*›
by (*simp add*: *mod-mult-div-eq*)
finally have ‹*?A* + (*?B* + (*?C* + *?D*)) = (*a mod c* + *?B*) + (*b mod c* + *?C*)›
by (*simp add*: *ac-simps*)
then show *?thesis*
by (*simp add*: *mod-mult-div-eq*)
qed
have ‹((*a* + *b*) *div c*, (*a* + *b*) *mod c*) = (*a div c* + *b div c* + (*a mod c* + *b mod c*) *div c*, (*a* + *b*) *mod c*)›
proof (*induction rule*: *euclidean-relationI*)
case *by0*
then show *?case*
by *simp*
next
case *divides*
with ∗ **show** *?case*
by (*simp add*: *algebra-simps*)
next
case *euclidean-relation*
with ∗ **show** *?case*
by (*simp add*: *division-segment-mod mod-size-less algebra-simps*)
qed
then show *?thesis*
by *simp*
qed

end

class *unique-euclidean-ring* = *euclidean-ring* + *unique-euclidean-semiring*
begin

subclass *euclidean-ring-cancel* **..**

end

56.4 Division on *nat*

instantiation *nat* :: *normalization-semidom*
begin

definition *normalize-nat* :: ‹*nat* ⇒ *nat*›
where [*simp*]: ‹*normalize* = (*id* :: *nat* ⇒ *nat*)›

definition *unit-factor-nat* :: ‹*nat* ⇒ *nat*›
where ‹*unit-factor n* = *of-bool* (*n* > *0*)› **for** *n* :: *nat*

lemma *unit-factor-simps* [*simp*]:
‹*unit-factor 0* = (*0*::*nat*)›
‹*unit-factor* (*Suc n*) = *1*›
by (*simp-all add*: *unit-factor-nat-def*)

definition *divide-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m div n* = (*if n* = *0 then 0 else Max* {*k*. *k* ∗ *n* ≤ *m*})› **for** *m n* :: *nat*

instance
by *standard* (*auto simp add*: *divide-nat-def ac-simps unit-factor-nat-def intro*: *Max-eqI*)

end

lemma *coprime-Suc-0-left* [*simp*]:
coprime (*Suc 0*) *n*
using *coprime-1-left* [*of n*] **by** *simp*

lemma *coprime-Suc-0-right* [*simp*]:
coprime n (*Suc 0*)
using *coprime-1-right* [*of n*] **by** *simp*

lemma *coprime-common-divisor-nat*: *coprime a b* ⟹ *x dvd a* ⟹ *x dvd b* ⟹ *x* = *1*
for *a b* :: *nat*
by (*drule coprime-common-divisor* [*of - - x*]) *simp-all*

instantiation *nat* :: *unique-euclidean-semiring*
begin

definition *euclidean-size-nat* :: ‹*nat* ⇒ *nat*›
where [*simp*]: ‹*euclidean-size-nat* = *id*›

definition *division-segment-nat* :: ‹*nat* ⇒ *nat*›
where [*simp*]: ‹*division-segment n* = *1*› **for** *n* :: *nat*

definition *modulo-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m mod n* = *m* − (*m div n* ∗ *n*)› **for** *m n* :: *nat*

instance proof
  fix *m n* :: *nat*
  have *ex*: $\exists k.\ k * n \le l$ **for** *l* :: *nat*
    by (*rule exI* [*of* - *0*]) *simp*
  have *fin*: *finite* $\{k.\ k * n \le l\}$ **if** $n > 0$ **for** *l*
  proof −
    from *that* **have** $\{k.\ k * n \le l\} \subseteq \{k.\ k \le l\}$
      by (*cases n*) *auto*
    then show *?thesis*
      by (*rule finite-subset*) *simp*
  qed
  have *mult-div-unfold*: $n * (m\ div\ n) = Max\ \{l.\ l \le m \wedge n\ dvd\ l\}$
  proof (*cases* $n = 0$)
    case *True*
    moreover have $\{l.\ l = 0 \wedge l \le m\} = \{0::nat\}$
      by *auto*
    ultimately show *?thesis*
      by *simp*
  next
    case *False*
    with *ex* [*of m*] *fin* **have** $n * Max\ \{k.\ k * n \le m\} = Max\ (times\ n\ \text{`}\ \{k.\ k * n$
$\le m\})$
      by (*auto simp add*: *nat-mult-max-right intro*: *hom-Max-commute*)
    also have $times\ n\ \text{`}\ \{k.\ k * n \le m\} = \{l.\ l \le m \wedge n\ dvd\ l\}$
      by (*auto simp add*: *ac-simps elim*!: *dvdE*)
    finally show *?thesis*
      using *False* **by** (*simp add*: *divide-nat-def ac-simps*)
  qed
  have *less-eq*: $m\ div\ n * n \le m$
    by (*auto simp add*: *mult-div-unfold ac-simps intro*: *Max.boundedI*)
  then show $m\ div\ n * n + m\ mod\ n = m$
    by (*simp add*: *modulo-nat-def*)
  assume $n \neq 0$
  show *euclidean-size* (*m mod n*) < *euclidean-size n*
  proof −
    have $m < Suc\ (m\ div\ n) * n$
    proof (*rule ccontr*)
      assume $\neg\ m < Suc\ (m\ div\ n) * n$
      then have $Suc\ (m\ div\ n) * n \le m$
        by (*simp add*: *not-less*)
      moreover from ‹$n \neq 0$› **have** $Max\ \{k.\ k * n \le m\} < Suc\ (m\ div\ n)$
        by (*simp add*: *divide-nat-def*)
      with ‹$n \neq 0$› *ex fin* **have** $\bigwedge k.\ k * n \le m \Longrightarrow k < Suc\ (m\ div\ n)$
        by *auto*
      ultimately have $Suc\ (m\ div\ n) < Suc\ (m\ div\ n)$
        by *blast*
      then show *False*
        by *simp*
    qed

```
    with ‹n ≠ 0› show ?thesis
      by (simp add: modulo-nat-def)
  qed
  show euclidean-size m ≤ euclidean-size (m ∗ n)
    using ‹n ≠ 0› by (cases n) simp-all
  fix q r :: nat
  show (q ∗ n + r) div n = q if euclidean-size r < euclidean-size n
  proof −
    from that have r < n
      by simp
    have k ≤ q if k ∗ n ≤ q ∗ n + r for k
    proof (rule ccontr)
      assume ¬ k ≤ q
      then have q < k
        by simp
      then obtain l where k = Suc (q + l)
        by (auto simp add: less-iff-Suc-add)
      with ‹r < n› that show False
        by (simp add: algebra-simps)
    qed
    with ‹n ≠ 0› ex fin show ?thesis
      by (auto simp add: divide-nat-def Max-eq-iff)
  qed
qed simp-all

end

lemma euclidean-relation-natI [case-names by0 divides euclidean-relation]:
  ‹(m div n, m mod n) = (q, r)›
    if by0: ‹n = 0 =⇒ q = 0 ∧ r = m›
    and divides: ‹n > 0 =⇒ n dvd m =⇒ r = 0 ∧ m = q ∗ n›
    and euclidean-relation: ‹n > 0 =⇒ ¬ n dvd m =⇒ r < n ∧ m = q ∗ n + r›
for m n q r :: nat
  by (rule euclidean-relationI) (use that in simp-all)

lemma div-nat-eqI:
  ‹m div n = q› if ‹n ∗ q ≤ m› and ‹m < n ∗ Suc q› for m n q :: nat
proof −
  have ‹(m div n, m mod n) = (q, m − n ∗ q)›
  proof (induction rule: euclidean-relation-natI)
    case by0
    with that show ?case
      by simp
  next
    case divides
    from ‹n dvd m› obtain s where ‹m = n ∗ s› ..
    with ‹n > 0› that have ‹s < Suc q›
      by (simp only: mult-less-cancel1)
    with ‹m = n ∗ s› ‹n > 0› that have ‹q = s›
```

```
      by simp
    with ‹m = n * s› show ?case
      by (simp add: ac-simps)
  next
    case euclidean-relation
    with that show ?case
      by (simp add: ac-simps)
  qed
  then show ?thesis
    by simp
qed

lemma mod-nat-eqI:
  ‹m mod n = r› if ‹r < n› and ‹r ≤ m› and ‹n dvd m − r› for m n r :: nat
proof −
  have ‹(m div n, m mod n) = ((m − r) div n, r)›
  proof (induction rule: euclidean-relation-natI)
    case by0
    with that show ?case
      by simp
  next
    case divides
    from that dvd-minus-add [of r ‹m› 1 n]
    have ‹n dvd m + (n − r)›
      by simp
    with divides have ‹n dvd n − r›
      by (simp add: dvd-add-right-iff)
    then have ‹n ≤ n − r›
      by (rule dvd-imp-le) (use ‹r < n› in simp)
    with ‹n > 0› have ‹r = 0›
      by simp
    with ‹n > 0› that show ?case
      by simp
  next
    case euclidean-relation
    with that show ?case
      by (simp add: ac-simps)
  qed
  then show ?thesis
    by simp
qed
```

Tool support

```
ML ‹
structure Cancel-Div-Mod-Nat = Cancel-Div-Mod
(
  val div-name = const-name ‹divide›;
  val mod-name = const-name ‹modulo›;
  val mk-binop = HOLogic.mk-binop;
```

```
  val dest-plus = HOLogic.dest-bin const-name ‹Groups.plus› HOLogic.natT;
  val mk-sum = Arith-Data.mk-sum;
  fun dest-sum tm =
    if HOLogic.is-zero tm then []
    else
      (case try HOLogic.dest-Suc tm of
        SOME t => HOLogic.Suc-zero :: dest-sum t
      | NONE =>
          (case try dest-plus tm of
            SOME (t, u) => dest-sum t @ dest-sum u
          | NONE => [tm]));

  val div-mod-eqs = map mk-meta-eq @{thms cancel-div-mod-rules};

  val prove-eq-sums = Arith-Data.prove-conv2 all-tac
    (Arith-Data.simp-all-tac @{thms add-0-left add-0-right ac-simps})
)
›
```

simproc-setup *cancel-div-mod-nat* ((*m*::*nat*) + *n*) =
‹*K Cancel-Div-Mod-Nat.proc*›

lemma *div-mult-self-is-m* [*simp*]:
m ∗ *n div n* = *m* **if** *n* > *0* **for** *m n* :: *nat*
using *that* **by** *simp*

lemma *div-mult-self1-is-m* [*simp*]:
n ∗ *m div n* = *m* **if** *n* > *0* **for** *m n* :: *nat*
using *that* **by** *simp*

lemma *mod-less-divisor* [*simp*]:
m mod n < *n* **if** *n* > *0* **for** *m n* :: *nat*
using *mod-size-less* [*of n m*] *that* **by** *simp*

lemma *mod-le-divisor* [*simp*]:
m mod n ≤ *n* **if** *n* > *0* **for** *m n* :: *nat*
using *that* **by** (*auto simp add*: *le-less*)

lemma *div-times-less-eq-dividend* [*simp*]:
m div n ∗ *n* ≤ *m* **for** *m n* :: *nat*
by (*simp add*: *minus-mod-eq-div-mult* [*symmetric*])

lemma *times-div-less-eq-dividend* [*simp*]:
n ∗ (*m div n*) ≤ *m* **for** *m n* :: *nat*
using *div-times-less-eq-dividend* [*of m n*]
by (*simp add*: *ac-simps*)

lemma *dividend-less-div-times*:
m < *n* + (*m div n*) ∗ *n* **if** *0* < *n* **for** *m n* :: *nat*

proof −
  from *that* **have** *m mod n* < *n*
    by *simp*
  then show *?thesis*
    by (*simp add*: *minus-mod-eq-div-mult* [*symmetric*])
qed

lemma *dividend-less-times-div*:
  m < *n* + *n* ∗ (*m div n*) **if** *0* < *n* **for** *m n* :: *nat*
  using *dividend-less-div-times* [*of n m*] *that*
  by (*simp add*: *ac-simps*)

lemma *mod-Suc-le-divisor* [*simp*]:
  m mod Suc n ≤ *n*
  using *mod-less-divisor* [*of Suc n m*] **by** *arith*

lemma *mod-less-eq-dividend* [*simp*]:
  m mod n ≤ *m* **for** *m n* :: *nat*
proof (*rule add-leD2*)
  from *div-mult-mod-eq* **have** *m div n* ∗ *n* + *m mod n* = *m* **.**
  then show *m div n* ∗ *n* + *m mod n* ≤ *m* **by** *auto*
qed

lemma
  div-less [*simp*]: *m div n* = *0*
  and *mod-less* [*simp*]: *m mod n* = *m*
  if *m* < *n* **for** *m n* :: *nat*
  using *that* **by** (*auto intro*: *div-nat-eqI mod-nat-eqI*)

lemma *split-div*:
  ‹*P* (*m div n*) ⟷
    (*n* = *0* ⟶ *P 0*) ∧
    (*n* ≠ *0* ⟶ (∀ *i j*. *j* < *n* ∧ *m* = *n* ∗ *i* + *j* ⟶ *P i*))› (**is** *?div*)
  and *split-mod*:
  ‹*Q* (*m mod n*) ⟷
    (*n* = *0* ⟶ *Q m*) ∧
    (*n* ≠ *0* ⟶ (∀ *i j*. *j* < *n* ∧ *m* = *n* ∗ *i* + *j* ⟶ *Q j*))› (**is** *?mod*)
  for *m n* :: *nat*
proof −
  have ∗: ‹*R* (*m div n*) (*m mod n*) ⟷
    (*n* = *0* ⟶ *R 0 m*) ∧
    (*n* ≠ *0* ⟶ (∀ *i j*. *j* < *n* ∧ *m* = *n* ∗ *i* + *j* ⟶ *R i j*))› **for** *R*
    by (*cases* ‹*n* = *0*›) *auto*
  from ∗ [*of* ‹λ*q* -. *P q*›] **show** *?div* **.**
  from ∗ [*of* ‹λ- *r*. *Q r*›] **show** *?mod* **.**
qed

declare *split-div* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*
declare *split-mod* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*

lemma *split-div'*:
  $P\ (m\ div\ n) \longleftrightarrow n = 0 \wedge P\ 0 \vee (\exists q.\ (n * q \leq m \wedge m < n * Suc\ q) \wedge P\ q)$
proof (*cases* $n = 0$)
  case *True*
  then show *?thesis*
    by *simp*
next
  case *False*
  then have $n * q \leq m \wedge m < n * Suc\ q \longleftrightarrow m\ div\ n = q$ **for** q
    by (*auto intro*: *div-nat-eqI dividend-less-times-div*)
  then show *?thesis*
    by *auto*
qed

lemma *le-div-geq*:
  $m\ div\ n = Suc\ ((m - n)\ div\ n)$ **if** $0 < n$ **and** $n \leq m$ **for** $m\ n :: nat$
proof −
  from ‹$n \leq m$› **obtain** q **where** $m = n + q$
    by (*auto simp add*: *le-iff-add*)
  with ‹$0 < n$› **show** *?thesis*
    by (*simp add*: *div-add-self1*)
qed

lemma *le-mod-geq*:
  $m\ mod\ n = (m - n)\ mod\ n$ **if** $n \leq m$ **for** $m\ n :: nat$
proof −
  from ‹$n \leq m$› **obtain** q **where** $m = n + q$
    by (*auto simp add*: *le-iff-add*)
  then show *?thesis*
    by *simp*
qed

lemma *div-if*:
  $m\ div\ n = (if\ m < n \vee n = 0\ then\ 0\ else\ Suc\ ((m - n)\ div\ n))$
  by (*simp add*: *le-div-geq*)

lemma *mod-if*:
  $m\ mod\ n = (if\ m < n\ then\ m\ else\ (m - n)\ mod\ n)$ **for** $m\ n :: nat$
  by (*simp add*: *le-mod-geq*)

lemma *div-eq-0-iff*:
  $m\ div\ n = 0 \longleftrightarrow m < n \vee n = 0$ **for** $m\ n :: nat$
  by (*simp add*: *div-eq-0-iff*)

lemma *div-greater-zero-iff*:
  $m\ div\ n > 0 \longleftrightarrow n \leq m \wedge n > 0$ **for** $m\ n :: nat$
  using *div-eq-0-iff* [*of m n*] **by** *auto*

lemma *mod-greater-zero-iff-not-dvd*:
m mod n > *0* ⟷ ¬ *n dvd m* **for** *m n* :: *nat*
by (*simp add*: *dvd-eq-mod-eq-0*)

lemma *div-by-Suc-0* [*simp*]:
m div Suc 0 = *m*
using *div-by-1* [*of m*] **by** *simp*

lemma *mod-by-Suc-0* [*simp*]:
m mod Suc 0 = *0*
using *mod-by-1* [*of m*] **by** *simp*

lemma *div2-Suc-Suc* [*simp*]:
Suc (*Suc m*) *div 2* = *Suc* (*m div 2*)
by (*simp add*: *numeral-2-eq-2 le-div-geq*)

lemma *Suc-n-div-2-gt-zero* [*simp*]:
0 < *Suc n div 2* **if** *n* > *0* **for** *n* :: *nat*
using *that* **by** (*cases n*) *simp-all*

lemma *div-2-gt-zero* [*simp*]:
0 < *n div 2* **if** *Suc 0* < *n* **for** *n* :: *nat*
using *that Suc-n-div-2-gt-zero* [*of n* − *1*] **by** *simp*

lemma *mod2-Suc-Suc* [*simp*]:
Suc (*Suc m*) *mod 2* = *m mod 2*
by (*simp add*: *numeral-2-eq-2 le-mod-geq*)

lemma *add-self-div-2* [*simp*]:
(*m* + *m*) *div 2* = *m* **for** *m* :: *nat*
by (*simp add*: *mult-2* [*symmetric*])

lemma *add-self-mod-2* [*simp*]:
(*m* + *m*) *mod 2* = *0* **for** *m* :: *nat*
by (*simp add*: *mult-2* [*symmetric*])

lemma *mod2-gr-0* [*simp*]:
0 < *m mod 2* ⟷ *m mod 2* = *1* **for** *m* :: *nat*
proof −
have *m mod 2* < *2*
by (*rule mod-less-divisor*) *simp*
then have *m mod 2* = *0* ∨ *m mod 2* = *1*
by *arith*
then show *?thesis*
by *auto*
qed

lemma *mod-Suc-eq* [*mod-simps*]:
Suc (*m mod n*) *mod n* = *Suc m mod n*

proof –
  have (*m mod n* + *1*) *mod n* = (*m* + *1*) *mod n*
    by (*simp only*: *mod-simps*)
  then show *?thesis*
    by *simp*
qed

lemma *mod-Suc-Suc-eq* [*mod-simps*]:
  Suc (*Suc* (*m mod n*)) *mod n* = *Suc* (*Suc m*) *mod n*
proof –
  have (*m mod n* + *2*) *mod n* = (*m* + *2*) *mod n*
    by (*simp only*: *mod-simps*)
  then show *?thesis*
    by *simp*
qed

lemma
  Suc-mod-mult-self1 [*simp*]: *Suc* (*m* + *k* ∗ *n*) *mod n* = *Suc m mod n*
  and *Suc-mod-mult-self2* [*simp*]: *Suc* (*m* + *n* ∗ *k*) *mod n* = *Suc m mod n*
  and *Suc-mod-mult-self3* [*simp*]: *Suc* (*k* ∗ *n* + *m*) *mod n* = *Suc m mod n*
  and *Suc-mod-mult-self4* [*simp*]: *Suc* (*n* ∗ *k* + *m*) *mod n* = *Suc m mod n*
  by (*subst mod-Suc-eq* [*symmetric*], *simp add*: *mod-simps*)+

lemma *Suc-0-mod-eq* [*simp*]:
  Suc 0 mod n = *of-bool* (*n* ≠ *Suc 0*)
  by (*cases n*) *simp-all*

lemma *div-mult2-eq*:
    ‹*m div* (*n* ∗ *q*) = (*m div n*) *div q*› (**is** *?Q*)
  and *mod-mult2-eq*:
    ‹*m mod* (*n* ∗ *q*) = *n* ∗ (*m div n mod q*) + *m mod n*› (**is** *?R*)
  for *m n q* :: *nat*
proof –
  have ‹(*m div* (*n* ∗ *q*), *m mod* (*n* ∗ *q*)) = ((*m div n*) *div q*, *n* ∗ (*m div n mod q*)
+ *m mod n*)›
  proof (*induction rule*: *euclidean-relation-natI*)
    case *by0*
    then show *?case*
      by *auto*
  next
    case *divides*
    from ‹*n* ∗ *q dvd m*› **obtain** *t* **where** ‹*m* = *n* ∗ *q* ∗ *t*› **..**
    with ‹*n* ∗ *q* > *0*› **show** *?case*
      by (*simp add*: *algebra-simps*)
  next
    case *euclidean-relation*
    then have ‹*n* > *0*› ‹*q* > *0*›
      by *simp-all*
    from ‹*n* > *0*› **have** ‹*m mod n* < *n*›

```
      by (rule mod-less-divisor)
    from ‹q > 0› have ‹m div n mod q < q›
      by (rule mod-less-divisor)
    then obtain s where ‹q = Suc (m div n mod q + s)›
      by (blast dest: less-imp-Suc-add)
    moreover have ‹m mod n + n ∗ (m div n mod q) < n ∗ Suc (m div n mod q
+ s)›
      using ‹m mod n < n› by (simp add: add-mult-distrib2)
    ultimately have ‹m mod n + n ∗ (m div n mod q) < n ∗ q›
      by simp
    then show ?case
      by (simp add: algebra-simps flip: add-mult-distrib2)
  qed
  then show ?Q and ?R
    by simp-all
qed

lemma div-le-mono:
  m div k ≤ n div k if m ≤ n for m n k :: nat
proof −
  from that obtain q where n = m + q
    by (auto simp add: le-iff-add)
  then show ?thesis
    by (simp add: div-add1-eq [of m q k])
qed
```

Antimonotonicity of (*div*) in second argument

```
lemma div-le-mono2:
  k div n ≤ k div m if 0 < m and m ≤ n for m n k :: nat
using that proof (induct k arbitrary: m rule: less-induct)
  case (less k)
  show ?case
  proof (cases n ≤ k)
    case False
    then show ?thesis
      by simp
  next
    case True
    have (k − n) div n ≤ (k − m) div n
      using less.prems
      by (blast intro: div-le-mono diff-le-mono2)
    also have . . . ≤ (k − m) div m
      using ‹n ≤ k› less.prems less.hyps [of k − m m]
      by simp
    finally show ?thesis
      using ‹n ≤ k› less.prems
      by (simp add: le-div-geq)
  qed
qed
```

lemma *div-le-dividend* [*simp*]:
  m div n ≤ *m* **for** *m n* :: *nat*
  using *div-le-mono2* [*of 1 n m*] **by** (*cases n* = *0*) *simp-all*

lemma *div-less-dividend* [*simp*]:
  m div n < *m* **if** *1* < *n* **and** *0* < *m* **for** *m n* :: *nat*
using *that* **proof** (*induct m rule*: *less-induct*)
  case (*less m*)
  show *?case*
  proof (*cases n* < *m*)
    case *False*
    with *less* **show** *?thesis*
      by (*cases n* = *m*) *simp-all*
  next
    case *True*
    then show *?thesis*
      using *less.hyps* [*of m* − *n*] *less.prems*
      by (*simp add*: *le-div-geq*)
  qed
qed

lemma *div-eq-dividend-iff*:
  m div n = *m* ⟷ *n* = *1* **if** *m* > *0* **for** *m n* :: *nat*
proof
  assume *n* = *1*
  then show *m div n* = *m*
    by *simp*
next
  assume *P*: *m div n* = *m*
  show *n* = *1*
  proof (*rule ccontr*)
    have *n* ≠ *0*
      by (*rule ccontr*) (*use that P* **in** *auto*)
    moreover assume *n* ≠ *1*
    ultimately have *n* > *1*
      by *simp*
    with *that* **have** *m div n* < *m*
      by *simp*
    with *P* **show** *False*
      by *simp*
  qed
qed

lemma *less-mult-imp-div-less*:
  m div n < *i* **if** *m* < *i* ∗ *n* **for** *m n i* :: *nat*
proof −
  from *that* **have** *i* ∗ *n* > *0*
    by (*cases i* ∗ *n* = *0*) *simp-all*

then have *i* > *0* **and** *n* > *0*
  by *simp-all*
have *m div n* ∗ *n* ≤ *m*
  by *simp*
then have *m div n* ∗ *n* < *i* ∗ *n*
  using *that* **by** (*rule le-less-trans*)
with ‹*n* > *0*› **show** *?thesis*
  by *simp*
qed

lemma *div-less-iff-less-mult*:
‹*m div q* < *n* ⟷ *m* < *n* ∗ *q*› (**is** ‹*?P* ⟷ *?Q*›)
if ‹*q* > *0*› **for** *m n q* :: *nat*
proof
  assume *?Q* **then show** *?P*
    by (*rule less-mult-imp-div-less*)
next
  assume *?P*
  then obtain *h* **where** ‹*n* = *Suc* (*m div q* + *h*)›
    using *less-natE* **by** *blast*
  moreover have ‹*m* < *m* + (*Suc h* ∗ *q* − *m mod q*)›
    using *that* **by** (*simp add*: *trans-less-add1*)
  ultimately show *?Q*
    by (*simp add*: *algebra-simps flip*: *minus-mod-eq-mult-div*)
qed

lemma *less-eq-div-iff-mult-less-eq*:
‹*m* ≤ *n div q* ⟷ *m* ∗ *q* ≤ *n*› **if** ‹*q* > *0*› **for** *m n q* :: *nat*
using *div-less-iff-less-mult* [*of q n m*] *that* **by** *auto*

lemma *div-Suc*:
‹*Suc m div n* = (*if Suc m mod n* = *0 then Suc* (*m div n*) *else m div n*)›
proof (*cases* ‹*n* = *0* ∨ *n* = *1*›)
  case *True*
  then show *?thesis* **by** *auto*
next
  case *False*
  then have ‹*n* > *1*›
    by *simp*
  then have ‹*Suc m div n* = *m div n* + *Suc* (*m mod n*) *div n*›
    using *div-add1-eq* [*of m 1 n*] **by** *simp*
  also have ‹*Suc* (*m mod n*) *div n* = *of-bool* (*n dvd Suc m*)›
  proof (*cases* ‹*n dvd Suc m*›)
    case *False*
    moreover have ‹*Suc* (*m mod n*) ≠ *n*›
    proof (*rule ccontr*)
      assume ‹¬ *Suc* (*m mod n*) ≠ *n*›
      then have ‹*m mod n* = *n* − *Suc 0*›
        by *simp*

```
    with ‹n > 1› have ‹(m + 1) mod n = 0›
      by (subst mod-add-left-eq [symmetric]) simp
    then have ‹n dvd Suc m›
      by auto
    with False show False ..
  qed
  moreover have ‹Suc (m mod n) ≤ n›
    using ‹n > 1› by (simp add: Suc-le-eq)
  ultimately show ?thesis
    by (simp add: div-eq-0-iff)
next
  case True
  then obtain q where q: ‹Suc m = n * q› ..
  moreover have ‹q > 0› by (rule ccontr)
    (use q in simp)
  ultimately have ‹m mod n = n − Suc 0›
    using ‹n > 1› mult-le-cancel1 [of n ‹Suc 0› q]
    by (auto intro: mod-nat-eqI)
  with True ‹n > 1› show ?thesis
    by simp
qed
finally show ?thesis
  by (simp add: mod-greater-zero-iff-not-dvd)
qed

lemma mod-Suc:
  ‹Suc m mod n = (if Suc (m mod n) = n then 0 else Suc (m mod n))›
proof (cases ‹n = 0›)
  case True
  then show ?thesis
    by simp
next
  case False
  moreover have ‹Suc m mod n = Suc (m mod n) mod n›
    by (simp add: mod-simps)
  ultimately show ?thesis
    by (auto intro!: mod-nat-eqI intro: neq-le-trans simp add: Suc-le-eq)
qed

lemma Suc-times-mod-eq:
  Suc (m * n) mod m = 1 if Suc 0 < m
  using that by (simp add: mod-Suc)

lemma Suc-times-numeral-mod-eq [simp]:
  Suc (numeral k * n) mod numeral k = 1 if numeral k ≠ (1::nat)
  by (rule Suc-times-mod-eq) (use that in simp)

lemma Suc-div-le-mono [simp]:
  m div n ≤ Suc m div n
```

by (*simp add*: *div-le-mono*)

These lemmas collapse some needless occurrences of Suc: at least three Sucs, since two and fewer are rewritten back to Suc again! We already have some rules to simplify operands smaller than 3.

lemma *div-Suc-eq-div-add3* [*simp*]:
m div Suc (*Suc* (*Suc n*)) = *m div* (*3* + *n*)
by (*simp add*: *Suc3-eq-add-3*)

lemma *mod-Suc-eq-mod-add3* [*simp*]:
m mod Suc (*Suc* (*Suc n*)) = *m mod* (*3* + *n*)
by (*simp add*: *Suc3-eq-add-3*)

lemma *Suc-div-eq-add3-div*:
Suc (*Suc* (*Suc m*)) *div n* = (*3* + *m*) *div n*
by (*simp add*: *Suc3-eq-add-3*)

lemma *Suc-mod-eq-add3-mod*:
Suc (*Suc* (*Suc m*)) *mod n* = (*3* + *m*) *mod n*
by (*simp add*: *Suc3-eq-add-3*)

lemmas *Suc-div-eq-add3-div-numeral* [*simp*] =
Suc-div-eq-add3-div [*of - numeral v*] **for** *v*

lemmas *Suc-mod-eq-add3-mod-numeral* [*simp*] =
Suc-mod-eq-add3-mod [*of - numeral v*] **for** *v*

lemma (**in** *field-char-0*) *of-nat-div*:
of-nat (*m div n*) = ((*of-nat m* − *of-nat* (*m mod n*)) / *of-nat n*)
proof −
have *of-nat* (*m div n*) = ((*of-nat* (*m div n* ∗ *n* + *m mod n*) − *of-nat* (*m mod n*)) / *of-nat n* :: *'a*)
unfolding *of-nat-add* **by** (*cases n* = *0*) *simp-all*
then show *?thesis*
by *simp*
qed

An "induction" law for modulus arithmetic.

lemma *mod-induct* [*consumes 3*, *case-names step*]:
P m **if** *P n* **and** *n* < *p* **and** *m* < *p*
and *step*: ⋀*n*. *n* < *p* ⟹ *P n* ⟹ *P* (*Suc n mod p*)
using ‹*m* < *p*› **proof** (*induct m*)
case *0*
show *?case*
proof (*rule ccontr*)
assume ¬ *P 0*
from ‹*n* < *p*› **have** *0* < *p*
by *simp*
from ‹*n* < *p*› **obtain** *m* **where** *0* < *m* **and** *p* = *n* + *m*

```
      by (blast dest: less-imp-add-positive)
    with ‹P n› have P (p − m)
      by simp
    moreover have ¬ P (p − m)
    using ‹0 < m› proof (induct m)
      case 0
      then show ?case
        by simp
    next
      case (Suc m)
      show ?case
      proof
        assume P: P (p − Suc m)
        with ‹¬ P 0› have Suc m < p
          by (auto intro: ccontr)
        then have Suc (p − Suc m) = p − m
          by arith
        moreover from ‹0 < p› have p − Suc m < p
          by arith
        with P step have P ((Suc (p − Suc m)) mod p)
          by blast
        ultimately show False
          using ‹¬ P 0› Suc.hyps by (cases m = 0) simp-all
      qed
    qed
    ultimately show False
      by blast
  qed
next
  case (Suc m)
  then have m < p and mod: Suc m mod p = Suc m
    by simp-all
  from ‹m < p› have P m
    by (rule Suc.hyps)
  with ‹m < p› have P (Suc m mod p)
    by (rule step)
  with mod show ?case
    by simp
qed

lemma funpow-mod-eq:
  ‹(f ^^ (m mod n)) x = (f ^^ m) x› if ‹(f ^^ n) x = x›
proof −
  have ‹(f ^^ m) x = (f ^^ (m mod n + m div n * n)) x›
    by simp
  also have ‹... = (f ^^ (m mod n)) (((f ^^ n) ^^ (m div n)) x)›
    by (simp only: funpow-add funpow-mult ac-simps) simp
  also have ‹((f ^^ n) ^^ q) x = x› for q
    by (induction q) (use ‹(f ^^ n) x = x› in simp-all)
```

finally show *?thesis*
by *simp*
qed

lemma *mod-eq-dvd-iff-nat*:
‹*m mod q* = *n mod q* ⟷ *q dvd m* − *n*› (**is** ‹*?P* ⟷ *?Q*›)
if ‹*m* ≥ *n*› **for** *m n q* :: *nat*
proof
assume *?Q*
then obtain *s* **where** ‹*m* − *n* = *q* ∗ *s*› **..**
with *that* **have** ‹*m* = *q* ∗ *s* + *n*›
by *simp*
then show *?P*
by *simp*
next
assume *?P*
have ‹*m* − *n* = *m div q* ∗ *q* + *m mod q* − (*n div q* ∗ *q* + *n mod q*)›
by *simp*
also have ‹. . . = *q* ∗ (*m div q* − *n div q*)›
by (*simp only*: *algebra-simps* ‹*?P*›)
finally show *?Q* **..**
qed

lemma *mod-eq-iff-dvd-symdiff-nat*:
‹*m mod q* = *n mod q* ⟷ *q dvd nat* |*int m* − *int n*|›
by (*auto simp add*: *abs-if mod-eq-dvd-iff-nat nat-diff-distrib dest*: *sym intro*: *sym*)

lemma *mod-eq-nat1E*:
fixes *m n q* :: *nat*
assumes *m mod q* = *n mod q* **and** *m* ≥ *n*
obtains *s* **where** *m* = *n* + *q* ∗ *s*
proof −
from *assms* **have** *q dvd m* − *n*
by (*simp add*: *mod-eq-dvd-iff-nat*)
then obtain *s* **where** *m* − *n* = *q* ∗ *s* **..**
with ‹*m* ≥ *n*› **have** *m* = *n* + *q* ∗ *s*
by *simp*
with *that* **show** *thesis* **.**
qed

lemma *mod-eq-nat2E*:
fixes *m n q* :: *nat*
assumes *m mod q* = *n mod q* **and** *n* ≥ *m*
obtains *s* **where** *n* = *m* + *q* ∗ *s*
using *assms mod-eq-nat1E* [*of n q m*] **by** (*auto simp add*: *ac-simps*)

lemma *nat-mod-eq-iff*:
(*x*::*nat*) *mod n* = *y mod n* ⟷ (∃ *q1 q2*. *x* + *n* ∗ *q1* = *y* + *n* ∗ *q2*) (**is** *?lhs* = *?rhs*)

```
proof
  assume H: x mod n = y mod n
  { assume xy: x ≤ y
    from H have th: y mod n = x mod n by simp
    from mod-eq-nat1E [OF th xy] obtain q where y = x + n * q .
    then have x + n * q = y + n * 0
      by simp
    then have ∃ q1 q2. x + n * q1 = y + n * q2
      by blast
  }
  moreover
  { assume xy: y ≤ x
    from mod-eq-nat1E [OF H xy] obtain q where x = y + n * q .
    then have x + n * 0 = y + n * q
      by simp
    then have ∃ q1 q2. x + n * q1 = y + n * q2
      by blast
  }
  ultimately show ?rhs using linear[of x y] by blast
next
  assume ?rhs then obtain q1 q2 where q12: x + n * q1 = y + n * q2 by blast
  hence (x + n * q1) mod n = (y + n * q2) mod n by simp
  thus ?lhs by simp
qed
```

56.5 Division on *int*

The following specification of integer division rounds towards minus infinity and is advocated by Donald Knuth. See [5] for an overview and terminology of different possibilities to specify integer division; there division rounding towards minus infinitiy is named "F-division".

56.5.1 Basic instantiation

```
instantiation int :: {normalization-semidom, idom-modulo}
begin

definition normalize-int :: ‹int ⇒ int›
  where [simp]: ‹normalize = (abs :: int ⇒ int)›

definition unit-factor-int :: ‹int ⇒ int›
  where [simp]: ‹unit-factor = (sgn :: int ⇒ int)›

definition divide-int :: ‹int ⇒ int ⇒ int›
  where ‹k div l = (sgn k * sgn l * int (nat |k| div nat |l|)
    − of-bool (l ≠ 0 ∧ sgn k ≠ sgn l ∧ ¬ l dvd k))›

lemma divide-int-unfold:
  ‹(sgn k * int m) div (sgn l * int n) = (sgn k * sgn l * int (m div n)
```

− *of-bool* ((*k* = *0* ⟷ *m* = *0*) ∧ *l* ≠ *0* ∧ *n* ≠ *0* ∧ *sgn k* ≠ *sgn l* ∧ ¬ *n dvd m*))›
by (*simp add*: *divide-int-def sgn-mult nat-mult-distrib abs-mult sgn-eq-0-iff ac-simps*)

definition *modulo-int* :: ‹*int* ⇒ *int* ⇒ *int*›
where ‹*k mod l* = *sgn k* ∗ *int* (*nat* |*k*| *mod nat* |*l*|) + *l* ∗ *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*)›

lemma *modulo-int-unfold*:
‹(*sgn k* ∗ *int m*) *mod* (*sgn l* ∗ *int n*) =
sgn k ∗ *int* (*m mod* (*of-bool* (*l* ≠ *0*) ∗ *n*)) + (*sgn l* ∗ *int n*) ∗ *of-bool* ((*k* = *0* ⟷ *m* = *0*) ∧ *sgn k* ≠ *sgn l* ∧ ¬ *n dvd m*)›
by (*auto simp add*: *modulo-int-def sgn-mult abs-mult*)

instance proof
fix *k* :: *int* **show** *k div 0* = *0*
by (*simp add*: *divide-int-def*)
next
fix *k l* :: *int*
assume *l* ≠ *0*
obtain *n m* **and** *s t* **where** *k*: *k* = *sgn s* ∗ *int n* **and** *l*: *l* = *sgn t* ∗ *int m*
by (*blast intro*: *int-sgnE elim*: *that*)
then have *k* ∗ *l* = *sgn* (*s* ∗ *t*) ∗ *int* (*n* ∗ *m*)
by (*simp add*: *ac-simps sgn-mult*)
with *k l* ‹*l* ≠ *0*› **show** *k* ∗ *l div l* = *k*
by (*simp only*: *divide-int-unfold*)
(*auto simp add*: *algebra-simps sgn-mult sgn-1-pos sgn-0-0*)
next
fix *k l* :: *int*
obtain *n m* **and** *s t* **where** *k* = *sgn s* ∗ *int n* **and** *l* = *sgn t* ∗ *int m*
by (*blast intro*: *int-sgnE elim*: *that*)
then show *k div l* ∗ *l* + *k mod l* = *k*
by (*simp add*: *divide-int-unfold modulo-int-unfold algebra-simps modulo-nat-def of-nat-diff*)
qed (*auto simp add*: *sgn-mult mult-sgn-abs abs-eq-iff'*)

end

56.5.2 Algebraic foundations

lemma *coprime-int-iff* [*simp*]:
coprime (*int m*) (*int n*) ⟷ *coprime m n* (**is** *?P* ⟷ *?Q*)
proof
assume *?P*
show *?Q*
proof (*rule coprimeI*)
fix *q*
assume *q dvd m q dvd n*
then have *int q dvd int m int q dvd int n*

```
      by simp-all
    with ‹?P› have is-unit (int q)
      by (rule coprime-common-divisor)
    then show is-unit q
      by simp
  qed
next
  assume ?Q
  show ?P
  proof (rule coprimeI)
    fix k
    assume k dvd int m k dvd int n
    then have nat |k| dvd m nat |k| dvd n
      by simp-all
    with ‹?Q› have is-unit (nat |k|)
      by (rule coprime-common-divisor)
    then show is-unit k
      by simp
  qed
qed

lemma coprime-abs-left-iff [simp]:
  coprime |k| l ⟷ coprime k l for k l :: int
  using coprime-normalize-left-iff [of k l] by simp

lemma coprime-abs-right-iff [simp]:
  coprime k |l| ⟷ coprime k l for k l :: int
  using coprime-abs-left-iff [of l k] by (simp add: ac-simps)

lemma coprime-nat-abs-left-iff [simp]:
  coprime (nat |k|) n ⟷ coprime k (int n)
proof −
  define m where m = nat |k|
  then have |k| = int m
    by simp
  moreover have coprime k (int n) ⟷ coprime |k| (int n)
    by simp
  ultimately show ?thesis
    by simp
qed

lemma coprime-nat-abs-right-iff [simp]:
  coprime n (nat |k|) ⟷ coprime (int n) k
  using coprime-nat-abs-left-iff [of k n] by (simp add: ac-simps)

lemma coprime-common-divisor-int: coprime a b ⟹ x dvd a ⟹ x dvd b ⟹ |x|
= 1
  for a b :: int
  by (drule coprime-common-divisor [of - - x]) simp-all
```

56.5.3 Basic conversions

lemma *div-abs-eq-div-nat*:
$|k|$ *div* $|l|$ = *int* (*nat* $|k|$ *div nat* $|l|$)
by (*auto simp add*: *divide-int-def*)

lemma *div-eq-div-abs*:
‹*k div l* = *sgn k* ∗ *sgn l* ∗ ($|k|$ *div* $|l|$)
− *of-bool* ($l \neq 0$ ∧ *sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*)›
for *k l* :: *int*
by (*simp add*: *divide-int-def* [*of k l*] *div-abs-eq-div-nat*)

lemma *div-abs-eq*:
‹$|k|$ *div* $|l|$ = *sgn k* ∗ *sgn l* ∗ (*k div l* + *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*))›
for *k l* :: *int*
by (*simp add*: *div-eq-div-abs* [*of k l*] *ac-simps*)

lemma *mod-abs-eq-div-nat*:
$|k|$ *mod* $|l|$ = *int* (*nat* $|k|$ *mod nat* $|l|$)
by (*simp add*: *modulo-int-def*)

lemma *mod-eq-mod-abs*:
‹*k mod l* = *sgn k* ∗ ($|k|$ *mod* $|l|$) + *l* ∗ *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*)›
for *k l* :: *int*
by (*simp add*: *modulo-int-def* [*of k l*] *mod-abs-eq-div-nat*)

lemma *mod-abs-eq*:
‹$|k|$ *mod* $|l|$ = *sgn k* ∗ (*k mod l* − *l* ∗ *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*))›
for *k l* :: *int*
by (*auto simp*: *mod-eq-mod-abs* [*of k l*])

lemma *div-sgn-abs-cancel*:
fixes *k l v* :: *int*
assumes $v \neq 0$
shows (*sgn v* ∗ $|k|$) *div* (*sgn v* ∗ $|l|$) = $|k|$ *div* $|l|$
using *assms* **by** (*simp add*: *sgn-mult abs-mult sgn-0-0*
divide-int-def [*of sgn v* ∗ $|k|$ *sgn v* ∗ $|l|$] *flip*: *div-abs-eq-div-nat*)

lemma *div-eq-sgn-abs*:
fixes *k l v* :: *int*
assumes *sgn k* = *sgn l*
shows *k div l* = $|k|$ *div* $|l|$
using *assms* **by** (*auto simp add*: *div-abs-eq*)

lemma *div-dvd-sgn-abs*:
fixes *k l* :: *int*
assumes *l dvd k*
shows *k div l* = (*sgn k* ∗ *sgn l*) ∗ ($|k|$ *div* $|l|$)
using *assms* **by** (*auto simp add*: *div-abs-eq ac-simps*)

lemma *div-noneq-sgn-abs*:
  fixes *k l* :: *int*
  assumes $l \neq 0$
  assumes *sgn k* $\neq$ *sgn l*
  shows *k div l* = − (|*k*| *div* |*l*|) − *of-bool* (¬ *l dvd k*)
 using *assms* **by** (*auto simp add*: *div-abs-eq ac-simps sgn-0-0 dest*!: *sgn-not-eq-imp*)

56.5.4 Euclidean division

instantiation *int* :: *unique-euclidean-ring*
begin

definition *euclidean-size-int* :: *int* ⇒ *nat*
  where [*simp*]: *euclidean-size-int* = (*nat* ∘ *abs* :: *int* ⇒ *nat*)

definition *division-segment-int* :: *int* ⇒ *int*
  where *division-segment-int k* = (*if k* ≥ *0 then 1 else* − *1*)

lemma *division-segment-eq-sgn*:
  division-segment k = *sgn k* **if** *k* ≠ *0* **for** *k* :: *int*
  using *that* **by** (*simp add*: *division-segment-int-def*)

lemma *abs-division-segment* [*simp*]:
  |*division-segment k*| = *1* **for** *k* :: *int*
  by (*simp add*: *division-segment-int-def*)

lemma *abs-mod-less*:
  |*k mod l*| < |*l*| **if** *l* ≠ *0* **for** *k l* :: *int*
proof −
  obtain *n m* **and** *s t* **where** *k* = *sgn s* ∗ *int n* **and** *l* = *sgn t* ∗ *int m*
    by (*blast intro*: *int-sgnE elim*: *that*)
  with *that* **show** *?thesis*
    by (*auto simp add*: *modulo-int-unfold abs-mult mod-greater-zero-iff-not-dvd*
        simp flip: *right-diff-distrib dest*!: *sgn-not-eq-imp*)
      (*simp add*: *sgn-0-0*)
qed

lemma *sgn-mod*:
  sgn (*k mod l*) = *sgn l* **if** *l* ≠ *0* ¬ *l dvd k* **for** *k l* :: *int*
proof −
  obtain *n m* **and** *s t* **where** *k* = *sgn s* ∗ *int n* **and** *l* = *sgn t* ∗ *int m*
    by (*blast intro*: *int-sgnE elim*: *that*)
  with *that* **show** *?thesis*
    by (*auto simp add*: *modulo-int-unfold sgn-mult mod-greater-zero-iff-not-dvd*
      simp flip: *right-diff-distrib dest*!: *sgn-not-eq-imp*)
qed

instance proof
  fix *k l* :: *int*

show *division-segment* (*k mod l*) = *division-segment l* **if**
l ≠ *0* **and** ¬ *l dvd k*
using *that* **by** (*simp add*: *division-segment-eq-sgn dvd-eq-mod-eq-0 sgn-mod*)
next
fix *l q r* :: *int*
obtain *n m* **and** *s t*
where *l*: *l* = *sgn s* ∗ *int n* **and** *q*: *q* = *sgn t* ∗ *int m*
by (*blast intro*: *int-sgnE elim*: *that*)
assume ‹*l* ≠ *0*›
with *l* **have** *s* ≠ *0* **and** *n* > *0*
by (*simp-all add*: *sgn-0-0*)
assume *division-segment r* = *division-segment l*
moreover have *r* = *sgn r* ∗ |*r*|
by (*simp add*: *sgn-mult-abs*)
moreover define *u* **where** *u* = *nat* |*r*|
ultimately have *r* = *sgn l* ∗ *int u*
using *division-segment-eq-sgn* ‹*l* ≠ *0*› **by** (*cases r* = *0*) *simp-all*
with *l* ‹*n* > *0*› **have** *r*: *r* = *sgn s* ∗ *int u*
by (*simp add*: *sgn-mult*)
assume *euclidean-size r* < *euclidean-size l*
with *l r* ‹*s* ≠ *0*› **have** *u* < *n*
by (*simp add*: *abs-mult*)
show (*q* ∗ *l* + *r*) *div l* = *q*
proof (*cases q* = *0* ∨ *r* = *0*)
case *True*
then show *?thesis*
proof
assume *q* = *0*
then show *?thesis*
using *l r* ‹*u* < *n*› **by** (*simp add*: *divide-int-unfold*)
next
assume *r* = *0*
from ‹*r* = *0*› **have** ∗: *q* ∗ *l* + *r* = *sgn* (*t* ∗ *s*) ∗ *int* (*n* ∗ *m*)
using *q l* **by** (*simp add*: *ac-simps sgn-mult*)
from ‹*s* ≠ *0*› ‹*n* > *0*› **show** *?thesis*
by (*simp only*: ∗, *simp only*: ∗ *q l divide-int-unfold*)
(*auto simp add*: *sgn-mult ac-simps*)
qed
next
case *False*
with *q r* **have** *t* ≠ *0* **and** *m* > *0* **and** *s* ≠ *0* **and** *u* > *0*
by (*simp-all add*: *sgn-0-0*)
moreover from ‹*0* < *m*› ‹*u* < *n*› **have** *u* ≤ *m* ∗ *n*
using *mult-le-less-imp-less* [*of 1 m u n*] **by** *simp*
ultimately have ∗: *q* ∗ *l* + *r* = *sgn* (*s* ∗ *t*)
∗ *int* (*if t* < *0 then m* ∗ *n* − *u else m* ∗ *n* + *u*)
using *l q r*
by (*simp add*: *sgn-mult algebra-simps of-nat-diff*)
have (*m* ∗ *n* − *u*) *div n* = *m* − *1* **if** *u* > *0*

using ‹*0* < *m*› ‹*u* < *n*› *that*
by (*auto intro*: *div-nat-eqI simp add*: *algebra-simps*)
moreover have *n dvd m* ∗ *n* − *u* ⟷ *n dvd u*
using ‹*u* ≤ *m* ∗ *n*› *dvd-diffD1* [*of n m* ∗ *n u*]
by *auto*
ultimately show *?thesis*
using ‹*s* ≠ *0*› ‹*m* > *0*› ‹*u* > *0*› ‹*u* < *n*› ‹*u* ≤ *m* ∗ *n*›
by (*simp only*: ∗, *simp only*: *l q divide-int-unfold*)
(*auto simp add*: *sgn-mult sgn-0-0 sgn-1-pos algebra-simps dest*: *dvd-imp-le*)
qed
qed (*use mult-le-mono2* [*of 1*] **in** ‹*auto simp add*: *division-segment-int-def not-le zero-less-mult-iff mult-less-0-iff abs-mult sgn-mult abs-mod-less sgn-mod nat-mult-distrib*›)

end

lemma *euclidean-relation-intI* [*case-names by0 divides euclidean-relation*]:
‹(*k div l*, *k mod l*) = (*q*, *r*)›
if *by0'*: ‹*l* = *0* ⟹ *q* = *0* ∧ *r* = *k*›
and *divides'*: ‹*l* ≠ *0* ⟹ *l dvd k* ⟹ *r* = *0* ∧ *k* = *q* ∗ *l*›
and *euclidean-relation'*: ‹*l* ≠ *0* ⟹ ¬ *l dvd k* ⟹ *sgn r* = *sgn l*
∧ |*r*| < |*l*| ∧ *k* = *q* ∗ *l* + *r*› **for** *k l* :: *int*
proof (*induction rule*: *euclidean-relationI*)
case *by0*
then show *?case*
by (*rule by0'*)
next
case *divides*
then show *?case*
by (*rule divides'*)
next
case *euclidean-relation*
with *euclidean-relation'* **have** ‹*sgn r* = *sgn l*› ‹|*r*| < |*l*|› ‹*k* = *q* ∗ *l* + *r*›
by *simp-all*
from ‹*sgn r* = *sgn l*› ‹*l* ≠ *0*› **have** ‹*division-segment r* = *division-segment l*›
by (*simp add*: *division-segment-int-def sgn-if split*: *if-splits*)
with ‹|*r*| < |*l*|› ‹*k* = *q* ∗ *l* + *r*›
show *?case*
by *simp*
qed

56.5.5 Trivial reduction steps

lemma *div-pos-pos-trivial* [*simp*]:
k div l = *0* **if** *k* ≥ *0* **and** *k* < *l* **for** *k l* :: *int*
using *that* **by** (*simp add*: *unique-euclidean-semiring-class.div-eq-0-iff division-segment-int-def*)

lemma *mod-pos-pos-trivial* [*simp*]:
k mod l = *k* **if** *k* ≥ *0* **and** *k* < *l* **for** *k l* :: *int*
using *that* **by** (*simp add*: *mod-eq-self-iff-div-eq-0*)

lemma *div-neg-neg-trivial* [*simp*]:
  k div l = *0* **if** *k* ≤ *0* **and** *l* < *k* **for** *k l* :: *int*
  using *that* **by** (*cases k* = *0*) (*simp*, *simp add*: *unique-euclidean-semiring-class.div-eq-0-iff division-segment-int-def*)

lemma *mod-neg-neg-trivial* [*simp*]:
  k mod l = *k* **if** *k* ≤ *0* **and** *l* < *k* **for** *k l* :: *int*
  using *that* **by** (*simp add*: *mod-eq-self-iff-div-eq-0*)

lemma
  div-pos-neg-trivial: ‹*k div l* = − *1*› (**is** *?Q*)
  and *mod-pos-neg-trivial*: ‹*k mod l* = *k* + *l*› (**is** *?R*)
    if ‹*0* < *k*› **and** ‹*k* + *l* ≤ *0*› **for** *k l* :: *int*
proof −
  from *that* **have** ‹*l* < *0*›
    by *simp*
  have ‹(*k div l*, *k mod l*) = (− *1*, *k* + *l*)›
  proof (*induction rule*: *euclidean-relation-intI*)
    case *by0*
    with ‹*l* < *0*› **show** *?case*
      by *simp*
  next
    case *divides*
    from ‹*l dvd k*› **obtain** *j* **where** ‹*k* = *l* ∗ *j*› **..**
    with ‹*l* < *0*› ‹*0* < *k*› **have** ‹*j* < *0*›
      by (*simp add*: *zero-less-mult-iff*)
    moreover from ‹*k* + *l* ≤ *0*› ‹*k* = *l* ∗ *j*› **have** ‹*l* ∗ (*j* + *1*) ≤ *0*›
      by (*simp add*: *algebra-simps*)
    with ‹*l* < *0*› **have** ‹*j* + *1* ≥ *0*›
      by (*simp add*: *mult-le-0-iff*)
    with ‹*j* < *0*› **have** ‹*j* = − *1*›
      by *simp*
    with ‹*k* = *l* ∗ *j*› **show** *?case*
      by *simp*
  next
    case *euclidean-relation*
    with ‹*k* + *l* ≤ *0*› **have** ‹*k* + *l* < *0*›
      by (*auto simp add*: *less-le add-eq-0-iff*)
    with ‹*0* < *k*› **show** *?case*
      by *simp*
  qed
  then show *?Q* **and** *?R*
    by *simp-all*
qed

There is neither *div-neg-pos-trivial* nor *mod-neg-pos-trivial* because *0 div l* = *0* would supersede it.

56.5.6 More uniqueness rules

lemma
fixes *a b q r* :: *int*
assumes ‹*a* = *b* ∗ *q* + *r*› ‹*0* ≤ *r*› ‹*r* < *b*›
shows *int-div-pos-eq*:
‹*a div b* = *q*› (**is** *?Q*)
and *int-mod-pos-eq*:
‹*a mod b* = *r*› (**is** *?R*)
proof −
have ‹(*a div b*, *a mod b*) = (*q*, *r*)›
by (*induction rule*: *euclidean-relation-intI*)
(*use assms* **in** ‹*auto simp add*: *ac-simps dvd-add-left-iff sgn-1-pos le-less dest*: *zdvd-imp-le*›)
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *int-div-neg-eq*:
‹*a div b* = *q*› **if** ‹*a* = *b* ∗ *q* + *r*› ‹*r* ≤ *0*› ‹*b* < *r*› **for** *a b q r* :: *int*
using *that int-div-pos-eq* [*of a* ‹− *b*› ‹− *q*› ‹− *r*›] **by** *simp-all*

lemma *int-mod-neg-eq*:
‹*a mod b* = *r*› **if** ‹*a* = *b* ∗ *q* + *r*› ‹*r* ≤ *0*› ‹*b* < *r*› **for** *a b q r* :: *int*
using *that int-div-neg-eq* [*of a b q r*] **by** *simp*

56.5.7 Laws for unary minus

lemma *zmod-zminus1-not-zero*:
fixes *k l* :: *int*
shows − *k mod l* ≠ *0* ⟹ *k mod l* ≠ *0*
by (*simp add*: *mod-eq-0-iff-dvd*)

lemma *zmod-zminus2-not-zero*:
fixes *k l* :: *int*
shows *k mod* − *l* ≠ *0* ⟹ *k mod l* ≠ *0*
by (*simp add*: *mod-eq-0-iff-dvd*)

lemma *zdiv-zminus1-eq-if*:
‹(− *a*) *div b* = (*if a mod b* = *0 then* − (*a div b*) *else* − (*a div b*) − *1*)›
if ‹*b* ≠ *0*› **for** *a b* :: *int*
using *that sgn-not-eq-imp* [*of b* ‹− *a*›]
by (*cases* ‹*a* = *0*›) (*auto simp add*: *div-eq-div-abs* [*of* ‹− *a*› *b*] *div-eq-div-abs* [*of a b*] *sgn-eq-0-iff*)

lemma *zdiv-zminus2-eq-if*:
‹*a div* (− *b*) = (*if a mod b* = *0 then* − (*a div b*) *else* − (*a div b*) − *1*)›
if ‹*b* ≠ *0*› **for** *a b* :: *int*
using *that* **by** (*auto simp add*: *zdiv-zminus1-eq-if div-minus-right*)

lemma *zmod-zminus1-eq-if*:
‹(− a) mod b = (if a mod b = 0 then 0 else b − (a mod b))›
for *a b* :: *int*
by (*cases* ‹b = 0›)
(*auto simp flip*: *minus-div-mult-eq-mod simp add*: *zdiv-zminus1-eq-if algebra-simps*)

lemma *zmod-zminus2-eq-if*:
‹a mod (− b) = (if a mod b = 0 then 0 else (a mod b) − b)›
for *a b* :: *int*
by (*auto simp add*: *zmod-zminus1-eq-if mod-minus-right*)

56.5.8 Borders

lemma *pos-mod-bound* [*simp*]:
k mod l < l **if** *l > 0* **for** *k l* :: *int*
proof −
obtain *m* **and** *s* **where** *k = sgn s ∗ int m*
by (*rule int-sgnE*)
moreover from *that* **obtain** *n* **where** *l = sgn 1 ∗ int n*
by (*cases l*) *simp-all*
moreover from *this that* **have** *n > 0*
by *simp*
ultimately show *?thesis*
by (*simp only*: *modulo-int-unfold*)
(*auto simp add*: *mod-greater-zero-iff-not-dvd sgn-1-pos*)
qed

lemma *neg-mod-bound* [*simp*]:
l < k mod l **if** *l < 0* **for** *k l* :: *int*
proof −
obtain *m* **and** *s* **where** *k = sgn s ∗ int m*
by (*rule int-sgnE*)
moreover from *that* **obtain** *q* **where** *l = sgn (− 1) ∗ int (Suc q)*
by (*cases l*) *simp-all*
moreover define *n* **where** *n = Suc q*
then have *Suc q = n*
by *simp*
ultimately show *?thesis*
by (*simp only*: *modulo-int-unfold*)
(*auto simp add*: *mod-greater-zero-iff-not-dvd sgn-1-neg*)
qed

lemma *pos-mod-sign* [*simp*]:
0 ≤ k mod l **if** *l > 0* **for** *k l* :: *int*
proof −
obtain *m* **and** *s* **where** *k = sgn s ∗ int m*
by (*rule int-sgnE*)
moreover from *that* **obtain** *n* **where** *l = sgn 1 ∗ int n*
by (*cases l*) *auto*

moreover from *this that* **have** *n > 0*
  by *simp*
ultimately show *?thesis*
  by (*simp only*: *modulo-int-unfold*) (*auto simp add*: *sgn-1-pos*)
qed

lemma *neg-mod-sign* [*simp*]:
k mod l ≤ 0 **if** *l < 0* **for** *k l* :: *int*
proof −
obtain *m* **and** *s* **where** *k = sgn s ∗ int m*
  by (*rule int-sgnE*)
moreover from *that* **obtain** *q* **where** *l = sgn* (− *1*) ∗ *int* (*Suc q*)
  by (*cases l*) *simp-all*
moreover define *n* **where** *n = Suc q*
then have *Suc q = n*
  by *simp*
moreover have ‹*int* (*m mod n*) ≤ *int n*›
  using ‹*Suc q = n*› **by** *simp*
then have ‹*sgn s ∗ int* (*m mod n*) ≤ *int n*›
  by (*cases s* ‹*0*::*int*› *rule*: *linorder-cases*) *simp-all*
ultimately show *?thesis*
  by (*simp only*: *modulo-int-unfold*) *auto*
qed

56.5.9 Splitting Rules for div and mod

lemma *split-zdiv*:
‹*P* (*n div k*) ⟷
  (*k = 0* ⟶ *P 0*) ∧
  (*0 < k* ⟶ (∀ *i j*. *0 ≤ j* ∧ *j < k* ∧ *n = k ∗ i + j* ⟶ *P i*)) ∧
  (*k < 0* ⟶ (∀ *i j*. *k < j* ∧ *j ≤ 0* ∧ *n = k ∗ i + j* ⟶ *P i*))› (**is** *?div*)
and *split-zmod*:
‹*Q* (*n mod k*) ⟷
  (*k = 0* ⟶ *Q n*) ∧
  (*0 < k* ⟶ (∀ *i j*. *0 ≤ j* ∧ *j < k* ∧ *n = k ∗ i + j* ⟶ *Q j*)) ∧
  (*k < 0* ⟶ (∀ *i j*. *k < j* ∧ *j ≤ 0* ∧ *n = k ∗ i + j* ⟶ *Q j*))› (**is** *?mod*)
for *n k* :: *int*
proof −
have ∗: ‹*R* (*n div k*) (*n mod k*) ⟷
  (*k = 0* ⟶ *R 0 n*) ∧
  (*0 < k* ⟶ (∀ *i j*. *0 ≤ j* ∧ *j < k* ∧ *n = k ∗ i + j* ⟶ *R i j*)) ∧
  (*k < 0* ⟶ (∀ *i j*. *k < j* ∧ *j ≤ 0* ∧ *n = k ∗ i + j* ⟶ *R i j*))› **for** *R*
  by (*cases* ‹*k = 0*›)
    (*auto simp add*: *linorder-class.neq-iff*)
from ∗ [*of* ‹λ*q* -. *P q*›] **show** *?div* **.**
from ∗ [*of* ‹λ- *r*. *Q r*›] **show** *?mod* **.**
qed

Enable (lin)arith to deal with (*div*) and (*mod*) when these are applied to some constant that is of the form *numeral k*:

declare *split-zdiv* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*
declare *split-zdiv* [*of* - - ‹− *numeral n*›, *linarith-split*] **for** *n*
declare *split-zmod* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*
declare *split-zmod* [*of* - - ‹− *numeral n*›, *linarith-split*] **for** *n*

lemma *zdiv-eq-0-iff*:
  $i\ div\ k = 0 \longleftrightarrow k = 0 \lor 0 \leq i \land i < k \lor i \leq 0 \land k < i$ (**is** *?L = ?R*)
  for *i k* :: *int*
proof
  assume *?L*
  moreover have *?L* ⟶ *?R*
    by (*rule split-zdiv* [*THEN iffD2*]) *simp*
  ultimately show *?R*
    by *blast*
next
  assume *?R* **then show** *?L*
    by *auto*
qed

lemma *zmod-trivial-iff*:
  fixes *i k* :: *int*
  shows $i\ mod\ k = i \longleftrightarrow k = 0 \lor 0 \leq i \land i < k \lor i \leq 0 \land k < i$
proof −
  have $i\ mod\ k = i \longleftrightarrow i\ div\ k = 0$
    using *div-mult-mod-eq* [*of i k*] **by** *safe auto*
  with *zdiv-eq-0-iff*
  show *?thesis*
    by *simp*
qed

56.5.10 Algebraic rewrites

lemma *zdiv-zmult2-eq*: ‹$a\ div\ (b * c) = (a\ div\ b)\ div\ c$› (**is** *?Q*)
  and *zmod-zmult2-eq*: ‹$a\ mod\ (b * c) = b * (a\ div\ b\ mod\ c) + a\ mod\ b$› (**is** *?P*)
  if ‹$c \geq 0$› **for** *a b c* :: *int*
proof −
  have ∗: ‹$(a\ div\ (b * c),\ a\ mod\ (b * c)) = ((a\ div\ b)\ div\ c,\ b * (a\ div\ b\ mod\ c) + a\ mod\ b)$›
    if ‹$b > 0$› **for** *a b*
  proof (*induction rule*: *euclidean-relationI*)
    case *by0*
    then show *?case* **by** *auto*
  next
    case *divides*
    then obtain *d* **where** ‹$a = b * c * d$›
      by *blast*
    with *divides that* **show** *?case*
      by (*simp add*: *ac-simps*)
  next

case *euclidean-relation*
with ‹$b > 0$› ‹$c \geq 0$› **have** ‹$0 < c$› ‹$b > 0$›
by *simp-all*
then have ‹a *mod* $b < b$›
by *simp*
moreover have ‹$1 \leq c - a$ *div* b *mod* c›
using ‹$c > 0$› **by** (*simp add*: *int-one-le-iff-zero-less*)
ultimately have ‹a *mod* $b * 1 < b * (c - a$ *div* b *mod* $c)$›
by (*rule mult-less-le-imp-less*) (*use* ‹$b > 0$› **in** *simp-all*)
with ‹$0 < b$› ‹$0 < c$› **show** *?case*
by (*simp add*: *division-segment-int-def algebra-simps flip*: *minus-mod-eq-mult-div*)
qed
show *?Q*
proof (*cases* ‹$b \geq 0$›)
case *True*
with ∗ [*of b a*] **show** *?thesis*
by (*cases* ‹$b = 0$›) *simp-all*
next
case *False*
with ∗ [*of* ‹$-$ b› ‹$-$ a›] **show** *?thesis*
by *simp*
qed
show *?P*
proof (*cases* ‹$b \geq 0$›)
case *True*
with ∗ [*of b a*] **show** *?thesis*
by (*cases* ‹$b = 0$›) *simp-all*
next
case *False*
with ∗ [*of* ‹$-$ b› ‹$-$ a›] **show** *?thesis*
by *simp*
qed
qed

lemma *zdiv-zmult2-eq′*:
‹k *div* $(l * j) = ((sgn\ j * k)$ *div* $l)$ *div* $|j|$› **for** $k\ l\ j$ :: *int*
proof −
have ‹k *div* $(l * j) = (sgn\ j * k)$ *div* $(sgn\ j * (l * j))$›
by (*simp add*: *sgn-0-0*)
also have ‹$sgn\ j * (l * j) = l * |j|$›
by (*simp add*: *mult.left-commute* [*of* - *l*] *abs-sgn*) (*simp add*: *ac-simps*)
also have ‹$(sgn\ j * k)$ *div* $(l * |j|) = ((sgn\ j * k)$ *div* $l)$ *div* $|j|$›
by (*simp add*: *zdiv-zmult2-eq*)
finally show *?thesis* **.**
qed

lemma *half-nonnegative-int-iff* [*simp*]:
‹k *div* $2 \geq 0 \longleftrightarrow k \geq 0$› **for** k :: *int*
by *auto*

lemma *half-negative-int-iff* [*simp*]:
‹*k div 2* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
by *auto*

56.5.11 Distributive laws for conversions.

lemma *zdiv-int*:
‹*int* (*m div n*) = *int m div int n*›
by (*cases* ‹*m* = *0*›) (*auto simp add*: *divide-int-def*)

lemma *zmod-int*:
‹*int* (*m mod n*) = *int m mod int n*›
by (*cases* ‹*m* = *0*›) (*auto simp add*: *modulo-int-def*)

lemma *nat-div-distrib*:
‹*nat* (*x div y*) = *nat x div nat y*› **if** ‹*0* ≤ *x*›
using *that* **by** (*simp add*: *divide-int-def sgn-if*)

lemma *nat-div-distrib′*:
‹*nat* (*x div y*) = *nat x div nat y*› **if** ‹*0* ≤ *y*›
using *that* **by** (*simp add*: *divide-int-def sgn-if*)

lemma *nat-mod-distrib*: — Fails if y<0: the LHS collapses to (nat z) but the RHS doesn't
‹*nat* (*x mod y*) = *nat x mod nat y*› **if** ‹*0* ≤ *x*› ‹*0* ≤ *y*›
using *that* **by** (*simp add*: *modulo-int-def sgn-if*)

56.5.12 Monotonicity in the First Argument (Dividend)

lemma *zdiv-mono1*:
‹*a div b* ≤ *a′ div b*›
if ‹*a* ≤ *a′*› ‹*0* < *b*›
for *a b b′* :: *int*
proof −
from ‹*a* ≤ *a′*› **have** ‹*b* ∗ (*a div b*) + *a mod b* ≤ *b* ∗ (*a′ div b*) + *a′ mod b*›
by *simp*
then have ‹*b* ∗ (*a div b*) ≤ (*a′ mod b* − *a mod b*) + *b* ∗ (*a′ div b*)›
by (*simp add*: *algebra-simps*)
moreover have ‹*a′ mod b* < *b* + *a mod b*›
by (*rule less-le-trans* [*of - b*]) (*use* ‹*0* < *b*› **in** *simp-all*)
ultimately have ‹*b* ∗ (*a div b*) < *b* ∗ (*1* + *a′ div b*)›
by (*simp add*: *distrib-left*)
with ‹*0* < *b*› **have** ‹*a div b* < *1* + *a′ div b*›
by (*simp add*: *mult-less-cancel-left*)
then show *?thesis*
by *simp*
qed

lemma *zdiv-mono1-neg*:

‹a' *div* $b \le a$ *div* b›
if ‹$a \le a'$› ‹$b < 0$›
for $a\ a'\ b$ :: *int*
using *that zdiv-mono1* [*of* ‹$- a'$› ‹$- a$› ‹$- b$›] **by** *simp*

56.5.13 Monotonicity in the Second Argument (Divisor)

lemma *zdiv-mono2*:
‹a *div* $b \le a$ *div* b'› **if** ‹$0 \le a$› ‹$0 < b'$› ‹$b' \le b$› **for** $a\ b\ b'$ :: *int*
proof −
define $q\ q'\ r\ r'$ **where** $**$: ‹$q = a$ *div* b› ‹$q' = a$ *div* b'› ‹$r = a$ *mod* b› ‹$r' = a$ *mod* b'›
then have $*$: ‹$b * q + r = b' * q' + r'$› ‹$0 \le b' * q' + r'$›
‹$r' < b'$› ‹$0 \le r$› ‹$0 < b'$› ‹$b' \le b$›
using *that* **by** *simp-all*
have ‹$0 < b' * (q' + 1)$›
using $*$ **by** (*simp add*: *distrib-left*)
with $*$ **have** ‹$0 \le q'$›
by (*simp add*: *zero-less-mult-iff*)
moreover have ‹$b * q = r' - r + b' * q'$›
using $*$ **by** *linarith*
ultimately have ‹$b * q < b * (q' + 1)$›
using *mult-right-mono* $*$ **unfolding** *distrib-left* **by** *fastforce*
with $*$ **have** ‹$q \le q'$›
by (*simp add*: *mult-less-cancel-left-pos*)
with $**$ **show** *?thesis*
by *simp*
qed

lemma *zdiv-mono2-neg*:
‹a *div* $b' \le a$ *div* b› **if** ‹$a < 0$› ‹$0 < b'$› ‹$b' \le b$› **for** $a\ b\ b'$ :: *int*
proof −
define $q\ q'\ r\ r'$ **where** $**$: ‹$q = a$ *div* b› ‹$q' = a$ *div* b'› ‹$r = a$ *mod* b› ‹$r' = a$ *mod* b'›
then have $*$: ‹$b * q + r = b' * q' + r'$› ‹$b' * q' + r' < 0$›
‹$r < b$› ‹$0 \le r'$› ‹$0 < b'$› ‹$b' \le b$›
using *that* **by** *simp-all*
have ‹$b' * q' < 0$›
using $*$ **by** *linarith*
with $*$ **have** ‹$q' \le 0$›
by (*simp add*: *mult-less-0-iff*)
have ‹$b * q' \le b' * q'$›
by (*simp add*: ‹$q' \le 0$› $*$ *mult-right-mono-neg*)
then have $b * q' < b * (q + 1)$
using $*$ **by** (*simp add*: *distrib-left*)
then have ‹$q' \le q$›
using $*$ **by** (*simp add*: *mult-less-cancel-left*)
then show *?thesis*
by (*simp add*: $**$)

qed

56.5.14 Quotients of Signs

lemma *div-eq-minus1*:
‹*0* < *b* ⟹ − *1 div b* = − *1*› **for** *b* :: *int*
by (*simp add*: *divide-int-def*)

lemma *zmod-minus1*:
‹*0* < *b* ⟹ − *1 mod b* = *b* − *1*› **for** *b* :: *int*
by (*auto simp add*: *modulo-int-def*)

lemma *minus-mod-int-eq*:
‹− *k mod l* = *l* − *1* − (*k* − *1*) *mod l*› **if** ‹*l* ≥ *0*› **for** *k l* :: *int*
proof (*cases* ‹*l* = *0*›)
case *True*
then show *?thesis*
by *simp*
next
case *False*
with *that* **have** ‹*l* > *0*›
by *simp*
then show *?thesis*
proof (*cases* ‹*l dvd k*›)
case *True*
then obtain *j* **where** ‹*k* = *l* ∗ *j*› **..**
moreover have ‹(*l* ∗ *j mod l* − *1*) *mod l* = *l* − *1*›
using ‹*l* > *0*› **by** (*simp add*: *zmod-minus1*)
then have ‹(*l* ∗ *j* − *1*) *mod l* = *l* − *1*›
by (*simp only*: *mod-simps*)
ultimately show *?thesis*
by *simp*
next
case *False*
moreover have *1*: ‹*0* < *k mod l*›
using ‹*0* < *l*› *False le-less* **by** *fastforce*
moreover have *2*: ‹*k mod l* < *1* + *l*›
using ‹*0* < *l*› *pos-mod-bound*[*of l k*] **by** *linarith*
from *1 2* ‹*l* > *0*› **have** ‹(*k mod l* − *1*) *mod l* = *k mod l* − *1*›
by (*simp add*: *zmod-trivial-iff*)
ultimately show *?thesis*
by (*simp only*: *zmod-zminus1-eq-if*)
(*simp add*: *mod-eq-0-iff-dvd algebra-simps mod-simps*)
qed
qed

lemma *div-neg-pos-less0*:
‹*a div b* < *0*› **if** ‹*a* < *0*› ‹*0* < *b*› **for** *a b* :: *int*
proof −

have *a div b* ≤ − *1 div b*
  using *zdiv-mono1 that* **by** *auto*
also have ... ≤ −*1*
  by (*simp add*: *that*(*2*) *div-eq-minus1*)
finally show *?thesis*
  by *force*
qed

lemma *div-nonneg-neg-le0*:
‹*a div b* ≤ *0*› **if** ‹*0* ≤ *a*› ‹*b* < *0*› **for** *a b* :: *int*
using *that* **by** (*auto dest*: *zdiv-mono1-neg*)

lemma *div-nonpos-pos-le0*:
‹*a div b* ≤ *0*› **if** ‹*a* ≤ *0*› ‹*0* < *b*› **for** *a b* :: *int*
using *that* **by** (*auto dest*: *zdiv-mono1*)

Now for some equivalences of the form *a div b* >=< *0* ⟷ . . . conditional upon the sign of *a* or *b*. There are many more. They should all be simp rules unless that causes too much search.

lemma *pos-imp-zdiv-nonneg-iff*:
‹*0* ≤ *a div b* ⟷ *0* ≤ *a*›
if ‹*0* < *b*› **for** *a b* :: *int*
proof
  assume ‹*0* ≤ *a div b*›
  show ‹*0* ≤ *a*›
  proof (*rule ccontr*)
    assume ‹¬ *0* ≤ *a*›
    then have ‹*a* < *0*›
      by *simp*
    then have ‹*a div b* < *0*›
      using *that* **by** (*rule div-neg-pos-less0*)
    with ‹*0* ≤ *a div b*› **show** *False*
      by *simp*
  qed
next
  assume *0* ≤ *a*
  then have *0 div b* ≤ *a div b*
    using *zdiv-mono1 that* **by** *blast*
  then show *0* ≤ *a div b*
    by *auto*
qed

lemma *neg-imp-zdiv-nonneg-iff*:
‹*0* ≤ *a div b* ⟷ *a* ≤ *0*› **if** ‹*b* < *0*› **for** *a b* :: *int*
using *that pos-imp-zdiv-nonneg-iff* [*of* ‹− *b*› ‹− *a*›] **by** *simp*

lemma *pos-imp-zdiv-pos-iff*:
‹*0* < (*i*::*int*) *div k* ⟷ *k* ≤ *i*› **if** ‹*0* < *k*› **for** *i k* :: *int*
using *that pos-imp-zdiv-nonneg-iff* [*of k i*] *zdiv-eq-0-iff* [*of i k*] **by** *arith*

lemma *pos-imp-zdiv-neg-iff*:
‹$a\ div\ b < 0 \longleftrightarrow a < 0$› **if** ‹$0 < b$› **for** $a\ b$ :: *int*
— But not $(a\ div\ b \leq 0) = (a \leq 0)$; consider $a = 1$, $b = 2$ when $a\ div\ b = 0$.
using *that* **by** (*simp add*: *pos-imp-zdiv-nonneg-iff flip*: *linorder-not-le*)

lemma *neg-imp-zdiv-neg-iff*:
— But not $(a\ div\ b \leq 0) = (0 \leq a)$; consider $a = -1$, $b = -2$ when $a\ div\ b = 0$.
‹$a\ div\ b < 0 \longleftrightarrow 0 < a$› **if** ‹$b < 0$› **for** $a\ b$ :: *int*
using *that* **by** (*simp add*: *neg-imp-zdiv-nonneg-iff flip*: *linorder-not-le*)

lemma *nonneg1-imp-zdiv-pos-iff*:
‹$a\ div\ b > 0 \longleftrightarrow a \geq b \wedge b > 0$› **if** ‹$0 \leq a$› **for** $a\ b$ :: *int*
proof −
have $0 < a\ div\ b \Longrightarrow b \leq a$
using *div-pos-pos-trivial*[*of a b*] *that* **by** *arith*
moreover have $0 < a\ div\ b \Longrightarrow b > 0$
using *that div-nonneg-neg-le0*[*of a b*] **by** (*cases b=0*; *force*)
moreover have $b \leq a \wedge 0 < b \Longrightarrow 0 < a\ div\ b$
using *int-one-le-iff-zero-less*[*of a div b*] *zdiv-mono1*[*of b a b*] **by** *simp*
ultimately show *?thesis*
by *blast*
qed

lemma *zmod-le-nonneg-dividend*:
‹$m\ mod\ k \leq m$› **if** ‹$(m::int) \geq 0$› **for** $m\ k$ :: *int*
proof −
from *that* **have** ‹$m > 0 \vee m = 0$›
by *auto*
then show *?thesis* **proof**
assume ‹$m = 0$› **then show** *?thesis*
by *simp*
next
assume ‹$m > 0$› **then show** *?thesis*
proof (*cases k* ‹*0::int*› *rule*: *linorder-cases*)
case *less*
moreover define l **where** ‹$l = -k$›
ultimately have ‹$l > 0$›
by *simp*
with ‹$m > 0$› **have** ‹$int\ (nat\ m\ mod\ nat\ l) \leq m$›
by (*simp flip*: *le-nat-iff*)
then have ‹$int\ (nat\ m\ mod\ nat\ l) - l \leq m$›
using ‹$l > 0$› **by** *simp*
with ‹$m > 0$› ‹$l > 0$› **show** *?thesis*
by (*simp add*: *modulo-int-def l-def flip*: *le-nat-iff*)
qed (*simp-all add*: *modulo-int-def flip*: *le-nat-iff*)
qed
qed

lemma *sgn-div-eq-sgn-mult*:
‹*sgn* (*k div l*) = *of-bool* (*k div l* ≠ *0*) ∗ *sgn* (*k* ∗ *l*)›
for *k l* :: *int*
proof (*cases* ‹*k div l* = *0*›)
case *True*
then show *?thesis*
by *simp*
next
case *False*
have ‹*0* ≤ |*k*| *div* |*l*|›
by (*cases* ‹*l* = *0*›) (*simp-all add*: *pos-imp-zdiv-nonneg-iff*)
then have ‹|*k*| *div* |*l*| ≠ *0* ⟷ *0* < |*k*| *div* |*l*|›
by (*simp add*: *less-le*)
also have ‹. . . ⟷ |*k*| ≥ |*l*|›
using *False nonneg1-imp-zdiv-pos-iff* **by** *auto*
finally have ∗: ‹|*k*| *div* |*l*| ≠ *0* ⟷ |*l*| ≤ |*k*|› **.**
show *?thesis*
using ‹*0* ≤ |*k*| *div* |*l*|› *False*
by (*auto simp add*: *div-eq-div-abs* [*of k l*] *div-eq-sgn-abs* [*of k l*]
sgn-mult sgn-1-pos sgn-1-neg sgn-eq-0-iff nonneg1-imp-zdiv-pos-iff ∗ *dest*: *sgn-not-eq-imp*)
qed

56.5.15 Further properties

lemma *div-int-pos-iff*:
k div l ≥ *0* ⟷ *k* = *0* ∨ *l* = *0* ∨ *k* ≥ *0* ∧ *l* ≥ *0*
∨ *k* < *0* ∧ *l* < *0*
for *k l* :: *int*
proof (*cases k* = *0* ∨ *l* = *0*)
case *False*
then have ∗: *k* ≠ *0 l* ≠ *0*
by *auto*
then have *0* ≤ *k div l* ⟹ ¬ *k* < *0* ⟹ *0* ≤ *l*
by (*meson neg-imp-zdiv-neg-iff not-le not-less-iff-gr-or-eq*)
then show *?thesis*
using ∗ **by** (*auto simp add*: *pos-imp-zdiv-nonneg-iff neg-imp-zdiv-nonneg-iff*)
qed *auto*

lemma *mod-int-pos-iff*:
‹*k mod l* ≥ *0* ⟷ *l dvd k* ∨ *l* = *0* ∧ *k* ≥ *0* ∨ *l* > *0*›
for *k l* :: *int*
proof (*cases l* > *0*)
case *False*
then show *?thesis*
by (*simp add*: *dvd-eq-mod-eq-0*) (*use neg-mod-sign* [*of l k*] **in** ‹*auto simp add*:
le-less not-less›)
qed *auto*

lemma *abs-div*:
‹|*x div y*| = |*x*| *div* |*y*|› **if** ‹*y dvd x*› **for** *x y* :: *int*
using *that* **by** (*cases* ‹*y* = *0*›) (*auto simp add*: *abs-mult*)

lemma *int-power-div-base*:
‹*k* ^ *m div k* = *k* ^ (*m* − *Suc 0*)› **if** ‹*0* < *m*› ‹*0* < *k*› **for** *k* :: *int*
using *that* **by** (*cases m*) *simp-all*

lemma *int-div-less-self*:
‹*x div k* < *x*› **if** ‹*0* < *x*› ‹*1* < *k*› **for** *x k* :: *int*
proof −
from *that* **have** ‹*nat* (*x div k*) = *nat x div nat k*›
by (*simp add*: *nat-div-distrib*)
also from *that* **have** ‹*nat x div nat k* < *nat x*›
by *simp*
finally show *?thesis*
by *simp*
qed

56.5.16 Computing *div* and *mod* by shifting

lemma *div-pos-geq*:
‹*k div l* = (*k* − *l*) *div l* + *1*› **if** ‹*0* < *l*› ‹*l* ≤ *k*› **for** *k l* :: *int*
proof −
have *k* = (*k* − *l*) + *l* **by** *simp*
then obtain *j* **where** *k*: *k* = *j* + *l* **..**
with *that* **show** *?thesis* **by** (*simp add*: *div-add-self2*)
qed

lemma *mod-pos-geq*:
‹*k mod l* = (*k* − *l*) *mod l*› **if** ‹*0* < *l*› ‹*l* ≤ *k*› **for** *k l* :: *int*
proof −
have *k* = (*k* − *l*) + *l* **by** *simp*
then obtain *j* **where** *k*: *k* = *j* + *l* **..**
with *that* **show** *?thesis* **by** *simp*
qed

lemma *pos-zdiv-mult-2*: ‹(*1* + *2* ∗ *b*) *div* (*2* ∗ *a*) = *b div a*› (**is** *?Q*)
and *pos-zmod-mult-2*: ‹(*1* + *2* ∗ *b*) *mod* (*2* ∗ *a*) = *1* + *2* ∗ (*b mod a*)› (**is** *?R*)
if ‹*0* ≤ *a*› **for** *a b* :: *int*
proof −
have ‹((*1* + *2* ∗ *b*) *div* (*2* ∗ *a*), (*1* + *2* ∗ *b*) *mod* (*2* ∗ *a*)) = (*b div a*, *1* + *2* ∗ (*b mod a*))›
proof (*induction rule*: *euclidean-relation-intI*)
case *by0*
then show *?case*
by *simp*
next
case *divides*

have ‹*2 dvd* (*2* ∗ *a*)›
by *simp*
then have ‹*2 dvd* (*1* + *2* ∗ *b*)›
using ‹*2* ∗ *a dvd 1* + *2* ∗ *b*› **by** (*rule dvd-trans*)
then have ‹*2 dvd* (*1* + *b* ∗ *2*)›
by (*simp add*: *ac-simps*)
then have ‹*is-unit* (*2* :: *int*)›
by *simp*
then show *?case*
by *simp*
next
case *euclidean-relation*
with *that* **have** ‹*a* > *0*›
by *simp*
moreover have ‹*b mod a* < *a*›
using ‹*a* > *0*› **by** *simp*
then have ‹*1* + *2* ∗ (*b mod a*) < *2* ∗ *a*›
by *simp*
moreover have ‹*2* ∗ (*b mod a*) + *a* ∗ (*2* ∗ (*b div a*)) = *2* ∗ (*b div a* ∗ *a* + *b mod a*)›
by (*simp only*: *algebra-simps*)
moreover have ‹*0* ≤ *2* ∗ (*b mod a*)›
using ‹*a* > *0*› **by** *simp*
ultimately show *?case*
by (*simp add*: *algebra-simps*)
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *neg-zdiv-mult-2*: ‹(*1* + *2* ∗ *b*) *div* (*2* ∗ *a*) = (*b* + *1*) *div a*› (**is** *?Q*)
and *neg-zmod-mult-2*: ‹(*1* + *2* ∗ *b*) *mod* (*2* ∗ *a*) = *2* ∗ ((*b* + *1*) *mod a*) − *1*› (**is** *?R*)
if ‹*a* ≤ *0*› **for** *a b* :: *int*
proof −
have ‹((*1* + *2* ∗ *b*) *div* (*2* ∗ *a*), (*1* + *2* ∗ *b*) *mod* (*2* ∗ *a*)) = ((*b* + *1*) *div a*, *2* ∗ ((*b* + *1*) *mod a*) − *1*)›
proof (*induction rule*: *euclidean-relation-intI*)
case *by0*
then show *?case*
by *simp*
next
case *divides*
have ‹*2 dvd* (*2* ∗ *a*)›
by *simp*
then have ‹*2 dvd* (*1* + *2* ∗ *b*)›
using ‹*2* ∗ *a dvd 1* + *2* ∗ *b*› **by** (*rule dvd-trans*)
then have ‹*2 dvd* (*1* + *b* ∗ *2*)›
by (*simp add*: *ac-simps*)

then have ‹*is-unit* (*2* :: *int*)›
by *simp*
then show *?case*
by *simp*
next
case *euclidean-relation*
with *that* **have** ‹*a* < *0*›
by *simp*
moreover have ‹(*b* + *1*) *mod a* > *a*›
using ‹*a* < *0*› **by** *simp*
then have ‹*2* ∗ ((*b* + *1*) *mod a*) > *1* + *2* ∗ *a*›
by *simp*
moreover have ‹((*1* + *b*) *mod a*) ≤ *0*›
using ‹*a* < *0*› **by** *simp*
then have ‹*2* ∗ ((*1* + *b*) *mod a*) ≤ *0*›
by *simp*
moreover have ‹*2* ∗ ((*1* + *b*) *mod a*) + *a* ∗ (*2* ∗ ((*1* + *b*) *div a*)) =
2 ∗ ((*1* + *b*) *div a* ∗ *a* + (*1* + *b*) *mod a*)›
by (*simp only*: *algebra-simps*)
ultimately show *?case*
by (*simp add*: *algebra-simps sgn-mult abs-mult*)
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *zdiv-numeral-Bit0* [*simp*]:
‹*numeral* (*Num.Bit0 v*) *div numeral* (*Num.Bit0 w*) =
numeral v div (*numeral w* :: *int*)›
unfolding *numeral.simps* **unfolding** *mult-2* [*symmetric*]
by (*rule div-mult-mult1*) *simp*

lemma *zdiv-numeral-Bit1* [*simp*]:
‹*numeral* (*Num.Bit1 v*) *div numeral* (*Num.Bit0 w*) =
(*numeral v div* (*numeral w* :: *int*))›
unfolding *numeral.simps*
unfolding *mult-2* [*symmetric*] *add.commute* [*of - 1*]
by (*rule pos-zdiv-mult-2*) *simp*

lemma *zmod-numeral-Bit0* [*simp*]:
‹*numeral* (*Num.Bit0 v*) *mod numeral* (*Num.Bit0 w*) =
(*2*::*int*) ∗ (*numeral v mod numeral w*)›
unfolding *numeral-Bit0* [*of v*] *numeral-Bit0* [*of w*]
unfolding *mult-2* [*symmetric*] **by** (*rule mod-mult-mult1*)

lemma *zmod-numeral-Bit1* [*simp*]:
‹*numeral* (*Num.Bit1 v*) *mod numeral* (*Num.Bit0 w*) =
2 ∗ (*numeral v mod numeral w*) + (*1*::*int*)›
unfolding *numeral-Bit1* [*of v*] *numeral-Bit0* [*of w*]

unfolding *mult-2* [*symmetric*] *add.commute* [*of* - *1*]
by (*rule pos-zmod-mult-2*) *simp*

56.6 Code generation

context
begin

qualified definition *divmod-nat* :: *nat* ⇒ *nat* ⇒ *nat* × *nat*
where *divmod-nat m n* = (*m div n*, *m mod n*)

qualified lemma *divmod-nat-if* [*code*]:
divmod-nat m n = (*if n* = *0* ∨ *m* < *n then* (*0*, *m*) *else*
let (*q*, *r*) = *divmod-nat* (*m* − *n*) *n in* (*Suc q*, *r*))
by (*simp add*: *divmod-nat-def prod-eq-iff case-prod-beta not-less le-div-geq le-mod-geq*)

qualified lemma [*code*]:
m div n = *fst* (*divmod-nat m n*)
m mod n = *snd* (*divmod-nat m n*)
by (*simp-all add*: *divmod-nat-def*)

end

code-identifier
code-module *Euclidean-Rings* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

57 Parity in rings and semirings

theory *Parity*
imports *Euclidean-Rings*
begin

57.1 Ring structures with parity and *even*/*odd* predicates

class *semiring-parity* = *comm-semiring-1* + *semiring-modulo* +
assumes *mod-2-eq-odd*: ‹*a mod 2* = *of-bool* (¬ *2 dvd a*)›
and *odd-one* [*simp*]: ‹¬ *2 dvd 1*›
and *even-half-succ-eq* [*simp*]: ‹*2 dvd a* ⟹ (*1* + *a*) *div 2* = *a div 2*›
begin

abbreviation *even* :: ′*a* ⇒ *bool*
where ‹*even a* ≡ *2 dvd a*›

abbreviation *odd* :: ′*a* ⇒ *bool*
where ‹*odd a* ≡ ¬ *2 dvd a*›

end

class *ring-parity* = *ring* + *semiring-parity*
begin

subclass *comm-ring-1* **..**

end

instance *nat* :: *semiring-parity*
  by *standard* (*auto simp add*: *dvd-eq-mod-eq-0*)

instance *int* :: *ring-parity*
  by *standard* (*auto simp add*: *dvd-eq-mod-eq-0*)

context *semiring-parity*
begin

lemma *evenE* [*elim?*]:
  assumes ‹*even a*›
  obtains *b* **where** ‹*a* = *2* ∗ *b*›
  using *assms* **by** (*rule dvdE*)

lemma *oddE* [*elim?*]:
  assumes ‹*odd a*›
  obtains *b* **where** ‹*a* = *2* ∗ *b* + *1*›
proof −
  have ‹*a* = *2* ∗ (*a div 2*) + *a mod 2*›
    by (*simp add*: *mult-div-mod-eq*)
  with *assms* **have** ‹*a* = *2* ∗ (*a div 2*) + *1*›
    by (*simp add*: *mod-2-eq-odd*)
  then show *thesis* **..**
qed

lemma *of-bool-odd-eq-mod-2*:
  ‹*of-bool* (*odd a*) = *a mod 2*›
  by (*simp add*: *mod-2-eq-odd*)

lemma *odd-of-bool-self* [*simp*]:
  ‹*odd* (*of-bool p*) ⟷ *p*›
  by (*cases p*) *simp-all*

lemma *not-mod-2-eq-0-eq-1* [*simp*]:
  ‹*a mod 2* ≠ *0* ⟷ *a mod 2* = *1*›
  by (*simp add*: *mod-2-eq-odd*)

lemma *not-mod-2-eq-1-eq-0* [*simp*]:
  ‹*a mod 2* ≠ *1* ⟷ *a mod 2* = *0*›
  by (*simp add*: *mod-2-eq-odd*)

lemma *even-iff-mod-2-eq-zero*:
‹*2 dvd a* ⟷ *a mod 2* = *0*›
by (*simp add*: *mod-2-eq-odd*)

lemma *odd-iff-mod-2-eq-one*:
‹¬ *2 dvd a* ⟷ *a mod 2* = *1*›
by (*simp add*: *mod-2-eq-odd*)

lemma *even-mod-2-iff* [*simp*]:
‹*even* (*a mod 2*) ⟷ *even a*›
by (*simp add*: *mod-2-eq-odd*)

lemma *mod2-eq-if*:
a mod 2 = (*if even a then 0 else 1*)
by (*simp add*: *mod-2-eq-odd*)

lemma *zero-mod-two-eq-zero* [*simp*]:
‹*0 mod 2* = *0*›
by (*simp add*: *mod-2-eq-odd*)

lemma *one-mod-two-eq-one* [*simp*]:
‹*1 mod 2* = *1*›
by (*simp add*: *mod-2-eq-odd*)

lemma *parity-cases* [*case-names even odd*]:
assumes ‹*even a* ⟹ *a mod 2* = *0* ⟹ *P*›
assumes ‹*odd a* ⟹ *a mod 2* = *1* ⟹ *P*›
shows *P*
using *assms* **by** (*auto simp add*: *mod-2-eq-odd*)

lemma *even-zero* [*simp*]:
‹*even 0*›
by (*fact dvd-0-right*)

lemma *odd-even-add*:
even (*a* + *b*) **if** *odd a* **and** *odd b*
proof −
from *that* **obtain** *c d* **where** *a* = *2* ∗ *c* + *1* **and** *b* = *2* ∗ *d* + *1*
by (*blast elim*: *oddE*)
then have *a* + *b* = *2* ∗ *c* + *2* ∗ *d* + (*1* + *1*)
by (*simp only*: *ac-simps*)
also have . . . = *2* ∗ (*c* + *d* + *1*)
by (*simp add*: *algebra-simps*)
finally show *?thesis* **..**
qed

lemma *even-add* [*simp*]:
even (*a* + *b*) ⟷ (*even a* ⟷ *even b*)

by (*auto simp add*: *dvd-add-right-iff dvd-add-left-iff odd-even-add*)

lemma *odd-add* [*simp*]:
odd (*a* + *b*) ⟷ ¬ (*odd a* ⟷ *odd b*)
by *simp*

lemma *even-plus-one-iff* [*simp*]:
even (*a* + *1*) ⟷ *odd a*
by (*auto simp add*: *dvd-add-right-iff intro*: *odd-even-add*)

lemma *even-mult-iff* [*simp*]:
even (*a* ∗ *b*) ⟷ *even a* ∨ *even b* (**is** *?P* ⟷ *?Q*)
proof
assume *?Q*
then show *?P*
by *auto*
next
assume *?P*
show *?Q*
proof (*rule ccontr*)
assume ¬ (*even a* ∨ *even b*)
then have *odd a* **and** *odd b*
by *auto*
then obtain *r s* **where** *a* = *2* ∗ *r* + *1* **and** *b* = *2* ∗ *s* + *1*
by (*blast elim*: *oddE*)
then have *a* ∗ *b* = (*2* ∗ *r* + *1*) ∗ (*2* ∗ *s* + *1*)
by *simp*
also have ... = *2* ∗ (*2* ∗ *r* ∗ *s* + *r* + *s*) + *1*
by (*simp add*: *algebra-simps*)
finally have *odd* (*a* ∗ *b*)
by *simp*
with ‹*?P*› **show** *False*
by *auto*
qed
qed

lemma *even-numeral* [*simp*]: *even* (*numeral* (*Num.Bit0 n*))
proof −
have *even* (*2* ∗ *numeral n*)
unfolding *even-mult-iff* **by** *simp*
then have *even* (*numeral n* + *numeral n*)
unfolding *mult-2* **.**
then show *?thesis*
unfolding *numeral.simps* **.**
qed

lemma *odd-numeral* [*simp*]: *odd* (*numeral* (*Num.Bit1 n*))
proof
assume *even* (*numeral* (*num.Bit1 n*))

then have *even* (*numeral n* + *numeral n* + *1*)
unfolding *numeral.simps* **.**
then have *even* (*2* ∗ *numeral n* + *1*)
unfolding *mult-2* **.**
then have *2 dvd numeral n* ∗ *2* + *1*
by (*simp add*: *ac-simps*)
then have *2 dvd 1*
using *dvd-add-times-triv-left-iff* [*of 2 numeral n 1*] **by** *simp*
then show *False* **by** *simp*
qed

lemma *odd-numeral-BitM* [*simp*]:
‹*odd* (*numeral* (*Num.BitM w*))›
by (*cases w*) *simp-all*

lemma *even-power* [*simp*]: *even* (*a* ^ *n*) ⟷ *even a* ∧ *n* > *0*
by (*induct n*) *auto*

lemma *even-prod-iff*:
‹*even* (*prod f A*) ⟷ (∃ *a*∈*A*. *even* (*f a*))› **if** ‹*finite A*›
using *that* **by** (*induction A*) *simp-all*

lemma *even-half-maybe-succ-eq* [*simp*]:
‹*even a* ⟹ (*of-bool b* + *a*) *div 2* = *a div 2*›
by *simp*

lemma *even-half-maybe-succ′-eq* [*simp*]:
‹*even a* ⟹ (*b mod 2* + *a*) *div 2* = *a div 2*›
by (*simp add*: *mod2-eq-if*)

lemma *mask-eq-sum-exp*:
‹*2* ^ *n* − *1* = (∑ *m*∈{*q*. *q* < *n*}. *2* ^ *m*)›
proof −
have ∗: ‹{*q*. *q* < *Suc m*} = *insert m* {*q*. *q* < *m*}› **for** *m*
by *auto*
have ‹*2* ^ *n* = (∑ *m*∈{*q*. *q* < *n*}. *2* ^ *m*) + *1*›
by (*induction n*) (*simp-all add*: *ac-simps mult-2* ∗)
then have ‹*2* ^ *n* − *1* = (∑ *m*∈{*q*. *q* < *n*}. *2* ^ *m*) + *1* − *1*›
by *simp*
then show *?thesis*
by *simp*
qed

lemma (**in** −) *mask-eq-sum-exp-nat*:
‹*2* ^ *n* − *Suc 0* = (∑ *m*∈{*q*. *q* < *n*}. *2* ^ *m*)›
using *mask-eq-sum-exp* [**where** *?′a* = *nat*] **by** *simp*

end

context *ring-parity*
begin

lemma *even-minus*:
even $(- a) \longleftrightarrow$ *even a*
by (*fact dvd-minus-iff*)

lemma *even-diff* [*simp*]:
even $(a - b) \longleftrightarrow$ *even* $(a + b)$
using *even-add* [*of* $a - b$] **by** *simp*

end

57.2 Instance for *nat*

lemma *even-Suc-Suc-iff* [*simp*]:
even (*Suc* (*Suc n*)) $\longleftrightarrow$ *even n*
using *dvd-add-triv-right-iff* [*of 2 n*] **by** *simp*

lemma *even-Suc* [*simp*]: *even* (*Suc n*) $\longleftrightarrow$ *odd n*
using *even-plus-one-iff* [*of n*] **by** *simp*

lemma *even-diff-nat* [*simp*]:
even $(m - n) \longleftrightarrow m < n \lor$ *even* $(m + n)$ **for** *m n* :: *nat*
proof (*cases* $n \leq m$)
case *True*
then have $m - n + n * 2 = m + n$ **by** (*simp add*: *mult-2-right*)
moreover have *even* $(m - n) \longleftrightarrow$ *even* $(m - n + n * 2)$ **by** *simp*
ultimately have *even* $(m - n) \longleftrightarrow$ *even* $(m + n)$ **by** (*simp only*:)
then show *?thesis* **by** *auto*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *odd-pos*:
odd n $\Longrightarrow 0 < n$ **for** *n* :: *nat*
by (*auto elim*: *oddE*)

lemma *Suc-double-not-eq-double*:
Suc $(2 * m) \neq 2 * n$
proof
assume *Suc* $(2 * m) = 2 * n$
moreover have *odd* (*Suc* $(2 * m)$) **and** *even* $(2 * n)$
by *simp-all*
ultimately show *False* **by** *simp*
qed

lemma *double-not-eq-Suc-double*:

2 ∗ m ≠ Suc (2 ∗ n)
using *Suc-double-not-eq-double* [*of n m*] **by** *simp*

lemma *odd-Suc-minus-one* [*simp*]: *odd n* ⟹ *Suc* (*n* − *Suc 0*) = *n*
by (*auto elim*: *oddE*)

lemma *even-Suc-div-two* [*simp*]:
even n ⟹ *Suc n div 2* = *n div 2*
by *auto*

lemma *odd-Suc-div-two* [*simp*]:
odd n ⟹ *Suc n div 2* = *Suc* (*n div 2*)
by (*auto elim*: *oddE*)

lemma *odd-two-times-div-two-nat* [*simp*]:
assumes *odd n*
shows *2* ∗ (*n div 2*) = *n* − (*1* :: *nat*)
proof −
from *assms* **have** *2* ∗ (*n div 2*) + *1* = *n*
by (*auto elim*: *oddE*)
then have *Suc* (*2* ∗ (*n div 2*)) − *1* = *n* − *1*
by *simp*
then show *?thesis*
by *simp*
qed

lemma *not-mod2-eq-Suc-0-eq-0* [*simp*]:
n mod 2 ≠ *Suc 0* ⟷ *n mod 2* = *0*
using *not-mod-2-eq-1-eq-0* [*of n*] **by** *simp*

lemma *odd-card-imp-not-empty*:
‹*A* ≠ {}› **if** ‹*odd* (*card A*)›
using *that* **by** *auto*

lemma *nat-induct2* [*case-names 0 1 step*]:
assumes *P 0 P 1* **and** *step*: ⋀*n*::*nat*. *P n* ⟹ *P* (*n* + *2*)
shows *P n*
proof (*induct n rule*: *less-induct*)
case (*less n*)
show *?case*
proof (*cases n* < *Suc* (*Suc 0*))
case *True*
then show *?thesis*
using *assms* **by** (*auto simp*: *less-Suc-eq*)
next
case *False*
then obtain *k* **where** *k*: *n* = *Suc* (*Suc k*)
by (*force simp*: *not-less nat-le-iff-add*)
then have *k*<*n*

by *simp*
with *less assms* **have** *P* (*k*+*2*)
by *blast*
then show *?thesis*
by (*simp add*: *k*)
qed
qed

lemma *mod-double-nat*:
‹n *mod* $(2 * m) = n$ *mod* $m \vee n$ *mod* $(2 * m) = n$ *mod* $m + m$›
for *m n* :: *nat*
by (*cases* ‹*even* (*n div m*)›)
(*simp-all add*: *mod-mult2-eq ac-simps even-iff-mod-2-eq-zero*)

context *semiring-parity*
begin

lemma *even-sum-iff*:
‹*even* (*sum f A*) ⟷ *even* (*card* {*a*∈*A*. *odd* (*f a*)})› **if** ‹*finite A*›
using *that* **proof** (*induction A*)
case *empty*
then show *?case*
by *simp*
next
case (*insert a A*)
moreover have ‹{*b* ∈ *insert a A*. *odd* (*f b*)} = (*if odd* (*f a*) *then* {*a*} *else* {}) ∪ {*b* ∈ *A*. *odd* (*f b*)}›
by *auto*
ultimately show *?case*
by *simp*
qed

lemma *even-mask-iff* [*simp*]:
‹*even* $(2$ ^ $n - 1)$ ⟷ $n = 0$›
proof (*cases* ‹$n = 0$›)
case *True*
then show *?thesis*
by *simp*
next
case *False*
then have ‹$\{a.\ a = 0 \wedge a < n\} = \{0\}$›
by *auto*
then show *?thesis*
by (*auto simp add*: *mask-eq-sum-exp even-sum-iff*)
qed

lemma *even-of-nat-iff* [*simp*]:
even (*of-nat n*) ⟷ *even n*
by (*induction n*) *simp-all*

end

57.3 Parity and powers

context *ring-1*
begin

lemma *power-minus-even* [*simp*]: *even n* $\Longrightarrow$ (− *a*) ^ *n* = *a* ^ *n*
  by (*auto elim*: *evenE*)

lemma *power-minus-odd* [*simp*]: *odd n* $\Longrightarrow$ (− *a*) ^ *n* = − (*a* ^ *n*)
  by (*auto elim*: *oddE*)

lemma *uminus-power-if*:
  (− *a*) ^ *n* = (*if even n then a* ^ *n else* − (*a* ^ *n*))
  by *auto*

lemma *neg-one-even-power* [*simp*]: *even n* $\Longrightarrow$ (− *1*) ^ *n* = *1*
  by *simp*

lemma *neg-one-odd-power* [*simp*]: *odd n* $\Longrightarrow$ (− *1*) ^ *n* = − *1*
  by *simp*

lemma *neg-one-power-add-eq-neg-one-power-diff*: *k* ≤ *n* $\Longrightarrow$ (− *1*) ^ (*n* + *k*) = (− *1*) ^ (*n* − *k*)
  by (*cases even* (*n* + *k*)) *auto*

lemma *minus-one-power-iff*: (− *1*) ^ *n* = (*if even n then 1 else* − *1*)
  by (*induct n*) *auto*

end

context *linordered-idom*
begin

lemma *zero-le-even-power*: *even n* $\Longrightarrow$ *0* ≤ *a* ^ *n*
  by (*auto elim*: *evenE*)

lemma *zero-le-odd-power*: *odd n* $\Longrightarrow$ *0* ≤ *a* ^ *n* ⟷ *0* ≤ *a*
  by (*auto simp add*: *power-even-eq zero-le-mult-iff elim*: *oddE*)

lemma *zero-le-power-eq*: *0* ≤ *a* ^ *n* ⟷ *even n* ∨ *odd n* ∧ *0* ≤ *a*
  by (*auto simp add*: *zero-le-even-power zero-le-odd-power*)

lemma *zero-less-power-eq*: *0* < *a* ^ *n* ⟷ *n* = *0* ∨ *even n* ∧ *a* ≠ *0* ∨ *odd n* ∧ *0* < *a*
proof −
  have [*simp*]: *0* = *a* ^ *n* ⟷ *a* = *0* ∧ *n* > *0*

```
    unfolding power-eq-0-iff [of a n, symmetric] by blast
  show ?thesis
    unfolding less-le zero-le-power-eq by auto
qed

lemma power-less-zero-eq [simp]: a ^ n < 0 ⟷ odd n ∧ a < 0
  unfolding not-le [symmetric] zero-le-power-eq by auto

lemma power-le-zero-eq: a ^ n ≤ 0 ⟷ n > 0 ∧ (odd n ∧ a ≤ 0 ∨ even n ∧ a
= 0)
  unfolding not-less [symmetric] zero-less-power-eq by auto

lemma power-even-abs: even n ⟹ |a| ^ n = a ^ n
  using power-abs [of a n] by (simp add: zero-le-even-power)

lemma power-mono-even:
  assumes even n and |a| ≤ |b|
  shows a ^ n ≤ b ^ n
proof −
  have 0 ≤ |a| by auto
  with ‹|a| ≤ |b|› have |a| ^ n ≤ |b| ^ n
    by (rule power-mono)
  with ‹even n› show ?thesis
    by (simp add: power-even-abs)
qed

lemma power-mono-odd:
  assumes odd n and a ≤ b
  shows a ^ n ≤ b ^ n
proof (cases b < 0)
  case True
  with ‹a ≤ b› have − b ≤ − a and 0 ≤ − b by auto
  then have (− b) ^ n ≤ (− a) ^ n by (rule power-mono)
  with ‹odd n› show ?thesis by simp
next
  case False
  then have 0 ≤ b by auto
  show ?thesis
  proof (cases a < 0)
    case True
    then have n ≠ 0 and a ≤ 0 using ‹odd n› [THEN odd-pos] by auto
    then have a ^ n ≤ 0 unfolding power-le-zero-eq using ‹odd n› by auto
    moreover from ‹0 ≤ b› have 0 ≤ b ^ n by auto
    ultimately show ?thesis by auto
  next
    case False
    then have 0 ≤ a by auto
    with ‹a ≤ b› show ?thesis
      using power-mono by auto
```

qed
qed

Simplify, when the exponent is a numeral

lemma *zero-le-power-eq-numeral* [*simp*]:
$0 \le a$ ^ $numeral\ w \longleftrightarrow even\ (numeral\ w :: nat) \lor odd\ (numeral\ w :: nat) \land 0 \le a$
by (*fact zero-le-power-eq*)

lemma *zero-less-power-eq-numeral* [*simp*]:
$0 < a$ ^ $numeral\ w \longleftrightarrow$
$numeral\ w = (0 :: nat) \lor$
$even\ (numeral\ w :: nat) \land a \neq 0 \lor$
$odd\ (numeral\ w :: nat) \land 0 < a$
by (*fact zero-less-power-eq*)

lemma *power-le-zero-eq-numeral* [*simp*]:
a ^ $numeral\ w \le 0 \longleftrightarrow$
$(0 :: nat) < numeral\ w \land$
$(odd\ (numeral\ w :: nat) \land a \le 0 \lor even\ (numeral\ w :: nat) \land a = 0)$
by (*fact power-le-zero-eq*)

lemma *power-less-zero-eq-numeral* [*simp*]:
a ^ $numeral\ w < 0 \longleftrightarrow odd\ (numeral\ w :: nat) \land a < 0$
by (*fact power-less-zero-eq*)

lemma *power-even-abs-numeral* [*simp*]:
$even\ (numeral\ w :: nat) \Longrightarrow |a|$ ^ $numeral\ w = a$ ^ $numeral\ w$
by (*fact power-even-abs*)

end

57.4 Instance for *int*

lemma *even-diff-iff*:
$even\ (k - l) \longleftrightarrow even\ (k + l)$ **for** $k\ l :: int$
by (*fact even-diff*)

lemma *even-abs-add-iff*:
$even\ (|k| + l) \longleftrightarrow even\ (k + l)$ **for** $k\ l :: int$
by *simp*

lemma *even-add-abs-iff*:
$even\ (k + |l|) \longleftrightarrow even\ (k + l)$ **for** $k\ l :: int$
by *simp*

lemma *even-nat-iff*: $0 \le k \Longrightarrow even\ (nat\ k) \longleftrightarrow even\ k$
by (*simp add*: *even-of-nat-iff* [*of nat k*, **where** *?'a = int*, *symmetric*])

context
 assumes *SORT-CONSTRAINT*(*'a*::*division-ring*)
begin

lemma *power-int-minus-left*:
 power-int $(-a :: 'a)$ *n* = (*if even n then power-int a n else* −*power-int a n*)
 by (*auto simp*: *power-int-def minus-one-power-iff even-nat-iff*)

lemma *power-int-minus-left-even* [*simp*]: *even n* ⟹ *power-int* $(-a :: 'a)$ *n* = *power-int a n*
 by (*simp add*: *power-int-minus-left*)

lemma *power-int-minus-left-odd* [*simp*]: *odd n* ⟹ *power-int* $(-a :: 'a)$ *n* = −*power-int a n*
 by (*simp add*: *power-int-minus-left*)

lemma *power-int-minus-left-distrib*:
 NO-MATCH (−*1*) *x* ⟹ *power-int* $(-a :: 'a)$ *n* = *power-int* (−*1*) *n* ∗ *power-int a n*
 by (*simp add*: *power-int-minus-left*)

lemma *power-int-minus-one-minus*: *power-int* (−*1* :: *'a*) (−*n*) = *power-int* (−*1*) *n*
 by (*simp add*: *power-int-minus-left*)

lemma *power-int-minus-one-diff-commute*: *power-int* (−*1* :: *'a*) (*a* − *b*) = *power-int* (−*1*) (*b* − *a*)
 by (*subst power-int-minus-one-minus* [*symmetric*]) *auto*

lemma *power-int-minus-one-mult-self* [*simp*]:
 power-int (−*1* :: *'a*) *m* ∗ *power-int* (−*1*) *m* = *1*
 by (*simp add*: *power-int-minus-left*)

lemma *power-int-minus-one-mult-self'* [*simp*]:
 power-int (−*1* :: *'a*) *m* ∗ (*power-int* (−*1*) *m* ∗ *b*) = *b*
 by (*simp add*: *power-int-minus-left*)

end

57.5 Special case: euclidean rings structurally containing the natural numbers

class *linordered-euclidean-semiring* = *discrete-linordered-semidom* + *unique-euclidean-semiring* +
 assumes *of-nat-div*: *of-nat* (*m div n*) = *of-nat m div of-nat n*
 and *division-segment-of-nat* [*simp*]: *division-segment* (*of-nat n*) = *1*
 and *division-segment-euclidean-size* [*simp*]: *division-segment a* ∗ *of-nat* (*euclidean-size a*) = *a*
begin

lemma *division-segment-eq-iff*:
a = *b* **if** *division-segment a* = *division-segment b*
and *euclidean-size a* = *euclidean-size b*
using *that division-segment-euclidean-size* [*of a*] **by** *simp*

lemma *euclidean-size-of-nat* [*simp*]:
euclidean-size (*of-nat n*) = *n*
proof −
have *division-segment* (*of-nat n*) ∗ *of-nat* (*euclidean-size* (*of-nat n*)) = *of-nat n*
by (*fact division-segment-euclidean-size*)
then show *?thesis* **by** *simp*
qed

lemma *of-nat-euclidean-size*:
of-nat (*euclidean-size a*) = *a div division-segment a*
proof −
have *of-nat* (*euclidean-size a*) = *division-segment a* ∗ *of-nat* (*euclidean-size a*) *div division-segment a*
by (*subst nonzero-mult-div-cancel-left*) *simp-all*
also have . . . = *a div division-segment a*
by *simp*
finally show *?thesis* **.**
qed

lemma *division-segment-1* [*simp*]:
division-segment 1 = *1*
using *division-segment-of-nat* [*of 1*] **by** *simp*

lemma *division-segment-numeral* [*simp*]:
division-segment (*numeral k*) = *1*
using *division-segment-of-nat* [*of numeral k*] **by** *simp*

lemma *euclidean-size-1* [*simp*]:
euclidean-size 1 = *1*
using *euclidean-size-of-nat* [*of 1*] **by** *simp*

lemma *euclidean-size-numeral* [*simp*]:
euclidean-size (*numeral k*) = *numeral k*
using *euclidean-size-of-nat* [*of numeral k*] **by** *simp*

lemma *of-nat-dvd-iff*:
of-nat m dvd of-nat n ⟷ *m dvd n* (**is** *?P* ⟷ *?Q*)
proof (*cases m* = *0*)
case *True*
then show *?thesis*
by *simp*
next
case *False*

```
  show ?thesis
  proof
    assume ?Q
    then show ?P
      by auto
  next
    assume ?P
    with False have of-nat n = of-nat n div of-nat m * of-nat m
      by simp
    then have of-nat n = of-nat (n div m * m)
      by (simp add: of-nat-div)
    then have n = n div m * m
      by (simp only: of-nat-eq-iff)
    then have n = m * (n div m)
      by (simp add: ac-simps)
    then show ?Q ..
  qed
qed

lemma of-nat-mod:
  of-nat (m mod n) = of-nat m mod of-nat n
proof -
  have of-nat m div of-nat n * of-nat n + of-nat m mod of-nat n = of-nat m
    by (simp add: div-mult-mod-eq)
  also have of-nat m = of-nat (m div n * n + m mod n)
    by simp
  finally show ?thesis
    by (simp only: of-nat-div of-nat-mult of-nat-add) simp
qed

lemma one-div-two-eq-zero [simp]:
  1 div 2 = 0
proof -
  from of-nat-div [symmetric] have of-nat 1 div of-nat 2 = of-nat 0
    by (simp only:) simp
  then show ?thesis
    by simp
qed

lemma one-mod-2-pow-eq [simp]:
  1 mod (2 ^ n) = of-bool (n > 0)
proof -
  have 1 mod (2 ^ n) = of-nat (1 mod (2 ^ n))
    using of-nat-mod [of 1 2 ^ n] by simp
  also have ... = of-bool (n > 0)
    by simp
  finally show ?thesis .
qed
```

lemma *one-div-2-pow-eq* [*simp*]:
  1 div (*2* ^ *n*) = *of-bool* (*n* = *0*)
  using *div-mult-mod-eq* [*of 1 2* ^ *n*] **by** *auto*

lemma *div-mult2-eq'*:
  ‹*a div* (*of-nat m* ∗ *of-nat n*) = *a div of-nat m div of-nat n*›
proof (*cases* ‹*m* = *0* ∨ *n* = *0*›)
  case *True*
  then show *?thesis*
    by *auto*
next
  case *False*
  then have ‹*m* > *0*› ‹*n* > *0*›
    by *simp-all*
  show *?thesis*
  proof (*cases* ‹*of-nat m* ∗ *of-nat n dvd a*›)
    case *True*
    then obtain *b* **where** ‹*a* = (*of-nat m* ∗ *of-nat n*) ∗ *b*› **..**
    then have ‹*a* = *of-nat m* ∗ (*of-nat n* ∗ *b*)›
      by (*simp add*: *ac-simps*)
    then show *?thesis*
      by *simp*
  next
    case *False*
    define *q* **where** ‹*q* = *a div* (*of-nat m* ∗ *of-nat n*)›
    define *r* **where** ‹*r* = *a mod* (*of-nat m* ∗ *of-nat n*)›
    from ‹*m* > *0*› ‹*n* > *0*› ‹¬ *of-nat m* ∗ *of-nat n dvd a*› *r-def* **have** *division-segment r* = *1*
      using *division-segment-of-nat* [*of m* ∗ *n*] **by** (*simp add*: *division-segment-mod*)
    with *division-segment-euclidean-size* [*of r*]
    have *of-nat* (*euclidean-size r*) = *r*
      by *simp*
    have *a mod* (*of-nat m* ∗ *of-nat n*) *div* (*of-nat m* ∗ *of-nat n*) = *0*
      by *simp*
    with ‹*m* > *0*› ‹*n* > *0*› *r-def* **have** *r div* (*of-nat m* ∗ *of-nat n*) = *0*
      by *simp*
    with ‹*of-nat* (*euclidean-size r*) = *r*›
    have *of-nat* (*euclidean-size r*) *div* (*of-nat m* ∗ *of-nat n*) = *0*
      by *simp*
    then have *of-nat* (*euclidean-size r div* (*m* ∗ *n*)) = *0*
      by (*simp add*: *of-nat-div*)
    then have *of-nat* (*euclidean-size r div m div n*) = *0*
      by (*simp add*: *div-mult2-eq*)
    with ‹*of-nat* (*euclidean-size r*) = *r*› **have** *r div of-nat m div of-nat n* = *0*
      by (*simp add*: *of-nat-div*)
    with ‹*m* > *0*› ‹*n* > *0*› *q-def*
    have *q* = (*r div of-nat m* + *q* ∗ *of-nat n* ∗ *of-nat m div of-nat m*) *div of-nat n*
      by *simp*
    moreover have ‹*a* = *q* ∗ (*of-nat m* ∗ *of-nat n*) + *r*›

by (*simp add*: *q-def r-def div-mult-mod-eq*)
ultimately show ‹*a div* (*of-nat m* ∗ *of-nat n*) = *a div of-nat m div of-nat n*›
using *q-def* [*symmetric*] *div-plus-div-distrib-dvd-right* [*of* ‹*of-nat m*› ‹*q* ∗ (*of-nat m* ∗ *of-nat n*)› *r*]
by (*simp add*: *ac-simps*)
qed
qed

lemma *mod-mult2-eq′*:
a mod (*of-nat m* ∗ *of-nat n*) = *of-nat m* ∗ (*a div of-nat m mod of-nat n*) + *a mod of-nat m*
proof −
have *a div* (*of-nat m* ∗ *of-nat n*) ∗ (*of-nat m* ∗ *of-nat n*) + *a mod* (*of-nat m* ∗ *of-nat n*) = *a div of-nat m div of-nat n* ∗ *of-nat n* ∗ *of-nat m* + (*a div of-nat m mod of-nat n* ∗ *of-nat m* + *a mod of-nat m*)
by (*simp add*: *combine-common-factor div-mult-mod-eq*)
moreover have *a div of-nat m div of-nat n* ∗ *of-nat n* ∗ *of-nat m* = *of-nat n* ∗ *of-nat m* ∗ (*a div of-nat m div of-nat n*)
by (*simp add*: *ac-simps*)
ultimately show *?thesis*
by (*simp add*: *div-mult2-eq′ mult-commute*)
qed

lemma *div-mult2-numeral-eq*:
a div numeral k div numeral l = *a div numeral* (*k* ∗ *l*) (**is** *?A* = *?B*)
proof −
have *?A* = *a div of-nat* (*numeral k*) *div of-nat* (*numeral l*)
by *simp*
also have ... = *a div* (*of-nat* (*numeral k*) ∗ *of-nat* (*numeral l*))
by (*fact div-mult2-eq′* [*symmetric*])
also have ... = *?B*
by *simp*
finally show *?thesis* **.**
qed

lemma *numeral-Bit0-div-2*:
numeral (*num.Bit0 n*) *div 2* = *numeral n*
proof −
have *numeral* (*num.Bit0 n*) = *numeral n* + *numeral n*
by (*simp only*: *numeral.simps*)
also have ... = *numeral n* ∗ *2*
by (*simp add*: *mult-2-right*)
finally have *numeral* (*num.Bit0 n*) *div 2* = *numeral n* ∗ *2 div 2*
by *simp*
also have ... = *numeral n*
by (*rule nonzero-mult-div-cancel-right*) *simp*
finally show *?thesis* **.**
qed

lemma *numeral-Bit1-div-2*:
numeral (*num.Bit1 n*) *div 2* = *numeral n*
proof –
have *numeral* (*num.Bit1 n*) = *numeral n* + *numeral n* + *1*
by (*simp only*: *numeral.simps*)
also have ... = *numeral n* ∗ *2* + *1*
by (*simp add*: *mult-2-right*)
finally have *numeral* (*num.Bit1 n*) *div 2* = (*numeral n* ∗ *2* + *1*) *div 2*
by *simp*
also have ... = *numeral n* ∗ *2 div 2* + *1 div 2*
using *dvd-triv-right* **by** (*rule div-plus-div-distrib-dvd-left*)
also have ... = *numeral n* ∗ *2 div 2*
by *simp*
also have ... = *numeral n*
by (*rule nonzero-mult-div-cancel-right*) *simp*
finally show *?thesis* **.**
qed

lemma *exp-mod-exp*:
‹*2* ^ *m mod 2* ^ *n* = *of-bool* (*m* < *n*) ∗ *2* ^ *m*›
proof –
have ‹(*2*::*nat*) ^ *m mod 2* ^ *n* = *of-bool* (*m* < *n*) ∗ *2* ^ *m*› (**is** ‹*?lhs* = *?rhs*›)
by (*auto simp add*: *linorder-class.not-less monoid-mult-class.power-add dest*!: *le-Suc-ex*)
then have ‹*of-nat ?lhs* = *of-nat ?rhs*›
by *simp*
then show *?thesis*
by (*simp add*: *of-nat-mod*)
qed

lemma *mask-mod-exp*:
‹(*2* ^ *n* − *1*) *mod 2* ^ *m* = *2* ^ *min m n* − *1*›
proof –
have ‹(*2* ^ *n* − *1*) *mod 2* ^ *m* = *2* ^ *min m n* − (*1*::*nat*)› (**is** ‹*?lhs* = *?rhs*›)
proof (*cases* ‹*n* ≤ *m*›)
case *True*
then show *?thesis*
by (*simp add*: *Suc-le-lessD*)
next
case *False*
then have ‹*m* < *n*›
by *simp*
then obtain *q* **where** *n*: ‹*n* = *Suc q* + *m*›
by (*auto dest*: *less-imp-Suc-add*)
then have ‹*min m n* = *m*›
by *simp*
moreover have ‹(*2*::*nat*) ^ *m* ≤ *2* ∗ *2* ^ *q* ∗ *2* ^ *m*›
using *mult-le-mono1* [*of 1* ‹*2* ∗ *2* ^ *q*› ‹*2* ^ *m*›] **by** *simp*
with *n* **have** ‹*2* ^ *n* − *1* = (*2* ^ *Suc q* − *1*) ∗ *2* ^ *m* + (*2* ^ *m* − (*1*::*nat*))›

by (*simp add*: *monoid-mult-class.power-add algebra-simps*)
ultimately show *?thesis*
by (*simp only*: *euclidean-semiring-cancel-class.mod-mult-self3*) *simp*
qed
then have ‹*of-nat ?lhs* = *of-nat ?rhs*›
by *simp*
then show *?thesis*
by (*simp add*: *of-nat-mod of-nat-diff*)
qed

lemma *of-bool-half-eq-0* [*simp*]:
‹*of-bool b div 2* = *0*›
by *simp*

lemma *of-nat-mod-double*:
‹*of-nat n mod* (*2* ∗ *of-nat m*) = *of-nat n mod of-nat m* ∨ *of-nat n mod* (*2* ∗ *of-nat m*) = *of-nat n mod of-nat m* + *of-nat m*›
by (*simp add*: *mod-double-nat flip*: *of-nat-mod of-nat-add of-nat-mult of-nat-numeral*)

end

instance *nat* :: *linordered-euclidean-semiring*
by *standard simp-all*

instance *int* :: *linordered-euclidean-semiring*
by *standard* (*auto simp add*: *divide-int-def division-segment-int-def elim*: *contrapos-np*)

context *linordered-euclidean-semiring*
begin

subclass *semiring-parity*
proof
show ‹*a mod 2* = *of-bool* (¬ *2 dvd a*)› **for** *a*
proof (*cases* ‹*2 dvd a*›)
case *True*
then show *?thesis*
by (*simp add*: *dvd-eq-mod-eq-0*)
next
case *False*
have *eucl*: *euclidean-size* (*a mod 2*) = *1*
proof (*rule Orderings.order-antisym*)
show *euclidean-size* (*a mod 2*) ≤ *1*
using *mod-size-less* [*of 2 a*] **by** *simp*
show *1* ≤ *euclidean-size* (*a mod 2*)
using ‹¬ *2 dvd a*› **by** (*simp add*: *Suc-le-eq dvd-eq-mod-eq-0*)
qed
from ‹¬ *2 dvd a*› **have** ¬ *of-nat 2 dvd division-segment a* ∗ *of-nat* (*euclidean-size a*)

by *simp*
then have ¬ *of-nat 2 dvd of-nat* (*euclidean-size a*)
by (*auto simp only*: *dvd-mult-unit-iff ' is-unit-division-segment*)
then have ¬ *2 dvd euclidean-size a*
using *of-nat-dvd-iff* [*of 2*] **by** *simp*
then have *euclidean-size a mod 2 = 1*
by (*simp add*: *semidom-modulo-class.dvd-eq-mod-eq-0*)
then have *of-nat* (*euclidean-size a mod 2*) = *of-nat 1*
by *simp*
then have *of-nat* (*euclidean-size a*) *mod 2 = 1*
by (*simp add*: *of-nat-mod*)
from ‹¬ *2 dvd a*› *eucl*
have *a mod 2 = 1*
by (*auto intro*: *division-segment-eq-iff simp add*: *division-segment-mod*)
with ‹¬ *2 dvd a*› **show** *?thesis*
by *simp*
qed
show ‹¬ *is-unit 2*›
proof
assume ‹*is-unit 2*›
then have ‹*of-nat 2 dvd of-nat 1*›
by *simp*
then have ‹*is-unit* (*2::nat*)›
by (*simp only*: *of-nat-dvd-iff*)
then show *False*
by *simp*
qed
show ‹(*1* + *a*) *div 2 = a div 2*› **if** ‹*2 dvd a*› **for** *a*
using *that* **by** *auto*
qed

lemma *even-succ-div-two* [*simp*]:
even a ⟹ (*a* + *1*) *div 2 = a div 2*
by (*cases a = 0*) (*auto elim*!: *evenE dest*: *mult-not-zero*)

lemma *odd-succ-div-two* [*simp*]:
odd a ⟹ (*a* + *1*) *div 2 = a div 2 + 1*
by (*auto elim*!: *oddE simp add*: *add.assoc*)

lemma *even-two-times-div-two*:
even a ⟹ *2* ∗ (*a div 2*) = *a*
by (*fact dvd-mult-div-cancel*)

lemma *odd-two-times-div-two-succ* [*simp*]:
odd a ⟹ *2* ∗ (*a div 2*) + *1* = *a*
using *mult-div-mod-eq* [*of 2 a*]
by (*simp add*: *even-iff-mod-2-eq-zero*)

lemma *coprime-left-2-iff-odd* [*simp*]:

coprime 2 a ⟷ *odd a*
proof
assume *odd a*
show *coprime 2 a*
proof (*rule coprimeI*)
fix *b*
assume *b dvd 2 b dvd a*
then have *b dvd a mod 2*
by (*auto intro*: *dvd-mod*)
with ‹*odd a*› **show** *is-unit b*
by (*simp add*: *mod-2-eq-odd*)
qed
next
assume *coprime 2 a*
show *odd a*
proof (*rule notI*)
assume *even a*
then obtain *b* **where** *a* = *2* ∗ *b* **..**
with ‹*coprime 2 a*› **have** *coprime 2* (*2* ∗ *b*)
by *simp*
moreover have ¬ *coprime 2* (*2* ∗ *b*)
by (*rule not-coprimeI* [*of 2*]) *simp-all*
ultimately show *False*
by *blast*
qed
qed

lemma *coprime-right-2-iff-odd* [*simp*]:
coprime a 2 ⟷ *odd a*
using *coprime-left-2-iff-odd* [*of a*] **by** (*simp add*: *ac-simps*)

end

lemma *minus-1-mod-2-eq* [*simp*]:
‹− *1 mod 2* = (*1*::*int*)›
by (*simp add*: *mod-2-eq-odd*)

lemma *minus-1-div-2-eq* [*simp*]:
− *1 div 2* = − (*1*::*int*)
proof −
from *div-mult-mod-eq* [*of* − *1* :: *int 2*]
have − *1 div 2* ∗ *2* = − *1* ∗ (*2* :: *int*)
using *add-implies-diff* **by** *fastforce*
then show *?thesis*
using *mult-right-cancel* [*of 2* − *1 div 2* − (*1* :: *int*)] **by** *simp*
qed

context *linordered-euclidean-semiring*
begin

lemma *even-decr-exp-div-exp-iff′*:
‹*even* ((*2* ^ *m* − *1*) *div* *2* ^ *n*) ⟷ *m* ≤ *n*›
proof −
have ‹*even* ((*2* ^ *m* − *1*) *div* *2* ^ *n*) ⟷ *even* (*of-nat* ((*2* ^ *m* − *Suc 0*) *div* *2* ^ *n*))›
by (*simp only*: *of-nat-div*) (*simp add*: *of-nat-diff*)
also have ‹. . . ⟷ *even* ((*2* ^ *m* − *Suc 0*) *div* *2* ^ *n*)›
by *simp*
also have ‹. . . ⟷ *m* ≤ *n*›
proof (*cases* ‹*m* ≤ *n*›)
case *True*
then show *?thesis*
by (*simp add*: *Suc-le-lessD*)
next
case *False*
then obtain *r* **where** *r*: ‹*m* = *n* + *Suc r*›
using *less-imp-Suc-add* **by** *fastforce*
from *r* **have** ‹{*q*. *q* < *m*} ∩ {*q*. *2* ^ *n* *dvd* (*2*::*nat*) ^ *q*} = {*q*. *n* ≤ *q* ∧ *q* < *m*}›
by (*auto simp add*: *dvd-power-iff-le*)
moreover from *r* **have** ‹{*q*. *q* < *m*} ∩ {*q*. ¬ *2* ^ *n* *dvd* (*2*::*nat*) ^ *q*} = {*q*. *q* < *n*}›
by (*auto simp add*: *dvd-power-iff-le*)
moreover from *False* **have** ‹{*q*. *n* ≤ *q* ∧ *q* < *m* ∧ *q* ≤ *n*} = {*n*}›
by *auto*
then have ‹*odd* ((∑ *a*∈{*q*. *n* ≤ *q* ∧ *q* < *m*}. *2* ^ *a* *div* (*2*::*nat*) ^ *n*) + *sum* ((^) *2*) {*q*. *q* < *n*} *div* *2* ^ *n*)›
by (*simp-all add*: *euclidean-semiring-cancel-class.power-diff-power-eq semiring-parity-class.even-sum-iff*
linorder-class.not-less mask-eq-sum-exp-nat [*symmetric*])
ultimately have ‹*odd* (*sum* ((^) (*2*::*nat*)) {*q*. *q* < *m*} *div* *2* ^ *n*)›
by (*subst euclidean-semiring-cancel-class.sum-div-partition*) *simp-all*
with *False* **show** *?thesis*
by (*simp add*: *mask-eq-sum-exp-nat*)
qed
finally show *?thesis* **.**
qed

end

57.6 Generic symbolic computations

The following type class contains everything necessary to formulate a division algorithm in ring structures with numerals, restricted to its positive segments.

class *linordered-euclidean-semiring-division* = *linordered-euclidean-semiring* +
fixes *divmod* :: ‹*num* ⇒ *num* ⇒ ′*a* × ′*a*›

and *divmod-step* :: ‹'a ⇒ 'a × 'a ⇒ 'a × 'a› — These are conceptually definitions but force generated code to be monomorphic wrt. particular instances of this class which yields a significant speedup.
assumes *divmod-def*: ‹*divmod m n* = (*numeral m div numeral n*, *numeral m mod numeral n*)›
and *divmod-step-def* [*simp*]: ‹*divmod-step l* (*q*, *r*) =
(*if euclidean-size l* ≤ *euclidean-size r then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))› — This is a formulation of one step (referring to one digit position) in school-method division: compare the dividend at the current digit position with the remainder from previous division steps and evaluate accordingly.
begin

lemma *fst-divmod*:
‹*fst* (*divmod m n*) = *numeral m div numeral n*›
by (*simp add*: *divmod-def*)

lemma *snd-divmod*:
‹*snd* (*divmod m n*) = *numeral m mod numeral n*›
by (*simp add*: *divmod-def*)

Following a formulation of school-method division. If the divisor is smaller than the dividend, terminate. If not, shift the dividend to the right until termination occurs and then reiterate single division steps in the opposite direction.

lemma *divmod-divmod-step*:
‹*divmod m n* = (*if m* < *n then* (*0*, *numeral m*)
else divmod-step (*numeral n*) (*divmod m* (*Num.Bit0 n*)))›
proof (*cases* ‹*m* < *n*›)
case *True*
then show *?thesis*
by (*simp add*: *prod-eq-iff fst-divmod snd-divmod flip*: *of-nat-numeral of-nat-div of-nat-mod*)
next
case *False*
define *r s t* **where** ‹*r* = (*numeral m* :: *nat*)› ‹*s* = (*numeral n* :: *nat*)› ‹*t* = *2* ∗ *s*›
then have ∗: ‹*numeral m* = *of-nat r*› ‹*numeral n* = *of-nat s*› ‹*numeral* (*num.Bit0 n*) = *of-nat t*›
and ‹¬ *s* ≤ *r mod s*›
by (*simp-all add*: *linorder-class.not-le*)
have *t*: ‹*2* ∗ (*r div t*) = *r div s* − *r div s mod 2*›
‹*r mod t* = *s* ∗ (*r div s mod 2*) + *r mod s*›
by (*simp add*: *Rings.minus-mod-eq-mult-div Groups.mult.commute* [*of 2*] *Euclidean-Rings.div-mult2-eq* ‹*t* = *2* ∗ *s*›)
(*use mod-mult2-eq* [*of r s 2*] **in** ‹*simp add*: *ac-simps* ‹*t* = *2* ∗ *s*››)
have *rs*: ‹*r div s mod 2* = *0* ∨ *r div s mod 2* = *Suc 0*›
by *auto*
from ‹¬ *s* ≤ *r mod s*› **have** ‹*s* ≤ *r mod t* ⟹
r div s = *Suc* (*2* ∗ (*r div t*)) ∧

r mod s = r mod t − s›
using *rs*
by (*auto simp add*: *t*)
moreover have ‹*r mod t* < *s* ⟹
r div s = *2* ∗ (*r div t*) ∧
r mod s = *r mod t*›
using *rs*
by (*auto simp add*: *t*)
ultimately show *?thesis*
by (*simp add*: *divmod-def prod-eq-iff split-def Let-def*
not-less mod-eq-0-iff-dvd Rings.mod-eq-0-iff-dvd False not-le ∗)
(*simp add*: *flip*: *of-nat-numeral of-nat-mult add.commute* [*of 1*] *of-nat-div*
of-nat-mod of-nat-Suc of-nat-diff)
qed

The division rewrite proper – first, trivial results involving *1*

lemma *divmod-trivial* [*simp*]:
divmod m Num.One = (*numeral m*, *0*)
divmod num.One (*num.Bit0 n*) = (*0*, *Numeral1*)
divmod num.One (*num.Bit1 n*) = (*0*, *Numeral1*)
using *divmod-divmod-step* [*of Num.One*] **by** (*simp-all add*: *divmod-def*)

Division by an even number is a right-shift

lemma *divmod-cancel* [*simp*]:
‹*divmod* (*Num.Bit0 m*) (*Num.Bit0 n*) = (*case divmod m n of* (*q*, *r*) ⇒ (*q*, *2* ∗ *r*))› (**is** *?P*)
‹*divmod* (*Num.Bit1 m*) (*Num.Bit0 n*) = (*case divmod m n of* (*q*, *r*) ⇒ (*q*, *2* ∗ *r* + *1*))› (**is** *?Q*)
proof −
define *r s* **where** ‹*r* = (*numeral m* :: *nat*)› ‹*s* = (*numeral n* :: *nat*)›
then have ∗: ‹*numeral m* = *of-nat r*› ‹*numeral n* = *of-nat s*›
‹*numeral* (*num.Bit0 m*) = *of-nat* (*2* ∗ *r*)› ‹*numeral* (*num.Bit0 n*) = *of-nat* (*2* ∗ *s*)›
‹*numeral* (*num.Bit1 m*) = *of-nat* (*Suc* (*2* ∗ *r*))›
by *simp-all*
have ∗∗: ‹*Suc* (*2* ∗ *r*) *div 2* = *r*›
by *simp*
show *?P* **and** *?Q*
by (*simp-all add*: *divmod-def* ∗)
(*simp-all flip*: *of-nat-numeral of-nat-div of-nat-mod of-nat-mult add.commute* [*of 1*] *of-nat-Suc*
add: *Euclidean-Rings.mod-mult-mult1 div-mult2-eq* [*of - 2*] *mod-mult2-eq* [*of - 2*] ∗∗)
qed

The really hard work

lemma *divmod-steps* [*simp*]:
divmod (*num.Bit0 m*) (*num.Bit1 n*) =
(*if m* ≤ *n then* (*0*, *numeral* (*num.Bit0 m*))

else divmod-step (*numeral* (*num.Bit1 n*))
(*divmod* (*num.Bit0 m*)
(*num.Bit0* (*num.Bit1 n*))))
divmod (*num.Bit1 m*) (*num.Bit1 n*) =
(*if m* < *n then* (*0*, *numeral* (*num.Bit1 m*))
else divmod-step (*numeral* (*num.Bit1 n*))
(*divmod* (*num.Bit1 m*)
(*num.Bit0* (*num.Bit1 n*))))
by (*simp-all add*: *divmod-divmod-step*)

lemmas *divmod-algorithm-code* = *divmod-trivial divmod-cancel divmod-steps*

Special case: divisibility

definition *divides-aux* :: *'a* × *'a* ⇒ *bool*
where
divides-aux qr ⟷ *snd qr* = *0*

lemma *divides-aux-eq* [*simp*]:
divides-aux (*q*, *r*) ⟷ *r* = *0*
by (*simp add*: *divides-aux-def*)

lemma *dvd-numeral-simp* [*simp*]:
numeral m dvd numeral n ⟷ *divides-aux* (*divmod n m*)
by (*simp add*: *divmod-def mod-eq-0-iff-dvd*)

Generic computation of quotient and remainder

lemma *numeral-div-numeral* [*simp*]:
numeral k div numeral l = *fst* (*divmod k l*)
by (*simp add*: *fst-divmod*)

lemma *numeral-mod-numeral* [*simp*]:
numeral k mod numeral l = *snd* (*divmod k l*)
by (*simp add*: *snd-divmod*)

lemma *one-div-numeral* [*simp*]:
1 div numeral n = *fst* (*divmod num.One n*)
by (*simp add*: *fst-divmod*)

lemma *one-mod-numeral* [*simp*]:
1 mod numeral n = *snd* (*divmod num.One n*)
by (*simp add*: *snd-divmod*)

end

instantiation *nat* :: *linordered-euclidean-semiring-division*
begin

definition *divmod-nat* :: *num* ⇒ *num* ⇒ *nat* × *nat*
where

divmod′-nat-def: *divmod-nat m n* = (*numeral m div numeral n*, *numeral m mod numeral n*)

definition *divmod-step-nat* :: *nat* ⇒ *nat* × *nat* ⇒ *nat* × *nat*
where
divmod-step-nat l qr = (*let* (*q*, *r*) = *qr*
in if r ≥ *l then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))

instance
by *standard* (*simp-all add*: *divmod′-nat-def divmod-step-nat-def*)

end

declare *divmod-algorithm-code* [**where** *?′a* = *nat*, *code*]

lemma *Suc-0-div-numeral* [*simp*]:
‹*Suc 0 div numeral Num.One* = *1*›
‹*Suc 0 div numeral* (*Num.Bit0 n*) = *0*›
‹*Suc 0 div numeral* (*Num.Bit1 n*) = *0*›
by *simp-all*

lemma *Suc-0-mod-numeral* [*simp*]:
‹*Suc 0 mod numeral Num.One* = *0*›
‹*Suc 0 mod numeral* (*Num.Bit0 n*) = *1*›
‹*Suc 0 mod numeral* (*Num.Bit1 n*) = *1*›
by *simp-all*

instantiation *int* :: *linordered-euclidean-semiring-division*
begin

definition *divmod-int* :: *num* ⇒ *num* ⇒ *int* × *int*
where
divmod-int m n = (*numeral m div numeral n*, *numeral m mod numeral n*)

definition *divmod-step-int* :: *int* ⇒ *int* × *int* ⇒ *int* × *int*
where
divmod-step-int l qr = (*let* (*q*, *r*) = *qr*
in if |*l*| ≤ |*r*| *then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))

instance
by *standard* (*auto simp add*: *divmod-int-def divmod-step-int-def*)

end

declare *divmod-algorithm-code* [**where** *?′a* = *int*, *code*]

context

begin

qualified definition *adjust-div* :: *int* × *int* ⇒ *int*
where
adjust-div qr = (*let* (*q*, *r*) = *qr in q* + *of-bool* (*r* ≠ *0*))

qualified lemma *adjust-div-eq* [*simp*, *code*]:
adjust-div (*q*, *r*) = *q* + *of-bool* (*r* ≠ *0*)
by (*simp add*: *adjust-div-def*)

qualified definition *adjust-mod* :: *num* ⇒ *int* ⇒ *int*
where
[*simp*]: *adjust-mod l r* = (*if r* = *0 then 0 else numeral l* − *r*)

lemma *minus-numeral-div-numeral* [*simp*]:
− *numeral m div numeral n* = − (*adjust-div* (*divmod m n*) :: *int*)
proof −
have *int* (*fst* (*divmod m n*)) = *fst* (*divmod m n*)
by (*simp only*: *fst-divmod divide-int-def*) *auto*
then show *?thesis*
by (*auto simp add*: *split-def Let-def adjust-div-def divides-aux-def divide-int-def*)
qed

lemma *minus-numeral-mod-numeral* [*simp*]:
− *numeral m mod numeral n* = *adjust-mod n* (*snd* (*divmod m n*) :: *int*)
proof (*cases snd* (*divmod m n*) = (*0*::*int*))
case *True*
then show *?thesis*
by (*simp add*: *mod-eq-0-iff-dvd divides-aux-def*)
next
case *False*
then have *int* (*snd* (*divmod m n*)) = *snd* (*divmod m n*) **if** *snd* (*divmod m n*) ≠ (*0*::*int*)
by (*simp only*: *snd-divmod modulo-int-def*) *auto*
then show *?thesis*
by (*simp add*: *divides-aux-def adjust-div-def*)
(*simp add*: *divides-aux-def modulo-int-def*)
qed

lemma *numeral-div-minus-numeral* [*simp*]:
numeral m div − *numeral n* = − (*adjust-div* (*divmod m n*) :: *int*)
proof −
have *int* (*fst* (*divmod m n*)) = *fst* (*divmod m n*)
by (*simp only*: *fst-divmod divide-int-def*) *auto*
then show *?thesis*
by (*auto simp add*: *split-def Let-def adjust-div-def divides-aux-def divide-int-def*)
qed

lemma *numeral-mod-minus-numeral* [*simp*]:

numeral m mod − *numeral n* = − *adjust-mod n* (*snd* (*divmod m n*) :: *int*)
proof (*cases snd* (*divmod m n*) = (*0*::*int*))
case *True*
then show *?thesis*
by (*simp add*: *mod-eq-0-iff-dvd divides-aux-def*)
next
case *False*
then have *int* (*snd* (*divmod m n*)) = *snd* (*divmod m n*) **if** *snd* (*divmod m n*) ≠ (*0*::*int*)
by (*simp only*: *snd-divmod modulo-int-def*) *auto*
then show *?thesis*
by (*simp add*: *divides-aux-def adjust-div-def*)
(*simp add*: *divides-aux-def modulo-int-def*)
qed

lemma *minus-one-div-numeral* [*simp*]:
− *1 div numeral n* = − (*adjust-div* (*divmod Num.One n*) :: *int*)
using *minus-numeral-div-numeral* [*of Num.One n*] **by** *simp*

lemma *minus-one-mod-numeral* [*simp*]:
− *1 mod numeral n* = *adjust-mod n* (*snd* (*divmod Num.One n*) :: *int*)
using *minus-numeral-mod-numeral* [*of Num.One n*] **by** *simp*

lemma *one-div-minus-numeral* [*simp*]:
1 div − *numeral n* = − (*adjust-div* (*divmod Num.One n*) :: *int*)
using *numeral-div-minus-numeral* [*of Num.One n*] **by** *simp*

lemma *one-mod-minus-numeral* [*simp*]:
1 mod − *numeral n* = − *adjust-mod n* (*snd* (*divmod Num.One n*) :: *int*)
using *numeral-mod-minus-numeral* [*of Num.One n*] **by** *simp*

lemma [*code*]:
fixes *k* :: *int*
shows
k div 0 = *0*
k mod 0 = *k*
0 div k = *0*
0 mod k = *0*
k div Int.Pos Num.One = *k*
k mod Int.Pos Num.One = *0*
k div Int.Neg Num.One = − *k*
k mod Int.Neg Num.One = *0*
Int.Pos m div Int.Pos n = (*fst* (*divmod m n*) :: *int*)
Int.Pos m mod Int.Pos n = (*snd* (*divmod m n*) :: *int*)
Int.Neg m div Int.Pos n = − (*adjust-div* (*divmod m n*) :: *int*)
Int.Neg m mod Int.Pos n = *adjust-mod n* (*snd* (*divmod m n*) :: *int*)
Int.Pos m div Int.Neg n = − (*adjust-div* (*divmod m n*) :: *int*)
Int.Pos m mod Int.Neg n = − *adjust-mod n* (*snd* (*divmod m n*) :: *int*)
Int.Neg m div Int.Neg n = (*fst* (*divmod m n*) :: *int*)

Int.Neg m mod Int.Neg n = − (*snd* (*divmod m n*) :: *int*)
by *simp-all*

end

lemma *divmod-BitM-2-eq* [*simp*]:
‹*divmod* (*Num.BitM m*) (*Num.Bit0 Num.One*) = (*numeral m* − *1*, (*1* :: *int*))›
by (*cases m*) *simp-all*

57.6.1 Computation by simplification

lemma *euclidean-size-nat-less-eq-iff*:
‹*euclidean-size m* ≤ *euclidean-size n* ⟷ *m* ≤ *n*› **for** *m n* :: *nat*
by *simp*

lemma *euclidean-size-int-less-eq-iff*:
‹*euclidean-size k* ≤ *euclidean-size l* ⟷ |*k*| ≤ |*l*|› **for** *k l* :: *int*
by *auto*

simproc-setup *numeral-divmod*
(*0 div 0* :: *'a* :: *linordered-euclidean-semiring-division* | *0 mod 0* :: *'a* :: *linordered-euclidean-semiring-division* |
0 div 1 :: *'a* :: *linordered-euclidean-semiring-division* | *0 mod 1* :: *'a* :: *linordered-euclidean-semiring-division* |
0 div − *1* :: *int* | *0 mod* − *1* :: *int* |
0 div numeral b :: *'a* :: *linordered-euclidean-semiring-division* | *0 mod numeral b* :: *'a* :: *linordered-euclidean-semiring-division* |
0 div − *numeral b* :: *int* | *0 mod* − *numeral b* :: *int* |
1 div 0 :: *'a* :: *linordered-euclidean-semiring-division* | *1 mod 0* :: *'a* :: *linordered-euclidean-semiring-division* |
1 div 1 :: *'a* :: *linordered-euclidean-semiring-division* | *1 mod 1* :: *'a* :: *linordered-euclidean-semiring-division* |
1 div − *1* :: *int* | *1 mod* − *1* :: *int* |
1 div numeral b :: *'a* :: *linordered-euclidean-semiring-division* | *1 mod numeral b* :: *'a* :: *linordered-euclidean-semiring-division* |
1 div − *numeral b* :: *int* |*1 mod* − *numeral b* :: *int* |
− *1 div 0* :: *int* | − *1 mod 0* :: *int* | − *1 div 1* :: *int* | − *1 mod 1* :: *int* |
− *1 div* − *1* :: *int* | − *1 mod* − *1* :: *int* | − *1 div numeral b* :: *int* | − *1 mod numeral b* :: *int* |
− *1 div* − *numeral b* :: *int* | − *1 mod* − *numeral b* :: *int* |
numeral a div 0 :: *'a* :: *linordered-euclidean-semiring-division* | *numeral a mod 0* :: *'a* :: *linordered-euclidean-semiring-division* |
numeral a div 1 :: *'a* :: *linordered-euclidean-semiring-division* | *numeral a mod 1* :: *'a* :: *linordered-euclidean-semiring-division* |
numeral a div − *1* :: *int* | *numeral a mod* − *1* :: *int* |
numeral a div numeral b :: *'a* :: *linordered-euclidean-semiring-division* | *numeral a mod numeral b* :: *'a* :: *linordered-euclidean-semiring-division* |
numeral a div − *numeral b* :: *int* | *numeral a mod* − *numeral b* :: *int* |
− *numeral a div 0* :: *int* | − *numeral a mod 0* :: *int* |

− *numeral a div 1 :: int* | − *numeral a mod 1 :: int* |
− *numeral a div − 1 :: int* | − *numeral a mod − 1 :: int* |
− *numeral a div numeral b :: int* | − *numeral a mod numeral b :: int* |
− *numeral a div − numeral b :: int* | − *numeral a mod − numeral b :: int*) = ‹
let
val if-cong = the (Code.get-case-cong ***theory const-name*** *‹If›);*
fun successful-rewrite ctxt ct =
let
val thm = Simplifier.rewrite ctxt ct
in if Thm.is-reflexive thm then NONE else SOME thm end;
val simps = @{thms div-0 mod-0 div-by-0 mod-by-0 div-by-1 mod-by-1
one-div-numeral one-mod-numeral minus-one-div-numeral minus-one-mod-numeral
one-div-minus-numeral one-mod-minus-numeral
numeral-div-numeral numeral-mod-numeral minus-numeral-div-numeral minus-numeral-mod-numeral
numeral-div-minus-numeral numeral-mod-minus-numeral
div-minus-minus mod-minus-minus Parity.adjust-div-eq of-bool-eq one-neq-zero
numeral-neq-zero neg-equal-0-iff-equal arith-simps arith-special divmod-trivial
divmod-cancel divmod-steps divmod-step-def fst-conv snd-conv numeral-One
case-prod-beta rel-simps Parity.adjust-mod-def div-minus1-right mod-minus1-right
minus-minus numeral-times-numeral mult-zero-right mult-1-right
euclidean-size-nat-less-eq-iff euclidean-size-int-less-eq-iff diff-nat-numeral nat-numeral}
@ [@{*lemma 0 = 0* ⟷ *True by simp*}];
val simpset =
HOL-ss |> Simplifier.simpset-map ***context***
(Simplifier.add-cong if-cong #> fold Simplifier.add-simp simps);
in K (fn ctxt => successful-rewrite (Simplifier.put-simpset simpset ctxt)) end
› — There is space for improvement here: the calculation itself could be carried out outside the logic, and a generic simproc (simplifier setup) for generic calculation would be helpful.

57.7 Computing congruences modulo *2 ^ q*

context *linordered-euclidean-semiring-division*
begin

lemma *cong-exp-iff-simps*:
numeral n mod numeral Num.One = 0
⟷ *True*
numeral (Num.Bit0 n) mod numeral (Num.Bit0 q) = 0
⟷ *numeral n mod numeral q = 0*
numeral (Num.Bit1 n) mod numeral (Num.Bit0 q) = 0
⟷ *False*
numeral m mod numeral Num.One = (numeral n mod numeral Num.One)
⟷ *True*
numeral Num.One mod numeral (Num.Bit0 q) = (numeral Num.One mod numeral (Num.Bit0 q))
⟷ *True*
numeral Num.One mod numeral (Num.Bit0 q) = (numeral (Num.Bit0 n) mod

numeral (*Num.Bit0 q*))
⟷ *False*
numeral Num.One mod numeral (*Num.Bit0 q*) = (*numeral* (*Num.Bit1 n*) *mod numeral* (*Num.Bit0 q*))
⟷ (*numeral n mod numeral q*) = *0*
numeral (*Num.Bit0 m*) *mod numeral* (*Num.Bit0 q*) = (*numeral Num.One mod numeral* (*Num.Bit0 q*))
⟷ *False*
numeral (*Num.Bit0 m*) *mod numeral* (*Num.Bit0 q*) = (*numeral* (*Num.Bit0 n*) *mod numeral* (*Num.Bit0 q*))
⟷ *numeral m mod numeral q* = (*numeral n mod numeral q*)
numeral (*Num.Bit0 m*) *mod numeral* (*Num.Bit0 q*) = (*numeral* (*Num.Bit1 n*) *mod numeral* (*Num.Bit0 q*))
⟷ *False*
numeral (*Num.Bit1 m*) *mod numeral* (*Num.Bit0 q*) = (*numeral Num.One mod numeral* (*Num.Bit0 q*))
⟷ (*numeral m mod numeral q*) = *0*
numeral (*Num.Bit1 m*) *mod numeral* (*Num.Bit0 q*) = (*numeral* (*Num.Bit0 n*) *mod numeral* (*Num.Bit0 q*))
⟷ *False*
numeral (*Num.Bit1 m*) *mod numeral* (*Num.Bit0 q*) = (*numeral* (*Num.Bit1 n*) *mod numeral* (*Num.Bit0 q*))
⟷ *numeral m mod numeral q* = (*numeral n mod numeral q*)
by (*auto simp add*: *case-prod-beta dest*: *arg-cong* [*of - - even*])

end

code-identifier
code-module *Parity* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

lemmas *even-of-nat* = *even-of-nat-iff*

end

58 Combination and Cancellation Simprocs for Numeral Expressions

theory *Numeral-Simprocs*
imports *Parity*
begin

ML-file ‹~~/*src/Provers/Arith/assoc-fold.ML*›
ML-file ‹~~/*src/Provers/Arith/cancel-numerals.ML*›
ML-file ‹~~/*src/Provers/Arith/combine-numerals.ML*›
ML-file ‹~~/*src/Provers/Arith/cancel-numeral-factor.ML*›
ML-file ‹~~/*src/Provers/Arith/extract-common-term.ML*›

lemmas *semiring-norm* =
Let-def arith-simps diff-nat-numeral rel-simps
if-False if-True
add-Suc add-numeral-left
add-neg-numeral-left mult-numeral-left
numeral-One [*symmetric*] *uminus-numeral-One* [*symmetric*] *Suc-eq-plus1*
eq-numeral-iff-iszero not-iszero-Numeral1

For *combine-numerals*

lemma *left-add-mult-distrib*: *i*∗*u* + (*j*∗*u* + *k*) = (*i*+*j*)∗*u* + (*k*::*nat*)
by (*simp add*: *add-mult-distrib*)

For *cancel-numerals*

lemma *nat-diff-add-eq1*:
j <= (*i*::*nat*) ==> ((*i*∗*u* + *m*) − (*j*∗*u* + *n*)) = (((*i*−*j*)∗*u* + *m*) − *n*)
by (*simp split*: *nat-diff-split add*: *add-mult-distrib*)

lemma *nat-diff-add-eq2*:
i <= (*j*::*nat*) ==> ((*i*∗*u* + *m*) − (*j*∗*u* + *n*)) = (*m* − ((*j*−*i*)∗*u* + *n*))
by (*simp split*: *nat-diff-split add*: *add-mult-distrib*)

lemma *nat-eq-add-iff1*:
j <= (*i*::*nat*) ==> (*i*∗*u* + *m* = *j*∗*u* + *n*) = ((*i*−*j*)∗*u* + *m* = *n*)
by (*auto split*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-eq-add-iff2*:
i <= (*j*::*nat*) ==> (*i*∗*u* + *m* = *j*∗*u* + *n*) = (*m* = (*j*−*i*)∗*u* + *n*)
by (*auto split*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-less-add-iff1*:
j <= (*i*::*nat*) ==> (*i*∗*u* + *m* < *j*∗*u* + *n*) = ((*i*−*j*)∗*u* + *m* < *n*)
by (*auto split*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-less-add-iff2*:
i <= (*j*::*nat*) ==> (*i*∗*u* + *m* < *j*∗*u* + *n*) = (*m* < (*j*−*i*)∗*u* + *n*)
by (*auto split*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-le-add-iff1*:
j <= (*i*::*nat*) ==> (*i*∗*u* + *m* <= *j*∗*u* + *n*) = ((*i*−*j*)∗*u* + *m* <= *n*)
by (*auto split*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-le-add-iff2*:
i <= (*j*::*nat*) ==> (*i*∗*u* + *m* <= *j*∗*u* + *n*) = (*m* <= (*j*−*i*)∗*u* + *n*)
by (*auto split*: *nat-diff-split simp add*: *add-mult-distrib*)

For *cancel-numeral-factors*

lemma *nat-mult-le-cancel1*: (*0*::*nat*) < *k* ==> (*k*∗*m* <= *k*∗*n*) = (*m*<=*n*)
by *auto*

lemma *nat-mult-less-cancel1*: (*0*::*nat*) < *k* ==> (*k*∗*m* < *k*∗*n*) = (*m*<*n*)
by *auto*

lemma *nat-mult-eq-cancel1*: (*0*::*nat*) < *k* ==> (*k*∗*m* = *k*∗*n*) = (*m*=*n*)
by *auto*

lemma *nat-mult-div-cancel1*: (*0*::*nat*) < *k* ==> (*k*∗*m*) *div* (*k*∗*n*) = (*m div n*)
by *auto*

lemma *nat-mult-dvd-cancel-disj*[*simp*]:
(*k*∗*m*) *dvd* (*k*∗*n*) = (*k*=*0* ∨ *m dvd* (*n*::*nat*))
by (*auto simp*: *dvd-eq-mod-eq-0 mod-mult-mult1*)

lemma *nat-mult-dvd-cancel1*: *0* < *k* ⟹ (*k*∗*m*) *dvd* (*k*∗*n*::*nat*) = (*m dvd n*)
by(*auto*)

For *cancel-factor*

lemmas *nat-mult-le-cancel-disj* = *mult-le-cancel1*

lemmas *nat-mult-less-cancel-disj* = *mult-less-cancel1*

lemma *nat-mult-eq-cancel-disj*:
fixes *k m n* :: *nat*
shows *k* ∗ *m* = *k* ∗ *n* ⟷ *k* = *0* ∨ *m* = *n*
by (*fact mult-cancel-left*)

lemma *nat-mult-div-cancel-disj*:
fixes *k m n* :: *nat*
shows (*k* ∗ *m*) *div* (*k* ∗ *n*) = (*if k* = *0 then 0 else m div n*)
by (*fact div-mult-mult1-if*)

lemma *numeral-times-minus-swap*:
fixes *x*:: *'a*::*comm-ring-1* **shows** *numeral w* ∗ −*x* = *x* ∗ − *numeral w*
by (*simp add*: *ac-simps*)

ML-file ‹*Tools*/*numeral-simprocs.ML*›

simproc-setup *semiring-assoc-fold*
((*a*::*'a*::*comm-semiring-1-cancel*) ∗ *b*) =
‹*K Numeral-Simprocs.assoc-fold*›

simproc-setup *int-combine-numerals*
((*i*::*'a*::*comm-ring-1*) + *j* | (*i*::*'a*::*comm-ring-1*) − *j*) =
‹*K Numeral-Simprocs.combine-numerals*›

simproc-setup *field-combine-numerals*
((*i*::*'a*::{*field*,*ring-char-0*}) + *j*
|(*i*::*'a*::{*field*,*ring-char-0*}) − *j*) =

‹K Numeral-Simprocs.field-combine-numerals›

simproc-setup *inteq-cancel-numerals*
((l::'a::comm-ring-1) + m = n
|(l::'a::comm-ring-1) = m + n
|(l::'a::comm-ring-1) − m = n
|(l::'a::comm-ring-1) = m − n
|(l::'a::comm-ring-1) ∗ m = n
|(l::'a::comm-ring-1) = m ∗ n
|− (l::'a::comm-ring-1) = m
|(l::'a::comm-ring-1) = − m) =
‹K Numeral-Simprocs.eq-cancel-numerals›

simproc-setup *intless-cancel-numerals*
((l::'a::linordered-idom) + m < n
|(l::'a::linordered-idom) < m + n
|(l::'a::linordered-idom) − m < n
|(l::'a::linordered-idom) < m − n
|(l::'a::linordered-idom) ∗ m < n
|(l::'a::linordered-idom) < m ∗ n
|− (l::'a::linordered-idom) < m
|(l::'a::linordered-idom) < − m) =
‹K Numeral-Simprocs.less-cancel-numerals›

simproc-setup *intle-cancel-numerals*
((l::'a::linordered-idom) + m ≤ n
|(l::'a::linordered-idom) ≤ m + n
|(l::'a::linordered-idom) − m ≤ n
|(l::'a::linordered-idom) ≤ m − n
|(l::'a::linordered-idom) ∗ m ≤ n
|(l::'a::linordered-idom) ≤ m ∗ n
|− (l::'a::linordered-idom) ≤ m
|(l::'a::linordered-idom) ≤ − m) =
‹K Numeral-Simprocs.le-cancel-numerals›

simproc-setup *ring-eq-cancel-numeral-factor*
((l::'a::{idom,ring-char-0}) ∗ m = n
|(l::'a::{idom,ring-char-0}) = m ∗ n) =
‹K Numeral-Simprocs.eq-cancel-numeral-factor›

simproc-setup *ring-less-cancel-numeral-factor*
((l::'a::linordered-idom) ∗ m < n
|(l::'a::linordered-idom) < m ∗ n) =
‹K Numeral-Simprocs.less-cancel-numeral-factor›

simproc-setup *ring-le-cancel-numeral-factor*
((l::'a::linordered-idom) ∗ m <= n
|(l::'a::linordered-idom) <= m ∗ n) =
‹K Numeral-Simprocs.le-cancel-numeral-factor›

simproc-setup *int-div-cancel-numeral-factors*
((((l::'a::{euclidean-semiring-cancel,comm-ring-1,ring-char-0}) * m) div n
|(l::'a::{euclidean-semiring-cancel,comm-ring-1,ring-char-0}) div (m * n)) =
‹K Numeral-Simprocs.div-cancel-numeral-factor›

simproc-setup *divide-cancel-numeral-factor*
((((l::'a::{field,ring-char-0}) * m) / n
|(l::'a::{field,ring-char-0}) / (m * n)
|((numeral v)::'a::{field,ring-char-0}) / (numeral w)) =
‹K Numeral-Simprocs.divide-cancel-numeral-factor›

simproc-setup *ring-eq-cancel-factor*
((l::'a::idom) * m = n | (l::'a::idom) = m * n) =
‹K Numeral-Simprocs.eq-cancel-factor›

simproc-setup *linordered-ring-le-cancel-factor*
((l::'a::linordered-idom) * m <= n
|(l::'a::linordered-idom) <= m * n) =
‹K Numeral-Simprocs.le-cancel-factor›

simproc-setup *linordered-ring-less-cancel-factor*
((l::'a::linordered-idom) * m < n
|(l::'a::linordered-idom) < m * n) =
‹K Numeral-Simprocs.less-cancel-factor›

simproc-setup *int-div-cancel-factor*
((((l::'a::euclidean-semiring-cancel) * m) div n
|(l::'a::euclidean-semiring-cancel) div (m * n)) =
‹K Numeral-Simprocs.div-cancel-factor›

simproc-setup *int-mod-cancel-factor*
((((l::'a::euclidean-semiring-cancel) * m) mod n
|(l::'a::euclidean-semiring-cancel) mod (m * n)) =
‹K Numeral-Simprocs.mod-cancel-factor›

simproc-setup *dvd-cancel-factor*
((((l::'a::idom) * m) dvd n
|(l::'a::idom) dvd (m * n)) =
‹K Numeral-Simprocs.dvd-cancel-factor›

simproc-setup *divide-cancel-factor*
((((l::'a::field) * m) / n
|(l::'a::field) / (m * n)) =
‹K Numeral-Simprocs.divide-cancel-factor›

ML-file ‹*Tools/nat-numeral-simprocs.ML*›

simproc-setup *nat-combine-numerals*
$((i{::}nat) + j \mid Suc\ (i + j)) =$
‹*K Nat-Numeral-Simprocs.combine-numerals*›

simproc-setup *nateq-cancel-numerals*
$((l{::}nat) + m = n \mid (l{::}nat) = m + n \mid$
$(l{::}nat) * m = n \mid (l{::}nat) = m * n \mid$
$Suc\ m = n \mid m = Suc\ n) =$
‹*K Nat-Numeral-Simprocs.eq-cancel-numerals*›

simproc-setup *natless-cancel-numerals*
$((l{::}nat) + m < n \mid (l{::}nat) < m + n \mid$
$(l{::}nat) * m < n \mid (l{::}nat) < m * n \mid$
$Suc\ m < n \mid m < Suc\ n) =$
‹*K Nat-Numeral-Simprocs.less-cancel-numerals*›

simproc-setup *natle-cancel-numerals*
$((l{::}nat) + m \leq n \mid (l{::}nat) \leq m + n \mid$
$(l{::}nat) * m \leq n \mid (l{::}nat) \leq m * n \mid$
$Suc\ m \leq n \mid m \leq Suc\ n) =$
‹*K Nat-Numeral-Simprocs.le-cancel-numerals*›

simproc-setup *natdiff-cancel-numerals*
$(((l{::}nat) + m) - n \mid (l{::}nat) - (m + n) \mid$
$(l{::}nat) * m - n \mid (l{::}nat) - m * n \mid$
$Suc\ m - n \mid m - Suc\ n) =$
‹*K Nat-Numeral-Simprocs.diff-cancel-numerals*›

simproc-setup *nat-eq-cancel-numeral-factor*
$((l{::}nat) * m = n \mid (l{::}nat) = m * n) =$
‹*K Nat-Numeral-Simprocs.eq-cancel-numeral-factor*›

simproc-setup *nat-less-cancel-numeral-factor*
$((l{::}nat) * m < n \mid (l{::}nat) < m * n) =$
‹*K Nat-Numeral-Simprocs.less-cancel-numeral-factor*›

simproc-setup *nat-le-cancel-numeral-factor*
$((l{::}nat) * m <= n \mid (l{::}nat) <= m * n) =$
‹*K Nat-Numeral-Simprocs.le-cancel-numeral-factor*›

simproc-setup *nat-div-cancel-numeral-factor*
$(((l{::}nat) * m)\ div\ n \mid (l{::}nat)\ div\ (m * n)) =$
‹*K Nat-Numeral-Simprocs.div-cancel-numeral-factor*›

simproc-setup *nat-dvd-cancel-numeral-factor*
$(((l{::}nat) * m)\ dvd\ n \mid (l{::}nat)\ dvd\ (m * n)) =$
‹*K Nat-Numeral-Simprocs.dvd-cancel-numeral-factor*›

simproc-setup *nat-eq-cancel-factor*

((*l*::*nat*) ∗ *m* = *n* | (*l*::*nat*) = *m* ∗ *n*) =
‹*K Nat-Numeral-Simprocs.eq-cancel-factor*›

simproc-setup *nat-less-cancel-factor*
((*l*::*nat*) ∗ *m* < *n* | (*l*::*nat*) < *m* ∗ *n*) =
‹*K Nat-Numeral-Simprocs.less-cancel-factor*›

simproc-setup *nat-le-cancel-factor*
((*l*::*nat*) ∗ *m* <= *n* | (*l*::*nat*) <= *m* ∗ *n*) =
‹*K Nat-Numeral-Simprocs.le-cancel-factor*›

simproc-setup *nat-div-cancel-factor*
(((*l*::*nat*) ∗ *m*) *div n* | (*l*::*nat*) *div* (*m* ∗ *n*)) =
‹*K Nat-Numeral-Simprocs.div-cancel-factor*›

simproc-setup *nat-dvd-cancel-factor*
(((*l*::*nat*) ∗ *m*) *dvd n* | (*l*::*nat*) *dvd* (*m* ∗ *n*)) =
‹*K Nat-Numeral-Simprocs.dvd-cancel-factor*›

declaration ‹
K (*Lin-Arith.add-simprocs*
[***simproc*** ‹*semiring-assoc-fold*›,
simproc ‹*int-combine-numerals*›,
simproc ‹*inteq-cancel-numerals*›,
simproc ‹*intless-cancel-numerals*›,
simproc ‹*intle-cancel-numerals*›,
simproc ‹*field-combine-numerals*›,
simproc ‹*nat-combine-numerals*›,
simproc ‹*nateq-cancel-numerals*›,
simproc ‹*natless-cancel-numerals*›,
simproc ‹*natle-cancel-numerals*›,
simproc ‹*natdiff-cancel-numerals*›,
Numeral-Simprocs.field-divide-cancel-numeral-factor])
›

end

59 Semiring normalization

theory *Semiring-Normalization*
imports *Numeral-Simprocs*
begin

Prelude

class *comm-semiring-1-cancel-crossproduct* = *comm-semiring-1-cancel* +
assumes *crossproduct-eq*: $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$
begin

lemma *crossproduct-noteq*:

$a \neq b \wedge c \neq d \longleftrightarrow a * c + b * d \neq a * d + b * c$
by (*simp add*: *crossproduct-eq*)

lemma *add-scale-eq-noteq*:
$r \neq 0 \Longrightarrow a = b \wedge c \neq d \Longrightarrow a + r * c \neq b + r * d$
proof (*rule notI*)
assume *nz*: $r \neq 0$ **and** *cnd*: $a = b \wedge c \neq d$
and *eq*: $a + (r * c) = b + (r * d)$
have $(0 * d) + (r * c) = (0 * c) + (r * d)$
using *add-left-imp-eq eq mult-zero-left* **by** (*simp add*: *cnd*)
then show *False* **using** *crossproduct-eq* [*of 0 d*] *nz cnd* **by** *simp*
qed

lemma *add-0-iff*:
$b = b + a \longleftrightarrow a = 0$
using *add-left-imp-eq* [*of b a 0*] **by** *auto*

end

subclass (**in** *idom*) *comm-semiring-1-cancel-crossproduct*
proof
fix *w x y z*
show $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$
proof
assume $w * y + x * z = w * z + x * y$
then have $w * y + x * z - w * z - x * y = 0$ **by** (*simp add*: *algebra-simps*)
then have $w * (y - z) - x * (y - z) = 0$ **by** (*simp add*: *algebra-simps*)
then have $(y - z) * (w - x) = 0$ **by** (*simp add*: *algebra-simps*)
then have $y - z = 0 \vee w - x = 0$ **by** (*rule divisors-zero*)
then show $w = x \vee y = z$ **by** *auto*
qed (*auto simp add*: *ac-simps*)
qed

instance *nat* :: *comm-semiring-1-cancel-crossproduct*
proof
fix *w x y z* :: *nat*
have *aux*: $\bigwedge y\ z.\ y < z \Longrightarrow w * y + x * z = w * z + x * y \Longrightarrow w = x$
proof −
fix *y z* :: *nat*
assume $y < z$ **then have** $\exists k.\ z = y + k \wedge k \neq 0$ **by** (*intro exI* [*of - z − y*]) *auto*
then obtain *k* **where** $z = y + k$ **and** $k \neq 0$ **by** *blast*
assume $w * y + x * z = w * z + x * y$
then have $(w * y + x * y) + x * k = (w * y + x * y) + w * k$ **by** (*simp add*: ‹$z = y + k$› *algebra-simps*)
then have $x * k = w * k$ **by** *simp*
then show $w = x$ **using** ‹$k \neq 0$› **by** *simp*
qed
show $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$

by (*auto simp add*: *neq-iff dest*!: *aux*)
qed

Semiring normalization proper

ML-file ‹*Tools/semiring-normalizer.ML*›

context *comm-semiring-1*
begin

lemma *semiring-normalization-rules* [*no-atp*]:
$(a * m) + (b * m) = (a + b) * m$
$(a * m) + m = (a + 1) * m$
$m + (a * m) = (a + 1) * m$
$m + m = (1 + 1) * m$
$0 + a = a$
$a + 0 = a$
$a * b = b * a$
$(a + b) * c = (a * c) + (b * c)$
$0 * a = 0$
$a * 0 = 0$
$1 * a = a$
$a * 1 = a$
$(lx * ly) * (rx * ry) = (lx * rx) * (ly * ry)$
$(lx * ly) * (rx * ry) = lx * (ly * (rx * ry))$
$(lx * ly) * (rx * ry) = rx * ((lx * ly) * ry)$
$(lx * ly) * rx = (lx * rx) * ly$
$(lx * ly) * rx = lx * (ly * rx)$
$lx * (rx * ry) = (lx * rx) * ry$
$lx * (rx * ry) = rx * (lx * ry)$
$(a + b) + (c + d) = (a + c) + (b + d)$
$(a + b) + c = a + (b + c)$
$a + (c + d) = c + (a + d)$
$(a + b) + c = (a + c) + b$
$a + c = c + a$
$a + (c + d) = (a + c) + d$
$(x \hat{} p) * (x \hat{} q) = x \hat{} (p + q)$
$x * (x \hat{} q) = x \hat{} (Suc\ q)$
$(x \hat{} q) * x = x \hat{} (Suc\ q)$
$x * x = x^2$
$(x * y) \hat{} q = (x \hat{} q) * (y \hat{} q)$
$(x \hat{} p) \hat{} q = x \hat{} (p * q)$
$x \hat{} 0 = 1$
$x \hat{} 1 = x$
$x * (y + z) = (x * y) + (x * z)$
$x \hat{} (Suc\ q) = x * (x \hat{} q)$
$x \hat{} (2*n) = (x \hat{} n) * (x \hat{} n)$
by (*simp-all add*: *algebra-simps power-add power2-eq-square power-mult-distrib power-mult del*: *one-add-one*)

local-setup ‹
Semiring-Normalizer.declare @{*thm comm-semiring-1-axioms*}
{*semiring* = ([***term*** ‹$x + y$›, ***term*** ‹$x * y$›, ***term*** ‹$x \,\widehat{}\, n$›, ***term*** ‹*0*›, ***term*** ‹*1*›],
@{*thms semiring-normalization-rules*}),
ring = ([], []),
field = ([], []),
idom = [],
ideal = []}
›

end

context *comm-ring-1*
begin

lemma *ring-normalization-rules* [*no-atp*]:
$- x = (- 1) * x$
$x - y = x + (- y)$
by *simp-all*

local-setup ‹
Semiring-Normalizer.declare @{*thm comm-ring-1-axioms*}
{*semiring* = ([***term*** ‹$x + y$›, ***term*** ‹$x * y$›, ***term*** ‹$x \,\widehat{}\, n$›, ***term*** ‹*0*›, ***term*** ‹*1*›],
@{*thms semiring-normalization-rules*}),
ring = ([***term*** ‹$x - y$›, ***term*** ‹$- x$›], @{*thms ring-normalization-rules*}),
field = ([], []),
idom = [],
ideal = []}
›

end

context *comm-semiring-1-cancel-crossproduct*
begin

local-setup ‹
Semiring-Normalizer.declare @{*thm comm-semiring-1-cancel-crossproduct-axioms*}
{*semiring* = ([***term*** ‹$x + y$›, ***term*** ‹$x * y$›, ***term*** ‹$x \,\widehat{}\, n$›, ***term*** ‹*0*›, ***term*** ‹*1*›],
@{*thms semiring-normalization-rules*}),
ring = ([], []),
field = ([], []),
idom = @{*thms crossproduct-noteq add-scale-eq-noteq*},
ideal = []}
›

end

context *idom*
begin

local-setup ‹
Semiring-Normalizer.declare @{*thm idom-axioms*}
{*semiring* = ([***term*** ‹*x* + *y*›, ***term*** ‹*x* ∗ *y*›, ***term*** ‹*x* ^ *n*›, ***term*** ‹*0*›, ***term*** ‹*1*›],
@{*thms semiring-normalization-rules*}),
ring = ([***term*** ‹*x* − *y*›, ***term*** ‹− *x*›], @{*thms ring-normalization-rules*}),
field = ([], []),
idom = @{*thms crossproduct-noteq add-scale-eq-noteq*},
ideal = @{*thms right-minus-eq add-0-iff*}}
›

end

context *field*
begin

local-setup ‹
Semiring-Normalizer.declare @{*thm field-axioms*}
{*semiring* = ([***term*** ‹*x* + *y*›, ***term*** ‹*x* ∗ *y*›, ***term*** ‹*x* ^ *n*›, ***term*** ‹*0*›, ***term*** ‹*1*›],
@{*thms semiring-normalization-rules*}),
ring = ([***term*** ‹*x* − *y*›, ***term*** ‹− *x*›], @{*thms ring-normalization-rules*}),
field = ([***term*** ‹*x* / *y*›, ***term*** ‹*inverse x*›], @{*thms divide-inverse inverse-eq-divide*}),
idom = @{*thms crossproduct-noteq add-scale-eq-noteq*},
ideal = @{*thms right-minus-eq add-0-iff*}}
›

end

code-identifier
code-module *Semiring-Normalization* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

60 Groebner bases

theory *Groebner-Basis*
imports *Semiring-Normalization Parity*
begin

60.1 Groebner Bases

lemmas *bool-simps* = *simp-thms*(*1*−*34*) — FIXME move to *HOL.HOL*

lemma *nnf-simps*: — FIXME shadows fact binding in *HOL.HOL*
(¬(*P* ∧ *Q*)) = (¬*P* ∨ ¬*Q*) (¬(*P* ∨ *Q*)) = (¬*P* ∧ ¬*Q*)
(*P* ⟶ *Q*) = (¬*P* ∨ *Q*)
(*P* = *Q*) = ((*P* ∧ *Q*) ∨ (¬*P* ∧ ¬ *Q*)) (¬ ¬(*P*)) = *P*
by *blast*+

lemma *dnf*:
$(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$
$((Q \vee R) \wedge P) = ((Q \wedge P) \vee (R \wedge P))$
$(P \wedge Q) = (Q \wedge P)$
$(P \vee Q) = (Q \vee P)$
by *blast+*

lemmas *weak-dnf-simps* = *dnf bool-simps*

lemma *PFalse*:
$P \equiv False \Longrightarrow \neg P$
$\neg P \Longrightarrow (P \equiv False)$
by *auto*

named-theorems *algebra pre−simplification rules for algebraic methods*
ML-file ‹*Tools/groebner.ML*›

method-setup *algebra* = ‹
```
let
  fun keyword k = Scan.lift (Args.$$$ k -- Args.colon) >> K ()
  val addN = add
  val delN = del
  val any-keyword = keyword addN || keyword delN
  val thms = Scan.repeats (Scan.unless any-keyword Attrib.multi-thm);
in
  Scan.optional (keyword addN |-- thms) [] --
   Scan.optional (keyword delN |-- thms) [] >>
  (fn (add-ths, del-ths) => fn ctxt =>
    SIMPLE-METHOD' (Groebner.algebra-tac add-ths del-ths ctxt))
end
```
› *solve polynomial equations over* (*semi*)*rings and ideal membership problems using Groebner bases*

declare *dvd-def*[*algebra*]
declare *mod-eq-0-iff-dvd*[*algebra*]
declare *mod-div-trivial*[*algebra*]
declare *mod-mod-trivial*[*algebra*]
declare *div-by-0*[*algebra*]
declare *mod-by-0*[*algebra*]
declare *mult-div-mod-eq*[*algebra*]
declare *div-minus-minus*[*algebra*]
declare *mod-minus-minus*[*algebra*]
declare *div-minus-right*[*algebra*]
declare *mod-minus-right*[*algebra*]
declare *div-0*[*algebra*]
declare *mod-0*[*algebra*]
declare *mod-by-1*[*algebra*]
declare *div-by-1*[*algebra*]

declare *mod-minus1-right*[*algebra*]
declare *div-minus1-right*[*algebra*]
declare *mod-mult-self2-is-0*[*algebra*]
declare *mod-mult-self1-is-0*[*algebra*]

lemma *zmod-eq-0-iff* [*algebra*]:
‹*m mod d* = *0* ⟷ (∃ *q*. *m* = *d* ∗ *q*)› **for** *m d* :: *int*
by (*auto simp add*: *mod-eq-0-iff-dvd*)

declare *dvd-0-left-iff*[*algebra*]
declare *zdvd1-eq*[*algebra*]
declare *mod-eq-dvd-iff*[*algebra*]
declare *nat-mod-eq-iff*[*algebra*]

context *semiring-parity*
begin

declare *even-mult-iff* [*algebra*]
declare *even-power* [*algebra*]

end

context *ring-parity*
begin

declare *even-minus* [*algebra*]

end

declare *even-Suc* [*algebra*]
declare *even-diff-nat* [*algebra*]

end

61 Set intervals

theory *Set-Interval*
imports *Parity*
begin

lemma *card-2-iff*: *card S* = *2* ⟷ (∃ *x y*. *S* = {*x*,*y*} ∧ *x* ≠ *y*)
by (*auto simp*: *card-Suc-eq numeral-eq-Suc*)

lemma *card-2-iff'*: *card S* = *2* ⟷ (∃ *x*∈*S*. ∃ *y*∈*S*. *x* ≠ *y* ∧ (∀ *z*∈*S*. *z* = *x* ∨ *z* = *y*))
by (*auto simp*: *card-Suc-eq numeral-eq-Suc*)

lemma *card-3-iff*: *card S* = *3* ⟷ (∃ *x y z*. *S* = {*x*,*y*,*z*} ∧ *x* ≠ *y* ∧ *y* ≠ *z* ∧ *x* ≠

z)
by (*fastforce simp*: *card-Suc-eq numeral-eq-Suc*)

context *ord*
begin

definition
lessThan :: *'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{..<-})›)
where
{..<u} == {x. x < u}

definition
atMost :: *'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{..-})›)
where
{..u} == {x. x ≤ u}

definition
greaterThan :: *'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{-<..})›)
where
{l<..} == {x. l<x}

definition
atLeast :: *'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{-..})›)
where
{l..} == {x. l≤x}

definition
greaterThanLessThan :: *'a => 'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{-/<..<-})›) **where**
{l<..<u} == {l<..} *Int* {..<u}

definition
atLeastLessThan :: *'a => 'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{-/..<-})›) **where**
{l..<u} == {l..} *Int* {..<u}

definition
greaterThanAtMost :: *'a => 'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{-/<..-})›) **where**
{l<..u} == {l<..} *Int* {..u}

definition
atLeastAtMost :: *'a => 'a => 'a set* (‹(‹indent=1 notation=‹mixfix set interval››{-/..-})›) **where**
{l..u} == {l..} *Int* {..u}

end

A note of warning when using {..<n} on type *nat*: it is equivalent to {0..<n}

but some lemmas involving $\{m..<n\}$ may not exist in $\{..<n\}$-form as well.

syntax (*ASCII*)
-UNION-le :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder UN›› UN -<=-./ -)› [0, 0, 10] 10)
-UNION-less :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder UN›› UN -<-./ -)› [0, 0, 10] 10)
-INTER-le :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder INT›› INT -<=-./ -)› [0, 0, 10] 10)
-INTER-less :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder INT›› INT -<-./ -)› [0, 0, 10] 10)

syntax (*latex* **output**)
-UNION-le :: *'a ⇒ 'a => 'b set => 'b set* (‹(3⋃(‹unbreakable›- ≤ -)/ -)› [0, 0, 10] 10)
-UNION-less :: *'a ⇒ 'a => 'b set => 'b set* (‹(3⋃(‹unbreakable›- < -)/ -)› [0, 0, 10] 10)
-INTER-le :: *'a ⇒ 'a => 'b set => 'b set* (‹(3⋂(‹unbreakable›- ≤ -)/ -)› [0, 0, 10] 10)
-INTER-less :: *'a ⇒ 'a => 'b set => 'b set* (‹(3⋂(‹unbreakable›- < -)/ -)› [0, 0, 10] 10)

syntax
-UNION-le :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder ⋃››⋃ -≤-./ -)› [0, 0, 10] 10)
-UNION-less :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder ⋃››⋃ -<-./ -)› [0, 0, 10] 10)
-INTER-le :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder ⋂››⋂ -≤-./ -)› [0, 0, 10] 10)
-INTER-less :: *'a => 'a => 'b set => 'b set* (‹(‹indent=3 notation=‹binder ⋂››⋂ -<-./ -)› [0, 0, 10] 10)

syntax-consts
-UNION-le -UNION-less ⇌ *Union* **and**
-INTER-le -INTER-less ⇌ *Inter*

translations
⋃*i*≤*n*. *A* ⇌ ⋃*i*∈{..*n*}. *A*
⋃*i*<*n*. *A* ⇌ ⋃*i*∈{..<*n*}. *A*
⋂*i*≤*n*. *A* ⇌ ⋂*i*∈{..*n*}. *A*
⋂*i*<*n*. *A* ⇌ ⋂*i*∈{..<*n*}. *A*

61.1 Various equivalences

lemma (**in** *ord*) *lessThan-iff* [*iff*]: (*i* ∈ *lessThan k*) = (*i*<*k*)
by (*simp add*: *lessThan-def*)

lemma *Compl-lessThan* [*simp*]:
!!*k*:: *'a*::*linorder*. −*lessThan k* = *atLeast k*
by (*auto simp add*: *lessThan-def atLeast-def*)

lemma *single-Diff-lessThan* [*simp*]: !!*k*:: *'a*::*preorder*. {*k*} − *lessThan k* = {*k*}
by *auto*

lemma (**in** *ord*) *greaterThan-iff* [*iff*]: (*i* ∈ *greaterThan k*) = (*k*<*i*)
by (*simp add*: *greaterThan-def*)

lemma *Compl-greaterThan* [*simp*]:
!!*k*:: *'a*::*linorder*. −*greaterThan k* = *atMost k*
by (*auto simp add*: *greaterThan-def atMost-def*)

lemma *Compl-atMost* [*simp*]: !!*k*:: *'a*::*linorder*. −*atMost k* = *greaterThan k*
by (*metis Compl-greaterThan double-complement*)

lemma (**in** *ord*) *atLeast-iff* [*iff*]: (*i* ∈ *atLeast k*) = (*k*<=*i*)
by (*simp add*: *atLeast-def*)

lemma *Compl-atLeast* [*simp*]: !!*k*:: *'a*::*linorder*. −*atLeast k* = *lessThan k*
by (*auto simp add*: *lessThan-def atLeast-def*)

lemma (**in** *ord*) *atMost-iff* [*iff*]: (*i* ∈ *atMost k*) = (*i*<=*k*)
by (*simp add*: *atMost-def*)

lemma *atMost-Int-atLeast*: !!*n*:: *'a*::*order*. *atMost n Int atLeast n* = {*n*}
by (*blast intro*: *order-antisym*)

lemma (**in** *linorder*) *lessThan-Int-lessThan*: { *a* <..} ∩ { *b* <..} = { *max a b* <..}
by *auto*

lemma (**in** *linorder*) *greaterThan-Int-greaterThan*: {..< *a*} ∩ {..< *b*} = {..< *min a b*}
by *auto*

61.2 Logical Equivalences for Set Inclusion and Equality

lemma *atLeast-empty-triv* [*simp*]: {{}..} = *UNIV*
by *auto*

lemma *atMost-UNIV-triv* [*simp*]: {..*UNIV*} = *UNIV*
by *auto*

lemma *atLeast-subset-iff* [*iff*]:
(*atLeast x* ⊆ *atLeast y*) = (*y* ≤ (*x*::*'a*::*preorder*))
by (*blast intro*: *order-trans*)

lemma *atLeast-eq-iff* [*iff*]:
(*atLeast x* = *atLeast y*) = (*x* = (*y*::*'a*::*order*))
by (*blast intro*: *order-antisym order-trans*)

lemma *greaterThan-subset-iff* [*iff*]:
$(greaterThan\ x \subseteq greaterThan\ y) = (y \le (x{::}'a{::}linorder))$
unfolding *greaterThan-def* **by** (*auto simp*: *linorder-not-less* [*symmetric*])

lemma *greaterThan-eq-iff* [*iff*]:
$(greaterThan\ x = greaterThan\ y) = (x = (y{::}'a{::}linorder))$
by (*auto simp*: *elim*!: *equalityE*)

lemma *atMost-subset-iff* [*iff*]: $(atMost\ x \subseteq atMost\ y) = (x \le (y{::}'a{::}preorder))$
by (*blast intro*: *order-trans*)

lemma *atMost-eq-iff* [*iff*]: $(atMost\ x = atMost\ y) = (x = (y{::}'a{::}order))$
by (*blast intro*: *order-antisym order-trans*)

lemma *lessThan-subset-iff* [*iff*]:
$(lessThan\ x \subseteq lessThan\ y) = (x \le (y{::}'a{::}linorder))$
unfolding *lessThan-def* **by** (*auto simp*: *linorder-not-less* [*symmetric*])

lemma *lessThan-eq-iff* [*iff*]:
$(lessThan\ x = lessThan\ y) = (x = (y{::}'a{::}linorder))$
by (*auto simp*: *elim*!: *equalityE*)

lemma *lessThan-strict-subset-iff*:
fixes $m\ n :: 'a{::}linorder$
shows $\{..<m\} < \{..<n\} \longleftrightarrow m < n$
by (*metis leD lessThan-subset-iff linorder-linear not-less-iff-gr-or-eq psubset-eq*)

lemma (**in** *linorder*) *Ici-subset-Ioi-iff*: $\{a\ ..\} \subseteq \{b\ <..\} \longleftrightarrow b < a$
by *auto*

lemma (**in** *linorder*) *Iic-subset-Iio-iff*: $\{..\ a\} \subseteq \{..<\ b\} \longleftrightarrow a < b$
by *auto*

lemma (**in** *preorder*) *Ioi-le-Ico*: $\{a\ <..\} \subseteq \{a\ ..\}$
by (*auto intro*: *less-imp-le*)

61.3 Two-sided intervals

context *ord*
begin

lemma *greaterThanLessThan-iff* [*simp*]: $(i \in \{l<..<u\}) = (l < i \wedge i < u)$
by (*simp add*: *greaterThanLessThan-def*)

lemma *atLeastLessThan-iff* [*simp*]: $(i \in \{l..<u\}) = (l \le i \wedge i < u)$
by (*simp add*: *atLeastLessThan-def*)

lemma *greaterThanAtMost-iff* [*simp*]: $(i \in \{l<..u\}) = (l < i \wedge i \le u)$
by (*simp add*: *greaterThanAtMost-def*)

lemma *atLeastAtMost-iff* [*simp*]: $(i \in \{l..u\}) = (l \leq i \wedge i \leq u)$
by (*simp add*: *atLeastAtMost-def*)

The above four lemmas could be declared as iffs. Unfortunately this breaks many proofs. Since it only helps blast, it is better to leave them alone.

lemma *greaterThanLessThan-eq*: $\{ a <..< b\} = \{ a <..\} \cap \{..< b \}$
by *auto*

lemma (**in** *order*) *atLeastLessThan-eq-atLeastAtMost-diff*:
$\{a..<b\} = \{a..b\} - \{b\}$
by (*auto simp add*: *atLeastLessThan-def atLeastAtMost-def*)

lemma (**in** *order*) *greaterThanAtMost-eq-atLeastAtMost-diff*:
$\{a<..b\} = \{a..b\} - \{a\}$
by (*auto simp add*: *greaterThanAtMost-def atLeastAtMost-def*)

end

61.3.1 Emptyness, singletons, subset

context *preorder*
begin

lemma *atLeastatMost-empty-iff*[*simp*]:
$\{a..b\} = \{\} \longleftrightarrow (\neg\ a \leq b)$
by *auto* (*blast intro*: *order-trans*)

lemma *atLeastatMost-empty-iff2*[*simp*]:
$\{\} = \{a..b\} \longleftrightarrow (\neg\ a \leq b)$
by *auto* (*blast intro*: *order-trans*)

lemma *atLeastLessThan-empty-iff*[*simp*]:
$\{a..<b\} = \{\} \longleftrightarrow (\neg\ a < b)$
by *auto* (*blast intro*: *le-less-trans*)

lemma *atLeastLessThan-empty-iff2*[*simp*]:
$\{\} = \{a..<b\} \longleftrightarrow (\neg\ a < b)$
by *auto* (*blast intro*: *le-less-trans*)

lemma *greaterThanAtMost-empty-iff*[*simp*]: $\{k<..l\} = \{\} \longleftrightarrow \neg\ k < l$
by *auto* (*blast intro*: *less-le-trans*)

lemma *greaterThanAtMost-empty-iff2*[*simp*]: $\{\} = \{k<..l\} \longleftrightarrow \neg\ k < l$
by *auto* (*blast intro*: *less-le-trans*)

lemma *atLeastatMost-subset-iff*[*simp*]:
$\{a..b\} \leq \{c..d\} \longleftrightarrow (\neg\ a \leq b) \vee c \leq a \wedge b \leq d$
unfolding *atLeastAtMost-def atLeast-def atMost-def*

by (*blast intro*: *order-trans*)

lemma *atLeastatMost-psubset-iff*:
$\{a..b\} < \{c..d\} \longleftrightarrow$
$((\neg\ a \le b) \lor c \le a \land b \le d \land (c < a \lor b < d)) \land c \le d$
by(*simp add*: *psubset-eq set-eq-iff less-le-not-le*)(*blast intro*: *order-trans*)

lemma *atLeastAtMost-subseteq-atLeastLessThan-iff*:
$\{a..b\} \subseteq \{c\ ..< d\} \longleftrightarrow (a \le b \longrightarrow c \le a \land b < d)$
by *auto* (*blast intro*: *local.order-trans local.le-less-trans elim*:)+

lemma *Icc-subset-Ici-iff*[*simp*]:
$\{l..h\} \subseteq \{l'..\} = (\neg\ l{\le}h \lor l{\ge}l')$
by(*auto simp*: *subset-eq intro*: *order-trans*)

lemma *Icc-subset-Iic-iff*[*simp*]:
$\{l..h\} \subseteq \{..h'\} = (\neg\ l{\le}h \lor h{\le}h')$
by(*auto simp*: *subset-eq intro*: *order-trans*)

lemma *not-Ici-eq-empty*[*simp*]: $\{l..\} \ne \{\}$
by(*auto simp*: *set-eq-iff*)

lemma *not-Iic-eq-empty*[*simp*]: $\{..h\} \ne \{\}$
by(*auto simp*: *set-eq-iff*)

lemmas *not-empty-eq-Ici-eq-empty*[*simp*] = *not-Ici-eq-empty*[*symmetric*]
lemmas *not-empty-eq-Iic-eq-empty*[*simp*] = *not-Iic-eq-empty*[*symmetric*]

end

context *order*
begin

lemma *atLeastatMost-empty*[*simp*]: $b < a \Longrightarrow \{a..b\} = \{\}$
and *atLeastatMost-empty'*[*simp*]: $\neg\ a \le b \Longrightarrow \{a..b\} = \{\}$
by(*auto simp*: *atLeastAtMost-def atLeast-def atMost-def*)

lemma *atLeastLessThan-empty*[*simp*]:
$b \le a \Longrightarrow \{a..{<}b\} = \{\}$
by(*auto simp*: *atLeastLessThan-def*)

lemma *greaterThanAtMost-empty*[*simp*]: $l \le k$ ==> $\{k{<}..l\} = \{\}$
by(*auto simp*:*greaterThanAtMost-def greaterThan-def atMost-def*)

lemma *greaterThanLessThan-empty*[*simp*]:$l \le k$ ==> $\{k{<}..{<}l\} = \{\}$
by(*auto simp*:*greaterThanLessThan-def greaterThan-def lessThan-def*)

lemma *atLeastAtMost-singleton* [*simp*]: $\{a..a\} = \{a\}$
by (*auto simp add*: *atLeastAtMost-def atMost-def atLeast-def*)

lemma *atLeastAtMost-singleton'*: $a = b \Longrightarrow \{a\ ..\ b\} = \{a\}$ **by** *simp*

lemma *Icc-eq-Icc*[*simp*]:
  $\{l..h\} = \{l'..h'\} = (l{=}l' \wedge h{=}h' \vee \neg\ l{\le}h \wedge \neg\ l'{\le}h')$
  by (*simp add*: *order-class.order.eq-iff*) (*auto intro*: *order-trans*)

lemma (**in** *linorder*) *Ico-eq-Ico*:
  $\{l..{<}h\} = \{l'..{<}h'\} = (l{=}l' \wedge h{=}h' \vee \neg\ l{<}h \wedge \neg\ l'{<}h')$
  by (*metis atLeastLessThan-empty-iff2 nle-le not-less ord.atLeastLessThan-iff*)

lemma *atLeastAtMost-singleton-iff*[*simp*]:
  $\{a\ ..\ b\} = \{c\} \longleftrightarrow a = b \wedge b = c$
proof
  assume $\{a..b\} = \{c\}$
  hence $*$: $\neg\ (\neg\ a \le b)$ **unfolding** *atLeastatMost-empty-iff*[*symmetric*] **by** *simp*
  with ‹$\{a..b\} = \{c\}$› **have** $c \le a \wedge b \le c$ **by** *auto*
  with $*$ **show** $a = b \wedge b = c$ **by** *auto*
qed *simp*

The following results generalise their namesakes in *HOL.Nat* to intervals

lemma *lift-Suc-mono-le-ivl*:
  assumes *mono*: $\bigwedge n.\ n{\in}N \Longrightarrow f\ n \le f\ (Suc\ n)$
    and $n \le n'$ **and** *subN*: $\{n..{<}n'\} \subseteq N$
  shows $f\ n \le f\ n'$
proof (*cases* $n < n'$)
  case *True*
  then show *?thesis*
    using *subN*
  proof (*induction* $n\ n'$ *rule*: *less-Suc-induct*)
    case (*1 i*)
    then show *?case*
      by (*simp add*: *mono subsetD*)
  next
    case (*2 i j k*)
    have $f\ i \le f\ j\ f\ j \le f\ k$
      using *2* **by** *force+*
    then show *?case* **by** *auto*
  qed
next
  case *False*
  with ‹$n \le n'$› **show** *?thesis* **by** *auto*
qed

lemma *lift-Suc-antimono-le-ivl*:
  assumes *mono*: $\bigwedge n.\ n{\in}N \Longrightarrow f\ n \ge f\ (Suc\ n)$
    and $n \le n'$ **and** *subN*: $\{n..{<}n'\} \subseteq N$
  shows $f\ n \ge f\ n'$
proof (*cases* $n < n'$)

case *True*
then show *?thesis*
using *subN*
proof (*induction n n′ rule*: *less-Suc-induct*)
case (*1 i*)
then show *?case*
by (*simp add*: *mono subsetD*)
next
case (*2 i j k*)
have $f\ i \geq f\ j\ f\ j \geq f\ k$
using *2* **by** *force+*
then show *?case* **by** *auto*
qed
next
case *False*
with ‹$n \leq n'$› **show** *?thesis* **by** *auto*
qed

lemma *lift-Suc-mono-less-ivl*:
assumes *mono*: $\bigwedge n.\ n \in N \Longrightarrow f\ n < f\ (Suc\ n)$
and $n < n'$ **and** *subN*: $\{n..<n'\} \subseteq N$
shows $f\ n < f\ n'$
using ‹$n < n'$›
using *subN*
proof (*induction n n′ rule*: *less-Suc-induct*)
case (*1 i*)
then show *?case*
by (*simp add*: *mono subsetD*)
next
case (*2 i j k*)
have $f\ i < f\ j\ f\ j < f\ k$
using *2* **by** *force+*
then show *?case* **by** *auto*
qed

end

context *no-top*
begin

lemma *not-UNIV-le-Icc*[*simp*]: $\neg\ UNIV \subseteq \{l..h\}$
using *gt-ex*[*of h*] **by**(*auto simp*: *subset-eq less-le-not-le*)

lemma *not-UNIV-le-Iic*[*simp*]: $\neg\ UNIV \subseteq \{..h\}$
using *gt-ex*[*of h*] **by**(*auto simp*: *subset-eq less-le-not-le*)

lemma *not-Ici-le-Icc*[*simp*]: $\neg\ \{l..\} \subseteq \{l'..h'\}$
using *gt-ex*[*of h′*]

by(*auto simp*: *subset-eq less-le*)(*blast dest*:*antisym-conv intro*: *order-trans*)

lemma *not-Ici-le-Iic*[*simp*]: ¬ {*l*..} ⊆ {..*h'*}
using *gt-ex*[*of h'*]
by(*auto simp*: *subset-eq less-le*)(*blast dest*:*antisym-conv intro*: *order-trans*)

end

context *no-bot*
begin

lemma *not-UNIV-le-Ici*[*simp*]: ¬ *UNIV* ⊆ {*l*..}
using *lt-ex*[*of l*] **by**(*auto simp*: *subset-eq less-le-not-le*)

lemma *not-Iic-le-Icc*[*simp*]: ¬ {..*h*} ⊆ {*l'*..*h'*}
using *lt-ex*[*of l'*]
by(*auto simp*: *subset-eq less-le*)(*blast dest*:*antisym-conv intro*: *order-trans*)

lemma *not-Iic-le-Ici*[*simp*]: ¬ {..*h*} ⊆ {*l'*..}
using *lt-ex*[*of l'*]
by(*auto simp*: *subset-eq less-le*)(*blast dest*:*antisym-conv intro*: *order-trans*)

end

context *no-top*
begin

lemma *not-UNIV-eq-Icc*[*simp*]: ¬ *UNIV* = {*l'*..*h'*}
using *gt-ex*[*of h'*] **by**(*auto simp*: *set-eq-iff less-le-not-le*)

lemmas *not-Icc-eq-UNIV*[*simp*] = *not-UNIV-eq-Icc*[*symmetric*]

lemma *not-UNIV-eq-Iic*[*simp*]: ¬ *UNIV* = {..*h'*}
using *gt-ex*[*of h'*] **by**(*auto simp*: *set-eq-iff less-le-not-le*)

lemmas *not-Iic-eq-UNIV*[*simp*] = *not-UNIV-eq-Iic*[*symmetric*]

lemma *not-Icc-eq-Ici*[*simp*]: ¬ {*l*..*h*} = {*l'*..}
unfolding *atLeastAtMost-def* **using** *not-Ici-le-Iic*[*of l'*] **by** *blast*

lemmas *not-Ici-eq-Icc*[*simp*] = *not-Icc-eq-Ici*[*symmetric*]

lemma *not-Iic-eq-Ici*[*simp*]: ¬ {..*h*} = {*l'*..}
using *not-Ici-le-Iic*[*of l' h*] **by** *blast*

lemmas *not-Ici-eq-Iic*[*simp*] = *not-Iic-eq-Ici*[*symmetric*]

end

context *no-bot*
begin

lemma *not-UNIV-eq-Ici*[*simp*]: $\neg$ *UNIV* = $\{l'..\}$
using *lt-ex*[*of l'*] **by**(*auto simp*: *set-eq-iff less-le-not-le*)

lemmas *not-Ici-eq-UNIV*[*simp*] = *not-UNIV-eq-Ici*[*symmetric*]

lemma *not-Icc-eq-Iic*[*simp*]: $\neg$ $\{l..h\} = \{..h'\}$
unfolding *atLeastAtMost-def* **using** *not-Iic-le-Ici*[*of h'*] **by** *blast*

lemmas *not-Iic-eq-Icc*[*simp*] = *not-Icc-eq-Iic*[*symmetric*]

end

context *dense-linorder*
begin

lemma *greaterThanLessThan-empty-iff*[*simp*]:
 $\{ a <..< b \} = \{\} \longleftrightarrow b \leq a$
 using *dense*[*of a b*] **by** (*cases a < b*) *auto*

lemma *greaterThanLessThan-empty-iff2*[*simp*]:
 $\{\} = \{ a <..< b \} \longleftrightarrow b \leq a$
 using *dense*[*of a b*] **by** (*cases a < b*) *auto*

lemma *atLeastLessThan-subseteq-atLeastAtMost-iff*:
 $\{a ..< b\} \subseteq \{ c .. d \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
 using *dense*[*of max a d b*]
 by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

lemma *greaterThanAtMost-subseteq-atLeastAtMost-iff*:
 $\{a <.. b\} \subseteq \{ c .. d \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
 using *dense*[*of a min c b*]
 by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

lemma *greaterThanLessThan-subseteq-atLeastAtMost-iff*:
 $\{a <..< b\} \subseteq \{ c .. d \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
 using *dense*[*of a min c b*] *dense*[*of max a d b*]
 by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

lemma *greaterThanLessThan-subseteq-greaterThanLessThan*:
 $\{a <..< b\} \subseteq \{c <..< d\} \longleftrightarrow (a < b \longrightarrow a \geq c \wedge b \leq d)$
 using *dense*[*of a min c b*] *dense*[*of max a d b*]
 by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

lemma *greaterThanAtMost-subseteq-atLeastLessThan-iff*:
$\{a <.. b\} \subseteq \{ c ..< d \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b < d)$
using *dense*[*of a min c b*]
by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

lemma *greaterThanLessThan-subseteq-atLeastLessThan-iff*:
$\{a <..< b\} \subseteq \{ c ..< d \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
using *dense*[*of a min c b*] *dense*[*of max a d b*]
by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

lemma *greaterThanLessThan-subseteq-greaterThanAtMost-iff*:
$\{a <..< b\} \subseteq \{ c <.. d \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
using *dense*[*of a min c b*] *dense*[*of max a d b*]
by (*force simp*: *subset-eq Ball-def not-less*[*symmetric*])

end

context *no-top*
begin

lemma *greaterThan-non-empty*[*simp*]: $\{x <..\} \neq \{\}$
using *gt-ex*[*of x*] **by** *auto*

end

context *no-bot*
begin

lemma *lessThan-non-empty*[*simp*]: $\{..< x\} \neq \{\}$
using *lt-ex*[*of x*] **by** *auto*

end

lemma (**in** *linorder*) *atLeastLessThan-subset-iff*:
$\{a..<b\} \subseteq \{c..<d\} \Longrightarrow b \leq a \vee c \leq a \wedge b \leq d$
proof (*cases* $a < b$)
case *True*
assume *assm*: $\{a..<b\} \subseteq \{c..<d\}$
then have *1*: $c \leq a \wedge a \leq d$
using *True* **by** (*auto simp add*: *subset-eq Ball-def*)
then have *2*: $b \leq d$
using *assm* **by** (*auto simp add*: *subset-eq*)
from *1 2* **show** *?thesis*
by *simp*
qed (*auto*)

lemma *atLeastLessThan-inj*:
fixes *a b c d* :: *'a*::*linorder*

assumes *eq*: $\{a ..< b\} = \{c ..< d\}$ **and** $a < b$ $c < d$
shows $a = c$ $b = d$
using *assms* **by** (*metis atLeastLessThan-subset-iff eq less-le-not-le antisym-conv2 subset-refl*)+

lemma *atLeastLessThan-eq-iff*:
fixes *a b c d* :: *'a::linorder*
assumes $a < b$ $c < d$
shows $\{a ..< b\} = \{c ..< d\} \longleftrightarrow a = c \wedge b = d$
using *atLeastLessThan-inj assms* **by** *auto*

lemma (**in** *linorder*) *Ioc-inj*:
‹$\{a <.. b\} = \{c <.. d\} \longleftrightarrow (b \leq a \wedge d \leq c) \vee a = c \wedge b = d$› (**is** ‹$?P \longleftrightarrow ?Q$›)
proof
assume *?Q*
then show *?P*
by *auto*
next
assume *?P*
then have ‹$a < x \wedge x \leq b \longleftrightarrow c < x \wedge x \leq d$› **for** *x*
by (*simp add*: *set-eq-iff*)
from *this* [*of a*] *this* [*of b*] *this* [*of c*] *this* [*of d*] **show** *?Q*
by *auto*
qed

lemma (**in** *order*) *Iio-Int-singleton*: $\{..<k\} \cap \{x\} = (\textit{if } x < k \textit{ then } \{x\} \textit{ else } \{\})$
by *auto*

lemma (**in** *linorder*) *Ioc-subset-iff*: $\{a<..b\} \subseteq \{c<..d\} \longleftrightarrow (b \leq a \vee c \leq a \wedge b \leq d)$
by (*auto simp*: *subset-eq Ball-def*) (*metis less-le not-less*)

lemma (**in** *order-bot*) *atLeast-eq-UNIV-iff*: $\{x..\} = UNIV \longleftrightarrow x = bot$
by (*auto simp*: *set-eq-iff intro*: *le-bot*)

lemma (**in** *order-top*) *atMost-eq-UNIV-iff*: $\{..x\} = UNIV \longleftrightarrow x = top$
by (*auto simp*: *set-eq-iff intro*: *top-le*)

lemma (**in** *bounded-lattice*) *atLeastAtMost-eq-UNIV-iff*:
$\{x..y\} = UNIV \longleftrightarrow (x = bot \wedge y = top)$
by (*auto simp*: *set-eq-iff intro*: *top-le le-bot*)

lemma *Iio-eq-empty-iff*: $\{..< n::'a::\{linorder, order\text{-}bot\}\} = \{\} \longleftrightarrow n = bot$
by (*auto simp*: *set-eq-iff not-less le-bot*)

lemma *lessThan-empty-iff*: $\{..< n::nat\} = \{\} \longleftrightarrow n = 0$
by (*simp add*: *Iio-eq-empty-iff bot-nat-def*)

lemma *mono-image-least*:

assumes *f-mono*: *mono f* **and** *f-img*: f ' $\{m\ ..< n\} = \{m'\ ..< n'\}$ $m < n$
shows $f\ m = m'$
proof −
from *f-img* **have** $\{m'\ ..< n'\} \neq \{\}$
by (*metis atLeastLessThan-empty-iff image-is-empty*)
with *f-img* **have** $m' \in f$ ' $\{m\ ..< n\}$ **by** *auto*
then obtain k **where** $f\ k = m'$ $m \leq k$ **by** *auto*
moreover have $m' \leq f\ m$ **using** *f-img* **by** *auto*
ultimately show $f\ m = m'$
using *f-mono* **by** (*auto elim*: *monoE*[**where** *x*=*m* **and** *y*=*k*])
qed

61.4 Infinite intervals

context *dense-linorder*
begin

lemma *infinite-Ioo*:
assumes $a < b$
shows ¬ *finite* $\{a<..<b\}$
proof
assume *fin*: *finite* $\{a<..<b\}$
moreover have *ne*: $\{a<..<b\} \neq \{\}$
using ‹$a < b$› **by** *auto*
ultimately have $a <$ *Max* $\{a <..< b\}$ *Max* $\{a <..< b\} < b$
using *Max-in*[*of* $\{a <..< b\}$] **by** *auto*
then obtain x **where** *Max* $\{a <..< b\} < x$ $x < b$
using *dense*[*of Max* $\{a<..<b\}$ *b*] **by** *auto*
then have $x \in \{a <..< b\}$
using ‹$a <$ *Max* $\{a <..< b\}$› **by** *auto*
then have $x \leq$ *Max* $\{a <..< b\}$
using *fin* **by** *auto*
with ‹*Max* $\{a <..< b\} < x$› **show** *False* **by** *auto*
qed

lemma *infinite-Icc*: $a < b \Longrightarrow$ ¬ *finite* $\{a\ ..\ b\}$
using *greaterThanLessThan-subseteq-atLeastAtMost-iff*[*of a b a b*] *infinite-Ioo*[*of a b*]
by (*auto dest*: *finite-subset*)

lemma *infinite-Ico*: $a < b \Longrightarrow$ ¬ *finite* $\{a\ ..< b\}$
using *greaterThanLessThan-subseteq-atLeastLessThan-iff*[*of a b a b*] *infinite-Ioo*[*of a b*]
by (*auto dest*: *finite-subset*)

lemma *infinite-Ioc*: $a < b \Longrightarrow$ ¬ *finite* $\{a <..\ b\}$
using *greaterThanLessThan-subseteq-greaterThanAtMost-iff*[*of a b a b*] *infinite-Ioo*[*of a b*]
by (*auto dest*: *finite-subset*)

lemma *infinite-Ioo-iff* [*simp*]: *infinite* {*a*<..<*b*} ⟷ *a* < *b*
  using *not-less-iff-gr-or-eq* **by** (*fastforce simp*: *infinite-Ioo*)

lemma *infinite-Icc-iff* [*simp*]: *infinite* {*a* .. *b*} ⟷ *a* < *b*
  using *not-less-iff-gr-or-eq* **by** (*fastforce simp*: *infinite-Icc*)

lemma *infinite-Ico-iff* [*simp*]: *infinite* {*a*..<*b*} ⟷ *a* < *b*
  using *not-less-iff-gr-or-eq* **by** (*fastforce simp*: *infinite-Ico*)

lemma *infinite-Ioc-iff* [*simp*]: *infinite* {*a*<..*b*} ⟷ *a* < *b*
  using *not-less-iff-gr-or-eq* **by** (*fastforce simp*: *infinite-Ioc*)

end

lemma *infinite-Iio*: ¬ *finite* {..< *a* :: ′*a* :: {*no-bot*, *linorder*}}
proof
  assume *finite* {..< *a*}
  then have ∗: ⋀*x*. *x* < *a* ⟹ *Min* {..< *a*} ≤ *x*
    by *auto*
  obtain *x* **where** *x* < *a*
    using *lt-ex* **by** *auto*

  obtain *y* **where** *y* < *Min* {..< *a*}
    using *lt-ex* **by** *auto*
  also have *Min* {..< *a*} ≤ *x*
    using ‹*x* < *a*› **by** *fact*
  also note ‹*x* < *a*›
  finally have *Min* {..< *a*} ≤ *y*
    by *fact*
  with ‹*y* < *Min* {..< *a*}› **show** *False* **by** *auto*
qed

lemma *infinite-Iic*: ¬ *finite* {.. *a* :: ′*a* :: {*no-bot*, *linorder*}}
  using *infinite-Iio*[*of a*] *finite-subset*[*of* {..< *a*} {.. *a*}]
  by (*auto simp*: *subset-eq less-imp-le*)

lemma *infinite-Ioi*: ¬ *finite* {*a* :: ′*a* :: {*no-top*, *linorder*} <..}
proof
  assume *finite* {*a* <..}
  then have ∗: ⋀*x*. *a* < *x* ⟹ *x* ≤ *Max* {*a* <..}
    by *auto*

  obtain *y* **where** *Max* {*a* <..} < *y*
    using *gt-ex* **by** *auto*

  obtain *x* **where** *x*: *a* < *x*
    using *gt-ex* **by** *auto*
  also from *x* **have** *x* ≤ *Max* {*a* <..}

by *fact*
also note ‹*Max* {*a* <..} < *y*›
finally have *y* ≤ *Max* { *a* <..}
by *fact*
with ‹*Max* {*a* <..} < *y*› **show** *False* **by** *auto*
qed

lemma *infinite-Ici*: ¬ *finite* {*a* :: ′*a* :: {*no-top*, *linorder*} ..}
using *infinite-Ioi*[*of a*] *finite-subset*[*of* {*a* <..} {*a* ..}]
by (*auto simp*: *subset-eq less-imp-le*)

61.4.1 Intersection

context *linorder*
begin

lemma *Int-atLeastAtMost*[*simp*]: {*a*..*b*} *Int* {*c*..*d*} = {*max a c* .. *min b d*}
by *auto*

lemma *Int-atLeastAtMostR1*[*simp*]: {..*b*} *Int* {*c*..*d*} = {*c* .. *min b d*}
by *auto*

lemma *Int-atLeastAtMostR2*[*simp*]: {*a*..} *Int* {*c*..*d*} = {*max a c* .. *d*}
by *auto*

lemma *Int-atLeastAtMostL1*[*simp*]: {*a*..*b*} *Int* {..*d*} = {*a* .. *min b d*}
by *auto*

lemma *Int-atLeastAtMostL2*[*simp*]: {*a*..*b*} *Int* {*c*..} = {*max a c* .. *b*}
by *auto*

lemma *Int-atLeastLessThan*[*simp*]: {*a*..<*b*} *Int* {*c*..<*d*} = {*max a c* ..< *min b d*}
by *auto*

lemma *Int-greaterThanAtMost*[*simp*]: {*a*<..*b*} *Int* {*c*<..*d*} = {*max a c* <.. *min b d*}
by *auto*

lemma *Int-greaterThanLessThan*[*simp*]: {*a*<..<*b*} *Int* {*c*<..<*d*} = {*max a c* <..< *min b d*}
by *auto*

lemma *Int-atMost*[*simp*]: {..*a*} ∩ {..*b*} = {.. *min a b*}
by (*auto simp*: *min-def*)

lemma *Ioc-disjoint*: {*a*<..*b*} ∩ {*c*<..*d*} = {} ⟷ *b* ≤ *a* ∨ *d* ≤ *c* ∨ *b* ≤ *c* ∨ *d* ≤ *a*
by *auto*

end

context *complete-lattice*
begin

lemma
shows *Sup-atLeast*[*simp*]: *Sup* {*x* ..} = *top*
and *Sup-greaterThanAtLeast*[*simp*]: *x* < *top* ⟹ *Sup* {*x* <..} = *top*
and *Sup-atMost*[*simp*]: *Sup* {.. *y*} = *y*
and *Sup-atLeastAtMost*[*simp*]: *x* ≤ *y* ⟹ *Sup* { *x* .. *y*} = *y*
and *Sup-greaterThanAtMost*[*simp*]: *x* < *y* ⟹ *Sup* { *x* <.. *y*} = *y*
by (*auto intro*!: *Sup-eqI*)

lemma
shows *Inf-atMost*[*simp*]: *Inf* {.. *x*} = *bot*
and *Inf-atMostLessThan*[*simp*]: *top* < *x* ⟹ *Inf* {..< *x*} = *bot*
and *Inf-atLeast*[*simp*]: *Inf* {*x* ..} = *x*
and *Inf-atLeastAtMost*[*simp*]: *x* ≤ *y* ⟹ *Inf* { *x* .. *y*} = *x*
and *Inf-atLeastLessThan*[*simp*]: *x* < *y* ⟹ *Inf* { *x* ..< *y*} = *x*
by (*auto intro*!: *Inf-eqI*)

end

lemma
fixes *x y* :: *'a* :: {*complete-lattice*, *dense-linorder*}
shows *Sup-lessThan*[*simp*]: *Sup* {..< *y*} = *y*
and *Sup-atLeastLessThan*[*simp*]: *x* < *y* ⟹ *Sup* { *x* ..< *y*} = *y*
and *Sup-greaterThanLessThan*[*simp*]: *x* < *y* ⟹ *Sup* { *x* <..< *y*} = *y*
and *Inf-greaterThan*[*simp*]: *Inf* {*x* <..} = *x*
and *Inf-greaterThanAtMost*[*simp*]: *x* < *y* ⟹ *Inf* { *x* <.. *y*} = *x*
and *Inf-greaterThanLessThan*[*simp*]: *x* < *y* ⟹ *Inf* { *x* <..< *y*} = *x*
by (*auto intro*!: *Inf-eqI Sup-eqI intro*: *dense-le dense-le-bounded dense-ge dense-ge-bounded*)

61.5 Intervals of natural numbers

61.5.1 The Constant *lessThan*

lemma *lessThan-0* [*simp*]: *lessThan* (*0*::*nat*) = {}
by (*simp add*: *lessThan-def*)

lemma *lessThan-Suc*: *lessThan* (*Suc k*) = *insert k* (*lessThan k*)
by (*simp add*: *lessThan-def less-Suc-eq*, *blast*)

The following proof is convenient in induction proofs where new elements get indices at the beginning. So it is used to transform {..<*Suc n*} to *0* and {..<*n*}.

lemma *zero-notin-Suc-image* [*simp*]: *0* ∉ *Suc* ' *A*
by *auto*

lemma *lessThan-Suc-eq-insert-0*: {..<*Suc n*} = *insert 0* (*Suc* ' {..<*n*})
by (*auto simp*: *image-iff less-Suc-eq-0-disj*)

lemma *lessThan-Suc-atMost*: *lessThan* (*Suc k*) = *atMost k*
by (*simp add*: *lessThan-def atMost-def less-Suc-eq-le*)

lemma *atMost-Suc-eq-insert-0*: {.. *Suc n*} = *insert 0* (*Suc* ' {.. *n*})
unfolding *lessThan-Suc-atMost*[*symmetric*] *lessThan-Suc-eq-insert-0*[*of Suc n*]
..

lemma *UN-lessThan-UNIV*: (⋃ *m*::*nat*. *lessThan m*) = *UNIV*
by *blast*

61.5.2 The Constant *greaterThan*

lemma *greaterThan-0*: *greaterThan 0* = *range Suc*
unfolding *greaterThan-def*
by (*blast dest*: *gr0-conv-Suc* [*THEN iffD1*])

lemma *greaterThan-Suc*: *greaterThan* (*Suc k*) = *greaterThan k* − {*Suc k*}
unfolding *greaterThan-def*
by (*auto elim*: *linorder-neqE*)

lemma *INT-greaterThan-UNIV*: (⋂ *m*::*nat*. *greaterThan m*) = {}
by *blast*

61.5.3 The Constant *atLeast*

lemma *atLeast-0* [*simp*]: *atLeast* (*0*::*nat*) = *UNIV*
by (*unfold atLeast-def UNIV-def*, *simp*)

lemma *atLeast-Suc*: *atLeast* (*Suc k*) = *atLeast k* − {*k*}
unfolding *atLeast-def* **by** (*auto simp*: *order-le-less Suc-le-eq*)

lemma *atLeast-Suc-greaterThan*: *atLeast* (*Suc k*) = *greaterThan k*
by (*auto simp add*: *greaterThan-def atLeast-def less-Suc-eq-le*)

lemma *UN-atLeast-UNIV*: (⋃ *m*::*nat*. *atLeast m*) = *UNIV*
by *blast*

61.5.4 The Constant *atMost*

lemma *atMost-0* [*simp*]: *atMost* (*0*::*nat*) = {*0*}
by (*simp add*: *atMost-def*)

lemma *atMost-Suc*: *atMost* (*Suc k*) = *insert* (*Suc k*) (*atMost k*)
unfolding *atMost-def* **by** (*auto simp add*: *less-Suc-eq order-le-less*)

lemma *UN-atMost-UNIV*: (⋃ *m*::*nat*. *atMost m*) = *UNIV*
by *blast*

61.5.5 The Constant *atLeastLessThan*

The orientation of the following 2 rules is tricky. The lhs is defined in terms of the rhs. Hence the chosen orientation makes sense in this theory — the reverse orientation complicates proofs (eg nontermination). But outside, when the definition of the lhs is rarely used, the opposite orientation seems preferable because it reduces a specific concept to a more general one.

lemma *atLeast0LessThan* [*code-abbrev*]: $\{0::nat..<n\} = \{..<n\}$
by(*simp add*:*lessThan-def atLeastLessThan-def*)

lemma *atLeast0AtMost* [*code-abbrev*]: $\{0..n::nat\} = \{..n\}$
by(*simp add*:*atMost-def atLeastAtMost-def*)

lemma *lessThan-atLeast0*: $\{..<n\} = \{0::nat..<n\}$
by (*simp add*: *atLeast0LessThan*)

lemma *atMost-atLeast0*: $\{..n\} = \{0::nat..n\}$
by (*simp add*: *atLeast0AtMost*)

lemma *atLeastLessThan0*: $\{m..<0::nat\} = \{\}$
by (*simp add*: *atLeastLessThan-def*)

lemma *atLeast0-lessThan-Suc*: $\{0..<Suc\ n\} = insert\ n\ \{0..<n\}$
by (*simp add*: *atLeast0LessThan lessThan-Suc*)

lemma *atLeast0-lessThan-Suc-eq-insert-0*: $\{0..<Suc\ n\} = insert\ 0\ (Suc\ `\ \{0..<n\})$
by (*simp add*: *atLeast0LessThan lessThan-Suc-eq-insert-0*)

61.5.6 The Constant *atLeastAtMost*

lemma *Icc-eq-insert-lb-nat*: $m \leq n \Longrightarrow \{m..n\} = insert\ m\ \{Suc\ m..n\}$
by *auto*

lemma *atLeast0-atMost-Suc*:
$\{0..Suc\ n\} = insert\ (Suc\ n)\ \{0..n\}$
by (*simp add*: *atLeast0AtMost atMost-Suc*)

lemma *atLeast0-atMost-Suc-eq-insert-0*:
$\{0..Suc\ n\} = insert\ 0\ (Suc\ `\ \{0..n\})$
by (*simp add*: *atLeast0AtMost atMost-Suc-eq-insert-0*)

61.5.7 Intervals of nats with *Suc*

Not a simprule because the RHS is too messy.

lemma *atLeastLessThanSuc*:
$\{m..<Suc\ n\} = (if\ m \leq n\ then\ insert\ n\ \{m..<n\}\ else\ \{\})$
by (*auto simp add*: *atLeastLessThan-def*)

lemma *atLeastLessThan-singleton* [*simp*]: {*m*..<*Suc m*} = {*m*}
by (*auto simp add*: *atLeastLessThan-def*)

lemma *atLeastLessThanSuc-atLeastAtMost*: {*l*..<*Suc u*} = {*l*..*u*}
by (*simp add*: *lessThan-Suc-atMost atLeastAtMost-def atLeastLessThan-def*)

lemma *atLeastSucAtMost-greaterThanAtMost*: {*Suc l*..*u*} = {*l*<..*u*}
by (*simp add*: *atLeast-Suc-greaterThan atLeastAtMost-def greaterThanAtMost-def*)

lemma *atLeastSucLessThan-greaterThanLessThan*: {*Suc l*..<*u*} = {*l*<..<*u*}
by (*simp add*: *atLeast-Suc-greaterThan atLeastLessThan-def greaterThanLessThan-def*)

lemma *atLeastAtMostSuc-conv*: $m \leq Suc\ n \Longrightarrow \{m..Suc\ n\} = insert\ (Suc\ n)\ \{m..n\}$
by *auto*

lemma *atLeastAtMost-insertL*: $m \leq n \Longrightarrow insert\ m\ \{Suc\ m..n\} = \{m\ ..n\}$
by *auto*

The analogous result is useful on *int*:

lemma *atLeastAtMostPlus1-int-conv*:
$m \leq 1{+}n \Longrightarrow \{m..1{+}n\} = insert\ (1{+}n)\ \{m..n{::}int\}$
by (*auto intro*: *set-eqI*)

lemma *atLeastLessThan-add-Un*: $i \leq j \Longrightarrow \{i..{<}j{+}k\} = \{i..{<}j\} \cup \{j..{<}j{+}k{::}nat\}$
by (*induct k*) (*simp-all add*: *atLeastLessThanSuc*)

61.5.8 Intervals and numerals

lemma *lessThan-nat-numeral*: — Evaluation for specific numerals
lessThan (*numeral k* :: *nat*) = *insert* (*pred-numeral k*) (*lessThan* (*pred-numeral k*))
by (*simp add*: *numeral-eq-Suc lessThan-Suc*)

lemma *atMost-nat-numeral*: — Evaluation for specific numerals
atMost (*numeral k* :: *nat*) = *insert* (*numeral k*) (*atMost* (*pred-numeral k*))
by (*simp add*: *numeral-eq-Suc atMost-Suc*)

lemma *atLeastLessThan-nat-numeral*: — Evaluation for specific numerals
atLeastLessThan m (*numeral k* :: *nat*) =
(*if m* ≤ (*pred-numeral k*) *then insert* (*pred-numeral k*) (*atLeastLessThan m* (*pred-numeral k*))
else {})
by (*simp add*: *numeral-eq-Suc atLeastLessThanSuc*)

61.5.9 Image

context *linordered-semidom*

```
begin

lemma image-add-atLeast[simp]: plus k ` {i..} = {k + i..}
proof −
  have n = k + (n − k) if i + k ≤ n for n
  proof −
    have n = (n − (k + i)) + (k + i) using that
      by (metis add-commute le-add-diff-inverse)
    then show n = k + (n − k)
      by (metis local.add-diff-cancel-left' add-assoc add-commute)
  qed
  then show ?thesis
    by (fastforce simp: add-le-imp-le-diff add.commute)
qed

lemma image-add-atLeastAtMost [simp]:
  plus k ` {i..j} = {i + k..j + k} (is ?A = ?B)
proof
  show ?A ⊆ ?B
    by (auto simp add: ac-simps)
next
  show ?B ⊆ ?A
  proof
    fix n
    assume n ∈ ?B
    then have i ≤ n − k
      by (simp add: add-le-imp-le-diff)
    have n = n − k + k
    proof −
      from ‹n ∈ ?B› have n = n − (i + k) + (i + k)
        by simp
      also have ... = n − k − i + i + k
        by (simp add: algebra-simps)
      also have ... = n − k + k
        using ‹i ≤ n − k› by simp
      finally show ?thesis .
    qed
    moreover have n − k ∈ {i..j}
      using ‹n ∈ ?B›
      by (auto simp: add-le-imp-le-diff add-le-add-imp-diff-le)
    ultimately show n ∈ ?A
      by (simp add: ac-simps)
  qed
qed

lemma image-add-atLeastAtMost' [simp]:
  (λn. n + k) ` {i..j} = {i + k..j + k}
  by (simp add: add.commute [of - k])
```

lemma *image-add-atLeastLessThan* [*simp*]:
  plus k ‘ $\{i..<j\} = \{i + k..<j + k\}$
  by (*simp add*: *image-set-diff atLeastLessThan-eq-atLeastAtMost-diff ac-simps*)

lemma *image-add-atLeastLessThan′* [*simp*]:
  $(\lambda n.\ n + k)$ ‘ $\{i..<j\} = \{i + k..<j + k\}$
  by (*simp add*: *add.commute* [*of - k*])

lemma *image-add-greaterThanAtMost*[*simp*]: $(+)\ c$ ‘ $\{a<..b\} = \{c + a<..c + b\}$
  by (*simp add*: *image-set-diff greaterThanAtMost-eq-atLeastAtMost-diff ac-simps*)

end

context *ordered-ab-group-add*
begin

lemma
  fixes $x :: {'a}$
  shows *image-uminus-greaterThan*[*simp*]: *uminus* ‘ $\{x<..\} = \{..<-x\}$
  and *image-uminus-atLeast*[*simp*]: *uminus* ‘ $\{x..\} = \{..-x\}$
proof *safe*
  fix y **assume** $y < -x$
  hence $*$: $x < -y$ **using** *neg-less-iff-less*[*of* $-y\ x$] **by** *simp*
  have $-(-y) \in$ *uminus* ‘ $\{x<..\}$
    by (*rule imageI*) (*simp add*: $*$)
  thus $y \in$ *uminus* ‘ $\{x<..\}$ **by** *simp*
next
  fix y **assume** $y \le -x$
  have $-(-y) \in$ *uminus* ‘ $\{x..\}$
    by (*rule imageI*) (*use* ‹$y \le -x$›[*THEN le-imp-neg-le*] **in** ‹*simp*›)
  thus $y \in$ *uminus* ‘ $\{x..\}$ **by** *simp*
qed *simp-all*

lemma
  fixes $x :: {'a}$
  shows *image-uminus-lessThan*[*simp*]: *uminus* ‘ $\{..<x\} = \{-x<..\}$
  and *image-uminus-atMost*[*simp*]: *uminus* ‘ $\{..x\} = \{-x..\}$
proof −
  have *uminus* ‘ $\{..<x\}$ = *uminus* ‘ *uminus* ‘ $\{-x<..\}$
    and *uminus* ‘ $\{..x\}$ = *uminus* ‘ *uminus* ‘ $\{-x..\}$ **by** *simp-all*
  thus *uminus* ‘ $\{..<x\} = \{-x<..\}$ **and** *uminus* ‘ $\{..x\} = \{-x..\}$
    by (*simp-all add*: *image-image*
        del: *image-uminus-greaterThan image-uminus-atLeast*)
qed

lemma
  fixes $x :: {'a}$
  shows *image-uminus-atLeastAtMost*[*simp*]: *uminus* ‘ $\{x..y\} = \{-y..-x\}$
  and *image-uminus-greaterThanAtMost*[*simp*]: *uminus* ‘ $\{x<..y\} = \{-y..<-x\}$

and *image-uminus-atLeastLessThan*[*simp*]: *uminus* ' {*x*..<*y*} = {−*y*<..−*x*}
and *image-uminus-greaterThanLessThan*[*simp*]: *uminus* ' {*x*<..<*y*} = {−*y*<..<−*x*}
by (*simp-all add*: *atLeastAtMost-def greaterThanAtMost-def atLeastLessThan-def greaterThanLessThan-def image-Int*[*OF inj-uminus*] *Int-commute*)

lemma *image-add-atMost*[*simp*]: (+) *c* ' {..*a*} = {..*c* + *a*}
by (*auto intro*!: *image-eqI*[**where** *x*=*x* − *c* **for** *x*] *simp*: *algebra-simps*)

end

lemma *image-Suc-atLeastAtMost* [*simp*]:
Suc ' {*i*..*j*} = {*Suc i*..*Suc j*}
using *image-add-atLeastAtMost* [*of 1 i j*]
by (*simp only*: *plus-1-eq-Suc*) *simp*

lemma *image-Suc-atLeastLessThan* [*simp*]:
Suc ' {*i*..<*j*} = {*Suc i*..<*Suc j*}
using *image-add-atLeastLessThan* [*of 1 i j*]
by (*simp only*: *plus-1-eq-Suc*) *simp*

corollary *image-Suc-atMost*:
Suc ' {..*n*} = {*1*..*Suc n*}
by (*simp add*: *atMost-atLeast0 atLeastLessThanSuc-atLeastAtMost*)

corollary *image-Suc-lessThan*:
Suc ' {..<*n*} = {*1*..*n*}
by (*simp add*: *lessThan-atLeast0 atLeastLessThanSuc-atLeastAtMost*)

lemma *image-diff-atLeastAtMost* [*simp*]:
fixes *d*::′*a*::*linordered-idom* **shows** ((−) *d* ' {*a*..*b*}) = {*d*−*b*..*d*−*a*}
proof
show {*d* − *b*..*d* − *a*} ⊆ (−) *d* ' {*a*..*b*}
proof
fix *x*
assume *x* ∈ {*d* − *b*..*d* − *a*}
then have *d* − *x* ∈ {*a*..*b*} **and** *x* = *d* − (*d* − *x*)
by *auto*
then show *x* ∈ (−) *d* ' {*a*..*b*}
by (*rule rev-image-eqI*)
qed
qed(*auto*)

lemma *image-diff-atLeastLessThan* [*simp*]:
fixes *a b c*::′*a*::*linordered-idom*
shows (−) *c* ' {*a*..<*b*} = {*c* − *b*<..*c* − *a*}
proof −
have (−) *c* ' {*a*..<*b*} = (+) *c* ' *uminus* ' {*a* ..<*b*}
unfolding *image-image* **by** *simp*
also have ... = {*c* − *b*<..*c* − *a*} **by** *simp*

finally show *?thesis* **by** *simp*
qed

lemma *image-minus-const-greaterThanAtMost*[*simp*]:
fixes *a b c*::'*a*::*linordered-idom*
shows $(-)\ c\ `\ \{a<..b\} = \{c - b..<c - a\}$
proof −
have $(-)\ c\ `\ \{a<..b\} = (+)\ c\ `\ uminus\ `\ \{a<..b\}$
unfolding *image-image* **by** *simp*
also have $\ldots = \{c - b..<c - a\}$ **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *image-minus-const-atLeast*[*simp*]:
fixes *a c*::'*a*::*linordered-idom*
shows $(-)\ c\ `\ \{a..\} = \{..c - a\}$
proof −
have $(-)\ c\ `\ \{a..\} = (+)\ c\ `\ uminus\ `\ \{a\ ..\}$
unfolding *image-image* **by** *simp*
also have $\ldots = \{..c - a\}$ **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *image-minus-const-AtMost*[*simp*]:
fixes *b c*::'*a*::*linordered-idom*
shows $(-)\ c\ `\ \{..b\} = \{c - b..\}$
proof −
have $(-)\ c\ `\ \{..b\} = (+)\ c\ `\ uminus\ `\ \{..b\}$
unfolding *image-image* **by** *simp*
also have $\ldots = \{c - b..\}$ **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *image-minus-const-atLeastAtMost'* [*simp*]:
$(\lambda t.\ t-d)`\{a..b\} = \{a-d..b-d\}$ **for** *d*::'*a*::*linordered-idom*
by (*metis* (*no-types*, *lifting*) *diff-conv-add-uminus image-add-atLeastAtMost' image-cong*)

context *linordered-field*
begin

lemma *image-mult-atLeastAtMost* [*simp*]:
$((*)\ d\ `\ \{a..b\}) = \{d*a..d*b\}$ **if** $d>0$
using *that*
by (*auto simp*: *field-simps mult-le-cancel-right intro*: *rev-image-eqI* [**where** $x=x/d$ **for** *x*])

lemma *image-divide-atLeastAtMost* [*simp*]:
$((\lambda c.\ c\ /\ d)\ `\ \{a..b\}) = \{a/d..b/d\}$ **if** $d>0$

proof –
  from *that* **have** *inverse d > 0*
    by *simp*
  with *image-mult-atLeastAtMost* [*of inverse d a b*]
  have (∗) (*inverse d*) ‘ {*a..b*} = {*inverse d* ∗ *a..inverse d* ∗ *b*}
    by *blast*
  moreover have (∗) (*inverse d*) = (λ*c. c / d*)
    by (*simp add*: *fun-eq-iff field-simps*)
  ultimately show *?thesis*
    by *simp*
qed

lemma *image-mult-atLeastAtMost-if*:
  (∗) *c* ‘ {*x* .. *y*} =
    (*if c* > *0 then* {*c* ∗ *x* .. *c* ∗ *y*} *else if x* ≤ *y then* {*c* ∗ *y* .. *c* ∗ *x*} *else* {})
proof (*cases c* = *0* ∨ *x* > *y*)
  case *True*
  then show *?thesis*
    by *auto*
next
  case *False*
  then have *x* ≤ *y*
    by *auto*
  from *False* **consider** *c* < *0*| *c* > *0*
    by (*auto simp add*: *neq-iff*)
  then show *?thesis*
  proof *cases*
    case *1*
    have (∗) *c* ‘ {*x..y*} = {*c* ∗ *y..c* ∗ *x*}
    proof (*rule set-eqI*)
      fix *d*
      from *1* **have** *inj* (λ*z. z / c*)
        by (*auto intro*: *injI*)
      then have *d* ∈ (∗) *c* ‘ {*x..y*} ⟷ *d / c* ∈ (λ*z. z div c*) ‘ (∗) *c* ‘ {*x..y*}
        by (*subst inj-image-mem-iff*) *simp-all*
      also have ... ⟷ *d / c* ∈ {*x..y*}
        using *1* **by** (*simp add*: *image-image*)
      also have ... ⟷ *d* ∈ {*c* ∗ *y..c* ∗ *x*}
        by (*auto simp add*: *field-simps 1*)
      finally show *d* ∈ (∗) *c* ‘ {*x..y*} ⟷ *d* ∈ {*c* ∗ *y..c* ∗ *x*} **.**
    qed
    with ‹*x* ≤ *y*› **show** *?thesis*
      by *auto*
  qed (*simp add*: *mult-left-mono-neg*)
qed

lemma *image-mult-atLeastAtMost-if′*:
  (λ*x. x* ∗ *c*) ‘ {*x..y*} =
    (*if x* ≤ *y then if c* > *0 then* {*x* ∗ *c* .. *y* ∗ *c*} *else* {*y* ∗ *c* .. *x* ∗ *c*} *else* {})

using *image-mult-atLeastAtMost-if* [*of c x y*] **by** (*auto simp add*: *ac-simps*)

lemma *image-affinity-atLeastAtMost*:
$((\lambda x.\ m * x + c)$ ‘ $\{a..b\}) = (if\ \{a..b\} = \{\}\ then\ \{\}$
$else\ if\ 0 \leq m\ then\ \{m * a + c\ ..\ m * b + c\}$
$else\ \{m * b + c\ ..\ m * a + c\})$
proof −
have ∗: $(\lambda x.\ m * x + c) = ((\lambda x.\ x + c) \circ (*)\ m)$
by (*simp add*: *fun-eq-iff*)
show *?thesis* **by** (*simp only*: ∗ *image-comp* [*symmetric*] *image-mult-atLeastAtMost-if*)
(*auto simp add*: *mult-le-cancel-left*)
qed

lemma *image-affinity-atLeastAtMost-diff*:
$((\lambda x.\ m*x - c)$ ‘ $\{a..b\}) = (if\ \{a..b\}=\{\}\ then\ \{\}$
$else\ if\ 0 \leq m\ then\ \{m*a - c\ ..\ m*b - c\}$
$else\ \{m*b - c\ ..\ m*a - c\})$
using *image-affinity-atLeastAtMost* [*of m* −*c a b*]
by *simp*

lemma *image-affinity-atLeastAtMost-div*:
$((\lambda x.\ x/m + c)$ ‘ $\{a..b\}) = (if\ \{a..b\}=\{\}\ then\ \{\}$
$else\ if\ 0 \leq m\ then\ \{a/m + c\ ..\ b/m + c\}$
$else\ \{b/m + c\ ..\ a/m + c\})$
using *image-affinity-atLeastAtMost* [*of inverse m c a b*]
by (*simp add*: *field-class.field-divide-inverse algebra-simps inverse-eq-divide*)

lemma *image-affinity-atLeastAtMost-div-diff*:
$((\lambda x.\ x/m - c)$ ‘ $\{a..b\}) = (if\ \{a..b\}=\{\}\ then\ \{\}$
$else\ if\ 0 \leq m\ then\ \{a/m - c\ ..\ b/m - c\}$
$else\ \{b/m - c\ ..\ a/m - c\})$
using *image-affinity-atLeastAtMost-diff* [*of inverse m c a b*]
by (*simp add*: *field-class.field-divide-inverse algebra-simps inverse-eq-divide*)

end

lemma *atLeast1-lessThan-eq-remove0*:
$\{Suc\ 0..<n\} = \{..<n\} - \{0\}$
by *auto*

lemma *atLeast1-atMost-eq-remove0*:
$\{Suc\ 0..n\} = \{..n\} - \{0\}$
by *auto*

lemma *image-add-int-atLeastLessThan*:
$(\lambda x.\ x + (l::int))$ ‘ $\{0..<u-l\} = \{l..<u\}$
by *safe auto*

lemma *image-minus-const-atLeastLessThan-nat*:

fixes *c* :: *nat*
shows $(\lambda i.\ i - c)$ ' $\{x ..< y\}$ =
($if\ c < y\ then\ \{x - c ..< y - c\}\ else\ if\ x < y\ then\ \{0\}\ else\ \{\}$)
(**is** - = *?right*)
proof *safe*
fix *a* **assume** *a*: $a \in$ *?right*
show $a \in (\lambda i.\ i - c)$ ' $\{x ..< y\}$
proof *cases*
assume $c < y$ **with** *a* **show** *?thesis*
by (*auto intro*!: *image-eqI*[*of* - - $a + c$])
next
assume $\neg\ c < y$ **with** *a* **show** *?thesis*
by (*auto intro*!: *image-eqI*[*of* - - *x*] *split*: *if-split-asm*)
qed
qed *auto*

lemma *image-int-atLeastLessThan*:
int ' $\{a..<b\} = \{int\ a..<int\ b\}$
by (*auto intro*!: *image-eqI* [**where** $x = nat\ x$ **for** *x*])

lemma *image-int-atLeastAtMost*:
int ' $\{a..b\} = \{int\ a..int\ b\}$
by (*auto intro*!: *image-eqI* [**where** $x = nat\ x$ **for** *x*])

61.5.10 Finiteness

lemma *finite-lessThan* [*iff*]: **fixes** *k* :: *nat* **shows** *finite* $\{..<k\}$
by (*induct k*) (*simp-all add*: *lessThan-Suc*)

lemma *finite-atMost* [*iff*]: **fixes** *k* :: *nat* **shows** *finite* $\{..k\}$
by (*induct k*) (*simp-all add*: *atMost-Suc*)

lemma *finite-greaterThanLessThan* [*iff*]:
fixes *l* :: *nat* **shows** *finite* $\{l<..<u\}$
by (*simp add*: *greaterThanLessThan-def*)

lemma *finite-atLeastLessThan* [*iff*]:
fixes *l* :: *nat* **shows** *finite* $\{l..<u\}$
by (*simp add*: *atLeastLessThan-def*)

lemma *finite-greaterThanAtMost* [*iff*]:
fixes *l* :: *nat* **shows** *finite* $\{l<..u\}$
by (*simp add*: *greaterThanAtMost-def*)

lemma *finite-atLeastAtMost* [*iff*]:
fixes *l* :: *nat* **shows** *finite* $\{l..u\}$
by (*simp add*: *atLeastAtMost-def*)

A bounded set of natural numbers is finite.

lemma *bounded-nat-set-is-finite*: $(\forall\, i{\in}N.\ i < (n{::}nat)) \Longrightarrow$ *finite N*
by (*rule finite-subset* [*OF - finite-lessThan*]) *auto*

A set of natural numbers is finite iff it is bounded.

lemma *finite-nat-set-iff-bounded*:
finite$(N{::}nat\ set) = (\exists\, m.\ \forall\, n{\in}N.\ n{<}m)$ (**is** *?F = ?B*)
proof
assume *f*:*?F* **show** *?B*
using *Max-ge*[*OF* ‹*?F*›, *simplified less-Suc-eq-le*[*symmetric*]] **by** *blast*
next
assume *?B* **show** *?F* **using** ‹*?B*› **by**(*blast intro*:*bounded-nat-set-is-finite*)
qed

lemma *finite-nat-set-iff-bounded-le*: *finite*$(N{::}nat\ set) = (\exists\, m.\ \forall\, n{\in}N.\ n{\le}m)$
unfolding *finite-nat-set-iff-bounded*
by (*blast dest*:*less-imp-le-nat le-imp-less-Suc*)

lemma *finite-less-ub*:
$\bigwedge f{::}nat{\Rightarrow}nat.\ (!!n.\ n \le f\ n) \Longrightarrow$ *finite* $\{n.\ f\ n \le u\}$
by (*rule finite-subset*[*of* - $\{..u\}$])
(*auto intro*: *order-trans*)

lemma *bounded-Max-nat*:
fixes $P :: nat \Rightarrow bool$
assumes x: $P\ x$ **and** M: $\bigwedge x.\ P\ x \Longrightarrow x \le M$
obtains m **where** $P\ m$ $\bigwedge x.\ P\ x \Longrightarrow x \le m$
proof −
have *finite* $\{x.\ P\ x\}$
using *M finite-nat-set-iff-bounded-le* **by** *auto*
then have *Max* $\{x.\ P\ x\} \in \{x.\ P\ x\}$
using *Max-in x* **by** *auto*
then show *?thesis*
by (*simp add*: ‹*finite* $\{x.\ P\ x\}$› *that*)
qed

Any subset of an interval of natural numbers the size of the subset is exactly that interval.

lemma *subset-card-intvl-is-intvl*:
assumes $A \subseteq \{k..{<}k + card\ A\}$
shows $A = \{k..{<}k + card\ A\}$
proof (*cases finite A*)
case *True*
from *this* **and** *assms* **show** *?thesis*
proof (*induct A rule*: *finite-linorder-max-induct*)
case *empty* **thus** *?case* **by** *auto*
next
case (*insert b A*)
hence $*$: $b \notin A$ **by** *auto*
with *insert* **have** $A \le \{k..{<}k + card\ A\}$ **and** $b = k + card\ A$

    by *fastforce+*
  with *insert* ∗ **show** *?case* **by** *auto*
 qed
next
 case *False*
 with *assms* **show** *?thesis* **by** *simp*
qed

61.5.11 Proving Inclusions and Equalities between Unions

lemma *UN-le-eq-Un0*:
 $(\bigcup i \leq n::nat.\ M\ i) = (\bigcup i \in \{1..n\}.\ M\ i) \cup M\ 0$ (**is** $?A = ?B$)
proof
 show $?A \subseteq ?B$
 proof
  fix x **assume** $x \in ?A$
  then obtain i **where** i: $i \leq n$ $x \in M\ i$ **by** *auto*
  show $x \in ?B$
  proof(*cases i*)
   case *0* **with** i **show** *?thesis* **by** *simp*
  next
   case (*Suc j*) **with** i **show** *?thesis* **by** *auto*
  qed
 qed
next
 show $?B \subseteq ?A$ **by** *fastforce*
qed

lemma *UN-le-add-shift*:
 $(\bigcup i \leq n::nat.\ M(i+k)) = (\bigcup i \in \{k..n+k\}.\ M\ i)$ (**is** $?A = ?B$)
proof
 show $?A \subseteq ?B$ **by** *fastforce*
next
 show $?B \subseteq ?A$
 proof
  fix x **assume** $x \in ?B$
  then obtain i **where** i: $i \in \{k..n+k\}$ $x \in M(i)$ **by** *auto*
  hence $i-k \leq n \wedge x \in M((i-k)+k)$ **by** *auto*
  thus $x \in ?A$ **by** *blast*
 qed
qed

lemma *UN-le-add-shift-strict*:
 $(\bigcup i < n::nat.\ M(i+k)) = (\bigcup i \in \{k..<n+k\}.\ M\ i)$ (**is** $?A = ?B$)
proof
 show $?B \subseteq ?A$
 proof
  fix x **assume** $x \in ?B$
  then obtain i **where** i: $i \in \{k..<n+k\}$ $x \in M(i)$ **by** *auto*

then have $i - k < n \wedge x \in M((i-k) + k)$ **by** *auto*
then show $x \in ?A$ **using** *UN-le-add-shift* **by** *blast*
qed
qed (*fastforce*)

lemma *UN-UN-finite-eq*: $(\bigcup n::nat. \bigcup i\in\{0..<n\}. A\ i) = (\bigcup n. A\ n)$
by (*auto simp add*: *atLeast0LessThan*)

lemma *UN-finite-subset*:
$(\bigwedge n::nat. (\bigcup i\in\{0..<n\}. A\ i) \subseteq C) \Longrightarrow (\bigcup n. A\ n) \subseteq C$
by (*subst UN-UN-finite-eq* [*symmetric*]) *blast*

lemma *UN-finite2-subset*:
assumes $\bigwedge n::nat. (\bigcup i\in\{0..<n\}. A\ i) \subseteq (\bigcup i\in\{0..<n + k\}. B\ i)$
shows $(\bigcup n. A\ n) \subseteq (\bigcup n. B\ n)$
proof (*rule UN-finite-subset*, *rule subsetI*)
fix n **and** a
from *assms* **have** $(\bigcup i\in\{0..<n\}. A\ i) \subseteq (\bigcup i\in\{0..<n + k\}. B\ i)$ **.**
moreover assume $a \in (\bigcup i\in\{0..<n\}. A\ i)$
ultimately have $a \in (\bigcup i\in\{0..<n + k\}. B\ i)$ **by** *blast*
then show $a \in (\bigcup i. B\ i)$ **by** (*auto simp add*: *UN-UN-finite-eq*)
qed

lemma *UN-finite2-eq*:
assumes $(\bigwedge n::nat. (\bigcup i\in\{0..<n\}. A\ i) = (\bigcup i\in\{0..<n + k\}. B\ i))$
shows $(\bigcup n. A\ n) = (\bigcup n. B\ n)$
proof (*rule subset-antisym* [*OF UN-finite-subset UN-finite2-subset*])
fix n
show $\bigcup (A\ `\ \{0..<n\}) \subseteq (\bigcup n. B\ n)$
using *assms* **by** *auto*
next
fix n
show $\bigcup (B\ `\ \{0..<n\}) \subseteq \bigcup (A\ `\ \{0..<n + k\})$
using *assms* **by** (*force simp add*: *atLeastLessThan-add-Un* [*of 0*])+
qed

61.5.12 Cardinality

lemma *card-lessThan* [*simp*]: *card* $\{..<u\} = u$
by (*induct u*, *simp-all add*: *lessThan-Suc*)

lemma *card-atMost* [*simp*]: *card* $\{..u\} = Suc\ u$
by (*simp add*: *lessThan-Suc-atMost* [*THEN sym*])

lemma *card-atLeastLessThan* [*simp*]: *card* $\{l..<u\} = u - l$
proof −
have $(\lambda x. x + l)\ `\ \{..<u - l\} \subseteq \{l..<u\}$
by *auto*
moreover have $\{l..<u\} \subseteq (\lambda x. x + l)\ `\ \{..<u-l\}$

proof
fix *x*
assume ∗: $x \in \{l..<u\}$
then have $x - l \in \{..< u - l\}$
by *auto*
then have $(x - l) + l \in (\lambda x.\ x + l)$ ` $\{..< u - l\}$
by *auto*
then show $x \in (\lambda x.\ x + l)$ ` $\{..<u - l\}$
using ∗ **by** *auto*
qed
ultimately have $\{l..<u\} = (\lambda x.\ x + l)$ ` $\{..<u-l\}$
by *auto*
then have *card* $\{l..<u\}$ = *card* $\{..<u-l\}$
by (*simp add*: *card-image inj-on-def*)
then show *?thesis*
by *simp*
qed

lemma *card-atLeastAtMost* [*simp*]: *card* $\{l..u\}$ = *Suc* $u - l$
by (*subst atLeastLessThanSuc-atLeastAtMost* [*THEN sym*], *simp*)

lemma *card-greaterThanAtMost* [*simp*]: *card* $\{l<..u\} = u - l$
by (*subst atLeastSucAtMost-greaterThanAtMost* [*THEN sym*], *simp*)

lemma *card-greaterThanLessThan* [*simp*]: *card* $\{l<..<u\} = u -$ *Suc* l
by (*subst atLeastSucLessThan-greaterThanLessThan* [*THEN sym*], *simp*)

lemma *subset-eq-atLeast0-lessThan-finite*:
fixes *n* :: *nat*
assumes $N \subseteq \{0..<n\}$
shows *finite N*
using *assms finite-atLeastLessThan* **by** (*rule finite-subset*)

lemma *subset-eq-atLeast0-atMost-finite*:
fixes *n* :: *nat*
assumes $N \subseteq \{0..n\}$
shows *finite N*
using *assms finite-atLeastAtMost* **by** (*rule finite-subset*)

lemma *ex-bij-betw-nat-finite*:
finite M $\Longrightarrow \exists h.$ *bij-betw h* $\{0..<$*card M*$\}$ *M*
apply(*drule finite-imp-nat-seg-image-inj-on*)
apply(*auto simp*:*atLeast0LessThan*[*symmetric*] *lessThan-def*[*symmetric*] *card-image bij-betw-def*)
done

lemma *ex-bij-betw-finite-nat*:
finite M $\Longrightarrow \exists h.$ *bij-betw h M* $\{0..<$*card M*$\}$
by (*blast dest*: *ex-bij-betw-nat-finite bij-betw-inv*)

lemma *finite-same-card-bij*:
finite A $\Longrightarrow$ *finite B* $\Longrightarrow$ *card A* = *card B* $\Longrightarrow$ $\exists$ *h. bij-betw h A B*
apply(*drule ex-bij-betw-finite-nat*)
apply(*drule ex-bij-betw-nat-finite*)
apply(*auto intro*!:*bij-betw-trans*)
done

lemma *ex-bij-betw-nat-finite-1*:
finite M $\Longrightarrow$ $\exists$ *h. bij-betw h* {*1* .. *card M*} *M*
by (*rule finite-same-card-bij*) *auto*

lemma *bij-betw-iff-card*:
assumes *finite A finite B*
shows ($\exists$ *f. bij-betw f A B*) $\longleftrightarrow$ (*card A* = *card B*)
proof
assume *card A* = *card B*
moreover obtain *f* **where** *bij-betw f A* {*0* ..< *card A*}
using *assms ex-bij-betw-finite-nat* **by** *blast*
moreover obtain *g* **where** *bij-betw g* {*0* ..< *card B*} *B*
using *assms ex-bij-betw-nat-finite* **by** *blast*
ultimately have *bij-betw* (*g* $\circ$ *f*) *A B*
by (*auto simp*: *bij-betw-trans*)
thus ($\exists$ *f. bij-betw f A B*) **by** *blast*
qed (*auto simp*: *bij-betw-same-card*)

lemma *subset-eq-atLeast0-lessThan-card*:
fixes *n* :: *nat*
assumes *N* $\subseteq$ {*0*..<*n*}
shows *card N* $\leq$ *n*
proof −
from *assms finite-lessThan* **have** *card N* $\leq$ *card* {*0*..<*n*}
using *card-mono* **by** *blast*
then show *?thesis* **by** *simp*
qed

Relational version of *card-inj-on-le*:

lemma *card-le-if-inj-on-rel*:
assumes *finite B*
$\bigwedge$*a. a* $\in$ *A* $\Longrightarrow$ $\exists$ *b. b*$\in$*B* $\wedge$ *r a b*
$\bigwedge$*a1 a2 b.* $[\![$ *a1* $\in$ *A*; *a2* $\in$ *A*; *b* $\in$ *B*; *r a1 b*; *r a2 b* $]\!]$ $\Longrightarrow$ *a1* = *a2*
shows *card A* $\leq$ *card B*
proof −
let *?P* = λ*a b. b* $\in$ *B* $\wedge$ *r a b*
let *?f* = λ*a. SOME b. ?P a b*
have *1*: *?f* ` *A* $\subseteq$ *B* **by** (*auto intro*: *someI2-ex*[*OF assms*(*2*)])
have *inj-on ?f A*
unfolding *inj-on-def*
proof *safe*

fix *a1 a2* **assume** *asms*: $a1 \in A$ $a2 \in A$ $?f\ a1 = ?f\ a2$
have *0*: $?f\ a1 \in B$ **using** *1* ‹$a1 \in A$› **by** *blast*
have *1*: *r a1* (*?f a1*) **using** *someI-ex*[*OF assms*(*2*)[*OF* ‹$a1 \in A$›]] **by** *blast*
have *2*: *r a2* (*?f a1*) **using** *someI-ex*[*OF assms*(*2*)[*OF* ‹$a2 \in A$›]] *asms*(*3*) **by** *auto*
show $a1 = a2$ **using** *assms*(*3*)[*OF asms*(*1*,*2*) *0 1 2*] **.**
qed
with *1* **show** *?thesis* **using** *card-inj-on-le*[*of ?f A B*] *assms*(*1*) **by** *simp*
qed

lemma *inj-on-funpow-least*:
‹*inj-on* ($\lambda k.$ (f ^^ k) s) {*0*..<n}›
if ‹(f ^^ n) $s = s$› ‹$\bigwedge m.\ 0 < m \Longrightarrow m < n \Longrightarrow$ (f ^^ m) $s \neq s$›
proof −
{ **fix** *k l* **assume** *A*: $k < n$ $l < n$ $k \neq l$ (f ^^ k) $s =$ (f ^^ l) s
define k' l' **where** $k' = min\ k\ l$ **and** $l' = max\ k\ l$
with *A* **have** A': $k' < l'$ (f ^^ k') $s =$ (f ^^ l') s $l' < n$
by (*auto simp*: *min-def max-def*)

have $s =$ (f ^^ (($n - l'$) + l')) s **using** *that* ‹$l' < n$› **by** *simp*
also have ... = (f ^^ ($n - l'$)) ((f ^^ l') s) **by** (*simp add*: *funpow-add*)
also have (f ^^ l') $s =$ (f ^^ k') s **by** (*simp add*: A')
also have (f ^^ ($n - l'$)) ... = (f ^^ ($n - l' + k'$)) s **by** (*simp add*: *funpow-add*)
finally have (f ^^ ($n - l' + k'$)) $s = s$ **by** *simp*
moreover have $n - l' + k' < n$ $0 < n - l' + k'$**using** A' **by** *linarith*+
ultimately have *False* **using** *that*(*2*) **by** *auto*
}
then show *?thesis* **by** (*intro inj-onI*) *auto*
qed

61.6 Intervals of integers

lemma *atLeastLessThanPlusOne-atLeastAtMost-int*: {*l*..<*u*+*1*} = {*l*..(*u*::*int*)}
by (*auto simp add*: *atLeastAtMost-def atLeastLessThan-def*)

lemma *atLeastPlusOneAtMost-greaterThanAtMost-int*: {*l*+*1*..*u*} = {*l*<..(*u*::*int*)}
by (*auto simp add*: *atLeastAtMost-def greaterThanAtMost-def*)

lemma *atLeastPlusOneLessThan-greaterThanLessThan-int*:
{*l*+*1*..<*u*} = {*l*<..<*u*::*int*}
by (*auto simp add*: *atLeastLessThan-def greaterThanLessThan-def*)

61.6.1 Finiteness

lemma *image-atLeastZeroLessThan-int*:
assumes $0 \leq u$
shows {(*0*::*int*)..<*u*} = *int* ' {..<*nat u*}
unfolding *image-def lessThan-def*
proof
show {*0*..<*u*} $\subseteq$ {*y*. $\exists x \in$ {*x*. $x <$ *nat u*}. $y =$ *int x*}

proof
fix *x*
assume $x \in \{0..<u\}$
then have $x = int\ (nat\ x)$ **and** $nat\ x < nat\ u$
by (*auto simp add*: *zless-nat-eq-int-zless* [*THEN sym*])
then have $\exists xa<nat\ u.\ x = int\ xa$
using *exI*[*of* - (*nat x*)] **by** *simp*
then show $x \in \{y.\ \exists x\in\{x.\ x < nat\ u\}.\ y = int\ x\}$
by *simp*
qed
qed (*auto*)

lemma *finite-atLeastZeroLessThan-int*: *finite* $\{(0::int)..<u\}$
proof (*cases* $0 \le u$)
case *True*
then show *?thesis*
by (*auto simp*: *image-atLeastZeroLessThan-int*)
qed *auto*

lemma *finite-atLeastLessThan-int* [*iff*]: *finite* $\{l..<u::int\}$
by (*simp only*: *image-add-int-atLeastLessThan* [*symmetric*, *of l*] *finite-imageI* *finite-atLeastZeroLessThan-int*)

lemma *finite-atLeastAtMost-int* [*iff*]: *finite* $\{l..(u::int)\}$
by (*subst atLeastLessThanPlusOne-atLeastAtMost-int* [*THEN sym*], *simp*)

lemma *finite-greaterThanAtMost-int* [*iff*]: *finite* $\{l<..(u::int)\}$
by (*subst atLeastPlusOneAtMost-greaterThanAtMost-int* [*THEN sym*], *simp*)

lemma *finite-greaterThanLessThan-int* [*iff*]: *finite* $\{l<..<u::int\}$
by (*subst atLeastPlusOneLessThan-greaterThanLessThan-int* [*THEN sym*], *simp*)

61.6.2 Cardinality

lemma *card-atLeastZeroLessThan-int*: *card* $\{(0::int)..<u\} = nat\ u$
proof (*cases* $0 \le u$)
case *True*
then show *?thesis*
by (*auto simp*: *image-atLeastZeroLessThan-int card-image inj-on-def*)
qed *auto*

lemma *card-atLeastLessThan-int* [*simp*]: *card* $\{l..<u\} = nat\ (u - l)$
proof −
have *card* $\{l..<u\}$ = *card* $\{0..<u-l\}$
apply (*subst image-add-int-atLeastLessThan* [*symmetric*])
apply (*rule card-image*)
apply (*simp add*: *inj-on-def*)
done

then show *?thesis*
by (*simp add*: *card-atLeastZeroLessThan-int*)
qed

lemma *card-atLeastAtMost-int* [*simp*]: *card* {*l..u*} = *nat* (*u* − *l* + *1*)
apply (*subst atLeastLessThanPlusOne-atLeastAtMost-int* [*THEN sym*])
apply (*auto simp add*: *algebra-simps*)
done

lemma *card-greaterThanAtMost-int* [*simp*]: *card* {*l*<..*u*} = *nat* (*u* − *l*)
by (*subst atLeastPlusOneAtMost-greaterThanAtMost-int* [*THEN sym*], *simp*)

lemma *card-greaterThanLessThan-int* [*simp*]: *card* {*l*<..<*u*} = *nat* (*u* − (*l* + *1*))
by (*subst atLeastPlusOneLessThan-greaterThanLessThan-int* [*THEN sym*], *simp*)

lemma *finite-M-bounded-by-nat*: *finite* {*k*. *P k* ∧ *k* < (*i*::*nat*)}
proof −
have {*k*. *P k* ∧ *k* < *i*} ⊆ {..<*i*} **by** *auto*
with *finite-lessThan*[*of i*] **show** *?thesis* **by** (*simp add*: *finite-subset*)
qed

lemma *card-less*:
assumes *zero-in-M*: *0* ∈ *M*
shows *card* {*k* ∈ *M*. *k* < *Suc i*} ≠ *0*
proof −
from *zero-in-M* **have** {*k* ∈ *M*. *k* < *Suc i*} ≠ {} **by** *auto*
with *finite-M-bounded-by-nat* **show** *?thesis* **by** (*auto simp add*: *card-eq-0-iff*)
qed

lemma *card-less-Suc2*:
assumes *0* ∉ *M* **shows** *card* {*k*. *Suc k* ∈ *M* ∧ *k* < *i*} = *card* {*k* ∈ *M*. *k* < *Suc i*}
proof −
have ∗: ⟦*j* ∈ *M*; *j* < *Suc i*⟧ ⟹ *j* − *Suc 0* < *i* ∧ *Suc* (*j* − *Suc 0*) ∈ *M* ∧ *Suc 0* ≤ *j* **for** *j*
by (*cases j*) (*use assms* **in** *auto*)
show *?thesis*
proof (*rule card-bij-eq*)
show *inj-on Suc* {*k*. *Suc k* ∈ *M* ∧ *k* < *i*}
by *force*
show *inj-on* (λ*x*. *x* − *Suc 0*) {*k* ∈ *M*. *k* < *Suc i*}
by (*rule inj-on-diff-nat*) (*use* ∗ **in** *blast*)
qed (*use* ∗ **in** *auto*)
qed

lemma *card-less-Suc*:
assumes *0* ∈ *M*
shows *Suc* (*card* {*k*. *Suc k* ∈ *M* ∧ *k* < *i*}) = *card* {*k* ∈ *M*. *k* < *Suc i*}
proof −

have *Suc* (*card* {*k*. *Suc k* ∈ *M* ∧ *k* < *i*}) = *Suc* (*card* {*k*. *Suc k* ∈ *M* − {*0*} ∧ *k* < *i*})
  by *simp*
 also have ... = *Suc* (*card* {*k* ∈ *M* − {*0*}. *k* < *Suc i*})
  apply (*subst card-less-Suc2*)
  using *assms* **by** *auto*
 also have ... = *Suc* (*card* ({*k* ∈ *M*. *k* < *Suc i*} − {*0*}))
  by (*force intro*: *arg-cong* [**where** *f*=*card*])
 also have ... = *card* (*insert 0* ({*k* ∈ *M*. *k* < *Suc i*} − {*0*}))
  by (*simp add*: *card.insert-remove*)
 also have ... = *card* {*k* ∈ *M*. *k* < *Suc i*}
  using *assms*
  by (*force simp add*: *intro*: *arg-cong* [**where** *f*=*card*])
 finally show *?thesis***.**
qed

lemma *card-le-Suc-Max*: *finite S* ⟹ *card S* ≤ *Suc* (*Max S*)
proof (*rule classical*)
 assume *finite S* **and** ¬ *Suc* (*Max S*) ≥ *card S*
 then have *Suc* (*Max S*) < *card S*
  by *simp*
 with ‹*finite S*› **have** *S* ⊆ {*0*..*Max S*}
  by *auto*
 hence *card S* ≤ *card* {*0*..*Max S*}
  by (*intro card-mono*; *auto*)
 thus *card S* ≤ *Suc* (*Max S*)
  by *simp*
qed

lemma *finite-countable-subset*:
 assumes *finite A* **and** *A*: *A* ⊆ (⋃*i*::*nat*. *B i*)
 obtains *n* **where** *A* ⊆ (⋃*i*<*n*. *B i*)
proof −
 obtain *f* **where** *f*: ⋀*x*. *x* ∈ *A* ⟹ *x* ∈ *B*(*f x*)
  by (*metis in-mono UN-iff A*)
 define *n* **where** *n* = *Suc* (*Max* (*f*‘*A*))
 have *finite* (*f* ‘ *A*)
  by (*simp add*: ‹*finite A*›)
 then have *A* ⊆ (⋃*i*<*n*. *B i*)
  unfolding *UN-iff f n-def subset-iff*
  by (*meson Max-ge f imageI le-imp-less-Suc lessThan-iff*)
 then show *?thesis* **..**
qed

lemma *finite-countable-equals*:
 assumes *finite A A* = (⋃*i*::*nat*. *B i*)
 obtains *n* **where** *A* = (⋃*i*<*n*. *B i*)
proof −
 obtain *n* **where** *A* ⊆ (⋃*i*<*n*. *B i*)

proof (*rule finite-countable-subset*)
  show *A* ⊆ ⋃ (*range B*)
    by (*force simp*: *assms*)
qed (*use assms* **in** *auto*)
with *that* **show** *?thesis*
  by (*force simp*: *assms*)
qed

61.7 Lemmas useful with the summation operator sum

For examples, see Algebra/poly/UnivPoly2.thy

61.7.1 Disjoint Unions

Singletons and open intervals

lemma *ivl-disj-un-singleton*:
  {l::'a::linorder} Un {l<..} = {l..}
  {..<u} Un {u::'a::linorder} = {..u}
  (l::'a::linorder) < u ==> {l} Un {l<..<u} = {l..<u}
  (l::'a::linorder) < u ==> {l<..<u} Un {u} = {l<..u}
  (l::'a::linorder) ≤ u ==> {l} Un {l<..u} = {l..u}
  (l::'a::linorder) ≤ u ==> {l..<u} Un {u} = {l..u}
by *auto*

One- and two-sided intervals

lemma *ivl-disj-un-one*:
  (l::'a::linorder) < u ==> {..l} Un {l<..<u} = {..<u}
  (l::'a::linorder) ≤ u ==> {..<l} Un {l..<u} = {..<u}
  (l::'a::linorder) ≤ u ==> {..l} Un {l<..u} = {..u}
  (l::'a::linorder) ≤ u ==> {..<l} Un {l..u} = {..u}
  (l::'a::linorder) ≤ u ==> {l<..u} Un {u<..} = {l<..}
  (l::'a::linorder) < u ==> {l<..<u} Un {u..} = {l<..}
  (l::'a::linorder) ≤ u ==> {l..u} Un {u<..} = {l..}
  (l::'a::linorder) ≤ u ==> {l..<u} Un {u..} = {l..}
by *auto*

Two- and two-sided intervals

lemma *ivl-disj-un-two*:
  [| (l::'a::linorder) < m; m ≤ u |] ==> {l<..<m} Un {m..<u} = {l<..<u}
  [| (l::'a::linorder) ≤ m; m < u |] ==> {l<..m} Un {m<..<u} = {l<..<u}
  [| (l::'a::linorder) ≤ m; m ≤ u |] ==> {l..<m} Un {m..<u} = {l..<u}
  [| (l::'a::linorder) ≤ m; m < u |] ==> {l..m} Un {m<..<u} = {l..<u}
  [| (l::'a::linorder) < m; m ≤ u |] ==> {l<..<m} Un {m..u} = {l<..u}
  [| (l::'a::linorder) ≤ m; m ≤ u |] ==> {l<..m} Un {m<..u} = {l<..u}
  [| (l::'a::linorder) ≤ m; m ≤ u |] ==> {l..<m} Un {m..u} = {l..u}
  [| (l::'a::linorder) ≤ m; m ≤ u |] ==> {l..m} Un {m<..u} = {l..u}
by *auto*

lemma *ivl-disj-un-two-touch*:
[| (*l*::'*a*::*linorder*) < *m*; *m* < *u* |] ==> {*l*<..*m*} *Un* {*m*..<*u*} = {*l*<..<*u*}
[| (*l*::'*a*::*linorder*) ≤ *m*; *m* < *u* |] ==> {*l*..*m*} *Un* {*m*..<*u*} = {*l*..<*u*}
[| (*l*::'*a*::*linorder*) < *m*; *m* ≤ *u* |] ==> {*l*<..*m*} *Un* {*m*..*u*} = {*l*<..*u*}
[| (*l*::'*a*::*linorder*) ≤ *m*; *m* ≤ *u* |] ==> {*l*..*m*} *Un* {*m*..*u*} = {*l*..*u*}
by *auto*

lemmas *ivl-disj-un* = *ivl-disj-un-singleton ivl-disj-un-one ivl-disj-un-two ivl-disj-un-two-touch*

61.7.2 Disjoint Intersections

One- and two-sided intervals

lemma *ivl-disj-int-one*:
{..*l*::'*a*::*order*} *Int* {*l*<..<*u*} = {}
{..<*l*} *Int* {*l*..<*u*} = {}
{..*l*} *Int* {*l*<..*u*} = {}
{..<*l*} *Int* {*l*..*u*} = {}
{*l*<..*u*} *Int* {*u*<..} = {}
{*l*<..<*u*} *Int* {*u*..} = {}
{*l*..*u*} *Int* {*u*<..} = {}
{*l*..<*u*} *Int* {*u*..} = {}
by *auto*

Two- and two-sided intervals

lemma *ivl-disj-int-two*:
{*l*::'*a*::*order*<..<*m*} *Int* {*m*..<*u*} = {}
{*l*<..*m*} *Int* {*m*<..<*u*} = {}
{*l*..<*m*} *Int* {*m*..<*u*} = {}
{*l*..*m*} *Int* {*m*<..<*u*} = {}
{*l*<..<*m*} *Int* {*m*..*u*} = {}
{*l*<..*m*} *Int* {*m*<..*u*} = {}
{*l*..<*m*} *Int* {*m*..*u*} = {}
{*l*..*m*} *Int* {*m*<..*u*} = {}
by *auto*

lemmas *ivl-disj-int* = *ivl-disj-int-one ivl-disj-int-two*

61.7.3 Some Differences

lemma *ivl-diff*[*simp*]:
i ≤ *n* ⟹ {*i*..<*m*} − {*i*..<*n*} = {*n*..<(*m*::'*a*::*linorder*)}
by(*auto*)

lemma (**in** *linorder*) *lessThan-minus-lessThan* [*simp*]:
{..< *n*} − {..< *m*} = {*m* ..< *n*}
by *auto*

lemma (**in** *linorder*) *atLeastAtMost-diff-ends*:
{*a*..*b*} − {*a*, *b*} = {*a*<..<*b*}

by *auto*

61.7.4 Some Subset Conditions

lemma *ivl-subset* [*simp*]: $(\{i..<j\} \subseteq \{m..<n\}) = (j \leq i \vee m \leq i \wedge j \leq (n::'a::linorder))$
using *linorder-class.le-less-linear*[*of i n*]
by *safe* (*force intro*: *leI*)+

61.8 Generic big monoid operation over intervals

context *semiring-char-0*
begin

lemma *inj-on-of-nat* [*simp*]:
inj-on of-nat N
by (*rule inj-onI*) *simp*

lemma *bij-betw-of-nat* [*simp*]:
bij-betw of-nat N A $\longleftrightarrow$ *of-nat* ' $N = A$
by (*simp add*: *bij-betw-def*)

lemma *Nats-infinite*: *infinite* ($\mathbb{N}$ :: *'a set*)
by (*metis Nats-def finite-imageD infinite-UNIV-char-0 inj-on-of-nat*)

end

context *comm-monoid-set*
begin

lemma *atLeastLessThan-reindex*:
$F\ g\ \{h\ m..<h\ n\} = F\ (g \circ h)\ \{m..<n\}$
if *bij-betw h* $\{m..<n\}$ $\{h\ m..<h\ n\}$ **for** *m n* ::*nat*
proof −
from *that* **have** *inj-on h* $\{m..<n\}$ **and** *h* ' $\{m..<n\} = \{h\ m..<h\ n\}$
by (*simp-all add*: *bij-betw-def*)
then show *?thesis*
using *reindex* [*of h* $\{m..<n\}$ *g*] **by** *simp*
qed

lemma *atLeastAtMost-reindex*:
$F\ g\ \{h\ m..h\ n\} = F\ (g \circ h)\ \{m..n\}$
if *bij-betw h* $\{m..n\}$ $\{h\ m..h\ n\}$ **for** *m n* ::*nat*
proof −
from *that* **have** *inj-on h* $\{m..n\}$ **and** *h* ' $\{m..n\} = \{h\ m..h\ n\}$
by (*simp-all add*: *bij-betw-def*)
then show *?thesis*
using *reindex* [*of h* $\{m..n\}$ *g*] **by** *simp*
qed

lemma *atLeastLessThan-shift-bounds*:

F g {*m* + *k*..<*n* + *k*} = *F* (*g* ∘ *plus k*) {*m*..<*n*}
for *m n k* :: *nat*
using *atLeastLessThan-reindex* [*of plus k m n g*]
by (*simp add*: *ac-simps*)

lemma *atLeastAtMost-shift-bounds*:
F g {*m* + *k*..*n* + *k*} = *F* (*g* ∘ *plus k*) {*m*..*n*}
for *m n k* :: *nat*
using *atLeastAtMost-reindex* [*of plus k m n g*]
by (*simp add*: *ac-simps*)

lemma *atLeast-Suc-lessThan-Suc-shift*:
F g {*Suc m*..<*Suc n*} = *F* (*g* ∘ *Suc*) {*m*..<*n*}
using *atLeastLessThan-shift-bounds* [*of - - 1*]
by (*simp add*: *plus-1-eq-Suc*)

lemma *atLeast-Suc-atMost-Suc-shift*:
F g {*Suc m*..*Suc n*} = *F* (*g* ∘ *Suc*) {*m*..*n*}
using *atLeastAtMost-shift-bounds* [*of - - 1*]
by (*simp add*: *plus-1-eq-Suc*)

lemma *atLeast-atMost-pred-shift*:
F (*g* ∘ (λ*n*. *n* − *Suc 0*)) {*Suc m*..*Suc n*} = *F g* {*m*..*n*}
unfolding *atLeast-Suc-atMost-Suc-shift* **by** *simp*

lemma *atLeast-lessThan-pred-shift*:
F (*g* ∘ (λ*n*. *n* − *Suc 0*)) {*Suc m*..<*Suc n*} = *F g* {*m*..<*n*}
unfolding *atLeast-Suc-lessThan-Suc-shift* **by** *simp*

lemma *atLeast-int-lessThan-int-shift*:
F g {*int m*..<*int n*} = *F* (*g* ∘ *int*) {*m*..<*n*}
by (*rule atLeastLessThan-reindex*)
(*simp add*: *image-int-atLeastLessThan*)

lemma *atLeast-int-atMost-int-shift*:
F g {*int m*..*int n*} = *F* (*g* ∘ *int*) {*m*..*n*}
by (*rule atLeastAtMost-reindex*)
(*simp add*: *image-int-atLeastAtMost*)

lemma *atLeast0-lessThan-Suc*:
F g {*0*..<*Suc n*} = *F g* {*0*..<*n*} ∗ *g n*
by (*simp add*: *atLeast0-lessThan-Suc ac-simps*)

lemma *atLeast0-atMost-Suc*:
F g {*0*..*Suc n*} = *F g* {*0*..*n*} ∗ *g* (*Suc n*)
by (*simp add*: *atLeast0-atMost-Suc ac-simps*)

lemma *atLeast0-lessThan-Suc-shift*:
F g {*0*..<*Suc n*} = *g 0* ∗ *F* (*g* ∘ *Suc*) {*0*..<*n*}

by (*simp add*: *atLeast0-lessThan-Suc-eq-insert-0 atLeast-Suc-lessThan-Suc-shift*)

lemma *atLeast0-atMost-Suc-shift*:
$F\ g\ \{0..Suc\ n\} = g\ 0 * F\ (g \circ Suc)\ \{0..n\}$
by (*simp add*: *atLeast0-atMost-Suc-eq-insert-0 atLeast-Suc-atMost-Suc-shift*)

lemma *atLeast-Suc-lessThan*:
$F\ g\ \{m..<n\} = g\ m * F\ g\ \{Suc\ m..<n\}$ **if** $m < n$
proof −
from *that* **have** $\{m..<n\} = insert\ m\ \{Suc\ m..<n\}$
by *auto*
then show *?thesis* **by** *simp*
qed

lemma *atLeast-Suc-atMost*:
$F\ g\ \{m..n\} = g\ m * F\ g\ \{Suc\ m..n\}$ **if** $m \leq n$
proof −
from *that* **have** $\{m..n\} = insert\ m\ \{Suc\ m..n\}$
by *auto*
then show *?thesis* **by** *simp*
qed

lemma *ivl-cong*:
$a = c \Longrightarrow b = d \Longrightarrow (\bigwedge x.\ c \leq x \Longrightarrow x < d \Longrightarrow g\ x = h\ x)$
$\Longrightarrow F\ g\ \{a..<b\} = F\ h\ \{c..<d\}$
by (*rule cong*) *simp-all*

lemma *atLeastLessThan-shift-0*:
fixes *m n p* :: *nat*
shows $F\ g\ \{m..<n\} = F\ (g \circ plus\ m)\ \{0..<n - m\}$
using *atLeastLessThan-shift-bounds* [*of g 0 m n − m*]
by (*cases* $m \leq n$) *simp-all*

lemma *atLeastAtMost-shift-0*:
fixes *m n p* :: *nat*
assumes $m \leq n$
shows $F\ g\ \{m..n\} = F\ (g \circ plus\ m)\ \{0..n - m\}$
using *assms atLeastAtMost-shift-bounds* [*of g 0 m n − m*] **by** *simp*

lemma *atLeastLessThan-concat*:
fixes *m n p* :: *nat*
shows $m \leq n \Longrightarrow n \leq p \Longrightarrow F\ g\ \{m..<n\} * F\ g\ \{n..<p\} = F\ g\ \{m..<p\}$
by (*simp add*: *union-disjoint* [*symmetric*] *ivl-disj-un*)

lemma *atLeastLessThan-rev*:
$F\ g\ \{n..<m\} = F\ (\lambda i.\ g\ (m + n - Suc\ i))\ \{n..<m\}$
by (*rule reindex-bij-witness* [**where** $i=\lambda i.\ m + n - Suc\ i$ **and** $j=\lambda i.\ m + n - Suc\ i$], *auto*)

lemma *atLeastAtMost-rev*:
fixes *n m* :: *nat*
shows *F g* {*n..m*} = *F* (λ*i*. *g* (*m* + *n* − *i*)) {*n..m*}
by (*rule reindex-bij-witness* [**where** *i*=λ*i*. *m* + *n* − *i* **and** *j*=λ*i*. *m* + *n* − *i*]) *auto*

lemma *atLeastLessThan-rev-at-least-Suc-atMost*:
F g {*n..<m*} = *F* (λ*i*. *g* (*m* + *n* − *i*)) {*Suc n..m*}
unfolding *atLeastLessThan-rev* [*of g n m*]
by (*cases m*) (*simp-all add*: *atLeast-Suc-atMost-Suc-shift atLeastLessThanSuc-atLeastAtMost*)

end

61.9 Summation indexed over intervals

syntax (*ASCII*)
-from-to-sum :: *idt* ⇒ ′*a* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*notation*=‹*binder SUM*››*SUM* - = -..-./ -)› [*0*,*0*,*0*,*10*] *10*)
-from-upto-sum :: *idt* ⇒ ′*a* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*notation*=‹*binder SUM*››*SUM* - = -..<-./ -)› [*0*,*0*,*0*,*10*] *10*)
-upt-sum :: *idt* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*notation*=‹*binder SUM*››*SUM* -<-./ -)› [*0*,*0*,*10*] *10*)
-upto-sum :: *idt* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*notation*=‹*binder SUM*››*SUM* -<=-./ -)› [*0*,*0*,*10*] *10*)

syntax (*latex-sum* **output**)
-from-to-sum :: *idt* ⇒ ′*a* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b*
(‹(*3*$\sum_{- = -}^{-}$ -)› [*0*,*0*,*0*,*10*] *10*)
-from-upto-sum :: *idt* ⇒ ′*a* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b*
(‹(*3*$\sum_{- = -}^{<-}$ -)› [*0*,*0*,*0*,*10*] *10*)
-upt-sum :: *idt* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b*
(‹(*3*$\sum_{- < -}$ -)› [*0*,*0*,*10*] *10*)
-upto-sum :: *idt* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b*
(‹(*3*$\sum_{- \leq -}$ -)› [*0*,*0*,*10*] *10*)

syntax
-from-to-sum :: *idt* ⇒ ′*a* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*indent*=*3* *notation*=‹*binder* ∑››∑ - = -..-./ -)› [*0*,*0*,*0*,*10*] *10*)
-from-upto-sum :: *idt* ⇒ ′*a* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*indent*=*3* *notation*=‹*binder* ∑››∑ - = -..<-./ -)› [*0*,*0*,*0*,*10*] *10*)
-upt-sum :: *idt* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*indent*=*3* *notation*=‹*binder* ∑››∑ -<-./ -)› [*0*,*0*,*10*] *10*)
-upto-sum :: *idt* ⇒ ′*a* ⇒ ′*b* ⇒ ′*b* (‹(‹*indent*=*3* *notation*=‹*binder* ∑››∑ -≤-./ -)› [*0*,*0*,*10*] *10*)

syntax-consts
-from-to-sum -from-upto-sum -upt-sum -upto-sum == *sum*

translations

$\sum x{=}a..b.\ t == CONST\ sum\ (\lambda x.\ t)\ \{a..b\}$
$\sum x{=}a..{<}b.\ t == CONST\ sum\ (\lambda x.\ t)\ \{a..{<}b\}$
$\sum i{\le}n.\ t == CONST\ sum\ (\lambda i.\ t)\ \{..n\}$
$\sum i{<}n.\ t == CONST\ sum\ (\lambda i.\ t)\ \{..{<}n\}$

The above introduces some pretty alternative syntaxes for summation over intervals:

Old	New	LaTeX
$\sum x{\in}\{a..b\}.\ e$	$\sum x = a..b.\ e$	$\sum_{x = a}^{b} e$
$\sum x{\in}\{a..{<}b\}.\ e$	$\sum x = a..{<}b.\ e$	$\sum_{x = a}^{<b} e$
$\sum x{\in}\{..b\}.\ e$	$\sum x{\le}b.\ e$	$\sum_{x \le b} e$
$\sum x{\in}\{..{<}b\}.\ e$	$\sum x{<}b.\ e$	$\sum_{x < b} e$

The left column shows the term before introduction of the new syntax, the middle column shows the new (default) syntax, and the right column shows a special syntax. The latter is only meaningful for latex output and has to be activated explicitly by setting the print mode to *latex-sum* (e.g. via *mode* = *latex-sum* in antiquotations). It is not the default LaTeX output because it only works well with italic-style formulae, not tt-style.

Note that for uniformity on *nat* it is better to use $\sum x = 0..{<}n.\ e$ rather than $\sum x{<}n.\ e$: *sum* may not provide all lemmas available for $\{m..{<}n\}$ also in the special form for $\{..{<}n\}$.

This congruence rule should be used for sums over intervals as the standard theorem *sum.cong* does not work well with the simplifier who adds the unsimplified premise $x \in B$ to the context.

context *comm-monoid-set*
begin

lemma *zero-middle*:
  assumes $1 \le p\ k \le p$
  shows $F\ (\lambda j.\ if\ j < k\ then\ g\ j\ else\ if\ j = k\ then\ \mathbf{1}\ else\ h\ (j - Suc\ 0))\ \{..p\}$
    $= F\ (\lambda j.\ if\ j < k\ then\ g\ j\ else\ h\ j)\ \{..p - Suc\ 0\}$ (**is** *?lhs* = *?rhs*)
proof −
  have [*simp*]: $\{..p - Suc\ 0\} \cap \{j.\ j < k\} = \{..{<}k\}$ $\{..p - Suc\ 0\} \cap - \{j.\ j < k\} = \{k..p - Suc\ 0\}$
    using *assms* **by** *auto*
  have $?lhs = F\ g\ \{..{<}k\} * F\ (\lambda j.\ if\ j = k\ then\ \mathbf{1}\ else\ h\ (j - Suc\ 0))\ \{k..p\}$
    using *union-disjoint* [*of* $\{..{<}k\}$ $\{k..p\}$] *assms*
    by (*simp add*: *ivl-disj-int-one ivl-disj-un-one*)
  also have $\ldots = F\ g\ \{..{<}k\} * F\ (\lambda j.\ h\ (j - Suc\ 0))\ \{Suc\ k..p\}$
    by (*simp add*: *atLeast-Suc-atMost* [*of k p*] *assms*)
  also have $\ldots = F\ g\ \{..{<}k\} * F\ h\ \{k\ ..\ p - Suc\ 0\}$
    using *reindex* [*of Suc* $\{k..p - Suc\ 0\}$] *assms* **by** *simp*
  also have $\ldots = ?rhs$
    by (*simp add*: *If-cases*)

finally show *?thesis* **.**
qed

lemma *atMost-Suc* [*simp*]:
$F\ g\ \{..Suc\ n\} = F\ g\ \{..n\} * g\ (Suc\ n)$
by (*simp add*: *atMost-Suc ac-simps*)

lemma *lessThan-Suc* [*simp*]:
$F\ g\ \{..<Suc\ n\} = F\ g\ \{..<n\} * g\ n$
by (*simp add*: *lessThan-Suc ac-simps*)

lemma *cl-ivl-Suc* [*simp*]:
$F\ g\ \{m..Suc\ n\} = (\mathit{if}\ Suc\ n < m\ \mathit{then}\ \mathbf{1}\ \mathit{else}\ F\ g\ \{m..n\} * g(Suc\ n))$
by (*auto simp*: *ac-simps atLeastAtMostSuc-conv*)

lemma *op-ivl-Suc* [*simp*]:
$F\ g\ \{m..<Suc\ n\} = (\mathit{if}\ n < m\ \mathit{then}\ \mathbf{1}\ \mathit{else}\ F\ g\ \{m..<n\} * g(n))$
by (*auto simp*: *ac-simps atLeastLessThanSuc*)

lemma *head*:
fixes *n* :: *nat*
assumes *mn*: $m \le n$
shows $F\ g\ \{m..n\} = g\ m * F\ g\ \{m<..n\}$ (**is** *?lhs* = *?rhs*)
proof −
from *mn*
have $\{m..n\} = \{m\} \cup \{m<..n\}$
by (*auto intro*: *ivl-disj-un-singleton*)
hence *?lhs* = $F\ g\ (\{m\} \cup \{m<..n\})$
by (*simp add*: *atLeast0LessThan*)
also have ... = *?rhs* **by** *simp*
finally show *?thesis* **.**
qed

lemma *last-plus*:
fixes *n*::*nat* **shows** $m \le n \Longrightarrow F\ g\ \{m..n\} = g\ n * F\ g\ \{m..<n\}$
by (*cases n*) (*auto simp*: *atLeastLessThanSuc-atLeastAtMost commute*)

lemma *head-if*:
fixes *n* :: *nat*
shows $F\ g\ \{m..n\} = (\mathit{if}\ n < m\ \mathit{then}\ \mathbf{1}\ \mathit{else}\ \ F\ g\ \{m..<n\} * g(n))$
by (*simp add*: *commute last-plus*)

lemma *ub-add-nat*:
assumes $(m::nat) \le n + 1$
shows $F\ g\ \{m..n + p\} = F\ g\ \{m..n\} * F\ g\ \{n + 1..n + p\}$
proof−
have $\{m\ ..\ n+p\} = \{m..n\} \cup \{n+1..n+p\}$ **using** ‹$m \le n+1$› **by** *auto*
thus *?thesis* **by** (*auto simp*: *ivl-disj-int union-disjoint atLeastSucAtMost-greaterThanAtMost*)
qed

lemma *nat-group*:
fixes *k*::*nat* **shows** F (λm. F g {$m * k$ ..< $m*k + k$}) {..<n} = F g {..< $n * k$}
proof (*cases k*)
case (*Suc l*)
then have $k > 0$
by *auto*
then show *?thesis*
by (*induct n*) (*simp-all add*: *atLeastLessThan-concat add.commute atLeast0LessThan*[*symmetric*])
qed *auto*

lemma *triangle-reindex*:
fixes *n* :: *nat*
shows F ($\lambda(i,j)$. g i j) {(i,j). $i+j < n$} = F (λk. F (λi. g i ($k - i$)) {..k}) {..<n}
apply (*simp add*: *Sigma*)
apply (*rule reindex-bij-witness*[**where** j=$\lambda(i, j)$. ($i+j$, i) **and** i=$\lambda(k, i)$. (i, $k - i$)])
apply *auto*
done

lemma *triangle-reindex-eq*:
fixes *n* :: *nat*
shows F ($\lambda(i,j)$. g i j) {(i,j). $i+j \leq n$} = F (λk. F (λi. g i ($k - i$)) {..k}) {..n}
using *triangle-reindex* [*of g Suc n*]
by (*simp only*: *Nat.less-Suc-eq-le lessThan-Suc-atMost*)

lemma *nat-diff-reindex*: F (λi. g (n − *Suc* i)) {..<n} = F g {..<n}
by (*rule reindex-bij-witness*[**where** i=λi. n − *Suc* i **and** j=λi. n − *Suc* i]) *auto*

lemma *shift-bounds-nat-ivl*:
F g {$m+k$..<$n+k$} = F (λi. $g(i + k)$){m..<n::*nat*}
by (*induct n*, *auto simp*: *atLeastLessThanSuc*)

lemma *shift-bounds-cl-nat-ivl*:
F g {$m+k$..$n+k$} = F (λi. $g(i + k)$){m..n::*nat*}
by (*rule reindex-bij-witness*[**where** i=λi. $i + k$ **and** j=λi. $i - k$]) *auto*

corollary *shift-bounds-cl-Suc-ivl*:
F g {*Suc* m..*Suc* n} = F (λi. g(*Suc* i)){m..n}
by (*simp add*: *shift-bounds-cl-nat-ivl*[**where** k=*Suc 0*, *simplified*])

corollary *Suc-reindex-ivl*: $m \leq n \Longrightarrow F$ g {m..n} $*$ g (*Suc* n) = g m $*$ F (λi. g (*Suc* i)) {m..n}
by (*simp add*: *assoc atLeast-Suc-atMost flip*: *shift-bounds-cl-Suc-ivl*)

corollary *shift-bounds-Suc-ivl*:
F g {*Suc* m..<*Suc* n} = F (λi. g(*Suc* i)){m..<n}
by (*simp add*: *shift-bounds-nat-ivl*[**where** k=*Suc 0*, *simplified*])

lemma *atMost-Suc-shift*:
  shows $F\ g\ \{..Suc\ n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..n\}$
proof (*induct n*)
  case *0* **show** *?case* **by** *simp*
next
  case (*Suc n*) **note** *IH* = *this*
  have $F\ g\ \{..Suc\ (Suc\ n)\} = F\ g\ \{..Suc\ n\} * g\ (Suc\ (Suc\ n))$
    by (*rule atMost-Suc*)
  also have $F\ g\ \{..Suc\ n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..n\}$
    by (*rule IH*)
  also have $g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..n\} * g\ (Suc\ (Suc\ n)) =$
           $g\ 0 * (F\ (\lambda i.\ g\ (Suc\ i))\ \{..n\} * g\ (Suc\ (Suc\ n)))$
    by (*rule assoc*)
  also have $F\ (\lambda i.\ g\ (Suc\ i))\ \{..n\} * g\ (Suc\ (Suc\ n)) = F\ (\lambda i.\ g\ (Suc\ i))\ \{..Suc$
$n\}$
    by (*rule atMost-Suc* [*symmetric*])
  finally show *?case* **.**
qed

lemma *lessThan-Suc-shift*:
  $F\ g\ \{..<Suc\ n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..<n\}$
  by (*induction n*) (*simp-all add*: *ac-simps*)

lemma *atMost-shift*:
  $F\ g\ \{..n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..<n\}$
  by (*metis atLeast0AtMost atLeast0LessThan atLeastLessThanSuc-atLeastAtMost*

    atLeastSucAtMost-greaterThanAtMost le0 head shift-bounds-Suc-ivl)

lemma *nested-swap*:
   $F\ (\lambda i.\ F\ (\lambda j.\ a\ i\ j)\ \{0..<i\})\ \{0..n\} = F\ (\lambda j.\ F\ (\lambda i.\ a\ i\ j)\ \{Suc\ j..n\})\ \{0..<n\}$
  by (*induction n*) (*auto simp*: *distrib*)

lemma *nested-swap'*:
   $F\ (\lambda i.\ F\ (\lambda j.\ a\ i\ j)\ \{..<i\})\ \{..n\} = F\ (\lambda j.\ F\ (\lambda i.\ a\ i\ j)\ \{Suc\ j..n\})\ \{..<n\}$
  by (*induction n*) (*auto simp*: *distrib*)

lemma *atLeast1-atMost-eq*:
  $F\ g\ \{Suc\ 0..n\} = F\ (\lambda k.\ g\ (Suc\ k))\ \{..<n\}$
proof −
  have $F\ g\ \{Suc\ 0..n\} = F\ g\ (Suc\ `\ \{..<n\})$
    by (*simp add*: *image-Suc-lessThan*)
  also have $\ldots = F\ (\lambda k.\ g\ (Suc\ k))\ \{..<n\}$
    by (*simp add*: *reindex*)
  finally show *?thesis* **.**
qed

lemma *atLeastLessThan-Suc*: $a \le b \Longrightarrow F\ g\ \{a..<Suc\ b\} = F\ g\ \{a..<b\} * g\ b$

by (*simp add*: *atLeastLessThanSuc commute*)

lemma *nat-ivl-Suc′*:
assumes $m \leq Suc\ n$
shows $F\ g\ \{m..Suc\ n\} = g\ (Suc\ n) * F\ g\ \{m..n\}$
proof −
from *assms* **have** $\{m..Suc\ n\} = insert\ (Suc\ n)\ \{m..n\}$ **by** *auto*
also have $F\ g \ldots = g\ (Suc\ n) * F\ g\ \{m..n\}$ **by** *simp*
finally show *?thesis* **.**
qed

lemma *in-pairs*: $F\ g\ \{2*m..Suc(2*n)\} = F\ (\lambda i.\ g(2*i) * g(Suc(2*i)))\ \{m..n\}$
proof (*induction n*)
case *0*
show *?case*
by (*cases m=0*) *auto*
next
case (*Suc n*)
then show *?case*
by (*auto simp*: *assoc split*: *if-split-asm*)
qed

lemma *in-pairs-0*: $F\ g\ \{..Suc(2*n)\} = F\ (\lambda i.\ g(2*i) * g(Suc(2*i)))\ \{..n\}$
using *in-pairs* [*of - 0 n*] **by** (*simp add*: *atLeast0AtMost*)

end

lemma *card-sum-le-nat-sum*: $\sum\ \{0..<card\ S\} \leq \sum\ S$
proof (*cases finite S*)
case *True*
then show *?thesis*
proof (*induction card S arbitrary*: *S*)
case (*Suc x*)
then have $Max\ S \geq x$ **using** *card-le-Suc-Max* **by** *fastforce*
let $?S' = S - \{Max\ S\}$
from *Suc* **have** $Max\ S \in S$ **by** (*auto intro*: *Max-in*)
hence *cards*: $card\ S = Suc\ (card\ ?S')$
using ‹*finite S*› **by** (*intro card.remove*; *auto*)
hence $\sum\ \{0..<card\ ?S'\} \leq \sum\ ?S'$
using *Suc* **by** (*intro Suc*; *auto*)

hence $\sum\ \{0..<card\ ?S'\} + x \leq \sum\ ?S' + Max\ S$
using ‹$Max\ S \geq x$› **by** *simp*
also have ... $= \sum\ S$
using *sum.remove*[*OF* ‹*finite S*› ‹$Max\ S \in S$›, **where** $g=\lambda x.\ x$]
by *simp*
finally show *?case*
using *cards Suc* **by** *auto*
qed *simp*

qed *simp*

lemma *sum-natinterval-diff*:
fixes f:: *nat* ⇒ ('*a*::*ab-group-add*)
shows *sum* $(\lambda k.\ f\ k - f(k + 1))$ $\{(m::nat)\ ..\ n\}$ =
(*if* $m \le n$ *then* $f\ m - f(n + 1)$ *else* *0*)
by (*induct n*, *auto simp add*: *algebra-simps not-le le-Suc-eq*)

lemma *sum-diff-nat-ivl*:
fixes f :: *nat* ⇒ '*a*::*ab-group-add*
shows ⟦ $m \le n$; $n \le p$ ⟧ ⟹ *sum f* $\{m..<p\}$ − *sum f* $\{m..<n\}$ = *sum f* $\{n..<p\}$
using *sum.atLeastLessThan-concat* [*of m n p f*,*symmetric*]
by (*simp add*: *ac-simps*)

lemma *sum-diff-distrib*: $\forall x.\ Q\ x \le P\ x \implies (\sum x{<}n.\ P\ x) - (\sum x{<}n.\ Q\ x) = (\sum x{<}n.\ P\ x - Q\ x :: nat)$
by (*subst sum-subtractf-nat*) *auto*

61.9.1 Shifting bounds

context *comm-monoid-add*
begin

context
fixes f :: *nat* ⇒ '*a*
assumes $f\ 0 = 0$
begin

lemma *sum-shift-lb-Suc0-0-upt*:
sum f $\{Suc\ 0..<k\}$ = *sum f* $\{0..<k\}$
proof (*cases k*)
case *0*
then show *?thesis*
by *simp*
next
case (*Suc k*)
moreover have $\{0..<Suc\ k\}$ = *insert 0* $\{Suc\ 0..<Suc\ k\}$
by *auto*
ultimately show *?thesis*
using ‹$f\ 0 = 0$› **by** *simp*
qed

lemma *sum-shift-lb-Suc0-0*: *sum f* $\{Suc\ 0..k\}$ = *sum f* $\{0..k\}$
proof (*cases k*)
case *0*
with ‹$f\ 0 = 0$› **show** *?thesis*
by *simp*
next
case (*Suc k*)

moreover have {*0..Suc k*} = *insert 0* {*Suc 0..Suc k*}
by *auto*
ultimately show *?thesis*
using ‹*f 0* = *0*› **by** *simp*
qed

end

end

lemma *sum-Suc-diff*:
fixes *f* :: *nat* ⇒ ′*a*::*ab-group-add*
assumes *m* ≤ *Suc n*
shows (∑ *i* = *m..n*. *f*(*Suc i*) − *f i*) = *f* (*Suc n*) − *f m*
using *assms* **by** (*induct n*) (*auto simp*: *le-Suc-eq*)

lemma *sum-Suc-diff* ′:
fixes *f* :: *nat* ⇒ ′*a*::*ab-group-add*
assumes *m* ≤ *n*
shows (∑ *i* = *m..<n*. *f* (*Suc i*) − *f i*) = *f n* − *f m*
using *assms* **by** (*induct n*) (*auto simp*: *le-Suc-eq*)

lemma *sum-diff-split*:
fixes *f*:: *nat* ⇒ ′*a*::*ab-group-add*
assumes *m* ≤ *n*
shows (∑ *i*≤*n*. *f i*) − (∑ *i*<*m*. *f i*) = (∑ *i*≤*n* − *m*. *f*(*n* − *i*))
proof −
have ⋀*i*. *i* ≤ *n*−*m* ⟹ ∃ *k*≥*m*. *k* ≤ *n* ∧ *i* = *n*−*k*
by (*metis Nat.le-diff-conv2 add.commute* ‹*m*≤*n*› *diff-diff-cancel diff-le-self order.trans*)
then have *eq*: {..*n*−*m*} = (−)*n* ‘ {*m..n*}
by *force*
have *inj*: *inj-on* ((−)*n*) {*m..n*}
by (*auto simp*: *inj-on-def*)
have (∑ *i*≤*n* − *m*. *f*(*n* − *i*)) = (∑ *i*=*m..n*. *f i*)
by (*simp add*: *eq sum.reindex-cong* [*OF inj*])
also have . . . = (∑ *i*≤*n*. *f i*) − (∑ *i*<*m*. *f i*)
using *sum-diff-nat-ivl*[*of 0 m Suc n f*] *assms*
by (*simp only*: *atLeast0AtMost atLeast0LessThan atLeastLessThanSuc-atLeastAtMost*)
finally show *?thesis* **by** *metis*
qed

lemma *prod-divide-nat-ivl*:
fixes *f* :: *nat* ⇒ ′*a*::*idom-divide*
shows ⟦ *m* ≤ *n*; *n* ≤ *p*; *prod f* {*m..<n*} ≠ *0*⟧ ⟹ *prod f* {*m..<p*} *div prod f* {*m..<n*} = *prod f* {*n..<p*}
using *prod.atLeastLessThan-concat* [*of m n p f*,*symmetric*]
by (*simp add*: *ac-simps*)

lemma *prod-divide-split*:
  fixes f:: $nat \Rightarrow$ $'a$::*idom-divide*
  assumes $m \leq n$ *prod* f $\{..<m\} \neq 0$
  shows (*prod* f $\{..n\}$) *div* (*prod* f $\{..<m\}$) = *prod* ($\lambda i.$ $f(n - i)$) $\{..n - m\}$
proof –
  have $\bigwedge i.$ $i \leq n-m \Longrightarrow \exists k \geq m.$ $k \leq n \wedge i = n-k$
    by (*metis Nat.le-diff-conv2 add.commute* ‹$m \leq n$› *diff-diff-cancel diff-le-self order.trans*)
  then have *eq*: $\{..n-m\} = (-)n$ ‘ $\{m..n\}$
    by *force*
  have *inj*: *inj-on* $((-)n)$ $\{m..n\}$
    by (*auto simp*: *inj-on-def*)
  have *prod* ($\lambda i.$ $f(n - i)$) $\{..n - m\}$ = *prod* f $\{m..n\}$
    by (*simp add*: *eq prod.reindex-cong* [*OF inj*])
  also have ... = *prod* f $\{..n\}$ *div prod* f $\{..<m\}$
    using *prod-divide-nat-ivl*[*of 0 m Suc n f*] *assms*
    by (*force simp*: *atLeast0AtMost atLeast0LessThan atLeastLessThanSuc-atLeastAtMost*)
  finally show *?thesis* **by** *metis*
qed

61.9.2 Telescoping sums

lemma *sum-telescope*:
  fixes f::$nat \Rightarrow$ $'a$::*ab-group-add*
  shows *sum* ($\lambda i.$ f i $-$ f (*Suc* i)) $\{..\ i\}$ = f 0 $-$ f (*Suc* i)
  by (*induct* i) *simp-all*

lemma *sum-telescope''*:
  assumes $m \leq n$
  shows ($\sum k \in \{Suc\ m..n\}.$ f k $-$ f $(k - 1)$) = f n $-$ (f m :: $'a$ :: *ab-group-add*)
  by (*rule dec-induct*[*OF assms*]) (*simp-all add*: *algebra-simps*)

lemma *sum-lessThan-telescope*:
  ($\sum n<m.$ f (*Suc* n) $-$ f n :: $'a$ :: *ab-group-add*) = f m $-$ f 0
  by (*induction* m) (*simp-all add*: *algebra-simps*)

lemma *sum-lessThan-telescope'*:
  ($\sum n<m.$ f n $-$ f (*Suc* n) :: $'a$ :: *ab-group-add*) = f 0 $-$ f m
  by (*induction* m) (*simp-all add*: *algebra-simps*)

61.9.3 The formula for geometric sums

lemma *sum-power2*: ($\sum i=0..<k.$ $(2::nat)\hat{\ }i$) = $2\hat{\ }k-1$
  by (*induction* k) (*auto simp*: *mult-2*)

lemma *geometric-sum*:
  assumes $x \neq 1$
  shows ($\sum i<n.$ x ^ i) = (x ^ n $-$ 1) / (x $-$ 1::$'a$::*field*)
proof –
  from *assms* **obtain** y **where** $y = x - 1$ **and** $y \neq 0$ **by** *simp-all*

moreover have $(\sum i{<}n.\ (y + 1)\ \hat{}\ i) = ((y + 1)\ \hat{}\ n - 1)\ /\ y$
by (*induct n*) (*simp-all add*: *field-simps* ‹$y \neq 0$›)
ultimately show *?thesis* **by** *simp*
qed

lemma *geometric-sum-less*:
assumes $0 < x\ x < 1$ *finite* S
shows $(\sum i{\in}S.\ x\ \hat{}\ i) < 1\ /\ (1 - x{::}'a{::}linordered\text{-}field)$
proof −
define n **where** $n \equiv Suc\ (Max\ S)$
have $(\sum i{\in}S.\ x\ \hat{}\ i) \leq (\sum i{<}n.\ x\ \hat{}\ i)$
unfolding *n-def* **using** *assms* **by** (*fastforce intro*!: *sum-mono2 le-imp-less-Suc*)
also have $\ldots = (1 - x\ \hat{}\ n)\ /\ (1 - x)$
using *assms* **by** (*simp add*: *geometric-sum field-simps*)
also have $\ldots < 1\ /\ (1{-}x)$
using *assms* **by** (*simp add*: *field-simps power-Suc-less*)
finally show *?thesis* **.**
qed

lemma *diff-power-eq-sum*:
fixes y :: $'a{::}\{comm\text{-}ring,monoid\text{-}mult\}$
shows
$x\ \hat{}\ (Suc\ n) - y\ \hat{}\ (Suc\ n) =$
$(x - y) * (\sum p{<}Suc\ n.\ (x\ \hat{}\ p) * y\ \hat{}\ (n - p))$
proof (*induct n*)
case (*Suc n*)
have $x\ \hat{}\ Suc\ (Suc\ n) - y\ \hat{}\ Suc\ (Suc\ n) = x * (x * x\hat{}n) - y * (y * y\ \hat{}\ n)$
by *simp*
also have ... $= y * (x\ \hat{}\ (Suc\ n) - y\ \hat{}\ (Suc\ n)) + (x - y) * (x * x\hat{}n)$
by (*simp add*: *algebra-simps*)
also have ... $= y * ((x - y) * (\sum p{<}Suc\ n.\ (x\ \hat{}\ p) * y\ \hat{}\ (n - p))) + (x - y)$
$* (x * x\hat{}n)$
by (*simp only*: *Suc*)
also have ... $= (x - y) * (y * (\sum p{<}Suc\ n.\ (x\ \hat{}\ p) * y\ \hat{}\ (n - p))) + (x - y)$
$* (x * x\hat{}n)$
by (*simp only*: *mult.left-commute*)
also have ... $= (x - y) * (\sum p{<}Suc\ (Suc\ n).\ x\ \hat{}\ p * y\ \hat{}\ (Suc\ n - p))$
by (*simp add*: *field-simps Suc-diff-le sum-distrib-right sum-distrib-left*)
finally show *?case* **.**
qed *simp*

corollary *power-diff-sumr2*: — *COMPLEX-POLYFUN* in HOL Light
fixes x :: $'a{::}\{comm\text{-}ring,monoid\text{-}mult\}$
shows $x\hat{}n - y\hat{}n = (x - y) * (\sum i{<}n.\ y\hat{}(n - Suc\ i) * x\hat{}i)$
using *diff-power-eq-sum*[*of x n* − *1 y*]
by (*cases n* = *0*) (*simp-all add*: *field-simps*)

lemma *power-diff-1-eq*:
fixes x :: $'a{::}\{comm\text{-}ring,monoid\text{-}mult\}$

shows $x^\frown n - 1 = (x - 1) * (\sum i<n. (x^\frown i))$
using *diff-power-eq-sum* [*of x - 1*]
by (*cases n*) *auto*

lemma *one-diff-power-eq'*:
fixes x :: *'a::{comm-ring,monoid-mult}*
shows $1 - x^\frown n = (1 - x) * (\sum i<n. x^\frown(n - Suc\ i))$
using *diff-power-eq-sum* [*of 1 - x*]
by (*cases n*) *auto*

lemma *one-diff-power-eq*:
fixes x :: *'a::{comm-ring,monoid-mult}*
shows $1 - x^\frown n = (1 - x) * (\sum i<n. x^\frown i)$
by (*metis one-diff-power-eq' sum.nat-diff-reindex*)

lemma *sum-gp-basic*:
fixes x :: *'a::{comm-ring,monoid-mult}*
shows $(1 - x) * (\sum i\leq n. x^\frown i) = 1 - x^\frown Suc\ n$
by (*simp only*: *one-diff-power-eq lessThan-Suc-atMost*)

lemma *sum-power-shift*:
fixes x :: *'a::{comm-ring,monoid-mult}*
assumes $m \leq n$
shows $(\sum i=m..n. x^\frown i) = x^\frown m * (\sum i\leq n-m. x^\frown i)$
proof −
have $(\sum i=m..n. x^\frown i) = x^\frown m * (\sum i=m..n. x^\frown(i-m))$
by (*simp add*: *sum-distrib-left power-add* [*symmetric*])
also have $(\sum i=m..n. x^\frown(i-m)) = (\sum i\leq n-m. x^\frown i)$
using ‹$m \leq n$› **by** (*intro sum.reindex-bij-witness*[**where** $j=\lambda i.\ i - m$ **and** $i=\lambda i.\ i + m$]) *auto*
finally show *?thesis* **.**
qed

lemma *sum-gp-multiplied*:
fixes x :: *'a::{comm-ring,monoid-mult}*
assumes $m \leq n$
shows $(1 - x) * (\sum i=m..n. x^\frown i) = x^\frown m - x^\frown Suc\ n$
proof −
have $(1 - x) * (\sum i=m..n. x^\frown i) = x^\frown m * (1 - x) * (\sum i\leq n-m. x^\frown i)$
by (*metis mult.assoc mult.commute assms sum-power-shift*)
also have ... $=x^\frown m * (1 - x^\frown Suc(n-m))$
by (*metis mult.assoc sum-gp-basic*)
also have ... $= x^\frown m - x^\frown Suc\ n$
using *assms*
by (*simp add*: *algebra-simps*) (*metis le-add-diff-inverse power-add*)
finally show *?thesis* **.**
qed

lemma *sum-gp*:

fixes x :: $'a$::{*comm-ring*,*division-ring*}
shows $(\sum i{=}m..n.\ x\hat{\ }i) =$
$(if\ n < m\ then\ 0$
$else\ if\ x = 1\ then\ of\text{-}nat((n + 1) - m)$
$else\ (x\hat{\ }m - x\hat{\ }Suc\ n) / (1 - x))$
proof (*cases* $n < m$)
case *False*
assume $*$: $\neg\ n < m$
then show *?thesis*
proof (*cases* $x = 1$)
case *False*
assume $x \neq 1$
then have *not-zero*: $1 - x \neq 0$
by *auto*
have $(1 - x) * (\sum i{=}m..n.\ x\hat{\ }i) = x\ \hat{\ }\ m - x * x\ \hat{\ }\ n$
using *sum-gp-multiplied* [*of m n x*] $*$ **by** *auto*
then have $(\sum i{=}m..n.\ x\hat{\ }i) = (x\ \hat{\ }\ m - x * x\ \hat{\ }\ n) / (1 - x)$
using *nonzero-divide-eq-eq mult.commute not-zero*
by *metis*
then show *?thesis*
by *auto*
qed (*auto*)
qed (*auto*)

61.9.4 Geometric progressions

lemma *sum-gp0*:
fixes x :: $'a$::{*comm-ring*,*division-ring*}
shows $(\sum i{\le}n.\ x\hat{\ }i) = (if\ x = 1\ then\ of\text{-}nat(n + 1)\ else\ (1 - x\hat{\ }Suc\ n) / (1 - x))$
using *sum-gp-basic*[*of x n*]
by (*simp add*: *mult.commute field-split-simps*)

lemma *sum-power-add*:
fixes x :: $'a$::{*comm-ring*,*monoid-mult*}
shows $(\sum i{\in}I.\ x\hat{\ }(m{+}i)) = x\hat{\ }m * (\sum i{\in}I.\ x\hat{\ }i)$
by (*simp add*: *sum-distrib-left power-add*)

lemma *sum-gp-offset*:
fixes x :: $'a$::{*comm-ring*,*division-ring*}
shows $(\sum i{=}m..m{+}n.\ x\hat{\ }i) =$
$(if\ x = 1\ then\ of\text{-}nat\ n + 1\ else\ x\hat{\ }m * (1 - x\hat{\ }Suc\ n) / (1 - x))$
using *sum-gp* [*of x m m+n*]
by (*auto simp*: *power-add algebra-simps*)

lemma *sum-gp-strict*:
fixes x :: $'a$::{*comm-ring*,*division-ring*}
shows $(\sum i{<}n.\ x\hat{\ }i) = (if\ x = 1\ then\ of\text{-}nat\ n\ else\ (1 - x\hat{\ }n) / (1 - x))$
by (*induct n*) (*auto simp*: *algebra-simps field-split-simps*)

61.9.5 The formulae for arithmetic sums

context *comm-semiring-1*
begin

lemma *double-gauss-sum*:
 $2 * (\sum i = 0..n.\ \textit{of-nat}\ i) = \textit{of-nat}\ n * (\textit{of-nat}\ n + 1)$
 by (*induct n*) (*simp-all add*: *sum.atLeast0-atMost-Suc algebra-simps left-add-twice*)

lemma *double-gauss-sum-from-Suc-0*:
 $2 * (\sum i = \textit{Suc}\ 0..n.\ \textit{of-nat}\ i) = \textit{of-nat}\ n * (\textit{of-nat}\ n + 1)$
proof −
 have *sum of-nat* {*Suc 0..n*} = *sum of-nat* (*insert 0* {*Suc 0..n*})
  by *simp*
 also have ... = *sum of-nat* {*0..n*}
  by (*cases n*) (*simp-all add*: *atLeast0-atMost-Suc-eq-insert-0*)
 finally show *?thesis*
  by (*simp add*: *double-gauss-sum*)
qed

lemma *double-arith-series*:
 $2 * (\sum i = 0..n.\ a + \textit{of-nat}\ i * d) = (\textit{of-nat}\ n + 1) * (2 * a + \textit{of-nat}\ n * d)$
proof −
 have $(\sum i = 0..n.\ a + \textit{of-nat}\ i * d) = ((\sum i = 0..n.\ a) + (\sum i = 0..n.\ \textit{of-nat}\ i * d))$
  by (*rule sum.distrib*)
 also have ... $= (\textit{of-nat}\ (\textit{Suc}\ n) * a + d * (\sum i = 0..n.\ \textit{of-nat}\ i))$
  by (*simp add*: *sum-distrib-left algebra-simps*)
 finally show *?thesis*
  by (*simp add*: *algebra-simps double-gauss-sum left-add-twice*)
qed

end

context *linordered-euclidean-semiring*
begin

lemma *gauss-sum*:
 $(\sum i = 0..n.\ \textit{of-nat}\ i) = \textit{of-nat}\ n * (\textit{of-nat}\ n + 1)\ \textit{div}\ 2$
 using *double-gauss-sum* [*of n*, *symmetric*] **by** *simp*

lemma *gauss-sum-from-Suc-0*:
 $(\sum i = \textit{Suc}\ 0..n.\ \textit{of-nat}\ i) = \textit{of-nat}\ n * (\textit{of-nat}\ n + 1)\ \textit{div}\ 2$
 using *double-gauss-sum-from-Suc-0* [*of n*, *symmetric*] **by** *simp*

lemma *arith-series*:
 $(\sum i = 0..n.\ a + \textit{of-nat}\ i * d) = (\textit{of-nat}\ n + 1) * (2 * a + \textit{of-nat}\ n * d)\ \textit{div}\ 2$
 using *double-arith-series* [*of a d n*, *symmetric*] **by** *simp*

end

lemma *gauss-sum-nat*:
$\sum\{0..n\} = (n * Suc\ n)\ div\ 2$
using *gauss-sum* [*of n*, **where** *?'a = nat*] **by** *simp*

lemma *arith-series-nat*:
$(\sum i = 0..n.\ a + i * d) = Suc\ n * (2 * a + n * d)\ div\ 2$
using *arith-series* [*of a d n*] **by** *simp*

lemma *Sum-Icc-int*:
$\sum\{m..n\} = (n * (n + 1) - m * (m - 1))\ div\ 2$
if $m \le n$ **for** *m n* :: *int*
using *that* **proof** (*induct* $i \equiv nat\ (n - m)$ *arbitrary*: *m n*)
case *0*
then have $m = n$
by *arith*
then show *?case*
by (*simp add*: *algebra-simps mult-2* [*symmetric*])
next
case (*Suc i*)
have *0*: $i = nat((n-1) - m)$ $m \le n-1$ **using** *Suc*(*2*,*3*) **by** *arith+*
have $\sum\ \{m..n\} = \sum\ \{m..1+(n-1)\}$ **by** *simp*
also have $\ldots = \sum\ \{m..n-1\} + n$ **using** ‹$m \le n$›
by(*subst atLeastAtMostPlus1-int-conv*) *simp-all*
also have $\ldots = ((n-1)*(n-1+1) - m*(m-1))\ div\ 2 + n$
by(*simp add*: *Suc*(*1*)[*OF 0*])
also have $\ldots = ((n-1)*(n-1+1) - m*(m-1) + 2*n)\ div\ 2$ **by** *simp*
also have $\ldots = (n*(n+1) - m*(m-1))\ div\ 2$
by (*simp add*: *algebra-simps mult-2-right*)
finally show *?case* **.**
qed

lemma *Sum-Icc-nat*:
$\sum\{m..n\} = (n * (n + 1) - m * (m - 1))\ div\ 2$ **for** *m n* :: *nat*
proof (*cases* $m \le n$)
case *True*
then have $*$: $m * (m - 1) \le n * (n + 1)$
by (*meson diff-le-self order-trans le-add1 mult-le-mono*)
have *int* $(\sum\{m..n\}) = (\sum\{int\ m..int\ n\})$
by (*simp add*: *sum.atLeast-int-atMost-int-shift*)
also have $\ldots = (int\ n * (int\ n + 1) - int\ m * (int\ m - 1))\ div\ 2$
using ‹$m \le n$› **by** (*simp add*: *Sum-Icc-int*)
also have $\ldots = int\ ((n * (n + 1) - m * (m - 1))\ div\ 2)$
using *le-square* $*$ **by** (*simp add*: *algebra-simps of-nat-div of-nat-diff*)
finally show *?thesis*
by (*simp only*: *of-nat-eq-iff*)
next
case *False*
then show *?thesis*

by (*auto dest*: *less-imp-Suc-add simp add*: *not-le algebra-simps*)
qed

lemma *Sum-Ico-nat*:
$\sum\{m..<n\} = (n * (n - 1) - m * (m - 1))$ *div 2* **for** *m n* :: *nat*
by (*cases n*) (*simp-all add*: *atLeastLessThanSuc-atLeastAtMost Sum-Icc-nat*)

61.9.6 Division remainder

lemma *range-mod*:
fixes *n* :: *nat*
assumes $n > 0$
shows *range* ($\lambda m.$ *m mod n*) = $\{0..<n\}$ (**is** *?A* = *?B*)
proof (*rule set-eqI*)
fix *m*
show $m \in$ *?A* $\longleftrightarrow$ $m \in$ *?B*
proof
assume $m \in$ *?A*
with *assms* **show** $m \in$ *?B*
by *auto*
next
assume $m \in$ *?B*
moreover have *m mod n* $\in$ *?A*
by (*rule rangeI*)
ultimately show $m \in$ *?A*
by *simp*
qed
qed

61.10 Products indexed over intervals

syntax (*ASCII*)
-from-to-prod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*notation*=‹*binder PROD*››*PROD* - = -..-./ -)› [*0*,*0*,*0*,*10*] *10*)
-from-upto-prod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*notation*=‹*binder PROD*››*PROD* - = -..<-./ -)› [*0*,*0*,*0*,*10*] *10*)
-upt-prod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*notation*=‹*binder PROD*››*PROD* -<-./ -)› [*0*,*0*,*10*] *10*)
-upto-prod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*notation*=‹*binder PROD*››*PROD* -<=-./ -)› [*0*,*0*,*10*] *10*)

syntax (*latex-prod* **output**)
-from-to-prod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(‹($3\prod_{- = -}^{-}$ -)› [*0*,*0*,*0*,*10*] *10*)
-from-upto-prod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(‹($3\prod_{- = -}^{<-}$ -)› [*0*,*0*,*0*,*10*] *10*)
-upt-prod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(‹($3\prod_{- < -}$ -)› [*0*,*0*,*10*] *10*)
-upto-prod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(‹($3\prod_{- \leq -}$ -)› [*0*,*0*,*10*] *10*)

syntax
-from-to-prod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*indent=3 notation=*‹*binder* ∏››∏ -= -..-./ -)› [*0*,*0*,*0*,*10*] *10*)
-from-upto-prod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*indent=3 notation=*‹*binder* ∏››∏ - = -..<-./ -)› [*0*,*0*,*0*,*10*] *10*)
-upt-prod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*indent=3 notation=*‹*binder* ∏››∏ -<-./ -)› [*0*,*0*,*10*] *10*)
-upto-prod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* (‹(‹*indent=3 notation=*‹*binder* ∏››∏ -≤-./ -)› [*0*,*0*,*10*] *10*)

syntax-consts
-from-to-prod -from-upto-prod -upt-prod -upto-prod ⇌ *prod*

translations
∏ *x*=*a*..*b*. *t* ⇌ *CONST prod* (λ*x*. *t*) {*a*..*b*}
∏ *x*=*a*..<*b*. *t* ⇌ *CONST prod* (λ*x*. *t*) {*a*..<*b*}
∏ *i*≤*n*. *t* ⇌ *CONST prod* (λ*i*. *t*) {..*n*}
∏ *i*<*n*. *t* ⇌ *CONST prod* (λ*i*. *t*) {..<*n*}

lemma *prod-int-plus-eq*: *prod int* {*i*..*i*+*j*} = ∏{*int i*..*int* (*i*+*j*)}
by (*induct j*) (*auto simp add*: *atLeastAtMostSuc-conv atLeastAtMostPlus1-int-conv*)

lemma *prod-int-eq*: *prod int* {*i*..*j*} = ∏{*int i*..*int j*}
proof (*cases i* ≤ *j*)
case *True*
then show *?thesis*
by (*metis le-iff-add prod-int-plus-eq*)
next
case *False*
then show *?thesis*
by *auto*
qed

61.10.1 Telescoping products

lemma *prod-telescope*:
fixes *f*::*nat* ⇒ *'a*::*field*
assumes ⋀*i*. *i*≤*n* ⟹ *f* (*Suc i*) ≠ *0*
shows (∏ *i*≤*n*. *f i* / *f* (*Suc i*)) = *f 0* / *f* (*Suc n*)
using *assms* **by** (*induction n*) *auto*

lemma *prod-telescope''*:
fixes *f*::*nat* ⇒ *'a*::*field*
assumes *m* ≤ *n*
assumes ⋀*i*. *i* ∈ {*m*..*n*} ⟹ *f i* ≠ *0*
shows (∏ *i* = *Suc m*..*n*. *f i* / *f* (*i* − *1*)) = *f n* / *f m*
by (*rule dec-induct*[*OF* ‹*m* ≤ *n*›]) (*auto simp add*: *assms*)

lemma *prod-lessThan-telescope*:
fixes f::*nat* ⇒ ′*a*::*field*
assumes $\bigwedge i.\ i \le n \Longrightarrow f\ i \ne 0$
shows $(\prod i<n.\ f\ (Suc\ i)\ /\ f\ i) = f\ n\ /\ f\ 0$
using *assms* **by** (*induction n*) *auto*

lemma *prod-lessThan-telescope′*:
fixes f::*nat* ⇒ ′*a*::*field*
assumes $\bigwedge i.\ i \le n \Longrightarrow f\ i \ne 0$
shows $(\prod i<n.\ f\ i\ /\ f\ (Suc\ i)) = f\ 0\ /\ f\ n$
using *assms* **by** (*induction n*) *auto*

61.11 Efficient folding over intervals

function *fold-atLeastAtMost-nat* **where**
[*simp del*]: *fold-atLeastAtMost-nat f a* (*b*::*nat*) *acc* =
(*if a > b then acc else fold-atLeastAtMost-nat f* (*a+1*) *b* (*f a acc*))
by *pat-completeness auto*
termination by (*relation measure* (λ(-,*a*,*b*,-). *Suc b − a*)) *auto*

lemma *fold-atLeastAtMost-nat*:
assumes *comp-fun-commute f*
shows *fold-atLeastAtMost-nat f a b acc = Finite-Set.fold f acc* {*a..b*}
using *assms*
proof (*induction f a b acc rule*: *fold-atLeastAtMost-nat.induct*, *goal-cases*)
case (*1 f a b acc*)
interpret *comp-fun-commute f* **by** *fact*
show *?case*
proof (*cases a > b*)
case *True*
thus *?thesis* **by** (*subst fold-atLeastAtMost-nat.simps*) *auto*
next
case *False*
with *1* **show** *?thesis*
by (*subst fold-atLeastAtMost-nat.simps*)
(*auto simp*: *atLeastAtMost-insertL*[*symmetric*] *fold-fun-left-comm*)
qed
qed

lemma *sum-atLeastAtMost-code*:
sum f {*a..b*} = *fold-atLeastAtMost-nat* (λ*a acc. f a + acc*) *a b 0*
proof −
have *comp-fun-commute* (λ*a*. (+) (*f a*))
by *unfold-locales* (*auto simp*: *o-def add-ac*)
thus *?thesis*
by (*simp add*: *sum.eq-fold fold-atLeastAtMost-nat o-def*)
qed

lemma *prod-atLeastAtMost-code*:

prod f $\{a..b\}$ = *fold-atLeastAtMost-nat* (λa *acc*. *f a* $*$ *acc*) *a b 1*
proof −
  have *comp-fun-commute* (λa. ($*$) (*f a*))
    by *unfold-locales* (*auto simp*: *o-def mult-ac*)
  thus *?thesis*
    by (*simp add*: *prod.eq-fold fold-atLeastAtMost-nat o-def*)
qed

lemma *pairs-le-eq-Sigma*: $\{(i, j).\ i + j \le m\}$ = *Sigma* (*atMost m*) (λr. *atMost* ($m - r$))
  for *m* :: *nat*
  by *auto*

lemma *sum-up-index-split*: ($\sum k \le m + n.\ f\ k$) = ($\sum k \le m.\ f\ k$) + ($\sum k$ = *Suc* $m..m + n.\ f\ k$)
  by (*metis atLeast0AtMost Suc-eq-plus1 le0 sum.ub-add-nat*)

lemma *Sigma-interval-disjoint*: (*SIGMA i*:*A*. $\{..v\ i\}$) $\cap$ (*SIGMA i*:*A*.$\{v\ i<..w\}$) = $\{\}$
  for *w* :: *'a*::*order*
  by *auto*

lemma *product-atMost-eq-Un*: $A \times \{..m\}$ = (*SIGMA i*:*A*.$\{..m - i\}$) $\cup$ (*SIGMA i*:*A*.$\{m - i<..m\}$)
  for *m* :: *nat*
  by *auto*

lemma *polynomial-product*:
  fixes *x* :: *'a*::*idom*
  assumes *m*: $\bigwedge i.\ i > m \Longrightarrow a\ i = 0$
    and *n*: $\bigwedge j.\ j > n \Longrightarrow b\ j = 0$
  shows ($\sum i \le m.\ (a\ i) * x\ \hat{}\ i$) $*$ ($\sum j \le n.\ (b\ j) * x\ \hat{}\ j$) =
        ($\sum r \le m + n.\ (\sum k \le r.\ (a\ k) * (b\ (r - k))) * x\ \hat{}\ r$)
proof −
  have $\bigwedge i\ j.\ [\![m + n - i < j;\ a\ i \ne 0]\!] \Longrightarrow b\ j = 0$
    by (*meson le-add-diff leI le-less-trans m n*)
  then have §: ($\sum (i,j) \in$(*SIGMA i*:$\{..m+n\}$. $\{m+n - i<..m+n\}$). $a\ i * x\ \hat{}\ i * (b\ j * x\ \hat{}\ j)$) = *0*
    by (*clarsimp simp add*: *sum-Un Sigma-interval-disjoint intro*!: *sum.neutral*)
  have ($\sum i \le m.\ (a\ i) * x\ \hat{}\ i$) $*$ ($\sum j \le n.\ (b\ j) * x\ \hat{}\ j$) = ($\sum i \le m.\ \sum j \le n.\ (a\ i * x\ \hat{}\ i) * (b\ j * x\ \hat{}\ j)$)
    by (*rule sum-product*)
  also have ... = ($\sum i \le m + n.\ \sum j \le n + m.\ a\ i * x\ \hat{}\ i * (b\ j * x\ \hat{}\ j)$)
    using *assms* **by** (*auto simp*: *sum-up-index-split*)
  also have ... = ($\sum r \le m + n.\ \sum j \le m + n - r.\ a\ r * x\ \hat{}\ r * (b\ j * x\ \hat{}\ j)$)
   by (*simp add*: *add-ac sum.Sigma product-atMost-eq-Un sum-Un Sigma-interval-disjoint* §)

also have . . . = ($\sum$ *(i,j)*∈{*(i,j)*. *i+j* ≤ *m+n*}. (*a i* ∗ *x* ^ *i*) ∗ (*b j* ∗ *x* ^ *j*))
  by (*auto simp*: *pairs-le-eq-Sigma sum.Sigma*)
also have ... = ($\sum$ *k*≤*m* + *n*. $\sum$ *i*≤*k*. *a i* ∗ *x* ^ *i* ∗ (*b* (*k* − *i*) ∗ *x* ^ (*k* − *i*)))
  by (*rule sum.triangle-reindex-eq*)
also have . . . = ($\sum$ *r*≤*m* + *n*. ($\sum$ *k*≤*r*. (*a k*) ∗ (*b* (*r* − *k*))) ∗ *x* ^ *r*)
  by (*auto simp*: *algebra-simps sum-distrib-left simp flip*: *power-add intro*!: *sum.cong*)
finally show *?thesis* .
qed

end

62 Decision Procedure for Presburger Arithmetic

theory *Presburger*
imports *Groebner-Basis Set-Interval*
keywords *try0* :: *diag*
begin

ML-file ‹*Tools/Qelim/qelim.ML*›
ML-file ‹*Tools/Qelim/cooper-procedure.ML*›

62.1 The −∞ and +∞ Properties

lemma *minf*:
  ⟦∃ (*z* ::'*a*::*linorder*).∀ *x*<*z*. *P x* = *P' x*; ∃ *z*.∀ *x*<*z*. *Q x* = *Q' x*⟧
    ⟹ ∃ *z*.∀ *x*<*z*. (*P x* ∧ *Q x*) = (*P' x* ∧ *Q' x*)
  ⟦∃ (*z* ::'*a*::*linorder*).∀ *x*<*z*. *P x* = *P' x*; ∃ *z*.∀ *x*<*z*. *Q x* = *Q' x*⟧
    ⟹ ∃ *z*.∀ *x*<*z*. (*P x* ∨ *Q x*) = (*P' x* ∨ *Q' x*)
  ∃ (*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* = *t*) = *False*
  ∃ (*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≠ *t*) = *True*
  ∃ (*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* < *t*) = *True*
  ∃ (*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≤ *t*) = *True*
  ∃ (*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* > *t*) = *False*
  ∃ (*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≥ *t*) = *False*
  ∃ *z*.∀ (*x*::'*b*::{*linorder*,*plus*,*Rings.dvd*})<*z*. (*d dvd x* + *s*) = (*d dvd x* + *s*)
  ∃ *z*.∀ (*x*::'*b*::{*linorder*,*plus*,*Rings.dvd*})<*z*. (¬ *d dvd x* + *s*) = (¬ *d dvd x* + *s*)
  ∃ *z*.∀ *x*<*z*. *F* = *F*
proof *safe*
  fix *z1 z2*
  assume ∀ *x*<*z1*. *P x* = *P' x* **and** ∀ *x*<*z2*. *Q x* = *Q' x*
  then have ∀ *x* < *min z1 z2*. (*P x* ∧ *Q x*) = (*P' x* ∧ *Q' x*)
    by *simp*
  then show ∃ *z*. ∀ *x*<*z*. (*P x* ∧ *Q x*) = (*P' x* ∧ *Q' x*)
    by *blast*
next
  fix *z1 z2*
  assume ∀ *x*<*z1*. *P x* = *P' x* **and** ∀ *x*<*z2*. *Q x* = *Q' x*
  then have ∀ *x* < *min z1 z2*. (*P x* ∨ *Q x*) = (*P' x* ∨ *Q' x*)
    by *simp*

then show $\exists z.\ \forall x{<}z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
  by *blast*
next
  have $\forall x{<}t.\ x \leq t$
  by *fastforce*
  then show $\exists z.\ \forall x{<}z.\ (x \leq t) = \mathit{True}$
  by *auto*
next
  have $\forall x{<}t.\ \neg\ t < x$
  by *fastforce*
  then show $\exists z.\ \forall x{<}z.\ (t < x) = \mathit{False}$
  by *auto*
next
  have $\forall x{<}t.\ \neg\ t \leq x$
  by *fastforce*
  then show $\exists z.\ \forall x{<}z.\ (t \leq x) = \mathit{False}$
  by *auto*
qed *auto*

lemma *pinf*:
$[\![\exists (z ::'a::\mathit{linorder}).\forall x{>}z.\ P\ x = P'\ x;\ \exists z.\forall x{>}z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x{>}z.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
$[\![\exists (z ::'a::\mathit{linorder}).\forall x{>}z.\ P\ x = P'\ x;\ \exists z.\forall x{>}z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x{>}z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
$\exists (z ::'a::\{\mathit{linorder}\}).\forall x{>}z.(x = t) = \mathit{False}$
$\exists (z ::'a::\{\mathit{linorder}\}).\forall x{>}z.(x \neq t) = \mathit{True}$
$\exists (z ::'a::\{\mathit{linorder}\}).\forall x{>}z.(x < t) = \mathit{False}$
$\exists (z ::'a::\{\mathit{linorder}\}).\forall x{>}z.(x \leq t) = \mathit{False}$
$\exists (z ::'a::\{\mathit{linorder}\}).\forall x{>}z.(x > t) = \mathit{True}$
$\exists (z ::'a::\{\mathit{linorder}\}).\forall x{>}z.(x \geq t) = \mathit{True}$
$\exists z.\forall (x::'b::\{\mathit{linorder},\mathit{plus},\mathit{Rings.dvd}\}){>}z.\ (d\ \mathit{dvd}\ x + s) = (d\ \mathit{dvd}\ x + s)$
$\exists z.\forall (x::'b::\{\mathit{linorder},\mathit{plus},\mathit{Rings.dvd}\}){>}z.\ (\neg\ d\ \mathit{dvd}\ x + s) = (\neg\ d\ \mathit{dvd}\ x + s)$
$\exists z.\forall x{>}z.\ F = F$
proof *safe*
  fix *z1 z2*
  assume $\forall x{>}z1.\ P\ x = P'\ x$ **and** $\forall x{>}z2.\ Q\ x = Q'\ x$
  then have $\forall x > \mathit{max}\ z1\ z2.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
  by *simp*
  then show $\exists z.\ \forall x{>}z.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
  by *blast*
next
  fix *z1 z2*
  assume $\forall x{>}z1.\ P\ x = P'\ x$ **and** $\forall x{>}z2.\ Q\ x = Q'\ x$
  then have $\forall x > \mathit{max}\ z1\ z2.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
  by *simp*
  then show $\exists z.\ \forall x{>}z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
  by *blast*
next
  have $\forall x{>}t.\ \neg\ x < t$

by *fastforce*
then show $\exists z.\ \forall x>z.\ x < t = False$
by *blast*
next
have $\forall x>t.\ \neg\ x \leq t$
by *fastforce*
then show $\exists z.\ \forall x>z.\ x \leq t = False$
by *blast*
next
have $\forall x>t.\ t \leq x$
by *fastforce*
then show $\exists z.\ \forall x>z.\ t \leq x = True$
by *blast*
qed *auto*

lemma *inf-period*:
$[\![\forall x\ k.\ P\ x = P\ (x - k*D);\ \forall x\ k.\ Q\ x = Q\ (x - k*D)]\!]$
$\Longrightarrow \forall x\ k.\ (P\ x \wedge Q\ x) = (P\ (x - k*D) \wedge Q\ (x - k*D))$
$[\![\forall x\ k.\ P\ x = P\ (x - k*D);\ \forall x\ k.\ Q\ x = Q\ (x - k*D)]\!]$
$\Longrightarrow \forall x\ k.\ (P\ x \vee Q\ x) = (P\ (x - k*D) \vee Q\ (x - k*D))$
$(d::'a::\{comm\text{-}ring,Rings.dvd\})\ dvd\ D \Longrightarrow \forall x\ k.\ (d\ dvd\ x + t) = (d\ dvd\ (x - k*D) + t)$
$(d::'a::\{comm\text{-}ring,Rings.dvd\})\ dvd\ D \Longrightarrow \forall x\ k.\ (\neg d\ dvd\ x + t) = (\neg d\ dvd\ (x - k*D) + t)$
$\forall x\ k.\ F = F$
apply (*auto elim*!: *dvdE simp add*: *algebra-simps*)
unfolding *mult.assoc* [*symmetric*] *distrib-right* [*symmetric*] *left-diff-distrib* [*symmetric*]
unfolding *dvd-def mult.commute* [*of d*]
by *auto*

62.2 The A and B sets

lemma *bset*:
$[\![\forall x.(\forall j \in \{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x - D) \wedge Q\ (x - D))$
$[\![\forall x.(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x - D) \vee Q\ (x - D))$
$[\![D>0;\ t - 1\in B]\!] \Longrightarrow (\forall x.(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow (x = t) \longrightarrow (x - D = t))$
$[\![D>0\ ;\ t \in B]\!] \Longrightarrow(\forall (x::int).(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow (x \neq t) \longrightarrow (x - D \neq t))$
$D>0 \Longrightarrow (\forall (x::int).(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow (x < t) \longrightarrow (x - D < t))$
$D>0 \Longrightarrow (\forall (x::int).(\forall j\in\{1\ ..\ D\}.\ \forall b\in B.\ x \neq b + j)\longrightarrow (x \leq t) \longrightarrow (x - D \leq t))$

$[\![D>0 ; t \in B]\!] \Longrightarrow (\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x > t) \longrightarrow (x - D > t))$
$[\![D>0 ; t - 1 \in B]\!] \Longrightarrow (\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x \geq t) \longrightarrow (x - D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (d\ dvd\ x+t) \longrightarrow (d\ dvd\ (x - D) + t))$
$d\ dvd\ D \Longrightarrow (\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (\neg d\ dvd\ x+t) \longrightarrow (\neg d\ dvd\ (x - D) + t))$
$\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow F \longrightarrow F$
proof (*blast*, *blast*)
assume *dp*: $D > 0$ **and** *tB*: $t - 1\in B$
show $(\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x = t) \longrightarrow (x - D = t))$
apply (*rule allI*, *rule impI*,*erule ballE*[**where** *x=1*],*erule ballE*[**where** $x=t - 1$])
apply *algebra* **using** *dp tB* **by** *simp-all*
next
assume *dp*: $D > 0$ **and** *tB*: $t \in B$
show $(\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x \neq t) \longrightarrow (x - D \neq t))$
apply (*rule allI*, *rule impI*,*erule ballE*[**where** *x=D*],*erule ballE*[**where** *x=t*])
apply *algebra*
using *dp tB* **by** *simp-all*
next
assume *dp*: $D > 0$ **thus** $(\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x < t) \longrightarrow (x - D < t))$ **by** *arith*
next
assume *dp*: $D > 0$ **thus** $\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x \leq t) \longrightarrow (x - D \leq t)$ **by** *arith*
next
assume *dp*: $D > 0$ **and** *tB*:$t \in B$
{fix *x* **assume** *nob*: $\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j$ **and** *g*: $x > t$ **and** *ng*: $\neg (x - D) > t$
hence $x - t \leq D$ **and** $1 \leq x - t$ **by** *simp+*
hence $\exists j \in \{1 .. D\}. x - t = j$ **by** *auto*
hence $\exists j \in \{1 .. D\}. x = t + j$ **by** (*simp add*: *algebra-simps*)
with *nob tB* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x > t) \longrightarrow (x - D > t)$ **by** *blast*
next
assume *dp*: $D > 0$ **and** *tB*:$t - 1\in B$
{fix *x* **assume** *nob*: $\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j$ **and** *g*: $x \geq t$ **and** *ng*: $\neg (x - D) \geq t$
hence $x - (t - 1) \leq D$ **and** $1 \leq x - (t - 1)$ **by** *simp+*
hence $\exists j \in \{1 .. D\}. x - (t - 1) = j$ **by** *auto*
hence $\exists j \in \{1 .. D\}. x = (t - 1) + j$ **by** (*simp add*: *algebra-simps*)
with *nob tB* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (x \geq t) \longrightarrow (x - D \geq t)$ **by** *blast*
next
assume *d*: *d dvd D*
{fix *x* **assume** *H*: $d\ dvd\ x + t$ **with** *d* **have** $d\ dvd\ (x - D) + t$ **by** *algebra***}**
thus $\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in B. x \neq b + j) \longrightarrow (d\ dvd\ x+t) \longrightarrow (d\ dvd\ (x$

$-$ *D*) + *t*) **by** *simp*
next
assume *d*: *d dvd D*
{fix *x* **assume** *H*: ¬(*d dvd x* + *t*) **with** *d* **have** ¬ *d dvd* (*x* − *D*) + *t*
by (*clarsimp simp add*: *dvd-def*,*erule-tac x*= *ka* + *k* **in** *allE*,*simp add*: *algebra-simps*)**}**
thus ∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*B*. *x* ≠ *b* + *j*)⟶ (¬*d dvd x*+*t*) ⟶ (¬*d dvd* (*x* − *D*) + *t*) **by** *auto*
qed *blast*

lemma *aset*:
⟦∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ *P x* ⟶ *P*(*x* + *D*) ;
∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ *Q x* ⟶ *Q*(*x* + *D*)⟧ ⟹
∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*) ⟶ (*P x* ∧ *Q x*) ⟶ (*P*(*x* + *D*) ∧ *Q* (*x* + *D*))
⟦∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ *P x* ⟶ *P*(*x* + *D*) ;
∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ *Q x* ⟶ *Q*(*x* + *D*)⟧ ⟹
∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*P x* ∨ *Q x*) ⟶ (*P*(*x* + *D*) ∨ *Q* (*x* + *D*))
⟦*D*>*0*; *t* + *1*∈ *A*⟧ ⟹ (∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* = *t*) ⟶ (*x* + *D* = *t*))
⟦*D*>*0* ; *t* ∈ *A*⟧ ⟹(∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* ≠ *t*) ⟶ (*x* + *D* ≠ *t*))
⟦*D*>*0*; *t*∈ *A*⟧ ⟹(∀(*x*::*int*). (∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* < *t*) ⟶ (*x* + *D* < *t*))
⟦*D*>*0*; *t* + *1* ∈ *A*⟧ ⟹ (∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* ≤ *t*) ⟶ (*x* + *D* ≤ *t*))
D>*0* ⟹(∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* > *t*) ⟶ (*x* + *D* > *t*))
D>*0* ⟹(∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* ≥ *t*) ⟶ (*x* + *D* ≥ *t*))
d dvd D ⟹(∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*d dvd x*+*t*) ⟶ (*d dvd* (*x* + *D*) + *t*))
d dvd D ⟹(∀(*x*::*int*).(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (¬*d dvd x*+*t*) ⟶ (¬ *d dvd* (*x* + *D*) + *t*))
∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*) ⟶ *F* ⟶ *F*
proof (*blast*, *blast*)
assume *dp*: *D* > *0* **and** *tA*: *t* + *1* ∈ *A*
show (∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* = *t*) ⟶ (*x* + *D* = *t*))
apply (*rule allI*, *rule impI*,*erule ballE*[**where** *x*=*1*],*erule ballE*[**where** *x*=*t* + *1*])
using *dp tA* **by** *simp-all*
next
assume *dp*: *D* > *0* **and** *tA*: *t* ∈ *A*
show (∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* ≠ *t*) ⟶ (*x* + *D* ≠ *t*))
apply (*rule allI*, *rule impI*,*erule ballE*[**where** *x*=*D*],*erule ballE*[**where** *x*=*t*])
using *dp tA* **by** *simp-all*
next
assume *dp*: *D* > *0* **thus** (∀*x*.(∀*j*∈{*1* .. *D*}. ∀*b*∈*A*. *x* ≠ *b* − *j*)⟶ (*x* > *t*) ⟶

$(x + D > t))$ **by** *arith*
next
assume *dp*: $D > 0$ **thus** $\forall x.(\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j) \longrightarrow (x \geq t) \longrightarrow (x + D \geq t)$ **by** *arith*
next
assume *dp*: $D > 0$ **and** *tA*:$t \in A$
{fix x **assume** *nob*: $\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j$ **and** *g*: $x < t$ **and** *ng*: $\neg (x + D) < t$
hence $t - x \leq D$ **and** $1 \leq t - x$ **by** *simp+*
hence $\exists j \in \{1 .. D\}. t - x = j$ **by** *auto*
hence $\exists j \in \{1 .. D\}. x = t - j$ **by** (*auto simp add*: *algebra-simps*)
with *nob tA* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j) \longrightarrow (x < t) \longrightarrow (x + D < t)$ **by** *blast*
next
assume *dp*: $D > 0$ **and** *tA*:$t + 1\in A$
{fix x **assume** *nob*: $\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j$ **and** *g*: $x \leq t$ **and** *ng*: $\neg (x + D) \leq t$
hence $(t + 1) - x \leq D$ **and** $1 \leq (t + 1) - x$ **by** (*simp-all add*: *algebra-simps*)
hence $\exists j \in \{1 .. D\}. (t + 1) - x = j$ **by** *auto*
hence $\exists j \in \{1 .. D\}. x = (t + 1) - j$ **by** (*auto simp add*: *algebra-simps*)
with *nob tA* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j) \longrightarrow (x \leq t) \longrightarrow (x + D \leq t)$ **by** *blast*
next
assume *d*: *d dvd D*
have $\bigwedge x.$ *d dvd* $x + t \Longrightarrow$ *d dvd* $x + D + t$
proof −
fix x
assume *H*: *d dvd* $x + t$
then obtain *ka* **where** $x + t = d * ka$
unfolding *dvd-def* **by** *blast*
moreover from *d* **obtain** *k* **where** $*$:$D = d * k$
unfolding *dvd-def* **by** *blast*
ultimately have $x + d * k + t = d * (ka + k)$
by (*simp add*: *algebra-simps*)
then show *d dvd* $(x + D) + t$
using $*$ **unfolding** *dvd-def* **by** *blast*
qed
thus $\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j) \longrightarrow (d\ dvd\ x+t) \longrightarrow (d\ dvd\ (x + D) + t)$ **by** *simp*
next
assume *d*: *d dvd D*
{fix x **assume** *H*: $\neg(d\ dvd\ x + t)$ **with** *d* **have** $\neg d\ dvd\ (x + D) + t$
using *dvd-add-left-iff*[*OF d*, *of x+t*] **by** (*simp add*: *algebra-simps*)**}**
thus $\forall (x::int).(\forall j\in\{1 .. D\}. \forall b\in A. x \neq b - j) \longrightarrow (\neg d\ dvd\ x+t) \longrightarrow (\neg d\ dvd\ (x + D) + t)$ **by** *auto*
qed *blast*

62.3 Cooper's Theorem $-\infty$ and $+\infty$ Version

62.3.1 First some trivial facts about periodic sets or predicates

lemma *periodic-finite-ex*:
  assumes *dpos*: $(0::int) < d$ **and** *modd*: $\forall x\ k.\ P\ x = P(x - k*d)$
  shows $(\exists x.\ P\ x) = (\exists j \in \{1..d\}.\ P\ j)$
  (**is** *?LHS* = *?RHS*)
proof
  assume *?LHS*
  then obtain *x* **where** *P*: *P x* ..
   have $x\ mod\ d = x - (x\ div\ d)*d$ **by**(*simp add*:*mult-div-mod-eq* [*symmetric*] *ac-simps eq-diff-eq*)
  hence *Pmod*: $P\ x = P(x\ mod\ d)$ **using** *modd* **by** *simp*
  show *?RHS*
  proof (*cases*)
    assume $x\ mod\ d = 0$
    hence *P 0* **using** *P Pmod* **by** *simp*
    moreover have $P\ 0 = P(0 - (-1)*d)$ **using** *modd* **by** *blast*
    ultimately have *P d* **by** *simp*
    moreover have $d \in \{1..d\}$ **using** *dpos* **by** *simp*
    ultimately show *?RHS* ..
  next
    assume *not0*: $x\ mod\ d \neq 0$
    have $P(x\ mod\ d)$ **using** *dpos P Pmod* **by** *simp*
    moreover have $x\ mod\ d \in \{1..d\}$
    proof −
      from *dpos* **have** $0 \leq x\ mod\ d$ **by**(*rule pos-mod-sign*)
      moreover from *dpos* **have** $x\ mod\ d < d$ **by**(*rule pos-mod-bound*)
      ultimately show *?thesis* **using** *not0* **by** *simp*
    qed
    ultimately show *?RHS* ..
  qed
qed *auto*

62.3.2 The $-\infty$ Version

lemma *decr-lemma*: $0 < (d::int) \Longrightarrow x - (|x - z| + 1) * d < z$
  by (*induct rule*: *int-gr-induct*) (*simp-all add*: *int-distrib*)

lemma *incr-lemma*: $0 < (d::int) \Longrightarrow z < x + (|x - z| + 1) * d$
  by (*induct rule*: *int-gr-induct*) (*simp-all add*: *int-distrib*)

lemma *decr-mult-lemma*:
  assumes *dpos*: $(0::int) < d$ **and** *minus*: $\forall x.\ P\ x \longrightarrow P(x - d)$ **and** *knneg*: $0 <= k$
  shows $\forall x.\ P\ x \longrightarrow P(x - k*d)$
using *knneg*
proof (*induct rule*:*int-ge-induct*)
  case *base* **thus** *?case* **by** *simp*

next
 case (*step i*)
 {fix x
  have $P\ x \longrightarrow P\ (x - i * d)$ **using** *step.hyps* **by** *blast*
  also have $\ldots \longrightarrow P(x - (i + 1) * d)$ **using** *minus*[*THEN spec*, *of* $x - i * d$]
   by (*simp add*: *algebra-simps*)
  ultimately have $P\ x \longrightarrow P(x - (i + 1) * d)$ **by** *blast***}**
 thus *?case* **..**
qed

lemma *minusinfinity*:
 assumes *dpos*: $0 < d$ **and**
  P1eqP1: $\forall x\ k.\ P1\ x = P1(x - k{*}d)$ **and** *ePeqP1*: $\exists z{::}int.\ \forall x.\ x < z \longrightarrow (P\ x = P1\ x)$
 shows $(\exists x.\ P1\ x) \longrightarrow (\exists x.\ P\ x)$
proof
 assume *eP1*: $\exists x.\ P1\ x$
 then obtain x **where** *P1*: $P1\ x$ **..**
 from *ePeqP1* **obtain** z **where** *P1eqP*: $\forall x.\ x < z \longrightarrow (P\ x = P1\ x)$ **..**
 let $?w = x - (|x - z| + 1) * d$
 from *dpos* **have** *w*: $?w < z$ **by**(*rule decr-lemma*)
 have $P1\ x = P1\ ?w$ **using** *P1eqP1* **by** *blast*
 also have $\ldots = P(?w)$ **using** *w P1eqP* **by** *blast*
 finally have $P\ ?w$ **using** *P1* **by** *blast*
 thus $\exists x.\ P\ x$ **..**
qed

lemma *cpmi*:
 assumes *dp*: $0 < D$ **and** *p1*:$\exists z.\ \forall\ x< z.\ P\ x = P'\ x$
 and *nb*:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b{::}int) \in B.\ x \neq b{+}j) \longrightarrow P\ (x) \longrightarrow P\ (x - D)$
 and *pd*: $\forall\ x\ k.\ P'\ x = P'\ (x{-}k{*}D)$
 shows $(\exists x.\ P\ x) = ((\exists j \in \{1..D\}\ .\ P'\ j) \vee (\exists j \in \{1..D\}.\ \exists\ b \in B.\ P\ (b{+}j)))$
   (**is** $?L = (?R1 \vee ?R2)$)
proof−
 {assume *?R2* **hence** *?L* **by** *blast***}**
 moreover
 {assume *H*:*?R1* **hence** *?L* **using** *minusinfinity*[*OF dp pd p1*] *periodic-finite-ex*[*OF dp pd*] **by** *simp***}**
 moreover
 { fix x
  assume *P*: $P\ x$ **and** *H*: $\neg\ ?R2$
  {fix y **assume** $\neg\ (\exists j\in\{1..D\}.\ \exists b\in B.\ P\ (b + j))$ **and** *P*: $P\ y$
   hence $\neg(\exists (j{::}int) \in \{1..D\}.\ \exists (b{::}int) \in B.\ y = b{+}j)$ **by** *auto*
   with *nb P* **have** $P\ (y - D)$ **by** *auto* **}**
  hence $\forall x.\ \neg(\exists (j{::}int) \in \{1..D\}.\ \exists (b{::}int) \in B.\ P(b{+}j)) \longrightarrow P\ (x) \longrightarrow P\ (x - D)$ **by** *blast*
  with *H P* **have** *th*: $\forall x.\ P\ x \longrightarrow P\ (x - D)$ **by** *auto*
  from *p1* **obtain** z **where** *z*: $\forall x.\ x < z \longrightarrow (P\ x = P'\ x)$ **by** *blast*
  let $?y = x - (|x - z| + 1){*}D$

have *zp*: $0 <= (|x - z| + 1)$ **by** *arith*
from *dp* **have** *yz*: $?y < z$ **using** *decr-lemma*[*OF dp*] **by** *simp*
from *z*[*rule-format*, *OF yz*] *decr-mult-lemma*[*OF dp th zp*, *rule-format*, *OF P*]
have *th2*: $P'\ ?y$ **by** *auto*
with *periodic-finite-ex*[*OF dp pd*]
have *?R1* **by** *blast*}
ultimately show *?thesis* **by** *blast*
qed

62.3.3 The $+\infty$ Version

lemma *plusinfinity*:
assumes *dpos*: $(0::int) < d$ **and**
P1eqP1: $\forall x\ k.\ P'\ x = P'(x - k*d)$ **and** *ePeqP1*: $\exists\ z.\ \forall\ x>z.\ P\ x = P'\ x$
shows $(\exists\ x.\ P'\ x) \longrightarrow (\exists\ x.\ P\ x)$
proof
assume *eP1*: $\exists x.\ P'\ x$
then obtain *x* **where** *P1*: $P'\ x$ **..**
from *ePeqP1* **obtain** *z* **where** *P1eqP*: $\forall x>z.\ P\ x = P'\ x$ **..**
let $?w' = x + (|x - z| + 1) * d$
let $?w = x - (- (|x - z| + 1)) * d$
have *ww'*[*simp*]: $?w = ?w'$ **by** (*simp add*: *algebra-simps*)
from *dpos* **have** *w*: $?w > z$ **by**(*simp only*: *ww' incr-lemma*)
hence $P'\ x = P'\ ?w$ **using** *P1eqP1* **by** *blast*
also have $\ldots = P(?w)$ **using** *w P1eqP* **by** *blast*
finally have $P\ ?w$ **using** *P1* **by** *blast*
thus $\exists x.\ P\ x$ **..**
qed

lemma *incr-mult-lemma*:
assumes *dpos*: $(0::int) < d$ **and** *plus*: $\forall x::int.\ P\ x \longrightarrow P(x + d)$ **and** *knneg*: $0 <= k$
shows $\forall x.\ P\ x \longrightarrow P(x + k*d)$
using *knneg*
proof (*induct rule*:*int-ge-induct*)
case *base* **thus** *?case* **by** *simp*
next
case (*step i*)
{**fix** *x*
have $P\ x \longrightarrow P\ (x + i * d)$ **using** *step.hyps* **by** *blast*
also have $\ldots \longrightarrow P(x + (i + 1) * d)$ **using** *plus*[*THEN spec*, *of* $x + i * d$]
by (*simp add*:*int-distrib ac-simps*)
ultimately have $P\ x \longrightarrow P(x + (i + 1) * d)$ **by** *blast*}
thus *?case* **..**
qed

lemma *cppi*:
assumes *dp*: $0 < D$ **and** *p1*:$\exists z.\ \forall\ x> z.\ P\ x = P'\ x$
and *nb*:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b::int) \in A.\ x \neq b - j) \longrightarrow P\ (x) \longrightarrow P\ (x + D)$

and *pd*: ∀ *x k*. *P′ x*= *P′* (*x*−*k*∗*D*)
shows (∃ *x*. *P x*) = ((∃ *j* ∈ {*1*..*D*} . *P′ j*) ∨ (∃ *j* ∈ {*1*..*D*}. ∃ *b*∈ *A*. *P* (*b* − *j*)))
(**is** *?L* = (*?R1* ∨ *?R2*))
proof−
{**assume** *?R2* **hence** *?L* **by** *blast*}
moreover
{**assume** *H*:*?R1* **hence** *?L* **using** *plusinfinity*[*OF dp pd p1*] *periodic-finite-ex*[*OF dp pd*] **by** *simp*}
moreover
{ **fix** *x*
assume *P*: *P x* **and** *H*: ¬ *?R2*
{**fix** *y* **assume** ¬ (∃ *j*∈{*1*..*D*}. ∃ *b*∈*A*. *P* (*b* − *j*)) **and** *P*: *P y*
hence ¬(∃ (*j*::*int*) ∈ {*1*..*D*}. ∃ (*b*::*int*) ∈ *A*. *y* = *b* − *j*) **by** *auto*
with *nb P* **have** *P* (*y* + *D*) **by** *auto* }
hence ∀ *x*. ¬(∃ (*j*::*int*) ∈ {*1*..*D*}. ∃ (*b*::*int*) ∈ *A*. *P*(*b*−*j*)) ⟶ *P* (*x*) ⟶ *P* (*x* + *D*) **by** *blast*
with *H P* **have** *th*: ∀ *x*. *P x* ⟶ *P* (*x* + *D*) **by** *auto*
from *p1* **obtain** *z* **where** *z*: ∀ *x*. *x* > *z* ⟶ (*P x* = *P′ x*) **by** *blast*
let *?y* = *x* + (|*x* − *z*| + *1*)∗*D*
have *zp*: *0* <= (|*x* − *z*| + *1*) **by** *arith*
from *dp* **have** *yz*: *?y* > *z* **using** *incr-lemma*[*OF dp*] **by** *simp*
from *z*[*rule-format*, *OF yz*] *incr-mult-lemma*[*OF dp th zp*, *rule-format*, *OF P*] **have** *th2*: *P′ ?y* **by** *auto*
with *periodic-finite-ex*[*OF dp pd*]
have *?R1* **by** *blast*}
ultimately show *?thesis* **by** *blast*
qed

lemma *simp-from-to*: {*i*..*j*::*int*} = (*if j* < *i then* {} *else insert i* {*i*+*1*..*j*})
apply(*simp add*:*atLeastAtMost-def atLeast-def atMost-def*)
apply(*fastforce*)
done

theorem *unity-coeff-ex*: (∃ (*x*::′*a*::{*semiring-0*,*Rings.dvd*}). *P* (*l* ∗ *x*)) ≡ (∃ *x*. *l dvd* (*x* + *0*) ∧ *P x*)
unfolding *dvd-def* **by** (*rule eq-reflection*, *rule iffI*) *auto*

lemma *zdvd-mono*:
fixes *k m t* :: *int*
assumes *k* ≠ *0*
shows *m dvd t* ≡ *k* ∗ *m dvd k* ∗ *t*
using *assms* **by** *simp*

lemma *uminus-dvd-conv*:
fixes *d t* :: *int*
shows *d dvd t* ≡ − *d dvd t* **and** *d dvd t* ≡ *d dvd* − *t*
by *simp-all*

Theorems for transforming predicates on nat to predicates on *int*

lemma *zdiff-int-split*: P (*int* $(x - y)) =$
$((y \leq x \longrightarrow P$ (*int* $x -$ *int* $y)) \wedge (x < y \longrightarrow P\ 0))$
by (*cases* $y \leq x$) (*simp-all add*: *of-nat-diff*)

Specific instances of congruence rules, to prevent simplifier from looping.

theorem *imp-le-cong*:
$[\![x = x'; 0 \leq x' \Longrightarrow P = P']\!] \Longrightarrow (0 \leq (x::int) \longrightarrow P) = (0 \leq x' \longrightarrow P')$
by *simp*

theorem *conj-le-cong*:
$[\![x = x'; 0 \leq x' \Longrightarrow P = P']\!] \Longrightarrow (0 \leq (x::int) \wedge P) = (0 \leq x' \wedge P')$
by (*simp cong*: *conj-cong*)

ML-file ‹*Tools/Qelim/cooper.ML*›

method-setup *presburger* = ‹
```
let
  fun keyword k = Scan.lift (Args.$$$ k -- Args.colon) >> K ()
  fun simple-keyword k = Scan.lift (Args.$$$ k) >> K ()
  val addN = add
  val delN = del
  val elimN = elim
  val any-keyword = keyword addN || keyword delN || simple-keyword elimN
  val thms = Scan.repeats (Scan.unless any-keyword Attrib.multi-thm)
in
  Scan.optional (simple-keyword elimN >> K false) true --
  Scan.optional (keyword addN |-- thms) [] --
  Scan.optional (keyword delN |-- thms) [] >>
  (fn ((elim, add-ths), del-ths) => fn ctxt =>
    SIMPLE-METHOD' (Cooper.tac elim add-ths del-ths ctxt))
end
```
› *Cooper's algorithm for Presburger arithmetic*

declare *mod-eq-0-iff-dvd* [*presburger*]
declare *mod-by-Suc-0* [*presburger*]
declare *mod-0* [*presburger*]
declare *mod-by-1* [*presburger*]
declare *mod-self* [*presburger*]
declare *div-by-0* [*presburger*]
declare *mod-by-0* [*presburger*]
declare *mod-div-trivial* [*presburger*]
declare *mult-div-mod-eq* [*presburger*]
declare *div-mult-mod-eq* [*presburger*]
declare *mod-mult-self1* [*presburger*]
declare *mod-mult-self2* [*presburger*]
declare *mod2-Suc-Suc* [*presburger*]
declare *not-mod-2-eq-0-eq-1* [*presburger*]
declare *nat-zero-less-power-iff* [*presburger*]

lemma [*presburger*, *algebra*]: *m mod 2* = (*1*::*nat*) ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod 2* = *Suc 0* ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod* (*Suc* (*Suc 0*)) = (*1*::*nat*) ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod* (*Suc* (*Suc 0*)) = *Suc 0* ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod 2* = (*1*::*int*) ⟷ ¬ *2 dvd m* **by** *presburger*

context *semiring-parity*
begin

declare *even-mult-iff* [*presburger*]

declare *even-power* [*presburger*]

lemma [*presburger*]:
even (*a* + *b*) ⟷ *even a* ∧ *even b* ∨ *odd a* ∧ *odd b*
by *auto*

end

context *ring-parity*
begin

declare *even-minus* [*presburger*]

end

context *linordered-idom*
begin

declare *zero-le-power-eq* [*presburger*]

declare *zero-less-power-eq* [*presburger*]

declare *power-less-zero-eq* [*presburger*]

declare *power-le-zero-eq* [*presburger*]

end

declare *even-Suc* [*presburger*]

lemma [*presburger*]:
Suc n div Suc (*Suc 0*) = *n div Suc* (*Suc 0*) ⟷ *even n*
by *presburger*

declare *even-diff-nat* [*presburger*]

lemma [*presburger*]:
fixes k :: *int*
shows $(k + 1)$ *div* $2 = k$ *div* $2 \longleftrightarrow$ *even* k
by *presburger*

lemma [*presburger*]:
fixes k :: *int*
shows $(k + 1)$ *div* $2 = k$ *div* $2 + 1 \longleftrightarrow$ *odd* k
by *presburger*

lemma [*presburger*]:
even $n \longleftrightarrow$ *even* (*int* n)
by *simp*

62.4 Nice facts about division by $4::'a$

lemma *even-even-mod-4-iff*:
even (n::*nat*) $\longleftrightarrow$ *even* (n *mod* 4)
by *presburger*

lemma *odd-mod-4-div-2*:
n *mod* $4 = (3$::*nat*$) \Longrightarrow$ *odd* ((n − *Suc* 0) *div* 2)
by *presburger*

lemma *even-mod-4-div-2*:
n *mod* $4 =$ *Suc* $0 \Longrightarrow$ *even* ((n − *Suc* 0) *div* 2)
by *presburger*

62.5 Try0

ML-file ‹*Tools/try0.ML*›

end

63 Bindings to Satisfiability Modulo Theories (SMT) solvers based on SMT-LIB 2

theory *SMT*
imports *Numeral-Simprocs*
keywords *smt-status* :: *diag*
begin

63.1 A skolemization tactic and proof method

lemma *ex-iff-push*: $(\exists\, y.\ P \longleftrightarrow Q\ y) \longleftrightarrow (P \longrightarrow (\exists\, y.\ Q\ y)) \wedge ((\forall\, y.\ Q\ y) \longrightarrow P)$
by *metis*

ML ‹

fun moura-tac ctxt =
TRY o Atomize-Elim.atomize-elim-tac ctxt THEN′
REPEAT o EqSubst.eqsubst-tac ctxt [0]
@{thms choice-iff[symmetric] bchoice-iff[symmetric]} THEN′
TRY o Simplifier.asm-full-simp-tac
(clear-simpset ctxt addsimps @{thms all-simps ex-simps ex-iff-push}) THEN-ALL-NEW
Metis-Tactic.metis-tac (take 1 ATP-Proof-Reconstruct.partial-type-encs)
ATP-Proof-Reconstruct.default-metis-lam-trans ctxt []
›

method-setup *moura* = ‹
Scan.succeed (SIMPLE-METHOD′ o moura-tac)
› *solve skolemization goals, especially those arising from Z3 proofs*

hide-fact (**open**) *ex-iff-push*

63.2 Triggers for quantifier instantiation

Some SMT solvers support patterns as a quantifier instantiation heuristics. Patterns may either be positive terms (tagged by "pat") triggering quantifier instantiations – when the solver finds a term matching a positive pattern, it instantiates the corresponding quantifier accordingly – or negative terms (tagged by "nopat") inhibiting quantifier instantiations. A list of patterns of the same kind is called a multipattern, and all patterns in a multipattern are considered conjunctively for quantifier instantiation. A list of multipatterns is called a trigger, and their multipatterns act disjunctively during quantifier instantiation. Each multipattern should mention at least all quantified variables of the preceding quantifier block.

typedecl *′a symb-list*

consts
Symb-Nil :: *′a symb-list*
Symb-Cons :: *′a* ⇒ *′a symb-list* ⇒ *′a symb-list*

typedecl *pattern*

consts
pat :: *′a* ⇒ *pattern*
nopat :: *′a* ⇒ *pattern*

definition *trigger* :: *pattern symb-list symb-list* ⇒ *bool* ⇒ *bool* **where**
trigger - P = P

63.3 Higher-order encoding

Application is made explicit for constants occurring with varying numbers of arguments. This is achieved by the introduction of the following constant.

definition *fun-app* :: $'a \Rightarrow 'a$ **where** *fun-app f* = *f*

Some solvers support a theory of arrays which can be used to encode higher-order functions. The following set of lemmas specifies the properties of such (extensional) arrays.

lemmas *array-rules* = *ext fun-upd-apply fun-upd-same fun-upd-other fun-upd-upd fun-app-def*

63.4 Normalization

lemma *case-bool-if*[*abs-def*]: *case-bool x y P* = (*if P then x else y*)
  by *simp*

lemmas *Ex1-def-raw* = *Ex1-def*[*abs-def*]
lemmas *Ball-def-raw* = *Ball-def*[*abs-def*]
lemmas *Bex-def-raw* = *Bex-def*[*abs-def*]
lemmas *abs-if-raw* = *abs-if*[*abs-def*]
lemmas *min-def-raw* = *min-def*[*abs-def*]
lemmas *max-def-raw* = *max-def*[*abs-def*]

lemma *nat-zero-as-int*:
  0 = *nat 0*
  by *simp*

lemma *nat-one-as-int*:
  1 = *nat 1*
  by *simp*

lemma *nat-numeral-as-int*: *numeral* = (λi. *nat* (*numeral i*)) **by** *simp*
lemma *nat-less-as-int*: ($<$) = ($\lambda a\ b$. *int a* $<$ *int b*) **by** *simp*
lemma *nat-leq-as-int*: ($\leq$) = ($\lambda a\ b$. *int a* $\leq$ *int b*) **by** *simp*
lemma *Suc-as-int*: *Suc* = (λa. *nat* (*int a* + *1*)) **by** (*rule ext*) *simp*
lemma *nat-plus-as-int*: ($+$) = ($\lambda a\ b$. *nat* (*int a* + *int b*)) **by** (*rule ext*)+ *simp*
lemma *nat-minus-as-int*: ($-$) = ($\lambda a\ b$. *nat* (*int a* $-$ *int b*)) **by** (*rule ext*)+ *simp*
lemma *nat-times-as-int*: ($*$) = ($\lambda a\ b$. *nat* (*int a* $*$ *int b*)) **by** (*simp add*: *nat-mult-distrib*)
lemma *nat-div-as-int*: (*div*) = ($\lambda a\ b$. *nat* (*int a div int b*)) **by** (*simp add*: *nat-div-distrib*)
lemma *nat-mod-as-int*: (*mod*) = ($\lambda a\ b$. *nat* (*int a mod int b*)) **by** (*simp add*: *nat-mod-distrib*)

lemma *int-Suc*: *int* (*Suc n*) = *int n* + *1* **by** *simp*
lemma *int-plus*: *int* (*n* + *m*) = *int n* + *int m* **by** (*rule of-nat-add*)
lemma *int-minus*: *int* (*n* $-$ *m*) = *int* (*nat* (*int n* $-$ *int m*)) **by** *auto*

lemma *nat-int-comparison*:
  fixes *a b* :: *nat*
  shows (*a* = *b*) = (*int a* = *int b*)
    and (*a* $<$ *b*) = (*int a* $<$ *int b*)
    and (*a* $\leq$ *b*) = (*int a* $\leq$ *int b*)
  by *simp-all*

lemma *int-ops*:
  fixes *a b* :: *nat*
  shows *int 0* = *0*
    and *int 1* = *1*
    and *int* (*numeral n*) = *numeral n*
    and *int* (*Suc a*) = *int a* + *1*
    and *int* (*a* + *b*) = *int a* + *int b*
    and *int* (*a* − *b*) = (*if int a* < *int b then 0 else int a* − *int b*)
    and *int* (*a* ∗ *b*) = *int a* ∗ *int b*
    and *int* (*a div b*) = *int a div int b*
    and *int* (*a mod b*) = *int a mod int b*
  by (*auto intro*: *zdiv-int zmod-int*)

lemma *int-if*:
  fixes *a b* :: *nat*
  shows *int* (*if P then a else b*) = (*if P then int a else int b*)
  by *simp*

63.5 Integer division and modulo for Z3

The following Z3-inspired definitions are overspecified for the case where *l* = *0*. This Schönheitsfehler is corrected in the *div-as-z3div* and *mod-as-z3mod* theorems.

definition *z3div* :: *int* ⇒ *int* ⇒ *int* **where**
  z3div k l = (*if l* ≥ *0 then k div l else* − (*k div* − *l*))

definition *z3mod* :: *int* ⇒ *int* ⇒ *int* **where**
  z3mod k l = *k mod* (*if l* ≥ *0 then l else* − *l*)

lemma *div-as-z3div*:
  ∀ *k l*. *k div l* = (*if l* = *0 then 0 else if l* > *0 then z3div k l else z3div* (− *k*) (− *l*))
  by (*simp add*: *z3div-def*)

lemma *mod-as-z3mod*:
  ∀ *k l*. *k mod l* = (*if l* = *0 then k else if l* > *0 then z3mod k l else* − *z3mod* (− *k*) (− *l*))
  by (*simp add*: *z3mod-def*)

63.6 Extra theorems for veriT reconstruction

lemma *verit-sko-forall*: ‹(∀ *x*. *P x*) ⟷ *P* (*SOME x*. ¬*P x*)›
  using *someI*[*of* ‹λ*x*. ¬*P x*›]
  by *auto*

lemma *verit-sko-forall'*: ‹*P* (*SOME x*. ¬*P x*) = *A* ⟹ (∀ *x*. *P x*) = *A*›
  by (*subst verit-sko-forall*)

lemma *verit-sko-forall''*: ‹$B = A \Longrightarrow (SOME\ x.\ P\ x) = A \equiv (SOME\ x.\ P\ x) = B$›
by *auto*

lemma *verit-sko-forall-indirect*: ‹$x = (SOME\ x.\ \neg P\ x) \Longrightarrow (\forall x.\ P\ x) \longleftrightarrow P\ x$›
using *someI*[*of* ‹$\lambda x.\ \neg P\ x$›]
by *auto*

lemma *verit-sko-forall-indirect2*:
‹$x = (SOME\ x.\ \neg P\ x) \Longrightarrow (\bigwedge x :: 'a.\ (P\ x = P'\ x)) \Longrightarrow (\forall x.\ P'\ x) \longleftrightarrow P\ x$›
using *someI*[*of* ‹$\lambda x.\ \neg P\ x$›]
by *auto*

lemma *verit-sko-ex*: ‹$(\exists x.\ P\ x) \longleftrightarrow P\ (SOME\ x.\ P\ x)$›
using *someI*[*of* ‹$\lambda x.\ P\ x$›]
by *auto*

lemma *verit-sko-ex'*: ‹$P\ (SOME\ x.\ P\ x) = A \Longrightarrow (\exists x.\ P\ x) = A$›
by (*subst verit-sko-ex*)

lemma *verit-sko-ex-indirect*: ‹$x = (SOME\ x.\ P\ x) \Longrightarrow (\exists x.\ P\ x) \longleftrightarrow P\ x$›
using *someI*[*of* ‹$\lambda x.\ P\ x$›]
by *auto*

lemma *verit-sko-ex-indirect2*: ‹$x = (SOME\ x.\ P\ x) \Longrightarrow (\bigwedge x.\ P\ x = P'\ x) \Longrightarrow (\exists x.\ P'\ x) \longleftrightarrow P\ x$›
using *someI*[*of* ‹$\lambda x.\ P\ x$›]
by *auto*

lemma *verit-Pure-trans*:
‹$P \equiv Q \Longrightarrow Q \Longrightarrow P$›
by *auto*

lemma *verit-if-cong*:
assumes ‹$b \equiv c$›
and ‹$c \Longrightarrow x \equiv u$›
and ‹$\neg\ c \Longrightarrow y \equiv v$›
shows ‹(*if* b *then* x *else* y) $\equiv$ (*if* c *then* u *else* v)›
using *assms if-cong*[*of* $b\ c\ x\ u$] **by** *auto*

lemma *verit-if-weak-cong'*:
‹$b \equiv c \Longrightarrow$ (*if* b *then* x *else* y) $\equiv$ (*if* c *then* x *else* y)›
by *auto*

lemma *verit-or-neg*:
‹$(A \Longrightarrow B) \Longrightarrow B \lor \neg A$›
‹$(\neg A \Longrightarrow B) \Longrightarrow B \lor A$›
by *auto*

lemma *verit-subst-bool*: ‹$P \Longrightarrow f\ True \Longrightarrow f\ P$›
by *auto*

lemma *verit-and-pos*:
‹$(a \Longrightarrow \neg(b \wedge c) \vee A) \Longrightarrow \neg(a \wedge b \wedge c) \vee A$›
‹$(a \Longrightarrow b \Longrightarrow A) \Longrightarrow \neg(a \wedge b) \vee A$›
by *blast+*

lemma *verit-farkas*:
‹$(a \Longrightarrow A) \Longrightarrow \neg a \vee A$›
‹$(\neg a \Longrightarrow A) \Longrightarrow a \vee A$›
by *blast+*

lemma *verit-or-pos*:
‹$A \wedge A' \Longrightarrow (c \wedge A) \vee (\neg c \wedge A')$›
‹$A \wedge A' \Longrightarrow (\neg c \wedge A) \vee (c \wedge A')$›
by *blast+*

lemma *verit-la-generic*:
‹$(a{::}int) \leq x \vee a = x \vee a \geq x$›
by *linarith*

lemma *verit-bfun-elim*:
‹$(\mathit{if}\ b\ \mathit{then}\ P\ True\ \mathit{else}\ P\ False) = P\ b$›
‹$(\forall b.\ P'\ b) = (P'\ False \wedge P'\ True)$›
‹$(\exists b.\ P'\ b) = (P'\ False \vee P'\ True)$›
by (*cases b*) (*auto simp*: *all-bool-eq ex-bool-eq*)

lemma *verit-eq-true-simplify*:
‹$(P = True) \equiv P$›
by *auto*

lemma *verit-and-neg*:
‹$(a \Longrightarrow \neg b \vee A) \Longrightarrow \neg(a \wedge b) \vee A$›
‹$(a \Longrightarrow A) \Longrightarrow \neg a \vee A$›
‹$(\neg a \Longrightarrow A) \Longrightarrow a \vee A$›
by *blast+*

lemma *verit-forall-inst*:
‹$A \longleftrightarrow B \Longrightarrow \neg A \vee B$›
‹$\neg A \longleftrightarrow B \Longrightarrow A \vee B$›
‹$A \longleftrightarrow B \Longrightarrow \neg B \vee A$›
‹$A \longleftrightarrow \neg B \Longrightarrow B \vee A$›
‹$A \longrightarrow B \Longrightarrow \neg A \vee B$›
‹$\neg A \longrightarrow B \Longrightarrow A \vee B$›
by *blast+*

lemma *verit-eq-transitive*:

‹$A = B \Longrightarrow B = C \Longrightarrow A = C$›
‹$A = B \Longrightarrow C = B \Longrightarrow A = C$›
‹$B = A \Longrightarrow B = C \Longrightarrow A = C$›
‹$B = A \Longrightarrow C = B \Longrightarrow A = C$›
by *auto*

lemma *verit-bool-simplify*:
‹$\neg(P \longrightarrow Q) \longleftrightarrow P \wedge \neg Q$›
‹$\neg(P \vee Q) \longleftrightarrow \neg P \wedge \neg Q$›
‹$\neg(P \wedge Q) \longleftrightarrow \neg P \vee \neg Q$›
‹$(P \longrightarrow (Q \longrightarrow R)) \longleftrightarrow ((P \wedge Q) \longrightarrow R)$›
‹$((P \longrightarrow Q) \longrightarrow Q) \longleftrightarrow P \vee Q$›
‹$(Q \longleftrightarrow (P \vee Q)) \longleftrightarrow (P \longrightarrow Q)$› — This rule was inverted
‹$P \wedge (P \longrightarrow Q) \longleftrightarrow P \wedge Q$›
‹$(P \longrightarrow Q) \wedge P \longleftrightarrow P \wedge Q$›

unfolding *not-imp imp-conjL*
by *auto*

We need the last equation for $\neg\ (\forall\, a\ b.\ \neg\ P\ a\ b)$

lemma *verit-connective-def*: — the definition of XOR is missing as the operator is not generated by Isabelle
‹$(A = B) \longleftrightarrow ((A \longrightarrow B) \wedge (B \longrightarrow A))$›
‹$(\mathit{If}\ A\ B\ C) = ((A \longrightarrow B) \wedge (\neg A \longrightarrow C))$›
‹$(\exists\, x.\ P\ x) \longleftrightarrow \neg(\forall\, x.\ \neg P\ x)$›
‹$\neg(\exists\, x.\ P\ x) \longleftrightarrow (\forall\, x.\ \neg P\ x)$›
by *auto*

lemma *verit-ite-simplify*:
‹$(\mathit{If}\ \mathit{True}\ B\ C) = B$›
‹$(\mathit{If}\ \mathit{False}\ B\ C) = C$›
‹$(\mathit{If}\ A'\ B\ B) = B$›
‹$(\mathit{If}\ (\neg A')\ B\ C) = (\mathit{If}\ A'\ C\ B)$›
‹$(\mathit{If}\ c\ (\mathit{If}\ c\ A\ B)\ C) = (\mathit{If}\ c\ A\ C)$›
‹$(\mathit{If}\ c\ C\ (\mathit{If}\ c\ A\ B)) = (\mathit{If}\ c\ C\ B)$›
‹$(\mathit{If}\ A'\ \mathit{True}\ \mathit{False}) = A'$›
‹$(\mathit{If}\ A'\ \mathit{False}\ \mathit{True}) \longleftrightarrow \neg A'$›
‹$(\mathit{If}\ A'\ \mathit{True}\ B') \longleftrightarrow A' \vee B'$›
‹$(\mathit{If}\ A'\ B'\ \mathit{False}) \longleftrightarrow A' \wedge B'$›
‹$(\mathit{If}\ A'\ \mathit{False}\ B') \longleftrightarrow \neg A' \wedge B'$›
‹$(\mathit{If}\ A'\ B'\ \mathit{True}) \longleftrightarrow \neg A' \vee B'$›
‹$x \wedge \mathit{True} \longleftrightarrow x$›
‹$x \vee \mathit{False} \longleftrightarrow x$›
for $B\ C$ :: *'a* **and** $A'\ B'\ C'$ :: *bool*
by *auto*

lemma *verit-and-simplify1*:
‹$\mathit{True} \wedge b \longleftrightarrow b$› ‹$b \wedge \mathit{True} \longleftrightarrow b$›
‹$\mathit{False} \wedge b \longleftrightarrow \mathit{False}$› ‹$b \wedge \mathit{False} \longleftrightarrow \mathit{False}$›

‹$(c \wedge \neg c) \longleftrightarrow$ *False*› ‹$(\neg c \wedge c) \longleftrightarrow$ *False*›
‹$\neg\neg a = a$›
by *auto*

lemmas *verit-and-simplify* = *conj-ac de-Morgan-conj disj-not1*

lemma *verit-or-simplify-1*:
‹*False* $\vee$ $b \longleftrightarrow b$› ‹$b \vee$ *False* $\longleftrightarrow b$›
‹$b \vee \neg b$›
‹$\neg b \vee b$›
by *auto*

lemmas *verit-or-simplify* = *disj-ac*

lemma *verit-not-simplify*:
‹$\neg \neg b \longleftrightarrow b$› ‹$\neg$*True* $\longleftrightarrow$ *False*› ‹$\neg$*False* $\longleftrightarrow$ *True*›
by *auto*

lemma *verit-implies-simplify*:
‹$(\neg a \longrightarrow \neg b) \longleftrightarrow (b \longrightarrow a)$›
‹(*False* $\longrightarrow a) \longleftrightarrow$ *True*›
‹($a \longrightarrow$ *True*) $\longleftrightarrow$ *True*›
‹(*True* $\longrightarrow a) \longleftrightarrow a$›
‹($a \longrightarrow$ *False*) $\longleftrightarrow \neg a$›
‹$(a \longrightarrow a) \longleftrightarrow$ *True*›
‹$(\neg a \longrightarrow a) \longleftrightarrow a$›
‹$(a \longrightarrow \neg a) \longleftrightarrow \neg a$›
‹$((a \longrightarrow b) \longrightarrow b) \longleftrightarrow a \vee b$›
by *auto*

lemma *verit-equiv-simplify*:
‹$((\neg a) = (\neg b)) \longleftrightarrow (a = b)$›
‹$(a = a) \longleftrightarrow$ *True*›
‹$(a = (\neg a)) \longleftrightarrow$ *False*›
‹$((\neg a) = a) \longleftrightarrow$ *False*›
‹(*True* $= a) \longleftrightarrow a$›
‹($a =$ *True*) $\longleftrightarrow a$›
‹(*False* $= a) \longleftrightarrow \neg a$›
‹($a =$ *False*) $\longleftrightarrow \neg a$›
‹$\neg\neg a \longleftrightarrow a$›
‹($\neg$ *False*) = *True*›
for a b :: *bool*
by *auto*

lemmas *verit-eq-simplify* =
semiring-char-0-class.eq-numeral-simps eq-refl zero-neq-one num.simps
neg-equal-zero equal-neg-zero one-neq-zero neg-equal-iff-equal

lemma *verit-minus-simplify*:
‹$(a :: 'a :: cancel\text{-}comm\text{-}monoid\text{-}add) - a = 0$›
‹$(a :: 'a :: cancel\text{-}comm\text{-}monoid\text{-}add) - 0 = a$›
‹$0 - (b :: 'b :: \{group\text{-}add\}) = -b$›
‹$- (- (b :: 'b :: group\text{-}add)) = b$›
by *auto*

lemma *verit-sum-simplify*:
‹$(a :: 'a :: cancel\text{-}comm\text{-}monoid\text{-}add) + 0 = a$›
by *auto*

lemmas *verit-prod-simplify* =

mult-1
mult-1-right

lemma *verit-comp-simplify1*:
‹$(a :: 'a ::order) < a \longleftrightarrow False$›
‹$a \le a$›
‹$\neg(b' \le a') \longleftrightarrow (a' :: 'b :: linorder) < b'$›
by *auto*

lemmas *verit-comp-simplify* =
verit-comp-simplify1
le-numeral-simps
le-num-simps
less-numeral-simps
less-num-simps
zero-less-one
zero-le-one
less-neg-numeral-simps

lemma *verit-la-disequality*:
‹$(a :: 'a ::linorder) = b \lor \neg a \le b \lor \neg b \le a$›
by *auto*

context
begin

For the reconstruction, we need to keep the order of the arguments.

named-theorems *smt-arith-multiplication* ‹*Theorems to reconstruct arithmetic theorems.*›

named-theorems *smt-arith-combine* ‹*Theorems to reconstruct arithmetic theorems.*›

named-theorems *smt-arith-simplify* ‹*Theorems to combine theorems in the LA procedure*›

lemmas [*smt-arith-simplify*] =
div-add dvd-numeral-simp divmod-steps less-num-simps le-num-simps if-True if-False divmod-cancel
dvd-mult dvd-mult2 less-irrefl prod.case numeral-plus-one divmod-step-def order.refl le-zero-eq
le-numeral-simps less-numeral-simps mult.right-neutral simp-thms divides-aux-eq
mult-nonneg-nonneg dvd-imp-mod-0 dvd-add zero-less-one mod-mult-self4 numeral-mod-numeral
divmod-trivial prod.sel mult.left-neutral div-pos-pos-trivial arith-simps div-add div-mult-self1
add-le-cancel-left add-le-same-cancel2 not-one-le-zero le-numeral-simps add-le-same-cancel1
zero-neq-one zero-le-one le-num-simps add-Suc mod-div-trivial nat.distinct mult-minus-right
add.inverse-inverse distrib-left-numeral mult-num-simps numeral-times-numeral add-num-simps
divmod-steps rel-simps if-True if-False numeral-div-numeral divmod-cancel prod.case
add-num-simps one-plus-numeral fst-conv arith-simps sub-num-simps dbl-inc-simps
dbl-simps mult-1 add-le-cancel-right left-diff-distrib-numeral add-uminus-conv-diff zero-neq-one
zero-le-one One-nat-def add-Suc mod-div-trivial nat.distinct of-int-1 numerals numeral-One
of-int-numeral add-uminus-conv-diff zle-diff1-eq add-less-same-cancel2 minus-add-distrib
add-uminus-conv-diff mult.left-neutral semiring-class.distrib-right
add-diff-cancel-left′ add-diff-eq ring-distribs mult-minus-left minus-diff-eq

lemma [*smt-arith-simplify*]:
‹$\neg$ (a' :: $'a$ :: *linorder*) $<$ b' $\longleftrightarrow$ b' $\leq$ a'›
‹$\neg$ (a' :: $'a$ :: *linorder*) $\leq$ b' $\longleftrightarrow$ b' $<$ a'›
‹(c::*int*) *mod Numeral1* $=$ *0*›
‹(a::*nat*) *mod Numeral1* $=$ *0*›
‹(c::*int*) *div Numeral1* $=$ c›
‹a *div Numeral1* $=$ a›
‹(c::*int*) *mod 1* $=$ *0*›
‹a *mod 1* $=$ *0*›
‹(c::*int*) *div 1* $=$ c›
‹a *div 1* $=$ a›
‹$\neg(a' \neq b') \longleftrightarrow a' = b'$›
by *auto*

lemma *div-mod-decomp*: $A = (A\ div\ n) * n + (A\ mod\ n)$ **for** A :: *nat*
by *auto*

lemma *div-less-mono*:
fixes A B :: *nat*
assumes $A < B$ $0 < n$ **and**
mod: $A\ mod\ n = 0 B\ mod\ n = 0$
shows $(A\ div\ n) < (B\ div\ n)$
proof −

show *?thesis*
using *assms*(*1*)
apply (*subst* (*asm*) *div-mod-decomp*[*of A n*])
apply (*subst* (*asm*) *div-mod-decomp*[*of B n*])
unfolding *mod*
by (*use assms*(*2*,*3*) **in** ‹*auto simp*: *ac-simps*›)
qed

lemma *verit-le-mono-div*:
fixes *A B* :: *nat*
assumes *A* < *B 0* < *n*
shows (*A div n*) + (*if B mod n* = *0 then 1 else 0*) ≤ (*B div n*)
by (*auto simp*: *ac-simps Suc-leI assms less-mult-imp-div-less div-le-mono less-imp-le-nat*)

lemmas [*smt-arith-multiplication*] =
verit-le-mono-div[*THEN mult-le-mono1*, *unfolded add-mult-distrib*]
div-le-mono[*THEN mult-le-mono2*, *unfolded add-mult-distrib*]

lemma *div-mod-decomp-int*: *A* = (*A div n*) ∗ *n* + (*A mod n*) **for** *A* :: *int*
by *auto*

lemma *zdiv-mono-strict*:
fixes *A B* :: *int*
assumes *A* < *B 0* < *n* **and**
mod: *A mod n* = *0B mod n* = *0*
shows (*A div n*) < (*B div n*)
proof −
show *?thesis*
using *assms*(*1*)
apply (*subst* (*asm*) *div-mod-decomp-int*[*of A n*])
apply (*subst* (*asm*) *div-mod-decomp-int*[*of B n*])
unfolding *mod*
by (*use assms*(*2*,*3*) **in** ‹*auto simp*: *ac-simps*›)
qed

lemma *verit-le-mono-div-int*:
‹*A div n* + (*if B mod n* = *0 then 1 else 0*) ≤ *B div n*›
if ‹*A* < *B*› ‹*0* < *n*›
for *A B n* :: *int*
proof −
from ‹*A* < *B*› ‹*0* < *n*› **have** ‹*A div n* ≤ *B div n*›
by (*auto intro*: *zdiv-mono1*)
show *?thesis*
proof (*cases* ‹*n dvd B*›)
case *False*
with ‹*A div n* ≤ *B div n*› **show** *?thesis*
by *auto*
next
case *True*

then obtain C **where** ‹$B = n * C$› **..**
then have ‹B *div* $n = C$›
using ‹$0 < n$› **by** *simp*
from ‹$0 < n$› **have** ‹A *mod* $n \geq 0$›
by *simp*
have ‹A *div* $n < C$›
proof (*rule ccontr*)
assume ‹$\neg$ A *div* $n < C$›
then have ‹$C \leq A$ *div* n›
by *simp*
with ‹B *div* $n = C$› ‹A *div* $n \leq B$ *div* n›
have ‹A *div* $n = C$›
by *simp*
moreover from ‹$A < B$› **have** ‹$n * (A$ *div* $n) + A$ *mod* $n < B$›
by *simp*
ultimately have ‹$n * C + A$ *mod* $n < n * C$›
using ‹$B = n * C$› **by** *simp*
moreover have ‹A *mod* $n \geq 0$›
using ‹$0 < n$› **by** *simp*
ultimately show *False*
by *simp*
qed
with ‹n *dvd* B› ‹B *div* $n = C$› **show** *?thesis*
by *simp*
qed
qed

lemma *verit-less-mono-div-int2*:
fixes A B :: *int*
assumes $A \leq B$ $0 < -n$
shows (A *div* n) $\geq$ (B *div* n)
using *assms*(*1*) *assms*(*2*) *zdiv-mono1-neg* **by** *auto*

lemmas [*smt-arith-multiplication*] =
verit-le-mono-div-int[*THEN mult-left-mono*, *unfolded int-distrib*]
zdiv-mono1[*THEN mult-left-mono*, *unfolded int-distrib*]

lemmas [*smt-arith-multiplication*] =
arg-cong[*of - -* ‹λa :: *nat*. a *div* $n * p$› **for** n p :: *nat*, *THEN sym*]
arg-cong[*of - -* ‹λa :: *int*. a *div* $n * p$› **for** n p :: *int*, *THEN sym*]

lemma [*smt-arith-combine*]:
$a < b \Longrightarrow c < d \Longrightarrow a + c + 2 \leq b + d$
$a < b \Longrightarrow c \leq d \Longrightarrow a + c + 1 \leq b + d$
$a \leq b \Longrightarrow c < d \Longrightarrow a + c + 1 \leq b + d$ **for** a b c :: *int*
by *auto*

lemma [*smt-arith-combine*]:
$a < b \Longrightarrow c < d \Longrightarrow a + c + 2 \leq b + d$

$a < b \Longrightarrow c \le d \Longrightarrow a + c + 1 \le b + d$
$a \le b \Longrightarrow c < d \Longrightarrow a + c + 1 \le b + d$ **for** *a b c* :: *nat*
by *auto*

lemmas [*smt-arith-combine*] =
add-strict-mono
add-less-le-mono
add-mono
add-le-less-mono

lemma [*smt-arith-combine*]:
‹$m < n \Longrightarrow c = d \Longrightarrow m + c < n + d$›
‹$m \le n \Longrightarrow c = d \Longrightarrow m + c \le n + d$›
‹$c = d \Longrightarrow m < n \Longrightarrow m + c < n + d$›
‹$c = d \Longrightarrow m \le n \Longrightarrow m + c \le n + d$›
for *m* :: ‹*'a* :: *ordered-cancel-ab-semigroup-add*›
by (*auto intro*: *ordered-cancel-ab-semigroup-add-class.add-strict-right-mono*
ordered-ab-semigroup-add-class.add-right-mono)

lemma *verit-negate-coefficient*:
‹$a \le (b :: 'a :: \{ordered\text{-}ab\text{-}group\text{-}add\}) \Longrightarrow -a \ge -b$›
‹$a < b \Longrightarrow -a > -b$›
‹$a = b \Longrightarrow -a = -b$›
by *auto*

end

lemma *verit-ite-intro*:
‹(*if a then P* (*if a then a′ else b′*) *else Q*) ⟷ (*if a then P a′ else Q*)›
‹(*if a then P′ else Q′* (*if a then a′ else b′*)) ⟷ (*if a then P′ else Q′ b′*)›
‹*A* = *f* (*if a then R else S*) ⟷ (*if a then A* = *f R else A* = *f S*)›
by *auto*

lemma *verit-ite-if-cong*:
fixes *x y* :: *bool*
assumes *b*=*c*
and $c \equiv$ *True* $\Longrightarrow x = u$
and $c \equiv$ *False* $\Longrightarrow y = v$
shows (*if b then x else y*) ≡ (*if c then u else v*)
proof −
have *H*: (*if b then x else y*) = (*if c then u else v*)
using *assms* **by** (*auto split*: *if-splits*)

show (*if b then x else y*) ≡ (*if c then u else v*)
by (*subst H*) *auto*
qed

63.7 Setup

ML-file ‹*Tools/SMT/smt-util.ML*›
ML-file ‹*Tools/SMT/smt-failure.ML*›
ML-file ‹*Tools/SMT/smt-config.ML*›
ML-file ‹*Tools/SMT/smt-builtin.ML*›
ML-file ‹*Tools/SMT/smt-datatypes.ML*›
ML-file ‹*Tools/SMT/smt-normalize.ML*›
ML-file ‹*Tools/SMT/smt-translate.ML*›
ML-file ‹*Tools/SMT/smtlib.ML*›
ML-file ‹*Tools/SMT/smtlib-interface.ML*›
ML-file ‹*Tools/SMT/smtlib-proof.ML*›
ML-file ‹*Tools/SMT/smtlib-isar.ML*›
ML-file ‹*Tools/SMT/z3-proof.ML*›
ML-file ‹*Tools/SMT/z3-isar.ML*›
ML-file ‹*Tools/SMT/smt-solver.ML*›
ML-file ‹*Tools/SMT/cvc-interface.ML*›
ML-file ‹*Tools/SMT/lethe-proof.ML*›
ML-file ‹*Tools/SMT/lethe-isar.ML*›
ML-file ‹*Tools/SMT/lethe-proof-parse.ML*›
ML-file ‹*Tools/SMT/cvc-proof-parse.ML*›
ML-file ‹*Tools/SMT/conj-disj-perm.ML*›
ML-file ‹*Tools/SMT/smt-replay-methods.ML*›
ML-file ‹*Tools/SMT/smt-replay.ML*›
ML-file ‹*Tools/SMT/smt-replay-arith.ML*›
ML-file ‹*Tools/SMT/z3-interface.ML*›
ML-file ‹*Tools/SMT/z3-replay-rules.ML*›
ML-file ‹*Tools/SMT/z3-replay-methods.ML*›
ML-file ‹*Tools/SMT/z3-replay.ML*›
ML-file ‹*Tools/SMT/lethe-replay-methods.ML*›
ML-file ‹*Tools/SMT/cvc5-replay-methods.ML*›
ML-file ‹*Tools/SMT/verit-replay-methods.ML*›
ML-file ‹*Tools/SMT/verit-strategies.ML*›
ML-file ‹*Tools/SMT/verit-replay.ML*›
ML-file ‹*Tools/SMT/cvc5-replay.ML*›
ML-file ‹*Tools/SMT/smt-systems.ML*›

63.8 Configuration

The current configuration can be printed by the command *smt-status*, which shows the values of most options.

63.9 General configuration options

The option *smt-solver* can be used to change the target SMT solver. The possible values can be obtained from the *smt-status* command.

declare [[*smt-solver* = *z3*]]

Since SMT solvers are potentially nonterminating, there is a timeout (given in seconds) to restrict their runtime.

declare [[*smt-timeout* = *0*]]

SMT solvers apply randomized heuristics. In case a problem is not solvable by an SMT solver, changing the following option might help.

declare [[*smt-random-seed* = *1*]]

In general, the binding to SMT solvers runs as an oracle, i.e, the SMT solvers are fully trusted without additional checks. The following option can cause the SMT solver to run in proof-producing mode, giving a checkable certificate. This is currently implemented only for veriT and Z3.

declare [[*smt-oracle* = *false*]]

Each SMT solver provides several command-line options to tweak its behaviour. They can be passed to the solver by setting the following options.

declare [[*cvc4-options* =]]
declare [[*cvc5-options* =]]
declare [[*cvc5-proof-options* = *−−proof−format−mode=alethe −−proof−granularity=dsl−rewrite*]]
declare [[*verit-options* =]]
declare [[*z3-options* =]]

The SMT method provides an inference mechanism to detect simple triggers in quantified formulas, which might increase the number of problems solvable by SMT solvers (note: triggers guide quantifier instantiations in the SMT solver). To turn it on, set the following option.

declare [[*smt-infer-triggers* = *false*]]

Enable the following option to use built-in support for datatypes, codatatypes, and records in CVC4 and cvc5. Currently, this is implemented only in oracle mode.

declare [[*cvc-extensions* = *false*]]

Enable the following option to use built-in support for div/mod, datatypes, and records in Z3. Currently, this is implemented only in oracle mode.

declare [[*z3-extensions* = *false*]]

63.10 Certificates

By setting the option *smt-certificates* to the name of a file, all following applications of an SMT solver are cached in that file. Any further application of the same SMT solver (using the very same configuration) re-uses the cached certificate instead of invoking the solver. An empty string disables caching certificates.

The filename should be given as an explicit path. It is good practice to use the name of the current theory (with ending *.certs* instead of *.thy*) as the certificates file. Certificate files should be used at most once in a certain theory context, to avoid race conditions with other concurrent accesses.

declare [[*smt-certificates* =]]

The option *smt-read-only-certificates* controls whether only stored certificates should be used or invocation of an SMT solver is allowed. When set to *true*, no SMT solver will ever be invoked and only the existing certificates found in the configured cache are used; when set to *false* and there is no cached certificate for some proposition, then the configured SMT solver is invoked.

declare [[*smt-read-only-certificates* = *false*]]

63.11 Tracing

The SMT method, when applied, traces important information. To make it entirely silent, set the following option to *false*.

declare [[*smt-verbose* = *true*]]

For tracing the generated problem file given to the SMT solver as well as the returned result of the solver, the option *smt-trace* should be set to *true*.

declare [[*smt-trace* = *false*]]

63.12 Schematic rules for Z3 proof reconstruction

Several prof rules of Z3 are not very well documented. There are two lemma groups which can turn failing Z3 proof reconstruction attempts into succeeding ones: the facts in *z3-rule* are tried prior to any implemented reconstruction procedure for all uncertain Z3 proof rules; the facts in *z3-simp* are only fed to invocations of the simplifier when reconstructing theory-specific proof steps.

lemmas [*z3-rule*] =
refl eq-commute conj-commute disj-commute simp-thms nnf-simps
ring-distribs field-simps times-divide-eq-right times-divide-eq-left
if-True if-False not-not
NO-MATCH-def

lemma [*z3-rule*]:
$(P \wedge Q) = (\neg (\neg P \vee \neg Q))$
$(P \wedge Q) = (\neg (\neg Q \vee \neg P))$
$(\neg P \wedge Q) = (\neg (P \vee \neg Q))$
$(\neg P \wedge Q) = (\neg (\neg Q \vee P))$
$(P \wedge \neg Q) = (\neg (\neg P \vee Q))$
$(P \wedge \neg Q) = (\neg (Q \vee \neg P))$

$(\neg\ P \wedge \neg\ Q) = (\neg\ (P \vee Q))$
$(\neg\ P \wedge \neg\ Q) = (\neg\ (Q \vee P))$
by *auto*

lemma [*z3-rule*]:
$(P \longrightarrow Q) = (Q \vee \neg\ P)$
$(\neg\ P \longrightarrow Q) = (P \vee Q)$
$(\neg\ P \longrightarrow Q) = (Q \vee P)$
$(\mathit{True} \longrightarrow P) = P$
$(P \longrightarrow \mathit{True}) = \mathit{True}$
$(\mathit{False} \longrightarrow P) = \mathit{True}$
$(P \longrightarrow P) = \mathit{True}$
$(\neg\ (A \longleftrightarrow \neg\ B)) \longleftrightarrow (A \longleftrightarrow B)$
by *auto*

lemma [*z3-rule*]:
$((P = Q) \longrightarrow R) = (R \vee (Q = (\neg\ P)))$
by *auto*

lemma [*z3-rule*]:
$(\neg\ \mathit{True}) = \mathit{False}$
$(\neg\ \mathit{False}) = \mathit{True}$
$(x = x) = \mathit{True}$
$(P = \mathit{True}) = P$
$(\mathit{True} = P) = P$
$(P = \mathit{False}) = (\neg\ P)$
$(\mathit{False} = P) = (\neg\ P)$
$((\neg\ P) = P) = \mathit{False}$
$(P = (\neg\ P)) = \mathit{False}$
$((\neg\ P) = (\neg\ Q)) = (P = Q)$
$\neg\ (P = (\neg\ Q)) = (P = Q)$
$\neg\ ((\neg\ P) = Q) = (P = Q)$
$(P \neq Q) = (Q = (\neg\ P))$
$(P = Q) = ((\neg\ P \vee Q) \wedge (P \vee \neg\ Q))$
$(P \neq Q) = ((\neg\ P \vee \neg\ Q) \wedge (P \vee Q))$
by *auto*

lemma [*z3-rule*]:
$(\mathit{if}\ P\ \mathit{then}\ P\ \mathit{else}\ \neg\ P) = \mathit{True}$
$(\mathit{if}\ \neg\ P\ \mathit{then}\ \neg\ P\ \mathit{else}\ P) = \mathit{True}$
$(\mathit{if}\ P\ \mathit{then}\ \mathit{True}\ \mathit{else}\ \mathit{False}) = P$
$(\mathit{if}\ P\ \mathit{then}\ \mathit{False}\ \mathit{else}\ \mathit{True}) = (\neg\ P)$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ \mathit{True}) = ((\neg\ P) \vee Q)$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ \mathit{True}) = (Q \vee (\neg\ P))$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ \neg\ Q) = (P = Q)$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ \neg\ Q) = (Q = P)$
$(\mathit{if}\ P\ \mathit{then}\ \neg\ Q\ \mathit{else}\ Q) = (P = (\neg\ Q))$
$(\mathit{if}\ P\ \mathit{then}\ \neg\ Q\ \mathit{else}\ Q) = ((\neg\ Q) = P)$
$(\mathit{if}\ \neg\ P\ \mathit{then}\ x\ \mathit{else}\ y) = (\mathit{if}\ P\ \mathit{then}\ y\ \mathit{else}\ x)$

$(\mathit{if}\ P\ \mathit{then}\ (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)\ \mathit{else}\ x) = (\mathit{if}\ P \wedge (\neg\ Q)\ \mathit{then}\ y\ \mathit{else}\ x)$
$(\mathit{if}\ P\ \mathit{then}\ (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)\ \mathit{else}\ x) = (\mathit{if}\ (\neg\ Q) \wedge P\ \mathit{then}\ y\ \mathit{else}\ x)$
$(\mathit{if}\ P\ \mathit{then}\ (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)\ \mathit{else}\ y) = (\mathit{if}\ P \wedge Q\ \mathit{then}\ x\ \mathit{else}\ y)$
$(\mathit{if}\ P\ \mathit{then}\ (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)\ \mathit{else}\ y) = (\mathit{if}\ Q \wedge P\ \mathit{then}\ x\ \mathit{else}\ y)$
$(\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ \mathit{if}\ P\ \mathit{then}\ y\ \mathit{else}\ z) = (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ z)$
$(\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ \mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) = (\mathit{if}\ P \vee Q\ \mathit{then}\ x\ \mathit{else}\ y)$
$(\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ \mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) = (\mathit{if}\ Q \vee P\ \mathit{then}\ x\ \mathit{else}\ y)$
$(\mathit{if}\ P\ \mathit{then}\ x = y\ \mathit{else}\ x = z) = (x = (\mathit{if}\ P\ \mathit{then}\ y\ \mathit{else}\ z))$
$(\mathit{if}\ P\ \mathit{then}\ x = y\ \mathit{else}\ y = z) = (y = (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ z))$
$(\mathit{if}\ P\ \mathit{then}\ x = y\ \mathit{else}\ z = y) = (y = (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ z))$
by *auto*

lemma [*z3-rule*]:
$0 + (x::\mathit{int}) = x$
$x + 0 = x$
$x + x = 2 * x$
$0 * x = 0$
$1 * x = x$
$x + y = y + x$
by *auto*

lemma [*z3-rule*]:
$P = Q \vee P \vee Q$
$P = Q \vee \neg\ P \vee \neg\ Q$
$(\neg\ P) = Q \vee \neg\ P \vee Q$
$(\neg\ P) = Q \vee P \vee \neg\ Q$
$P = (\neg\ Q) \vee \neg\ P \vee Q$
$P = (\neg\ Q) \vee P \vee \neg\ Q$
$P \neq Q \vee P \vee \neg\ Q$
$P \neq Q \vee \neg\ P \vee Q$
$P \neq (\neg\ Q) \vee P \vee Q$
$(\neg\ P) \neq Q \vee P \vee Q$
$P \vee Q \vee P \neq (\neg\ Q)$
$P \vee Q \vee (\neg\ P) \neq Q$
$P \vee \neg\ Q \vee P \neq Q$
$\neg\ P \vee Q \vee P \neq Q$
$P \vee y = (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y)$
$P \vee (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y) = y$
$\neg\ P \vee x = (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y)$
$\neg\ P \vee (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y) = x$
$P \vee R \vee \neg\ (\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R)$
$\neg\ P \vee Q \vee \neg\ (\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R)$
$\neg\ (\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R) \vee \neg\ P \vee Q$
$\neg\ (\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R) \vee P \vee R$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R) \vee \neg\ P \vee \neg\ Q$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R) \vee P \vee \neg\ R$
$(\mathit{if}\ P\ \mathit{then}\ \neg\ Q\ \mathit{else}\ R) \vee \neg\ P \vee Q$
$(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ \neg\ R) \vee P \vee R$
by *auto*

hide-type (**open**) *symb-list pattern*
hide-const (**open**) *Symb-Nil Symb-Cons trigger pat nopat fun-app z3div z3mod*

end

64 Sledgehammer: Isabelle–ATP Linkup

theory *Sledgehammer*
imports *Presburger SMT*
keywords
sledgehammer :: *diag* **and**
sledgehammer-params :: *thy-decl*
begin

ML-file ‹*Tools/ATP/system-on-tptp.ML*›
ML-file ‹*Tools/Sledgehammer/async-manager-legacy.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-util.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-fact.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-proof-methods.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-instantiations.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-isar-annotate.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-isar-proof.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-isar-preplay.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-isar-compress.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-isar-minimize.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-isar.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-atp-systems.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-prover.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-prover-atp.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-prover-smt.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-prover-minimize.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-mepo.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-mash.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-commands.ML*›
ML-file ‹*Tools/Sledgehammer/sledgehammer-tactics.ML*›

end

65 Setup for Lifting/Transfer for the set type

theory *Lifting-Set*
imports *Lifting Groups-Big*
begin

65.1 Relator and predicator properties

lemma *rel-setD1*: ⟦ *rel-set R A B*; $x \in A$ ⟧ $\Longrightarrow \exists\, y \in B.\ R\ x\ y$
and *rel-setD2*: ⟦ *rel-set R A B*; $y \in B$ ⟧ $\Longrightarrow \exists\, x \in A.\ R\ x\ y$
by (*simp-all add*: *rel-set-def*)

lemma *rel-set-conversep* [*simp*]: *rel-set* A^{-1-1} = (*rel-set* $A)^{-1-1}$
unfolding *rel-set-def* **by** *auto*

lemma *rel-set-eq* [*relator-eq*]: *rel-set* (=) = (=)
unfolding *rel-set-def fun-eq-iff* **by** *auto*

lemma *rel-set-mono*[*relator-mono*]:
assumes $A \leq B$
shows *rel-set* $A \leq$ *rel-set* B
using *assms* **unfolding** *rel-set-def* **by** *blast*

lemma *rel-set-OO*[*relator-distr*]: *rel-set R OO rel-set S* = *rel-set* (*R OO S*)
apply (*rule sym*)
apply (*intro ext*)
subgoal for *X Z*
apply (*rule iffI*)
apply (*rule relcomppI* [**where** $b=\{y.\ (\exists\, x{\in}X.\ R\ x\ y) \wedge (\exists\, z{\in}Z.\ S\ y\ z)\}$])
apply (*simp add*: *rel-set-def*, *fast*)+
done
done

lemma *Domainp-set*[*relator-domain*]:
Domainp (*rel-set T*) = ($\lambda A.$ *Ball A* (*Domainp T*))
unfolding *rel-set-def Domainp-iff*[*abs-def*]
apply (*intro ext*)
apply (*rule iffI*)
apply *blast*
subgoal for *A* **by** (*rule exI* [**where** $x=\{y.\ \exists\, x{\in}A.\ T\ x\ y\}$]) *fast*
done

lemma *left-total-rel-set*[*transfer-rule*]:
left-total A $\Longrightarrow$ *left-total* (*rel-set A*)
unfolding *left-total-def rel-set-def*
apply *safe*
subgoal for *X* **by** (*rule exI* [**where** $x=\{y.\ \exists\, x{\in}X.\ A\ x\ y\}$]) *fast*
done

lemma *left-unique-rel-set*[*transfer-rule*]:
left-unique A $\Longrightarrow$ *left-unique* (*rel-set A*)
unfolding *left-unique-def rel-set-def*
by *fast*

lemma *right-total-rel-set* [*transfer-rule*]:
right-total A $\Longrightarrow$ *right-total* (*rel-set A*)

using *left-total-rel-set*[*of* A^{-1-1}] **by** *simp*

lemma *right-unique-rel-set* [*transfer-rule*]:
right-unique A $\Longrightarrow$ *right-unique* (*rel-set A*)
unfolding *right-unique-def rel-set-def* **by** *fast*

lemma *bi-total-rel-set* [*transfer-rule*]:
bi-total A $\Longrightarrow$ *bi-total* (*rel-set A*)
by(*simp add*: *bi-total-alt-def left-total-rel-set right-total-rel-set*)

lemma *bi-unique-rel-set* [*transfer-rule*]:
bi-unique A $\Longrightarrow$ *bi-unique* (*rel-set A*)
unfolding *bi-unique-def rel-set-def* **by** *fast*

lemma *set-relator-eq-onp* [*relator-eq-onp*]:
rel-set (*eq-onp P*) = *eq-onp* (λ*A*. *Ball A P*)
unfolding *fun-eq-iff rel-set-def eq-onp-def Ball-def* **by** *fast*

lemma *bi-unique-rel-set-lemma*:
assumes *bi-unique R* **and** *rel-set R X Y*
obtains *f* **where** Y = *image f X* **and** *inj-on f X* **and** $\forall x \in X.\ R\ x\ (f\ x)$
proof
define *f* **where** *f x* = (*THE y*. *R x y*) **for** *x*
{ fix *x* **assume** $x \in X$
with ‹*rel-set R X Y*› ‹*bi-unique R*› **have** *R x* (*f x*)
by (*simp add*: *bi-unique-def rel-set-def f-def*) (*metis theI*)
with *assms* ‹$x \in X$›
have $R\ x\ (f\ x)\ \forall x' \in X.\ R\ x'\ (f\ x) \longrightarrow x = x'\ \forall y \in Y.\ R\ x\ y \longrightarrow y = f\ x\ f\ x \in$
Y
by (*fastforce simp add*: *bi-unique-def rel-set-def*)+ **}**
note $*$ = *this*
moreover
{ fix *y* **assume** $y \in Y$
with ‹*rel-set R X Y*› $*$(*3*) ‹$y \in Y$› **have** $\exists x \in X.\ y = f\ x$
by (*fastforce simp*: *rel-set-def*) **}**
ultimately show $\forall x \in X.\ R\ x\ (f\ x)$ Y = *image f X inj-on f X*
by (*auto simp*: *inj-on-def image-iff*)
qed

65.2 Quotient theorem for the Lifting package

lemma *Quotient-set*[*quot-map*]:
assumes *Quotient R Abs Rep T*
shows *Quotient* (*rel-set R*) (*image Abs*) (*image Rep*) (*rel-set T*)
using *assms* **unfolding** *Quotient-alt-def4*
apply (*simp add*: *rel-set-OO*[*symmetric*])
apply (*simp add*: *rel-set-def*)
apply *fast*
done

65.3 Transfer rules for the Transfer package

65.3.1 Unconditional transfer rules

context includes *lifting-syntax*
begin

lemma *empty-transfer* [*transfer-rule*]: (*rel-set A*) {} {}
  unfolding *rel-set-def* **by** *simp*

lemma *insert-transfer* [*transfer-rule*]:
  (*A* ===> *rel-set A* ===> *rel-set A*) *insert insert*
  unfolding *rel-fun-def rel-set-def* **by** *auto*

lemma *union-transfer* [*transfer-rule*]:
  (*rel-set A* ===> *rel-set A* ===> *rel-set A*) *union union*
  unfolding *rel-fun-def rel-set-def* **by** *auto*

lemma *Union-transfer* [*transfer-rule*]:
  (*rel-set* (*rel-set A*) ===> *rel-set A*) *Union Union*
  unfolding *rel-fun-def rel-set-def* **by** *simp fast*

lemma *image-transfer* [*transfer-rule*]:
  ((*A* ===> *B*) ===> *rel-set A* ===> *rel-set B*) *image image*
  unfolding *rel-fun-def rel-set-def* **by** *simp fast*

lemma *UNION-transfer* [*transfer-rule*]: — TODO deletion candidate
  (*rel-set A* ===> (*A* ===> *rel-set B*) ===> *rel-set B*) ($\lambda A\ f.\ \bigcup(f\ `\ A)$) ($\lambda A\ f.\ \bigcup(f\ `\ A)$)
  by *transfer-prover*

lemma *Ball-transfer* [*transfer-rule*]:
  (*rel-set A* ===> (*A* ===> (=)) ===> (=)) *Ball Ball*
  unfolding *rel-set-def rel-fun-def* **by** *fast*

lemma *Bex-transfer* [*transfer-rule*]:
  (*rel-set A* ===> (*A* ===> (=)) ===> (=)) *Bex Bex*
  unfolding *rel-set-def rel-fun-def* **by** *fast*

lemma *Pow-transfer* [*transfer-rule*]:
  (*rel-set A* ===> *rel-set* (*rel-set A*)) *Pow Pow*
  apply (*rule rel-funI*)
  apply (*rule rel-setI*)
  subgoal for $X\ Y\ X'$
    apply (*rule rev-bexI* [**where** $x=\{y\in Y.\ \exists x\in X'.\ A\ x\ y\}$])
    apply *clarsimp*
    apply (*simp add*: *rel-set-def*)
    apply *fast*
    done
  subgoal for $X\ Y\ Y'$

apply (*rule rev-bexI* [**where** $x=\{x{\in}X.\ \exists\, y{\in}Y'.\ A\ x\ y\}$])
apply *clarsimp*
apply (*simp add*: *rel-set-def*)
apply *fast*
done
done

lemma *rel-set-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> (=)) ===> *rel-set A* ===> *rel-set B* ===> (=)) *rel-set rel-set*
unfolding *rel-fun-def rel-set-def* **by** *fast*

lemma *bind-transfer* [*transfer-rule*]:
(*rel-set A* ===> (*A* ===> *rel-set B*) ===> *rel-set B*) *Set.bind Set.bind*
unfolding *bind-UNION* [*abs-def*] **by** *transfer-prover*

lemma *INF-parametric* [*transfer-rule*]: — TODO deletion candidate
(*rel-set A* ===> (*A* ===> *HOL.eq*) ===> *HOL.eq*) (λ*A f*. *Inf* (*f* ' *A*)) (λ*A f*. *Inf* (*f* ' *A*))
by *transfer-prover*

lemma *SUP-parametric* [*transfer-rule*]: — TODO deletion candidate
(*rel-set R* ===> (*R* ===> *HOL.eq*) ===> *HOL.eq*) (λ*A f*. *Sup* (*f* ' *A*)) (λ*A f*. *Sup* (*f* ' *A*))
by *transfer-prover*

65.3.2 Rules requiring bi-unique, bi-total or right-total relations

lemma *member-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*A* ===> *rel-set A* ===> (=)) (∈) (∈)
using *assms* **unfolding** *rel-fun-def rel-set-def bi-unique-def* **by** *fast*

lemma *right-total-Collect-transfer*[*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> *rel-set A*) (λ*P*. *Collect* (λ*x*. *P x* ∧ *Domainp A x*)) *Collect*
using *assms* **unfolding** *right-total-def rel-set-def rel-fun-def Domainp-iff* **by** *fast*

lemma *Collect-transfer* [*transfer-rule*]:
assumes *bi-total A*
shows ((*A* ===> (=)) ===> *rel-set A*) *Collect Collect*
using *assms* **unfolding** *rel-fun-def rel-set-def bi-total-def* **by** *fast*

lemma *inter-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> *rel-set A*) *inter inter*
using *assms* **unfolding** *rel-fun-def rel-set-def bi-unique-def* **by** *fast*

lemma *Diff-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> *rel-set A*) (−) (−)
using *assms* **unfolding** *rel-fun-def rel-set-def bi-unique-def*
unfolding *Ball-def Bex-def Diff-eq*
by (*safe*, *simp*, *metis*, *simp*, *metis*)

lemma *subset-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> (=)) (⊆) (⊆)
unfolding *subset-eq* [*abs-def*] **by** *transfer-prover*

context
includes *lifting-syntax*
begin

lemma *strict-subset-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> (=)) (⊂) (⊂)
unfolding *subset-not-subset-eq* **by** *transfer-prover*

end

declare *right-total-UNIV-transfer*[*transfer-rule*]

lemma *UNIV-transfer* [*transfer-rule*]:
assumes *bi-total A*
shows (*rel-set A*) *UNIV UNIV*
using *assms* **unfolding** *rel-set-def bi-total-def* **by** *simp*

lemma *right-total-Compl-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *right-total A*
shows (*rel-set A* ===> *rel-set A*) (λ*S*. *uminus S* ∩ *Collect* (*Domainp A*)) *uminus*
unfolding *Compl-eq* [*abs-def*]
by (*subst Collect-conj-eq*[*symmetric*]) *transfer-prover*

lemma *Compl-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *bi-total A*
shows (*rel-set A* ===> *rel-set A*) *uminus uminus*
unfolding *Compl-eq* [*abs-def*] **by** *transfer-prover*

lemma *right-total-Inter-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *right-total A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*) (λ*S*. ⋂ *S* ∩ *Collect* (*Domainp A*)) *Inter*
unfolding *Inter-eq*[*abs-def*]
by (*subst Collect-conj-eq*[*symmetric*]) *transfer-prover*

lemma *Inter-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *bi-total A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*) *Inter Inter*
unfolding *Inter-eq* [*abs-def*] **by** *transfer-prover*

lemma *filter-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows ((*A* ===> (=)) ===> *rel-set A* ===> *rel-set A*) *Set.filter Set.filter*
unfolding *Set.filter-def*[*abs-def*] *rel-fun-def rel-set-def* **by** *blast*

lemma *finite-transfer* [*transfer-rule*]:
bi-unique A ⟹ (*rel-set A* ===> (=)) *finite finite*
by (*rule rel-funI*, *erule* (*1*) *bi-unique-rel-set-lemma*)
(*auto dest*: *finite-imageD*)

lemma *card-transfer* [*transfer-rule*]:
bi-unique A ⟹ (*rel-set A* ===> (=)) *card card*
by (*rule rel-funI*, *erule* (*1*) *bi-unique-rel-set-lemma*)
(*simp add*: *card-image*)

context
includes *lifting-syntax*
begin

lemma *vimage-right-total-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique B right-total A*
shows ((*A* ===> *B*) ===> *rel-set B* ===> *rel-set A*) ($\lambda f\ X.\ f\ -`\ X \cap$ *Collect* (*Domainp A*)) *vimage*
proof −
let *?vimage* = ($\lambda f\ B.\ \{x.\ f\ x \in B \wedge$ *Domainp A x*$\}$)
have ((*A* ===> *B*) ===> *rel-set B* ===> *rel-set A*) *?vimage vimage*
unfolding *vimage-def*
by *transfer-prover*
also have *?vimage* = ($\lambda f\ X.\ f\ -`\ X \cap$ *Collect* (*Domainp A*))
by *auto*
finally show *?thesis* **.**
qed

end

lemma *vimage-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A bi-unique B*
shows ((*A* ===> *B*) ===> *rel-set B* ===> *rel-set A*) *vimage vimage*
unfolding *vimage-def*[*abs-def*] **by** *transfer-prover*

lemma *Image-parametric* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-set* (*rel-prod A B*) ===> *rel-set A* ===> *rel-set B*) (``) (``)
by (*intro rel-funI rel-setI*)

(*force dest*: *rel-setD1 bi-uniqueDr*[*OF assms*], *force dest*: *rel-setD2 bi-uniqueDl*[*OF assms*])

lemma *inj-on-transfer*[*transfer-rule*]:
((*A* ===> *B*) ===> *rel-set A* ===> (=)) *inj-on inj-on*
if [*transfer-rule*]: *bi-unique A bi-unique B*
unfolding *inj-on-def*
by *transfer-prover*

end

lemma (**in** *comm-monoid-set*) *F-parametric* [*transfer-rule*]:
fixes *A* :: $'b \Rightarrow 'c \Rightarrow$ *bool*
assumes *bi-unique A*
shows *rel-fun* (*rel-fun A* (=)) (*rel-fun* (*rel-set A*) (=)) *F F*
proof (*rule rel-funI*)+
fix *f* :: $'b \Rightarrow 'a$ **and** *g S T*
assume *rel-fun A* (=) *f g rel-set A S T*
with ‹*bi-unique A*› **obtain** *i* **where** *bij-betw i S T* $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ (i\ x)$
by (*auto elim*: *bi-unique-rel-set-lemma simp*: *rel-fun-def bij-betw-def*)
then show *F f S* = *F g T*
by (*simp add*: *reindex-bij-betw*)
qed

lemmas *sum-parametric* = *sum.F-parametric*
lemmas *prod-parametric* = *prod.F-parametric*

lemma *rel-set-UNION*:
assumes [*transfer-rule*]: *rel-set Q A B rel-fun Q* (*rel-set R*) *f g*
shows *rel-set R* ($\bigcup$(*f* ‘ *A*)) ($\bigcup$(*g* ‘ *B*))
by *transfer-prover*

context
includes *lifting-syntax*
begin

lemma *fold-graph-transfer*[*transfer-rule*]:
assumes *bi-unique R right-total R*
shows ((*R* ===> (=) ===> (=)) ===> (=) ===> *rel-set R* ===> (=) ===> (=)) *fold-graph fold-graph*
proof(*intro rel-funI*)
fix *f1* :: $'a \Rightarrow 'c \Rightarrow 'c$ **and** *f2* :: $'b \Rightarrow 'c \Rightarrow 'c$
assume *rel-f*: (*R* ===> (=) ===> (=)) *f1 f2*
fix *z1 z2* :: $'c$ **assume** [*simp*]: *z1* = *z2*
fix *A1 A2* **assume** *rel-A*: *rel-set R A1 A2*
fix *y1 y2* :: $'c$ **assume** [*simp*]: *y1* = *y2*

from ‹*bi-unique R*› ‹*right-total R*› **have** *The-y*: $\forall y.\ \exists!x.\ R\ x\ y$
unfolding *bi-unique-def right-total-def* **by** *auto*

define *r* **where** *r* ≡ *λy. THE x. R x y*

from *The-y* **have** *r-y*: *R* (*r y*) *y* **for** *y*
unfolding *r-def* **using** *the-equality* **by** *fastforce*
with *assms rel-A* **have** *inj-on r A2 A1* = *r* ‘ *A2*
unfolding *r-def rel-set-def inj-on-def bi-unique-def*
apply(*auto simp*: *image-iff*) **by** *metis+*
with ‹*bi-unique R*› *rel-f r-y* **have** (*f1 o r*) *y* = *f2 y* **for** *y*
unfolding *bi-unique-def rel-fun-def* **by** *auto*
then have (*f1 o r*) = *f2*
by *blast*
then show *fold-graph f1 z1 A1 y1* = *fold-graph f2 z2 A2 y2*
by (*fastforce simp*: *fold-graph-image*[*OF* ‹*inj-on r A2*›] ‹*A1* = *r* ‘ *A2*›)
qed

lemma *fold-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique R right-total R*
shows ((*R* ===> (=) ===> (=)) ===> (=) ===> *rel-set R* ===> (=)) *Finite-Set.fold Finite-Set.fold*
unfolding *Finite-Set.fold-def*
by *transfer-prover*

end

end

66 The datatype of finite lists

theory *List*
imports *Sledgehammer Lifting-Set*
begin

datatype (*set*: ′*a*) *list* =
Nil (‹[]›)
| *Cons* (*hd*: ′*a*) (*tl*: ′*a list*) (**infixr** ‹#› *65*)
for
map: *map*
rel: *list-all2*
pred: *list-all*
where
tl [] = []

bundle *list-syntax*
begin
notation *Nil* (‹[]›)
and *Cons* (**infixr** ‹#› *65*)
end

datatype-compat *list*

lemma [*case-names Nil Cons*, *cases type*: *list*]:
— for backward compatibility – names of variables differ
$(y = [] \Longrightarrow P) \Longrightarrow (\bigwedge a\ list.\ y = a \ \#\ list \Longrightarrow P) \Longrightarrow P$
by (*rule list.exhaust*)

lemma [*case-names Nil Cons*, *induct type*: *list*]:
— for backward compatibility – names of variables differ
$P\ [] \Longrightarrow (\bigwedge a\ list.\ P\ list \Longrightarrow P\ (a \ \#\ list)) \Longrightarrow P\ list$
by (*rule list.induct*)

Compatibility:

setup ‹*Sign.mandatory-path list*›

lemmas *inducts* = *list.induct*
lemmas *recs* = *list.rec*
lemmas *cases* = *list.case*

setup ‹*Sign.parent-path*›

lemmas *set-simps* = *list.set*

List enumeration

open-bundle *list-enumeration-syntax*
begin

syntax
-*list* :: *args* ⇒ ′*a list* (‹(‹*indent=1 notation=*‹*mixfix list enumeration*››[-])›)
syntax-consts
-*list* ⇌ *Cons*
translations
[*x*, *xs*] ⇌ *x*#[*xs*]
[*x*] ⇌ *x*#[]

end

66.1 Basic list processing functions

primrec (*nonexhaustive*) *last* :: ′*a list* ⇒ ′*a* **where**
last (*x* # *xs*) = (*if xs* = [] *then x else last xs*)

primrec *butlast* :: ′*a list* ⇒ ′*a list* **where**
butlast [] = [] |
butlast (*x* # *xs*) = (*if xs* = [] *then* [] *else x* # *butlast xs*)

lemma *set-rec*: *set xs* = *rec-list* {} (λ*x* -. *insert x*) *xs*
by (*induct xs*) *auto*

definition *coset* :: *'a list* ⇒ *'a set* **where**
[*simp*]: *coset xs* = − *set xs*

primrec *append* :: *'a list* ⇒ *'a list* ⇒ *'a list* (**infixr** ‹@› *65*) **where**
append-Nil: [] @ *ys* = *ys* |
append-Cons: (*x#xs*) @ *ys* = *x* # *xs* @ *ys*

primrec *rev* :: *'a list* ⇒ *'a list* **where**
rev [] = [] |
rev (*x* # *xs*) = *rev xs* @ [*x*]

primrec *filter*:: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list* **where**
filter P [] = [] |
filter P (*x* # *xs*) = (*if P x then x* # *filter P xs else filter P xs*)

open-bundle *filter-syntax* — Special input syntax for filter
begin

syntax (*ASCII*)
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list* (‹(‹*indent=1 notation*=‹*mixfix filter*››[-<−-./ -])›)
syntax
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list* (‹(‹*indent=1 notation*=‹*mixfix filter*››[-←-./ -])›)
syntax-consts
-filter ⇌ *filter*
translations
[*x*<−*xs* . *P*] ⇀ *CONST filter* (λ*x*. *P*) *xs*

end

primrec *fold* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a list* ⇒ *'b* ⇒ *'b* **where**
fold-Nil: *fold f* [] = *id* |
fold-Cons: *fold f* (*x* # *xs*) = *fold f xs* ∘ *f x*

primrec *foldr* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a list* ⇒ *'b* ⇒ *'b* **where**
foldr-Nil: *foldr f* [] = *id* |
foldr-Cons: *foldr f* (*x* # *xs*) = *f x* ∘ *foldr f xs*

primrec *foldl* :: (*'b* ⇒ *'a* ⇒ *'b*) ⇒ *'b* ⇒ *'a list* ⇒ *'b* **where**
foldl-Nil: *foldl f a* [] = *a* |
foldl-Cons: *foldl f a* (*x* # *xs*) = *foldl f* (*f a x*) *xs*

primrec *concat*:: *'a list list* ⇒ *'a list* **where**
concat [] = [] |
concat (*x* # *xs*) = *x* @ *concat xs*

primrec *drop*:: *nat* ⇒ *'a list* ⇒ *'a list* **where**
drop-Nil: *drop n* [] = [] |

drop-Cons: *drop n* (*x* # *xs*) = (*case n of 0* ⇒ *x* # *xs* | *Suc m* ⇒ *drop m xs*)
— Warning: simpset does not contain this definition, but separate theorems for $n = 0$ and $n = Suc\ k$

primrec *take*:: *nat* ⇒ *'a list* ⇒ *'a list* **where**
take-Nil:*take n* [] = [] |
take-Cons: *take n* (*x* # *xs*) = (*case n of 0* ⇒ [] | *Suc m* ⇒ *x* # *take m xs*)
— Warning: simpset does not contain this definition, but separate theorems for $n = 0$ and $n = Suc\ k$

primrec (*nonexhaustive*) *nth* :: *'a list* => *nat* => *'a* (**infixl** ‹!› *100*) **where**
nth-Cons: (*x* # *xs*) ! *n* = (*case n of 0* ⇒ *x* | *Suc k* ⇒ *xs* ! *k*)
— Warning: simpset does not contain this definition, but separate theorems for $n = 0$ and $n = Suc\ k$

primrec *list-update* :: *'a list* ⇒ *nat* ⇒ *'a* ⇒ *'a list* **where**
list-update [] *i v* = [] |
list-update (*x* # *xs*) *i v* =
(*case i of 0* ⇒ *v* # *xs* | *Suc j* ⇒ *x* # *list-update xs j v*)

nonterminal *lupdbinds* **and** *lupdbind*

open-bundle *list-update-syntax*
begin

syntax
-lupdbind:: [*'a*, *'a*] => *lupdbind* (‹(‹*indent=2 notation=*‹*mixfix update*››- :=/ -)›)
:: *lupdbind* => *lupdbinds* (‹-›)
-lupdbinds :: [*lupdbind*, *lupdbinds*] => *lupdbinds* (‹-,/ -›)
-LUpdate :: [*'a*, *lupdbinds*] => *'a*
(‹(‹*open-block notation=*‹*mixfix list update*››-/[(-)])› [*1000*,*0*] *900*)
syntax-consts
-LUpdate ⇌ *list-update*
translations
-LUpdate xs (*-lupdbinds b bs*) == *-LUpdate* (*-LUpdate xs b*) *bs*
xs[*i*:=*x*] == *CONST list-update xs i x*

end

primrec *takeWhile* :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list* **where**
takeWhile P [] = [] |
takeWhile P (*x* # *xs*) = (*if P x then x* # *takeWhile P xs else* [])

primrec *dropWhile* :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list* **where**
dropWhile P [] = [] |
dropWhile P (*x* # *xs*) = (*if P x then dropWhile P xs else x* # *xs*)

primrec *zip* :: *'a list* ⇒ *'b list* ⇒ (*'a* × *'b*) *list* **where**

zip xs [] = [] |
zip-Cons: *zip xs (y # ys) =*
(case xs of [] ⇒ [] | z # zs ⇒ (z, y) # zip zs ys)
— Warning: simpset does not contain this definition, but separate theorems for *xs = []* and *xs = z # zs*

abbreviation *map2 :: ('a ⇒ 'b ⇒ 'c) ⇒ 'a list ⇒ 'b list ⇒ 'c list* **where**
map2 f xs ys ≡ map (λ(x,y). f x y) (zip xs ys)

primrec *product :: 'a list ⇒ 'b list ⇒ ('a × 'b) list* **where**
product [] - = [] |
product (x#xs) ys = map (Pair x) ys @ product xs ys

hide-const (**open**) *product*

primrec *product-lists :: 'a list list ⇒ 'a list list* **where**
product-lists [] = [[]] |
product-lists (xs # xss) = concat (map (λx. map (Cons x) (product-lists xss)) xs)

primrec *upt :: nat ⇒ nat ⇒ nat list (‹(‹indent=1 notation=‹mixfix list interval››[-..<-'])›)* **where**
upt-0: *[i..<0] = [] |*
upt-Suc: *[i..<(Suc j)] = (if i ≤ j then [i..<j] @ [j] else [])*

definition *insert :: 'a ⇒ 'a list ⇒ 'a list* **where**
insert x xs = (if x ∈ set xs then xs else x # xs)

definition *union :: 'a list ⇒ 'a list ⇒ 'a list* **where**
union = fold insert

hide-const (**open**) *insert union*
hide-fact (**open**) *insert-def union-def*

primrec *find :: ('a ⇒ bool) ⇒ 'a list ⇒ 'a option* **where**
find - [] = None |
find P (x#xs) = (if P x then Some x else find P xs)

In the context of multisets, *count-list* is equivalent to *count ∘ mset* and it is advisable to use the latter.

primrec *count-list :: 'a list ⇒ 'a ⇒ nat* **where**
count-list [] y = 0 |
count-list (x#xs) y = (if x=y then count-list xs y + 1 else count-list xs y)

definition
*extract :: ('a ⇒ bool) ⇒ 'a list ⇒ ('a list * 'a * 'a list) option*
where *extract P xs =*
(case dropWhile (Not ∘ P) xs of
[] ⇒ None |
y#ys ⇒ Some(takeWhile (Not ∘ P) xs, y, ys))

hide-const (**open**) *extract*

primrec *those* :: $'a$ *option list* $\Rightarrow$ $'a$ *list option*
where
those [] = *Some* [] |
those (x # xs) = (*case* x *of*
None $\Rightarrow$ *None*
| *Some* y $\Rightarrow$ *map-option* (*Cons* y) (*those* xs))

primrec *remove1* :: $'a \Rightarrow 'a$ *list* $\Rightarrow 'a$ *list* **where**
remove1 x [] = [] |
remove1 x (y # xs) = (*if* $x = y$ *then* xs *else* y # *remove1* x xs)

primrec *removeAll* :: $'a \Rightarrow 'a$ *list* $\Rightarrow 'a$ *list* **where**
removeAll x [] = [] |
removeAll x (y # xs) = (*if* $x = y$ *then* *removeAll* x xs *else* y # *removeAll* x xs)

primrec *distinct* :: $'a$ *list* $\Rightarrow$ *bool* **where**
distinct [] $\longleftrightarrow$ *True* |
distinct (x # xs) $\longleftrightarrow$ $x \notin$ *set* xs $\wedge$ *distinct* xs

fun *successively* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow 'a$ *list* $\Rightarrow$ *bool* **where**
successively P [] = *True* |
successively P [x] = *True* |
successively P (x # y # xs) = (P x y $\wedge$ *successively* P (y#xs))

definition *distinct-adj* **where**
distinct-adj = *successively* ($\neq$)

primrec *remdups* :: $'a$ *list* $\Rightarrow 'a$ *list* **where**
remdups [] = [] |
remdups (x # xs) = (*if* $x \in$ *set* xs *then* *remdups* xs *else* x # *remdups* xs)

fun *remdups-adj* :: $'a$ *list* $\Rightarrow 'a$ *list* **where**
remdups-adj [] = [] |
remdups-adj [x] = [x] |
remdups-adj (x # y # xs) = (*if* $x = y$ *then* *remdups-adj* (x # xs) *else* x # *remdups-adj* (y # xs))

primrec *replicate* :: *nat* $\Rightarrow 'a \Rightarrow 'a$ *list* **where**
replicate-0: *replicate* *0* x = [] |
replicate-Suc: *replicate* (*Suc* n) x = x # *replicate* n x

Function *size* is overloaded for all datatypes. Users may refer to the list version as *length*.

abbreviation *length* :: $'a$ *list* $\Rightarrow$ *nat* **where**
length $\equiv$ *size*

definition *enumerate* :: *nat* ⇒ *'a list* ⇒ (*nat* × *'a*) *list* **where**
enumerate-eq-zip: *enumerate n xs* = *zip* [*n*..<*n* + *length xs*] *xs*

primrec *rotate1* :: *'a list* ⇒ *'a list* **where**
rotate1 [] = [] |
rotate1 (*x* # *xs*) = *xs* @ [*x*]

definition *rotate* :: *nat* ⇒ *'a list* ⇒ *'a list* **where**
rotate n = *rotate1* ^^ *n*

definition *nths* :: *'a list* => *nat set* => *'a list* **where**
nths xs A = *map fst* (*filter* (λ*p*. *snd p* ∈ *A*) (*zip xs* [*0*..<*size xs*]))

primrec *subseqs* :: *'a list* ⇒ *'a list list* **where**
subseqs [] = [[]] |
subseqs (*x*#*xs*) = (*let xss* = *subseqs xs in map* (*Cons x*) *xss* @ *xss*)

primrec *n-lists* :: *nat* ⇒ *'a list* ⇒ *'a list list* **where**
n-lists 0 xs = [[]] |
n-lists (*Suc n*) *xs* = *concat* (*map* (λ*ys*. *map* (λ*y*. *y* # *ys*) *xs*) (*n-lists n xs*))

hide-const (**open**) *n-lists*

function *splice* :: *'a list* ⇒ *'a list* ⇒ *'a list* **where**
splice [] *ys* = *ys* |
splice (*x*#*xs*) *ys* = *x* # *splice ys xs*
by *pat-completeness auto*

termination
by(*relation measure*(λ(*xs*,*ys*). *size xs* + *size ys*)) *auto*

function *shuffles* **where**
shuffles [] *ys* = {*ys*}
| *shuffles xs* [] = {*xs*}
| *shuffles* (*x* # *xs*) (*y* # *ys*) = (#) *x* ' *shuffles xs* (*y* # *ys*) ∪ (#) *y* ' *shuffles* (*x* # *xs*) *ys*
by *pat-completeness simp-all*
termination by *lexicographic-order*

Use only if you cannot use *Min* instead:

fun *min-list* :: *'a*::*ord list* ⇒ *'a* **where**
min-list (*x* # *xs*) = (*case xs of* [] ⇒ *x* | - ⇒ *min x* (*min-list xs*))

Returns first minimum:

fun *arg-min-list* :: (*'a* ⇒ (*'b*::*linorder*)) ⇒ *'a list* ⇒ *'a* **where**
arg-min-list f [*x*] = *x* |
arg-min-list f (*x*#*y*#*zs*) = (*let m* = *arg-min-list f* (*y*#*zs*) *in if f x* ≤ *f m then x else m*)

[a, b] @ [c, d] = [a, b, c, d]
length [a, b, c] = 3
set [a, b, c] = {a, b, c}
map f [a, b, c] = [f a, f b, f c]
rev [a, b, c] = [c, b, a]
hd [a, b, c, d] = a
tl [a, b, c, d] = [b, c, d]
last [a, b, c, d] = d
butlast [a, b, c, d] = [a, b, c]
filter (λn::nat. n<2) [0,2,1] = [0,1]
concat [[a, b], [c, d, e], [], [f]] = [a, b, c, d, e, f]
fold f [a, b, c] x = f c (f b (f a x))
foldr f [a, b, c] x = f a (f b (f c x))
foldl f x [a, b, c] = f (f (f x a) b) c
successively (≠) [True, False, True, False]
zip [a, b, c] [x, y, z] = [(a, x), (b, y), (c, z)]
zip [a, b] [x, y, z] = [(a, x), (b, y)]
enumerate 3 [a, b, c] = [(3, a), (4, b), (5, c)]
List.product [a, b] [c, d] = [(a, c), (a, d), (b, c), (b, d)]
product-lists [[a, b], [c], [d, e]] = [[a, c, d], [a, c, e], [b, c, d], [b, c, e]]
splice [a, b, c] [x, y, z] = [a, x, b, y, c, z]
splice [a, b, c, d] [x, y] = [a, x, b, y, c, d]
shuffles [a, b] [c, d] = {[a, b, c, d], [a, c, b, d], [a, c, d, b], [c, a, b, d], [c, a, d, b], [c, d, a, b]}
take 2 [a, b, c, d] = [a, b]
take 6 [a, b, c, d] = [a, b, c, d]
drop 2 [a, b, c, d] = [c, d]
drop 6 [a, b, c, d] = []
takeWhile (λn. n < 3) [1, 2, 3, 0] = [1, 2]
dropWhile (λn. n < 3) [1, 2, 3, 0] = [3, 0]
distinct [2, 0, 1]
remdups [2, 0, 2, 1, 2] = [0, 1, 2]
remdups-adj [2, 2, 3, 1, 1, 2, 1] = [2, 3, 1, 2, 1]
List.insert 2 [0, 1, 2] = [0, 1, 2]
List.insert 3 [0, 1, 2] = [3, 0, 1, 2]
List.union [2, 3, 4] [0, 1, 2] = [4, 3, 0, 1, 2]
find ((<) 0) [0, 0] = None
find ((<) 0) [0, 1, 0, 2] = Some 1
count-list [0, 1, 0, 2] 0 = 2
List.extract ((<) 0) [0, 0] = None
List.extract ((<) 0) [0, 1, 0, 2] = Some ([0], 1, [0, 2])
remove1 2 [2, 0, 2, 1, 2] = [0, 2, 1, 2]
removeAll 2 [2, 0, 2, 1, 2] = [0, 1]
[a, b, c, d] ! 2 = c
[a, b, c, d][2 := x] = [a, b, x, d]
nths [a, b, c, d, e] {0, 2, 3} = [a, c, d]
subseqs [a, b] = [[a, b], [a], [b], []]
List.n-lists 2 [a, b, c] = [[a, a], [b, a], [c, a], [a, b], [b, b], [c, b], [a, c], [b, c], [c, c]]
rotate1 [a, b, c, d] = [b, c, d, a]
rotate 3 [a, b, c, d] = [d, a, b, c]
replicate 4 a = [a, a, a, a]
[2..<5] = [2, 3, 4]
min-list [3, 1, − 2] = − 2
*arg-min-list (λi. i * i) [3, − 1, 1, − 2] = − 1*

Figure 1 shows characteristic examples that should give an intuitive understanding of the above functions.

The following simple sort(ed) functions are intended for proofs, not for efficient implementations.

A sorted predicate w.r.t. a relation:

fun *sorted-wrt* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
$sorted\text{-}wrt\ P\ [] = True\ |$
$sorted\text{-}wrt\ P\ (x\ \#\ ys) = ((\forall y \in set\ ys.\ P\ x\ y) \wedge sorted\text{-}wrt\ P\ ys)$

A class-based sorted predicate:

context *linorder*
begin

abbreviation *sorted* :: $'a\ list \Rightarrow bool$ **where**
$sorted \equiv sorted\text{-}wrt\ (\leq)$

lemma *sorted-simps*: $sorted\ [] = True$ $sorted\ (x\ \#\ ys) = ((\forall y \in set\ ys.\ x \leq y) \wedge sorted\ ys)$
by *auto*

lemma *strict-sorted-simps*: $sorted\text{-}wrt\ (<)\ [] = True$ $sorted\text{-}wrt\ (<)\ (x\ \#\ ys) = ((\forall y \in set\ ys.\ x<y) \wedge sorted\text{-}wrt\ (<)\ ys)$
by *auto*

primrec *insort-key* :: $('b \Rightarrow 'a) \Rightarrow 'b \Rightarrow 'b\ list \Rightarrow 'b\ list$ **where**
$insort\text{-}key\ f\ x\ [] = [x]\ |$
$insort\text{-}key\ f\ x\ (y\#ys) =$
$(if\ f\ x \leq f\ y\ then\ (x\#y\#ys)\ else\ y\#(insort\text{-}key\ f\ x\ ys))$

definition *sort-key* :: $('b \Rightarrow 'a) \Rightarrow 'b\ list \Rightarrow 'b\ list$ **where**
$sort\text{-}key\ f\ xs = foldr\ (insort\text{-}key\ f)\ xs\ []$

definition *insort-insert-key* :: $('b \Rightarrow 'a) \Rightarrow 'b \Rightarrow 'b\ list \Rightarrow 'b\ list$ **where**
$insort\text{-}insert\text{-}key\ f\ x\ xs =$
$(if\ f\ x \in f\ `\ set\ xs\ then\ xs\ else\ insort\text{-}key\ f\ x\ xs)$

abbreviation $sort \equiv sort\text{-}key\ (\lambda x.\ x)$
abbreviation $insort \equiv insort\text{-}key\ (\lambda x.\ x)$
abbreviation $insort\text{-}insert \equiv insort\text{-}insert\text{-}key\ (\lambda x.\ x)$

definition *stable-sort-key* :: $(('b \Rightarrow 'a) \Rightarrow 'b\ list \Rightarrow 'b\ list) \Rightarrow bool$ **where**
$stable\text{-}sort\text{-}key\ sk =$
$(\forall f\ xs\ k.\ filter\ (\lambda y.\ f\ y = k)\ (sk\ f\ xs) = filter\ (\lambda y.\ f\ y = k)\ xs)$

lemma *strict-sorted-iff*: $sorted\text{-}wrt\ (<)\ l \longleftrightarrow sorted\ l \wedge distinct\ l$
by (*induction l*) (*auto iff*: *antisym-conv1*)

lemma *strict-sorted-imp-sorted*: *sorted-wrt* (<) *xs* ⟹ *sorted xs*
  by (*auto simp*: *strict-sorted-iff*)

end

66.1.1 List comprehension

Input syntax for Haskell-like list comprehension notation. Typical example: [(*x*,*y*). *x* ← *xs*, *y* ← *ys*, *x* ≠ *y*], the list of all pairs of distinct elements from *xs* and *ys*. The syntax is as in Haskell, except that | becomes a dot (like in Isabelle's set comprehension): [*e*. *x* ← *xs*, ...] rather than `[e| x <- xs, ...]`.
The qualifiers after the dot are

generators *p* ← *xs*, where *p* is a pattern and *xs* an expression of list type, or

guards *b*, where *b* is a boolean expression.

Just like in Haskell, list comprehension is just a shorthand. To avoid misunderstandings, the translation into desugared form is not reversed upon output. Note that the translation of [*e*. *x* ← *xs*] is optmized to *map* (*λx*. *e*) *xs*.
It is easy to write short list comprehensions which stand for complex expressions. During proofs, they may become unreadable (and mangled). In such cases it can be advisable to introduce separate definitions for the list comprehensions in question.

nonterminal *lc-qual* **and** *lc-quals*

open-bundle *list-comprehension-syntax*
begin
syntax
  -listcompr :: *'a* ⇒ *lc-qual* ⇒ *lc-quals* ⇒ *'a list* (‹[- . --›)
  -lc-gen :: *'a* ⇒ *'a list* ⇒ *lc-qual* (‹- ← -›)
  -lc-test :: *bool* ⇒ *lc-qual* (‹-›)

  -lc-end :: *lc-quals* (‹]›)
  -lc-quals :: *lc-qual* ⇒ *lc-quals* ⇒ *lc-quals* (‹, --›)

syntax (*ASCII*)
  -lc-gen :: *'a* ⇒ *'a list* ⇒ *lc-qual* (‹- <− -›)
end

parse-translation ‹
let
  val NilC = *Syntax.const* ***const-syntax***‹*Nil*›;
  val ConsC = *Syntax.const* ***const-syntax***‹*Cons*›;

```
val mapC = Syntax.const const-syntax‹map›;
val concatC = Syntax.const const-syntax‹concat›;
val IfC = Syntax.const const-syntax‹If›;
val dummyC = Syntax.const const-syntax‹Pure.dummy-pattern›

fun single x = ConsC $ x $ NilC;

fun pat-tr ctxt p e opti = (* %x. case x of p => e | - => [] *)
  let
    (* FIXME proper name context!? *)
    val x =
      Free (singleton (Name.variant-list (fold Term.add-free-names [p, e] [])) x,
dummyT);
    val e = if opti then single e else e;
    val case1 = Syntax.const syntax-const‹-case1› $ p $ e;
    val case2 =
      Syntax.const syntax-const‹-case1› $ dummyC $ NilC;
    val cs = Syntax.const syntax-const‹-case2› $ case1 $ case2;
  in Syntax-Trans.abs-tr [x, Case-Translation.case-tr false ctxt [x, cs]] end;

fun pair-pat-tr (x as Free -) e = Syntax-Trans.abs-tr [x, e]
  | pair-pat-tr (- $ p1 $ p2) e =
      Syntax.const const-syntax‹case-prod› $ pair-pat-tr p1 (pair-pat-tr p2 e)
  | pair-pat-tr dummy e = Syntax-Trans.abs-tr [Syntax.const -idtdummy, e]

fun pair-pat ctxt (Const (const-syntax‹Pair›,-) $ s $ t) =
      pair-pat ctxt s andalso pair-pat ctxt t
  | pair-pat ctxt (Free (s,-)) =
      let
        val thy = Proof-Context.theory-of ctxt;
        val s' = Proof-Context.intern-const ctxt s;
      in not (Sign.declared-const thy s') end
  | pair-pat - t = (t = dummyC);

fun abs-tr ctxt p e opti =
  let val p = Term-Position.strip-positions p
  in if pair-pat ctxt p
     then (pair-pat-tr p e, true)
     else (pat-tr ctxt p e opti, false)
  end

fun lc-tr ctxt [e, Const (syntax-const‹-lc-test›, -) $ b, qs] =
  let
    val res =
      (case qs of
         Const (syntax-const‹-lc-end›, -) => single e
       | Const (syntax-const‹-lc-quals›, -) $ q $ qs => lc-tr ctxt [e, q, qs]);
  in IfC $ b $ res $ NilC end
| lc-tr ctxt
```

```
    [e, Const (syntax-const ‹-lc-gen›, -) $ p $ es,
        Const(syntax-const ‹-lc-end›, -)] =
    (case abs-tr ctxt p e true of
       (f, true) => mapC $ f $ es
     | (f, false) => concatC $ (mapC $ f $ es))
  | lc-tr ctxt
    [e, Const (syntax-const ‹-lc-gen›, -) $ p $ es,
        Const (syntax-const ‹-lc-quals›, -) $ q $ qs] =
    let val e' = lc-tr ctxt [e, q, qs];
    in concatC $ (mapC $ (fst (abs-tr ctxt p e' false)) $ es) end;

in [(syntax-const ‹-listcompr›, lc-tr)] end
›
```

ML-val ‹

```
  let
    val read = Syntax.read-term context o Syntax.implode-input;
    fun check s1 s2 =
      read s1 aconv read s2 orelse
        error (Check failed: ^
          quote (#1 (Input.source-content s1)) ^ Position.here-list [Input.pos-of s1,
Input.pos-of s2]);
  in
    check ‹[(x,y,z). b]› ‹if b then [(x, y, z)] else []›;
    check ‹[(x,y,z). (x,-,y)←xs]› ‹map (λ(x,-,y). (x, y, z)) xs›;
    check ‹[e x y. (x,-)←xs, y←ys]› ‹concat (map (λ(x,-). map (λy. e x y) ys) xs)›;
    check ‹[(x,y,z). x<a, x>b]› ‹if x < a then if b < x then [(x, y, z)] else [] else
[]›;
    check ‹[(x,y,z). x←xs, x>b]› ‹concat (map (λx. if b < x then [(x, y, z)] else [])
xs)›;
    check ‹[(x,y,z). x<a, x←xs]› ‹if x < a then map (λx. (x, y, z)) xs else []›;
    check ‹[(x,y). Cons True x ← xs]›
      ‹concat (map (λxa. case xa of [] ⇒ [] | True # x ⇒ [(x, y)] | False # x ⇒ [])
xs)›;
    check ‹[(x,y,z). Cons x [] ← xs]›
      ‹concat (map (λxa. case xa of [] ⇒ [] | [x] ⇒ [(x, y, z)] | x # aa # lista ⇒
[]) xs)›;
    check ‹[(x,y,z). x<a, x>b, x=d]›
      ‹if x < a then if b < x then if x = d then [(x, y, z)] else [] else [] else []›;
    check ‹[(x,y,z). x<a, x>b, y←ys]›
      ‹if x < a then if b < x then map (λy. (x, y, z)) ys else [] else []›;
    check ‹[(x,y,z). x<a, (-,x)←xs,y>b]›
      ‹if x < a then concat (map (λ(-,x). if b < y then [(x, y, z)] else []) xs) else []›;
    check ‹[(x,y,z). x<a, x←xs, y←ys]›
      ‹if x < a then concat (map (λx. map (λy. (x, y, z)) ys) xs) else []›;
    check ‹[(x,y,z). x←xs, x>b, y<a]›
      ‹concat (map (λx. if b < x then if y < a then [(x, y, z)] else [] else []) xs)›;
    check ‹[(x,y,z). x←xs, x>b, y←ys]›
      ‹concat (map (λx. if b < x then map (λy. (x, y, z)) ys else []) xs)›;
```

```
    check ‹[(x,y,z). x←xs, (y,-)←ys,y>x]›
      ‹concat (map (λx. concat (map (λ(y,-). if x < y then [(x, y, z)] else []) ys))
xs)›;
    check ‹[(x,y,z). x←xs, y←ys,z←zs]›
      ‹concat (map (λx. concat (map (λy. map (λz. (x, y, z)) zs) ys)) xs)›
  end;
›
```

ML ‹

```
(* Simproc for rewriting list comprehensions applied to List.set to set
   comprehension. *)

signature LIST-TO-SET-COMPREHENSION =
sig
  val proc: Simplifier.proc
end

structure List-to-Set-Comprehension : LIST-TO-SET-COMPREHENSION =
struct

(* conversion *)

fun all-exists-conv cv ctxt ct =
  (case Thm.term-of ct of
    Const (const-name ‹Ex›, -) $ Abs - =>
      Conv.arg-conv (Conv.abs-conv (all-exists-conv cv o #2) ctxt) ct
  | - => cv ctxt ct)

fun all-but-last-exists-conv cv ctxt ct =
  (case Thm.term-of ct of
    Const (const-name ‹Ex›, -) $ Abs (-, -, Const (const-name ‹Ex›, -) $ -) =>
      Conv.arg-conv (Conv.abs-conv (all-but-last-exists-conv cv o #2) ctxt) ct
  | - => cv ctxt ct)

fun Collect-conv cv ctxt ct =
  (case Thm.term-of ct of
    Const (const-name ‹Collect›, -) $ Abs - => Conv.arg-conv (Conv.abs-conv cv
ctxt) ct
  | - => raise CTERM (Collect-conv, [ct]))

fun rewr-conv′ th = Conv.rewr-conv (mk-meta-eq th)

fun conjunct-assoc-conv ct =
  Conv.try-conv
   (rewr-conv′ @{thm conj-assoc} then-conv HOLogic.conj-conv Conv.all-conv con-
junct-assoc-conv) ct

fun right-hand-set-comprehension-conv conv ctxt =
  HOLogic.Trueprop-conv (HOLogic.eq-conv Conv.all-conv
```

```
  (Collect-conv (all-exists-conv conv o #2) ctxt))


(* term abstraction of list comprehension patterns *)

datatype termlets = If | Case of typ * int

local

val set-Nil-I = @{lemma set [] = {x. False} by (simp add: empty-def [symmetric])}
val set-singleton = @{lemma set [a] = {x. x = a} by simp}
val inst-Collect-mem-eq = @{lemma set A = {x. x ∈ set A} by simp}
val del-refl-eq = @{lemma (t = t ∧ P) ≡ P by simp}

fun mk-set T = Const (const-name ‹set›, HOLogic.listT T --> HOLogic.mk-setT
T)
fun dest-set (Const (const-name ‹set›, -) $ xs) = xs

fun dest-singleton-list (Const (const-name ‹Cons›, -) $ t $ (Const (const-name ‹Nil›,
-))) = t
  | dest-singleton-list t = raise TERM (dest-singleton-list, [t])

(*We check that one case returns a singleton list and all other cases
  return [], and return the index of the one singleton list case.*)
fun possible-index-of-singleton-case cases =
  let
    fun check (i, case-t) s =
      (case strip-abs-body case-t of
        (Const (const-name ‹Nil›, -)) => s
      | - => (case s of SOME NONE => SOME (SOME i) | - => NONE))
  in
    fold-index check cases (SOME NONE) |> the-default NONE
  end

(*returns condition continuing term option*)
fun dest-if (Const (const-name ‹If›, -) $ cond $ then-t $ Const (const-name ‹Nil›,
-)) =
      SOME (cond, then-t)
  | dest-if - = NONE

(*returns (case-expr type index chosen-case constr-name) option*)
fun dest-case ctxt case-term =
  let
    val (case-const, args) = strip-comb case-term
  in
    (case try dest-Const case-const of
      SOME (c, T) =>
        (case Ctr-Sugar.ctr-sugar-of-case ctxt c of
          SOME {ctrs, ...} =>
```

```
          (case possible-index-of-singleton-case (fst (split-last args)) of
            SOME i =>
              let
                val constr-names = map dest-Const-name ctrs
                val (Ts, -) = strip-type T
                val T' = List.last Ts
              in SOME (List.last args, T', i, nth args i, nth constr-names i) end
          | NONE => NONE)
      | NONE => NONE)
  | NONE => NONE)
  end

fun tac ctxt [] =
      resolve-tac ctxt [set-singleton] 1 ORELSE
      resolve-tac ctxt [inst-Collect-mem-eq] 1
  | tac ctxt (If :: cont) =
      Splitter.split-tac ctxt @{thms if-split} 1
      THEN resolve-tac ctxt @{thms conjI} 1
      THEN resolve-tac ctxt @{thms impI} 1
      THEN Subgoal.FOCUS (fn {prems, context = ctxt', ...} =>
        CONVERSION (right-hand-set-comprehension-conv (K
       (HOLogic.conj-conv (Conv.rewr-conv (List.last prems RS @{thm Eq-TrueI}))
Conv.all-conv
           then-conv
           rewr-conv' @{lemma (True ∧ P) = P by simp})) ctxt') 1) ctxt 1
      THEN tac ctxt cont
      THEN resolve-tac ctxt @{thms impI} 1
      THEN Subgoal.FOCUS (fn {prems, context = ctxt', ...} =>
          CONVERSION (right-hand-set-comprehension-conv (K
                (HOLogic.conj-conv (Conv.rewr-conv (List.last prems RS @{thm
Eq-FalseI})) Conv.all-conv
            then-conv rewr-conv' @{lemma (False ∧ P) = False by simp})) ctxt') 1)
ctxt 1
      THEN resolve-tac ctxt [set-Nil-I] 1
  | tac ctxt (Case (T, i) :: cont) =
      let
        val SOME {injects, distincts, case-thms, split, ...} =
          Ctr-Sugar.ctr-sugar-of ctxt (dest-Type-name T)
      in
        (* do case distinction *)
        Splitter.split-tac ctxt [split] 1
        THEN EVERY (map-index (fn (i', -) =>
          (if i' < length case-thms − 1 then resolve-tac ctxt @{thms conjI} 1 else
all-tac)
          THEN REPEAT-DETERM (resolve-tac ctxt @{thms allI} 1)
          THEN resolve-tac ctxt @{thms impI} 1
          THEN (if i' = i then
            (* continue recursively *)
            Subgoal.FOCUS (fn {prems, context = ctxt', ...} =>
```

```
        CONVERSION (Thm.eta-conversion then-conv right-hand-set-comprehension-conv
(K
              ((HOLogic.conj-conv
                   (HOLogic.eq-conv Conv.all-conv (rewr-conv' (List.last prems))
then-conv
                 (Conv.try-conv (Conv.rewrs-conv (map mk-meta-eq injects))))
                Conv.all-conv)
                then-conv (Conv.try-conv (Conv.rewr-conv del-refl-eq))
                then-conv conjunct-assoc-conv)) ctxt'
            then-conv
              (HOLogic.Trueprop-conv
                (HOLogic.eq-conv Conv.all-conv (Collect-conv (fn (-, ctxt'') =>
                  Conv.repeat-conv
                    (all-but-last-exists-conv
                      (K (rewr-conv'
                           @{lemma (∃x. x = t ∧ P x) = P t by simp})) ctxt''))
ctxt')))) 1) ctxt 1
          THEN tac ctxt cont
        else
          Subgoal.FOCUS (fn {prems, context = ctxt', ...} =>
            CONVERSION
              (right-hand-set-comprehension-conv (K
                (HOLogic.conj-conv
                  ((HOLogic.eq-conv Conv.all-conv
                    (rewr-conv' (List.last prems))) then-conv
                      (Conv.rewrs-conv (map (fn th => th RS @{thm Eq-FalseI})
distincts)))
                  Conv.all-conv then-conv
                      (rewr-conv' @{lemma (False ∧ P) = False by simp}))) ctxt'
then-conv
              HOLogic.Trueprop-conv
                (HOLogic.eq-conv Conv.all-conv
                  (Collect-conv (fn (-, ctxt'') =>
                    Conv.repeat-conv
                      (Conv.bottom-conv
                         (K (rewr-conv' @{lemma (∃x. P) = P by simp})) ctxt''))
ctxt'))) 1) ctxt 1
          THEN resolve-tac ctxt [set-Nil-I] 1)) case-thms)
    end

in

fun proc ctxt redex =
  let
    fun make-inner-eqs bound-vs Tis eqs t =
      (case dest-case ctxt t of
        SOME (x, T, i, cont, constr-name) =>
          let
            val (vs, body) = strip-abs (Envir.eta-long (map snd bound-vs) cont)
```

```
          val x' = incr-boundvars (length vs) x
          val eqs' = map (incr-boundvars (length vs)) eqs
          val constr-t =
            list-comb
              (Const (constr-name, map snd vs ---> T), map Bound (((length vs) - 1) downto 0))
          val constr-eq = Const (const-name ‹HOL.eq›, T --> T --> typ ‹bool›) $ constr-t $ x'
        in
          make-inner-eqs (rev vs @ bound-vs) (Case (T, i) :: Tis) (constr-eq :: eqs') body
        end
    | NONE =>
        (case dest-if t of
        SOME (condition, cont) => make-inner-eqs bound-vs (If :: Tis) (condition :: eqs) cont
        | NONE =>
          if null eqs then NONE (*no rewriting, nothing to be done*)
          else
            let
             val Type (type-name ‹list›, [rT]) = fastype-of1 (map snd bound-vs, t)
              val pat-eq =
                (case try dest-singleton-list t of
                  SOME t' =>
                    Const (const-name ‹HOL.eq›, rT --> rT --> typ ‹bool›) $
                      Bound (length bound-vs) $ t'
                | NONE =>
                    Const (const-name ‹Set.member›, rT --> HOLogic.mk-setT rT --> typ ‹bool›) $
                      Bound (length bound-vs) $ (mk-set rT $ t))
              val reverse-bounds = curry subst-bounds
                ((map Bound ((length bound-vs - 1) downto 0)) @ [Bound (length bound-vs)])
              val eqs' = map reverse-bounds eqs
              val pat-eq' = reverse-bounds pat-eq
              val inner-t =
              fold (fn (-, T) => fn t => HOLogic.exists-const T $ absdummy T t)
                  (rev bound-vs) (fold (curry HOLogic.mk-conj) eqs' pat-eq')
              val lhs = Thm.term-of redex
              val rhs = HOLogic.mk-Collect (x, rT, inner-t)
             val rewrite-rule-t = HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs, rhs))
            in
              SOME
                ((Goal.prove ctxt [] [] rewrite-rule-t
                    (fn {context = ctxt', ...} => tac ctxt' (rev Tis))) RS @{thm eq-reflection})
            end))
  in
    make-inner-eqs [] [] [] (dest-set (Thm.term-of redex))
```

end

end

end
›

simproc-setup *list-to-set-comprehension* (*set xs*) =
‹*K List-to-Set-Comprehension.proc*›

code-datatype *set coset*
hide-const (**open**) *coset*

66.1.2 [] and (#)

lemma *not-Cons-self* [*simp*]:
$xs \neq x \# xs$
by (*induct xs*) *auto*

lemma *not-Cons-self2* [*simp*]: $x \# xs \neq xs$
by (*rule not-Cons-self* [*symmetric*])

lemma *neq-Nil-conv*: $(xs \neq []) = (\exists y\ ys.\ xs = y \# ys)$
by (*induct xs*) *auto*

lemma *tl-Nil*: $tl\ xs = [] \longleftrightarrow xs = [] \vee (\exists x.\ xs = [x])$
by (*cases xs*) *auto*

lemmas *Nil-tl* = *tl-Nil*[*THEN eq-iff-swap*]

lemma *length-induct*:
$(\bigwedge xs.\ \forall ys.\ length\ ys < length\ xs \longrightarrow P\ ys \Longrightarrow P\ xs) \Longrightarrow P\ xs$
by (*fact measure-induct*)

lemma *induct-list012*:
$[\![P\ [];\ \bigwedge x.\ P\ [x];\ \bigwedge x\ y\ zs.\ [\![\ P\ zs;\ P\ (y \# zs)\ ]\!] \Longrightarrow P\ (x \# y \# zs)]\!] \Longrightarrow P\ xs$
by *induction-schema* (*pat-completeness*, *lexicographic-order*)

lemma *list-nonempty-induct* [*consumes 1*, *case-names single cons*]:
$[\![\ xs \neq [];\ \bigwedge x.\ P\ [x];\ \bigwedge x\ xs.\ xs \neq [] \Longrightarrow P\ xs \Longrightarrow P\ (x \# xs)]\!] \Longrightarrow P\ xs$
by(*induction xs rule*: *induct-list012*) *auto*

lemma *inj-split-Cons*: *inj-on* $(\lambda(xs, n).\ n\#xs)\ X$
by (*auto intro*!: *inj-onI*)

lemma *inj-on-Cons1* [*simp*]: *inj-on* ((#) *x*) *A*
by(*simp add*: *inj-on-def*)

66.1.3 *length*

Needs to come before @ because of theorem *append-eq-append-conv*.

lemma *length-append* [*simp*]: *length* (*xs* @ *ys*) = *length xs* + *length ys*
by (*induct xs*) *auto*

lemma *length-map* [*simp*]: *length* (*map f xs*) = *length xs*
by (*induct xs*) *auto*

lemma *length-rev* [*simp*]: *length* (*rev xs*) = *length xs*
by (*induct xs*) *auto*

lemma *length-tl* [*simp*]: *length* (*tl xs*) = *length xs* − *1*
by (*cases xs*) *auto*

lemma *length-0-conv* [*iff*]: (*length xs* = *0*) = (*xs* = [])
by (*induct xs*) *auto*

lemma *length-greater-0-conv* [*iff*]: (*0* < *length xs*) = (*xs* ≠ [])
by (*induct xs*) *auto*

lemma *length-pos-if-in-set*: *x* ∈ *set xs* ⟹ *length xs* > *0*
by *auto*

lemma *length-Suc-conv*: (*length xs* = *Suc n*) = (∃ *y ys*. *xs* = *y* # *ys* ∧ *length ys* = *n*)
by (*induct xs*) *auto*

lemmas *Suc-length-conv* = *length-Suc-conv*[*THEN eq-iff-swap*]

lemma *Suc-le-length-iff*:
(*Suc n* ≤ *length xs*) = (∃ *x ys*. *xs* = *x* # *ys* ∧ *n* ≤ *length ys*)
by (*metis Suc-le-D*[*of n*] *Suc-le-mono*[*of n*] *Suc-length-conv*[*of - xs*])

lemma *impossible-Cons*: *length xs* ≤ *length ys* ⟹ *xs* = *x* # *ys* = *False*
by (*induct xs*) *auto*

lemma *list-induct2* [*consumes 1*, *case-names Nil Cons*]:
length xs = *length ys* ⟹ *P* [] [] ⟹
(⋀*x xs y ys*. *length xs* = *length ys* ⟹ *P xs ys* ⟹ *P* (*x*#*xs*) (*y*#*ys*))
⟹ *P xs ys*
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs ys*) **then show** *?case* **by** (*cases ys*) *simp-all*
qed *simp*

lemma *list-induct3* [*consumes 2*, *case-names Nil Cons*]:
length xs = *length ys* ⟹ *length ys* = *length zs* ⟹ *P* [] [] [] ⟹
(⋀*x xs y ys z zs*. *length xs* = *length ys* ⟹ *length ys* = *length zs* ⟹ *P xs ys zs* ⟹ *P* (*x*#*xs*) (*y*#*ys*) (*z*#*zs*))

$\Longrightarrow$ *P xs ys zs*
proof (*induct xs arbitrary*: *ys zs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs ys zs*) **then show** *?case* **by** (*cases ys*, *simp-all*)
(*cases zs*, *simp-all*)
qed

lemma *list-induct4* [*consumes 3*, *case-names Nil Cons*]:
length xs = *length ys* $\Longrightarrow$ *length ys* = *length zs* $\Longrightarrow$ *length zs* = *length ws* $\Longrightarrow$
P [] [] [] [] $\Longrightarrow$ ($\bigwedge$*x xs y ys z zs w ws*. *length xs* = *length ys* $\Longrightarrow$
length ys = *length zs* $\Longrightarrow$ *length zs* = *length ws* $\Longrightarrow$ *P xs ys zs ws* $\Longrightarrow$
P (*x#xs*) (*y#ys*) (*z#zs*) (*w#ws*)) $\Longrightarrow$ *P xs ys zs ws*
proof (*induct xs arbitrary*: *ys zs ws*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs ys zs ws*) **then show** *?case* **by** ((*cases ys*, *simp-all*), (*cases zs*,*simp-all*)) (*cases ws*, *simp-all*)
qed

lemma *list-induct2′*:
⟦ *P* [] [];
$\bigwedge$*x xs*. *P* (*x#xs*) [];
$\bigwedge$*y ys*. *P* [] (*y#ys*);
$\bigwedge$*x xs y ys*. *P xs ys* $\Longrightarrow$ *P* (*x#xs*) (*y#ys*) ⟧
$\Longrightarrow$ *P xs ys*
by (*induct xs arbitrary*: *ys*) (*case-tac x*, *auto*)+

lemma *list-all2-iff*:
list-all2 P xs ys $\longleftrightarrow$ *length xs* = *length ys* $\wedge$ ($\forall$(*x*, *y*) $\in$ *set* (*zip xs ys*). *P x y*)
by (*induct xs ys rule*: *list-induct2′*) *auto*

lemma *neq-if-length-neq*: *length xs* $\neq$ *length ys* $\Longrightarrow$ (*xs* = *ys*) == *False*
by (*rule Eq-FalseI*) *auto*

66.1.4 @ – append

global-interpretation *append*: *monoid append Nil*
proof
fix *xs ys zs* :: *′a list*
show (*xs* @ *ys*) @ *zs* = *xs* @ (*ys* @ *zs*)
by (*induct xs*) *simp-all*
show *xs* @ [] = *xs*
by (*induct xs*) *simp-all*
qed *simp*

lemma *append-assoc* [*simp*]: (*xs* @ *ys*) @ *zs* = *xs* @ (*ys* @ *zs*)
by (*fact append.assoc*)

lemma *append-Nil2*: *xs* @ [] = *xs*
by (*fact append.right-neutral*)

lemma *append-is-Nil-conv* [*iff*]: (*xs* @ *ys* = []) = (*xs* = [] ∧ *ys* = [])
by (*induct xs*) *auto*

lemmas *Nil-is-append-conv* [*iff*] = *append-is-Nil-conv*[*THEN eq-iff-swap*]

lemma *append-self-conv* [*iff*]: (*xs* @ *ys* = *xs*) = (*ys* = [])
by (*induct xs*) *auto*

lemmas *self-append-conv* [*iff*] = *append-self-conv*[*THEN eq-iff-swap*]

lemma *append-eq-append-conv* [*simp*]:
length xs = *length ys* ∨ *length us* = *length vs*
⟹ (*xs*@*us* = *ys*@*vs*) = (*xs*=*ys* ∧ *us*=*vs*)
by (*induct xs arbitrary*: *ys*; *case-tac ys*; *force*)

lemma *append-eq-append-conv2*: (*xs* @ *ys* = *zs* @ *ts*) =
(∃ *us*. *xs* = *zs* @ *us* ∧ *us* @ *ys* = *ts* ∨ *xs* @ *us* = *zs* ∧ *ys* = *us* @ *ts*)
proof (*induct xs arbitrary*: *ys zs ts*)
case (*Cons x xs*)
then show *?case*
by (*cases zs*) *auto*
qed *fastforce*

lemma *same-append-eq* [*iff*, *induct-simp*]: (*xs* @ *ys* = *xs* @ *zs*) = (*ys* = *zs*)
by *simp*

lemma *append1-eq-conv* [*iff*]: (*xs* @ [*x*] = *ys* @ [*y*]) = (*xs* = *ys* ∧ *x* = *y*)
by *simp*

lemma *append-same-eq* [*iff*, *induct-simp*]: (*ys* @ *xs* = *zs* @ *xs*) = (*ys* = *zs*)
by *simp*

lemma *append-self-conv2* [*iff*]: (*xs* @ *ys* = *ys*) = (*xs* = [])
using *append-same-eq* [*of* - - []] **by** *auto*

lemmas *self-append-conv2* [*iff*] = *append-self-conv2*[*THEN eq-iff-swap*]

lemma *hd-Cons-tl*: *xs* ≠ [] ⟹ *hd xs* # *tl xs* = *xs*
by (*fact list.collapse*)

lemma *hd-append*: *hd* (*xs* @ *ys*) = (*if xs* = [] *then hd ys else hd xs*)
by (*induct xs*) *auto*

lemma *hd-append2* [*simp*]: *xs* ≠ [] ⟹ *hd* (*xs* @ *ys*) = *hd xs*
by (*simp add*: *hd-append split*: *list.split*)

lemma *tl-append*: *tl* (*xs* @ *ys*) = (*case xs of* [] ⇒ *tl ys* | *z*#*zs* ⇒ *zs* @ *ys*)
by (*simp split*: *list.split*)

lemma *tl-append2* [*simp*]: *xs* ≠ [] ⟹ *tl* (*xs* @ *ys*) = *tl xs* @ *ys*
by (*simp add*: *tl-append split*: *list.split*)

lemma *tl-append-if*: *tl* (*xs* @ *ys*) = (*if xs* = [] *then tl ys else tl xs* @ *ys*)
by (*simp*)

lemma *Cons-eq-append-conv*: *x*#*xs* = *ys*@*zs* =
(*ys* = [] ∧ *x*#*xs* = *zs* ∨ (∃ *ys'*. *x*#*ys'* = *ys* ∧ *xs* = *ys'*@*zs*))
by(*cases ys*) *auto*

lemma *append-eq-Cons-conv*: (*ys*@*zs* = *x*#*xs*) =
(*ys* = [] ∧ *zs* = *x*#*xs* ∨ (∃ *ys'*. *ys* = *x*#*ys'* ∧ *ys'*@*zs* = *xs*))
by(*cases ys*) *auto*

lemma *longest-common-prefix*:
∃ *ps xs' ys'*. *xs* = *ps* @ *xs'* ∧ *ys* = *ps* @ *ys'*
∧ (*xs'* = [] ∨ *ys'* = [] ∨ *hd xs'* ≠ *hd ys'*)
by (*induct xs ys rule*: *list-induct2'*)
(*blast*, *blast*, *blast*,
metis (*no-types*, *opaque-lifting*) *append-Cons append-Nil list.sel*(*1*))

Trivial rules for solving @-equations automatically.

lemma *eq-Nil-appendI*: *xs* = *ys* ⟹ *xs* = [] @ *ys*
by *simp*

lemma *Cons-eq-appendI*: ⟦*x* # *xs1* = *ys*; *xs* = *xs1* @ *zs*⟧ ⟹ *x* # *xs* = *ys* @ *zs*
by *auto*

lemma *append-eq-appendI*: ⟦*xs* @ *xs1* = *zs*; *ys* = *xs1* @ *us*⟧ ⟹ *xs* @ *ys* = *zs* @ *us*
by *auto*

Simplification procedure for all list equalities. Currently only tries to rearrange @ to see if - both lists end in a singleton list, - or both lists end in the same list.

simproc-setup *list-eq* ((*xs*::'*a list*) = *ys*) = ‹
let
fun last (*cons as Const* (***const-name*** ‹*Cons*›, -) $ - $ *xs*) =
(*case xs of Const* (***const-name*** ‹*Nil*›, -) => *cons* | - => *last xs*)
| *last* (*Const*(***const-name*** ‹*append*›,-) $ - $ *ys*) = *last ys*
| *last t* = *t*;

fun list1 (*Const*(***const-name*** ‹*Cons*›,-) $ - $ *Const*(***const-name*** ‹*Nil*›,-)) = *true*
| *list1* - = *false*;

```
  fun butlast ((cons as Const(const-name ‹Cons›,-) $ x) $ xs) =
        (case xs of Const (const-name ‹Nil›, -) => xs | - => cons $ butlast xs)
    | butlast ((app as Const (const-name ‹append›, -) $ xs) $ ys) = app $ butlast
ys
    | butlast xs = Const(const-name ‹Nil›, fastype-of xs);

  val rearr-ss =
    simpset-of (put-simpset HOL-basic-ss context
    addsimps [@{thm append-assoc}, @{thm append-Nil}, @{thm append-Cons}]);

  fun list-eq ctxt (F as (eq as Const(-,eqT)) $ lhs $ rhs) =
    let
      val lastl = last lhs and lastr = last rhs;
      fun rearr conv =
        let
          val lhs1 = butlast lhs and rhs1 = butlast rhs;
          val Type(-,listT::-) = eqT
          val appT = [listT,listT] ---> listT
          val app = Const(const-name ‹append›,appT)
          val F2 = eq $ (app$lhs1$lastl) $ (app$rhs1$lastr)
          val eq = HOLogic.mk-Trueprop (HOLogic.mk-eq (F,F2));
          val thm = Goal.prove ctxt [] [] eq
            (K (simp-tac (put-simpset rearr-ss ctxt) 1));
        in SOME ((conv RS (thm RS trans)) RS eq-reflection) end;
    in
      if list1 lastl andalso list1 lastr then rearr @{thm append1-eq-conv}
      else if lastl aconv lastr then rearr @{thm append-same-eq}
      else NONE
    end;
 in K (fn ctxt => fn ct => list-eq ctxt (Thm.term-of ct)) end
›
```

66.1.5 *map*

lemma *hd-map*: $xs \neq [] \Longrightarrow hd\ (map\ f\ xs) = f\ (hd\ xs)$
by (*cases xs*) *simp-all*

lemma *map-tl*: $map\ f\ (tl\ xs) = tl\ (map\ f\ xs)$
by (*cases xs*) *simp-all*

lemma *map-ext*: $(\bigwedge x.\ x \in set\ xs \longrightarrow f\ x = g\ x) \Longrightarrow map\ f\ xs = map\ g\ xs$
by (*induct xs*) *simp-all*

lemma *map-ident* [*simp*]: $map\ (\lambda x.\ x) = (\lambda xs.\ xs)$
by (*rule ext*, *induct-tac xs*) *auto*

lemma *map-append* [*simp*]: $map\ f\ (xs\ @\ ys) = map\ f\ xs\ @\ map\ f\ ys$
by (*induct xs*) *auto*

lemma *map-map* [*simp*]: *map f* (*map g xs*) = *map* (*f* ∘ *g*) *xs*
by (*induct xs*) *auto*

lemma *map-comp-map*[*simp*]: ((*map f*) ∘ (*map g*)) = *map*(*f* ∘ *g*)
by (*rule ext*) *simp*

lemma *rev-map*: *rev* (*map f xs*) = *map f* (*rev xs*)
by (*induct xs*) *auto*

lemma *map-eq-conv*[*simp*]: (*map f xs* = *map g xs*) = (∀ *x* ∈ *set xs*. *f x* = *g x*)
by (*induct xs*) *auto*

lemma *map-cong* [*fundef-cong*]:
  xs = *ys* ⟹ (⋀*x*. *x* ∈ *set ys* ⟹ *f x* = *g x*) ⟹ *map f xs* = *map g ys*
by *simp*

lemma *map-is-Nil-conv* [*iff*]: (*map f xs* = []) = (*xs* = [])
by (*rule list.map-disc-iff*)

lemmas *Nil-is-map-conv* [*iff*] = *map-is-Nil-conv*[*THEN eq-iff-swap*]

lemma *map-eq-Cons-conv*:
  (*map f xs* = *y*#*ys*) = (∃ *z zs*. *xs* = *z*#*zs* ∧ *f z* = *y* ∧ *map f zs* = *ys*)
by (*cases xs*) *auto*

lemma *Cons-eq-map-conv*:
  (*x*#*xs* = *map f ys*) = (∃ *z zs*. *ys* = *z*#*zs* ∧ *x* = *f z* ∧ *xs* = *map f zs*)
by (*cases ys*) *auto*

lemmas *map-eq-Cons-D* = *map-eq-Cons-conv* [*THEN iffD1*]
lemmas *Cons-eq-map-D* = *Cons-eq-map-conv* [*THEN iffD1*]
declare *map-eq-Cons-D* [*dest*!] *Cons-eq-map-D* [*dest*!]

lemma *ex-map-conv*:
  (∃ *xs*. *ys* = *map f xs*) = (∀ *y* ∈ *set ys*. ∃ *x*. *y* = *f x*)
by(*induct ys*, *auto simp add*: *Cons-eq-map-conv*)

lemma *map-eq-imp-length-eq*:
  assumes *map f xs* = *map g ys*
  shows *length xs* = *length ys*
  using *assms*
proof (*induct ys arbitrary*: *xs*)
  case *Nil* **then show** *?case* **by** *simp*
next
  case (*Cons y ys*) **then obtain** *z zs* **where** *xs*: *xs* = *z* # *zs* **by** *auto*
  from *Cons xs* **have** *map f zs* = *map g ys* **by** *simp*
  with *Cons* **have** *length zs* = *length ys* **by** *blast*
  with *xs* **show** *?case* **by** *simp*
qed

lemma *map-inj-on*:
assumes *map*: *map f xs = map f ys* **and** *inj*: *inj-on f (set xs Un set ys)*
shows *xs = ys*
using *map-eq-imp-length-eq* [*OF map*] *assms*
proof (*induct rule*: *list-induct2*)
case (*Cons x xs y ys*)
then show *?case*
by (*auto intro*: *sym*)
qed *auto*

lemma *inj-on-map-eq-map*:
inj-on f (set xs Un set ys) $\Longrightarrow$ (*map f xs = map f ys*) = (*xs = ys*)
by(*blast dest*:*map-inj-on*)

lemma *map-injective*:
map f xs = map f ys $\Longrightarrow$ *inj f* $\Longrightarrow$ *xs = ys*
by (*induct ys arbitrary*: *xs*) (*auto dest*!:*injD*)

lemma *inj-map-eq-map*[*simp*]: *inj f* $\Longrightarrow$ (*map f xs = map f ys*) = (*xs = ys*)
by(*blast dest*:*map-injective*)

lemma *inj-mapI*: *inj f* $\Longrightarrow$ *inj* (*map f*)
by (*rule list.inj-map*)

lemma *inj-mapD*: *inj* (*map f*) $\Longrightarrow$ *inj f*
by (*metis* (*no-types*, *opaque-lifting*) *injI list.inject list.simps*(*9*) *the-inv-f-f*)

lemma *inj-map*[*iff*]: *inj* (*map f*) = *inj f*
by (*blast dest*: *inj-mapD intro*: *inj-mapI*)

lemma *inj-on-mapI*: *inj-on f* ($\bigcup$(*set* ' *A*)) $\Longrightarrow$ *inj-on* (*map f*) *A*
by (*blast intro*:*inj-onI dest*:*inj-onD map-inj-on*)

lemma *map-idI*: ($\bigwedge$*x*. *x* $\in$ *set xs* $\Longrightarrow$ *f x = x*) $\Longrightarrow$ *map f xs = xs*
by (*rule list.map-ident-strong*)

lemma *map-fun-upd* [*simp*]: *y* $\notin$ *set xs* $\Longrightarrow$ *map* (*f*(*y*:=*v*)) *xs = map f xs*
by (*induct xs*) *auto*

lemma *map-fst-zip*[*simp*]:
length xs = length ys $\Longrightarrow$ *map fst* (*zip xs ys*) = *xs*
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *map-snd-zip*[*simp*]:
length xs = length ys $\Longrightarrow$ *map snd* (*zip xs ys*) = *ys*
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *map-fst-zip-take*:

map fst (*zip xs ys*) = *take* (*min* (*length xs*) (*length ys*)) *xs*
by (*induct xs ys rule*: *list-induct2'*) *simp-all*

lemma *map-snd-zip-take*:
map snd (*zip xs ys*) = *take* (*min* (*length xs*) (*length ys*)) *ys*
by (*induct xs ys rule*: *list-induct2'*) *simp-all*

lemma *map2-map-map*: *map2 h* (*map f xs*) (*map g xs*) = *map* (λ*x*. *h* (*f x*) (*g x*)) *xs*
by (*induction xs*) (*auto*)

functor *map*: *map*
by (*simp-all add*: *id-def*)

declare *map.id* [*simp*]

66.1.6 *rev*

lemma *rev-append* [*simp*]: *rev* (*xs* @ *ys*) = *rev ys* @ *rev xs*
by (*induct xs*) *auto*

lemma *rev-rev-ident* [*simp*]: *rev* (*rev xs*) = *xs*
by (*induct xs*) *auto*

lemma *rev-involution*[*simp*]: *rev* ∘ *rev* = *id*
by *auto*

lemma *rev-swap*: (*rev xs* = *ys*) = (*xs* = *rev ys*)
by *auto*

lemma *rev-is-Nil-conv* [*iff*]: (*rev xs* = []) = (*xs* = [])
by (*induct xs*) *auto*

lemmas *Nil-is-rev-conv* [*iff*] = *rev-is-Nil-conv*[*THEN eq-iff-swap*]

lemma *rev-singleton-conv* [*simp*]: (*rev xs* = [*x*]) = (*xs* = [*x*])
by (*cases xs*) *auto*

lemma *singleton-rev-conv* [*simp*]: ([*x*] = *rev xs*) = ([*x*] = *xs*)
by (*cases xs*) *auto*

lemma *rev-is-rev-conv* [*iff*]: (*rev xs* = *rev ys*) = (*xs* = *ys*)
proof (*induct xs arbitrary*: *ys*)
  case *Nil*
  then show *?case* **by** *force*
next
  case *Cons*
  then show *?case* **by** (*cases ys*) *auto*
qed

lemma *rev-eq-append-conv*: *rev xs* = *ys* @ *zs* ⟷ *xs* = *rev zs* @ *rev ys*
by (*metis rev-append rev-rev-ident*)

lemma *append-eq-rev-conv*: *ys* @ *zs* = *rev xs* ⟷ *rev zs* @ *rev ys* = *xs*
using *rev-eq-append-conv*[*THEN eq-iff-swap*] **by** *metis*

lemma *rev-eq-Cons-iff* [*iff*]: (*rev xs* = *y*#*ys*) = (*xs* = *rev ys* @ [*y*])
by (*simp add*: *rev-swap*)

lemmas *Cons-eq-rev-iff* = *rev-eq-Cons-iff* [*THEN eq-iff-swap*]

lemma *inj-on-rev*[*iff*]: *inj-on rev A*
by(*simp add*:*inj-on-def*)

lemma *rev-induct* [*case-names Nil snoc*]:
  assumes *P* [] **and** ⋀*x xs*. *P xs* ⟹ *P* (*xs* @ [*x*])
  shows *P xs*
proof −
  have *P* (*rev* (*rev xs*))
    by (*rule-tac list* = *rev xs* **in** *list.induct*, *simp-all add*: *assms*)
  then show *?thesis* **by** *simp*
qed

lemma *rev-exhaust* [*case-names Nil snoc*]:
  (*xs* = [] ⟹ *P*) ⟹(⋀*ys y*. *xs* = *ys* @ [*y*] ⟹ *P*) ⟹ *P*
by (*induct xs rule*: *rev-induct*) *auto*

lemmas *rev-cases* = *rev-exhaust*

lemma *rev-nonempty-induct* [*consumes 1*, *case-names single snoc*]:
  assumes *xs* ≠ []
  and *single*: ⋀*x*. *P* [*x*]
  and *snoc′*: ⋀*x xs*. *xs* ≠ [] ⟹ *P xs* ⟹ *P* (*xs*@[*x*])
  shows *P xs*
using ‹*xs* ≠ []› **proof** (*induct xs rule*: *rev-induct*)
  case (*snoc x xs*) **then show** *?case*
  proof (*cases xs*)
    case *Nil* **thus** *?thesis* **by** (*simp add*: *single*)
  next
    case *Cons* **with** *snoc* **show** *?thesis* **by** (*fastforce intro*!: *snoc′*)
  qed
qed *simp*

lemma *rev-induct2*:
  ⟦ *P* [] [];
  ⋀*x xs*. *P* (*xs* @ [*x*]) [];
  ⋀*y ys*. *P* [] (*ys* @ [*y*]);
  ⋀*x xs y ys*. *P xs ys* ⟹ *P* (*xs* @ [*x*]) (*ys* @ [*y*]) ⟧

$\Longrightarrow$ *P xs ys*
proof (*induct xs arbitrary*: *ys rule*: *rev-induct*)
case *Nil*
then show *?case* **using** *rev-induct*[*of P* []] **by** *presburger*
next
case (*snoc x xs*)
hence *P xs ys'* **for** *ys'* **by** *simp*
then show *?case* **by** (*simp add*: *rev-induct snoc.prems*(*2*,*4*))
qed

lemma *length-Suc-conv-rev*: (*length xs* = *Suc n*) = ($\exists$ *y ys*. *xs* = *ys* @ [*y*] $\wedge$ *length ys* = *n*)
by (*induct xs rule*: *rev-induct*) *auto*

66.1.7 *set*

declare *list.set*[*code-post*] — pretty output

lemma *finite-set* [*iff*]: *finite* (*set xs*)
by (*induct xs*) *auto*

lemma *set-append* [*simp*]: *set* (*xs* @ *ys*) = (*set xs* $\cup$ *set ys*)
by (*induct xs*) *auto*

lemma *hd-in-set*[*simp*]: *xs* $\neq$ [] $\Longrightarrow$ *hd xs* $\in$ *set xs*
by(*cases xs*) *auto*

lemma *set-subset-Cons*: *set xs* $\subseteq$ *set* (*x* # *xs*)
by *auto*

lemma *set-ConsD*: *y* $\in$ *set* (*x* # *xs*) $\Longrightarrow$ *y*=*x* $\vee$ *y* $\in$ *set xs*
by *auto*

lemma *set-empty* [*iff*]: (*set xs* = {}) = (*xs* = [])
by (*induct xs*) *auto*

lemmas *set-empty2*[*iff*] = *set-empty*[*THEN eq-iff-swap*]

lemma *set-rev* [*simp*]: *set* (*rev xs*) = *set xs*
by (*induct xs*) *auto*

lemma *set-map* [*simp*]: *set* (*map f xs*) = *f*'(*set xs*)
by (*induct xs*) *auto*

lemma *set-filter* [*simp*]: *set* (*filter P xs*) = {*x*. *x* $\in$ *set xs* $\wedge$ *P x*}
by (*induct xs*) *auto*

lemma *set-upt* [*simp*]: *set*[*i*..<*j*] = {*i*..<*j*}
by (*induct j*) *auto*

lemma *split-list*: $x \in set\ xs \Longrightarrow \exists ys\ zs.\ xs = ys\ @\ x\ \#\ zs$
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case *Cons* **thus** *?case* **by** (*auto intro*: *Cons-eq-appendI*)
qed

lemma *in-set-conv-decomp*: $x \in set\ xs \longleftrightarrow (\exists ys\ zs.\ xs = ys\ @\ x\ \#\ zs)$
  by (*auto elim*: *split-list*)

lemma *split-list-first*: $x \in set\ xs \Longrightarrow \exists ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ ys$
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons a xs*)
  show *?case*
  proof *cases*
    assume $x = a$ **thus** *?case* **using** *Cons* **by** *fastforce*
  next
    assume $x \neq a$ **thus** *?case* **using** *Cons* **by**(*fastforce intro*!: *Cons-eq-appendI*)
  qed
qed

lemma *in-set-conv-decomp-first*:
  $(x \in set\ xs) = (\exists ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ ys)$
  by (*auto dest*!: *split-list-first*)

lemma *split-list-last*: $x \in set\ xs \Longrightarrow \exists ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ zs$
proof (*induct xs rule*: *rev-induct*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*snoc a xs*)
  show *?case*
  proof *cases*
    assume $x = a$ **thus** *?case* **using** *snoc* **by** (*auto intro*!: *exI*)
  next
    assume $x \neq a$ **thus** *?case* **using** *snoc* **by** *fastforce*
  qed
qed

lemma *in-set-conv-decomp-last*:
  $(x \in set\ xs) = (\exists ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ zs)$
  by (*auto dest*!: *split-list-last*)

lemma *split-list-prop*: $\exists x \in set\ xs.\ P\ x \Longrightarrow \exists ys\ x\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge P\ x$
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*

next
  case *Cons* **thus** *?case*
    by(*simp add*:*Bex-def*)(*metis append-Cons append.simps*(*1*))
qed

lemma *split-list-propE*:
  assumes ∃ *x* ∈ *set xs*. *P x*
  obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x*
using *split-list-prop* [*OF assms*] **by** *blast*

lemma *split-list-first-prop*:
  ∃ *x* ∈ *set xs*. *P x* ⟹
   ∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *y* ∈ *set ys*. ¬ *P y*)
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons x xs*)
  show *?case*
  proof *cases*
    assume *P x*
    hence *x* # *xs* = [] @ *x* # *xs* ∧ *P x* ∧ (∀ *y*∈*set* []. ¬ *P y*) **by** *simp*
    thus *?thesis* **by** *fast*
  next
    assume ¬ *P x*
    hence ∃ *x*∈*set xs*. *P x* **using** *Cons*(*2*) **by** *simp*
    thus *?thesis* **using** ‹¬ *P x*› *Cons*(*1*) **by** (*metis append-Cons set-ConsD*)
  qed
qed

lemma *split-list-first-propE*:
  assumes ∃ *x* ∈ *set xs*. *P x*
  obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x* **and** ∀ *y* ∈ *set ys*. ¬ *P y*
using *split-list-first-prop* [*OF assms*] **by** *blast*

lemma *split-list-first-prop-iff*:
  (∃ *x* ∈ *set xs*. *P x*) ⟷
   (∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *y* ∈ *set ys*. ¬ *P y*))
by (*rule*, *erule split-list-first-prop*) *auto*

lemma *split-list-last-prop*:
  ∃ *x* ∈ *set xs*. *P x* ⟹
   ∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *z* ∈ *set zs*. ¬ *P z*)
proof(*induct xs rule*:*rev-induct*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*snoc x xs*)
  show *?case*
  proof *cases*
    assume *P x* **thus** *?thesis* **by** (*auto intro*!: *exI*)

next
assume ¬ *P x*
hence ∃ *x*∈*set xs*. *P x* **using** *snoc*(*2*) **by** *simp*
thus *?thesis* **using** ‹¬ *P x*› *snoc*(*1*) **by** *fastforce*
qed
qed

lemma *split-list-last-propE*:
assumes ∃ *x* ∈ *set xs*. *P x*
obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x* **and** ∀ *z* ∈ *set zs*. ¬ *P z*
using *split-list-last-prop* [*OF assms*] **by** *blast*

lemma *split-list-last-prop-iff*:
(∃ *x* ∈ *set xs*. *P x*) ⟷
(∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *z* ∈ *set zs*. ¬ *P z*))
by *rule* (*erule split-list-last-prop*, *auto*)

lemma *finite-list*: *finite A* ⟹ ∃ *xs*. *set xs* = *A*
by (*erule finite-induct*) (*auto simp add*: *list.set*(*2*)[*symmetric*] *simp del*: *list.set*(*2*))

lemma *card-length*: *card* (*set xs*) ≤ *length xs*
by (*induct xs*) (*auto simp add*: *card-insert-if*)

lemma *set-minus-filter-out*:
set xs − {*y*} = *set* (*filter* (λ*x*. ¬ (*x* = *y*)) *xs*)
by (*induct xs*) *auto*

lemma *append-Cons-eq-iff*:
⟦ *x* ∉ *set xs*; *x* ∉ *set ys* ⟧ ⟹
xs @ *x* # *ys* = *xs'* @ *x* # *ys'* ⟷ (*xs* = *xs'* ∧ *ys* = *ys'*)
by(*auto simp*: *append-eq-Cons-conv Cons-eq-append-conv append-eq-append-conv2*)

66.1.8 *concat*

lemma *concat-append* [*simp*]: *concat* (*xs* @ *ys*) = *concat xs* @ *concat ys*
by (*induct xs*) *auto*

lemma *concat-eq-Nil-conv* [*simp*]: (*concat xss* = []) = (∀ *xs* ∈ *set xss*. *xs* = [])
by (*induct xss*) *auto*

lemmas *Nil-eq-concat-conv* [*simp*] = *concat-eq-Nil-conv*[*THEN eq-iff-swap*]

lemma *set-concat* [*simp*]: *set* (*concat xs*) = (⋃ *x*∈*set xs*. *set x*)
by (*induct xs*) *auto*

lemma *concat-map-singleton*[*simp*]: *concat*(*map* (%*x*. [*f x*]) *xs*) = *map f xs*
by (*induct xs*) *auto*

lemma *map-concat*: *map f* (*concat xs*) = *concat* (*map* (*map f*) *xs*)
 by (*induct xs*) *auto*

lemma *rev-concat*: *rev* (*concat xs*) = *concat* (*map rev* (*rev xs*))
 by (*induct xs*) *auto*

lemma *length-concat-rev*[*simp*]: *length* (*concat* (*rev xs*)) = *length* (*concat xs*)
by (*induction xs*) *auto*

lemma *concat-eq-concat-iff*: ∀ (*x*, *y*) ∈ *set* (*zip xs ys*). *length x* = *length y* ⟹ *length xs* = *length ys* ⟹ (*concat xs* = *concat ys*) = (*xs* = *ys*)
proof (*induct xs arbitrary*: *ys*)
 case (*Cons x xs ys*)
 thus *?case* **by** (*cases ys*) *auto*
qed (*auto*)

lemma *concat-injective*: *concat xs* = *concat ys* ⟹ *length xs* = *length ys* ⟹ ∀ (*x*, *y*) ∈ *set* (*zip xs ys*). *length x* = *length y* ⟹ *xs* = *ys*
 by (*simp add*: *concat-eq-concat-iff*)

lemma *concat-eq-appendD*:
 assumes *concat xss* = *ys* @ *zs xss* ≠ []
 shows ∃ *xss1 xs xs′ xss2*. *xss* = *xss1* @ (*xs* @ *xs′*) # *xss2* ∧ *ys* = *concat xss1* @ *xs* ∧ *zs* = *xs′* @ *concat xss2*
 using *assms*
proof(*induction xss arbitrary*: *ys*)
 case (*Cons xs xss*)
 from *Cons.prems* **consider**
  us **where** *xs* @ *us* = *ys concat xss* = *us* @ *zs* |
  us **where** *xs* = *ys* @ *us us* @ *concat xss* = *zs*
  by(*auto simp add*: *append-eq-append-conv2*)
 then show *?case*
 proof *cases*
  case *1*
  then show *?thesis* **using** *Cons.IH*[*OF 1*(*2*)]
   by(*cases xss*)(*auto intro*: *exI*[**where** *x*=[]], *metis append.assoc append-Cons concat.simps*(*2*))
 qed(*auto intro*: *exI*[**where** *x*=[]])
qed *simp*

lemma *concat-eq-append-conv*:
 concat xss = *ys* @ *zs* ⟷
 (*if xss* = [] *then ys* = [] ∧ *zs* = []
  else ∃ *xss1 xs xs′ xss2*. *xss* = *xss1* @ (*xs* @ *xs′*) # *xss2* ∧ *ys* = *concat xss1* @ *xs* ∧ *zs* = *xs′* @ *concat xss2*)
 by(*auto dest*: *concat-eq-appendD*)

lemma *hd-concat*: ⟦*xs* ≠ []; *hd xs* ≠ []⟧ ⟹ *hd* (*concat xs*) = *hd* (*hd xs*)
 by (*metis concat.simps*(*2*) *hd-Cons-tl hd-append2*)

simproc-setup *list-neq* ((*xs*::'*a list*) = *ys*) = ‹

```
(*
Reduces xs=ys to False if xs and ys cannot be of the same length.
This is the case if the atomic sublists of one are a submultiset
of those of the other list and there are fewer Cons's in one than the other.
*)

let

fun len (Const(const-name ‹Nil›,-)) acc = acc
  | len (Const(const-name ‹Cons›,-) $ - $ xs) (ts,n) = len xs (ts,n+1)
  | len (Const(const-name ‹append›,-) $ xs $ ys) acc = len xs (len ys acc)
  | len (Const(const-name ‹rev›,-) $ xs) acc = len xs acc
  | len (Const(const-name ‹map›,-) $ - $ xs) acc = len xs acc
  | len (Const(const-name ‹concat›,T) $ (Const(const-name ‹rev›,-) $ xss)) acc
    = len (Const(const-name ‹concat›,T) $ xss) acc
  | len t (ts,n) = (t::ts,n);

val ss = simpset-of context;

fun list-neq ctxt ct =
  let
    val (Const(-,eqT) $ lhs $ rhs) = Thm.term-of ct;
    val (ls,m) = len lhs ([],0) and (rs,n) = len rhs ([],0);
    fun prove-neq() =
      let
        val Type(-,listT::-) = eqT;
        val size = HOLogic.size-const listT;
        val eq-len = HOLogic.mk-eq (size $ lhs, size $ rhs);
        val neq-len = HOLogic.mk-Trueprop (HOLogic.Not $ eq-len);
        val thm = Goal.prove ctxt [] [] neq-len
          (K (simp-tac (put-simpset ss ctxt) 1));
      in SOME (thm RS @{thm neq-if-length-neq}) end
  in
    if m < n andalso submultiset (op aconv) (ls,rs) orelse
       n < m andalso submultiset (op aconv) (rs,ls)
    then prove-neq() else NONE
  end;
in K list-neq end
›
```

66.1.9 *filter*

lemma *filter-append* [*simp*]: *filter P* (*xs* @ *ys*) = *filter P xs* @ *filter P ys*
by (*induct xs*) *auto*

lemma *rev-filter*: *rev* (*filter P xs*) = *filter P* (*rev xs*)

by (*induct xs*) *simp-all*

lemma *filter-filter* [*simp*]: *filter P* (*filter Q xs*) = *filter* (λ*x*. *Q x* ∧ *P x*) *xs*
by (*induct xs*) *auto*

lemma *filter-concat*: *filter p* (*concat xs*) = *concat* (*map* (*filter p*) *xs*)
by (*induct xs*) *auto*

lemma *length-filter-le* [*simp*]: *length* (*filter P xs*) ≤ *length xs*
by (*induct xs*) (*auto simp add*: *le-SucI*)

lemma *sum-length-filter-compl*:
length(*filter P xs*) + *length*(*filter* (λ*x*. ¬*P x*) *xs*) = *length xs*
by(*induct xs*) *simp-all*

lemma *filter-True* [*simp*]: ∀ *x* ∈ *set xs*. *P x* ⟹ *filter P xs* = *xs*
by (*induct xs*) *auto*

lemma *filter-False* [*simp*]: ∀ *x* ∈ *set xs*. ¬ *P x* ⟹ *filter P xs* = []
by (*induct xs*) *auto*

lemma *filter-empty-conv*: (*filter P xs* = []) = (∀ *x*∈*set xs*. ¬ *P x*)
by (*induct xs*) *simp-all*

lemmas *empty-filter-conv* = *filter-empty-conv*[*THEN eq-iff-swap*]

lemma *filter-id-conv*: (*filter P xs* = *xs*) = (∀ *x*∈*set xs*. *P x*)
proof (*induct xs*)
  case (*Cons x xs*)
  then show *?case*
    using *length-filter-le*
    by (*simp add*: *impossible-Cons*)
qed *auto*

lemma *filter-map*: *filter P* (*map f xs*) = *map f* (*filter* (*P* ∘ *f*) *xs*)
by (*induct xs*) *simp-all*

lemma *length-filter-map*[*simp*]:
length (*filter P* (*map f xs*)) = *length*(*filter* (*P* ∘ *f*) *xs*)
by (*simp add*:*filter-map*)

lemma *filter-is-subset* [*simp*]: *set* (*filter P xs*) ≤ *set xs*
by *auto*

lemma *length-filter-less*:
⟦ *x* ∈ *set xs*; ¬ *P x* ⟧ ⟹ *length*(*filter P xs*) < *length xs*
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*
next

case (*Cons x xs*) **thus** *?case*
using *Suc-le-eq* **by** *fastforce*
qed

lemma *length-filter-conv-card*:
length(*filter p xs*) = *card*{*i*. *i* < *length xs* ∧ *p*(*xs*!*i*)}
proof (*induct xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*)
let *?S* = {*i*. *i* < *length xs* ∧ *p*(*xs*!*i*)}
have *fin*: *finite ?S* **by**(*fast intro*: *bounded-nat-set-is-finite*)
show *?case* (**is** *?l* = *card ?S'*)
proof (*cases*)
assume *p x*
hence *eq*: *?S'* = *insert 0* (*Suc* ‘ *?S*)
by(*auto simp*: *image-def split*:*nat.split dest*:*gr0-implies-Suc*)
have *length* (*filter p* (*x* # *xs*)) = *Suc*(*card ?S*)
using *Cons* ‹*p x*› **by** *simp*
also have ... = *Suc*(*card*(*Suc* ‘ *?S*)) **using** *fin*
by (*simp add*: *card-image*)
also have ... = *card ?S'* **using** *eq fin*
by (*simp add*:*card-insert-if*)
finally show *?thesis* **.**
next
assume ¬ *p x*
hence *eq*: *?S'* = *Suc* ‘ *?S*
by(*auto simp add*: *image-def split*:*nat.split elim*:*lessE*)
have *length* (*filter p* (*x* # *xs*)) = *card ?S*
using *Cons* ‹¬ *p x*› **by** *simp*
also have ... = *card*(*Suc* ‘ *?S*) **using** *fin*
by (*simp add*: *card-image*)
also have ... = *card ?S'* **using** *eq fin*
by (*simp add*:*card-insert-if*)
finally show *?thesis* **.**
qed
qed

lemma *Cons-eq-filterD*:
x#*xs* = *filter P ys* ⟹
∃ *us vs*. *ys* = *us* @ *x* # *vs* ∧ (∀ *u*∈*set us*. ¬ *P u*) ∧ *P x* ∧ *xs* = *filter P vs*
(**is** - ⟹ ∃ *us vs*. *?P ys us vs*)
proof(*induct ys*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons y ys*)
show *?case* (**is** ∃ *x*. *?Q x*)
proof *cases*
assume *Py*: *P y*

show *?thesis*
proof *cases*
assume $x = y$
with *Py Cons.prems* **have** *?Q* [] **by** *simp*
then show *?thesis* **..**
next
assume $x \neq y$
with *Py Cons.prems* **show** *?thesis* **by** *simp*
qed
next
assume $\neg$ *P y*
with *Cons* **obtain** *us vs* **where** *?P* (*y#ys*) (*y#us*) *vs* **by** *fastforce*
then have *?Q* (*y#us*) **by** *simp*
then show *?thesis* **..**
qed
qed

lemma *filter-eq-ConsD*:
filter P ys = *x#xs* $\Longrightarrow$
$\exists$ *us vs. ys* = *us* @ *x* # *vs* $\wedge$ ($\forall$ *u*$\in$*set us.* $\neg$ *P u*) $\wedge$ *P x* $\wedge$ *xs* = *filter P vs*
by(*rule Cons-eq-filterD*) *simp*

lemma *filter-eq-Cons-iff*:
(*filter P ys* = *x#xs*) =
($\exists$ *us vs. ys* = *us* @ *x* # *vs* $\wedge$ ($\forall$ *u*$\in$*set us.* $\neg$ *P u*) $\wedge$ *P x* $\wedge$ *xs* = *filter P vs*)
by(*auto dest*:*filter-eq-ConsD*)

lemmas *Cons-eq-filter-iff* = *filter-eq-Cons-iff*[*THEN eq-iff-swap*]

lemma *inj-on-filter-key-eq*:
assumes *inj-on f* (*insert y* (*set xs*))
shows *filter* ($\lambda x.$ *f y* = *f x*) *xs* = *filter* (*HOL.eq y*) *xs*
using *assms* **by** (*induct xs*) *auto*

lemma *filter-cong*[*fundef-cong*]:
xs = *ys* $\Longrightarrow$ ($\bigwedge x.$ *x* $\in$ *set ys* $\Longrightarrow$ *P x* = *Q x*) $\Longrightarrow$ *filter P xs* = *filter Q ys*
by (*induct ys arbitrary*: *xs*) *auto*

66.1.10 List partitioning

primrec *partition* :: ($'a \Rightarrow$ *bool*) $\Rightarrow$$'a$ *list* $\Rightarrow$ $'a$ *list* $\times$ $'a$ *list* **where**
partition P [] = ([], []) |
partition P (*x* # *xs*) =
(*let* (*yes*, *no*) = *partition P xs*
in if P x then (*x* # *yes*, *no*) *else* (*yes*, *x* # *no*))

lemma *partition-filter1*: *fst* (*partition P xs*) = *filter P xs*
by (*induct xs*) (*auto simp add*: *Let-def split-def*)

lemma *partition-filter2*: *snd* (*partition P xs*) = *filter* (*Not* ∘ *P*) *xs*
  by (*induct xs*) (*auto simp add*: *Let-def split-def*)

lemma *partition-P*:
  assumes *partition P xs* = (*yes*, *no*)
  shows (∀ *p* ∈ *set yes*. *P p*) ∧ (∀ *p* ∈ *set no*. ¬ *P p*)
proof −
  from *assms* **have** *yes* = *fst* (*partition P xs*) **and** *no* = *snd* (*partition P xs*)
    by *simp-all*
  then show *?thesis* **by** (*simp-all add*: *partition-filter1 partition-filter2*)
qed

lemma *partition-set*:
  assumes *partition P xs* = (*yes*, *no*)
  shows *set yes* ∪ *set no* = *set xs*
proof −
  from *assms* **have** *yes* = *fst* (*partition P xs*) **and** *no* = *snd* (*partition P xs*)
    by *simp-all*
  then show *?thesis* **by** (*auto simp add*: *partition-filter1 partition-filter2*)
qed

lemma *partition-filter-conv*[*simp*]:
  partition f xs = (*filter f xs*,*filter* (*Not* ∘ *f*) *xs*)
  unfolding *partition-filter2*[*symmetric*]
  unfolding *partition-filter1*[*symmetric*] **by** *simp*

declare *partition.simps*[*simp del*]

66.1.11 (!)

lemma *nth-Cons-0* [*simp*, *code*]: (*x* # *xs*)!*0* = *x*
  by *auto*

lemma *nth-Cons-Suc* [*simp*, *code*]: (*x* # *xs*)!(*Suc n*) = *xs*!*n*
  by *auto*

declare *nth.simps* [*simp del*]

lemma *nth-Cons-pos*[*simp*]: *0* < *n* ⟹ (*x*#*xs*) ! *n* = *xs* ! (*n* − *1*)
  by(*auto simp*: *Nat.gr0-conv-Suc*)

lemma *nth-append*:
  (*xs* @ *ys*)!*n* = (*if n* < *length xs then xs*!*n else ys*!(*n* − *length xs*))
proof (*induct xs arbitrary*: *n*)
  case (*Cons x xs*)
  then show *?case*
    using *less-Suc-eq-0-disj* **by** *auto*
qed *simp*

lemma *nth-append-left*: $i < length\ xs \Longrightarrow (xs\ @\ ys)\ !\ i = xs\ !\ i$
by (*auto simp*: *nth-append*)

lemma *nth-append-right*: $i \geq length\ xs \Longrightarrow (xs\ @\ ys)\ !\ i = ys\ !\ (i - length\ xs)$
by (*auto simp*: *nth-append*)

lemma *nth-append-length* [*simp*]: $(xs\ @\ x\ \#\ ys)\ !\ length\ xs = x$
by (*induct xs*) *auto*

lemma *nth-append-length-plus*[*simp*]: $(xs\ @\ ys)\ !\ (length\ xs + n) = ys\ !\ n$
by (*induct xs*) *auto*

lemma *nth-map* [*simp*]: $n < length\ xs \Longrightarrow (map\ f\ xs)!n = f(xs!n)$
proof (*induct xs arbitrary*: *n*)
case (*Cons x xs*)
then show *?case*
using *less-Suc-eq-0-disj* **by** *auto*
qed *simp*

lemma *nth-tl*: $n < length\ (tl\ xs) \Longrightarrow tl\ xs\ !\ n = xs\ !\ Suc\ n$
by (*induction xs*) *auto*

lemma *hd-conv-nth*: $xs \neq [] \Longrightarrow hd\ xs = xs!0$
by(*cases xs*) *simp-all*

lemma *list-eq-iff-nth-eq*:
$(xs = ys) = (length\ xs = length\ ys \wedge (\forall i{<}length\ xs.\ xs!i = ys!i))$
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs ys*)
show *?case*
proof (*cases ys*)
case (*Cons y ys*)
with *Cons.hyps* **show** *?thesis* **by** *fastforce*
qed *simp*
qed *force*

lemma *map-equality-iff*:
$map\ f\ xs = map\ g\ ys \longleftrightarrow length\ xs = length\ ys \wedge (\forall i{<}length\ ys.\ f\ (xs!i) = g\ (ys!i))$
by (*fastforce simp*: *list-eq-iff-nth-eq*)

lemma *set-conv-nth*: $set\ xs = \{xs!i \mid i.\ i < length\ xs\}$
proof (*induct xs*)
case (*Cons x xs*)
have $insert\ x\ \{xs\ !\ i\ |i.\ i < length\ xs\} = \{(x\ \#\ xs)\ !\ i\ |i.\ i < Suc\ (length\ xs)\}$ (**is** *?L=?R*)
proof
show $?L \subseteq ?R$
by *force*

```
    show ?R ⊆ ?L
      using less-Suc-eq-0-disj by auto
  qed
  with Cons show ?case
    by simp
qed simp

lemma in-set-conv-nth: (x ∈ set xs) = (∃ i < length xs. xs!i = x)
  by(auto simp:set-conv-nth)

lemma nth-equal-first-eq:
  assumes x ∉ set xs
  assumes n ≤ length xs
  shows (x # xs) ! n = x ⟷ n = 0 (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  show ?rhs
  proof (rule ccontr)
    assume n ≠ 0
    then have n > 0 by simp
    with ‹?lhs› have xs ! (n − 1) = x by simp
    moreover from ‹n > 0› ‹n ≤ length xs› have n − 1 < length xs by simp
    ultimately have ∃ i<length xs. xs ! i = x by auto
    with ‹x ∉ set xs› in-set-conv-nth [of x xs] show False by simp
  qed
next
  assume ?rhs then show ?lhs by simp
qed

lemma nth-non-equal-first-eq:
  assumes x ≠ y
  shows (x # xs) ! n = y ⟷ xs ! (n − 1) = y ∧ n > 0 (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs with assms have n > 0 by (cases n) simp-all
  with ‹?lhs› show ?rhs by simp
next
  assume ?rhs then show ?lhs by simp
qed

lemma list-ball-nth: ⟦n < length xs; ∀ x ∈ set xs. P x⟧ ⟹ P(xs!n)
  by (auto simp add: set-conv-nth)

lemma nth-mem [simp]: n < length xs ⟹ xs!n ∈ set xs
  by (auto simp add: set-conv-nth)

lemma all-nth-imp-all-set:
  ⟦∀ i < length xs. P(xs!i); x ∈ set xs⟧ ⟹ P x
  by (auto simp add: set-conv-nth)
```

lemma *all-set-conv-all-nth*:
  $(\forall x \in set\ xs.\ P\ x) = (\forall i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$
  by (*auto simp add*: *set-conv-nth*)

lemma *rev-nth*:
  $n < size\ xs \Longrightarrow rev\ xs\ !\ n = xs\ !\ (length\ xs - Suc\ n)$
proof (*induct xs arbitrary*: *n*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons x xs*)
  hence *n*: $n < Suc\ (length\ xs)$ **by** *simp*
  moreover
  { assume $n < length\ xs$
    with *n* **obtain** n' **where** n': $length\ xs - n = Suc\ n'$
      by (*cases length xs* − *n*, *auto*)
    moreover
    from n' **have** $length\ xs - Suc\ n = n'$ **by** *simp*
    ultimately
    have $xs\ !\ (length\ xs - Suc\ n) = (x\ \#\ xs)\ !\ (length\ xs - n)$ **by** *simp*
  }
  ultimately
  show *?case* **by** (*clarsimp simp add*: *Cons nth-append*)
qed

lemma *Skolem-list-nth*:
  $(\forall i<k.\ \exists x.\ P\ i\ x) = (\exists xs.\ size\ xs = k \wedge (\forall i<k.\ P\ i\ (xs!i)))$
  (**is** - = ($\exists xs.$ *?P k xs*))
proof(*induct k*)
  case *0* **show** *?case* **by** *simp*
next
  case (*Suc k*)
  show *?case* (**is** *?L* = *?R* **is** - = ($\exists xs.$ *?P' xs*))
  proof
    assume *?R* **thus** *?L* **using** *Suc* **by** *auto*
  next
    assume *?L*
    with *Suc* **obtain** *x xs* **where** *?P k xs* ∧ *P k x* **by** (*metis less-Suc-eq*)
    hence *?P'*(*xs*@[*x*]) **by**(*simp add*:*nth-append less-Suc-eq*)
    thus *?R* **..**
  qed
qed

66.1.12 *list-update*

lemma *length-list-update* [*simp*]: $length(xs[i:=x]) = length\ xs$
  by (*induct xs arbitrary*: *i*) (*auto split*: *nat.split*)

lemma *nth-list-update*:
  $i < length\ xs \Longrightarrow (xs[i:=x])!j = (if\ i = j\ then\ x\ else\ xs!j)$

by (*induct xs arbitrary*: *i j*) (*auto simp add*: *nth-Cons split*: *nat.split*)

lemma *nth-list-update-eq* [*simp*]: $i < length\ xs \Longrightarrow (xs[i{:=}x])!i = x$
by (*simp add*: *nth-list-update*)

lemma *nth-list-update-neq* [*simp*]: $i \neq j \Longrightarrow xs[i{:=}x]!j = xs!j$
by (*induct xs arbitrary*: *i j*) (*auto simp add*: *nth-Cons split*: *nat.split*)

lemma *list-update-id*[*simp*]: $xs[i := xs!i] = xs$
by (*induct xs arbitrary*: *i*) (*simp-all split*:*nat.splits*)

lemma *list-update-beyond*[*simp*]: $length\ xs \leq i \Longrightarrow xs[i{:=}x] = xs$
proof (*induct xs arbitrary*: *i*)
case (*Cons x xs i*)
then show *?case*
by (*metis leD length-list-update list-eq-iff-nth-eq nth-list-update-neq*)
qed *simp*

lemma *list-update-nonempty*[*simp*]: $xs[k{:=}x] = [] \longleftrightarrow xs{=}[]$
by (*simp only*: *length-0-conv*[*symmetric*] *length-list-update*)

lemma *list-update-same-conv*:
$i < length\ xs \Longrightarrow (xs[i := x] = xs) = (xs!i = x)$
by (*induct xs arbitrary*: *i*) (*auto split*: *nat.split*)

lemma *list-update-append1*:
$i < size\ xs \Longrightarrow (xs\ @\ ys)[i{:=}x] = xs[i{:=}x]\ @\ ys$
by (*induct xs arbitrary*: *i*)(*auto split*:*nat.split*)

lemma *list-update-append*:
$(xs\ @\ ys)\ [n{:=}\ x] =$
$(if\ n < length\ xs\ then\ xs[n{:=}\ x]\ @\ ys\ else\ xs\ @\ (ys\ [n{-}length\ xs{:=}\ x]))$
by (*induct xs arbitrary*: *n*) (*auto split*:*nat.splits*)

lemma *list-update-length* [*simp*]:
$(xs\ @\ x\ \#\ ys)[length\ xs := y] = (xs\ @\ y\ \#\ ys)$
by (*induct xs*, *auto*)

lemma *map-update*: $map\ f\ (xs[k{:=}\ y]) = (map\ f\ xs)[k := f\ y]$
by(*induct xs arbitrary*: *k*)(*auto split*:*nat.splits*)

lemma *rev-update*:
$k < length\ xs \Longrightarrow rev\ (xs[k{:=}\ y]) = (rev\ xs)[length\ xs - k - 1 := y]$
by (*induct xs arbitrary*: *k*) (*auto simp*: *list-update-append split*:*nat.splits*)

lemma *update-zip*:
$(zip\ xs\ ys)[i{:=}xy] = zip\ (xs[i{:=}fst\ xy])\ (ys[i{:=}snd\ xy])$
by (*induct ys arbitrary*: *i xy xs*) (*auto*, *case-tac xs*, *auto split*: *nat.split*)

lemma *set-update-subset-insert*: $set(xs[i{:=}x]) \leq insert\ x\ (set\ xs)$
by (*induct xs arbitrary*: *i*) (*auto split*: *nat.split*)

lemma *set-update-subsetI*: $[\![set\ xs \subseteq A;\ x \in A]\!] \Longrightarrow set(xs[i := x]) \subseteq A$
by (*blast dest*!: *set-update-subset-insert* [*THEN subsetD*])

lemma *set-update-memI*: $n < length\ xs \Longrightarrow x \in set\ (xs[n := x])$
by (*induct xs arbitrary*: *n*) (*auto split*:*nat.splits*)

lemma *list-update-overwrite*[*simp*]:
$xs\ [i := x,\ i := y] = xs\ [i := y]$
by (*induct xs arbitrary*: *i*) (*simp-all split*: *nat.split*)

lemma *list-update-swap*:
$i \neq i' \Longrightarrow xs\ [i := x,\ i' := x'] = xs\ [i' := x',\ i := x]$
by (*induct xs arbitrary*: *i i'*) (*simp-all split*: *nat.split*)

lemma *list-update-code* [*code*]:
$[][i := y] = []$
$(x\ \#\ xs)[0 := y] = y\ \#\ xs$
$(x\ \#\ xs)[Suc\ i := y] = x\ \#\ xs[i := y]$
by *simp-all*

66.1.13 *last* and *butlast*

lemma *hd-Nil-eq-last*: $hd\ Nil = last\ Nil$
unfolding *hd-def last-def* **by** *simp*

lemma *last-snoc* [*simp*]: $last\ (xs\ @\ [x]) = x$
by (*induct xs*) *auto*

lemma *butlast-snoc* [*simp*]: $butlast\ (xs\ @\ [x]) = xs$
by (*induct xs*) *auto*

lemma *last-ConsL*: $xs = [] \Longrightarrow last(x\#xs) = x$
by *simp*

lemma *last-ConsR*: $xs \neq [] \Longrightarrow last(x\#xs) = last\ xs$
by *simp*

lemma *last-append*: $last(xs\ @\ ys) = (if\ ys = []\ then\ last\ xs\ else\ last\ ys)$
by (*induct xs*) (*auto*)

lemma *last-appendL*[*simp*]: $ys = [] \Longrightarrow last(xs\ @\ ys) = last\ xs$
by(*simp add*:*last-append*)

lemma *last-appendR*[*simp*]: $ys \neq [] \Longrightarrow last(xs\ @\ ys) = last\ ys$
by(*simp add*:*last-append*)

lemma *last-tl*: *xs* = [] ∨ *tl xs* ≠ [] ⟹*last* (*tl xs*) = *last xs*
by (*induct xs*) *simp-all*

lemma *butlast-tl*: *butlast* (*tl xs*) = *tl* (*butlast xs*)
by (*induct xs*) *simp-all*

lemma *hd-rev*: *hd*(*rev xs*) = *last xs*
by (*metis hd-Cons-tl hd-Nil-eq-last last-snoc rev-eq-Cons-iff rev-is-Nil-conv*)

lemma *last-rev*: *last*(*rev xs*) = *hd xs*
by (*metis hd-rev rev-swap*)

lemma *last-in-set*[*simp*]: *as* ≠ [] ⟹ *last as* ∈ *set as*
by (*induct as*) *auto*

lemma *length-butlast* [*simp*]: *length* (*butlast xs*) = *length xs* − *1*
by (*induct xs rule*: *rev-induct*) *auto*

lemma *butlast-append*:
butlast (*xs* @ *ys*) = (*if ys* = [] *then butlast xs else xs* @ *butlast ys*)
by (*induct xs arbitrary*: *ys*) *auto*

lemma *append-butlast-last-id* [*simp*]:
xs ≠ [] ⟹ *butlast xs* @ [*last xs*] = *xs*
by (*induct xs*) *auto*

lemma *in-set-butlastD*: *x* ∈ *set* (*butlast xs*) ⟹ *x* ∈ *set xs*
by (*induct xs*) (*auto split*: *if-split-asm*)

lemma *in-set-butlast-appendI*:
x ∈ *set* (*butlast xs*) ∨ *x* ∈ *set* (*butlast ys*) ⟹ *x* ∈ *set* (*butlast* (*xs* @ *ys*))
by (*auto dest*: *in-set-butlastD simp add*: *butlast-append*)

lemma *last-drop*[*simp*]: *n* < *length xs* ⟹ *last* (*drop n xs*) = *last xs*
by (*induct xs arbitrary*: *n*)(*auto split*:*nat.split*)

lemma *nth-butlast*:
assumes *n* < *length* (*butlast xs*) **shows** *butlast xs* ! *n* = *xs* ! *n*
proof (*cases xs*)
case (*Cons y ys*)
moreover from *assms* **have** *butlast xs* ! *n* = (*butlast xs* @ [*last xs*]) ! *n*
by (*simp add*: *nth-append*)
ultimately show *?thesis* **using** *append-butlast-last-id* **by** *simp*
qed *simp*

lemma *last-conv-nth*: *xs*≠[] ⟹ *last xs* = *xs*!(*length xs* − *1*)
by(*induct xs*)(*auto simp*:*neq-Nil-conv*)

lemma *butlast-conv-take*: *butlast xs* = *take* (*length xs* − *1*) *xs*

by (*induction xs rule*: *induct-list012*) *simp-all*

lemma *last-list-update*:
$xs \neq [] \Longrightarrow last(xs[k:=x]) = (if\ k = size\ xs - 1\ then\ x\ else\ last\ xs)$
by (*auto simp*: *last-conv-nth*)

lemma *butlast-list-update*:
$butlast(xs[k:=x]) =$
$(if\ k = size\ xs - 1\ then\ butlast\ xs\ else\ (butlast\ xs)[k:=x])$
by(*cases xs rule*:*rev-cases*)(*auto simp*: *list-update-append split*: *nat.splits*)

lemma *last-map*: $xs \neq [] \Longrightarrow last\ (map\ f\ xs) = f\ (last\ xs)$
by (*cases xs rule*: *rev-cases*) *simp-all*

lemma *map-butlast*: $map\ f\ (butlast\ xs) = butlast\ (map\ f\ xs)$
by (*induct xs*) *simp-all*

lemma *snoc-eq-iff-butlast*:
$xs\ @\ [x] = ys \longleftrightarrow (ys \neq [] \wedge butlast\ ys = xs \wedge last\ ys = x)$
by *fastforce*

corollary *longest-common-suffix*:
$\exists\, ss\ xs'\ ys'.\ xs = xs'\ @\ ss \wedge ys = ys'\ @\ ss$
$\wedge\ (xs' = [] \vee ys' = [] \vee last\ xs' \neq last\ ys')$
using *longest-common-prefix*[*of rev xs rev ys*]
unfolding *rev-swap rev-append* **by** (*metis last-rev rev-is-Nil-conv*)

lemma *butlast-rev* [*simp*]: $butlast\ (rev\ xs) = rev\ (tl\ xs)$
by (*cases xs*) *simp-all*

66.1.14 *take* and *drop*

lemma *take-0*: $take\ 0\ xs = []$
by (*induct xs*) *auto*

lemma *drop-0*: $drop\ 0\ xs = xs$
by (*induct xs*) *auto*

lemma *take0*[*simp*]: $take\ 0 = (\lambda xs.\ [])$
by(*rule ext*) (*rule take-0*)

lemma *drop0*[*simp*]: $drop\ 0 = (\lambda x.\ x)$
by(*rule ext*) (*rule drop-0*)

lemma *take-Suc-Cons* [*simp*]: $take\ (Suc\ n)\ (x\ \#\ xs) = x\ \#\ take\ n\ xs$
by *simp*

lemma *drop-Suc-Cons* [*simp*]: $drop\ (Suc\ n)\ (x\ \#\ xs) = drop\ n\ xs$
by *simp*

declare *take-Cons* [*simp del*] **and** *drop-Cons* [*simp del*]

lemma *take-Suc*: *xs* ≠ [] ⟹ *take* (*Suc n*) *xs* = *hd xs* # *take n* (*tl xs*)
by(*clarsimp simp add*:*neq-Nil-conv*)

lemma *drop-Suc*: *drop* (*Suc n*) *xs* = *drop n* (*tl xs*)
by(*cases xs*, *simp-all*)

lemma *hd-take*[*simp*]: *j* > *0* ⟹ *hd* (*take j xs*) = *hd xs*
by (*metis gr0-conv-Suc list.sel*(*1*) *take.simps*(*1*) *take-Suc*)

lemma *take-tl*: *take n* (*tl xs*) = *tl* (*take* (*Suc n*) *xs*)
by (*induct xs arbitrary*: *n*) *simp-all*

lemma *drop-tl*: *drop n* (*tl xs*) = *tl*(*drop n xs*)
by(*induct xs arbitrary*: *n*, *simp-all add*:*drop-Cons drop-Suc split*:*nat.split*)

lemma *tl-take*: *tl* (*take n xs*) = *take* (*n* − *1*) (*tl xs*)
by (*cases n*, *simp*, *cases xs*, *auto*)

lemma *tl-drop*: *tl* (*drop n xs*) = *drop n* (*tl xs*)
by (*simp only*: *drop-tl*)

lemma *nth-via-drop*: *drop n xs* = *y*#*ys* ⟹ *xs*!*n* = *y*
by (*induct xs arbitrary*: *n*, *simp*)(*auto simp*: *drop-Cons nth-Cons split*: *nat.splits*)

lemma *take-Suc-conv-app-nth*:
i < *length xs* ⟹ *take* (*Suc i*) *xs* = *take i xs* @ [*xs*!*i*]
proof (*induct xs arbitrary*: *i*)
case *Nil*
then show *?case* **by** *simp*
next
case *Cons*
then show *?case* **by** (*cases i*) *auto*
qed

lemma *Cons-nth-drop-Suc*:
i < *length xs* ⟹ (*xs*!*i*) # (*drop* (*Suc i*) *xs*) = *drop i xs*
proof (*induct xs arbitrary*: *i*)
case *Nil*
then show *?case* **by** *simp*
next
case *Cons*
then show *?case* **by** (*cases i*) *auto*
qed

lemma *length-take* [*simp*]: *length* (*take n xs*) = *min* (*length xs*) *n*
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *length-drop* [*simp*]: *length* (*drop n xs*) = (*length xs* − *n*)
  by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *take-all* [*simp*]: *length xs* ≤ *n* ⟹ *take n xs* = *xs*
  by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *drop-all* [*simp*]: *length xs* ≤ *n* ⟹ *drop n xs* = []
  by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *take-all-iff* [*simp*]: *take n xs* = *xs* ⟷ *length xs* ≤ *n*
by (*metis length-take min.order-iff take-all*)

lemma *take-eq-Nil*[*simp*]: (*take n xs* = []) = (*n* = *0* ∨ *xs* = [])
  by(*induct xs arbitrary*: *n*)(*auto simp*: *take-Cons split*:*nat.split*)

lemmas *take-eq-Nil2*[*simp*] = *take-eq-Nil*[*THEN eq-iff-swap*]

lemma *drop-eq-Nil* [*simp*]: *drop n xs* = [] ⟷ *length xs* ≤ *n*
by (*metis diff-is-0-eq drop-all length-drop list.size*(*3*))

lemmas *drop-eq-Nil2* [*simp*] = *drop-eq-Nil*[*THEN eq-iff-swap*]

lemma *take-append* [*simp*]:
  take n (*xs* @ *ys*) = (*take n xs* @ *take* (*n* − *length xs*) *ys*)
  by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *drop-append* [*simp*]:
  drop n (*xs* @ *ys*) = *drop n xs* @ *drop* (*n* − *length xs*) *ys*
  by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *take-take* [*simp*]: *take n* (*take m xs*) = *take* (*min n m*) *xs*
proof (*induct m arbitrary*: *xs n*)
  case *0*
  then show *?case* **by** *simp*
next
  case *Suc*
  then show *?case* **by** (*cases xs*; *cases n*) *simp-all*
qed

lemma *drop-drop* [*simp*]: *drop n* (*drop m xs*) = *drop* (*n* + *m*) *xs*
proof (*induct m arbitrary*: *xs*)
  case *0*
  then show *?case* **by** *simp*
next
  case *Suc*
  then show *?case* **by** (*cases xs*) *simp-all*

qed

lemma *take-drop*: *take n* (*drop m xs*) = *drop m* (*take* (*n* + *m*) *xs*)
proof (*induct m arbitrary*: *xs n*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case* **by** (*cases xs*; *cases n*) *simp-all*
qed

lemma *drop-take*: *drop n* (*take m xs*) = *take* (*m*−*n*) (*drop n xs*)
by(*induct xs arbitrary*: *m n*)(*auto simp*: *take-Cons drop-Cons split*: *nat.split*)

lemma *append-take-drop-id* [*simp*]: *take n xs* @ *drop n xs* = *xs*
proof (*induct n arbitrary*: *xs*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case* **by** (*cases xs*) *simp-all*
qed

lemma *take-map*: *take n* (*map f xs*) = *map f* (*take n xs*)
proof (*induct n arbitrary*: *xs*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case* **by** (*cases xs*) *simp-all*
qed

lemma *drop-map*: *drop n* (*map f xs*) = *map f* (*drop n xs*)
proof (*induct n arbitrary*: *xs*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case* **by** (*cases xs*) *simp-all*
qed

lemma *rev-take*: *rev* (*take i xs*) = *drop* (*length xs* − *i*) (*rev xs*)
proof (*induct xs arbitrary*: *i*)
case *Nil*
then show *?case* **by** *simp*
next
case *Cons*
then show *?case* **by** (*cases i*) *auto*
qed

lemma *rev-drop*: *rev* (*drop i xs*) = *take* (*length xs* − *i*) (*rev xs*)
proof (*induct xs arbitrary*: *i*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case *Cons*
  then show *?case* **by** (*cases i*) *auto*
qed

lemma *drop-rev*: *drop n* (*rev xs*) = *rev* (*take* (*length xs* − *n*) *xs*)
  by (*cases length xs* < *n*) (*auto simp*: *rev-take*)

lemma *take-rev*: *take n* (*rev xs*) = *rev* (*drop* (*length xs* − *n*) *xs*)
  by (*cases length xs* < *n*) (*auto simp*: *rev-drop*)

lemma *nth-take* [*simp*]: *i* < *n* ⟹ (*take n xs*)!*i* = *xs*!*i*
proof (*induct xs arbitrary*: *i n*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case *Cons*
  then show *?case* **by** (*cases n*; *cases i*) *simp-all*
qed

lemma *nth-drop* [*simp*]:
  n ≤ *length xs* ⟹ (*drop n xs*)!*i* = *xs*!(*n* + *i*)
proof (*induct n arbitrary*: *xs*)
  case *0*
  then show *?case* **by** *simp*
next
  case *Suc*
  then show *?case* **by** (*cases xs*) *simp-all*
qed

lemma *butlast-take*:
  n ≤ *length xs* ⟹ *butlast* (*take n xs*) = *take* (*n* − *1*) *xs*
  by (*simp add*: *butlast-conv-take*)

lemma *butlast-drop*: *butlast* (*drop n xs*) = *drop n* (*butlast xs*)
  by (*simp add*: *butlast-conv-take drop-take ac-simps*)

lemma *take-butlast*: *n* < *length xs* ⟹ *take n* (*butlast xs*) = *take n xs*
  by (*simp add*: *butlast-conv-take*)

lemma *drop-butlast*: *drop n* (*butlast xs*) = *butlast* (*drop n xs*)
  by (*simp add*: *butlast-conv-take drop-take ac-simps*)

lemma *butlast-power*: (*butlast* ^^ *n*) *xs* = *take* (*length xs* − *n*) *xs*

by (*induct n*) (*auto simp*: *butlast-take*)

lemma *hd-drop-conv-nth*: $n < length\ xs \Longrightarrow hd(drop\ n\ xs) = xs!n$
by(*simp add*: *hd-conv-nth*)

lemma *set-take-subset-set-take*:
$m \leq n \Longrightarrow set(take\ m\ xs) \leq set(take\ n\ xs)$
proof (*induct xs arbitrary*: *m n*)
case (*Cons x xs m n*) **then show** *?case*
by (*cases n*) (*auto simp*: *take-Cons*)
qed *simp*

lemma *set-take-subset*: $set(take\ n\ xs) \subseteq set\ xs$
by(*induct xs arbitrary*: *n*)(*auto simp*:*take-Cons split*:*nat.split*)

lemma *set-drop-subset*: $set(drop\ n\ xs) \subseteq set\ xs$
by(*induct xs arbitrary*: *n*)(*auto simp*:*drop-Cons split*:*nat.split*)

lemma *set-drop-subset-set-drop*:
$m \geq n \Longrightarrow set(drop\ m\ xs) \leq set(drop\ n\ xs)$
proof (*induct xs arbitrary*: *m n*)
case (*Cons x xs m n*)
then show *?case*
by (*clarsimp simp*: *drop-Cons split*: *nat.split*) (*metis set-drop-subset subset-iff*)
qed *simp*

lemma *in-set-takeD*: $x \in set(take\ n\ xs) \Longrightarrow x \in set\ xs$
using *set-take-subset* **by** *fast*

lemma *in-set-dropD*: $x \in set(drop\ n\ xs) \Longrightarrow x \in set\ xs$
using *set-drop-subset* **by** *fast*

lemma *append-eq-conv-conj*:
$(xs\ @\ ys = zs) = (xs = take\ (length\ xs)\ zs \wedge ys = drop\ (length\ xs)\ zs)$
proof (*induct xs arbitrary*: *zs*)
case (*Cons x xs zs*) **then show** *?case*
by (*cases zs*, *auto*)
qed *auto*

lemma *map-eq-append-conv*:
$map\ f\ xs = ys\ @\ zs \longleftrightarrow (\exists\, us\ vs.\ xs = us\ @\ vs \wedge ys = map\ f\ us \wedge zs = map\ f\ vs)$
proof –
have $map\ f\ xs \neq ys\ @\ zs \wedge map\ f\ xs \neq ys\ @\ zs \vee map\ f\ xs \neq ys\ @\ zs \vee map\ f\ xs = ys\ @\ zs \wedge$
$(\exists\, bs\ bsa.\ xs = bs\ @\ bsa \wedge ys = map\ f\ bs \wedge zs = map\ f\ bsa)$
by (*metis append-eq-conv-conj append-take-drop-id drop-map take-map*)
then show *?thesis*
using *map-append* **by** *blast*

qed

lemmas *append-eq-map-conv* = *map-eq-append-conv*[*THEN eq-iff-swap*]

lemma *take-add*: *take* (*i*+*j*) *xs* = *take i xs* @ *take j* (*drop i xs*)
proof (*induct xs arbitrary*: *i*)
  case (*Cons x xs i*) **then show** *?case*
    by (*cases i*, *auto*)
qed *auto*

lemma *append-eq-append-conv-if*:
  (xs_1 @ xs_2 = ys_1 @ ys_2) =
  (*if size* $xs_1 \leq$ *size* ys_1
   then xs_1 = *take* (*size* xs_1) $ys_1 \wedge xs_2$ = *drop* (*size* xs_1) ys_1 @ ys_2
   else take (*size* ys_1) $xs_1 = ys_1 \wedge$ *drop* (*size* ys_1) xs_1 @ $xs_2 = ys_2$)
proof (*induct* xs_1 *arbitrary*: ys_1)
  case (*Cons a* xs_1 ys_1) **then show** *?case*
    by (*cases* ys_1, *auto*)
qed *auto*

lemma *take-hd-drop*:
  n < *length xs* ⟹ *take n xs* @ [*hd* (*drop n xs*)] = *take* (*Suc n*) *xs*
  by (*induct xs arbitrary*: *n*) (*simp-all add*:*drop-Cons split*:*nat.split*)

lemma *id-take-nth-drop*:
  i < *length xs* ⟹ *xs* = *take i xs* @ *xs*!*i* # *drop* (*Suc i*) *xs*
proof −
  assume *si*: *i* < *length xs*
  hence *xs* = *take* (*Suc i*) *xs* @ *drop* (*Suc i*) *xs* **by** *auto*
  moreover
  from *si* **have** *take* (*Suc i*) *xs* = *take i xs* @ [*xs*!*i*]
    using *take-Suc-conv-app-nth* **by** *blast*
  ultimately show *?thesis* **by** *auto*
qed

lemma *take-update-cancel*[*simp*]: *n* ≤ *m* ⟹ *take n* (*xs*[*m* := *y*]) = *take n xs*
  by(*simp add*: *list-eq-iff-nth-eq*)

lemma *drop-update-cancel*[*simp*]: *n* < *m* ⟹ *drop m* (*xs*[*n* := *x*]) = *drop m xs*
  by(*simp add*: *list-eq-iff-nth-eq*)

lemma *upd-conv-take-nth-drop*:
  i < *length xs* ⟹ *xs*[*i*:=*a*] = *take i xs* @ *a* # *drop* (*Suc i*) *xs*
proof −
  assume *i*: *i* < *length xs*
  have *xs*[*i*:=*a*] = (*take i xs* @ *xs*!*i* # *drop* (*Suc i*) *xs*)[*i*:=*a*]
    by(*rule arg-cong*[*OF id-take-nth-drop*[*OF i*]])
  also have ... = *take i xs* @ *a* # *drop* (*Suc i*) *xs*
    using *i* **by** (*simp add*: *list-update-append*)

finally show *?thesis* **.**
qed

lemma *take-update-swap*: *take m* (*xs*[*n* := *x*]) = (*take m xs*)[*n* := *x*]
proof (*cases n* ≥ *length xs*)
case *False*
then show *?thesis*
by (*simp add*: *upd-conv-take-nth-drop take-Cons drop-take min-def diff-Suc split*: *nat.split*)
qed *auto*

lemma *drop-update-swap*:
assumes *m* ≤ *n* **shows** *drop m* (*xs*[*n* := *x*]) = (*drop m xs*)[*n*−*m* := *x*]
proof (*cases n* ≥ *length xs*)
case *False*
with *assms* **show** *?thesis*
by (*simp add*: *upd-conv-take-nth-drop drop-take*)
qed *auto*

lemma *nth-image*: *l* ≤ *size xs* ⟹ *nth xs* ‘ {*0*..<*l*} = *set*(*take l xs*)
by (*simp add*: *set-conv-nth*) *force*

66.1.15 *takeWhile* and *dropWhile*

lemma *length-takeWhile-le*: *length* (*takeWhile P xs*) ≤ *length xs*
by (*induct xs*) *auto*

lemma *takeWhile-dropWhile-id* [*simp*]: *takeWhile P xs* @ *dropWhile P xs* = *xs*
by (*induct xs*) *auto*

lemma *takeWhile-append1* [*simp*]:
⟦*x* ∈ *set xs*; ¬*P*(*x*)⟧ ⟹ *takeWhile P* (*xs* @ *ys*) = *takeWhile P xs*
by (*induct xs*) *auto*

lemma *takeWhile-append2* [*simp*]:
(⋀*x*. *x* ∈ *set xs* ⟹ *P x*) ⟹ *takeWhile P* (*xs* @ *ys*) = *xs* @ *takeWhile P ys*
by (*induct xs*) *auto*

lemma *takeWhile-append*:
takeWhile P (*xs* @ *ys*) = (*if* ∀*x*∈*set xs*. *P x then xs* @ *takeWhile P ys else takeWhile P xs*)
using *takeWhile-append1*[*of - xs P ys*] *takeWhile-append2*[*of xs P ys*] **by** *auto*

lemma *takeWhile-tail*: ¬ *P x* ⟹ *takeWhile P* (*xs* @ (*x*#*l*)) = *takeWhile P xs*
by (*induct xs*) *auto*

lemma *takeWhile-eq-Nil-iff*: *takeWhile P xs* = [] ⟷ *xs* = [] ∨ ¬*P* (*hd xs*)
by (*cases xs*) *auto*

lemma *takeWhile-nth*: $j < length\ (takeWhile\ P\ xs) \Longrightarrow takeWhile\ P\ xs\ !\ j = xs\ !\ j$
by (*metis nth-append takeWhile-dropWhile-id*)

lemma *takeWhile-takeWhile*: $takeWhile\ Q\ (takeWhile\ P\ xs) = takeWhile\ (\lambda x.\ P\ x \wedge Q\ x)\ xs$
by(*induct xs*) *simp-all*

lemma *dropWhile-nth*: $j < length\ (dropWhile\ P\ xs) \Longrightarrow$
$dropWhile\ P\ xs\ !\ j = xs\ !\ (j + length\ (takeWhile\ P\ xs))$
by (*metis add.commute nth-append-length-plus takeWhile-dropWhile-id*)

lemma *length-dropWhile-le*: $length\ (dropWhile\ P\ xs) \leq length\ xs$
by (*induct xs*) *auto*

lemma *dropWhile-append1* [*simp*]:
$[\![x \in set\ xs;\ \neg P(x)]\!] \Longrightarrow dropWhile\ P\ (xs\ @\ ys) = (dropWhile\ P\ xs)@ys$
by (*induct xs*) *auto*

lemma *dropWhile-append2* [*simp*]:
$(\bigwedge x.\ x \in set\ xs \Longrightarrow P(x)) \Longrightarrow dropWhile\ P\ (xs\ @\ ys) = dropWhile\ P\ ys$
by (*induct xs*) *auto*

lemma *dropWhile-append3*:
$\neg\ P\ y \Longrightarrow dropWhile\ P\ (xs\ @\ y\ \#\ ys) = dropWhile\ P\ xs\ @\ y\ \#\ ys$
by (*induct xs*) *auto*

lemma *dropWhile-append*:
$dropWhile\ P\ (xs\ @\ ys) = (if\ \forall x{\in}set\ xs.\ P\ x\ then\ dropWhile\ P\ ys\ else\ dropWhile\ P\ xs\ @\ ys)$
using *dropWhile-append1*[*of - xs P ys*] *dropWhile-append2*[*of xs P ys*] **by** *auto*

lemma *dropWhile-last*:
$x \in set\ xs \Longrightarrow \neg\ P\ x \Longrightarrow last\ (dropWhile\ P\ xs) = last\ xs$
by (*auto simp add*: *dropWhile-append3 in-set-conv-decomp*)

lemma *set-dropWhileD*: $x \in set\ (dropWhile\ P\ xs) \Longrightarrow x \in set\ xs$
by (*induct xs*) (*auto split*: *if-split-asm*)

lemma *set-takeWhileD*: $x \in set\ (takeWhile\ P\ xs) \Longrightarrow x \in set\ xs \wedge P\ x$
by (*induct xs*) (*auto split*: *if-split-asm*)

lemma *takeWhile-eq-all-conv*[*simp*]:
$(takeWhile\ P\ xs = xs) = (\forall x \in set\ xs.\ P\ x)$
by(*induct xs*, *auto*)

lemma *dropWhile-eq-Nil-conv*[*simp*]:
$(dropWhile\ P\ xs = []) = (\forall x \in set\ xs.\ P\ x)$
by(*induct xs*, *auto*)

lemma *dropWhile-eq-Cons-conv*:
$(dropWhile\ P\ xs = y\#ys) = (xs = takeWhile\ P\ xs\ @\ y\ \#\ ys \wedge \neg\ P\ y)$
by(*induct xs*, *auto*)

lemma *dropWhile-eq-self-iff*: $dropWhile\ P\ xs = xs \longleftrightarrow xs = [] \vee \neg P\ (hd\ xs)$
by (*cases xs*) (*auto simp*: *dropWhile-eq-Cons-conv*)

lemma *dropWhile-dropWhile1*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x) \Longrightarrow dropWhile\ Q\ (dropWhile\ P\ xs) = dropWhile\ P\ xs$
by(*induct xs*) *simp-all*

lemma *dropWhile-dropWhile2*: $(\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow takeWhile\ P\ (takeWhile\ Q\ xs) = takeWhile\ P\ xs$
by(*induct xs*) *simp-all*

lemma *dropWhile-takeWhile*:
$(\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow dropWhile\ P\ (takeWhile\ Q\ xs) = takeWhile\ Q\ (dropWhile\ P\ xs)$
by (*induction xs*) *auto*

lemma *distinct-takeWhile*[*simp*]: $distinct\ xs \Longrightarrow distinct\ (takeWhile\ P\ xs)$
by (*induct xs*) (*auto dest*: *set-takeWhileD*)

lemma *distinct-dropWhile*[*simp*]: $distinct\ xs \Longrightarrow distinct\ (dropWhile\ P\ xs)$
by (*induct xs*) *auto*

lemma *takeWhile-map*: $takeWhile\ P\ (map\ f\ xs) = map\ f\ (takeWhile\ (P \circ f)\ xs)$
by (*induct xs*) *auto*

lemma *dropWhile-map*: $dropWhile\ P\ (map\ f\ xs) = map\ f\ (dropWhile\ (P \circ f)\ xs)$
by (*induct xs*) *auto*

lemma *takeWhile-eq-take*: $takeWhile\ P\ xs = take\ (length\ (takeWhile\ P\ xs))\ xs$
by (*induct xs*) *auto*

lemma *dropWhile-eq-drop*: $dropWhile\ P\ xs = drop\ (length\ (takeWhile\ P\ xs))\ xs$
by (*induct xs*) *auto*

lemma *hd-dropWhile*: $dropWhile\ P\ xs \neq [] \Longrightarrow \neg\ P\ (hd\ (dropWhile\ P\ xs))$
by (*induct xs*) *auto*

lemma *takeWhile-eq-filter*:
assumes $\bigwedge x.\ x \in set\ (dropWhile\ P\ xs) \Longrightarrow \neg\ P\ x$
shows $takeWhile\ P\ xs = filter\ P\ xs$
proof −
have *A*: $filter\ P\ xs = filter\ P\ (takeWhile\ P\ xs\ @\ dropWhile\ P\ xs)$
by *simp*
have *B*: $filter\ P\ (dropWhile\ P\ xs) = []$

unfolding *filter-empty-conv* **using** *assms* **by** *blast*
have *filter P xs = takeWhile P xs*
unfolding *A filter-append B*
by (*auto simp add*: *filter-id-conv dest*: *set-takeWhileD*)
thus *?thesis* **..**
qed

lemma *takeWhile-eq-take-P-nth*:
$[\![\bigwedge i.\ [\![i < n ; i < length\ xs]\!] \Longrightarrow P\ (xs\ !\ i) ; n < length\ xs \Longrightarrow \neg P\ (xs\ !\ n)]\!] \Longrightarrow$
takeWhile P xs = take n xs
proof (*induct xs arbitrary*: *n*)
case *Nil*
thus *?case* **by** *simp*
next
case (*Cons x xs*)
show *?case*
proof (*cases n*)
case *0*
with *Cons* **show** *?thesis* **by** *simp*
next
case [*simp*]: (*Suc n'*)
have *P x* **using** *Cons.prems*(*1*)[*of 0*] **by** *simp*
moreover have *takeWhile P xs = take n' xs*
proof (*rule Cons.hyps*)
fix *i*
assume *i < n' i < length xs*
thus *P* (*xs ! i*) **using** *Cons.prems*(*1*)[*of Suc i*] **by** *simp*
next
assume *n' < length xs*
thus ¬ *P* (*xs ! n'*) **using** *Cons* **by** *auto*
qed
ultimately show *?thesis* **by** *simp*
qed
qed

lemma *nth-length-takeWhile*:
length (*takeWhile P xs*) < *length xs* ⟹ ¬ *P* (*xs ! length* (*takeWhile P xs*))
by (*induct xs*) *auto*

lemma *length-takeWhile-less-P-nth*:
assumes *all*: $\bigwedge i.\ i < j \Longrightarrow P\ (xs\ !\ i)$ **and** $j \le length\ xs$
shows $j \le length\ (takeWhile\ P\ xs)$
proof (*rule classical*)
assume ¬ *?thesis*
hence *length* (*takeWhile P xs*) < *length xs* **using** *assms* **by** *simp*
thus *?thesis* **using** *all* ‹¬ *?thesis*› *nth-length-takeWhile*[*of P xs*] **by** *auto*
qed

lemma *takeWhile-neq-rev*: ⟦*distinct xs*; $x \in$ *set xs*⟧ $\Longrightarrow$
takeWhile ($\lambda y.\ y \neq x$) (*rev xs*) = *rev* (*tl* (*dropWhile* ($\lambda y.\ y \neq x$) *xs*))
by(*induct xs*) (*auto simp*: *takeWhile-tail*[**where** *l*=[]])

lemma *dropWhile-neq-rev*: ⟦*distinct xs*; $x \in$ *set xs*⟧ $\Longrightarrow$
dropWhile ($\lambda y.\ y \neq x$) (*rev xs*) = *x* # *rev* (*takeWhile* ($\lambda y.\ y \neq x$) *xs*)
proof (*induct xs*)
case (*Cons a xs*)
then show *?case*
by(*auto*, *subst dropWhile-append2*, *auto*)
qed *simp*

lemma *takeWhile-not-last*:
distinct xs $\Longrightarrow$ *takeWhile* ($\lambda y.\ y \neq$ *last xs*) *xs* = *butlast xs*
by(*induction xs rule*: *induct-list012*) *auto*

lemma *takeWhile-cong* [*fundef-cong*]:
⟦*l* = *k*; $\bigwedge x.\ x \in$ *set l* $\Longrightarrow$ *P x* = *Q x*⟧
$\Longrightarrow$ *takeWhile P l* = *takeWhile Q k*
by (*induct k arbitrary*: *l*) (*simp-all*)

lemma *dropWhile-cong* [*fundef-cong*]:
⟦*l* = *k*; $\bigwedge x.\ x \in$ *set l* $\Longrightarrow$ *P x* = *Q x*⟧
$\Longrightarrow$ *dropWhile P l* = *dropWhile Q k*
by (*induct k arbitrary*: *l*, *simp-all*)

lemma *takeWhile-idem* [*simp*]:
takeWhile P (*takeWhile P xs*) = *takeWhile P xs*
by (*induct xs*) *auto*

lemma *dropWhile-idem* [*simp*]:
dropWhile P (*dropWhile P xs*) = *dropWhile P xs*
by (*induct xs*) *auto*

66.1.16 *zip*

lemma *zip-Nil* [*simp*]: *zip* [] *ys* = []
by (*induct ys*) *auto*

lemma *zip-Cons-Cons* [*simp*]: *zip* (*x* # *xs*) (*y* # *ys*) = (*x*, *y*) # *zip xs ys*
by *simp*

declare *zip-Cons* [*simp del*]

lemma [*code*]:
zip [] *ys* = []
zip xs [] = []
zip (*x* # *xs*) (*y* # *ys*) = (*x*, *y*) # *zip xs ys*
by (*fact zip-Nil zip.simps*(*1*) *zip-Cons-Cons*)+

lemma *zip-Cons1*:
zip (*x*#*xs*) *ys* = (*case ys of* [] ⇒ [] | *y*#*ys* ⇒ (*x*,*y*)#*zip xs ys*)
by(*auto split*:*list.split*)

lemma *length-zip* [*simp*]:
length (*zip xs ys*) = *min* (*length xs*) (*length ys*)
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *zip-obtain-same-length*:
assumes ⋀*zs ws n*. *length zs* = *length ws* ⟹ *n* = *min* (*length xs*) (*length ys*)
⟹ *zs* = *take n xs* ⟹ *ws* = *take n ys* ⟹ *P* (*zip zs ws*)
shows *P* (*zip xs ys*)
proof −
let *?n* = *min* (*length xs*) (*length ys*)
have *P* (*zip* (*take ?n xs*) (*take ?n ys*))
by (*rule assms*) *simp-all*
moreover have *zip xs ys* = *zip* (*take ?n xs*) (*take ?n ys*)
proof (*induct xs arbitrary*: *ys*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **then show** *?case* **by** (*cases ys*) *simp-all*
qed
ultimately show *?thesis* **by** *simp*
qed

lemma *zip-append1*:
zip (*xs* @ *ys*) *zs* =
zip xs (*take* (*length xs*) *zs*) @ *zip ys* (*drop* (*length xs*) *zs*)
by (*induct xs zs rule*:*list-induct2'*) *auto*

lemma *zip-append2*:
zip xs (*ys* @ *zs*) =
zip (*take* (*length ys*) *xs*) *ys* @ *zip* (*drop* (*length ys*) *xs*) *zs*
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *zip-append* [*simp*]:
⟦*length xs* = *length us*⟧ ⟹
zip (*xs*@*ys*) (*us*@*vs*) = *zip xs us* @ *zip ys vs*
by (*simp add*: *zip-append1*)

lemma *zip-rev*:
length xs = *length ys* ⟹ *zip* (*rev xs*) (*rev ys*) = *rev* (*zip xs ys*)
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *zip-map-map*:
zip (*map f xs*) (*map g ys*) = *map* (λ (*x*, *y*). (*f x*, *g y*)) (*zip xs ys*)
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs*) **note** *Cons-x-xs* = *Cons.hyps*

show *?case*
proof (*cases ys*)
case (*Cons y ys'*)
show *?thesis* **unfolding** *Cons* **using** *Cons-x-xs* **by** *simp*
qed *simp*
qed *simp*

lemma *zip-map1*:
zip (*map f xs*) *ys* = *map* (λ(*x*, *y*). (*f x*, *y*)) (*zip xs ys*)
using *zip-map-map*[*of f xs* λ*x*. *x ys*] **by** *simp*

lemma *zip-map2*:
zip xs (*map f ys*) = *map* (λ(*x*, *y*). (*x*, *f y*)) (*zip xs ys*)
using *zip-map-map*[*of* λ*x*. *x xs f ys*] **by** *simp*

lemma *map-zip-map*:
map f (*zip* (*map g xs*) *ys*) = *map* (%(*x*,*y*). *f*(*g x*, *y*)) (*zip xs ys*)
by (*auto simp*: *zip-map1*)

lemma *map-zip-map2*:
map f (*zip xs* (*map g ys*)) = *map* (%(*x*,*y*). *f*(*x*, *g y*)) (*zip xs ys*)
by (*auto simp*: *zip-map2*)

Courtesy of Andreas Lochbihler:

lemma *zip-same-conv-map*: *zip xs xs* = *map* (λ*x*. (*x*, *x*)) *xs*
by(*induct xs*) *auto*

lemma *nth-zip* [*simp*]:
⟦*i* < *length xs*; *i* < *length ys*⟧ ⟹ (*zip xs ys*)!*i* = (*xs*!*i*, *ys*!*i*)
proof (*induct ys arbitrary*: *i xs*)
case (*Cons y ys*)
then show *?case*
by (*cases xs*) (*simp-all add*: *nth.simps split*: *nat.split*)
qed *auto*

lemma *set-zip*:
set (*zip xs ys*) = {(*xs*!*i*, *ys*!*i*) | *i*. *i* < *min* (*length xs*) (*length ys*)}
by(*simp add*: *set-conv-nth cong*: *rev-conj-cong*)

lemma *zip-same*: ((*a*,*b*) ∈ *set* (*zip xs xs*)) = (*a* ∈ *set xs* ∧ *a* = *b*)
by(*induct xs*) *auto*

lemma *zip-update*: *zip* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*]) = (*zip xs ys*)[*i*:=(*x*,*y*)]
by (*simp add*: *update-zip*)

lemma *zip-replicate* [*simp*]:
zip (*replicate i x*) (*replicate j y*) = *replicate* (*min i j*) (*x*,*y*)
proof (*induct i arbitrary*: *j*)
case (*Suc i*)

then show *?case*
by (*cases j*, *auto*)
qed *auto*

lemma *zip-replicate1*: *zip* (*replicate n x*) *ys* = *map* (*Pair x*) (*take n ys*)
by(*induction ys arbitrary*: *n*)(*case-tac* [*2*] *n*, *simp-all*)

lemma *take-zip*: *take n* (*zip xs ys*) = *zip* (*take n xs*) (*take n ys*)
proof (*induct n arbitrary*: *xs ys*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case* **by** (*cases xs*; *cases ys*) *simp-all*
qed

lemma *drop-zip*: *drop n* (*zip xs ys*) = *zip* (*drop n xs*) (*drop n ys*)
proof (*induct n arbitrary*: *xs ys*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case* **by** (*cases xs*; *cases ys*) *simp-all*
qed

lemma *zip-takeWhile-fst*: *zip* (*takeWhile P xs*) *ys* = *takeWhile* (*P* ∘ *fst*) (*zip xs ys*)
proof (*induct xs arbitrary*: *ys*)
case *Nil*
then show *?case* **by** *simp*
next
case *Cons*
then show *?case* **by** (*cases ys*) *auto*
qed

lemma *zip-takeWhile-snd*: *zip xs* (*takeWhile P ys*) = *takeWhile* (*P* ∘ *snd*) (*zip xs ys*)
proof (*induct xs arbitrary*: *ys*)
case *Nil*
then show *?case* **by** *simp*
next
case *Cons*
then show *?case* **by** (*cases ys*) *auto*
qed

lemma *set-zip-leftD*: (*x*,*y*)∈ *set* (*zip xs ys*) ⟹ *x* ∈ *set xs*
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *set-zip-rightD*: (*x*,*y*)∈ *set* (*zip xs ys*) ⟹ *y* ∈ *set ys*

by (*induct xs ys rule*:*list-induct2′*) *auto*

lemma *in-set-zipE*:
$(x,y) \in set(zip\ xs\ ys) \Longrightarrow (\llbracket\ x \in set\ xs;\ y \in set\ ys\ \rrbracket \Longrightarrow R) \Longrightarrow R$
by(*blast dest*: *set-zip-leftD set-zip-rightD*)

lemma *zip-map-fst-snd*: $zip\ (map\ fst\ zs)\ (map\ snd\ zs) = zs$
by (*induct zs*) *simp-all*

lemma *zip-eq-conv*:
$length\ xs = length\ ys \Longrightarrow zip\ xs\ ys = zs \longleftrightarrow map\ fst\ zs = xs \wedge map\ snd\ zs = ys$
by (*auto simp add*: *zip-map-fst-snd*)

lemma *in-set-zip*:
$p \in set\ (zip\ xs\ ys) \longleftrightarrow (\exists\, n.\ xs\ !\ n = fst\ p \wedge ys\ !\ n = snd\ p$
$\wedge\ n < length\ xs \wedge n < length\ ys)$
by (*cases p*) (*auto simp add*: *set-zip*)

lemma *in-set-impl-in-set-zip1*:
assumes $length\ xs = length\ ys$
assumes $x \in set\ xs$
obtains *y* **where** $(x,\ y) \in set\ (zip\ xs\ ys)$
proof −
from *assms* **have** $x \in set\ (map\ fst\ (zip\ xs\ ys))$ **by** *simp*
from *this that* **show** *?thesis* **by** *fastforce*
qed

lemma *in-set-impl-in-set-zip2*:
assumes $length\ xs = length\ ys$
assumes $y \in set\ ys$
obtains *x* **where** $(x,\ y) \in set\ (zip\ xs\ ys)$
proof −
from *assms* **have** $y \in set\ (map\ snd\ (zip\ xs\ ys))$ **by** *simp*
from *this that* **show** *?thesis* **by** *fastforce*
qed

lemma *zip-eq-Nil-iff* [*simp*]:
$zip\ xs\ ys = [] \longleftrightarrow xs = [] \vee ys = []$
by (*cases xs*; *cases ys*) *simp-all*

lemmas *Nil-eq-zip-iff* [*simp*] = *zip-eq-Nil-iff* [*THEN eq-iff-swap*]

lemma *zip-eq-ConsE*:
assumes $zip\ xs\ ys = xy\ \#\ xys$
obtains $x\ xs'\ y\ ys'$ **where** $xs = x\ \#\ xs'$
and $ys = y\ \#\ ys'$ **and** $xy = (x,\ y)$
and $xys = zip\ xs'\ ys'$
proof −
from *assms* **have** $xs \neq []$ **and** $ys \neq []$

using *zip-eq-Nil-iff* [*of xs ys*] **by** *simp-all*
then obtain *x xs' y ys'* **where** *xs*: *xs* = *x* # *xs'*
and *ys*: *ys* = *y* # *ys'*
by (*cases xs*; *cases ys*) *auto*
with *assms* **have** *xy* = (*x*, *y*) **and** *xys* = *zip xs' ys'*
by *simp-all*
with *xs ys* **show** *?thesis* **..**
qed

lemma *semilattice-map2*:
semilattice (*map2* (∗)) **if** *semilattice* (∗)
for *f* (**infixl** ‹∗› *70*)
proof −
from *that* **interpret** *semilattice f* **.**
show *?thesis*
proof
show *map2* (∗) (*map2* (∗) *xs ys*) *zs* = *map2* (∗) *xs* (*map2* (∗) *ys zs*)
for *xs ys zs* :: *'a list*
proof (*induction zip xs* (*zip ys zs*) *arbitrary*: *xs ys zs*)
case *Nil*
from *Nil* [*symmetric*] **show** *?case*
by *auto*
next
case (*Cons xyz xyzs*)
from *Cons.hyps*(*2*) [*symmetric*] **show** *?case*
by (*rule zip-eq-ConsE*) (*erule zip-eq-ConsE*,
auto intro: *Cons.hyps*(*1*) *simp add*: *ac-simps*)
qed
show *map2* (∗) *xs ys* = *map2* (∗) *ys xs*
for *xs ys* :: *'a list*
proof (*induction zip xs ys arbitrary*: *xs ys*)
case *Nil*
then show *?case*
by *auto*
next
case (*Cons xy xys*)
from *Cons.hyps*(*2*) [*symmetric*] **show** *?case*
by (*rule zip-eq-ConsE*) (*auto intro*: *Cons.hyps*(*1*) *simp add*: *ac-simps*)
qed
show *map2* (∗) *xs xs* = *xs*
for *xs* :: *'a list*
by (*induction xs*) *simp-all*
qed
qed

lemma *pair-list-eqI*:
assumes *map fst xs* = *map fst ys* **and** *map snd xs* = *map snd ys*
shows *xs* = *ys*
proof −

from *assms*(*1*) **have** *length xs* = *length ys* **by** (*rule map-eq-imp-length-eq*)
from *this assms* **show** *?thesis*
by (*induct xs ys rule*: *list-induct2*) (*simp-all add*: *prod-eqI*)
qed

lemma *hd-zip*:
‹*hd* (*zip xs ys*) = (*hd xs*, *hd ys*)› **if** ‹*xs* ≠ []› **and** ‹*ys* ≠ []›
using *that* **by** (*cases xs*; *cases ys*) *simp-all*

lemma *last-zip*:
‹*last* (*zip xs ys*) = (*last xs*, *last ys*)› **if** ‹*xs* ≠ []› **and** ‹*ys* ≠ []›
and ‹*length xs* = *length ys*›
using *that* **by** (*cases xs rule*: *rev-cases*; *cases ys rule*: *rev-cases*) *simp-all*

66.1.17 *list-all2*

lemma *list-all2-lengthD* [*intro?*]:
list-all2 P xs ys ⟹ *length xs* = *length ys*
by (*simp add*: *list-all2-iff*)

lemma *list-all2-Nil* [*iff*, *code*]: *list-all2 P* [] *ys* = (*ys* = [])
by (*simp add*: *list-all2-iff*)

lemma *list-all2-Nil2* [*iff*, *code*]: *list-all2 P xs* [] = (*xs* = [])
by (*simp add*: *list-all2-iff*)

lemma *list-all2-Cons* [*iff*, *code*]:
list-all2 P (*x* # *xs*) (*y* # *ys*) = (*P x y* ∧ *list-all2 P xs ys*)
by (*auto simp add*: *list-all2-iff*)

lemma *list-all2-Cons1*:
list-all2 P (*x* # *xs*) *ys* = (∃ *z zs*. *ys* = *z* # *zs* ∧ *P x z* ∧ *list-all2 P xs zs*)
by (*cases ys*) *auto*

lemma *list-all2-Cons2*:
list-all2 P xs (*y* # *ys*) = (∃ *z zs*. *xs* = *z* # *zs* ∧ *P z y* ∧ *list-all2 P zs ys*)
by (*cases xs*) *auto*

lemma *list-all2-induct*
[*consumes 1*, *case-names Nil Cons*, *induct set*: *list-all2*]:
assumes *P*: *list-all2 P xs ys*
assumes *Nil*: *R* [] []
assumes *Cons*: ⋀*x xs y ys*.
⟦*P x y*; *list-all2 P xs ys*; *R xs ys*⟧ ⟹ *R* (*x* # *xs*) (*y* # *ys*)
shows *R xs ys*
using *P*
by (*induct xs arbitrary*: *ys*) (*auto simp add*: *list-all2-Cons1 Nil Cons*)

lemma *list-all2-rev* [*iff*]:

list-all2 P (*rev xs*) (*rev ys*) = *list-all2 P xs ys*
by (*simp add*: *list-all2-iff zip-rev cong*: *conj-cong*)

lemma *list-all2-rev1*:
list-all2 P (*rev xs*) *ys* = *list-all2 P xs* (*rev ys*)
by (*subst list-all2-rev* [*symmetric*]) *simp*

lemma *list-all2-append1*:
list-all2 P (*xs* @ *ys*) *zs* =
(∃ *us vs*. *zs* = *us* @ *vs* ∧ *length us* = *length xs* ∧ *length vs* = *length ys* ∧
list-all2 P xs us ∧ *list-all2 P ys vs*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*rule-tac x* = *take* (*length xs*) *zs* **in** *exI*)
apply (*rule-tac x* = *drop* (*length xs*) *zs* **in** *exI*)
apply (*force split*: *nat-diff-split simp add*: *list-all2-iff zip-append1*)
done
next
assume *?rhs*
then show *?lhs*
by (*auto simp add*: *list-all2-iff*)
qed

lemma *list-all2-append2*:
list-all2 P xs (*ys* @ *zs*) =
(∃ *us vs*. *xs* = *us* @ *vs* ∧ *length us* = *length ys* ∧ *length vs* = *length zs* ∧
list-all2 P us ys ∧ *list-all2 P vs zs*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*rule-tac x* = *take* (*length ys*) *xs* **in** *exI*)
apply (*rule-tac x* = *drop* (*length ys*) *xs* **in** *exI*)
apply (*force split*: *nat-diff-split simp add*: *list-all2-iff zip-append2*)
done
next
assume *?rhs*
then show *?lhs*
by (*auto simp add*: *list-all2-iff*)
qed

lemma *list-all2-append*:
length xs = *length ys* ⟹
list-all2 P (*xs*@*us*) (*ys*@*vs*) = (*list-all2 P xs ys* ∧ *list-all2 P us vs*)
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *list-all2-appendI* [*intro?*, *trans*]:
⟦ *list-all2 P a b*; *list-all2 P c d* ⟧ ⟹ *list-all2 P* (*a*@*c*) (*b*@*d*)
by (*simp add*: *list-all2-append list-all2-lengthD*)

lemma *list-all2-conv-all-nth*:
list-all2 P xs ys =
(*length xs* = *length ys* ∧ (∀ *i* < *length xs*. *P* (*xs*!*i*) (*ys*!*i*)))
by (*force simp add*: *list-all2-iff set-zip*)

lemma *list-all2-trans*:
assumes *tr*: !!*a b c*. *P1 a b* ⟹ *P2 b c* ⟹ *P3 a c*
shows !!*bs cs*. *list-all2 P1 as bs* ⟹ *list-all2 P2 bs cs* ⟹ *list-all2 P3 as cs*
(**is** !!*bs cs*. *PROP ?Q as bs cs*)
proof (*induct as*)
fix *x xs bs* **assume** *I1*: !!*bs cs*. *PROP ?Q xs bs cs*
show !!*cs*. *PROP ?Q* (*x* # *xs*) *bs cs*
proof (*induct bs*)
fix *y ys cs* **assume** *I2*: !!*cs*. *PROP ?Q* (*x* # *xs*) *ys cs*
show *PROP ?Q* (*x* # *xs*) (*y* # *ys*) *cs*
by (*induct cs*) (*auto intro*: *tr I1 I2*)
qed *simp*
qed *simp*

lemma *list-all2-all-nthI* [*intro?*]:
length a = *length b* ⟹ (⋀*n*. *n* < *length a* ⟹ *P* (*a*!*n*) (*b*!*n*)) ⟹ *list-all2 P a b*
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2I*:
∀ *x* ∈ *set* (*zip a b*). *case-prod P x* ⟹ *length a* = *length b* ⟹ *list-all2 P a b*
by (*simp add*: *list-all2-iff*)

lemma *list-all2-nthD*:
⟦ *list-all2 P xs ys*; *p* < *size xs* ⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-nthD2*:
⟦*list-all2 P xs ys*; *p* < *size ys*⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
by (*frule list-all2-lengthD*) (*auto intro*: *list-all2-nthD*)

lemma *list-all2-map1*:
list-all2 P (*map f as*) *bs* = *list-all2* (λ*x y*. *P* (*f x*) *y*) *as bs*
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-map2*:
list-all2 P as (*map f bs*) = *list-all2* (λ*x y*. *P x* (*f y*)) *as bs*
by (*auto simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-refl* [*intro?*]:
(⋀*x*. *P x x*) ⟹ *list-all2 P xs xs*
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-update-cong*:

⟦ *list-all2 P xs ys*; *P x y* ⟧ ⟹ *list-all2 P* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*])
by (*cases i* < *length ys*) (*auto simp add*: *list-all2-conv-all-nth nth-list-update*)

lemma *list-all2-takeI* [*simp*,*intro*?]:
list-all2 P xs ys ⟹ *list-all2 P* (*take n xs*) (*take n ys*)
proof (*induct xs arbitrary*: *n ys*)
case (*Cons x xs*)
then show *?case*
by (*cases n*) (*auto simp*: *list-all2-Cons1*)
qed *auto*

lemma *list-all2-dropI* [*simp*,*intro*?]:
list-all2 P xs ys ⟹ *list-all2 P* (*drop n xs*) (*drop n ys*)
proof (*induct xs arbitrary*: *n ys*)
case (*Cons x xs*)
then show *?case*
by (*cases n*) (*auto simp*: *list-all2-Cons1*)
qed *auto*

lemma *list-all2-mono* [*intro*?]:
list-all2 P xs ys ⟹ (⋀*xs ys*. *P xs ys* ⟹ *Q xs ys*) ⟹ *list-all2 Q xs ys*
by (*rule list.rel-mono-strong*)

lemma *list-all2-eq*:
xs = *ys* ⟷ *list-all2* (=) *xs ys*
by (*induct xs ys rule*: *list-induct2'*) *auto*

lemma *list-eq-iff-zip-eq*:
xs = *ys* ⟷ *length xs* = *length ys* ∧ (∀ (*x*,*y*) ∈ *set* (*zip xs ys*). *x* = *y*)
by(*auto simp add*: *set-zip list-all2-eq list-all2-conv-all-nth cong*: *conj-cong*)

lemma *list-all2-same*: *list-all2 P xs xs* ⟷ (∀ *x*∈*set xs*. *P x x*)
by(*auto simp add*: *list-all2-conv-all-nth set-conv-nth*)

lemma *zip-assoc*:
zip xs (*zip ys zs*) = *map* (λ((*x*, *y*), *z*). (*x*, *y*, *z*)) (*zip* (*zip xs ys*) *zs*)
by(*rule list-all2-all-nthI*[**where** *P*=(=), *unfolded list.rel-eq*]) *simp-all*

lemma *zip-commute*: *zip xs ys* = *map* (λ(*x*, *y*). (*y*, *x*)) (*zip ys xs*)
by(*rule list-all2-all-nthI*[**where** *P*=(=), *unfolded list.rel-eq*]) *simp-all*

lemma *zip-left-commute*:
zip xs (*zip ys zs*) = *map* (λ(*y*, (*x*, *z*)). (*x*, *y*, *z*)) (*zip ys* (*zip xs zs*))
by(*rule list-all2-all-nthI*[**where** *P*=(=), *unfolded list.rel-eq*]) *simp-all*

lemma *zip-replicate2*: *zip xs* (*replicate n y*) = *map* (λ*x*. (*x*, *y*)) (*take n xs*)
by(*subst zip-commute*)(*simp add*: *zip-replicate1*)

66.1.18 *List.product* and *product-lists*

lemma *product-concat-map*:
List.product xs ys = *concat* (*map* (λ*x*. *map* (λ*y*. (*x*,*y*)) *ys*) *xs*)
by(*induction xs*) (*simp*)+

lemma *set-product*[*simp*]: *set* (*List.product xs ys*) = *set xs* × *set ys*
by (*induct xs*) *auto*

lemma *length-product* [*simp*]:
length (*List.product xs ys*) = *length xs* ∗ *length ys*
by (*induct xs*) *simp-all*

lemma *product-nth*:
assumes *n* < *length xs* ∗ *length ys*
shows *List.product xs ys* ! *n* = (*xs* ! (*n div length ys*), *ys* ! (*n mod length ys*))
using *assms* **proof** (*induct xs arbitrary*: *n*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs n*)
then have *length ys* > *0* **by** *auto*
with *Cons* **show** *?case*
by (*auto simp add*: *nth-append not-less le-mod-geq le-div-geq*)
qed

lemma *in-set-product-lists-length*:
xs ∈ *set* (*product-lists xss*) =⇒ *length xs* = *length xss*
by (*induct xss arbitrary*: *xs*) *auto*

lemma *product-lists-set*:
set (*product-lists xss*) = {*xs*. *list-all2* (λ*x ys*. *x* ∈ *set ys*) *xs xss*} (**is** *?L* = *Collect ?R*)
proof (*intro equalityI subsetI*, *unfold mem-Collect-eq*)
fix *xs* **assume** *xs* ∈ *?L*
then have *length xs* = *length xss* **by** (*rule in-set-product-lists-length*)
from *this* ‹*xs* ∈ *?L*› **show** *?R xs* **by** (*induct xs xss rule*: *list-induct2*) *auto*
next
fix *xs* **assume** *?R xs*
then show *xs* ∈ *?L* **by** *induct auto*
qed

66.1.19 *fold* with natural argument order

lemma *fold-simps* [*code*]: — eta-expanded variant for generated code – enables tail-recursion optimisation in Scala
fold f [] *s* = *s*
fold f (*x* # *xs*) *s* = *fold f xs* (*f x s*)
by *simp-all*

lemma *fold-remove1-split*:

$[\![\bigwedge x\ y.\ x \in set\ xs \Longrightarrow y \in set\ xs \Longrightarrow f\ x \circ f\ y = f\ y \circ f\ x;$
$x \in set\ xs]\!]$
$\Longrightarrow fold\ f\ xs = fold\ f\ (remove1\ x\ xs) \circ f\ x$
by (*induct xs*) (*auto simp add*: *comp-assoc*)

lemma *fold-cong* [*fundef-cong*]:
$a = b \Longrightarrow xs = ys \Longrightarrow (\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x = g\ x)$
$\Longrightarrow fold\ f\ xs\ a = fold\ g\ ys\ b$
by (*induct ys arbitrary*: *a b xs*) *simp-all*

lemma *fold-id*: $(\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x = id) \Longrightarrow fold\ f\ xs = id$
by (*induct xs*) *simp-all*

lemma *fold-commute*:
$(\bigwedge x.\ x \in set\ xs \Longrightarrow h \circ g\ x = f\ x \circ h) \Longrightarrow h \circ fold\ g\ xs = fold\ f\ xs \circ h$
by (*induct xs*) (*simp-all add*: *fun-eq-iff*)

lemma *fold-commute-apply*:
assumes $\bigwedge x.\ x \in set\ xs \Longrightarrow h \circ g\ x = f\ x \circ h$
shows $h\ (fold\ g\ xs\ s) = fold\ f\ xs\ (h\ s)$
proof −
from *assms* **have** $h \circ fold\ g\ xs = fold\ f\ xs \circ h$ **by** (*rule fold-commute*)
then show *?thesis* **by** (*simp add*: *fun-eq-iff*)
qed

lemma *fold-invariant*:
$[\![\bigwedge x.\ x \in set\ xs \Longrightarrow Q\ x;\ \ P\ s;\ \ \bigwedge x\ s.\ Q\ x \Longrightarrow P\ s \Longrightarrow P\ (f\ x\ s)]\!]$
$\Longrightarrow P\ (fold\ f\ xs\ s)$
by (*induct xs arbitrary*: *s*) *simp-all*

lemma *fold-append* [*simp*]: $fold\ f\ (xs\ @\ ys) = fold\ f\ ys \circ fold\ f\ xs$
by (*induct xs*) *simp-all*

lemma *fold-map* [*code-unfold*]: $fold\ g\ (map\ f\ xs) = fold\ (g \circ f)\ xs$
by (*induct xs*) *simp-all*

lemma *fold-filter*:
$fold\ f\ (filter\ P\ xs) = fold\ (\lambda x.\ if\ P\ x\ then\ f\ x\ else\ id)\ xs$
by (*induct xs*) *simp-all*

lemma *fold-rev*:
$(\bigwedge x\ y.\ x \in set\ xs \Longrightarrow y \in set\ xs \Longrightarrow f\ y \circ f\ x = f\ x \circ f\ y)$
$\Longrightarrow fold\ f\ (rev\ xs) = fold\ f\ xs$
by (*induct xs*) (*simp-all add*: *fold-commute-apply fun-eq-iff*)

lemma *fold-Cons-rev*: $fold\ Cons\ xs = append\ (rev\ xs)$
by (*induct xs*) *simp-all*

lemma *rev-conv-fold* [*code*]: $rev\ xs = fold\ Cons\ xs\ []$

by (*simp add*: *fold-Cons-rev*)

lemma *fold-append-concat-rev*: *fold append xss* = *append* (*concat* (*rev xss*))
by (*induct xss*) *simp-all*

Finite-Set.fold and *fold*

lemma (**in** *comp-fun-commute-on*) *fold-set-fold-remdups*:
assumes *set xs* ⊆ *S*
shows *Finite-Set.fold f y* (*set xs*) = *fold f* (*remdups xs*) *y*
by (*rule sym*, *use assms* **in** ‹*induct xs arbitrary*: *y*›)
(*simp-all add*: *insert-absorb fold-fun-left-comm*)

lemma (**in** *comp-fun-idem-on*) *fold-set-fold*:
assumes *set xs* ⊆ *S*
shows *Finite-Set.fold f y* (*set xs*) = *fold f xs y*
by (*rule sym*, *use assms* **in** ‹*induct xs arbitrary*: *y*›) (*simp-all add*: *fold-fun-left-comm*)

lemma *union-set-fold* [*code*]: *set xs* ∪ *A* = *fold Set.insert xs A*
proof −
interpret *comp-fun-idem Set.insert*
by (*fact comp-fun-idem-insert*)
show *?thesis* **by** (*simp add*: *union-fold-insert fold-set-fold*)
qed

lemma *union-coset-filter* [*code*]:
List.coset xs ∪ *A* = *List.coset* (*List.filter* (λ*x*. *x* ∉ *A*) *xs*)
by *auto*

lemma *minus-set-fold* [*code*]: *A* − *set xs* = *fold Set.remove xs A*
proof −
interpret *comp-fun-idem Set.remove*
by (*fact comp-fun-idem-remove*)
show *?thesis*
by (*simp add*: *minus-fold-remove* [*of - A*] *fold-set-fold*)
qed

lemma *minus-coset-filter* [*code*]:
A − *List.coset xs* = *set* (*List.filter* (λ*x*. *x* ∈ *A*) *xs*)
by *auto*

lemma *inter-set-filter* [*code*]:
A ∩ *set xs* = *set* (*List.filter* (λ*x*. *x* ∈ *A*) *xs*)
by *auto*

lemma *inter-coset-fold* [*code*]:
A ∩ *List.coset xs* = *fold Set.remove xs A*
by (*simp add*: *Diff-eq* [*symmetric*] *minus-set-fold*)

lemma (**in** *semilattice-set*) *set-eq-fold* [*code*]:

F (*set* (*x* # *xs*)) = *fold f xs x*
proof −
interpret *comp-fun-idem f*
by *standard* (*simp-all add*: *fun-eq-iff left-commute*)
show *?thesis* **by** (*simp add*: *eq-fold fold-set-fold*)
qed

lemma (**in** *complete-lattice*) *Inf-set-fold*:
Inf (*set xs*) = *fold inf xs top*
proof −
interpret *comp-fun-idem inf* :: $'a \Rightarrow 'a \Rightarrow 'a$
by (*fact comp-fun-idem-inf*)
show *?thesis* **by** (*simp add*: *Inf-fold-inf fold-set-fold inf-commute*)
qed

declare *Inf-set-fold* [**where** $'a = 'a$ *set*, *code*]

lemma (**in** *complete-lattice*) *Sup-set-fold*:
Sup (*set xs*) = *fold sup xs bot*
proof −
interpret *comp-fun-idem sup* :: $'a \Rightarrow 'a \Rightarrow 'a$
by (*fact comp-fun-idem-sup*)
show *?thesis* **by** (*simp add*: *Sup-fold-sup fold-set-fold sup-commute*)
qed

declare *Sup-set-fold* [**where** $'a = 'a$ *set*, *code*]

lemma (**in** *complete-lattice*) *INF-set-fold*:
$\bigsqcap$(*f* ' *set xs*) = *fold* (*inf* ∘ *f*) *xs top*
using *Inf-set-fold* [*of map f xs*] **by** (*simp add*: *fold-map*)

lemma (**in** *complete-lattice*) *SUP-set-fold*:
$\bigsqcup$(*f* ' *set xs*) = *fold* (*sup* ∘ *f*) *xs bot*
using *Sup-set-fold* [*of map f xs*] **by** (*simp add*: *fold-map*)

66.1.20 Fold variants: *foldr* and *foldl*

Correspondence

lemma *foldr-conv-fold* [*code-abbrev*]: *foldr f xs* = *fold f* (*rev xs*)
by (*induct xs*) *simp-all*

lemma *foldl-conv-fold*: *foldl f s xs* = *fold* (λ*x s*. *f s x*) *xs s*
by (*induct xs arbitrary*: *s*) *simp-all*

lemma *foldr-conv-foldl*: — The "Third Duality Theorem" in Bird & Wadler:
foldr f xs a = *foldl* (λ*x y*. *f y x*) *a* (*rev xs*)
by (*simp add*: *foldr-conv-fold foldl-conv-fold*)

lemma *foldl-conv-foldr*:

foldl f a xs = foldr (*λx y. f y x*) (*rev xs*) *a*
by (*simp add*: *foldr-conv-fold foldl-conv-fold*)

lemma *foldr-fold*:
(⋀*x y. x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *f y* ∘ *f x* = *f x* ∘ *f y*)
⟹ *foldr f xs* = *fold f xs*
unfolding *foldr-conv-fold* **by** (*rule fold-rev*)

lemma *foldr-cong* [*fundef-cong*]:
a = *b* ⟹ *l* = *k* ⟹ (⋀*a x. x* ∈ *set l* ⟹ *f x a* = *g x a*) ⟹ *foldr f l a* = *foldr g k b*
by (*auto simp add*: *foldr-conv-fold intro*!: *fold-cong*)

lemma *foldl-cong* [*fundef-cong*]:
a = *b* ⟹ *l* = *k* ⟹ (⋀*a x. x* ∈ *set l* ⟹ *f a x* = *g a x*) ⟹ *foldl f a l* = *foldl g b k*
by (*auto simp add*: *foldl-conv-fold intro*!: *fold-cong*)

lemma *foldr-append* [*simp*]: *foldr f* (*xs* @ *ys*) *a* = *foldr f xs* (*foldr f ys a*)
by (*simp add*: *foldr-conv-fold*)

lemma *foldl-append* [*simp*]: *foldl f a* (*xs* @ *ys*) = *foldl f* (*foldl f a xs*) *ys*
by (*simp add*: *foldl-conv-fold*)

lemma *foldr-map* [*code-unfold*]: *foldr g* (*map f xs*) *a* = *foldr* (*g* ∘ *f*) *xs a*
by (*simp add*: *foldr-conv-fold fold-map rev-map*)

lemma *foldr-filter*:
foldr f (*filter P xs*) = *foldr* (*λx. if P x then f x else id*) *xs*
by (*simp add*: *foldr-conv-fold rev-filter fold-filter*)

lemma *foldl-map* [*code-unfold*]:
foldl g a (*map f xs*) = *foldl* (*λa x. g a* (*f x*)) *a xs*
by (*simp add*: *foldl-conv-fold fold-map comp-def*)

lemma *concat-conv-foldr* [*code*]:
concat xss = *foldr append xss* []
by (*simp add*: *fold-append-concat-rev foldr-conv-fold*)

66.1.21 *upt*

lemma *upt-rec*[*code*]: [*i..*<*j*] = (*if i*<*j then i*#[*Suc i..*<*j*] *else* [])
— simp does not terminate!
by (*induct j*) *auto*

lemmas *upt-rec-numeral*[*simp*] = *upt-rec*[*of numeral m numeral n*] **for** *m n*

lemma *upt-conv-Nil* [*simp*]: *j* ≤ *i* ⟹ [*i..*<*j*] = []
by (*subst upt-rec*) *simp*

lemma *upt-eq-Nil-conv*[*simp*]: ([*i*..<*j*] = []) = (*j* = *0* ∨ *j* ≤ *i*)
by(*induct j*)*simp-all*

lemma *upt-eq-Cons-conv*:
([*i*..<*j*] = *x*#*xs*) = (*i* < *j* ∧ *i* = *x* ∧ [*i*+*1*..<*j*] = *xs*)
proof (*induct j arbitrary*: *x xs*)
case (*Suc j*)
then show *?case*
by (*simp add*: *upt-rec*)
qed *simp*

lemma *upt-Suc-append*: *i* ≤ *j* ⟹ [*i*..<(*Suc j*)] = [*i*..<*j*]@[*j*]
— Only needed if *upt-Suc* is deleted from the simpset.
by *simp*

lemma *upt-conv-Cons*: *i* < *j* ⟹ [*i*..<*j*] = *i* # [*Suc i*..<*j*]
by (*simp add*: *upt-rec*)

lemma *upt-conv-Cons-Cons*: — no precondition
m # *n* # *ns* = [*m*..<*q*] ⟷ *n* # *ns* = [*Suc m*..<*q*]
proof (*cases m* < *q*)
case *False* **then show** *?thesis* **by** *simp*
next
case *True* **then show** *?thesis* **by** (*simp add*: *upt-conv-Cons*)
qed

lemma *upt-add-eq-append*: *i*<=*j* ⟹ [*i*..<*j*+*k*] = [*i*..<*j*]@[*j*..<*j*+*k*]
— LOOPS as a simprule, since *j* ≤ *j*.
by (*induct k*) *auto*

lemma *length-upt* [*simp*]: *length* [*i*..<*j*] = *j* − *i*
by (*induct j*) (*auto simp add*: *Suc-diff-le*)

lemma *nth-upt* [*simp*]: *i* + *k* < *j* ⟹ [*i*..<*j*] ! *k* = *i* + *k*
by (*induct j*) (*auto simp add*: *less-Suc-eq nth-append split*: *nat-diff-split*)

lemma *hd-upt*[*simp*]: *i* < *j* ⟹ *hd*[*i*..<*j*] = *i*
by(*simp add*:*upt-conv-Cons*)

lemma *tl-upt* [*simp*]: *tl* [*m*..<*n*] = [*Suc m*..<*n*]
by (*simp add*: *upt-rec*)

lemma *last-upt*[*simp*]: *i* < *j* ⟹ *last*[*i*..<*j*] = *j* − *1*
by(*cases j*)(*auto simp*: *upt-Suc-append*)

lemma *take-upt* [*simp*]: *i*+*m* ≤ *n* ⟹ *take m* [*i*..<*n*] = [*i*..<*i*+*m*]
proof (*induct m arbitrary*: *i*)
case (*Suc m*)

then show *?case*
by (*subst take-Suc-conv-app-nth*) *auto*
qed *simp*

lemma *drop-upt*[*simp*]: *drop m* [*i*..<*j*] = [*i*+*m*..<*j*]
by(*induct j*) *auto*

lemma *map-Suc-upt*: *map Suc* [*m*..<*n*] = [*Suc m*..<*Suc n*]
by (*induct n*) *auto*

lemma *map-add-upt*: *map* (λ*i*. *i* + *n*) [*0*..<*m*] = [*n*..<*m* + *n*]
by (*induct m*) *simp-all*

lemma *nth-map-upt*: *i* < *n*−*m* ⟹ (*map f* [*m*..<*n*]) ! *i* = *f*(*m*+*i*)
proof (*induct n m arbitrary*: *i rule*: *diff-induct*)
case (*3 x y*)
then show *?case*
by (*metis add.commute length-upt less-diff-conv nth-map nth-upt*)
qed *auto*

lemma *map-decr-upt*: *map* (λ*n*. *n* − *Suc 0*) [*Suc m*..<*Suc n*] = [*m*..<*n*]
by (*induct n*) *simp-all*

lemma *map-upt-Suc*: *map f* [*0* ..< *Suc n*] = *f 0* # *map* (λ*i*. *f* (*Suc i*)) [*0* ..< *n*]
by (*induct n arbitrary*: *f*) *auto*

lemma *nth-take-lemma*:
k ≤ *length xs* ⟹ *k* ≤ *length ys* ⟹
(⋀*i*. *i* < *k* ⟶ *xs*!*i* = *ys*!*i*) ⟹ *take k xs* = *take k ys*
by (*induct k arbitrary*: *xs ys*) (*simp-all add*: *take-Suc-conv-app-nth*)

lemma *nth-equalityI*:
⟦*length xs* = *length ys*; ⋀*i*. *i* < *length xs* ⟹ *xs*!*i* = *ys*!*i*⟧ ⟹ *xs* = *ys*
by (*frule nth-take-lemma* [*OF le-refl eq-imp-le*]) *simp-all*

lemma *map-nth*:
map (λ*i*. *xs* ! *i*) [*0*..<*length xs*] = *xs*
by (*rule nth-equalityI*, *auto*)

lemma *list-all2-antisym*:
⟦ (⋀*x y*. ⟦*P x y*; *Q y x*⟧ ⟹ *x* = *y*); *list-all2 P xs ys*; *list-all2 Q ys xs* ⟧
⟹ *xs* = *ys*
by (*simp add*: *list-all2-conv-all-nth nth-equalityI*)

lemma *take-equalityI*: (∀ *i*. *take i xs* = *take i ys*) ⟹ *xs* = *ys*
— The famous take-lemma.
by (*metis length-take min.commute order-refl take-all*)

lemma *take-Cons'*:

take n (*x* # *xs*) = (*if n* = *0 then* [] *else x* # *take* (*n* − *1*) *xs*)
by (*cases n*) *simp-all*

lemma *drop-Cons'*:
drop n (*x* # *xs*) = (*if n* = *0 then x* # *xs else drop* (*n* − *1*) *xs*)
by (*cases n*) *simp-all*

lemma *nth-Cons'*: (*x* # *xs*)!*n* = (*if n* = *0 then x else xs*!(*n* − *1*))
by (*cases n*) *simp-all*

lemma *take-Cons-numeral* [*simp*]:
take (*numeral v*) (*x* # *xs*) = *x* # *take* (*numeral v* − *1*) *xs*
by (*simp add*: *take-Cons'*)

lemma *drop-Cons-numeral* [*simp*]:
drop (*numeral v*) (*x* # *xs*) = *drop* (*numeral v* − *1*) *xs*
by (*simp add*: *drop-Cons'*)

lemma *nth-Cons-numeral* [*simp*]:
(*x* # *xs*) ! *numeral v* = *xs* ! (*numeral v* − *1*)
by (*simp add*: *nth-Cons'*)

lemma *map-upt-eqI*:
‹*map f* [*m*..<*n*] = *xs*› **if** ‹*length xs* = *n* − *m*›
‹⋀*i*. *i* < *length xs* ⟹ *xs* ! *i* = *f* (*m* + *i*)›
proof (*rule nth-equalityI*)
from ‹*length xs* = *n* − *m*› **show** ‹*length* (*map f* [*m*..<*n*]) = *length xs*›
by *simp*
next
fix *i*
assume ‹*i* < *length* (*map f* [*m*..<*n*])›
then have ‹*i* < *n* − *m*›
by *simp*
with *that* **have** ‹*xs* ! *i* = *f* (*m* + *i*)›
by *simp*
with ‹*i* < *n* − *m*› **show** ‹*map f* [*m*..<*n*] ! *i* = *xs* ! *i*›
by *simp*
qed

66.1.22 *upto*: interval-list on *int*

function *upto* :: *int* ⇒ *int* ⇒ *int list* (‹(‹*indent=1 notation=*‹*mixfix list interval*››[-../-])›) **where**
upto i j = (*if i* ≤ *j then i* # [*i*+*1*..*j*] *else* [])
by *auto*
termination
by(*relation measure*(%(*i*::*int*,*j*). *nat*(*j* − *i* + *1*))) *auto*

declare *upto.simps*[*simp del*]

lemmas *upto-rec-numeral* [*simp*] =
  upto.simps[*of numeral m numeral n*]
  upto.simps[*of numeral m* − *numeral n*]
  upto.simps[*of* − *numeral m numeral n*]
  upto.simps[*of* − *numeral m* − *numeral n*] **for** *m n*

lemma *upto-empty*[*simp*]: $j < i \Longrightarrow [i..j] = []$
by(*simp add*: *upto.simps*)

lemma *upto-single*[*simp*]: $[i..i] = [i]$
by(*simp add*: *upto.simps*)

lemma *upto-Nil*[*simp*]: $[i..j] = [] \longleftrightarrow j < i$
by (*simp add*: *upto.simps*)

lemmas *upto-Nil2*[*simp*] = *upto-Nil*[*THEN eq-iff-swap*]

lemma *upto-rec1*: $i \le j \Longrightarrow [i..j] = i\#[i+1..j]$
by(*simp add*: *upto.simps*)

lemma *upto-rec2*: $i \le j \Longrightarrow [i..j] = [i..j - 1]@[j]$
proof(*induct nat*(*j*−*i*) *arbitrary*: *i j*)
  case *0* **thus** *?case* **by**(*simp add*: *upto.simps*)
next
  case (*Suc n*)
  hence $n = nat\ (j - (i + 1))\ i < j$ **by** *linarith+*
  from *this*(*2*) *Suc.hyps*(*1*)[*OF this*(*1*)] *Suc*(*2*,*3*) *upto-rec1* **show** *?case* **by** *simp*
qed

lemma *length-upto*[*simp*]: *length* $[i..j] = nat(j - i + 1)$
by(*induction i j rule*: *upto.induct*) (*auto simp*: *upto.simps*)

lemma *set-upto*[*simp*]: *set*$[i..j] = \{i..j\}$
proof(*induct i j rule*:*upto.induct*)
  case (*1 i j*)
  from *this* **show** *?case*
    unfolding *upto.simps*[*of i j*] **by** *auto*
qed

lemma *nth-upto*[*simp*]: $i + int\ k \le j \Longrightarrow [i..j]\ !\ k = i + int\ k$
proof(*induction i j arbitrary*: *k rule*: *upto.induct*)
  case (*1 i j*)
  then show *?case*
    by (*auto simp add*: *upto-rec1* [*of i j*] *nth-Cons′*)
qed

lemma *upto-split1*:
  $i \le j \Longrightarrow j \le k \Longrightarrow [i..k] = [i..j{-}1]\ @\ [j..k]$

proof (*induction j rule*: *int-ge-induct*)
  case *base* **thus** *?case* **by** (*simp add*: *upto-rec1*)
next
  case *step* **thus** *?case* **using** *upto-rec1 upto-rec2* **by** *simp*
qed

lemma *upto-split2*:
  $i \le j \Longrightarrow j \le k \Longrightarrow [i..k] = [i..j]\ @\ [j+1..k]$
using *upto-rec1 upto-rec2 upto-split1* **by** *auto*

lemma *upto-split3*: $[\![\ i \le j;\ j \le k\ ]\!] \Longrightarrow [i..k] = [i..j-1]\ @\ j\ \#\ [j+1..k]$
using *upto-rec1 upto-split1* **by** *auto*

Tail recursive version for code generation:

definition *upto-aux* :: *int* ⇒ *int* ⇒ *int list* ⇒ *int list* **where**
  upto-aux i j js = [*i..j*] @ *js*

lemma *upto-aux-rec* [*code*]:
  upto-aux i j js = (*if j<i then js else upto-aux i* (*j* − *1*) (*j#js*))
by (*simp add*: *upto-aux-def upto-rec2*)

lemma *upto-code*[*code*]: [*i..j*] = *upto-aux i j* []
by(*simp add*: *upto-aux-def*)

66.1.23 *successively*

lemma *successively-Cons*:
  successively P (*x* # *xs*) ⟷ *xs* = [] ∨ *P x* (*hd xs*) ∧ *successively P xs*
by (*cases xs*) *auto*

lemma *successively-cong* [*cong*]:
  assumes ⋀*x y*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *P x y* ⟷ *Q x y xs* = *ys*
  shows *successively P xs* ⟷ *successively Q ys*
  unfolding *assms*(*2*) [*symmetric*] **using** *assms*(*1*)
  by (*induction xs*) (*auto simp*: *successively-Cons*)

lemma *successively-append-iff*:
  successively P (*xs* @ *ys*) ⟷
    successively P xs ∧ *successively P ys* ∧
    (*xs* = [] ∨ *ys* = [] ∨ *P* (*last xs*) (*hd ys*))
by (*induction xs*) (*auto simp*: *successively-Cons*)

lemma *successively-if-sorted-wrt*: *sorted-wrt P xs* ⟹ *successively P xs*
by (*induction xs rule*: *induct-list012*) *auto*

lemma *successively-iff-sorted-wrt-strong*:
  assumes ⋀*x y z*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *z* ∈ *set xs* ⟹

$P\ x\ y \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$
shows *successively P xs* ⟷ *sorted-wrt P xs*
proof
assume *successively P xs*
from *this* **and** *assms* **show** *sorted-wrt P xs*
proof (*induction xs rule*: *induct-list012*)
case (*3 x y xs*)
from *3.prems* **have** *P x y*
by *auto*
have *IH*: *sorted-wrt P* (*y* # *xs*)
using *3.prems*
by(*intro 3.IH*(*2*) *list.set-intros*(*2*))(*simp*, *blast intro*: *list.set-intros*(*2*))
have *P x z* **if** *asm*: *z* ∈ *set xs* **for** *z*
proof −
from *IH* **and** *asm* **have** *P y z*
by *auto*
with ‹*P x y*› **show** *P x z*
using *3.prems asm* **by** *auto*
qed
with *IH* **and** ‹*P x y*› **show** *?case* **by** *auto*
qed *auto*
qed (*use successively-if-sorted-wrt* **in** *blast*)

lemma *successively-conv-sorted-wrt*:
assumes *transp P*
shows *successively P xs* ⟷ *sorted-wrt P xs*
using *assms* **unfolding** *transp-def*
by (*intro successively-iff-sorted-wrt-strong*) *blast*

lemma *successively-rev* [*simp*]: *successively P* (*rev xs*) ⟷ *successively* (λ*x y*. *P y x*) *xs*
by (*induction xs rule*: *remdups-adj.induct*)
(*auto simp*: *successively-append-iff successively-Cons*)

lemma *successively-map*: *successively P* (*map f xs*) ⟷ *successively* (λ*x y*. *P* (*f x*) (*f y*)) *xs*
by (*induction xs rule*: *induct-list012*) *auto*

lemma *successively-mono*:
assumes *successively P xs*
assumes ⋀*x y*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *P x y* ⟹ *Q x y*
shows *successively Q xs*
using *assms* **by** (*induction Q xs rule*: *successively.induct*) *auto*

lemma *successively-altdef*:
successively = (λ*P*. *rec-list True* (λ*x xs b*. *case xs of* [] ⇒ *True* | *y* # - ⇒ *P x y* ∧ *b*))
proof (*intro ext*)
fix *P* **and** *xs* :: *'a list*

show *successively P xs = rec-list True* (λx *xs b. case xs of* [] $\Rightarrow$ *True* | *y* # - $\Rightarrow$ *P x y* $\wedge$ *b*) *xs*
by (*induction xs*) (*auto simp*: *successively-Cons split*: *list.splits*)
qed

66.1.24 *distinct* and *remdups* and *remdups-adj*

lemma *distinct-tl*: *distinct xs* $\Longrightarrow$ *distinct* (*tl xs*)
by (*cases xs*) *simp-all*

lemma *distinct-append* [*simp*]:
distinct (*xs* @ *ys*) = (*distinct xs* $\wedge$ *distinct ys* $\wedge$ *set xs* $\cap$ *set ys* = {})
by (*induct xs*) *auto*

lemma *distinct-rev*[*simp*]: *distinct*(*rev xs*) = *distinct xs*
by(*induct xs*) *auto*

lemma *set-remdups* [*simp*]: *set* (*remdups xs*) = *set xs*
by (*induct xs*) (*auto simp add*: *insert-absorb*)

lemma *distinct-remdups* [*iff*]: *distinct* (*remdups xs*)
by (*induct xs*) *auto*

lemma *distinct-remdups-id*: *distinct xs* $\Longrightarrow$ *remdups xs* = *xs*
by (*induct xs*, *auto*)

lemma *remdups-id-iff-distinct* [*simp*]: *remdups xs* = *xs* $\longleftrightarrow$ *distinct xs*
by (*metis distinct-remdups distinct-remdups-id*)

lemma *finite-distinct-list*: *finite A* $\Longrightarrow$ $\exists$ *xs. set xs* = *A* $\wedge$ *distinct xs*
by (*metis distinct-remdups finite-list set-remdups*)

lemma *remdups-eq-nil-iff* [*simp*]: (*remdups x* = []) = (*x* = [])
by (*induct x*, *auto*)

lemmas *remdups-eq-nil-right-iff* [*simp*] = *remdups-eq-nil-iff*[*THEN eq-iff-swap*]

lemma *length-remdups-leq*[*iff*]: *length*(*remdups xs*) $\leq$ *length xs*
by (*induct xs*) *auto*

lemma *length-remdups-eq*[*iff*]:
(*length* (*remdups xs*) = *length xs*) = (*remdups xs* = *xs*)
proof (*induct xs*)
case (*Cons a xs*)
then show *?case*
by *simp* (*metis Suc-n-not-le-n impossible-Cons length-remdups-leq*)
qed *auto*

lemma *remdups-filter*: *remdups*(*filter P xs*) = *filter P* (*remdups xs*)

by (*induct xs*) *auto*

lemma *distinct-map*:
distinct(*map f xs*) = (*distinct xs* ∧ *inj-on f* (*set xs*))
by (*induct xs*) *auto*

lemma *distinct-map-filter*:
distinct (*map f xs*) ⟹ *distinct* (*map f* (*filter P xs*))
by (*induct xs*) *auto*

lemma *distinct-filter* [*simp*]: *distinct xs* ⟹ *distinct* (*filter P xs*)
by (*induct xs*) *auto*

lemma *distinct-upt*[*simp*]: *distinct*[*i*..<*j*]
by (*induct j*) *auto*

lemma *distinct-upto*[*simp*]: *distinct*[*i*..*j*]
proof (*induction i j rule*: *upto.induct*)
case (*1 i j*)
then show *?case*
by (*simp add*: *upto.simps* [*of i*])
qed

lemma *distinct-take*[*simp*]: *distinct xs* ⟹ *distinct* (*take i xs*)
proof (*induct xs arbitrary*: *i*)
case (*Cons a xs*)
then show *?case*
by (*metis Cons.prems append-take-drop-id distinct-append*)
qed *auto*

lemma *distinct-drop*[*simp*]: *distinct xs* ⟹ *distinct* (*drop i xs*)
proof (*induct xs arbitrary*: *i*)
case (*Cons a xs*)
then show *?case*
by (*metis Cons.prems append-take-drop-id distinct-append*)
qed *auto*

lemma *distinct-list-update*:
assumes *d*: *distinct xs* **and** *a*: *a* ∉ *set xs* − {*xs*!*i*}
shows *distinct* (*xs*[*i*:=*a*])
proof (*cases i* < *length xs*)
case *True*
with *a* **have** *anot*: *a* ∉ *set* (*take i xs* @ *xs* ! *i* # *drop* (*Suc i*) *xs*) − {*xs*!*i*}
by *simp* (*metis in-set-dropD in-set-takeD*)
show *?thesis*
proof (*cases a* = *xs*!*i*)
case *True*
with *d* **show** *?thesis*
by *auto*

next
case *False*
have *set* (*take i xs*) ∩ *set* (*drop* (*Suc i*) *xs*) = {}
by (*metis True d disjoint-insert*(*1*) *distinct-append id-take-nth-drop list.set*(*2*))
then show *?thesis*
using *d False anot* ‹*i* < *length xs*› **by** (*simp add*: *upd-conv-take-nth-drop*)
qed
next
case *False* **with** *d* **show** *?thesis* **by** *auto*
qed

lemma *distinct-concat*:
⟦ *distinct xs*;
⋀ *ys*. *ys* ∈ *set xs* ⟹ *distinct ys*;
⋀ *ys zs*. ⟦ *ys* ∈ *set xs* ; *zs* ∈ *set xs* ; *ys* ≠ *zs* ⟧ ⟹ *set ys* ∩ *set zs* = {}
⟧ ⟹ *distinct* (*concat xs*)
by (*induct xs*) *auto*

An iff-version of ⟦*distinct ?xs*; ⋀*ys*. *ys* ∈ *set ?xs* ⟹ *distinct ys*; ⋀*ys zs*. ⟦*ys* ∈ *set ?xs*; *zs* ∈ *set ?xs*; *ys* ≠ *zs*⟧ ⟹ *set ys* ∩ *set zs* = {}⟧ ⟹ *distinct* (*concat ?xs*) is available further down as *distinct-concat-iff*.

It is best to avoid the following indexed version of distinct, but sometimes it is useful.

lemma *distinct-conv-nth*: *distinct xs* = (∀ *i* < *size xs*. ∀ *j* < *size xs*. *i* ≠ *j* ⟶ *xs*!*i* ≠ *xs*!*j*)
proof (*induct xs*)
case (*Cons x xs*)
show *?case*
apply (*auto simp add*: *Cons nth-Cons less-Suc-eq-le split*: *nat.split-asm*)
apply (*metis Suc-leI in-set-conv-nth length-pos-if-in-set lessI less-imp-le-nat less-nat-zero-code*)
apply (*metis Suc-le-eq*)
done
qed *auto*

lemma *nth-eq-iff-index-eq*:
⟦ *distinct xs*; *i* < *length xs*; *j* < *length xs* ⟧ ⟹ (*xs*!*i* = *xs*!*j*) = (*i* = *j*)
by(*auto simp*: *distinct-conv-nth*)

lemma *distinct-Ex1*:
distinct xs ⟹ *x* ∈ *set xs* ⟹ (∃!*i*. *i* < *length xs* ∧ *xs* ! *i* = *x*)
by (*auto simp*: *in-set-conv-nth nth-eq-iff-index-eq*)

lemma *inj-on-nth*: *distinct xs* ⟹ ∀ *i* ∈ *I*. *i* < *length xs* ⟹ *inj-on* (*nth xs*) *I*
by (*rule inj-onI*) (*simp add*: *nth-eq-iff-index-eq*)

lemma *bij-betw-nth*:
assumes *distinct xs A* = {..<*length xs*} *B* = *set xs*

shows *bij-betw* ((!) *xs*) *A B*
using *assms* **unfolding** *bij-betw-def*
by (*auto intro*!: *inj-on-nth simp*: *set-conv-nth*)

lemma *set-update-distinct*: ⟦ *distinct xs*; *n* < *length xs* ⟧ ⟹
set(*xs*[*n* := *x*]) = *insert x* (*set xs* − {*xs*!*n*})
by(*auto simp*: *set-eq-iff in-set-conv-nth nth-list-update nth-eq-iff-index-eq*)

lemma *distinct-swap*[*simp*]: ⟦ *i* < *size xs*; *j* < *size xs*⟧ ⟹
distinct(*xs*[*i* := *xs*!*j*, *j* := *xs*!*i*]) = *distinct xs*
by (*smt* (*verit*, *del-insts*) *distinct-conv-nth length-list-update nth-list-update*)

lemma *set-swap*[*simp*]:
⟦ *i* < *size xs*; *j* < *size xs* ⟧ ⟹ *set*(*xs*[*i* := *xs*!*j*, *j* := *xs*!*i*]) = *set xs*
by(*simp add*: *set-conv-nth nth-list-update*) *metis*

lemma *distinct-card*: *distinct xs* ⟹ *card* (*set xs*) = *size xs*
by (*induct xs*) *auto*

lemma *card-distinct*: *card* (*set xs*) = *size xs* ⟹ *distinct xs*
proof (*induct xs*)
case (*Cons x xs*)
show *?case*
proof (*cases x* ∈ *set xs*)
case *False* **with** *Cons* **show** *?thesis* **by** *simp*
next
case *True* **with** *Cons.prems*
have *card* (*set xs*) = *Suc* (*length xs*)
by (*simp add*: *card-insert-if split*: *if-split-asm*)
moreover have *card* (*set xs*) ≤ *length xs* **by** (*rule card-length*)
ultimately have *False* **by** *simp*
thus *?thesis* **..**
qed
qed *simp*

lemma *distinct-length-filter*: *distinct xs* ⟹ *length* (*filter P xs*) = *card* ({*x*. *P x*} *Int set xs*)
by (*induct xs*) (*auto*)

lemma *not-distinct-decomp*: ¬ *distinct ws* ⟹ ∃ *xs ys zs y*. *ws* = *xs*@[*y*]@*ys*@[*y*]@*zs*
proof (*induct n* == *length ws arbitrary*:*ws*)
case (*Suc n ws*)
then show *?case*
using *length-Suc-conv* [*of ws n*]
apply (*auto simp*: *eq-commute*)
apply (*metis append-Nil in-set-conv-decomp-first*)
by (*metis append-Cons*)
qed *simp*

lemma *not-distinct-conv-prefix*:
  defines *dec as xs y ys* ≡ *y* ∈ *set xs* ∧ *distinct xs* ∧ *as* = *xs* @ *y* # *ys*
  shows ¬*distinct as* ⟷ (∃ *xs y ys. dec as xs y ys*) (**is** *?L* = *?R*)
proof
  assume *?L* **then show** *?R*
  proof (*induct length as arbitrary*: *as rule*: *less-induct*)
    case *less*
    obtain *xs ys zs y* **where** *decomp*: *as* = (*xs* @ *y* # *ys*) @ *y* # *zs*
      using *not-distinct-decomp*[*OF less.prems*] **by** *auto*
    show *?case*
    proof (*cases distinct* (*xs* @ *y* # *ys*))
      case *True*
      with *decomp* **have** *dec as* (*xs* @ *y* # *ys*) *y zs* **by** (*simp add*: *dec-def*)
      then show *?thesis* **by** *blast*
    next
      case *False*
      with *less decomp* **obtain** *xs′ y′ ys′* **where** *dec* (*xs* @ *y* # *ys*) *xs′ y′ ys′*
        by *atomize-elim auto*
      with *decomp* **have** *dec as xs′ y′* (*ys′* @ *y* # *zs*) **by** (*simp add*: *dec-def*)
      then show *?thesis* **by** *blast*
    qed
  qed
qed (*auto simp*: *dec-def*)

lemma *distinct-product*:
  distinct xs ⟹ *distinct ys* ⟹ *distinct* (*List.product xs ys*)
by (*induct xs*) (*auto intro*: *inj-onI simp add*: *distinct-map*)

lemma *distinct-product-lists*:
  assumes ∀ *xs* ∈ *set xss. distinct xs*
  shows *distinct* (*product-lists xss*)
using *assms* **proof** (*induction xss*)
  case (*Cons xs xss*) **note** ∗ = *this*
  then show *?case*
  proof (*cases product-lists xss*)
    case *Nil* **then show** *?thesis* **by** (*induct xs*) *simp-all*
  next
    case (*Cons ps pss*) **with** ∗ **show** *?thesis*
      by (*auto intro*!: *inj-onI distinct-concat simp add*: *distinct-map*)
  qed
qed *simp*

lemma *length-remdups-concat*:
  length (*remdups* (*concat xss*)) = *card* (⋃*xs*∈*set xss. set xs*)
by (*simp add*: *distinct-card* [*symmetric*])

lemma *remdups-append2*:
  remdups (*xs* @ *remdups ys*) = *remdups* (*xs* @ *ys*)
by(*induction xs*) *auto*

lemma *length-remdups-card-conv*: *length*(*remdups xs*) = *card*(*set xs*)
proof −
  have *xs*: *concat*[*xs*] = *xs* **by** *simp*
  from *length-remdups-concat*[*of* [*xs*]] **show** *?thesis* **unfolding** *xs* **by** *simp*
qed

lemma *remdups-remdups*: *remdups* (*remdups xs*) = *remdups xs*
by (*induct xs*) *simp-all*

lemma *distinct-butlast*:
  assumes *distinct xs*
  shows *distinct* (*butlast xs*)
proof (*cases xs* = [])
  case *False*
    from ‹*xs* ≠ []› **obtain** *ys y* **where** *xs* = *ys* @ [*y*] **by** (*cases xs rule*: *rev-cases*) *auto*
    with ‹*distinct xs*› **show** *?thesis* **by** *simp*
qed (*auto*)

lemma *remdups-map-remdups*:
  remdups (*map f* (*remdups xs*)) = *remdups* (*map f xs*)
by (*induct xs*) *simp-all*

lemma *distinct-zipI1*:
  assumes *distinct xs*
  shows *distinct* (*zip xs ys*)
proof (*rule zip-obtain-same-length*)
  fix *xs'* :: *'a list* **and** *ys'* :: *'b list* **and** *n*
  assume *length xs'* = *length ys'*
  assume *xs'* = *take n xs*
  with *assms* **have** *distinct xs'* **by** *simp*
  with ‹*length xs'* = *length ys'*› **show** *distinct* (*zip xs' ys'*)
    by (*induct xs' ys' rule*: *list-induct2*) (*auto elim*: *in-set-zipE*)
qed

lemma *distinct-zipI2*:
  assumes *distinct ys*
  shows *distinct* (*zip xs ys*)
proof (*rule zip-obtain-same-length*)
  fix *xs'* :: *'b list* **and** *ys'* :: *'a list* **and** *n*
  assume *length xs'* = *length ys'*
  assume *ys'* = *take n ys*
  with *assms* **have** *distinct ys'* **by** *simp*
  with ‹*length xs'* = *length ys'*› **show** *distinct* (*zip xs' ys'*)
    by (*induct xs' ys' rule*: *list-induct2*) (*auto elim*: *in-set-zipE*)
qed

lemma *set-take-disj-set-drop-if-distinct*:

distinct vs ⟹ *i* ≤ *j* ⟹ *set* (*take i vs*) ∩ *set* (*drop j vs*) = {}
by (*auto simp*: *in-set-conv-nth distinct-conv-nth*)

lemma *distinct-singleton*: *distinct* [*x*] **by** *simp*

lemma *distinct-length-2-or-more*:
distinct (*a* # *b* # *xs*) ⟷ (*a* ≠ *b* ∧ *distinct* (*a* # *xs*) ∧ *distinct* (*b* # *xs*))
by *force*

lemma *remdups-adj-altdef*: (*remdups-adj xs* = *ys*) ⟷
(∃*f*::*nat* => *nat*. *mono f* ∧ *f* ' {*0* ..< *size xs*} = {*0* ..< *size ys*}
∧ (∀ *i* < *size xs*. *xs*!*i* = *ys*!(*f i*))
∧ (∀ *i*. *i* + *1* < *size xs* ⟶ (*xs*!*i* = *xs*!(*i*+*1*) ⟷ *f i* = *f*(*i*+*1*)))) (**is** *?L* ⟷ (∃*f*. *?p f xs ys*))
proof
assume *?L*
then show ∃*f*. *?p f xs ys*
proof (*induct xs arbitrary*: *ys rule*: *remdups-adj.induct*)
case (*1 ys*)
thus *?case* **by** (*intro exI*[*of* - *id*]) (*auto simp*: *mono-def*)
next
case (*2 x ys*)
thus *?case* **by** (*intro exI*[*of* - *id*]) (*auto simp*: *mono-def*)
next
case (*3 x1 x2 xs ys*)
let *?xs* = *x1* # *x2* # *xs*
let *?cond* = *x1* = *x2*
define *zs* **where** *zs* = *remdups-adj* (*x2* # *xs*)
from *3*(*1*−*2*)[*of zs*]
obtain *f* **where** *p*: *?p f* (*x2* # *xs*) *zs* **unfolding** *zs-def* **by** (*cases ?cond*) *auto*
then have *f0*: *f 0* = *0*
by (*intro mono-image-least*[**where** *f*=*f*]) *blast*+
from *p* **have** *mono*: *mono f* **and** *f-xs-zs*: *f* ' {*0*..<*length* (*x2* # *xs*)} = {*0*..<*length zs*} **by** *auto*
have *ys*: *ys* = (*if x1* = *x2 then zs else x1* # *zs*)
unfolding *3*(*3*)[*symmetric*] *zs-def* **by** *auto*
have *zs0*: *zs* ! *0* = *x2* **unfolding** *zs-def* **by** (*induct xs*) *auto*
have *zsne*: *zs* ≠ [] **unfolding** *zs-def* **by** (*induct xs*) *auto*
let *?Succ* = *if ?cond then id else Suc*
let *?x1* = *if ?cond then id else Cons x1*
let *?f* = λ *i*. *if i* = *0 then 0 else ?Succ* (*f* (*i* − *1*))
have *ys*: *ys* = *?x1 zs* **unfolding** *ys* **by** (*cases ?cond*, *auto*)
have *mono*: *mono ?f* **using** ‹*mono f*› **unfolding** *mono-def* **by** *auto*
show *?case* **unfolding** *ys*
proof (*intro exI*[*of* - *?f*] *conjI allI impI*)
show *mono ?f* **by** *fact*
next

```
      fix i assume i: i < length ?xs
      with p show ?xs ! i = ?x1 zs ! (?f i) using zs0 by auto
    next
      fix i assume i: i + 1 < length ?xs
      with p show (?xs ! i = ?xs ! (i + 1)) = (?f i = ?f (i + 1))
        by (cases i) (auto simp: f0)
    next
      have id: {0 ..< length (?x1 zs)} = insert 0 (?Succ ` {0 ..< length zs})
        using zsne by (cases ?cond, auto)
      { fix i  assume i < Suc (length xs)
        hence Suc i ∈ {0..<Suc (Suc (length xs))} ∩ Collect ((<) 0) by auto
        from imageI[OF this, of λi. ?Succ (f (i - Suc 0))]
         have ?Succ (f i) ∈ (λi. ?Succ (f (i - Suc 0))) ` ({0..<Suc (Suc (length
xs))} ∩ Collect ((<) 0)) by auto
      }
      then show ?f ` {0 ..< length ?xs} = {0 ..< length (?x1  zs)}
        unfolding id f-xs-zs[symmetric] by auto
    qed
  qed
next
  assume ∃ f. ?p f xs ys
  then show ?L
  proof (induct xs arbitrary: ys rule: remdups-adj.induct)
    case 1 then show ?case by auto
  next
    case (2 x) then obtain f where f-img: f ` {0 ..< size [x]} = {0 ..< size ys}
        and f-nth: ⋀i. i < size [x] ⟹ [x]!i = ys!(f i)
      by blast

    have length ys = card (f ` {0 ..< size [x]})
      using f-img by auto
    then have *: length ys = 1 by auto
    then have f 0 = 0 using f-img by auto
    with * show ?case using f-nth by (cases ys) auto
  next
    case (3 x1 x2 xs)
    from 3.prems obtain f where f-mono: mono f
      and f-img: f ` {0..<length (x1 # x2 # xs)} = {0..<length ys}
      and f-nth:
        ⋀i. i < length (x1 # x2 # xs) ⟹ (x1 # x2 # xs) ! i = ys ! f i
        ⋀i. i + 1 < length (x1 # x2 #xs) ⟹
          ((x1 # x2 # xs) ! i = (x1 # x2 # xs) ! (i + 1)) = (f i = f (i + 1))
      by blast

    show ?case
    proof cases
      assume x1 = x2

      let ?f' = f ∘ Suc
```

```
    have remdups-adj (x1 # xs) = ys
    proof (intro 3.hyps exI conjI impI allI)
      show mono ?f′
        using f-mono by (simp add: mono-iff-le-Suc)
    next
       have ?f′ ‘ {0 ..< length (x1 # xs)} = f ‘ {Suc 0 ..< length (x1 # x2 #
xs)}
        using less-Suc-eq-0-disj by auto
      also have . . . = f ‘ {0 ..< length (x1 # x2 # xs)}
      proof −
        have f 0 = f (Suc 0) using ‹x1 = x2› f-nth[of 0] by simp
        then show ?thesis
          using less-Suc-eq-0-disj by auto
      qed
      also have . . . = {0 ..< length ys} by fact
      finally show  ?f′ ‘ {0 ..< length (x1 # xs)} = {0 ..< length ys} .
    qed (insert f-nth[of Suc i for i], auto simp: ‹x1 = x2›)
    then show ?thesis using ‹x1 = x2› by simp
  next
    assume x1 ≠ x2

    have two: Suc (Suc 0) ≤ length ys
    proof −
      have 2 = card {f 0, f 1} using ‹x1 ≠ x2› f-nth[of 0] by auto
      also have . . . ≤ card (f ‘ {0..< length (x1 # x2 # xs)})
        by (rule card-mono) auto
      finally show ?thesis using f-img by simp
    qed

    have f 0 = 0 using f-mono f-img by (rule mono-image-least) simp

    have f (Suc 0) = Suc 0
    proof (rule ccontr)
      assume f (Suc 0) ≠ Suc 0
       then have Suc 0 < f (Suc 0) using f-nth[of 0] ‹x1 ≠ x2› ‹f 0 = 0› by
auto
      then have ⋀i. Suc 0 < f (Suc i)
        using f-mono
        by (meson Suc-le-mono le0 less-le-trans monoD)
      then have Suc 0 ≠ f i for i using ‹f 0 = 0›
        by (cases i) fastforce+
      then have Suc 0 ∉ f ‘ {0 ..< length (x1 # x2 # xs)} by auto
      then show False using f-img two by auto
    qed
    obtain ys′ where ys = x1 # x2 # ys′
      using two f-nth[of 0] f-nth[of 1]
      by (auto simp: Suc-le-length-iff ‹f 0 = 0› ‹f (Suc 0) = Suc 0›)
```

have *Suc0-le-f-Suc*: *Suc 0* ≤ *f* (*Suc i*) **for** *i*
by (*metis Suc-le-mono* ‹*f* (*Suc 0*) = *Suc 0*› *f-mono le0 mono-def*)

define *f'* **where** *f' x* = *f* (*Suc x*) − *1* **for** *x*
have *f-Suc*: *f* (*Suc i*) = *Suc* (*f' i*) **for** *i*
using *Suc0-le-f-Suc*[*of i*] **by** (*auto simp*: *f'-def*)

have *remdups-adj* (*x2* # *xs*) = (*x2* # *ys'*)
proof (*intro 3.hyps exI conjI impI allI*)
show *mono f'*
using *Suc0-le-f-Suc f-mono* **by** (*auto simp*: *f'-def mono-iff-le-Suc le-diff-iff*)
next
have *f'* ‘ {*0* ..< *length* (*x2* # *xs*)} = (λ*x*. *f x* − *1*) ‘ {*0* ..< *length* (*x1* # *x2* #*xs*)}
by (*auto simp*: *f'-def* ‹*f 0* = *0*› ‹*f* (*Suc 0*) = *Suc 0*› *image-def Bex-def less-Suc-eq-0-disj*)
also have ... = (λ*x*. *x* − *1*) ‘ *f* ‘ {*0* ..< *length* (*x1* # *x2* #*xs*)}
by (*auto simp*: *image-comp*)
also have ... = (λ*x*. *x* − *1*) ‘ {*0* ..< *length ys*}
by (*simp only*: *f-img*)
also have ... = {*0* ..< *length* (*x2* # *ys'*)}
using ‹*ys* = -› **by** (*fastforce intro*: *rev-image-eqI*)
finally show *f'* ‘ {*0* ..< *length* (*x2* # *xs*)} = {*0* ..< *length* (*x2* # *ys'*)} **.**
qed (*insert f-nth*[*of Suc i* **for** *i*] ‹*x1* ≠ *x2*›, *auto simp add*: *f-Suc* ‹*ys* = -›)
then show *?case* **using** ‹*ys* = -› ‹*x1* ≠ *x2*› **by** *simp*
qed
qed
qed

lemma *hd-remdups-adj*[*simp*]: *hd* (*remdups-adj xs*) = *hd xs*
by (*induction xs rule*: *remdups-adj.induct*) *simp-all*

lemma *remdups-adj-Cons*: *remdups-adj* (*x* # *xs*) =
(*case remdups-adj xs of* [] ⇒ [*x*] | *y* # *xs* ⇒ *if x* = *y then y* # *xs else x* # *y* # *xs*)
by (*induct xs arbitrary*: *x*) (*auto split*: *list.splits*)

lemma *remdups-adj-append-two*:
remdups-adj (*xs* @ [*x*,*y*]) = *remdups-adj* (*xs* @ [*x*]) @ (*if x* = *y then* [] *else* [*y*])
by (*induct xs rule*: *remdups-adj.induct*, *simp-all*)

lemma *remdups-adj-adjacent*:
Suc i < *length* (*remdups-adj xs*) ⟹ *remdups-adj xs* ! *i* ≠ *remdups-adj xs* ! *Suc i*
proof (*induction xs arbitrary*: *i rule*: *remdups-adj.induct*)
case (*3 x y xs i*)
thus *?case* **by** (*cases i*, *cases x* = *y*) (*simp*, *auto simp*: *hd-conv-nth*[*symmetric*])
qed *simp-all*

lemma *remdups-adj-rev*[*simp*]: *remdups-adj* (*rev xs*) = *rev* (*remdups-adj xs*)

by (*induct xs rule*: *remdups-adj.induct*, *simp-all add*: *remdups-adj-append-two*)

lemma *remdups-adj-length*[*simp*]: *length* (*remdups-adj xs*) ≤ *length xs*
by (*induct xs rule*: *remdups-adj.induct*, *auto*)

lemma *remdups-adj-length-ge1*[*simp*]: *xs* ≠ [] ⟹ *length* (*remdups-adj xs*) ≥ *Suc 0*
by (*induct xs rule*: *remdups-adj.induct*, *simp-all*)

lemma *remdups-adj-Nil-iff*[*simp*]: *remdups-adj xs* = [] ⟷ *xs* = []
by (*induct xs rule*: *remdups-adj.induct*, *simp-all*)

lemma *remdups-adj-set*[*simp*]: *set* (*remdups-adj xs*) = *set xs*
by (*induct xs rule*: *remdups-adj.induct*, *simp-all*)

lemma *last-remdups-adj* [*simp*]: *last* (*remdups-adj xs*) = *last xs*
  by (*induction xs rule*: *remdups-adj.induct*) *auto*

lemma *remdups-adj-Cons-alt*[*simp*]: *x* # *tl* (*remdups-adj* (*x* # *xs*)) = *remdups-adj* (*x* # *xs*)
by (*induct xs rule*: *remdups-adj.induct*, *auto*)

lemma *remdups-adj-distinct*: *distinct xs* ⟹ *remdups-adj xs* = *xs*
by (*induct xs rule*: *remdups-adj.induct*, *simp-all*)

lemma *remdups-adj-append*:
  remdups-adj (xs_1 @ *x* # xs_2) = *remdups-adj* (xs_1 @ [*x*]) @ *tl* (*remdups-adj* (*x* # xs_2))
by (*induct* xs_1 *rule*: *remdups-adj.induct*, *simp-all*)

lemma *remdups-adj-singleton*:
  remdups-adj xs = [*x*] ⟹ *xs* = *replicate* (*length xs*) *x*
by (*induct xs rule*: *remdups-adj.induct*, *auto split*: *if-split-asm*)

lemma *remdups-adj-map-injective*:
  assumes *inj f*
  shows *remdups-adj* (*map f xs*) = *map f* (*remdups-adj xs*)
by (*induct xs rule*: *remdups-adj.induct*) (*auto simp add*: *injD*[*OF assms*])

lemma *remdups-adj-replicate*:
  remdups-adj (*replicate n x*) = (*if n* = *0 then* [] *else* [*x*])
  by (*induction n*) (*auto simp*: *remdups-adj-Cons*)

lemma *remdups-upt* [*simp*]: *remdups* [*m*..<*n*] = [*m*..<*n*]
proof (*cases m* ≤ *n*)
  case *False* **then show** *?thesis* **by** *simp*
next
  case *True* **then obtain** *q* **where** *n* = *m* + *q*
    by (*auto simp add*: *le-iff-add*)

moreover have $\mathit{remdups}\ [m..{<}m + q] = [m..{<}m + q]$
by (*induct q*) *simp-all*
ultimately show *?thesis* **by** *simp*
qed

lemma *successively-remdups-adjI*:
$\mathit{successively}\ P\ \mathit{xs} \Longrightarrow \mathit{successively}\ P\ (\mathit{remdups\text{-}adj}\ \mathit{xs})$
by (*induction xs rule*: *remdups-adj.induct*) (*auto simp*: *successively-Cons*)

lemma *successively-remdups-adj-iff*:
$(\bigwedge x.\ x \in \mathit{set}\ \mathit{xs} \Longrightarrow P\ x\ x) \Longrightarrow$
$\mathit{successively}\ P\ (\mathit{remdups\text{-}adj}\ \mathit{xs}) \longleftrightarrow \mathit{successively}\ P\ \mathit{xs}$
by (*induction xs rule*: *remdups-adj.induct*)(*auto simp*: *successively-Cons*)

lemma *successively-conv-nth*:
$\mathit{successively}\ P\ \mathit{xs} \longleftrightarrow (\forall i.\ \mathit{Suc}\ i < \mathit{length}\ \mathit{xs} \longrightarrow P\ (\mathit{xs}\ !\ i)\ (\mathit{xs}\ !\ \mathit{Suc}\ i))$
by (*induction P xs rule*: *successively.induct*)
(*force simp*: *nth-Cons split*: *nat.splits*)+

lemma *successively-nth*: $\mathit{successively}\ P\ \mathit{xs} \Longrightarrow \mathit{Suc}\ i < \mathit{length}\ \mathit{xs} \Longrightarrow P\ (\mathit{xs}\ !\ i)\ (\mathit{xs}\ !\ \mathit{Suc}\ i)$
unfolding *successively-conv-nth* **by** *blast*

lemma *distinct-adj-conv-nth*:
$\mathit{distinct\text{-}adj}\ \mathit{xs} \longleftrightarrow (\forall i.\ \mathit{Suc}\ i < \mathit{length}\ \mathit{xs} \longrightarrow \mathit{xs}\ !\ i \neq \mathit{xs}\ !\ \mathit{Suc}\ i)$
by (*simp add*: *distinct-adj-def successively-conv-nth*)

lemma *distinct-adj-nth*: $\mathit{distinct\text{-}adj}\ \mathit{xs} \Longrightarrow \mathit{Suc}\ i < \mathit{length}\ \mathit{xs} \Longrightarrow \mathit{xs}\ !\ i \neq \mathit{xs}\ !\ \mathit{Suc}\ i$
unfolding *distinct-adj-conv-nth* **by** *blast*

lemma *remdups-adj-Cons'*:
$\mathit{remdups\text{-}adj}\ (x\ \#\ \mathit{xs}) = x\ \#\ \mathit{remdups\text{-}adj}\ (\mathit{dropWhile}\ (\lambda y.\ y = x)\ \mathit{xs})$
by (*induction xs*) *auto*

lemma *remdups-adj-singleton-iff*:
$\mathit{length}\ (\mathit{remdups\text{-}adj}\ \mathit{xs}) = \mathit{Suc}\ 0 \longleftrightarrow \mathit{xs} \neq [] \wedge \mathit{xs} = \mathit{replicate}\ (\mathit{length}\ \mathit{xs})\ (\mathit{hd}\ \mathit{xs})$
proof *safe*
assume $*$: $\mathit{xs} = \mathit{replicate}\ (\mathit{length}\ \mathit{xs})\ (\mathit{hd}\ \mathit{xs})$ **and** [*simp*]: $\mathit{xs} \neq []$
show $\mathit{length}\ (\mathit{remdups\text{-}adj}\ \mathit{xs}) = \mathit{Suc}\ 0$
by (*subst* $*$) (*auto simp*: *remdups-adj-replicate*)
next
assume $\mathit{length}\ (\mathit{remdups\text{-}adj}\ \mathit{xs}) = \mathit{Suc}\ 0$
thus $\mathit{xs} = \mathit{replicate}\ (\mathit{length}\ \mathit{xs})\ (\mathit{hd}\ \mathit{xs})$
by (*induction xs rule*: *remdups-adj.induct*) (*auto split*: *if-splits*)
qed *auto*

lemma *tl-remdups-adj*:
$\mathit{ys} \neq [] \Longrightarrow \mathit{tl}\ (\mathit{remdups\text{-}adj}\ \mathit{ys}) = \mathit{remdups\text{-}adj}\ (\mathit{dropWhile}\ (\lambda x.\ x = \mathit{hd}\ \mathit{ys})\ (\mathit{tl}$

ys))
by (*cases ys*) (*simp-all add*: *remdups-adj-Cons′*)

lemma *remdups-adj-append-dropWhile*:
remdups-adj (*xs* @ *y* # *ys*) = *remdups-adj* (*xs* @ [*y*]) @ *remdups-adj* (*dropWhile* (λ*x*. *x* = *y*) *ys*)
by (*subst remdups-adj-append*) (*simp add*: *tl-remdups-adj*)

lemma *remdups-adj-append′*:
assumes *xs* = [] ∨ *ys* = [] ∨ *last xs* ≠ *hd ys*
shows *remdups-adj* (*xs* @ *ys*) = *remdups-adj xs* @ *remdups-adj ys*
proof −
have *?thesis* **if** [*simp*]: *xs* ≠ [] *ys* ≠ [] **and** *last xs* ≠ *hd ys*
proof −
obtain *x xs′* **where** *xs*: *xs* = *xs′* @ [*x*]
by (*cases xs rule*: *rev-cases*) *auto*
have *remdups-adj* (*xs′* @ *x* # *ys*) = *remdups-adj* (*xs′* @ [*x*]) @ *remdups-adj ys*
using ‹*last xs* ≠ *hd ys*› **unfolding** *xs*
by (*metis* (*full-types*) *dropWhile-eq-self-iff last-snoc remdups-adj-append-dropWhile*)
thus *?thesis* **by** (*simp add*: *xs*)
qed
thus *?thesis* **using** *assms*
by (*cases xs* = []; *cases ys* = []) *auto*
qed

lemma *remdups-adj-append″*: *xs* ≠ []
⟹ *remdups-adj* (*xs* @ *ys*) = *remdups-adj xs* @ *remdups-adj* (*dropWhile* (λ*y*. *y* = *last xs*) *ys*)
by (*induction xs rule*: *remdups-adj.induct*) (*auto simp*: *remdups-adj-Cons′*)

lemma *remdups-filter-last*:
last [*x*←*remdups xs*. *P x*] = *last* [*x*←*xs*. *P x*]
by (*induction xs*, *auto simp*: *filter-empty-conv*)

lemma *remdups-append*:
set xs ⊆ *set ys* ⟹ *remdups* (*xs* @ *ys*) = *remdups ys*
by (*induction xs*, *simp-all*)

lemma *remdups-concat*:
remdups (*concat* (*remdups xs*)) = *remdups* (*concat xs*)
proof (*induction xs*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a xs*)
show *?case*
proof (*cases a* ∈ *set xs*)
case *True*
then have *remdups* (*concat xs*) = *remdups* (*a* @ *concat xs*)

by (*metis remdups-append concat.simps*(*2*) *insert-absorb set-simps*(*2*) *set-append set-concat sup-ge1*)
then show *?thesis*
by (*simp add*: *Cons True*)
next
case *False*
then show *?thesis*
by (*metis Cons remdups-append2 concat.simps*(*2*) *remdups.simps*(*2*))
qed
qed

66.2 *distinct-adj*

lemma *distinct-adj-Nil* [*simp*]: *distinct-adj* []
and *distinct-adj-singleton* [*simp*]: *distinct-adj* [*x*]
and *distinct-adj-Cons-Cons* [*simp*]: *distinct-adj* (*x* # *y* # *xs*) ⟷ *x* ≠ *y* ∧ *distinct-adj* (*y* # *xs*)
by (*auto simp*: *distinct-adj-def*)

lemma *distinct-adj-Cons*: *distinct-adj* (*x* # *xs*) ⟷ *xs* = [] ∨ *x* ≠ *hd xs* ∧ *distinct-adj xs*
by (*cases xs*) *auto*

lemma *distinct-adj-ConsD*: *distinct-adj* (*x* # *xs*) ⟹ *distinct-adj xs*
by (*cases xs*) *auto*

lemma *distinct-adj-remdups-adj*[*simp*]: *distinct-adj* (*remdups-adj xs*)
by (*induction xs rule*: *remdups-adj.induct*) (*auto simp*: *distinct-adj-Cons*)

lemma *distinct-adj-altdef*: *distinct-adj xs* ⟷ *remdups-adj xs* = *xs*
proof
assume *remdups-adj xs* = *xs*
with *distinct-adj-remdups-adj*[*of xs*] **show** *distinct-adj xs*
by *simp*
next
assume *distinct-adj xs*
thus *remdups-adj xs* = *xs*
by (*induction xs rule*: *induct-list012*) *auto*
qed

lemma *distinct-adj-rev* [*simp*]: *distinct-adj* (*rev xs*) ⟷ *distinct-adj xs*
by (*simp add*: *distinct-adj-def eq-commute*)

lemma *distinct-adj-append-iff*:
distinct-adj (*xs* @ *ys*) ⟷
distinct-adj xs ∧ *distinct-adj ys* ∧ (*xs* = [] ∨ *ys* = [] ∨ *last xs* ≠ *hd ys*)
by (*auto simp*: *distinct-adj-def successively-append-iff*)

lemma *distinct-adj-appendD1* [*dest*]: *distinct-adj* (*xs* @ *ys*) ⟹ *distinct-adj xs*

and *distinct-adj-appendD2* [*dest*]: *distinct-adj* (*xs* @ *ys*) $\Longrightarrow$ *distinct-adj ys*
by (*auto simp*: *distinct-adj-append-iff*)

lemma *distinct-adj-mapI*: *distinct-adj xs* $\Longrightarrow$ *inj-on f* (*set xs*) $\Longrightarrow$ *distinct-adj* (*map f xs*)
unfolding *distinct-adj-def successively-map*
by (*erule successively-mono*) (*auto simp*: *inj-on-def*)

lemma *distinct-adj-mapD*: *distinct-adj* (*map f xs*) $\Longrightarrow$ *distinct-adj xs*
unfolding *distinct-adj-def successively-map* **by** (*erule successively-mono*) *auto*

lemma *distinct-adj-map-iff*: *inj-on f* (*set xs*) $\Longrightarrow$ *distinct-adj* (*map f xs*) $\longleftrightarrow$ *distinct-adj xs*
using *distinct-adj-mapD distinct-adj-mapI* **by** *blast*

lemma *distinct-adj-conv-length-remdups-adj*:
distinct-adj xs $\longleftrightarrow$ *length* (*remdups-adj xs*) = *length xs*
proof (*induction xs rule*: *remdups-adj.induct*)
case (*3 x y xs*)
thus *?case*
using *remdups-adj-length*[*of y # xs*] **by** *auto*
qed *auto*

66.2.1 *insert*

lemma *in-set-insert* [*simp*]:
$x \in$ *set xs* $\Longrightarrow$ *List.insert x xs* = *xs*
by (*simp add*: *List.insert-def*)

lemma *not-in-set-insert* [*simp*]:
$x \notin$ *set xs* $\Longrightarrow$ *List.insert x xs* = *x # xs*
by (*simp add*: *List.insert-def*)

lemma *insert-Nil* [*simp*]: *List.insert x* [] = [*x*]
by *simp*

lemma *set-insert* [*simp*]: *set* (*List.insert x xs*) = *insert x* (*set xs*)
by (*auto simp add*: *List.insert-def*)

lemma *distinct-insert* [*simp*]: *distinct* (*List.insert x xs*) = *distinct xs*
by (*simp add*: *List.insert-def*)

lemma *insert-remdups*:
List.insert x (*remdups xs*) = *remdups* (*List.insert x xs*)
by (*simp add*: *List.insert-def*)

66.2.2 *List.union*

This is all one should need to know about union:

lemma *set-union*[*simp*]: *set* (*List.union xs ys*) = *set xs* ∪ *set ys*
unfolding *List.union-def*
by(*induct xs arbitrary*: *ys*) *simp-all*

lemma *distinct-union*[*simp*]: *distinct*(*List.union xs ys*) = *distinct ys*
unfolding *List.union-def*
by(*induct xs arbitrary*: *ys*) *simp-all*

66.2.3 *find*

lemma *find-None-iff*: *List.find P xs* = *None* ⟷ ¬ (∃ *x*. *x* ∈ *set xs* ∧ *P x*)
proof (*induction xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons x xs*) **thus** *?case* **by** (*fastforce split*: *if-splits*)
qed

lemmas *find-None-iff2* = *find-None-iff*[*THEN eq-iff-swap*]

lemma *find-Some-iff*:
  List.find P xs = *Some x* ⟷
  (∃ *i*<*length xs*. *P* (*xs*!*i*) ∧ *x* = *xs*!*i* ∧ (∀ *j*<*i*. ¬ *P* (*xs*!*j*)))
proof (*induction xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons x xs*) **thus** *?case*
    apply(*auto simp*: *nth-Cons′ split*: *if-splits*)
    using *diff-Suc-1 less-Suc-eq-0-disj* **by** *fastforce*
qed

lemmas *find-Some-iff2* = *find-Some-iff*[*THEN eq-iff-swap*]

lemma *find-cong*[*fundef-cong*]:
  assumes *xs* = *ys* **and** ⋀*x*. *x* ∈ *set ys* ⟹ *P x* = *Q x*
  shows *List.find P xs* = *List.find Q ys*
proof (*cases List.find P xs*)
  case *None* **thus** *?thesis* **by** (*metis find-None-iff assms*)
next
  case (*Some x*)
  hence *List.find Q ys* = *Some x* **using** *assms*
    by (*auto simp add*: *find-Some-iff*)
  thus *?thesis* **using** *Some* **by** *auto*
qed

lemma *find-dropWhile*:
  List.find P xs = (*case dropWhile* (*Not* ∘ *P*) *xs*
   of [] ⇒ *None*
   | *x* # - ⇒ *Some x*)
by (*induct xs*) *simp-all*

66.2.4 *count-list*

This library is intentionally minimal. See the remark about multisets at the point above where *count-list* is defined.

lemma *count-list-append*[*simp*]: *count-list* (*xs* @ *ys*) *x* = *count-list xs x* + *count-list ys x*
by (*induction xs*) *simp-all*

lemma *count-list-0-iff*: *count-list xs x* = *0* ⟷ *x* ∉ *set xs*
by (*induction xs*) *simp-all*

lemma *count-notin*[*simp*]: *x* ∉ *set xs* ⟹ *count-list xs x* = *0*
by(*simp add*: *count-list-0-iff*)

lemma *count-le-length*: *count-list xs x* ≤ *length xs*
by (*induction xs*) *auto*

lemma *count-list-map-ge*: *count-list xs x* ≤ *count-list* (*map f xs*) (*f x*)
by (*induction xs*) *auto*

lemma *count-list-inj-map*:
⟦ *inj-on f* (*set xs*); *x* ∈ *set xs* ⟧ ⟹ *count-list* (*map f xs*) (*f x*) = *count-list xs x*
by (*induction xs*) (*simp*, *fastforce*)

lemma *count-list-map-conv*:
assumes *inj f* **shows** *count-list* (*map f xs*) (*f x*) = *count-list xs x*
by (*induction xs*) (*simp-all add*: *inj-eq*[*OF assms*])

lemma *count-list-rev*[*simp*]: *count-list* (*rev xs*) *x* = *count-list xs x*
by (*induction xs*) *auto*

lemma *sum-count-set*:
set xs ⊆ *X* ⟹ *finite X* ⟹ *sum* (*count-list xs*) *X* = *length xs*
proof (*induction xs arbitrary*: *X*)
case (*Cons x xs*)
then show *?case*
using *sum.remove* [*of X x count-list xs*]
by (*auto simp*: *sum.If-cases simp flip*: *diff-eq*)
qed *simp*

lemma *count-list-Suc-split-first*:
assumes *count-list xs x* = *Suc n*
shows ∃ *pref rest*. *xs* = *pref* @ *x* # *rest* ∧ *x* ∉ *set pref* ∧ *count-list rest x* = *n*
proof −
let *?pref* = *takeWhile* (λ*u*. *u* ≠ *x*) *xs*
let *?rest* = *drop* (*length ?pref*) *xs*
have *x* ∈ *set xs* **using** *assms count-notin* **by** *fastforce*
hence *rest*: *?rest* ≠ [] ∧ *hd ?rest* = *x*
by (*metis* (*mono-tags*, *lifting*) *append-Nil2 dropWhile-eq-drop hd-dropWhile*

takeWhile-dropWhile-id takeWhile-eq-all-conv)
have *1*: $x \notin$ *set ?pref* **by** (*metis* (*full-types*) *set-takeWhileD*)
have *2*: *xs* = *?pref* @ *x* # *tl ?rest*
by (*metis rest append-eq-conv-conj hd-Cons-tl takeWhile-eq-take*)
have *count-list* (*tl ?rest*) *x* = *n*
using *assms rest 1 2 count-notin count-list-append*[*of ?pref x # tl ?rest x*] **by** *simp*
with *1 2* **show** *?thesis* **by** *blast*
qed

lemma *split-list-cycles*:
$\exists$ *pref xss. xs* = *pref* @ *concat xss* $\wedge$ $x \notin$ *set pref* $\wedge$ ($\forall$ *ys* $\in$ *set xss.* $\exists$ *zs. ys* = *x* # *zs*)
proof (*induction count-list xs x arbitrary*: *xs*)
case *0*
show *?case* **using** *0*[*symmetric*] *concat.simps*(*1*) *count-list-0-iff* **by** *fastforce*
next
case (*Suc n*)
from *Suc.hyps*(*2*) **obtain** *pref rest* **where**
∗: *xs* = *pref* @ *x* # *rest* $x \notin$ *set pref count-list rest x* = *n*
by (*metis count-list-Suc-split-first*)
from *Suc.hyps*(*1*)[*OF* ∗(*3*)[*symmetric*]] **obtain** *pref1 xss* **where**
∗∗: *rest* = *pref1* @ *concat xss* $x \notin$ *set pref1* $\forall$ *ys*$\in$*set xss.* $\exists$ *zs. ys* = *x* # *zs*
by *blast*
let *?xss* = (*x* # *pref1*) # *xss*
have *xs* = *pref* @ *concat ?xss* $\wedge$ $x \notin$ *set pref* $\wedge$ ($\forall$ *ys* $\in$ *set ?xss.* $\exists$ *zs. ys* = *x* # *zs*)
using ∗(*1*,*2*) ∗∗ **by** *auto*
thus *?case* **by** *blast*
qed

66.2.5 *List.extract*

lemma *extract-None-iff*: *List.extract P xs* = *None* $\longleftrightarrow$ $\neg$ ($\exists$ *x*$\in$*set xs. P x*)
by(*auto simp*: *extract-def dropWhile-eq-Cons-conv split*: *list.splits*)
(*metis in-set-conv-decomp*)

lemma *extract-SomeE*:
List.extract P xs = *Some* (*ys*, *y*, *zs*) $\Longrightarrow$
xs = *ys* @ *y* # *zs* $\wedge$ *P y* $\wedge$ $\neg$ ($\exists$ *y* $\in$ *set ys. P y*)
by(*auto simp*: *extract-def dropWhile-eq-Cons-conv split*: *list.splits*)

lemma *extract-Some-iff*:
List.extract P xs = *Some* (*ys*, *y*, *zs*) $\longleftrightarrow$
xs = *ys* @ *y* # *zs* $\wedge$ *P y* $\wedge$ $\neg$ ($\exists$ *y* $\in$ *set ys. P y*)
by(*auto simp*: *extract-def dropWhile-eq-Cons-conv dest*: *set-takeWhileD split*: *list.splits*)

lemma *extract-Nil-code*[*code*]: *List.extract P* [] = *None*
by(*simp add*: *extract-def*)

lemma *extract-Cons-code*[*code*]:
List.extract P (*x* # *xs*) = (*if P x then Some* ([], *x*, *xs*) *else*
(*case List.extract P xs of*
None ⇒ *None* |
Some (*ys*, *y*, *zs*) ⇒ *Some* (*x*#*ys*, *y*, *zs*)))
by(*auto simp add*: *extract-def comp-def split*: *list.splits*)
(*metis dropWhile-eq-Nil-conv list.distinct*(*1*))

66.2.6 *remove1*

lemma *remove1-append*:
remove1 x (*xs* @ *ys*) =
(*if x* ∈ *set xs then remove1 x xs* @ *ys else xs* @ *remove1 x ys*)
by (*induct xs*) *auto*

lemma *remove1-commute*: *remove1 x* (*remove1 y zs*) = *remove1 y* (*remove1 x zs*)
by (*induct zs*) *auto*

lemma *in-set-remove1*[*simp*]:
$a \neq b \Longrightarrow a \in set(remove1\ b\ xs) = (a \in set\ xs)$
by (*induct xs*) *auto*

lemma *set-remove1-subset*: $set(remove1\ x\ xs) \subseteq set\ xs$
by (*induct xs*) *auto*

lemma *set-remove1-eq* [*simp*]: $distinct\ xs \Longrightarrow set(remove1\ x\ xs) = set\ xs - \{x\}$
by (*induct xs*) *auto*

lemma *length-remove1*:
length(*remove1 x xs*) = (*if x* ∈ *set xs then length xs* − *1 else length xs*)
by (*induct xs*) (*auto dest*!:*length-pos-if-in-set*)

lemma *remove1-filter-not*[*simp*]:
¬ *P x* ⟹ *remove1 x* (*filter P xs*) = *filter P xs*
by(*induct xs*) *auto*

lemma *filter-remove1*:
filter Q (*remove1 x xs*) = *remove1 x* (*filter Q xs*)
by (*induct xs*) *auto*

lemma *notin-set-remove1*[*simp*]: $x \notin set\ xs \Longrightarrow x \notin set(remove1\ y\ xs)$
by(*insert set-remove1-subset*) *fast*

lemma *distinct-remove1*[*simp*]: *distinct xs* ⟹ *distinct*(*remove1 x xs*)
by (*induct xs*) *simp-all*

lemma *remove1-remdups*:
distinct xs ⟹ *remove1 x* (*remdups xs*) = *remdups* (*remove1 x xs*)

by (*induct xs*) *simp-all*

lemma *remove1-idem*: $x \notin set\ xs \Longrightarrow remove1\ x\ xs = xs$
by (*induct xs*) *simp-all*

lemma *remove1-split*:
$a \in set\ xs \Longrightarrow remove1\ a\ xs = ys \longleftrightarrow (\exists\ ls\ rs.\ xs = ls\ @\ a\ \#\ rs \wedge a \notin set\ ls \wedge ys = ls\ @\ rs)$
by (*metis remove1.simps*(*2*) *remove1-append split-list-first*)

66.2.7 *removeAll*

lemma *removeAll-filter-not-eq*:
$removeAll\ x = filter\ (\lambda y.\ x \neq y)$
proof
fix *xs*
show $removeAll\ x\ xs = filter\ (\lambda y.\ x \neq y)\ xs$
by (*induct xs*) *auto*
qed

lemma *removeAll-append*[*simp*]:
$removeAll\ x\ (xs\ @\ ys) = removeAll\ x\ xs\ @\ removeAll\ x\ ys$
by (*induct xs*) *auto*

lemma *set-removeAll*[*simp*]: $set(removeAll\ x\ xs) = set\ xs - \{x\}$
by (*induct xs*) *auto*

lemma *removeAll-id*[*simp*]: $x \notin set\ xs \Longrightarrow removeAll\ x\ xs = xs$
by (*induct xs*) *auto*

lemma *removeAll-filter-not*[*simp*]:
$\neg\ P\ x \Longrightarrow removeAll\ x\ (filter\ P\ xs) = filter\ P\ xs$
by(*induct xs*) *auto*

lemma *distinct-removeAll*:
$distinct\ xs \Longrightarrow distinct\ (removeAll\ x\ xs)$
by (*simp add*: *removeAll-filter-not-eq*)

lemma *distinct-remove1-removeAll*:
$distinct\ xs \Longrightarrow remove1\ x\ xs = removeAll\ x\ xs$
by (*induct xs*) *simp-all*

lemma *map-removeAll-inj-on*: $inj\text{-}on\ f\ (insert\ x\ (set\ xs)) \Longrightarrow map\ f\ (removeAll\ x\ xs) = removeAll\ (f\ x)\ (map\ f\ xs)$
by (*induct xs*) (*simp-all add*:*inj-on-def*)

lemma *map-removeAll-inj*: $inj\ f \Longrightarrow$

map f (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
by (*rule map-removeAll-inj-on*, *erule subset-inj-on*, *rule subset-UNIV*)

lemma *length-removeAll-less-eq* [*simp*]:
length (*removeAll x xs*) ≤ *length xs*
by (*simp add*: *removeAll-filter-not-eq*)

lemma *length-removeAll-less* [*termination-simp*]:
x ∈ *set xs* ⟹ *length* (*removeAll x xs*) < *length xs*
by (*auto dest*: *length-filter-less simp add*: *removeAll-filter-not-eq*)

lemma *distinct-concat-iff*: *distinct* (*concat xs*) ⟷
distinct (*removeAll* [] *xs*) ∧
(∀ *ys*. *ys* ∈ *set xs* ⟶ *distinct ys*) ∧
(∀ *ys zs*. *ys* ∈ *set xs* ∧ *zs* ∈ *set xs* ∧ *ys* ≠ *zs* ⟶ *set ys* ∩ *set zs* = {})
proof (*induct xs*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons a xs*)
have ⟦*set a* ∩ ⋃ (*set* ' *set xs*) = {}; *a* ∈ *set xs*⟧ ⟹ *a*=[]
by (*metis Int-iff UN-I empty-iff equals0I set-empty*)
then show *?case*
by (*auto simp*: *Cons*)
qed

66.2.8 *replicate*

lemma *length-replicate* [*simp*]: *length* (*replicate n x*) = *n*
by (*induct n*) *auto*

lemma *replicate-eqI*:
assumes *length xs* = *n* **and** ⋀*y*. *y* ∈ *set xs* ⟹ *y* = *x*
shows *xs* = *replicate n x*
using *assms*
proof (*induct xs arbitrary*: *n*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **then show** *?case* **by** (*cases n*) *simp-all*
qed

lemma *Ex-list-of-length*: ∃ *xs*. *length xs* = *n*
by (*rule exI*[*of* - *replicate n undefined*]) *simp*

lemma *map-replicate* [*simp*]: *map f* (*replicate n x*) = *replicate n* (*f x*)
by (*induct n*) *auto*

lemma *map-replicate-const*:
map (λ *x*. *k*) *lst* = *replicate* (*length lst*) *k*

by (*induct lst*) *auto*

lemma *replicate-app-Cons-same*:
(*replicate n x*) @ (*x* # *xs*) = *x* # *replicate n x* @ *xs*
by (*induct n*) *auto*

lemma *rev-replicate* [*simp*]: *rev* (*replicate n x*) = *replicate n x*
by (*metis length-rev map-replicate map-replicate-const rev-map*)

lemma *replicate-add*: *replicate* (*n* + *m*) *x* = *replicate n x* @ *replicate m x*
by (*induct n*) *auto*

Courtesy of Matthias Daum:

lemma *append-replicate-commute*:
replicate n x @ *replicate k x* = *replicate k x* @ *replicate n x*
by (*metis add.commute replicate-add*)

Courtesy of Andreas Lochbihler:

lemma *filter-replicate*:
filter P (*replicate n x*) = (*if P x then replicate n x else* [])
by(*induct n*) *auto*

lemma *hd-replicate* [*simp*]: *n* ≠ *0* ⟹ *hd* (*replicate n x*) = *x*
by (*induct n*) *auto*

lemma *tl-replicate* [*simp*]: *tl* (*replicate n x*) = *replicate* (*n* − *1*) *x*
by (*induct n*) *auto*

lemma *last-replicate* [*simp*]: *n* ≠ *0* ⟹ *last* (*replicate n x*) = *x*
by (*atomize* (*full*), *induct n*) *auto*

lemma *nth-replicate*[*simp*]: *i* < *n* ⟹ (*replicate n x*)!*i* = *x*
by (*induct n arbitrary*: *i*)(*auto simp*: *nth-Cons split*: *nat.split*)

Courtesy of Matthias Daum (2 lemmas):

lemma *take-replicate*[*simp*]: *take i* (*replicate k x*) = *replicate* (*min i k*) *x*
proof (*cases k* ≤ *i*)
case *True*
then show *?thesis*
by (*simp add*: *min-def*)
next
case *False*
then have *replicate k x* = *replicate i x* @ *replicate* (*k* − *i*) *x*
by (*simp add*: *replicate-add* [*symmetric*])
then show *?thesis*
by (*simp add*: *min-def*)
qed

lemma *drop-replicate*[*simp*]: *drop i* (*replicate k x*) = *replicate* (*k*−*i*) *x*

proof (*induct k arbitrary*: *i*)
  case (*Suc k*)
  then show *?case*
    by (*simp add*: *drop-Cons'*)
qed *simp*

lemma *set-replicate-Suc*: *set* (*replicate* (*Suc n*) *x*) = {*x*}
by (*induct n*) *auto*

lemma *set-replicate* [*simp*]: $n \neq 0 \Longrightarrow$ *set* (*replicate n x*) = {*x*}
by (*fast dest*!: *not0-implies-Suc intro*!: *set-replicate-Suc*)

lemma *set-replicate-conv-if*: *set* (*replicate n x*) = (*if n* = *0 then* {} *else* {*x*})
by *auto*

lemma *in-set-replicate*[*simp*]: ($x \in$ *set* (*replicate n y*)) = ($x = y \wedge n \neq 0$)
by (*simp add*: *set-replicate-conv-if*)

lemma *card-set-1-iff-replicate*:
  card(*set xs*) = *Suc 0* $\longleftrightarrow$ *xs* $\neq$ [] $\wedge$ ($\exists$ *x*. *xs* = *replicate* (*length xs*) *x*)
by (*metis card-1-singleton-iff empty-iff insert-iff replicate-eqI set-empty2 set-replicate*)

lemma *Ball-set-replicate*[*simp*]:
  ($\forall x \in$ *set*(*replicate n a*). *P x*) = (*P a* $\vee$ *n=0*)
by(*simp add*: *set-replicate-conv-if*)

lemma *Bex-set-replicate*[*simp*]:
  ($\exists x \in$ *set*(*replicate n a*). *P x*) = (*P a* $\wedge$ *n*$\neq$*0*)
by(*simp add*: *set-replicate-conv-if*)

lemma *replicate-append-same*:
  replicate i x @ [*x*] = *x* # *replicate i x*
  by (*induct i*) *simp-all*

lemma *map-replicate-trivial*:
  map (λ*i*. *x*) [*0*..<*i*] = *replicate i x*
by (*induct i*) (*simp-all add*: *replicate-append-same*)

lemma *concat-replicate-trivial*[*simp*]:
  concat (*replicate i* []) = []
  by (*induct i*) (*auto simp add*: *map-replicate-const*)

lemma *concat-replicate-single*[*simp*]: *concat* (*replicate m* [*a*]) = *replicate m a*
by(*induction m*) *auto*

lemma *replicate-empty*[*simp*]: (*replicate n x* = []) $\longleftrightarrow$ *n=0*
by (*induct n*) *auto*

lemmas *empty-replicate*[*simp*] = *replicate-empty*[*THEN eq-iff-swap*]

lemma *replicate-eq-replicate*[*simp*]:
  (*replicate m x* = *replicate n y*) ⟷ (*m*=*n* ∧ (*m*≠*0* ⟶ *x*=*y*))
proof (*induct m arbitrary*: *n*)
  case (*Suc m n*)
  then show *?case*
    by (*induct n*) *auto*
qed *simp*

lemma *takeWhile-replicate*[*simp*]:
  takeWhile P (*replicate n x*) = (*if P x then replicate n x else* [])
using *takeWhile-eq-Nil-iff* **by** *fastforce*

lemma *dropWhile-replicate*[*simp*]:
  dropWhile P (*replicate n x*) = (*if P x then* [] *else replicate n x*)
using *dropWhile-eq-self-iff* **by** *fastforce*

lemma *replicate-length-filter*:
  replicate (*length* (*filter* (λ*y*. *x* = *y*) *xs*)) *x* = *filter* (λ*y*. *x* = *y*) *xs*
  by (*induct xs*) *auto*

lemma *comm-append-are-replicate*:
  xs @ *ys* = *ys* @ *xs* ⟹ ∃ *m n zs*. *concat* (*replicate m zs*) = *xs* ∧ *concat* (*replicate n zs*) = *ys*
proof (*induction length* (*xs* @ *ys*) + *length xs arbitrary*: *xs ys rule*: *less-induct*)
  case *less*
  consider (*1*) *length ys* < *length xs* | (*2*) *xs* = [] | (*3*) *length xs* ≤ *length ys* ∧ *xs* ≠ []
    by *linarith*
  then show *?case*
  proof (*cases*)
    case *1*
    then show *?thesis*
      using *less.hyps*[*OF* - *less.prems*[*symmetric*]] *nat-add-left-cancel-less* **by** *auto*
  next
    case *2*
    then have *concat* (*replicate 0 ys*) = *xs* ∧ *concat* (*replicate 1 ys*) = *ys*
      by *simp*
    then show *?thesis*
      by *blast*
  next
    case *3*
    then have *length xs* ≤ *length ys* **and** *xs* ≠ []
      by *blast*+
    from ‹*length xs* ≤ *length ys*› **and** ‹*xs* @ *ys* = *ys* @ *xs*›
    obtain *ws* **where** *ys* = *xs* @ *ws*
      by (*auto simp*: *append-eq-append-conv2*)
    from *this* **and** ‹*xs* ≠ []›

have *length ws* < *length ys*
by *simp*
from ‹*xs* @ *ys* = *ys* @ *xs*›[*unfolded* ‹*ys* = *xs* @ *ws*›]
have *xs* @ *ws* = *ws* @ *xs*
by *simp*
from *less.hyps*[*OF* - *this*] ‹*length ws* < *length ys*›
obtain *m n′ zs* **where** *concat* (*replicate m zs*) = *xs* **and** *concat* (*replicate n′ zs*) = *ws*
by *auto*
then have *concat* (*replicate* (*m*+*n′*) *zs*) = *ys*
using ‹*ys* = *xs* @ *ws*›
by (*simp add*: *replicate-add*)
then show *?thesis*
using ‹*concat* (*replicate m zs*) = *xs*› **by** *blast*
qed
qed

lemma *comm-append-is-replicate*:
fixes *xs ys* :: *′a list*
assumes *xs* ≠ [] *ys* ≠ []
assumes *xs* @ *ys* = *ys* @ *xs*
shows ∃ *n zs*. *n* > *1* ∧ *concat* (*replicate n zs*) = *xs* @ *ys*
proof −
obtain *m n zs* **where** *concat* (*replicate m zs*) = *xs*
and *concat* (*replicate n zs*) = *ys*
using *comm-append-are-replicate*[*OF assms*(*3*)] **by** *blast*
then have *m* + *n* > *1* **and** *concat* (*replicate* (*m*+*n*) *zs*) = *xs* @ *ys*
using ‹*xs* ≠ []› **and** ‹*ys* ≠ []›
by (*auto simp*: *replicate-add*)
then show *?thesis* **by** *blast*
qed

lemma *Cons-replicate-eq*:
x # *xs* = *replicate n y* ⟷ *x* = *y* ∧ *n* > *0* ∧ *xs* = *replicate* (*n* − *1*) *x*
by (*induct n*) *auto*

lemma *replicate-length-same*:
(∀ *y*∈*set xs*. *y* = *x*) ⟹ *replicate* (*length xs*) *x* = *xs*
by (*induct xs*) *simp-all*

lemma *foldr-replicate* [*simp*]:
foldr f (*replicate n x*) = *f x* ^^ *n*
by (*induct n*) (*simp-all*)

lemma *fold-replicate* [*simp*]:
fold f (*replicate n x*) = *f x* ^^ *n*
by (*subst foldr-fold* [*symmetric*]) *simp-all*

66.2.9 *enumerate*

lemma *enumerate-simps* [*simp*, *code*]:
enumerate n [] = []
enumerate n (*x* # *xs*) = (*n*, *x*) # *enumerate* (*Suc n*) *xs*
by (*simp-all add*: *enumerate-eq-zip upt-rec*)

lemma *length-enumerate* [*simp*]:
length (*enumerate n xs*) = *length xs*
by (*simp add*: *enumerate-eq-zip*)

lemma *map-fst-enumerate* [*simp*]:
map fst (*enumerate n xs*) = [*n*..<*n* + *length xs*]
by (*simp add*: *enumerate-eq-zip*)

lemma *map-snd-enumerate* [*simp*]:
map snd (*enumerate n xs*) = *xs*
by (*simp add*: *enumerate-eq-zip*)

lemma *in-set-enumerate-eq*:
$p \in set\ (enumerate\ n\ xs) \longleftrightarrow n \leq fst\ p \wedge fst\ p < length\ xs + n \wedge nth\ xs\ (fst\ p - n) = snd\ p$
proof −
{ fix *m*
assume $n \leq m$
moreover assume $m < length\ xs + n$
ultimately have [*n*..<*n* + *length xs*] ! ($m - n$) = $m \wedge$
xs ! ($m - n$) = *xs* ! ($m - n$) $\wedge$ $m - n < length\ xs$ **by** *auto*
then have $\exists q$. [*n*..<*n* + *length xs*] ! *q* = $m \wedge$
xs ! *q* = *xs* ! ($m - n$) $\wedge$ $q < length\ xs$ **..**
} then show *?thesis* **by** (*cases p*) (*auto simp add*: *enumerate-eq-zip in-set-zip*)
qed

lemma *nth-enumerate-eq*: $m < length\ xs \Longrightarrow enumerate\ n\ xs\ !\ m = (n + m, xs\ !\ m)$
by (*simp add*: *enumerate-eq-zip*)

lemma *enumerate-replicate-eq*:
enumerate n (*replicate m a*) = *map* (λq. (*q*, *a*)) [*n*..<*n* + *m*]
by (*rule pair-list-eqI*)
(*simp-all add*: *enumerate-eq-zip comp-def map-replicate-const*)

lemma *enumerate-Suc-eq*:
enumerate (*Suc n*) *xs* = *map* (*apfst Suc*) (*enumerate n xs*)
by (*rule pair-list-eqI*)
(*simp-all add*: *not-le*, *simp del*: *map-map add*: *map-Suc-upt map-map* [*symmetric*])

lemma *distinct-enumerate* [*simp*]:
distinct (*enumerate n xs*)
by (*simp add*: *enumerate-eq-zip distinct-zipI1*)

lemma *enumerate-append-eq*:
enumerate n (xs @ ys) = enumerate n xs @ enumerate (n + length xs) ys
by (*simp add*: *enumerate-eq-zip add.assoc zip-append2*)

lemma *enumerate-map-upt*:
enumerate n (map f [n..<m]) = map (λk. (k, f k)) [n..<m]
by (*cases n ≤ m*) (*simp-all add*: *zip-map2 zip-same-conv-map enumerate-eq-zip*)

66.2.10 *rotate1* and *rotate*

lemma *rotate0*[*simp*]: *rotate 0 = id*
by(*simp add*:*rotate-def*)

lemma *rotate-Suc*[*simp*]: *rotate (Suc n) xs = rotate1(rotate n xs)*
by(*simp add*:*rotate-def*)

lemma *rotate-add*:
rotate (m+n) = rotate m ∘ rotate n
by(*simp add*:*rotate-def funpow-add*)

lemma *rotate-rotate*: *rotate m (rotate n xs) = rotate (m+n) xs*
by(*simp add*:*rotate-add*)

lemma *rotate1-map*: *rotate1 (map f xs) = map f (rotate1 xs)*
by(*cases xs*) *simp-all*

lemma *rotate1-rotate-swap*: *rotate1 (rotate n xs) = rotate n (rotate1 xs)*
by(*simp add*:*rotate-def funpow-swap1*)

lemma *rotate1-length01*[*simp*]: *length xs ≤ 1 ⟹ rotate1 xs = xs*
by(*cases xs*) *simp-all*

lemma *rotate-length01*[*simp*]: *length xs ≤ 1 ⟹ rotate n xs = xs*
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate1-hd-tl*: *xs ≠ [] ⟹ rotate1 xs = tl xs @ [hd xs]*
by (*cases xs*) *simp-all*

lemma *rotate-drop-take*:
rotate n xs = drop (n mod length xs) xs @ take (n mod length xs) xs
proof (*induct n*)
case (*Suc n*)
show *?case*
proof (*cases xs = []*)
case *False*
then show *?thesis*
proof (*cases n mod length xs = 0*)
case *True*

then show *?thesis*
by (*auto simp add*: *mod-Suc False Suc.hyps drop-Suc rotate1-hd-tl take-Suc Suc-length-conv*)
next
case *False*
with ‹*xs* ≠ []› *Suc*
show *?thesis*
by (*simp add*: *rotate-def mod-Suc rotate1-hd-tl drop-Suc*[*symmetric*] *drop-tl*[*symmetric*] *take-hd-drop linorder-not-le*)
qed
qed *simp*
qed *simp*

lemma *rotate-conv-mod*: *rotate n xs* = *rotate* (*n mod length xs*) *xs*
by(*simp add*:*rotate-drop-take*)

lemma *rotate-id*[*simp*]: *n mod length xs* = *0* ⟹ *rotate n xs* = *xs*
by(*simp add*:*rotate-drop-take*)

lemma *length-rotate1* [*simp*]: *length*(*rotate1 xs*) = *length xs*
by (*cases xs*) *simp-all*

lemma *length-rotate*[*simp*]: *length*(*rotate n xs*) = *length xs*
by (*induct n arbitrary*: *xs*) (*simp-all add*:*rotate-def*)

lemma *distinct1-rotate*[*simp*]: *distinct*(*rotate1 xs*) = *distinct xs*
by (*cases xs*) *auto*

lemma *distinct-rotate*[*simp*]: *distinct*(*rotate n xs*) = *distinct xs*
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate-map*: *rotate n* (*map f xs*) = *map f* (*rotate n xs*)
by(*simp add*:*rotate-drop-take take-map drop-map*)

lemma *set-rotate1* [*simp*]: *set*(*rotate1 xs*) = *set xs*
by (*cases xs*) *auto*

lemma *set-rotate*[*simp*]: *set*(*rotate n xs*) = *set xs*
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate1-replicate*[*simp*]: *rotate1* (*replicate n a*) = *replicate n a*
by (*cases n*) (*simp-all add*: *replicate-append-same*)

lemma *rotate1-is-Nil-conv*[*simp*]: (*rotate1 xs* = []) = (*xs* = [])
by (*cases xs*) *auto*

lemma *rotate-is-Nil-conv*[*simp*]: (*rotate n xs* = []) = (*xs* = [])
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate-rev*:
rotate n (*rev xs*) = *rev*(*rotate* (*length xs* − (*n mod length xs*)) *xs*)
proof (*cases length xs* = *0* ∨ *n mod length xs* = *0*)
case *False*
then show *?thesis*
by(*simp add*:*rotate-drop-take rev-drop rev-take*)
qed *force*

lemma *hd-rotate-conv-nth*:
assumes *xs* ≠ [] **shows** *hd*(*rotate n xs*) = *xs*!(*n mod length xs*)
proof −
have *n mod length xs* < *length xs*
using *assms* **by** *simp*
then show *?thesis*
by (*metis drop-eq-Nil hd-append2 hd-drop-conv-nth leD rotate-drop-take*)
qed

lemma *rotate-append*: *rotate* (*length l*) (*l* @ *q*) = *q* @ *l*
by (*induct l arbitrary*: *q*) (*auto simp add*: *rotate1-rotate-swap*)

lemma *nth-rotate*:
‹*rotate m xs* ! *n* = *xs* ! ((*m* + *n*) *mod length xs*)› **if** ‹*n* < *length xs*›
by (*smt* (*verit*) *add.commute hd-rotate-conv-nth length-rotate not-less0 list.size*(*3*) *mod-less rotate-rotate that*)

lemma *nth-rotate1*:
‹*rotate1 xs* ! *n* = *xs* ! (*Suc n mod length xs*)› **if** ‹*n* < *length xs*›
using *that nth-rotate* [*of n xs 1*] **by** *simp*

lemma *inj-rotate1*: *inj rotate1*
proof
fix *xs ys* :: *'a list* **show** *rotate1 xs* = *rotate1 ys* ⟹ *xs* = *ys*
by (*cases xs*; *cases ys*; *simp*)
qed

lemma *surj-rotate1*: *surj rotate1*
proof (*safe*, *simp-all*)
fix *xs* :: *'a list* **show** *xs* ∈ *range rotate1*
proof (*cases xs rule*: *rev-exhaust*)
case *Nil*
hence *xs* = *rotate1* [] **by** *auto*
thus *?thesis* **by** *fast*
next
case (*snoc as a*)
hence *xs* = *rotate1* (*a*#*as*) **by** *force*
thus *?thesis* **by** *fast*
qed
qed

lemma *bij-rotate1*: *bij* (*rotate1* :: *'a list* ⇒ *'a list*)
using *bijI inj-rotate1 surj-rotate1* **by** *blast*

lemma *rotate1-fixpoint-card*: *rotate1 xs* = *xs* ⟹ *xs* = [] ∨ *card*(*set xs*) = *1*
by(*induction xs*) (*auto simp*: *card-insert-if*[*OF finite-set*] *append-eq-Cons-conv*)

66.2.11 *nths* — a generalization of (!) to sets

lemma *nths-empty* [*simp*]: *nths xs* {} = []
  by (*auto simp add*: *nths-def*)

lemma *nths-nil* [*simp*]: *nths* [] *A* = []
  by (*auto simp add*: *nths-def*)

lemma *nths-all*: ∀ *i* < *length xs*. *i* ∈ *I* ⟹ *nths xs I* = *xs*
  unfolding *nths-def*
  by (*metis add-0 diff-zero filter-True in-set-zip length-upt nth-upt zip-eq-conv*)

lemma *length-nths*:
  length (*nths xs I*) = *card*{*i*. *i* < *length xs* ∧ *i* ∈ *I*}
  by(*simp add*: *nths-def length-filter-conv-card cong*:*conj-cong*)

lemma *nths-shift-lemma-Suc*:
  map fst (*filter* (λ*p*. *P*(*Suc*(*snd p*))) (*zip xs is*)) =
   map fst (*filter* (λ*p*. *P*(*snd p*)) (*zip xs* (*map Suc is*)))
proof (*induct xs arbitrary*: *is*)
  case (*Cons x xs is*)
  show *?case*
    by (*cases is*) (*auto simp add*: *Cons.hyps*)
qed *simp*

lemma *nths-shift-lemma*:
  map fst (*filter* (λ*p*. *snd p* ∈ *A*) (*zip xs* [*i*..<*i* + *length xs*])) =
   map fst (*filter* (λ*p*. *snd p* + *i* ∈ *A*) (*zip xs* [*0*..<*length xs*]))
by (*induct xs rule*: *rev-induct*) (*simp-all add*: *add.commute*)

lemma *nths-append*:
  nths (*l* @ *l'*) *A* = *nths l A* @ *nths l'* {*j*. *j* + *length l* ∈ *A*}
  unfolding *nths-def*
proof (*induct l' rule*: *rev-induct*)
  case (*snoc x xs*)
  then show *?case*
    by (*simp add*: *upt-add-eq-append*[*of 0*] *nths-shift-lemma add.commute*)
qed *auto*

lemma *nths-Cons*:
  nths (*x* # *l*) *A* = (*if 0* ∈ *A then* [*x*] *else* []) @ *nths l* {*j*. *Suc j* ∈ *A*}
proof (*induct l rule*: *rev-induct*)
  case (*snoc x xs*)

then show *?case*
by (*simp flip*: *append-Cons add*: *nths-append*)
qed (*auto simp*: *nths-def*)

lemma *nths-map*: *nths* (*map f xs*) *I* = *map f* (*nths xs I*)
by(*induction xs arbitrary*: *I*) (*simp-all add*: *nths-Cons*)

lemma *set-nths*: *set*(*nths xs I*) = {*xs*!*i*|*i*. *i*<*size xs* ∧ *i* ∈ *I*}
by (*induct xs arbitrary*: *I*) (*auto simp*: *nths-Cons nth-Cons split*:*nat.split dest*!: *gr0-implies-Suc*)

lemma *set-nths-subset*: *set*(*nths xs I*) ⊆ *set xs*
by(*auto simp add*:*set-nths*)

lemma *notin-set-nthsI*[*simp*]: *x* ∉ *set xs* ⟹ *x* ∉ *set*(*nths xs I*)
by(*auto simp add*:*set-nths*)

lemma *in-set-nthsD*: *x* ∈ *set*(*nths xs I*) ⟹ *x* ∈ *set xs*
by(*auto simp add*:*set-nths*)

lemma *nths-singleton* [*simp*]: *nths* [*x*] *A* = (*if 0* ∈ *A then* [*x*] *else* [])
by (*simp add*: *nths-Cons*)

lemma *distinct-nthsI*[*simp*]: *distinct xs* ⟹ *distinct* (*nths xs I*)
by (*induct xs arbitrary*: *I*) (*auto simp*: *nths-Cons*)

lemma *nths-upt-eq-take* [*simp*]: *nths l* {..<*n*} = *take n l*
by (*induct l rule*: *rev-induct*) (*simp-all split*: *nat-diff-split add*: *nths-append*)

lemma *nths-nths*: *nths* (*nths xs A*) *B* = *nths xs* {*i* ∈ *A*. ∃*j* ∈ *B*. *card* {*i'* ∈ *A*. *i'* < *i*} = *j*}
by (*induction xs arbitrary*: *A B*) (*auto simp add*: *nths-Cons card-less-Suc card-less-Suc2*)

lemma *drop-eq-nths*: *drop n xs* = *nths xs* {*i*. *i* ≥ *n*}
by (*induction xs arbitrary*: *n*) (*auto simp add*: *nths-Cons nths-all drop-Cons'* *intro*: *arg-cong2*[**where** *f*=*nths*, *OF refl*])

lemma *nths-drop*: *nths* (*drop n xs*) *I* = *nths xs* ((+) *n* ' *I*)
by(*force simp*: *drop-eq-nths nths-nths simp flip*: *atLeastLessThan-iff*
intro: *arg-cong2*[**where** *f*=*nths*, *OF refl*])

lemma *filter-eq-nths*: *filter P xs* = *nths xs* {*i*. *i*<*length xs* ∧ *P*(*xs*!*i*)}
by(*induction xs*) (*auto simp*: *nths-Cons*)

lemma *filter-in-nths*:
distinct xs ⟹ *filter* (%*x*. *x* ∈ *set* (*nths xs s*)) *xs* = *nths xs s*
proof (*induct xs arbitrary*: *s*)
case *Nil* **thus** *?case* **by** *simp*
next

case (*Cons a xs*)
then have $\forall x.\ x \in set\ xs \longrightarrow x \neq a$ **by** *auto*
with *Cons* **show** *?case* **by**(*simp add*: *nths-Cons cong*:*filter-cong*)
qed

66.2.12 *subseqs* and *List.n-lists*

lemma *length-subseqs*: $length\ (subseqs\ xs) = 2\ \hat{}\ length\ xs$
by (*induct xs*) (*simp-all add*: *Let-def*)

lemma *subseqs-powset*: *set ‘ set* (*subseqs xs*) = *Pow* (*set xs*)
proof −
have *aux*: $\bigwedge x\ A.\ set\ `\ Cons\ x\ `\ A = insert\ x\ `\ set\ `\ A$
by (*auto simp add*: *image-def*)
have *set* (*map set* (*subseqs xs*)) = *Pow* (*set xs*)
by (*induct xs*) (*simp-all add*: *aux Let-def Pow-insert Un-commute comp-def del*: *map-map*)
then show *?thesis* **by** *simp*
qed

lemma *distinct-set-subseqs*:
assumes *distinct xs*
shows *distinct* (*map set* (*subseqs xs*))
by (*simp add*: *assms card-Pow card-distinct distinct-card length-subseqs subseqs-powset*)

lemma *n-lists-Nil* [*simp*]: *List.n-lists n* [] = (*if n* = *0 then* [[]] *else* [])
by (*induct n*) *simp-all*

lemma *length-n-lists-elem*: $ys \in set\ (List.n\text{-}lists\ n\ xs) \Longrightarrow length\ ys = n$
by (*induct n arbitrary*: *ys*) *auto*

lemma *set-n-lists*: $set\ (List.n\text{-}lists\ n\ xs) = \{ys.\ length\ ys = n \wedge set\ ys \subseteq set\ xs\}$
proof (*rule set-eqI*)
fix *ys* :: *'a list*
show $ys \in set\ (List.n\text{-}lists\ n\ xs) \longleftrightarrow ys \in \{ys.\ length\ ys = n \wedge set\ ys \subseteq set\ xs\}$
proof −
have $ys \in set\ (List.n\text{-}lists\ n\ xs) \Longrightarrow length\ ys = n$
by (*induct n arbitrary*: *ys*) *auto*
moreover have $\bigwedge x.\ ys \in set\ (List.n\text{-}lists\ n\ xs) \Longrightarrow x \in set\ ys \Longrightarrow x \in set\ xs$
by (*induct n arbitrary*: *ys*) *auto*
moreover have $set\ ys \subseteq set\ xs \Longrightarrow ys \in set\ (List.n\text{-}lists\ (length\ ys)\ xs)$
by (*induct ys*) *auto*
ultimately show *?thesis* **by** *auto*
qed
qed

lemma *subseqs-refl*: $xs \in set\ (subseqs\ xs)$
by (*induct xs*) (*simp-all add*: *Let-def*)

lemma *subset-subseqs*: $X \subseteq set\ xs \Longrightarrow X \in set\ `\ set\ (subseqs\ xs)$
unfolding *subseqs-powset* **by** *simp*

lemma *Cons-in-subseqsD*: $y\ \#\ ys \in set\ (subseqs\ xs) \Longrightarrow ys \in set\ (subseqs\ xs)$
by (*induct xs*) (*auto simp*: *Let-def*)

lemma *subseqs-distinctD*: $[\![\ ys \in set\ (subseqs\ xs);\ distinct\ xs\ ]\!] \Longrightarrow distinct\ ys$
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs ys*)
then show *?case*
by (*auto simp*: *Let-def*) (*metis Pow-iff contra-subsetD image-eqI subseqs-powset*)
qed *simp*

66.2.13 *splice*

lemma *splice-Nil2* [*simp*]: *splice xs* [] = *xs*
by (*cases xs*) *simp-all*

lemma *length-splice*[*simp*]: *length*(*splice xs ys*) = *length xs* + *length ys*
by (*induct xs ys rule*: *splice.induct*) *auto*

lemma *split-Nil-iff* [*simp*]: *splice xs ys* = [] $\longleftrightarrow$ *xs* = [] $\wedge$ *ys* = []
by (*induct xs ys rule*: *splice.induct*) *auto*

lemma *splice-replicate*[*simp*]: *splice* (*replicate m x*) (*replicate n x*) = *replicate* (*m*+*n*) *x*
proof (*induction replicate m x replicate n x arbitrary*: *m n rule*: *splice.induct*)
case (*2 x xs*)
then show *?case*
by (*auto simp add*: *Cons-replicate-eq dest*: *gr0-implies-Suc*)
qed *auto*

66.2.14 *shuffles*

lemma *shuffles-commutes*: *shuffles xs ys* = *shuffles ys xs*
by (*induction xs ys rule*: *shuffles.induct*) (*simp-all add*: *Un-commute*)

lemma *Nil-in-shuffles*[*simp*]: [] $\in$ *shuffles xs ys* $\longleftrightarrow$ *xs* = [] $\wedge$ *ys* = []
by (*induct xs ys rule*: *shuffles.induct*) *auto*

lemma *shufflesE*:
$zs \in shuffles\ xs\ ys \Longrightarrow$
$(zs = xs \Longrightarrow ys = [] \Longrightarrow P) \Longrightarrow$
$(zs = ys \Longrightarrow xs = [] \Longrightarrow P) \Longrightarrow$
$(\bigwedge x\ xs'\ z\ zs'.\ xs = x\ \#\ xs' \Longrightarrow zs = z\ \#\ zs' \Longrightarrow x = z \Longrightarrow zs' \in shuffles\ xs'\ ys \Longrightarrow P) \Longrightarrow$
$(\bigwedge y\ ys'\ z\ zs'.\ ys = y\ \#\ ys' \Longrightarrow zs = z\ \#\ zs' \Longrightarrow y = z \Longrightarrow zs' \in shuffles\ xs\ ys' \Longrightarrow P) \Longrightarrow P$
by (*induct xs ys rule*: *shuffles.induct*) *auto*

lemma *Cons-in-shuffles-iff*:
 $z \# zs \in shuffles\ xs\ ys \longleftrightarrow$
 $(xs \neq [] \wedge hd\ xs = z \wedge zs \in shuffles\ (tl\ xs)\ ys \vee$
 $ys \neq [] \wedge hd\ ys = z \wedge zs \in shuffles\ xs\ (tl\ ys))$
 by (*induct xs ys rule*: *shuffles.induct*) *auto*

lemma *splice-in-shuffles* [*simp*, *intro*]: *splice xs ys* ∈ *shuffles xs ys*
 by (*induction xs ys rule*: *splice.induct*) (*simp-all add*: *Cons-in-shuffles-iff shuffles-commutes*)

lemma *Nil-in-shufflesI*: $xs = [] \Longrightarrow ys = [] \Longrightarrow [] \in shuffles\ xs\ ys$
 by *simp*

lemma *Cons-in-shuffles-leftI*: $zs \in shuffles\ xs\ ys \Longrightarrow z \# zs \in shuffles\ (z \# xs)\ ys$
 by (*cases ys*) *auto*

lemma *Cons-in-shuffles-rightI*: $zs \in shuffles\ xs\ ys \Longrightarrow z \# zs \in shuffles\ xs\ (z \# ys)$
 by (*cases xs*) *auto*

lemma *finite-shuffles* [*simp*, *intro*]: *finite* (*shuffles xs ys*)
 by (*induction xs ys rule*: *shuffles.induct*) *simp-all*

lemma *length-shuffles*: $zs \in shuffles\ xs\ ys \Longrightarrow length\ zs = length\ xs + length\ ys$
 by (*induction xs ys arbitrary*: *zs rule*: *shuffles.induct*) *auto*

lemma *set-shuffles*: $zs \in shuffles\ xs\ ys \Longrightarrow set\ zs = set\ xs \cup set\ ys$
 by (*induction xs ys arbitrary*: *zs rule*: *shuffles.induct*) *auto*

lemma *distinct-disjoint-shuffles*:
 assumes *distinct xs distinct ys* $set\ xs \cap set\ ys = \{\}$ $zs \in shuffles\ xs\ ys$
 shows *distinct zs*
using *assms*
proof (*induction xs ys arbitrary*: *zs rule*: *shuffles.induct*)
 case (*3 x xs y ys*)
 show *?case*
 proof (*cases zs*)
  case (*Cons z zs'*)
  with *3.prems* **and** *3.IH*[*of zs'*] **show** *?thesis* **by** (*force dest*: *set-shuffles*)
 qed *simp-all*
qed *simp-all*

lemma *Cons-shuffles-subset1*: $(\#)\ x$ ‘ $shuffles\ xs\ ys \subseteq shuffles\ (x \# xs)\ ys$
 by (*cases ys*) *auto*

lemma *Cons-shuffles-subset2*: $(\#)\ y$ ‘ $shuffles\ xs\ ys \subseteq shuffles\ xs\ (y \# ys)$
 by (*cases xs*) *auto*

lemma *filter-shuffles*:
filter P ' shuffles xs ys = shuffles (filter P xs) (filter P ys)
proof –
have ∗: *filter P ' (#) x ' A = (if P x then (#) x ' filter P ' A else filter P ' A)*
for *x A*
by (*auto simp*: *image-image*)
show *?thesis*
by (*induction xs ys rule*: *shuffles.induct*)
(*simp-all split*: *if-splits add*: *image-Un* ∗ *Un-absorb1 Un-absorb2*
Cons-shuffles-subset1 Cons-shuffles-subset2)
qed

lemma *filter-shuffles-disjoint1*:
assumes *set xs* ∩ *set ys* = {} *zs* ∈ *shuffles xs ys*
shows *filter* (λ*x*. *x* ∈ *set xs*) *zs* = *xs* (**is** *filter ?P - = -*)
and *filter* (λ*x*. *x* ∉ *set xs*) *zs* = *ys* (**is** *filter ?Q - = -*)
using *assms*
proof –
from *assms* **have** *filter ?P zs* ∈ *filter ?P ' shuffles xs ys* **by** *blast*
also have *filter ?P ' shuffles xs ys = shuffles (filter ?P xs) (filter ?P ys)*
by (*rule filter-shuffles*)
also have *filter ?P xs = xs* **by** (*rule filter-True*) *simp-all*
also have *filter ?P ys* = [] **by** (*rule filter-False*) (*insert assms*(*1*), *auto*)
also have *shuffles xs* [] = {*xs*} **by** *simp*
finally show *filter ?P zs = xs* **by** *simp*
next
from *assms* **have** *filter ?Q zs* ∈ *filter ?Q ' shuffles xs ys* **by** *blast*
also have *filter ?Q ' shuffles xs ys = shuffles (filter ?Q xs) (filter ?Q ys)*
by (*rule filter-shuffles*)
also have *filter ?Q ys = ys* **by** (*rule filter-True*) (*insert assms*(*1*), *auto*)
also have *filter ?Q xs* = [] **by** (*rule filter-False*) (*insert assms*(*1*), *auto*)
also have *shuffles* [] *ys* = {*ys*} **by** *simp*
finally show *filter ?Q zs = ys* **by** *simp*
qed

lemma *filter-shuffles-disjoint2*:
assumes *set xs* ∩ *set ys* = {} *zs* ∈ *shuffles xs ys*
shows *filter* (λ*x*. *x* ∈ *set ys*) *zs* = *ys filter* (λ*x*. *x* ∉ *set ys*) *zs* = *xs*
using *filter-shuffles-disjoint1*[*of ys xs zs*] *assms*
by (*simp-all add*: *shuffles-commutes Int-commute*)

lemma *partition-in-shuffles*:
xs ∈ *shuffles* (*filter P xs*) (*filter* (λ*x*. ¬*P x*) *xs*)
proof (*induction xs*)
case (*Cons x xs*)
show *?case*
proof (*cases P x*)
case *True*

hence $x \# xs \in (\#)\ x\ `\ shuffles\ (filter\ P\ xs)\ (filter\ (\lambda x.\ \neg P\ x)\ xs)$
by (*intro imageI Cons.IH*)
also have ... $\subseteq shuffles\ (filter\ P\ (x \# xs))\ (filter\ (\lambda x.\ \neg P\ x)\ (x \# xs))$
by (*simp add*: *True Cons-shuffles-subset1*)
finally show *?thesis* **.**
next
case *False*
hence $x \# xs \in (\#)\ x\ `\ shuffles\ (filter\ P\ xs)\ (filter\ (\lambda x.\ \neg P\ x)\ xs)$
by (*intro imageI Cons.IH*)
also have ... $\subseteq shuffles\ (filter\ P\ (x \# xs))\ (filter\ (\lambda x.\ \neg P\ x)\ (x \# xs))$
by (*simp add*: *False Cons-shuffles-subset2*)
finally show *?thesis* **.**
qed
qed *auto*

lemma *inv-image-partition*:
assumes $\bigwedge x.\ x \in set\ xs \Longrightarrow P\ x\ \bigwedge y.\ y \in set\ ys \Longrightarrow \neg P\ y$
shows $partition\ P\ -`\ \{(xs,\ ys)\} = shuffles\ xs\ ys$
proof (*intro equalityI subsetI*)
fix *zs* **assume** *zs*: $zs \in shuffles\ xs\ ys$
hence [*simp*]: $set\ zs = set\ xs \cup set\ ys$ **by** (*rule set-shuffles*)
from *assms* **have** $filter\ P\ zs = filter\ (\lambda x.\ x \in set\ xs)\ zs$
$filter\ (\lambda x.\ \neg P\ x)\ zs = filter\ (\lambda x.\ x \in set\ ys)\ zs$
by (*intro filter-cong refl*; *force*)+
moreover from *assms* **have** $set\ xs \cap set\ ys = \{\}$ **by** *auto*
ultimately show $zs \in partition\ P\ -`\ \{(xs,\ ys)\}$ **using** *zs*
by (*simp add*: *o-def filter-shuffles-disjoint1 filter-shuffles-disjoint2*)
next
fix *zs* **assume** $zs \in partition\ P\ -`\ \{(xs,\ ys)\}$
thus $zs \in shuffles\ xs\ ys$ **using** *partition-in-shuffles*[*of zs*] **by** (*auto simp*: *o-def*)
qed

66.2.15 Transpose

function *transpose* **where**
$transpose\ []\ = []\ |$
$transpose\ ([]\ \#\ xss) = transpose\ xss\ |$
$transpose\ ((x\#xs)\ \#\ xss) =$
$(x\ \#\ [h.\ (h\#t) \leftarrow xss])\ \#\ transpose\ (xs\ \#\ [t.\ (h\#t) \leftarrow xss])$
by *pat-completeness auto*

lemma *transpose-aux-filter-head*:
$concat\ (map\ (case\text{-}list\ []\ (\lambda h\ t.\ [h]))\ xss) =$
$map\ (\lambda xs.\ hd\ xs)\ (filter\ (\lambda ys.\ ys \neq [])\ xss)$
by (*induct xss*) (*auto split*: *list.split*)

lemma *transpose-aux-filter-tail*:
$concat\ (map\ (case\text{-}list\ []\ (\lambda h\ t.\ [t]))\ xss) =$
$map\ (\lambda xs.\ tl\ xs)\ (filter\ (\lambda ys.\ ys \neq [])\ xss)$

by (*induct xss*) (*auto split*: *list.split*)

lemma *transpose-aux-max*:
max (*Suc* (*length xs*)) (*foldr* (λ*xs*. *max* (*length xs*)) *xss 0*) =
Suc (*max* (*length xs*) (*foldr* (λ*x*. *max* (*length x* − *Suc 0*)) (*filter* (λ*ys*. *ys* ≠ [])
xss) *0*))
(**is** *max* - *?foldB* = *Suc* (*max* - *?foldA*))
proof (*cases* (*filter* (λ*ys*. *ys* ≠ []) *xss*) = [])
case *True*
hence *foldr* (λ*xs*. *max* (*length xs*)) *xss 0* = *0*
proof (*induct xss*)
case (*Cons x xs*)
then have *x* = [] **by** (*cases x*) *auto*
with *Cons* **show** *?case* **by** *auto*
qed *simp*
thus *?thesis* **using** *True* **by** *simp*
next
case *False*

have *foldA*: *?foldA* = *foldr* (λ*x*. *max* (*length x*)) (*filter* (λ*ys*. *ys* ≠ []) *xss*) *0* − *1*
by (*induct xss*) *auto*
have *foldB*: *?foldB* = *foldr* (λ*x*. *max* (*length x*)) (*filter* (λ*ys*. *ys* ≠ []) *xss*) *0*
by (*induct xss*) *auto*

have *0* < *?foldB*
proof −
from *False*
obtain *z zs* **where** *zs*: (*filter* (λ*ys*. *ys* ≠ []) *xss*) = *z#zs* **by** (*auto simp*:
neq-Nil-conv)
hence *z* ∈ *set* (*filter* (λ*ys*. *ys* ≠ []) *xss*) **by** *auto*
hence *z* ≠ [] **by** *auto*
thus *?thesis*
unfolding *foldB zs*
by (*auto simp*: *max-def intro*: *less-le-trans*)
qed
thus *?thesis*
unfolding *foldA foldB max-Suc-Suc*[*symmetric*]
by *simp*
qed

termination *transpose*
by (*relation measure* (λ*xs*. *foldr* (λ*xs*. *max* (*length xs*)) *xs 0* + *length xs*))
(*auto simp*: *transpose-aux-filter-tail foldr-map comp-def transpose-aux-max*
less-Suc-eq-le)

lemma *transpose-empty*: (*transpose xs* = []) ⟷ (∀ *x* ∈ *set xs*. *x* = [])
by (*induct rule*: *transpose.induct*) *simp-all*

lemma *length-transpose*:

fixes *xs* :: *'a list list*
shows *length* (*transpose xs*) = *foldr* (λ*xs*. *max* (*length xs*)) *xs 0*
by (*induct rule*: *transpose.induct*)
(*auto simp*: *transpose-aux-filter-tail foldr-map comp-def transpose-aux-max max-Suc-Suc*[*symmetric*] *simp del*: *max-Suc-Suc*)

lemma *nth-transpose*:
fixes *xs* :: *'a list list*
assumes *i* < *length* (*transpose xs*)
shows *transpose xs* ! *i* = *map* (λ*xs*. *xs* ! *i*) (*filter* (λ*ys*. *i* < *length ys*) *xs*)
using *assms* **proof** (*induct arbitrary*: *i rule*: *transpose.induct*)
case (*3 x xs xss*)
define *XS* **where** *XS* = (*x* # *xs*) # *xss*
hence [*simp*]: *XS* ≠ [] **by** *auto*
thus *?case*
proof (*cases i*)
case *0*
thus *?thesis* **by** (*simp add*: *transpose-aux-filter-head hd-conv-nth*)
next
case (*Suc j*)
have ∗: ⋀*xss*. *xs* # *map tl xss* = *map tl* ((*x*#*xs*)#*xss*) **by** *simp*
have ∗∗: ⋀*xss*. (*x*#*xs*) # *filter* (λ*ys*. *ys* ≠ []) *xss* = *filter* (λ*ys*. *ys* ≠ []) ((*x*#*xs*)#*xss*) **by** *simp*
{ fix *xs* :: ‹*'a list*› **have** *Suc j* < *length xs* ⟷ *xs* ≠ [] ∧ *j* < *length xs* − *Suc 0*
by (*cases xs*) *simp-all*
} note ∗∗∗ = *this*

have *j-less*: *j* < *length* (*transpose* (*xs* # *concat* (*map* (*case-list* [] (λ*h t*. [*t*])) *xss*)))
using *3.prems* **by** (*simp add*: *transpose-aux-filter-tail length-transpose Suc*)

show *?thesis*
unfolding *transpose.simps* ‹*i* = *Suc j*› *nth-Cons-Suc 3.hyps*[*OF j-less*]
by (*auto simp*: *nth-tl transpose-aux-filter-tail filter-map comp-def length-transpose* ∗ ∗∗ ∗∗∗ *XS-def*[*symmetric*])
qed
qed *simp-all*

lemma *transpose-map-map*:
transpose (*map* (*map f*) *xs*) = *map* (*map f*) (*transpose xs*)
proof (*rule nth-equalityI*)
have [*simp*]: *length* (*transpose* (*map* (*map f*) *xs*)) = *length* (*transpose xs*)
by (*simp add*: *length-transpose foldr-map comp-def*)
show *length* (*transpose* (*map* (*map f*) *xs*)) = *length* (*map* (*map f*) (*transpose xs*)) **by** *simp*

fix *i* **assume** *i* < *length* (*transpose* (*map* (*map f*) *xs*))
thus *transpose* (*map* (*map f*) *xs*) ! *i* = *map* (*map f*) (*transpose xs*) ! *i*
by (*simp add*: *nth-transpose filter-map comp-def*)

qed

66.2.16 *min* and *arg-min*

lemma *min-list-Min*: *xs* ≠ [] ⟹ *min-list xs* = *Min* (*set xs*)
by (*induction xs rule*: *induct-list012*)(*auto*)

lemma *f-arg-min-list-f*: *xs* ≠ [] ⟹ *f* (*arg-min-list f xs*) = *Min* (*f* ' (*set xs*))
by(*induction f xs rule*: *arg-min-list.induct*) (*auto simp*: *min-def intro*!: *antisym*)

lemma *arg-min-list-in*: *xs* ≠ [] ⟹ *arg-min-list f xs* ∈ *set xs*
by(*induction xs rule*: *induct-list012*) (*auto simp*: *Let-def*)

66.2.17 (In)finiteness

lemma *finite-list-length*: *finite* {*xs*::('*a*::*finite*) *list*. *length xs* = *n*}
proof(*induction n*)
case (*Suc n*)
have {*xs*::'*a list*. *length xs* = *Suc n*} = (⋃*x*. (#) *x* ' {*xs*. *length xs* = *n*})
by (*auto simp*: *length-Suc-conv*)
then show *?case* **using** *Suc* **by** *simp*
qed *simp*

lemma *finite-maxlen*:
finite (*M*::'*a list set*) ⟹ ∃ *n*. ∀ *s*∈*M*. *size s* < *n*
proof (*induct rule*: *finite.induct*)
case *emptyI* **show** *?case* **by** *simp*
next
case (*insertI M xs*)
then obtain *n* **where** ∀ *s*∈*M*. *length s* < *n* **by** *blast*
hence ∀ *s*∈*insert xs M*. *size s* < *max n* (*size xs*) + *1* **by** *auto*
thus *?case* **..**
qed

lemma *lists-length-Suc-eq*:
{*xs*. *set xs* ⊆ *A* ∧ *length xs* = *Suc n*} =
(λ(*xs*, *n*). *n*#*xs*) ' ({*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*} × *A*)
by (*auto simp*: *length-Suc-conv*)

lemma
assumes *finite A*
shows *finite-lists-length-eq*: *finite* {*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*}
and *card-lists-length-eq*: *card* {*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*} = (*card A*)^*n*
using ‹*finite A*›
by (*induct n*)
(*auto simp*: *card-image inj-split-Cons lists-length-Suc-eq cong*: *conj-cong*)

lemma *finite-lists-length-le*:
assumes *finite A* **shows** *finite* {*xs*. *set xs* ⊆ *A* ∧ *length xs* ≤ *n*}
(**is** *finite ?S*)

proof−
  have *?S* = ($\bigcup$ *n*∈{*0..n*}. {*xs. set xs* ⊆ *A* ∧ *length xs* = *n*}) **by** *auto*
  thus *?thesis* **by** (*auto intro*!: *finite-lists-length-eq*[*OF* ‹*finite A*›] *simp only*:)
qed

lemma *card-lists-length-le*:
  assumes *finite A* **shows** *card* {*xs. set xs* ⊆ *A* ∧ *length xs* ≤ *n*} = ($\sum$ *i*≤*n. card A*^*i*)
proof −
  have ($\sum$ *i*≤*n. card A*^*i*) = *card* ($\bigcup$ *i*≤*n.* {*xs. set xs* ⊆ *A* ∧ *length xs* = *i*})
    using ‹*finite A*›
    by (*subst card-UN-disjoint*)
      (*auto simp add*: *card-lists-length-eq finite-lists-length-eq*)
  also have ($\bigcup$ *i*≤*n.* {*xs. set xs* ⊆ *A* ∧ *length xs* = *i*}) = {*xs. set xs* ⊆ *A* ∧ *length xs* ≤ *n*}
    by *auto*
  finally show *?thesis* **by** *simp*
qed

lemma *finite-subset-distinct*:
  assumes *finite A*
  shows *finite* {*xs. set xs* ⊆ *A* ∧ *distinct xs*} (**is** *finite ?S*)
proof (*rule finite-subset*)
  from *assms* **show** *?S* ⊆ {*xs. set xs* ⊆ *A* ∧ *length xs* ≤ *card A*}
    by *clarsimp* (*metis distinct-card card-mono*)
  from *assms* **show** *finite* ... **by** (*rule finite-lists-length-le*)
qed

lemma *card-lists-distinct-length-eq*:
  assumes *finite A k* ≤ *card A*
  shows *card* {*xs. length xs* = *k* ∧ *distinct xs* ∧ *set xs* ⊆ *A*} = $\prod$ {*card A* − *k* + *1* .. *card A*}
using *assms*
proof (*induct k*)
  case *0*
  then have {*xs. length xs* = *0* ∧ *distinct xs* ∧ *set xs* ⊆ *A*} = {[]} **by** *auto*
  then show *?case* **by** *simp*
next
  case (*Suc k*)
  let *?k-list* = λ*k xs. length xs* = *k* ∧ *distinct xs* ∧ *set xs* ⊆ *A*
  have *inj-Cons*: $\bigwedge$*A. inj-on* (λ(*xs*, *n*). *n* # *xs*) *A* **by** (*rule inj-onI*) *auto*

  from *Suc* **have** *k* ≤ *card A* **by** *simp*
  moreover note ‹*finite A*›
  moreover have *finite* {*xs. ?k-list k xs*}
    by (*rule finite-subset*) (*use finite-lists-length-eq*[*OF* ‹*finite A*›, *of k*] **in** *auto*)
  moreover have $\bigwedge$*i j. i* ≠ *j* ⟶ {*i*} × (*A* − *set i*) ∩ {*j*} × (*A* − *set j*) = {}
    by *auto*
  moreover have $\bigwedge$*i. i* ∈ {*xs. ?k-list k xs*} ⟹ *card* (*A* − *set i*) = *card A* − *k*

by (*simp add*: *card-Diff-subset distinct-card*)
moreover have {*xs*. *?k-list* (*Suc k*) *xs*} =
(λ(*xs*, *n*). *n*#*xs*) ' ⋃((λ*xs*. {*xs*} × (*A* − *set xs*)) ' {*xs*. *?k-list k xs*})
by (*auto simp*: *length-Suc-conv*)
moreover have *Suc* (*card A* − *Suc k*) = *card A* − *k* **using** *Suc.prems* **by** *simp*
then have (*card A* − *k*) ∗ ∏{*Suc* (*card A* − *k*)..*card A*} = ∏{*Suc* (*card A* − *Suc k*)..*card A*}
by (*subst prod.insert*[*symmetric*]) (*simp add*: *atLeastAtMost-insertL*)+
ultimately show *?case*
by (*simp add*: *card-image inj-Cons card-UN-disjoint Suc.hyps algebra-simps*)
qed

lemma *card-lists-distinct-length-eq′*:
assumes *k* < *card A*
shows *card* {*xs*. *length xs* = *k* ∧ *distinct xs* ∧ *set xs* ⊆ *A*} = ∏{*card A* − *k* + *1* .. *card A*}
proof −
from ‹*k* < *card A*› **have** *finite A* **and** *k* ≤ *card A* **using** *card.infinite* **by** *force*+
from *this* **show** *?thesis* **by** (*rule card-lists-distinct-length-eq*)
qed

lemma *infinite-UNIV-listI*: ¬ *finite*(*UNIV*::′*a list set*)
by (*metis UNIV-I finite-maxlen length-replicate less-irrefl*)

lemma *same-length-different*:
assumes *xs* ≠ *ys* **and** *length xs* = *length ys*
shows ∃ *pre x xs′ y ys′*. *x*≠*y* ∧ *xs* = *pre* @ [*x*] @ *xs′* ∧ *ys* = *pre* @ [*y*] @ *ys′*
using *assms*
proof (*induction xs arbitrary*: *ys*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons x xs*)
then obtain *z zs* **where** *ys*: *ys* = *Cons z zs*
by (*metis length-Suc-conv*)
show *?case*
proof (*cases x*=*z*)
case *True*
then have *xs* ≠ *zs length xs* = *length zs*
using *Cons.prems ys* **by** *auto*
then obtain *pre u xs′ v ys′* **where** *u*≠*v* **and** *xs*: *xs* = *pre* @ [*u*] @ *xs′* **and** *zs*: *zs* = *pre* @ [*v*] @*ys′*
using *Cons.IH* **by** *meson*
then have *x* # *xs* = (*z*#*pre*) @ [*u*] @ *xs′* ∧ *ys* = (*z*#*pre*) @ [*v*] @ *ys′*
by (*simp add*: *True ys*)
with ‹*u*≠*v*› **show** *?thesis*
by *blast*
next
case *False*

```
    then have x # xs = [] @ [x] @ xs ∧ ys = [] @ [z] @ zs
      by (simp add: ys)
    then show ?thesis
      using False by blast
  qed
qed
```

66.3 Sorting

66.3.1 *sorted-wrt*

Sometimes the second equation in the definition of *sorted-wrt* is too aggressive because it relates each list element to *all* its successors. Then this equation should be removed and *sorted-wrt2-simps* should be added instead.

```
lemma sorted-wrt1: sorted-wrt P [x] = True
by(simp)

lemma sorted-wrt2: transp P ⟹ sorted-wrt P (x # y # zs) = (P x y ∧ sorted-wrt
P (y # zs))
proof (induction zs arbitrary: x y)
  case (Cons z zs)
  then show ?case
    by simp (meson transpD)+
qed auto

lemmas sorted-wrt2-simps = sorted-wrt1 sorted-wrt2

lemma sorted-wrt-true [simp]:
  sorted-wrt (λ- -. True) xs
by (induction xs) simp-all

lemma sorted-wrt-append:
  sorted-wrt P (xs @ ys) ⟷
  sorted-wrt P xs ∧ sorted-wrt P ys ∧ (∀x∈set xs. ∀y∈set ys. P x y)
by (induction xs) auto

lemma sorted-wrt-map:
  sorted-wrt R (map f xs) = sorted-wrt (λx y. R (f x) (f y)) xs
by (induction xs) simp-all

lemma
  assumes sorted-wrt f xs
  shows sorted-wrt-take: sorted-wrt f (take n xs)
  and   sorted-wrt-drop: sorted-wrt f (drop n xs)
proof −
  from assms have sorted-wrt f (take n xs @ drop n xs) by simp
  thus sorted-wrt f (take n xs) and sorted-wrt f (drop n xs)
    unfolding sorted-wrt-append by simp-all
qed
```

lemma *sorted-wrt-filter*:
sorted-wrt f xs $\Longrightarrow$ *sorted-wrt f* (*filter P xs*)
by (*induction xs*) *auto*

lemma *sorted-wrt-rev*:
sorted-wrt P (*rev xs*) = *sorted-wrt* ($\lambda x\ y.\ P\ y\ x$) *xs*
by (*induction xs*) (*auto simp add*: *sorted-wrt-append*)

lemma *sorted-wrt-mono-rel*:
($\bigwedge x\ y.$ ⟦ $x \in$ *set xs*; $y \in$ *set xs*; $P\ x\ y$ ⟧ $\Longrightarrow Q\ x\ y$) $\Longrightarrow$ *sorted-wrt P xs* $\Longrightarrow$ *sorted-wrt Q xs*
by(*induction xs*)(*auto*)

lemma *sorted-wrt01*: *length xs* $\leq$ *1* $\Longrightarrow$ *sorted-wrt P xs*
by(*auto simp*: *le-Suc-eq length-Suc-conv*)

lemma *sorted-wrt-iff-nth-less*:
sorted-wrt P xs = ($\forall i\ j.\ i < j \longrightarrow j <$ *length xs* $\longrightarrow P$ (*xs* ! *i*) (*xs* ! *j*))
by (*induction xs*) (*auto simp add*: *in-set-conv-nth Ball-def nth-Cons split*: *nat.split*)

lemma *sorted-wrt-nth-less*:
⟦ *sorted-wrt P xs*; $i < j$; $j <$ *length xs* ⟧ $\Longrightarrow P$ (*xs* ! *i*) (*xs* ! *j*)
by(*auto simp*: *sorted-wrt-iff-nth-less*)

lemma *sorted-wrt-iff-nth-Suc-transp*: **assumes** *transp P*
shows *sorted-wrt P xs* $\longleftrightarrow$ ($\forall i.$ *Suc i* $<$ *length xs* $\longrightarrow P$ (*xs*!*i*) (*xs*!(*Suc i*))) (**is** *?L* = *?R*)
proof
assume *?L*
thus *?R*
by (*simp add*: *sorted-wrt-iff-nth-less*)
next
assume *?R*
have $i < j \Longrightarrow j <$ *length xs* $\Longrightarrow P$ (*xs* ! *i*) (*xs* ! *j*) **for** *i j*
by(*induct i j rule*: *less-Suc-induct*)(*simp add*: ‹*?R*›, *meson assms transpE transp-on-less*)
thus *?L*
by (*simp add*: *sorted-wrt-iff-nth-less*)
qed

lemma *sorted-wrt-upt*[*simp*]: *sorted-wrt* (<) [*m*..<*n*]
by(*induction n*) (*auto simp*: *sorted-wrt-append*)

lemma *sorted-wrt-upto*[*simp*]: *sorted-wrt* (<) [*i*..*j*]
proof(*induct i j rule*:*upto.induct*)
case (*1 i j*)
from *this* **show** *?case*
unfolding *upto.simps*[*of i j*] **by** *auto*

qed

Each element is greater or equal to its index:

lemma *sorted-wrt-less-idx*:
sorted-wrt $(<)$ *ns* $\Longrightarrow i <$ *length ns* $\Longrightarrow i \leq ns!i$
proof (*induction ns arbitrary*: *i rule*: *rev-induct*)
case *Nil* **thus** *?case* **by** *simp*
next
case *snoc*
thus *?case*
by (*simp add*: *nth-append sorted-wrt-append*)
(*metis less-antisym not-less nth-mem*)
qed

66.3.2 *sorted*

context *linorder*
begin

Sometimes the second equation in the definition of *sorted* is too aggressive because it relates each list element to *all* its successors. Then this equation should be removed and *sorted2-simps* should be added instead. Executable code is one such use case.

lemma *sorted0*: *sorted* [] = *True*
by *simp*

lemma *sorted1*: *sorted* $[x]$ = *True*
by *simp*

lemma *sorted2*: *sorted* $(x \# y \# zs) = (x \leq y \wedge$ *sorted* $(y \# zs))$
by *auto*

lemmas *sorted2-simps* = *sorted1 sorted2*

lemma *sorted-append*:
sorted $(xs@ys) = ($*sorted xs* $\wedge$ *sorted ys* $\wedge (\forall x \in$ *set xs*. $\forall y \in$ *set ys*. $x \leq y))$
by (*simp add*: *sorted-wrt-append*)

lemma *sorted-map*:
sorted (*map f xs*) = *sorted-wrt* $(\lambda x\ y.\ f\ x \leq f\ y)$ *xs*
by (*simp add*: *sorted-wrt-map*)

lemma *sorted01*: *length xs* $\leq 1 \Longrightarrow$ *sorted xs*
by (*simp add*: *sorted-wrt01*)

lemma *sorted-tl*:
sorted xs $\Longrightarrow$ *sorted* (*tl xs*)
by (*cases xs*) (*simp-all*)

lemma *sorted-iff-nth-mono-less*:
sorted xs = (∀ *i j*. *i* < *j* ⟶ *j* < *length xs* ⟶ *xs* ! *i* ≤ *xs* ! *j*)
by (*simp add*: *sorted-wrt-iff-nth-less*)

lemma *sorted-iff-nth-mono*:
sorted xs = (∀ *i j*. *i* ≤ *j* ⟶ *j* < *length xs* ⟶ *xs* ! *i* ≤ *xs* ! *j*)
by (*auto simp*: *sorted-iff-nth-mono-less nat-less-le*)

lemma *sorted-nth-mono*:
sorted xs ⟹ *i* ≤ *j* ⟹ *j* < *length xs* ⟹ *xs*!*i* ≤ *xs*!*j*
by (*auto simp*: *sorted-iff-nth-mono*)

lemma *sorted-iff-nth-Suc*:
sorted xs ⟷ (∀ *i*. *Suc i* < *length xs* ⟶ *xs*!*i* ≤ *xs*!(*Suc i*))
by(*simp add*: *sorted-wrt-iff-nth-Suc-transp*)

lemma *sorted-rev-nth-mono*:
sorted (*rev xs*) ⟹ *i* ≤ *j* ⟹ *j* < *length xs* ⟹ *xs*!*j* ≤ *xs*!*i*
by (*metis local.nle-le order-class.antisym-conv1 sorted-wrt-iff-nth-less sorted-wrt-rev*)

lemma *sorted-rev-iff-nth-mono*:
sorted (*rev xs*) ⟷ (∀ *i j*. *i* ≤ *j* ⟶ *j* < *length xs* ⟶ *xs*!*j* ≤ *xs*!*i*) (**is** *?L* = *?R*)
proof
assume *?L* **thus** *?R*
by (*blast intro*: *sorted-rev-nth-mono*)
next
assume *?R*
have *rev xs* ! *k* ≤ *rev xs* ! *l* **if** *asms*: *k* ≤ *l l* < *length*(*rev xs*) **for** *k l*
proof −
have *k* < *length xs* *l* < *length xs*
length xs − *Suc l* ≤ *length xs* − *Suc k length xs* − *Suc k* < *length xs*
using *asms* **by** *auto*
thus *rev xs* ! *k* ≤ *rev xs* ! *l*
by (*simp add*: ‹*?R*› *rev-nth*)
qed
thus *?L* **by** (*simp add*: *sorted-iff-nth-mono*)
qed

lemma *sorted-rev-iff-nth-Suc*:
sorted (*rev xs*) ⟷ (∀ *i*. *Suc i* < *length xs* ⟶ *xs*!(*Suc i*) ≤ *xs*!*i*)
proof−
interpret *dual*: *linorder* (λ*x y*. *y* ≤ *x*) (λ*x y*. *y* < *x*)
using *dual-linorder* **.**
show *?thesis*
using *dual-linorder dual.sorted-iff-nth-Suc dual.sorted-iff-nth-mono*
unfolding *sorted-rev-iff-nth-mono* **by** *simp*
qed

lemma *sorted-map-remove1*:

sorted (*map f xs*) $\Longrightarrow$ *sorted* (*map f* (*remove1 x xs*))
by (*induct xs*) (*auto*)

lemma *sorted-remove1*: *sorted xs* $\Longrightarrow$ *sorted* (*remove1 a xs*)
using *sorted-map-remove1* [*of* $\lambda x.\ x$] **by** *simp*

lemma *sorted-butlast*:
assumes *sorted xs*
shows *sorted* (*butlast xs*)
by (*simp add*: *assms butlast-conv-take sorted-wrt-take*)

lemma *sorted-replicate* [*simp*]: *sorted*(*replicate n x*)
by(*induction n*) (*auto*)

lemma *sorted-remdups*[*simp*]:
sorted xs $\Longrightarrow$ *sorted* (*remdups xs*)
by (*induct xs*) (*auto*)

lemma *sorted-remdups-adj*[*simp*]:
sorted xs $\Longrightarrow$ *sorted* (*remdups-adj xs*)
by (*induct xs rule*: *remdups-adj.induct*, *simp-all split*: *if-split-asm*)

lemma *sorted-nths*: *sorted xs* $\Longrightarrow$ *sorted* (*nths xs I*)
by(*induction xs arbitrary*: *I*)(*auto simp*: *nths-Cons*)

lemma *sorted-distinct-set-unique*:
assumes *sorted xs distinct xs sorted ys distinct ys set xs* = *set ys*
shows *xs* = *ys*
proof −
from *assms* **have** *1*: *length xs* = *length ys* **by** (*auto dest*!: *distinct-card*)
from *assms* **show** *?thesis*
proof(*induct rule*:*list-induct2*[*OF 1*])
case *1* **show** *?case* **by** *simp*
next
case (*2 x xs y ys*)
then show *?case*
by (*cases* ‹*x* = *y*›) (*auto simp add*: *insert-eq-iff*)
qed
qed

lemma *map-sorted-distinct-set-unique*:
assumes *inj-on f* (*set xs* ∪ *set ys*)
assumes *sorted* (*map f xs*) *distinct* (*map f xs*)
sorted (*map f ys*) *distinct* (*map f ys*)
assumes *set xs* = *set ys*
shows *xs* = *ys*
using *assms map-inj-on sorted-distinct-set-unique* **by** *fastforce*

lemma *sorted-dropWhile*: *sorted xs* $\Longrightarrow$ *sorted* (*dropWhile P xs*)

by (*auto dest*: *sorted-wrt-drop simp add*: *dropWhile-eq-drop*)

lemma *sorted-takeWhile*: *sorted xs* $\Longrightarrow$ *sorted* (*takeWhile P xs*)
by (*subst takeWhile-eq-take*) (*auto dest*: *sorted-wrt-take*)

lemma *sorted-filter*:
sorted (*map f xs*) $\Longrightarrow$ *sorted* (*map f* (*filter P xs*))
by (*induct xs*) *simp-all*

lemma *foldr-max-sorted*:
assumes *sorted* (*rev xs*)
shows *foldr max xs y* = (*if xs* = [] *then y else max* (*xs* ! *0*) *y*)
using *assms*
proof (*induct xs*)
case (*Cons x xs*)
then have *sorted* (*rev xs*) **using** *sorted-append* **by** *auto*
with *Cons* **show** *?case*
by (*cases xs*) (*auto simp add*: *sorted-append max-def*)
qed *simp*

lemma *filter-equals-takeWhile-sorted-rev*:
assumes *sorted*: *sorted* (*rev* (*map f xs*))
shows *filter* ($\lambda x.\ t < f\ x$) *xs* = *takeWhile* ($\lambda\ x.\ t < f\ x$) *xs*
(**is** *filter ?P xs* = *?tW*)
proof (*rule takeWhile-eq-filter*[*symmetric*])
let *?dW* = *dropWhile ?P xs*
fix *x* **assume** *x*: $x \in$ *set ?dW*
then obtain *i* **where** *i*: *i* < *length ?dW* **and** *nth-i*: *x* = *?dW* ! *i*
unfolding *in-set-conv-nth* **by** *auto*
hence *length ?tW* + *i* < *length* (*?tW* @ *?dW*)
unfolding *length-append* **by** *simp*
hence *i'*: *length* (*map f ?tW*) + *i* < *length* (*map f xs*) **by** *simp*
have (*map f ?tW* @ *map f ?dW*) ! (*length* (*map f ?tW*) + *i*) $\le$
(*map f ?tW* @ *map f ?dW*) ! (*length* (*map f ?tW*) + *0*)
using *sorted-rev-nth-mono*[*OF sorted - i'*, *of length ?tW*]
unfolding *map-append*[*symmetric*] **by** *simp*
hence *f x* $\le$ *f* (*?dW* ! *0*)
unfolding *nth-append-length-plus nth-i*
using *i preorder-class.le-less-trans*[*OF le0 i*] **by** *simp*
also have ... $\le$ *t*
by (*metis hd-conv-nth hd-dropWhile length-greater-0-conv length-pos-if-in-set local.leI x*)
finally show $\neg$ *t* < *f x* **by** *simp*
qed

lemma *sorted-map-same*:
sorted (*map f* (*filter* ($\lambda x.\ f\ x = g\ xs$) *xs*))
proof (*induct xs arbitrary*: *g*)
case *Nil* **then show** *?case* **by** *simp*

next
case (*Cons x xs*)
then have *sorted* (*map f* (*filter* ($\lambda y.$ *f y* = ($\lambda xs.$ *f x*) *xs*) *xs*)) **.**
moreover from *Cons* **have** *sorted* (*map f* (*filter* ($\lambda y.$ *f y* = (*g* ∘ *Cons x*) *xs*) *xs*)) **.**
ultimately show *?case* **by** *simp-all*
qed

lemma *sorted-same*:
sorted (*filter* ($\lambda x.$ *x* = *g xs*) *xs*)
using *sorted-map-same* [*of* $\lambda x.$ *x*] **by** *simp*

end

lemma *sorted-upt*[*simp*]: *sorted* [*m*..<*n*]
by(*simp add*: *sorted-wrt-mono-rel*[*OF - sorted-wrt-upt*])

lemma *sorted-upto*[*simp*]: *sorted* [*m*..*n*]
by(*simp add*: *sorted-wrt-mono-rel*[*OF - sorted-wrt-upto*])

66.3.3 Sorting functions

Currently it is not shown that *sort* returns a permutation of its input because the nicest proof is via multisets, which are not part of Main. Alternatively one could define a function that counts the number of occurrences of an element in a list and use that instead of multisets to state the correctness property.

context *linorder*
begin

lemma *set-insort-key*:
set (*insort-key f x xs*) = *insert x* (*set xs*)
by (*induct xs*) *auto*

lemma *length-insort* [*simp*]:
length (*insort-key f x xs*) = *Suc* (*length xs*)
by (*induct xs*) *simp-all*

lemma *insort-key-left-comm*:
assumes $f\,x \neq f\,y$
shows *insort-key f y* (*insort-key f x xs*) = *insort-key f x* (*insort-key f y xs*)
by (*induct xs*) (*auto simp add*: *assms dest*: *order.antisym*)

lemma *insort-left-comm*:
insort x (*insort y xs*) = *insort y* (*insort x xs*)
by (*cases x* = *y*) (*auto intro*: *insort-key-left-comm*)

lemma *comp-fun-commute-insort*: *comp-fun-commute insort*

proof
qed (*simp add*: *insort-left-comm fun-eq-iff*)

lemma *sort-key-simps* [*simp*]:
  sort-key f [] = []
  sort-key f (*x#xs*) = *insort-key f x* (*sort-key f xs*)
by (*simp-all add*: *sort-key-def*)

lemma *sort-key-conv-fold*:
  assumes *inj-on f* (*set xs*)
  shows *sort-key f xs* = *fold* (*insort-key f*) *xs* []
proof −
  have *fold* (*insort-key f*) (*rev xs*) = *fold* (*insort-key f*) *xs*
  proof (*rule fold-rev*, *rule ext*)
    fix *zs*
    fix *x y*
    assume $x \in$ *set xs* $y \in$ *set xs*
    with *assms* **have** ∗: *f y* = *f x* $\Longrightarrow$ *y* = *x* **by** (*auto dest*: *inj-onD*)
    have ∗∗: *x* = *y* $\longleftrightarrow$ *y* = *x* **by** *auto*
    show (*insort-key f y* ∘ *insort-key f x*) *zs* = (*insort-key f x* ∘ *insort-key f y*) *zs*
      by (*induct zs*) (*auto intro*: ∗ *simp add*: ∗∗)
  qed
  then show *?thesis* **by** (*simp add*: *sort-key-def foldr-conv-fold*)
qed

lemma *sort-conv-fold*:
  sort xs = *fold insort xs* []
  by (*rule sort-key-conv-fold*) *simp*

lemma *length-sort*[*simp*]: *length* (*sort-key f xs*) = *length xs*
  by (*induct xs*, *auto*)

lemma *set-sort*[*simp*]: *set*(*sort-key f xs*) = *set xs*
  by (*induct xs*) (*simp-all add*: *set-insort-key*)

lemma *distinct-insort*: *distinct* (*insort-key f x xs*) = (*x* ∉ *set xs* ∧ *distinct xs*)
  by(*induct xs*)(*auto simp*: *set-insort-key*)

lemma *distinct-insort-key*:
  distinct (*map f* (*insort-key f x xs*)) = (*f x* ∉ *f* ‘ *set xs* ∧ (*distinct* (*map f xs*)))
  by (*induct xs*) (*auto simp*: *set-insort-key*)

lemma *distinct-sort*[*simp*]: *distinct* (*sort-key f xs*) = *distinct xs*
  by (*induct xs*) (*simp-all add*: *distinct-insort*)

lemma *sorted-insort-key*: *sorted* (*map f* (*insort-key f x xs*)) = *sorted* (*map f xs*)
  by (*induct xs*) (*auto simp*: *set-insort-key*)

lemma *sorted-insort*: *sorted* (*insort x xs*) = *sorted xs*

using *sorted-insort-key* [**where** *f*=*λx. x*] **by** *simp*

theorem *sorted-sort-key* [*simp*]: *sorted* (*map f* (*sort-key f xs*))
by (*induct xs*) (*auto simp*:*sorted-insort-key*)

theorem *sorted-sort* [*simp*]: *sorted* (*sort xs*)
using *sorted-sort-key* [**where** *f*=*λx. x*] **by** *simp*

lemma *insort-not-Nil* [*simp*]:
insort-key f a xs ≠ []
by (*induction xs*) *simp-all*

lemma *insort-is-Cons*: ∀ *x*∈*set xs. f a* ≤ *f x* ⟹ *insort-key f a xs* = *a* # *xs*
by (*cases xs*) *auto*

lemma *sort-key-id-if-sorted*: *sorted* (*map f xs*) ⟹ *sort-key f xs* = *xs*
by (*induction xs*) (*auto simp add*: *insort-is-Cons*)

Subsumed by *sorted* (*map ?f ?xs*) ⟹ *sort-key ?f ?xs* = *?xs* but easier to find:

lemma *sorted-sort-id*: *sorted xs* ⟹ *sort xs* = *xs*
by (*simp add*: *sort-key-id-if-sorted*)

lemma *sort-replicate* [*simp*]: *sort* (*replicate n x*) = *replicate n x*
using *sorted-replicate sorted-sort-id*
by *presburger*

lemma *insort-key-remove1*:
assumes *a* ∈ *set xs* **and** *sorted* (*map f xs*) **and** *hd* (*filter* (*λx. f a* = *f x*) *xs*) = *a*
shows *insort-key f a* (*remove1 a xs*) = *xs*
using *assms* **proof** (*induct xs*)
case (*Cons x xs*)
then show *?case*
proof (*cases x* = *a*)
case *False*
then have *f x* ≠ *f a* **using** *Cons.prems* **by** *auto*
then have *f x* < *f a* **using** *Cons.prems* **by** *auto*
with ‹*f x* ≠ *f a*› **show** *?thesis* **using** *Cons* **by** (*auto simp*: *insort-is-Cons*)
qed (*auto simp*: *insort-is-Cons*)
qed *simp*

lemma *insort-remove1*:
assumes *a* ∈ *set xs* **and** *sorted xs*
shows *insort a* (*remove1 a xs*) = *xs*
proof (*rule insort-key-remove1*)
define *n* **where** *n* = *length* (*filter* ((=) *a*) *xs*) − *1*
from ‹*a* ∈ *set xs*› **show** *a* ∈ *set xs* **.**
from ‹*sorted xs*› **show** *sorted* (*map* (*λx. x*) *xs*) **by** *simp*
from ‹*a* ∈ *set xs*› **have** *a* ∈ *set* (*filter* ((=) *a*) *xs*) **by** *auto*

then have *set* (*filter* ((=) *a*) *xs*) ≠ {} **by** *auto*
then have *filter* ((=) *a*) *xs* ≠ [] **by** (*auto simp only*: *set-empty*)
then have *length* (*filter* ((=) *a*) *xs*) > *0* **by** *simp*
then have *n*: *Suc n* = *length* (*filter* ((=) *a*) *xs*) **by** (*simp add*: *n-def*)
moreover have *replicate* (*Suc n*) *a* = *a* # *replicate n a*
by *simp*
ultimately show *hd* (*filter* ((=) *a*) *xs*) = *a* **by** (*simp add*: *replicate-length-filter*)
qed

lemma *finite-sorted-distinct-unique*:
assumes *finite A* **shows** ∃!*xs*. *set xs* = *A* ∧ *sorted xs* ∧ *distinct xs*
proof −
obtain *xs* **where** *distinct xs A* = *set xs*
using *finite-distinct-list* [*OF assms*] **by** *metis*
then show *?thesis*
by (*rule-tac a*=*sort xs* **in** *ex1I*) (*auto simp*: *sorted-distinct-set-unique*)
qed

lemma *insort-insert-key-triv*:
f x ∈ *f* ' *set xs* ⟹ *insort-insert-key f x xs* = *xs*
by (*simp add*: *insort-insert-key-def*)

lemma *insort-insert-triv*:
x ∈ *set xs* ⟹ *insort-insert x xs* = *xs*
using *insort-insert-key-triv* [*of* λ*x*. *x*] **by** *simp*

lemma *insort-insert-insort-key*:
f x ∉ *f* ' *set xs* ⟹ *insort-insert-key f x xs* = *insort-key f x xs*
by (*simp add*: *insort-insert-key-def*)

lemma *insort-insert-insort*:
x ∉ *set xs* ⟹ *insort-insert x xs* = *insort x xs*
using *insort-insert-insort-key* [*of* λ*x*. *x*] **by** *simp*

lemma *set-insort-insert*:
set (*insort-insert x xs*) = *insert x* (*set xs*)
by (*auto simp add*: *insort-insert-key-def set-insort-key*)

lemma *distinct-insort-insert*:
assumes *distinct xs*
shows *distinct* (*insort-insert-key f x xs*)
using *assms* **by** (*induct xs*) (*auto simp add*: *insort-insert-key-def set-insort-key*)

lemma *sorted-insort-insert-key*:
assumes *sorted* (*map f xs*)
shows *sorted* (*map f* (*insort-insert-key f x xs*))
using *assms* **by** (*simp add*: *insort-insert-key-def sorted-insort-key*)

lemma *sorted-insort-insert*:

assumes *sorted xs*
shows *sorted* (*insort-insert x xs*)
using *assms sorted-insort-insert-key* [*of* $\lambda x.\ x$] **by** *simp*

lemma *filter-insort-triv*:
$\neg P\ x \Longrightarrow$ *filter P* (*insort-key f x xs*) = *filter P xs*
by (*induct xs*) *simp-all*

lemma *filter-insort*:
sorted (*map f xs*) $\Longrightarrow P\ x \Longrightarrow$ *filter P* (*insort-key f x xs*) = *insort-key f x* (*filter P xs*)
by (*induct xs*) (*auto*, *subst insort-is-Cons*, *auto*)

lemma *filter-sort*:
filter P (*sort-key f xs*) = *sort-key f* (*filter P xs*)
by (*induct xs*) (*simp-all add*: *filter-insort-triv filter-insort*)

lemma *remove1-insort-key* [*simp*]:
remove1 x (*insort-key f x xs*) = *xs*
by (*induct xs*) *simp-all*

end

lemma *sort-upt* [*simp*]: *sort* [*m*..<*n*] = [*m*..<*n*]
by (*rule sort-key-id-if-sorted*) *simp*

lemma *sort-upto* [*simp*]: *sort* [*i*..*j*] = [*i*..*j*]
by (*rule sort-key-id-if-sorted*) *simp*

lemma *sorted-find-Min*:
sorted xs $\Longrightarrow \exists x \in$ *set xs. P x* $\Longrightarrow$ *List.find P xs* = *Some* (*Min* {*x*∈*set xs. P x*})
proof (*induct xs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **show** *?case* **proof** (*cases P x*)
case *True*
with *Cons* **show** *?thesis* **by** (*auto intro*: *Min-eqI* [*symmetric*])
next
case *False* **then have** {*y*. (*y* = *x* ∨ *y* ∈ *set xs*) ∧ *P y*} = {*y* ∈ *set xs*. *P y*}
by *auto*
with *Cons False* **show** *?thesis* **by** (*simp-all*)
qed
qed

lemma *sorted-enumerate* [*simp*]: *sorted* (*map fst* (*enumerate n xs*))
by (*simp add*: *enumerate-eq-zip*)

lemma *sorted-insort-is-snoc*: *sorted xs* $\Longrightarrow \forall x \in$ *set xs*. $a \geq x \Longrightarrow$ *insort a xs* = *xs* @ [*a*]

by (*induct xs*) (*auto dest*!: *insort-is-Cons*)

Stability of *sort-key*:

lemma *sort-key-stable*: *filter* (λ*y*. *f y* = *k*) (*sort-key f xs*) = *filter* (λ*y*. *f y* = *k*) *xs*
by (*induction xs*) (*auto simp*: *filter-insort insort-is-Cons filter-insort-triv*)

corollary *stable-sort-key-sort-key*: *stable-sort-key sort-key*
by(*simp add*: *stable-sort-key-def sort-key-stable*)

lemma *sort-key-const*: *sort-key* (λ*x*. *c*) *xs* = *xs*
by (*metis* (*mono-tags*) *filter-True sort-key-stable*)

66.3.4 *transpose* on sorted lists

lemma *sorted-transpose*[*simp*]: *sorted* (*rev* (*map length* (*transpose xs*)))
by (*auto simp*: *sorted-iff-nth-mono rev-nth nth-transpose*
length-filter-conv-card intro: *card-mono*)

lemma *transpose-max-length*:
foldr (λ*xs*. *max* (*length xs*)) (*transpose xs*) *0* = *length* (*filter* (λ*x*. *x* ≠ []) *xs*)
(**is** *?L* = *?R*)
proof (*cases transpose xs* = [])
case *False*
have *?L* = *foldr max* (*map length* (*transpose xs*)) *0*
by (*simp add*: *foldr-map comp-def*)
also have ... = *length* (*transpose xs* ! *0*)
using *False sorted-transpose* **by** (*simp add*: *foldr-max-sorted*)
finally show *?thesis*
using *False* **by** (*simp add*: *nth-transpose*)
next
case *True*
hence *filter* (λ*x*. *x* ≠ []) *xs* = []
by (*auto intro*!: *filter-False simp*: *transpose-empty*)
thus *?thesis* **by** (*simp add*: *transpose-empty True*)
qed

lemma *length-transpose-sorted*:
fixes *xs* :: ′*a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*))
shows *length* (*transpose xs*) = (*if xs* = [] *then 0 else length* (*xs* ! *0*))
proof (*cases xs* = [])
case *False*
thus *?thesis*
using *foldr-max-sorted*[*OF sorted*] *False*
unfolding *length-transpose foldr-map comp-def*
by *simp*
qed *simp*

lemma *nth-nth-transpose-sorted*[*simp*]:

```
  fixes xs :: 'a list list
  assumes sorted: sorted (rev (map length xs))
  and i: i < length (transpose xs)
  and j: j < length (filter (λys. i < length ys) xs)
  shows transpose xs ! i ! j = xs ! j ! i
using j filter-equals-takeWhile-sorted-rev[OF sorted, of i]
    nth-transpose[OF i] nth-map[OF j]
by (simp add: takeWhile-nth)

lemma transpose-column-length:
  fixes xs :: 'a list list
  assumes sorted: sorted (rev (map length xs)) and i < length xs
  shows length (filter (λys. i < length ys) (transpose xs)) = length (xs ! i)
proof −
  have xs ≠ [] using ‹i < length xs› by auto
  note filter-equals-takeWhile-sorted-rev[OF sorted, simp]
  { fix j assume j ≤ i
    note sorted-rev-nth-mono[OF sorted, of j i, simplified, OF this ‹i < length xs›]
  } note sortedE = this[consumes 1]

  have {j. j < length (transpose xs) ∧ i < length (transpose xs ! j)}
    = {..< length (xs ! i)}
  proof safe
    fix j
    assume j < length (transpose xs) and i < length (transpose xs ! j)
    with this(2) nth-transpose[OF this(1)]
    have i < length (takeWhile (λys. j < length ys) xs) by simp
    from nth-mem[OF this] takeWhile-nth[OF this]
    show j < length (xs ! i) by (auto dest: set-takeWhileD)
  next
    fix j assume j < length (xs ! i)
    thus j < length (transpose xs)
      using foldr-max-sorted[OF sorted] ‹xs ≠ []› sortedE[OF le0]
      by (auto simp: length-transpose comp-def foldr-map)

    have Suc i ≤ length (takeWhile (λys. j < length ys) xs)
      using ‹i < length xs› ‹j < length (xs ! i)› less-Suc-eq-le
      by (auto intro!: length-takeWhile-less-P-nth dest!: sortedE)
    with nth-transpose[OF ‹j < length (transpose xs)›]
    show i < length (transpose xs ! j) by simp
  qed
  thus ?thesis by (simp add: length-filter-conv-card)
qed

lemma transpose-column:
  fixes xs :: 'a list list
  assumes sorted: sorted (rev (map length xs)) and i < length xs
  shows map (λys. ys ! i) (filter (λys. i < length ys) (transpose xs))
    = xs ! i (is ?R = -)
```

proof (*rule nth-equalityI*)
  show *length*: *length ?R* = *length* (*xs* ! *i*)
    using *transpose-column-length*[*OF assms*] **by** *simp*

  fix *j* **assume** *j*: *j* < *length ?R*
  note ∗ = *less-le-trans*[*OF this*, *unfolded length-map*, *OF length-filter-le*]
  from *j* **have** *j-less*: *j* < *length* (*xs* ! *i*) **using** *length* **by** *simp*
  have *i-less-tW*: *Suc i* ≤ *length* (*takeWhile* (λ*ys*. *Suc j* ≤ *length ys*) *xs*)
  proof (*rule length-takeWhile-less-P-nth*)
    show *Suc i* ≤ *length xs* **using** ‹*i* < *length xs*› **by** *simp*
    fix *k* **assume** *k* < *Suc i*
    hence *k* ≤ *i* **by** *auto*
    with *sorted-rev-nth-mono*[*OF sorted this*] ‹*i* < *length xs*›
    have *length* (*xs* ! *i*) ≤ *length* (*xs* ! *k*) **by** *simp*
    thus *Suc j* ≤ *length* (*xs* ! *k*) **using** *j-less* **by** *simp*
  qed
  have *i-less-filter*: *i* < *length* (*filter* (λ*ys*. *j* < *length ys*) *xs*)
    unfolding *filter-equals-takeWhile-sorted-rev*[*OF sorted*, *of j*]
    using *i-less-tW* **by** (*simp-all add*: *Suc-le-eq*)
  from *j* **show** *?R* ! *j* = *xs* ! *i* ! *j*
    unfolding *filter-equals-takeWhile-sorted-rev*[*OF sorted-transpose*, *of i*]
    by (*simp add*: *takeWhile-nth nth-nth-transpose-sorted*[*OF sorted* ∗ *i-less-filter*])
qed

lemma *transpose-transpose*:
  fixes *xs* :: *'a list list*
  assumes *sorted*: *sorted* (*rev* (*map length xs*))
  shows *transpose* (*transpose xs*) = *takeWhile* (λ*x*. *x* ≠ []) *xs* (**is** *?L* = *?R*)
proof −
  have *len*: *length ?L* = *length ?R*
    unfolding *length-transpose transpose-max-length*
    using *filter-equals-takeWhile-sorted-rev*[*OF sorted*, *of 0*]
    by *simp*

  { fix *i* **assume** *i* < *length ?R*
    with *less-le-trans*[*OF - length-takeWhile-le*[*of - xs*]]
    have *i* < *length xs* **by** *simp*
  } note ∗ = *this*
  show *?thesis*
    by (*rule nth-equalityI*)
      (*simp-all add*: *len nth-transpose transpose-column*[*OF sorted*] ∗ *takeWhile-nth*)
qed

theorem *transpose-rectangle*:
  assumes *xs* = [] ⟹ *n* = *0*
  assumes *rect*: ⋀ *i*. *i* < *length xs* ⟹ *length* (*xs* ! *i*) = *n*
  shows *transpose xs* = *map* (λ *i*. *map* (λ *j*. *xs* ! *j* ! *i*) [*0*..<*length xs*]) [*0*..<*n*]
    (**is** *?trans* = *?map*)
proof (*rule nth-equalityI*)

have *sorted* (*rev* (*map length xs*))
  by (*auto simp*: *rev-nth rect sorted-iff-nth-mono*)
from *foldr-max-sorted*[*OF this*] *assms*
show *len*: *length ?trans* = *length ?map*
  by (*simp-all add*: *length-transpose foldr-map comp-def*)
moreover
{ fix *i* **assume** *i* < *n* **hence** *filter* (λ*ys*. *i* < *length ys*) *xs* = *xs*
  using *rect* **by** (*auto simp*: *in-set-conv-nth intro*!: *filter-True*) **}**
ultimately show ⋀*i*. *i* < *length* (*transpose xs*) ⟹ *?trans* ! *i* = *?map* ! *i*
  by (*auto simp*: *nth-transpose intro*: *nth-equalityI*)
qed

66.3.5 *sorted-key-list-of-set*

This function maps (finite) linearly ordered sets to sorted lists. The linear order is obtained by a key function that maps the elements of the set to a type that is linearly ordered. Warning: in most cases it is not a good idea to convert from sets to lists but one should convert in the other direction (via *set*).

Note: this is a generalisation of the older *sorted-list-of-set* that is obtained by setting the key function to the identity. Consequently, new theorems should be added to the locale below. They should also be aliased to more convenient names for use with *sorted-list-of-set* as seen further below.

definition (**in** *linorder*) *sorted-key-list-of-set* :: (′*b* ⇒ ′*a*) ⇒ ′*b set* ⇒ ′*b list*
  where *sorted-key-list-of-set f* ≡ *folding-on*.*F* (*insort-key f*) []

locale *folding-insort-key* = *lo?*: *linorder less-eq* :: ′*a* ⇒ ′*a* ⇒ *bool less*
  for *less-eq* (**infix** ‹⪯› *50*) **and** *less* (**infix** ‹≺› *50*) +
  fixes *S*
  fixes *f* :: ′*b* ⇒ ′*a*
  assumes *inj-on*: *inj-on f S*
begin

lemma *insort-key-commute*:
  x ∈ *S* ⟹ *y* ∈ *S* ⟹ *insort-key f y o insort-key f x* = *insort-key f x o insort-key f y*
proof(*rule ext*, *goal-cases*)
  case (*1 xs*)
  with *inj-on* **show** *?case* **by** (*induction xs*) (*auto simp*: *inj-onD*)
qed

sublocale *fold-insort-key*: *folding-on S insort-key f* []
  rewrites *folding-on*.*F* (*insort-key f*) [] = *sorted-key-list-of-set f*
proof −
  show *folding-on S* (*insort-key f*)
    by *standard* (*simp add*: *insort-key-commute*)
qed (*simp add*: *sorted-key-list-of-set-def*)

lemma *idem-if-sorted-distinct*:
  assumes *set xs* ⊆ *S* **and** *sorted* (*map f xs*) *distinct xs*
  shows *sorted-key-list-of-set f* (*set xs*) = *xs*
proof(*cases S* = {})
  case *True*
  then show *?thesis* **using** ‹*set xs* ⊆ *S*› **by** *auto*
next
  case *False*
  with *assms* **show** *?thesis*
  proof(*induction xs*)
    case (*Cons a xs*)
    with *Cons* **show** *?case* **by** (*cases xs*) *auto*
  qed *simp*
qed

lemma *sorted-key-list-of-set-empty*:
  sorted-key-list-of-set f {} = []
  by (*fact fold-insort-key.empty*)

lemma *sorted-key-list-of-set-insert*:
  assumes *insert x A* ⊆ *S* **and** *finite A x* ∉ *A*
  shows *sorted-key-list-of-set f* (*insert x A*)
        = *insort-key f x* (*sorted-key-list-of-set f A*)
  using *assms* **by** (*fact fold-insort-key.insert*)

lemma *sorted-key-list-of-set-insert-remove* [*simp*]:
  assumes *insert x A* ⊆ *S* **and** *finite A*
  shows *sorted-key-list-of-set f* (*insert x A*)
        = *insort-key f x* (*sorted-key-list-of-set f* (*A* − {*x*}))
  using *assms* **by** (*fact fold-insort-key.insert-remove*)

lemma *sorted-key-list-of-set-eq-Nil-iff* [*simp*]:
  assumes *A* ⊆ *S* **and** *finite A*
  shows *sorted-key-list-of-set f A* = [] ⟷ *A* = {}
  using *assms* **by** (*auto simp*: *fold-insort-key.remove*)

lemma *set-sorted-key-list-of-set* [*simp*]:
  assumes *A* ⊆ *S* **and** *finite A*
  shows *set* (*sorted-key-list-of-set f A*) = *A*
  using *assms*(*2*,*1*)
  by (*induct A rule*: *finite-induct*) (*simp-all add*: *set-insort-key*)

lemma *sorted-sorted-key-list-of-set* [*simp*]:
  assumes *A* ⊆ *S*
  shows *sorted* (*map f* (*sorted-key-list-of-set f A*))
proof (*cases finite A*)
  case *True* **thus** *?thesis* **using** ‹*A* ⊆ *S*›
    by (*induction A*) (*simp-all add*: *sorted-insort-key*)
next

case *False* **thus** *?thesis* **by** *simp*
qed

lemma *distinct-if-distinct-map*: *distinct* (*map f xs*) $\Longrightarrow$ *distinct xs*
using *inj-on* **by** (*simp add*: *distinct-map*)

lemma *distinct-sorted-key-list-of-set* [*simp*]:
assumes $A \subseteq S$
shows *distinct* (*map f* (*sorted-key-list-of-set f A*))
proof (*cases finite A*)
case *True* **thus** *?thesis* **using** ‹$A \subseteq S$› *inj-on*
by (*induction A*) (*force simp*: *distinct-insort-key dest*: *inj-onD*)+
next
case *False* **thus** *?thesis* **by** *simp*
qed

lemma *length-sorted-key-list-of-set* [*simp*]:
assumes $A \subseteq S$
shows *length* (*sorted-key-list-of-set f A*) = *card A*
proof (*cases finite A*)
case *True*
with *assms inj-on* **show** *?thesis*
using *distinct-card*[*symmetric*, *OF distinct-sorted-key-list-of-set*]
by (*auto simp*: *subset-inj-on intro*!: *card-image*)
qed *auto*

lemmas *sorted-key-list-of-set* =
set-sorted-key-list-of-set sorted-sorted-key-list-of-set distinct-sorted-key-list-of-set

lemma *sorted-key-list-of-set-remove*:
assumes *insert x A* $\subseteq S$ **and** *finite A*
shows *sorted-key-list-of-set f* ($A - \{x\}$) = *remove1 x* (*sorted-key-list-of-set f A*)
proof (*cases* $x \in A$)
case *False* **with** *assms* **have** $x \notin$ *set* (*sorted-key-list-of-set f A*) **by** *simp*
with *False* **show** *?thesis* **by** (*simp add*: *remove1-idem*)
next
case *True* **then obtain** *B* **where** *A*: *A* = *insert x B* **by** (*rule Set.set-insert*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *strict-sorted-key-list-of-set* [*simp*]:
$A \subseteq S \Longrightarrow$ *sorted-wrt* ($\prec$) (*map f* (*sorted-key-list-of-set f A*))
by (*cases finite A*) (*auto simp*: *strict-sorted-iff subset-inj-on*[*OF inj-on*])

lemma *finite-set-strict-sorted*:
assumes $A \subseteq S$ **and** *finite A*
obtains *l* **where** *sorted-wrt* ($\prec$) (*map f l*) *set l* = *A length l* = *card A*
using *assms*
by (*meson length-sorted-key-list-of-set set-sorted-key-list-of-set strict-sorted-key-list-of-set*)

lemma (**in** *linorder*) *strict-sorted-equal*:
  assumes *sorted-wrt* (<) *xs*
    and *sorted-wrt* (<) *ys*
    and *set ys* = *set xs*
  shows *ys* = *xs*
  using *assms*
proof (*induction xs arbitrary*: *ys*)
  case (*Cons x xs*)
  show *?case*
  proof (*cases ys*)
    case *Nil*
    then show *?thesis*
      using *Cons.prems* **by** *auto*
  next
    case (*Cons y ys'*)
    then have *xs* = *ys'*
      by (*metis Cons.prems list.inject sorted-distinct-set-unique strict-sorted-iff*)
    moreover have *x* = *y*
      using *Cons.prems* ‹*xs* = *ys'*› *local.Cons* **by** *fastforce*
    ultimately show *?thesis*
      using *local.Cons* **by** *blast*
  qed
qed *auto*

lemma (**in** *linorder*) *strict-sorted-equal-Uniq*: $\exists_{\leq 1}$*xs*. *sorted-wrt* (<) *xs* ∧ *set xs* = *A*
  by (*simp add*: *Uniq-def strict-sorted-equal*)

lemma *sorted-key-list-of-set-inject*:
  assumes *A* ⊆ *S B* ⊆ *S*
  assumes *sorted-key-list-of-set f A* = *sorted-key-list-of-set f B finite A finite B*
  shows *A* = *B*
  using *assms set-sorted-key-list-of-set* **by** *metis*

lemma *sorted-key-list-of-set-unique*:
  assumes *A* ⊆ *S* **and** *finite A*
  shows *sorted-wrt* (≺) (*map f l*) ∧ *set l* = *A* ∧ *length l* = *card A*
      ⟷ *sorted-key-list-of-set f A* = *l*
  using *assms*
  by (*auto simp*: *strict-sorted-iff card-distinct idem-if-sorted-distinct*)

end

context *linorder*
begin

definition *sorted-list-of-set* ≡ *sorted-key-list-of-set* (λ*x*::'*a*. *x*)

We abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the key function to the identity.

sublocale *sorted-list-of-set*: *folding-insort-key* (≤) (<) *UNIV* (λ*x*. *x*)
rewrites *sorted-key-list-of-set* (λ*x*. *x*) = *sorted-list-of-set*
and ⋀*xs*. *map* (λ*x*. *x*) *xs* ≡ *xs*
and ⋀*X*. (*X* ⊆ *UNIV*) ≡ *True*
and ⋀*x*. *x* ∈ *UNIV* ≡ *True*
and ⋀*P*. (*True* ⟹ *P*) ≡ *Trueprop P*
and ⋀*P Q*. (*True* ⟹ *PROP P* ⟹ *PROP Q*) ≡ (*PROP P* ⟹ *True* ⟹ *PROP Q*)
proof −
show *folding-insort-key* (≤) (<) *UNIV* (λ*x*. *x*)
by *standard simp*
qed (*simp-all add*: *sorted-list-of-set-def*)

Alias theorems for backwards compatibility and ease of use.

lemmas *sorted-list-of-set* = *sorted-list-of-set.sorted-key-list-of-set* **and**
sorted-list-of-set-empty = *sorted-list-of-set.sorted-key-list-of-set-empty* **and**
sorted-list-of-set-insert = *sorted-list-of-set.sorted-key-list-of-set-insert* **and**
sorted-list-of-set-insert-remove = *sorted-list-of-set.sorted-key-list-of-set-insert-remove* **and**
sorted-list-of-set-eq-Nil-iff = *sorted-list-of-set.sorted-key-list-of-set-eq-Nil-iff* **and**
set-sorted-list-of-set = *sorted-list-of-set.set-sorted-key-list-of-set* **and**
sorted-sorted-list-of-set = *sorted-list-of-set.sorted-sorted-key-list-of-set* **and**
distinct-sorted-list-of-set = *sorted-list-of-set.distinct-sorted-key-list-of-set* **and**
length-sorted-list-of-set = *sorted-list-of-set.length-sorted-key-list-of-set* **and**
sorted-list-of-set-remove = *sorted-list-of-set.sorted-key-list-of-set-remove* **and**
strict-sorted-list-of-set = *sorted-list-of-set.strict-sorted-key-list-of-set* **and**
sorted-list-of-set-inject = *sorted-list-of-set.sorted-key-list-of-set-inject* **and**
sorted-list-of-set-unique = *sorted-list-of-set.sorted-key-list-of-set-unique* **and**
finite-set-strict-sorted = *sorted-list-of-set.finite-set-strict-sorted*

lemma *sorted-list-of-set-sort-remdups* [*code*]:
sorted-list-of-set (*set xs*) = *sort* (*remdups xs*)
proof −
interpret *comp-fun-commute insort* **by** (*fact comp-fun-commute-insort*)
show *?thesis*
by (*simp add*: *sorted-list-of-set.fold-insort-key.eq-fold sort-conv-fold fold-set-fold-remdups*)
qed

end

lemma *sorted-list-of-set-range* [*simp*]:
sorted-list-of-set {*m*..<*n*} = [*m*..<*n*]
by (*rule sorted-distinct-set-unique*) *simp-all*

lemma *sorted-list-of-set-lessThan-Suc* [*simp*]:
sorted-list-of-set {..<*Suc k*} = *sorted-list-of-set* {..<*k*} @ [*k*]

using *le0 lessThan-atLeast0 sorted-list-of-set-range upt-Suc-append* **by** *presburger*

lemma *sorted-list-of-set-atMost-Suc* [*simp*]:
sorted-list-of-set {..*Suc k*} = *sorted-list-of-set* {..*k*} @ [*Suc k*]
using *lessThan-Suc-atMost sorted-list-of-set-lessThan-Suc* **by** *fastforce*

lemma *sorted-list-of-set-nonempty*:
assumes *finite A A* ≠ {}
shows *sorted-list-of-set A* = *Min A* # *sorted-list-of-set* (*A* − {*Min A*})
using *assms*
by (*auto simp*: *less-le simp flip*: *sorted-list-of-set.sorted-key-list-of-set-unique intro*: *Min-in*)

lemma *sorted-list-of-set-greaterThanLessThan*:
assumes *Suc i* < *j*
shows *sorted-list-of-set* {*i*<..<*j*} = *Suc i* # *sorted-list-of-set* {*Suc i*<..<*j*}
proof −
have {*i*<..<*j*} = *insert* (*Suc i*) {*Suc i*<..<*j*}
using *assms* **by** *auto*
then show *?thesis*
by (*metis assms atLeastSucLessThan-greaterThanLessThan sorted-list-of-set-range upt-conv-Cons*)
qed

lemma *sorted-list-of-set-greaterThanAtMost*:
assumes *Suc i* ≤ *j*
shows *sorted-list-of-set* {*i*<..*j*} = *Suc i* # *sorted-list-of-set* {*Suc i*<..*j*}
using *sorted-list-of-set-greaterThanLessThan* [*of i Suc j*]
by (*metis assms greaterThanAtMost-def greaterThanLessThan-eq le-imp-less-Suc lessThan-Suc-atMost*)

lemma *nth-sorted-list-of-set-greaterThanLessThan*:
n < *j* − *Suc i* ⟹ *sorted-list-of-set* {*i*<..<*j*} ! *n* = *Suc* (*i*+*n*)
by (*induction n arbitrary*: *i*) (*auto simp*: *sorted-list-of-set-greaterThanLessThan*)

lemma *nth-sorted-list-of-set-greaterThanAtMost*:
n < *j* − *i* ⟹ *sorted-list-of-set* {*i*<..*j*} ! *n* = *Suc* (*i*+*n*)
using *nth-sorted-list-of-set-greaterThanLessThan* [*of n Suc j i*]
by (*simp add*: *greaterThanAtMost-def greaterThanLessThan-eq lessThan-Suc-atMost*)

lemma *sorted-wrt-induct* [*consumes 1*, *case-names Nil Cons*]:
assumes *sorted-wrt R xs*
assumes *P* []
⋀*x xs*. (⋀*y*. *y* ∈ *set xs* ⟹ *R x y*) ⟹ *P xs* ⟹ *P* (*x* # *xs*)
shows *P xs*
using *assms*(*1*) **by** (*induction xs*) (*auto intro*: *assms*)

lemma *sorted-wrt-trans-induct* [*consumes 2*, *case-names Nil single Cons*]:

assumes *sorted-wrt R xs transp R*
assumes *P* [] $\bigwedge x.\ P\ [x]$
$\bigwedge x\ y\ xs.\ R\ x\ y \Longrightarrow P\ (y\ \#\ xs) \Longrightarrow P\ (x\ \#\ y\ \#\ xs)$
shows *P xs*
using *assms*(*1*)
by (*induction xs rule*: *induct-list012*)
(*auto intro*: *assms simp*: *sorted-wrt2*[*OF assms*(*2*)])

lemmas *sorted-induct* [*consumes 1*, *case-names Nil single Cons*] =
sorted-wrt-trans-induct[*OF - preorder-class.transp-on-le*]

lemma *sorted-wrt-map-mono*:
assumes *sorted-wrt R xs*
assumes $\bigwedge x\ y.\ x \in set\ xs \Longrightarrow y \in set\ xs \Longrightarrow R\ x\ y \Longrightarrow R'\ (f\ x)\ (f\ y)$
shows *sorted-wrt R′* (*map f xs*)
using *assms* **by** (*induction rule*: *sorted-wrt-induct*) *auto*

lemma *sorted-map-mono*:
assumes *sorted xs* **and** *mono-on* (*set xs*) *f*
shows *sorted* (*map f xs*)
using *assms*(*1*)
by (*rule sorted-wrt-map-mono*) (*use assms* **in** ‹*auto simp*: *mono-on-def*›)

66.3.6 *lists*: the list-forming operator over sets

inductive-set
lists :: *′a set* => *′a list set*
for *A* :: *′a set*
where
Nil [*intro*!, *simp*]: [] ∈ *lists A*
| *Cons* [*intro*!, *simp*]: ⟦*a* ∈ *A*; *l* ∈ *lists A*⟧ ⟹ *a*#*l* ∈ *lists A*

inductive-cases *listsE* [*elim*!]: *x*#*l* ∈ *lists A*
inductive-cases *listspE* [*elim*!]: *listsp A* (*x* # *l*)

inductive-simps *listsp-simps*[*code*]:
listsp A []
listsp A (*x* # *xs*)

lemma *listsp-mono* [*mono*]: *A* ≤ *B* ⟹ *listsp A* ≤ *listsp B*
by (*rule predicate1I*, *erule listsp.induct*, *blast*+)

lemmas *lists-mono* = *listsp-mono* [*to-set*]

lemma *listsp-infI*:
assumes *l*: *listsp A l* **shows** *listsp B l* ⟹ *listsp* (*inf A B*) *l* **using** *l*
by *induct blast*+

lemmas *lists-IntI* = *listsp-infI* [*to-set*]

lemma *listsp-inf-eq* [*simp*]: *listsp* (*inf A B*) = *inf* (*listsp A*) (*listsp B*)
proof (*rule mono-inf* [**where** *f=listsp*, *THEN order-antisym*])
  show *mono listsp* **by** (*simp add*: *mono-def listsp-mono*)
  show *inf* (*listsp A*) (*listsp B*) ≤ *listsp* (*inf A B*) **by** (*blast intro*!: *listsp-infI*)
qed

lemmas *listsp-conj-eq* [*simp*] = *listsp-inf-eq* [*simplified inf-fun-def inf-bool-def*]

lemmas *lists-Int-eq* [*simp*] = *listsp-inf-eq* [*to-set*]

lemma *Cons-in-lists-iff* [*simp*]: *x#xs* ∈ *lists A* ⟷ *x* ∈ *A* ∧ *xs* ∈ *lists A*
by *auto*

lemma *append-in-listsp-conv* [*iff*]: (*listsp A* (*xs* @ *ys*)) = (*listsp A xs* ∧ *listsp A ys*)
by (*induct xs*) *auto*

lemmas *append-in-lists-conv* [*iff*] = *append-in-listsp-conv* [*to-set*]

lemma *in-listsp-conv-set*: (*listsp A xs*) = (∀ *x* ∈ *set xs*. *A x*)
— eliminate *listsp* in favour of *set*
by (*induct xs*) *auto*

lemmas *in-lists-conv-set* [*code-unfold*] = *in-listsp-conv-set* [*to-set*]

lemma *in-listspD* [*dest*!]: *listsp A xs* ⟹ ∀ *x*∈*set xs*. *A x*
by (*rule in-listsp-conv-set* [*THEN iffD1*])

lemmas *in-listsD* [*dest*!] = *in-listspD* [*to-set*]

lemma *in-listspI* [*intro*!]: ∀ *x*∈*set xs*. *A x* ⟹ *listsp A xs*
by (*rule in-listsp-conv-set* [*THEN iffD2*])

lemmas *in-listsI* [*intro*!] = *in-listspI* [*to-set*]

lemma *mono-lists*: *mono lists*
unfolding *mono-def* **by** *auto*

lemma *lists-eq-set*: *lists A* = {*xs*. *set xs* ≤ *A*}
by *auto*

lemma *lists-empty* [*simp*]: *lists* {} = {[]}
by *auto*

lemma *lists-UNIV* [*simp*]: *lists UNIV* = *UNIV*
by *auto*

lemma *lists-image*: *lists* (*f*`*A*) = *map f* ` *lists A*

proof –
 { fix *xs* **have** ∀ *x*∈*set xs*. *x* ∈ *f* ' *A* ⟹ *xs* ∈ *map f* ' *lists A*
  by (*induct xs*) (*auto simp del*: *list.map simp add*: *list.map*[*symmetric*] *intro*!: *imageI*) **}**
 then show *?thesis* **by** *auto*
qed

lemma *inj-on-map-lists*: **assumes** *inj-on f A*
 shows *inj-on* (*map f*) (*lists A*)
proof
 fix *xs ys*
 assume *xs* ∈ *lists A* **and** *ys* ∈ *lists A* **and** *map f xs* = *map f ys*
 have *x* = *y* **if** *x* ∈ *set xs* **and** *y* ∈ *set ys* **and** *f x* = *f y* **for** *x y*
  using *in-listsD*[*OF* ‹*xs* ∈ *lists A*›, *rule-format*, *OF* ‹*x* ∈ *set xs*›]
   in-listsD[*OF* ‹*ys* ∈ *lists A*›, *rule-format*, *OF* ‹*y* ∈ *set ys*›]
   ‹*inj-on f A*›[*unfolded inj-on-def*, *rule-format*, *OF* - - ‹*f x* = *f y*›] **by** *blast*
 from *list.inj-map-strong*[*OF this* ‹*map f xs* = *map f ys*›]
 show *xs* = *ys*.
qed

lemma *bij-lists*: *bij-betw f X Y* ⟹ *bij-betw* (*map f*) (*lists X*) (*lists Y*)
unfolding *bij-betw-def* **using** *inj-on-map-lists lists-image* **by** *metis*

lemma *replicate-in-lists*: *a* ∈ *A* ⟹ *replicate k a* ∈ *lists A*
by (*induction k*) *auto*

66.3.7 Inductive definition for membership

inductive *ListMem* :: ′*a* ⇒ ′*a list* ⇒ *bool*
where
  elem: *ListMem x* (*x* # *xs*)
 | *insert*: *ListMem x xs* ⟹ *ListMem x* (*y* # *xs*)

lemma *ListMem-iff*: (*ListMem x xs*) = (*x* ∈ *set xs*)
proof
 show *ListMem x xs* ⟹ *x* ∈ *set xs*
  by (*induct set*: *ListMem*) *auto*
 show *x* ∈ *set xs* ⟹ *ListMem x xs*
  by (*induct xs*) (*auto intro*: *ListMem.intros*)
qed

66.3.8 Lists as Cartesian products

set-Cons A Xs: the set of lists with head drawn from *A* and tail drawn from *Xs*.

definition *set-Cons* :: ′*a set* ⇒ ′*a list set* ⇒ ′*a list set* **where**
set-Cons A XS = {*z*. ∃ *x xs*. *z* = *x* # *xs* ∧ *x* ∈ *A* ∧ *xs* ∈ *XS*}

lemma *set-Cons-sing-Nil* [*simp*]: *set-Cons A* {[]} = (%*x*. [*x*])'*A*

by (*auto simp add*: *set-Cons-def*)

Yields the set of lists, all of the same length as the argument and with elements drawn from the corresponding element of the argument.

primrec *listset* :: *'a set list* ⇒ *'a list set* **where**
listset [] = {[]} |
listset (*A* # *As*) = *set-Cons A* (*listset As*)

66.4 Relations on Lists

66.4.1 Length Lexicographic Ordering

These orderings preserve well-foundedness: shorter lists precede longer lists. These ordering are not used in dictionaries.

primrec — The lexicographic ordering for lists of the specified length
  lexn :: (*'a* × *'a*) *set* ⇒ *nat* ⇒ (*'a list* × *'a list*) *set* **where**
lexn r 0 = {} |
lexn r (*Suc n*) =
  (*map-prod* (%(*x*, *xs*). *x*#*xs*) (%(*x*, *xs*). *x*#*xs*) ‘ (*r* <*lex*> *lexn r n*)) *Int*
  {(*xs*, *ys*). *length xs* = *Suc n* ∧ *length ys* = *Suc n*}

definition *lex* :: (*'a* × *'a*) *set* ⇒ (*'a list* × *'a list*) *set* **where**
lex r = (⋃ *n*. *lexn r n*) — Holds only between lists of the same length

definition *lenlex* :: (*'a* × *'a*) *set* => (*'a list* × *'a list*) *set* **where**
lenlex r = *inv-image* (*less-than* <*lex*> *lex r*) (λ*xs*. (*length xs*, *xs*))
  — Compares lists by their length and then lexicographically

lemma *wf-lexn*: **assumes** *wf r* **shows** *wf* (*lexn r n*)
proof (*induct n*)
  case (*Suc n*)
  have *inj*: *inj* (λ(*x*, *xs*). *x* # *xs*)
    using *assms* **by** (*auto simp*: *inj-on-def*)
  have *wf*: *wf* (*map-prod* (λ(*x*, *xs*). *x* # *xs*) (λ(*x*, *xs*). *x* # *xs*) ‘ (*r* <*lex*> *lexn r n*))
    by (*simp add*: *Suc.hyps assms wf-lex-prod wf-map-prod-image* [*OF* - *inj*])
  then show *?case*
    by (*rule wf-subset*) *auto*
qed *auto*

lemma *lexn-length*:
  (*xs*, *ys*) ∈ *lexn r n* ⟹ *length xs* = *n* ∧ *length ys* = *n*
  by (*induct n arbitrary*: *xs ys*) *auto*

lemma *wf-lex* [*intro*!]:
  assumes *wf r* **shows** *wf* (*lex r*)
  unfolding *lex-def*
proof (*rule wf-UN*)
  show *wf* (*lexn r i*) **for** *i*

by (*simp add*: *assms wf-lexn*)
show $\bigwedge i\ j.\ lexn\ r\ i \neq lexn\ r\ j \Longrightarrow Domain\ (lexn\ r\ i) \cap Range\ (lexn\ r\ j) = \{\}$
by (*metis DomainE Int-emptyI RangeE lexn-length*)
qed

lemma *lexn-conv*:
$lexn\ r\ n =$
$\{(xs,ys).\ length\ xs = n \wedge length\ ys = n \wedge$
$(\exists xys\ x\ y\ xs'\ ys'.\ xs = xys\ @\ x\#xs' \wedge ys = xys\ @\ y\ \#\ ys' \wedge (x,\ y) \in r)\}$
(**is** $?L\ n = ?R\ n$ **is** $- = \{(xs,ys).\ ?len\ n\ xs \wedge ?len\ n\ ys \wedge (\exists xys.\ ?P\ xs\ ys\ xys)\}$)
proof (*induction n*)
case (*Suc n*)

have $(xs,ys) \in ?L\ (Suc\ n)$ **if** r: $(xs,ys) \in ?R\ (Suc\ n)$ **for** *xs ys*
proof −
from *r* **obtain** *xys* **where** r': $?len\ (Suc\ n)\ xs\ ?len\ (Suc\ n)\ ys\ ?P\ xs\ ys\ xys$ **by** *auto*
then show *?thesis*
using r' *Suc*
by (*cases xys*; *fastforce simp*: *image-Collect lex-prod-def*)
qed
moreover have $(xs,ys) \in ?L\ (Suc\ n) \Longrightarrow (xs,ys) \in ?R\ (Suc\ n)$ **for** *xs ys*
using *Suc* **by** (*auto simp add*: *image-Collect lex-prod-def*)(*blast*, *meson Cons-eq-appendI*)
ultimately show *?case* **by** (*meson pred-equals-eq2*)
qed *auto*

By Mathias Fleury:

proposition *lexn-transI*:
assumes *trans r* **shows** *trans* (*lexn r n*)
unfolding *trans-def*
proof (*intro allI impI*)
fix *as bs cs*
assume *asbs*: $(as,\ bs) \in lexn\ r\ n$ **and** *bscs*: $(bs,\ cs) \in lexn\ r\ n$
obtain *abs a b as′ bs′* **where**
n: $length\ as = n$ **and** $length\ bs = n$ **and**
as: $as = abs\ @\ a\ \#\ as'$ **and**
bs: $bs = abs\ @\ b\ \#\ bs'$ **and**
abr: $(a,\ b) \in r$
using *asbs* **unfolding** *lexn-conv* **by** *blast*
obtain *bcs b′ c′ cs′ bs′* **where**
n': $length\ cs = n$ **and** $length\ bs = n$ **and**
bs': $bs = bcs\ @\ b'\ \#\ bs'$ **and**
cs: $cs = bcs\ @\ c'\ \#\ cs'$ **and**
$b'c'r$: $(b',\ c') \in r$
using *bscs* **unfolding** *lexn-conv* **by** *blast*
consider (*le*) $length\ bcs < length\ abs$
| (*eq*) $length\ bcs = length\ abs$
| (*ge*) $length\ bcs > length\ abs$ **by** *linarith*
thus $(as,\ cs) \in lexn\ r\ n$

proof *cases*
  let *?k = length bcs*
  case *le*
  hence *as ! ?k = bs ! ?k* **unfolding** *as bs* **by** (*simp add*: *nth-append*)
  hence (*as ! ?k*, *cs ! ?k*) ∈ *r* **using** *b′c′r* **unfolding** *bs′ cs* **by** *auto*
  moreover
  have *length bcs < length as* **using** *le* **unfolding** *as* **by** *simp*
  from *id-take-nth-drop*[*OF this*]
  have *as = take ?k as @ as ! ?k # drop (Suc ?k) as* **.**
  moreover
  have *length bcs < length cs* **unfolding** *cs* **by** *simp*
  from *id-take-nth-drop*[*OF this*]
  have *cs = take ?k cs @ cs ! ?k # drop (Suc ?k) cs* **.**
  moreover have *take ?k as = take ?k cs*
    using *le arg-cong*[*OF bs*, *of take* (*length bcs*)]
    unfolding *cs as bs′* **by** *auto*
  ultimately show *?thesis* **using** *n n′* **unfolding** *lexn-conv* **by** *auto*
next
  let *?k = length abs*
  case *ge*
  hence *bs ! ?k = cs ! ?k* **unfolding** *bs′ cs* **by** (*simp add*: *nth-append*)
  hence (*as ! ?k*, *cs ! ?k*) ∈ *r* **using** *abr* **unfolding** *as bs* **by** *auto*
  moreover
  have *length abs < length as* **using** *ge* **unfolding** *as* **by** *simp*
  from *id-take-nth-drop*[*OF this*]
  have *as = take ?k as @ as ! ?k # drop (Suc ?k) as* **.**
  moreover have *length abs < length cs* **using** *n n′* **unfolding** *as* **by** *simp*
  from *id-take-nth-drop*[*OF this*]
  have *cs = take ?k cs @ cs ! ?k # drop (Suc ?k) cs* **.**
  moreover have *take ?k as = take ?k cs*
    using *ge arg-cong*[*OF bs′*, *of take* (*length abs*)]
    unfolding *cs as bs* **by** *auto*
  ultimately show *?thesis* **using** *n n′* **unfolding** *lexn-conv* **by** *auto*
next
  let *?k = length abs*
  case *eq*
  hence ∗: *abs = bcs b = b′* **using** *bs bs′* **by** *auto*
  hence (*a*, *c′*) ∈ *r*
    using *abr b′c′r assms* **unfolding** *trans-def* **by** *blast*
  with ∗ **show** *?thesis* **using** *n n′* **unfolding** *lexn-conv as bs cs* **by** *auto*
qed
qed

corollary *lex-transI*:
  assumes *trans r* **shows** *trans* (*lex r*)
  using *lexn-transI* [*OF assms*]
  by (*clarsimp simp add*: *lex-def trans-def*) (*metis lexn-length*)

lemma *lex-conv*:

lex r =
{(*xs*,*ys*). *length xs* = *length ys* ∧
(∃ *xys x y xs′ ys′*. *xs* = *xys* @ *x* # *xs′* ∧ *ys* = *xys* @ *y* # *ys′* ∧ (*x*, *y*) ∈ *r*)}
by (*force simp add*: *lex-def lexn-conv*)

lemma *wf-lenlex* [*intro*!]: *wf r* ⟹ *wf* (*lenlex r*)
by (*unfold lenlex-def*) *blast*

lemma *lenlex-conv*:
lenlex r = {(*xs*,*ys*). *length xs* < *length ys* ∨
length xs = *length ys* ∧ (*xs*, *ys*) ∈ *lex r*}
by (*auto simp add*: *lenlex-def Id-on-def lex-prod-def inv-image-def*)

lemma *total-lenlex*:
assumes *total r*
shows *total* (*lenlex r*)
proof −
have (*xs*,*ys*) ∈ *lexn r* (*length xs*) ∨ (*ys*,*xs*) ∈ *lexn r* (*length xs*)
if *xs* ≠ *ys* **and** *len*: *length xs* = *length ys* **for** *xs ys*
proof −
obtain *pre x xs′ y ys′* **where** *x*≠*y* **and** *xs*: *xs* = *pre* @ [*x*] @ *xs′* **and** *ys*: *ys* = *pre* @ [*y*] @*ys′*
by (*meson len* ‹*xs* ≠ *ys*› *same-length-different*)
then consider (*x*,*y*) ∈ *r* | (*y*,*x*) ∈ *r*
by (*meson UNIV-I assms total-on-def*)
then show *?thesis*
by *cases* (*use len* **in** ‹(*force simp add*: *lexn-conv xs ys*)+›)
qed
then show *?thesis*
by (*fastforce simp*: *lenlex-def total-on-def lex-def*)
qed

lemma *lenlex-transI* [*intro*]: *trans r* ⟹ *trans* (*lenlex r*)
unfolding *lenlex-def*
by (*meson lex-transI trans-inv-image trans-less-than trans-lex-prod*)

lemma *Nil-notin-lex* [*iff*]: ([], *ys*) ∉ *lex r*
by (*simp add*: *lex-conv*)

lemma *Nil2-notin-lex* [*iff*]: (*xs*, []) ∉ *lex r*
by (*simp add*:*lex-conv*)

lemma *Cons-in-lex* [*simp*]:
(*x* # *xs*, *y* # *ys*) ∈ *lex r* ⟷ (*x*, *y*) ∈ *r* ∧ *length xs* = *length ys* ∨ *x* = *y* ∧ (*xs*, *ys*) ∈ *lex r*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*simp add*: *lex-conv*) (*metis hd-append list.sel*(*1*) *list.sel*(*3*) *tl-append2*)

next
  assume *?rhs* **then show** *?lhs*
    by (*simp add*: *lex-conv*) (*blast intro*: *Cons-eq-appendI*)
qed

lemma *Nil-lenlex-iff1* [*simp*]: ([], *ns*) $\in$ *lenlex r* $\longleftrightarrow$ *ns* $\neq$ []
  and *Nil-lenlex-iff2* [*simp*]: (*ns*,[]) $\notin$ *lenlex r*
  by (*auto simp*: *lenlex-def*)

lemma *Cons-lenlex-iff*:
  ((*m* # *ms*, *n* # *ns*) $\in$ *lenlex r*) $\longleftrightarrow$
    length ms < *length ns*
  $\vee$ *length ms* = *length ns* $\wedge$ (*m*,*n*) $\in$ *r*
  $\vee$ (*m* = *n* $\wedge$ (*ms*,*ns*) $\in$ *lenlex r*)
  by (*auto simp*: *lenlex-def*)

lemma *lenlex-irreflexive*: ($\bigwedge$*x*. (*x*,*x*) $\notin$ *r*) $\Longrightarrow$ (*xs*,*xs*) $\notin$ *lenlex r*
  by (*induction xs*) (*auto simp add*: *Cons-lenlex-iff*)

lemma *lenlex-trans*:
    $[\![$(*x*,*y*) $\in$ *lenlex r*; (*y*,*z*) $\in$ *lenlex r*; *trans r*$]\!]$ $\Longrightarrow$ (*x*,*z*) $\in$ *lenlex r*
  by (*meson lenlex-transI transD*)

lemma *lenlex-length*: (*ms*, *ns*) $\in$ *lenlex r* $\Longrightarrow$ *length ms* $\leq$ *length ns*
  by (*auto simp*: *lenlex-def*)

lemma *lex-append-rightI*:
  (*xs*, *ys*) $\in$ *lex r* $\Longrightarrow$ *length vs* = *length us* $\Longrightarrow$ (*xs* @ *us*, *ys* @ *vs*) $\in$ *lex r*
  by (*fastforce simp*: *lex-def lexn-conv*)

lemma *lex-append-leftI*:
  (*ys*, *zs*) $\in$ *lex r* $\Longrightarrow$ (*xs* @ *ys*, *xs* @ *zs*) $\in$ *lex r*
  by (*induct xs*) *auto*

lemma *lex-append-leftD*:
  $\forall$*x*. (*x*,*x*) $\notin$ *r* $\Longrightarrow$ (*xs* @ *ys*, *xs* @ *zs*) $\in$ *lex r* $\Longrightarrow$ (*ys*, *zs*) $\in$ *lex r*
  by (*induct xs*) *auto*

lemma *lex-append-left-iff*:
  $\forall$*x*. (*x*,*x*) $\notin$ *r* $\Longrightarrow$ (*xs* @ *ys*, *xs* @ *zs*) $\in$ *lex r* $\longleftrightarrow$ (*ys*, *zs*) $\in$ *lex r*
  by(*metis lex-append-leftD lex-append-leftI*)

lemma *lex-take-index*:
  assumes (*xs*, *ys*) $\in$ *lex r*
  obtains *i* **where** *i* < *length xs* **and** *i* < *length ys* **and** *take i xs* = *take i ys*
    and (*xs* ! *i*, *ys* ! *i*) $\in$ *r*
proof −
  obtain *n us x xs'* *y ys'* **where** (*xs*, *ys*) $\in$ *lexn r n* **and** *length xs* = *n* **and** *length ys* = *n*

and *xs* = *us* @ *x* # *xs'* **and** *ys* = *us* @ *y* # *ys'* **and** (*x*, *y*) ∈ *r*
using *assms* **by** (*fastforce simp*: *lex-def lexn-conv*)
then show *?thesis* **by** (*intro that* [*of length us*]) *auto*
qed

lemma *irrefl-lex*: *irrefl r* ⟹ *irrefl* (*lex r*)
by (*meson irrefl-def lex-take-index*)

lemma *lexl-not-refl* [*simp*]: *irrefl r* ⟹ (*x*,*x*) ∉ *lex r*
by (*meson irrefl-def lex-take-index*)

66.4.2 Lexicographic Ordering

Classical lexicographic ordering on lists, ie. "a" < "ab" < "b". This ordering does *not* preserve well-foundedness. Author: N. Voelker, March 2005.

definition *lexord* :: (*'a* × *'a*) *set* ⇒ (*'a list* × *'a list*) *set* **where**
lexord r = {(*x*,*y*). ∃ *a v*. *y* = *x* @ *a* # *v* ∨
(∃ *u a b v w*. (*a*,*b*) ∈ *r* ∧ *x* = *u* @ (*a* # *v*) ∧ *y* = *u* @ (*b* # *w*))}

lemma *lexord-Nil-left*[*simp*]: ([],*y*) ∈ *lexord r* = (∃ *a x*. *y* = *a* # *x*)
by (*unfold lexord-def*, *induct-tac y*, *auto*)

lemma *lexord-Nil-right*[*simp*]: (*x*,[]) ∉ *lexord r*
by (*unfold lexord-def*, *induct-tac x*, *auto*)

lemma *lexord-cons-cons*[*simp*]:
(*a* # *x*, *b* # *y*) ∈ *lexord r* ⟷ (*a*,*b*)∈ *r* ∨ (*a* = *b* ∧ (*x*,*y*)∈ *lexord r*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*simp add*: *lexord-def*)
apply (*metis hd-append list.sel*(*1*) *list.sel*(*3*) *tl-append2*)
done
qed (*auto simp add*: *lexord-def*; (*blast* | *meson Cons-eq-appendI*))

lemmas *lexord-simps* = *lexord-Nil-left lexord-Nil-right lexord-cons-cons*

lemma *lexord-same-pref-iff*:
(*xs* @ *ys*, *xs* @ *zs*) ∈ *lexord r* ⟷ (∃ *x* ∈ *set xs*. (*x*,*x*) ∈ *r*) ∨ (*ys*, *zs*) ∈ *lexord r*
by(*induction xs*) *auto*

lemma *lexord-same-pref-if-irrefl*[*simp*]:
irrefl r ⟹ (*xs* @ *ys*, *xs* @ *zs*) ∈ *lexord r* ⟷ (*ys*, *zs*) ∈ *lexord r*
by (*simp add*: *irrefl-def lexord-same-pref-iff*)

lemma *lexord-append-rightI*: ∃ *b z*. *y* = *b* # *z* ⟹ (*x*, *x* @ *y*) ∈ *lexord r*
by (*metis append-Nil2 lexord-Nil-left lexord-same-pref-iff*)

lemma *lexord-append-left-rightI*:
$(a,b) \in r \Longrightarrow (u @ a \# x, u @ b \# y) \in lexord\ r$
by (*simp add*: *lexord-same-pref-iff*)

lemma *lexord-append-leftI*: $(u,v) \in lexord\ r \Longrightarrow (x @ u, x @ v) \in lexord\ r$
by (*simp add*: *lexord-same-pref-iff*)

lemma *lexord-append-leftD*:
$[\![(x @ u, x @ v) \in lexord\ r; (\forall a. (a,a) \notin r)]\!] \Longrightarrow (u,v) \in lexord\ r$
by (*simp add*: *lexord-same-pref-iff*)

lemma *lexord-take-index-conv*:
$((x,y) \in lexord\ r) =$
$((length\ x < length\ y \wedge take\ (length\ x)\ y = x) \vee$
$(\exists i.\ i < min(length\ x)(length\ y) \wedge take\ i\ x = take\ i\ y \wedge (x!i,y!i) \in r))$
proof −
have $(\exists a\ v.\ y = x @ a \# v) = (length\ x < length\ y \wedge take\ (length\ x)\ y = x)$
by (*metis Cons-nth-drop-Suc append-eq-conv-conj drop-all list.simps*(*3*) *not-le*)
moreover
have $(\exists u\ a\ b.\ (a, b) \in r \wedge (\exists v.\ x = u @ a \# v) \wedge (\exists w.\ y = u @ b \# w)) =$
$(\exists i<length\ x.\ i < length\ y \wedge take\ i\ x = take\ i\ y \wedge (x\ !\ i, y\ !\ i) \in r)$
(**is** *?L=?R*)
proof
show *?L⟹?R*
by (*metis append-eq-conv-conj drop-all leI list.simps*(*3*) *nth-append-length*)
show *?R⟹?L*
by (*metis id-take-nth-drop*)
qed
ultimately show *?thesis*
by (*auto simp*: *lexord-def Let-def*)
qed

— lexord is extension of partial ordering List.lex
lemma *lexord-lex*: $(x,y) \in lex\ r = ((x,y) \in lexord\ r \wedge length\ x = length\ y)$
proof (*induction x arbitrary*: *y*)
case (*Cons a x y*) **then show** *?case*
by (*cases y*) (*force+*)
qed *auto*

lemma *lexord-sufI*:
assumes $(u,w) \in lexord\ r$ $length\ w \le length\ u$
shows $(u@v,w@z) \in lexord\ r$
proof−
from *leD*[*OF assms*(*2*)] *assms*(*1*)[*unfolded lexord-take-index-conv*[*of u w r*] *min-absorb2*[*OF assms*(*2*)]]
obtain i **where** $take\ i\ u = take\ i\ w$ **and** $(u!i,w!i) \in r$ **and** $i < length\ w$
by *blast*
hence $((u@v)!i, (w@z)!i) \in r$
unfolding *nth-append* **using** *less-le-trans*[*OF* ‹$i < length\ w$› *assms*(*2*)] ‹$(u!i,w!i)$

∈ *r*›
  by *presburger*
 moreover have $i < min\ (length\ (u@v))\ (length\ (w@z))$
  using *assms*(*2*) ‹$i < length\ w$› **by** *simp*
 moreover have $take\ i\ (u@v) = take\ i\ (w@z)$
  using *assms*(*2*) ‹$i < length\ w$› ‹$take\ i\ u = take\ i\ w$› **by** *simp*
 ultimately show *?thesis*
  using *lexord-take-index-conv* **by** *blast*
qed

lemma *lexord-sufE*:
 assumes $(xs@zs,ys@qs) \in lexord\ r\ xs \neq ys\ length\ xs = length\ ys\ length\ zs = length\ qs$
 shows $(xs,ys) \in lexord\ r$
proof−
 obtain *i* **where** $i < length\ (xs@zs)$ **and** $i < length\ (ys@qs)$ **and** $take\ i\ (xs@zs) = take\ i\ (ys@qs)$
 and $((xs@zs)\ !\ i,\ (ys@qs)\ !\ i) \in r$
  using *assms*(*1*) *lex-take-index*[*unfolded lexord-lex*,*of xs* @ *zs ys* @ *qs r*]
   length-append[*of xs zs*, *unfolded assms*(*3*,*4*), *folded length-append*[*of ys qs*]]
  by *blast*
 have $length\ (take\ i\ xs) = length\ (take\ i\ ys)$
  by (*simp add*: *assms*(*3*))
 have $i < length\ xs$
  using *assms*(*2*,*3*) *le-less-linear take-all*[*of xs i*] *take-all*[*of ys i*]
   ‹$take\ i\ (xs\ @\ zs) = take\ i\ (ys\ @\ qs)$› *append-eq-append-conv take-append*
  by *metis*
 hence $(xs\ !\ i,\ ys\ !\ i) \in r$
  using ‹$((xs\ @\ zs)\ !\ i,\ (ys\ @\ qs)\ !\ i) \in r$› *assms*(*3*) **by** (*simp add*: *nth-append*)
 moreover have $take\ i\ xs = take\ i\ ys$
  using *assms*(*3*) ‹$take\ i\ (xs\ @\ zs) = take\ i\ (ys\ @\ qs)$› **by** *auto*
 ultimately show *?thesis*
  unfolding *lexord-take-index-conv* **using** ‹$i < length\ xs$› *assms*(*3*) **by** *fastforce*
qed

lemma *lexord-irreflexive*: $\forall x.\ (x,x) \notin r \Longrightarrow (xs,xs) \notin lexord\ r$
by (*induct xs*) *auto*

By René Thiemann:

lemma *lexord-partial-trans*:
 $(\bigwedge x\ y\ z.\ x \in set\ xs \Longrightarrow (x,y) \in r \Longrightarrow (y,z) \in r \Longrightarrow (x,z) \in r)$
  $\Longrightarrow (xs,ys) \in lexord\ r \Longrightarrow (ys,zs) \in lexord\ r \Longrightarrow (xs,zs) \in lexord\ r$
proof (*induct xs arbitrary*: *ys zs*)
 case *Nil*
 from *Nil*(*3*) **show** *?case* **unfolding** *lexord-def* **by** (*cases zs*, *auto*)
next
 case (*Cons x xs yys zzs*)
 from *Cons*(*3*) **obtain** *y ys* **where** *yys*: $yys = y\ \#\ ys$ **unfolding** *lexord-def*
  by (*cases yys*, *auto*)

note *Cons* = *Cons*[*unfolded yys*]
from *Cons*(*3*) **have** *one*: (*x*,*y*) ∈ *r* ∨ *x* = *y* ∧ (*xs*,*ys*) ∈ *lexord r* **by** *auto*
from *Cons*(*4*) **obtain** *z zs* **where** *zzs*: *zzs* = *z* # *zs* **unfolding** *lexord-def*
by (*cases zzs*, *auto*)
note *Cons* = *Cons*[*unfolded zzs*]
from *Cons*(*4*) **have** *two*: (*y*,*z*) ∈ *r* ∨ *y* = *z* ∧ (*ys*,*zs*) ∈ *lexord r* **by** *auto*
{
assume (*xs*,*ys*) ∈ *lexord r* **and** (*ys*,*zs*) ∈ *lexord r*
from *Cons*(*1*)[*OF* - *this*] *Cons*(*2*)
have (*xs*,*zs*) ∈ *lexord r* **by** *auto*
} **note** *ind1* = *this*
{
assume (*x*,*y*) ∈ *r* **and** (*y*,*z*) ∈ *r*
from *Cons*(*2*)[*OF* - *this*] **have** (*x*,*z*) ∈ *r* **by** *auto*
} **note** *ind2* = *this*
from *one two ind1 ind2*
have (*x*,*z*) ∈ *r* ∨ *x* = *z* ∧ (*xs*,*zs*) ∈ *lexord r* **by** *blast*
thus *?case* **unfolding** *zzs* **by** *auto*
qed

lemma *lexord-trans*:
⟦ (*x*, *y*) ∈ *lexord r*; (*y*, *z*) ∈ *lexord r*; *trans r* ⟧ ⟹ (*x*, *z*) ∈ *lexord r*
by(*auto simp*: *trans-def intro*:*lexord-partial-trans*)

lemma *lexord-transI*: *trans r* ⟹ *trans* (*lexord r*)
by (*meson lexord-trans transI*)

lemma *total-lexord*: *total r* ⟹ *total* (*lexord r*)
unfolding *total-on-def*
proof *clarsimp*
fix *x y*
assume ∀ *x y*. *x* ≠ *y* ⟶ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*
and (*x*::*'a list*) ≠ *y*
and (*y*, *x*) ∉ *lexord r*
then
show (*x*, *y*) ∈ *lexord r*
proof (*induction x arbitrary*: *y*)
case *Nil*
then show *?case*
by (*metis lexord-Nil-left list.exhaust*)
next
case (*Cons a x y*) **then show** *?case*
by (*cases y*) (*force*+)
qed
qed

corollary *lexord-linear*: (∀ *a b*. (*a*,*b*) ∈ *r* ∨ *a* = *b* ∨ (*b*,*a*) ∈ *r*) ⟹ (*x*,*y*) ∈ *lexord r* ∨ *x* = *y* ∨ (*y*,*x*) ∈ *lexord r*
using *total-lexord* **by** (*metis UNIV-I total-on-def*)

lemma *lexord-irrefl*:
  irrefl R ⟹ *irrefl* (*lexord R*)
by (*simp add*: *irrefl-def lexord-irreflexive*)

lemma *lexord-asym*:
  assumes *asym R*
  shows *asym* (*lexord R*)
proof
  fix *xs ys*
  assume (*xs*, *ys*) ∈ *lexord R*
  then show (*ys*, *xs*) ∉ *lexord R*
  proof (*induct xs arbitrary*: *ys*)
    case *Nil*
    then show *?case* **by** *simp*
  next
    case (*Cons x xs*)
    then obtain *z zs* **where** *ys*: *ys* = *z* # *zs* **by** (*cases ys*) *auto*
    with *assms Cons* **show** *?case* **by** (*auto dest*: *asymD*)
  qed
qed

lemma *lexord-asymmetric*:
  assumes *asym R*
  assumes *hyp*: (*a*, *b*) ∈ *lexord R*
  shows (*b*, *a*) ∉ *lexord R*
proof −
  from ‹*asym R*› **have** *asym* (*lexord R*) **by** (*rule lexord-asym*)
  then show *?thesis* **by** (*auto simp*: *hyp dest*: *asymD*)
qed

lemma *asym-lex*: *asym R* ⟹ *asym* (*lex R*)
  by (*meson asymI asymD irrefl-lex lexord-asym lexord-lex*)

lemma *asym-lenlex*: *asym R* ⟹ *asym* (*lenlex R*)
  by (*simp add*: *lenlex-def asym-inv-image asym-less-than asym-lex asym-lex-prod*)

lemma *lenlex-append1*:
  assumes *len*: (*us*,*xs*) ∈ *lenlex R* **and** *eq*: *length vs* = *length ys*
  shows (*us* @ *vs*, *xs* @ *ys*) ∈ *lenlex R*
  using *len*
proof (*induction us*)
  case *Nil*
  then show *?case*
    by (*simp add*: *lenlex-def eq*)
next
  case (*Cons u us*)
  with *lex-append-rightI* **show** *?case*
    by (*fastforce simp add*: *lenlex-def eq*)

qed

lemma *lenlex-append2* [*simp*]:
  assumes *irrefl R*
  shows (*us* @ *xs*, *us* @ *ys*) ∈ *lenlex R* ⟷ (*xs*, *ys*) ∈ *lenlex R*
proof (*induction us*)
  case *Nil*
  then show *?case*
    by (*simp add*: *lenlex-def*)
next
  case (*Cons u us*)
  with *assms* **show** *?case*
    by (*auto simp*: *lenlex-def irrefl-def*)
qed

Predicate version of lexicographic order integrated with Isabelle's order type classes. Author: Andreas Lochbihler

context *ord*
begin

context
  notes [[*inductive-internals*]]
begin

inductive *lexordp* :: *'a list* ⇒ *'a list* ⇒ *bool*
where
  Nil: *lexordp* [] (*y* # *ys*)
| *Cons*: *x* < *y* ⟹ *lexordp* (*x* # *xs*) (*y* # *ys*)
| *Cons-eq*:
  ⟦ ¬ *x* < *y*; ¬ *y* < *x*; *lexordp xs ys* ⟧ ⟹ *lexordp* (*x* # *xs*) (*y* # *ys*)

end

lemma *lexordp-simps* [*simp*, *code*]:
  lexordp [] *ys* = (*ys* ≠ [])
  lexordp xs [] = *False*
  lexordp (*x* # *xs*) (*y* # *ys*) ⟷ *x* < *y* ∨ ¬ *y* < *x* ∧ *lexordp xs ys*
by(*subst lexordp.simps*, *fastforce simp add*: *neq-Nil-conv*)+

inductive *lexordp-eq* :: *'a list* ⇒ *'a list* ⇒ *bool* **where**
  Nil: *lexordp-eq* [] *ys*
| *Cons*: *x* < *y* ⟹ *lexordp-eq* (*x* # *xs*) (*y* # *ys*)
| *Cons-eq*: ⟦ ¬ *x* < *y*; ¬ *y* < *x*; *lexordp-eq xs ys* ⟧ ⟹ *lexordp-eq* (*x* # *xs*) (*y* # *ys*)

lemma *lexordp-eq-simps* [*simp*, *code*]:
  lexordp-eq [] *ys* = *True*
  lexordp-eq xs [] ⟷ *xs* = []
  lexordp-eq (*x* # *xs*) [] = *False*
  lexordp-eq (*x* # *xs*) (*y* # *ys*) ⟷ *x* < *y* ∨ ¬ *y* < *x* ∧ *lexordp-eq xs ys*

by(*subst lexordp-eq.simps*, *fastforce*)+

lemma *lexordp-append-rightI*: $ys \neq Nil \Longrightarrow$ *lexordp xs* (*xs* @ *ys*)
  by(*induct xs*)(*auto simp add*: *neq-Nil-conv*)

lemma *lexordp-append-left-rightI*: $x < y \Longrightarrow$ *lexordp* (*us* @ *x* # *xs*) (*us* @ *y* # *ys*)
  by(*induct us*) *auto*

lemma *lexordp-eq-refl*: *lexordp-eq xs xs*
  by(*induct xs*) *simp-all*

lemma *lexordp-append-leftI*: *lexordp us vs* $\Longrightarrow$ *lexordp* (*xs* @ *us*) (*xs* @ *vs*)
  by(*induct xs*) *auto*

lemma *lexordp-append-leftD*: ⟦ *lexordp* (*xs* @ *us*) (*xs* @ *vs*); $\forall a. \neg a < a$ ⟧ $\Longrightarrow$ *lexordp us vs*
  by(*induct xs*) *auto*

lemma *lexordp-irreflexive*:
  assumes *irrefl*: $\forall x. \neg x < x$
  shows ¬ *lexordp xs xs*
proof
  assume *lexordp xs xs*
  thus *False* **by**(*induct xs ys≡xs*)(*simp-all add*: *irrefl*)
qed

lemma *lexordp-into-lexordp-eq*:
  lexordp xs ys $\Longrightarrow$ *lexordp-eq xs ys*
by (*induction rule*: *lexordp.induct*) *simp-all*

lemma *lexordp-eq-pref*: *lexordp-eq u* (*u* @ *v*)
by (*metis append-Nil2 lexordp-append-rightI lexordp-eq-refl lexordp-into-lexordp-eq*)

end

declare *ord.lexordp-simps* [*simp*, *code*]
declare *ord.lexordp-eq-simps* [*simp*, *code*]

context *order*
begin

lemma *lexordp-antisym*:
  assumes *lexordp xs ys lexordp ys xs*
  shows *False*
  using *assms* **by** *induct auto*

lemma *lexordp-irreflexive'*: ¬ *lexordp xs xs*
by(*rule lexordp-irreflexive*) *simp*

end

context *linorder* **begin**

lemma *lexordp-cases* [*consumes 1*, *case-names Nil Cons Cons-eq*, *cases pred*: *lexordp*]:
assumes *lexordp xs ys*
obtains (*Nil*) *y ys'* **where** *xs* = [] *ys* = *y* # *ys'*
| (*Cons*) *x xs' y ys'* **where** *xs* = *x* # *xs' ys* = *y* # *ys' x* < *y*
| (*Cons-eq*) *x xs' ys'* **where** *xs* = *x* # *xs' ys* = *x* # *ys' lexordp xs' ys'*
using *assms* **by** *cases* (*fastforce simp add*: *not-less-iff-gr-or-eq*)+

lemma *lexordp-induct* [*consumes 1*, *case-names Nil Cons Cons-eq*, *induct pred*: *lexordp*]:
assumes *major*: *lexordp xs ys*
and *Nil*: ⋀*y ys*. *P* [] (*y* # *ys*)
and *Cons*: ⋀*x xs y ys*. *x* < *y* ⟹ *P* (*x* # *xs*) (*y* # *ys*)
and *Cons-eq*: ⋀*x xs ys*. ⟦ *lexordp xs ys*; *P xs ys* ⟧ ⟹ *P* (*x* # *xs*) (*x* # *ys*)
shows *P xs ys*
using *major* **by** *induct* (*simp-all add*: *Nil Cons not-less-iff-gr-or-eq Cons-eq*)

lemma *lexordp-iff*:
lexordp xs ys ⟷ (∃ *x vs*. *ys* = *xs* @ *x* # *vs*) ∨ (∃ *us a b vs ws*. *a* < *b* ∧ *xs* = *us* @ *a* # *vs* ∧ *ys* = *us* @ *b* # *ws*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **thus** *?rhs*
proof *induct*
case *Cons-eq* **thus** *?case* **by** *simp* (*metis append.simps*(*2*))
qed(*fastforce intro*: *disjI2 del*: *disjCI intro*: *exI*[**where** *x*=[]])+
next
assume *?rhs* **thus** *?lhs*
by(*auto intro*: *lexordp-append-leftI*[**where** *us*=[], *simplified*] *lexordp-append-leftI*)
qed

lemma *lexordp-conv-lexord*:
lexordp xs ys ⟷ (*xs*, *ys*) ∈ *lexord* {(*x*, *y*). *x* < *y*}
by(*simp add*: *lexordp-iff lexord-def*)

lemma *lexordp-eq-antisym*:
assumes *lexordp-eq xs ys lexordp-eq ys xs*
shows *xs* = *ys*
using *assms* **by** *induct simp-all*

lemma *lexordp-eq-trans*:
assumes *lexordp-eq xs ys* **and** *lexordp-eq ys zs*
shows *lexordp-eq xs zs*
using *assms*
by (*induct arbitrary*: *zs*) (*case-tac zs*; *auto*)+

lemma *lexordp-trans*:
assumes *lexordp xs ys lexordp ys zs*
shows *lexordp xs zs*
using *assms*
by (*induct arbitrary*: *zs*) (*case-tac zs*; *auto*)+

lemma *lexordp-linear*: *lexordp xs ys* $\vee$ *xs* = *ys* $\vee$ *lexordp ys xs*
by(*induct xs arbitrary*: *ys*; *case-tac ys*; *fastforce*)

lemma *lexordp-conv-lexordp-eq*: *lexordp xs ys* $\longleftrightarrow$ *lexordp-eq xs ys* $\wedge$ $\neg$ *lexordp-eq ys xs*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
hence $\neg$ *lexordp-eq ys xs* **by** *induct simp-all*
with ‹*?lhs*› **show** *?rhs* **by** (*simp add*: *lexordp-into-lexordp-eq*)
next
assume *?rhs*
hence *lexordp-eq xs ys* $\neg$ *lexordp-eq ys xs* **by** *simp-all*
thus *?lhs* **by** *induct simp-all*
qed

lemma *lexordp-eq-conv-lexord*: *lexordp-eq xs ys* $\longleftrightarrow$ *xs* = *ys* $\vee$ *lexordp xs ys*
by(*auto simp add*: *lexordp-conv-lexordp-eq lexordp-eq-refl dest*: *lexordp-eq-antisym*)

lemma *lexordp-eq-linear*: *lexordp-eq xs ys* $\vee$ *lexordp-eq ys xs*
by (*induct xs arbitrary*: *ys*) (*case-tac ys*; *auto*)+

lemma *lexordp-linorder*: *class.linorder lexordp-eq lexordp*
by *unfold-locales*
(*auto simp add*: *lexordp-conv-lexordp-eq lexordp-eq-refl lexordp-eq-antisym intro*: *lexordp-eq-trans del*: *disjCI intro*: *lexordp-eq-linear*)

end

66.4.3 Lexicographic combination of measure functions

These are useful for termination proofs

definition *measures fs* = *inv-image* (*lex less-than*) (%*a*. *map* (%*f*. *f a*) *fs*)

lemma *wf-measures*[*simp*]: *wf* (*measures fs*)
unfolding *measures-def*
by *blast*

lemma *in-measures*[*simp*]:
(*x*, *y*) $\in$ *measures* [] = *False*
(*x*, *y*) $\in$ *measures* (*f* # *fs*)
= (*f x* < *f y* $\vee$ (*f x* = *f y* $\wedge$ (*x*, *y*) $\in$ *measures fs*))

unfolding *measures-def*
by *auto*

lemma *measures-less*: $f\ x < f\ y \Longrightarrow (x,\ y) \in measures\ (f\#fs)$
by *simp*

lemma *measures-lesseq*: $f\ x \leq f\ y \Longrightarrow (x,\ y) \in measures\ fs \Longrightarrow (x,\ y) \in measures\ (f\#fs)$
by *auto*

66.4.4 Lifting Relations to Lists: one element

definition *listrel1* :: $('a \times 'a)\ set \Rightarrow ('a\ list \times 'a\ list)\ set$ **where**
$listrel1\ r = \{(xs,ys).$
$\exists us\ z\ z'\ vs.\ xs = us\ @\ z\ \#\ vs \wedge (z,z') \in r \wedge ys = us\ @\ z'\ \#\ vs\}$

lemma *listrel1I*:
$[\![\ (x,\ y) \in r;\ \ xs = us\ @\ x\ \#\ vs;\ \ ys = us\ @\ y\ \#\ vs\ ]\!] \Longrightarrow$
$(xs,\ ys) \in listrel1\ r$
unfolding *listrel1-def* **by** *auto*

lemma *listrel1E*:
$[\![\ (xs,\ ys) \in listrel1\ r;$
$!!x\ y\ us\ vs.\ [\![\ (x,\ y) \in r;\ \ xs = us\ @\ x\ \#\ vs;\ \ ys = us\ @\ y\ \#\ vs\ ]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
unfolding *listrel1-def* **by** *auto*

lemma *not-Nil-listrel1* [*iff*]: $([],\ xs) \notin listrel1\ r$
unfolding *listrel1-def* **by** *blast*

lemma *not-listrel1-Nil* [*iff*]: $(xs,\ []) \notin listrel1\ r$
unfolding *listrel1-def* **by** *blast*

lemma *Cons-listrel1-Cons* [*iff*]:
$(x\ \#\ xs,\ y\ \#\ ys) \in listrel1\ r \longleftrightarrow$
$(x,y) \in r \wedge xs = ys \vee x = y \wedge (xs,\ ys) \in listrel1\ r$
by (*simp add*: *listrel1-def Cons-eq-append-conv*) (*blast*)

lemma *listrel1I1*: $(x,y) \in r \Longrightarrow (x\ \#\ xs,\ y\ \#\ xs) \in listrel1\ r$
by *fast*

lemma *listrel1I2*: $(xs,\ ys) \in listrel1\ r \Longrightarrow (x\ \#\ xs,\ x\ \#\ ys) \in listrel1\ r$
by *fast*

lemma *append-listrel1I*:
$(xs,\ ys) \in listrel1\ r \wedge us = vs \vee xs = ys \wedge (us,\ vs) \in listrel1\ r$
$\Longrightarrow (xs\ @\ us,\ ys\ @\ vs) \in listrel1\ r$
unfolding *listrel1-def*
by *auto* (*blast intro*: *append-eq-appendI*)+

lemma *Cons-listrel1E1* [*elim*!]:
assumes $(x \# xs,\ ys) \in listrel1\ r$
and $\bigwedge y.\ ys = y \# xs \Longrightarrow (x,\ y) \in r \Longrightarrow R$
and $\bigwedge zs.\ ys = x \# zs \Longrightarrow (xs,\ zs) \in listrel1\ r \Longrightarrow R$
shows R
using *assms* **by** (*cases ys*) *blast+*

lemma *Cons-listrel1E2* [*elim*!]:
assumes $(xs,\ y \# ys) \in listrel1\ r$
and $\bigwedge x.\ xs = x \# ys \Longrightarrow (x,\ y) \in r \Longrightarrow R$
and $\bigwedge zs.\ xs = y \# zs \Longrightarrow (zs,\ ys) \in listrel1\ r \Longrightarrow R$
shows R
using *assms* **by** (*cases xs*) *blast+*

lemma *snoc-listrel1-snoc-iff*:
$(xs\ @\ [x],\ ys\ @\ [y]) \in listrel1\ r$
$\longleftrightarrow (xs,\ ys) \in listrel1\ r \wedge x = y \vee xs = ys \wedge (x,y) \in r$ (**is** *?L* $\longleftrightarrow$ *?R*)
proof
assume *?L* **thus** *?R*
by (*fastforce simp*: *listrel1-def snoc-eq-iff-butlast butlast-append*)
next
assume *?R* **then show** *?L* **unfolding** *listrel1-def* **by** *force*
qed

lemma *listrel1-eq-len*: $(xs,ys) \in listrel1\ r \Longrightarrow length\ xs = length\ ys$
unfolding *listrel1-def* **by** *auto*

lemma *listrel1-mono*:
$r \subseteq s \Longrightarrow listrel1\ r \subseteq listrel1\ s$
unfolding *listrel1-def* **by** *blast*

lemma *listrel1-converse*: $listrel1\ (r^{-1}) = (listrel1\ r)^{-1}$
unfolding *listrel1-def* **by** *blast*

lemma *in-listrel1-converse*:
$(x,y) \in listrel1\ (r^{-1}) \longleftrightarrow (x,y) \in (listrel1\ r)^{-1}$
unfolding *listrel1-def* **by** *blast*

lemma *listrel1-iff-update*:
$(xs,ys) \in (listrel1\ r)$
$\longleftrightarrow (\exists\, y\ n.\ (xs\ !\ n,\ y) \in r \wedge n < length\ xs \wedge ys = xs[n{:=}y])$ (**is** *?L* $\longleftrightarrow$ *?R*)
proof
assume *?L*
then obtain $x\ y\ u\ v$ **where** $xs = u\ @\ x \# v\ \ ys = u\ @\ y \# v\ \ (x,y) \in r$
unfolding *listrel1-def* **by** *auto*
then have $ys = xs[length\ u := y]$ **and** $length\ u < length\ xs$
and $(xs\ !\ length\ u,\ y) \in r$ **by** *auto*

then show *?R* **by** *auto*
next
assume *?R*
then obtain *x y n* **where** $(xs!n,\ y) \in r$ $n < size\ xs$ $ys = xs[n{:=}y]$ $x = xs!n$
by *auto*
then obtain *u v* **where** $xs = u\ @\ x\ \#\ v$ **and** $ys = u\ @\ y\ \#\ v$ **and** $(x,\ y) \in r$
by (*auto intro*: *upd-conv-take-nth-drop id-take-nth-drop*)
then show *?L* **by** (*auto simp*: *listrel1-def*)
qed

Accessible part and wellfoundedness:

lemma *Cons-acc-listrel1I* [*intro*!]:
$x \in Wellfounded.acc\ r \Longrightarrow xs \in Wellfounded.acc\ (listrel1\ r) \Longrightarrow (x\ \#\ xs) \in Wellfounded.acc\ (listrel1\ r)$
proof (*induction arbitrary*: *xs set*: *Wellfounded.acc*)
case *outer*: (*1 u*)
show *?case*
proof (*induct xs rule*: *acc-induct*)
case *1*
show $xs \in Wellfounded.acc\ (listrel1\ r)$
by (*simp add*: *outer.prems*)
qed (*metis* (*no-types*, *lifting*) *Cons-listrel1E2 acc.simps outer.IH*)
qed

lemma *lists-accD*: $xs \in lists\ (Wellfounded.acc\ r) \Longrightarrow xs \in Wellfounded.acc\ (listrel1\ r)$
proof (*induct set*: *lists*)
case *Nil*
then show *?case*
by (*meson acc.intros not-listrel1-Nil*)
next
case (*Cons a l*)
then show *?case*
by *blast*
qed

lemma *lists-accI*: $xs \in Wellfounded.acc\ (listrel1\ r) \Longrightarrow xs \in lists\ (Wellfounded.acc\ r)$
proof (*induction set*: *Wellfounded.acc*)
case (*1 x*)
then have $\bigwedge u\ v.\ [\![u \in set\ x;\ (v,\ u) \in r]\!] \Longrightarrow v \in Wellfounded.acc\ r$
by (*metis in-lists-conv-set in-set-conv-decomp listrel1I*)
then show *?case*
by (*meson acc.intros in-listsI*)
qed

lemma *wf-listrel1-iff*[*simp*]: $wf(listrel1\ r) = wf\ r$
by (*auto simp*: *wf-iff-acc*
intro: *lists-accD lists-accI*[*THEN Cons-in-lists-iff*[*THEN iffD1*, *THEN con-*

junct1]])

66.4.5 Lifting Relations to Lists: all elements

inductive-set
listrel :: ($'a \times 'b$) *set* ⇒ ($'a$ *list* × $'b$ *list*) *set*
for *r* :: ($'a \times 'b$) *set*
where
Nil: ([],[]) ∈ *listrel r*
| *Cons*: ⟦(*x*,*y*) ∈ *r*; (*xs*,*ys*) ∈ *listrel r*⟧ ⟹ (*x*#*xs*, *y*#*ys*) ∈ *listrel r*

inductive-cases *listrel-Nil1* [*elim*!]: ([],*xs*) ∈ *listrel r*
inductive-cases *listrel-Nil2* [*elim*!]: (*xs*,[]) ∈ *listrel r*
inductive-cases *listrel-Cons1* [*elim*!]: (*y*#*ys*,*xs*) ∈ *listrel r*
inductive-cases *listrel-Cons2* [*elim*!]: (*xs*,*y*#*ys*) ∈ *listrel r*

lemma *listrel-eq-len*: (*xs*, *ys*) ∈ *listrel r* ⟹ *length xs* = *length ys*
by(*induct rule*: *listrel.induct*) *auto*

lemma *listrel-iff-zip* [*code-unfold*]: (*xs*,*ys*) ∈ *listrel r* ⟷
length xs = *length ys* ∧ (∀ (*x*,*y*) ∈ *set*(*zip xs ys*). (*x*,*y*) ∈ *r*) (**is** *?L* ⟷ *?R*)
proof
assume *?L* **thus** *?R* **by** *induct* (*auto intro*: *listrel-eq-len*)
next
assume *?R* **thus** *?L*
apply (*clarify*)
by (*induct rule*: *list-induct2*) (*auto intro*: *listrel.intros*)
qed

lemma *listrel-iff-nth*: (*xs*,*ys*) ∈ *listrel r* ⟷
length xs = *length ys* ∧ (∀ *n* < *length xs*. (*xs*!*n*, *ys*!*n*) ∈ *r*) (**is** *?L* ⟷ *?R*)
by (*auto simp add*: *all-set-conv-all-nth listrel-iff-zip*)

lemma *listrel-mono*: *r* ⊆ *s* ⟹ *listrel r* ⊆ *listrel s*
by (*meson listrel-iff-nth subrelI subset-eq*)

lemma *listrel-subset*:
assumes *r* ⊆ *A* × *A* **shows** *listrel r* ⊆ *lists A* × *lists A*
proof *clarify*
show *a* ∈ *lists A* ∧ *b* ∈ *lists A* **if** (*a*, *b*) ∈ *listrel r* **for** *a b*
using *that assms* **by** (*induction rule*: *listrel.induct*, *auto*)
qed

lemma *listrel-refl-on*:
assumes *refl-on A r* **shows** *refl-on* (*lists A*) (*listrel r*)
proof −
have *l* ∈ *lists A* ⟹ (*l*, *l*) ∈ *listrel r* **for** *l*
using *assms* **unfolding** *refl-on-def*

by (*induction l*, *auto intro*: *listrel.intros*)
then show *?thesis*
by (*meson assms listrel-subset refl-on-def*)
qed

lemma *listrel-sym*: *sym r* $\Longrightarrow$ *sym* (*listrel r*)
by (*simp add*: *listrel-iff-nth sym-def*)

lemma *listrel-trans*:
assumes *trans r* **shows** *trans* (*listrel r*)
proof −
have $(x, z) \in$ *listrel r* **if** $(x, y) \in$ *listrel r* $(y, z) \in$ *listrel r* **for** *x y z*
using *that*
proof *induction*
case (*Cons x y xs ys*)
then show *?case*
by *clarsimp* (*metis assms listrel.Cons listrel-iff-nth transD*)
qed *auto*
then show *?thesis*
using *transI* **by** *blast*
qed

theorem *equiv-listrel*: *equiv A r* $\Longrightarrow$ *equiv* (*lists A*) (*listrel r*)
by (*simp add*: *equiv-def listrel-refl-on listrel-sym listrel-trans*)

lemma *listrel-rtrancl-refl*[*iff*]: (*xs*,*xs*) $\in$ *listrel*(r^*)
using *listrel-refl-on*[*of UNIV*, *OF refl-rtrancl*]
by(*auto simp*: *refl-on-def*)

lemma *listrel-rtrancl-trans*:
⟦(*xs*,*ys*) $\in$ *listrel*(r^*); (*ys*,*zs*) $\in$ *listrel*(r^*)⟧ $\Longrightarrow$ (*xs*,*zs*) $\in$ *listrel*(r^*)
by (*metis listrel-trans trans-def trans-rtrancl*)

lemma *listrel-Nil* [*simp*]: *listrel r* `` {[]} = {[]}
by (*blast intro*: *listrel.intros*)

lemma *listrel-Cons*:
listrel r `` {*x*#*xs*} = *set-Cons* (*r*``{*x*}) (*listrel r* `` {*xs*})
by (*auto simp add*: *set-Cons-def intro*: *listrel.intros*)

Relating *listrel1*, *listrel* and closures:

lemma *listrel1-rtrancl-subset-rtrancl-listrel1*: *listrel1* (r^*) $\subseteq$ (*listrel1 r*)*
proof (*rule subrelI*)
fix *xs ys* **assume** *1*: (*xs*,*ys*) $\in$ *listrel1* (r^*)
{ fix *x y us vs*
have (*x*,*y*) $\in r^*$ $\Longrightarrow$ (*us* @ *x* # *vs*, *us* @ *y* # *vs*) $\in$ (*listrel1 r*)*
proof(*induct rule*: *rtrancl.induct*)
case *rtrancl-refl* **show** *?case* **by** *simp*
next

case *rtrancl-into-rtrancl* **thus** *?case*
by (*metis listrel1I rtrancl.rtrancl-into-rtrancl*)
qed **}**
thus (*xs*,*ys*) ∈ (*listrel1 r*)* **using** *1* **by**(*blast elim*: *listrel1E*)
qed

lemma *rtrancl-listrel1-eq-len*: (*x*,*y*) ∈ (*listrel1 r*)* ⟹ *length x* = *length y*
by (*induct rule*: *rtrancl.induct*) (*auto intro*: *listrel1-eq-len*)

lemma *rtrancl-listrel1-ConsI1*:
(*xs*,*ys*) ∈ (*listrel1 r*)* ⟹ (*x*#*xs*,*x*#*ys*) ∈ (*listrel1 r*)*
proof (*induction rule*: *rtrancl.induct*)
case (*rtrancl-into-rtrancl a b c*)
then show *?case*
by (*metis listrel1I2 rtrancl.rtrancl-into-rtrancl*)
qed *auto*

lemma *rtrancl-listrel1-ConsI2*:
(*x*,*y*) ∈ r^* ⟹ (*xs*, *ys*) ∈ (*listrel1 r*)* ⟹ (*x* # *xs*, *y* # *ys*) ∈ (*listrel1 r*)*
by (*meson in-mono listrel1I1 listrel1-rtrancl-subset-rtrancl-listrel1 rtrancl-listrel1-ConsI1 rtrancl-trans*)

lemma *listrel1-subset-listrel*:
r ⊆ *r'* ⟹ *refl r'* ⟹ *listrel1 r* ⊆ *listrel*(*r'*)
by(*auto elim*!: *listrel1E simp add*: *listrel-iff-zip set-zip refl-on-def*)

lemma *listrel-reflcl-if-listrel1*:
(*xs*,*ys*) ∈ *listrel1 r* ⟹ (*xs*,*ys*) ∈ *listrel*(r^*)
by(*erule listrel1E*)(*auto simp add*: *listrel-iff-zip set-zip*)

lemma *listrel-rtrancl-eq-rtrancl-listrel1*: *listrel* (r^*) = (*listrel1 r*)*
proof
{ **fix** *x y* **assume** (*x*,*y*) ∈ *listrel* (r^*)
then have (*x*,*y*) ∈ (*listrel1 r*)*
by *induct* (*auto intro*: *rtrancl-listrel1-ConsI2*) **}**
then show *listrel* (r^*) ⊆ (*listrel1 r*)*
by (*rule subrelI*)
next
show *listrel* (r^*) ⊇ (*listrel1 r*)*
proof(*rule subrelI*)
fix *xs ys* **assume** (*xs*,*ys*) ∈ (*listrel1 r*)*
then show (*xs*,*ys*) ∈ *listrel* (r^*)
proof *induct*
case *base* **show** *?case* **by**(*auto simp add*: *listrel-iff-zip set-zip*)
next
case (*step ys zs*)
thus *?case* **by** (*metis listrel-reflcl-if-listrel1 listrel-rtrancl-trans*)
qed
qed

qed

lemma *rtrancl-listrel1-if-listrel*:
$(xs,ys) \in listrel\ r \Longrightarrow (xs,ys) \in (listrel1\ r)^*$
by(*metis listrel-rtrancl-eq-rtrancl-listrel1 subsetD*[*OF listrel-mono*] *r-into-rtrancl subsetI*)

lemma *listrel-subset-rtrancl-listrel1*: $listrel\ r \subseteq (listrel1\ r)^*$
by(*fast intro*:*rtrancl-listrel1-if-listrel*)

66.5 Size function

lemma [*measure-function*]: $is\text{-}measure\ f \Longrightarrow is\text{-}measure\ (size\text{-}list\ f)$
by (*rule is-measure-trivial*)

lemma [*measure-function*]: $is\text{-}measure\ f \Longrightarrow is\text{-}measure\ (size\text{-}option\ f)$
by (*rule is-measure-trivial*)

lemma *size-list-estimation*[*termination-simp*]:
$x \in set\ xs \Longrightarrow y < f\ x \Longrightarrow y < size\text{-}list\ f\ xs$
by (*induct xs*) *auto*

lemma *size-list-estimation'*[*termination-simp*]:
$x \in set\ xs \Longrightarrow y \le f\ x \Longrightarrow y \le size\text{-}list\ f\ xs$
by (*induct xs*) *auto*

lemma *size-list-map*[*simp*]: $size\text{-}list\ f\ (map\ g\ xs) = size\text{-}list\ (f \circ g)\ xs$
by (*induct xs*) *auto*

lemma *size-list-append*[*simp*]: $size\text{-}list\ f\ (xs\ @\ ys) = size\text{-}list\ f\ xs + size\text{-}list\ f\ ys$
by (*induct xs*, *auto*)

lemma *size-list-pointwise*[*termination-simp*]:
$(\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x \le g\ x) \Longrightarrow size\text{-}list\ f\ xs \le size\text{-}list\ g\ xs$
by (*induct xs*) *force+*

66.6 Monad operation

definition $bind :: 'a\ list \Rightarrow ('a \Rightarrow 'b\ list) \Rightarrow 'b\ list$ **where**
$bind\ xs\ f = concat\ (map\ f\ xs)$

hide-const (**open**) *bind*

lemma *bind-simps* [*simp*]:
$List.bind\ []\ f = []$
$List.bind\ (x\ \#\ xs)\ f = f\ x\ @\ List.bind\ xs\ f$
by (*simp-all add*: *bind-def*)

lemma *list-bind-cong* [*fundef-cong*]:
assumes $xs = ys\ (\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x = g\ x)$

shows *List.bind xs f* = *List.bind ys g*
proof −
from *assms*(*2*) **have** *List.bind xs f* = *List.bind xs g*
by (*induction xs*) *simp-all*
with *assms*(*1*) **show** *?thesis* **by** *simp*
qed

lemma *set-list-bind*: *set* (*List.bind xs f*) = ($\bigcup$ *x*∈*set xs*. *set* (*f x*))
by (*induction xs*) *simp-all*

66.7 Code generation

Optional tail recursive version of *map*. Can avoid stack overflow in some target languages.

fun *map-tailrec-rev* :: (*'a* ⇒ *'b*) ⇒ *'a list* ⇒ *'b list* ⇒ *'b list* **where**
map-tailrec-rev f [] *bs* = *bs* |
map-tailrec-rev f (*a*#*as*) *bs* = *map-tailrec-rev f as* (*f a* # *bs*)

lemma *map-tailrec-rev*:
map-tailrec-rev f as bs = *rev*(*map f as*) @ *bs*
by(*induction as arbitrary*: *bs*) *simp-all*

definition *map-tailrec* :: (*'a* ⇒ *'b*) ⇒ *'a list* ⇒ *'b list* **where**
map-tailrec f as = *rev* (*map-tailrec-rev f as* [])

Code equation:

lemma *map-eq-map-tailrec*: *map* = *map-tailrec*
by(*simp add*: *fun-eq-iff map-tailrec-def map-tailrec-rev*)

66.7.1 Counterparts for set-related operations

definition *member* :: *'a list* ⇒ *'a* ⇒ *bool* **where**
[*code-abbrev*]: *member xs x* ⟷ *x* ∈ *set xs*

Use *member* only for generating executable code. Otherwise use *x* ∈ *set xs* instead — it is much easier to reason about.

lemma *member-rec* [*code*]:
member (*x* # *xs*) *y* ⟷ *x* = *y* ∨ *member xs y*
member [] *y* ⟷ *False*
by (*auto simp add*: *member-def*)

lemma *in-set-member* :
x ∈ *set xs* ⟷ *member xs x*
by (*simp add*: *member-def*)

lemmas *list-all-iff* [*code-abbrev*] = *fun-cong*[*OF list.pred-set*]

definition *list-ex* :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *bool* **where**

list-ex-iff [*code-abbrev*]: *list-ex P xs* ⟷ *Bex* (*set xs*) *P*

definition *list-ex1* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ *bool* **where**
list-ex1-iff [*code-abbrev*]: *list-ex1 P xs* ⟷ (∃! *x*. *x* ∈ *set xs* ∧ *P x*)

Usually you should prefer ∀ *x*∈*set xs*, ∃ *x*∈*set xs* and ∃!*x*. *x*∈*set xs* ∧ - over *list-all*, *list-ex* and *list-ex1* in specifications.

lemma *list-all-simps* [*code*]:
list-all P (*x* # *xs*) ⟷ *P x* ∧ *list-all P xs*
list-all P [] ⟷ *True*
by (*simp-all add*: *list-all-iff*)

lemma *list-ex-simps* [*simp*, *code*]:
list-ex P (*x* # *xs*) ⟷ *P x* ∨ *list-ex P xs*
list-ex P [] ⟷ *False*
by (*simp-all add*: *list-ex-iff*)

lemma *list-ex1-simps* [*simp*, *code*]:
list-ex1 P [] = *False*
list-ex1 P (*x* # *xs*) = (*if P x then list-all* (λ*y*. ¬ *P y* ∨ *x* = *y*) *xs else list-ex1 P xs*)
by (*auto simp add*: *list-ex1-iff list-all-iff*)

lemma *Ball-set-list-all*:
Ball (*set xs*) *P* ⟷ *list-all P xs*
by (*simp add*: *list-all-iff*)

lemma *Bex-set-list-ex*:
Bex (*set xs*) *P* ⟷ *list-ex P xs*
by (*simp add*: *list-ex-iff*)

lemma *list-all-append* [*simp*]:
list-all P (*xs* @ *ys*) ⟷ *list-all P xs* ∧ *list-all P ys*
by (*auto simp add*: *list-all-iff*)

lemma *list-ex-append* [*simp*]:
list-ex P (*xs* @ *ys*) ⟷ *list-ex P xs* ∨ *list-ex P ys*
by (*auto simp add*: *list-ex-iff*)

lemma *list-all-rev* [*simp*]:
list-all P (*rev xs*) ⟷ *list-all P xs*
by (*simp add*: *list-all-iff*)

lemma *list-ex-rev* [*simp*]:
list-ex P (*rev xs*) ⟷ *list-ex P xs*
by (*simp add*: *list-ex-iff*)

lemma *list-all-length*:
list-all P xs ⟷ (∀ *n* < *length xs*. *P* (*xs* ! *n*))

by (*auto simp add*: *list-all-iff set-conv-nth*)

lemma *list-ex-length*:
list-ex P xs $\longleftrightarrow$ ($\exists$ *n* < *length xs*. *P* (*xs* ! *n*))
by (*auto simp add*: *list-ex-iff set-conv-nth*)

lemmas *list-all-cong* [*fundef-cong*] = *list.pred-cong*

lemma *list-ex-cong* [*fundef-cong*]:
xs = *ys* $\Longrightarrow$ ($\bigwedge$*x*. *x* $\in$ *set ys* $\Longrightarrow$ *f x* = *g x*) $\Longrightarrow$ *list-ex f xs* = *list-ex g ys*
by (*simp add*: *list-ex-iff*)

definition *can-select* :: ('*a* $\Rightarrow$ *bool*) $\Rightarrow$ '*a set* $\Rightarrow$ *bool* **where**
[*code-abbrev*]: *can-select P A* = ($\exists$!*x*$\in$*A*. *P x*)

lemma *can-select-set-list-ex1* [*code*]:
can-select P (*set A*) = *list-ex1 P A*
by (*simp add*: *list-ex1-iff can-select-def*)

Executable checks for relations on sets

definition *listrel1p* :: ('*a* $\Rightarrow$ '*a* $\Rightarrow$ *bool*) $\Rightarrow$ '*a list* $\Rightarrow$ '*a list* $\Rightarrow$ *bool* **where**
listrel1p r xs ys = ((*xs*, *ys*) $\in$ *listrel1* {(*x*, *y*). *r x y*})

lemma [*code-unfold*]:
(*xs*, *ys*) $\in$ *listrel1 r* = *listrel1p* (λ*x y*. (*x*, *y*) $\in$ *r*) *xs ys*
unfolding *listrel1p-def* **by** *auto*

lemma [*code*]:
listrel1p r [] *xs* = *False*
listrel1p r xs [] = *False*
listrel1p r (*x* # *xs*) (*y* # *ys*) $\longleftrightarrow$
r x y $\wedge$ *xs* = *ys* $\vee$ *x* = *y* $\wedge$ *listrel1p r xs ys*
by (*simp add*: *listrel1p-def*)+

definition
lexordp :: ('*a* $\Rightarrow$ '*a* $\Rightarrow$ *bool*) $\Rightarrow$ '*a list* $\Rightarrow$ '*a list* $\Rightarrow$ *bool* **where**
lexordp r xs ys = ((*xs*, *ys*) $\in$ *lexord* {(*x*, *y*). *r x y*})

lemma [*code-unfold*]:
(*xs*, *ys*) $\in$ *lexord r* = *lexordp* (λ*x y*. (*x*, *y*) $\in$ *r*) *xs ys*
unfolding *lexordp-def* **by** *auto*

lemma [*code*]:
lexordp r xs [] = *False*
lexordp r [] (*y*#*ys*) = *True*
lexordp r (*x* # *xs*) (*y* # *ys*) = (*r x y* $\vee$ (*x* = *y* $\wedge$ *lexordp r xs ys*))
unfolding *lexordp-def* **by** *auto*

Bounded quantification and summation over nats.

lemma *atMost-upto* [*code-unfold*]:
{..*n*} = *set* [*0*..<*Suc n*]
by *auto*

lemma *atLeast-upt* [*code-unfold*]:
{..<*n*} = *set* [*0*..<*n*]
by *auto*

lemma *greaterThanLessThan-upt* [*code-unfold*]:
{*n*<..<*m*} = *set* [*Suc n*..<*m*]
by *auto*

lemmas *atLeastLessThan-upt* [*code-unfold*] = *set-upt* [*symmetric*]

lemma *greaterThanAtMost-upt* [*code-unfold*]:
{*n*<..*m*} = *set* [*Suc n*..<*Suc m*]
by *auto*

lemma *atLeastAtMost-upt* [*code-unfold*]:
{*n*..*m*} = *set* [*n*..<*Suc m*]
by *auto*

lemma *all-nat-less-eq* [*code-unfold*]:
$(\forall m<n::nat.\ P\ m) \longleftrightarrow (\forall m \in \{0..<n\}.\ P\ m)$
by *auto*

lemma *ex-nat-less-eq* [*code-unfold*]:
$(\exists m<n::nat.\ P\ m) \longleftrightarrow (\exists m \in \{0..<n\}.\ P\ m)$
by *auto*

lemma *all-nat-less* [*code-unfold*]:
$(\forall m\leq n::nat.\ P\ m) \longleftrightarrow (\forall m \in \{0..n\}.\ P\ m)$
by *auto*

lemma *ex-nat-less* [*code-unfold*]:
$(\exists m\leq n::nat.\ P\ m) \longleftrightarrow (\exists m \in \{0..n\}.\ P\ m)$
by *auto*

Bounded *LEAST* operator:

definition *Bleast S P* = (*LEAST x*. $x \in S \wedge P\ x$)

definition *abort-Bleast S P* = (*LEAST x*. $x \in S \wedge P\ x$)

declare [[*code abort*: *abort-Bleast*]]

lemma *Bleast-code* [*code*]:
Bleast (*set xs*) *P* = (*case filter P* (*sort xs*) *of*
x#*xs* ⇒ *x* |
[] ⇒ *abort-Bleast* (*set xs*) *P*)

proof (*cases filter P* (*sort xs*))
  case *Nil* **thus** *?thesis* **by** (*simp add*: *Bleast-def abort-Bleast-def*)
next
  case (*Cons x ys*)
  have (*LEAST x*. $x \in$ *set xs* $\wedge$ *P x*) = *x*
  proof (*rule Least-equality*)
    show $x \in$ *set xs* $\wedge$ *P x*
      by (*metis Cons Cons-eq-filter-iff in-set-conv-decomp set-sort*)
    next
      fix *y* **assume** $y \in$ *set xs* $\wedge$ *P y*
      hence $y \in$ *set* (*filter P xs*) **by** *auto*
      thus $x \leq y$
          by (*metis Cons eq-iff filter-sort set-ConsD set-sort sorted-wrt.simps*(*2*) *sorted-sort*)
  qed
  thus *?thesis* **using** *Cons* **by** (*simp add*: *Bleast-def*)
qed

declare *Bleast-def*[*symmetric*, *code-unfold*]

Summation over ints.

lemma *greaterThanLessThan-upto* [*code-unfold*]:
  {*i*<..<*j*::*int*} = *set* [*i*+*1*..*j* − *1*]
by *auto*

lemma *atLeastLessThan-upto* [*code-unfold*]:
  {*i*..<*j*::*int*} = *set* [*i*..*j* − *1*]
by *auto*

lemma *greaterThanAtMost-upto* [*code-unfold*]:
  {*i*<..*j*::*int*} = *set* [*i*+*1*..*j*]
by *auto*

lemmas *atLeastAtMost-upto* [*code-unfold*] = *set-upto* [*symmetric*]

66.7.2 Optimizing by rewriting

definition *null* :: *'a list* ⇒ *bool* **where**
  [*code-abbrev*]: *null xs* ⟷ *xs* = []

Efficient emptyness check is implemented by *null*.

lemma *null-rec* [*code*]:
  null (*x* # *xs*) ⟷ *False*
  null [] ⟷ *True*
  by (*simp-all add*: *null-def*)

lemma *eq-Nil-null*:
  xs = [] ⟷ *null xs*
  by (*simp add*: *null-def*)

lemma *equal-Nil-null* [*code-unfold*]:
HOL.equal xs [] ⟷ *null xs*
HOL.equal [] = *null*
by (*auto simp add*: *equal null-def*)

definition *maps* :: (*'a* ⇒ *'b list*) ⇒ *'a list* ⇒ *'b list* **where**
[*code-abbrev*]: *maps f xs* = *concat* (*map f xs*)

definition *map-filter* :: (*'a* ⇒ *'b option*) ⇒ *'a list* ⇒ *'b list* **where**
[*code-post*]: *map-filter f xs* = *map* (*the* ∘ *f*) (*filter* (λ*x*. *f x* ≠ *None*) *xs*)

Operations *maps* and *map-filter* avoid intermediate lists on execution – do not use for proving.

lemma *maps-simps* [*code*]:
maps f (*x* # *xs*) = *f x* @ *maps f xs*
maps f [] = []
by (*simp-all add*: *maps-def*)

lemma *map-filter-simps* [*code*]:
map-filter f (*x* # *xs*) = (*case f x of None* ⇒ *map-filter f xs* | *Some y* ⇒ *y* # *map-filter f xs*)
map-filter f [] = []
by (*simp-all add*: *map-filter-def split*: *option.split*)

lemma *concat-map-maps*:
concat (*map f xs*) = *maps f xs*
by (*simp add*: *maps-def*)

lemma *map-filter-map-filter* [*code-unfold*]:
map f (*filter P xs*) = *map-filter* (λ*x*. *if P x then Some* (*f x*) *else None*) *xs*
by (*simp add*: *map-filter-def*)

Optimized code for ∀ *i*∈{*a*..*b*::*int*} and ∀ *n*:{*a*..<*b*::*nat*} and similiarly for ∃.

definition *all-interval-nat* :: (*nat* ⇒ *bool*) ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
all-interval-nat P i j ⟷ (∀ *n* ∈ {*i*..<*j*}. *P n*)

lemma [*code*]:
all-interval-nat P i j ⟷ *i* ≥ *j* ∨ *P i* ∧ *all-interval-nat P* (*Suc i*) *j*
proof –
have ∗: ⋀*n*. *P i* ⟹ ∀ *n*∈{*Suc i*..<*j*}. *P n* ⟹ *i* ≤ *n* ⟹ *n* < *j* ⟹ *P n*
using *le-less-Suc-eq* **by** *fastforce*
show *?thesis* **by** (*auto simp add*: *all-interval-nat-def intro*: ∗)
qed

lemma *list-all-iff-all-interval-nat* [*code-unfold*]:
list-all P [*i*..<*j*] ⟷ *all-interval-nat P i j*
by (*simp add*: *list-all-iff all-interval-nat-def*)

lemma *list-ex-iff-not-all-inverval-nat* [*code-unfold*]:
list-ex P [*i..<j*] $\longleftrightarrow$ $\neg$ (*all-interval-nat* (*Not* $\circ$ *P*) *i j*)
by (*simp add*: *list-ex-iff all-interval-nat-def*)

definition *all-interval-int* :: (*int* $\Rightarrow$ *bool*) $\Rightarrow$ *int* $\Rightarrow$ *int* $\Rightarrow$ *bool* **where**
all-interval-int P i j $\longleftrightarrow$ ($\forall$ *k* $\in$ {*i..j*}. *P k*)

lemma [*code*]:
all-interval-int P i j $\longleftrightarrow$ *i* > *j* $\vee$ *P i* $\wedge$ *all-interval-int P* (*i* + *1*) *j*
proof −
have ∗: $\bigwedge$*k*. *P i* $\Longrightarrow$ $\forall$ *k*$\in${*i*+*1* ..*j*}. *P k* $\Longrightarrow$ *i* $\leq$ *k* $\Longrightarrow$ *k* $\leq$ *j* $\Longrightarrow$ *P k*
by (*smt* (*verit*, *best*) *atLeastAtMost-iff*)
show *?thesis* **by** (*auto simp add*: *all-interval-int-def intro*: ∗)
qed

lemma *list-all-iff-all-interval-int* [*code-unfold*]:
list-all P [*i..j*] $\longleftrightarrow$ *all-interval-int P i j*
by (*simp add*: *list-all-iff all-interval-int-def*)

lemma *list-ex-iff-not-all-inverval-int* [*code-unfold*]:
list-ex P [*i..j*] $\longleftrightarrow$ $\neg$ (*all-interval-int* (*Not* $\circ$ *P*) *i j*)
by (*simp add*: *list-ex-iff all-interval-int-def*)

optimized code (tail-recursive) for *length*

definition *gen-length* :: *nat* $\Rightarrow$ *'a list* $\Rightarrow$ *nat*
where *gen-length n xs* = *n* + *length xs*

lemma *gen-length-code* [*code*]:
gen-length n [] = *n*
gen-length n (*x* # *xs*) = *gen-length* (*Suc n*) *xs*
by(*simp-all add*: *gen-length-def*)

declare *list.size*(*3*−*4*)[*code del*]

lemma *length-code* [*code*]: *length* = *gen-length 0*
by(*simp add*: *gen-length-def fun-eq-iff*)

hide-const (**open**) *member null maps map-filter all-interval-nat all-interval-int gen-length*

66.7.3 Pretty lists

ML ‹
(∗ *Code generation for list literals.* ∗)

signature LIST-CODE =
sig
val add-literal-list: *string* −> *theory* −> *theory*
end;

```
structure List-Code : LIST-CODE =
struct

open Basic-Code-Thingol;

fun implode-list t =
  let
    fun dest-cons (IConst { sym = Code-Symbol.Constant const-name ‹Cons›, ...
} `$ t1 `$ t2) = SOME (t1, t2)
      | dest-cons - = NONE;
    val (ts, t') = Code-Thingol.unfoldr dest-cons t;
  in case t'
   of IConst { sym = Code-Symbol.Constant const-name ‹Nil›, ... } => SOME
ts
    | - => NONE
  end;

fun print-list (target-fxy, target-cons) pr fxy t1 t2 =
  Code-Printer.brackify-infix (target-fxy, Code-Printer.R) fxy (
    pr (Code-Printer.INFX (target-fxy, Code-Printer.X)) t1,
    Code-Printer.str target-cons,
    pr (Code-Printer.INFX (target-fxy, Code-Printer.R)) t2
  );

fun add-literal-list target =
  let
    fun pretty literals pr - vars fxy [(t1, -), (t2, -)] =
      case Option.map (cons t1) (implode-list t2)
       of SOME ts =>
            Code-Printer.literal-list literals (map (pr vars Code-Printer.NOBR) ts)
        | NONE =>
            print-list (Code-Printer.infix-cons literals) (pr vars) fxy t1 t2;
  in
    Code-Target.set-printings (Code-Symbol.Constant (const-name ‹Cons›,
      [(target, SOME (Code-Printer.complex-const-syntax (2, pretty)))]))
  end

end;
›
```

code-printing
  type-constructor *list* ⇀
    (*SML*) - *list*
    and (*OCaml*) - *list*
    and (*Haskell*) ![(-)]
    and (*Scala*) *List*[(-)]
| **constant** *Nil* ⇀
    (*SML*) []

and (*OCaml*) []
and (*Haskell*) []
and (*Scala*) !*Nil*
| **class-instance** *list* :: *equal* ⇀
(*Haskell*) −
| **constant** *HOL.equal* :: *'a list* ⇒ *'a list* ⇒ *bool* ⇀
(*Haskell*) **infix** *4* ==

setup ‹*fold* (*List-Code.add-literal-list*) [*SML*, *OCaml*, *Haskell*, *Scala*]›

code-reserved
(*SML*) *list*
and (*OCaml*) *list*

66.7.4 Use convenient predefined operations

code-printing
constant (@) ⇀
(*SML*) **infixr** *7* @
and (*OCaml*) **infixr** *6* @
and (*Haskell*) **infixr** *5* ++
and (*Scala*) **infixl** *7* ++
| **constant** *map* ⇀
(*Haskell*) *map*
| **constant** *filter* ⇀
(*Haskell*) *filter*
| **constant** *concat* ⇀
(*Haskell*) *concat*
| **constant** *List.maps* ⇀
(*Haskell*) *concatMap*
| **constant** *rev* ⇀
(*Haskell*) *reverse*
| **constant** *zip* ⇀
(*Haskell*) *zip*
| **constant** *List.null* ⇀
(*Haskell*) *null*
| **constant** *takeWhile* ⇀
(*Haskell*) *takeWhile*
| **constant** *dropWhile* ⇀
(*Haskell*) *dropWhile*
| **constant** *list-all* ⇀
(*Haskell*) *all*
| **constant** *list-ex* ⇀
(*Haskell*) *any*

66.7.5 Implementation of sets by lists

lemma *is-empty-set* [*code*]:
Set.is-empty (*set xs*) ⟷ *List.null xs*
by (*simp add*: *Set.is-empty-def null-def*)

lemma *empty-set* [*code*]:
$\{\} = set\ []$
by *simp*

lemma *UNIV-coset* [*code*]:
$UNIV = List.coset\ []$
by *simp*

lemma *compl-set* [*code*]:
$-\ set\ xs = List.coset\ xs$
by *simp*

lemma *compl-coset* [*code*]:
$-\ List.coset\ xs = set\ xs$
by *simp*

lemma [*code*]:
$x \in set\ xs \longleftrightarrow List.member\ xs\ x$
$x \in List.coset\ xs \longleftrightarrow \neg\ List.member\ xs\ x$
by (*simp-all add*: *member-def*)

lemma *insert-code* [*code*]:
$insert\ x\ (set\ xs) = set\ (List.insert\ x\ xs)$
$insert\ x\ (List.coset\ xs) = List.coset\ (removeAll\ x\ xs)$
by *simp-all*

lemma *remove-code* [*code*]:
$Set.remove\ x\ (set\ xs) = set\ (removeAll\ x\ xs)$
$Set.remove\ x\ (List.coset\ xs) = List.coset\ (List.insert\ x\ xs)$
by (*simp-all add*: *remove-def Compl-insert*)

lemma *filter-set* [*code*]:
$Set.filter\ P\ (set\ xs) = set\ (filter\ P\ xs)$
by *auto*

lemma *image-set* [*code*]:
$image\ f\ (set\ xs) = set\ (map\ f\ xs)$
by *simp*

lemma *subset-code* [*code*]:
$set\ xs \leq B \longleftrightarrow (\forall x{\in}set\ xs.\ x \in B)$
$A \leq List.coset\ ys \longleftrightarrow (\forall y{\in}set\ ys.\ y \notin A)$
$List.coset\ [] \subseteq set\ [] \longleftrightarrow False$
by *auto*

A frequent case – avoid intermediate sets

lemma [*code-unfold*]:
$set\ xs \subseteq set\ ys \longleftrightarrow list\text{-}all\ (\lambda x.\ x \in set\ ys)\ xs$

by (*auto simp*: *list-all-iff*)

lemma *Ball-set* [*code*]:
Ball (*set xs*) *P* ⟷ *list-all P xs*
by (*simp add*: *list-all-iff*)

lemma *Bex-set* [*code*]:
Bex (*set xs*) *P* ⟷ *list-ex P xs*
by (*simp add*: *list-ex-iff*)

lemma *card-set* [*code*]:
card (*set xs*) = *length* (*remdups xs*)
by (*simp add*: *length-remdups-card-conv*)

lemma *the-elem-set* [*code*]:
the-elem (*set* [*x*]) = *x*
by *simp*

lemma *Pow-set* [*code*]:
Pow (*set* []) = {{}}
Pow (*set* (*x* # *xs*)) = (*let A* = *Pow* (*set xs*) *in A* ∪ *insert x* ' *A*)
by (*simp-all add*: *Pow-insert Let-def*)

definition *map-project* :: (*'a* ⇒ *'b option*) ⇒ *'a set* ⇒ *'b set* **where**
map-project f A = {*b*. ∃ *a* ∈ *A*. *f a* = *Some b*}

lemma [*code*]:
map-project f (*set xs*) = *set* (*List.map-filter f xs*)
by (*auto simp add*: *map-project-def map-filter-def image-def*)

hide-const (**open**) *map-project*

Operations on relations

lemma *product-code* [*code*]:
Product-Type.product (*set xs*) (*set ys*) = *set* [(*x*, *y*). *x* ← *xs*, *y* ← *ys*]
by (*auto simp add*: *Product-Type.product-def*)

lemma *Id-on-set* [*code*]:
Id-on (*set xs*) = *set* [(*x*, *x*). *x* ← *xs*]
by (*auto simp add*: *Id-on-def*)

lemma [*code*]:
R '' *S* = *List.map-project* (λ(*x*, *y*). *if x* ∈ *S then Some y else None*) *R*
unfolding *map-project-def* **by** (*auto split*: *prod.split if-split-asm*)

lemma *trancl-set-ntrancl* [*code*]:
trancl (*set xs*) = *ntrancl* (*card* (*set xs*) − *1*) (*set xs*)
by (*simp add*: *finite-trancl-ntranl*)

lemma *set-relcomp* [*code*]:
set xys O set yzs = *set* ([(*fst xy*, *snd yz*). *xy* ← *xys*, *yz* ← *yzs*, *snd xy* = *fst yz*])
by *auto* (*auto simp add*: *Bex-def image-def*)

lemma *wf-set*:
wf (*set xs*) = *acyclic* (*set xs*)
by (*simp add*: *wf-iff-acyclic-if-finite*)

lemma *wf-code-set*[*code*]: *wf-code* (*set xs*) = *acyclic* (*set xs*)
unfolding *wf-code-def* **using** *wf-set* **.**

66.8 Setup for Lifting/Transfer

66.8.1 Transfer rules for the Transfer package

context includes *lifting-syntax*
begin

lemma *tl-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *tl tl*
unfolding *tl-def*[*abs-def*] **by** *transfer-prover*

lemma *butlast-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *butlast butlast*
by (*rule rel-funI*, *erule list-all2-induct*, *auto*)

lemma *map-rec*: *map f xs* = *rec-list Nil* (%*x* - *y*. *Cons* (*f x*) *y*) *xs*
by (*induct xs*) *auto*

lemma *append-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A* ===> *list-all2 A*) *append append*
unfolding *List.append-def* **by** *transfer-prover*

lemma *rev-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *rev rev*
unfolding *List.rev-def* **by** *transfer-prover*

lemma *filter-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *list-all2 A*) *filter filter*
unfolding *List.filter-def* **by** *transfer-prover*

lemma *fold-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> *B*) ===> *list-all2 A* ===> *B* ===> *B*) *fold fold*
unfolding *List.fold-def* **by** *transfer-prover*

lemma *foldr-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> *B*) ===> *list-all2 A* ===> *B* ===> *B*) *foldr foldr*
unfolding *List.foldr-def* **by** *transfer-prover*

lemma *foldl-transfer* [*transfer-rule*]:

((B ===> A ===> B) ===> B ===> list-all2 A ===> B) foldl foldl
unfolding List.foldl-def **by** transfer-prover

lemma concat-transfer [transfer-rule]:
(list-all2 (list-all2 A) ===> list-all2 A) concat concat
unfolding List.concat-def **by** transfer-prover

lemma drop-transfer [transfer-rule]:
((=) ===> list-all2 A ===> list-all2 A) drop drop
unfolding List.drop-def **by** transfer-prover

lemma take-transfer [transfer-rule]:
((=) ===> list-all2 A ===> list-all2 A) take take
unfolding List.take-def **by** transfer-prover

lemma list-update-transfer [transfer-rule]:
(list-all2 A ===> (=) ===> A ===> list-all2 A) list-update list-update
unfolding list-update-def **by** transfer-prover

lemma takeWhile-transfer [transfer-rule]:
((A ===> (=)) ===> list-all2 A ===> list-all2 A) takeWhile takeWhile
unfolding takeWhile-def **by** transfer-prover

lemma dropWhile-transfer [transfer-rule]:
((A ===> (=)) ===> list-all2 A ===> list-all2 A) dropWhile dropWhile
unfolding dropWhile-def **by** transfer-prover

lemma zip-transfer [transfer-rule]:
(list-all2 A ===> list-all2 B ===> list-all2 (rel-prod A B)) zip zip
unfolding zip-def **by** transfer-prover

lemma product-transfer [transfer-rule]:
(list-all2 A ===> list-all2 B ===> list-all2 (rel-prod A B)) List.product List.product
unfolding List.product-def **by** transfer-prover

lemma product-lists-transfer [transfer-rule]:
(list-all2 (list-all2 A) ===> list-all2 (list-all2 A)) product-lists product-lists
unfolding product-lists-def **by** transfer-prover

lemma insert-transfer [transfer-rule]:
assumes [transfer-rule]: bi-unique A
shows (A ===> list-all2 A ===> list-all2 A) List.insert List.insert
unfolding List.insert-def [abs-def] **by** transfer-prover

lemma find-transfer [transfer-rule]:
((A ===> (=)) ===> list-all2 A ===> rel-option A) List.find List.find
unfolding List.find-def **by** transfer-prover

lemma those-transfer [transfer-rule]:

(*list-all2* (*rel-option P*) ===> *rel-option* (*list-all2 P*)) *those those*
unfolding *List.those-def* **by** *transfer-prover*

lemma *remove1-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> *list-all2 A* ===> *list-all2 A*) *remove1 remove1*
unfolding *remove1-def* **by** *transfer-prover*

lemma *removeAll-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> *list-all2 A* ===> *list-all2 A*) *removeAll removeAll*
unfolding *removeAll-def* **by** *transfer-prover*

lemma *successively-transfer* [*transfer-rule*]:
((*A* ===> *A* ===> (=)) ===> *list-all2 A* ===> (=)) *successively successively*
unfolding *successively-altdef* **by** *transfer-prover*

lemma *distinct-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> (=)) *distinct distinct*
unfolding *distinct-def* **by** *transfer-prover*

lemma *distinct-adj-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*list-all2 A* ===> (=)) *distinct-adj distinct-adj*
unfolding *rel-fun-def*
proof (*intro allI impI*)
fix *xs ys* **assume** *list-all2 A xs ys*
thus *distinct-adj xs* ⟷ *distinct-adj ys*
proof (*induction rule*: *list-all2-induct*)
case (*Cons x xs y ys*)
show *?case*
by (*metis Cons assms bi-unique-def distinct-adj-Cons list.rel-sel*)
qed *auto*
qed

lemma *remdups-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A*) *remdups remdups*
unfolding *remdups-def* **by** *transfer-prover*

lemma *remdups-adj-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A*) *remdups-adj remdups-adj*
proof (*rule rel-funI*, *erule list-all2-induct*)
qed (*auto simp*: *remdups-adj-Cons assms*[*unfolded bi-unique-def*] *split*: *list.splits*)

lemma *replicate-transfer* [*transfer-rule*]:
((=) ===> *A* ===> *list-all2 A*) *replicate replicate*

unfolding *replicate-def* **by** *transfer-prover*

lemma *length-transfer* [*transfer-rule*]:
(*list-all2 A* ===> (=)) *length length*
unfolding *size-list-overloaded-def size-list-def* **by** *transfer-prover*

lemma *rotate1-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *rotate1 rotate1*
unfolding *rotate1-def* **by** *transfer-prover*

lemma *rotate-transfer* [*transfer-rule*]:
((=) ===> *list-all2 A* ===> *list-all2 A*) *rotate rotate*
unfolding *rotate-def* [*abs-def*] **by** *transfer-prover*

lemma *nths-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *rel-set* (=) ===> *list-all2 A*) *nths nths*
unfolding *nths-def* [*abs-def*] **by** *transfer-prover*

lemma *subseqs-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2* (*list-all2 A*)) *subseqs subseqs*
unfolding *subseqs-def* [*abs-def*] **by** *transfer-prover*

lemma *partition-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *rel-prod* (*list-all2 A*) (*list-all2 A*))
partition partition
unfolding *partition-def* **by** *transfer-prover*

lemma *lists-transfer* [*transfer-rule*]:
(*rel-set A* ===> *rel-set* (*list-all2 A*)) *lists lists*
proof (*rule rel-funI*, *rule rel-setI*)
show $[\![l \in lists\ X;\ rel\text{-}set\ A\ X\ Y]\!] \Longrightarrow \exists y{\in}lists\ Y.\ list\text{-}all2\ A\ l\ y$ **for** *X Y l*
proof (*induction l rule*: *lists.induct*)
case (*Cons a l*)
then show *?case*
by (*simp only*: *rel-set-def list-all2-Cons1*, *metis lists.Cons*)
qed *auto*
show $[\![l \in lists\ Y;\ rel\text{-}set\ A\ X\ Y]\!] \Longrightarrow \exists x{\in}lists\ X.\ list\text{-}all2\ A\ x\ l$ **for** *X Y l*
proof (*induction l rule*: *lists.induct*)
case (*Cons a l*)
then show *?case*
by (*simp only*: *rel-set-def list-all2-Cons2*, *metis lists.Cons*)
qed *auto*
qed

lemma *set-Cons-transfer* [*transfer-rule*]:
(*rel-set A* ===> *rel-set* (*list-all2 A*) ===> *rel-set* (*list-all2 A*))
set-Cons set-Cons
unfolding *rel-fun-def rel-set-def set-Cons-def*
by (*fastforce simp add*: *list-all2-Cons1 list-all2-Cons2*)

lemma *listset-transfer* [*transfer-rule*]:
  (*list-all2* (*rel-set A*) ===> *rel-set* (*list-all2 A*)) *listset listset*
  unfolding *listset-def* **by** *transfer-prover*

lemma *null-transfer* [*transfer-rule*]:
  (*list-all2 A* ===> (=)) *List.null List.null*
  unfolding *rel-fun-def List.null-def* **by** *auto*

lemma *list-all-transfer* [*transfer-rule*]:
  ((*A* ===> (=)) ===> *list-all2 A* ===> (=)) *list-all list-all*
  using *list.pred-transfer* **by** *blast*

lemma *list-ex-transfer* [*transfer-rule*]:
  ((*A* ===> (=)) ===> *list-all2 A* ===> (=)) *list-ex list-ex*
  unfolding *list-ex-iff* [*abs-def*] **by** *transfer-prover*

lemma *splice-transfer* [*transfer-rule*]:
  (*list-all2 A* ===> *list-all2 A* ===> *list-all2 A*) *splice splice*
  apply (*rule rel-funI*, *erule list-all2-induct*, *simp add*: *rel-fun-def*, *simp*)
  apply (*rule rel-funI*)
  apply (*erule-tac xs=x* **in** *list-all2-induct*, *simp*, *simp add*: *rel-fun-def*)
  done

lemma *shuffles-transfer* [*transfer-rule*]:
  (*list-all2 A* ===> *list-all2 A* ===> *rel-set* (*list-all2 A*)) *shuffles shuffles*
proof (*intro rel-funI*, *goal-cases*)
  case (*1 xs xs′ ys ys′*)
  thus *?case*
  proof (*induction xs ys arbitrary*: *xs′ ys′ rule*: *shuffles.induct*)
    case (*3 x xs y ys xs′ ys′*)
    from *3.prems* **obtain** *x′ xs″* **where** *xs′*: *xs′* = *x′* # *xs″* **by** (*cases xs′*) *auto*
    from *3.prems* **obtain** *y′ ys″* **where** *ys′*: *ys′* = *y′* # *ys″* **by** (*cases ys′*) *auto*
    have [*transfer-rule*]: *A x x′ A y y′ list-all2 A xs xs″ list-all2 A ys ys″*
      using *3.prems* **by** (*simp-all add*: *xs′ ys′*)
    have [*transfer-rule*]: *rel-set* (*list-all2 A*) (*shuffles xs* (*y* # *ys*)) (*shuffles xs″ ys′*)
and
        [*transfer-rule*]: *rel-set* (*list-all2 A*) (*shuffles* (*x* # *xs*) *ys*) (*shuffles xs′ ys″*)
      using *3.prems* **by** (*auto intro*!: *3.IH simp*: *xs′ ys′*)
    have *rel-set* (*list-all2 A*) ((#) *x* ‘ *shuffles xs* (*y* # *ys*) ∪ (#) *y* ‘ *shuffles* (*x* # *xs*) *ys*)
          ((#) *x′* ‘ *shuffles xs″ ys′* ∪ (#) *y′* ‘ *shuffles xs′ ys″*) **by** *transfer-prover*
    thus *?case* **by** (*simp add*: *xs′ ys′*)
  qed (*auto simp*: *rel-set-def*)
qed

lemma *rtrancl-parametric* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-unique A bi-total A*
  shows (*rel-set* (*rel-prod A A*) ===> *rel-set* (*rel-prod A A*)) *rtrancl rtrancl*

unfolding *rtrancl-def* **by** *transfer-prover*

lemma *monotone-parametric* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-total A*
  shows ((*A* ===> *A* ===> (=)) ===> (*B* ===> *B* ===> (=)) ===> (*A* ===> *B*) ===> (=)) *monotone monotone*
unfolding *monotone-def*[*abs-def*] **by** *transfer-prover*

lemma *fun-ord-parametric* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-total C*
  shows ((*A* ===> *B* ===> (=)) ===> (*C* ===> *A*) ===> (*C* ===> *B*) ===> (=)) *fun-ord fun-ord*
unfolding *fun-ord-def*[*abs-def*] **by** *transfer-prover*

lemma *fun-lub-parametric* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-total A* *bi-unique A*
  shows ((*rel-set A* ===> *B*) ===> *rel-set* (*C* ===> *A*) ===> *C* ===> *B*) *fun-lub fun-lub*
unfolding *fun-lub-def*[*abs-def*] **by** *transfer-prover*

end

end

67 Sum and product over lists

theory *Groups-List*
imports *List*
begin

locale *monoid-list* = *monoid*
begin

definition *F* :: *'a list* ⇒ *'a*
where
  eq-foldr [*code*]: *F xs* = *foldr f xs* **1**

lemma *Nil* [*simp*]:
  F [] = **1**
  by (*simp add*: *eq-foldr*)

lemma *Cons* [*simp*]:
  F (*x* # *xs*) = *x* ∗ *F xs*
  by (*simp add*: *eq-foldr*)

lemma *append* [*simp*]:
  F (*xs* @ *ys*) = *F xs* ∗ *F ys*
  by (*induct xs*) (*simp-all add*: *assoc*)

end

locale *comm-monoid-list* = *comm-monoid* + *monoid-list*
begin

lemma *rev* [*simp*]:
F (*rev xs*) = *F xs*
by (*simp add*: *eq-foldr foldr-fold fold-rev fun-eq-iff assoc left-commute*)

end

locale *comm-monoid-list-set* = *list*: *comm-monoid-list* + *set*: *comm-monoid-set*
begin

lemma *distinct-set-conv-list*:
distinct xs ⟹ *set.F g* (*set xs*) = *list.F* (*map g xs*)
by (*induct xs*) *simp-all*

lemma *set-conv-list* [*code*]:
set.F g (*set xs*) = *list.F* (*map g* (*remdups xs*))
by (*simp add*: *distinct-set-conv-list* [*symmetric*])

lemma *list-conv-set-nth*:
list.F xs = *set.F* (λ*i*. *xs* ! *i*) {*0*..<*length xs*}
proof −
have *xs* = *map* (λ*i*. *xs* ! *i*) [*0*..<*length xs*]
by (*simp add*: *map-nth*)
also have *list.F* ... = *set.F* (λ*i*. *xs* ! *i*) {*0*..<*length xs*}
by (*subst distinct-set-conv-list* [*symmetric*]) *auto*
finally show *?thesis* **.**
qed

end

67.1 List summation

context *monoid-add*
begin

sublocale *sum-list*: *monoid-list plus 0*
defines
sum-list = *sum-list.F* **..**

end

context *comm-monoid-add*
begin

sublocale *sum-list*: *comm-monoid-list plus 0*

rewrites
monoid-list.F plus 0 = sum-list
proof −
show *comm-monoid-list plus 0* **..**
then interpret *sum-list*: *comm-monoid-list plus 0* **.**
from *sum-list-def* **show** *monoid-list.F plus 0 = sum-list* **by** *simp*
qed

sublocale *sum*: *comm-monoid-list-set plus 0*
rewrites
monoid-list.F plus 0 = sum-list
and *comm-monoid-set.F plus 0 = sum*
proof −
show *comm-monoid-list-set plus 0* **..**
then interpret *sum*: *comm-monoid-list-set plus 0* **.**
from *sum-list-def* **show** *monoid-list.F plus 0 = sum-list* **by** *simp*
from *sum-def* **show** *comm-monoid-set.F plus 0 = sum* **by** (*auto intro*: *sym*)
qed

end

Some syntactic sugar for summing a function over a list:

open-bundle *sum-list-syntax*
begin

syntax (*ASCII*)
-sum-list :: *pttrn => 'a list => 'b => 'b* (‹(‹indent=3 notation=‹binder SUM››SUM -<−-. -)› [0, 51, 10] 10)
syntax
-sum-list :: *pttrn => 'a list => 'b => 'b* (‹(‹indent=3 notation=‹binder ∑››∑ -←-. -)› [0, 51, 10] 10)
syntax-consts
-sum-list == sum-list
translations — Beware of argument permutation!
∑*x*←*xs. b == CONST sum-list* (*CONST map* (λ*x. b*) *xs*)

end

context
includes *lifting-syntax*
begin

lemma *sum-list-transfer* [*transfer-rule*]:
(*list-all2 A ===> A*) *sum-list sum-list*
if [*transfer-rule*]: *A 0 0* (*A ===> A ===> A*) (+) (+)
unfolding *sum-list.eq-foldr* [*abs-def*]
by *transfer-prover*

end

TODO duplicates

lemmas *sum-list-simps* = *sum-list.Nil sum-list.Cons*
lemmas *sum-list-append* = *sum-list.append*
lemmas *sum-list-rev* = *sum-list.rev*

lemma (**in** *monoid-add*) *fold-plus-sum-list-rev*:
fold plus xs = *plus* (*sum-list* (*rev xs*))
proof
fix *x*
have *fold plus xs x* = *sum-list* (*rev xs* @ [*x*])
by (*simp add*: *foldr-conv-fold sum-list.eq-foldr*)
also have ... = *sum-list* (*rev xs*) + *x*
by *simp*
finally show *fold plus xs x* = *sum-list* (*rev xs*) + *x*
.
qed

lemma *sum-list-of-nat*: *sum-list* (*map of-nat xs*) = *of-nat* (*sum-list xs*)
by (*induction xs*) *auto*

lemma *sum-list-of-int*: *sum-list* (*map of-int xs*) = *of-int* (*sum-list xs*)
by (*induction xs*) *auto*

lemma (**in** *comm-monoid-add*) *sum-list-map-remove1*:
$x \in$ *set xs* $\Longrightarrow$ *sum-list* (*map f xs*) = *f x* + *sum-list* (*map f* (*remove1 x xs*))
by (*induct xs*) (*auto simp add*: *ac-simps*)

lemma (**in** *monoid-add*) *size-list-conv-sum-list*:
size-list f xs = *sum-list* (*map f xs*) + *size xs*
by (*induct xs*) *auto*

lemma (**in** *monoid-add*) *length-concat*:
length (*concat xss*) = *sum-list* (*map length xss*)
by (*induct xss*) *simp-all*

lemma (**in** *monoid-add*) *length-product-lists*:
length (*product-lists xss*) = *foldr* (∗) (*map length xss*) *1*
proof (*induct xss*)
case (*Cons xs xss*) **then show** *?case* **by** (*induct xs*) (*auto simp*: *length-concat o-def*)
qed *simp*

lemma (**in** *monoid-add*) *sum-list-map-filter*:
assumes $\bigwedge x.\ x \in$ *set xs* $\Longrightarrow \neg$ *P x* $\Longrightarrow$ *f x* = *0*
shows *sum-list* (*map f* (*filter P xs*)) = *sum-list* (*map f xs*)
using *assms* **by** (*induct xs*) *auto*

lemma *sum-list-filter-le-nat*:
fixes *f* :: *'a* ⇒ *nat*

shows *sum-list* (*map f* (*filter P xs*)) ≤ *sum-list* (*map f xs*)
by(*induction xs*; *simp*)

lemma (**in** *comm-monoid-add*) *distinct-sum-list-conv-Sum*:
distinct xs ⟹ *sum-list xs* = *Sum* (*set xs*)
by (*metis local.sum.set-conv-list local.sum-list-def map-ident remdups-id-iff-distinct*)

lemma *sum-list-upt*[*simp*]:
$m \leq n \Longrightarrow$ *sum-list* [*m..<n*] = ∑ {*m..<n*}
by(*simp add*: *distinct-sum-list-conv-Sum*)

context *ordered-comm-monoid-add*
begin

lemma *sum-list-nonneg*: (⋀*x*. *x* ∈ *set xs* ⟹ *0* ≤ *x*) ⟹ *0* ≤ *sum-list xs*
by (*induction xs*) *auto*

lemma *sum-list-nonpos*: (⋀*x*. *x* ∈ *set xs* ⟹ *x* ≤ *0*) ⟹ *sum-list xs* ≤ *0*
by (*induction xs*) (*auto simp*: *add-nonpos-nonpos*)

lemma *sum-list-nonneg-eq-0-iff*:
(⋀*x*. *x* ∈ *set xs* ⟹ *0* ≤ *x*) ⟹ *sum-list xs* = *0* ⟷ (∀ *x*∈ *set xs*. *x* = *0*)
by (*induction xs*) (*simp-all add*: *add-nonneg-eq-0-iff sum-list-nonneg*)

end

context *canonically-ordered-monoid-add*
begin

lemma *sum-list-eq-0-iff* [*simp*]:
sum-list ns = *0* ⟷ (∀ *n* ∈ *set ns*. *n* = *0*)
by (*simp add*: *sum-list-nonneg-eq-0-iff*)

lemma *member-le-sum-list*:
x ∈ *set xs* ⟹ *x* ≤ *sum-list xs*
by (*induction xs*) (*auto simp*: *add-increasing add-increasing2*)

lemma *elem-le-sum-list*:
k < *size ns* ⟹ *ns* ! *k* ≤ *sum-list* (*ns*)
by (*simp add*: *member-le-sum-list*)

end

lemma (**in** *ordered-cancel-comm-monoid-diff*) *sum-list-update*:
k < *size xs* ⟹ *sum-list* (*xs*[*k* := *x*]) = *sum-list xs* + *x* − *xs* ! *k*
proof (*induction xs arbitrary*:*k*)
case *Nil*
then show *?case* **by** *auto*
next

case (*Cons a xs*)
then show *?case*
apply (*simp add*: *add-ac split*: *nat.split*)
using *add-increasing diff-add-assoc elem-le-sum-list zero-le* **by** *force*
qed

lemma (**in** *monoid-add*) *sum-list-triv*:
$(\sum x \leftarrow xs.\ r) = of\text{-}nat\ (length\ xs) * r$
by (*induct xs*) (*simp-all add*: *distrib-right*)

lemma (**in** *monoid-add*) *sum-list-0* [*simp*]:
$(\sum x \leftarrow xs.\ 0) = 0$
by (*induct xs*) (*simp-all add*: *distrib-right*)

For non-Abelian groups *xs* needs to be reversed on one side:

lemma (**in** *ab-group-add*) *uminus-sum-list-map*:
$-\ sum\text{-}list\ (map\ f\ xs) = sum\text{-}list\ (map\ (uminus \circ f)\ xs)$
by (*induct xs*) *simp-all*

lemma (**in** *comm-monoid-add*) *sum-list-addf*:
$(\sum x \leftarrow xs.\ f\ x + g\ x) = sum\text{-}list\ (map\ f\ xs) + sum\text{-}list\ (map\ g\ xs)$
by (*induct xs*) (*simp-all add*: *algebra-simps*)

lemma (**in** *ab-group-add*) *sum-list-subtractf*:
$(\sum x \leftarrow xs.\ f\ x - g\ x) = sum\text{-}list\ (map\ f\ xs) - sum\text{-}list\ (map\ g\ xs)$
by (*induct xs*) (*simp-all add*: *algebra-simps*)

lemma (**in** *semiring-0*) *sum-list-const-mult*:
$(\sum x \leftarrow xs.\ c * f\ x) = c * (\sum x \leftarrow xs.\ f\ x)$
by (*induct xs*) (*simp-all add*: *algebra-simps*)

lemma (**in** *semiring-0*) *sum-list-mult-const*:
$(\sum x \leftarrow xs.\ f\ x * c) = (\sum x \leftarrow xs.\ f\ x) * c$
by (*induct xs*) (*simp-all add*: *algebra-simps*)

lemma (**in** *ordered-ab-group-add-abs*) *sum-list-abs*:
$|sum\text{-}list\ xs| \le sum\text{-}list\ (map\ abs\ xs)$
by (*induct xs*) (*simp-all add*: *order-trans* [*OF abs-triangle-ineq*])

lemma *sum-list-mono*:
fixes $f\ g :: 'a \Rightarrow 'b::\{monoid\text{-}add,\ ordered\text{-}ab\text{-}semigroup\text{-}add\}$
shows $(\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x \le g\ x) \Longrightarrow (\sum x \leftarrow xs.\ f\ x) \le (\sum x \leftarrow xs.\ g\ x)$
by (*induct xs*) (*simp*, *simp add*: *add-mono*)

lemma *sum-list-strict-mono*:
fixes $f\ g :: 'a \Rightarrow 'b::\{monoid\text{-}add,\ strict\text{-}ordered\text{-}ab\text{-}semigroup\text{-}add\}$
shows $[\![\ xs \ne [];\ \ \bigwedge x.\ x \in set\ xs \Longrightarrow f\ x < g\ x\ ]\!]$
$\Longrightarrow sum\text{-}list\ (map\ f\ xs) < sum\text{-}list\ (map\ g\ xs)$
proof (*induction xs*)

case *Nil* **thus** *?case* **by** *simp*
next
case *C*: (*Cons* - *xs*)
show *?case*
proof (*cases xs*)
case *Nil* **thus** *?thesis* **using** *C.prems* **by** *simp*
next
case *Cons* **thus** *?thesis* **using** *C* **by**(*simp add*: *add-strict-mono*)
qed
qed

A much more general version of this monotonicity lemma can be formulated with multisets and the multiset order

lemma *sum-list-mono2*: **fixes** *xs* :: *'a* ::*ordered-comm-monoid-add list*
shows $\llbracket$ *length xs* = *length ys*; $\bigwedge i.$ *i* < *length xs* $\longrightarrow$ *xs*!*i* $\leq$ *ys*!*i* $\rrbracket$
$\Longrightarrow$ *sum-list xs* $\leq$ *sum-list ys*
by (*induction xs ys rule*: *list-induct2*) (*auto simp*: *nth-Cons' less-Suc-eq-0-disj imp-ex add-mono*)

lemma (**in** *monoid-add*) *sum-list-distinct-conv-sum-set*:
distinct xs $\Longrightarrow$ *sum-list* (*map f xs*) = *sum f* (*set xs*)
by (*induct xs*) *simp-all*

lemma (**in** *monoid-add*) *interv-sum-list-conv-sum-set-nat*:
sum-list (*map f* [*m*..<*n*]) = *sum f* (*set* [*m*..<*n*])
by (*simp add*: *sum-list-distinct-conv-sum-set*)

lemma (**in** *monoid-add*) *interv-sum-list-conv-sum-set-int*:
sum-list (*map f* [*k*..*l*]) = *sum f* (*set* [*k*..*l*])
by (*simp add*: *sum-list-distinct-conv-sum-set*)

General equivalence between *sum-list* and *sum*

lemma (**in** *monoid-add*) *sum-list-sum-nth*:
sum-list xs = ($\sum$ *i* = *0* ..< *length xs*. *xs* ! *i*)
using *interv-sum-list-conv-sum-set-nat* [*of* (!) *xs 0 length xs*] **by** (*simp add*: *map-nth*)

lemma *sum-list-map-eq-sum-count*:
sum-list (*map f xs*) = *sum* ($\lambda x.$ *count-list xs x* $*$ *f x*) (*set xs*)
proof(*induction xs*)
case (*Cons x xs*)
show *?case* (**is** *?l* = *?r*)
proof *cases*
assume *x* $\in$ *set xs*
have *?l* = *f x* + ($\sum$ *x*$\in$*set xs*. *count-list xs x* $*$ *f x*) **by** (*simp add*: *Cons.IH*)
also have *set xs* = *insert x* (*set xs* $-$ {*x*}) **using** ‹*x* $\in$ *set xs*›**by** *blast*
also have *f x* + ($\sum$ *x*$\in$*insert x* (*set xs* $-$ {*x*}). *count-list xs x* $*$ *f x*) = *?r*
by (*simp add*: *sum.insert-remove eq-commute*)
finally show *?thesis* **.**

next
assume *x* ∉ *set xs*
hence ⋀*xa*. *xa* ∈ *set xs* ⟹ *x* ≠ *xa* **by** *blast*
thus *?thesis* **by** (*simp add*: *Cons.IH* ‹*x* ∉ *set xs*›)
qed
qed *simp*

lemma *sum-list-map-eq-sum-count2*:
assumes *set xs* ⊆ *X finite X*
shows *sum-list* (*map f xs*) = *sum* (λ*x*. *count-list xs x* ∗ *f x*) *X*
proof−
let *?F* = λ*x*. *count-list xs x* ∗ *f x*
have *sum ?F X* = *sum ?F* (*set xs* ∪ (*X* − *set xs*))
using *Un-absorb1*[*OF assms*(*1*)] **by**(*simp*)
also have ... = *sum ?F* (*set xs*)
using *assms*(*2*)
by(*simp add*: *sum.union-disjoint*[*OF - - Diff-disjoint*] *del*: *Un-Diff-cancel*)
finally show *?thesis* **by**(*simp add*:*sum-list-map-eq-sum-count*)
qed

lemma *sum-list-replicate*: *sum-list* (*replicate n c*) = *of-nat n* ∗ *c*
by(*induction n*)(*auto simp add*: *distrib-right*)

lemma *sum-list-nonneg*:
(⋀*x*. *x* ∈ *set xs* ⟹ (*x* :: ′*a* :: *ordered-comm-monoid-add*) ≥ *0*) ⟹ *sum-list xs* ≥ *0*
by (*induction xs*) *simp-all*

lemma *sum-list-Suc*:
sum-list (*map* (λ*x*. *Suc*(*f x*)) *xs*) = *sum-list* (*map f xs*) + *length xs*
by(*induction xs*; *simp*)

lemma (**in** *monoid-add*) *sum-list-map-filter′*:
sum-list (*map f* (*filter P xs*)) = *sum-list* (*map* (λ*x*. *if P x then f x else 0*) *xs*)
by (*induction xs*) *simp-all*

Summation of a strictly ascending sequence with length *n* can be upper-bounded by summation over {*0*..<*n*}.

lemma *sorted-wrt-less-sum-mono-lowerbound*:
fixes *f* :: *nat* ⇒ (′*b*::*ordered-comm-monoid-add*)
assumes *mono*: ⋀*x y*. *x*≤*y* ⟹ *f x* ≤ *f y*
shows *sorted-wrt* (<) *ns* ⟹
(∑ *i*∈{*0*..<*length ns*}. *f i*) ≤ (∑ *i*←*ns*. *f i*)
proof (*induction ns rule*: *rev-induct*)
case *Nil*
then show *?case* **by** *simp*
next
case (*snoc n ns*)

```
  have sum f {0..<length (ns @ [n])}
      = sum f {0..<length ns} + f (length ns)
    by simp
  also have sum f {0..<length ns} ≤ sum-list (map f ns)
    using snoc by (auto simp: sorted-wrt-append)
  also have length ns ≤ n
    using sorted-wrt-less-idx[OF snoc.prems(1), of length ns] by auto
  finally have sum f {0..<length (ns @ [n])} ≤ sum-list (map f ns) + f n
    using mono add-mono by blast
  thus ?case by simp
qed
```

```
lemma member-le-sum-list:
  fixes x :: 'a :: ordered-comm-monoid-add
  assumes x ∈ set xs ⋀x. x ∈ set xs ⟹ x ≥ 0
  shows   x ≤ sum-list xs
  using assms
proof (induction xs)
  case (Cons y xs)
  show ?case
  proof (cases y = x)
    case True
    have x + 0 ≤ x + sum-list xs
      by (intro add-mono order.refl sum-list-nonneg) (use Cons in auto)
    thus ?thesis
      using True by auto
  next
    case False
    have 0 + x ≤ y + sum-list xs
      by (intro add-mono Cons.IH Cons.prems) (use Cons.prems False in auto)
    thus ?thesis
      by auto
  qed
qed auto
```

67.2 Horner sums

```
context comm-semiring-0
begin

definition horner-sum :: ‹('b ⇒ 'a) ⇒ 'a ⇒ 'b list ⇒ 'a›
  where horner-sum-foldr: ‹horner-sum f a xs = foldr (λx b. f x + a * b) xs 0›

lemma horner-sum-simps [simp]:
  ‹horner-sum f a [] = 0›
  ‹horner-sum f a (x # xs) = f x + a * horner-sum f a xs›
  by (simp-all add: horner-sum-foldr)
```

lemma *horner-sum-eq-sum-funpow*:
‹*horner-sum f a xs* = ($\sum$ *n* = *0*..<*length xs*. ((∗) *a* ^^ *n*) (*f* (*xs* ! *n*)))›
proof (*induction xs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons x xs*)
then show *?case*
by (*simp add*: *sum.atLeast0-lessThan-Suc-shift sum-distrib-left del*: *sum.op-ivl-Suc*)
qed

end

context
includes *lifting-syntax*
begin

lemma *horner-sum-transfer* [*transfer-rule*]:
‹((*B* ===> *A*) ===> *A* ===> *list-all2 B* ===> *A*) *horner-sum horner-sum*›
if [*transfer-rule*]: ‹*A 0 0*›
and [*transfer-rule*]: ‹(*A* ===> *A* ===> *A*) (+) (+)›
and [*transfer-rule*]: ‹(*A* ===> *A* ===> *A*) (∗) (∗)›
by (*unfold horner-sum-foldr*) *transfer-prover*

end

context *comm-semiring-1*
begin

lemma *horner-sum-eq-sum*:
‹*horner-sum f a xs* = ($\sum$ *n* = *0*..<*length xs*. *f* (*xs* ! *n*) ∗ *a* ^ *n*)›
proof −
have ‹(∗) *a* ^^ *n* = (∗) (*a* ^ *n*)› **for** *n*
by (*induction n*) (*simp-all add*: *ac-simps*)
then show *?thesis*
by (*simp add*: *horner-sum-eq-sum-funpow ac-simps*)
qed

lemma *horner-sum-append*:
‹*horner-sum f a* (*xs* @ *ys*) = *horner-sum f a xs* + *a* ^ *length xs* ∗ *horner-sum f a ys*›
using *sum.atLeastLessThan-shift-bounds* [*of - 0* ‹*length xs*› ‹*length ys*›]
atLeastLessThan-add-Un [*of 0* ‹*length xs*› ‹*length ys*›]
by (*simp add*: *horner-sum-eq-sum sum-distrib-left sum.union-disjoint ac-simps nth-append power-add*)

end

context *linordered-semidom*
begin

lemma *horner-sum-nonnegative*:
‹*0* ≤ *horner-sum of-bool 2 bs*›
by (*induction bs*) *simp-all*

end

context *discrete-linordered-semidom*
begin

lemma *horner-sum-bound*:
‹*horner-sum of-bool 2 bs* < *2* ^ *length bs*›
proof (*induction bs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons b bs*)
moreover define *a* **where** ‹*a* = *2* ^ *length bs* − *horner-sum of-bool 2 bs*›
ultimately have ∗: ‹*2* ^ *length bs* = *horner-sum of-bool 2 bs* + *a*›
by *simp*
have ‹*0* < *a*›
using *Cons* ∗ **by** *simp*
moreover have ‹*1* ≤ *a*›
using ‹*0* < *a*› **by** (*simp add*: *less-eq-iff-succ-less*)
ultimately have ‹*0* + *1* < *a* + *a*›
by (*rule add-less-le-mono*)
then have ‹*1* < *a* ∗ *2*›
by (*simp add*: *mult-2-right*)
with *Cons* **show** *?case*
by (*simp add*: ∗ *algebra-simps*)
qed

lemma *horner-sum-of-bool-2-less*:
‹(*horner-sum of-bool 2 bs*) < *2* ^ *length bs*›
by (*fact horner-sum-bound*)

end

lemma *nat-horner-sum* [*simp*]:
‹*nat* (*horner-sum of-bool 2 bs*) = *horner-sum of-bool 2 bs*›
by (*induction bs*) (*auto simp add*: *nat-add-distrib horner-sum-nonnegative*)

context *discrete-linordered-semidom*
begin

lemma *horner-sum-less-eq-iff-lexordp-eq*:

‹*horner-sum of-bool 2 bs* ≤ *horner-sum of-bool 2 cs* ⟷ *lexordp-eq* (*rev bs*) (*rev cs*)›
if ‹*length bs* = *length cs*›
proof −
have ‹*horner-sum of-bool 2* (*rev bs*) ≤ *horner-sum of-bool 2* (*rev cs*) ⟷ *lexordp-eq bs cs*›
if ‹*length bs* = *length cs*› **for** *bs cs*
using *that* **proof** (*induction bs cs rule*: *list-induct2*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons b bs c cs*)
with *horner-sum-nonnegative* [*of* ‹*rev bs*›] *horner-sum-nonnegative* [*of* ‹*rev cs*›]
horner-sum-bound [*of* ‹*rev bs*›] *horner-sum-bound* [*of* ‹*rev cs*›]
show *?case*
by (*auto simp add*: *horner-sum-append not-le Cons intro*: *add-strict-increasing2 add-increasing*)
qed
from *that this* [*of* ‹*rev bs*› ‹*rev cs*›] **show** *?thesis*
by *simp*
qed

lemma *horner-sum-less-iff-lexordp*:
‹*horner-sum of-bool 2 bs* < *horner-sum of-bool 2 cs* ⟷ *ord-class.lexordp* (*rev bs*) (*rev cs*)›
if ‹*length bs* = *length cs*›
proof −
have ‹*horner-sum of-bool 2* (*rev bs*) < *horner-sum of-bool 2* (*rev cs*) ⟷ *ord-class.lexordp bs cs*›
if ‹*length bs* = *length cs*› **for** *bs cs*
using *that* **proof** (*induction bs cs rule*: *list-induct2*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons b bs c cs*)
with *horner-sum-nonnegative* [*of* ‹*rev bs*›] *horner-sum-nonnegative* [*of* ‹*rev cs*›]
horner-sum-bound [*of* ‹*rev bs*›] *horner-sum-bound* [*of* ‹*rev cs*›]
show *?case*
by (*auto simp add*: *horner-sum-append not-less Cons intro*: *add-strict-increasing2 add-increasing*)
qed
from *that this* [*of* ‹*rev bs*› ‹*rev cs*›] **show** *?thesis*
by *simp*
qed

end

67.3 Further facts about *List.n-lists*

lemma *length-n-lists*: *length* (*List.n-lists n xs*) = *length xs* ^ *n*
by (*induct n*) (*auto simp add*: *comp-def length-concat sum-list-triv*)

lemma *distinct-n-lists*:
assumes *distinct xs*
shows *distinct* (*List.n-lists n xs*)
proof (*rule card-distinct*)
from *assms* **have** *card-length*: *card* (*set xs*) = *length xs* **by** (*rule distinct-card*)
have *card* (*set* (*List.n-lists n xs*)) = *card* (*set xs*) ^ *n*
proof (*induct n*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
moreover have *card* ($\bigcup$ *ys*∈*set* (*List.n-lists n xs*). (λ*y*. *y* # *ys*) ' *set xs*)
= ($\sum$ *ys*∈*set* (*List.n-lists n xs*). *card* ((λ*y*. *y* # *ys*) ' *set xs*))
by (*rule card-UN-disjoint*) *auto*
moreover have $\bigwedge$*ys*. *card* ((λ*y*. *y* # *ys*) ' *set xs*) = *card* (*set xs*)
by (*rule card-image*) (*simp add*: *inj-on-def*)
ultimately show *?case* **by** *auto*
qed
also have ... = *length xs* ^ *n* **by** (*simp add*: *card-length*)
finally show *card* (*set* (*List.n-lists n xs*)) = *length* (*List.n-lists n xs*)
by (*simp add*: *length-n-lists*)
qed

67.4 Tools setup

lemmas *sum-code* = *sum.set-conv-list*

lemma *sum-set-upto-conv-sum-list-int* [*code-unfold*]:
sum f (*set* [*i..j*::*int*]) = *sum-list* (*map f* [*i..j*])
by (*simp add*: *interv-sum-list-conv-sum-set-int*)

lemma *sum-set-upt-conv-sum-list-nat* [*code-unfold*]:
sum f (*set* [*m..*<*n*]) = *sum-list* (*map f* [*m..*<*n*])
by (*simp add*: *interv-sum-list-conv-sum-set-nat*)

67.5 List product

context *monoid-mult*
begin

sublocale *prod-list*: *monoid-list times 1*
defines
prod-list = *prod-list.F* **..**

end

context *comm-monoid-mult*
begin

sublocale *prod-list*: *comm-monoid-list times 1*
rewrites
  monoid-list.F times 1 = prod-list
proof −
  show *comm-monoid-list times 1* **..**
  then interpret *prod-list*: *comm-monoid-list times 1* **.**
  from *prod-list-def* **show** *monoid-list.F times 1 = prod-list* **by** *simp*
qed

sublocale *prod*: *comm-monoid-list-set times 1*
rewrites
  monoid-list.F times 1 = prod-list
  and *comm-monoid-set.F times 1 = prod*
proof −
  show *comm-monoid-list-set times 1* **..**
  then interpret *prod*: *comm-monoid-list-set times 1* **.**
  from *prod-list-def* **show** *monoid-list.F times 1 = prod-list* **by** *simp*
  from *prod-def* **show** *comm-monoid-set.F times 1 = prod* **by** (*auto intro*: *sym*)
qed

end

Some syntactic sugar:

open-bundle *prod-list-syntax*
begin

syntax (*ASCII*)
  -prod-list :: *pttrn => 'a list => 'b => 'b* (‹(‹*indent=3 notation=*‹*binder PROD*››*PROD* -<−-. -)› [*0*, *51*, *10*] *10*)
syntax
  -prod-list :: *pttrn => 'a list => 'b => 'b* (‹(‹*indent=3 notation=*‹*binder* ∏››∏ -←-. -)› [*0*, *51*, *10*] *10*)
syntax-consts
  -prod-list ⇌ *prod-list*
translations — Beware of argument permutation!
  ∏*x*←*xs*. *b* ⇌ *CONST prod-list* (*CONST map* (λ*x*. *b*) *xs*)

end

context
  includes *lifting-syntax*
begin

lemma *prod-list-transfer* [*transfer-rule*]:
  (*list-all2 A ===> A*) *prod-list prod-list*
    if [*transfer-rule*]: *A 1 1* (*A ===> A ===> A*) (∗) (∗)

unfolding *prod-list.eq-foldr* [*abs-def*]
by *transfer-prover*

end

lemma *prod-list-zero-iff*:
prod-list xs = *0* $\longleftrightarrow$ (*0* :: *'a* :: {*semiring-no-zero-divisors*, *semiring-1*}) $\in$ *set xs*
by (*induction xs*) *simp-all*

end

68 Bit operations in suitable algebraic structures

theory *Bit-Operations*
imports *Presburger Groups-List*
begin

68.1 Abstract bit structures

class *semiring-bits* = *semiring-parity* + *semiring-modulo-trivial* +
assumes *bit-induct* [*case-names stable rec*]:
‹($\bigwedge$*a*. *a div 2* = *a* $\Longrightarrow$ *P a*)
$\Longrightarrow$ ($\bigwedge$*a b*. *P a* $\Longrightarrow$ (*of-bool b* + *2* ∗ *a*) *div 2* = *a* $\Longrightarrow$ *P* (*of-bool b* + *2* ∗ *a*))
$\Longrightarrow$ *P a*›
assumes *bits-mod-div-trivial* [*simp*]: ‹*a mod b div b* = *0*›
and *half-div-exp-eq*: ‹*a div 2 div 2* ^ *n* = *a div 2* ^ *Suc n*›
and *even-double-div-exp-iff*: ‹*2* ^ *Suc n* $\neq$ *0* $\Longrightarrow$ *even* (*2* ∗ *a div 2* ^ *Suc n*) $\longleftrightarrow$ *even* (*a div 2* ^ *n*)›
fixes *bit* :: ‹*'a* $\Rightarrow$ *nat* $\Rightarrow$ *bool*›
assumes *bit-iff-odd*: ‹*bit a n* $\longleftrightarrow$ *odd* (*a div 2* ^ *n*)›
begin

Having *bit* as definitional class operation takes into account that specific instances can be implemented differently wrt. code generation.

lemma *half-1* [*simp*]:
‹*1 div 2* = *0*›
using *even-half-succ-eq* [*of 0*] **by** *simp*

lemma *div-exp-eq-funpow-half*:
‹*a div 2* ^ *n* = ((λ*a*. *a div 2*) ^^ *n*) *a*›
proof −
have ‹((λ*a*. *a div 2*) ^^ *n*) = (λ*a*. *a div 2* ^ *n*)›
by (*induction n*)
(*simp-all del*: *funpow.simps power.simps add*: *power-0 funpow-Suc-right half-div-exp-eq*)
then show *?thesis*
by *simp*
qed

lemma *div-exp-eq*:

‹a div 2 ^ m div 2 ^ n = a div 2 ^ (m + n)›
by (*simp add*: *div-exp-eq-funpow-half Groups.add.commute* [*of m*] *funpow-add*)

lemma *bit-0*:
‹bit a 0 ⟷ odd a›
by (*simp add*: *bit-iff-odd*)

lemma *bit-Suc*:
‹bit a (Suc n) ⟷ bit (a div 2) n›
using *div-exp-eq* [*of a 1 n*] **by** (*simp add*: *bit-iff-odd*)

lemma *bit-rec*:
‹bit a n ⟷ (if n = 0 then odd a else bit (a div 2) (n − 1))›
by (*cases n*) (*simp-all add*: *bit-Suc bit-0*)

context
fixes *a*
assumes *stable*: ‹a div 2 = a›
begin

lemma *bits-stable-imp-add-self*:
‹a + a mod 2 = 0›
proof −
have ‹a div 2 ∗ 2 + a mod 2 = a›
by (*fact div-mult-mod-eq*)
then have ‹a ∗ 2 + a mod 2 = a›
by (*simp add*: *stable*)
then show *?thesis*
by (*simp add*: *mult-2-right ac-simps*)
qed

lemma *stable-imp-bit-iff-odd*:
‹bit a n ⟷ odd a›
by (*induction n*) (*simp-all add*: *stable bit-Suc bit-0*)

end

lemma *bit-iff-odd-imp-stable*:
‹a div 2 = a› **if** ‹⋀n. bit a n ⟷ odd a›
using *that* **proof** (*induction a rule*: *bit-induct*)
case (*stable a*)
then show *?case*
by *simp*
next
case (*rec a b*)
from *rec.prems* [*of 1*] **have** [*simp*]: ‹b = odd a›
by (*simp add*: *rec.hyps bit-Suc bit-0*)
from *rec.hyps* **have** *hyp*: ‹(of-bool (odd a) + 2 ∗ a) div 2 = a›
by *simp*

have ‹*bit a n* ⟷ *odd a*› **for** *n*
using *rec.prems* [*of* ‹*Suc n*›] **by** (*simp add*: *hyp bit-Suc*)
then have ‹*a div 2* = *a*›
by (*rule rec.IH*)
then have ‹*of-bool* (*odd a*) + *2* ∗ *a* = *2* ∗ (*a div 2*) + *of-bool* (*odd a*)›
by (*simp add*: *ac-simps*)
also have ‹. . . = *a*›
using *mult-div-mod-eq* [*of 2 a*]
by (*simp add*: *of-bool-odd-eq-mod-2*)
finally show *?case*
using ‹*a div 2* = *a*› **by** (*simp add*: *hyp*)
qed

lemma *even-succ-div-exp* [*simp*]:
‹(*1* + *a*) *div 2* ^ *n* = *a div 2* ^ *n*› **if** ‹*even a*› **and** ‹*n* > *0*›
proof (*cases n*)
case *0*
with *that* **show** *?thesis*
by *simp*
next
case (*Suc n*)
with ‹*even a*› **have** ‹(*1* + *a*) *div 2* ^ *Suc n* = *a div 2* ^ *Suc n*›
proof (*induction n*)
case *0*
then show *?case*
by *simp*
next
case (*Suc n*)
then show *?case*
using *div-exp-eq* [*of - 1* ‹*Suc n*›, *symmetric*]
by *simp*
qed
with *Suc* **show** *?thesis*
by *simp*
qed

lemma *even-succ-mod-exp* [*simp*]:
‹(*1* + *a*) *mod 2* ^ *n* = *1* + (*a mod 2* ^ *n*)› **if** ‹*even a*› **and** ‹*n* > *0*›
using *div-mult-mod-eq* [*of* ‹*1* + *a*› ‹*2* ^ *n*›] *div-mult-mod-eq* [*of a* ‹*2* ^ *n*›] *that*
by *simp* (*metis* (*full-types*) *add.left-commute add-left-imp-eq*)

lemma *half-numeral-Bit1-eq* [*simp*]:
‹*numeral* (*num.Bit1 m*) *div 2* = *numeral* (*num.Bit0 m*) *div 2*›
using *even-half-succ-eq* [*of* ‹*2* ∗ *numeral m*›]
by *simp*

lemma *double-half-numeral-Bit-0-eq* [*simp*]:
‹*2* ∗ (*numeral* (*num.Bit0 m*) *div 2*) = *numeral* (*num.Bit0 m*)›
‹(*numeral* (*num.Bit0 m*) *div 2*) ∗ *2* = *numeral* (*num.Bit0 m*)›

using *mod-mult-div-eq* [*of* ‹*numeral* (*Num.Bit0 m*)› *2*]
by (*simp-all add*: *mod2-eq-if ac-simps*)

named-theorems *bit-simps* ‹*Simplification rules for* ***const*** ‹*bit*››

definition *possible-bit* :: ‹*'a itself* ⇒ *nat* ⇒ *bool*›
where ‹*possible-bit TYPE('a) n* ⟷ *2* ^ *n* ≠ *0*›
— This auxiliary avoids non-termination with extensionality.

lemma *possible-bit-0* [*simp*]:
‹*possible-bit TYPE('a) 0*›
by (*simp add*: *possible-bit-def*)

lemma *fold-possible-bit*:
‹*2* ^ *n* = *0* ⟷ ¬ *possible-bit TYPE('a) n*›
by (*simp add*: *possible-bit-def*)

lemma *bit-imp-possible-bit*:
‹*possible-bit TYPE('a) n*› **if** ‹*bit a n*›
by (*rule ccontr*) (*use that* **in** ‹*auto simp*: *bit-iff-odd possible-bit-def*›)

lemma *impossible-bit*:
‹¬ *bit a n*› **if** ‹¬ *possible-bit TYPE('a) n*›
using *that* **by** (*blast dest*: *bit-imp-possible-bit*)

lemma *possible-bit-less-imp*:
‹*possible-bit TYPE('a) j*› **if** ‹*possible-bit TYPE('a) i*› ‹*j* ≤ *i*›
using *power-add* [*of 2 j* ‹*i* − *j*›] *that mult-not-zero* [*of* ‹*2* ^ *j*› ‹*2* ^ (*i* − *j*)›]
by (*simp add*: *possible-bit-def*)

lemma *possible-bit-min* [*simp*]:
‹*possible-bit TYPE('a)* (*min i j*) ⟷ *possible-bit TYPE('a) i* ∨ *possible-bit TYPE('a) j*›
by (*auto simp*: *min-def elim*: *possible-bit-less-imp*)

lemma *bit-eqI*:
‹*a* = *b*› **if** ‹⋀*n*. *possible-bit TYPE('a) n* ⟹ *bit a n* ⟷ *bit b n*›
proof −
have ‹*bit a n* ⟷ *bit b n*› **for** *n*
proof (*cases* ‹*possible-bit TYPE('a) n*›)
case *False*
then show *?thesis*
by (*simp add*: *impossible-bit*)
next
case *True*
then show *?thesis*
by (*rule that*)
qed
then show *?thesis* **proof** (*induction a arbitrary*: *b rule*: *bit-induct*)

```
    case (stable a)
    from stable(2) [of 0] have **: ‹even b ⟷ even a›
      by (simp add: bit-0)
    have ‹b div 2 = b›
    proof (rule bit-iff-odd-imp-stable)
      fix n
      from stable have *: ‹bit b n ⟷ bit a n›
        by simp
      also have ‹bit a n ⟷ odd a›
        using stable by (simp add: stable-imp-bit-iff-odd)
      finally show ‹bit b n ⟷ odd b›
        by (simp add: **)
    qed
    from ** have ‹a mod 2 = b mod 2›
      by (simp add: mod2-eq-if)
    then have ‹a mod 2 + (a + b) = b mod 2 + (a + b)›
      by simp
    then have ‹a + a mod 2 + b = b + b mod 2 + a›
      by (simp add: ac-simps)
    with ‹a div 2 = a› ‹b div 2 = b› show ?case
      by (simp add: bits-stable-imp-add-self)
  next
    case (rec a p)
    from rec.prems [of 0] have [simp]: ‹p = odd b›
      by (simp add: bit-0)
    from rec.hyps have ‹bit a n ⟷ bit (b div 2) n› for n
      using rec.prems [of ‹Suc n›] by (simp add: bit-Suc)
    then have ‹a = b div 2›
      by (rule rec.IH)
    then have ‹2 * a = 2 * (b div 2)›
      by simp
    then have ‹b mod 2 + 2 * a = b mod 2 + 2 * (b div 2)›
      by simp
    also have ‹... = b›
      by (fact mod-mult-div-eq)
    finally show ?case
      by (auto simp: mod2-eq-if)
  qed
qed

lemma bit-eq-rec:
  ‹a = b ⟷ (even a ⟷ even b) ∧ a div 2 = b div 2› (is ‹?P = ?Q›)
proof
  assume ?P
  then show ?Q
    by simp
next
  assume ?Q
  then have ‹even a ⟷ even b› and ‹a div 2 = b div 2›
```

by *simp-all*
show *?P*
proof (*rule bit-eqI*)
fix *n*
show ‹*bit a n* ⟷ *bit b n*›
proof (*cases n*)
case *0*
with ‹*even a* ⟷ *even b*› **show** *?thesis*
by (*simp add*: *bit-0*)
next
case (*Suc n*)
moreover from ‹*a div 2* = *b div 2*› **have** ‹*bit* (*a div 2*) *n* = *bit* (*b div 2*) *n*›
by *simp*
ultimately show *?thesis*
by (*simp add*: *bit-Suc*)
qed
qed
qed

lemma *bit-eq-iff*:
‹*a* = *b* ⟷ (∀ *n*. *possible-bit TYPE*(′*a*) *n* ⟶ *bit a n* ⟷ *bit b n*)›
by (*auto intro*: *bit-eqI simp add*: *possible-bit-def*)

lemma *bit-0-eq* [*simp*]:
‹*bit 0* = ⊥›
proof −
have ‹*0 div 2* ^ *n* = *0*› **for** *n*
unfolding *div-exp-eq-funpow-half* **by** (*induction n*) *simp-all*
then show *?thesis*
by (*simp add*: *fun-eq-iff bit-iff-odd*)
qed

lemma *bit-double-Suc-iff*:
‹*bit* (*2* ∗ *a*) (*Suc n*) ⟷ *possible-bit TYPE*(′*a*) (*Suc n*) ∧ *bit a n*›
using *even-double-div-exp-iff* [*of n a*]
by (*cases* ‹*possible-bit TYPE*(′*a*) (*Suc n*)›)
(*auto simp*: *bit-iff-odd possible-bit-def*)

lemma *bit-double-iff* [*bit-simps*]:
‹*bit* (*2* ∗ *a*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ *n* ≠ *0* ∧ *bit a* (*n* − *1*)›
by (*cases n*) (*simp-all add*: *bit-0 bit-double-Suc-iff*)

lemma *even-bit-succ-iff*:
‹*bit* (*1* + *a*) *n* ⟷ *bit a n* ∨ *n* = *0*› **if** ‹*even a*›
using *that* **by** (*cases* ‹*n* = *0*›) (*simp-all add*: *bit-iff-odd*)

lemma *odd-bit-iff-bit-pred*:
‹*bit a n* ⟷ *bit* (*a* − *1*) *n* ∨ *n* = *0*› **if** ‹*odd a*›
proof −

```
  from ‹odd a› obtain b where ‹a = 2 * b + 1› ..
  moreover have ‹bit (2 * b) n ∨ n = 0 ⟷ bit (1 + 2 * b) n›
    using even-bit-succ-iff by simp
  ultimately show ?thesis by (simp add: ac-simps)
qed

lemma bit-exp-iff [bit-simps]:
  ‹bit (2 ^ m) n ⟷ possible-bit TYPE('a) n ∧ n = m›
proof (cases ‹possible-bit TYPE('a) n›)
  case False
  then show ?thesis
    by (simp add: impossible-bit)
next
  case True
  then show ?thesis
  proof (induction n arbitrary: m)
    case 0
    show ?case
      by (simp add: bit-0)
  next
    case (Suc n)
    then have ‹possible-bit TYPE('a) n›
      by (simp add: possible-bit-less-imp)
    show ?case
    proof (cases m)
      case 0
      then show ?thesis
        by (simp add: bit-Suc)
    next
      case (Suc m)
      with Suc.IH [of m] ‹possible-bit TYPE('a) n› show ?thesis
        by (simp add: bit-double-Suc-iff)
    qed
  qed
qed

lemma bit-1-iff [bit-simps]:
  ‹bit 1 n ⟷ n = 0›
  using bit-exp-iff [of 0 n] by auto

lemma bit-2-iff [bit-simps]:
  ‹bit 2 n ⟷ possible-bit TYPE('a) 1 ∧ n = 1›
  using bit-exp-iff [of 1 n] by auto

lemma bit-of-bool-iff [bit-simps]:
  ‹bit (of-bool b) n ⟷ n = 0 ∧ b›
  by (simp add: bit-1-iff)

lemma bit-mod-2-iff [simp]:
```

```
  ‹bit (a mod 2) n ⟷ n = 0 ∧ odd a›
  by (simp add: mod-2-eq-odd bit-simps)

end

lemma nat-bit-induct [case-names zero even odd]:
  ‹P n› if zero: ‹P 0›
    and even: ‹⋀n. P n ⟹ n > 0 ⟹ P (2 * n)›
    and odd: ‹⋀n. P n ⟹ P (Suc (2 * n))›
proof (induction n rule: less-induct)
  case (less n)
  show ‹P n›
  proof (cases ‹n = 0›)
    case True with zero show ?thesis by simp
  next
    case False
    with less have hyp: ‹P (n div 2)› by simp
    show ?thesis
    proof (cases ‹even n›)
      case True
      then have ‹n ≠ 1›
        by auto
      with ‹n ≠ 0› have ‹n div 2 > 0›
        by simp
      with ‹even n› hyp even [of ‹n div 2›] show ?thesis
        by simp
    next
      case False
      with hyp odd [of ‹n div 2›] show ?thesis
        by simp
    qed
  qed
qed

instantiation nat :: semiring-bits
begin

definition bit-nat :: ‹nat ⇒ nat ⇒ bool›
  where ‹bit-nat m n ⟷ odd (m div 2 ^ n)›

instance
proof
  show ‹P n› if stable: ‹⋀n. n div 2 = n ⟹ P n›
    and rec: ‹⋀n b. P n ⟹ (of-bool b + 2 * n) div 2 = n ⟹ P (of-bool b + 2 *
n)›
    for P and n :: nat
  proof (induction n rule: nat-bit-induct)
    case zero
    from stable [of 0] show ?case
```

```
      by simp
  next
    case (even n)
    with rec [of n False] show ?case
      by simp
  next
    case (odd n)
    with rec [of n True] show ?case
      by simp
  qed
qed (auto simp: div-mult2-eq bit-nat-def)

end

lemma possible-bit-nat [simp]:
  ‹possible-bit TYPE(nat) n›
  by (simp add: possible-bit-def)

lemma bit-Suc-0-iff [bit-simps]:
  ‹bit (Suc 0) n ⟷ n = 0›
  using bit-1-iff [of n, where ?'a = nat] by simp

lemma not-bit-Suc-0-Suc [simp]:
  ‹¬ bit (Suc 0) (Suc n)›
  by (simp add: bit-Suc)

lemma not-bit-Suc-0-numeral [simp]:
  ‹¬ bit (Suc 0) (numeral n)›
  by (simp add: numeral-eq-Suc)

context semiring-bits
begin

lemma bit-of-nat-iff [bit-simps]:
  ‹bit (of-nat m) n ⟷ possible-bit TYPE('a) n ∧ bit m n›
proof (cases ‹possible-bit TYPE('a) n›)
  case False
  then show ?thesis
    by (simp add: impossible-bit)
next
  case True
  then have ‹bit (of-nat m) n ⟷ bit m n›
  proof (induction m arbitrary: n rule: nat-bit-induct)
    case zero
    then show ?case
      by simp
  next
    case (even m)
    then show ?case
```

by (*cases n*)
(*auto simp*: *bit-double-iff Bit-Operations.bit-double-iff possible-bit-def bit-0*
dest: *mult-not-zero*)
next
case (*odd m*)
then show *?case*
by (*cases n*)
(*auto simp*: *bit-double-iff even-bit-succ-iff possible-bit-def*
Bit-Operations.bit-Suc Bit-Operations.bit-0 dest: *mult-not-zero*)
qed
with *True* **show** *?thesis*
by *simp*
qed

end

lemma *int-bit-induct* [*case-names zero minus even odd*]:
‹*P k*› **if** *zero-int*: ‹*P 0*›
and *minus-int*: ‹$P\ (-\ 1)$›
and *even-int*: ‹$\bigwedge k.\ P\ k \Longrightarrow k \neq 0 \Longrightarrow P\ (k * 2)$›
and *odd-int*: ‹$\bigwedge k.\ P\ k \Longrightarrow k \neq -\ 1 \Longrightarrow P\ (1 + (k * 2))$› **for** *k* :: *int*
proof (*cases* ‹$k \geq 0$›)
case *True*
define *n* **where** ‹*n* = *nat k*›
with *True* **have** ‹*k* = *int n*›
by *simp*
then show ‹*P k*›
proof (*induction n arbitrary*: *k rule*: *nat-bit-induct*)
case *zero*
then show *?case*
by (*simp add*: *zero-int*)
next
case (*even n*)
have ‹$P\ (int\ n * 2)$›
by (*rule even-int*) (*use even* **in** *simp-all*)
with *even* **show** *?case*
by (*simp add*: *ac-simps*)
next
case (*odd n*)
have ‹$P\ (1 + (int\ n * 2))$›
by (*rule odd-int*) (*use odd* **in** *simp-all*)
with *odd* **show** *?case*
by (*simp add*: *ac-simps*)
qed
next
case *False*
define *n* **where** ‹$n = nat\ (-\ k - 1)$›
with *False* **have** ‹$k = -\ int\ n - 1$›
by *simp*

then show ‹*P k*›
proof (*induction n arbitrary*: *k rule*: *nat-bit-induct*)
case *zero*
then show *?case*
by (*simp add*: *minus-int*)
next
case (*even n*)
have ‹*P* (*1* + (− *int* (*Suc n*) ∗ *2*))›
by (*rule odd-int*) (*use even* **in** ‹*simp-all add*: *algebra-simps*›)
also have ‹. . . = − *int* (*2* ∗ *n*) − *1*›
by (*simp add*: *algebra-simps*)
finally show *?case*
using *even.prems* **by** *simp*
next
case (*odd n*)
have ‹*P* (− *int* (*Suc n*) ∗ *2*)›
by (*rule even-int*) (*use odd* **in** ‹*simp-all add*: *algebra-simps*›)
also have ‹. . . = − *int* (*Suc* (*2* ∗ *n*)) − *1*›
by (*simp add*: *algebra-simps*)
finally show *?case*
using *odd.prems* **by** *simp*
qed
qed

instantiation *int* :: *semiring-bits*
begin

definition *bit-int* :: ‹*int* ⇒ *nat* ⇒ *bool*›
where ‹*bit-int k n* ⟷ *odd* (*k div 2* ^ *n*)›

instance
proof
show ‹*P k*› **if** *stable*: ‹⋀*k*. *k div 2* = *k* ⟹ *P k*›
and *rec*: ‹⋀*k b*. *P k* ⟹ (*of-bool b* + *2* ∗ *k*) *div 2* = *k* ⟹ *P* (*of-bool b* + *2* ∗ *k*)›
for *P* **and** *k* :: *int*
proof (*induction k rule*: *int-bit-induct*)
case *zero*
from *stable* [*of 0*] **show** *?case*
by *simp*
next
case *minus*
from *stable* [*of* ‹− *1*›] **show** *?case*
by *simp*
next
case (*even k*)
with *rec* [*of k False*] **show** *?case*
by (*simp add*: *ac-simps*)
next

case (*odd k*)
with *rec* [*of k True*] **show** *?case*
by (*simp add*: *ac-simps*)
qed
qed (*auto simp*: *zdiv-zmult2-eq bit-int-def*)

end

lemma *possible-bit-int* [*simp*]:
‹*possible-bit TYPE*(*int*) *n*›
by (*simp add*: *possible-bit-def*)

lemma *bit-nat-iff* [*bit-simps*]:
‹*bit* (*nat k*) *n* ⟷ *k* ≥ *0* ∧ *bit k n*›
proof (*cases* ‹*k* ≥ *0*›)
case *True*
moreover define *m* **where** ‹*m* = *nat k*›
ultimately have ‹*k* = *int m*›
by *simp*
then show *?thesis*
by (*simp add*: *bit-simps*)
next
case *False*
then show *?thesis*
by *simp*
qed

68.2 Bit operations

class *semiring-bit-operations* = *semiring-bits* +
fixes *and* :: ‹′*a* ⇒ ′*a* ⇒ ′*a*› (**infixr** ‹*AND*› *64*)
and *or* :: ‹′*a* ⇒ ′*a* ⇒ ′*a*› (**infixr** ‹*OR*› *59*)
and *xor* :: ‹′*a* ⇒ ′*a* ⇒ ′*a*› (**infixr** ‹*XOR*› *59*)
and *mask* :: ‹*nat* ⇒ ′*a*›
and *set-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
and *unset-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
and *flip-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
and *push-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
and *drop-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
and *take-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
assumes *and-rec*: ‹*a AND b* = *of-bool* (*odd a* ∧ *odd b*) + *2* ∗ ((*a div 2*) *AND* (*b div 2*))›
and *or-rec*: ‹*a OR b* = *of-bool* (*odd a* ∨ *odd b*) + *2* ∗ ((*a div 2*) *OR* (*b div 2*))›
and *xor-rec*: ‹*a XOR b* = *of-bool* (*odd a* ≠ *odd b*) + *2* ∗ ((*a div 2*) *XOR* (*b div 2*))›
and *mask-eq-exp-minus-1*: ‹*mask n* = *2* ^ *n* − *1*›
and *set-bit-eq-or*: ‹*set-bit n a* = *a OR push-bit n 1*›
and *unset-bit-eq-or-xor*: ‹*unset-bit n a* = (*a OR push-bit n 1*) *XOR push-bit n 1*›

and *flip-bit-eq-xor*: ‹*flip-bit n a = a XOR push-bit n 1*›
and *push-bit-eq-mult*: ‹*push-bit n a = a * 2 ^ n*›
and *drop-bit-eq-div*: ‹*drop-bit n a = a div 2 ^ n*›
and *take-bit-eq-mod*: ‹*take-bit n a = a mod 2 ^ n*›
begin

We want the bitwise operations to bind slightly weaker than + and −.

Logically, *push-bit*, *drop-bit* and *take-bit* are just aliases; having them as separate operations makes proofs easier, otherwise proof automation would fiddle with concrete expressions $(2::'a)^n$ in a way obfuscating the basic algebraic relationships between those operations.

For the sake of code generation operations are specified as definitional class operations, taking into account that specific instances of these can be implemented differently wrt. code generation.

lemma *bit-iff-odd-drop-bit*:
‹*bit a n* ⟷ *odd* (*drop-bit n a*)›
by (*simp add*: *bit-iff-odd drop-bit-eq-div*)

lemma *even-drop-bit-iff-not-bit*:
‹*even* (*drop-bit n a*) ⟷ ¬ *bit a n*›
by (*simp add*: *bit-iff-odd-drop-bit*)

lemma *bit-and-iff* [*bit-simps*]:
‹*bit* (*a AND b*) *n* ⟷ *bit a n* ∧ *bit b n*›
proof (*induction n arbitrary*: *a b*)
case *0*
show *?case*
by (*simp add*: *bit-0 and-rec* [*of a b*] *even-bit-succ-iff*)
next
case (*Suc n*)
from *Suc* [*of* ‹*a div 2*› ‹*b div 2*›]
show *?case*
by (*simp add*: *and-rec* [*of a b*] *bit-Suc*)
(*auto simp flip*: *bit-Suc simp add*: *bit-double-iff dest*: *bit-imp-possible-bit*)
qed

lemma *bit-or-iff* [*bit-simps*]:
‹*bit* (*a OR b*) *n* ⟷ *bit a n* ∨ *bit b n*›
proof (*induction n arbitrary*: *a b*)
case *0*
show *?case*
by (*simp add*: *bit-0 or-rec* [*of a b*] *even-bit-succ-iff*)
next
case (*Suc n*)
from *Suc* [*of* ‹*a div 2*› ‹*b div 2*›]
show *?case*
by (*simp add*: *or-rec* [*of a b*] *bit-Suc*)
(*auto simp flip*: *bit-Suc simp add*: *bit-double-iff dest*: *bit-imp-possible-bit*)

qed

lemma *bit-xor-iff* [*bit-simps*]:
‹*bit* (*a XOR b*) *n* ⟷ *bit a n* ≠ *bit b n*›
proof (*induction n arbitrary*: *a b*)
case *0*
show *?case*
by (*simp add*: *bit-0 xor-rec* [*of a b*] *even-bit-succ-iff*)
next
case (*Suc n*)
from *Suc* [*of* ‹*a div 2*› ‹*b div 2*›]
show *?case*
by (*simp add*: *xor-rec* [*of a b*] *bit-Suc*)
(*auto simp flip*: *bit-Suc simp add*: *bit-double-iff dest*: *bit-imp-possible-bit*)
qed

sublocale *and*: *semilattice* ‹(*AND*)›
by *standard* (*auto simp*: *bit-eq-iff bit-and-iff*)

sublocale *or*: *semilattice-neutr* ‹(*OR*)› *0*
by *standard* (*auto simp*: *bit-eq-iff bit-or-iff*)

sublocale *xor*: *comm-monoid* ‹(*XOR*)› *0*
by *standard* (*auto simp*: *bit-eq-iff bit-xor-iff*)

lemma *even-and-iff*:
‹*even* (*a AND b*) ⟷ *even a* ∨ *even b*›
using *bit-and-iff* [*of a b 0*] **by** (*auto simp*: *bit-0*)

lemma *even-or-iff*:
‹*even* (*a OR b*) ⟷ *even a* ∧ *even b*›
using *bit-or-iff* [*of a b 0*] **by** (*auto simp*: *bit-0*)

lemma *even-xor-iff*:
‹*even* (*a XOR b*) ⟷ (*even a* ⟷ *even b*)›
using *bit-xor-iff* [*of a b 0*] **by** (*auto simp*: *bit-0*)

lemma *zero-and-eq* [*simp*]:
‹*0 AND a* = *0*›
by (*simp add*: *bit-eq-iff bit-and-iff*)

lemma *and-zero-eq* [*simp*]:
‹*a AND 0* = *0*›
by (*simp add*: *bit-eq-iff bit-and-iff*)

lemma *one-and-eq*:
‹*1 AND a* = *a mod 2*›
by (*simp add*: *bit-eq-iff bit-and-iff*) (*auto simp*: *bit-1-iff bit-0*)

lemma *and-one-eq*:
‹*a AND 1 = a mod 2*›
using *one-and-eq* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *one-or-eq*:
‹*1 OR a = a + of-bool* (*even a*)›
by (*simp add*: *bit-eq-iff bit-or-iff add.commute* [*of - 1*] *even-bit-succ-iff*)
(*auto simp*: *bit-1-iff bit-0*)

lemma *or-one-eq*:
‹*a OR 1 = a + of-bool* (*even a*)›
using *one-or-eq* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *one-xor-eq*:
‹*1 XOR a = a + of-bool* (*even a*) − *of-bool* (*odd a*)›
by (*simp add*: *bit-eq-iff bit-xor-iff add.commute* [*of - 1*] *even-bit-succ-iff*)
(*auto simp*: *bit-1-iff odd-bit-iff-bit-pred bit-0 elim*: *oddE*)

lemma *xor-one-eq*:
‹*a XOR 1 = a + of-bool* (*even a*) − *of-bool* (*odd a*)›
using *one-xor-eq* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *xor-self-eq* [*simp*]:
‹*a XOR a = 0*›
by (*rule bit-eqI*) (*simp add*: *bit-simps*)

lemma *mask-0* [*simp*]:
‹*mask 0 = 0*›
by (*simp add*: *mask-eq-exp-minus-1*)

lemma *inc-mask-eq-exp*:
‹*mask n + 1 = 2 ^ n*›
proof (*induction n*)
case *0*
then show *?case*
by *simp*
next
case (*Suc n*)
from *Suc.IH* [*symmetric*] **have** ‹*2 ^ Suc n = 2 ∗ mask n + 2*›
by (*simp add*: *algebra-simps*)
also have ‹*. . . = 2 ∗ mask n + 1 + 1*›
by (*simp add*: *add.assoc*)
finally have ∗: ‹*2 ^ Suc n = 2 ∗ mask n + 1 + 1*› **.**
then show *?case*
by (*simp add*: *mask-eq-exp-minus-1*)
qed

lemma *mask-Suc-double*:
‹*mask* (*Suc n*) = *1 OR 2 ∗ mask n*›

proof −
  have ‹*mask* (*Suc n*) + *1* = (*mask n* + *1*) + (*mask n* + *1*)›
    by (*simp add*: *inc-mask-eq-exp mult-2*)
  also have ‹. . . = (*1 OR 2* ∗ *mask n*) + *1*›
    by (*simp add*: *one-or-eq mult-2-right algebra-simps*)
  finally show *?thesis*
    by *simp*
qed

lemma *bit-mask-iff* [*bit-simps*]:
  ‹*bit* (*mask m*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ *n* < *m*›
proof (*cases* ‹*possible-bit TYPE*(′*a*) *n*›)
  case *False*
  then show *?thesis*
    by (*simp add*: *impossible-bit*)
next
  case *True*
  then have ‹*bit* (*mask m*) *n* ⟷ *n* < *m*›
  proof (*induction m arbitrary*: *n*)
    case *0*
    then show *?case*
      by (*simp add*: *bit-iff-odd*)
  next
    case (*Suc m*)
    show *?case*
    proof (*cases n*)
      case *0*
      then show *?thesis*
        by (*simp add*: *bit-0 mask-Suc-double even-or-iff*)
    next
      case (*Suc n*)
      with *Suc.prems* **have** ‹*possible-bit TYPE*(′*a*) *n*›
        using *possible-bit-less-imp* **by** *auto*
      with *Suc.IH* [*of n*] **have** ‹*bit* (*mask m*) *n* ⟷ *n* < *m*› **.**
      with *Suc.prems* **show** *?thesis*
        by (*simp add*: *Suc mask-Suc-double bit-simps*)
    qed
  qed
  with *True* **show** *?thesis*
    by *simp*
qed

lemma *even-mask-iff*:
  ‹*even* (*mask n*) ⟷ *n* = *0*›
  using *bit-mask-iff* [*of n 0*] **by** (*auto simp*: *bit-0*)

lemma *mask-Suc-0* [*simp*]:
  ‹*mask* (*Suc 0*) = *1*›
  by (*simp add*: *mask-Suc-double*)

lemma *mask-Suc-exp*:
‹*mask* (*Suc n*) = *2* ^ *n OR mask n*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *mask-numeral*:
‹*mask* (*numeral n*) = *1* + *2* ∗ *mask* (*pred-numeral n*)›
by (*simp add*: *numeral-eq-Suc mask-Suc-double one-or-eq ac-simps*)

lemma *push-bit-0-id* [*simp*]:
‹*push-bit 0* = *id*›
by (*simp add*: *fun-eq-iff push-bit-eq-mult*)

lemma *push-bit-Suc* [*simp*]:
‹*push-bit* (*Suc n*) *a* = *push-bit n* (*a* ∗ *2*)›
by (*simp add*: *push-bit-eq-mult ac-simps*)

lemma *push-bit-double*:
‹*push-bit n* (*a* ∗ *2*) = *push-bit n a* ∗ *2*›
by (*simp add*: *push-bit-eq-mult ac-simps*)

lemma *bit-push-bit-iff* [*bit-simps*]:
‹*bit* (*push-bit m a*) *n* ⟷ *m* ≤ *n* ∧ *possible-bit TYPE*(*'a*) *n* ∧ *bit a* (*n* − *m*)›
proof (*induction n arbitrary*: *m*)
case *0*
then show *?case*
by (*auto simp*: *bit-0 push-bit-eq-mult*)
next
case (*Suc n*)
show *?case*
proof (*cases m*)
case *0*
then show *?thesis*
by (*auto simp*: *bit-imp-possible-bit*)
next
case (*Suc m'*)
with *Suc.prems Suc.IH* [*of m'*] **show** *?thesis*
apply (*simp add*: *push-bit-double*)
apply (*auto simp*: *possible-bit-less-imp bit-simps mult.commute* [*of* - *2*])
done
qed
qed

lemma *funpow-double-eq-push-bit*:
‹*times 2* ^^ *n* = *push-bit n*›
by (*induction n*) (*simp-all add*: *fun-eq-iff push-bit-double ac-simps*)

lemma *div-push-bit-of-1-eq-drop-bit*:
‹*a div push-bit n 1* = *drop-bit n a*›

by (*simp add*: *push-bit-eq-mult drop-bit-eq-div*)

lemma *bits-ident*:
‹*push-bit n* (*drop-bit n a*) + *take-bit n a* = *a*›
using *div-mult-mod-eq* **by** (*simp add*: *push-bit-eq-mult take-bit-eq-mod drop-bit-eq-div*)

lemma *push-bit-push-bit* [*simp*]:
‹*push-bit m* (*push-bit n a*) = *push-bit* (*m* + *n*) *a*›
by (*simp add*: *push-bit-eq-mult power-add ac-simps*)

lemma *push-bit-of-0* [*simp*]:
‹*push-bit n 0* = *0*›
by (*simp add*: *push-bit-eq-mult*)

lemma *push-bit-of-1* [*simp*]:
‹*push-bit n 1* = *2* ^ *n*›
by (*simp add*: *push-bit-eq-mult*)

lemma *push-bit-add*:
‹*push-bit n* (*a* + *b*) = *push-bit n a* + *push-bit n b*›
by (*simp add*: *push-bit-eq-mult algebra-simps*)

lemma *push-bit-numeral* [*simp*]:
‹*push-bit* (*numeral l*) (*numeral k*) = *push-bit* (*pred-numeral l*) (*numeral* (*Num.Bit0* *k*))›
by (*simp add*: *numeral-eq-Suc mult-2-right*) (*simp add*: *numeral-Bit0*)

lemma *bit-drop-bit-eq* [*bit-simps*]:
‹*bit* (*drop-bit n a*) = *bit a* ∘ (+) *n*›
by *rule* (*simp add*: *drop-bit-eq-div bit-iff-odd div-exp-eq*)

lemma *disjunctive-xor-eq-or*:
‹*a XOR b* = *a OR b*› **if** ‹*a AND b* = *0*›
using *that* **by** (*auto simp*: *bit-eq-iff bit-simps*)

lemma *disjunctive-add-eq-or*:
‹*a* + *b* = *a OR b*› **if** ‹*a AND b* = *0*›
proof (*rule bit-eqI*)
fix *n*
assume ‹*possible-bit TYPE*(*'a*) *n*›
moreover from *that* **have** ‹⋀*n*. ¬ *bit* (*a AND b*) *n*›
by *simp*
then have ‹⋀*n*. ¬ *bit a n* ∨ ¬ *bit b n*›
by (*simp add*: *bit-simps*)
ultimately show ‹*bit* (*a* + *b*) *n* ⟷ *bit* (*a OR b*) *n*›
proof (*induction n arbitrary*: *a b*)
case *0*
from *0*(*2*)[*of 0*] **show** *?case*
by (*auto simp*: *even-or-iff bit-0*)

next
case (*Suc n*)
from *Suc.prems*(*2*) [*of 0*] **have** *even*: ‹*even a* ∨ *even b*›
by (*auto simp*: *bit-0*)
have *bit*: ‹¬ *bit* (*a div 2*) *n* ∨ ¬ *bit* (*b div 2*) *n*› **for** *n*
using *Suc.prems*(*2*) [*of* ‹*Suc n*›] **by** (*simp add*: *bit-Suc*)
from *Suc.prems* **have** ‹*possible-bit TYPE*(′*a*) *n*›
using *possible-bit-less-imp* **by** *force*
with ‹⋀*n*. ¬ *bit* (*a div 2*) *n* ∨ ¬ *bit* (*b div 2*) *n*› *Suc.IH* [*of* ‹*a div 2*› ‹*b div 2*›]
have *IH*: ‹*bit* (*a div 2* + *b div 2*) *n* ⟷ *bit* (*a div 2 OR b div 2*) *n*›
by (*simp add*: *bit-Suc*)
have ‹*a* + *b* = (*a div 2* ∗ *2* + *a mod 2*) + (*b div 2* ∗ *2* + *b mod 2*)›
using *div-mult-mod-eq* [*of a 2*] *div-mult-mod-eq* [*of b 2*] **by** *simp*
also have ‹. . . = *of-bool* (*odd a* ∨ *odd b*) + *2* ∗ (*a div 2* + *b div 2*)›
using *even* **by** (*auto simp*: *algebra-simps mod2-eq-if*)
finally have ‹*bit* ((*a* + *b*) *div 2*) *n* ⟷ *bit* (*a div 2* + *b div 2*) *n*›
using ‹*possible-bit TYPE*(′*a*) (*Suc n*)› **by** *simp* (*simp-all flip*: *bit-Suc add*: *bit-double-iff possible-bit-def*)
also have ‹. . . ⟷ *bit* (*a div 2 OR b div 2*) *n*›
by (*rule IH*)
finally show *?case*
by (*simp add*: *bit-simps flip*: *bit-Suc*)
qed
qed

lemma *disjunctive-add-eq-xor*:
‹*a* + *b* = *a XOR b*› **if** ‹*a AND b* = *0*›
using *that* **by** (*simp add*: *disjunctive-add-eq-or disjunctive-xor-eq-or*)

lemma *take-bit-0* [*simp*]:
take-bit 0 a = *0*
by (*simp add*: *take-bit-eq-mod*)

lemma *bit-take-bit-iff* [*bit-simps*]:
‹*bit* (*take-bit m a*) *n* ⟷ *n* < *m* ∧ *bit a n*›
proof −
have ‹*push-bit m* (*drop-bit m a*) *AND take-bit m a* = *0*› (**is** ‹*?lhs* = -›)
proof (*rule bit-eqI*)
fix *n*
show ‹*bit ?lhs n* ⟷ *bit 0 n*›
proof (*cases* ‹*m* ≤ *n*›)
case *False*
then show *?thesis*
by (*simp add*: *bit-simps*)
next
case *True*
moreover define *q* **where** ‹*q* = *n* − *m*›
ultimately have ‹*n* = *m* + *q*› **by** *simp*

moreover have ‹¬ *bit* (*take-bit m a*) (*m* + *q*)›
by (*simp add*: *take-bit-eq-mod bit-iff-odd flip*: *div-exp-eq*)
ultimately show *?thesis*
by (*simp add*: *bit-simps*)
qed
qed
then have ‹*push-bit m* (*drop-bit m a*) *XOR take-bit m a* = *push-bit m* (*drop-bit m a*) + *take-bit m a*›
by (*simp add*: *disjunctive-add-eq-xor*)
also have ‹. . . = *a*›
by (*simp add*: *bits-ident*)
finally have ‹*bit* (*push-bit m* (*drop-bit m a*) *XOR take-bit m a*) *n* ⟷ *bit a n*›
by *simp*
also have ‹. . . ⟷ (*m* ≤ *n* ∨ *n* < *m*) ∧ *bit a n*›
by *auto*
also have ‹. . . ⟷ *m* ≤ *n* ∧ *bit a n* ∨ *n* < *m* ∧ *bit a n*›
by *auto*
also have ‹*m* ≤ *n* ∧ *bit a n* ⟷ *bit* (*push-bit m* (*drop-bit m a*)) *n*›
by (*auto simp*: *bit-simps bit-imp-possible-bit*)
finally show *?thesis*
by (*auto simp*: *bit-simps*)
qed

lemma *take-bit-Suc*:
‹*take-bit* (*Suc n*) *a* = *take-bit n* (*a div 2*) ∗ *2* + *a mod 2*› (**is** ‹*?lhs* = *?rhs*›)
proof (*rule bit-eqI*)
fix *m*
assume ‹*possible-bit TYPE*(*'a*) *m*›
then show ‹*bit ?lhs m* ⟷ *bit ?rhs m*›
apply (*cases a rule*: *parity-cases*; *cases m*)
apply (*simp-all add*: *bit-simps even-bit-succ-iff mult.commute* [*of - 2*] *add.commute* [*of - 1*] *flip*: *bit-Suc*)
apply (*simp-all add*: *bit-0*)
done
qed

lemma *take-bit-rec*:
‹*take-bit n a* = (*if n* = *0 then 0 else take-bit* (*n* − *1*) (*a div 2*) ∗ *2* + *a mod 2*)›
by (*cases n*) (*simp-all add*: *take-bit-Suc*)

lemma *take-bit-Suc-0* [*simp*]:
‹*take-bit* (*Suc 0*) *a* = *a mod 2*›
by (*simp add*: *take-bit-eq-mod*)

lemma *take-bit-of-0* [*simp*]:
‹*take-bit n 0* = *0*›
by (*rule bit-eqI*) (*simp add*: *bit-simps*)

lemma *take-bit-of-1* [*simp*]:

‹*take-bit n 1 = of-bool (n > 0)*›
by (*cases n*) (*simp-all add*: *take-bit-Suc*)

lemma *drop-bit-of-0* [*simp*]:
‹*drop-bit n 0 = 0*›
by (*rule bit-eqI*) (*simp add*: *bit-simps*)

lemma *drop-bit-of-1* [*simp*]:
‹*drop-bit n 1 = of-bool (n = 0)*›
by (*rule bit-eqI*) (*simp add*: *bit-simps ac-simps*)

lemma *drop-bit-0* [*simp*]:
‹*drop-bit 0 = id*›
by (*simp add*: *fun-eq-iff drop-bit-eq-div*)

lemma *drop-bit-Suc*:
‹*drop-bit (Suc n) a = drop-bit n (a div 2)*›
using *div-exp-eq* [*of a 1*] **by** (*simp add*: *drop-bit-eq-div*)

lemma *drop-bit-rec*:
‹*drop-bit n a = (if n = 0 then a else drop-bit (n − 1) (a div 2))*›
by (*cases n*) (*simp-all add*: *drop-bit-Suc*)

lemma *drop-bit-half*:
‹*drop-bit n (a div 2) = drop-bit n a div 2*›
by (*induction n arbitrary*: *a*) (*simp-all add*: *drop-bit-Suc*)

lemma *drop-bit-of-bool* [*simp*]:
‹*drop-bit n (of-bool b) = of-bool (n = 0 ∧ b)*›
by (*cases n*) *simp-all*

lemma *even-take-bit-eq* [*simp*]:
‹*even (take-bit n a) ⟷ n = 0 ∨ even a*›
by (*simp add*: *take-bit-rec* [*of n a*])

lemma *take-bit-take-bit* [*simp*]:
‹*take-bit m (take-bit n a) = take-bit (min m n) a*›
by (*rule bit-eqI*) (*simp add*: *bit-simps*)

lemma *drop-bit-drop-bit* [*simp*]:
‹*drop-bit m (drop-bit n a) = drop-bit (m + n) a*›
by (*simp add*: *drop-bit-eq-div power-add div-exp-eq ac-simps*)

lemma *push-bit-take-bit*:
‹*push-bit m (take-bit n a) = take-bit (m + n) (push-bit m a)*›
by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *take-bit-push-bit*:
‹*take-bit m (push-bit n a) = push-bit n (take-bit (m − n) a)*›

by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *take-bit-drop-bit*:
‹*take-bit m* (*drop-bit n a*) = *drop-bit n* (*take-bit* (*m* + *n*) *a*)›
by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *drop-bit-take-bit*:
‹*drop-bit m* (*take-bit n a*) = *take-bit* (*n* − *m*) (*drop-bit m a*)›
by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *even-push-bit-iff* [*simp*]:
‹*even* (*push-bit n a*) ⟷ *n* ≠ *0* ∨ *even a*›
by (*simp add*: *push-bit-eq-mult*) *auto*

lemma *stable-imp-drop-bit-eq*:
‹*drop-bit n a* = *a*›
if ‹*a div 2* = *a*›
by (*induction n*) (*simp-all add*: *that drop-bit-Suc*)

lemma *stable-imp-take-bit-eq*:
‹*take-bit n a* = (*if even a then 0 else mask n*)›
if ‹*a div 2* = *a*›
by (*rule bit-eqI*) (*use that* **in** ‹*simp add*: *bit-simps stable-imp-bit-iff-odd*›)

lemma *exp-dvdE*:
assumes ‹*2* ^ *n dvd a*›
obtains *b* **where** ‹*a* = *push-bit n b*›
proof −
from *assms* **obtain** *b* **where** ‹*a* = *2* ^ *n* ∗ *b*› **..**
then have ‹*a* = *push-bit n b*›
by (*simp add*: *push-bit-eq-mult ac-simps*)
with *that* **show** *thesis* **.**
qed

lemma *take-bit-eq-0-iff*:
‹*take-bit n a* = *0* ⟷ *2* ^ *n dvd a*› (**is** ‹*?P* ⟷ *?Q*›)
proof
assume *?P*
then show *?Q*
by (*simp add*: *take-bit-eq-mod mod-0-imp-dvd*)
next
assume *?Q*
then obtain *b* **where** ‹*a* = *push-bit n b*›
by (*rule exp-dvdE*)
then show *?P*
by (*simp add*: *take-bit-push-bit*)
qed

lemma *take-bit-tightened*:

‹*take-bit m a* = *take-bit m b*› **if** ‹*take-bit n a* = *take-bit n b*› **and** ‹*m* ≤ *n*›
proof −
  from *that* **have** ‹*take-bit m* (*take-bit n a*) = *take-bit m* (*take-bit n b*)›
    by *simp*
  then have ‹*take-bit* (*min m n*) *a* = *take-bit* (*min m n*) *b*›
    by *simp*
  with *that* **show** *?thesis*
    by (*simp add*: *min-def*)
qed

lemma *take-bit-eq-self-iff-drop-bit-eq-0*:
  ‹*take-bit n a* = *a* ⟷ *drop-bit n a* = *0*› (**is** ‹*?P* ⟷ *?Q*›)
proof
  assume *?P*
  show *?Q*
  proof (*rule bit-eqI*)
    fix *m*
    from ‹*?P*› **have** ‹*a* = *take-bit n a*› **..**
    also have ‹¬ *bit* (*take-bit n a*) (*n* + *m*)›
      unfolding *bit-simps*
      by (*simp add*: *bit-simps*)
    finally show ‹*bit* (*drop-bit n a*) *m* ⟷ *bit 0 m*›
      by (*simp add*: *bit-simps*)
  qed
next
  assume *?Q*
  show *?P*
  proof (*rule bit-eqI*)
    fix *m*
    from ‹*?Q*› **have** ‹¬ *bit* (*drop-bit n a*) (*m* − *n*)›
      by *simp*
    then have ‹ ¬ *bit a* (*n* + (*m* − *n*))›
      by (*simp add*: *bit-simps*)
    then show ‹*bit* (*take-bit n a*) *m* ⟷ *bit a m*›
      by (*cases* ‹*m* < *n*›) (*auto simp*: *bit-simps*)
  qed
qed

lemma *drop-bit-exp-eq*:
  ‹*drop-bit m* (*2* ^ *n*) = *of-bool* (*m* ≤ *n* ∧ *possible-bit TYPE*(*'a*) *n*) ∗ *2* ^ (*n* − *m*)›
  by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *take-bit-and* [*simp*]:
  ‹*take-bit n* (*a AND b*) = *take-bit n a AND take-bit n b*›
  by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *take-bit-or* [*simp*]:
  ‹*take-bit n* (*a OR b*) = *take-bit n a OR take-bit n b*›
  by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *take-bit-xor* [*simp*]:
‹*take-bit n* (*a XOR b*) = *take-bit n a XOR take-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *push-bit-and* [*simp*]:
‹*push-bit n* (*a AND b*) = *push-bit n a AND push-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *push-bit-or* [*simp*]:
‹*push-bit n* (*a OR b*) = *push-bit n a OR push-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *push-bit-xor* [*simp*]:
‹*push-bit n* (*a XOR b*) = *push-bit n a XOR push-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *drop-bit-and* [*simp*]:
‹*drop-bit n* (*a AND b*) = *drop-bit n a AND drop-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *drop-bit-or* [*simp*]:
‹*drop-bit n* (*a OR b*) = *drop-bit n a OR drop-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *drop-bit-xor* [*simp*]:
‹*drop-bit n* (*a XOR b*) = *drop-bit n a XOR drop-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *take-bit-of-mask* [*simp*]:
‹*take-bit m* (*mask n*) = *mask* (*min m n*)›
by (*rule bit-eqI*) (*simp add*: *bit-simps*)

lemma *take-bit-eq-mask*:
‹*take-bit n a* = *a AND mask n*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *or-eq-0-iff*:
‹*a OR b* = *0* ⟷ *a* = *0* ∧ *b* = *0*›
by (*auto simp*: *bit-eq-iff bit-or-iff*)

lemma *bit-iff-and-drop-bit-eq-1*:
‹*bit a n* ⟷ *drop-bit n a AND 1* = *1*›
by (*simp add*: *bit-iff-odd-drop-bit and-one-eq odd-iff-mod-2-eq-one*)

lemma *bit-iff-and-push-bit-not-eq-0*:
‹*bit a n* ⟷ *a AND push-bit n 1* ≠ *0*›
by (*cases* ‹*possible-bit TYPE*(*'a*) *n*›) (*simp-all add*: *bit-eq-iff bit-simps impossible-bit*)

lemma *bit-set-bit-iff* [*bit-simps*]:
‹*bit* (*set-bit m a*) *n* ⟷ *bit a n* ∨ (*m* = *n* ∧ *possible-bit TYPE*(*'a*) *n*)›
by (*auto simp*: *set-bit-eq-or bit-or-iff bit-exp-iff*)

lemma *even-set-bit-iff*:
‹*even* (*set-bit m a*) ⟷ *even a* ∧ *m* ≠ *0*›
using *bit-set-bit-iff* [*of m a 0*] **by** (*auto simp*: *bit-0*)

lemma *bit-unset-bit-iff* [*bit-simps*]:
‹*bit* (*unset-bit m a*) *n* ⟷ *bit a n* ∧ *m* ≠ *n*›
by (*auto simp*: *unset-bit-eq-or-xor bit-simps dest*: *bit-imp-possible-bit*)

lemma *even-unset-bit-iff*:
‹*even* (*unset-bit m a*) ⟷ *even a* ∨ *m* = *0*›
using *bit-unset-bit-iff* [*of m a 0*] **by** (*auto simp*: *bit-0*)

lemma *bit-flip-bit-iff* [*bit-simps*]:
‹*bit* (*flip-bit m a*) *n* ⟷ (*m* = *n* ⟷ ¬ *bit a n*) ∧ *possible-bit TYPE*(*'a*) *n*›
by (*auto simp*: *bit-eq-iff bit-simps flip-bit-eq-xor bit-imp-possible-bit*)

lemma *even-flip-bit-iff*:
‹*even* (*flip-bit m a*) ⟷ ¬ (*even a* ⟷ *m* = *0*)›
using *bit-flip-bit-iff* [*of m a 0*] **by** (*auto simp*: *possible-bit-def bit-0*)

lemma *and-exp-eq-0-iff-not-bit*:
‹*a AND 2* ^ *n* = *0* ⟷ ¬ *bit a n*› (**is** ‹*?P* ⟷ *?Q*›)
using *bit-imp-possible-bit*[*of a n*]
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *bit-sum-mult-2-cases*:
assumes *a*: ‹∀ *j*. ¬ *bit a* (*Suc j*)›
shows ‹*bit* (*a* + *2* ∗ *b*) *n* = (*if n* = *0 then odd a else bit* (*2* ∗ *b*) *n*)›
proof −
from *a* **have** ‹*n* = *0*› **if** ‹*bit a n*› **for** *n* **using** *that*
by (*cases n*) *simp-all*
then have ‹*a* = *0* ∨ *a* = *1*›
by (*auto simp*: *bit-eq-iff bit-1-iff*)
then show *?thesis*
by (*cases n*) (*auto simp*: *bit-0 bit-double-iff even-bit-succ-iff*)
qed

lemma *set-bit-0*:
‹*set-bit 0 a* = *1* + *2* ∗ (*a div 2*)›
by (*auto simp*: *bit-eq-iff bit-simps even-bit-succ-iff simp flip*: *bit-Suc*)

lemma *set-bit-Suc*:
‹*set-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *set-bit n* (*a div 2*)›
by (*auto simp*: *bit-eq-iff bit-sum-mult-2-cases bit-simps bit-0 simp flip*: *bit-Suc*

elim: *possible-bit-less-imp*)

lemma *unset-bit-0*:
‹*unset-bit 0 a* = *2* ∗ (*a div 2*)›
by (*auto simp*: *bit-eq-iff bit-simps simp flip*: *bit-Suc*)

lemma *unset-bit-Suc*:
‹*unset-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *unset-bit n* (*a div 2*)›
by (*auto simp*: *bit-eq-iff bit-sum-mult-2-cases bit-simps bit-0 simp flip*: *bit-Suc*)

lemma *flip-bit-0*:
‹*flip-bit 0 a* = *of-bool* (*even a*) + *2* ∗ (*a div 2*)›
by (*auto simp*: *bit-eq-iff bit-simps even-bit-succ-iff bit-0 simp flip*: *bit-Suc*)

lemma *flip-bit-Suc*:
‹*flip-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *flip-bit n* (*a div 2*)›
by (*auto simp*: *bit-eq-iff bit-sum-mult-2-cases bit-simps bit-0 simp flip*: *bit-Suc*
elim: *possible-bit-less-imp*)

lemma *flip-bit-eq-if*:
‹*flip-bit n a* = (*if bit a n then unset-bit else set-bit*) *n a*›
by (*rule bit-eqI*) (*auto simp*: *bit-set-bit-iff bit-unset-bit-iff bit-flip-bit-iff*)

lemma *take-bit-set-bit-eq*:
‹*take-bit n* (*set-bit m a*) = (*if n* ≤ *m then take-bit n a else set-bit m* (*take-bit n a*))›
by (*rule bit-eqI*) (*auto simp*: *bit-take-bit-iff bit-set-bit-iff*)

lemma *take-bit-unset-bit-eq*:
‹*take-bit n* (*unset-bit m a*) = (*if n* ≤ *m then take-bit n a else unset-bit m* (*take-bit n a*))›
by (*rule bit-eqI*) (*auto simp*: *bit-take-bit-iff bit-unset-bit-iff*)

lemma *take-bit-flip-bit-eq*:
‹*take-bit n* (*flip-bit m a*) = (*if n* ≤ *m then take-bit n a else flip-bit m* (*take-bit n a*))›
by (*rule bit-eqI*) (*auto simp*: *bit-take-bit-iff bit-flip-bit-iff*)

lemma *push-bit-Suc-numeral* [*simp*]:
‹*push-bit* (*Suc n*) (*numeral k*) = *push-bit n* (*numeral* (*Num.Bit0 k*))›
by (*simp add*: *numeral-eq-Suc mult-2-right*) (*simp add*: *numeral-Bit0*)

lemma *mask-eq-0-iff* [*simp*]:
‹*mask n* = *0* ⟷ *n* = *0*›
by (*cases n*) (*simp-all add*: *mask-Suc-double or-eq-0-iff*)

lemma *bit-horner-sum-bit-iff* [*bit-simps*]:
‹*bit* (*horner-sum of-bool 2 bs*) *n* ⟷ *possible-bit TYPE*(*'a*) *n* ∧ *n* < *length bs* ∧ *bs* ! *n*›

proof (*induction bs arbitrary*: *n*)
  case *Nil*
  then show *?case*
    by *simp*
next
  case (*Cons b bs*)
  show *?case*
  proof (*cases n*)
    case *0*
    then show *?thesis*
      by (*simp add*: *bit-0*)
  next
    case (*Suc m*)
    with *bit-rec* [*of* - *n*] *Cons.prems Cons.IH* [*of m*]
    show *?thesis*
      by (*simp add*: *bit-simps*)
        (*auto simp*: *possible-bit-less-imp bit-simps simp flip*: *bit-Suc*)
  qed
qed

lemma *horner-sum-bit-eq-take-bit*:
  ‹*horner-sum of-bool 2* (*map* (*bit a*) [*0*..<*n*]) = *take-bit n a*›
  by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *take-bit-horner-sum-bit-eq*:
  ‹*take-bit n* (*horner-sum of-bool 2 bs*) = *horner-sum of-bool 2* (*take n bs*)›
  by (*auto simp*: *bit-eq-iff bit-take-bit-iff bit-horner-sum-bit-iff*)

lemma *take-bit-sum*:
  ‹*take-bit n a* = ($\sum$ *k* = *0*..<*n*. *push-bit k* (*of-bool* (*bit a k*)))›
  by (*simp flip*: *horner-sum-bit-eq-take-bit add*: *horner-sum-eq-sum push-bit-eq-mult*)

lemma *set-bit-eq*:
  ‹*set-bit n a* = *a* + *of-bool* (¬ *bit a n*) ∗ *2* ^ *n*›
proof −
  have ‹*a AND of-bool* (¬ *bit a n*) ∗ *2* ^ *n* = *0*›
    by (*auto simp*: *bit-eq-iff bit-simps*)
  then show *?thesis*
    by (*auto simp*: *bit-eq-iff bit-simps disjunctive-add-eq-or*)
qed

end

class *ring-bit-operations* = *semiring-bit-operations* + *ring-parity* +
  fixes *not* :: ‹*'a* ⇒ *'a*› (‹*NOT*›)
  assumes *not-eq-complement*: ‹*NOT a* = − *a* − *1*›
begin

For the sake of code generation *NOT* is specified as definitional class operation. Note that *NOT* has no sensible definition for unlimited but only positive bit strings (type *nat*).

lemma *bits-minus-1-mod-2-eq* [*simp*]:
‹(− *1*) *mod 2* = *1*›
by (*simp add*: *mod-2-eq-odd*)

lemma *minus-eq-not-plus-1*:
‹− *a* = *NOT a* + *1*›
using *not-eq-complement* [*of a*] **by** *simp*

lemma *minus-eq-not-minus-1*:
‹− *a* = *NOT* (*a* − *1*)›
using *not-eq-complement* [*of* ‹*a* − *1*›] **by** *simp* (*simp add*: *algebra-simps*)

lemma *not-rec*:
‹*NOT a* = *of-bool* (*even a*) + *2* ∗ *NOT* (*a div 2*)›
by (*simp add*: *not-eq-complement algebra-simps mod-2-eq-odd flip*: *minus-mod-eq-mult-div*)

lemma *decr-eq-not-minus*:
‹*a* − *1* = *NOT* (− *a*)›
using *not-eq-complement* [*of* ‹− *a*›] **by** *simp*

lemma *even-not-iff* [*simp*]:
‹*even* (*NOT a*) ⟷ *odd a*›
by (*simp add*: *not-eq-complement*)

lemma *bit-not-iff* [*bit-simps*]:
‹*bit* (*NOT a*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ ¬ *bit a n*›
proof (*cases* ‹*possible-bit TYPE*(′*a*) *n*›)
case *False*
then show *?thesis*
by (*auto dest*: *bit-imp-possible-bit*)
next
case *True*
moreover have ‹*bit* (*NOT a*) *n* ⟷ ¬ *bit a n*›
using ‹*possible-bit TYPE*(′*a*) *n*› **proof** (*induction n arbitrary*: *a*)
case *0*
then show *?case*
by (*simp add*: *bit-0*)
next
case (*Suc n*)
from *Suc.prems Suc.IH* [*of* ‹*a div 2*›]
show *?case*
by (*simp add*: *impossible-bit possible-bit-less-imp not-rec* [*of a*] *even-bit-succ-iff bit-double-iff flip*: *bit-Suc*)
qed
ultimately show *?thesis*
by *simp*
qed

lemma *bit-not-exp-iff* [*bit-simps*]:
‹*bit* (*NOT* (*2* ^ *m*)) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ *n* ≠ *m*›
by (*auto simp*: *bit-not-iff bit-exp-iff*)

lemma *bit-minus-iff* [*bit-simps*]:
‹*bit* (− *a*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ ¬ *bit* (*a* − *1*) *n*›
by (*simp add*: *minus-eq-not-minus-1 bit-not-iff*)

lemma *bit-minus-1-iff* [*simp*]:
‹*bit* (− *1*) *n* ⟷ *possible-bit TYPE*(′*a*) *n*›
by (*simp add*: *bit-minus-iff*)

lemma *bit-minus-exp-iff* [*bit-simps*]:
‹*bit* (− (*2* ^ *m*)) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ *n* ≥ *m*›
by (*auto simp*: *bit-simps simp flip*: *mask-eq-exp-minus-1*)

lemma *bit-minus-2-iff* [*simp*]:
‹*bit* (− *2*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ *n* > *0*›
by (*simp add*: *bit-minus-iff bit-1-iff*)

lemma *bit-decr-iff*:
‹*bit* (*a* − *1*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ ¬ *bit* (− *a*) *n*›
by (*simp add*: *decr-eq-not-minus bit-not-iff*)

lemma *bit-not-iff-eq*:
‹*bit* (*NOT a*) *n* ⟷ *2* ^ *n* ≠ *0* ∧ ¬ *bit a n*›
by (*simp add*: *bit-simps possible-bit-def*)

lemma *not-one-eq* [*simp*]:
‹*NOT 1* = − *2*›
by (*rule bit-eqI*, *simp add*: *bit-simps*)

sublocale *and*: *semilattice-neutr* ‹(*AND*)› ‹− *1*›
by *standard* (*rule bit-eqI*, *simp add*: *bit-and-iff*)

sublocale *bit*: *abstract-boolean-algebra* ‹(*AND*)› ‹(*OR*)› *NOT 0* ‹− *1*›
by *standard* (*auto simp*: *bit-and-iff bit-or-iff bit-not-iff intro*: *bit-eqI*)

sublocale *bit*: *abstract-boolean-algebra-sym-diff* ‹(*AND*)› ‹(*OR*)› *NOT 0* ‹− *1*› ‹(*XOR*)›
proof
show ‹⋀*x y*. *x XOR y* = *x AND NOT y OR NOT x AND y*›
by (*intro bit-eqI*) (*auto simp*: *bit-simps*)
qed

lemma *and-eq-not-not-or*:
‹*a AND b* = *NOT* (*NOT a OR NOT b*)›
by *simp*

lemma *or-eq-not-not-and*:
‹*a OR b* = *NOT* (*NOT a AND NOT b*)›
by *simp*

lemma *not-add-distrib*:
‹*NOT* (*a* + *b*) = *NOT a* − *b*›
by (*simp add*: *not-eq-complement algebra-simps*)

lemma *not-diff-distrib*:
‹*NOT* (*a* − *b*) = *NOT a* + *b*›
using *not-add-distrib* [*of a* ‹− *b*›] **by** *simp*

lemma *and-eq-minus-1-iff*:
‹*a AND b* = − *1* ⟷ *a* = − *1* ∧ *b* = − *1*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *disjunctive-and-not-eq-xor*:
‹*a AND NOT b* = *a XOR b*› **if** ‹*NOT a AND b* = *0*›
using *that* **by** (*auto simp*: *bit-eq-iff bit-simps*)

lemma *disjunctive-diff-eq-and-not*:
‹*a* − *b* = *a AND NOT b*› **if** ‹*NOT a AND b* = *0*›
proof −
from *that* **have** ‹*NOT a* + *b* = *NOT a OR b*›
by (*rule disjunctive-add-eq-or*)
then have ‹*NOT* (*NOT a* + *b*) = *NOT* (*NOT a OR b*)›
by *simp*
then show *?thesis*
by (*simp add*: *not-add-distrib*)
qed

lemma *disjunctive-diff-eq-xor*:
‹*a AND NOT b* = *a XOR b*› **if** ‹*NOT a AND b* = *0*›
using *that* **by** (*simp add*: *disjunctive-and-not-eq-xor disjunctive-diff-eq-and-not*)

lemma *push-bit-minus*:
‹*push-bit n* (− *a*) = − *push-bit n a*›
by (*simp add*: *push-bit-eq-mult*)

lemma *take-bit-not-take-bit*:
‹*take-bit n* (*NOT* (*take-bit n a*)) = *take-bit n* (*NOT a*)›
by (*auto simp*: *bit-eq-iff bit-take-bit-iff bit-not-iff*)

lemma *take-bit-not-iff*:
‹*take-bit n* (*NOT a*) = *take-bit n* (*NOT b*) ⟷ *take-bit n a* = *take-bit n b*›
by (*auto simp*: *bit-eq-iff bit-simps*)

lemma *take-bit-not-eq-mask-diff*:

‹*take-bit n* (*NOT a*) = *mask n* − *take-bit n a*›
proof −
have ‹*NOT* (*mask n*) *AND take-bit n a* = *0*›
by (*simp add*: *bit-eq-iff bit-simps*)
moreover have ‹*take-bit n* (*NOT a*) = *mask n AND NOT* (*take-bit n a*)›
by (*auto simp*: *bit-eq-iff bit-simps*)
ultimately show *?thesis*
by (*simp add*: *disjunctive-diff-eq-and-not*)
qed

lemma *mask-eq-take-bit-minus-one*:
‹*mask n* = *take-bit n* (− *1*)›
by (*simp add*: *bit-eq-iff bit-mask-iff bit-take-bit-iff conj-commute*)

lemma *take-bit-minus-one-eq-mask* [*simp*]:
‹*take-bit n* (− *1*) = *mask n*›
by (*simp add*: *mask-eq-take-bit-minus-one*)

lemma *minus-exp-eq-not-mask*:
‹− (*2* ^ *n*) = *NOT* (*mask n*)›
by (*rule bit-eqI*) (*simp add*: *bit-minus-iff bit-not-iff flip*: *mask-eq-exp-minus-1*)

lemma *push-bit-minus-one-eq-not-mask* [*simp*]:
‹*push-bit n* (− *1*) = *NOT* (*mask n*)›
by (*simp add*: *push-bit-eq-mult minus-exp-eq-not-mask*)

lemma *take-bit-not-mask-eq-0*:
‹*take-bit m* (*NOT* (*mask n*)) = *0*› **if** ‹*n* ≥ *m*›
by (*rule bit-eqI*) (*use that* **in** ‹*simp add*: *bit-take-bit-iff bit-not-iff bit-mask-iff*›)

lemma *unset-bit-eq-and-not*:
‹*unset-bit n a* = *a AND NOT* (*push-bit n 1*)›
by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *push-bit-Suc-minus-numeral* [*simp*]:
‹*push-bit* (*Suc n*) (− *numeral k*) = *push-bit n* (− *numeral* (*Num.Bit0 k*))›
using *local.push-bit-Suc-numeral push-bit-minus* **by** *presburger*

lemma *push-bit-minus-numeral* [*simp*]:
‹*push-bit* (*numeral l*) (− *numeral k*) = *push-bit* (*pred-numeral l*) (− *numeral* (*Num.Bit0 k*))›
by (*simp only*: *numeral-eq-Suc push-bit-Suc-minus-numeral*)

lemma *take-bit-Suc-minus-1-eq*:
‹*take-bit* (*Suc n*) (− *1*) = *2* ^ *Suc n* − *1*›
by (*simp add*: *mask-eq-exp-minus-1*)

lemma *take-bit-numeral-minus-1-eq*:
‹*take-bit* (*numeral k*) (− *1*) = *2* ^ *numeral k* − *1*›

by (*simp add*: *mask-eq-exp-minus-1*)

lemma *push-bit-mask-eq*:
‹*push-bit m* (*mask n*) = *mask* (*n* + *m*) *AND NOT* (*mask m*)›
by (*rule bit-eqI*) (*auto simp*: *bit-simps not-less possible-bit-less-imp*)

lemma *slice-eq-mask*:
‹*push-bit n* (*take-bit m* (*drop-bit n a*)) = *a AND mask* (*m* + *n*) *AND NOT* (*mask* *n*)›
by (*rule bit-eqI*) (*auto simp*: *bit-simps*)

lemma *push-bit-numeral-minus-1* [*simp*]:
‹*push-bit* (*numeral n*) (− *1*) = − (*2* ^ *numeral n*)›
by (*simp add*: *push-bit-eq-mult*)

lemma *unset-bit-eq*:
‹*unset-bit n a* = *a* − *of-bool* (*bit a n*) ∗ *2* ^ *n*›
proof −
have ‹*NOT a AND of-bool* (*bit a n*) ∗ *2* ^ *n* = *0*›
by (*auto simp*: *bit-eq-iff bit-simps*)
moreover have ‹*unset-bit n a* = *a AND NOT* (*of-bool* (*bit a n*) ∗ *2* ^ *n*)›
by (*auto simp*: *bit-eq-iff bit-simps*)
ultimately show *?thesis*
by (*simp add*: *disjunctive-diff-eq-and-not*)
qed

end

68.3 Common algebraic structure

class *linordered-euclidean-semiring-bit-operations* =
linordered-euclidean-semiring + *semiring-bit-operations*
begin

lemma *possible-bit* [*simp*]:
‹*possible-bit TYPE*(*'a*) *n*›
by (*simp add*: *possible-bit-def*)

lemma *take-bit-of-exp* [*simp*]:
‹*take-bit m* (*2* ^ *n*) = *of-bool* (*n* < *m*) ∗ *2* ^ *n*›
by (*simp add*: *take-bit-eq-mod exp-mod-exp*)

lemma *take-bit-of-2* [*simp*]:
‹*take-bit n 2* = *of-bool* (*2* ≤ *n*) ∗ *2*›
using *take-bit-of-exp* [*of n 1*] **by** *simp*

lemma *push-bit-eq-0-iff* [*simp*]:
‹*push-bit n a* = *0* ⟷ *a* = *0*›
by (*simp add*: *push-bit-eq-mult*)

lemma *take-bit-add*:
‹*take-bit n* (*take-bit n a* + *take-bit n b*) = *take-bit n* (*a* + *b*)›
by (*simp add*: *take-bit-eq-mod mod-simps*)

lemma *take-bit-of-1-eq-0-iff* [*simp*]:
‹*take-bit n 1* = *0* ⟷ *n* = *0*›
by (*simp add*: *take-bit-eq-mod*)

lemma *drop-bit-Suc-bit0* [*simp*]:
‹*drop-bit* (*Suc n*) (*numeral* (*Num.Bit0 k*)) = *drop-bit n* (*numeral k*)›
by (*simp add*: *drop-bit-Suc numeral-Bit0-div-2*)

lemma *drop-bit-Suc-bit1* [*simp*]:
‹*drop-bit* (*Suc n*) (*numeral* (*Num.Bit1 k*)) = *drop-bit n* (*numeral k*)›
by (*simp add*: *drop-bit-Suc numeral-Bit0-div-2*)

lemma *drop-bit-numeral-bit0* [*simp*]:
‹*drop-bit* (*numeral l*) (*numeral* (*Num.Bit0 k*)) = *drop-bit* (*pred-numeral l*) (*numeral k*)›
by (*simp add*: *drop-bit-rec numeral-Bit0-div-2*)

lemma *drop-bit-numeral-bit1* [*simp*]:
‹*drop-bit* (*numeral l*) (*numeral* (*Num.Bit1 k*)) = *drop-bit* (*pred-numeral l*) (*numeral k*)›
by (*simp add*: *drop-bit-rec numeral-Bit0-div-2*)

lemma *take-bit-Suc-1* [*simp*]:
‹*take-bit* (*Suc n*) *1* = *1*›
by (*simp add*: *take-bit-Suc*)

lemma *take-bit-Suc-bit0*:
‹*take-bit* (*Suc n*) (*numeral* (*Num.Bit0 k*)) = *take-bit n* (*numeral k*) ∗ *2*›
by (*simp add*: *take-bit-Suc numeral-Bit0-div-2*)

lemma *take-bit-Suc-bit1*:
‹*take-bit* (*Suc n*) (*numeral* (*Num.Bit1 k*)) = *take-bit n* (*numeral k*) ∗ *2* + *1*›
by (*simp add*: *take-bit-Suc numeral-Bit0-div-2 mod-2-eq-odd*)

lemma *take-bit-numeral-1* [*simp*]:
‹*take-bit* (*numeral l*) *1* = *1*›
by (*simp add*: *take-bit-rec* [*of* ‹*numeral l*› *1*])

lemma *take-bit-numeral-bit0*:
‹*take-bit* (*numeral l*) (*numeral* (*Num.Bit0 k*)) = *take-bit* (*pred-numeral l*) (*numeral k*) ∗ *2*›
by (*simp add*: *take-bit-rec numeral-Bit0-div-2*)

lemma *take-bit-numeral-bit1*:

‹*take-bit* (*numeral l*) (*numeral* (*Num.Bit1 k*)) = *take-bit* (*pred-numeral l*) (*numeral k*) ∗ *2* + *1*›
by (*simp add*: *take-bit-rec numeral-Bit0-div-2 mod-2-eq-odd*)

lemma *bit-of-nat-iff-bit* [*bit-simps*]:
‹*bit* (*of-nat m*) *n* ⟷ *bit m n*›
proof −
have ‹*even* (*m div 2* ^ *n*) ⟷ *even* (*of-nat* (*m div 2* ^ *n*))›
by *simp*
also have ‹*of-nat* (*m div 2* ^ *n*) = *of-nat m div of-nat* (*2* ^ *n*)›
by (*simp add*: *of-nat-div*)
finally show *?thesis*
by (*simp add*: *bit-iff-odd semiring-bits-class.bit-iff-odd*)
qed

lemma *drop-bit-mask-eq*:
‹*drop-bit m* (*mask n*) = *mask* (*n* − *m*)›
by (*rule bit-eqI*) (*auto simp*: *bit-simps possible-bit-def*)

lemma *bit-push-bit-iff'*:
‹*bit* (*push-bit m a*) *n* ⟷ *m* ≤ *n* ∧ *bit a* (*n* − *m*)›
by (*simp add*: *bit-simps*)

lemma *mask-half*:
‹*mask n div 2* = *mask* (*n* − *1*)›
by (*cases n*) (*simp-all add*: *mask-Suc-double one-or-eq*)

lemma *take-bit-Suc-from-most*:
‹*take-bit* (*Suc n*) *a* = *2* ^ *n* ∗ *of-bool* (*bit a n*) + *take-bit n a*›
using *mod-mult2-eq'* [*of a* ‹*2* ^ *n*› *2*]
by (*simp only*: *take-bit-eq-mod power-Suc2*)
(*simp-all add*: *bit-iff-odd odd-iff-mod-2-eq-one*)

lemma *take-bit-nonnegative* [*simp*]:
‹*0* ≤ *take-bit n a*›
using *horner-sum-nonnegative* **by** (*simp flip*: *horner-sum-bit-eq-take-bit*)

lemma *not-take-bit-negative* [*simp*]:
‹¬ *take-bit n a* < *0*›
by (*simp add*: *not-less*)

lemma *bit-imp-take-bit-positive*:
‹*0* < *take-bit m a*› **if** ‹*n* < *m*› **and** ‹*bit a n*›
proof (*rule ccontr*)
assume ‹¬ *0* < *take-bit m a*›
then have ‹*take-bit m a* = *0*›
by (*auto simp*: *not-less intro*: *order-antisym*)
then have ‹*bit* (*take-bit m a*) *n* = *bit 0 n*›
by *simp*

with *that* **show** *False*
by (*simp add*: *bit-take-bit-iff*)
qed

lemma *take-bit-mult*:
‹*take-bit n* (*take-bit n a* ∗ *take-bit n b*) = *take-bit n* (*a* ∗ *b*)›
by (*simp add*: *take-bit-eq-mod mod-mult-eq*)

lemma *drop-bit-push-bit*:
‹*drop-bit m* (*push-bit n a*) = *drop-bit* (*m* − *n*) (*push-bit* (*n* − *m*) *a*)›
by (*cases* ‹*m* ≤ *n*›)
(*auto simp*: *mult.left-commute* [*of* - ‹*2* ^ *n*›] *mult.commute* [*of* - ‹*2* ^ *n*›] *mult.assoc*
mult.commute [*of a*] *drop-bit-eq-div push-bit-eq-mult not-le power-add Orderings.not-le dest*!: *le-Suc-ex less-imp-Suc-add*)

end

68.4 Instance *int*

locale *fold2-bit-int* =
fixes *f* :: ‹*bool* ⇒ *bool* ⇒ *bool*›
begin

context
begin

function *F* :: ‹*int* ⇒ *int* ⇒ *int*›
where ‹*F k l* = (*if k* ∈ {*0*, − *1*} ∧ *l* ∈ {*0*, − *1*}
then − *of-bool* (*f* (*odd k*) (*odd l*))
else of-bool (*f* (*odd k*) (*odd l*)) + *2* ∗ (*F* (*k div 2*) (*l div 2*)))›
by *auto*

private termination proof (*relation* ‹*measure* (λ(*k*, *l*). *nat* (|*k*| + |*l*|))›)
have *less-eq*: ‹|*k div 2*| ≤ |*k*|› **for** *k* :: *int*
by (*cases k*) (*simp-all add*: *divide-int-def nat-add-distrib*)
then have *less*: ‹|*k div 2*| < |*k*|› **if** ‹*k* ∉ {*0*, − *1*}› **for** *k* :: *int*
using *that* **by** (*auto simp*: *less-le* [*of k*])
show ‹*wf* (*measure* (λ(*k*, *l*). *nat* (|*k*| + |*l*|)))›
by *simp*
show ‹((*k div 2*, *l div 2*), *k*, *l*) ∈ *measure* (λ(*k*, *l*). *nat* (|*k*| + |*l*|))›
if ‹¬ (*k* ∈ {*0*, − *1*} ∧ *l* ∈ {*0*, − *1*})› **for** *k l*
proof −
from *that* **have** ∗: ‹*k* ∉ {*0*, − *1*} ∨ *l* ∉ {*0*, − *1*}›
by *simp*
then have ‹*0* < |*k*| + |*l*|›
by *auto*
moreover from ∗ **have** ‹|*k div 2*| + |*l div 2*| < |*k*| + |*l*|›
proof

```
      assume ‹k ∉ {0, − 1}›
      then have ‹|k div 2| < |k|›
        by (rule less)
      with less-eq [of l] show ?thesis
        by auto
    next
      assume ‹l ∉ {0, − 1}›
      then have ‹|l div 2| < |l|›
        by (rule less)
      with less-eq [of k] show ?thesis
        by auto
    qed
    ultimately show ?thesis
      by (simp only: in-measure split-def fst-conv snd-conv nat-mono-iff)
  qed
qed

declare F.simps [simp del]

lemma rec:
  ‹F k l = of-bool (f (odd k) (odd l)) + 2 ∗ (F (k div 2) (l div 2))›
    for k l :: int
proof (cases ‹k ∈ {0, − 1} ∧ l ∈ {0, − 1}›)
  case True
  then show ?thesis
    by (auto simp: F.simps [of 0] F.simps [of ‹− 1›])
next
  case False
  then show ?thesis
    by (auto simp: ac-simps F.simps [of k l])
qed

lemma bit-iff:
  ‹bit (F k l) n ⟷ f (bit k n) (bit l n)› for k l :: int
proof (induction n arbitrary: k l)
  case 0
  then show ?case
    by (simp add: rec [of k l] bit-0)
next
  case (Suc n)
  then show ?case
    by (simp add: rec [of k l] bit-Suc)
qed

end

end

instantiation int :: ring-bit-operations
```

begin

definition *not-int* :: ‹*int* ⇒ *int*›
  where ‹*not-int k* = − *k* − *1*›

global-interpretation *and-int*: *fold2-bit-int* ‹(∧)›
  defines *and-int* = *and-int.F* **.**

global-interpretation *or-int*: *fold2-bit-int* ‹(∨)›
  defines *or-int* = *or-int.F* **.**

global-interpretation *xor-int*: *fold2-bit-int* ‹(≠)›
  defines *xor-int* = *xor-int.F* **.**

definition *mask-int* :: ‹*nat* ⇒ *int*›
  where ‹*mask n* = (*2* :: *int*) ^ *n* − *1*›

definition *push-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
  where ‹*push-bit-int n k* = *k* ∗ *2* ^ *n*›

definition *drop-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
  where ‹*drop-bit-int n k* = *k div 2* ^ *n*›

definition *take-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
  where ‹*take-bit-int n k* = *k mod 2* ^ *n*›

definition *set-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
  where ‹*set-bit n k* = *k OR push-bit n 1*› **for** *k* :: *int*

definition *unset-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
  where ‹*unset-bit n k* = *k AND NOT* (*push-bit n 1*)› **for** *k* :: *int*

definition *flip-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
  where ‹*flip-bit n k* = *k XOR push-bit n 1*› **for** *k* :: *int*

lemma *not-int-div-2*:
  ‹*NOT k div 2* = *NOT* (*k div 2*)› **for** *k* :: *int*
  by (*simp add*: *not-int-def*)

lemma *bit-not-int-iff*:
  ‹*bit* (*NOT k*) *n* ⟷ ¬ *bit k n*› **for** *k* :: *int*
proof (*rule sym*, *induction n arbitrary*: *k*)
  case *0*
  then show *?case*
    by (*simp add*: *bit-0 not-int-def*)
next
  case (*Suc n*)
  then show *?case*
    by (*simp add*: *bit-Suc not-int-div-2*)

qed

instance proof
fix *k l* :: *int* **and** *m n* :: *nat*
show ‹*unset-bit n k* = (*k OR push-bit n 1*) *XOR push-bit n 1*›
by (*rule bit-eqI*)
(*auto simp*: *unset-bit-int-def*
and-int.bit-iff or-int.bit-iff xor-int.bit-iff bit-not-int-iff push-bit-int-def bit-simps)
qed (*fact and-int.rec or-int.rec xor-int.rec mask-int-def set-bit-int-def flip-bit-int-def*
push-bit-int-def drop-bit-int-def take-bit-int-def not-int-def)+

end

instance *int* :: *linordered-euclidean-semiring-bit-operations* **..**

context *ring-bit-operations*
begin

lemma *even-of-int-iff*:
‹*even* (*of-int k*) ⟷ *even k*›
by (*induction k rule*: *int-bit-induct*) *simp-all*

lemma *bit-of-int-iff* [*bit-simps*]:
‹*bit* (*of-int k*) *n* ⟷ *possible-bit TYPE*(*'a*) *n* ∧ *bit k n*›
proof (*cases* ‹*possible-bit TYPE*(*'a*) *n*›)
case *False*
then show *?thesis*
by (*simp add*: *impossible-bit*)
next
case *True*
then have ‹*bit* (*of-int k*) *n* ⟷ *bit k n*›
proof (*induction k arbitrary*: *n rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*
by *simp*
next
case (*even k*)
then show *?case*
using *bit-double-iff* [*of* ‹*of-int k*› *n*] *Bit-Operations.bit-double-iff* [*of k n*]
by (*cases n*) (*auto simp*: *ac-simps possible-bit-def dest*: *mult-not-zero*)
next
case (*odd k*)
then show *?case*
using *bit-double-iff* [*of* ‹*of-int k*› *n*]
by (*cases n*)

(*auto simp*: *ac-simps bit-double-iff even-bit-succ-iff Bit-Operations.bit-0 Bit-Operations.bit-Suc*
possible-bit-def dest: *mult-not-zero*)
qed
with *True* **show** *?thesis*
by *simp*
qed

lemma *push-bit-of-int*:
‹*push-bit n* (*of-int k*) = *of-int* (*push-bit n k*)›
by (*simp add*: *push-bit-eq-mult Bit-Operations.push-bit-eq-mult*)

lemma *of-int-push-bit*:
‹*of-int* (*push-bit n k*) = *push-bit n* (*of-int k*)›
by (*simp add*: *push-bit-eq-mult Bit-Operations.push-bit-eq-mult*)

lemma *take-bit-of-int*:
‹*take-bit n* (*of-int k*) = *of-int* (*take-bit n k*)›
by (*rule bit-eqI*) (*simp add*: *bit-take-bit-iff Bit-Operations.bit-take-bit-iff bit-of-int-iff*)

lemma *of-int-take-bit*:
‹*of-int* (*take-bit n k*) = *take-bit n* (*of-int k*)›
by (*rule bit-eqI*) (*simp add*: *bit-take-bit-iff Bit-Operations.bit-take-bit-iff bit-of-int-iff*)

lemma *of-int-not-eq*:
‹*of-int* (*NOT k*) = *NOT* (*of-int k*)›
by (*rule bit-eqI*) (*simp add*: *bit-not-iff Bit-Operations.bit-not-iff bit-of-int-iff*)

lemma *of-int-not-numeral*:
‹*of-int* (*NOT* (*numeral k*)) = *NOT* (*numeral k*)›
by (*simp add*: *local.of-int-not-eq*)

lemma *of-int-and-eq*:
‹*of-int* (*k AND l*) = *of-int k AND of-int l*›
by (*rule bit-eqI*) (*simp add*: *bit-of-int-iff bit-and-iff Bit-Operations.bit-and-iff*)

lemma *of-int-or-eq*:
‹*of-int* (*k OR l*) = *of-int k OR of-int l*›
by (*rule bit-eqI*) (*simp add*: *bit-of-int-iff bit-or-iff Bit-Operations.bit-or-iff*)

lemma *of-int-xor-eq*:
‹*of-int* (*k XOR l*) = *of-int k XOR of-int l*›
by (*rule bit-eqI*) (*simp add*: *bit-of-int-iff bit-xor-iff Bit-Operations.bit-xor-iff*)

lemma *of-int-mask-eq*:
‹*of-int* (*mask n*) = *mask n*›
by (*induction n*) (*simp-all add*: *mask-Suc-double Bit-Operations.mask-Suc-double of-int-or-eq*)

end

lemma *take-bit-int-less-exp* [*simp*]:
‹*take-bit n k* < *2* ^ *n*› **for** *k* :: *int*
by (*simp add*: *take-bit-eq-mod*)

lemma *take-bit-int-eq-self-iff*:
‹*take-bit n k* = *k* ⟷ *0* ≤ *k* ∧ *k* < *2* ^ *n*› (**is** ‹*?P* ⟷ *?Q*›)
for *k* :: *int*
proof
assume *?P*
moreover note *take-bit-int-less-exp* [*of n k*] *take-bit-nonnegative* [*of n k*]
ultimately show *?Q*
by *simp*
next
assume *?Q*
then show *?P*
by (*simp add*: *take-bit-eq-mod*)
qed

lemma *take-bit-int-eq-self*:
‹*take-bit n k* = *k*› **if** ‹*0* ≤ *k*› ‹*k* < *2* ^ *n*› **for** *k* :: *int*
using *that* **by** (*simp add*: *take-bit-int-eq-self-iff*)

lemma *mask-nonnegative-int* [*simp*]:
‹*mask n* ≥ (*0*::*int*)›
by (*simp add*: *mask-eq-exp-minus-1 add-le-imp-le-diff*)

lemma *not-mask-negative-int* [*simp*]:
‹¬ *mask n* < (*0*::*int*)›
by (*simp add*: *not-less*)

lemma *not-nonnegative-int-iff* [*simp*]:
‹*NOT k* ≥ *0* ⟷ *k* < *0*› **for** *k* :: *int*
by (*simp add*: *not-int-def*)

lemma *not-negative-int-iff* [*simp*]:
‹*NOT k* < *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
by (*subst Not-eq-iff* [*symmetric*]) (*simp add*: *not-less not-le*)

lemma *and-nonnegative-int-iff* [*simp*]:
‹*k AND l* ≥ *0* ⟷ *k* ≥ *0* ∨ *l* ≥ *0*› **for** *k l* :: *int*
proof (*induction k arbitrary*: *l rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*

by *simp*
next
case (*even k*)
then show *?case*
using *and-int.rec* [*of* ‹$k * 2$› *l*]
by (*simp add*: *pos-imp-zdiv-nonneg-iff zero-le-mult-iff*)
next
case (*odd k*)
from *odd* **have** ‹$0 \leq k$ *AND* l *div* $2 \longleftrightarrow 0 \leq k \vee 0 \leq l$ *div* 2›
by *simp*
then have ‹$0 \leq (1 + k * 2)$ *div* 2 *AND* l *div* $2 \longleftrightarrow 0 \leq (1 + k * 2)$ *div* $2 \vee 0 \leq l$ *div* 2›
by *simp*
with *and-int.rec* [*of* ‹$1 + k * 2$› *l*]
show *?case*
by (*auto simp*: *zero-le-mult-iff not-le*)
qed

lemma *and-negative-int-iff* [*simp*]:
‹k *AND* $l < 0 \longleftrightarrow k < 0 \wedge l < 0$› **for** k l :: *int*
by (*subst Not-eq-iff* [*symmetric*]) (*simp add*: *not-less*)

lemma *and-less-eq*:
‹k *AND* $l \leq k$› **if** ‹$l < 0$› **for** k l :: *int*
using *that* **proof** (*induction k arbitrary*: *l rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*
by *simp*
next
case (*even k*)
from *even.IH* [*of* ‹l *div* 2›] *even.hyps even.prems*
show *?case*
by (*simp add*: *and-int.rec* [*of* - *l*])
next
case (*odd k*)
from *odd.IH* [*of* ‹l *div* 2›] *odd.hyps odd.prems*
show *?case*
by (*simp add*: *and-int.rec* [*of* - *l*])
qed

lemma *or-nonnegative-int-iff* [*simp*]:
‹k *OR* $l \geq 0 \longleftrightarrow k \geq 0 \wedge l \geq 0$› **for** k l :: *int*
by (*simp only*: *or-eq-not-not-and not-nonnegative-int-iff*) *simp*

lemma *or-negative-int-iff* [*simp*]:

‹*k OR l* < *0* ⟷ *k* < *0* ∨ *l* < *0*› **for** *k l* :: *int*
by (*subst Not-eq-iff* [*symmetric*]) (*simp add*: *not-less*)

lemma *or-greater-eq*:
‹*k OR l* ≥ *k*› **if** ‹*l* ≥ *0*› **for** *k l* :: *int*
using *that* **proof** (*induction k arbitrary*: *l rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*
by *simp*
next
case (*even k*)
from *even.IH* [*of* ‹*l div 2*›] *even.hyps even.prems*
show *?case*
by (*simp add*: *or-int.rec* [*of - l*])
next
case (*odd k*)
from *odd.IH* [*of* ‹*l div 2*›] *odd.hyps odd.prems*
show *?case*
by (*simp add*: *or-int.rec* [*of - l*])
qed

lemma *xor-nonnegative-int-iff* [*simp*]:
‹*k XOR l* ≥ *0* ⟷ (*k* ≥ *0* ⟷ *l* ≥ *0*)› **for** *k l* :: *int*
by (*simp only*: *bit.xor-def or-nonnegative-int-iff*) *auto*

lemma *xor-negative-int-iff* [*simp*]:
‹*k XOR l* < *0* ⟷ (*k* < *0*) ≠ (*l* < *0*)› **for** *k l* :: *int*
by (*subst Not-eq-iff* [*symmetric*]) (*auto simp*: *not-less*)

lemma *OR-upper*:
‹*x OR y* < *2* ^ *n*› **if** ‹*0* ≤ *x*› ‹*x* < *2* ^ *n*› ‹*y* < *2* ^ *n*› **for** *x y* :: *int*
using *that* **proof** (*induction x arbitrary*: *y n rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*
by *simp*
next
case (*even x*)
from *even.IH* [*of* ‹*n* − *1*› ‹*y div 2*›] *even.prems even.hyps*
show *?case*
by (*cases n*) (*auto simp*: *or-int.rec* [*of* ‹*- ∗ 2*›] *elim*: *oddE*)
next

case (*odd x*)
from *odd.IH* [*of* ‹*n* − *1*› ‹*y div 2*›] *odd.prems odd.hyps*
show *?case*
by (*cases n*) (*auto simp*: *or-int.rec* [*of* ‹*1* + - ∗ *2*›], *linarith*)
qed

lemma *XOR-upper*:
‹*x XOR y* < *2* ^ *n*› **if** ‹*0* ≤ *x*› ‹*x* < *2* ^ *n*› ‹*y* < *2* ^ *n*› **for** *x y* :: *int*
using *that* **proof** (*induction x arbitrary*: *y n rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*
by *simp*
next
case (*even x*)
from *even.IH* [*of* ‹*n* − *1*› ‹*y div 2*›] *even.prems even.hyps*
show *?case*
by (*cases n*) (*auto simp*: *xor-int.rec* [*of* ‹- ∗ *2*›] *elim*: *oddE*)
next
case (*odd x*)
from *odd.IH* [*of* ‹*n* − *1*› ‹*y div 2*›] *odd.prems odd.hyps*
show *?case*
by (*cases n*) (*auto simp*: *xor-int.rec* [*of* ‹*1* + - ∗ *2*›])
qed

lemma *AND-lower* [*simp*]:
‹*0* ≤ *x AND y*› **if** ‹*0* ≤ *x*› **for** *x y* :: *int*
using *that* **by** *simp*

lemma *OR-lower* [*simp*]:
‹*0* ≤ *x OR y*› **if** ‹*0* ≤ *x*› ‹*0* ≤ *y*› **for** *x y* :: *int*
using *that* **by** *simp*

lemma *XOR-lower* [*simp*]:
‹*0* ≤ *x XOR y*› **if** ‹*0* ≤ *x*› ‹*0* ≤ *y*› **for** *x y* :: *int*
using *that* **by** *simp*

lemma *AND-upper1* [*simp*]:
‹*x AND y* ≤ *x*› **if** ‹*0* ≤ *x*› **for** *x y* :: *int*
using *that* **proof** (*induction x arbitrary*: *y rule*: *int-bit-induct*)
case (*odd k*)
then have ‹*k AND y div 2* ≤ *k*›
by *simp*
then show *?case*
by (*simp add*: *and-int.rec* [*of* ‹*1* + - ∗ *2*›])
qed (*simp-all add*: *and-int.rec* [*of* ‹- ∗ *2*›])

lemma *AND-upper1′* [*simp*]:
‹*y AND x* ≤ *z*› **if** ‹*0* ≤ *y*› ‹*y* ≤ *z*› **for** *x y z* :: *int*
using - ‹*y* ≤ *z*› **by** (*rule order-trans*) (*use* ‹*0* ≤ *y*› **in** *simp*)

lemma *AND-upper1″* [*simp*]:
‹*y AND x* < *z*› **if** ‹*0* ≤ *y*› ‹*y* < *z*› **for** *x y z* :: *int*
using - ‹*y* < *z*› **by** (*rule order-le-less-trans*) (*use* ‹*0* ≤ *y*› **in** *simp*)

lemma *AND-upper2* [*simp*]:
‹*x AND y* ≤ *y*› **if** ‹*0* ≤ *y*› **for** *x y* :: *int*
using *that AND-upper1* [*of y x*] **by** (*simp add*: *ac-simps*)

lemma *AND-upper2′* [*simp*]:
‹*x AND y* ≤ *z*› **if** ‹*0* ≤ *y*› ‹*y* ≤ *z*› **for** *x y* :: *int*
using *that AND-upper1′* [*of y z x*] **by** (*simp add*: *ac-simps*)

lemma *AND-upper2″* [*simp*]:
‹*x AND y* < *z*› **if** ‹*0* ≤ *y*› ‹*y* < *z*› **for** *x y* :: *int*
using *that AND-upper1″* [*of y z x*] **by** (*simp add*: *ac-simps*)

lemma *plus-and-or*:
‹(*x AND y*) + (*x OR y*) = *x* + *y*› **for** *x y* :: *int*
proof (*induction x arbitrary*: *y rule*: *int-bit-induct*)
case *zero*
then show *?case*
by *simp*
next
case *minus*
then show *?case*
by *simp*
next
case (*even x*)
from *even.IH* [*of* ‹*y div 2*›]
show *?case*
by (*auto simp*: *and-int.rec* [*of - y*] *or-int.rec* [*of - y*] *elim*: *oddE*)
next
case (*odd x*)
from *odd.IH* [*of* ‹*y div 2*›]
show *?case*
by (*auto simp*: *and-int.rec* [*of - y*] *or-int.rec* [*of - y*] *elim*: *oddE*)
qed

lemma *push-bit-minus-one*:
‹*push-bit n* (− *1* :: *int*) = − (*2* ^ *n*)›
by (*simp add*: *push-bit-eq-mult*)

lemma *minus-1-div-exp-eq-int*:
‹− *1 div* (*2* :: *int*) ^ *n* = − *1*›

by (*induction n*) (*use div-exp-eq* [*symmetric*, *of* ‹− *1* :: *int*› *1*] **in** ‹*simp-all add*: *ac-simps*›)

lemma *drop-bit-minus-one* [*simp*]:
‹*drop-bit n* (− *1* :: *int*) = − *1*›
by (*simp add*: *drop-bit-eq-div minus-1-div-exp-eq-int*)

lemma *take-bit-minus*:
‹*take-bit n* (− *take-bit n k*) = *take-bit n* (− *k*)›
for *k* :: *int*
by (*simp add*: *take-bit-eq-mod mod-minus-eq*)

lemma *take-bit-diff*:
‹*take-bit n* (*take-bit n k* − *take-bit n l*) = *take-bit n* (*k* − *l*)›
for *k l* :: *int*
by (*simp add*: *take-bit-eq-mod mod-diff-eq*)

lemma (**in** *ring-1*) *of-nat-nat-take-bit-eq* [*simp*]:
‹*of-nat* (*nat* (*take-bit n k*)) = *of-int* (*take-bit n k*)›
by *simp*

lemma *take-bit-minus-small-eq*:
‹*take-bit n* (− *k*) = *2* ^ *n* − *k*› **if** ‹*0* < *k*› ‹*k* ≤ *2* ^ *n*› **for** *k* :: *int*
proof −
define *m* **where** ‹*m* = *nat k*›
with *that* **have** ‹*k* = *int m*› **and** ‹*0* < *m*› **and** ‹*m* ≤ *2* ^ *n*›
by *simp-all*
have ‹(*2* ^ *n* − *m*) *mod 2* ^ *n* = *2* ^ *n* − *m*›
using ‹*0* < *m*› **by** *simp*
then have ‹*int* ((*2* ^ *n* − *m*) *mod 2* ^ *n*) = *int* (*2* ^ *n* − *m*)›
by *simp*
then have ‹(*2* ^ *n* − *int m*) *mod 2* ^ *n* = *2* ^ *n* − *int m*›
using ‹*m* ≤ *2* ^ *n*› **by** (*simp only*: *of-nat-mod of-nat-diff*) *simp*
with ‹*k* = *int m*› **have** ‹(*2* ^ *n* − *k*) *mod 2* ^ *n* = *2* ^ *n* − *k*›
by *simp*
then show *?thesis*
by (*simp add*: *take-bit-eq-mod*)
qed

lemma *push-bit-nonnegative-int-iff* [*simp*]:
‹*push-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
by (*simp add*: *push-bit-eq-mult zero-le-mult-iff power-le-zero-eq*)

lemma *push-bit-negative-int-iff* [*simp*]:
‹*push-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
by (*subst Not-eq-iff* [*symmetric*]) (*simp add*: *not-less*)

lemma *drop-bit-nonnegative-int-iff* [*simp*]:
‹*drop-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*

by (*induction n*) (*auto simp*: *drop-bit-Suc drop-bit-half*)

lemma *drop-bit-negative-int-iff* [*simp*]:
‹*drop-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
by (*subst Not-eq-iff* [*symmetric*]) (*simp add*: *not-less*)

lemma *set-bit-nonnegative-int-iff* [*simp*]:
‹*set-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
by (*simp add*: *set-bit-eq-or*)

lemma *set-bit-negative-int-iff* [*simp*]:
‹*set-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
by (*simp add*: *set-bit-eq-or*)

lemma *unset-bit-nonnegative-int-iff* [*simp*]:
‹*unset-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
by (*simp add*: *unset-bit-eq-and-not*)

lemma *unset-bit-negative-int-iff* [*simp*]:
‹*unset-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
by (*simp add*: *unset-bit-eq-and-not*)

lemma *flip-bit-nonnegative-int-iff* [*simp*]:
‹*flip-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
by (*simp add*: *flip-bit-eq-xor*)

lemma *flip-bit-negative-int-iff* [*simp*]:
‹*flip-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
by (*simp add*: *flip-bit-eq-xor*)

lemma *set-bit-greater-eq*:
‹*set-bit n k* ≥ *k*› **for** *k* :: *int*
by (*simp add*: *set-bit-eq-or or-greater-eq*)

lemma *unset-bit-less-eq*:
‹*unset-bit n k* ≤ *k*› **for** *k* :: *int*
by (*simp add*: *unset-bit-eq-and-not and-less-eq*)

lemma *and-int-unfold*:
‹*k AND l* = (*if k* = *0* ∨ *l* = *0 then 0 else if k* = − *1 then l else if l* = − *1 then k*
else (*k mod 2*) ∗ (*l mod 2*) + *2* ∗ ((*k div 2*) *AND* (*l div 2*)))› **for** *k l* :: *int*
by (*auto simp*: *and-int.rec* [*of k l*] *zmult-eq-1-iff elim*: *oddE*)

lemma *or-int-unfold*:
‹*k OR l* = (*if k* = − *1* ∨ *l* = − *1 then* − *1 else if k* = *0 then l else if l* = *0 then*
k
else max (*k mod 2*) (*l mod 2*) + *2* ∗ ((*k div 2*) *OR* (*l div 2*)))› **for** *k l* :: *int*
by (*auto simp*: *or-int.rec* [*of k l*] *elim*: *oddE*)

lemma *xor-int-unfold*:
‹k *XOR* l = (*if* $k = -1$ *then NOT* l *else if* $l = -1$ *then NOT* k *else if* $k = 0$ *then* l *else if* $l = 0$ *then* k
else $|k$ *mod* $2 - l$ *mod* $2| + 2 * ((k$ *div* $2)$ *XOR* $(l$ *div* $2)))$› **for** $k\ l$ :: *int*
by (*auto simp*: *xor-int.rec* [*of k l*] *not-int-def elim*!: *oddE*)

lemma *bit-minus-int-iff*:
‹*bit* $(- k)$ $n \longleftrightarrow$ *bit* (*NOT* $(k - 1)$) n› **for** k :: *int*
by (*simp add*: *bit-simps*)

lemma *take-bit-incr-eq*:
‹*take-bit* n $(k + 1) = 1 +$ *take-bit* n k› **if** ‹*take-bit* n $k \neq 2$ ^ $n - 1$› **for** k :: *int*
proof −
from *that* **have** ‹2 ^ $n \neq k$ *mod* 2 ^ $n + 1$›
by (*simp add*: *take-bit-eq-mod*)
moreover have ‹k *mod* 2 ^ $n < 2$ ^ n›
by *simp*
ultimately have $*$: ‹k *mod* 2 ^ $n + 1 < 2$ ^ n›
by *linarith*
have ‹$(k + 1)$ *mod* 2 ^ $n = (k$ *mod* 2 ^ $n + 1)$ *mod* 2 ^ n›
by (*simp add*: *mod-simps*)
also have ‹$\ldots = k$ *mod* 2 ^ $n + 1$›
using $*$ **by** (*simp add*: *zmod-trivial-iff*)
finally have ‹$(k + 1)$ *mod* 2 ^ $n = k$ *mod* 2 ^ $n + 1$› **.**
then show *?thesis*
by (*simp add*: *take-bit-eq-mod*)
qed

lemma *take-bit-decr-eq*:
‹*take-bit* n $(k - 1) =$ *take-bit* n $k - 1$› **if** ‹*take-bit* n $k \neq 0$› **for** k :: *int*
proof −
from *that* **have** ‹k *mod* 2 ^ $n \neq 0$›
by (*simp add*: *take-bit-eq-mod*)
moreover have ‹k *mod* 2 ^ $n \geq 0$› ‹k *mod* 2 ^ $n < 2$ ^ n›
by *simp-all*
ultimately have $*$: ‹k *mod* 2 ^ $n > 0$›
by *linarith*
have ‹$(k - 1)$ *mod* 2 ^ $n = (k$ *mod* 2 ^ $n - 1)$ *mod* 2 ^ n›
by (*simp add*: *mod-simps*)
also have ‹$\ldots = k$ *mod* 2 ^ $n - 1$›
by (*simp add*: *zmod-trivial-iff*)
(*use* ‹k *mod* 2 ^ $n < 2$ ^ n› $*$ **in** *linarith*)
finally have ‹$(k - 1)$ *mod* 2 ^ $n = k$ *mod* 2 ^ $n - 1$› **.**
then show *?thesis*
by (*simp add*: *take-bit-eq-mod*)
qed

lemma *take-bit-int-greater-eq*:
‹$k + 2$ ^ $n \leq$ *take-bit* n k› **if** ‹$k < 0$› **for** k :: *int*

proof –
  have ‹$k + 2 \char`^ n \leq$ *take-bit n* $(k + 2 \char`^ n)$›
  proof (*cases* ‹$k > -(2 \char`^ n)$›)
    case *False*
    then have ‹$k + 2 \char`^ n \leq 0$›
      by *simp*
    also note *take-bit-nonnegative*
    finally show *?thesis* **.**
  next
    case *True*
    with *that* **have** ‹$0 \leq k + 2 \char`^ n$› **and** ‹$k + 2 \char`^ n < 2 \char`^ n$›
      by *simp-all*
    then show *?thesis*
      by (*simp only*: *take-bit-eq-mod mod-pos-pos-trivial*)
  qed
  then show *?thesis*
    by (*simp add*: *take-bit-eq-mod*)
qed

lemma *take-bit-int-less-eq*:
  ‹*take-bit n k* $\leq k - 2 \char`^ n$› **if** ‹$2 \char`^ n \leq k$› **and** ‹$n > 0$› **for** *k* :: *int*
  using *that zmod-le-nonneg-dividend* [*of* ‹$k - 2 \char`^ n$› ‹$2 \char`^ n$›]
  by (*simp add*: *take-bit-eq-mod*)

lemma *take-bit-int-less-eq-self-iff*:
  ‹*take-bit n k* $\leq k \longleftrightarrow 0 \leq k$› (**is** ‹*?P* $\longleftrightarrow$ *?Q*›) **for** *k* :: *int*
proof
  assume *?P*
  show *?Q*
  proof (*rule ccontr*)
    assume ‹$\neg\ 0 \leq k$›
    then have ‹$k < 0$›
      by *simp*
    with ‹*?P*›
    have ‹*take-bit n k* < 0›
      by (*rule le-less-trans*)
    then show *False*
      by *simp*
  qed
next
  assume *?Q*
  then show *?P*
    by (*simp add*: *take-bit-eq-mod zmod-le-nonneg-dividend*)
qed

lemma *take-bit-int-less-self-iff*:
  ‹*take-bit n k* $< k \longleftrightarrow 2 \char`^ n \leq k$› **for** *k* :: *int*
  by (*auto simp*: *less-le take-bit-int-less-eq-self-iff take-bit-int-eq-self-iff*
    intro: *order-trans* [*of* 0 ‹$2 \char`^ n$› *k*])

lemma *take-bit-int-greater-self-iff*:
‹$k < take\text{-}bit\ n\ k \longleftrightarrow k < 0$› **for** k :: *int*
using *take-bit-int-less-eq-self-iff* [*of n k*] **by** *auto*

lemma *take-bit-int-greater-eq-self-iff*:
‹$k \leq take\text{-}bit\ n\ k \longleftrightarrow k < 2\ \widehat{}\ n$› **for** k :: *int*
by (*auto simp*: *le-less take-bit-int-greater-self-iff take-bit-int-eq-self-iff*
dest: *sym not-sym intro*: *less-trans* [*of k 0* ‹$2\ \widehat{}\ n$›])

lemma *take-bit-tightened-less-eq-int*:
‹$take\text{-}bit\ m\ k \leq take\text{-}bit\ n\ k$› **if** ‹$m \leq n$› **for** k :: *int*
proof −
have ‹$take\text{-}bit\ m\ (take\text{-}bit\ n\ k) \leq take\text{-}bit\ n\ k$›
by (*simp only*: *take-bit-int-less-eq-self-iff take-bit-nonnegative*)
with *that* **show** *?thesis*
by *simp*
qed

lemma *not-exp-less-eq-0-int* [*simp*]:
‹$\neg\ 2\ \widehat{}\ n \leq (0::int)$›
by (*simp add*: *power-le-zero-eq*)

lemma *int-bit-bound*:
fixes k :: *int*
obtains n **where** ‹$\bigwedge m.\ n \leq m \Longrightarrow bit\ k\ m \longleftrightarrow bit\ k\ n$›
and ‹$n > 0 \Longrightarrow bit\ k\ (n - 1) \neq bit\ k\ n$›
proof −
obtain q **where** ∗: ‹$\bigwedge m.\ q \leq m \Longrightarrow bit\ k\ m \longleftrightarrow bit\ k\ q$›
proof (*cases* ‹$k \geq 0$›)
case *True*
moreover from *power-gt-expt* [*of 2* ‹*nat k*›]
have ‹$nat\ k < 2\ \widehat{}\ nat\ k$›
by *simp*
then have ‹$int\ (nat\ k) < int\ (2\ \widehat{}\ nat\ k)$›
by (*simp only*: *of-nat-less-iff*)
ultimately have ∗: ‹$k\ div\ 2\ \widehat{}\ nat\ k = 0$›
by *simp*
show *thesis*
proof (*rule that* [*of* ‹*nat k*›])
fix m
assume ‹$nat\ k \leq m$›
then show ‹$bit\ k\ m \longleftrightarrow bit\ k\ (nat\ k)$›
by (*auto simp*: ∗ *bit-iff-odd power-add zdiv-zmult2-eq dest*!: *le-Suc-ex*)
qed
next
case *False*
moreover from *power-gt-expt* [*of 2* ‹$nat\ (-\ k)$›]
have ‹$nat\ (-\ k) < 2\ \widehat{}\ nat\ (-\ k)$›

by *simp*
then have ‹*int* (*nat* (− *k*)) < *int* (*2* ^ *nat* (− *k*))›
by (*simp only*: *of-nat-less-iff*)
ultimately have ‹− *k div* − (*2* ^ *nat* (− *k*)) = − *1*›
by (*subst div-pos-neg-trivial*) *simp-all*
then have ∗: ‹*k div 2* ^ *nat* (− *k*) = − *1*›
by *simp*
show *thesis*
proof (*rule that* [*of* ‹*nat* (− *k*)›])
fix *m*
assume ‹*nat* (− *k*) ≤ *m*›
then show ‹*bit k m* ⟷ *bit k* (*nat* (− *k*))›
by (*auto simp*: ∗ *bit-iff-odd power-add zdiv-zmult2-eq minus-1-div-exp-eq-int dest*!: *le-Suc-ex*)
qed
qed
show *thesis*
proof (*cases* ‹∀ *m*. *bit k m* ⟷ *bit k q*›)
case *True*
then have ‹*bit k 0* ⟷ *bit k q*›
by *blast*
with *True that* [*of 0*] **show** *thesis*
by *simp*
next
case *False*
then obtain *r* **where** ∗∗: ‹*bit k r* ≠ *bit k q*›
by *blast*
have ‹*r* < *q*›
by (*rule ccontr*) (*use* ∗ [*of r*] ∗∗ **in** *simp*)
define *N* **where** ‹*N* = {*n*. *n* < *q* ∧ *bit k n* ≠ *bit k q*}›
moreover have ‹*finite N*› ‹*r* ∈ *N*›
using ∗∗ *N-def* ‹*r* < *q*› **by** *auto*
moreover define *n* **where** ‹*n* = *Suc* (*Max N*)›
ultimately have †: ‹⋀*m*. *n* ≤ *m* ⟹ *bit k m* ⟷ *bit k n*›
by (*smt* (*verit*) ∗ *Max-ge Suc-n-not-le-n linorder-not-less mem-Collect-eq not-less-eq-eq*)
have ‹*bit k* (*Max N*) ≠ *bit k n*›
by (*metis* (*mono-tags*, *lifting*) ∗ *Max-in N-def* ‹⋀*m*. *n* ≤ *m* ⟹ *bit k m* = *bit k n*› ‹*finite N*› ‹*r* ∈ *N*› *empty-iff le-cases mem-Collect-eq*)
with † *n-def that* [*of n*] **show** *thesis*
by *fastforce*
qed
qed

68.5 Instance *nat*

instantiation *nat* :: *semiring-bit-operations*
begin

definition *and-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m AND n* = *nat* (*int m AND int n*)› **for** *m n* :: *nat*

definition *or-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m OR n* = *nat* (*int m OR int n*)› **for** *m n* :: *nat*

definition *xor-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m XOR n* = *nat* (*int m XOR int n*)› **for** *m n* :: *nat*

definition *mask-nat* :: ‹*nat* ⇒ *nat*›
where ‹*mask n* = (*2* :: *nat*) ^ *n* − *1*›

definition *push-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*push-bit-nat n m* = *m* ∗ *2* ^ *n*›

definition *drop-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*drop-bit-nat n m* = *m div 2* ^ *n*›

definition *take-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*take-bit-nat n m* = *m mod 2* ^ *n*›

definition *set-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*set-bit m n* = *n OR push-bit m 1*› **for** *m n* :: *nat*

definition *unset-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*unset-bit m n* = (*n OR push-bit m 1*) *XOR push-bit m 1*› **for** *m n* :: *nat*

definition *flip-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*flip-bit m n* = *n XOR push-bit m 1*› **for** *m n* :: *nat*

instance proof
fix *m n* :: *nat*
show ‹*m AND n* = *of-bool* (*odd m* ∧ *odd n*) + *2* ∗ (*m div 2 AND n div 2*)›
by (*simp add*: *and-nat-def and-rec* [*of* ‹*int m*› ‹*int n*›] *nat-add-distrib of-nat-div*)
show ‹*m OR n* = *of-bool* (*odd m* ∨ *odd n*) + *2* ∗ (*m div 2 OR n div 2*)›
by (*simp add*: *or-nat-def or-rec* [*of* ‹*int m*› ‹*int n*›] *nat-add-distrib of-nat-div*)
show ‹*m XOR n* = *of-bool* (*odd m* ≠ *odd n*) + *2* ∗ (*m div 2 XOR n div 2*)›
by (*simp add*: *xor-nat-def xor-rec* [*of* ‹*int m*› ‹*int n*›] *nat-add-distrib of-nat-div*)
qed (*simp-all add*: *mask-nat-def set-bit-nat-def unset-bit-nat-def flip-bit-nat-def push-bit-nat-def drop-bit-nat-def take-bit-nat-def*)

end

instance *nat* :: *linordered-euclidean-semiring-bit-operations* **..**

context *semiring-bit-operations*
begin

lemma *push-bit-of-nat*:

‹*push-bit n* (*of-nat m*) = *of-nat* (*push-bit n m*)›
by (*simp add*: *push-bit-eq-mult Bit-Operations.push-bit-eq-mult*)

lemma *of-nat-push-bit*:
‹*of-nat* (*push-bit m n*) = *push-bit m* (*of-nat n*)›
by (*simp add*: *push-bit-eq-mult Bit-Operations.push-bit-eq-mult*)

lemma *take-bit-of-nat*:
‹*take-bit n* (*of-nat m*) = *of-nat* (*take-bit n m*)›
by (*rule bit-eqI*) (*simp add*: *bit-take-bit-iff Bit-Operations.bit-take-bit-iff bit-of-nat-iff*)

lemma *of-nat-take-bit*:
‹*of-nat* (*take-bit n m*) = *take-bit n* (*of-nat m*)›
by (*rule bit-eqI*) (*simp add*: *bit-take-bit-iff Bit-Operations.bit-take-bit-iff bit-of-nat-iff*)

lemma *of-nat-and-eq*:
‹*of-nat* (*m AND n*) = *of-nat m AND of-nat n*›
by (*rule bit-eqI*) (*simp add*: *bit-of-nat-iff bit-and-iff Bit-Operations.bit-and-iff*)

lemma *of-nat-or-eq*:
‹*of-nat* (*m OR n*) = *of-nat m OR of-nat n*›
by (*rule bit-eqI*) (*simp add*: *bit-of-nat-iff bit-or-iff Bit-Operations.bit-or-iff*)

lemma *of-nat-xor-eq*:
‹*of-nat* (*m XOR n*) = *of-nat m XOR of-nat n*›
by (*rule bit-eqI*) (*simp add*: *bit-of-nat-iff bit-xor-iff Bit-Operations.bit-xor-iff*)

lemma *of-nat-mask-eq*:
‹*of-nat* (*mask n*) = *mask n*›
by (*induction n*) (*simp-all add*: *mask-Suc-double Bit-Operations.mask-Suc-double of-nat-or-eq*)

lemma *of-nat-set-bit-eq*:
‹*of-nat* (*set-bit n m*) = *set-bit n* (*of-nat m*)›
by (*simp add*: *set-bit-eq-or Bit-Operations.set-bit-eq-or of-nat-or-eq Bit-Operations.push-bit-eq-mult*)

lemma *of-nat-unset-bit-eq*:
‹*of-nat* (*unset-bit n m*) = *unset-bit n* (*of-nat m*)›
by (*simp add*: *unset-bit-eq-or-xor Bit-Operations.unset-bit-eq-or-xor of-nat-or-eq of-nat-xor-eq Bit-Operations.push-bit-eq-mult*)

lemma *of-nat-flip-bit-eq*:
‹*of-nat* (*flip-bit n m*) = *flip-bit n* (*of-nat m*)›
by (*simp add*: *flip-bit-eq-xor Bit-Operations.flip-bit-eq-xor of-nat-xor-eq Bit-Operations.push-bit-eq-mult*)

end

context *linordered-euclidean-semiring-bit-operations*
begin

lemma *drop-bit-of-nat*:
drop-bit n (*of-nat m*) = *of-nat* (*drop-bit n m*)
by (*simp add*: *drop-bit-eq-div Bit-Operations.drop-bit-eq-div of-nat-div* [*of m 2* ^ *n*])

lemma *of-nat-drop-bit*:
‹*of-nat* (*drop-bit m n*) = *drop-bit m* (*of-nat n*)›
by (*simp add*: *drop-bit-eq-div Bit-Operations.drop-bit-eq-div of-nat-div*)

end

lemma *take-bit-nat-less-exp* [*simp*]:
‹*take-bit n m* < *2* ^ *n*› **for** *n m* :: *nat*
by (*simp add*: *take-bit-eq-mod*)

lemma *take-bit-nat-eq-self-iff*:
‹*take-bit n m* = *m* ⟷ *m* < *2* ^ *n*› (**is** ‹*?P* ⟷ *?Q*›) **for** *n m* :: *nat*
proof
assume *?P*
moreover note *take-bit-nat-less-exp* [*of n m*]
ultimately show *?Q*
by *simp*
next
assume *?Q*
then show *?P*
by (*simp add*: *take-bit-eq-mod*)
qed

lemma *take-bit-nat-eq-self*:
‹*take-bit n m* = *m*› **if** ‹*m* < *2* ^ *n*› **for** *m n* :: *nat*
using *that* **by** (*simp add*: *take-bit-nat-eq-self-iff*)

lemma *take-bit-nat-less-eq-self* [*simp*]:
‹*take-bit n m* ≤ *m*› **for** *n m* :: *nat*
by (*simp add*: *take-bit-eq-mod*)

lemma *take-bit-nat-less-self-iff*:
‹*take-bit n m* < *m* ⟷ *2* ^ *n* ≤ *m*› (**is** ‹*?P* ⟷ *?Q*›) **for** *m n* :: *nat*
proof
assume *?P*
then have ‹*take-bit n m* ≠ *m*›
by *simp*
then show ‹*?Q*›
by (*simp add*: *take-bit-nat-eq-self-iff*)
next
have ‹*take-bit n m* < *2* ^ *n*›
by (*fact take-bit-nat-less-exp*)
also assume *?Q*

finally show *?P* **.**
qed

lemma *Suc-0-and-eq* [*simp*]:
‹*Suc 0 AND n* = *n mod 2*›
using *one-and-eq* [*of n*] **by** *simp*

lemma *and-Suc-0-eq* [*simp*]:
‹*n AND Suc 0* = *n mod 2*›
using *and-one-eq* [*of n*] **by** *simp*

lemma *Suc-0-or-eq*:
‹*Suc 0 OR n* = *n* + *of-bool* (*even n*)›
using *one-or-eq* [*of n*] **by** *simp*

lemma *or-Suc-0-eq*:
‹*n OR Suc 0* = *n* + *of-bool* (*even n*)›
using *or-one-eq* [*of n*] **by** *simp*

lemma *Suc-0-xor-eq*:
‹*Suc 0 XOR n* = *n* + *of-bool* (*even n*) − *of-bool* (*odd n*)›
using *one-xor-eq* [*of n*] **by** *simp*

lemma *xor-Suc-0-eq*:
‹*n XOR Suc 0* = *n* + *of-bool* (*even n*) − *of-bool* (*odd n*)›
using *xor-one-eq* [*of n*] **by** *simp*

lemma *and-nat-unfold* [*code*]:
‹*m AND n* = (*if m* = *0* ∨ *n* = *0 then 0 else* (*m mod 2*) ∗ (*n mod 2*) + *2* ∗ ((*m div 2*) *AND* (*n div 2*)))›
for *m n* :: *nat*
by (*auto simp*: *and-rec* [*of m n*] *elim*: *oddE*)

lemma *or-nat-unfold* [*code*]:
‹*m OR n* = (*if m* = *0 then n else if n* = *0 then m*
else max (*m mod 2*) (*n mod 2*) + *2* ∗ ((*m div 2*) *OR* (*n div 2*)))› **for** *m n* :: *nat*
by (*auto simp*: *or-rec* [*of m n*] *elim*: *oddE*)

lemma *xor-nat-unfold* [*code*]:
‹*m XOR n* = (*if m* = *0 then n else if n* = *0 then m*
else (*m mod 2* + *n mod 2*) *mod 2* + *2* ∗ ((*m div 2*) *XOR* (*n div 2*)))› **for** *m n* :: *nat*
by (*auto simp*: *xor-rec* [*of m n*] *elim*!: *oddE*)

lemma [*code*]:
‹*unset-bit 0 m* = *2* ∗ (*m div 2*)›
‹*unset-bit* (*Suc n*) *m* = *m mod 2* + *2* ∗ *unset-bit n* (*m div 2*)› **for** *m n* :: *nat*
by (*simp-all add*: *unset-bit-0 unset-bit-Suc*)

lemma *push-bit-of-Suc-0* [*simp*]:
‹*push-bit n* (*Suc 0*) = *2* ^ *n*›
using *push-bit-of-1* [**where** *?'a* = *nat*] **by** *simp*

lemma *take-bit-of-Suc-0* [*simp*]:
‹*take-bit n* (*Suc 0*) = *of-bool* (*0* < *n*)›
using *take-bit-of-1* [**where** *?'a* = *nat*] **by** *simp*

lemma *drop-bit-of-Suc-0* [*simp*]:
‹*drop-bit n* (*Suc 0*) = *of-bool* (*n* = *0*)›
using *drop-bit-of-1* [**where** *?'a* = *nat*] **by** *simp*

lemma *Suc-mask-eq-exp*:
‹*Suc* (*mask n*) = *2* ^ *n*›
by (*simp add*: *mask-eq-exp-minus-1*)

lemma *less-eq-mask*:
‹*n* ≤ *mask n*›
by (*simp add*: *mask-eq-exp-minus-1 le-diff-conv2*)
(*metis Suc-mask-eq-exp diff-Suc-1 diff-le-diff-pow diff-zero le-refl not-less-eq-eq power-0*)

lemma *less-mask*:
‹*n* < *mask n*› **if** ‹*Suc 0* < *n*›
proof −
define *m* **where** ‹*m* = *n* − *2*›
with *that* **have** ∗: ‹*n* = *m* + *2*›
by *simp*
have ‹*Suc* (*Suc* (*Suc m*)) < *4* ∗ *2* ^ *m*›
by (*induction m*) *simp-all*
then have ‹*Suc* (*m* + *2*) < *Suc* (*mask* (*m* + *2*))›
by (*simp add*: *Suc-mask-eq-exp*)
then have ‹*m* + *2* < *mask* (*m* + *2*)›
by (*simp add*: *less-le*)
with ∗ **show** *?thesis*
by *simp*
qed

lemma *mask-nat-less-exp* [*simp*]:
‹(*mask n* :: *nat*) < *2* ^ *n*›
by (*simp add*: *mask-eq-exp-minus-1*)

lemma *mask-nat-positive-iff* [*simp*]:
‹(*0*::*nat*) < *mask n* ⟷ *0* < *n*›
proof (*cases* ‹*n* = *0*›)
case *True*
then show *?thesis*
by *simp*

next
  case *False*
  then have ‹*0 < n*›
    by *simp*
  then have ‹(*0*::*nat*) < *mask n*›
    using *less-eq-mask* [*of n*] **by** (*rule order-less-le-trans*)
  with ‹*0 < n*› **show** *?thesis*
    by *simp*
qed

lemma *take-bit-tightened-less-eq-nat*:
  ‹*take-bit m q* ≤ *take-bit n q*› **if** ‹*m* ≤ *n*› **for** *q* :: *nat*
proof −
  have ‹*take-bit m* (*take-bit n q*) ≤ *take-bit n q*›
    by (*rule take-bit-nat-less-eq-self*)
  with *that* **show** *?thesis*
    by *simp*
qed

lemma *push-bit-nat-eq*:
  ‹*push-bit n* (*nat k*) = *nat* (*push-bit n k*)›
  by (*cases* ‹*k* ≥ *0*›) (*simp-all add*: *push-bit-eq-mult nat-mult-distrib not-le mult-nonneg-nonpos2*)

lemma *drop-bit-nat-eq*:
  ‹*drop-bit n* (*nat k*) = *nat* (*drop-bit n k*)›
proof (*cases* ‹*k* ≥ *0*›)
  case *True*
  then show *?thesis*
    by (*metis drop-bit-of-nat int-nat-eq nat-int*)
qed (*simp add*: *nat-eq-iff2*)

lemma *take-bit-nat-eq*:
  ‹*take-bit n* (*nat k*) = *nat* (*take-bit n k*)› **if** ‹*k* ≥ *0*›
  using *that* **by** (*simp add*: *take-bit-eq-mod nat-mod-distrib nat-power-eq*)

lemma *nat-take-bit-eq*:
  ‹*nat* (*take-bit n k*) = *take-bit n* (*nat k*)›
  if ‹*k* ≥ *0*›
  using *that* **by** (*simp add*: *take-bit-eq-mod nat-mod-distrib nat-power-eq*)

lemma *nat-mask-eq*:
  ‹*nat* (*mask n*) = *mask n*›
  by (*simp add*: *nat-eq-iff of-nat-mask-eq*)

68.6 Symbolic computations on numeral expressions

context *semiring-bits*
begin

lemma *bit-1-0* [*simp*]:
‹*bit 1 0*›
by (*simp add*: *bit-0*)

lemma *not-bit-1-Suc* [*simp*]:
‹¬ *bit 1* (*Suc n*)›
by (*simp add*: *bit-Suc*)

lemma *not-bit-1-numeral* [*simp*]:
‹¬ *bit 1* (*numeral m*)›
by (*simp add*: *numeral-eq-Suc*)

lemma *not-bit-numeral-Bit0-0* [*simp*]:
‹¬ *bit* (*numeral* (*Num.Bit0 m*)) *0*›
by (*simp add*: *bit-0*)

lemma *bit-numeral-Bit1-0* [*simp*]:
‹*bit* (*numeral* (*Num.Bit1 m*)) *0*›
by (*simp add*: *bit-0*)

lemma *bit-numeral-Bit0-iff*:
‹*bit* (*numeral* (*num.Bit0 m*)) *n*
⟷ *possible-bit TYPE*(′*a*) *n* ∧ *n* > *0* ∧ *bit* (*numeral m*) (*n* − *1*)›
by (*simp only*: *numeral-Bit0-eq-double* [*of m*] *bit-simps*) *simp*

lemma *bit-numeral-Bit1-Suc-iff*:
‹*bit* (*numeral* (*num.Bit1 m*)) (*Suc n*)
⟷ *possible-bit TYPE*(′*a*) (*Suc n*) ∧ *bit* (*numeral m*) *n*›
using *even-bit-succ-iff* [*of* ‹*2* ∗ *numeral m*› ‹*Suc n*›]
by (*simp only*: *numeral-Bit1-eq-inc-double* [*of m*] *bit-simps*) *simp*

end

context *ring-bit-operations*
begin

lemma *not-bit-minus-numeral-Bit0-0* [*simp*]:
‹¬ *bit* (− *numeral* (*Num.Bit0 m*)) *0*›
by (*simp add*: *bit-0*)

lemma *bit-minus-numeral-Bit1-0* [*simp*]:
‹*bit* (− *numeral* (*Num.Bit1 m*)) *0*›
by (*simp add*: *bit-0*)

lemma *bit-minus-numeral-Bit0-Suc-iff*:
‹*bit* (− *numeral* (*num.Bit0 m*)) (*Suc n*)
⟷ *possible-bit TYPE*(′*a*) (*Suc n*) ∧ *bit* (− *numeral m*) *n*›
by (*simp only*: *numeral-Bit0-eq-double* [*of m*] *minus-mult-right bit-simps*) *auto*

lemma *bit-minus-numeral-Bit1-Suc-iff*:
‹*bit* (− *numeral* (*num.Bit1 m*)) (*Suc n*)
⟷ *possible-bit TYPE*(′*a*) (*Suc n*) ∧ ¬ *bit* (*numeral m*) *n*›
by (*simp only*: *numeral-Bit1-eq-inc-double* [*of m*] *minus-add-distrib minus-mult-right add-uminus-conv-diff*
bit-decr-iff bit-double-iff)
auto

lemma *bit-numeral-BitM-0* [*simp*]:
‹*bit* (*numeral* (*Num.BitM m*)) *0*›
by (*simp only*: *numeral-BitM bit-decr-iff not-bit-minus-numeral-Bit0-0*) *simp*

lemma *bit-numeral-BitM-Suc-iff*:
‹*bit* (*numeral* (*Num.BitM m*)) (*Suc n*) ⟷ *possible-bit TYPE*(′*a*) (*Suc n*) ∧ ¬ *bit* (− *numeral m*) *n*›
by (*simp-all only*: *numeral-BitM bit-decr-iff bit-minus-numeral-Bit0-Suc-iff*) *auto*

end

context *linordered-euclidean-semiring-bit-operations*
begin

lemma *bit-numeral-iff*:
‹*bit* (*numeral m*) *n* ⟷ *bit* (*numeral m* :: *nat*) *n*›
using *bit-of-nat-iff-bit* [*of* ‹*numeral m*› *n*] **by** *simp*

lemma *bit-numeral-Bit0-Suc-iff* [*simp*]:
‹*bit* (*numeral* (*Num.Bit0 m*)) (*Suc n*) ⟷ *bit* (*numeral m*) *n*›
by (*simp add*: *bit-Suc numeral-Bit0-div-2*)

lemma *bit-numeral-Bit1-Suc-iff* [*simp*]:
‹*bit* (*numeral* (*Num.Bit1 m*)) (*Suc n*) ⟷ *bit* (*numeral m*) *n*›
by (*simp add*: *bit-Suc numeral-Bit0-div-2*)

lemma *bit-numeral-rec*:
‹*bit* (*numeral* (*Num.Bit0 w*)) *n* ⟷ (*case n of 0* ⇒ *False* | *Suc m* ⇒ *bit* (*numeral w*) *m*)›
‹*bit* (*numeral* (*Num.Bit1 w*)) *n* ⟷ (*case n of 0* ⇒ *True* | *Suc m* ⇒ *bit* (*numeral w*) *m*)›
by (*cases n*; *simp add*: *bit-0*)+

lemma *bit-numeral-simps* [*simp*]:
‹*bit* (*numeral* (*Num.Bit0 w*)) (*numeral n*) ⟷ *bit* (*numeral w*) (*pred-numeral n*)›
‹*bit* (*numeral* (*Num.Bit1 w*)) (*numeral n*) ⟷ *bit* (*numeral w*) (*pred-numeral n*)›
by (*simp-all add*: *bit-1-iff numeral-eq-Suc*)

lemma *and-numerals* [*simp*]:
‹*1 AND numeral* (*Num.Bit0 y*) = *0*›
‹*1 AND numeral* (*Num.Bit1 y*) = *1*›

‹numeral (Num.Bit0 x) AND numeral (Num.Bit0 y) = 2 * (numeral x AND numeral y)›
‹numeral (Num.Bit0 x) AND numeral (Num.Bit1 y) = 2 * (numeral x AND numeral y)›
‹numeral (Num.Bit0 x) AND 1 = 0›
‹numeral (Num.Bit1 x) AND numeral (Num.Bit0 y) = 2 * (numeral x AND numeral y)›
‹numeral (Num.Bit1 x) AND numeral (Num.Bit1 y) = 1 + 2 * (numeral x AND numeral y)›
‹numeral (Num.Bit1 x) AND 1 = 1›
by (*simp-all add*: *bit-eq-iff*) (*simp-all add*: *bit-0 bit-simps bit-Suc bit-numeral-rec split*: *nat.splits*)

lemma *or-numerals* [*simp*]:
‹1 OR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)›
‹1 OR numeral (Num.Bit1 y) = numeral (Num.Bit1 y)›
‹numeral (Num.Bit0 x) OR numeral (Num.Bit0 y) = 2 * (numeral x OR numeral y)›
‹numeral (Num.Bit0 x) OR numeral (Num.Bit1 y) = 1 + 2 * (numeral x OR numeral y)›
‹numeral (Num.Bit0 x) OR 1 = numeral (Num.Bit1 x)›
‹numeral (Num.Bit1 x) OR numeral (Num.Bit0 y) = 1 + 2 * (numeral x OR numeral y)›
‹numeral (Num.Bit1 x) OR numeral (Num.Bit1 y) = 1 + 2 * (numeral x OR numeral y)›
‹numeral (Num.Bit1 x) OR 1 = numeral (Num.Bit1 x)›
by (*simp-all add*: *bit-eq-iff*) (*simp-all add*: *bit-0 bit-simps bit-Suc bit-numeral-rec split*: *nat.splits*)

lemma *xor-numerals* [*simp*]:
‹1 XOR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)›
‹1 XOR numeral (Num.Bit1 y) = numeral (Num.Bit0 y)›
‹numeral (Num.Bit0 x) XOR numeral (Num.Bit0 y) = 2 * (numeral x XOR numeral y)›
‹numeral (Num.Bit0 x) XOR numeral (Num.Bit1 y) = 1 + 2 * (numeral x XOR numeral y)›
‹numeral (Num.Bit0 x) XOR 1 = numeral (Num.Bit1 x)›
‹numeral (Num.Bit1 x) XOR numeral (Num.Bit0 y) = 1 + 2 * (numeral x XOR numeral y)›
‹numeral (Num.Bit1 x) XOR numeral (Num.Bit1 y) = 2 * (numeral x XOR numeral y)›
‹numeral (Num.Bit1 x) XOR 1 = numeral (Num.Bit0 x)›
by (*simp-all add*: *bit-eq-iff*) (*simp-all add*: *bit-0 bit-simps bit-Suc bit-numeral-rec split*: *nat.splits*)

end

lemma *drop-bit-Suc-minus-bit0* [*simp*]:
‹drop-bit (Suc n) (− numeral (Num.Bit0 k)) = drop-bit n (− numeral k :: int)›

by (*simp add*: *drop-bit-Suc numeral-Bit0-div-2*)

lemma *drop-bit-Suc-minus-bit1* [*simp*]:
‹*drop-bit* (*Suc n*) (− *numeral* (*Num.Bit1 k*)) = *drop-bit n* (− *numeral* (*Num.inc k*) :: *int*)›
by (*simp add*: *drop-bit-Suc numeral-Bit1-div-2 add-One*)

lemma *drop-bit-numeral-minus-bit0* [*simp*]:
‹*drop-bit* (*numeral l*) (− *numeral* (*Num.Bit0 k*)) = *drop-bit* (*pred-numeral l*) (− *numeral k* :: *int*)›
by (*simp add*: *numeral-eq-Suc numeral-Bit0-div-2*)

lemma *drop-bit-numeral-minus-bit1* [*simp*]:
‹*drop-bit* (*numeral l*) (− *numeral* (*Num.Bit1 k*)) = *drop-bit* (*pred-numeral l*) (− *numeral* (*Num.inc k*) :: *int*)›
by (*simp add*: *numeral-eq-Suc numeral-Bit1-div-2*)

lemma *take-bit-Suc-minus-bit0*:
‹*take-bit* (*Suc n*) (− *numeral* (*Num.Bit0 k*)) = *take-bit n* (− *numeral k*) ∗ (*2* :: *int*)›
by (*simp add*: *take-bit-Suc numeral-Bit0-div-2*)

lemma *take-bit-Suc-minus-bit1*:
‹*take-bit* (*Suc n*) (− *numeral* (*Num.Bit1 k*)) = *take-bit n* (− *numeral* (*Num.inc k*)) ∗ *2* + (*1* :: *int*)›
by (*simp add*: *take-bit-Suc numeral-Bit1-div-2 add-One*)

lemma *take-bit-numeral-minus-bit0*:
‹*take-bit* (*numeral l*) (− *numeral* (*Num.Bit0 k*)) = *take-bit* (*pred-numeral l*) (− *numeral k*) ∗ (*2* :: *int*)›
by (*simp add*: *numeral-eq-Suc numeral-Bit0-div-2 take-bit-Suc-minus-bit0*)

lemma *take-bit-numeral-minus-bit1*:
‹*take-bit* (*numeral l*) (− *numeral* (*Num.Bit1 k*)) = *take-bit* (*pred-numeral l*) (− *numeral* (*Num.inc k*)) ∗ *2* + (*1* :: *int*)›
by (*simp add*: *numeral-eq-Suc numeral-Bit1-div-2 take-bit-Suc-minus-bit1*)

lemma *and-nat-numerals* [*simp*]:
‹*Suc 0 AND numeral* (*Num.Bit0 y*) = *0*›
‹*Suc 0 AND numeral* (*Num.Bit1 y*) = *1*›
‹*numeral* (*Num.Bit0 x*) *AND Suc 0* = *0*›
‹*numeral* (*Num.Bit1 x*) *AND Suc 0* = *1*›
by (*simp-all only*: *and-numerals flip*: *One-nat-def*)

lemma *or-nat-numerals* [*simp*]:
‹*Suc 0 OR numeral* (*Num.Bit0 y*) = *numeral* (*Num.Bit1 y*)›
‹*Suc 0 OR numeral* (*Num.Bit1 y*) = *numeral* (*Num.Bit1 y*)›
‹*numeral* (*Num.Bit0 x*) *OR Suc 0* = *numeral* (*Num.Bit1 x*)›
‹*numeral* (*Num.Bit1 x*) *OR Suc 0* = *numeral* (*Num.Bit1 x*)›

by (*simp-all only*: *or-numerals flip*: *One-nat-def*)

lemma *xor-nat-numerals* [*simp*]:
‹*Suc 0 XOR numeral* (*Num.Bit0 y*) = *numeral* (*Num.Bit1 y*)›
‹*Suc 0 XOR numeral* (*Num.Bit1 y*) = *numeral* (*Num.Bit0 y*)›
‹*numeral* (*Num.Bit0 x*) *XOR Suc 0* = *numeral* (*Num.Bit1 x*)›
‹*numeral* (*Num.Bit1 x*) *XOR Suc 0* = *numeral* (*Num.Bit0 x*)›
by (*simp-all only*: *xor-numerals flip*: *One-nat-def*)

context *ring-bit-operations*
begin

lemma *minus-numeral-inc-eq*:
‹− *numeral* (*Num.inc n*) = *NOT* (*numeral n*)›
by (*simp add*: *not-eq-complement sub-inc-One-eq add-One*)

lemma *sub-one-eq-not-neg*:
‹*Num.sub n num.One* = *NOT* (− *numeral n*)›
by (*simp add*: *not-eq-complement*)

lemma *minus-numeral-eq-not-sub-one*:
‹− *numeral n* = *NOT* (*Num.sub n num.One*)›
by (*simp add*: *not-eq-complement*)

lemma *not-numeral-eq* [*simp*]:
‹*NOT* (*numeral n*) = − *numeral* (*Num.inc n*)›
by (*simp add*: *minus-numeral-inc-eq*)

lemma *not-minus-numeral-eq* [*simp*]:
‹*NOT* (− *numeral n*) = *Num.sub n num.One*›
by (*simp add*: *sub-one-eq-not-neg*)

lemma *minus-not-numeral-eq* [*simp*]:
‹− (*NOT* (*numeral n*)) = *numeral* (*Num.inc n*)›
by *simp*

lemma *not-numeral-BitM-eq*:
‹*NOT* (*numeral* (*Num.BitM n*)) = − *numeral* (*num.Bit0 n*)›
by (*simp add*: *inc-BitM-eq*)

lemma *not-numeral-Bit0-eq*:
‹*NOT* (*numeral* (*Num.Bit0 n*)) = − *numeral* (*num.Bit1 n*)›
by *simp*

end

lemma *bit-minus-numeral-int* [*simp*]:
‹*bit* (− *numeral* (*num.Bit0 w*) :: *int*) (*numeral n*) ⟷ *bit* (− *numeral w* :: *int*) (*pred-numeral n*)›

‹bit (− numeral (num.Bit1 w) :: int) (numeral n) ⟷ ¬ bit (numeral w :: int) (pred-numeral n)›
by (simp-all add: bit-minus-iff bit-not-iff numeral-eq-Suc bit-Suc add-One sub-inc-One-eq)

lemma bit-minus-numeral-Bit0-Suc-iff [simp]:
‹bit (− numeral (num.Bit0 w) :: int) (Suc n) ⟷ bit (− numeral w :: int) n›
by (simp add: bit-Suc)

lemma bit-minus-numeral-Bit1-Suc-iff [simp]:
‹bit (− numeral (num.Bit1 w) :: int) (Suc n) ⟷ ¬ bit (numeral w :: int) n›
by (simp add: bit-Suc add-One flip: bit-not-int-iff)

lemma and-not-numerals:
‹1 AND NOT 1 = (0 :: int)›
‹1 AND NOT (numeral (Num.Bit0 n)) = (1 :: int)›
‹1 AND NOT (numeral (Num.Bit1 n)) = (0 :: int)›
‹numeral (Num.Bit0 m) AND NOT (1 :: int) = numeral (Num.Bit0 m)›
‹numeral (Num.Bit0 m) AND NOT (numeral (Num.Bit0 n)) = (2 :: int) ∗ (numeral m AND NOT (numeral n))›
‹numeral (Num.Bit0 m) AND NOT (numeral (Num.Bit1 n)) = (2 :: int) ∗ (numeral m AND NOT (numeral n))›
‹numeral (Num.Bit1 m) AND NOT (1 :: int) = numeral (Num.Bit0 m)›
‹numeral (Num.Bit1 m) AND NOT (numeral (Num.Bit0 n)) = 1 + (2 :: int) ∗ (numeral m AND NOT (numeral n))›
‹numeral (Num.Bit1 m) AND NOT (numeral (Num.Bit1 n)) = (2 :: int) ∗ (numeral m AND NOT (numeral n))›
by (simp-all add: bit-eq-iff)
(auto simp: bit-0 bit-simps bit-Suc bit-numeral-rec BitM-inc-eq sub-inc-One-eq split: nat.split)

fun and-not-num :: ‹num ⇒ num ⇒ num option›
where
‹and-not-num num.One num.One = None›
| ‹and-not-num num.One (num.Bit0 n) = Some num.One›
| ‹and-not-num num.One (num.Bit1 n) = None›
| ‹and-not-num (num.Bit0 m) num.One = Some (num.Bit0 m)›
| ‹and-not-num (num.Bit0 m) (num.Bit0 n) = map-option num.Bit0 (and-not-num m n)›
| ‹and-not-num (num.Bit0 m) (num.Bit1 n) = map-option num.Bit0 (and-not-num m n)›
| ‹and-not-num (num.Bit1 m) num.One = Some (num.Bit0 m)›
| ‹and-not-num (num.Bit1 m) (num.Bit0 n) = (case and-not-num m n of None ⇒ Some num.One | Some n′ ⇒ Some (num.Bit1 n′))›
| ‹and-not-num (num.Bit1 m) (num.Bit1 n) = map-option num.Bit0 (and-not-num m n)›

lemma int-numeral-and-not-num:
‹numeral m AND NOT (numeral n) = (case and-not-num m n of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)›

by (*induction m n rule*: *and-not-num.induct*) (*simp-all del*: *not-numeral-eq not-one-eq add*: *and-not-numerals split*: *option.splits*)

lemma *int-numeral-not-and-num*:
‹*NOT* (*numeral m*) *AND numeral n* = (*case and-not-num n m of None* ⇒ *0* :: *int* | *Some n′* ⇒ *numeral n′*)›
using *int-numeral-and-not-num* [*of n m*] **by** (*simp add*: *ac-simps*)

lemma *and-not-num-eq-None-iff*:
‹*and-not-num m n* = *None* ⟷ *numeral m AND NOT* (*numeral n*) = (*0* :: *int*)›
by (*simp del*: *not-numeral-eq add*: *int-numeral-and-not-num split*: *option.split*)

lemma *and-not-num-eq-Some-iff*:
‹*and-not-num m n* = *Some q* ⟷ *numeral m AND NOT* (*numeral n*) = (*numeral q* :: *int*)›
by (*simp del*: *not-numeral-eq add*: *int-numeral-and-not-num split*: *option.split*)

lemma *and-minus-numerals* [*simp*]:
‹*1 AND* − (*numeral* (*num.Bit0 n*)) = (*0*::*int*)›
‹*1 AND* − (*numeral* (*num.Bit1 n*)) = (*1*::*int*)›
‹*numeral m AND* − (*numeral* (*num.Bit0 n*)) = (*case and-not-num m* (*Num.BitM n*) *of None* ⇒ *0* :: *int* | *Some n′* ⇒ *numeral n′*)›
‹*numeral m AND* − (*numeral* (*num.Bit1 n*)) = (*case and-not-num m* (*Num.Bit0 n*) *of None* ⇒ *0* :: *int* | *Some n′* ⇒ *numeral n′*)›
‹− (*numeral* (*num.Bit0 n*)) *AND 1* = (*0*::*int*)›
‹− (*numeral* (*num.Bit1 n*)) *AND 1* = (*1*::*int*)›
‹− (*numeral* (*num.Bit0 n*)) *AND numeral m* = (*case and-not-num m* (*Num.BitM n*) *of None* ⇒ *0* :: *int* | *Some n′* ⇒ *numeral n′*)›
‹− (*numeral* (*num.Bit1 n*)) *AND numeral m* = (*case and-not-num m* (*Num.Bit0 n*) *of None* ⇒ *0* :: *int* | *Some n′* ⇒ *numeral n′*)›
by (*simp-all del*: *not-numeral-eq add*: *ac-simps*
and-not-numerals one-and-eq not-numeral-BitM-eq not-numeral-Bit0-eq and-not-num-eq-None-iff and-not-num-eq-Some-iff split: *option.split*)

lemma *and-minus-minus-numerals* [*simp*]:
‹− (*numeral m* :: *int*) *AND* − (*numeral n* :: *int*) = *NOT* ((*numeral m* − *1*) *OR* (*numeral n* − *1*))›
by (*simp add*: *minus-numeral-eq-not-sub-one*)

lemma *or-not-numerals*:
‹*1 OR NOT 1* = *NOT* (*0* :: *int*)›
‹*1 OR NOT* (*numeral* (*Num.Bit0 n*)) = *NOT* (*numeral* (*Num.Bit0 n*) :: *int*)›
‹*1 OR NOT* (*numeral* (*Num.Bit1 n*)) = *NOT* (*numeral* (*Num.Bit0 n*) :: *int*)›
‹*numeral* (*Num.Bit0 m*) *OR NOT* (*1* :: *int*) = *NOT* (*1* :: *int*)›
‹*numeral* (*Num.Bit0 m*) *OR NOT* (*numeral* (*Num.Bit0 n*)) = *1* + (*2* :: *int*) ∗ (*numeral m OR NOT* (*numeral n*))›
‹*numeral* (*Num.Bit0 m*) *OR NOT* (*numeral* (*Num.Bit1 n*)) = (*2* :: *int*) ∗ (*numeral m OR NOT* (*numeral n*))›
‹*numeral* (*Num.Bit1 m*) *OR NOT* (*1* :: *int*) = *NOT* (*0* :: *int*)›

⟨*numeral* (*Num.Bit1 m*) *OR NOT* (*numeral* (*Num.Bit0 n*)) = *1* + (*2* :: *int*) ∗ (*numeral m OR NOT* (*numeral n*))⟩
⟨*numeral* (*Num.Bit1 m*) *OR NOT* (*numeral* (*Num.Bit1 n*)) = *1* + (*2* :: *int*) ∗ (*numeral m OR NOT* (*numeral n*))⟩
by (*simp-all add*: *bit-eq-iff*)
(*auto simp*: *bit-0 bit-simps bit-Suc bit-numeral-rec sub-inc-One-eq split*: *nat.split*)

fun *or-not-num-neg* :: ⟨*num* ⇒ *num* ⇒ *num*⟩
where
⟨*or-not-num-neg num.One num.One* = *num.One*⟩
| ⟨*or-not-num-neg num.One* (*num.Bit0 m*) = *num.Bit1 m*⟩
| ⟨*or-not-num-neg num.One* (*num.Bit1 m*) = *num.Bit1 m*⟩
| ⟨*or-not-num-neg* (*num.Bit0 n*) *num.One* = *num.Bit0 num.One*⟩
| ⟨*or-not-num-neg* (*num.Bit0 n*) (*num.Bit0 m*) = *Num.BitM* (*or-not-num-neg n m*)⟩
| ⟨*or-not-num-neg* (*num.Bit0 n*) (*num.Bit1 m*) = *num.Bit0* (*or-not-num-neg n m*)⟩
| ⟨*or-not-num-neg* (*num.Bit1 n*) *num.One* = *num.One*⟩
| ⟨*or-not-num-neg* (*num.Bit1 n*) (*num.Bit0 m*) = *Num.BitM* (*or-not-num-neg n m*)⟩
| ⟨*or-not-num-neg* (*num.Bit1 n*) (*num.Bit1 m*) = *Num.BitM* (*or-not-num-neg n m*)⟩

lemma *int-numeral-or-not-num-neg*:
⟨*numeral m OR NOT* (*numeral n* :: *int*) = − *numeral* (*or-not-num-neg m n*)⟩
by (*induction m n rule*: *or-not-num-neg.induct*) (*simp-all del*: *not-numeral-eq not-one-eq add*: *or-not-numerals*, *simp-all*)

lemma *int-numeral-not-or-num-neg*:
⟨*NOT* (*numeral m*) *OR* (*numeral n* :: *int*) = − *numeral* (*or-not-num-neg n m*)⟩
using *int-numeral-or-not-num-neg* [*of n m*] **by** (*simp add*: *ac-simps*)

lemma *numeral-or-not-num-eq*:
⟨*numeral* (*or-not-num-neg m n*) = − (*numeral m OR NOT* (*numeral n* :: *int*))⟩
using *int-numeral-or-not-num-neg* [*of m n*] **by** *simp*

lemma *or-minus-numerals* [*simp*]:
⟨*1 OR* − (*numeral* (*num.Bit0 n*)) = − (*numeral* (*or-not-num-neg num.One* (*Num.BitM n*)) :: *int*)⟩
⟨*1 OR* − (*numeral* (*num.Bit1 n*)) = − (*numeral* (*num.Bit1 n*) :: *int*)⟩
⟨*numeral m OR* − (*numeral* (*num.Bit0 n*)) = − (*numeral* (*or-not-num-neg m* (*Num.BitM n*)) :: *int*)⟩
⟨*numeral m OR* − (*numeral* (*num.Bit1 n*)) = − (*numeral* (*or-not-num-neg m* (*Num.Bit0 n*)) :: *int*)⟩
⟨− (*numeral* (*num.Bit0 n*)) *OR 1* = − (*numeral* (*or-not-num-neg num.One* (*Num.BitM n*)) :: *int*)⟩
⟨− (*numeral* (*num.Bit1 n*)) *OR 1* = − (*numeral* (*num.Bit1 n*) :: *int*)⟩
⟨− (*numeral* (*num.Bit0 n*)) *OR numeral m* = − (*numeral* (*or-not-num-neg m* (*Num.BitM n*)) :: *int*)⟩
⟨− (*numeral* (*num.Bit1 n*)) *OR numeral m* = − (*numeral* (*or-not-num-neg m*

(*Num.Bit0 n*)) :: *int*)›
by (*simp-all only*: *or.commute* [*of - 1*] *or.commute* [*of -* ‹*numeral m*›]
minus-numeral-eq-not-sub-one or-not-numerals
numeral-or-not-num-eq arith-simps minus-minus numeral-One)

lemma *or-minus-minus-numerals* [*simp*]:
‹− (*numeral m* :: *int*) *OR* − (*numeral n* :: *int*) = *NOT* ((*numeral m* − *1*) *AND* (*numeral n* − *1*))›
by (*simp add*: *minus-numeral-eq-not-sub-one*)

lemma *xor-minus-numerals* [*simp*]:
‹− *numeral n XOR k* = *NOT* (*neg-numeral-class.sub n num.One XOR k*)›
‹*k XOR* − *numeral n* = *NOT* (*k XOR* (*neg-numeral-class.sub n num.One*))› **for** *k* :: *int*
by (*simp-all add*: *minus-numeral-eq-not-sub-one*)

definition *take-bit-num* :: ‹*nat* ⇒ *num* ⇒ *num option*›
where ‹*take-bit-num n m* =
(*if take-bit n* (*numeral m* :: *nat*) = *0 then None else Some* (*num-of-nat* (*take-bit n* (*numeral m* :: *nat*))))›

lemma *take-bit-num-simps*:
‹*take-bit-num 0 m* = *None*›
‹*take-bit-num* (*Suc n*) *Num.One* =
Some Num.One›
‹*take-bit-num* (*Suc n*) (*Num.Bit0 m*) =
(*case take-bit-num n m of None* ⇒ *None* | *Some q* ⇒ *Some* (*Num.Bit0 q*))›
‹*take-bit-num* (*Suc n*) (*Num.Bit1 m*) =
Some (*case take-bit-num n m of None* ⇒ *Num.One* | *Some q* ⇒ *Num.Bit1 q*)›
‹*take-bit-num* (*numeral r*) *Num.One* =
Some Num.One›
‹*take-bit-num* (*numeral r*) (*Num.Bit0 m*) =
(*case take-bit-num* (*pred-numeral r*) *m of None* ⇒ *None* | *Some q* ⇒ *Some* (*Num.Bit0 q*))›
‹*take-bit-num* (*numeral r*) (*Num.Bit1 m*) =
Some (*case take-bit-num* (*pred-numeral r*) *m of None* ⇒ *Num.One* | *Some q* ⇒ *Num.Bit1 q*)›
by (*auto simp*: *take-bit-num-def ac-simps mult-2 num-of-nat-double*
take-bit-Suc-bit0 take-bit-Suc-bit1 take-bit-numeral-bit0 take-bit-numeral-bit1)

lemma *take-bit-num-code* [*code*]:
— Ocaml-style pattern matching is more robust wrt. different representations of *nat*
‹*take-bit-num n m* = (*case* (*n*, *m*)
of (*0*, -) ⇒ *None*
| (*Suc n*, *Num.One*) ⇒ *Some Num.One*
| (*Suc n*, *Num.Bit0 m*) ⇒ (*case take-bit-num n m of None* ⇒ *None* | *Some q* ⇒ *Some* (*Num.Bit0 q*))
| (*Suc n*, *Num.Bit1 m*) ⇒ *Some* (*case take-bit-num n m of None* ⇒ *Num.One*

| *Some q* ⇒ *Num.Bit1 q*))›
  by (*cases n*; *cases m*) (*simp-all add*: *take-bit-num-simps*)

context *semiring-bit-operations*
begin

lemma *take-bit-num-eq-None-imp*:
  ‹*take-bit m* (*numeral n*) = *0*› **if** ‹*take-bit-num m n* = *None*›
proof −
  from *that* **have** ‹*take-bit m* (*numeral n* :: *nat*) = *0*›
    by (*simp add*: *take-bit-num-def split*: *if-splits*)
  then have ‹*of-nat* (*take-bit m* (*numeral n*)) = *of-nat 0*›
    by *simp*
  then show *?thesis*
    by (*simp add*: *of-nat-take-bit*)
qed

lemma *take-bit-num-eq-Some-imp*:
  ‹*take-bit m* (*numeral n*) = *numeral q*› **if** ‹*take-bit-num m n* = *Some q*›
proof −
  from *that* **have** ‹*take-bit m* (*numeral n* :: *nat*) = *numeral q*›
    by (*auto simp*: *take-bit-num-def Num.numeral-num-of-nat-unfold split*: *if-splits*)
  then have ‹*of-nat* (*take-bit m* (*numeral n*)) = *of-nat* (*numeral q*)›
    by *simp*
  then show *?thesis*
    by (*simp add*: *of-nat-take-bit*)
qed

lemma *take-bit-numeral-numeral*:
  ‹*take-bit* (*numeral m*) (*numeral n*) =
    (*case take-bit-num* (*numeral m*) *n of None* ⇒ *0* | *Some q* ⇒ *numeral q*)›
  by (*auto split*: *option.split dest*: *take-bit-num-eq-None-imp take-bit-num-eq-Some-imp*)

end

lemma *take-bit-numeral-minus-numeral-int*:
  ‹*take-bit* (*numeral m*) (− *numeral n* :: *int*) =
    (*case take-bit-num* (*numeral m*) *n of None* ⇒ *0* | *Some q* ⇒ *take-bit* (*numeral m*) (*2* ^ *numeral m* − *numeral q*))› (**is** ‹*?lhs* = *?rhs*›)
proof (*cases* ‹*take-bit-num* (*numeral m*) *n*›)
  case *None*
  then show *?thesis*
    by (*auto dest*: *take-bit-num-eq-None-imp* [**where** *?'a* = *int*] *simp add*: *take-bit-eq-0-iff*)
next
  case (*Some q*)
  then have *q*: ‹*take-bit* (*numeral m*) (*numeral n* :: *int*) = *numeral q*›
    by (*auto dest*: *take-bit-num-eq-Some-imp*)
  let *?T* = ‹*take-bit* (*numeral m*) :: *int* ⇒ *int*›
  have ∗: ‹*?T* (*2* ^ *numeral m*) = *?T* (*?T 0*)›

by (*simp add*: *take-bit-eq-0-iff*)
have ‹*?lhs* = *?T* (*0* − *numeral n*)›
by *simp*
also have ‹... = *?T* (*?T* (*?T 0*) − *?T* (*?T* (*numeral n*)))›
by (*simp only*: *take-bit-diff*)
also have ‹... = *?T* (*2* ^ *numeral m* − *?T* (*numeral n*))›
by (*simp only*: *take-bit-diff flip*: ∗)
also have ‹... = *?rhs*›
by (*simp add*: *q Some*)
finally show *?thesis* **.**
qed

declare *take-bit-num-simps* [*simp*]
take-bit-numeral-numeral [*simp*]
take-bit-numeral-minus-numeral-int [*simp*]

68.7 Symbolic computations for code generation

lemma *bit-int-code* [*code*]:
‹*bit* (*0*::*int*) *n* ⟷ *False*›
‹*bit* (*Int.Neg num.One*) *n* ⟷ *True*›
‹*bit* (*Int.Pos num.One*) *0* ⟷ *True*›
‹*bit* (*Int.Pos* (*num.Bit0 m*)) *0* ⟷ *False*›
‹*bit* (*Int.Pos* (*num.Bit1 m*)) *0* ⟷ *True*›
‹*bit* (*Int.Neg* (*num.Bit0 m*)) *0* ⟷ *False*›
‹*bit* (*Int.Neg* (*num.Bit1 m*)) *0* ⟷ *True*›
‹*bit* (*Int.Pos num.One*) (*Suc n*) ⟷ *False*›
‹*bit* (*Int.Pos* (*num.Bit0 m*)) (*Suc n*) ⟷ *bit* (*Int.Pos m*) *n*›
‹*bit* (*Int.Pos* (*num.Bit1 m*)) (*Suc n*) ⟷ *bit* (*Int.Pos m*) *n*›
‹*bit* (*Int.Neg* (*num.Bit0 m*)) (*Suc n*) ⟷ *bit* (*Int.Neg m*) *n*›
‹*bit* (*Int.Neg* (*num.Bit1 m*)) (*Suc n*) ⟷ *bit* (*Int.Neg* (*Num.inc m*)) *n*›
by (*simp-all add*: *Num.add-One bit-0 bit-Suc*)

lemma *not-int-code* [*code*]:
‹*NOT* (*0* :: *int*) = − *1*›
‹*NOT* (*Int.Pos n*) = *Int.Neg* (*Num.inc n*)›
‹*NOT* (*Int.Neg n*) = *Num.sub n num.One*›
by (*simp-all add*: *Num.add-One not-int-def*)

fun *and-num* :: ‹*num* ⇒ *num* ⇒ *num option*›
where
‹*and-num num.One num.One* = *Some num.One*›
| ‹*and-num num.One* (*num.Bit0 n*) = *None*›
| ‹*and-num num.One* (*num.Bit1 n*) = *Some num.One*›
| ‹*and-num* (*num.Bit0 m*) *num.One* = *None*›
| ‹*and-num* (*num.Bit0 m*) (*num.Bit0 n*) = *map-option num.Bit0* (*and-num m n*)›
| ‹*and-num* (*num.Bit0 m*) (*num.Bit1 n*) = *map-option num.Bit0* (*and-num m n*)›
| ‹*and-num* (*num.Bit1 m*) *num.One* = *Some num.One*›
| ‹*and-num* (*num.Bit1 m*) (*num.Bit0 n*) = *map-option num.Bit0* (*and-num m n*)›

| ‹*and-num* (*num.Bit1 m*) (*num.Bit1 n*) = (*case and-num m n of None* ⇒ *Some num.One* | *Some n′* ⇒ *Some* (*num.Bit1 n′*))›

context *linordered-euclidean-semiring-bit-operations*
begin

lemma *numeral-and-num*:
‹*numeral m AND numeral n* = (*case and-num m n of None* ⇒ *0* | *Some n′* ⇒ *numeral n′*)›
by (*induction m n rule*: *and-num.induct*) (*simp-all add*: *split*: *option.split*)

lemma *and-num-eq-None-iff*:
‹*and-num m n* = *None* ⟷ *numeral m AND numeral n* = *0*›
by (*simp add*: *numeral-and-num split*: *option.split*)

lemma *and-num-eq-Some-iff*:
‹*and-num m n* = *Some q* ⟷ *numeral m AND numeral n* = *numeral q*›
by (*simp add*: *numeral-and-num split*: *option.split*)

end

lemma *and-int-code* [*code*]:
fixes *i j* :: *int* **shows**
‹*0 AND j* = *0*›
‹*i AND 0* = *0*›
‹*Int.Pos n AND Int.Pos m* = (*case and-num n m of None* ⇒ *0* | *Some n′* ⇒ *Int.Pos n′*)›
‹*Int.Neg n AND Int.Neg m* = *NOT* (*Num.sub n num.One OR Num.sub m num.One*)›
‹*Int.Pos n AND Int.Neg num.One* = *Int.Pos n*›
‹*Int.Pos n AND Int.Neg* (*num.Bit0 m*) = *Num.sub* (*or-not-num-neg* (*Num.BitM m*) *n*) *num.One*›
‹*Int.Pos n AND Int.Neg* (*num.Bit1 m*) = *Num.sub* (*or-not-num-neg* (*num.Bit0 m*) *n*) *num.One*›
‹*Int.Neg num.One AND Int.Pos m* = *Int.Pos m*›
‹*Int.Neg* (*num.Bit0 n*) *AND Int.Pos m* = *Num.sub* (*or-not-num-neg* (*Num.BitM n*) *m*) *num.One*›
‹*Int.Neg* (*num.Bit1 n*) *AND Int.Pos m* = *Num.sub* (*or-not-num-neg* (*num.Bit0 n*) *m*) *num.One*›
apply (*auto simp*: *and-num-eq-None-iff* [**where** *?′a* = *int*] *and-num-eq-Some-iff* [**where** *?′a* = *int*]
split: *option.split*)
apply (*simp-all only*: *sub-one-eq-not-neg numeral-or-not-num-eq minus-minus and-not-numerals*
bit.de-Morgan-disj bit.double-compl and-not-num-eq-None-iff and-not-num-eq-Some-iff ac-simps)
done

context *linordered-euclidean-semiring-bit-operations*

begin

fun *or-num* :: ‹*num* ⇒ *num* ⇒ *num*›
where
‹*or-num num.One num.One* = *num.One*›
| ‹*or-num num.One* (*num.Bit0 n*) = *num.Bit1 n*›
| ‹*or-num num.One* (*num.Bit1 n*) = *num.Bit1 n*›
| ‹*or-num* (*num.Bit0 m*) *num.One* = *num.Bit1 m*›
| ‹*or-num* (*num.Bit0 m*) (*num.Bit0 n*) = *num.Bit0* (*or-num m n*)›
| ‹*or-num* (*num.Bit0 m*) (*num.Bit1 n*) = *num.Bit1* (*or-num m n*)›
| ‹*or-num* (*num.Bit1 m*) *num.One* = *num.Bit1 m*›
| ‹*or-num* (*num.Bit1 m*) (*num.Bit0 n*) = *num.Bit1* (*or-num m n*)›
| ‹*or-num* (*num.Bit1 m*) (*num.Bit1 n*) = *num.Bit1* (*or-num m n*)›

lemma *numeral-or-num*:
‹*numeral m OR numeral n* = *numeral* (*or-num m n*)›
by (*induction m n rule*: *or-num.induct*) *simp-all*

lemma *numeral-or-num-eq*:
‹*numeral* (*or-num m n*) = *numeral m OR numeral n*›
by (*simp add*: *numeral-or-num*)

end

lemma *or-int-code* [*code*]:
fixes *i j* :: *int* **shows**
‹*0 OR j* = *j*›
‹*i OR 0* = *i*›
‹*Int.Pos n OR Int.Pos m* = *Int.Pos* (*or-num n m*)›
‹*Int.Neg n OR Int.Neg m* = *NOT* (*Num.sub n num.One AND Num.sub m num.One*)›
‹*Int.Pos n OR Int.Neg num.One* = *Int.Neg num.One*›
‹*Int.Pos n OR Int.Neg* (*num.Bit0 m*) = (*case and-not-num* (*Num.BitM m*) *n of None* ⇒ −*1* | *Some n*′ ⇒ *Int.Neg* (*Num.inc n*′))›
‹*Int.Pos n OR Int.Neg* (*num.Bit1 m*) = (*case and-not-num* (*num.Bit0 m*) *n of None* ⇒ −*1* | *Some n*′ ⇒ *Int.Neg* (*Num.inc n*′))›
‹*Int.Neg num.One OR Int.Pos m* = *Int.Neg num.One*›
‹*Int.Neg* (*num.Bit0 n*) *OR Int.Pos m* = (*case and-not-num* (*Num.BitM n*) *m of None* ⇒ −*1* | *Some n*′ ⇒ *Int.Neg* (*Num.inc n*′))›
‹*Int.Neg* (*num.Bit1 n*) *OR Int.Pos m* = (*case and-not-num* (*num.Bit0 n*) *m of None* ⇒ −*1* | *Some n*′ ⇒ *Int.Neg* (*Num.inc n*′))›
apply (*auto simp*: *numeral-or-num-eq split*: *option.splits*)
apply (*simp-all only*: *and-not-num-eq-None-iff and-not-num-eq-Some-iff and-not-numerals*
numeral-or-not-num-eq or-eq-not-not-and bit.double-compl ac-simps flip: *numeral-eq-iff* [**where** *?'a* = *int*])
apply *simp-all*
done

fun *xor-num* :: ‹*num* ⇒ *num* ⇒ *num option*›
where
‹*xor-num num.One num.One* = *None*›
| ‹*xor-num num.One* (*num.Bit0 n*) = *Some* (*num.Bit1 n*)›
| ‹*xor-num num.One* (*num.Bit1 n*) = *Some* (*num.Bit0 n*)›
| ‹*xor-num* (*num.Bit0 m*) *num.One* = *Some* (*num.Bit1 m*)›
| ‹*xor-num* (*num.Bit0 m*) (*num.Bit0 n*) = *map-option num.Bit0* (*xor-num m n*)›
| ‹*xor-num* (*num.Bit0 m*) (*num.Bit1 n*) = *Some* (*case xor-num m n of None* ⇒ *num.One* | *Some n'* ⇒ *num.Bit1 n'*)›
| ‹*xor-num* (*num.Bit1 m*) *num.One* = *Some* (*num.Bit0 m*)›
| ‹*xor-num* (*num.Bit1 m*) (*num.Bit0 n*) = *Some* (*case xor-num m n of None* ⇒ *num.One* | *Some n'* ⇒ *num.Bit1 n'*)›
| ‹*xor-num* (*num.Bit1 m*) (*num.Bit1 n*) = *map-option num.Bit0* (*xor-num m n*)›

context *linordered-euclidean-semiring-bit-operations*
begin

lemma *numeral-xor-num*:
‹*numeral m XOR numeral n* = (*case xor-num m n of None* ⇒ *0* | *Some n'* ⇒ *numeral n'*)›
by (*induction m n rule*: *xor-num.induct*) (*simp-all split*: *option.split*)

lemma *xor-num-eq-None-iff*:
‹*xor-num m n* = *None* ⟷ *numeral m XOR numeral n* = *0*›
by (*simp add*: *numeral-xor-num split*: *option.split*)

lemma *xor-num-eq-Some-iff*:
‹*xor-num m n* = *Some q* ⟷ *numeral m XOR numeral n* = *numeral q*›
by (*simp add*: *numeral-xor-num split*: *option.split*)

end

context *semiring-bit-operations*
begin

lemma *push-bit-eq-pow*:
‹*push-bit* (*numeral n*) *1* = *numeral* (*Num.pow* (*Num.Bit0 Num.One*) *n*)›
by *simp*

lemma *set-bit-of-0* [*simp*]:
‹*set-bit n 0* = *2* ^ *n*›
by (*simp add*: *set-bit-eq-or*)

lemma *unset-bit-of-0* [*simp*]:
‹*unset-bit n 0* = *0*›
by (*simp add*: *unset-bit-eq-or-xor*)

lemma *flip-bit-of-0* [*simp*]:
‹*flip-bit n 0* = *2* ^ *n*›

by (*simp add*: *flip-bit-eq-xor*)

lemma *set-bit-0-numeral-eq* [*simp*]:
‹*set-bit 0* (*numeral Num.One*) = *1*›
‹*set-bit 0* (*numeral* (*Num.Bit0 m*)) = *numeral* (*Num.Bit1 m*)›
‹*set-bit 0* (*numeral* (*Num.Bit1 m*)) = *numeral* (*Num.Bit1 m*)›
by (*simp-all add*: *set-bit-0*)

lemma *set-bit-numeral-eq-or* [*simp*]:
‹*set-bit* (*numeral n*) (*numeral m*) = *numeral m OR push-bit* (*numeral n*) *1*›
by (*fact set-bit-eq-or*)

lemma *unset-bit-0-numeral-eq-and-not′* [*simp*]:
‹*unset-bit 0* (*numeral Num.One*) = *0*›
‹*unset-bit 0* (*numeral* (*Num.Bit0 m*)) = *numeral* (*Num.Bit0 m*)›
‹*unset-bit 0* (*numeral* (*Num.Bit1 m*)) = *numeral* (*Num.Bit0 m*)›
by (*simp-all add*: *unset-bit-0*)

lemma *unset-bit-numeral-eq-or* [*simp*]:
‹*unset-bit* (*numeral n*) (*numeral m*) =
(*case and-not-num m* (*Num.pow* (*Num.Bit0 Num.One*) *n*)
of None ⇒ *0*
| *Some q* ⇒ *numeral q*)› (**is** ‹*?lhs* = -›)
proof −
have ‹*?lhs* = *of-nat* (*unset-bit* (*numeral n*) (*numeral m*))›
by (*simp add*: *of-nat-unset-bit-eq*)
also have ‹*unset-bit* (*numeral n*) (*numeral m*) = *nat* (*unset-bit* (*numeral n*) (*numeral m*))›
by (*simp flip*: *int-int-eq add*: *Bit-Operations.of-nat-unset-bit-eq*)
finally have ∗: ‹*?lhs* = *of-nat* (*nat* (*unset-bit* (*numeral n*) (*numeral m*)))› **.**
show *?thesis*
by (*simp only*: ∗ *unset-bit-eq-and-not Bit-Operations.push-bit-eq-pow int-numeral-and-not-num*)
(*auto split*: *option.splits*)
qed

lemma *flip-bit-0-numeral-eq-or* [*simp*]:
‹*flip-bit 0* (*numeral Num.One*) = *0*›
‹*flip-bit 0* (*numeral* (*Num.Bit0 m*)) = *numeral* (*Num.Bit1 m*)›
‹*flip-bit 0* (*numeral* (*Num.Bit1 m*)) = *numeral* (*Num.Bit0 m*)›
by (*simp-all add*: *flip-bit-0*)

lemma *flip-bit-numeral-eq-xor* [*simp*]:
‹*flip-bit* (*numeral n*) (*numeral m*) = *numeral m XOR push-bit* (*numeral n*) *1*›
by (*fact flip-bit-eq-xor*)

end

context *ring-bit-operations*
begin

lemma *set-bit-minus-numeral-eq-or* [*simp*]:
‹*set-bit* (*numeral n*) (− *numeral m*) = − *numeral m OR push-bit* (*numeral n*) *1*›
by (*fact set-bit-eq-or*)

lemma *unset-bit-minus-numeral-eq-and-not* [*simp*]:
‹*unset-bit* (*numeral n*) (− *numeral m*) = − *numeral m AND NOT* (*push-bit* (*numeral n*) *1*)›
by (*fact unset-bit-eq-and-not*)

lemma *flip-bit-minus-numeral-eq-xor* [*simp*]:
‹*flip-bit* (*numeral n*) (− *numeral m*) = − *numeral m XOR push-bit* (*numeral n*) *1*›
by (*fact flip-bit-eq-xor*)

end

lemma *xor-int-code* [*code*]:
fixes *i j* :: *int* **shows**
‹*0 XOR j* = *j*›
‹*i XOR 0* = *i*›
‹*Int.Pos n XOR Int.Pos m* = (*case xor-num n m of None* ⇒ *0* | *Some n′* ⇒ *Int.Pos n′*)›
‹*Int.Neg n XOR Int.Neg m* = *Num.sub n num.One XOR Num.sub m num.One*›
‹*Int.Neg n XOR Int.Pos m* = *NOT* (*Num.sub n num.One XOR Int.Pos m*)›
‹*Int.Pos n XOR Int.Neg m* = *NOT* (*Int.Pos n XOR Num.sub m num.One*)›
by (*simp-all add*: *xor-num-eq-None-iff* [**where** *?′a* = *int*] *xor-num-eq-Some-iff* [**where** *?′a* = *int*] *split*: *option.split*)

lemma *push-bit-int-code* [*code*]:
‹*push-bit 0 i* = *i*›
‹*push-bit* (*Suc n*) *i* = *push-bit n* (*Int.dup i*)›
by (*simp-all add*: *ac-simps*)

lemma *drop-bit-int-code* [*code*]:
fixes *i* :: *int* **shows**
‹*drop-bit 0 i* = *i*›
‹*drop-bit* (*Suc n*) *0* = (*0* :: *int*)›
‹*drop-bit* (*Suc n*) (*Int.Pos num.One*) = *0*›
‹*drop-bit* (*Suc n*) (*Int.Pos* (*num.Bit0 m*)) = *drop-bit n* (*Int.Pos m*)›
‹*drop-bit* (*Suc n*) (*Int.Pos* (*num.Bit1 m*)) = *drop-bit n* (*Int.Pos m*)›
‹*drop-bit* (*Suc n*) (*Int.Neg num.One*) = − *1*›
‹*drop-bit* (*Suc n*) (*Int.Neg* (*num.Bit0 m*)) = *drop-bit n* (*Int.Neg m*)›
‹*drop-bit* (*Suc n*) (*Int.Neg* (*num.Bit1 m*)) = *drop-bit n* (*Int.Neg* (*Num.inc m*))›
by (*simp-all add*: *drop-bit-Suc add-One*)

68.8 More properties

lemma *take-bit-eq-mask-iff*:

‹*take-bit n k = mask n* ⟷ *take-bit n* (*k* + *1*) = *0*› (**is** ‹*?P* ⟷ *?Q*›)
for *k* :: *int*
proof
assume *?P*
then have ‹*take-bit n* (*take-bit n k* + *take-bit n 1*) = *0*›
by (*simp add*: *mask-eq-exp-minus-1 take-bit-eq-0-iff*)
then show *?Q*
by (*simp only*: *take-bit-add*)
next
assume *?Q*
then have ‹*take-bit n* (*k* + *1*) − *1* = − *1*›
by *simp*
then have ‹*take-bit n* (*take-bit n* (*k* + *1*) − *1*) = *take-bit n* (− *1*)›
by *simp*
moreover have ‹*take-bit n* (*take-bit n* (*k* + *1*) − *1*) = *take-bit n k*›
by (*simp add*: *take-bit-eq-mod mod-simps*)
ultimately show *?P*
by *simp*
qed

lemma *take-bit-eq-mask-iff-exp-dvd*:
‹*take-bit n k = mask n* ⟷ *2* ^ *n dvd k* + *1*›
for *k* :: *int*
by (*simp add*: *take-bit-eq-mask-iff flip*: *take-bit-eq-0-iff*)

68.9 Bit concatenation

definition *concat-bit* :: ‹*nat* ⇒ *int* ⇒ *int* ⇒ *int*›
where ‹*concat-bit n k l = take-bit n k OR push-bit n l*›

lemma *bit-concat-bit-iff* [*bit-simps*]:
‹*bit* (*concat-bit m k l*) *n* ⟷ *n* < *m* ∧ *bit k n* ∨ *m* ≤ *n* ∧ *bit l* (*n* − *m*)›
by (*simp add*: *concat-bit-def bit-or-iff bit-and-iff bit-take-bit-iff bit-push-bit-iff ac-simps*)

lemma *concat-bit-eq*:
‹*concat-bit n k l = take-bit n k* + *push-bit n l*›
proof −
have ‹*take-bit n k AND push-bit n l* = *0*›
by (*simp add*: *bit-eq-iff bit-simps*)
then show *?thesis*
by (*simp add*: *bit-eq-iff bit-simps disjunctive-add-eq-or*)
qed

lemma *concat-bit-0* [*simp*]:
‹*concat-bit 0 k l = l*›
by (*simp add*: *concat-bit-def*)

lemma *concat-bit-Suc*:

‹*concat-bit* (*Suc n*) *k l* = *k mod 2* + *2* ∗ *concat-bit n* (*k div 2*) *l*›
by (*simp add*: *concat-bit-eq take-bit-Suc push-bit-double*)

lemma *concat-bit-of-zero-1* [*simp*]:
‹*concat-bit n 0 l* = *push-bit n l*›
by (*simp add*: *concat-bit-def*)

lemma *concat-bit-of-zero-2* [*simp*]:
‹*concat-bit n k 0* = *take-bit n k*›
by (*simp add*: *concat-bit-def take-bit-eq-mask*)

lemma *concat-bit-nonnegative-iff* [*simp*]:
‹*concat-bit n k l* ≥ *0* ⟷ *l* ≥ *0*›
by (*simp add*: *concat-bit-def*)

lemma *concat-bit-negative-iff* [*simp*]:
‹*concat-bit n k l* < *0* ⟷ *l* < *0*›
by (*simp add*: *concat-bit-def*)

lemma *concat-bit-assoc*:
‹*concat-bit n k* (*concat-bit m l r*) = *concat-bit* (*m* + *n*) (*concat-bit n k l*) *r*›
by (*rule bit-eqI*) (*auto simp*: *bit-concat-bit-iff ac-simps*)

lemma *concat-bit-assoc-sym*:
‹*concat-bit m* (*concat-bit n k l*) *r* = *concat-bit* (*min m n*) *k* (*concat-bit* (*m* − *n*) *l r*)›
by (*rule bit-eqI*) (*auto simp*: *bit-concat-bit-iff ac-simps min-def*)

lemma *concat-bit-eq-iff*:
‹*concat-bit n k l* = *concat-bit n r s*
⟷ *take-bit n k* = *take-bit n r* ∧ *l* = *s*› (**is** ‹*?P* ⟷ *?Q*›)
proof
assume *?Q*
then show *?P*
by (*simp add*: *concat-bit-def*)
next
assume *?P*
then have ∗: ‹*bit* (*concat-bit n k l*) *m* = *bit* (*concat-bit n r s*) *m*› **for** *m*
by (*simp add*: *bit-eq-iff*)
have ‹*take-bit n k* = *take-bit n r*›
proof (*rule bit-eqI*)
fix *m*
from ∗ [*of m*]
show ‹*bit* (*take-bit n k*) *m* ⟷ *bit* (*take-bit n r*) *m*›
by (*auto simp*: *bit-take-bit-iff bit-concat-bit-iff*)
qed
moreover have ‹*push-bit n l* = *push-bit n s*›
proof (*rule bit-eqI*)
fix *m*

from ∗ [*of m*]
show ‹*bit* (*push-bit n l*) *m* ⟷ *bit* (*push-bit n s*) *m*›
by (*auto simp*: *bit-push-bit-iff bit-concat-bit-iff*)
qed
then have ‹*l* = *s*›
by (*simp add*: *push-bit-eq-mult*)
ultimately show *?Q*
by (*simp add*: *concat-bit-def*)
qed

lemma *take-bit-concat-bit-eq*:
‹*take-bit m* (*concat-bit n k l*) = *concat-bit* (*min m n*) *k* (*take-bit* (*m* − *n*) *l*)›
by (*rule bit-eqI*)
(*auto simp*: *bit-take-bit-iff bit-concat-bit-iff min-def*)

lemma *concat-bit-take-bit-eq*:
‹*concat-bit n* (*take-bit n b*) = *concat-bit n b*›
by (*simp add*: *concat-bit-def* [*abs-def*])

68.10 Taking bits with sign propagation

context *ring-bit-operations*
begin

definition *signed-take-bit* :: ‹*nat* ⇒ ′*a* ⇒ ′*a*›
where ‹*signed-take-bit n a* = *take-bit n a OR* (*of-bool* (*bit a n*) ∗ *NOT* (*mask n*))›

lemma *signed-take-bit-eq-if-positive*:
‹*signed-take-bit n a* = *take-bit n a*› **if** ‹¬ *bit a n*›
using *that* **by** (*simp add*: *signed-take-bit-def*)

lemma *signed-take-bit-eq-if-negative*:
‹*signed-take-bit n a* = *take-bit n a OR NOT* (*mask n*)› **if** ‹*bit a n*›
using *that* **by** (*simp add*: *signed-take-bit-def*)

lemma *even-signed-take-bit-iff*:
‹*even* (*signed-take-bit m a*) ⟷ *even a*›
by (*auto simp*: *bit-0 signed-take-bit-def even-or-iff even-mask-iff bit-double-iff*)

lemma *bit-signed-take-bit-iff* [*bit-simps*]:
‹*bit* (*signed-take-bit m a*) *n* ⟷ *possible-bit TYPE*(′*a*) *n* ∧ *bit a* (*min m n*)›
by (*simp add*: *signed-take-bit-def bit-take-bit-iff bit-or-iff bit-not-iff bit-mask-iff min-def not-le*)
(*blast dest*: *bit-imp-possible-bit*)

lemma *signed-take-bit-0* [*simp*]:
‹*signed-take-bit 0 a* = − (*a mod 2*)›
by (*simp add*: *bit-0 signed-take-bit-def odd-iff-mod-2-eq-one*)

lemma *signed-take-bit-Suc*:
‹*signed-take-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *signed-take-bit n* (*a div 2*)›
by (*simp add*: *bit-eq-iff bit-sum-mult-2-cases bit-simps bit-0 possible-bit-less-imp flip*: *bit-Suc min-Suc-Suc*)

lemma *signed-take-bit-of-0* [*simp*]:
‹*signed-take-bit n 0* = *0*›
by (*simp add*: *signed-take-bit-def*)

lemma *signed-take-bit-of-minus-1* [*simp*]:
‹*signed-take-bit n* (− *1*) = − *1*›
by (*simp add*: *signed-take-bit-def mask-eq-exp-minus-1 possible-bit-def*)

lemma *signed-take-bit-Suc-1* [*simp*]:
‹*signed-take-bit* (*Suc n*) *1* = *1*›
by (*simp add*: *signed-take-bit-Suc*)

lemma *signed-take-bit-numeral-of-1* [*simp*]:
‹*signed-take-bit* (*numeral k*) *1* = *1*›
by (*simp add*: *bit-1-iff signed-take-bit-eq-if-positive*)

lemma *signed-take-bit-rec*:
‹*signed-take-bit n a* = (*if n* = *0 then* − (*a mod 2*) *else a mod 2* + *2* ∗ *signed-take-bit* (*n* − *1*) (*a div 2*))›
by (*cases n*) (*simp-all add*: *signed-take-bit-Suc*)

lemma *signed-take-bit-eq-iff-take-bit-eq*:
‹*signed-take-bit n a* = *signed-take-bit n b* ⟷ *take-bit* (*Suc n*) *a* = *take-bit* (*Suc n*) *b*›
proof −
have ‹*bit* (*signed-take-bit n a*) = *bit* (*signed-take-bit n b*) ⟷ *bit* (*take-bit* (*Suc n*) *a*) = *bit* (*take-bit* (*Suc n*) *b*)›
by (*simp add*: *fun-eq-iff bit-signed-take-bit-iff bit-take-bit-iff not-le less-Suc-eq-le min-def*)
(*use bit-imp-possible-bit* **in** *fastforce*)
then show *?thesis*
by (*auto simp*: *fun-eq-iff intro*: *bit-eqI*)
qed

lemma *signed-take-bit-signed-take-bit* [*simp*]:
‹*signed-take-bit m* (*signed-take-bit n a*) = *signed-take-bit* (*min m n*) *a*›
by (*auto simp*: *bit-eq-iff bit-simps ac-simps*)

lemma *signed-take-bit-take-bit*:
‹*signed-take-bit m* (*take-bit n a*) = (*if n* ≤ *m then take-bit n else signed-take-bit m*) *a*›
by (*rule bit-eqI*) (*auto simp*: *bit-signed-take-bit-iff min-def bit-take-bit-iff*)

lemma *take-bit-signed-take-bit*:
‹*take-bit m* (*signed-take-bit n a*) = *take-bit m a*› **if** ‹$m \leq$ *Suc n*›
using *that* **by** (*rule le-SucE*; *intro bit-eqI*)
(*auto simp*: *bit-take-bit-iff bit-signed-take-bit-iff min-def less-Suc-eq*)

lemma *signed-take-bit-eq-take-bit-add*:
‹*signed-take-bit n k* = *take-bit* (*Suc n*) *k* + *NOT* (*mask* (*Suc n*)) ∗ *of-bool* (*bit k n*)›
proof (*cases* ‹*bit k n*›)
case *False*
show *?thesis*
by (*rule bit-eqI*) (*simp add*: *False bit-simps min-def less-Suc-eq*)
next
case *True*
have ‹*signed-take-bit n k* = *take-bit* (*Suc n*) *k OR NOT* (*mask* (*Suc n*))›
by (*rule bit-eqI*) (*auto simp*: *bit-signed-take-bit-iff min-def bit-take-bit-iff bit-or-iff bit-not-iff bit-mask-iff less-Suc-eq True*)
also have ‹... = *take-bit* (*Suc n*) *k* + *NOT* (*mask* (*Suc n*))›
by (*simp add*: *disjunctive-add-eq-or bit-eq-iff bit-simps*)
finally show *?thesis*
by (*simp add*: *True*)
qed

lemma *signed-take-bit-eq-take-bit-minus*:
‹*signed-take-bit n k* = *take-bit* (*Suc n*) *k* − *2* ^ *Suc n* ∗ *of-bool* (*bit k n*)›
by (*simp add*: *signed-take-bit-eq-take-bit-add flip*: *minus-exp-eq-not-mask*)

end

Modulus centered around 0

lemma *signed-take-bit-eq-concat-bit*:
‹*signed-take-bit n k* = *concat-bit n k* (− *of-bool* (*bit k n*))›
by (*simp add*: *concat-bit-def signed-take-bit-def*)

lemma *signed-take-bit-add*:
‹*signed-take-bit n* (*signed-take-bit n k* + *signed-take-bit n l*) = *signed-take-bit n* (*k* + *l*)›
for *k l* :: *int*
proof −
have ‹*take-bit* (*Suc n*)
(*take-bit* (*Suc n*) (*signed-take-bit n k*) +
take-bit (*Suc n*) (*signed-take-bit n l*)) =
take-bit (*Suc n*) (*k* + *l*)›
by (*simp add*: *take-bit-signed-take-bit take-bit-add*)
then show *?thesis*
by (*simp only*: *signed-take-bit-eq-iff-take-bit-eq take-bit-add*)
qed

lemma *signed-take-bit-diff*:

‹*signed-take-bit n* (*signed-take-bit n k* − *signed-take-bit n l*) = *signed-take-bit n* (*k* − *l*)›
for *k l* :: *int*
proof −
have ‹*take-bit* (*Suc n*)
(*take-bit* (*Suc n*) (*signed-take-bit n k*) −
take-bit (*Suc n*) (*signed-take-bit n l*)) =
take-bit (*Suc n*) (*k* − *l*)›
by (*simp add*: *take-bit-signed-take-bit take-bit-diff*)
then show *?thesis*
by (*simp only*: *signed-take-bit-eq-iff-take-bit-eq take-bit-diff*)
qed

lemma *signed-take-bit-minus*:
‹*signed-take-bit n* (− *signed-take-bit n k*) = *signed-take-bit n* (− *k*)›
for *k* :: *int*
proof −
have ‹*take-bit* (*Suc n*)
(− *take-bit* (*Suc n*) (*signed-take-bit n k*)) =
take-bit (*Suc n*) (− *k*)›
by (*simp add*: *take-bit-signed-take-bit take-bit-minus*)
then show *?thesis*
by (*simp only*: *signed-take-bit-eq-iff-take-bit-eq take-bit-minus*)
qed

lemma *signed-take-bit-mult*:
‹*signed-take-bit n* (*signed-take-bit n k* ∗ *signed-take-bit n l*) = *signed-take-bit n* (*k* ∗ *l*)›
for *k l* :: *int*
proof −
have ‹*take-bit* (*Suc n*)
(*take-bit* (*Suc n*) (*signed-take-bit n k*) ∗
take-bit (*Suc n*) (*signed-take-bit n l*)) =
take-bit (*Suc n*) (*k* ∗ *l*)›
by (*simp add*: *take-bit-signed-take-bit take-bit-mult*)
then show *?thesis*
by (*simp only*: *signed-take-bit-eq-iff-take-bit-eq take-bit-mult*)
qed

lemma *signed-take-bit-eq-take-bit-shift*:
‹*signed-take-bit n k* = *take-bit* (*Suc n*) (*k* + *2* ^ *n*) − *2* ^ *n*› (**is** ‹*?lhs* = *?rhs*›)
for *k* :: *int*
proof −
have ‹*take-bit n k AND 2* ^ *n* = *0*›
by (*rule bit-eqI*) (*simp add*: *bit-simps*)
then have ∗: ‹*take-bit n k OR 2* ^ *n* = *take-bit n k* + *2* ^ *n*›
by (*simp add*: *disjunctive-add-eq-or*)
have ‹*take-bit n k* − *2* ^ *n* = *take-bit n k* + *NOT* (*mask n*)›
by (*simp add*: *minus-exp-eq-not-mask*)

also have ‹... = *take-bit n k OR NOT* (*mask n*)›
by (*rule disjunctive-add-eq-or*) (*simp add*: *bit-eq-iff bit-simps*)
finally have ∗∗: ‹*take-bit n k* − *2* ^ *n* = *take-bit n k OR NOT* (*mask n*)› **.**
have ‹*take-bit* (*Suc n*) (*k* + *2* ^ *n*) = *take-bit* (*Suc n*) (*take-bit* (*Suc n*) *k* + *take-bit* (*Suc n*) (*2* ^ *n*))›
by (*simp only*: *take-bit-add*)
also have ‹*take-bit* (*Suc n*) *k* = *2* ^ *n* ∗ *of-bool* (*bit k n*) + *take-bit n k*›
by (*simp add*: *take-bit-Suc-from-most*)
finally have ‹*take-bit* (*Suc n*) (*k* + *2* ^ *n*) = *take-bit* (*Suc n*) (*2* ^ (*n* + *of-bool* (*bit k n*)) + *take-bit n k*)›
by (*simp add*: *ac-simps*)
also have ‹*2* ^ (*n* + *of-bool* (*bit k n*)) + *take-bit n k* = *2* ^ (*n* + *of-bool* (*bit k n*)) *OR take-bit n k*›
by (*rule disjunctive-add-eq-or*, *rule bit-eqI*) (*simp add*: *bit-simps*)
finally show *?thesis*
using ∗ ∗∗ **by** (*simp add*: *signed-take-bit-def concat-bit-Suc min-def ac-simps*)
qed

lemma *signed-take-bit-nonnegative-iff* [*simp*]:
‹*0* ≤ *signed-take-bit n k* ⟷ ¬ *bit k n*›
for *k* :: *int*
by (*simp add*: *signed-take-bit-def not-less concat-bit-def*)

lemma *signed-take-bit-negative-iff* [*simp*]:
‹*signed-take-bit n k* < *0* ⟷ *bit k n*›
for *k* :: *int*
by (*simp add*: *signed-take-bit-def not-less concat-bit-def*)

lemma *signed-take-bit-int-greater-eq-minus-exp* [*simp*]:
‹− (*2* ^ *n*) ≤ *signed-take-bit n k*›
for *k* :: *int*
by (*simp add*: *signed-take-bit-eq-take-bit-shift*)

lemma *signed-take-bit-int-less-exp* [*simp*]:
‹*signed-take-bit n k* < *2* ^ *n*›
for *k* :: *int*
using *take-bit-int-less-exp* [*of* ‹*Suc n*›]
by (*simp add*: *signed-take-bit-eq-take-bit-shift*)

lemma *signed-take-bit-int-eq-self-iff*:
‹*signed-take-bit n k* = *k* ⟷ − (*2* ^ *n*) ≤ *k* ∧ *k* < *2* ^ *n*›
for *k* :: *int*
by (*auto simp*: *signed-take-bit-eq-take-bit-shift take-bit-int-eq-self-iff algebra-simps*)

lemma *signed-take-bit-int-eq-self*:
‹*signed-take-bit n k* = *k*› **if** ‹− (*2* ^ *n*) ≤ *k*› ‹*k* < *2* ^ *n*›
for *k* :: *int*
using *that* **by** (*simp add*: *signed-take-bit-int-eq-self-iff*)

lemma *signed-take-bit-int-less-eq-self-iff*:
‹*signed-take-bit n k* ≤ *k* ⟷ − (*2* ^ *n*) ≤ *k*›
for *k* :: *int*
by (*simp add*: *signed-take-bit-eq-take-bit-shift take-bit-int-less-eq-self-iff algebra-simps*)
linarith

lemma *signed-take-bit-int-less-self-iff*:
‹*signed-take-bit n k* < *k* ⟷ *2* ^ *n* ≤ *k*›
for *k* :: *int*
by (*simp add*: *signed-take-bit-eq-take-bit-shift take-bit-int-less-self-iff algebra-simps*)

lemma *signed-take-bit-int-greater-self-iff*:
‹*k* < *signed-take-bit n k* ⟷ *k* < − (*2* ^ *n*)›
for *k* :: *int*
by (*simp add*: *signed-take-bit-eq-take-bit-shift take-bit-int-greater-self-iff algebra-simps*)
linarith

lemma *signed-take-bit-int-greater-eq-self-iff*:
‹*k* ≤ *signed-take-bit n k* ⟷ *k* < *2* ^ *n*›
for *k* :: *int*
by (*simp add*: *signed-take-bit-eq-take-bit-shift take-bit-int-greater-eq-self-iff algebra-simps*)

lemma *signed-take-bit-int-greater-eq*:
‹*k* + *2* ^ *Suc n* ≤ *signed-take-bit n k*› **if** ‹*k* < − (*2* ^ *n*)›
for *k* :: *int*
using *that take-bit-int-greater-eq* [*of* ‹*k* + *2* ^ *n*› ‹*Suc n*›]
by (*simp add*: *signed-take-bit-eq-take-bit-shift*)

lemma *signed-take-bit-int-less-eq*:
‹*signed-take-bit n k* ≤ *k* − *2* ^ *Suc n*› **if** ‹*k* ≥ *2* ^ *n*›
for *k* :: *int*
using *that take-bit-int-less-eq* [*of* ‹*Suc n*› ‹*k* + *2* ^ *n*›]
by (*simp add*: *signed-take-bit-eq-take-bit-shift*)

lemma *signed-take-bit-Suc-bit0* [*simp*]:
‹*signed-take-bit* (*Suc n*) (*numeral* (*Num.Bit0 k*)) = *signed-take-bit n* (*numeral k*) ∗ (*2* :: *int*)›
by (*simp add*: *signed-take-bit-Suc*)

lemma *signed-take-bit-Suc-bit1* [*simp*]:
‹*signed-take-bit* (*Suc n*) (*numeral* (*Num.Bit1 k*)) = *signed-take-bit n* (*numeral k*) ∗ *2* + (*1* :: *int*)›
by (*simp add*: *signed-take-bit-Suc*)

lemma *signed-take-bit-Suc-minus-bit0* [*simp*]:
‹*signed-take-bit* (*Suc n*) (− *numeral* (*Num.Bit0 k*)) = *signed-take-bit n* (− *numeral k*) ∗ (*2* :: *int*)›
by (*simp add*: *signed-take-bit-Suc*)

lemma *signed-take-bit-Suc-minus-bit1* [*simp*]:
‹*signed-take-bit* (*Suc n*) (− *numeral* (*Num.Bit1 k*)) = *signed-take-bit n* (− *numeral k* − *1*) ∗ *2* + (*1* :: *int*)›
by (*simp add*: *signed-take-bit-Suc*)

lemma *signed-take-bit-numeral-bit0* [*simp*]:
‹*signed-take-bit* (*numeral l*) (*numeral* (*Num.Bit0 k*)) = *signed-take-bit* (*pred-numeral l*) (*numeral k*) ∗ (*2* :: *int*)›
by (*simp add*: *signed-take-bit-rec*)

lemma *signed-take-bit-numeral-bit1* [*simp*]:
‹*signed-take-bit* (*numeral l*) (*numeral* (*Num.Bit1 k*)) = *signed-take-bit* (*pred-numeral l*) (*numeral k*) ∗ *2* + (*1* :: *int*)›
by (*simp add*: *signed-take-bit-rec*)

lemma *signed-take-bit-numeral-minus-bit0* [*simp*]:
‹*signed-take-bit* (*numeral l*) (− *numeral* (*Num.Bit0 k*)) = *signed-take-bit* (*pred-numeral l*) (− *numeral k*) ∗ (*2* :: *int*)›
by (*simp add*: *signed-take-bit-rec*)

lemma *signed-take-bit-numeral-minus-bit1* [*simp*]:
‹*signed-take-bit* (*numeral l*) (− *numeral* (*Num.Bit1 k*)) = *signed-take-bit* (*pred-numeral l*) (− *numeral k* − *1*) ∗ *2* + (*1* :: *int*)›
by (*simp add*: *signed-take-bit-rec*)

lemma *signed-take-bit-code* [*code*]:
‹*signed-take-bit n a* =
(*let l* = *take-bit* (*Suc n*) *a*
in if bit l n then l + *push-bit* (*Suc n*) (− *1*) *else l*)›
by (*simp add*: *signed-take-bit-eq-take-bit-add bit-simps*)

68.11 Key ideas of bit operations

When formalizing bit operations, it is tempting to represent bit values as explicit lists over a binary type. This however is a bad idea, mainly due to the inherent ambiguities in representation concerning repeating leading bits.

Hence this approach avoids such explicit lists altogether following an algebraic path:

- Bit values are represented by numeric types: idealized unbounded bit values can be represented by type *int*, bounded bit values by quotient types over *int*.
- (A special case are idealized unbounded bit values ending in *0* which can be represented by type *nat* but only support a restricted set of operations).

- From this idea follows that
  - multiplication by 2 is a bit shift to the left and
  - division by 2 is a bit shift to the right.
- Concerning bounded bit values, iterated shifts to the left may result in eliminating all bits by shifting them all beyond the boundary. The property $2^n \neq 0$ represents that n is *not* beyond that boundary.
- The projection on a single bit is then $bit\ a\ n = odd\ (a\ div\ 2^n)$.
- This leads to the most fundamental properties of bit values:
  - Equality rule: $(\bigwedge n.\ possible\text{-}bit\ TYPE(int)\ n \Longrightarrow bit\ a\ n = bit\ b\ n) \Longrightarrow a = b$
  - Induction rule: $[\![\bigwedge a.\ a\ div\ 2 = a \Longrightarrow P\ a;\ \bigwedge a\ b.\ [\![P\ a;\ (of\text{-}bool\ b + 2 * a)\ div\ 2 = a]\!] \Longrightarrow P\ (of\text{-}bool\ b + 2 * a)]\!] \Longrightarrow P\ a$
- Typical operations are characterized as follows:
  - Singleton nth bit: 2^n
  - Bit mask upto bit n: $mask\ n = 2^n - 1$
  - Left shift: $push\text{-}bit\ n\ a = a * 2^n$
  - Right shift: $drop\text{-}bit\ n\ a = a\ div\ 2^n$
  - Truncation: $take\text{-}bit\ n\ a = a\ mod\ 2^n$
  - Negation: $bit\ (NOT\ a)\ n = (possible\text{-}bit\ TYPE(int)\ n \wedge \neg\ bit\ a\ n)$
  - And: $bit\ (a\ AND\ b)\ n = (bit\ a\ n \wedge bit\ b\ n)$
  - Or: $bit\ (a\ OR\ b)\ n = (bit\ a\ n \vee bit\ b\ n)$
  - Xor: $bit\ (a\ XOR\ b)\ n = (bit\ a\ n \neq bit\ b\ n)$
  - Set a single bit: $set\text{-}bit\ n\ a = a\ OR\ push\text{-}bit\ n\ 1$
  - Unset a single bit: $unset\text{-}bit\ n\ a = a\ AND\ NOT\ (push\text{-}bit\ n\ 1)$
  - Flip a single bit: $flip\text{-}bit\ n\ a = a\ XOR\ push\text{-}bit\ n\ 1$
  - Signed truncation, or modulus centered around 0: $signed\text{-}take\text{-}bit\ n\ a = take\text{-}bit\ n\ a\ OR\ of\text{-}bool\ (bit\ a\ n) * NOT\ (mask\ n)$
  - Bit concatenation: $concat\text{-}bit\ n\ k\ l = take\text{-}bit\ n\ k\ OR\ push\text{-}bit\ n\ l$
  - (Bounded) conversion from and to a list of bits: $horner\text{-}sum\ of\text{-}bool\ 2\ (map\ (bit\ a)\ [0..<n]) = take\text{-}bit\ n\ a$

68.12 Lemma duplicates and other

context *semiring-bits*
begin

lemma *exp-div-exp-eq*:
‹2 ^ m *div* 2 ^ n = *of-bool* (2 ^ $m \neq 0 \wedge m \geq n$) ∗ 2 ^ ($m - n$)›
using *bit-exp-iff div-exp-eq*
by (*intro bit-eqI*) (*auto simp*: *bit-iff-odd possible-bit-def*)

lemma *bits-1-div-2*:
‹1 *div* $2 = 0$›
by (*fact half-1*)

lemma *bits-1-div-exp*:
‹1 *div* 2 ^ n = *of-bool* ($n = 0$)›
using *div-exp-eq* [*of 1 1*] **by** (*cases n*) *simp-all*

lemma *exp-add-not-zero-imp*:
‹2 ^ $m \neq 0$› **and** ‹2 ^ $n \neq 0$› **if** ‹2 ^ ($m + n$) $\neq 0$›
proof −
have ‹¬ (2 ^ $m = 0 \vee 2$ ^ $n = 0$)›
proof (*rule notI*)
assume ‹2 ^ $m = 0 \vee 2$ ^ $n = 0$›
then have ‹2 ^ ($m + n$) $= 0$›
by (*rule disjE*) (*simp-all add*: *power-add*)
with *that* **show** *False* **..**
qed
then show ‹2 ^ $m \neq 0$› **and** ‹2 ^ $n \neq 0$›
by *simp-all*
qed

lemma
exp-add-not-zero-imp-left: ‹2 ^ $m \neq 0$›
and *exp-add-not-zero-imp-right*: ‹2 ^ $n \neq 0$›
if ‹2 ^ ($m + n$) $\neq 0$›
proof −
have ‹¬ (2 ^ $m = 0 \vee 2$ ^ $n = 0$)›
proof (*rule notI*)
assume ‹2 ^ $m = 0 \vee 2$ ^ $n = 0$›
then have ‹2 ^ ($m + n$) $= 0$›
by (*rule disjE*) (*simp-all add*: *power-add*)
with *that* **show** *False* **..**
qed
then show ‹2 ^ $m \neq 0$› **and** ‹2 ^ $n \neq 0$›
by *simp-all*
qed

lemma *exp-not-zero-imp-exp-diff-not-zero*:
‹2 ^ ($n - m$) $\neq 0$› **if** ‹2 ^ $n \neq 0$›

proof (*cases* ‹$m \le n$›)
  case *True*
  moreover define q **where** ‹$q = n - m$›
  ultimately have ‹$n = m + q$›
    by *simp*
  with *that* **show** *?thesis*
    by (*simp add*: *exp-add-not-zero-imp-right*)
next
  case *False*
  with *that* **show** *?thesis*
    by *simp*
qed

lemma *exp-eq-0-imp-not-bit*:
  ‹¬ *bit a n*› **if** ‹2 ^ $n = 0$›
  using *that* **by** (*simp add*: *bit-iff-odd*)

lemma *bit-disjunctive-add-iff*:
  ‹*bit* (*a* + *b*) *n* ⟷ *bit a n* ∨ *bit b n*›
  if ‹⋀*n*. ¬ *bit a n* ∨ ¬ *bit b n*›
proof (*cases* ‹*possible-bit TYPE*('*a*) *n*›)
  case *False*
  then show *?thesis*
    by (*auto dest*: *impossible-bit*)
next
  case *True*
  with *that* **show** *?thesis* **proof** (*induction n arbitrary*: *a b*)
    case *0*
    from *0.prems*(*1*) [*of 0*] **show** *?case*
      by (*auto simp*: *bit-0*)
  next
    case (*Suc n*)
    from *Suc.prems*(*1*) [*of 0*] **have** *even*: ‹*even a* ∨ *even b*›
      by (*auto simp*: *bit-0*)
    have *bit*: ‹¬ *bit* (*a div 2*) *n* ∨ ¬ *bit* (*b div 2*) *n*› **for** *n*
      using *Suc.prems*(*1*) [*of* ‹*Suc n*›] **by** (*simp add*: *bit-Suc*)
   from *Suc.prems*(*2*) **have** ‹*possible-bit TYPE*('*a*) (*Suc n*)› ‹*possible-bit TYPE*('*a*) *n*›
      by (*simp-all add*: *possible-bit-less-imp*)
    have ‹*a* + *b* = (*a div 2* ∗ *2* + *a mod 2*) + (*b div 2* ∗ *2* + *b mod 2*)›
      using *div-mult-mod-eq* [*of a 2*] *div-mult-mod-eq* [*of b 2*] **by** *simp*
    also have ‹... = *of-bool* (*odd a* ∨ *odd b*) + *2* ∗ (*a div 2* + *b div 2*)›
      using *even* **by** (*auto simp*: *algebra-simps mod2-eq-if*)
    finally have ‹*bit* ((*a* + *b*) *div 2*) *n* ⟷ *bit* (*a div 2* + *b div 2*) *n*›
      using ‹*possible-bit TYPE*('*a*) (*Suc n*)› **by** *simp* (*simp-all flip*: *bit-Suc add*: *bit-double-iff possible-bit-def*)
    also have ‹... ⟷ *bit* (*a div 2*) *n* ∨ *bit* (*b div 2*) *n*›
      using *bit* ‹*possible-bit TYPE*('*a*) *n*› **by** (*rule Suc.IH*)
    finally show *?case*

by (*simp add*: *bit-Suc*)
qed
qed

end

context *semiring-bit-operations*
begin

lemma *even-mask-div-iff*:
‹*even* ((*2* ^ *m* − *1*) *div* *2* ^ *n*) ⟷ *2* ^ *n* = *0* ∨ *m* ≤ *n*›
using *bit-mask-iff* [*of m n*] **by** (*auto simp*: *mask-eq-exp-minus-1 bit-iff-odd possible-bit-def*)

lemma *mod-exp-eq*:
‹*a mod 2* ^ *m mod 2* ^ *n* = *a mod 2* ^ *min m n*›
by (*simp flip*: *take-bit-eq-mod add*: *ac-simps*)

lemma *mult-exp-mod-exp-eq*:
‹*m* ≤ *n* ⟹ (*a* ∗ *2* ^ *m*) *mod* (*2* ^ *n*) = (*a mod 2* ^ (*n* − *m*)) ∗ *2* ^ *m*›
by (*simp flip*: *push-bit-eq-mult take-bit-eq-mod add*: *push-bit-take-bit*)

lemma *div-exp-mod-exp-eq*:
‹*a div 2* ^ *n mod 2* ^ *m* = *a mod* (*2* ^ (*n* + *m*)) *div 2* ^ *n*›
by (*simp flip*: *drop-bit-eq-div take-bit-eq-mod add*: *drop-bit-take-bit*)

lemma *even-mult-exp-div-exp-iff*:
‹*even* (*a* ∗ *2* ^ *m div 2* ^ *n*) ⟷ *m* > *n* ∨ *2* ^ *n* = *0* ∨ (*m* ≤ *n* ∧ *even* (*a div 2* ^ (*n* − *m*)))›
by (*simp flip*: *push-bit-eq-mult drop-bit-eq-div add*: *even-drop-bit-iff-not-bit bit-simps possible-bit-def*) *auto*

lemma *mod-exp-div-exp-eq-0*:
‹*a mod 2* ^ *n div 2* ^ *n* = *0*›
by (*simp flip*: *take-bit-eq-mod drop-bit-eq-div add*: *drop-bit-take-bit*)

lemmas *bits-one-mod-two-eq-one* = *one-mod-two-eq-one*

lemmas *set-bit-def* = *set-bit-eq-or*

lemmas *unset-bit-def* = *unset-bit-eq-and-not*

lemmas *flip-bit-def* = *flip-bit-eq-xor*

lemma *disjunctive-add*:
‹*a* + *b* = *a OR b*› **if** ‹⋀*n*. ¬ *bit a n* ∨ ¬ *bit b n*›
by (*rule disjunctive-add-eq-or*) (*use that* **in** ‹*simp add*: *bit-eq-iff bit-simps*›)

lemma *even-mod-exp-div-exp-iff*:

‹*even* (*a mod 2* ^ *m div 2* ^ *n*) ⟷ *m* ≤ *n* ∨ *even* (*a div 2* ^ *n*)›
by (*auto simp*: *even-drop-bit-iff-not-bit bit-simps simp flip*: *drop-bit-eq-div take-bit-eq-mod*)

end

context *ring-bit-operations*
begin

lemma *disjunctive-diff*:
‹*a* − *b* = *a AND NOT b*› **if** ‹⋀*n*. *bit b n* ⟹ *bit a n*›
proof −
have ‹*NOT a* + *b* = *NOT a OR b*›
by (*rule disjunctive-add*) (*auto simp*: *bit-not-iff dest*: *that*)
then have ‹*NOT* (*NOT a* + *b*) = *NOT* (*NOT a OR b*)›
by *simp*
then show *?thesis*
by (*simp add*: *not-add-distrib*)
qed

end

lemma *and-nat-rec*:
‹*m AND n* = *of-bool* (*odd m* ∧ *odd n*) + *2* ∗ ((*m div 2*) *AND* (*n div 2*))› **for** *m n* :: *nat*
by (*fact and-rec*)

lemma *or-nat-rec*:
‹*m OR n* = *of-bool* (*odd m* ∨ *odd n*) + *2* ∗ ((*m div 2*) *OR* (*n div 2*))› **for** *m n* :: *nat*
by (*fact or-rec*)

lemma *xor-nat-rec*:
‹*m XOR n* = *of-bool* (*odd m* ≠ *odd n*) + *2* ∗ ((*m div 2*) *XOR* (*n div 2*))› **for** *m n* :: *nat*
by (*fact xor-rec*)

lemma *bit-push-bit-iff-nat*:
‹*bit* (*push-bit m q*) *n* ⟷ *m* ≤ *n* ∧ *bit q* (*n* − *m*)› **for** *q* :: *nat*
by (*fact bit-push-bit-iff′*)

lemma *mask-half-int*:
‹*mask n div 2* = (*mask* (*n* − *1*) :: *int*)›
by (*fact mask-half*)

lemma *not-int-rec*:
‹*NOT k* = *of-bool* (*even k*) + *2* ∗ *NOT* (*k div 2*)› **for** *k* :: *int*
by (*fact not-rec*)

lemma *even-not-iff-int*:

‹*even* (*NOT k*) ⟷ *odd k*› **for** *k* :: *int*
by (*fact even-not-iff*)

lemma *bit-not-int-iff'*:
‹*bit* (− *k* − *1*) *n* ⟷ ¬ *bit k n*› **for** *k* :: *int*
by (*simp flip*: *not-eq-complement add*: *bit-simps*)

lemmas *and-int-rec* = *and-int.rec*

lemma *even-and-iff-int*:
‹*even* (*k AND l*) ⟷ *even k* ∨ *even l*› **for** *k l* :: *int*
by (*fact even-and-iff*)

lemmas *bit-and-int-iff* = *and-int.bit-iff*

lemmas *or-int-rec* = *or-int.rec*

lemmas *bit-or-int-iff* = *or-int.bit-iff*

lemmas *xor-int-rec* = *xor-int.rec*

lemmas *bit-xor-int-iff* = *xor-int.bit-iff*

lemma *drop-bit-push-bit-int*:
‹*drop-bit m* (*push-bit n k*) = *drop-bit* (*m* − *n*) (*push-bit* (*n* − *m*) *k*)› **for** *k* :: *int*
by (*fact drop-bit-push-bit*)

lemma *bit-push-bit-iff-int*:
‹*bit* (*push-bit m k*) *n* ⟷ *m* ≤ *n* ∧ *bit k* (*n* − *m*)› **for** *k* :: *int*
by (*fact bit-push-bit-iff'*)

bundle *bit-operations-syntax*
begin
notation
not (‹*NOT*›)
and *and* (**infixr** ‹*AND*› *64*)
and *or* (**infixr** ‹*OR*› *59*)
and *xor* (**infixr** ‹*XOR*› *59*)
end

unbundle *no bit-operations-syntax*

end

69 Numeric types for code generation onto target language numerals only

theory *Code-Numeral*

imports *Lifting Bit-Operations*
begin

69.1 Type of target language integers

typedef *integer* = *UNIV* :: *int set*
morphisms *int-of-integer integer-of-int* **..**

setup-lifting *type-definition-integer*

lemma *integer-eq-iff*:
k = *l* ⟷ *int-of-integer k* = *int-of-integer l*
by *transfer rule*

lemma *integer-eqI*:
int-of-integer k = *int-of-integer l* ⟹ *k* = *l*
using *integer-eq-iff* [*of k l*] **by** *simp*

lemma *int-of-integer-integer-of-int* [*simp*]:
int-of-integer (*integer-of-int k*) = *k*
by *transfer rule*

lemma *integer-of-int-int-of-integer* [*simp*]:
integer-of-int (*int-of-integer k*) = *k*
by *transfer rule*

instantiation *integer* :: *ring-1*
begin

lift-definition *zero-integer* :: *integer*
is *0* :: *int*
.

declare *zero-integer.rep-eq* [*simp*]

lift-definition *one-integer* :: *integer*
is *1* :: *int*
.

declare *one-integer.rep-eq* [*simp*]

lift-definition *plus-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *plus* :: *int* ⇒ *int* ⇒ *int*
.

declare *plus-integer.rep-eq* [*simp*]

lift-definition *uminus-integer* :: *integer* ⇒ *integer*
is *uminus* :: *int* ⇒ *int*

.

declare *uminus-integer.rep-eq* [*simp*]

lift-definition *minus-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *minus* :: *int* ⇒ *int* ⇒ *int*
.

declare *minus-integer.rep-eq* [*simp*]

lift-definition *times-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *times* :: *int* ⇒ *int* ⇒ *int*
.

declare *times-integer.rep-eq* [*simp*]

instance proof
qed (*transfer*, *simp add*: *algebra-simps*)+

end

instance *integer* :: *Rings.dvd* **..**

context
includes *lifting-syntax*
notes *transfer-rule-numeral* [*transfer-rule*]
begin

lemma [*transfer-rule*]:
(*pcr-integer* ===> *pcr-integer* ===> (⟷)) (*dvd*) (*dvd*)
by (*unfold dvd-def*) *transfer-prover*

lemma [*transfer-rule*]:
((⟷) ===> *pcr-integer*) *of-bool of-bool*
by (*unfold of-bool-def*) *transfer-prover*

lemma [*transfer-rule*]:
((=) ===> *pcr-integer*) *int of-nat*
by (*rule transfer-rule-of-nat*) *transfer-prover*+

lemma [*transfer-rule*]:
((=) ===> *pcr-integer*) (λ*k*. *k*) *of-int*
proof −
have ((=) ===> *pcr-integer*) *of-int of-int*
by (*rule transfer-rule-of-int*) *transfer-prover*+
then show *?thesis* **by** (*simp add*: *id-def*)
qed

lemma [*transfer-rule*]:

((=) ===> *pcr-integer*) *numeral numeral*
by *transfer-prover*

lemma [*transfer-rule*]:
((=) ===> (=) ===> *pcr-integer*) *Num.sub Num.sub*
by (*unfold Num.sub-def*) *transfer-prover*

lemma [*transfer-rule*]:
(*pcr-integer* ===> (=) ===> *pcr-integer*) (^) (^)
by (*unfold power-def*) *transfer-prover*

end

lemma *int-of-integer-of-nat* [*simp*]:
int-of-integer (*of-nat n*) = *of-nat n*
by *transfer rule*

lift-definition *integer-of-nat* :: *nat* ⇒ *integer*
is *of-nat* :: *nat* ⇒ *int*
.

lemma *integer-of-nat-eq-of-nat* [*code*]:
integer-of-nat = *of-nat*
by *transfer rule*

lemma *int-of-integer-integer-of-nat* [*simp*]:
int-of-integer (*integer-of-nat n*) = *of-nat n*
by *transfer rule*

lift-definition *nat-of-integer* :: *integer* ⇒ *nat*
is *Int.nat*
.

lemma *nat-of-integer-0* [*simp*]:
‹*nat-of-integer 0* = *0*›
by *transfer simp*

lemma *nat-of-integer-1* [*simp*]:
‹*nat-of-integer 1* = *1*›
by *transfer simp*

lemma *nat-of-integer-numeral* [*simp*]:
‹*nat-of-integer* (*numeral n*) = *numeral n*›
by *transfer simp*

lemma *nat-of-integer-of-nat* [*simp*]:
nat-of-integer (*of-nat n*) = *n*
by *transfer simp*

lemma *int-of-integer-of-int* [*simp*]:
int-of-integer (*of-int k*) = *k*
by *transfer simp*

lemma *nat-of-integer-integer-of-nat* [*simp*]:
nat-of-integer (*integer-of-nat n*) = *n*
by *transfer simp*

lemma *integer-of-int-eq-of-int* [*simp*, *code-abbrev*]:
integer-of-int = *of-int*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *of-int-integer-of* [*simp*]:
of-int (*int-of-integer k*) = (*k* :: *integer*)
by *transfer rule*

lemma *int-of-integer-numeral* [*simp*]:
int-of-integer (*numeral k*) = *numeral k*
by *transfer rule*

lemma *int-of-integer-sub* [*simp*]:
int-of-integer (*Num.sub k l*) = *Num.sub k l*
by *transfer rule*

definition *integer-of-num* :: *num* ⇒ *integer*
where [*simp*]: *integer-of-num* = *numeral*

lemma *integer-of-num* [*code*]:
integer-of-num Num.One = *1*
integer-of-num (*Num.Bit0 n*) = (*let k* = *integer-of-num n in k* + *k*)
integer-of-num (*Num.Bit1 n*) = (*let k* = *integer-of-num n in k* + *k* + *1*)
by (*simp-all only*: *integer-of-num-def numeral.simps Let-def*)

lemma *integer-of-num-triv*:
integer-of-num Num.One = *1*
integer-of-num (*Num.Bit0 Num.One*) = *2*
by *simp-all*

instantiation *integer* :: *equal*
begin

lift-definition *equal-integer* :: ‹*integer* ⇒ *integer* ⇒ *bool*›
is ‹*HOL.equal* :: *int* ⇒ *int* ⇒ *bool*›
.

instance
by (*standard*; *transfer*) (*fact equal-eq*)

end

instantiation *integer* :: *linordered-idom*
begin

lift-definition *abs-integer* :: ‹*integer* ⇒ *integer*›
  is ‹*abs* :: *int* ⇒ *int*›
  .

declare *abs-integer.rep-eq* [*simp*]

lift-definition *sgn-integer* :: ‹*integer* ⇒ *integer*›
  is ‹*sgn* :: *int* ⇒ *int*›
  .

declare *sgn-integer.rep-eq* [*simp*]

lift-definition *less-eq-integer* :: ‹*integer* ⇒ *integer* ⇒ *bool*›
  is ‹*less-eq* :: *int* ⇒ *int* ⇒ *bool*›
  .

lemma *integer-less-eq-iff*:
  ‹$k \leq l \longleftrightarrow$ *int-of-integer* $k \leq$ *int-of-integer* l›
  by (*fact less-eq-integer.rep-eq*)

lift-definition *less-integer* :: ‹*integer* ⇒ *integer* ⇒ *bool*›
  is ‹*less* :: *int* ⇒ *int* ⇒ *bool*›
  .

lemma *integer-less-iff*:
  ‹$k < l \longleftrightarrow$ *int-of-integer* $k <$ *int-of-integer* l›
  by (*fact less-integer.rep-eq*)

instance
  by (*standard*; *transfer*)
    (*simp-all add*: *algebra-simps less-le-not-le* [*symmetric*] *mult-strict-right-mono linear*)

end

instance *integer* :: *discrete-linordered-semidom*
  by (*standard*; *transfer*)
    (*fact less-iff-succ-less-eq*)

context
  includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
  ‹(*pcr-integer* ===> *pcr-integer* ===> *pcr-integer*) *min min*›

by (*unfold min-def*) *transfer-prover*

lemma [*transfer-rule*]:
‹(*pcr-integer* ===> *pcr-integer* ===> *pcr-integer*) *max max*›
by (*unfold max-def*) *transfer-prover*

end

lemma *int-of-integer-min* [*simp*]:
int-of-integer (*min k l*) = *min* (*int-of-integer k*) (*int-of-integer l*)
by *transfer rule*

lemma *int-of-integer-max* [*simp*]:
int-of-integer (*max k l*) = *max* (*int-of-integer k*) (*int-of-integer l*)
by *transfer rule*

lemma *nat-of-integer-non-positive* [*simp*]:
$k \leq 0 \Longrightarrow$ *nat-of-integer k* = *0*
by *transfer simp*

lemma *of-nat-of-integer* [*simp*]:
of-nat (*nat-of-integer k*) = *max 0 k*
by *transfer auto*

instantiation *integer* :: *unique-euclidean-ring*
begin

lift-definition *divide-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *divide* :: *int* ⇒ *int* ⇒ *int*
.

declare *divide-integer.rep-eq* [*simp*]

lift-definition *modulo-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *modulo* :: *int* ⇒ *int* ⇒ *int*
.

declare *modulo-integer.rep-eq* [*simp*]

lift-definition *euclidean-size-integer* :: *integer* ⇒ *nat*
is *euclidean-size* :: *int* ⇒ *nat*
.

declare *euclidean-size-integer.rep-eq* [*simp*]

lift-definition *division-segment-integer* :: *integer* ⇒ *integer*
is *division-segment* :: *int* ⇒ *int*
.

declare *division-segment-integer.rep-eq* [*simp*]

instance
apply (*standard*; *transfer*)
apply (*use mult-le-mono2* [*of 1*] **in** ‹*auto simp add*: *sgn-mult-abs abs-mult sgn-mult abs-mod-less sgn-mod nat-mult-distrib*
division-segment-mult division-segment-mod›)
apply (*simp add*: *division-segment-int-def split*: *if-splits*)
done

end

lemma [*code*]:
euclidean-size = *nat-of-integer* ∘ *abs*
by (*simp add*: *fun-eq-iff nat-of-integer.rep-eq*)

lemma [*code*]:
division-segment (*k* :: *integer*) = (*if k* ≥ *0 then 1 else* − *1*)
by *transfer* (*simp add*: *division-segment-int-def*)

instance *integer* :: *linordered-euclidean-semiring*
by (*standard*; *transfer*) (*simp-all add*: *of-nat-div division-segment-int-def*)

instantiation *integer* :: *ring-bit-operations*
begin

lift-definition *bit-integer* :: ‹*integer* ⇒ *nat* ⇒ *bool*›
is *bit* .

lift-definition *not-integer* :: ‹*integer* ⇒ *integer*›
is *not* .

lift-definition *and-integer* :: ‹*integer* ⇒ *integer* ⇒ *integer*›
is ‹*and*› .

lift-definition *or-integer* :: ‹*integer* ⇒ *integer* ⇒ *integer*›
is *or* .

lift-definition *xor-integer* :: ‹*integer* ⇒ *integer* ⇒ *integer*›
is *xor* .

lift-definition *mask-integer* :: ‹*nat* ⇒ *integer*›
is *mask* .

lift-definition *set-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
is *set-bit* .

lift-definition *unset-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
is *unset-bit* .

lift-definition *flip-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
is *flip-bit* **.**

lift-definition *push-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
is *push-bit* **.**

lift-definition *drop-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
is *drop-bit* **.**

lift-definition *take-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
is *take-bit* **.**

instance by (*standard*; *transfer*)
(*fact bit-induct div-by-0 div-by-1 div-0 even-half-succ-eq*
half-div-exp-eq even-double-div-exp-iff bits-mod-div-trivial
bit-iff-odd push-bit-eq-mult drop-bit-eq-div take-bit-eq-mod
and-rec or-rec xor-rec mask-eq-exp-minus-1
set-bit-eq-or unset-bit-eq-or-xor flip-bit-eq-xor not-eq-complement)+

end

instance *integer* :: *linordered-euclidean-semiring-bit-operations* **..**

instantiation *integer* :: *linordered-euclidean-semiring-division*
begin

definition *divmod-integer* :: *num* ⇒ *num* ⇒ *integer* × *integer*
where
divmod-integer′-def: *divmod-integer m n* = (*numeral m div numeral n*, *numeral m mod numeral n*)

definition *divmod-step-integer* :: *integer* ⇒ *integer* × *integer* ⇒ *integer* × *integer*
where
divmod-step-integer l qr = (*let* (*q*, *r*) = *qr*
in if |*l*| ≤ |*r*| *then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))

instance by *standard*
(*auto simp add*: *divmod-integer′-def divmod-step-integer-def integer-less-eq-iff*)

end

lemma *integer-of-nat-0*: *integer-of-nat 0* = *0*
by *transfer simp*

lemma *integer-of-nat-1*: *integer-of-nat 1* = *1*

by *transfer simp*

lemma *integer-of-nat-numeral*:
integer-of-nat (*numeral n*) = *numeral n*
by *transfer simp*

69.2 Code theorems for target language integers

Constructors

definition *Pos* :: *num* ⇒ *integer*
where
[*simp*, *code-post*]: *Pos* = *numeral*

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹((=) ===> *pcr-integer*) *numeral Pos*›
by *simp transfer-prover*

end

lemma *Pos-fold* [*code-unfold*]:
numeral Num.One = *Pos Num.One*
numeral (*Num.Bit0 k*) = *Pos* (*Num.Bit0 k*)
numeral (*Num.Bit1 k*) = *Pos* (*Num.Bit1 k*)
by *simp-all*

definition *Neg* :: *num* ⇒ *integer*
where
[*simp*, *code-abbrev*]: *Neg n* = − *Pos n*

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹((=) ===> *pcr-integer*) (λ*n*. − *numeral n*) *Neg*›
by (*unfold Neg-def*) *transfer-prover*

end

code-datatype *0*::*integer Pos Neg*

A further pair of constructors for generated computations

context
begin

qualified definition *positive* :: *num* ⇒ *integer*
where [*simp*]: *positive* = *numeral*

qualified definition *negative* :: *num* ⇒ *integer*
where [*simp*]: *negative* = *uminus* ∘ *numeral*

lemma [*code-computation-unfold*]:
numeral = *positive*
Pos = *positive*
Neg = *negative*
by (*simp-all add*: *fun-eq-iff*)

end

Auxiliary operations

lift-definition *dup* :: *integer* ⇒ *integer*
is λ*k*::*int*. *k* + *k*
.

lemma *dup-code* [*code*]:
dup 0 = *0*
dup (*Pos n*) = *Pos* (*Num.Bit0 n*)
dup (*Neg n*) = *Neg* (*Num.Bit0 n*)
by (*transfer*; *simp only*: *numeral-Bit0 minus-add-distrib*)+

lift-definition *sub* :: *num* ⇒ *num* ⇒ *integer*
is λ*m n*. *numeral m* − *numeral n* :: *int*
.

lemma *sub-code* [*code*]:
sub Num.One Num.One = *0*
sub (*Num.Bit0 m*) *Num.One* = *Pos* (*Num.BitM m*)
sub (*Num.Bit1 m*) *Num.One* = *Pos* (*Num.Bit0 m*)
sub Num.One (*Num.Bit0 n*) = *Neg* (*Num.BitM n*)
sub Num.One (*Num.Bit1 n*) = *Neg* (*Num.Bit0 n*)
sub (*Num.Bit0 m*) (*Num.Bit0 n*) = *dup* (*sub m n*)
sub (*Num.Bit1 m*) (*Num.Bit1 n*) = *dup* (*sub m n*)
sub (*Num.Bit1 m*) (*Num.Bit0 n*) = *dup* (*sub m n*) + *1*
sub (*Num.Bit0 m*) (*Num.Bit1 n*) = *dup* (*sub m n*) − *1*
by (*transfer*; *simp add*: *dbl-def dbl-inc-def dbl-dec-def*)+

Implementations

lemma *one-integer-code* [*code*, *code-unfold*]:
1 = *Pos Num.One*
by *simp*

lemma *plus-integer-code* [*code*]:
k + *0* = (*k*::*integer*)
0 + *l* = (*l*::*integer*)

Pos m + *Pos n* = *Pos* (*m* + *n*)
Pos m + *Neg n* = *sub m n*
Neg m + *Pos n* = *sub n m*
Neg m + *Neg n* = *Neg* (*m* + *n*)
by (*transfer*, *simp*)+

lemma *uminus-integer-code* [*code*]:
uminus 0 = (*0*::*integer*)
uminus (*Pos m*) = *Neg m*
uminus (*Neg m*) = *Pos m*
by *simp-all*

lemma *minus-integer-code* [*code*]:
k − *0* = (*k*::*integer*)
0 − *l* = *uminus* (*l*::*integer*)
Pos m − *Pos n* = *sub m n*
Pos m − *Neg n* = *Pos* (*m* + *n*)
Neg m − *Pos n* = *Neg* (*m* + *n*)
Neg m − *Neg n* = *sub n m*
by (*transfer*, *simp*)+

lemma *abs-integer-code* [*code*]:
|*k*| = (*if* (*k*::*integer*) < *0 then* − *k else k*)
by *simp*

lemma *sgn-integer-code* [*code*]:
sgn k = (*if k* = *0 then 0 else if* (*k*::*integer*) < *0 then* − *1 else 1*)
by *simp*

lemma *times-integer-code* [*code*]:
k * *0* = (*0*::*integer*)
0 * *l* = (*0*::*integer*)
Pos m * *Pos n* = *Pos* (*m* * *n*)
Pos m * *Neg n* = *Neg* (*m* * *n*)
Neg m * *Pos n* = *Neg* (*m* * *n*)
Neg m * *Neg n* = *Pos* (*m* * *n*)
by *simp-all*

definition *divmod-integer* :: *integer* ⇒ *integer* ⇒ *integer* × *integer*
where
divmod-integer k l = (*k div l*, *k mod l*)

lemma *fst-divmod-integer* [*simp*]:
fst (*divmod-integer k l*) = *k div l*
by (*simp add*: *divmod-integer-def*)

lemma *snd-divmod-integer* [*simp*]:
snd (*divmod-integer k l*) = *k mod l*
by (*simp add*: *divmod-integer-def*)

definition *divmod-abs* :: *integer* ⇒ *integer* ⇒ *integer* × *integer*
where
divmod-abs k l = (|*k*| *div* |*l*|, |*k*| *mod* |*l*|)

lemma *fst-divmod-abs* [*simp*]:
fst (*divmod-abs k l*) = |*k*| *div* |*l*|
by (*simp add*: *divmod-abs-def*)

lemma *snd-divmod-abs* [*simp*]:
snd (*divmod-abs k l*) = |*k*| *mod* |*l*|
by (*simp add*: *divmod-abs-def*)

declare *divmod-algorithm-code* [**where** *?'a* = *integer*,
folded integer-of-num-def, *unfolded integer-of-num-triv*,
code]

lemma *divmod-abs-code* [*code*]:
divmod-abs (*Pos k*) (*Pos l*) = *divmod k l*
divmod-abs (*Neg k*) (*Neg l*) = *divmod k l*
divmod-abs (*Neg k*) (*Pos l*) = *divmod k l*
divmod-abs (*Pos k*) (*Neg l*) = *divmod k l*
divmod-abs j 0 = (*0*, |*j*|)
divmod-abs 0 j = (*0*, *0*)
by (*simp-all add*: *prod-eq-iff*)

lemma *divmod-integer-eq-cases*:
divmod-integer k l =
(*if k* = *0 then* (*0*, *0*) *else if l* = *0 then* (*0*, *k*) *else*
(*apsnd* ∘ *times* ∘ *sgn*) *l* (*if sgn k* = *sgn l*
then divmod-abs k l
else (*let* (*r*, *s*) = *divmod-abs k l in*
if s = *0 then* (− *r*, *0*) *else* (− *r* − *1*, |*l*| − *s*))))
proof −
have ∗: *sgn k* = *sgn l* ⟷ *k* = *0* ∧ *l* = *0* ∨ *0* < *l* ∧ *0* < *k* ∨ *l* < *0* ∧ *k* < *0*
for *k l* :: *int*
by (*auto simp add*: *sgn-if*)
have ∗∗: − *k* = *l* ∗ *q* ⟷ *k* = − (*l* ∗ *q*) **for** *k l q* :: *int*
by *auto*
show *?thesis*
by (*simp add*: *divmod-integer-def divmod-abs-def*)
(*transfer*, *auto simp add*: ∗ ∗∗ *not-less zdiv-zminus1-eq-if zmod-zminus1-eq-if*
div-minus-right mod-minus-right)
qed

lemma *divmod-integer-code* [*code*]:
divmod-integer k l =
(*if k* = *0 then* (*0*, *0*)
else if l > *0 then*

(*if k > 0 then divmod-abs k l*
else case divmod-abs k l of (*r*, *s*) ⇒
if s = *0 then* (− *r*, *0*) *else* (− *r* − *1*, *l* − *s*))
else if l = *0 then* (*0*, *k*)
else apsnd uminus
(*if k* < *0 then divmod-abs k l*
else case divmod-abs k l of (*r*, *s*) ⇒
if s = *0 then* (− *r*, *0*) *else* (− *r* − *1*, − *l* − *s*)))
by (*cases l 0* :: *integer rule*: *linorder-cases*)
(*auto split*: *prod.splits simp add*: *divmod-integer-eq-cases*)

lemma *div-integer-code* [*code*]:
k div l = *fst* (*divmod-integer k l*)
by *simp*

lemma *mod-integer-code* [*code*]:
k mod l = *snd* (*divmod-integer k l*)
by *simp*

context
includes *bit-operations-syntax*
begin

lemma *and-integer-code* [*code*]:
‹*Pos Num.One AND Pos Num.One* = *Pos Num.One*›
‹*Pos Num.One AND Pos* (*Num.Bit0 n*) = *0*›
‹*Pos* (*Num.Bit0 m*) *AND Pos Num.One* = *0*›
‹*Pos Num.One AND Pos* (*Num.Bit1 n*) = *Pos Num.One*›
‹*Pos* (*Num.Bit1 m*) *AND Pos Num.One* = *Pos Num.One*›
‹*Pos* (*Num.Bit0 m*) *AND Pos* (*Num.Bit0 n*) = *dup* (*Pos m AND Pos n*)›
‹*Pos* (*Num.Bit0 m*) *AND Pos* (*Num.Bit1 n*) = *dup* (*Pos m AND Pos n*)›
‹*Pos* (*Num.Bit1 m*) *AND Pos* (*Num.Bit0 n*) = *dup* (*Pos m AND Pos n*)›
‹*Pos* (*Num.Bit1 m*) *AND Pos* (*Num.Bit1 n*) = *Pos Num.One* + *dup* (*Pos m AND Pos n*)›
‹*Pos m AND Neg* (*num.Bit0 n*) = (*case and-not-num m* (*Num.BitM n*) *of None* ⇒ *0* | *Some n′* ⇒ *Pos n′*)›
‹*Neg* (*num.Bit0 m*) *AND Pos n* = (*case and-not-num n* (*Num.BitM m*) *of None* ⇒ *0* | *Some n′* ⇒ *Pos n′*)›
‹*Pos m AND Neg* (*num.Bit1 n*) = (*case and-not-num m* (*Num.Bit0 n*) *of None* ⇒ *0* | *Some n′* ⇒ *Pos n′*)›
‹*Neg* (*num.Bit1 m*) *AND Pos n* = (*case and-not-num n* (*Num.Bit0 m*) *of None* ⇒ *0* | *Some n′* ⇒ *Pos n′*)›
‹*Neg m AND Neg n* = *NOT* (*sub m Num.One OR sub n Num.One*)›
‹*Neg Num.One AND k* = *k*›
‹*k AND Neg Num.One* = *k*›
‹*0 AND k* = *0*›
‹*k AND 0* = *0*›
for *k* :: *integer*
by (*transfer*; *simp*)+

lemma *or-integer-code* [*code*]:
‹*Pos Num.One AND Pos Num.One = Pos Num.One*›
‹*Pos Num.One OR Pos* (*Num.Bit0 n*) = *Pos* (*Num.Bit1 n*)›
‹*Pos* (*Num.Bit0 m*) *OR Pos Num.One* = *Pos* (*Num.Bit1 m*)›
‹*Pos Num.One OR Pos* (*Num.Bit1 n*) = *Pos* (*Num.Bit1 n*)›
‹*Pos* (*Num.Bit1 m*) *OR Pos Num.One* = *Pos* (*Num.Bit1 m*)›
‹*Pos* (*Num.Bit0 m*) *OR Pos* (*Num.Bit0 n*) = *dup* (*Pos m OR Pos n*)›
‹*Pos* (*Num.Bit0 m*) *OR Pos* (*Num.Bit1 n*) = *Pos Num.One* + *dup* (*Pos m OR Pos n*)›
‹*Pos* (*Num.Bit1 m*) *OR Pos* (*Num.Bit0 n*) = *Pos Num.One* + *dup* (*Pos m OR Pos n*)›
‹*Pos* (*Num.Bit1 m*) *OR Pos* (*Num.Bit1 n*) = *Pos Num.One* + *dup* (*Pos m OR Pos n*)›
‹*Pos m OR Neg* (*num.Bit0 n*) = *Neg* (*or-not-num-neg m* (*Num.BitM n*))›
‹*Neg* (*num.Bit0 m*) *OR Pos n* = *Neg* (*or-not-num-neg n* (*Num.BitM m*))›
‹*Pos m OR Neg* (*num.Bit1 n*) = *Neg* (*or-not-num-neg m* (*Num.Bit0 n*))›
‹*Neg* (*num.Bit1 m*) *OR Pos n* = *Neg* (*or-not-num-neg n* (*Num.Bit0 m*))›
‹*Neg m OR Neg n* = *NOT* (*sub m Num.One AND sub n Num.One*)›
‹*Neg Num.One OR k* = *Neg Num.One*›
‹*k OR Neg Num.One* = *Neg Num.One*›
‹*0 OR k* = *k*›
‹*k OR 0* = *k*›
for *k* :: *integer*
by (*transfer*; *simp*)+

lemma *xor-integer-code* [*code*]:
‹*Pos Num.One XOR Pos Num.One* = *0*›
‹*Pos Num.One XOR numeral* (*Num.Bit0 n*) = *Pos* (*Num.Bit1 n*)›
‹*Pos* (*Num.Bit0 m*) *XOR Pos Num.One* = *Pos* (*Num.Bit1 m*)›
‹*Pos Num.One XOR numeral* (*Num.Bit1 n*) = *Pos* (*Num.Bit0 n*)›
‹*Pos* (*Num.Bit1 m*) *XOR Pos Num.One* = *Pos* (*Num.Bit0 m*)›
‹*Pos* (*Num.Bit0 m*) *XOR Pos* (*Num.Bit0 n*) = *dup* (*Pos m XOR Pos n*)›
‹*Pos* (*Num.Bit0 m*) *XOR Pos* (*Num.Bit1 n*) = *Pos Num.One* + *dup* (*Pos m XOR Pos n*)›
‹*Pos* (*Num.Bit1 m*) *XOR Pos* (*Num.Bit0 n*) = *Pos Num.One* + *dup* (*Pos m XOR Pos n*)›
‹*Pos* (*Num.Bit1 m*) *XOR Pos* (*Num.Bit1 n*) = *dup* (*Pos m XOR Pos n*)›
‹*Neg m XOR k* = *NOT* (*sub m num.One XOR k*)›
‹*k XOR Neg n* = *NOT* (*k XOR* (*sub n num.One*))›
‹*Neg Num.One XOR k* = *NOT k*›
‹*k XOR Neg Num.One* = *NOT k*›
‹*0 XOR k* = *k*›
‹*k XOR 0* = *k*›
for *k* :: *integer*
by (*transfer*; *simp*)+

lemma [*code*]:
‹*NOT k* = − *k* − *1*› **for** *k* :: *integer*

by (*fact not-eq-complement*)

lemma [*code*]:
‹*bit k n* ⟷ *k AND push-bit n 1* ≠ (*0* :: *integer*)›
by (*simp add*: *and-exp-eq-0-iff-not-bit*)

lemma [*code*]:
‹*mask n* = *push-bit n 1* − (*1* :: *integer*)›
by (*simp add*: *mask-eq-exp-minus-1*)

lemma [*code*]:
‹*set-bit n k* = *k OR push-bit n 1*› **for** *k* :: *integer*
by (*fact set-bit-def*)

lemma [*code*]:
‹*unset-bit n k* = *k AND NOT* (*push-bit n 1*)› **for** *k* :: *integer*
by (*fact unset-bit-def*)

lemma [*code*]:
‹*flip-bit n k* = *k XOR push-bit n 1*› **for** *k* :: *integer*
by (*fact flip-bit-def*)

lemma [*code*]:
‹*push-bit n k* = *k* ∗ *2* ^ *n*› **for** *k* :: *integer*
by (*fact push-bit-eq-mult*)

lemma [*code*]:
‹*drop-bit n k* = *k div 2* ^ *n*› **for** *k* :: *integer*
by (*fact drop-bit-eq-div*)

lemma [*code*]:
‹*take-bit n k* = *k AND mask n*› **for** *k* :: *integer*
by (*fact take-bit-eq-mask*)

end

definition *bit-cut-integer* :: *integer* ⇒ *integer* × *bool*
where *bit-cut-integer k* = (*k div 2*, *odd k*)

lemma *bit-cut-integer-code* [*code*]:
bit-cut-integer k = (*if k* = *0 then* (*0*, *False*)
else let (*r*, *s*) = *Code-Numeral.divmod-abs k 2*
in (*if k* > *0 then r else* − *r* − *s*, *s* = *1*))
proof −
have *bit-cut-integer k* = (*let* (*r*, *s*) = *divmod-integer k 2 in* (*r*, *s* = *1*))
by (*simp add*: *divmod-integer-def bit-cut-integer-def odd-iff-mod-2-eq-one*)
then show *?thesis*
by (*simp add*: *divmod-integer-code*) (*auto simp add*: *split-def*)
qed

lemma *equal-integer-code* [*code*]:
HOL.equal 0 (*0*::*integer*) ⟷ *True*
HOL.equal 0 (*Pos l*) ⟷ *False*
HOL.equal 0 (*Neg l*) ⟷ *False*
HOL.equal (*Pos k*) *0* ⟷ *False*
HOL.equal (*Pos k*) (*Pos l*) ⟷ *HOL.equal k l*
HOL.equal (*Pos k*) (*Neg l*) ⟷ *False*
HOL.equal (*Neg k*) *0* ⟷ *False*
HOL.equal (*Neg k*) (*Pos l*) ⟷ *False*
HOL.equal (*Neg k*) (*Neg l*) ⟷ *HOL.equal k l*
by (*simp-all add*: *equal*)

lemma *equal-integer-refl* [*code nbe*]:
HOL.equal (*k*::*integer*) *k* ⟷ *True*
by (*fact equal-refl*)

lemma *less-eq-integer-code* [*code*]:
0 ≤ (*0*::*integer*) ⟷ *True*
0 ≤ *Pos l* ⟷ *True*
0 ≤ *Neg l* ⟷ *False*
Pos k ≤ *0* ⟷ *False*
Pos k ≤ *Pos l* ⟷ *k* ≤ *l*
Pos k ≤ *Neg l* ⟷ *False*
Neg k ≤ *0* ⟷ *True*
Neg k ≤ *Pos l* ⟷ *True*
Neg k ≤ *Neg l* ⟷ *l* ≤ *k*
by *simp-all*

lemma *less-integer-code* [*code*]:
0 < (*0*::*integer*) ⟷ *False*
0 < *Pos l* ⟷ *True*
0 < *Neg l* ⟷ *False*
Pos k < *0* ⟷ *False*
Pos k < *Pos l* ⟷ *k* < *l*
Pos k < *Neg l* ⟷ *False*
Neg k < *0* ⟷ *True*
Neg k < *Pos l* ⟷ *True*
Neg k < *Neg l* ⟷ *l* < *k*
by *simp-all*

lift-definition *num-of-integer* :: *integer* ⇒ *num*
is *num-of-nat* ∘ *nat*
.

lemma *num-of-integer-code* [*code*]:
num-of-integer k = (*if k* ≤ *1 then Num.One*
else let
(*l*, *j*) = *divmod-integer k 2*;

l′ = num-of-integer l;
l′′ = l′ + l′
in if j = 0 then l′′ else l′′ + Num.One)
proof −
{
assume *int-of-integer k mod 2 = 1*
then have *nat* (*int-of-integer k mod 2*) = *nat 1* **by** *simp*
moreover assume ∗: *1 < int-of-integer k*
ultimately have ∗∗: *nat* (*int-of-integer k*) *mod 2 = 1* **by** (*simp add*: *nat-mod-distrib*)
have *num-of-nat* (*nat* (*int-of-integer k*)) =
num-of-nat (*2* ∗ (*nat* (*int-of-integer k*) *div 2*) + *nat* (*int-of-integer k*) *mod 2*)
by *simp*
then have *num-of-nat* (*nat* (*int-of-integer k*)) =
num-of-nat (*nat* (*int-of-integer k*) *div 2* + *nat* (*int-of-integer k*) *div 2* + *nat* (*int-of-integer k*) *mod 2*)
by (*simp add*: *mult-2*)
with ∗∗ **have** *num-of-nat* (*nat* (*int-of-integer k*)) =
num-of-nat (*nat* (*int-of-integer k*) *div 2* + *nat* (*int-of-integer k*) *div 2* + *1*)
by *simp*
}
note *aux = this*
show *?thesis*
by (*auto simp add*: *num-of-integer-def nat-of-integer-def Let-def case-prod-beta not-le integer-eq-iff less-eq-integer-def nat-mult-distrib nat-div-distrib num-of-nat-One num-of-nat-plus-distrib mult-2* [**where** *′a=nat*] *aux add-One*)
qed

lemma *nat-of-integer-code* [*code*]:
nat-of-integer k = (*if k ≤ 0 then 0*
else let
(*l*, *j*) = *divmod-integer k 2*;
l′ = nat-of-integer l;
l′′ = l′ + l′
in if j = 0 then l′′ else l′′ + 1)
proof −
obtain *j* **where** *k*: *k = integer-of-int j*
proof
show *k = integer-of-int* (*int-of-integer k*) **by** *simp*
qed
have ∗: *nat j mod 2 = nat-of-integer* (*of-int j mod 2*) **if** *j ≥ 0*
using *that* **by** *transfer* (*simp add*: *nat-mod-distrib*)
from *k* **show** *?thesis*
by (*auto simp add*: *split-def Let-def nat-of-integer-def nat-div-distrib mult-2* [*symmetric*]
minus-mod-eq-mult-div [*symmetric*] ∗)
qed

lemma *int-of-integer-code* [*code*]:

int-of-integer k = (*if k* < *0 then* − (*int-of-integer* (− *k*))
else if k = *0 then 0*
else let
(*l*, *j*) = *divmod-integer k 2*;
l′ = *2* ∗ *int-of-integer l*
in if j = *0 then l*′ *else l*′ + *1*)
by (*auto simp add*: *split-def Let-def integer-eq-iff minus-mod-eq-mult-div* [*symmetric*])

lemma *integer-of-int-code* [*code*]:
integer-of-int k = (*if k* < *0 then* − (*integer-of-int* (− *k*))
else if k = *0 then 0*
else let
l = *2* ∗ *integer-of-int* (*k div 2*);
j = *k mod 2*
in if j = *0 then l else l* + *1*)
by (*auto simp add*: *split-def Let-def integer-eq-iff minus-mod-eq-mult-div* [*symmetric*])

hide-const (**open**) *Pos Neg sub dup divmod-abs*

69.3 Serializer setup for target language integers

code-reserved
(*Eval*) *int Integer abs*

code-printing
type-constructor *integer* ⇀
(*SML*) *IntInf.int*
and (*OCaml*) *Z.t*
and (*Haskell*) *Integer*
and (*Scala*) *BigInt*
and (*Eval*) *int*
| **class-instance** *integer* :: *equal* ⇀
(*Haskell*) −

code-printing
constant *0*::*integer* ⇀
(*SML*) !(*0*/ :/ *IntInf.int*)
and (*OCaml*) *Z.zero*
and (*Haskell*) !(*0*/ ::/ *Integer*)
and (*Scala*) *BigInt*(*0*)

setup ‹
fold (*fn target* =>
Numeral.add-code ***const-name*** ‹*Code-Numeral.Pos*› *I Code-Printer.literal-numeral target*
#> *Numeral.add-code* ***const-name*** ‹*Code-Numeral.Neg*› (~) *Code-Printer.literal-numeral target*)
[*SML*, *OCaml*, *Haskell*, *Scala*]
›

code-printing
constant *plus* :: *integer* ⇒ - ⇒ - ⇀
(*SML*) *IntInf*.+ ((-), (-))
and (*OCaml*) *Z.add*
and (*Haskell*) **infixl** *6* +
and (*Scala*) **infixl** *7* +
and (*Eval*) **infixl** *8* +
| **constant** *uminus* :: *integer* ⇒ - ⇀
(*SML*) *IntInf*.~
and (*OCaml*) *Z.neg*
and (*Haskell*) *negate*
and (*Scala*) !(− -)
and (*Eval*) ~/ -
| **constant** *minus* :: *integer* ⇒ - ⇀
(*SML*) *IntInf*.− ((-), (-))
and (*OCaml*) *Z.sub*
and (*Haskell*) **infixl** *6* −
and (*Scala*) **infixl** *7* −
and (*Eval*) **infixl** *8* −
| **constant** *Code-Numeral.dup* ⇀
(*SML*) *IntInf*.∗/ (*2*,/ (-))
and (*OCaml*) *Z.shift′-left*/ -/ *1*
and (*Haskell*) !(*2* ∗ -)
and (*Scala*) !(*2* ∗ -)
and (*Eval*) !(*2* ∗ -)
| **constant** *Code-Numeral.sub* ⇀
(*SML*) !(*raise*/ *Fail*/ *sub*)
and (*OCaml*) *failwith*/ *sub*
and (*Haskell*) *error*/ *sub*
and (*Scala*) !*sys.error*(*sub*)
| **constant** *times* :: *integer* ⇒ - ⇒ - ⇀
(*SML*) *IntInf*.∗ ((-), (-))
and (*OCaml*) *Z.mul*
and (*Haskell*) **infixl** *7* ∗
and (*Scala*) **infixl** *8* ∗
and (*Eval*) **infixl** *9* ∗
| **constant** *Code-Numeral.divmod-abs* ⇀
(*SML*) *IntInf.divMod*/ (*IntInf.abs* -,/ *IntInf.abs* -)
and (*OCaml*) !(*fun k l* −>/ *if Z.equal Z.zero l then*/ (*Z.zero*, *l*) *else*/ *Z.div′-rem*/ (*Z.abs k*)/ (*Z.abs l*))
and (*Haskell*) *divMod*/ (*abs* -)/ (*abs* -)
and (*Scala*) !((*k*: *BigInt*) => (*l*: *BigInt*) =>/ *l* == *0 match* { *case true* => (*BigInt*(*0*), *k*) *case false* => (*k.abs* ′/% *l.abs*) })
and (*Eval*) *Integer.div′-mod*/ (*abs* -)/ (*abs* -)
| **constant** *HOL.equal* :: *integer* ⇒ - ⇒ *bool* ⇀
(*SML*) !((- : *IntInf.int*) = -)
and (*OCaml*) *Z.equal*
and (*Haskell*) **infix** *4* ==

and (*Scala*) **infixl** *5* ==
and (*Eval*) **infixl** *6* =
| **constant** *less-eq* :: *integer* ⇒ - ⇒ *bool* ⇀
(*SML*) *IntInf*.<= ((-), (-))
and (*OCaml*) *Z.leq*
and (*Haskell*) **infix** *4* <=
and (*Scala*) **infixl** *4* <=
and (*Eval*) **infixl** *6* <=
| **constant** *less* :: *integer* ⇒ - ⇒ *bool* ⇀
(*SML*) *IntInf*.< ((-), (-))
and (*OCaml*) *Z.lt*
and (*Haskell*) **infix** *4* <
and (*Scala*) **infixl** *4* <
and (*Eval*) **infixl** *6* <
| **constant** *abs* :: *integer* ⇒ - ⇀
(*SML*) *IntInf.abs*
and (*OCaml*) *Z.abs*
and (*Haskell*) *Prelude.abs*
and (*Scala*) -.*abs*
and (*Eval*) *abs*
| **constant** *Bit-Operations.and* :: *integer* ⇒ *integer* ⇒ *integer* ⇀
(*SML*) *IntInf.andb* ((-),/ (-))
and (*OCaml*) *Z.logand*
and (*Haskell*) **infixl** *7* .&.
and (*Scala*) **infixl** *3* &
| **constant** *Bit-Operations.or* :: *integer* ⇒ *integer* ⇒ *integer* ⇀
(*SML*) *IntInf.orb* ((-),/ (-))
and (*OCaml*) *Z.logor*
and (*Haskell*) **infixl** *5* .|.
and (*Scala*) **infixl** *1* |
| **constant** *Bit-Operations.xor* :: *integer* ⇒ *integer* ⇒ *integer* ⇀
(*SML*) *IntInf.xorb* ((-),/ (-))
and (*OCaml*) *Z.logxor*
and (*Haskell*) **infixl** *6* .^.
and (*Scala*) **infixl** *2* ^
| **constant** *Bit-Operations.not* :: *integer* ⇒ *integer* ⇀
(*SML*) *IntInf.notb*
and (*OCaml*) *Z.lognot*
and (*Haskell*) *Data.Bits.complement*
and (*Scala*) -.*unary'*-~

code-identifier
code-module *Code-Numeral* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

69.4 Type of target language naturals

typedef *natural* = *UNIV* :: *nat set*
morphisms *nat-of-natural natural-of-nat* **..**

setup-lifting *type-definition-natural*

lemma *natural-eq-iff* [*termination-simp*]:
m = *n* ⟷ *nat-of-natural m* = *nat-of-natural n*
by *transfer rule*

lemma *natural-eqI*:
nat-of-natural m = *nat-of-natural n* ⟹ *m* = *n*
using *natural-eq-iff* [*of m n*] **by** *simp*

lemma *nat-of-natural-of-nat-inverse* [*simp*]:
nat-of-natural (*natural-of-nat n*) = *n*
by *transfer rule*

lemma *natural-of-nat-of-natural-inverse* [*simp*]:
natural-of-nat (*nat-of-natural n*) = *n*
by *transfer rule*

instantiation *natural* :: {*comm-monoid-diff*, *semiring-1*}
begin

lift-definition *zero-natural* :: *natural*
is *0* :: *nat*
.

declare *zero-natural.rep-eq* [*simp*]

lift-definition *one-natural* :: *natural*
is *1* :: *nat*
.

declare *one-natural.rep-eq* [*simp*]

lift-definition *plus-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *plus* :: *nat* ⇒ *nat* ⇒ *nat*
.

declare *plus-natural.rep-eq* [*simp*]

lift-definition *minus-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *minus* :: *nat* ⇒ *nat* ⇒ *nat*
.

declare *minus-natural.rep-eq* [*simp*]

lift-definition *times-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *times* :: *nat* ⇒ *nat* ⇒ *nat*
.

declare *times-natural.rep-eq* [*simp*]

instance proof
qed (*transfer*, *simp add*: *algebra-simps*)+

end

instance *natural* :: *Rings.dvd* **..**

context
  includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
  ‹(*pcr-natural* ===> *pcr-natural* ===> (⟷)) (*dvd*) (*dvd*)›
  by (*unfold dvd-def*) *transfer-prover*

lemma [*transfer-rule*]:
  ‹((⟷) ===> *pcr-natural*) *of-bool of-bool*›
  by (*unfold of-bool-def*) *transfer-prover*

lemma [*transfer-rule*]:
  ‹((=) ===> *pcr-natural*) (λ*n*. *n*) *of-nat*›
proof −
  have *rel-fun HOL.eq pcr-natural* (*of-nat* :: *nat* ⇒ *nat*) (*of-nat* :: *nat* ⇒ *natural*)
    by (*unfold of-nat-def*) *transfer-prover*
  then show *?thesis* **by** (*simp add*: *id-def*)
qed

lemma [*transfer-rule*]:
  ‹((=) ===> *pcr-natural*) *numeral numeral*›
proof −
  have ‹((=) ===> *pcr-natural*) *numeral* (λ*n*. *of-nat* (*numeral n*))›
    by *transfer-prover*
  then show *?thesis* **by** *simp*
qed

lemma [*transfer-rule*]:
  ‹(*pcr-natural* ===> (=) ===> *pcr-natural*) (^) (^)›
  by (*unfold power-def*) *transfer-prover*

end

lemma *nat-of-natural-of-nat* [*simp*]:
  nat-of-natural (*of-nat n*) = *n*
  by *transfer rule*

lemma *natural-of-nat-of-nat* [*simp*, *code-abbrev*]:

natural-of-nat = *of-nat*
by *transfer rule*

lemma *of-nat-of-natural* [*simp*]:
of-nat (*nat-of-natural n*) = *n*
by *transfer rule*

lemma *nat-of-natural-numeral* [*simp*]:
nat-of-natural (*numeral k*) = *numeral k*
by *transfer rule*

instantiation *natural* :: {*linordered-semiring*, *equal*}
begin

lift-definition *less-eq-natural* :: *natural* ⇒ *natural* ⇒ *bool*
is *less-eq* :: *nat* ⇒ *nat* ⇒ *bool*
.

declare *less-eq-natural.rep-eq* [*termination-simp*]

lift-definition *less-natural* :: *natural* ⇒ *natural* ⇒ *bool*
is *less* :: *nat* ⇒ *nat* ⇒ *bool*
.

declare *less-natural.rep-eq* [*termination-simp*]

lift-definition *equal-natural* :: *natural* ⇒ *natural* ⇒ *bool*
is *HOL.equal* :: *nat* ⇒ *nat* ⇒ *bool*
.

instance proof
qed (*transfer*, *simp add*: *algebra-simps equal less-le-not-le* [*symmetric*] *linear*)+

end

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> *pcr-natural* ===> *pcr-natural*) *min min*›
by (*unfold min-def*) *transfer-prover*

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> *pcr-natural* ===> *pcr-natural*) *max max*›
by (*unfold max-def*) *transfer-prover*

end

lemma *nat-of-natural-min* [*simp*]:
nat-of-natural (*min k l*) = *min* (*nat-of-natural k*) (*nat-of-natural l*)
by *transfer rule*

lemma *nat-of-natural-max* [*simp*]:
nat-of-natural (*max k l*) = *max* (*nat-of-natural k*) (*nat-of-natural l*)
by *transfer rule*

instantiation *natural* :: *unique-euclidean-semiring*
begin

lift-definition *divide-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *divide* :: *nat* ⇒ *nat* ⇒ *nat*
.

declare *divide-natural.rep-eq* [*simp*]

lift-definition *modulo-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *modulo* :: *nat* ⇒ *nat* ⇒ *nat*
.

declare *modulo-natural.rep-eq* [*simp*]

lift-definition *euclidean-size-natural* :: *natural* ⇒ *nat*
is *euclidean-size* :: *nat* ⇒ *nat*
.

declare *euclidean-size-natural.rep-eq* [*simp*]

lift-definition *division-segment-natural* :: *natural* ⇒ *natural*
is *division-segment* :: *nat* ⇒ *nat*
.

declare *division-segment-natural.rep-eq* [*simp*]

instance
by (*standard*; *transfer*)
(*auto simp add*: *algebra-simps unit-factor-nat-def gr0-conv-Suc*)

end

lemma [*code*]:
euclidean-size = *nat-of-natural*
by (*simp add*: *fun-eq-iff*)

lemma [*code*]:
division-segment (*n*::*natural*) = *1*
by (*simp add*: *natural-eq-iff*)

instance *natural* :: *discrete-linordered-semidom*
by (*standard*; *transfer*) (*simp-all add*: *Suc-le-eq*)

instance *natural* :: *linordered-euclidean-semiring*
by (*standard*; *transfer*) *simp-all*

instantiation *natural* :: *semiring-bit-operations*
begin

lift-definition *bit-natural* :: ‹*natural* ⇒ *nat* ⇒ *bool*›
is *bit* **.**

lift-definition *and-natural* :: ‹*natural* ⇒ *natural* ⇒ *natural*›
is ‹*and*› **.**

lift-definition *or-natural* :: ‹*natural* ⇒ *natural* ⇒ *natural*›
is *or* **.**

lift-definition *xor-natural* :: ‹*natural* ⇒ *natural* ⇒ *natural*›
is *xor* **.**

lift-definition *mask-natural* :: ‹*nat* ⇒ *natural*›
is *mask* **.**

lift-definition *set-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *set-bit* **.**

lift-definition *unset-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *unset-bit* **.**

lift-definition *flip-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *flip-bit* **.**

lift-definition *push-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *push-bit* **.**

lift-definition *drop-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *drop-bit* **.**

lift-definition *take-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *take-bit* **.**

instance by (*standard*; *transfer*)
(*fact bit-induct div-by-0 div-by-1 div-0 even-half-succ-eq*
half-div-exp-eq even-double-div-exp-iff bits-mod-div-trivial
bit-iff-odd push-bit-eq-mult drop-bit-eq-div take-bit-eq-mod
and-rec or-rec xor-rec mask-eq-exp-minus-1
set-bit-eq-or unset-bit-eq-or-xor flip-bit-eq-xor not-eq-complement)+

end

instance *natural* :: *linordered-euclidean-semiring-bit-operations* **..**

lift-definition *natural-of-integer* :: *integer* ⇒ *natural*
is *nat* :: *int* ⇒ *nat*
.

lift-definition *integer-of-natural* :: *natural* ⇒ *integer*
is *of-nat* :: *nat* ⇒ *int*
.

lemma *natural-of-integer-of-natural* [*simp*]:
natural-of-integer (*integer-of-natural n*) = *n*
by *transfer simp*

lemma *integer-of-natural-of-integer* [*simp*]:
integer-of-natural (*natural-of-integer k*) = *max 0 k*
by *transfer auto*

lemma *int-of-integer-of-natural* [*simp*]:
int-of-integer (*integer-of-natural n*) = *of-nat* (*nat-of-natural n*)
by *transfer rule*

lemma *integer-of-natural-of-nat* [*simp*]:
integer-of-natural (*of-nat n*) = *of-nat n*
by *transfer rule*

lemma [*measure-function*]:
is-measure nat-of-natural
by (*rule is-measure-trivial*)

69.5 Inductive representation of target language naturals

lift-definition *Suc* :: *natural* ⇒ *natural*
is *Nat.Suc*
.

declare *Suc.rep-eq* [*simp*]

old-rep-datatype *0*::*natural Suc*
by (*transfer*, *fact nat.induct nat.inject nat.distinct*)+

lemma *natural-cases* [*case-names nat*, *cases type*: *natural*]:
fixes *m* :: *natural*
assumes ⋀*n*. *m* = *of-nat n* ⟹ *P*
shows *P*
using *assms* **by** *transfer blast*

instantiation *natural* :: *size*
begin

definition *size-nat* **where** [*simp*, *code*]: *size-nat* = *nat-of-natural*

instance ..

end

lemma *natural-decr* [*termination-simp*]:
$n \neq 0 \Longrightarrow$ *nat-of-natural* n − *Nat.Suc* $0 <$ *nat-of-natural* n
by *transfer simp*

lemma *natural-zero-minus-one*: (*0*::*natural*) − *1* = *0*
by (*rule zero-diff*)

lemma *Suc-natural-minus-one*: *Suc n* − *1* = *n*
by *transfer simp*

hide-const (**open**) *Suc*

69.6 Code refinement for target language naturals

lift-definition *Nat* :: *integer* ⇒ *natural*
is *nat*
.

lemma [*code-post*]:
Nat 0 = *0*
Nat 1 = *1*
Nat (*numeral k*) = *numeral k*
by (*transfer*, *simp*)+

lemma [*code abstype*]:
Nat (*integer-of-natural n*) = *n*
by *transfer simp*

lemma [*code*]:
natural-of-nat n = *natural-of-integer* (*integer-of-nat n*)
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural (*natural-of-integer k*) = *max 0 k*
by *simp*

lemma [*code*]:
‹*integer-of-natural* (*mask n*) = *mask n*›
by *transfer* (*simp add*: *mask-eq-exp-minus-1*)

lemma [*code-abbrev*]:
natural-of-integer (*Code-Numeral.Pos k*) = *numeral k*
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural 0 = *0*
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural 1 = *1*
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural (*Code-Numeral.Suc n*) = *integer-of-natural n* + *1*
by *transfer simp*

lemma [*code*]:
nat-of-natural = *nat-of-integer* ∘ *integer-of-natural*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma [*code*, *code-unfold*]:
case-natural f g n = (*if n* = *0 then f else g* (*n* − *1*))
by (*cases n rule*: *natural.exhaust*) (*simp-all*, *simp add*: *Suc-def*)

declare *natural.rec* [*code del*]

lemma [*code abstract*]:
integer-of-natural (*m* + *n*) = *integer-of-natural m* + *integer-of-natural n*
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural (*m* − *n*) = *max 0* (*integer-of-natural m* − *integer-of-natural n*)
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural (*m* ∗ *n*) = *integer-of-natural m* ∗ *integer-of-natural n*
by *transfer simp*

lemma [*code abstract*]:
integer-of-natural (*m div n*) = *integer-of-natural m div integer-of-natural n*
by *transfer* (*simp add*: *zdiv-int*)

lemma [*code abstract*]:
integer-of-natural (*m mod n*) = *integer-of-natural m mod integer-of-natural n*
by *transfer* (*simp add*: *zmod-int*)

lemma [*code*]:
HOL.equal m n ⟷ *HOL.equal* (*integer-of-natural m*) (*integer-of-natural n*)

by *transfer* (*simp add*: *equal*)

lemma [*code nbe*]: *HOL.equal n* (*n*::*natural*) ⟷ *True*
by (*rule equal-class.equal-refl*)

lemma [*code*]: *m* ≤ *n* ⟷ *integer-of-natural m* ≤ *integer-of-natural n*
by *transfer simp*

lemma [*code*]: *m* < *n* ⟷ *integer-of-natural m* < *integer-of-natural n*
by *transfer simp*

context
includes *bit-operations-syntax*
begin

lemma [*code*]:
‹*bit m n* ⟷ *bit* (*integer-of-natural m*) *n*›
by *transfer* (*simp add*: *bit-simps*)

lemma [*code abstract*]:
‹*integer-of-natural* (*m AND n*) = *integer-of-natural m AND integer-of-natural n*›
by *transfer* (*simp add*: *of-nat-and-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*m OR n*) = *integer-of-natural m OR integer-of-natural n*›
by *transfer* (*simp add*: *of-nat-or-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*m XOR n*) = *integer-of-natural m XOR integer-of-natural n*›
by *transfer* (*simp add*: *of-nat-xor-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*mask n*) = *mask n*›
by *transfer* (*simp add*: *of-nat-mask-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*set-bit n m*) = *set-bit n* (*integer-of-natural m*)›
by *transfer* (*simp add*: *of-nat-set-bit-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*unset-bit n m*) = *unset-bit n* (*integer-of-natural m*)›
by *transfer* (*simp add*: *of-nat-unset-bit-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*flip-bit n m*) = *flip-bit n* (*integer-of-natural m*)›
by *transfer* (*simp add*: *of-nat-flip-bit-eq*)

lemma [*code abstract*]:
‹*integer-of-natural* (*push-bit n m*) = *push-bit n* (*integer-of-natural m*)›

by *transfer* (*simp add*: *of-nat-push-bit*)

lemma [*code abstract*]:
‹*integer-of-natural* (*drop-bit n m*) = *drop-bit n* (*integer-of-natural m*)›
by *transfer* (*simp add*: *of-nat-drop-bit*)

lemma [*code abstract*]:
‹*integer-of-natural* (*take-bit n m*) = *take-bit n* (*integer-of-natural m*)›
by *transfer* (*simp add*: *of-nat-take-bit*)

end

hide-const (**open**) *Nat*

code-reflect *Code-Numeral*
datatypes *natural*
functions *Code-Numeral.Suc 0* :: *natural 1* :: *natural*
plus :: *natural* ⇒ - *minus* :: *natural* ⇒ -
times :: *natural* ⇒ - *divide* :: *natural* ⇒ -
modulo :: *natural* ⇒ -
integer-of-natural natural-of-integer

lifting-update *integer.lifting*
lifting-forget *integer.lifting*

lifting-update *natural.lifting*
lifting-forget *natural.lifting*

end

70 A HOL random engine

theory *Random*
imports *List Groups-List Code-Numeral*
begin

70.1 Auxiliary functions

fun *log* :: *natural* ⇒ *natural* ⇒ *natural* **where**
log b i = (*if b* ≤ *1* ∨ *i* < *b then 1 else 1* + *log b* (*i div b*))

definition *inc-shift* :: *natural* ⇒ *natural* ⇒ *natural* **where**
inc-shift v k = (*if v* = *k then 1 else k* + *1*)

definition *minus-shift* :: *natural* ⇒ *natural* ⇒ *natural* ⇒ *natural* **where**
minus-shift r k l = (*if k* < *l then r* + *k* − *l else k* − *l*)

70.2 Random seeds

type-synonym *seed* = *natural* × *natural*

primrec *next* :: *seed* ⇒ *natural* × *seed* **where**
next (*v*, *w*) = (*let*
k = *v div 53668*;
v' = *minus-shift 2147483563* ((*v mod 53668*) ∗ *40014*) (*k* ∗ *12211*);
l = *w div 52774*;
w' = *minus-shift 2147483399* ((*w mod 52774*) ∗ *40692*) (*l* ∗ *3791*);
z = *minus-shift 2147483562 v'* (*w'* + *1*) + *1*
in (*z*, (*v'*, *w'*)))

definition *split-seed* :: *seed* ⇒ *seed* × *seed* **where**
split-seed s = (*let*
(*v*, *w*) = *s*;
(*v'*, *w'*) = *snd* (*next s*);
v'' = *inc-shift 2147483562 v*;
w'' = *inc-shift 2147483398 w*
in ((*v''*, *w'*), (*v'*, *w''*)))

70.3 Base selectors

context
includes *state-combinator-syntax*
begin

fun *iterate* :: *natural* ⇒ (*'b* ⇒ *'a* ⇒ *'b* × *'a*) ⇒ *'b* ⇒ *'a* ⇒ *'b* × *'a* **where**
iterate k f x = (*if k* = *0 then Pair x else f x* ∘→ *iterate* (*k* − *1*) *f*)

definition *range* :: *natural* ⇒ *seed* ⇒ *natural* × *seed* **where**
range k = *iterate* (*log 2147483561 k*)
(λ*l*. *next* ∘→ (λ*v*. *Pair* (*v* + *l* ∗ *2147483561*))) *1*
∘→ (λ*v*. *Pair* (*v mod k*))

lemma *range*:
k > *0* ⟹ *fst* (*range k s*) < *k*
by (*simp add*: *range-def split-def less-natural-def del*: *log.simps iterate.simps*)

definition *select* :: *'a list* ⇒ *seed* ⇒ *'a* × *seed* **where**
select xs = *range* (*natural-of-nat* (*length xs*))
∘→ (λ*k*. *Pair* (*nth xs* (*nat-of-natural k*)))

lemma *select*:
assumes *xs* ≠ []
shows *fst* (*select xs s*) ∈ *set xs*
proof −
from *assms* **have** *natural-of-nat* (*length xs*) > *0* **by** (*simp add*: *less-natural-def*)
with *range* **have**
fst (*range* (*natural-of-nat* (*length xs*)) *s*) < *natural-of-nat* (*length xs*) **by** *best*

then have
nat-of-natural (*fst* (*range* (*natural-of-nat* (*length xs*)) *s*)) < *length xs* **by** (*simp add*: *less-natural-def*)
then show *?thesis*
by (*simp add*: *split-beta select-def*)
qed

primrec *pick* :: (*natural* × *'a*) *list* ⇒ *natural* ⇒ *'a* **where**
pick (*x* # *xs*) *i* = (*if i* < *fst x then snd x else pick xs* (*i* − *fst x*))

lemma *pick-member*:
i < *sum-list* (*map fst xs*) ⟹ *pick xs i* ∈ *set* (*map snd xs*)
by (*induct xs arbitrary*: *i*) (*simp-all add*: *less-natural-def*)

lemma *pick-drop-zero*:
pick (*filter* (λ(*k*, -). *k* > *0*) *xs*) = *pick xs*
apply (*induct xs*)
apply (*auto simp*: *fun-eq-iff less-natural.rep-eq split*: *prod.split*)
by (*metis diff-zero of-nat-0 of-nat-of-natural*)

lemma *pick-same*:
l < *length xs* ⟹ *Random.pick* (*map* (*Pair 1*) *xs*) (*natural-of-nat l*) = *nth xs l*
proof (*induct xs arbitrary*: *l*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **then show** *?case* **by** (*cases l*) (*simp-all add*: *less-natural-def*)
qed

definition *select-weight* :: (*natural* × *'a*) *list* ⇒ *seed* ⇒ *'a* × *seed* **where**
select-weight xs = *range* (*sum-list* (*map fst xs*))
∘→ (λ*k*. *Pair* (*pick xs k*))

lemma *select-weight-member*:
assumes *0* < *sum-list* (*map fst xs*)
shows *fst* (*select-weight xs s*) ∈ *set* (*map snd xs*)
proof −
from *range assms*
have *fst* (*range* (*sum-list* (*map fst xs*)) *s*) < *sum-list* (*map fst xs*) **.**
with *pick-member*
have *pick xs* (*fst* (*range* (*sum-list* (*map fst xs*)) *s*)) ∈ *set* (*map snd xs*) **.**
then show *?thesis* **by** (*simp add*: *select-weight-def scomp-def split-def*)
qed

lemma *select-weight-cons-zero*:
select-weight ((*0*, *x*) # *xs*) = *select-weight xs*
by (*simp add*: *select-weight-def less-natural-def*)

lemma *select-weight-drop-zero*:
select-weight (*filter* (λ(*k*, -). *k* > *0*) *xs*) = *select-weight xs*

proof –
  have *sum-list* (*map fst* [(*k*, -)←*xs* . *0* < *k*]) = *sum-list* (*map fst xs*)
    by (*induct xs*) (*auto simp add*: *less-natural-def natural-eq-iff*)
  then show *?thesis* **by** (*simp only*: *select-weight-def pick-drop-zero*)
qed

lemma *select-weight-select*:
  assumes *xs* ≠ []
  shows *select-weight* (*map* (*Pair 1*) *xs*) = *select xs*
proof –
  have *less*: ⋀*s*. *fst* (*range* (*natural-of-nat* (*length xs*)) *s*) < *natural-of-nat* (*length xs*)
    using *assms* **by** (*intro range*) (*simp add*: *less-natural-def*)
  moreover have *sum-list* (*map fst* (*map* (*Pair 1*) *xs*)) = *natural-of-nat* (*length xs*)
    by (*induct xs*) *simp-all*
  ultimately show *?thesis*
    by (*auto simp add*: *select-weight-def select-def scomp-def split-def*
      fun-eq-iff pick-same [*symmetric*] *less-natural-def*)
qed

end

70.4 *ML* interface

code-reflect *Random-Engine*
  functions *range select select-weight*

ML ‹
structure Random-Engine =
struct

open Random-Engine;

type seed = *Code-Numeral.natural* ∗ *Code-Numeral.natural*;

local

val seed = *Unsynchronized.ref*
  (*let*
    val now = *Time.toMilliseconds* (*Time.now* ());
    val (*q*, *s1*) = *IntInf.divMod* (*now*, *2147483562*);
    val s2 = *q mod 2147483398*;
  in apply2 Code-Numeral.natural-of-integer (*s1* + *1*, *s2* + *1*) *end*);

in

fun next-seed () =
  let

```
    val (seed1, seed′) = @{code split-seed} (! seed)
    val - = seed := seed′
  in
    seed1
  end

fun run f =
  let
    val (x, seed′) = f (! seed);
    val - = seed := seed′
  in x end;

end;

end;
›
```

hide-type (**open**) *seed*
hide-const (**open**) *inc-shift minus-shift log next split-seed*
iterate range select pick select-weight
hide-fact (**open**) *range-def*

end

71 Maps

theory *Map*
imports *List*
abbrevs (= = $\subseteq_m$
begin

type-synonym ('a, 'b) *map* = 'a ⇒ 'b *option* (**infixr** ‹⇀› *0*)

abbreviation (*input*)
empty :: 'a ⇀ 'b **where**
empty ≡ λx. *None*

definition
map-comp :: ('b ⇀ 'c) ⇒ ('a ⇀ 'b) ⇒ ('a ⇀ 'c) (**infixl** ‹$\circ_m$› *55*) **where**
$f \circ_m g$ = (λk. *case g k of None* ⇒ *None* | *Some v* ⇒ *f v*)

definition
map-add :: ('a ⇀ 'b) ⇒ ('a ⇀ 'b) ⇒ ('a ⇀ 'b) (**infixl** ‹++› *100*) **where**
m1 ++ *m2* = (λx. *case m2 x of None* ⇒ *m1 x* | *Some y* ⇒ *Some y*)

definition
restrict-map :: ('a ⇀ 'b) ⇒ 'a *set* ⇒ ('a ⇀ 'b) (**infixl** ‹|`› *110*) **where**
m|`*A* = (λx. *if x* ∈ *A then m x else None*)

notation (*latex* **output**)
restrict-map (‹-↾-› [111,110] 110)

definition
dom :: ('a ⇀ 'b) ⇒ 'a set **where**
dom m = {a. m a ≠ None}

definition
ran :: ('a ⇀ 'b) ⇒ 'b set **where**
ran m = {b. ∃ a. m a = Some b}

definition
graph :: ('a ⇀ 'b) ⇒ ('a × 'b) set **where**
graph m = {(a, b) | a b. m a = Some b}

definition
map-le :: ('a ⇀ 'b) ⇒ ('a ⇀ 'b) ⇒ bool (**infix** ‹$\subseteq_m$› 50) **where**
($m_1 \subseteq_m m_2$) ⟷ (∀ a ∈ dom m_1. m_1 a = m_2 a)

Function update syntax $f(x := y, \ldots)$ is extended with $x \mapsto y$, which is short for $x :=$ *Some y*. := and ↦ can be mixed freely. The syntax $[x \mapsto y, \ldots]$ is short for *Map.empty*$(x \mapsto y, \ldots)$ but must only contain ↦, not :=, because $[x{:=}y]$ clashes with the list update syntax $xs[i{:=}x]$.

nonterminal *maplet* **and** *maplets*

open-bundle *maplet-syntax*
begin

syntax
-*maplet* :: ['a, 'a] ⇒ maplet (‹(‹open-block notation=‹mixfix maplet››- /↦/ -)›)
:: maplet ⇒ updbind (‹-›)
:: maplet ⇒ maplets (‹-›)
-*Maplets* :: [maplet, maplets] ⇒ maplets (‹-,/ -›)
-*Map* :: maplets ⇒ 'a ⇀ 'b (‹(‹indent=1 notation=‹mixfix map››[-])›)

syntax (*ASCII*)
-*maplet* :: ['a, 'a] ⇒ maplet (‹(‹open-block notation=‹mixfix maplet››- /|−>/ -)›)

syntax-consts
-*maplet* -*Maplets* -*Map* ⇌ *fun-upd*

translations
-*Update f* (-*maplet x y*) ⇌ *f*(*x* := *CONST Some y*)
-*Maplets m ms* ⇀ -*updbinds m ms*
-*Map ms* ⇀ -*Update* (*CONST empty*) *ms*

-*Map* (-*maplet* *x* *y*) ↽ -*Update* (*λu*. *CONST None*) (-*maplet* *x* *y*)
-*Map* (-*updbinds* *m* (-*maplet* *x* *y*)) ↽ -*Update* (-*Map* *m*) (-*maplet* *x* *y*)

end

Updating with lists:

primrec *map-of* :: ('*a* × '*b*) *list* ⇒ '*a* ⇀ '*b* **where**
map-of [] = *empty*
| *map-of* (*p* # *ps*) = (*map-of ps*)(*fst p* ↦ *snd p*)

lemma *map-of-Cons-code* [*code*]:
map-of [] *k* = *None*
map-of ((*l*, *v*) # *ps*) *k* = (*if l* = *k then Some v else map-of ps k*)
by *simp-all*

definition *map-upds* :: ('*a* ⇀ '*b*) ⇒ '*a list* ⇒ '*b list* ⇒ '*a* ⇀ '*b* **where**
map-upds m xs ys = *m* ++ *map-of* (*rev* (*zip xs ys*))

There is also the more specialized update syntax *xs* [↦] *ys* for lists *xs* and *ys*.

open-bundle *list-maplet-syntax*
begin

syntax
-*maplets* :: ['*a*, '*a*] ⇒ *maplet* (‹(‹*open-block notation*=‹*mixfix maplet*››- /[↦]/ -)›)

syntax (*ASCII*)
-*maplets* :: ['*a*, '*a*] ⇒ *maplet* (‹(‹*open-block notation*=‹*mixfix maplet*››- /[|−>]/ -)›)

syntax-consts
-*maplets* ⇌ *map-upds*

translations
-*Update* *m* (-*maplets xs ys*) ⇌ *CONST map-upds m xs ys*

-*Map* (-*maplets xs ys*) ↽ -*Update* (*λu*. *CONST None*) (-*maplets xs ys*)
-*Map* (-*updbinds* *m* (-*maplets xs ys*)) ↽ -*Update* (-*Map* *m*) (-*maplets xs ys*)

end

71.1 *empty*

lemma *empty-upd-none* [*simp*]: *empty*(*x* := *None*) = *empty*
by (*rule ext*) *simp*

71.2 *map-upd*

lemma *map-upd-triv*: *t k* = *Some x* ⟹ *t*(*k*↦*x*) = *t*

by (*rule ext*) *simp*

lemma *map-upd-nonempty* [*simp*]: $t(k{\mapsto}x) \neq empty$
proof
assume $t(k \mapsto x) = empty$
then have $(t(k \mapsto x))\ k = None$ **by** *simp*
then show *False* **by** *simp*
qed

lemma *map-upd-eqD1*:
assumes $m(a{\mapsto}x) = n(a{\mapsto}y)$
shows $x = y$
proof –
from *assms* **have** $(m(a{\mapsto}x))\ a = (n(a{\mapsto}y))\ a$ **by** *simp*
then show *?thesis* **by** *simp*
qed

lemma *map-upd-Some-unfold*:
$((m(a{\mapsto}b))\ x = Some\ y) = (x = a \wedge b = y \vee x \neq a \wedge m\ x = Some\ y)$
by *auto*

lemma *image-map-upd* [*simp*]: $x \notin A \Longrightarrow m(x \mapsto y)$ ‘ $A = m$ ‘ A
by *auto*

lemma *finite-range-updI*:
assumes *finite* (*range f*) **shows** *finite* (*range* $(f(a{\mapsto}b))$)
proof –
have *range* $(f(a{\mapsto}b)) \subseteq$ *insert* (*Some b*) (*range f*)
by *auto*
then show *?thesis*
by (*rule finite-subset*) (*use assms* **in** *auto*)
qed

71.3 *map-of*

lemma *map-of-eq-empty-iff* [*simp*]:
map-of xys = *empty* $\longleftrightarrow$ *xys* = []
proof
show *map-of xys* = *empty* $\Longrightarrow$ *xys* = []
by (*induction xys*) *simp-all*
qed *simp*

lemma *empty-eq-map-of-iff* [*simp*]:
empty = *map-of xys* $\longleftrightarrow$ *xys* = []
by(*subst eq-commute*) *simp*

lemma *map-of-eq-None-iff*:
(*map-of xys x* = *None*) = ($x \notin$ *fst* ‘ (*set xys*))
by (*induct xys*) *simp-all*

lemma *map-of-eq-Some-iff* [*simp*]:
distinct(*map fst xys*) $\Longrightarrow$ (*map-of xys x* = *Some y*) = ((*x*,*y*) $\in$ *set xys*)
proof (*induct xys*)
case (*Cons xy xys*)
then show *?case*
by (*cases xy*) (*auto simp flip*: *map-of-eq-None-iff*)
qed *auto*

lemma *Some-eq-map-of-iff* [*simp*]:
distinct(*map fst xys*) $\Longrightarrow$ (*Some y* = *map-of xys x*) = ((*x*,*y*) $\in$ *set xys*)
by (*auto simp del*: *map-of-eq-Some-iff simp*: *map-of-eq-Some-iff* [*symmetric*])

lemma *map-of-is-SomeI* [*simp*]:
$[\![$*distinct*(*map fst xys*); (*x*,*y*) $\in$ *set xys*$]\!]$ $\Longrightarrow$ *map-of xys x* = *Some y*
by *simp*

lemma *map-of-zip-is-None* [*simp*]:
length xs = *length ys* $\Longrightarrow$ (*map-of* (*zip xs ys*) *x* = *None*) = (*x* $\notin$ *set xs*)
by (*induct rule*: *list-induct2*) *simp-all*

lemma *map-of-zip-is-Some*:
assumes *length xs* = *length ys*
shows *x* $\in$ *set xs* $\longleftrightarrow$ ($\exists$ *y*. *map-of* (*zip xs ys*) *x* = *Some y*)
using *assms* **by** (*induct rule*: *list-induct2*) *simp-all*

lemma *map-of-zip-upd*:
fixes *x* :: *'a* **and** *xs* :: *'a list* **and** *ys zs* :: *'b list*
assumes *length ys* = *length xs*
and *length zs* = *length xs*
and *x* $\notin$ *set xs*
and (*map-of* (*zip xs ys*))(*x* $\mapsto$ *y*) = (*map-of* (*zip xs zs*))(*x* $\mapsto$ *z*)
shows *map-of* (*zip xs ys*) = *map-of* (*zip xs zs*)
proof
fix *x'* :: *'a*
show *map-of* (*zip xs ys*) *x'* = *map-of* (*zip xs zs*) *x'*
proof (*cases x* = *x'*)
case *True*
from *assms True map-of-zip-is-None* [*of xs ys x'*]
have *map-of* (*zip xs ys*) *x'* = *None* **by** *simp*
moreover from *assms True map-of-zip-is-None* [*of xs zs x'*]
have *map-of* (*zip xs zs*) *x'* = *None* **by** *simp*
ultimately show *?thesis* **by** *simp*
next
case *False* **from** *assms*
have ((*map-of* (*zip xs ys*))(*x* $\mapsto$ *y*)) *x'* = ((*map-of* (*zip xs zs*))(*x* $\mapsto$ *z*)) *x'* **by**
auto
with *False* **show** *?thesis* **by** *simp*
qed

qed

lemma *map-of-zip-inject*:
  assumes *length ys* = *length xs*
    and *length zs* = *length xs*
    and *dist*: *distinct xs*
    and *map-of*: *map-of* (*zip xs ys*) = *map-of* (*zip xs zs*)
  shows *ys* = *zs*
  using *assms*(*1*) *assms*(*2*)[*symmetric*]
  using *dist map-of*
proof (*induct ys xs zs rule*: *list-induct3*)
  case *Nil* **show** *?case* **by** *simp*
next
  case (*Cons y ys x xs z zs*)
  from ‹*map-of* (*zip* (*x#xs*) (*y#ys*)) = *map-of* (*zip* (*x#xs*) (*z#zs*))›
    have *map-of*: (*map-of* (*zip xs ys*))(*x* ↦ *y*) = (*map-of* (*zip xs zs*))(*x* ↦ *z*) **by** *simp*
  from *Cons* **have** *length ys* = *length xs* **and** *length zs* = *length xs*
    and *x* ∉ *set xs* **by** *simp-all*
  then have *map-of* (*zip xs ys*) = *map-of* (*zip xs zs*) **using** *map-of* **by** (*rule map-of-zip-upd*)
  with *Cons.hyps* ‹*distinct* (*x* # *xs*)› **have** *ys* = *zs* **by** *simp*
  moreover from *map-of* **have** *y* = *z* **by** (*rule map-upd-eqD1*)
  ultimately show *?case* **by** *simp*
qed

lemma *map-of-zip-nth*:
  assumes *length xs* = *length ys*
  assumes *distinct xs*
  assumes *i* < *length ys*
  shows *map-of* (*zip xs ys*) (*xs* ! *i*) = *Some* (*ys* ! *i*)
using *assms* **proof** (*induct arbitrary*: *i rule*: *list-induct2*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons x xs y ys*)
  then show *?case*
    using *less-Suc-eq-0-disj* **by** *auto*
qed

lemma *map-of-zip-map*:
  map-of (*zip xs* (*map f xs*)) = (λ*x*. *if x* ∈ *set xs then Some* (*f x*) *else None*)
  by (*induct xs*) (*simp-all add*: *fun-eq-iff*)

lemma *finite-range-map-of*: *finite* (*range* (*map-of xys*))
proof (*induct xys*)
  case (*Cons a xys*)
  then show *?case*
    using *finite-range-updI* **by** *fastforce*

qed *auto*

lemma *map-of-SomeD*: *map-of xs k* = *Some y* ⟹ (*k*, *y*) ∈ *set xs*
by (*induct xs*) (*auto split*: *if-splits*)

lemma *map-of-mapk-SomeI*:
inj f ⟹ *map-of t k* = *Some x* ⟹
map-of (*map* (*case-prod* (λ*k*. *Pair* (*f k*))) *t*) (*f k*) = *Some x*
by (*induct t*) (*auto simp*: *inj-eq*)

lemma *weak-map-of-SomeI*: (*k*, *x*) ∈ *set l* ⟹ ∃ *x*. *map-of l k* = *Some x*
by (*induct l*) *auto*

lemma *map-of-filter-in*:
map-of xs k = *Some z* ⟹ *P k z* ⟹ *map-of* (*filter* (*case-prod P*) *xs*) *k* = *Some z*
by (*induct xs*) *auto*

lemma *map-of-map*:
map-of (*map* (λ(*k*, *v*). (*k*, *f v*)) *xs*) = *map-option f* ∘ *map-of xs*
by (*induct xs*) (*auto simp*: *fun-eq-iff*)

lemma *dom-map-option*:
dom (λ*k*. *map-option* (*f k*) (*m k*)) = *dom m*
by (*simp add*: *dom-def*)

lemma *dom-map-option-comp* [*simp*]:
dom (*map-option g* ∘ *m*) = *dom m*
using *dom-map-option* [*of* λ-. *g m*] **by** (*simp add*: *comp-def*)

71.4 *map-option* related

lemma *map-option-o-empty* [*simp*]: *map-option f* ∘ *empty* = *empty*
by (*rule ext*) *simp*

lemma *map-option-o-map-upd* [*simp*]:
map-option f ∘ *m*(*a*↦*b*) = (*map-option f* ∘ *m*)(*a*↦*f b*)
by (*rule ext*) *simp*

71.5 *map-comp* related

lemma *map-comp-empty* [*simp*]:
$m \circ_m$ *empty* = *empty*
empty $\circ_m$ *m* = *empty*
by (*auto simp*: *map-comp-def split*: *option.splits*)

lemma *map-comp-simps* [*simp*]:
m2 k = *None* ⟹ (*m1* $\circ_m$ *m2*) *k* = *None*
m2 k = *Some k′* ⟹ (*m1* $\circ_m$ *m2*) *k* = *m1 k′*
by (*auto simp*: *map-comp-def*)

lemma *map-comp-Some-iff*:
$((m1 \circ_m m2)\ k = Some\ v) = (\exists k'.\ m2\ k = Some\ k' \wedge m1\ k' = Some\ v)$
by (*auto simp*: *map-comp-def split*: *option.splits*)

lemma *map-comp-None-iff*:
$((m1 \circ_m m2)\ k = None) = (m2\ k = None \vee (\exists k'.\ m2\ k = Some\ k' \wedge m1\ k' = None))$
by (*auto simp*: *map-comp-def split*: *option.splits*)

71.6 ++

lemma *map-add-empty*[*simp*]: m ++ $empty = m$
by(*simp add*: *map-add-def*)

lemma *empty-map-add*[*simp*]: $empty$ ++ $m = m$
by (*rule ext*) (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-assoc*[*simp*]: $m1$ ++ $(m2$ ++ $m3) = (m1$ ++ $m2)$ ++ $m3$
by (*rule ext*) (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-Some-iff*:
$((m$ ++ $n)\ k = Some\ x) = (n\ k = Some\ x \vee n\ k = None \wedge m\ k = Some\ x)$
by (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-SomeD* [*dest!*]:
$(m$ ++ $n)\ k = Some\ x \Longrightarrow n\ k = Some\ x \vee n\ k = None \wedge m\ k = Some\ x$
by (*rule map-add-Some-iff* [*THEN iffD1*])

lemma *map-add-find-right* [*simp*]: $n\ k = Some\ xx \Longrightarrow (m$ ++ $n)\ k = Some\ xx$
by (*subst map-add-Some-iff*) *fast*

lemma *map-add-None* [*iff*]: $((m$ ++ $n)\ k = None) = (n\ k = None \wedge m\ k = None)$
by (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-upd*[*simp*]: f ++ $g(x \mapsto y) = (f$ ++ $g)(x \mapsto y)$
by (*rule ext*) (*simp add*: *map-add-def*)

lemma *map-add-upds*[*simp*]: $m1$ ++ $(m2(xs[\mapsto]ys)) = (m1$++$m2)(xs[\mapsto]ys)$
by (*simp add*: *map-upds-def*)

lemma *map-add-upd-left*: $m \notin dom\ e2 \Longrightarrow e1(m \mapsto u1)$ ++ $e2 = (e1$ ++ $e2)(m \mapsto u1)$
by (*rule ext*) (*auto simp*: *map-add-def dom-def split*: *option.split*)

lemma *map-of-append*[*simp*]: *map-of* (*xs* @ *ys*) = *map-of ys* ++ *map-of xs*
unfolding *map-add-def*
proof (*induct xs*)
case (*Cons a xs*)

then show *?case*
by (*force split*: *option.split*)
qed *auto*

lemma *finite-range-map-of-map-add*:
finite (*range f*) $\Longrightarrow$ *finite* (*range* (*f* ++ *map-of l*))
proof (*induct l*)
case (*Cons a l*)
then show *?case*
by (*metis finite-range-updI map-add-upd map-of.simps*(*2*))
qed *auto*

lemma *inj-on-map-add-dom* [*iff*]:
inj-on (*m* ++ *m'*) (*dom m'*) = *inj-on m'* (*dom m'*)
by (*fastforce simp*: *map-add-def dom-def inj-on-def split*: *option.splits*)

lemma *map-upds-fold-map-upd*:
m(*ks*[$\mapsto$]*vs*) = *foldl* (λ*m* (*k*, *v*). *m*(*k* $\mapsto$ *v*)) *m* (*zip ks vs*)
unfolding *map-upds-def* **proof** (*rule sym*, *rule zip-obtain-same-length*)
fix *ks* :: *'a list* **and** *vs* :: *'b list*
assume *length ks* = *length vs*
then show *foldl* (λ*m* (*k*, *v*). *m*(*k*$\mapsto$*v*)) *m* (*zip ks vs*) = *m* ++ *map-of* (*rev* (*zip ks vs*))
by(*induct arbitrary*: *m rule*: *list-induct2*) *simp-all*
qed

lemma *map-add-map-of-foldr*:
m ++ *map-of ps* = *foldr* (λ(*k*, *v*) *m*. *m*(*k* $\mapsto$ *v*)) *ps m*
by (*induct ps*) (*auto simp*: *fun-eq-iff map-add-def*)

71.7 *restrict-map*

lemma *restrict-map-to-empty* [*simp*]: *m*|'{} = *empty*
by (*simp add*: *restrict-map-def*)

lemma *restrict-map-insert*: *f* |' (*insert a A*) = (*f* |' *A*)(*a* := *f a*)
by (*auto simp*: *restrict-map-def*)

lemma *restrict-map-empty* [*simp*]: *empty*|'*D* = *empty*
by (*simp add*: *restrict-map-def*)

lemma *restrict-in* [*simp*]: *x* $\in$ *A* $\Longrightarrow$ (*m*|'*A*) *x* = *m x*
by (*simp add*: *restrict-map-def*)

lemma *restrict-out* [*simp*]: *x* $\notin$ *A* $\Longrightarrow$ (*m*|'*A*) *x* = *None*
by (*simp add*: *restrict-map-def*)

lemma *ran-restrictD*: *y* $\in$ *ran* (*m*|'*A*) $\Longrightarrow$ $\exists$*x*$\in$*A*. *m x* = *Some y*
by (*auto simp*: *restrict-map-def ran-def split*: *if-split-asm*)

lemma *dom-restrict* [*simp*]: $dom\ (m|`A) = dom\ m \cap A$
by (*auto simp*: *restrict-map-def dom-def split*: *if-split-asm*)

lemma *restrict-upd-same* [*simp*]: $m(x\mapsto y)|`(-\{x\}) = m|`(-\{x\})$
by (*rule ext*) (*auto simp*: *restrict-map-def*)

lemma *restrict-restrict* [*simp*]: $m|`A|`B = m|`(A\cap B)$
by (*rule ext*) (*auto simp*: *restrict-map-def*)

lemma *restrict-fun-upd* [*simp*]:
$m(x := y)|`D = (if\ x \in D\ then\ (m|`(D-\{x\}))(x := y)\ else\ m|`D)$
by (*simp add*: *restrict-map-def fun-eq-iff*)

lemma *fun-upd-None-restrict* [*simp*]:
$(m|`D)(x := None) = (if\ x \in D\ then\ m|`(D - \{x\})\ else\ m|`D)$
by (*simp add*: *restrict-map-def fun-eq-iff*)

lemma *fun-upd-restrict*: $(m|`D)(x := y) = (m|`(D-\{x\}))(x := y)$
by (*simp add*: *restrict-map-def fun-eq-iff*)

lemma *fun-upd-restrict-conv* [*simp*]:
$x \in D \Longrightarrow (m|`D)(x := y) = (m|`(D-\{x\}))(x := y)$
by (*rule fun-upd-restrict*)

lemma *map-of-map-restrict*:
$map\text{-}of\ (map\ (\lambda k.\ (k, f\ k))\ ks) = (Some \circ f)\ |`\ set\ ks$
by (*induct ks*) (*simp-all add*: *fun-eq-iff restrict-map-insert*)

lemma *restrict-complement-singleton-eq*:
$f\ |`\ (-\ \{x\}) = f(x := None)$
by *auto*

71.8 *map-upds*

lemma *map-upds-Nil1* [*simp*]: $m([]\ [\mapsto]\ bs) = m$
by (*simp add*: *map-upds-def*)

lemma *map-upds-Nil2* [*simp*]: $m(as\ [\mapsto]\ []) = m$
by (*simp add*:*map-upds-def*)

lemma *map-upds-Cons* [*simp*]: $m(a\#as\ [\mapsto]\ b\#bs) = (m(a\mapsto b))(as[\mapsto]bs)$
by (*simp add*:*map-upds-def*)

lemma *map-upds-append1* [*simp*]:
$size\ xs < size\ ys \Longrightarrow m(xs@[x]\ [\mapsto]\ ys) = m(xs\ [\mapsto]\ ys,\ x \mapsto ys!size\ xs)$
proof (*induct xs arbitrary*: *ys m*)
case *Nil*
then show *?case*

by (*auto simp*: *neq-Nil-conv*)
next
case (*Cons a xs*)
then show *?case*
by (*cases ys*) *auto*
qed

lemma *map-upds-list-update2-drop* [*simp*]:
$size\ xs \leq i \Longrightarrow m(xs[\mapsto]ys[i:=y]) = m(xs[\mapsto]ys)$
proof (*induct xs arbitrary*: *m ys i*)
case *Nil*
then show *?case*
by *auto*
next
case (*Cons a xs*)
then show *?case*
by (*cases ys*) (*use Cons* **in** ‹*auto split*: *nat.split*›)
qed

Something weirdly sensitive about this proof, which needs only four lines in apply style

lemma *map-upd-upds-conv-if*:
$(f(x\mapsto y))(xs\ [\mapsto]\ ys) =$
$(if\ x \in set(take\ (length\ ys)\ xs)\ then\ f(xs\ [\mapsto]\ ys)$
$else\ (f(xs\ [\mapsto]\ ys))(x\mapsto y))$
proof (*induct xs arbitrary*: *x y ys f*)
case (*Cons a xs*)
show *?case*
proof (*cases ys*)
case (*Cons z zs*)
then show *?thesis*
using *Cons.hyps*
apply (*auto split*: *if-split simp*: *fun-upd-twist*)
using *Cons.hyps* **apply** *fastforce+*
done
qed *auto*
qed *auto*

lemma *map-upds-twist* [*simp*]:
$a \notin set\ as \Longrightarrow m(a\mapsto b,\ as[\mapsto]bs) = m(as[\mapsto]bs,\ a\mapsto b)$
using *set-take-subset* **by** (*fastforce simp add*: *map-upd-upds-conv-if*)

lemma *map-upds-apply-nontin* [*simp*]:
$x \notin set\ xs \Longrightarrow (f(xs[\mapsto]ys))\ x = f\ x$
proof (*induct xs arbitrary*: *ys*)
case (*Cons a xs*)
then show *?case*
by (*cases ys*) (*auto simp*: *map-upd-upds-conv-if*)

qed *auto*

lemma *fun-upds-append-drop* [*simp*]:
size xs = *size ys* $\Longrightarrow$ *m*(*xs*@*zs*[$\mapsto$]*ys*) = *m*(*xs*[$\mapsto$]*ys*)
proof (*induct xs arbitrary*: *ys*)
case (*Cons a xs*)
then show *?case*
by (*cases ys*) (*auto simp*: *map-upd-upds-conv-if*)
qed *auto*

lemma *fun-upds-append2-drop* [*simp*]:
size xs = *size ys* $\Longrightarrow$ *m*(*xs*[$\mapsto$]*ys*@*zs*) = *m*(*xs*[$\mapsto$]*ys*)
proof (*induct xs arbitrary*: *ys*)
case (*Cons a xs*)
then show *?case*
by (*cases ys*) (*auto simp*: *map-upd-upds-conv-if*)
qed *auto*

lemma *restrict-map-upds*[*simp*]:
$[\![$ *length xs* = *length ys*; *set xs* $\subseteq$ *D* $]\!]$
$\Longrightarrow$ *m*(*xs* [$\mapsto$] *ys*)|‘*D* = (*m*|‘(*D* − *set xs*))(*xs* [$\mapsto$] *ys*)
proof (*induct xs arbitrary*: *m ys*)
case (*Cons a xs*)
then show *?case*
proof (*cases ys*)
case (*Cons z zs*)
with *Cons.hyps Cons.prems* **show** *?thesis*
apply (*simp add*: *insert-absorb flip*: *Diff-insert*)
apply (*auto simp add*: *map-upd-upds-conv-if*)
done
qed *auto*
qed *auto*

71.9 *dom*

lemma *dom-eq-empty-conv* [*simp*]: *dom f* = {} $\longleftrightarrow$ *f* = *empty*
by (*auto simp*: *dom-def*)

lemma *domI*: *m a* = *Some b* $\Longrightarrow$ *a* $\in$ *dom m*
by (*simp add*: *dom-def*)

lemma *domD*: *a* $\in$ *dom m* $\Longrightarrow$ $\exists$ *b*. *m a* = *Some b*
by (*cases m a*) (*auto simp add*: *dom-def*)

lemma *domIff* [*iff*, *simp del*, *code-unfold*]: *a* $\in$ *dom m* $\longleftrightarrow$ *m a* $\neq$ *None*
by (*simp add*: *dom-def*)

lemma *dom-empty* [*simp*]: *dom empty* = {}

by (*simp add*: *dom-def*)

lemma *dom-fun-upd* [*simp*]:
dom(*f*(*x* := *y*)) = (*if y* = *None then dom f* − {*x*} *else insert x* (*dom f*))
by (*auto simp*: *dom-def*)

lemma *dom-if*:
dom (λ*x*. *if P x then f x else g x*) = *dom f* ∩ {*x*. *P x*} ∪ *dom g* ∩ {*x*. ¬ *P x*}
by (*auto split*: *if-splits*)

lemma *dom-map-of-conv-image-fst*:
dom (*map-of xys*) = *fst* ‘ *set xys*
by (*induct xys*) (*auto simp add*: *dom-if*)

lemma *dom-map-of-zip* [*simp*]: *length xs* = *length ys* ⟹ *dom* (*map-of* (*zip xs ys*)) = *set xs*
by (*induct rule*: *list-induct2*) (*auto simp*: *dom-if*)

lemma *finite-dom-map-of*: *finite* (*dom* (*map-of l*))
by (*induct l*) (*auto simp*: *dom-def insert-Collect* [*symmetric*])

lemma *dom-map-upds* [*simp*]:
dom(*m*(*xs*[↦]*ys*)) = *set*(*take* (*length ys*) *xs*) ∪ *dom m*
proof (*induct xs arbitrary*: *ys*)
case (*Cons a xs*)
then show *?case*
by (*cases ys*) (*auto simp*: *map-upd-upds-conv-if*)
qed *auto*

lemma *dom-map-add* [*simp*]: *dom* (*m* ++ *n*) = *dom n* ∪ *dom m*
by (*auto simp*: *dom-def*)

lemma *dom-override-on* [*simp*]:
dom (*override-on f g A*) =
(*dom f* − {*a*. *a* ∈ *A* − *dom g*}) ∪ {*a*. *a* ∈ *A* ∩ *dom g*}
by (*auto simp*: *dom-def override-on-def*)

lemma *map-add-comm*: *dom m1* ∩ *dom m2* = {} ⟹ *m1* ++ *m2* = *m2* ++ *m1*
by (*rule ext*) (*force simp*: *map-add-def dom-def split*: *option.split*)

lemma *map-add-dom-app-simps*:
m ∈ *dom l2* ⟹ (*l1* ++ *l2*) *m* = *l2 m*
m ∉ *dom l1* ⟹ (*l1* ++ *l2*) *m* = *l2 m*
m ∉ *dom l2* ⟹ (*l1* ++ *l2*) *m* = *l1 m*
by (*auto simp add*: *map-add-def split*: *option.split-asm*)

lemma *dom-const* [*simp*]:
dom (λ*x*. *Some* (*f x*)) = *UNIV*

by *auto*

lemma *finite-map-freshness*:
finite (*dom* (*f* :: '*a* ⇀ '*b*)) ⟹ ¬ *finite* (*UNIV* :: '*a set*) ⟹
∃ *x*. *f x* = *None*
by (*bestsimp dest*: *ex-new-if-finite*)

lemma *dom-minus*:
f x = *None* ⟹ *dom f* − *insert x A* = *dom f* − *A*
unfolding *dom-def* **by** *simp*

lemma *insert-dom*:
f x = *Some y* ⟹ *insert x* (*dom f*) = *dom f*
unfolding *dom-def* **by** *auto*

lemma *map-of-map-keys*:
set xs = *dom m* ⟹ *map-of* (*map* (λ*k*. (*k*, *the* (*m k*))) *xs*) = *m*
by (*rule ext*) (*auto simp add*: *map-of-map-restrict restrict-map-def*)

lemma *map-of-eqI*:
assumes *set-eq*: *set* (*map fst xs*) = *set* (*map fst ys*)
assumes *map-eq*: ∀ *k*∈*set* (*map fst xs*). *map-of xs k* = *map-of ys k*
shows *map-of xs* = *map-of ys*
proof (*rule ext*)
fix *k* **show** *map-of xs k* = *map-of ys k*
proof (*cases map-of xs k*)
case *None*
then have *k* ∉ *set* (*map fst xs*) **by** (*simp add*: *map-of-eq-None-iff*)
with *set-eq* **have** *k* ∉ *set* (*map fst ys*) **by** *simp*
then have *map-of ys k* = *None* **by** (*simp add*: *map-of-eq-None-iff*)
with *None* **show** *?thesis* **by** *simp*
next
case (*Some v*)
then have *k* ∈ *set* (*map fst xs*) **by** (*auto simp add*: *dom-map-of-conv-image-fst*
[*symmetric*])
with *map-eq* **show** *?thesis* **by** *auto*
qed
qed

lemma *map-of-eq-dom*:
assumes *map-of xs* = *map-of ys*
shows *fst* ‘ *set xs* = *fst* ‘ *set ys*
proof −
from *assms* **have** *dom* (*map-of xs*) = *dom* (*map-of ys*) **by** *simp*
then show *?thesis* **by** (*simp add*: *dom-map-of-conv-image-fst*)
qed

lemma *finite-set-of-finite-maps*:

assumes *finite A finite B*
shows *finite* $\{m.\ dom\ m = A \wedge ran\ m \subseteq B\}$ (**is** *finite ?S*)
proof −
let $?S' = \{m.\ \forall x.\ (x \in A \longrightarrow m\ x \in Some\ `\ B) \wedge (x \notin A \longrightarrow m\ x = None)\}$
have $?S = ?S'$
proof
show $?S \subseteq ?S'$ **by** (*auto simp*: *dom-def ran-def image-def*)
show $?S' \subseteq ?S$
proof
fix *m* **assume** $m \in ?S'$
hence *1*: $dom\ m = A$ **by** *force*
hence *2*: $ran\ m \subseteq B$ **using** ‹$m \in ?S'$› **by** (*auto simp*: *dom-def ran-def*)
from *1 2* **show** $m \in ?S$ **by** *blast*
qed
qed
with *assms* **show** *?thesis* **by**(*simp add*: *finite-set-of-finite-funs*)
qed

71.10 *ran*

lemma *ranI*: $m\ a = Some\ b \Longrightarrow b \in ran\ m$
by (*auto simp*: *ran-def*)

lemma *ran-empty* [*simp*]: $ran\ empty = \{\}$
by (*auto simp*: *ran-def*)

lemma *ran-map-upd* [*simp*]: $m\ a = None \Longrightarrow ran(m(a \mapsto b)) = insert\ b\ (ran\ m)$
unfolding *ran-def*
by *force*

lemma *fun-upd-None-if-notin-dom*[*simp*]: $k \notin dom\ m \Longrightarrow m(k := None) = m$
by *auto*

lemma *ran-map-upd-Some*:
$[\![\ m\ x = Some\ y;\ inj\text{-}on\ m\ (dom\ m);\ z \notin ran\ m\ ]\!] \Longrightarrow ran(m(x := Some\ z)) = ran\ m - \{y\} \cup \{z\}$
by(*force simp add*: *ran-def domI inj-onD*)

lemma *ran-map-add*:
assumes $dom\ m1 \cap dom\ m2 = \{\}$
shows $ran\ (m1 ++ m2) = ran\ m1 \cup ran\ m2$
proof
show $ran\ (m1 ++ m2) \subseteq ran\ m1 \cup ran\ m2$
unfolding *ran-def* **by** *auto*
next
show $ran\ m1 \cup ran\ m2 \subseteq ran\ (m1 ++ m2)$
proof −
have $(m1 ++ m2)\ x = Some\ y$ **if** $m1\ x = Some\ y$ **for** *x y*

using *assms map-add-comm that* **by** *fastforce*
moreover have (*m1* ++ *m2*) *x* = *Some y* **if** *m2 x* = *Some y* **for** *x y*
using *assms that* **by** *auto*
ultimately show *?thesis*
unfolding *ran-def* **by** *blast*
qed
qed

lemma *finite-ran*:
assumes *finite* (*dom p*)
shows *finite* (*ran p*)
proof −
have *ran p* = (λ*x*. *the* (*p x*)) ' *dom p*
unfolding *ran-def* **by** *force*
from *this* ‹*finite* (*dom p*)› **show** *?thesis* **by** *auto*
qed

lemma *ran-distinct*:
assumes *dist*: *distinct* (*map fst al*)
shows *ran* (*map-of al*) = *snd* ' *set al*
using *assms*
proof (*induct al*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons kv al*)
then have *ran* (*map-of al*) = *snd* ' *set al* **by** *simp*
moreover from *Cons.prems* **have** *map-of al* (*fst kv*) = *None*
by (*simp add*: *map-of-eq-None-iff*)
ultimately show *?case* **by** (*simp only*: *map-of.simps ran-map-upd*) *simp*
qed

lemma *ran-map-of-zip*:
assumes *length xs* = *length ys distinct xs*
shows *ran* (*map-of* (*zip xs ys*)) = *set ys*
using *assms* **by** (*simp add*: *ran-distinct set-map*[*symmetric*])

lemma *ran-map-option*: *ran* (λ*x*. *map-option f* (*m x*)) = *f* ' *ran m*
by (*auto simp add*: *ran-def*)

71.11 *graph*

lemma *graph-empty*[*simp*]: *graph empty* = {}
unfolding *graph-def* **by** *simp*

lemma *in-graphI*: *m k* = *Some v* ⟹ (*k*, *v*) ∈ *graph m*
unfolding *graph-def* **by** *blast*

lemma *in-graphD*: (*k*, *v*) ∈ *graph m* ⟹ *m k* = *Some v*

unfolding *graph-def* **by** *blast*

lemma *graph-map-upd*[*simp*]: *graph* (*m*(*k* ↦ *v*)) = *insert* (*k*, *v*) (*graph* (*m*(*k* := *None*)))
unfolding *graph-def* **by** (*auto split*: *if-splits*)

lemma *graph-fun-upd-None*: *graph* (*m*(*k* := *None*)) = {*e* ∈ *graph m*. *fst e* ≠ *k*}
unfolding *graph-def* **by** (*auto split*: *if-splits*)

lemma *graph-restrictD*:
assumes (*k*, *v*) ∈ *graph* (*m* |‘ *A*)
shows *k* ∈ *A* **and** *m k* = *Some v*
using *assms* **unfolding** *graph-def*
by (*auto simp*: *restrict-map-def split*: *if-splits*)

lemma *graph-map-comp*[*simp*]: *graph* (*m1* $\circ_m$ *m2*) = *graph m2 O graph m1*
unfolding *graph-def* **by** (*auto simp*: *map-comp-Some-iff relcomp-unfold*)

lemma *graph-map-add*: *dom m1* ∩ *dom m2* = {} ⟹ *graph* (*m1* ++ *m2*) = *graph m1* ∪ *graph m2*
unfolding *graph-def* **using** *map-add-comm* **by** *force*

lemma *graph-eq-to-snd-dom*: *graph m* = (λ*x*. (*x*, *the* (*m x*))) ‘ *dom m*
unfolding *graph-def dom-def* **by** *force*

lemma *fst-graph-eq-dom*: *fst* ‘ *graph m* = *dom m*
unfolding *graph-eq-to-snd-dom* **by** *force*

lemma *graph-domD*: *x* ∈ *graph m* ⟹ *fst x* ∈ *dom m*
using *fst-graph-eq-dom* **by** (*metis imageI*)

lemma *snd-graph-ran*: *snd* ‘ *graph m* = *ran m*
unfolding *graph-def ran-def* **by** *force*

lemma *graph-ranD*: *x* ∈ *graph m* ⟹ *snd x* ∈ *ran m*
using *snd-graph-ran* **by** (*metis imageI*)

lemma *finite-graph-map-of*: *finite* (*graph* (*map-of al*))
unfolding *graph-eq-to-snd-dom finite-dom-map-of*
using *finite-dom-map-of* **by** *blast*

lemma *graph-map-of-if-distinct-dom*: *distinct* (*map fst al*) ⟹ *graph* (*map-of al*) = *set al*
unfolding *graph-def* **by** *auto*

lemma *finite-graph-iff-finite-dom*[*simp*]: *finite* (*graph m*) = *finite* (*dom m*)
by (*metis graph-eq-to-snd-dom finite-imageI fst-graph-eq-dom*)

lemma *inj-on-fst-graph*: *inj-on fst* (*graph m*)

unfolding *graph-def inj-on-def* **by** *force*

71.12 *map-le*

lemma *map-le-empty* [*simp*]: *empty* $\subseteq_m$ *g*
by (*simp add*: *map-le-def*)

lemma *upd-None-map-le* [*simp*]: *f*(*x* := *None*) $\subseteq_m$ *f*
by (*force simp add*: *map-le-def*)

lemma *map-le-upd*[*simp*]: *f* $\subseteq_m$ *g* ==> *f*(*a* := *b*) $\subseteq_m$ *g*(*a* := *b*)
by (*fastforce simp add*: *map-le-def*)

lemma *map-le-imp-upd-le* [*simp*]: *m1* $\subseteq_m$ *m2* $\Longrightarrow$ *m1*(*x* := *None*) $\subseteq_m$ *m2*(*x* $\mapsto$ *y*)
by (*force simp add*: *map-le-def*)

lemma *map-le-upds* [*simp*]:
f $\subseteq_m$ *g* $\Longrightarrow$ *f*(*as* [$\mapsto$] *bs*) $\subseteq_m$ *g*(*as* [$\mapsto$] *bs*)
proof (*induct as arbitrary*: *f g bs*)
case (*Cons a as*)
then show *?case*
by (*cases bs*) (*use Cons* **in** *auto*)
qed *auto*

lemma *map-le-implies-dom-le*: (*f* $\subseteq_m$ *g*) $\Longrightarrow$ (*dom f* $\subseteq$ *dom g*)
by (*fastforce simp add*: *map-le-def dom-def*)

lemma *map-le-refl* [*simp*]: *f* $\subseteq_m$ *f*
by (*simp add*: *map-le-def*)

lemma *map-le-trans*[*trans*]: ⟦ *m1* $\subseteq_m$ *m2*; *m2* $\subseteq_m$ *m3*⟧ $\Longrightarrow$ *m1* $\subseteq_m$ *m3*
by (*auto simp add*: *map-le-def dom-def*)

lemma *map-le-antisym*: ⟦ *f* $\subseteq_m$ *g*; *g* $\subseteq_m$ *f* ⟧ $\Longrightarrow$ *f* = *g*
unfolding *map-le-def*
by (*metis ext domIff*)

lemma *map-le-map-add* [*simp*]: *f* $\subseteq_m$ *g* ++ *f*
by (*fastforce simp*: *map-le-def*)

lemma *map-le-iff-map-add-commute*: *f* $\subseteq_m$ *f* ++ *g* $\longleftrightarrow$ *f* ++ *g* = *g* ++ *f*
by (*fastforce simp*: *map-add-def map-le-def fun-eq-iff split*: *option.splits*)

lemma *map-add-le-mapE*: *f* ++ *g* $\subseteq_m$ *h* $\Longrightarrow$ *g* $\subseteq_m$ *h*
by (*fastforce simp*: *map-le-def map-add-def dom-def*)

lemma *map-add-le-mapI*: ⟦ *f* $\subseteq_m$ *h*; *g* $\subseteq_m$ *h* ⟧ $\Longrightarrow$ *f* ++ *g* $\subseteq_m$ *h*
by (*auto simp*: *map-le-def map-add-def dom-def split*: *option.splits*)

lemma *map-add-subsumed1*: $f \subseteq_m g \Longrightarrow f{+}{+}g = g$
by (*simp add*: *map-add-le-mapI map-le-antisym*)

lemma *map-add-subsumed2*: $f \subseteq_m g \Longrightarrow g{+}{+}f = g$
by (*metis map-add-subsumed1 map-le-iff-map-add-commute*)

lemma *dom-eq-singleton-conv*: *dom* $f = \{x\} \longleftrightarrow (\exists\, v.\ f = [x \mapsto v])$
  (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
  assume *?rhs*
  then show *?lhs* **by** (*auto split*: *if-split-asm*)
next
  assume *?lhs*
  then obtain v **where** v: $f\ x = Some\ v$ **by** *auto*
  show *?rhs*
  proof
    show $f = [x \mapsto v]$
    proof (*rule map-le-antisym*)
      show $[x \mapsto v] \subseteq_m f$
        using v **by** (*auto simp add*: *map-le-def*)
      show $f \subseteq_m [x \mapsto v]$
        using ‹*dom* $f = \{x\}$› ‹$f\ x = Some\ v$› **by** (*auto simp add*: *map-le-def*)
    qed
  qed
qed

lemma *map-add-eq-empty-iff* [*simp*]:
  $(f{+}{+}g = empty) \longleftrightarrow f = empty \wedge g = empty$
by (*metis map-add-None*)

lemma *empty-eq-map-add-iff* [*simp*]:
  $(empty = f{+}{+}g) \longleftrightarrow f = empty \wedge g = empty$
by(*subst map-add-eq-empty-iff* [*symmetric*])(*rule eq-commute*)

71.13 Various

lemma *set-map-of-compr*:
  assumes *distinct*: *distinct* (*map fst xs*)
  shows *set xs* $= \{(k, v).\ map\text{-}of\ xs\ k = Some\ v\}$
  using *assms*
proof (*induct xs*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons x xs*)
  obtain $k\ v$ **where** $x = (k, v)$ **by** (*cases x*) *blast*
  with *Cons.prems* **have** $k \notin dom\ (map\text{-}of\ xs)$
    by (*simp add*: *dom-map-of-conv-image-fst*)

then have ∗: *insert* (*k*, *v*) {(*k*, *v*). *map-of xs k* = *Some v*} =
{(*k*′, *v*′). ((*map-of xs*)(*k* ↦ *v*)) *k*′ = *Some v*′}
by (*auto split*: *if-splits*)
from *Cons* **have** *set xs* = {(*k*, *v*). *map-of xs k* = *Some v*} **by** *simp*
with ∗ ‹*x* = (*k*, *v*)› **show** *?case* **by** *simp*
qed

lemma *eq-key-imp-eq-value*:
v1 = *v2*
if *distinct* (*map fst xs*) (*k*, *v1*) ∈ *set xs* (*k*, *v2*) ∈ *set xs*
proof −
from *that* **have** *inj-on fst* (*set xs*)
by (*simp add*: *distinct-map*)
moreover have *fst* (*k*, *v1*) = *fst* (*k*, *v2*)
by *simp*
ultimately have (*k*, *v1*) = (*k*, *v2*)
by (*rule inj-onD*) (*fact that*)+
then show *?thesis*
by *simp*
qed

lemma *map-of-inject-set*:
assumes *distinct*: *distinct* (*map fst xs*) *distinct* (*map fst ys*)
shows *map-of xs* = *map-of ys* ⟷ *set xs* = *set ys* (**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
moreover from ‹*distinct* (*map fst xs*)› **have** *set xs* = {(*k*, *v*). *map-of xs k* = *Some v*}
by (*rule set-map-of-compr*)
moreover from ‹*distinct* (*map fst ys*)› **have** *set ys* = {(*k*, *v*). *map-of ys k* = *Some v*}
by (*rule set-map-of-compr*)
ultimately show *?rhs* **by** *simp*
next
assume *?rhs* **show** *?lhs*
proof
fix *k*
show *map-of xs k* = *map-of ys k*
proof (*cases map-of xs k*)
case *None*
with ‹*?rhs*› **have** *map-of ys k* = *None*
by (*simp add*: *map-of-eq-None-iff*)
with *None* **show** *?thesis* **by** *simp*
next
case (*Some v*)
with *distinct* ‹*?rhs*› **have** *map-of ys k* = *Some v*
by *simp*
with *Some* **show** *?thesis* **by** *simp*
qed

qed
qed

lemma *finite-Map-induct*[*consumes 1*, *case-names empty update*]:
assumes *finite* (*dom m*)
assumes *P Map.empty*
assumes $\bigwedge k\ v\ m.\ finite\ (dom\ m) \Longrightarrow k \notin dom\ m \Longrightarrow P\ m \Longrightarrow P\ (m(k \mapsto v))$
shows *P m*
using *assms*(*1*)
proof(*induction dom m arbitrary*: *m rule*: *finite-induct*)
case *empty*
then show *?case* **using** *assms*(*2*) **unfolding** *dom-def* **by** *simp*
next
case (*insert x F*)
then have *finite* (*dom* (*m*(*x*:=*None*))) $x \notin$ *dom* (*m*(*x*:=*None*)) *P* (*m*(*x*:=*None*))
by (*metis Diff-insert-absorb dom-fun-upd*)+
with *assms*(*3*)[*OF this*] **show** *?case*
by (*metis fun-upd-triv fun-upd-upd option.exhaust*)
qed

hide-const (**open**) *Map.empty Map.graph*

end

72 Finite types as explicit enumerations

theory *Enum*
imports *Map Groups-List*
begin

72.1 Class *enum*

class *enum* =
fixes *enum* :: *'a list*
fixes *enum-all* :: (*'a* ⇒ *bool*) ⇒ *bool*
fixes *enum-ex* :: (*'a* ⇒ *bool*) ⇒ *bool*
assumes *UNIV-enum*: *UNIV* = *set enum*
and *enum-distinct*: *distinct enum*
assumes *enum-all-UNIV*: *enum-all P* ⟷ *Ball UNIV P*
assumes *enum-ex-UNIV*: *enum-ex P* ⟷ *Bex UNIV P*
— tailored towards simple instantiation
begin

subclass *finite* **proof**
qed (*simp add*: *UNIV-enum*)

lemma *enum-UNIV*:
set enum = *UNIV*
by (*simp only*: *UNIV-enum*)

lemma *in-enum*: $x \in set\ enum$
  by (*simp add*: *enum-UNIV*)

lemma *enum-eq-I*:
  assumes $\bigwedge x.\ x \in set\ xs$
  shows *set enum* = *set xs*
proof −
  from *assms UNIV-eq-I* **have** *UNIV* = *set xs* **by** *auto*
  with *enum-UNIV* **show** *?thesis* **by** *simp*
qed

lemma *card-UNIV-length-enum*:
  card (*UNIV* :: *'a set*) = *length enum*
  by (*simp add*: *UNIV-enum distinct-card enum-distinct*)

lemma *enum-all* [*simp*]:
  enum-all = *HOL.All*
  by (*simp add*: *fun-eq-iff enum-all-UNIV*)

lemma *enum-ex* [*simp*]:
  enum-ex = *HOL.Ex*
  by (*simp add*: *fun-eq-iff enum-ex-UNIV*)

end

72.2 Implementations using *enum*

72.2.1 Unbounded operations and quantifiers

lemma *Collect-code* [*code*]:
  Collect P = *set* (*filter P enum*)
  by (*simp add*: *enum-UNIV*)

lemma *vimage-code* [*code*]:
  f −' *B* = *set* (*filter* ($\lambda x.\ f\ x \in B$) *enum-class.enum*)
  unfolding *vimage-def Collect-code* **..**

definition *card-UNIV* :: *'a itself* ⇒ *nat*
where
  [*code del*]: *card-UNIV TYPE*(*'a*) = *card* (*UNIV* :: *'a set*)

lemma [*code*]:
  card-UNIV TYPE(*'a* :: *enum*) = *card* (*set* (*Enum.enum* :: *'a list*))
  by (*simp only*: *card-UNIV-def enum-UNIV*)

lemma *all-code* [*code*]: ($\forall x.\ P\ x$) ⟷ *enum-all P*
  by *simp*

lemma *exists-code* [*code*]: ($\exists x.\ P\ x$) ⟷ *enum-ex P*

by *simp*

lemma *exists1-code* [*code*]: (∃!*x*. *P x*) ⟷ *list-ex1 P enum*
by (*auto simp add*: *list-ex1-iff enum-UNIV*)

72.2.2 An executable choice operator

definition
[*code del*]: *enum-the* = *The*

lemma [*code*]:
The P = (*case filter P enum of* [*x*] ⇒ *x* | - ⇒ *enum-the P*)
proof −
{
fix *a*
assume *filter-enum*: *filter P enum* = [*a*]
have *The P* = *a*
proof (*rule the-equality*)
fix *x*
assume *P x*
show *x* = *a*
proof (*rule ccontr*)
assume *x* ≠ *a*
from *filter-enum* **obtain** *us vs*
where *enum-eq*: *enum* = *us* @ [*a*] @ *vs*
and ∀ *x* ∈ *set us*. ¬ *P x*
and ∀ *x* ∈ *set vs*. ¬ *P x*
and *P a*
by (*auto simp add*: *filter-eq-Cons-iff*) (*simp only*: *filter-empty-conv*[*symmetric*])
with ‹*P x*› *in-enum*[*of x*, *unfolded enum-eq*] ‹*x* ≠ *a*› **show** *False* **by** *auto*
qed
next
from *filter-enum* **show** *P a* **by** (*auto simp add*: *filter-eq-Cons-iff*)
qed
}
from *this* **show** *?thesis*
unfolding *enum-the-def* **by** (*auto split*: *list.split*)
qed

declare [[*code abort*: *enum-the*]]

code-printing
constant *enum-the* ⇀ (*Eval*) (*fn* ′- => *raise Match*)

72.2.3 Equality and order on functions

instantiation *fun* :: (*enum*, *equal*) *equal*
begin

definition

HOL.equal f g $\longleftrightarrow$ ($\forall$ *x* $\in$ *set enum*. *f x* = *g x*)

instance proof
qed (*simp-all add*: *equal-fun-def fun-eq-iff enum-UNIV*)

end

lemma [*code*]:
HOL.equal f g $\longleftrightarrow$ *enum-all* (%*x*. *f x* = *g x*)
by (*auto simp add*: *equal fun-eq-iff*)

lemma [*code nbe*]:
HOL.equal (*f* :: - $\Rightarrow$ -) *f* $\longleftrightarrow$ *True*
by (*fact equal-refl*)

lemma *order-fun* [*code*]:
fixes *f g* :: *'a*::*enum* $\Rightarrow$ *'b*::*order*
shows *f* $\leq$ *g* $\longleftrightarrow$ *enum-all* (λ*x*. *f x* $\leq$ *g x*)
and *f* < *g* $\longleftrightarrow$ *f* $\leq$ *g* $\wedge$ *enum-ex* (λ*x*. *f x* $\neq$ *g x*)
by (*simp-all add*: *fun-eq-iff le-fun-def order-less-le*)

72.2.4 Operations on relations

lemma [*code*]:
Id = *image* (λ*x*. (*x*, *x*)) (*set Enum.enum*)
by (*auto intro*: *imageI in-enum*)

lemma *tranclp-unfold* [*code*]:
tranclp r a b $\longleftrightarrow$ (*a*, *b*) $\in$ *trancl* {(*x*, *y*). *r x y*}
by (*simp add*: *trancl-def*)

lemma *rtranclp-rtrancl-eq* [*code*]:
rtranclp r x y $\longleftrightarrow$ (*x*, *y*) $\in$ *rtrancl* {(*x*, *y*). *r x y*}
by (*simp add*: *rtrancl-def*)

lemma *max-ext-eq* [*code*]:
max-ext R = {(*X*, *Y*). *finite X* $\wedge$ *finite Y* $\wedge$ *Y* $\neq$ {} $\wedge$ ($\forall$ *x*. *x* $\in$ *X* $\longrightarrow$ ($\exists$ *xa* $\in$ *Y*. (*x*, *xa*) $\in$ *R*))}
by (*auto simp add*: *max-ext.simps*)

lemma *max-extp-eq* [*code*]:
max-extp r x y $\longleftrightarrow$ (*x*, *y*) $\in$ *max-ext* {(*x*, *y*). *r x y*}
by (*simp add*: *max-ext-def*)

lemma *mlex-eq* [*code*]:
f <*mlex*> *R* = {(*x*, *y*). *f x* < *f y* $\vee$ (*f x* $\leq$ *f y* $\wedge$ (*x*, *y*) $\in$ *R*)}
by (*auto simp add*: *mlex-prod-def*)

72.2.5 Bounded accessible part

primrec *bacc* :: ('*a* × '*a*) *set* ⇒ *nat* ⇒ '*a set*
where
  bacc r 0 = {*x*. ∀ *y*. (*y*, *x*) ∉ *r*}
| *bacc r* (*Suc n*) = (*bacc r n* ∪ {*x*. ∀ *y*. (*y*, *x*) ∈ *r* ⟶ *y* ∈ *bacc r n*})

lemma *bacc-subseteq-acc*:
  bacc r n ⊆ *Wellfounded.acc r*
  by (*induct n*) (*auto intro*: *acc.intros*)

lemma *bacc-mono*:
  n ≤ *m* ⟹ *bacc r n* ⊆ *bacc r m*
  by (*induct rule*: *dec-induct*) *auto*

lemma *bacc-upper-bound*:
  bacc (*r* :: ('*a* × '*a*) *set*) (*card* (*UNIV* :: '*a*::*finite set*)) = (⋃ *n*. *bacc r n*)
proof −
  have *mono* (*bacc r*) **unfolding** *mono-def* **by** (*simp add*: *bacc-mono*)
  moreover have ∀ *n*. *bacc r n* = *bacc r* (*Suc n*) ⟶ *bacc r* (*Suc n*) = *bacc r* (*Suc* (*Suc n*)) **by** *auto*
  moreover have *finite* (*range* (*bacc r*)) **by** *auto*
  ultimately show *?thesis*
   by (*intro finite-mono-strict-prefix-implies-finite-fixpoint*)
     (*auto intro*: *finite-mono-remains-stable-implies-strict-prefix*)
qed

lemma *acc-subseteq-bacc*:
  assumes *finite r*
  shows *Wellfounded.acc r* ⊆ (⋃ *n*. *bacc r n*)
proof
  fix *x*
  assume *x* ∈ *Wellfounded.acc r*
  then have ∃ *n*. *x* ∈ *bacc r n*
  proof (*induct x arbitrary*: *rule*: *acc.induct*)
    case (*accI x*)
    then have ∀ *y*. ∃ *n*. (*y*, *x*) ∈ *r* ⟶ *y* ∈ *bacc r n* **by** *simp*
    from *choice*[*OF this*] **obtain** *n* **where** *n*: ∀ *y*. (*y*, *x*) ∈ *r* ⟶ *y* ∈ *bacc r* (*n y*) **..**
    obtain *n* **where** ⋀*y*. (*y*, *x*) ∈ *r* ⟹ *y* ∈ *bacc r n*
    proof
      fix *y* **assume** *y*: (*y*, *x*) ∈ *r*
      with *n* **have** *y* ∈ *bacc r* (*n y*) **by** *auto*
      moreover have *n y* <= *Max* ((λ(*y*, *x*). *n y*) ' *r*)
        using *y* ‹*finite r*› **by** (*auto intro*!: *Max-ge*)
      note *bacc-mono*[*OF this*, *of r*]
      ultimately show *y* ∈ *bacc r* (*Max* ((λ(*y*, *x*). *n y*) ' *r*)) **by** *auto*
    qed
    then show *?case*
      by (*auto simp add*: *Let-def intro*!: *exI*[*of - Suc n*])

qed
then show $x \in (\bigcup n.\ bacc\ r\ n)$ **by** *auto*
qed

lemma *acc-bacc-eq*:
fixes $A :: ('a :: finite \times 'a)\ set$
assumes *finite A*
shows *Wellfounded.acc A* = *bacc A* (*card* (*UNIV* :: *'a set*))
using *assms* **by** (*metis acc-subseteq-bacc bacc-subseteq-acc bacc-upper-bound order-eq-iff*)

lemma [*code*]:
fixes $xs :: ('a::finite \times 'a)\ list$
shows *Wellfounded.acc* (*set xs*) = *bacc* (*set xs*) (*card-UNIV TYPE*(*'a*))
by (*simp add*: *card-UNIV-def acc-bacc-eq*)

72.3 Default instances for *enum*

lemma *map-of-zip-enum-is-Some*:
assumes *length ys* = *length* (*enum* :: *'a::enum list*)
shows ∃ *y. map-of* (*zip* (*enum* :: *'a::enum list*) *ys*) *x* = *Some y*
proof −
from *assms* **have** *x* ∈ *set* (*enum* :: *'a::enum list*) ⟷
(∃ *y. map-of* (*zip* (*enum* :: *'a::enum list*) *ys*) *x* = *Some y*)
by (*auto intro*!: *map-of-zip-is-Some*)
then show *?thesis* **using** *enum-UNIV* **by** *auto*
qed

lemma *map-of-zip-enum-inject*:
fixes *xs ys* :: *'b::enum list*
assumes *length*: *length xs* = *length* (*enum* :: *'a::enum list*)
length ys = *length* (*enum* :: *'a::enum list*)
and *map-of*: *the* ∘ *map-of* (*zip* (*enum* :: *'a::enum list*) *xs*) = *the* ∘ *map-of* (*zip* (*enum* :: *'a::enum list*) *ys*)
shows *xs* = *ys*
proof −
have *map-of* (*zip* (*enum* :: *'a list*) *xs*) = *map-of* (*zip* (*enum* :: *'a list*) *ys*)
proof
fix *x* :: *'a*
from *length map-of-zip-enum-is-Some* **obtain** *y1 y2*
where *map-of* (*zip* (*enum* :: *'a list*) *xs*) *x* = *Some y1*
and *map-of* (*zip* (*enum* :: *'a list*) *ys*) *x* = *Some y2* **by** *blast*
moreover from *map-of*
have *the* (*map-of* (*zip* (*enum* :: *'a::enum list*) *xs*) *x*) = *the* (*map-of* (*zip* (*enum* :: *'a::enum list*) *ys*) *x*)
by (*auto dest*: *fun-cong*)
ultimately show *map-of* (*zip* (*enum* :: *'a::enum list*) *xs*) *x* = *map-of* (*zip* (*enum* :: *'a::enum list*) *ys*) *x*
by *simp*

qed
with *length enum-distinct* **show** *xs = ys* **by** (*rule map-of-zip-inject*)
qed

definition *all-n-lists* :: ((*'a* :: *enum*) *list* ⇒ *bool*) ⇒ *nat* ⇒ *bool*
where
all-n-lists P n ⟷ (∀ *xs* ∈ *set* (*List.n-lists n enum*). *P xs*)

lemma [*code*]:
all-n-lists P n ⟷ (*if n* = *0 then P* [] *else enum-all* (%*x*. *all-n-lists* (%*xs*. *P* (*x* # *xs*)) (*n* − *1*)))
unfolding *all-n-lists-def enum-all*
by (*cases n*) (*auto simp add*: *enum-UNIV*)

definition *ex-n-lists* :: ((*'a* :: *enum*) *list* ⇒ *bool*) ⇒ *nat* ⇒ *bool*
where
ex-n-lists P n ⟷ (∃ *xs* ∈ *set* (*List.n-lists n enum*). *P xs*)

lemma [*code*]:
ex-n-lists P n ⟷ (*if n* = *0 then P* [] *else enum-ex* (%*x*. *ex-n-lists* (%*xs*. *P* (*x* # *xs*)) (*n* − *1*)))
unfolding *ex-n-lists-def enum-ex*
by (*cases n*) (*auto simp add*: *enum-UNIV*)

instantiation *fun* :: (*enum*, *enum*) *enum*
begin

definition
enum = *map* (λ*ys*. *the* ∘ *map-of* (*zip* (*enum*::*'a list*) *ys*)) (*List.n-lists* (*length* (*enum*::*'a*::*enum list*)) *enum*)

definition
enum-all P = *all-n-lists* (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum bs*))) (*length* (*enum* :: *'a list*))

definition
enum-ex P = *ex-n-lists* (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum bs*))) (*length* (*enum* :: *'a list*))

instance proof
show *UNIV* = *set* (*enum* :: (*'a* ⇒ *'b*) *list*)
proof (*rule UNIV-eq-I*)
fix *f* :: *'a* ⇒ *'b*
have *f* = *the* ∘ *map-of* (*zip* (*enum* :: *'a*::*enum list*) (*map f enum*))
by (*auto simp add*: *map-of-zip-map fun-eq-iff intro*: *in-enum*)
then show *f* ∈ *set enum*
by (*auto simp add*: *enum-fun-def set-n-lists intro*: *in-enum*)
qed
next

```
  from map-of-zip-enum-inject
  show distinct (enum :: ('a ⇒ 'b) list)
    by (auto intro!: inj-onI simp add: enum-fun-def
      distinct-map distinct-n-lists enum-distinct set-n-lists)
next
  fix P
  show enum-all (P :: ('a ⇒ 'b) ⇒ bool) = Ball UNIV P
  proof
    assume enum-all P
    show Ball UNIV P
    proof
      fix f :: 'a ⇒ 'b
      have f: f = the ∘ map-of (zip (enum :: 'a::enum list) (map f enum))
        by (auto simp add: map-of-zip-map fun-eq-iff intro: in-enum)
      from ‹enum-all P› have P (the ∘ map-of (zip enum (map f enum)))
        unfolding enum-all-fun-def all-n-lists-def
        apply (simp add: set-n-lists)
        apply (erule-tac x=map f enum in allE)
        apply (auto intro!: in-enum)
        done
      from this f show P f by auto
    qed
  next
    assume Ball UNIV P
    from this show enum-all P
      unfolding enum-all-fun-def all-n-lists-def by auto
  qed
next
  fix P
  show enum-ex (P :: ('a ⇒ 'b) ⇒ bool) = Bex UNIV P
  proof
    assume enum-ex P
    from this show Bex UNIV P
      unfolding enum-ex-fun-def ex-n-lists-def by auto
  next
    assume Bex UNIV P
    from this obtain f where P f ..
    have f: f = the ∘ map-of (zip (enum :: 'a::enum list) (map f enum))
      by (auto simp add: map-of-zip-map fun-eq-iff intro: in-enum)
     from ‹P f› this have P (the ∘ map-of (zip (enum :: 'a::enum list) (map f
enum)))
      by auto
    from  this show enum-ex P
      unfolding enum-ex-fun-def ex-n-lists-def
      apply (auto simp add: set-n-lists)
      apply (rule-tac x=map f enum in exI)
      apply (auto intro!: in-enum)
      done
  qed
```

qed

end

lemma *enum-fun-code* [*code*]: *enum* = (*let enum-a* = (*enum* :: 'a::{*enum*, *equal*} *list*)
in map (λ*ys*. *the* ∘ *map-of* (*zip enum-a ys*)) (*List.n-lists* (*length enum-a*) *enum*))
by (*simp add*: *enum-fun-def Let-def*)

lemma *enum-all-fun-code* [*code*]:
enum-all P = (*let enum-a* = (*enum* :: 'a::{*enum*, *equal*} *list*)
in all-n-lists (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum-a bs*))) (*length enum-a*))
by (*simp only*: *enum-all-fun-def Let-def*)

lemma *enum-ex-fun-code* [*code*]:
enum-ex P = (*let enum-a* = (*enum* :: 'a::{*enum*, *equal*} *list*)
in ex-n-lists (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum-a bs*))) (*length enum-a*))
by (*simp only*: *enum-ex-fun-def Let-def*)

instantiation *set* :: (*enum*) *enum*
begin

definition
enum = *map set* (*subseqs enum*)

definition
enum-all P ⟷ (∀ *A*∈*set enum*. *P* (*A*::'a *set*))

definition
enum-ex P ⟷ (∃ *A*∈*set enum*. *P* (*A*::'a *set*))

instance proof
qed (*simp-all add*: *enum-set-def enum-all-set-def enum-ex-set-def subseqs-powset distinct-set-subseqs*
enum-distinct enum-UNIV)

end

instantiation *unit* :: *enum*
begin

definition
enum = [()]

definition
enum-all P = *P* ()

definition
enum-ex P = *P* ()

instance proof
qed (*auto simp add*: *enum-unit-def enum-all-unit-def enum-ex-unit-def*)

end

instantiation *bool* :: *enum*
begin

definition
enum = [*False*, *True*]

definition
enum-all P ⟷ *P False* ∧ *P True*

definition
enum-ex P ⟷ *P False* ∨ *P True*

instance proof
qed (*simp-all only*: *enum-bool-def enum-all-bool-def enum-ex-bool-def UNIV-bool*, *simp-all*)

end

instantiation *prod* :: (*enum*, *enum*) *enum*
begin

definition
enum = *List.product enum enum*

definition
enum-all P = *enum-all* (%*x*. *enum-all* (%*y*. *P* (*x*, *y*)))

definition
enum-ex P = *enum-ex* (%*x*. *enum-ex* (%*y*. *P* (*x*, *y*)))

instance
by *standard*
(*simp-all add*: *enum-prod-def distinct-product*
enum-UNIV enum-distinct enum-all-prod-def enum-ex-prod-def)

end

instantiation *sum* :: (*enum*, *enum*) *enum*
begin

definition
enum = *map Inl enum* @ *map Inr enum*

definition
enum-all P $\longleftrightarrow$ *enum-all* (λx. *P* (*Inl x*)) $\wedge$ *enum-all* (λx. *P* (*Inr x*))

definition
enum-ex P $\longleftrightarrow$ *enum-ex* (λx. *P* (*Inl x*)) $\vee$ *enum-ex* (λx. *P* (*Inr x*))

instance proof
qed (*simp-all only*: *enum-sum-def enum-all-sum-def enum-ex-sum-def UNIV-sum*,
auto simp add: *enum-UNIV distinct-map enum-distinct*)

end

instantiation *option* :: (*enum*) *enum*
begin

definition
enum = *None* # *map Some enum*

definition
enum-all P $\longleftrightarrow$ *P None* $\wedge$ *enum-all* (λx. *P* (*Some x*))

definition
enum-ex P $\longleftrightarrow$ *P None* $\vee$ *enum-ex* (λx. *P* (*Some x*))

instance proof
qed (*simp-all only*: *enum-option-def enum-all-option-def enum-ex-option-def UNIV-option-conv*,
auto simp add: *distinct-map enum-UNIV enum-distinct*)

end

72.4 Small finite types

We define small finite types for use in Quickcheck

datatype (*plugins only*: *code quickcheck extraction*) *finite-1* =
a_1

notation (**output**) a_1 (‹a_1›)

lemma *UNIV-finite-1*:
UNIV = $\{a_1\}$
by (*auto intro*: *finite-1.exhaust*)

instantiation *finite-1* :: *enum*
begin

definition
enum = $[a_1]$

definition
enum-all $P = P\ a_1$

definition
enum-ex $P = P\ a_1$

instance proof
qed (*simp-all only*: *enum-finite-1-def enum-all-finite-1-def enum-ex-finite-1-def UNIV-finite-1*, *simp-all*)

end

instantiation *finite-1* :: *linorder*
begin

definition *less-finite-1* :: *finite-1* ⇒ *finite-1* ⇒ *bool*
where
$x < (y :: \mathit{finite\text{-}1}) \longleftrightarrow \mathit{False}$

definition *less-eq-finite-1* :: *finite-1* ⇒ *finite-1* ⇒ *bool*
where
$x \leq (y :: \mathit{finite\text{-}1}) \longleftrightarrow \mathit{True}$

instance
apply (*intro-classes*)
apply (*auto simp add*: *less-finite-1-def less-eq-finite-1-def*)
apply (*metis* (*full-types*) *finite-1.exhaust*)
done

end

instance *finite-1* :: {*dense-linorder*, *wellorder*}
by *intro-classes* (*simp-all add*: *less-finite-1-def*)

instantiation *finite-1* :: *complete-lattice*
begin

definition [*simp*]: *Inf* = (λ-. a_1)
definition [*simp*]: *Sup* = (λ-. a_1)
definition [*simp*]: *bot* = a_1
definition [*simp*]: *top* = a_1
definition [*simp*]: *inf* = (λ- -. a_1)
definition [*simp*]: *sup* = (λ- -. a_1)

instance by *intro-classes*(*simp-all add*: *less-eq-finite-1-def*)
end

instance *finite-1* :: *complete-distrib-lattice*
by *standard simp-all*

instance *finite-1* :: *complete-linorder* **..**

lemma *finite-1-eq*: $x = a_1$
by(*cases x*) *simp*

simproc-setup *finite-1-eq* (*x*::*finite-1*) = ‹
  K (*K* (*fn ct* =>
    (*case Thm.term-of ct of*
      Const (***const-name*** ‹a_1›, -) => *NONE*
    | - => *SOME* (*mk-meta-eq* @{*thm finite-1-eq*}))))
›

instantiation *finite-1* :: *complete-boolean-algebra*
begin
definition [*simp*]: (−) = (λ- -. a_1)
definition [*simp*]: *uminus* = (λ-. a_1)
instance by *intro-classes simp-all*
end

instantiation *finite-1* ::
  {*linordered-ring-strict*, *linordered-comm-semiring-strict*, *ordered-comm-ring*,
    ordered-cancel-comm-monoid-diff, *comm-monoid-mult*, *ordered-ring-abs*,
    one, *modulo*, *sgn*, *inverse*}
begin
definition [*simp*]: *Groups.zero* = a_1
definition [*simp*]: *Groups.one* = a_1
definition [*simp*]: (+) = (λ- -. a_1)
definition [*simp*]: (∗) = (λ- -. a_1)
definition [*simp*]: (*mod*) = (λ- -. a_1)
definition [*simp*]: *abs* = (λ-. a_1)
definition [*simp*]: *sgn* = (λ-. a_1)
definition [*simp*]: *inverse* = (λ-. a_1)
definition [*simp*]: *divide* = (λ- -. a_1)

instance by *intro-classes*(*simp-all add*: *less-finite-1-def*)
end

declare [[*simproc del*: *finite-1-eq*]]
hide-const (**open**) a_1

datatype (*plugins only*: *code quickcheck extraction*) *finite-2* =
  a_1 | a_2

notation (**output**) a_1 (‹a_1›)
notation (**output**) a_2 (‹a_2›)

lemma *UNIV-finite-2*:
  UNIV = {a_1, a_2}

by (*auto intro*: *finite-2.exhaust*)

instantiation *finite-2* :: *enum*
begin

definition
enum = $[a_1, a_2]$

definition
enum-all $P \longleftrightarrow P\ a_1 \wedge P\ a_2$

definition
enum-ex $P \longleftrightarrow P\ a_1 \vee P\ a_2$

instance proof
qed (*simp-all only*: *enum-finite-2-def enum-all-finite-2-def enum-ex-finite-2-def UNIV-finite-2*, *simp-all*)

end

instantiation *finite-2* :: *linorder*
begin

definition *less-finite-2* :: *finite-2* ⇒ *finite-2* ⇒ *bool*
where
$x < y \longleftrightarrow x = a_1 \wedge y = a_2$

definition *less-eq-finite-2* :: *finite-2* ⇒ *finite-2* ⇒ *bool*
where
$x \leq y \longleftrightarrow x = y \vee x < (y :: \textit{finite-2})$

instance
apply (*intro-classes*)
apply (*auto simp add*: *less-finite-2-def less-eq-finite-2-def*)
apply (*metis finite-2.nchotomy*)+
done

end

instance *finite-2* :: *wellorder*
by(*rule wf-wellorderI*)(*simp add*: *less-finite-2-def*, *intro-classes*)

instantiation *finite-2* :: *complete-lattice*
begin

definition $\bigsqcap A = (\textit{if } a_1 \in A \textit{ then } a_1 \textit{ else } a_2)$
definition $\bigsqcup A = (\textit{if } a_2 \in A \textit{ then } a_2 \textit{ else } a_1)$
definition [*simp*]: $bot = a_1$
definition [*simp*]: $top = a_2$

definition $x \sqcap y = (\textit{if } x = a_1 \vee y = a_1 \textit{ then } a_1 \textit{ else } a_2)$
definition $x \sqcup y = (\textit{if } x = a_2 \vee y = a_2 \textit{ then } a_2 \textit{ else } a_1)$

lemma *neq-finite-2-*a_1*-iff* [*simp*]: $x \neq a_1 \longleftrightarrow x = a_2$
by(*cases x*) *simp-all*

lemma *neq-finite-2-*a_1*-iff'* [*simp*]: $a_1 \neq x \longleftrightarrow x = a_2$
by(*cases x*) *simp-all*

lemma *neq-finite-2-*a_2*-iff* [*simp*]: $x \neq a_2 \longleftrightarrow x = a_1$
by(*cases x*) *simp-all*

lemma *neq-finite-2-*a_2*-iff'* [*simp*]: $a_2 \neq x \longleftrightarrow x = a_1$
by(*cases x*) *simp-all*

instance
proof
  fix x :: *finite-2* **and** A
  assume $x \in A$
  then show $\bigsqcap A \leq x$ $x \leq \bigsqcup A$
    by(*cases x*; *auto simp add*: *less-eq-finite-2-def less-finite-2-def Inf-finite-2-def Sup-finite-2-def*)+
qed(*auto simp add*: *less-eq-finite-2-def less-finite-2-def inf-finite-2-def sup-finite-2-def Inf-finite-2-def Sup-finite-2-def*)
end

instance *finite-2* :: *complete-linorder* **..**

instance *finite-2* :: *complete-distrib-lattice* **..**

instantiation *finite-2* :: {*field*, *idom-abs-sgn*, *idom-modulo*} **begin**
definition [*simp*]: $0 = a_1$
definition [*simp*]: $1 = a_2$
definition $x + y = (\textit{case } (x, y) \textit{ of } (a_1, a_1) \Rightarrow a_1 \mid (a_2, a_2) \Rightarrow a_1 \mid \text{-} \Rightarrow a_2)$
definition $\textit{uminus} = (\lambda x :: \textit{finite-2}.\ x)$
definition $(-) = ((+) :: \textit{finite-2} \Rightarrow \text{-})$
definition $x * y = (\textit{case } (x, y) \textit{ of } (a_2, a_2) \Rightarrow a_2 \mid \text{-} \Rightarrow a_1)$
definition $\textit{inverse} = (\lambda x :: \textit{finite-2}.\ x)$
definition $\textit{divide} = ((*) :: \textit{finite-2} \Rightarrow \text{-})$
definition $x \textit{ mod } y = (\textit{case } (x, y) \textit{ of } (a_2, a_1) \Rightarrow a_2 \mid \text{-} \Rightarrow a_1)$
definition $\textit{abs} = (\lambda x :: \textit{finite-2}.\ x)$
definition $\textit{sgn} = (\lambda x :: \textit{finite-2}.\ x)$
instance
  by *standard*
    (*subproofs*
      ‹*simp-all add*: *plus-finite-2-def uminus-finite-2-def minus-finite-2-def times-finite-2-def*
        inverse-finite-2-def divide-finite-2-def modulo-finite-2-def
        abs-finite-2-def sgn-finite-2-def

split: *finite-2.splits*›)
end

lemma *two-finite-2* [*simp*]:
2 = a_1
by (*simp add*: *numeral.simps plus-finite-2-def*)

lemma *dvd-finite-2-unfold*:
x dvd y ⟷ *x* = a_2 ∨ *y* = a_1
by (*auto simp add*: *dvd-def times-finite-2-def split*: *finite-2.splits*)

instantiation *finite-2* :: {*normalization-semidom*, *unique-euclidean-semiring*} **begin**
definition [*simp*]: *normalize* = (*id* :: *finite-2* ⇒ -)
definition [*simp*]: *unit-factor* = (*id* :: *finite-2* ⇒ -)
definition [*simp*]: *euclidean-size x* = (*case x of* a_1 ⇒ *0* | a_2 ⇒ *1*)
definition [*simp*]: *division-segment* (*x* :: *finite-2*) = *1*
instance
by *standard*
(*subproofs*
‹*auto simp add*: *divide-finite-2-def times-finite-2-def dvd-finite-2-unfold*
split: *finite-2.splits*›)
end

hide-const (**open**) a_1 a_2

datatype (*plugins only*: *code quickcheck extraction*) *finite-3* =
a_1 | a_2 | a_3

notation (**output**) a_1 (‹a_1›)
notation (**output**) a_2 (‹a_2›)
notation (**output**) a_3 (‹a_3›)

lemma *UNIV-finite-3*:
UNIV = {a_1, a_2, a_3}
by (*auto intro*: *finite-3.exhaust*)

instantiation *finite-3* :: *enum*
begin

definition
enum = [a_1, a_2, a_3]

definition
enum-all P ⟷ *P* a_1 ∧ *P* a_2 ∧ *P* a_3

definition
enum-ex P ⟷ *P* a_1 ∨ *P* a_2 ∨ *P* a_3

instance proof
qed (*simp-all only*: *enum-finite-3-def enum-all-finite-3-def enum-ex-finite-3-def UNIV-finite-3*, *simp-all*)

end

lemma *finite-3-not-eq-unfold*:
$x \neq a_1 \longleftrightarrow x \in \{a_2, a_3\}$
$x \neq a_2 \longleftrightarrow x \in \{a_1, a_3\}$
$x \neq a_3 \longleftrightarrow x \in \{a_1, a_2\}$
by (*cases x*; *simp*)+

instantiation *finite-3* :: *linorder*
begin

definition *less-finite-3* :: *finite-3* ⇒ *finite-3* ⇒ *bool*
where
$x < y = (case\ x\ of\ a_1 \Rightarrow y \neq a_1 \mid a_2 \Rightarrow y = a_3 \mid a_3 \Rightarrow False)$

definition *less-eq-finite-3* :: *finite-3* ⇒ *finite-3* ⇒ *bool*
where
$x \leq y \longleftrightarrow x = y \lor x < (y :: finite\text{-}3)$

instance proof (*intro-classes*)
qed (*auto simp add*: *less-finite-3-def less-eq-finite-3-def split*: *finite-3.split-asm*)

end

instance *finite-3* :: *wellorder*
proof(*rule wf-wellorderI*)
have *inv-image less-than* (*case-finite-3 0 1 2*) = {(*x*, *y*). *x* < *y*}
by(*auto simp add*: *less-finite-3-def split*: *finite-3.splits*)
from *this*[*symmetric*] **show** *wf* ... **by** *simp*
qed *intro-classes*

class *finite-lattice* = *finite* + *lattice* + *Inf* + *Sup* + *bot* + *top* +
assumes *Inf-finite-empty*: *Inf* {} = *Sup UNIV*
assumes *Inf-finite-insert*: *Inf* (*insert a A*) = $a \sqcap$ *Inf A*
assumes *Sup-finite-empty*: *Sup* {} = *Inf UNIV*
assumes *Sup-finite-insert*: *Sup* (*insert a A*) = $a \sqcup$ *Sup A*
assumes *bot-finite-def*: *bot* = *Inf UNIV*
assumes *top-finite-def*: *top* = *Sup UNIV*
begin

subclass *complete-lattice*
proof
fix *x A*
show $x \in A \Longrightarrow \bigsqcap A \leq x$

by (*metis Set.set-insert abel-semigroup.commute local.Inf-finite-insert local.inf.abel-semigroup-axioms local.inf.left-idem local.inf.orderI*)
show $x \in A \Longrightarrow x \le \bigsqcup A$
by (*metis Set.set-insert insert-absorb2 local.Sup-finite-insert local.sup.absorb-iff2*)
next
fix *A z*
have $\bigsqcup$ *UNIV* $= z \sqcup \bigsqcup$ *UNIV*
by (*subst Sup-finite-insert* [*symmetric*], *simp add*: *insert-UNIV*)
from *this* **have** [*simp*]: $z \le \bigsqcup$ *UNIV*
using *local.le-iff-sup* **by** *auto*
have $(\forall\ x.\ x \in A \longrightarrow z \le x) \longrightarrow z \le \bigsqcap A$
by (*rule finite-induct* [*of A* λ *A* . $(\forall\ x.\ x \in A \longrightarrow z \le x) \longrightarrow z \le \bigsqcap A$])
(*simp-all add*: *Inf-finite-empty Inf-finite-insert*)
from *this* **show** $(\bigwedge x.\ x \in A \Longrightarrow z \le x) \Longrightarrow z \le \bigsqcap A$
by *simp*

have $\bigsqcap$ *UNIV* $= z \sqcap \bigsqcap$ *UNIV*
by (*subst Inf-finite-insert* [*symmetric*], *simp add*: *insert-UNIV*)
from *this* **have** [*simp*]: $\bigsqcap$ *UNIV* $\le z$
by (*simp add*: *local.inf.absorb-iff2*)
have $(\forall\ x.\ x \in A \longrightarrow x \le z) \longrightarrow \bigsqcup A \le z$
by (*rule finite-induct* [*of A* λ *A* . $(\forall\ x.\ x \in A \longrightarrow x \le z) \longrightarrow \bigsqcup A \le z$], *simp-all add*: *Sup-finite-empty Sup-finite-insert*)
from *this* **show** $(\bigwedge x.\ x \in A \Longrightarrow x \le z) \Longrightarrow \bigsqcup A \le z$
by *blast*
next
show $\bigsqcap \{\} = \top$
by (*simp add*: *Inf-finite-empty top-finite-def*)
show $\bigsqcup \{\} = \bot$
by (*simp add*: *Sup-finite-empty bot-finite-def*)
qed
end

class *finite-distrib-lattice* = *finite-lattice* + *distrib-lattice*
begin
lemma *finite-inf-Sup*: $a \sqcap (Sup\ A) = Sup\ \{a \sqcap b \mid b\ .\ b \in A\}$
proof (*rule finite-induct* [*of A* λ *A* . $a \sqcap (Sup\ A) = Sup\ \{a \sqcap b \mid b\ .\ b \in A\}$], *simp-all*)
fix *x*::*'a*
fix *F*
assume $x \notin F$
assume [*simp*]: $a \sqcap \bigsqcup F = \bigsqcup \{a \sqcap b \mid b.\ b \in F\}$
have [*simp*]: *insert* $(a \sqcap x)\ \{a \sqcap b \mid b.\ b \in F\} = \{a \sqcap b \mid b.\ b = x \lor b \in F\}$
by *blast*
have $a \sqcap (x \sqcup \bigsqcup F) = a \sqcap x \sqcup a \sqcap \bigsqcup F$
by (*simp add*: *inf-sup-distrib1*)
also have ... $= a \sqcap x \sqcup \bigsqcup \{a \sqcap b \mid b.\ b \in F\}$
by *simp*
also have ... $= \bigsqcup \{a \sqcap b \mid b.\ b = x \lor b \in F\}$

by (*unfold Sup-insert*[*THEN sym*], *simp*)
finally show $a \sqcap (x \sqcup \bigsqcup F) = \bigsqcup\{a \sqcap b \mid b.\ b = x \lor b \in F\}$
by *simp*
qed

lemma *finite-Inf-Sup*: $\bigsqcap(Sup\ `\ A) \le \bigsqcup(Inf\ `\ \{f\ `\ A \mid f.\ \forall Y{\in}A.\ f\ Y \in Y\})$
proof (*rule finite-induct* [*of A* $\lambda A.\ \bigsqcap(Sup\ `\ A) \le \bigsqcup(Inf\ `\ \{f\ `\ A \mid f.\ \forall Y{\in}A.\ f\ Y \in Y\})$], *simp-all add*: *finite-UnionD*)
fix $x::'a\ set$
fix F
assume $x \notin F$
have [*simp*]: $\{\bigsqcup x \sqcap b \mid b\ .\ b \in Inf\ `\ \{f\ `\ F \mid f.\ \forall Y{\in}F.\ f\ Y \in Y\}\ \} = \{\bigsqcup x \sqcap (Inf\ (f\ `\ F)) \mid f\ .\ (\forall Y{\in}F.\ f\ Y \in Y)\}$
by *auto*
define *fa* **where** $fa = (\lambda\ (b::'a)\ f\ Y\ .\ (if\ Y = x\ then\ b\ else\ f\ Y))$
have $\bigwedge f\ b.\ \forall Y{\in}F.\ f\ Y \in Y \Longrightarrow b \in x \Longrightarrow insert\ b\ (f\ `\ (F \cap \{Y.\ Y \ne x\})) = insert\ (fa\ b\ f\ x)\ (fa\ b\ f\ `\ F) \land fa\ b\ f\ x \in x \land (\forall Y{\in}F.\ fa\ b\ f\ Y \in Y)$
by (*auto simp add*: *fa-def*)
from *this* **have** *B*: $\bigwedge f\ b.\ \forall Y{\in}F.\ f\ Y \in Y \Longrightarrow b \in x \Longrightarrow fa\ b\ f\ `\ (\{x\} \cup F) \in \{insert\ (f\ x)\ (f\ `\ F) \mid f.\ f\ x \in x \land (\forall Y{\in}F.\ f\ Y \in Y)\}$
by *blast*
have [*simp*]: $\bigwedge f\ b.\ \forall Y{\in}F.\ f\ Y \in Y \Longrightarrow b \in x \Longrightarrow b \sqcap (\bigsqcap x{\in}F.\ f\ x) \le \bigsqcup(Inf\ `\ \{insert\ (f\ x)\ (f\ `\ F) \mid f.\ f\ x \in x \land (\forall Y{\in}F.\ f\ Y \in Y)\})$
using *B* **apply** (*rule SUP-upper2*)
using ‹$x \notin F$› **apply** (*simp-all add*: *fa-def Inf-union-distrib*)
apply (*simp add*: *image-mono Inf-superset-mono inf.coboundedI2*)
done
assume $\bigsqcap(Sup\ `\ F) \le \bigsqcup(Inf\ `\ \{f\ `\ F \mid f.\ \forall Y{\in}F.\ f\ Y \in Y\})$

from *this* **have** $\bigsqcup x \sqcap \bigsqcap(Sup\ `\ F) \le \bigsqcup x \sqcap \bigsqcup(Inf\ `\ \{f\ `\ F \mid f.\ \forall Y{\in}F.\ f\ Y \in Y\})$
using *inf.coboundedI2* **by** *auto*
also have ... $= Sup\ \{\bigsqcup x \sqcap (Inf\ (f\ `\ F)) \mid f\ .\ (\forall Y{\in}F.\ f\ Y \in Y)\}$
by (*simp add*: *finite-inf-Sup*)

also have ... $= Sup\ \{Sup\ \{Inf\ (f\ `\ F) \sqcap b \mid b\ .\ b \in x\} \mid f\ .\ (\forall Y{\in}F.\ f\ Y \in Y)\}$
by (*subst inf-commute*) (*simp add*: *finite-inf-Sup*)

also have ... $\le \bigsqcup(Inf\ `\ \{insert\ (f\ x)\ (f\ `\ F) \mid f.\ f\ x \in x \land (\forall Y{\in}F.\ f\ Y \in Y)\})$
apply (*rule Sup-least*, *clarsimp*)+
apply (*subst inf-commute*, *simp*)
done

finally show $\bigsqcup x \sqcap \bigsqcap(Sup\ `\ F) \le \bigsqcup(Inf\ `\ \{insert\ (f\ x)\ (f\ `\ F) \mid f.\ f\ x \in x \land (\forall Y{\in}F.\ f\ Y \in Y)\})$
by *simp*
qed

subclass *complete-distrib-lattice*
by (*standard*, *rule finite-Inf-Sup*)

end

instantiation *finite-3* :: *finite-lattice*
begin

definition $\bigsqcap A = (\textit{if } a_1 \in A \textit{ then } a_1 \textit{ else if } a_2 \in A \textit{ then } a_2 \textit{ else } a_3)$
definition $\bigsqcup A = (\textit{if } a_3 \in A \textit{ then } a_3 \textit{ else if } a_2 \in A \textit{ then } a_2 \textit{ else } a_1)$
definition [*simp*]: $\textit{bot} = a_1$
definition [*simp*]: $\textit{top} = a_3$
definition [*simp*]: *inf* = (*min* :: *finite-3* ⇒ -)
definition [*simp*]: *sup* = (*max* :: *finite-3* ⇒ -)

instance
proof
qed (*auto simp add*: *Inf-finite-3-def Sup-finite-3-def max-def min-def less-eq-finite-3-def less-finite-3-def split*: *finite-3.split*)
end

instance *finite-3* :: *complete-lattice* **..**

instance *finite-3* :: *finite-distrib-lattice*
proof
qed (*auto simp add*: *min-def max-def*)

instance *finite-3* :: *complete-distrib-lattice* **..**

instance *finite-3* :: *complete-linorder* **..**

instantiation *finite-3* :: {*field*, *idom-abs-sgn*, *idom-modulo*} **begin**
definition [*simp*]: $0 = a_1$
definition [*simp*]: $1 = a_2$
definition
  $x + y = (\textit{case } (x, y) \textit{ of}$
    $(a_1, a_1) \Rightarrow a_1 \mid (a_2, a_3) \Rightarrow a_1 \mid (a_3, a_2) \Rightarrow a_1$
  $\mid (a_1, a_2) \Rightarrow a_2 \mid (a_2, a_1) \Rightarrow a_2 \mid (a_3, a_3) \Rightarrow a_2$
  $\mid \text{-} \Rightarrow a_3)$
definition $-\, x = (\textit{case } x \textit{ of } a_1 \Rightarrow a_1 \mid a_2 \Rightarrow a_3 \mid a_3 \Rightarrow a_2)$
definition $x - y = x + (-\, y :: \textit{finite-3})$
definition $x * y = (\textit{case } (x, y) \textit{ of } (a_2, a_2) \Rightarrow a_2 \mid (a_3, a_3) \Rightarrow a_2 \mid (a_2, a_3) \Rightarrow a_3$
$\mid (a_3, a_2) \Rightarrow a_3 \mid \text{-} \Rightarrow a_1)$
definition $\textit{inverse} = (\lambda x :: \textit{finite-3}.\ x)$
definition $x \textit{ div } y = x * \textit{inverse } (y :: \textit{finite-3})$
definition $x \textit{ mod } y = (\textit{case } y \textit{ of } a_1 \Rightarrow x \mid \text{-} \Rightarrow a_1)$
definition $\textit{abs} = (\lambda x.\ \textit{case } x \textit{ of } a_3 \Rightarrow a_2 \mid \text{-} \Rightarrow x)$
definition $\textit{sgn} = (\lambda x :: \textit{finite-3}.\ x)$
instance
  by *standard*
    (*subproofs*
      ‹*simp-all add*: *plus-finite-3-def uminus-finite-3-def minus-finite-3-def*

times-finite-3-def
inverse-finite-3-def divide-finite-3-def modulo-finite-3-def
abs-finite-3-def sgn-finite-3-def
less-finite-3-def
split: *finite-3.splits*›)
end

lemma *two-finite-3* [*simp*]:
$2 = a_3$
by (*simp add*: *numeral.simps plus-finite-3-def*)

lemma *dvd-finite-3-unfold*:
x *dvd* $y \longleftrightarrow x = a_2 \lor x = a_3 \lor y = a_1$
by (*cases x*) (*auto simp add*: *dvd-def times-finite-3-def split*: *finite-3.splits*)

instantiation *finite-3* :: {*normalization-semidom*, *unique-euclidean-semiring*} **begin**
definition [*simp*]: *normalize x* = (*case x of* $a_3 \Rightarrow a_2$ | - ⇒ *x*)
definition [*simp*]: *unit-factor* = (*id* :: *finite-3* ⇒ -)
definition [*simp*]: *euclidean-size x* = (*case x of* $a_1 \Rightarrow 0$ | - ⇒ *1*)
definition [*simp*]: *division-segment* (*x* :: *finite-3*) = *1*
instance
proof
fix *x* :: *finite-3*
assume $x \neq 0$
then show *is-unit* (*unit-factor x*)
by (*cases x*) (*simp-all add*: *dvd-finite-3-unfold*)
qed
(*subproofs*
‹*auto simp add*: *divide-finite-3-def times-finite-3-def*
dvd-finite-3-unfold inverse-finite-3-def plus-finite-3-def
split: *finite-3.splits*›)
end

hide-const (**open**) a_1 a_2 a_3

datatype (*plugins only*: *code quickcheck extraction*) *finite-4* =
a_1 | a_2 | a_3 | a_4

notation (**output**) a_1 (‹a_1›)
notation (**output**) a_2 (‹a_2›)
notation (**output**) a_3 (‹a_3›)
notation (**output**) a_4 (‹a_4›)

lemma *UNIV-finite-4*:
UNIV = $\{a_1, a_2, a_3, a_4\}$
by (*auto intro*: *finite-4.exhaust*)

instantiation *finite-4* :: *enum*

begin

definition
enum = $[a_1, a_2, a_3, a_4]$

definition
enum-all $P \longleftrightarrow P\ a_1 \wedge P\ a_2 \wedge P\ a_3 \wedge P\ a_4$

definition
enum-ex $P \longleftrightarrow P\ a_1 \vee P\ a_2 \vee P\ a_3 \vee P\ a_4$

instance proof
qed (*simp-all only*: *enum-finite-4-def enum-all-finite-4-def enum-ex-finite-4-def UNIV-finite-4*, *simp-all*)

end

instantiation *finite-4* :: *finite-distrib-lattice* **begin**

$a_1 < a_2$,$a_3 < a_4$, but a_2 and a_3 are incomparable.

definition
$x < y \longleftrightarrow$ (*case* (x, y) *of*
$(a_1, a_1) \Rightarrow$ *False* | $(a_1, \text{-}) \Rightarrow$ *True*
| $(a_2, a_4) \Rightarrow$ *True*
| $(a_3, a_4) \Rightarrow$ *True* | - $\Rightarrow$ *False*)

definition
$x \leq y \longleftrightarrow$ (*case* (x, y) *of*
$(a_1, \text{-}) \Rightarrow$ *True*
| $(a_2, a_2) \Rightarrow$ *True* | $(a_2, a_4) \Rightarrow$ *True*
| $(a_3, a_3) \Rightarrow$ *True* | $(a_3, a_4) \Rightarrow$ *True*
| $(a_4, a_4) \Rightarrow$ *True* | - $\Rightarrow$ *False*)

definition
$\bigsqcap A$ = (*if* $a_1 \in A \vee a_2 \in A \wedge a_3 \in A$ *then* a_1 *else if* $a_2 \in A$ *then* a_2 *else if* $a_3 \in A$ *then* a_3 *else* a_4)
definition
$\bigsqcup A$ = (*if* $a_4 \in A \vee a_2 \in A \wedge a_3 \in A$ *then* a_4 *else if* $a_2 \in A$ *then* a_2 *else if* $a_3 \in A$ *then* a_3 *else* a_1)
definition [*simp*]: *bot* = a_1
definition [*simp*]: *top* = a_4
definition
$x \sqcap y$ = (*case* (x, y) *of*
$(a_1, \text{-}) \Rightarrow a_1$ | $(\text{-}, a_1) \Rightarrow a_1$ | $(a_2, a_3) \Rightarrow a_1$ | $(a_3, a_2) \Rightarrow a_1$
| $(a_2, \text{-}) \Rightarrow a_2$ | $(\text{-}, a_2) \Rightarrow a_2$
| $(a_3, \text{-}) \Rightarrow a_3$ | $(\text{-}, a_3) \Rightarrow a_3$
| - $\Rightarrow a_4$)
definition
$x \sqcup y$ = (*case* (x, y) *of*

$(a_4, \text{-}) \Rightarrow a_4$ | $(\text{-}, a_4) \Rightarrow a_4$ | $(a_2, a_3) \Rightarrow a_4$ | $(a_3, a_2) \Rightarrow a_4$
| $(a_2, \text{-}) \Rightarrow a_2$ | $(\text{-}, a_2) \Rightarrow a_2$
| $(a_3, \text{-}) \Rightarrow a_3$ | $(\text{-}, a_3) \Rightarrow a_3$
| - $\Rightarrow a_1$)

instance
by *standard*
(*subproofs*
‹*auto simp add*: *less-finite-4-def less-eq-finite-4-def Inf-finite-4-def Sup-finite-4-def*

inf-finite-4-def sup-finite-4-def split: *finite-4.splits*›)
end

instance *finite-4* :: *complete-lattice* **..**

instance *finite-4* :: *complete-distrib-lattice* **..**

instantiation *finite-4* :: *complete-boolean-algebra* **begin**
definition $- x = ($*case* x *of* $a_1 \Rightarrow a_4$ | $a_2 \Rightarrow a_3$ | $a_3 \Rightarrow a_2$ | $a_4 \Rightarrow a_1)$
definition $x - y = x \sqcap - (y :: \textit{finite-4})$
instance
by *standard*
(*subproofs*
‹*simp-all add*: *inf-finite-4-def sup-finite-4-def uminus-finite-4-def minus-finite-4-def*

split: *finite-4.splits*›)
end

hide-const (**open**) a_1 a_2 a_3 a_4

datatype (*plugins only*: *code quickcheck extraction*) *finite-5* =
a_1 | a_2 | a_3 | a_4 | a_5

notation (**output**) a_1 (‹a_1›)
notation (**output**) a_2 (‹a_2›)
notation (**output**) a_3 (‹a_3›)
notation (**output**) a_4 (‹a_4›)
notation (**output**) a_5 (‹a_5›)

lemma *UNIV-finite-5*:
UNIV = $\{a_1, a_2, a_3, a_4, a_5\}$
by (*auto intro*: *finite-5.exhaust*)

instantiation *finite-5* :: *enum*
begin

definition
enum = $[a_1, a_2, a_3, a_4, a_5]$

definition
enum-all $P \longleftrightarrow P\ a_1 \wedge P\ a_2 \wedge P\ a_3 \wedge P\ a_4 \wedge P\ a_5$

definition
enum-ex $P \longleftrightarrow P\ a_1 \vee P\ a_2 \vee P\ a_3 \vee P\ a_4 \vee P\ a_5$

instance proof
qed (*simp-all only*: *enum-finite-5-def enum-all-finite-5-def enum-ex-finite-5-def UNIV-finite-5*, *simp-all*)

end

instantiation *finite-5* :: *finite-lattice*
begin

The non-distributive pentagon lattice N_5

definition
$x < y \longleftrightarrow (case\ (x,\ y)\ of$
$(a_1,\ a_1) \Rightarrow False \mid (a_1,\ \text{-}) \Rightarrow True$
$\mid (a_2,\ a_3) \Rightarrow True \mid (a_2,\ a_5) \Rightarrow True$
$\mid (a_3,\ a_5) \Rightarrow True$
$\mid (a_4,\ a_5) \Rightarrow True \mid \text{-} \Rightarrow False)$

definition
$x \leq y \longleftrightarrow (case\ (x,\ y)\ of$
$(a_1,\ \text{-}) \Rightarrow True$
$\mid (a_2,\ a_2) \Rightarrow True \mid (a_2,\ a_3) \Rightarrow True \mid (a_2,\ a_5) \Rightarrow True$
$\mid (a_3,\ a_3) \Rightarrow True \mid (a_3,\ a_5) \Rightarrow True$
$\mid (a_4,\ a_4) \Rightarrow True \mid (a_4,\ a_5) \Rightarrow True$
$\mid (a_5,\ a_5) \Rightarrow True \mid \text{-} \Rightarrow False)$

definition
$\bigsqcap A =$
$(if\ a_1 \in A \vee a_4 \in A \wedge (a_2 \in A \vee a_3 \in A)\ then\ a_1$
$else\ if\ a_2 \in A\ then\ a_2$
$else\ if\ a_3 \in A\ then\ a_3$
$else\ if\ a_4 \in A\ then\ a_4$
$else\ a_5)$
definition
$\bigsqcup A =$
$(if\ a_5 \in A \vee a_4 \in A \wedge (a_2 \in A \vee a_3 \in A)\ then\ a_5$
$else\ if\ a_3 \in A\ then\ a_3$
$else\ if\ a_2 \in A\ then\ a_2$
$else\ if\ a_4 \in A\ then\ a_4$
$else\ a_1)$
definition [*simp*]: $bot = a_1$
definition [*simp*]: $top = a_5$
definition
$x \sqcap y = (case\ (x,\ y)\ of$

(a_1, -) ⇒ a_1 | (-, a_1) ⇒ a_1 | (a_2, a_4) ⇒ a_1 | (a_4, a_2) ⇒ a_1 | (a_3, a_4) ⇒ a_1 |
(a_4, a_3) ⇒ a_1
| (a_2, -) ⇒ a_2 | (-, a_2) ⇒ a_2
| (a_3, -) ⇒ a_3 | (-, a_3) ⇒ a_3
| (a_4, -) ⇒ a_4 | (-, a_4) ⇒ a_4
| - ⇒ a_5)
definition
$x \sqcup y$ = (*case* (*x*, *y*) *of*
(a_5, -) ⇒ a_5 | (-, a_5) ⇒ a_5 | (a_2, a_4) ⇒ a_5 | (a_4, a_2) ⇒ a_5 | (a_3, a_4) ⇒ a_5 |
(a_4, a_3) ⇒ a_5
| (a_3, -) ⇒ a_3 | (-, a_3) ⇒ a_3
| (a_2, -) ⇒ a_2 | (-, a_2) ⇒ a_2
| (a_4, -) ⇒ a_4 | (-, a_4) ⇒ a_4
| - ⇒ a_1)

instance
by *standard*
(*subproofs*
‹*auto simp add*: *less-eq-finite-5-def less-finite-5-def inf-finite-5-def sup-finite-5-def*

Inf-finite-5-def Sup-finite-5-def split: *finite-5.splits if-split-asm*›)
end

instance *finite-5* :: *complete-lattice* **..**

hide-const (**open**) a_1 a_2 a_3 a_4 a_5

72.5 Closing up

hide-type (**open**) *finite-1 finite-2 finite-3 finite-4 finite-5*
hide-const (**open**) *enum enum-all enum-ex all-n-lists ex-n-lists ntrancl*

end

73 Character and string types

theory *String*
imports *Enum Bit-Operations Code-Numeral*
begin

73.1 Strings as list of bytes

When modelling strings, we follow the approach given in https://utf8everywhere.org/:

- Strings are a list of bytes (8 bit).

- Byte values from 0 to 127 are US-ASCII.
- Byte values from 128 to 255 are uninterpreted blobs.

73.1.1 Bytes as datatype

datatype *char* =
Char (*digit0*: *bool*) (*digit1*: *bool*) (*digit2*: *bool*) (*digit3*: *bool*)
(*digit4*: *bool*) (*digit5*: *bool*) (*digit6*: *bool*) (*digit7*: *bool*)

context *comm-semiring-1*
begin

definition *of-char* :: ‹*char* ⇒ *'a*›
where ‹*of-char c* = *horner-sum of-bool 2* [*digit0 c*, *digit1 c*, *digit2 c*, *digit3 c*, *digit4 c*, *digit5 c*, *digit6 c*, *digit7 c*]›

lemma *of-char-Char* [*simp*]:
‹*of-char* (*Char b0 b1 b2 b3 b4 b5 b6 b7*) =
horner-sum of-bool 2 [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*, *b7*]›
by (*simp add*: *of-char-def*)

end

lemma (**in** *comm-semiring-1*) *of-nat-of-char*:
‹*of-nat* (*of-char c*) = *of-char c*›
by (*cases c*) *simp*

lemma (**in** *comm-ring-1*) *of-int-of-char*:
‹*of-int* (*of-char c*) = *of-char c*›
by (*cases c*) *simp*

lemma *nat-of-char* [*simp*]:
‹*nat* (*of-char c*) = *of-char c*›
by (*cases c*) (*simp only*: *of-char-Char nat-horner-sum*)

context *linordered-euclidean-semiring-bit-operations*
begin

definition *char-of* :: ‹*'a* ⇒ *char*›
where ‹*char-of n* = *Char* (*bit n 0*) (*bit n 1*) (*bit n 2*) (*bit n 3*) (*bit n 4*) (*bit n 5*) (*bit n 6*) (*bit n 7*)›

lemma *char-of-take-bit-eq*:
‹*char-of* (*take-bit n m*) = *char-of m*› **if** ‹$n \geq 8$›
using *that* **by** (*simp add*: *char-of-def bit-take-bit-iff*)

lemma *char-of-char* [*simp*]:
‹*char-of* (*of-char c*) = *c*›
by (*simp only*: *of-char-def char-of-def bit-horner-sum-bit-iff*) *simp*

lemma *char-of-comp-of-char* [*simp*]:
char-of ∘ *of-char* = *id*
by (*simp add*: *fun-eq-iff*)

lemma *inj-of-char*:
‹*inj of-char*›
proof (*rule injI*)
fix *c d*
assume *of-char c* = *of-char d*
then have *char-of* (*of-char c*) = *char-of* (*of-char d*)
by *simp*
then show *c* = *d*
by *simp*
qed

lemma *of-char-eqI*:
‹*c* = *d*› **if** ‹*of-char c* = *of-char d*›
using *that inj-of-char* **by** (*simp add*: *inj-eq*)

lemma *of-char-eq-iff* [*simp*]:
‹*of-char c* = *of-char d* ⟷ *c* = *d*›
by (*auto intro*: *of-char-eqI*)

lemma *of-char-of* [*simp*]:
‹*of-char* (*char-of a*) = *a mod 256*›
proof −
have ‹[*0..<8*] = [*0*, *Suc 0*, *2*, *3*, *4*, *5*, *6*, *7* :: *nat*]›
by (*simp add*: *upt-eq-Cons-conv*)
then have ‹[*bit a 0*, *bit a 1*, *bit a 2*, *bit a 3*, *bit a 4*, *bit a 5*, *bit a 6*, *bit a 7*] = *map* (*bit a*) [*0..<8*]›
by *simp*
then have ‹*of-char* (*char-of a*) = *take-bit 8 a*›
by (*simp only*: *char-of-def of-char-def char.sel horner-sum-bit-eq-take-bit*)
then show *?thesis*
by (*simp add*: *take-bit-eq-mod*)
qed

lemma *char-of-mod-256* [*simp*]:
‹*char-of* (*n mod 256*) = *char-of n*›
by (*rule of-char-eqI*) *simp*

lemma *of-char-mod-256* [*simp*]:
‹*of-char c mod 256* = *of-char c*›
proof −
have ‹*of-char* (*char-of* (*of-char c*)) *mod 256* = *of-char* (*char-of* (*of-char c*))›

by (*simp only*: *of-char-of*) *simp*
then show *?thesis*
by *simp*
qed

lemma *char-of-quasi-inj* [*simp*]:
‹*char-of m* = *char-of n* ⟷ *m mod 256* = *n mod 256*› (**is** ‹*?P* ⟷ *?Q*›)
proof
assume *?Q*
then show *?P*
by (*auto intro*: *of-char-eqI*)
next
assume *?P*
then have ‹*of-char* (*char-of m*) = *of-char* (*char-of n*)›
by *simp*
then show *?Q*
by *simp*
qed

lemma *char-of-eq-iff*:
‹*char-of n* = *c* ⟷ *take-bit 8 n* = *of-char c*›
by (*auto intro*: *of-char-eqI simp add*: *take-bit-eq-mod*)

lemma *char-of-nat* [*simp*]:
‹*char-of* (*of-nat n*) = *char-of n*›
by (*simp add*: *char-of-def String.char-of-def drop-bit-of-nat bit-simps possible-bit-def*)

end

lemma *inj-on-char-of-nat* [*simp*]:
inj-on char-of {*0*::*nat*..<*256*}
by (*rule inj-onI*) *simp*

lemma *nat-of-char-less-256* [*simp*]:
of-char c < (*256* :: *nat*)
proof −
have *of-char c mod* (*256* :: *nat*) < *256*
by *arith*
then show *?thesis* **by** *simp*
qed

lemma *range-nat-of-char*:
range of-char = {*0*::*nat*..<*256*}
proof (*rule*; *rule*)
fix *n* :: *nat*
assume *n* ∈ *range of-char*
then show *n* ∈ {*0*..<*256*}
by *auto*
next

fix *n* :: *nat*
assume *n* ∈ {*0*..<*256*}
then have *n* = *of-char* (*char-of* *n*)
by *simp*
then show *n* ∈ *range of-char*
by (*rule range-eqI*)
qed

lemma *UNIV-char-of-nat*:
UNIV = *char-of* ' {*0*::*nat*..<*256*}
proof −
have *range* (*of-char* :: *char* ⇒ *nat*) = *of-char* ' *char-of* ' {*0*::*nat*..<*256*}
by (*simp add*: *image-image range-nat-of-char*)
with *inj-of-char* [**where** *?'a* = *nat*] **show** *?thesis*
by (*simp add*: *inj-image-eq-iff*)
qed

lemma *card-UNIV-char*:
card (*UNIV* :: *char set*) = *256*
by (*auto simp add*: *UNIV-char-of-nat card-image*)

context
includes *lifting-syntax* **and** *integer.lifting* **and** *natural.lifting*
begin

lemma [*transfer-rule*]:
‹(*pcr-integer* ===> (=)) *char-of char-of*›
by (*unfold char-of-def*) *transfer-prover*

lemma [*transfer-rule*]:
‹((=) ===> *pcr-integer*) *of-char of-char*›
by (*unfold of-char-def*) *transfer-prover*

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> (=)) *char-of char-of*›
by (*unfold char-of-def*) *transfer-prover*

lemma [*transfer-rule*]:
‹((=) ===> *pcr-natural*) *of-char of-char*›
by (*unfold of-char-def*) *transfer-prover*

end

lifting-update *integer.lifting*
lifting-forget *integer.lifting*

lifting-update *natural.lifting*
lifting-forget *natural.lifting*

lemma *size-char-eq-0* [*simp*, *code*]:
‹*size c* = *0*› **for** *c* :: *char*
by (*cases c*) *simp*

lemma *size′-char-eq-0* [*simp*, *code*]:
‹*size-char c* = *0*›
by (*cases c*) *simp*

syntax
-*Char* :: *str-position* ⇒ *char* (‹(‹*open-block notation*=‹*literal char*››*CHR* -)›)
-*Char-ord* :: *num-const* ⇒ *char* (‹(‹*open-block notation*=‹*literal char code*››*CHR* -)›)
syntax-consts
-*Char* -*Char-ord* ⇌ *Char*

type-synonym *string* = *char list*

syntax
-*String* :: *str-position* ⇒ *string* (‹(‹*open-block notation*=‹*literal string*››-)›)

ML-file ‹*Tools/string-syntax.ML*›

instantiation *char* :: *enum*
begin

definition
Enum.enum = [
CHR 0x00, *CHR 0x01*, *CHR 0x02*, *CHR 0x03*,
CHR 0x04, *CHR 0x05*, *CHR 0x06*, *CHR 0x07*,
CHR 0x08, *CHR 0x09*, *CHR* ''↩'', *CHR 0x0B*,
CHR 0x0C, *CHR 0x0D*, *CHR 0x0E*, *CHR 0x0F*,
CHR 0x10, *CHR 0x11*, *CHR 0x12*, *CHR 0x13*,
CHR 0x14, *CHR 0x15*, *CHR 0x16*, *CHR 0x17*,
CHR 0x18, *CHR 0x19*, *CHR 0x1A*, *CHR 0x1B*,
CHR 0x1C, *CHR 0x1D*, *CHR 0x1E*, *CHR 0x1F*,
CHR '' '', *CHR* ''!'', *CHR 0x22*, *CHR* ''#'',
CHR ''$'', *CHR* ''%'', *CHR* ''&'', *CHR 0x27*,
CHR ''('', *CHR* '')'', *CHR* ''*'', *CHR* ''+'',
CHR '','', *CHR* ''−'', *CHR* ''.'', *CHR* ''/'',
CHR ''0'', *CHR* ''1'', *CHR* ''2'', *CHR* ''3'',
CHR ''4'', *CHR* ''5'', *CHR* ''6'', *CHR* ''7'',
CHR ''8'', *CHR* ''9'', *CHR* '':'', *CHR* '';'',
CHR ''<'', *CHR* ''='', *CHR* ''>'', *CHR* ''?'',
CHR ''@'', *CHR* ''A'', *CHR* ''B'', *CHR* ''C'',
CHR ''D'', *CHR* ''E'', *CHR* ''F'', *CHR* ''G'',
CHR ''H'', *CHR* ''I'', *CHR* ''J'', *CHR* ''K'',
CHR ''L'', *CHR* ''M'', *CHR* ''N'', *CHR* ''O'',
CHR ''P'', *CHR* ''Q'', *CHR* ''R'', *CHR* ''S'',
CHR ''T'', *CHR* ''U'', *CHR* ''V'', *CHR* ''W'',

CHR ''X'', CHR ''Y'', CHR ''Z'', CHR ''['',
CHR 0x5C, CHR '']'', CHR ''^'', CHR ''-'',
CHR 0x60, CHR ''a'', CHR ''b'', CHR ''c'',
CHR ''d'', CHR ''e'', CHR ''f'', CHR ''g'',
CHR ''h'', CHR ''i'', CHR ''j'', CHR ''k'',
CHR ''l'', CHR ''m'', CHR ''n'', CHR ''o'',
CHR ''p'', CHR ''q'', CHR ''r'', CHR ''s'',
CHR ''t'', CHR ''u'', CHR ''v'', CHR ''w'',
CHR ''x'', CHR ''y'', CHR ''z'', CHR ''{'',
CHR ''|'', CHR ''}'', CHR ''~'', CHR 0x7F,
CHR 0x80, CHR 0x81, CHR 0x82, CHR 0x83,
CHR 0x84, CHR 0x85, CHR 0x86, CHR 0x87,
CHR 0x88, CHR 0x89, CHR 0x8A, CHR 0x8B,
CHR 0x8C, CHR 0x8D, CHR 0x8E, CHR 0x8F,
CHR 0x90, CHR 0x91, CHR 0x92, CHR 0x93,
CHR 0x94, CHR 0x95, CHR 0x96, CHR 0x97,
CHR 0x98, CHR 0x99, CHR 0x9A, CHR 0x9B,
CHR 0x9C, CHR 0x9D, CHR 0x9E, CHR 0x9F,
CHR 0xA0, CHR 0xA1, CHR 0xA2, CHR 0xA3,
CHR 0xA4, CHR 0xA5, CHR 0xA6, CHR 0xA7,
CHR 0xA8, CHR 0xA9, CHR 0xAA, CHR 0xAB,
CHR 0xAC, CHR 0xAD, CHR 0xAE, CHR 0xAF,
CHR 0xB0, CHR 0xB1, CHR 0xB2, CHR 0xB3,
CHR 0xB4, CHR 0xB5, CHR 0xB6, CHR 0xB7,
CHR 0xB8, CHR 0xB9, CHR 0xBA, CHR 0xBB,
CHR 0xBC, CHR 0xBD, CHR 0xBE, CHR 0xBF,
CHR 0xC0, CHR 0xC1, CHR 0xC2, CHR 0xC3,
CHR 0xC4, CHR 0xC5, CHR 0xC6, CHR 0xC7,
CHR 0xC8, CHR 0xC9, CHR 0xCA, CHR 0xCB,
CHR 0xCC, CHR 0xCD, CHR 0xCE, CHR 0xCF,
CHR 0xD0, CHR 0xD1, CHR 0xD2, CHR 0xD3,
CHR 0xD4, CHR 0xD5, CHR 0xD6, CHR 0xD7,
CHR 0xD8, CHR 0xD9, CHR 0xDA, CHR 0xDB,
CHR 0xDC, CHR 0xDD, CHR 0xDE, CHR 0xDF,
CHR 0xE0, CHR 0xE1, CHR 0xE2, CHR 0xE3,
CHR 0xE4, CHR 0xE5, CHR 0xE6, CHR 0xE7,
CHR 0xE8, CHR 0xE9, CHR 0xEA, CHR 0xEB,
CHR 0xEC, CHR 0xED, CHR 0xEE, CHR 0xEF,
CHR 0xF0, CHR 0xF1, CHR 0xF2, CHR 0xF3,
CHR 0xF4, CHR 0xF5, CHR 0xF6, CHR 0xF7,
CHR 0xF8, CHR 0xF9, CHR 0xFA, CHR 0xFB,
CHR 0xFC, CHR 0xFD, CHR 0xFE, CHR 0xFF]

definition
Enum.enum-all P ⟷ *list-all P* (*Enum.enum* :: *char list*)

definition
Enum.enum-ex P ⟷ *list-ex P* (*Enum.enum* :: *char list*)

lemma *enum-char-unfold*:
Enum.enum = *map char-of* [*0*..<*256*]
proof −
have *map* (*of-char* :: *char* ⇒ *nat*) *Enum.enum* = [*0*..<*256*]
by (*simp add*: *enum-char-def of-char-def upt-conv-Cons-Cons numeral-2-eq-2* [*symmetric*])
then have *map char-of* (*map* (*of-char* :: *char* ⇒ *nat*) *Enum.enum*) =
map char-of [*0*..<*256*]
by *simp*
then show *?thesis*
by *simp*
qed

instance proof
show *UNIV*: *UNIV* = *set* (*Enum.enum* :: *char list*)
by (*simp add*: *enum-char-unfold UNIV-char-of-nat atLeast0LessThan*)
show *distinct* (*Enum.enum* :: *char list*)
by (*auto simp add*: *enum-char-unfold distinct-map intro*: *inj-onI*)
show ⋀*P*. *Enum.enum-all P* ⟷ *Ball* (*UNIV* :: *char set*) *P*
by (*simp add*: *UNIV enum-all-char-def list-all-iff*)
show ⋀*P*. *Enum.enum-ex P* ⟷ *Bex* (*UNIV* :: *char set*) *P*
by (*simp add*: *UNIV enum-ex-char-def list-ex-iff*)
qed

end

lemma *linorder-char*:
class.linorder (λ*c d*. *of-char c* ≤ (*of-char d* :: *nat*)) (λ*c d*. *of-char c* < (*of-char d* :: *nat*))
by *standard auto*

Optimized version for execution

definition *char-of-integer* :: *integer* ⇒ *char*
where [*code-abbrev*]: *char-of-integer* = *char-of*

definition *integer-of-char* :: *char* ⇒ *integer*
where [*code-abbrev*]: *integer-of-char* = *of-char*

lemma *char-of-integer-code* [*code*]:
char-of-integer k = (*let*
(*q0*, *b0*) = *bit-cut-integer k*;
(*q1*, *b1*) = *bit-cut-integer q0*;
(*q2*, *b2*) = *bit-cut-integer q1*;
(*q3*, *b3*) = *bit-cut-integer q2*;
(*q4*, *b4*) = *bit-cut-integer q3*;
(*q5*, *b5*) = *bit-cut-integer q4*;
(*q6*, *b6*) = *bit-cut-integer q5*;
(-, *b7*) = *bit-cut-integer q6*
in Char b0 b1 b2 b3 b4 b5 b6 b7)

by (*simp add*: *bit-cut-integer-def char-of-integer-def char-of-def div-mult2-numeral-eq bit-iff-odd-drop-bit drop-bit-eq-div*)

lemma *integer-of-char-code* [*code*]:
integer-of-char (*Char b0 b1 b2 b3 b4 b5 b6 b7*) =
((((((*of-bool b7* ∗ *2* + *of-bool b6*) ∗ *2* +
of-bool b5) ∗ *2* + *of-bool b4*) ∗ *2* +
of-bool b3) ∗ *2* + *of-bool b2*) ∗ *2* +
of-bool b1) ∗ *2* + *of-bool b0*
by (*simp add*: *integer-of-char-def of-char-def*)

73.2 Strings as dedicated type for target language code generation

73.2.1 Logical specification

context
begin

qualified definition *ascii-of* :: *char* ⇒ *char*
where *ascii-of c* = *Char* (*digit0 c*) (*digit1 c*) (*digit2 c*) (*digit3 c*) (*digit4 c*) (*digit5 c*) (*digit6 c*) *False*

qualified lemma *ascii-of-Char* [*simp*]:
ascii-of (*Char b0 b1 b2 b3 b4 b5 b6 b7*) = *Char b0 b1 b2 b3 b4 b5 b6 False*
by (*simp add*: *ascii-of-def*)

qualified lemma *digit0-ascii-of-iff* [*simp*]:
digit0 (*String.ascii-of c*) ⟷ *digit0 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *digit1-ascii-of-iff* [*simp*]:
digit1 (*String.ascii-of c*) ⟷ *digit1 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *digit2-ascii-of-iff* [*simp*]:
digit2 (*String.ascii-of c*) ⟷ *digit2 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *digit3-ascii-of-iff* [*simp*]:
digit3 (*String.ascii-of c*) ⟷ *digit3 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *digit4-ascii-of-iff* [*simp*]:
digit4 (*String.ascii-of c*) ⟷ *digit4 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *digit5-ascii-of-iff* [*simp*]:
digit5 (*String.ascii-of c*) ⟷ *digit5 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *digit6-ascii-of-iff* [*simp*]:
digit6 (*String.ascii-of c*) ⟷ *digit6 c*
by (*simp add*: *String.ascii-of-def*)

qualified lemma *not-digit7-ascii-of* [*simp*]:
¬ *digit7* (*ascii-of c*)
by (*simp add*: *ascii-of-def*)

qualified lemma *ascii-of-idem*:
ascii-of c = *c* **if** ¬ *digit7 c*
using *that* **by** (*cases c*) *simp*

qualified typedef *literal* = {*cs*. ∀ *c*∈*set cs*. ¬ *digit7 c*}
morphisms *explode Abs-literal*
proof
show [] ∈ {*cs*. ∀ *c*∈*set cs*. ¬ *digit7 c*}
by *simp*
qed

qualified setup-lifting *type-definition-literal*

qualified lift-definition *implode* :: *string* ⇒ *literal*
is *map ascii-of*
by *auto*

qualified lemma *implode-explode-eq* [*simp*]:
String.implode (*String.explode s*) = *s*
proof *transfer*
fix *cs*
show *map ascii-of cs* = *cs* **if** ∀ *c*∈*set cs*. ¬ *digit7 c*
using *that*
by (*induction cs*) (*simp-all add*: *ascii-of-idem*)
qed

qualified lemma *explode-implode-eq* [*simp*]:
String.explode (*String.implode cs*) = *map ascii-of cs*
by *transfer rule*

end

context *linordered-euclidean-semiring-bit-operations*
begin

context
begin

qualified lemma *char-of-ascii-of* [*simp*]:
‹*of-char* (*String.ascii-of c*) = *take-bit 7* (*of-char c*)›

by (*cases c*) (*simp only*: *String.ascii-of-Char of-char-Char take-bit-horner-sum-bit-eq*, *simp*)

qualified lemma *ascii-of-char-of*:
‹*String.ascii-of* (*char-of a*) = *char-of* (*take-bit 7 a*)›
by (*simp add*: *char-of-def bit-simps*)

end

end

73.2.2 Syntactic representation

Logical ground representations for literals are:

1. *0* for the empty literal;
2. *Literal b0 ... b6 s* for a literal starting with one character and continued by another literal.

   Syntactic representations for literals are:

3. Printable text as string prefixed with *STR*;
4. A single ascii value as numerical hexadecimal value prefixed with *STR*.

instantiation *String.literal* :: *zero*
begin

context
begin

qualified lift-definition *zero-literal* :: *String.literal*
is *Nil*
by *simp*

instance ..

end

end

context
begin

qualified abbreviation (**output**) *empty-literal* :: *String.literal*
where *empty-literal* ≡ *0*

qualified lift-definition *Literal* :: *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *String.literal* ⇒ *String.literal*

is *λb0 b1 b2 b3 b4 b5 b6 cs. Char b0 b1 b2 b3 b4 b5 b6 False # cs*
by *auto*

qualified lemma *Literal-eq-iff* [*simp*]:
Literal b0 b1 b2 b3 b4 b5 b6 s = Literal c0 c1 c2 c3 c4 c5 c6 t
⟷ (*b0* ⟷ *c0*) ∧ (*b1* ⟷ *c1*) ∧ (*b2* ⟷ *c2*) ∧ (*b3* ⟷ *c3*)
∧ (*b4* ⟷ *c4*) ∧ (*b5* ⟷ *c5*) ∧ (*b6* ⟷ *c6*) ∧ *s* = *t*
by *transfer simp*

qualified lemma *empty-neq-Literal* [*simp*]:
empty-literal ≠ *Literal b0 b1 b2 b3 b4 b5 b6 s*
by *transfer simp*

qualified lemma *Literal-neq-empty* [*simp*]:
Literal b0 b1 b2 b3 b4 b5 b6 s ≠ *empty-literal*
by *transfer simp*

end

code-datatype *0* :: *String.literal String.Literal*

syntax
-Literal :: *str-position* ⇒ *String.literal*
(‹(‹*open-block notation*=‹*literal string*››*STR* -)›)
-Ascii :: *num-const* ⇒ *String.literal*
(‹(‹*open-block notation*=‹*literal char code*››*STR* -)›)
syntax-consts
-Literal -Ascii ⇌ *String.Literal*

ML-file ‹*Tools/literal.ML*›

73.2.3 Operations

instantiation *String.literal* :: *plus*
begin

context
begin

qualified lift-definition *plus-literal* :: *String.literal* ⇒ *String.literal* ⇒ *String.literal*
is (@)
by *auto*

instance ..

end

end

instance *String.literal* :: *monoid-add*
by (*standard*; *transfer*) *simp-all*

lemma *add-Literal-assoc*:
‹*String.Literal b0 b1 b2 b3 b4 b5 b6 t* + *s* = *String.Literal b0 b1 b2 b3 b4 b5 b6* (*t* + *s*)›
by *transfer simp*

instantiation *String.literal* :: *size*
begin

context
includes *literal.lifting*
begin

lift-definition *size-literal* :: *String.literal* ⇒ *nat*
is *length* **.**

end

instance ..

end

instantiation *String.literal* :: *equal*
begin

context
begin

qualified lift-definition *equal-literal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*
is *HOL.equal* **.**

instance
by (*standard*; *transfer*) (*simp add*: *equal*)

end

end

instantiation *String.literal* :: *linorder*
begin

context
begin

qualified lift-definition *less-eq-literal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*
is *ord.lexordp-eq* (λ*c d*. *of-char c* < (*of-char d* :: *nat*))
.

qualified lift-definition *less-literal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*
is *ord.lexordp* (λ*c d. of-char c* < (*of-char d* :: *nat*))
.

instance proof −
from *linorder-char* **interpret** *linorder ord.lexordp-eq* (λ*c d. of-char c* < (*of-char d* :: *nat*))
ord.lexordp (λ*c d. of-char c* < (*of-char d* :: *nat*)) :: *string* ⇒ *string* ⇒ *bool*
by (*rule linorder.lexordp-linorder*)
show *PROP ?thesis*
by (*standard*; *transfer*) (*simp-all add*: *less-le-not-le linear*)
qed

end

end

lemma *infinite-literal*:
infinite (*UNIV* :: *String.literal set*)
proof −
define *S* **where** *S* = *range* (λ*n. replicate n CHR ''A''*)
have *inj-on String.implode S*
proof (*rule inj-onI*)
fix *cs ds*
assume *String.implode cs* = *String.implode ds*
then have *String.explode* (*String.implode cs*) = *String.explode* (*String.implode ds*)
by *simp*
moreover assume *cs* ∈ *S* **and** *ds* ∈ *S*
ultimately show *cs* = *ds*
by (*auto simp add*: *S-def*)
qed
moreover have *infinite S*
by (*auto simp add*: *S-def dest*: *finite-range-imageI* [*of* - *length*])
ultimately have *infinite* (*String.implode* ‘ *S*)
by (*simp add*: *finite-image-iff*)
then show *?thesis*
by (*auto intro*: *finite-subset*)
qed

lemma *add-literal-code* [*code*]:
‹*STR* '''' + *s* = *s*›
‹*s* + *STR* '''' = *s*›
‹*String.Literal b0 b1 b2 b3 b4 b5 b6 t* + *s* = *String.Literal b0 b1 b2 b3 b4 b5 b6* (*t* + *s*)›
by (*simp-all add*: *add-Literal-assoc*)

73.2.4 Executable conversions

context
begin

qualified lift-definition *asciis-of-literal* :: *String.literal* ⇒ *integer list*
  is *map of-char*
  .

qualified lemma *asciis-of-zero* [*simp*, *code*]:
  asciis-of-literal 0 = []
  by *transfer simp*

qualified lemma *asciis-of-Literal* [*simp*, *code*]:
  asciis-of-literal (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*) =
    of-char (*Char b0 b1 b2 b3 b4 b5 b6 False*) # *asciis-of-literal s*
  by *transfer simp*

qualified lift-definition *literal-of-asciis* :: *integer list* ⇒ *String.literal*
  is *map* (*String.ascii-of* ∘ *char-of*)
  by *auto*

qualified lemma *literal-of-asciis-Nil* [*simp*, *code*]:
  literal-of-asciis [] = *0*
  by *transfer simp*

qualified lemma *literal-of-asciis-Cons* [*simp*, *code*]:
  literal-of-asciis (*k* # *ks*) = (*case char-of k*
     of Char b0 b1 b2 b3 b4 b5 b6 b7 ⇒ *String.Literal b0 b1 b2 b3 b4 b5 b6*
(*literal-of-asciis ks*))
  by (*simp add*: *char-of-def*) (*transfer*, *simp add*: *char-of-def*)

qualified lemma *literal-of-asciis-of-literal* [*simp*]:
  literal-of-asciis (*asciis-of-literal s*) = *s*
proof *transfer*
  fix *cs*
  assume ∀ *c*∈*set cs*. ¬ *digit7 c*
  then show *map* (*String.ascii-of* ∘ *char-of*) (*map of-char cs*) = *cs*
    by (*induction cs*) (*simp-all add*: *String.ascii-of-idem*)
qed

qualified lemma *explode-code* [*code*]:
  String.explode s = *map char-of* (*asciis-of-literal s*)
  by *transfer simp*

qualified lemma *implode-code* [*code*]:
  String.implode cs = *literal-of-asciis* (*map of-char cs*)
  by *transfer simp*

qualified lemma *equal-literal* [*code*]:

HOL.equal (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*)
(*String.Literal a0 a1 a2 a3 a4 a5 a6 r*)
$\longleftrightarrow$ (*b0* $\longleftrightarrow$ *a0*) $\wedge$ (*b1* $\longleftrightarrow$ *a1*) $\wedge$ (*b2* $\longleftrightarrow$ *a2*) $\wedge$ (*b3* $\longleftrightarrow$ *a3*)
$\wedge$ (*b4* $\longleftrightarrow$ *a4*) $\wedge$ (*b5* $\longleftrightarrow$ *a5*) $\wedge$ (*b6* $\longleftrightarrow$ *a6*) $\wedge$ (*s* = *r*)
by (*simp add*: *equal*)

end

73.2.5 Technical code generation setup

Alternative constructor for generated computations

context
begin

qualified definition *Literal′* :: *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *String.literal* ⇒ *String.literal*
where [*simp*]: *Literal′* = *String.Literal*

lemma [*code*]:
‹*Literal′ b0 b1 b2 b3 b4 b5 b6 s* = *String.literal-of-asciis*
[*foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*] *0*] + *s*›
proof −
have ‹*foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*] *0* = *of-char* (*Char b0 b1 b2 b3 b4 b5 b6 False*)›
by *simp*
moreover have ‹*Literal′ b0 b1 b2 b3 b4 b5 b6 s* = *String.literal-of-asciis*
[*of-char* (*Char b0 b1 b2 b3 b4 b5 b6 False*)] + *s*›
by (*unfold Literal′-def*) (*transfer*, *simp only*: *list.simps comp-apply char-of-char*, *simp*)
ultimately show *?thesis*
by *simp*
qed

lemma [*code-computation-unfold*]:
String.Literal = *Literal′*
by *simp*

end

code-reserved
(*SML*) *string String Char Str-Literal*
and (*OCaml*) *string String Char Str-Literal*
and (*Haskell*) *Str-Literal*
and (*Scala*) *String Str-Literal*

code-identifier
code-module *String* ⇀
(*SML*) *Str* **and** (*OCaml*) *Str* **and** (*Haskell*) *Str* **and** (*Scala*) *Str*

code-printing
type-constructor *String.literal* ⇀
(*SML*) *string*
and (*OCaml*) *string*
and (*Haskell*) *String*
and (*Scala*) *String*
| **constant** *STR ''''* ⇀
(*SML*)
and (*OCaml*)
and (*Haskell*)
and (*Scala*)

setup ‹
fold Literal.add-code [*SML*, *OCaml*, *Haskell*, *Scala*]
›

code-printing
code-module *Str-Literal* ⇀
(*SML*) ‹*structure Str-Literal* : *sig*
type int = *IntInf.int*
val literal-of-asciis : *int list* −> *string*
val asciis-of-literal : *string* −> *int list*
end = *struct*

open IntInf;

fun map f [] = []
| *map f* (*x* :: *xs*) = *f x* :: *map f xs*; (∗ *deliberate clone not relying on List.- structure* ∗)

fun check-ascii k =
if 0 <= *k andalso k* < *128*
then k
else raise Fail Non−*ASCII character in literal*;

val char-of-ascii = *Char.chr o toInt o* (*fn k* => *k mod 128*);

val ascii-of-char = *check-ascii o fromInt o Char.ord*;

val literal-of-asciis = *String.implode o map char-of-ascii*;

val asciis-of-literal = *map ascii-of-char o String.explode*;

end;› **for constant** *String.literal-of-asciis String.asciis-of-literal*
and (*OCaml*) ‹*module Str-Literal* : *sig*
val literal-of-asciis : *Z.t list* −> *string*
val asciis-of-literal: *string* −> *Z.t list*
end = *struct*

```
(* deliberate clones not relying on List.- module *)

let rec length xs = match xs with
    [] -> 0
  | x :: xs -> 1 + length xs;;

let rec nth xs n = match xs with
  (x :: xs) -> if n <= 0 then x else nth xs (n - 1);;

let rec map-range f n =
  if n <= 0
    then []
    else
      let m = n - 1
    in map-range f m @ [f m];;

let implode f xs =
  String.init (length xs) (fun n -> f (nth xs n));;

let explode f s =
  map-range (fun n -> f (String.get s n)) (String.length s);;

let z-128 = Z.of-int 128;;

let check-ascii k =
  if 0 <= k && k < 128
  then k
  else failwith Non-ASCII character in literal;;

let char-of-ascii k = Char.chr (Z.to-int (Z.rem k z-128));;

let ascii-of-char c = Z.of-int (check-ascii (Char.code c));;

let literal-of-asciis ks = implode char-of-ascii ks;;

let asciis-of-literal s = explode ascii-of-char s;;

end;;› for constant String.literal-of-asciis String.asciis-of-literal
    and (Haskell) ‹module Str-Literal(literalOfAsciis, asciisOfLiteral) where

check-ascii :: Int -> Int
check-ascii k
  | (0 <= k && k < 128) = k
  | otherwise = error Non-ASCII character in literal

charOfAscii :: Integer -> Char
charOfAscii = toEnum . Prelude.fromInteger . (\k -> k `mod ` 128)

asciiOfChar :: Char -> Integer
```

asciiOfChar = *toInteger* . *check-ascii* . *fromEnum*

literalOfAsciis :: [*Integer*] −> [*Char*]
literalOfAsciis = *map charOfAscii*

asciisOfLiteral :: [*Char*] −> [*Integer*]
asciisOfLiteral = *map asciiOfChar*
› **for constant** *String.literal-of-asciis String.asciis-of-literal*
  and (*Scala*) ‹*object Str-Literal* {

private def checkAscii(*k* : *Int*) : *Int* =
  0 <= *k* && *k* < *128 match* {
    case true => *k*
    case false => *sys.error*(*Non*−*ASCII character in literal*)
  }

private def charOfAscii(*k* : *BigInt*) : *Char* =
  (*k* % *128*).*charValue*

private def asciiOfChar(*c* : *Char*) : *BigInt* =
  BigInt(*checkAscii*(*c.toInt*))

def literalOfAsciis(*ks* : *List*[*BigInt*]) : *String* =
  ks.map(*charOfAscii*).*mkString*

def asciisOfLiteral(*s* : *String*) : *List*[*BigInt*] =
  s.toList.map(*asciiOfChar*)

}
› **for constant** *String.literal-of-asciis String.asciis-of-literal*
| **constant** ‹(+) :: *String.literal* ⇒ *String.literal* ⇒ *String.literal*› ⇀
  (*SML*) **infixl** *18* ^
  and (*OCaml*) **infixr** *6* ^
  and (*Haskell*) **infixr** *5* ++
  and (*Scala*) **infixl** *7* +
| **constant** *String.literal-of-asciis* ⇀
  (*SML*) *Str′-Literal.literal′-of′-asciis*
  and (*OCaml*) *Str′-Literal.literal′-of′-asciis*
  and (*Haskell*) *Str′-Literal.literalOfAsciis*
  and (*Scala*) *Str′-Literal.literalOfAsciis*
| **constant** *String.asciis-of-literal* ⇀
  (*SML*) *Str′-Literal.asciis′-of′-literal*
  and (*OCaml*) *Str′-Literal.asciis′-of′-literal*
  and (*Haskell*) *Str′-Literal.asciisOfLiteral*
  and (*Scala*) *Str′-Literal.asciisOfLiteral*
| **class-instance** *String.literal* :: *equal* ⇀
  (*Haskell*) −
| **constant** ‹*HOL.equal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*› ⇀
  (*SML*) !((- : *string*) = -)

and (*OCaml*) !((- : *string*) = -)
and (*Haskell*) **infix** *4* ==
and (*Scala*) **infixl** *5* ==
| **constant** ‹(≤) :: *String.literal* ⇒ *String.literal* ⇒ *bool*› ⇀
(*SML*) !((- : *string*) <= -)
and (*OCaml*) !((- : *string*) <= -)
and (*Haskell*) **infix** *4* <=

— Order operations for *String.literal* work in Haskell only if no type class instance needs to be generated, because String = [Char] in Haskell and *char list* need not have the same order as *String.literal*.

and (*Scala*) **infixl** *4* <=
and (*Eval*) **infixl** *6* <=
| **constant** (<) :: *String.literal* ⇒ *String.literal* ⇒ *bool* ⇀
(*SML*) !((- : *string*) < -)
and (*OCaml*) !((- : *string*) < -)
and (*Haskell*) **infix** *4* <
and (*Scala*) **infixl** *4* <
and (*Eval*) **infixl** *6* <

73.2.6 Code generation utility

setup ‹*Sign.map-naming* (*Name-Space.mandatory-path Code*)›

definition *abort* :: *String.literal* ⇒ (*unit* ⇒ *'a*) ⇒ *'a*
where [*simp*]: *abort - f* = *f* ()

declare [[*code drop*: *Code.abort*]]

lemma *abort-cong*:
msg = *msg'* ⟹ *Code.abort msg f* = *Code.abort msg' f*
by *simp*

setup ‹*Sign.map-naming Name-Space.parent-path*›

setup ‹*Code-Simp.map-ss* (*Simplifier.add-cong* @{*thm Code.abort-cong*})›

code-printing
constant *Code.abort* ⇀
(*SML*) !(*raise*/ *Fail*/ -)
and (*OCaml*) *failwith*
and (*Haskell*) !(*error*/ ::/ *forall a.*/ *String* −> (() −> *a*) −> *a*)
and (*Scala*) !{/ *sys.error*((-));/ ((-)).*apply*(())/ }

73.2.7 Finally

lifting-update *literal.lifting*
lifting-forget *literal.lifting*

end

74 Reflecting Pure types into HOL

theory *Typerep*
imports *String*
begin

datatype *typerep* = *Typerep String.literal typerep list*

class *typerep* =
  fixes *typerep* :: *'a itself* ⇒ *typerep*
begin

definition *typerep-of* :: *'a* ⇒ *typerep* **where**
  [*simp*]: *typerep-of x* = *typerep TYPE('a)*

end

syntax
  -TYPEREP :: *type* => *logic* (‹(‹indent=1 notation=‹mixfix TYPEREP›› TYPEREP/(1'(-')))›)
syntax-consts
  -TYPEREP ⇌ *typerep*

parse-translation ‹
  let
    fun typerep-tr (∗*-TYPEREP*∗) [*ty*] =
        Syntax.const ***const-syntax*** ‹*typerep*› $
          (*Syntax.const* ***syntax-const*** ‹*-constrain*› $ *Syntax.const* ***const-syntax*** ‹*Pure.type*› $
            (*Syntax.const* ***type-syntax*** ‹*itself*› $ *ty*))
      | *typerep-tr* (∗*-TYPEREP*∗) *ts* = *raise TERM* (*typerep-tr*, *ts*);
  in [(***syntax-const*** ‹*-TYPEREP*›, *K typerep-tr*)] *end*
›

typed-print-translation ‹
  let
    fun typerep-tr' ctxt (∗*typerep*∗) ***Type*** ‹*fun* ***Type*** ‹*itself T*› *-*›
          (*Const* (***const-syntax*** ‹*Pure.type*›, *-*) :: *ts*) =
        Term.list-comb
          (*Syntax.const* ***syntax-const*** ‹*-TYPEREP*› $ *Syntax-Phases.term-of-typ ctxt T*, *ts*)
      | *typerep-tr' - T ts* = *raise Match*;
  in [(***const-syntax*** ‹*typerep*›, *typerep-tr'*)] *end*
›

setup ‹
let

fun add-typerep tyco thy =
  let

```
    val sorts = replicate (Sign.arity-number thy tyco) sort ‹typerep›;
    val vs = Name.invent-types-global sorts;
    val ty = Type (tyco, map TFree vs);
    val lhs = Const ‹typerep ty› $ Free (T, Term.itselfT ty);
    val rhs = Const ‹Typerep› $ HOLogic.mk-literal tyco
      $ HOLogic.mk-list Type ‹typerep› (map (HOLogic.mk-typerep o TFree) vs);
    val eq = HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs, rhs));
  in
    thy
    |> Class.instantiation ([tyco], vs, sort ‹typerep›)
    |> `(fn lthy => Syntax.check-term lthy eq)
    |-> (fn eq => Specification.definition NONE [] [] (Binding.empty-atts, eq))
    |> snd
    |> Class.prove-instantiation-exit (fn ctxt => Class.intro-classes-tac ctxt [])
  end;

fun ensure-typerep tyco thy =
  if not (Sorts.has-instance (Sign.classes-of thy) tyco sort ‹typerep›)
    andalso Sorts.has-instance (Sign.classes-of thy) tyco sort ‹type›
  then add-typerep tyco thy else thy;

in

add-typerep type-name ‹fun›
#> Typedef.interpretation (Local-Theory.background-theory o ensure-typerep)
#> Code.type-interpretation ensure-typerep

end
›
```

lemma [*code*]:
 HOL.equal (*Typerep tyco1 tys1*) (*Typerep tyco2 tys2*) ⟷ *HOL.equal tyco1 tyco2*
  ∧ *list-all2 HOL.equal tys1 tys2*
 by (*auto simp add*: *eq-equal* [*symmetric*] *list-all2-eq* [*symmetric*])

lemma [*code nbe*]:
 HOL.equal (*x* :: *typerep*) *x* ⟷ *True*
 by (*fact equal-refl*)

code-printing
 type-constructor *typerep* ⇀ (*Eval*) *Term.typ*
| **constant** *Typerep* ⇀ (*Eval*) *Term.Type/* (-, -)

code-reserved
 (*Eval*) *Term*

hide-const (**open**) *typerep Typerep*

end

75 Predicates as enumerations

theory *Predicate*
imports *String*
begin

75.1 The type of predicate enumerations (a monad)

datatype (*plugins only*: *extraction*) (*dead 'a*) *pred* = *Pred* (*eval*: *'a* ⇒ *bool*)

lemma *pred-eqI*:
 $(\bigwedge w.\ eval\ P\ w \longleftrightarrow eval\ Q\ w) \Longrightarrow P = Q$
 by (*cases P*, *cases Q*) (*auto simp add*: *fun-eq-iff*)

lemma *pred-eq-iff*:
 $P = Q \Longrightarrow (\bigwedge w.\ eval\ P\ w \longleftrightarrow eval\ Q\ w)$
 by (*simp add*: *pred-eqI*)

instantiation *pred* :: (*type*) *complete-lattice*
begin

definition
 $P \le Q \longleftrightarrow eval\ P \le eval\ Q$

definition
 $P < Q \longleftrightarrow eval\ P < eval\ Q$

definition
 $\bot = Pred\ \bot$

lemma *eval-bot* [*simp*]:
 $eval\ \bot\ = \bot$
 by (*simp add*: *bot-pred-def*)

definition
 $\top = Pred\ \top$

lemma *eval-top* [*simp*]:
 $eval\ \top\ = \top$
 by (*simp add*: *top-pred-def*)

definition
 $P \sqcap Q = Pred\ (eval\ P \sqcap eval\ Q)$

lemma *eval-inf* [*simp*]:
 $eval\ (P \sqcap Q) = eval\ P \sqcap eval\ Q$
 by (*simp add*: *inf-pred-def*)

definition
 $P \sqcup Q = Pred\ (eval\ P \sqcup eval\ Q)$

lemma *eval-sup* [*simp*]:
$eval\ (P \sqcup Q) = eval\ P \sqcup eval\ Q$
by (*simp add*: *sup-pred-def*)

definition
$\bigsqcap A = Pred\ (\bigsqcap (eval\ `\ A))$

lemma *eval-Inf* [*simp*]:
$eval\ (\bigsqcap A) = \bigsqcap (eval\ `\ A)$
by (*simp add*: *Inf-pred-def*)

definition
$\bigsqcup A = Pred\ (\bigsqcup (eval\ `\ A))$

lemma *eval-Sup* [*simp*]:
$eval\ (\bigsqcup A) = \bigsqcup (eval\ `\ A)$
by (*simp add*: *Sup-pred-def*)

instance proof
qed (*auto intro*!: *pred-eqI simp add*: *less-eq-pred-def less-pred-def le-fun-def less-fun-def*)

end

lemma *eval-INF* [*simp*]:
$eval\ (\bigsqcap (f\ `\ A)) = \bigsqcap ((eval \circ f)\ `\ A)$
by (*simp add*: *image-comp*)

lemma *eval-SUP* [*simp*]:
$eval\ (\bigsqcup (f\ `\ A)) = \bigsqcup ((eval \circ f)\ `\ A)$
by (*simp add*: *image-comp*)

instantiation *pred* :: (*type*) *complete-boolean-algebra*
begin

definition
$-\ P = Pred\ (-\ eval\ P)$

lemma *eval-compl* [*simp*]:
$eval\ (-\ P) = -\ eval\ P$
by (*simp add*: *uminus-pred-def*)

definition
$P - Q = Pred\ (eval\ P - eval\ Q)$

lemma *eval-minus* [*simp*]:
$eval\ (P - Q) = eval\ P - eval\ Q$
by (*simp add*: *minus-pred-def*)

instance proof
 fix *A*::′*a pred set set*
 show $\bigsqcap$(*Sup* ‘ *A*) ≤ $\bigsqcup$(*Inf* ‘ {*f* ‘ *A* |*f*. ∀ *Y*∈*A*. *f Y* ∈ *Y*})
 proof (*simp add*: *less-eq-pred-def Sup-fun-def Inf-fun-def*, *safe*)
  fix *w*
  assume *A*: ∀ *x*∈*A*. ∃ *f*∈*x*. *eval f w*
  define *F* **where** *F* = (λ *x* . *SOME f* . *f* ∈ *x* ∧ *eval f w*)
  have [*simp*]: (∀ *f*∈ (*F* ‘ *A*). *eval f w*)
   by (*metis* (*no-types*, *lifting*) *A F-def image-iff some-eq-ex*)
  have (∃ *f*. *F* ‘ *A* = *f* ‘ *A* ∧ (∀ *Y*∈*A*. *f Y* ∈ *Y*)) ∧ (∀ *f*∈(*F* ‘ *A*). *eval f w*)
   using *A* **by** (*simp*, *metis* (*no-types*, *lifting*) *F-def someI*)+
  from *this* **show** ∃ *x*. (∃ *f*. *x* = *f* ‘ *A* ∧ (∀ *Y*∈*A*. *f Y* ∈ *Y*)) ∧ (∀ *f*∈*x*. *eval f w*)
   by (*rule exI* [*of* - *F* ‘ *A*])
 qed
qed (*auto intro*!: *pred-eqI*)

end

definition *single* :: ′*a* ⇒ ′*a pred* **where**
 single x = *Pred* ((=) *x*)

lemma *eval-single* [*simp*]:
 eval (*single x*) = (=) *x*
 by (*simp add*: *single-def*)

definition *bind* :: ′*a pred* ⇒ (′*a* ⇒ ′*b pred*) ⇒ ′*b pred* (**infixl** ‹≫=› *70*) **where**
 P ≫= *f* = ($\bigsqcup$(*f* ‘ {*x*. *eval P x*}))

lemma *eval-bind* [*simp*]:
 eval (*P* ≫= *f*) = *eval* ($\bigsqcup$(*f* ‘ {*x*. *eval P x*}))
 by (*simp add*: *bind-def*)

lemma *bind-bind*:
 (*P* ≫= *Q*) ≫= *R* = *P* ≫= (λ*x*. *Q x* ≫= *R*)
 by (*rule pred-eqI*) *auto*

lemma *bind-single*:
 P ≫= *single* = *P*
 by (*rule pred-eqI*) *auto*

lemma *single-bind*:
 single x ≫= *P* = *P x*
 by (*rule pred-eqI*) *auto*

lemma *bottom-bind*:
 ⊥ ≫= *P* = ⊥
 by (*rule pred-eqI*) *auto*

lemma *sup-bind*:

$(P \sqcup Q) \gg= R = P \gg= R \sqcup Q \gg= R$
by (*rule pred-eqI*) *auto*

lemma *Sup-bind*:
$(\bigsqcup A \gg= f) = \bigsqcup((\lambda x.\ x \gg= f)\ `\ A)$
by (*rule pred-eqI*) *auto*

lemma *pred-iffI*:
assumes $\bigwedge x.$ *eval* $A\ x \Longrightarrow$ *eval* $B\ x$
and $\bigwedge x.$ *eval* $B\ x \Longrightarrow$ *eval* $A\ x$
shows $A = B$
using *assms* **by** (*auto intro*: *pred-eqI*)

lemma *singleI*: *eval* (*single* x) x
by *simp*

lemma *singleI-unit*: *eval* (*single* ()) x
by *simp*

lemma *singleE*: *eval* (*single* x) $y \Longrightarrow (y = x \Longrightarrow P) \Longrightarrow P$
by *simp*

lemma *singleE'*: *eval* (*single* x) $y \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow P$
by *simp*

lemma *bindI*: *eval* $P\ x \Longrightarrow$ *eval* $(Q\ x)\ y \Longrightarrow$ *eval* $(P \gg= Q)\ y$
by *auto*

lemma *bindE*: *eval* $(R \gg= Q)\ y \Longrightarrow (\bigwedge x.$ *eval* $R\ x \Longrightarrow$ *eval* $(Q\ x)\ y \Longrightarrow P) \Longrightarrow$
P
by *auto*

lemma *botE*: *eval* $\bot\ x \Longrightarrow P$
by *auto*

lemma *supI1*: *eval* $A\ x \Longrightarrow$ *eval* $(A \sqcup B)\ x$
by *auto*

lemma *supI2*: *eval* $B\ x \Longrightarrow$ *eval* $(A \sqcup B)\ x$
by *auto*

lemma *supE*: *eval* $(A \sqcup B)\ x \Longrightarrow$ (*eval* $A\ x \Longrightarrow P) \Longrightarrow$ (*eval* $B\ x \Longrightarrow P) \Longrightarrow P$
by *auto*

lemma *single-not-bot* [*simp*]:
single $x \neq \bot$
by (*auto simp add*: *single-def bot-pred-def fun-eq-iff*)

lemma *not-bot*:

assumes $A \neq \bot$
obtains x **where** *eval A x*
using *assms* **by** (*cases A*) (*auto simp add*: *bot-pred-def*)

75.2 Emptiness check and definite choice

definition *is-empty* :: *'a pred* ⇒ *bool* **where**
is-empty A ⟷ $A = \bot$

lemma *is-empty-bot*:
is-empty ⊥
by (*simp add*: *is-empty-def*)

lemma *not-is-empty-single*:
¬ *is-empty* (*single x*)
by (*auto simp add*: *is-empty-def single-def bot-pred-def fun-eq-iff*)

lemma *is-empty-sup*:
is-empty (*A* ⊔ *B*) ⟷ *is-empty A* ∧ *is-empty B*
by (*auto simp add*: *is-empty-def*)

definition *singleton* :: (*unit* ⇒ *'a*) ⇒ *'a pred* ⇒ *'a* **where**
singleton default A = (*if* ∃!*x*. *eval A x then THE x*. *eval A x else default* ()) **for** *default*

lemma *singleton-eqI*:
∃!*x*. *eval A x* ⟹ *eval A x* ⟹ *singleton default A* = *x* **for** *default*
by (*auto simp add*: *singleton-def*)

lemma *eval-singletonI*:
∃!*x*. *eval A x* ⟹ *eval A* (*singleton default A*) **for** *default*
proof −
assume *assm*: ∃!*x*. *eval A x*
then obtain *x* **where** *x*: *eval A x* **..**
with *assm* **have** *singleton default A* = *x* **by** (*rule singleton-eqI*)
with *x* **show** *?thesis* **by** *simp*
qed

lemma *single-singleton*:
∃!*x*. *eval A x* ⟹ *single* (*singleton default A*) = *A* **for** *default*
proof −
assume *assm*: ∃!*x*. *eval A x*
then have *eval A* (*singleton default A*)
by (*rule eval-singletonI*)
moreover from *assm* **have** ⋀*x*. *eval A x* ⟹ *singleton default A* = *x*
by (*rule singleton-eqI*)
ultimately have *eval* (*single* (*singleton default A*)) = *eval A*
by (*simp* (*no-asm-use*) *add*: *single-def fun-eq-iff*) *blast*
then have ⋀*x*. *eval* (*single* (*singleton default A*)) *x* = *eval A x*

  by *simp*
 then show *?thesis* **by** (*rule pred-eqI*)
qed

lemma *singleton-undefinedI*:
 $\neg$ ($\exists$!*x*. *eval A x*) $\Longrightarrow$ *singleton default A* = *default* () **for** *default*
 by (*simp add*: *singleton-def*)

lemma *singleton-bot*:
 singleton default $\bot$ = *default* () **for** *default*
 by (*auto simp add*: *bot-pred-def intro*: *singleton-undefinedI*)

lemma *singleton-single*:
 singleton default (*single x*) = *x* **for** *default*
 by (*auto simp add*: *intro*: *singleton-eqI singleI elim*: *singleE*)

lemma *singleton-sup-single-single*:
 singleton default (*single x* $\sqcup$ *single y*) = (*if x* = *y then x else default* ()) **for** *default*
proof (*cases x* = *y*)
 case *True* **then show** *?thesis* **by** (*simp add*: *singleton-single*)
next
 case *False*
 have *eval* (*single x* $\sqcup$ *single y*) *x*
  and *eval* (*single x* $\sqcup$ *single y*) *y*
 by (*auto intro*: *supI1 supI2 singleI*)
 with *False* **have** $\neg$ ($\exists$!*z*. *eval* (*single x* $\sqcup$ *single y*) *z*)
  by *blast*
 then have *singleton default* (*single x* $\sqcup$ *single y*) = *default* ()
  by (*rule singleton-undefinedI*)
 with *False* **show** *?thesis* **by** *simp*
qed

lemma *singleton-sup-aux*:
 singleton default (*A* $\sqcup$ *B*) = (*if A* = $\bot$ *then singleton default B*
  else if B = $\bot$ *then singleton default A*
  else singleton default
   (*single* (*singleton default A*) $\sqcup$ *single* (*singleton default B*))) **for** *default*
proof (*cases* ($\exists$!*x*. *eval A x*) $\wedge$ ($\exists$!*y*. *eval B y*))
 case *True* **then show** *?thesis* **by** (*simp add*: *single-singleton*)
next
 case *False*
 from *False* **have** *A-or-B*:
  singleton default A = *default* () $\vee$ *singleton default B* = *default* ()
  by (*auto intro*!: *singleton-undefinedI*)
 then have *rhs*: *singleton default*
  (*single* (*singleton default A*) $\sqcup$ *single* (*singleton default B*)) = *default* ()
  by (*auto simp add*: *singleton-sup-single-single singleton-single*)
 from *False* **have** *not-unique*:

$\neg$ ($\exists$!*x*. *eval A x*) $\vee$ $\neg$ ($\exists$!*y*. *eval B y*) **by** *simp*
show *?thesis* **proof** (*cases A* $\neq \bot \wedge$ *B* $\neq \bot$)
case *True*
then obtain *a b* **where** *a*: *eval A a* **and** *b*: *eval B b*
by (*blast elim*: *not-bot*)
with *True not-unique* **have** $\neg$ ($\exists$!*x*. *eval* (*A* $\sqcup$ *B*) *x*)
by (*auto simp add*: *sup-pred-def bot-pred-def*)
then have *singleton default* (*A* $\sqcup$ *B*) = *default* () **by** (*rule singleton-undefinedI*)
with *True rhs* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis* **by** *auto*
qed
qed

lemma *singleton-sup*:
singleton default (*A* $\sqcup$ *B*) = (*if A* = $\bot$ *then singleton default B*
else if B = $\bot$ *then singleton default A*
else if singleton default A = *singleton default B then singleton default A else default* ()) **for** *default*
using *singleton-sup-aux* [*of default A B*] **by** (*simp only*: *singleton-sup-single-single*)

75.3 Derived operations

definition *if-pred* :: *bool* $\Rightarrow$ *unit pred* **where**
if-pred-eq: *if-pred b* = (*if b then single* () *else* $\bot$)

definition *holds* :: *unit pred* $\Rightarrow$ *bool* **where**
holds-eq: *holds P* = *eval P* ()

definition *not-pred* :: *unit pred* $\Rightarrow$ *unit pred* **where**
not-pred-eq: *not-pred P* = (*if eval P* () *then* $\bot$ *else single* ())

lemma *if-predI*: *P* $\Longrightarrow$ *eval* (*if-pred P*) ()
unfolding *if-pred-eq* **by** (*auto intro*: *singleI*)

lemma *if-predE*: *eval* (*if-pred b*) *x* $\Longrightarrow$ (*b* $\Longrightarrow$ *x* = () $\Longrightarrow$ *P*) $\Longrightarrow$ *P*
unfolding *if-pred-eq* **by** (*cases b*) (*auto elim*: *botE*)

lemma *not-predI*: $\neg$ *P* $\Longrightarrow$ *eval* (*not-pred* (*Pred* (λ*u*. *P*))) ()
unfolding *not-pred-eq* **by** (*auto intro*: *singleI*)

lemma *not-predI'*: $\neg$ *eval P* () $\Longrightarrow$ *eval* (*not-pred P*) ()
unfolding *not-pred-eq* **by** (*auto intro*: *singleI*)

lemma *not-predE*: *eval* (*not-pred* (*Pred* (λ*u*. *P*))) *x* $\Longrightarrow$ ($\neg$ *P* $\Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
unfolding *not-pred-eq*
by (*auto split*: *if-split-asm elim*: *botE*)

lemma *not-predE'*: *eval* (*not-pred P*) *x* ⟹ (¬ *eval P x* ⟹ *thesis*) ⟹ *thesis*
  unfolding *not-pred-eq*
  by (*auto split*: *if-split-asm elim*: *botE*)
lemma *f* () = *False* ∨ *f* () = *True*
by *simp*

lemma *closure-of-bool-cases* [*no-atp*]:
  fixes *f* :: *unit* ⇒ *bool*
  assumes *f* = (λ*u*. *False*) ⟹ *P f*
  assumes *f* = (λ*u*. *True*) ⟹ *P f*
  shows *P f*
proof −
  have *f* = (λ*u*. *False*) ∨ *f* = (λ*u*. *True*)
    apply (*cases f* ())
    apply (*rule disjI2*)
    apply (*rule ext*)
    apply (*simp add*: *unit-eq*)
    apply (*rule disjI1*)
    apply (*rule ext*)
    apply (*simp add*: *unit-eq*)
    done
  from *this assms* **show** *?thesis* **by** *blast*
qed

lemma *unit-pred-cases*:
  assumes *P* ⊥
  assumes *P* (*single* ())
  shows *P Q*
using *assms* **unfolding** *bot-pred-def bot-fun-def bot-bool-def empty-def single-def*
proof (*cases Q*)
  fix *f*
  assume *P* (*Pred* (λ*u*. *False*)) *P* (*Pred* (λ*u*. () = *u*))
  then have *P* (*Pred f*)
    by (*cases* - *f rule*: *closure-of-bool-cases*) *simp-all*
  moreover assume *Q* = *Pred f*
  ultimately show *P Q* **by** *simp*
qed

lemma *holds-if-pred*:
  holds (*if-pred b*) = *b*
unfolding *if-pred-eq holds-eq*
by (*cases b*) (*auto intro*: *singleI elim*: *botE*)

lemma *if-pred-holds*:
  if-pred (*holds P*) = *P*
unfolding *if-pred-eq holds-eq*
by (*rule unit-pred-cases*) (*auto intro*: *singleI elim*: *botE*)

lemma *is-empty-holds*:

is-empty P ⟷ ¬ *holds P*
unfolding *is-empty-def holds-eq*
by (*rule unit-pred-cases*) (*auto elim*: *botE intro*: *singleI*)

definition *map* :: ('*a* ⇒ '*b*) ⇒ '*a pred* ⇒ '*b pred* **where**
map f P = *P* ≫= (*single* ∘ *f*)

lemma *eval-map* [*simp*]:
eval (*map f P*) = (⨆*x*∈{*x*. *eval P x*}. (λ*y*. *f x* = *y*))
by (*simp add*: *map-def comp-def image-comp*)

functor *map*: *map*
by (*rule ext*, *rule pred-eqI*, *auto*)+

75.4 Implementation

datatype (*plugins only*: *code extraction*) (*dead* '*a*) *seq* =
Empty
| *Insert* '*a* '*a pred*
| *Join* '*a pred* '*a seq*

primrec *pred-of-seq* :: '*a seq* ⇒ '*a pred* **where**
pred-of-seq Empty = ⊥
| *pred-of-seq* (*Insert x P*) = *single x* ⊔ *P*
| *pred-of-seq* (*Join P xq*) = *P* ⊔ *pred-of-seq xq*

definition *Seq* :: (*unit* ⇒ '*a seq*) ⇒ '*a pred* **where**
Seq f = *pred-of-seq* (*f* ())

code-datatype *Seq*

primrec *member* :: '*a seq* ⇒ '*a* ⇒ *bool* **where**
member Empty x ⟷ *False*
| *member* (*Insert y P*) *x* ⟷ *x* = *y* ∨ *eval P x*
| *member* (*Join P xq*) *x* ⟷ *eval P x* ∨ *member xq x*

lemma *eval-member*:
member xq = *eval* (*pred-of-seq xq*)
proof (*induct xq*)
case *Empty* **show** *?case*
by (*auto simp add*: *fun-eq-iff elim*: *botE*)
next
case *Insert* **show** *?case*
by (*auto simp add*: *fun-eq-iff elim*: *supE singleE intro*: *supI1 supI2 singleI*)
next
case *Join* **then show** *?case*
by (*auto simp add*: *fun-eq-iff elim*: *supE intro*: *supI1 supI2*)
qed

lemma *eval-code* [*code*]: *eval* (*Seq f*) = *member* (*f* ())
  unfolding *Seq-def* **by** (*rule sym*, *rule eval-member*)

lemma *single-code* [*code*]:
  single x = *Seq* (λ*u*. *Insert x* ⊥)
  unfolding *Seq-def* **by** *simp*

primrec *apply* :: (′*a* ⇒ ′*b pred*) ⇒ ′*a seq* ⇒ ′*b seq* **where**
  apply f Empty = *Empty*
| *apply f* (*Insert x P*) = *Join* (*f x*) (*Join* (*P* ≫= *f*) *Empty*)
| *apply f* (*Join P xq*) = *Join* (*P* ≫= *f*) (*apply f xq*)

lemma *apply-bind*:
  pred-of-seq (*apply f xq*) = *pred-of-seq xq* ≫= *f*
proof (*induct xq*)
  case *Empty* **show** *?case*
    by (*simp add*: *bottom-bind*)
next
  case *Insert* **show** *?case*
    by (*simp add*: *single-bind sup-bind*)
next
  case *Join* **then show** *?case*
    by (*simp add*: *sup-bind*)
qed

lemma *bind-code* [*code*]:
  Seq g ≫= *f* = *Seq* (λ*u*. *apply f* (*g* ()))
  unfolding *Seq-def* **by** (*rule sym*, *rule apply-bind*)

lemma *bot-set-code* [*code*]:
  ⊥ = *Seq* (λ*u*. *Empty*)
  unfolding *Seq-def* **by** *simp*

primrec *adjunct* :: ′*a pred* ⇒ ′*a seq* ⇒ ′*a seq* **where**
  adjunct P Empty = *Join P Empty*
| *adjunct P* (*Insert x Q*) = *Insert x* (*Q* ⊔ *P*)
| *adjunct P* (*Join Q xq*) = *Join Q* (*adjunct P xq*)

lemma *adjunct-sup*:
  pred-of-seq (*adjunct P xq*) = *P* ⊔ *pred-of-seq xq*
  by (*induct xq*) (*simp-all add*: *sup-assoc sup-commute sup-left-commute*)

lemma *sup-code* [*code*]:
  Seq f ⊔ *Seq g* = *Seq* (λ*u*. *case f* ()
    of Empty ⇒ *g* ()
     | *Insert x P* ⇒ *Insert x* (*P* ⊔ *Seq g*)
     | *Join P xq* ⇒ *adjunct* (*Seq g*) (*Join P xq*))
proof (*cases f* ())
  case *Empty*

thus *?thesis*
unfolding *Seq-def* **by** (*simp add*: *sup-commute* [*of* ⊥])
next
case *Insert*
thus *?thesis*
unfolding *Seq-def* **by** (*simp add*: *sup-assoc*)
next
case *Join*
thus *?thesis*
unfolding *Seq-def*
by (*simp add*: *adjunct-sup sup-assoc sup-commute sup-left-commute*)
qed

primrec *contained* :: *'a seq* ⇒ *'a pred* ⇒ *bool* **where**
contained Empty Q ⟷ *True*
| *contained* (*Insert x P*) *Q* ⟷ *eval Q x* ∧ *P* ≤ *Q*
| *contained* (*Join P xq*) *Q* ⟷ *P* ≤ *Q* ∧ *contained xq Q*

lemma *single-less-eq-eval*:
single x ≤ *P* ⟷ *eval P x*
by (*auto simp add*: *less-eq-pred-def le-fun-def*)

lemma *contained-less-eq*:
contained xq Q ⟷ *pred-of-seq xq* ≤ *Q*
by (*induct xq*) (*simp-all add*: *single-less-eq-eval*)

lemma *less-eq-pred-code* [*code*]:
Seq f ≤ *Q* = (*case f* ()
of Empty ⇒ *True*
| *Insert x P* ⇒ *eval Q x* ∧ *P* ≤ *Q*
| *Join P xq* ⇒ *P* ≤ *Q* ∧ *contained xq Q*)
by (*cases f* ())
(*simp-all add*: *Seq-def single-less-eq-eval contained-less-eq*)

instantiation *pred* :: (*type*) *equal*
begin

definition *equal-pred*
where [*simp*]: *HOL.equal P Q* ⟷ *P* = (*Q* :: *'a pred*)

instance by *standard simp*

end

lemma [*code*]:
HOL.equal P Q ⟷ *P* ≤ *Q* ∧ *Q* ≤ *P* **for** *P Q* :: *'a pred*
by *auto*

lemma [*code nbe*]:

HOL.equal P P ⟷ *True* **for** *P* :: *'a pred*
by (*fact equal-refl*)

lemma [*code*]:
case-pred f P = *f* (*eval P*)
by (*fact pred.case-eq-if*)

lemma [*code*]:
rec-pred f P = *f* (*eval P*)
by (*cases P*) *simp*

inductive *eq* :: *'a* ⇒ *'a* ⇒ *bool* **where** *eq x x*

lemma *eq-is-eq*: *eq x y* ≡ (*x* = *y*)
by (*rule eq-reflection*) (*auto intro*: *eq.intros elim*: *eq.cases*)

primrec *null* :: *'a seq* ⇒ *bool* **where**
null Empty ⟷ *True*
| *null* (*Insert x P*) ⟷ *False*
| *null* (*Join P xq*) ⟷ *is-empty P* ∧ *null xq*

lemma *null-is-empty*:
null xq ⟷ *is-empty* (*pred-of-seq xq*)
by (*induct xq*) (*simp-all add*: *is-empty-bot not-is-empty-single is-empty-sup*)

lemma *is-empty-code* [*code*]:
is-empty (*Seq f*) ⟷ *null* (*f* ())
by (*simp add*: *null-is-empty Seq-def*)

primrec *the-only* :: (*unit* ⇒ *'a*) ⇒ *'a seq* ⇒ *'a* **where**
the-only default Empty = *default* () **for** *default*
| *the-only default* (*Insert x P*) =
(*if is-empty P then x else let y* = *singleton default P in if x* = *y then x else default* ()) **for** *default*
| *the-only default* (*Join P xq*) =
(*if is-empty P then the-only default xq else if null xq then singleton default P*
else let x = *singleton default P*; *y* = *the-only default xq in*
if x = *y then x else default* ()) **for** *default*

lemma *the-only-singleton*:
the-only default xq = *singleton default* (*pred-of-seq xq*) **for** *default*
by (*induct xq*)
(*auto simp add*: *singleton-bot singleton-single is-empty-def*
null-is-empty Let-def singleton-sup)

lemma *singleton-code* [*code*]:
singleton default (*Seq f*) =
(*case f* () *of*
Empty ⇒ *default* ()

```
  | Insert x P ⇒ if is-empty P then x
      else let y = singleton default P in
        if x = y then x else default ()
  | Join P xq ⇒ if is-empty P then the-only default xq
      else if null xq then singleton default P
      else let x = singleton default P; y = the-only default xq in
        if x = y then x else default ()) for default
by (cases f ())
 (auto simp add: Seq-def the-only-singleton is-empty-def
   null-is-empty singleton-bot singleton-single singleton-sup Let-def)

definition the :: 'a pred ⇒ 'a where
  the A = (THE x. eval A x)

lemma the-eqI:
  (THE x. eval P x) = x ⟹ the P = x
  by (simp add: the-def)

lemma the-eq [code]: the A = singleton (λx. Code.abort (STR ''not-unique'') (λ-.
the A)) A
  by (rule the-eqI) (simp add: singleton-def the-def)

code-reflect Predicate
  datatypes pred = Seq and seq = Empty | Insert | Join

ML ‹
signature PREDICATE =
sig
  val anamorph: ('a −> ('b ∗ 'a) option) −> int −> 'a −> 'b list ∗ 'a
  datatype 'a pred = Seq of (unit −> 'a seq)
  and 'a seq = Empty | Insert of 'a ∗ 'a pred | Join of 'a pred ∗ 'a seq
  val map: ('a −> 'b) −> 'a pred −> 'b pred
  val yield: 'a pred −> ('a ∗ 'a pred) option
  val yieldn: int −> 'a pred −> 'a list ∗ 'a pred
end;

structure Predicate : PREDICATE =
struct

fun anamorph f k x =
 (if k = 0 then ([], x)
  else case f x
   of NONE => ([], x)
    | SOME (v, y) => let
        val k' = k − 1;
        val (vs, z) = anamorph f k' y
      in (v :: vs, z) end);

datatype pred = datatype Predicate.pred
```

```
datatype seq = datatype Predicate.seq

fun map f = @{code Predicate.map} f;

fun yield (Seq f) = next (f ())
and next Empty = NONE
  | next (Insert (x, P)) = SOME (x, P)
  | next (Join (P, xq)) = (case yield P
     of NONE => next xq
      | SOME (x, Q) => SOME (x, Seq (fn - => Join (Q, xq))));

fun yieldn k = anamorph yield k;

end;
›
```

Conversion from and to sets

definition *pred-of-set* :: *'a set* ⇒ *'a pred* **where**
  pred-of-set = *Pred* ∘ (λ*A x*. *x* ∈ *A*)

lemma *eval-pred-of-set* [*simp*]:
  eval (*pred-of-set A*) *x* ⟷ *x* ∈*A*
  by (*simp add*: *pred-of-set-def*)

definition *set-of-pred* :: *'a pred* ⇒ *'a set* **where**
  set-of-pred = *Collect* ∘ *eval*

lemma *member-set-of-pred* [*simp*]:
  x ∈ *set-of-pred P* ⟷ *Predicate.eval P x*
  by (*simp add*: *set-of-pred-def*)

definition *set-of-seq* :: *'a seq* ⇒ *'a set* **where**
  set-of-seq = *set-of-pred* ∘ *pred-of-seq*

lemma *member-set-of-seq* [*simp*]:
  x ∈ *set-of-seq xq* = *Predicate.member xq x*
  by (*simp add*: *set-of-seq-def eval-member*)

lemma *of-pred-code* [*code*]:
  set-of-pred (*Predicate.Seq f*) = (*case f* () *of*
     Predicate.Empty ⇒ {}
   | *Predicate.Insert x P* ⇒ *insert x* (*set-of-pred P*)
   | *Predicate.Join P xq* ⇒ *set-of-pred P* ∪ *set-of-seq xq*)
  by (*auto split*: *seq.split simp add*: *eval-code*)

lemma *of-seq-code* [*code*]:
  set-of-seq Predicate.Empty = {}
  set-of-seq (*Predicate.Insert x P*) = *insert x* (*set-of-pred P*)
  set-of-seq (*Predicate.Join P xq*) = *set-of-pred P* ∪ *set-of-seq xq*

by *auto*

Lazy Evaluation of an indexed function

function *iterate-upto* :: (*natural* ⇒ *'a*) ⇒ *natural* ⇒ *natural* ⇒ *'a Predicate.pred*
where
iterate-upto f n m =
Predicate.Seq (%*u*. *if n* > *m then Predicate.Empty*
else Predicate.Insert (*f n*) (*iterate-upto f* (*n* + *1*) *m*))
by *pat-completeness auto*

termination by (*relation measure* (%(*f*, *n*, *m*). *nat-of-natural* (*m* + *1* − *n*)))
(*auto simp add*: *less-natural-def*)

Misc

declare *Inf-set-fold* [**where** *'a* = *'a Predicate.pred*, *code*]
declare *Sup-set-fold* [**where** *'a* = *'a Predicate.pred*, *code*]

lemma *pred-of-set-fold-sup*:
assumes *finite A*
shows *pred-of-set A* = *Finite-Set.fold sup bot* (*Predicate.single* ' *A*) (**is** *?lhs* = *?rhs*)
proof (*rule sym*)
interpret *comp-fun-idem sup* :: *'a Predicate.pred* ⇒ *'a Predicate.pred* ⇒ *'a Predicate.pred*
by (*fact comp-fun-idem-sup*)
from ‹*finite A*› **show** *?rhs* = *?lhs* **by** (*induct A*) (*auto intro*!: *pred-eqI*)
qed

lemma *pred-of-set-set-fold-sup*:
pred-of-set (*set xs*) = *fold sup* (*List.map Predicate.single xs*) *bot*
proof −
interpret *comp-fun-idem sup* :: *'a Predicate.pred* ⇒ *'a Predicate.pred* ⇒ *'a Predicate.pred*
by (*fact comp-fun-idem-sup*)
show *?thesis* **by** (*simp add*: *pred-of-set-fold-sup fold-set-fold* [*symmetric*])
qed

lemma *pred-of-set-set-foldr-sup* [*code*]:
pred-of-set (*set xs*) = *foldr sup* (*List.map Predicate.single xs*) *bot*
by (*simp add*: *pred-of-set-set-fold-sup ac-simps foldr-fold fun-eq-iff*)

no-notation *bind* (**infixl** ‹≫=› *70*)

hide-type (**open**) *pred seq*
hide-const (**open**) *Pred eval single bind is-empty singleton if-pred not-pred holds Empty Insert Join Seq member pred-of-seq apply adjunct null the-only eq map the iterate-upto*

hide-fact (**open**) *null-def member-def*

end

76 Lazy sequences

theory *Lazy-Sequence*
imports *Predicate*
begin

76.1 Type of lazy sequences

datatype (*plugins only*: *code extraction*) (*dead 'a*) *lazy-sequence* =
lazy-sequence-of-list 'a list

primrec *list-of-lazy-sequence* :: *'a lazy-sequence* ⇒ *'a list*
where
list-of-lazy-sequence (*lazy-sequence-of-list xs*) = *xs*

lemma *lazy-sequence-of-list-of-lazy-sequence* [*simp*]:
lazy-sequence-of-list (*list-of-lazy-sequence xq*) = *xq*
by (*cases xq*) *simp-all*

lemma *lazy-sequence-eqI*:
list-of-lazy-sequence xq = *list-of-lazy-sequence yq* ⟹ *xq* = *yq*
by (*cases xq*, *cases yq*) *simp*

lemma *lazy-sequence-eq-iff*:
xq = *yq* ⟷ *list-of-lazy-sequence xq* = *list-of-lazy-sequence yq*
by (*auto intro*: *lazy-sequence-eqI*)

lemma *case-lazy-sequence* [*simp*]:
case-lazy-sequence f xq = *f* (*list-of-lazy-sequence xq*)
by (*cases xq*) *auto*

lemma *rec-lazy-sequence* [*simp*]:
rec-lazy-sequence f xq = *f* (*list-of-lazy-sequence xq*)
by (*cases xq*) *auto*

definition *Lazy-Sequence* :: (*unit* ⇒ (*'a* × *'a lazy-sequence*) *option*) ⇒ *'a lazy-sequence*
where
Lazy-Sequence f = *lazy-sequence-of-list* (*case f* () *of*
None ⇒ []
| *Some* (*x*, *xq*) ⇒ *x* # *list-of-lazy-sequence xq*)

code-datatype *Lazy-Sequence*

declare *list-of-lazy-sequence.simps* [*code del*]
declare *lazy-sequence.case* [*code del*]

declare *lazy-sequence.rec* [*code del*]

lemma *list-of-Lazy-Sequence* [*simp*]:
list-of-lazy-sequence (*Lazy-Sequence f*) = (*case f* () *of*
None ⇒ []
| *Some* (*x*, *xq*) ⇒ *x* # *list-of-lazy-sequence xq*)
by (*simp add*: *Lazy-Sequence-def*)

definition *yield* :: *'a lazy-sequence* ⇒ (*'a* × *'a lazy-sequence*) *option*
where
yield xq = (*case list-of-lazy-sequence xq of*
[] ⇒ *None*
| *x* # *xs* ⇒ *Some* (*x*, *lazy-sequence-of-list xs*))

lemma *yield-Seq* [*simp*, *code*]:
yield (*Lazy-Sequence f*) = *f* ()
by (*cases f* ()) (*simp-all add*: *yield-def split-def*)

lemma *case-yield-eq* [*simp*]: *case-option g h* (*yield xq*) =
case-list g (λ*x*. *curry h x* ∘ *lazy-sequence-of-list*) (*list-of-lazy-sequence xq*)
by (*cases list-of-lazy-sequence xq*) (*simp-all add*: *yield-def*)

lemma *equal-lazy-sequence-code* [*code*]:
HOL.equal xq yq = (*case* (*yield xq*, *yield yq*) *of*
(*None*, *None*) ⇒ *True*
| (*Some* (*x*, *xq'*), *Some* (*y*, *yq'*)) ⇒ *HOL.equal x y* ∧ *HOL.equal xq yq*
| - ⇒ *False*)
by (*simp-all add*: *lazy-sequence-eq-iff equal-eq split*: *list.splits*)

lemma [*code nbe*]:
HOL.equal (*x* :: *'a lazy-sequence*) *x* ⟷ *True*
by (*fact equal-refl*)

definition *empty* :: *'a lazy-sequence*
where
empty = *lazy-sequence-of-list* []

lemma *list-of-lazy-sequence-empty* [*simp*]:
list-of-lazy-sequence empty = []
by (*simp add*: *empty-def*)

lemma *empty-code* [*code*]:
empty = *Lazy-Sequence* (λ-. *None*)
by (*simp add*: *lazy-sequence-eq-iff*)

definition *single* :: *'a* ⇒ *'a lazy-sequence*
where
single x = *lazy-sequence-of-list* [*x*]

lemma *list-of-lazy-sequence-single* [*simp*]:
list-of-lazy-sequence (*single x*) = [*x*]
by (*simp add*: *single-def*)

lemma *single-code* [*code*]:
single x = *Lazy-Sequence* (λ-. *Some* (*x*, *empty*))
by (*simp add*: *lazy-sequence-eq-iff*)

definition *append* :: *'a lazy-sequence* ⇒ *'a lazy-sequence* ⇒ *'a lazy-sequence*
where
append xq yq = *lazy-sequence-of-list* (*list-of-lazy-sequence xq* @ *list-of-lazy-sequence yq*)

lemma *list-of-lazy-sequence-append* [*simp*]:
list-of-lazy-sequence (*append xq yq*) = *list-of-lazy-sequence xq* @ *list-of-lazy-sequence yq*
by (*simp add*: *append-def*)

lemma *append-code* [*code*]:
append xq yq = *Lazy-Sequence* (λ-. *case yield xq of*
None ⇒ *yield yq*
| *Some* (*x*, *xq'*) ⇒ *Some* (*x*, *append xq' yq*))
by (*simp-all add*: *lazy-sequence-eq-iff split*: *list.splits*)

definition *map* :: (*'a* ⇒ *'b*) ⇒ *'a lazy-sequence* ⇒ *'b lazy-sequence*
where
map f xq = *lazy-sequence-of-list* (*List.map f* (*list-of-lazy-sequence xq*))

lemma *list-of-lazy-sequence-map* [*simp*]:
list-of-lazy-sequence (*map f xq*) = *List.map f* (*list-of-lazy-sequence xq*)
by (*simp add*: *map-def*)

lemma *map-code* [*code*]:
map f xq =
Lazy-Sequence (λ-. *map-option* (λ(*x*, *xq'*). (*f x*, *map f xq'*)) (*yield xq*))
by (*simp-all add*: *lazy-sequence-eq-iff split*: *list.splits*)

definition *flat* :: *'a lazy-sequence lazy-sequence* ⇒ *'a lazy-sequence*
where
flat xqq = *lazy-sequence-of-list* (*concat* (*List.map list-of-lazy-sequence* (*list-of-lazy-sequence xqq*)))

lemma *list-of-lazy-sequence-flat* [*simp*]:
list-of-lazy-sequence (*flat xqq*) = *concat* (*List.map list-of-lazy-sequence* (*list-of-lazy-sequence xqq*))
by (*simp add*: *flat-def*)

lemma *flat-code* [*code*]:
flat xqq = *Lazy-Sequence* (λ-. *case yield xqq of*

None ⇒ *None*
| *Some* (*xq*, *xqq'*) ⇒ *yield* (*append xq* (*flat xqq'*)))
by (*simp add*: *lazy-sequence-eq-iff split*: *list.splits*)

definition *bind* :: *'a lazy-sequence* ⇒ (*'a* ⇒ *'b lazy-sequence*) ⇒ *'b lazy-sequence*
where
bind xq f = *flat* (*map f xq*)

definition *if-seq* :: *bool* ⇒ *unit lazy-sequence*
where
if-seq b = (*if b then single* () *else empty*)

definition *those* :: *'a option lazy-sequence* ⇒ *'a lazy-sequence option*
where
those xq = *map-option lazy-sequence-of-list* (*List.those* (*list-of-lazy-sequence xq*))

function *iterate-upto* :: (*natural* ⇒ *'a*) ⇒ *natural* ⇒ *natural* ⇒ *'a lazy-sequence*
where
iterate-upto f n m =
Lazy-Sequence (λ-. *if n* > *m then None else Some* (*f n*, *iterate-upto f* (*n* + *1*) *m*))
by *pat-completeness auto*

termination by (*relation measure* (λ(*f*, *n*, *m*). *nat-of-natural* (*m* + *1* − *n*)))
(*auto simp add*: *less-natural-def*)

definition *not-seq* :: *unit lazy-sequence* ⇒ *unit lazy-sequence*
where
not-seq xq = (*case yield xq of*
None ⇒ *single* ()
| *Some* ((), *xq*) ⇒ *empty*)

76.2 Code setup

code-reflect *Lazy-Sequence*
datatypes *lazy-sequence* = *Lazy-Sequence*

ML ‹
signature LAZY-SEQUENCE =
sig
datatype 'a lazy-sequence = *Lazy-Sequence of* (*unit* −> (*'a* ∗ *'a Lazy-Sequence.lazy-sequence*) *option*)
val map: (*'a* −> *'b*) −> *'a lazy-sequence* −> *'b lazy-sequence*
val yield: *'a lazy-sequence* −> (*'a* ∗ *'a lazy-sequence*) *option*
val yieldn: *int* −> *'a lazy-sequence* −> *'a list* ∗ *'a lazy-sequence*
end;

structure Lazy-Sequence : *LAZY-SEQUENCE* =
struct

datatype lazy-sequence = datatype Lazy-Sequence.lazy-sequence;

fun map f = @{code Lazy-Sequence.map} f;

fun yield P = @{code Lazy-Sequence.yield} P;

fun yieldn k = Predicate.anamorph yield k;

end;
›

76.3 Generator Sequences

76.3.1 General lazy sequence operation

definition *product* :: *'a lazy-sequence* ⇒ *'b lazy-sequence* ⇒ (*'a* × *'b*) *lazy-sequence*
where
  product s1 s2 = *bind s1* (λ*a*. *bind s2* (λ*b*. *single* (*a*, *b*)))

76.3.2 Small lazy typeclasses

class *small-lazy* =
  fixes *small-lazy* :: *natural* ⇒ *'a lazy-sequence*

instantiation *unit* :: *small-lazy*
begin

definition *small-lazy d* = *single* ()

instance ..

end

instantiation *int* :: *small-lazy*
begin

maybe optimise this expression -> append (single x) xs == cons x xs Performance difference?

function *small-lazy'* :: *int* ⇒ *int* ⇒ *int lazy-sequence*
where
  small-lazy' d i = (*if d* < *i then empty*
    else append (*single i*) (*small-lazy' d* (*i* + *1*)))
    by *pat-completeness auto*

termination
  by (*relation measure* (%(*d*, *i*). *nat* (*d* + *1* − *i*))) *auto*

definition
  small-lazy d = *small-lazy'* (*int* (*nat-of-natural d*)) (− (*int* (*nat-of-natural d*)))

instance ..

end

instantiation *prod* :: (*small-lazy*, *small-lazy*) *small-lazy*
begin

definition
small-lazy d = *product* (*small-lazy d*) (*small-lazy d*)

instance ..

end

instantiation *list* :: (*small-lazy*) *small-lazy*
begin

fun *small-lazy-list* :: *natural* ⇒ *'a list lazy-sequence*
where
small-lazy-list d = *append* (*single* [])
(*if d* > *0 then bind* (*product* (*small-lazy* (*d* − *1*))
(*small-lazy* (*d* − *1*))) (λ(*x*, *xs*). *single* (*x* # *xs*)) *else empty*)

instance ..

end

76.4 With Hit Bound Value

assuming in negative context

type-synonym *'a hit-bound-lazy-sequence* = *'a option lazy-sequence*

definition *hit-bound* :: *'a hit-bound-lazy-sequence*
where
hit-bound = *Lazy-Sequence* (λ-. *Some* (*None*, *empty*))

lemma *list-of-lazy-sequence-hit-bound* [*simp*]:
list-of-lazy-sequence hit-bound = [*None*]
by (*simp add*: *hit-bound-def*)

definition *hb-single* :: *'a* ⇒ *'a hit-bound-lazy-sequence*
where
hb-single x = *Lazy-Sequence* (λ-. *Some* (*Some x*, *empty*))

definition *hb-map* :: (*'a* ⇒ *'b*) ⇒ *'a hit-bound-lazy-sequence* ⇒ *'b hit-bound-lazy-sequence*
where
hb-map f xq = *map* (*map-option f*) *xq*

lemma *hb-map-code* [*code*]:
hb-map f xq =
Lazy-Sequence (λ-. *map-option* (λ(*x*, *xq*′). (*map-option f x*, *hb-map f xq*′)) (*yield xq*))
using *map-code* [*of map-option f xq*] **by** (*simp add*: *hb-map-def*)

definition *hb-flat* :: ′*a hit-bound-lazy-sequence hit-bound-lazy-sequence* ⇒ ′*a hit-bound-lazy-sequence*
where
hb-flat xqq = *lazy-sequence-of-list* (*concat*
(*List.map* ((λ*x*. *case x of None* ⇒ [*None*] | *Some xs* ⇒ *xs*) ∘ *map-option list-of-lazy-sequence*) (*list-of-lazy-sequence xqq*)))

lemma *list-of-lazy-sequence-hb-flat* [*simp*]:
list-of-lazy-sequence (*hb-flat xqq*) =
concat (*List.map* ((λ*x*. *case x of None* ⇒ [*None*] | *Some xs* ⇒ *xs*) ∘ *map-option list-of-lazy-sequence*) (*list-of-lazy-sequence xqq*))
by (*simp add*: *hb-flat-def*)

lemma *hb-flat-code* [*code*]:
hb-flat xqq = *Lazy-Sequence* (λ-. *case yield xqq of*
None ⇒ *None*
| *Some* (*xq*, *xqq*′) ⇒ *yield*
(*append* (*case xq of None* ⇒ *hit-bound* | *Some xq* ⇒ *xq*) (*hb-flat xqq*′)))
by (*simp add*: *lazy-sequence-eq-iff split*: *list.splits option.splits*)

definition *hb-bind* :: ′*a hit-bound-lazy-sequence* ⇒ (′*a* ⇒ ′*b hit-bound-lazy-sequence*) ⇒ ′*b hit-bound-lazy-sequence*
where
hb-bind xq f = *hb-flat* (*hb-map f xq*)

definition *hb-if-seq* :: *bool* ⇒ *unit hit-bound-lazy-sequence*
where
hb-if-seq b = (*if b then hb-single* () *else empty*)

definition *hb-not-seq* :: *unit hit-bound-lazy-sequence* ⇒ *unit lazy-sequence*
where
hb-not-seq xq = (*case yield xq of*
None ⇒ *single* ()
| *Some* (*x*, *xq*) ⇒ *empty*)

hide-const (**open**) *yield empty single append flat map bind*
if-seq those iterate-upto not-seq product

hide-fact (**open**) *yield-def empty-def single-def append-def flat-def map-def bind-def*
if-seq-def those-def not-seq-def product-def

end

77 Depth-Limited Sequences with failure element

theory *Limited-Sequence*
imports *Lazy-Sequence*
begin

77.1 Depth-Limited Sequence

type-synonym *'a dseq = natural ⇒ bool ⇒ 'a lazy-sequence option*

definition *empty* :: *'a dseq*
where
empty = (λ- -. Some Lazy-Sequence.empty)

definition *single* :: *'a ⇒ 'a dseq*
where
single x = (λ- -. Some (Lazy-Sequence.single x))

definition *eval* :: *'a dseq ⇒ natural ⇒ bool ⇒ 'a lazy-sequence option*
where
[*simp*]: *eval f i pol = f i pol*

definition *yield* :: *'a dseq ⇒ natural ⇒ bool ⇒ ('a × 'a dseq) option*
where
yield f i pol = (case eval f i pol of
None ⇒ None
| Some s ⇒ (map-option ∘ apsnd) (λr - -. Some r) (Lazy-Sequence.yield s))

definition *map-seq* :: *('a ⇒ 'b dseq) ⇒ 'a lazy-sequence ⇒ 'b dseq*
where
map-seq f xq i pol = map-option Lazy-Sequence.flat
(Lazy-Sequence.those (Lazy-Sequence.map (λx. f x i pol) xq))

lemma *map-seq-code* [*code*]:
map-seq f xq i pol = (case Lazy-Sequence.yield xq of
None ⇒ Some Lazy-Sequence.empty
| Some (x, xq') ⇒ (case eval (f x) i pol of
None ⇒ None
| Some yq ⇒ (case map-seq f xq' i pol of
None ⇒ None
| Some zq ⇒ Some (Lazy-Sequence.append yq zq))))
by (*cases xq*)
(*auto simp add*: *map-seq-def Lazy-Sequence.those-def lazy-sequence-eq-iff split*: *list.splits option.splits*)

definition *bind* :: *'a dseq ⇒ ('a ⇒ 'b dseq) ⇒ 'b dseq*
where
bind x f = (λi pol.
if i = 0 then
(if pol then Some Lazy-Sequence.empty else None)

else
  (case x (i − 1) pol of
    None ⇒ None
  | Some xq ⇒ map-seq f xq i pol))

definition *union* :: *'a dseq ⇒ 'a dseq ⇒ 'a dseq*
where
union x y = (λi pol. case (x i pol, y i pol) of
  (Some xq, Some yq) ⇒ Some (Lazy-Sequence.append xq yq)
| - ⇒ None)

definition *if-seq* :: *bool ⇒ unit dseq*
where
if-seq b = (if b then single () else empty)

definition *not-seq* :: *unit dseq ⇒ unit dseq*
where
not-seq x = (λi pol. case x i (¬ pol) of
  None ⇒ Some Lazy-Sequence.empty
| Some xq ⇒ (case Lazy-Sequence.yield xq of
    None ⇒ Some (Lazy-Sequence.single ())
  | Some - ⇒ Some (Lazy-Sequence.empty)))

definition *map* :: *('a ⇒ 'b) ⇒ 'a dseq ⇒ 'b dseq*
where
map f g = (λi pol. case g i pol of
  None ⇒ None
| Some xq ⇒ Some (Lazy-Sequence.map f xq))

77.2 Positive Depth-Limited Sequence

type-synonym *'a pos-dseq = natural ⇒ 'a Lazy-Sequence.lazy-sequence*

definition *pos-empty* :: *'a pos-dseq*
where
pos-empty = (λi. Lazy-Sequence.empty)

definition *pos-single* :: *'a ⇒ 'a pos-dseq*
where
pos-single x = (λi. Lazy-Sequence.single x)

definition *pos-bind* :: *'a pos-dseq ⇒ ('a ⇒ 'b pos-dseq) ⇒ 'b pos-dseq*
where
pos-bind x f = (λi. Lazy-Sequence.bind (x i) (λa. f a i))

definition *pos-decr-bind* :: *'a pos-dseq ⇒ ('a ⇒ 'b pos-dseq) ⇒ 'b pos-dseq*
where
pos-decr-bind x f = (λi.
  if i = 0 then

Lazy-Sequence.empty
else
Lazy-Sequence.bind (*x* (*i* − *1*)) (λ*a*. *f a i*))

definition *pos-union* :: ′*a pos-dseq* ⇒ ′*a pos-dseq* ⇒ ′*a pos-dseq*
where
pos-union xq yq = (λ*i*. *Lazy-Sequence.append* (*xq i*) (*yq i*))

definition *pos-if-seq* :: *bool* ⇒ *unit pos-dseq*
where
pos-if-seq b = (*if b then pos-single* () *else pos-empty*)

definition *pos-iterate-upto* :: (*natural* ⇒ ′*a*) ⇒ *natural* ⇒ *natural* ⇒ ′*a pos-dseq*
where
pos-iterate-upto f n m = (λ*i*. *Lazy-Sequence.iterate-upto f n m*)

definition *pos-map* :: (′*a* ⇒ ′*b*) ⇒ ′*a pos-dseq* ⇒ ′*b pos-dseq*
where
pos-map f xq = (λ*i*. *Lazy-Sequence.map f* (*xq i*))

77.3 Negative Depth-Limited Sequence

type-synonym ′*a neg-dseq* = *natural* ⇒ ′*a Lazy-Sequence.hit-bound-lazy-sequence*

definition *neg-empty* :: ′*a neg-dseq*
where
neg-empty = (λ*i*. *Lazy-Sequence.empty*)

definition *neg-single* :: ′*a* ⇒ ′*a neg-dseq*
where
neg-single x = (λ*i*. *Lazy-Sequence.hb-single x*)

definition *neg-bind* :: ′*a neg-dseq* ⇒ (′*a* ⇒ ′*b neg-dseq*) ⇒ ′*b neg-dseq*
where
neg-bind x f = (λ*i*. *hb-bind* (*x i*) (λ*a*. *f a i*))

definition *neg-decr-bind* :: ′*a neg-dseq* ⇒ (′*a* ⇒ ′*b neg-dseq*) ⇒ ′*b neg-dseq*
where
neg-decr-bind x f = (λ*i*.
if i = *0 then*
Lazy-Sequence.hit-bound
else
hb-bind (*x* (*i* − *1*)) (λ*a*. *f a i*))

definition *neg-union* :: ′*a neg-dseq* ⇒ ′*a neg-dseq* ⇒ ′*a neg-dseq*
where
neg-union x y = (λ*i*. *Lazy-Sequence.append* (*x i*) (*y i*))

definition *neg-if-seq* :: *bool* ⇒ *unit neg-dseq*

where
neg-if-seq b = (if b then neg-single () else neg-empty)

definition *neg-iterate-upto*
where
neg-iterate-upto f n m = (λi. Lazy-Sequence.iterate-upto (λi. Some (f i)) n m)

definition *neg-map :: ('a ⇒ 'b) ⇒ 'a neg-dseq ⇒ 'b neg-dseq*
where
neg-map f xq = (λi. Lazy-Sequence.hb-map f (xq i))

77.4 Negation

definition *pos-not-seq :: unit neg-dseq ⇒ unit pos-dseq*
where
*pos-not-seq xq = (λi. Lazy-Sequence.hb-not-seq (xq (3 * i)))*

definition *neg-not-seq :: unit pos-dseq ⇒ unit neg-dseq*
where
neg-not-seq x = (λi. case Lazy-Sequence.yield (x i) of
None ⇒ Lazy-Sequence.hb-single ()
| Some ((), xq) ⇒ Lazy-Sequence.empty)

ML ‹

```
signature LIMITED-SEQUENCE =
sig
  type 'a dseq = Code-Numeral.natural -> bool -> 'a Lazy-Sequence.lazy-sequence option
  val map : ('a -> 'b) -> 'a dseq -> 'b dseq
  val yield : 'a dseq -> Code-Numeral.natural -> bool -> ('a * 'a dseq) option
  val yieldn : int -> 'a dseq -> Code-Numeral.natural -> bool -> 'a list * 'a dseq
end;

structure Limited-Sequence : LIMITED-SEQUENCE =
struct

type 'a dseq = Code-Numeral.natural -> bool -> 'a Lazy-Sequence.lazy-sequence option

fun map f = @{code Limited-Sequence.map} f;

fun yield f = @{code Limited-Sequence.yield} f;

fun yieldn n f i pol = (case f i pol of
    NONE => ([], fn - => fn - => NONE)
  | SOME s => let val (xs, s') = Lazy-Sequence.yieldn n s in (xs, fn - => fn - => SOME s') end);
```

end;
›

code-reserved
(*Eval*) *Limited-Sequence*

hide-const (**open**) *yield empty single eval map-seq bind union if-seq not-seq map*
pos-empty pos-single pos-bind pos-decr-bind pos-union pos-if-seq pos-iterate-upto pos-not-seq pos-map
neg-empty neg-single neg-bind neg-decr-bind neg-union neg-if-seq neg-iterate-upto neg-not-seq neg-map

hide-fact (**open**) *yield-def empty-def single-def eval-def map-seq-def bind-def union-def*
if-seq-def not-seq-def map-def
pos-empty-def pos-single-def pos-bind-def pos-union-def pos-if-seq-def pos-iterate-upto-def pos-not-seq-def pos-map-def
neg-empty-def neg-single-def neg-bind-def neg-union-def neg-if-seq-def neg-iterate-upto-def neg-not-seq-def neg-map-def

end

78 Term evaluation using the generic code generator

theory *Code-Evaluation*
imports *Typerep Limited-Sequence*
keywords *value* :: *diag*
begin

78.1 Term representation

78.1.1 Terms and class *term-of*

datatype (*plugins only*: *extraction*) *term* = *dummy-term*

definition *Const* :: *String.literal* ⇒ *typerep* ⇒ *term* **where**
Const - - = *dummy-term*

definition *App* :: *term* ⇒ *term* ⇒ *term* **where**
App - - = *dummy-term*

definition *Abs* :: *String.literal* ⇒ *typerep* ⇒ *term* ⇒ *term* **where**
Abs - - - = *dummy-term*

definition *Free* :: *String.literal* ⇒ *typerep* ⇒ *term* **where**
Free - - = *dummy-term*

code-datatype *Const App Abs Free*

class *term-of* = *typerep* +
  fixes *term-of* :: *'a* ⇒ *term*

lemma *term-of-anything*: *term-of x* ≡ *t*
  by (*rule eq-reflection*) (*cases term-of x*, *cases t*, *simp*)

definition *valapp* :: (*'a* ⇒ *'b*) × (*unit* ⇒ *term*)
  ⇒ *'a* × (*unit* ⇒ *term*) ⇒ *'b* × (*unit* ⇒ *term*) **where**
  valapp f x = (*fst f* (*fst x*), *λu. App* (*snd f* ()) (*snd x* ()))

lemma *valapp-code* [*code*, *code-unfold*]:
  valapp (*f*, *tf*) (*x*, *tx*) = (*f x*, *λu. App* (*tf* ()) (*tx* ()))
  by (*simp only*: *valapp-def fst-conv snd-conv*)

78.1.2 Syntax

definition *termify* :: *'a* ⇒ *term* **where**
  [*code del*]: *termify x* = *dummy-term*

abbreviation *valtermify* :: *'a* ⇒ *'a* × (*unit* ⇒ *term*) **where**
  valtermify x ≡ (*x*, *λu. termify x*)

bundle *term-syntax*
begin
notation *App* (**infixl** ‹<·>› *70*) **and** *valapp* (**infixl** ‹{·}› *70*)
end

78.2 Tools setup and evaluation

context
begin

qualified definition *TERM-OF* :: *'a::term-of itself*
where
  TERM-OF = *snd* (*Code-Evaluation.term-of* :: *'a* ⇒ -, *TYPE*(*'a*))

qualified definition *TERM-OF-EQUAL* :: *'a::term-of itself*
where
  TERM-OF-EQUAL = *snd* (*λ*(*a*::*'a*). (*Code-Evaluation.term-of a*, *HOL.eq a*), *TYPE*(*'a*))

end

lemma *eq-eq-TrueD*:
  fixes *x y* :: *'a*::{}
  assumes (*x* ≡ *y*) ≡ *Trueprop True*
  shows *x* ≡ *y*
  using *assms* **by** *simp*

code-printing
type-constructor *term* ⇀ (*Eval*) *Term.term*
| **constant** *Const* ⇀ (*Eval*) *Term.Const/* ((-), (-))
| **constant** *App* ⇀ (*Eval*) *Term.$/* ((-), (-))
| **constant** *Abs* ⇀ (*Eval*) *Term.Abs/* ((-), (-), (-))
| **constant** *Free* ⇀ (*Eval*) *Term.Free/* ((-), (-))

ML-file ‹*Tools/code-evaluation.ML*›

code-reserved
(*Eval*) *Code-Evaluation*

ML-file ‹~~/*src/HOL/Tools/value-command.ML*›

78.3 Dedicated *term-of* instances

instantiation *fun* :: (*typerep*, *typerep*) *term-of*
begin

definition
term-of (*f* :: ′*a* ⇒ ′*b*) =
Const (*STR* ′′*Pure.dummy-pattern*′′)
(*Typerep.Typerep* (*STR* ′′*fun*′′) [*Typerep.typerep* *TYPE*(′*a*), *Typerep.typerep* *TYPE*(′*b*)])

instance ..

end

declare [[*code drop*: *rec-term case-term*
term-of :: *typerep* ⇒ - *term-of* :: *term* ⇒ - *term-of* :: *String.literal* ⇒ -
term-of :: - *Predicate.pred* ⇒ *term term-of* :: - *Predicate.seq* ⇒ *term*]]

code-printing
constant *term-of* :: *integer* ⇒ *term* ⇀ (*Eval*) *HOLogic.mk*′*-number/ HOLogic.code*′*-integerT*
| **constant** *term-of* :: *String.literal* ⇒ *term* ⇀ (*Eval*) *HOLogic.mk*′*-literal*

declare [[*code drop*: *term-of* :: *integer* ⇒ -]]

lemma *term-of-integer* [*unfolded typerep-fun-def typerep-num-def typerep-integer-def*, *code*]:
term-of (*i* :: *integer*) =
(*if i* > *0 then*
App (*Const* (*STR* ′′*Num.numeral-class.numeral*′′) (*TYPEREP*(*num* ⇒ *integer*)))
(*term-of* (*num-of-integer i*))
else if i = *0 then Const* (*STR* ′′*Groups.zero-class.zero*′′) *TYPEREP*(*integer*)
else
App (*Const* (*STR* ′′*Groups.uminus-class.uminus*′′) *TYPEREP*(*integer* ⇒ *in-*

teger))
  (*term-of* (− *i*)))
 by (*rule term-of-anything* [*THEN meta-eq-to-obj-eq*])

code-reserved
 (*Eval*) *HOLogic*

78.4 Generic reification

ML-file ‹~~/*src*/*HOL*/*Tools*/*reification.ML*›

78.5 Diagnostic

definition *tracing* :: *String.literal* ⇒ ′*a* ⇒ ′*a* **where**
 [*code del*]: *tracing s x* = *x*

code-printing
 constant *tracing* :: *String.literal* => ′*a* => ′*a* ⇀ (*Eval*) *Code′-Evaluation.tracing*

hide-const *dummy-term valapp*
hide-const (**open**) *Const App Abs Free termify valtermify term-of tracing*

end

79 A simple counterexample generator performing random testing

theory *Quickcheck-Random*
imports *Random Code-Evaluation Enum*
begin

setup ‹*Code-Target.add-derived-target* (*Quickcheck*, [(*Code-Runtime.target*, *I*)])›

79.1 Catching Match exceptions

axiomatization *catch-match* :: ′*a* => ′*a* => ′*a*

code-printing
 constant *catch-match* ⇀ (*Quickcheck*) ((-) *handle Match* => -)

code-reserved
 (*Quickcheck*) *Match*

79.2 The *random* class

class *random* = *typerep* +
 fixes *random* :: *natural* ⇒ *Random.seed* ⇒ (′*a* × (*unit* ⇒ *term*)) × *Random.seed*

79.3 Fundamental and numeric types

instantiation *bool* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
random i = *Random.range 2* ∘→
(*λk. Pair* (*if k* = *0 then Code-Evaluation.valtermify False else Code-Evaluation.valtermify True*))

instance ..

end

end

instantiation *itself* :: (*typerep*) *random*
begin

definition
random-itself :: *natural* ⇒ *Random.seed* ⇒ (*'a itself* × (*unit* ⇒ *term*)) × *Random.seed*
where *random-itself* - = *Pair* (*Code-Evaluation.valtermify TYPE*(*'a*))

instance ..

end

instantiation *char* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
random - = *Random.select* (*Enum.enum* :: *char list*) ∘→ (*λc. Pair* (*c*, *λu. Code-Evaluation.term-of c*))

instance ..

end

end

instantiation *String.literal* :: *random*

begin

definition
random - = Pair (STR '''', λu. Code-Evaluation.term-of (STR ''''))

instance ..

end

instantiation *nat* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition *random-nat* :: *natural* ⇒ *Random.seed*
⇒ (*nat* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed*
where
random-nat i = Random.range (i + 1) ∘→ (λk. Pair (
let n = nat-of-natural k
in (n, λ-. Code-Evaluation.term-of n)))

instance ..

end

end

instantiation *int* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
random i = Random.range (2 ∗ i + 1) ∘→ (λk. Pair (
let j = (if k ≥ i then int (nat-of-natural (k − i)) else − (int (nat-of-natural (i − k))))
in (j, λ-. Code-Evaluation.term-of j)))

instance ..

end

end

instantiation *natural* :: *random*

begin

context
includes *state-combinator-syntax*
begin

definition *random-natural* :: *natural* ⇒ *Random.seed*
⇒ (*natural* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed*
where
random-natural i = *Random.range* (*i* + *1*) ∘→ (λ*n*. *Pair* (*n*, λ-. *Code-Evaluation.term-of n*))

instance ..

end

end

instantiation *integer* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition *random-integer* :: *natural* ⇒ *Random.seed*
⇒ (*integer* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed*
where
random-integer i = *Random.range* (*2* ∗ *i* + *1*) ∘→ (λ*k*. *Pair* (
let j = (*if k* ≥ *i then integer-of-natural* (*k* − *i*) *else* − (*integer-of-natural* (*i* − *k*)))
in (*j*, λ-. *Code-Evaluation.term-of j*)))

instance ..

end

end

79.4 Complex generators

Towards *'a* ⇒ *'b*

axiomatization *random-fun-aux* :: *typerep* ⇒ *typerep* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *term*)
⇒ (*Random.seed* ⇒ (*'b* × (*unit* ⇒ *term*)) × *Random.seed*)
⇒ (*Random.seed* ⇒ *Random.seed* × *Random.seed*)
⇒ *Random.seed* ⇒ ((*'a* ⇒ *'b*) × (*unit* ⇒ *term*)) × *Random.seed*

definition *random-fun-lift* :: (*Random.seed* ⇒ (*'b* × (*unit* ⇒ *term*)) × *Ran-*

dom.seed)
⇒ *Random.seed* ⇒ ((′*a*::*term-of* ⇒ ′*b*::*typerep*) × (*unit* ⇒ *term*)) × *Random.seed*
where
random-fun-lift f =
random-fun-aux TYPEREP(′*a*) *TYPEREP*(′*b*) (=) *Code-Evaluation.term-of f Random.split-seed*

instantiation *fun* :: ({*equal*, *term-of*}, *random*) *random*
begin

definition
random-fun :: *natural* ⇒ *Random.seed* ⇒ ((′*a* ⇒ ′*b*) × (*unit* ⇒ *term*)) × *Random.seed*
where *random i* = *random-fun-lift* (*random i*)

instance ..

end

Towards type copies and datatypes

context
includes *state-combinator-syntax*
begin

definition *collapse* :: (′*a* ⇒ (′*a* ⇒ ′*b* × ′*a*) × ′*a*) ⇒ ′*a* ⇒ ′*b* × ′*a*
where *collapse f* = (*f* ∘→ *id*)

end

definition *beyond* :: *natural* ⇒ *natural* ⇒ *natural*
where *beyond k l* = (*if l* > *k then l else 0*)

lemma *beyond-zero*: *beyond k 0* = *0*
by (*simp add*: *beyond-def*)

context
includes *term-syntax*
begin

definition [*code-unfold*]:
valterm-emptyset = *Code-Evaluation.valtermify* ({} :: (′*a* :: *typerep*) *set*)

definition [*code-unfold*]:
valtermify-insert x s = *Code-Evaluation.valtermify insert* {·} (*x* :: (′*a* :: *typerep* ∗ -)) {·} *s*

end

instantiation *set* :: (*random*) *random*

begin

context
includes *state-combinator-syntax*
begin

fun *random-aux-set*
where
random-aux-set 0 j = collapse (Random.select-weight [(1, Pair valterm-emptyset)])
| *random-aux-set (Code-Numeral.Suc i) j =*
collapse (Random.select-weight
[(1, Pair valterm-emptyset),
(Code-Numeral.Suc i,
random j ∘→ (%x. random-aux-set i j ∘→ (%s. Pair (valtermify-insert x s))))])

lemma [*code*]:
random-aux-set i j =
collapse (Random.select-weight [(1, Pair valterm-emptyset),
(i, random j ∘→ (%x. random-aux-set (i − 1) j ∘→ (%s. Pair (valtermify-insert x s))))])
proof (*induct i rule*: *natural.induct*)
case *zero*
show *?case* **by** (*subst select-weight-drop-zero* [*symmetric*])
(*simp add*: *random-aux-set.simps* [*simplified*] *less-natural-def*)
next
case (*Suc i*)
show *?case* **by** (*simp only*: *random-aux-set.simps*(*2*) [*of i*] *Suc-natural-minus-one*)
qed

definition *random-set i = random-aux-set i i*

instance ..

end

end

lemma *random-aux-rec*:
fixes *random-aux* :: *natural* ⇒ *'a*
assumes *random-aux 0 = rhs 0*
and $\bigwedge$*k. random-aux (Code-Numeral.Suc k) = rhs (Code-Numeral.Suc k)*
shows *random-aux k = rhs k*
using *assms* **by** (*rule natural.induct*)

79.5 Deriving random generators for datatypes

ML-file ‹*Tools/Quickcheck/quickcheck-common.ML*›
ML-file ‹*Tools/Quickcheck/random-generators.ML*›

79.6 Code setup

code-printing
constant *random-fun-aux* ⇀ (*Quickcheck*) *Random′-Generators.random′-fun*
— With enough criminal energy this can be abused to derive *False*; for this reason we use a distinguished target *Quickcheck* not spoiling the regular trusted code generation

code-reserved
(*Quickcheck*) *Random-Generators*

hide-const (**open**) *catch-match random collapse beyond random-fun-aux random-fun-lift*

hide-fact (**open**) *collapse-def beyond-def random-fun-lift-def*

end

80 The Random-Predicate Monad

theory *Random-Pred*
imports *Quickcheck-Random*
begin

fun *iter′* :: *′a itself* ⇒ *natural* ⇒ *natural* ⇒ *Random.seed* ⇒ (*′a*::*random*) *Predicate.pred*
where
iter′ T nrandom sz seed = (*if nrandom* = *0 then bot-class.bot else*
let ((*x*, -), *seed′*) = *Quickcheck-Random.random sz seed*
in Predicate.Seq (%*u*. *Predicate.Insert x* (*iter′ T* (*nrandom* − *1*) *sz seed′*)))

definition *iter* :: *natural* ⇒ *natural* ⇒ *Random.seed* ⇒ (*′a*::*random*) *Predicate.pred*
where
iter nrandom sz seed = *iter′* (*TYPE*(*′a*)) *nrandom sz seed*

lemma [*code*]:
iter nrandom sz seed = (*if nrandom* = *0 then bot-class.bot else*
let ((*x*, -), *seed′*) = *Quickcheck-Random.random sz seed*
in Predicate.Seq (%*u*. *Predicate.Insert x* (*iter* (*nrandom* − *1*) *sz seed′*)))
unfolding *iter-def iter′.simps* [*of* - *nrandom*] **..**

type-synonym *′a random-pred* = *Random.seed* ⇒ (*′a Predicate.pred* × *Random.seed*)

definition *empty* :: *′a random-pred*
where *empty* = *Pair bot*

definition *single* :: *′a* => *′a random-pred*
where *single x* = *Pair* (*Predicate.single x*)

definition *bind* :: *'a random-pred* ⇒ (*'a* ⇒ *'b random-pred*) ⇒ *'b random-pred*
  where
    bind R f = (λ*s. let*
      (*P*, *s'*) = *R s*;
      (*s1*, *s2*) = *Random.split-seed s'*
    in (*Predicate.bind P* (%*a. fst* (*f a s1*)), *s2*))

definition *union* :: *'a random-pred* ⇒ *'a random-pred* ⇒ *'a random-pred*
where
  union R1 R2 = (λ*s. let*
    (*P1*, *s'*) = *R1 s*; (*P2*, *s''*) = *R2 s'*
  in (*sup-class.sup P1 P2*, *s''*))

definition *if-randompred* :: *bool* ⇒ *unit random-pred*
where
  if-randompred b = (*if b then single* () *else empty*)

definition *iterate-upto* :: (*natural* ⇒ *'a*) => *natural* ⇒ *natural* ⇒ *'a random-pred*
where
  iterate-upto f n m = *Pair* (*Predicate.iterate-upto f n m*)

definition *not-randompred* :: *unit random-pred* ⇒ *unit random-pred*
where
  not-randompred P = (λ*s. let*
    (*P'*, *s'*) = *P s*
  in if Predicate.eval P' () *then* (*Orderings.bot*, *s'*) *else* (*Predicate.single* (), *s'*))

definition *Random* :: (*Random.seed* ⇒ (*'a* × (*unit* ⇒ *term*)) × *Random.seed*) ⇒ *'a random-pred*
  where *Random g* = *scomp g* (*Pair* ∘ (*Predicate.single* ∘ *fst*))

definition *map* :: (*'a* ⇒ *'b*) ⇒ *'a random-pred* ⇒ *'b random-pred*
  where *map f P* = *bind P* (*single* ∘ *f*)

hide-const (**open**) *iter' iter empty single bind union if-randompred iterate-upto not-randompred Random map*

hide-fact *iter'.simps*

hide-fact (**open**) *iter-def empty-def single-def bind-def union-def if-randompred-def iterate-upto-def not-randompred-def Random-def map-def*

end

81 Various kind of sequences inside the random monad

theory *Random-Sequence*

imports *Random-Pred*
begin

type-synonym *'a random-dseq = natural ⇒ natural ⇒ Random.seed ⇒ ('a Limited-Sequence.dseq × Random.seed)*

definition *empty* :: *'a random-dseq*
where
empty = (%nrandom size. Pair (Limited-Sequence.empty))

definition *single* :: *'a => 'a random-dseq*
where
single x = (%nrandom size. Pair (Limited-Sequence.single x))

definition *bind* :: *'a random-dseq => ('a ⇒ 'b random-dseq) ⇒ 'b random-dseq*
where
bind R f = (λnrandom size s. let
(P, s') = R nrandom size s;
(s1, s2) = Random.split-seed s'
in (Limited-Sequence.bind P (%a. fst (f a nrandom size s1)), s2))

definition *union* :: *'a random-dseq => 'a random-dseq => 'a random-dseq*
where
union R1 R2 = (λnrandom size s. let
(S1, s') = R1 nrandom size s; (S2, s'') = R2 nrandom size s'
in (Limited-Sequence.union S1 S2, s''))

definition *if-random-dseq* :: *bool => unit random-dseq*
where
if-random-dseq b = (if b then single () else empty)

definition *not-random-dseq* :: *unit random-dseq => unit random-dseq*
where
not-random-dseq R = (λnrandom size s. let
(S, s') = R nrandom size s
in (Limited-Sequence.not-seq S, s'))

definition *map* :: *('a => 'b) => 'a random-dseq => 'b random-dseq*
where
map f P = bind P (single ∘ f)

fun *Random* :: *(natural ⇒ Random.seed ⇒ (('a × (unit ⇒ term)) × Random.seed)) ⇒ 'a random-dseq*
where
Random g nrandom = (%size. if nrandom <= 0 then (Pair Limited-Sequence.empty) else
(scomp (g size) (%r. scomp (Random g (nrandom − 1) size) (%rs. Pair (Limited-Sequence.union (Limited-Sequence.single (fst r)) rs)))))

type-synonym *'a pos-random-dseq = natural ⇒ natural ⇒ Random.seed ⇒ 'a Limited-Sequence.pos-dseq*

definition *pos-empty* :: *'a pos-random-dseq*
where
pos-empty = (*%nrandom size seed. Limited-Sequence.pos-empty*)

definition *pos-single* :: *'a => 'a pos-random-dseq*
where
pos-single x = (*%nrandom size seed. Limited-Sequence.pos-single x*)

definition *pos-bind* :: *'a pos-random-dseq => ('a ⇒ 'b pos-random-dseq) ⇒ 'b pos-random-dseq*
where
pos-bind R f = (*λnrandom size seed. Limited-Sequence.pos-bind* (*R nrandom size seed*) (*%a. f a nrandom size seed*))

definition *pos-decr-bind* :: *'a pos-random-dseq => ('a ⇒ 'b pos-random-dseq) ⇒ 'b pos-random-dseq*
where
pos-decr-bind R f = (*λnrandom size seed. Limited-Sequence.pos-decr-bind* (*R nrandom size seed*) (*%a. f a nrandom size seed*))

definition *pos-union* :: *'a pos-random-dseq => 'a pos-random-dseq => 'a pos-random-dseq*
where
pos-union R1 R2 = (*λnrandom size seed. Limited-Sequence.pos-union* (*R1 nrandom size seed*) (*R2 nrandom size seed*))

definition *pos-if-random-dseq* :: *bool => unit pos-random-dseq*
where
pos-if-random-dseq b = (*if b then pos-single* () *else pos-empty*)

definition *pos-iterate-upto* :: (*natural => 'a*) *=> natural => natural => 'a pos-random-dseq*
where
pos-iterate-upto f n m = (*λnrandom size seed. Limited-Sequence.pos-iterate-upto f n m*)

definition *pos-map* :: (*'a => 'b*) *=> 'a pos-random-dseq => 'b pos-random-dseq*
where
pos-map f P = *pos-bind P* (*pos-single* ∘ *f*)

fun *iter* :: (*Random.seed* ⇒ (*'a* × (*unit* ⇒ *term*)) × *Random.seed*)
⇒ *natural* ⇒ *Random.seed* ⇒ *'a Lazy-Sequence.lazy-sequence*
where
iter random nrandom seed =
(*if nrandom* = *0 then Lazy-Sequence.empty else Lazy-Sequence.Lazy-Sequence* (*%u. let* ((*x*, -), *seed'*) = *random seed in Some* (*x*, *iter random* (*nrandom* − *1*)

seed')))

definition *pos-Random* :: (*natural* ⇒ *Random.seed* ⇒ (*'a* × (*unit* ⇒ *term*)) × *Random.seed*)
⇒ *'a pos-random-dseq*
where
pos-Random g = (%*nrandom size seed depth. iter* (*g size*) *nrandom seed*)

type-synonym *'a neg-random-dseq* = *natural* ⇒ *natural* ⇒ *Random.seed* ⇒ *'a Limited-Sequence.neg-dseq*

definition *neg-empty* :: *'a neg-random-dseq*
where
neg-empty = (%*nrandom size seed. Limited-Sequence.neg-empty*)

definition *neg-single* :: *'a* => *'a neg-random-dseq*
where
neg-single x = (%*nrandom size seed. Limited-Sequence.neg-single x*)

definition *neg-bind* :: *'a neg-random-dseq* => (*'a* ⇒ *'b neg-random-dseq*) ⇒ *'b neg-random-dseq*
where
neg-bind R f = (λ*nrandom size seed. Limited-Sequence.neg-bind* (*R nrandom size seed*) (%*a. f a nrandom size seed*))

definition *neg-decr-bind* :: *'a neg-random-dseq* => (*'a* ⇒ *'b neg-random-dseq*) ⇒ *'b neg-random-dseq*
where
neg-decr-bind R f = (λ*nrandom size seed. Limited-Sequence.neg-decr-bind* (*R nrandom size seed*) (%*a. f a nrandom size seed*))

definition *neg-union* :: *'a neg-random-dseq* => *'a neg-random-dseq* => *'a neg-random-dseq*
where
neg-union R1 R2 = (λ*nrandom size seed. Limited-Sequence.neg-union* (*R1 nrandom size seed*) (*R2 nrandom size seed*))

definition *neg-if-random-dseq* :: *bool* => *unit neg-random-dseq*
where
neg-if-random-dseq b = (*if b then neg-single* () *else neg-empty*)

definition *neg-iterate-upto* :: (*natural* => *'a*) => *natural* => *natural* => *'a neg-random-dseq*
where
neg-iterate-upto f n m = (λ*nrandom size seed. Limited-Sequence.neg-iterate-upto f n m*)

definition *neg-not-random-dseq* :: *unit pos-random-dseq* => *unit neg-random-dseq*
where

neg-not-random-dseq R = (λ*nrandom size seed. Limited-Sequence.neg-not-seq* (*R nrandom size seed*))

definition *neg-map* :: (′*a* => ′*b*) => ′*a neg-random-dseq* => ′*b neg-random-dseq*
where
neg-map f P = *neg-bind P* (*neg-single* ∘ *f*)

definition *pos-not-random-dseq* :: *unit neg-random-dseq* => *unit pos-random-dseq*
where
pos-not-random-dseq R = (λ*nrandom size seed. Limited-Sequence.pos-not-seq* (*R nrandom size seed*))

hide-const (**open**)
empty single bind union if-random-dseq not-random-dseq map Random
pos-empty pos-single pos-bind pos-decr-bind pos-union pos-if-random-dseq pos-iterate-upto
pos-not-random-dseq pos-map iter pos-Random
neg-empty neg-single neg-bind neg-decr-bind neg-union neg-if-random-dseq neg-iterate-upto
neg-not-random-dseq neg-map

hide-fact (**open**) *empty-def single-def bind-def union-def if-random-dseq-def not-random-dseq-def*
map-def Random.simps
pos-empty-def pos-single-def pos-bind-def pos-decr-bind-def pos-union-def pos-if-random-dseq-def
pos-iterate-upto-def pos-not-random-dseq-def pos-map-def iter.simps pos-Random-def
neg-empty-def neg-single-def neg-bind-def neg-decr-bind-def neg-union-def neg-if-random-dseq-def
neg-iterate-upto-def neg-not-random-dseq-def neg-map-def

end

82 A simple counterexample generator performing exhaustive testing

theory *Quickcheck-Exhaustive*
imports *Quickcheck-Random*
keywords *quickcheck-generator* :: *thy-decl*
begin

82.1 Basic operations for exhaustive generators

definition *orelse* :: ′*a option* ⇒ ′*a option* ⇒ ′*a option* (**infixr** ‹*orelse*› *55*)
where [*code-unfold*]: *x orelse y* = (*case x of Some x*′ ⇒ *Some x*′ | *None* ⇒ *y*)

82.2 Exhaustive generator type classes

class *exhaustive* = *term-of* +
fixes *exhaustive* :: (′*a* ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*

class *full-exhaustive* = *term-of* +

fixes *full-exhaustive* ::
('*a* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*

instantiation *natural* :: *full-exhaustive*
begin

function *full-exhaustive-natural'* ::
(*natural* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
natural ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where *full-exhaustive-natural' f d i* =
(*if d* < *i then None*
else (*f* (*i*, λ-. *Code-Evaluation.term-of i*)) *orelse* (*full-exhaustive-natural' f d* (*i* + *1*)))
by *pat-completeness auto*

termination
by (*relation measure* (λ(-, *d*, *i*). *nat-of-natural* (*d* + *1* − *i*))) (*auto simp add*: *less-natural-def*)

definition *full-exhaustive f d* = *full-exhaustive-natural' f d 0*

instance ..

end

instantiation *natural* :: *exhaustive*
begin

function *exhaustive-natural'* ::
(*natural* ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where *exhaustive-natural' f d i* =
(*if d* < *i then None*
else (*f i orelse exhaustive-natural' f d* (*i* + *1*)))
by *pat-completeness auto*

termination
by (*relation measure* (λ(-, *d*, *i*). *nat-of-natural* (*d* + *1* − *i*))) (*auto simp add*: *less-natural-def*)

definition *exhaustive f d* = *exhaustive-natural' f d 0*

instance ..

end

instantiation *integer* :: *exhaustive*
begin

function *exhaustive-integer′* ::
  (*integer* ⇒ (*bool* × *term list*) *option*) ⇒ *integer* ⇒ *integer* ⇒ (*bool* × *term list*) *option*
  where *exhaustive-integer′ f d i* =
  (*if d* < *i then None else* (*f i orelse exhaustive-integer′ f d* (*i* + *1*)))
by *pat-completeness auto*

termination
  by (*relation measure* (λ(-, *d*, *i*). *nat-of-integer* (*d* + *1* − *i*)))
  (*auto simp add*: *less-integer-def nat-of-integer-def*)

definition *exhaustive f d* = *exhaustive-integer′ f* (*integer-of-natural d*) (− (*integer-of-natural d*))

instance ..

end

instantiation *integer* :: *full-exhaustive*
begin

function *full-exhaustive-integer′* ::
  (*integer* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
    integer ⇒ *integer* ⇒ (*bool* × *term list*) *option*
  where *full-exhaustive-integer′ f d i* =
  (*if d* < *i then None*
   else
    (*case f* (*i*, λ-. *Code-Evaluation.term-of i*) *of*
     Some t ⇒ *Some t*
    | *None* ⇒ *full-exhaustive-integer′ f d* (*i* + *1*)))
by *pat-completeness auto*

termination
  by (*relation measure* (λ(-, *d*, *i*). *nat-of-integer* (*d* + *1* − *i*)))
  (*auto simp add*: *less-integer-def nat-of-integer-def*)

definition *full-exhaustive f d* =
 full-exhaustive-integer′ f (*integer-of-natural d*) (− (*integer-of-natural d*))

instance ..

end

instantiation *nat* :: *exhaustive*
begin

definition *exhaustive f d* = *exhaustive* (λ*x*. *f* (*nat-of-natural x*)) *d*

instance ..

end

instantiation *nat* :: *full-exhaustive*
begin

definition *full-exhaustive f d* =
full-exhaustive (λ(*x*, *xt*). *f* (*nat-of-natural x*, λ-. *Code-Evaluation.term-of* (*nat-of-natural x*))) *d*

instance ..

end

instantiation *int* :: *exhaustive*
begin

function *exhaustive-int′* ::
(*int* ⇒ (*bool* × *term list*) *option*) ⇒ *int* ⇒ *int* ⇒ (*bool* × *term list*) *option*
where *exhaustive-int′ f d i* =
(*if d* < *i then None else* (*f i orelse exhaustive-int′ f d* (*i* + *1*)))
by *pat-completeness auto*

termination
by (*relation measure* (λ(-, *d*, *i*). *nat* (*d* + *1* − *i*))) *auto*

definition *exhaustive f d* =
exhaustive-int′ f (*int-of-integer* (*integer-of-natural d*))
(− (*int-of-integer* (*integer-of-natural d*)))

instance ..

end

instantiation *int* :: *full-exhaustive*
begin

function *full-exhaustive-int′* ::
(*int* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
int ⇒ *int* ⇒ (*bool* × *term list*) *option*
where *full-exhaustive-int′ f d i* =
(*if d* < *i then None*
else
(*case f* (*i*, λ-. *Code-Evaluation.term-of i*) *of*
Some t ⇒ *Some t*
| *None* ⇒ *full-exhaustive-int′ f d* (*i* + *1*)))
by *pat-completeness auto*

termination
by (*relation measure* (λ(-, *d*, *i*). *nat* (*d* + *1* − *i*))) *auto*

definition *full-exhaustive f d* =
full-exhaustive-int′ f (*int-of-integer* (*integer-of-natural d*))
(− (*int-of-integer* (*integer-of-natural d*)))

instance ..

end

instantiation *prod* :: (*exhaustive*, *exhaustive*) *exhaustive*
begin

definition *exhaustive f d* = *exhaustive* (λ*x*. *exhaustive* (λ*y*. *f* ((*x*, *y*))) *d*) *d*

instance ..

end

context
includes *term-syntax*
begin

definition
[*code-unfold*]: *valtermify-pair x y* =
Code-Evaluation.valtermify (*Pair* :: ′*a*::*typerep* ⇒ ′*b*::*typerep* ⇒ ′*a* × ′*b*) {·} *x*
{·} *y*

end

instantiation *prod* :: (*full-exhaustive*, *full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive f d* =
full-exhaustive (λ*x*. *full-exhaustive* (λ*y*. *f* (*valtermify-pair x y*)) *d*) *d*

instance ..

end

instantiation *set* :: (*exhaustive*) *exhaustive*
begin

fun *exhaustive-set*
where
exhaustive-set f i =
(*if i* = *0 then None*
else

f {} *orelse*
exhaustive-set
(λA. *f A orelse exhaustive* (λx. *if* $x \in A$ *then None else f* (*insert x A*)) ($i - 1$)) ($i - 1$))

instance ..

end

instantiation *set* :: (*full-exhaustive*) *full-exhaustive*
begin

fun *full-exhaustive-set*
where
full-exhaustive-set f i =
(*if* $i = 0$ *then None*
else
f valterm-emptyset orelse
full-exhaustive-set
(λA. *f A orelse Quickcheck-Exhaustive.full-exhaustive*
(λx. *if fst* $x \in$ *fst A then None else f* (*valtermify-insert x A*)) ($i - 1$)) ($i - 1$))

instance ..

end

instantiation *fun* :: ({*equal*,*exhaustive*}, *exhaustive*) *exhaustive*
begin

fun *exhaustive-fun'* ::
(($'a \Rightarrow 'b$) $\Rightarrow$ (*bool* $\times$ *term list*) *option*) $\Rightarrow$ *natural* $\Rightarrow$ *natural* $\Rightarrow$ (*bool* $\times$ *term list*) *option*
where
exhaustive-fun' f i d =
(*exhaustive* (λb. *f* (λ-. *b*)) *d*) *orelse*
(*if* $i > 1$ *then*
exhaustive-fun'
(λg. *exhaustive* (λa. *exhaustive* (λb. *f* ($g(a := b)$)) *d*) *d*) ($i - 1$) *d else None*)

definition *exhaustive-fun* ::
(($'a \Rightarrow 'b$) $\Rightarrow$ (*bool* $\times$ *term list*) *option*) $\Rightarrow$ *natural* $\Rightarrow$ (*bool* $\times$ *term list*) *option*
where *exhaustive-fun f d* = *exhaustive-fun' f d d*

instance ..

end

definition [*code-unfold*]:
valtermify-absdummy =
(λ(*v*, *t*).
(λ-::'*a*. *v*,
λ*u*::*unit*. *Code-Evaluation.Abs* (*STR* ''*x*'') (*Typerep.typerep TYPE*('*a*::*typerep*))
(*t* ())))

context
includes *term-syntax*
begin

definition
[*code-unfold*]: *valtermify-fun-upd g a b* =
Code-Evaluation.valtermify
(*fun-upd* :: ('*a*::*typerep* ⇒ '*b*::*typerep*) ⇒ '*a* ⇒ '*b* ⇒ '*a* ⇒ '*b*) {·} *g* {·} *a* {·} *b*

end

instantiation *fun* :: ({*equal*,*full-exhaustive*}, *full-exhaustive*) *full-exhaustive*
begin

fun *full-exhaustive-fun'* ::
(('*a* ⇒ '*b*) × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
natural ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
full-exhaustive-fun' f i d =
full-exhaustive (λ*v*. *f* (*valtermify-absdummy v*)) *d orelse*
(*if i* > *1 then*
full-exhaustive-fun'
(λ*g*. *full-exhaustive*
(λ*a*. *full-exhaustive* (λ*b*. *f* (*valtermify-fun-upd g a b*)) *d*) *d*) (*i* − *1*) *d*
else None)

definition *full-exhaustive-fun* ::
(('*a* ⇒ '*b*) × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
natural ⇒ (*bool* × *term list*) *option*
where *full-exhaustive-fun f d* = *full-exhaustive-fun' f d d*

instance ..

end

82.2.1 A smarter enumeration scheme for functions over finite datatypes

class *check-all* = *enum* + *term-of* +
fixes *check-all* :: ('*a* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ (*bool* ∗ *term list*) *option*
fixes *enum-term-of* :: '*a itself* ⇒ *unit* ⇒ *term list*

fun *check-all-n-lists* :: (′a::check-all list × (unit ⇒ term list) ⇒
  (bool × term list) option) ⇒ natural ⇒ (bool ∗ term list) option
where
  check-all-n-lists f n =
    (*if n* = *0 then f* ([], (λ-. []))
     else check-all (λ(x, xt).
      check-all-n-lists (λ(xs, xst). f ((x # xs), (λ-. (xt () # xst ())))) (n − 1)))

context
  includes *term-syntax*
begin

definition
  [*code-unfold*]: *termify-fun-upd g a b* =
    (*Code-Evaluation.termify*
      (*fun-upd* :: (′a::typerep ⇒ ′b::typerep) ⇒ ′a ⇒ ′b ⇒ ′a ⇒ ′b) <·> g <·> a
<·> b)

end

definition *mk-map-term* ::
  (unit ⇒ typerep) ⇒ (unit ⇒ typerep) ⇒
    (unit ⇒ term list) ⇒ (unit ⇒ term list) ⇒ unit ⇒ term
  where *mk-map-term T1 T2 domm rng* =
    (λ-.
      let
        T1 = *T1* ();
        T2 = *T2* ();
        update-term =
          (λg (a, b).
            Code-Evaluation.App (*Code-Evaluation.App* (*Code-Evaluation.App*
             (*Code-Evaluation.Const* (*STR* ′′Fun.fun-upd′′)
                (*Typerep.Typerep* (*STR* ′′fun′′) [*Typerep.Typerep* (*STR* ′′fun′′) [T1,
T2],
                  Typerep.Typerep (*STR* ′′fun′′) [T1,
                   Typerep.Typerep (*STR* ′′fun′′) [T2, *Typerep.Typerep* (*STR* ′′fun′′)
[T1, T2]]]]))
                   g) a) b)
      in
        List.foldl update-term
          (*Code-Evaluation.Abs* (*STR* ′′x′′) T1
             (*Code-Evaluation.Const* (*STR* ′′HOL.undefined′′) T2)) (*zip* (*domm* ())
(*rng* ())))

instantiation *fun* :: ({equal,check-all}, check-all) check-all
begin

definition

```
  check-all f =
    (let
      mk-term =
        mk-map-term
          (λ-. Typerep.typerep (TYPE('a)))
          (λ-. Typerep.typerep (TYPE('b)))
          (enum-term-of (TYPE('a)));
      enum = (Enum.enum :: 'a list)
    in
      check-all-n-lists
        (λ(ys, yst). f (the ∘ map-of (zip enum ys), mk-term yst))
        (natural-of-nat (length enum)))

definition enum-term-of-fun :: ('a ⇒ 'b) itself ⇒ unit ⇒ term list
  where enum-term-of-fun =
    (λ- -.
      let
        enum-term-of-a = enum-term-of (TYPE('a));
        mk-term =
          mk-map-term
            (λ-. Typerep.typerep (TYPE('a)))
            (λ-. Typerep.typerep (TYPE('b)))
            enum-term-of-a
      in
        map (λys. mk-term (λ-. ys) ())
         (List.n-lists (length (enum-term-of-a ())) (enum-term-of (TYPE('b)) ())))

instance ..

end

context
  includes term-syntax
begin

fun check-all-subsets ::
  (('a::typerep) set × (unit ⇒ term) ⇒ (bool × term list) option) ⇒
    ('a × (unit ⇒ term)) list ⇒ (bool × term list) option
where
  check-all-subsets f [] = f valterm-emptyset
| check-all-subsets f (x # xs) =
   check-all-subsets (λs. case f s of Some ts ⇒ Some ts | None ⇒ f (valtermify-insert
x s)) xs

definition
  [code-unfold]: term-emptyset = Code-Evaluation.termify ({} :: ('a::typerep) set)

definition
  [code-unfold]: termify-insert x s =
```

Code-Evaluation.termify (*insert* :: (′*a*::*typerep*) ⇒ ′*a set* ⇒ ′*a set*) <·> *x* <·> *s*

definition *setify* :: (′*a*::*typerep*) *itself* ⇒ *term list* ⇒ *term*
where
setify T ts = *foldr* (*termify-insert T*) *ts* (*term-emptyset T*)

end

instantiation *set* :: (*check-all*) *check-all*
begin

definition
check-all-set f =
check-all-subsets f
(*zip* (*Enum.enum* :: ′*a list*)
(*map* (λ*a*. λ*u* :: *unit*. *a*) (*Quickcheck-Exhaustive.enum-term-of* (*TYPE* (′*a*)) ())))

definition *enum-term-of-set* :: ′*a set itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-set* - - =
map (*setify* (*TYPE*(′*a*))) (*subseqs* (*Quickcheck-Exhaustive.enum-term-of* (*TYPE*(′*a*)) ()))

instance ..

end

instantiation *unit* :: *check-all*
begin

definition *check-all f* = *f* (*Code-Evaluation.valtermify* ())

definition *enum-term-of-unit* :: *unit itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-unit* = (λ- -. [*Code-Evaluation.term-of* ()])

instance ..

end

instantiation *bool* :: *check-all*
begin

definition
check-all f =
(*case f* (*Code-Evaluation.valtermify False*) *of*
Some x′ ⇒ *Some x*′
| *None* ⇒ *f* (*Code-Evaluation.valtermify True*))

definition *enum-term-of-bool* :: *bool itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-bool* = (λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *bool list*))

instance ..

end

context
includes *term-syntax*
begin

definition [*code-unfold*]:
termify-pair x y =
Code-Evaluation.termify (*Pair* :: 'a::*typerep* ⇒ 'b :: *typerep* ⇒ 'a ∗ 'b) <·> *x* <·> *y*

end

instantiation *prod* :: (*check-all*, *check-all*) *check-all*
begin

definition *check-all f* = *check-all* (λ*x*. *check-all* (λ*y*. *f* (*valtermify-pair x y*)))

definition *enum-term-of-prod* :: ('a ∗ 'b) *itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-prod* =
(λ- -.
map (λ(*x*, *y*). *termify-pair TYPE*('a) *TYPE*('b) *x y*)
(*List.product* (*enum-term-of* (*TYPE*('a)) ()) (*enum-term-of* (*TYPE*('b)) ())))

instance ..

end

context
includes *term-syntax*
begin

definition
[*code-unfold*]: *valtermify-Inl x* =
Code-Evaluation.valtermify (*Inl* :: 'a::*typerep* ⇒ 'a + 'b :: *typerep*) {·} *x*

definition
[*code-unfold*]: *valtermify-Inr x* =
Code-Evaluation.valtermify (*Inr* :: 'b::*typerep* ⇒ 'a::*typerep* + 'b) {·} *x*

end

instantiation *sum* :: (*check-all*, *check-all*) *check-all*
begin

definition
check-all f = *check-all* (λ*a*. *f* (*valtermify-Inl a*)) *orelse check-all* (λ*b*. *f* (*valtermify-Inr b*))

definition *enum-term-of-sum* :: ('*a* + '*b*) *itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-sum* =
(λ- -.
let
T1 = *Typerep.typerep* (*TYPE*('*a*));
T2 = *Typerep.typerep* (*TYPE*('*b*))
in
map
(*Code-Evaluation.App* (*Code-Evaluation.Const* (*STR* ''*Sum-Type.Inl*'')
(*Typerep.Typerep* (*STR* ''*fun*'') [*T1*, *Typerep.Typerep* (*STR* ''*Sum-Type.sum*'') [*T1*, *T2*]])))
(*enum-term-of* (*TYPE*('*a*)) ()) @
map
(*Code-Evaluation.App* (*Code-Evaluation.Const* (*STR* ''*Sum-Type.Inr*'')
(*Typerep.Typerep* (*STR* ''*fun*'') [*T2*, *Typerep.Typerep* (*STR* ''*Sum-Type.sum*'') [*T1*, *T2*]])))
(*enum-term-of* (*TYPE*('*b*)) ()))

instance ..

end

instantiation *char* :: *check-all*
begin

primrec *check-all-char'* ::
(*char* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *char list* ⇒ (*bool* × *term list*) *option*
where *check-all-char' f* [] = *None*
| *check-all-char' f* (*c* # *cs*) = *f* (*c*, λ-. *Code-Evaluation.term-of c*)
orelse check-all-char' f cs

definition *check-all-char* ::
(*char* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ (*bool* × *term list*) *option*
where *check-all f* = *check-all-char' f Enum.enum*

definition *enum-term-of-char* :: *char itself* ⇒ *unit* ⇒ *term list*
where
enum-term-of-char = (λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *char list*))

instance ..

end

instantiation *option* :: (*check-all*) *check-all*
begin

definition
check-all f =
f (*Code-Evaluation.valtermify* (*None* :: *'a option*)) *orelse*
check-all
(λ(*x*, *t*).
f
(*Some x*,
λ-. *Code-Evaluation.App*
(*Code-Evaluation.Const* (*STR ''Option.option.Some''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.typerep TYPE*(*'a*),
Typerep.Typerep (*STR ''Option.option''*) [*Typerep.typerep TYPE*(*'a*)]]))
(*t* ())))

definition *enum-term-of-option* :: *'a option itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-option* =
(λ - -.
Code-Evaluation.term-of (*None* :: *'a option*) #
(*map*
(*Code-Evaluation.App*
(*Code-Evaluation.Const* (*STR ''Option.option.Some''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.typerep TYPE*(*'a*),
Typerep.Typerep (*STR ''Option.option''*) [*Typerep.typerep TYPE*(*'a*)]])))
(*enum-term-of* (*TYPE*(*'a*)) ())))

instance ..

end

instantiation *Enum.finite-1* :: *check-all*
begin

definition *check-all f* = *f* (*Code-Evaluation.valtermify Enum.finite-1.*a_1)

definition *enum-term-of-finite-1* :: *Enum.finite-1 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-1* = (λ- -. [*Code-Evaluation.term-of Enum.finite-1.*a_1])

instance ..

end

instantiation *Enum.finite-2* :: *check-all*
begin

definition
check-all f =
(*f* (*Code-Evaluation.valtermify Enum.finite-2.*a_1) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-2.*a_2))

definition *enum-term-of-finite-2* :: *Enum.finite-2 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-2* =
(λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *Enum.finite-2 list*))

instance ..

end

instantiation *Enum.finite-3* :: *check-all*
begin

definition
check-all f =
(*f* (*Code-Evaluation.valtermify Enum.finite-3.*a_1) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-3.*a_2) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-3.*a_3))

definition *enum-term-of-finite-3* :: *Enum.finite-3 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-3* =
(λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *Enum.finite-3 list*))

instance ..

end

instantiation *Enum.finite-4* :: *check-all*
begin

definition
check-all f =
f (*Code-Evaluation.valtermify Enum.finite-4.*a_1) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-4.*a_2) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-4.*a_3) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-4.*a_4)

definition *enum-term-of-finite-4* :: *Enum.finite-4 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-4* =
(λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *Enum.finite-4 list*))

instance ..

end

82.3 Bounded universal quantifiers

class *bounded-forall* =
  fixes *bounded-forall* :: ('*a* ⇒ *bool*) ⇒ *natural* ⇒ *bool*

82.4 Fast exhaustive combinators

class *fast-exhaustive* = *term-of* +
  fixes *fast-exhaustive* :: ('*a* ⇒ *unit*) ⇒ *natural* ⇒ *unit*

axiomatization *throw-Counterexample* :: *term list* ⇒ *unit*
axiomatization *catch-Counterexample* :: *unit* ⇒ *term list option*

code-printing
  constant *throw-Counterexample* ⇀
    (*Quickcheck*) *raise* (*Exhaustive'-Generators.Counterexample* -)
| **constant** *catch-Counterexample* ⇀
    (*Quickcheck*) (((-); *NONE*) *handle Exhaustive'-Generators.Counterexample ts* ⇒ *SOME ts*)

82.5 Continuation passing style functions as plus monad

type-synonym '*a cps* = ('*a* ⇒ *term list option*) ⇒ *term list option*

definition *cps-empty* :: '*a cps*
  where *cps-empty* = (λ*cont*. *None*)

definition *cps-single* :: '*a* ⇒ '*a cps*
  where *cps-single v* = (λ*cont*. *cont v*)

definition *cps-bind* :: '*a cps* ⇒ ('*a* ⇒ '*b cps*) ⇒ '*b cps*
  where *cps-bind m f* = (λ*cont*. *m* (λ*a*. (*f a*) *cont*))

definition *cps-plus* :: '*a cps* ⇒ '*a cps* ⇒ '*a cps*
  where *cps-plus a b* = (λ*c*. *case a c of None* ⇒ *b c* | *Some x* ⇒ *Some x*)

definition *cps-if* :: *bool* ⇒ *unit cps*
  where *cps-if b* = (*if b then cps-single* () *else cps-empty*)

definition *cps-not* :: *unit cps* ⇒ *unit cps*
  where *cps-not n* = (λ*c*. *case n* (λ*u*. *Some* []) *of None* ⇒ *c* () | *Some* - ⇒ *None*)

type-synonym '*a pos-bound-cps* =
  ('*a* ⇒ (*bool* ∗ *term list*) *option*) ⇒ *natural* ⇒ (*bool* ∗ *term list*) *option*

definition *pos-bound-cps-empty* :: '*a pos-bound-cps*
  where *pos-bound-cps-empty* = (λ*cont i*. *None*)

definition *pos-bound-cps-single* :: *'a ⇒ 'a pos-bound-cps*
  where *pos-bound-cps-single v* = (*λcont i. cont v*)

definition *pos-bound-cps-bind* :: *'a pos-bound-cps* ⇒ (*'a* ⇒ *'b pos-bound-cps*) ⇒ *'b pos-bound-cps*
  where *pos-bound-cps-bind m f* = (*λcont i. if i = 0 then None else* (*m* (*λa.* (*f a*) *cont i*) (*i* − *1*)))

definition *pos-bound-cps-plus* :: *'a pos-bound-cps ⇒ 'a pos-bound-cps ⇒ 'a pos-bound-cps*
  where *pos-bound-cps-plus a b* = (*λc i. case a c i of None ⇒ b c i* | *Some x* ⇒ *Some x*)

definition *pos-bound-cps-if* :: *bool ⇒ unit pos-bound-cps*
  where *pos-bound-cps-if b* = (*if b then pos-bound-cps-single* () *else pos-bound-cps-empty*)

datatype (*plugins only*: *code extraction*) (*dead 'a*) *unknown* =
  Unknown | *Known 'a*

datatype (*plugins only*: *code extraction*) (*dead 'a*) *three-valued* =
  Unknown-value | *Value 'a* | *No-value*

type-synonym *'a neg-bound-cps* =
  (*'a unknown ⇒ term list three-valued*) ⇒ *natural ⇒ term list three-valued*

definition *neg-bound-cps-empty* :: *'a neg-bound-cps*
  where *neg-bound-cps-empty* = (*λcont i. No-value*)

definition *neg-bound-cps-single* :: *'a ⇒ 'a neg-bound-cps*
  where *neg-bound-cps-single v* = (*λcont i. cont* (*Known v*))

definition *neg-bound-cps-bind* :: *'a neg-bound-cps* ⇒ (*'a* ⇒ *'b neg-bound-cps*) ⇒ *'b neg-bound-cps*
  where *neg-bound-cps-bind m f* =
    (*λcont i.*
      if i = 0 then cont Unknown
      else m (*λa. case a of Unknown ⇒ cont Unknown* | *Known a' ⇒ f a' cont i*) (*i* − *1*))

definition *neg-bound-cps-plus* :: *'a neg-bound-cps ⇒ 'a neg-bound-cps ⇒ 'a neg-bound-cps*
  where *neg-bound-cps-plus a b* =
    (*λc i.*
      case a c i of
        No-value ⇒ b c i
      | *Value x ⇒ Value x*
      | *Unknown-value ⇒*
          (*case b c i of*
            No-value ⇒ Unknown-value
          | *Value x ⇒ Value x*

| *Unknown-value* ⇒ *Unknown-value*))

definition *neg-bound-cps-if* :: *bool* ⇒ *unit neg-bound-cps*
where *neg-bound-cps-if b* = (*if b then neg-bound-cps-single* () *else neg-bound-cps-empty*)

definition *neg-bound-cps-not* :: *unit pos-bound-cps* ⇒ *unit neg-bound-cps*
where *neg-bound-cps-not n* =
(λ*c i. case n* (λ*u. Some* (*True*, [])) *i of None* ⇒ *c* (*Known* ()) | *Some* - ⇒ *No-value*)

definition *pos-bound-cps-not* :: *unit neg-bound-cps* ⇒ *unit pos-bound-cps*
where *pos-bound-cps-not n* =
(λ*c i. case n* (λ*u. Value* []) *i of No-value* ⇒ *c* () | *Value* - ⇒ *None* | *Unknown-value* ⇒ *None*)

82.6 Defining generators for any first-order data type

axiomatization *unknown* :: *'a*

notation (**output**) *unknown* (‹?›)

ML-file ‹*Tools/Quickcheck/exhaustive-generators.ML*›

declare [[*quickcheck-batch-tester* = *exhaustive*]]

82.7 Defining generators for abstract types

ML-file ‹*Tools/Quickcheck/abstract-generators.ML*›

hide-fact (**open**) *orelse-def*
no-notation *orelse* (**infixr** ‹*orelse*› *55*)

hide-const *valtermify-absdummy valtermify-fun-upd*
valterm-emptyset valtermify-insert
valtermify-pair valtermify-Inl valtermify-Inr
termify-fun-upd term-emptyset termify-insert termify-pair setify

hide-const (**open**)
exhaustive full-exhaustive
exhaustive-int' full-exhaustive-int'
exhaustive-integer' full-exhaustive-integer'
exhaustive-natural' full-exhaustive-natural'
throw-Counterexample catch-Counterexample
check-all enum-term-of
orelse unknown mk-map-term check-all-n-lists check-all-subsets

hide-type (**open**) *cps pos-bound-cps neg-bound-cps unknown three-valued*

hide-const (**open**) *cps-empty cps-single cps-bind cps-plus cps-if cps-not*
pos-bound-cps-empty pos-bound-cps-single pos-bound-cps-bind

pos-bound-cps-plus pos-bound-cps-if pos-bound-cps-not
neg-bound-cps-empty neg-bound-cps-single neg-bound-cps-bind
neg-bound-cps-plus neg-bound-cps-if neg-bound-cps-not
Unknown Known Unknown-value Value No-value

end

83 A compiler for predicates defined by introduction rules

theory *Predicate-Compile*
imports *Random-Sequence Quickcheck-Exhaustive*
keywords
code-pred :: *thy-goal* **and**
values :: *diag*
begin

ML-file ‹*Tools/Predicate-Compile/predicate-compile-aux.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-compilations.ML*›
ML-file ‹*Tools/Predicate-Compile/core-data.ML*›
ML-file ‹*Tools/Predicate-Compile/mode-inference.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-proof.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-core.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-data.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-fun.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-pred.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile-specialisation.ML*›
ML-file ‹*Tools/Predicate-Compile/predicate-compile.ML*›

83.1 Set membership as a generator predicate

Introduce a new constant for membership to allow fine-grained control in code equations.

definition *contains* :: *'a set => 'a => bool*
where *contains A x* ⟷ *x* ∈ *A*

definition *contains-pred* :: *'a set => 'a => unit Predicate.pred*
where *contains-pred A x* = (*if x* ∈ *A then Predicate.single* () *else bot*)

lemma *pred-of-setE*:
assumes *Predicate.eval* (*pred-of-set A*) *x*
obtains *contains A x*
using *assms* **by**(*simp add*: *contains-def*)

lemma *pred-of-setI*: *contains A x ==> Predicate.eval* (*pred-of-set A*) *x*
by(*simp add*: *contains-def*)

lemma *pred-of-set-eq*: *pred-of-set* ≡ λ*A*. *Predicate.Pred* (*contains A*)

by(*simp add*: *contains-def*[*abs-def*] *pred-of-set-def o-def*)

lemma *containsI*: $x \in A$ ==> *contains A x*
by(*simp add*: *contains-def*)

lemma *containsE*: **assumes** *contains A x*
obtains A' x' **where** $A = A'$ $x = x'$ $x \in A$
using *assms* **by**(*simp add*: *contains-def*)

lemma *contains-predI*: *contains A x* ==> *Predicate.eval* (*contains-pred A x*) ()
by(*simp add*: *contains-pred-def contains-def*)

lemma *contains-predE*:
assumes *Predicate.eval* (*contains-pred A x*) *y*
obtains *contains A x*
using *assms* **by**(*simp add*: *contains-pred-def contains-def split*: *if-split-asm*)

lemma *contains-pred-eq*: *contains-pred* $\equiv$ $\lambda A\ x.$ *Predicate.Pred* ($\lambda y.$ *contains A x*)
by(*rule eq-reflection*)(*auto simp add*: *contains-pred-def fun-eq-iff contains-def intro*: *pred-eqI*)

lemma *contains-pred-notI*:
¬ *contains A x* ==> *Predicate.eval* (*Predicate.not-pred* (*contains-pred A x*)) ()
by(*simp add*: *contains-pred-def contains-def not-pred-eq*)

setup ‹
```
let
  val Fun = Predicate-Compile-Aux.Fun
  val Input = Predicate-Compile-Aux.Input
  val Output = Predicate-Compile-Aux.Output
  val Bool = Predicate-Compile-Aux.Bool
  val io = Fun (Input, Fun (Output, Bool))
  val ii = Fun (Input, Fun (Input, Bool))
in
  Core-Data.PredData.map (Graph.new-node
   (const-name ‹contains›,
    Core-Data.PredData {
      pos = Position.thread-data (),
      intros = [(NONE, @{thm containsI})],
      elim = SOME @{thm containsE},
      preprocessed = true,
      function-names = [(Predicate-Compile-Aux.Pred,
        [(io, const-name ‹pred-of-set›), (ii, const-name ‹contains-pred›)])],
      predfun-data = [
        (io, Core-Data.PredfunData {
           elim = @{thm pred-of-setE}, intro = @{thm pred-of-setI},
           neg-intro = NONE, definition = @{thm pred-of-set-eq}
         }),
        (ii, Core-Data.PredfunData {
```

elim = @{*thm contains-predE*}, *intro* = @{*thm contains-predI*},
neg-intro = *SOME* @{*thm contains-pred-notI*}, *definition* = @{*thm contains-pred-eq*}
})],
needs-random = []}))
end
›

hide-const (**open**) *contains contains-pred*
hide-fact (**open**) *pred-of-setE pred-of-setI pred-of-set-eq*
containsI containsE contains-predI contains-predE contains-pred-eq contains-pred-notI

end

84 Counterexample generator performing narrowing-based testing

theory *Quickcheck-Narrowing*
imports *Quickcheck-Random*
keywords *find-unused-assms* :: *diag*
begin

84.1 Counterexample generator

84.1.1 Code generation setup

setup ‹*Code-Target.add-derived-target* (*Haskell-Quickcheck*, [(*Code-Haskell.target*, *I*)])›

code-printing
code-module *Typerep* ⇀ (*Haskell-Quickcheck*) ‹
module Typerep(*Typerep*(..)) *where*

data Typerep = *Typerep String* [*Typerep*]
› **for type-constructor** *typerep* **constant** *Typerep.Typerep*
| **type-constructor** *typerep* ⇀ (*Haskell-Quickcheck*) *Typerep.Typerep*
| **constant** *Typerep.Typerep* ⇀ (*Haskell-Quickcheck*) *Typerep.Typerep*

code-reserved
(*Haskell-Quickcheck*) *Typerep*

code-printing
type-constructor *integer* ⇀ (*Haskell-Quickcheck*) *Prelude.Int*
| **constant** *0*::*integer* ⇀
(*Haskell-Quickcheck*) !(*0*/ ::/ *Prelude.Int*)

setup ‹
let
val target = *Haskell-Quickcheck*;

fun print - = Code-Haskell.print-numeral Prelude.Int;
in
Numeral.add-code ***const-name*** ‹*Code-Numeral.Pos*› *I print target*
#> Numeral.add-code ***const-name*** ‹*Code-Numeral.Neg*› (~) *print target*
end
›

84.1.2 Narrowing's deep representation of types and terms

datatype (*plugins only*: *code extraction*) *narrowing-type* =
Narrowing-sum-of-products narrowing-type list list

datatype (*plugins only*: *code extraction*) *narrowing-term* =
Narrowing-variable integer list narrowing-type
| *Narrowing-constructor integer narrowing-term list*

datatype (*plugins only*: *code extraction*) (*dead 'a*) *narrowing-cons* =
Narrowing-cons narrowing-type (*narrowing-term list* ⇒ *'a*) *list*

primrec *map-cons* :: (*'a => 'b*) => *'a narrowing-cons => 'b narrowing-cons*
where
map-cons f (*Narrowing-cons ty cs*) = *Narrowing-cons ty* (*map* (*λc. f ∘ c*) *cs*)

84.1.3 From narrowing's deep representation of terms to *HOL.Code-Evaluation*'s terms

class *partial-term-of* = *typerep* +
fixes *partial-term-of* :: *'a itself => narrowing-term => Code-Evaluation.term*

lemma *partial-term-of-anything*: *partial-term-of x nt ≡ t*
by (*rule eq-reflection*) (*cases partial-term-of x nt*, *cases t*, *simp*)

84.1.4 Auxilary functions for Narrowing

consts *nth* :: *'a list => integer => 'a*

code-printing constant *nth* ⇀ (*Haskell-Quickcheck*) **infixl** *9* !!

consts *error* :: *char list => 'a*

code-printing constant *error* ⇀ (*Haskell-Quickcheck*) *error*

consts *toEnum* :: *integer => char*

code-printing constant *toEnum* ⇀ (*Haskell-Quickcheck*) *Prelude.toEnum*

consts *marker* :: *char*

code-printing constant *marker* ⇀ (*Haskell-Quickcheck*) *''\0'*

84.1.5 Narrowing's basic operations

type-synonym *'a narrowing = integer => 'a narrowing-cons*

definition *cons* :: *'a => 'a narrowing*
where
cons a d = (Narrowing-cons (Narrowing-sum-of-products [[]]) [(λ-. a)])

fun *conv* :: *(narrowing-term list => 'a) list => narrowing-term => 'a*
where
conv cs (Narrowing-variable p -) = error (marker # map toEnum p)
| *conv cs (Narrowing-constructor i xs) = (nth cs i) xs*

fun *non-empty* :: *narrowing-type => bool*
where
non-empty (Narrowing-sum-of-products ps) = (¬ (List.null ps))

definition *apply* :: *('a => 'b) narrowing => 'a narrowing => 'b narrowing*
where
apply f a d = (if d > 0 then
(case f d of Narrowing-cons (Narrowing-sum-of-products ps) cfs ⇒
case a (d − 1) of Narrowing-cons ta cas ⇒
let
shallow = non-empty ta;
cs = [(λ(x # xs) ⇒ cf xs (conv cas x)). shallow, cf ← cfs]
in Narrowing-cons (Narrowing-sum-of-products [ta # p. shallow, p ← ps])
cs)
else Narrowing-cons (Narrowing-sum-of-products []) [])

definition *sum* :: *'a narrowing => 'a narrowing => 'a narrowing*
where
sum a b d =
(case a d of Narrowing-cons (Narrowing-sum-of-products ssa) ca ⇒
case b d of Narrowing-cons (Narrowing-sum-of-products ssb) cb ⇒
Narrowing-cons (Narrowing-sum-of-products (ssa @ ssb)) (ca @ cb))

lemma [*fundef-cong*]:
assumes $a\ d = a'\ d\ b\ d = b'\ d\ d = d'$
shows $sum\ a\ b\ d = sum\ a'\ b'\ d'$
using *assms* **unfolding** *sum-def* **by** (*auto split*: *narrowing-cons.split narrowing-type.split*)

lemma [*fundef-cong*]:
assumes $f\ d = f'\ d\ (\bigwedge d'.\ 0 \leq d' \wedge d' < d \Longrightarrow a\ d' = a'\ d')$
assumes $d = d'$
shows $apply\ f\ a\ d = apply\ f'\ a'\ d'$
proof −
note *assms*
moreover have $0 < d' \Longrightarrow 0 \leq d' - 1$
by (*simp add*: *less-integer-def less-eq-integer-def*)
ultimately show *?thesis*

by (*auto simp add*: *apply-def Let-def*
split: *narrowing-cons.split narrowing-type.split*)
qed

84.1.6 Narrowing generator type class

class *narrowing* =
fixes *narrowing* :: *integer* => *'a narrowing-cons*

datatype (*plugins only*: *code extraction*) *property* =
Universal narrowing-type (*narrowing-term* => *property*) *narrowing-term* => *Code-Evaluation.term*
| *Existential narrowing-type* (*narrowing-term* => *property*) *narrowing-term* => *Code-Evaluation.term*
| *Property bool*

definition *exists* :: (*'a* :: {*narrowing*, *partial-term-of*} => *property*) => *property*
where
exists f = (*case narrowing* (*100* :: *integer*) *of Narrowing-cons ty cs* ⇒ *Existential ty* (λ *t*. *f* (*conv cs t*)) (*partial-term-of* (*TYPE*(*'a*))))

definition *all* :: (*'a* :: {*narrowing*, *partial-term-of*} => *property*) => *property*
where
all f = (*case narrowing* (*100* :: *integer*) *of Narrowing-cons ty cs* ⇒ *Universal ty* (λ*t*. *f* (*conv cs t*)) (*partial-term-of* (*TYPE*(*'a*))))

84.1.7 class *is-testable*

The class *is-testable* ensures that all necessary type instances are generated.

class *is-testable*

instance *bool* :: *is-testable* **..**

instance *fun* :: ({*term-of*, *narrowing*, *partial-term-of*}, *is-testable*) *is-testable* **..**

definition *ensure-testable* :: *'a* :: *is-testable* => *'a* :: *is-testable*
where
ensure-testable f = *f*

84.1.8 Defining a simple datatype to represent functions in an incomplete and redundant way

datatype (*plugins only*: *code quickcheck-narrowing extraction*) (*dead 'a*, *dead 'b*) *ffun* =
Constant 'b
| *Update 'a 'b* (*'a*, *'b*) *ffun*

primrec *eval-ffun* :: (*'a*, *'b*) *ffun* => *'a* => *'b*

where
  eval-ffun (*Constant c*) *x* = *c*
| *eval-ffun* (*Update x′ y f*) *x* = (*if x* = *x′ then y else eval-ffun f x*)

hide-type (**open**) *ffun*
hide-const (**open**) *Constant Update eval-ffun*

datatype (*plugins only*: *code quickcheck-narrowing extraction*) (*dead ′b*) *cfun* = *Constant ′b*

primrec *eval-cfun* :: *′b cfun* => *′a* => *′b*
where
  eval-cfun (*Constant c*) *y* = *c*

hide-type (**open**) *cfun*
hide-const (**open**) *Constant eval-cfun Abs-cfun Rep-cfun*

84.1.9 Setting up the counterexample generator

external-file ‹~~/*src*/*HOL*/*Tools*/*Quickcheck*/*Narrowing-Engine.hs*›
external-file ‹~~/*src*/*HOL*/*Tools*/*Quickcheck*/*PNF-Narrowing-Engine.hs*›
ML-file ‹*Tools*/*Quickcheck*/*narrowing-generators.ML*›

definition *narrowing-dummy-partial-term-of* :: (*′a* :: *partial-term-of*) *itself* => *narrowing-term* => *term*
where
  narrowing-dummy-partial-term-of = *partial-term-of*

definition *narrowing-dummy-narrowing* :: *integer* => (*′a* :: *narrowing*) *narrowing-cons*
where
  narrowing-dummy-narrowing = *narrowing*

lemma [*code*]:
  ensure-testable f =
    (*let*
      x = *narrowing-dummy-narrowing* :: *integer* => *bool narrowing-cons*;
      y = *narrowing-dummy-partial-term-of* :: *bool itself* => *narrowing-term* => *term*;
      z = (*conv* :: - => - => *unit*) *in f*)
unfolding *Let-def ensure-testable-def* **..**

84.2 Narrowing for sets

instantiation *set* :: (*narrowing*) *narrowing*
begin

definition *narrowing-set* = *Quickcheck-Narrowing.apply* (*Quickcheck-Narrowing.cons set*) *narrowing*

instance ..

end

84.3 Narrowing for integers

definition *drawn-from* :: *'a list* ⇒ *'a narrowing-cons*
where
drawn-from xs =
Narrowing-cons (*Narrowing-sum-of-products* (*map* (λ-. []) *xs*)) (*map* (λ*x* -. *x*) *xs*)

function *around-zero* :: *int* ⇒ *int list*
where
around-zero i = (*if i* < *0 then* [] *else* (*if i* = *0 then* [*0*] *else around-zero* (*i* − *1*) @ [*i*, −*i*]))
by *pat-completeness auto*
termination by (*relation measure nat*) *auto*

declare *around-zero.simps* [*simp del*]

lemma *length-around-zero*:
assumes *i* >= *0*
shows *length* (*around-zero i*) = *2* ∗ *nat i* + *1*
proof (*induct rule*: *int-ge-induct* [*OF assms*])
case *1*
from *1* **show** *?case* **by** (*simp add*: *around-zero.simps*)
next
case (*2 i*)
from *2* **show** *?case*
by (*simp add*: *around-zero.simps* [*of i* + *1*])
qed

instantiation *int* :: *narrowing*
begin

definition
narrowing-int d = (*let* (*u* :: - ⇒ - ⇒ *unit*) = *conv*; *i* = *int-of-integer d*
in drawn-from (*around-zero i*))

instance ..

end

declare [[*code drop*: *partial-term-of* :: *int itself* ⇒ -]]

lemma [*code*]:
partial-term-of (*ty* :: *int itself*) (*Narrowing-variable p t*) ≡
Code-Evaluation.Free (*STR* ''-'') (*Typerep.Typerep* (*STR* ''Int.int'') [])

partial-term-of (*ty* :: *int itself*) (*Narrowing-constructor i* []) ≡
  (*if i mod 2 = 0*
   then Code-Evaluation.term-of (− (*int-of-integer i*) *div 2*)
   else Code-Evaluation.term-of ((*int-of-integer i* + *1*) *div 2*))
by (*rule partial-term-of-anything*)+

instantiation *integer* :: *narrowing*
begin

definition
narrowing-integer d = (*let* (*u* :: - ⇒ - ⇒ *unit*) = *conv*; *i* = *int-of-integer d*
  in drawn-from (*map integer-of-int* (*around-zero i*)))

instance ..

end

declare [[*code drop*: *partial-term-of* :: *integer itself* ⇒ -]]

lemma [*code*]:
partial-term-of (*ty* :: *integer itself*) (*Narrowing-variable p t*) ≡
 Code-Evaluation.Free (*STR* ''-'') (*Typerep.Typerep* (*STR* ''*Code-Numeral.integer*'')
[])
partial-term-of (*ty* :: *integer itself*) (*Narrowing-constructor i* []) ≡
  (*if i mod 2 = 0*
   then Code-Evaluation.term-of (− *i div 2*)
   else Code-Evaluation.term-of ((*i* + *1*) *div 2*))
by (*rule partial-term-of-anything*)+

code-printing constant *Code-Evaluation.term-of* :: *integer* ⇒ *term* ⇀ (*Haskell-Quickcheck*)

```
(let { t = Typerep.Typerep Code′-Numeral.integer [];
   mkFunT s t = Typerep.Typerep fun [s, t];
   numT = Typerep.Typerep Num.num [];
   mkBit 0 = Generated′-Code.Const Num.num.Bit0 (mkFunT numT numT);
   mkBit 1 = Generated′-Code.Const Num.num.Bit1 (mkFunT numT numT);
   mkNumeral 1 = Generated′-Code.Const Num.num.One numT;
   mkNumeral i = let { q = i ‘Prelude.div‘ 2; r = i ‘Prelude.mod‘ 2 }
     in Generated′-Code.App (mkBit r) (mkNumeral q);
   mkNumber 0 = Generated′-Code.Const Groups.zero′-class.zero t;
   mkNumber 1 = Generated′-Code.Const Groups.one′-class.one t;
   mkNumber i = if i > 0 then
       Generated′-Code.App
         (Generated′-Code.Const Num.numeral′-class.numeral
           (mkFunT numT t))
         (mkNumeral i)
     else
       Generated′-Code.App
         (Generated′-Code.Const Groups.uminus′-class.uminus (mkFunT t t))
```

(*mkNumber* (− *i*)); } *in mkNumber*)

84.4 The *find-unused-assms* command

ML-file ‹*Tools/Quickcheck/find-unused-assms.ML*›

84.5 Closing up

hide-type *narrowing-type narrowing-term narrowing-cons property*
hide-const *map-cons nth error toEnum marker empty Narrowing-cons conv non-empty ensure-testable all exists drawn-from around-zero*
hide-const (**open**) *Narrowing-variable Narrowing-constructor apply sum cons*
hide-fact *empty-def cons-def conv.simps non-empty.simps apply-def sum-def ensure-testable-def all-def exists-def*

end

theory *Mirabelle*
imports *Sledgehammer Predicate-Compile Presburger*
begin

ML-file ‹*Tools/Mirabelle/mirabelle-util.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle.ML*›

ML ‹
signature MIRABELLE-ACTION = *sig*
val make-action : *Mirabelle.action-context* −> *string* ∗ *Mirabelle.action*
end
›

ML-file ‹*Tools/Mirabelle/mirabelle-arith.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-order.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-metis.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-presburger.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-quickcheck.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-sledgehammer-filter.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-sledgehammer.ML*›
ML-file ‹*Tools/Mirabelle/mirabelle-try0.ML*›

end

85 Program extraction for HOL

theory *Extraction*
imports *Option*
begin

85.1 Setup

setup ‹
  Extraction.add-types
    [(*bool*, ([], *NONE*))] #>
  Extraction.set-preprocessor (*fn thy* =>
    Proofterm.rewrite-proof-notypes
      ([], *Rewrite-HOL-Proof.elim-cong* :: *Proof-Rewrite-Rules.rprocs true*) *o*
    Proofterm.rewrite-proof thy
      (*Rewrite-HOL-Proof.rews*,
       Proof-Rewrite-Rules.rprocs true @ [*Proof-Rewrite-Rules.expand-of-class thy*])
o
    Proof-Rewrite-Rules.elim-vars (*curry Const* ***const-name*** ‹*default*›))
›

lemmas [*extraction-expand*] =
  meta-spec atomize-eq atomize-all atomize-imp atomize-conj
  allE rev-mp conjE Eq-TrueI Eq-FalseI eqTrueI eqTrueE eq-cong2
  notE′ impE′ impE iffE imp-cong simp-thms eq-True eq-False
  induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq
  induct-atomize induct-atomize′ induct-rulify induct-rulify′
  induct-rulify-fallback induct-trueI
  True-implies-equals implies-True-equals TrueE
  False-implies-equals implies-False-swap

lemmas [*extraction-expand-def*] =
  HOL.induct-forall-def HOL.induct-implies-def HOL.induct-equal-def HOL.induct-conj-def
  HOL.induct-true-def HOL.induct-false-def

datatype (*plugins only*: *code extraction*) *sumbool* = *Left* | *Right*

85.2 Type of extracted program

extract-type
  $typeof\ (Trueprop\ P) \equiv typeof\ P$

  $typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
    $typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE('Q))$

  $typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE(Null))$

  $typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
    $typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE('P \Rightarrow 'Q))$

  $(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
    $typeof\ (\forall x.\ P\ x) \equiv Type\ (TYPE(Null))$

  $(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE('P))) \Longrightarrow$
    $typeof\ (\forall x::'a.\ P\ x) \equiv Type\ (TYPE('a \Rightarrow 'P))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad typeof\ (\exists x::'a.\ P\ x) \equiv Type\ (TYPE('a))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE('P))) \Longrightarrow$
$\quad typeof\ (\exists x::'a.\ P\ x) \equiv Type\ (TYPE('a \times 'P))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad typeof\ (P \lor Q) \equiv Type\ (TYPE(sumbool))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \lor Q) \equiv Type\ (TYPE('Q\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad typeof\ (P \lor Q) \equiv Type\ (TYPE('P\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \lor Q) \equiv Type\ (TYPE('P + 'Q))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \land Q) \equiv Type\ (TYPE('Q))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad typeof\ (P \land Q) \equiv Type\ (TYPE('P))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \land Q) \equiv Type\ (TYPE('P \times 'Q))$

$typeof\ (P = Q) \equiv typeof\ ((P \longrightarrow Q) \land (Q \longrightarrow P))$

$typeof\ (x \in P) \equiv typeof\ P$

85.3 Realizability

realizability

$(realizes\ t\ (Trueprop\ P)) \equiv (Trueprop\ (realizes\ t\ P))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \longrightarrow Q)) \equiv (realizes\ Null\ P \longrightarrow realizes\ t\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE('P))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x::'P.\ realizes\ x\ P \longrightarrow realizes\ Null\ Q)$

$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x.\ realizes\ x\ P \longrightarrow realizes\ (t\ x)\ Q)$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ Null\ (P\ x))$

$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ (t\ x)\ (P\ x))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ Null\ (P\ t))$

$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ (snd\ t)\ (P\ (fst\ t)))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \lor Q)) \equiv$
$\quad (case\ t\ of\ Left \Rightarrow realizes\ Null\ P \mid Right \Rightarrow realizes\ Null\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \lor Q)) \equiv$
$\quad (case\ t\ of\ None \Rightarrow realizes\ Null\ P \mid Some\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \lor Q)) \equiv$
$\quad (case\ t\ of\ None \Rightarrow realizes\ Null\ Q \mid Some\ p \Rightarrow realizes\ p\ P)$

$(realizes\ t\ (P \lor Q)) \equiv$
$(case\ t\ of\ Inl\ p \Rightarrow realizes\ p\ P \mid Inr\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \land Q)) \equiv (realizes\ Null\ P \land realizes\ t\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \land Q)) \equiv (realizes\ t\ P \land realizes\ Null\ Q)$

$(realizes\ t\ (P \land Q)) \equiv (realizes\ (fst\ t)\ P \land realizes\ (snd\ t)\ Q)$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad realizes\ t\ (\neg\ P) \equiv \neg\ realizes\ Null\ P$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow$
$\quad realizes\ t\ (\neg\ P) \equiv (\forall x::'P.\ \neg\ realizes\ x\ P)$

$typeof\ (P::bool) \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad realizes\ t\ (P = Q) \equiv realizes\ Null\ P = realizes\ Null\ Q$

$(realizes\ t\ (P = Q)) \equiv (realizes\ t\ ((P \longrightarrow Q) \land (Q \longrightarrow P)))$

85.4 Computational content of basic inference rules

theorem *disjE-realizer*:
assumes *r*: $case\ x\ of\ Inl\ p \Rightarrow P\ p \mid Inr\ q \Rightarrow Q\ q$
and *r1*: $\bigwedge p.\ P\ p \Longrightarrow R\ (f\ p)$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
shows $R\ (case\ x\ of\ Inl\ p \Rightarrow f\ p \mid Inr\ q \Rightarrow g\ q)$
proof (*cases x*)

case *Inl*
with *r* **show** *?thesis* **by** *simp* (*rule r1*)
next
case *Inr*
with *r* **show** *?thesis* **by** *simp* (*rule r2*)
qed

theorem *disjE-realizer2*:
assumes *r*: *case x of None* ⇒ *P* | *Some q* ⇒ *Q q*
and *r1*: *P* ⟹ *R f* **and** *r2*: ⋀*q*. *Q q* ⟹ *R* (*g q*)
shows *R* (*case x of None* ⇒ *f* | *Some q* ⇒ *g q*)
proof (*cases x*)
case *None*
with *r* **show** *?thesis* **by** *simp* (*rule r1*)
next
case *Some*
with *r* **show** *?thesis* **by** *simp* (*rule r2*)
qed

theorem *disjE-realizer3*:
assumes *r*: *case x of Left* ⇒ *P* | *Right* ⇒ *Q*
and *r1*: *P* ⟹ *R f* **and** *r2*: *Q* ⟹ *R g*
shows *R* (*case x of Left* ⇒ *f* | *Right* ⇒ *g*)
proof (*cases x*)
case *Left*
with *r* **show** *?thesis* **by** *simp* (*rule r1*)
next
case *Right*
with *r* **show** *?thesis* **by** *simp* (*rule r2*)
qed

theorem *conjI-realizer*:
P p ⟹ *Q q* ⟹ *P* (*fst* (*p*, *q*)) ∧ *Q* (*snd* (*p*, *q*))
by *simp*

theorem *exI-realizer*:
P y x ⟹ *P* (*snd* (*x*, *y*)) (*fst* (*x*, *y*)) **by** *simp*

theorem *exE-realizer*: *P* (*snd p*) (*fst p*) ⟹
(⋀*x y*. *P y x* ⟹ *Q* (*f x y*)) ⟹ *Q* (*let* (*x*, *y*) = *p in f x y*)
by (*cases p*) (*simp add*: *Let-def*)

theorem *exE-realizer'*: *P* (*snd p*) (*fst p*) ⟹
(⋀*x y*. *P y x* ⟹ *Q*) ⟹ *Q* **by** (*cases p*) *simp*

realizers
impI (*P*, *Q*): *λpq*. *pq*
λ(*c*: -) (*d*: -) *P Q pq* (*h*: -). *allI* · - · *c* · (**λ***x*. *impI* · - · - · (*h* · *x*))

impI (*P*): *Null*
$\boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ (h\colon \text{-}).\ allI \cdot \text{-} \bullet c \bullet (\boldsymbol{\lambda}x.\ impI \cdot \text{-} \cdot \text{-} \bullet (h \cdot x))$

impI (*Q*): $\lambda q.\ q\ \boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ q.\ impI \cdot \text{-} \cdot \text{-}$

impI: *Null impI*

mp (*P*, *Q*): $\lambda pq.\ pq$
$\boldsymbol{\lambda}(c\colon \text{-})\ (d\colon \text{-})\ P\ Q\ pq\ (h\colon \text{-})\ p.\ mp \cdot \text{-} \cdot \text{-} \bullet (spec \cdot \text{-} \cdot p \bullet c \bullet h)$

mp (*P*): *Null*
$\boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ (h\colon \text{-})\ p.\ mp \cdot \text{-} \cdot \text{-} \bullet (spec \cdot \text{-} \cdot p \bullet c \bullet h)$

mp (*Q*): $\lambda q.\ q\ \boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ q.\ mp \cdot \text{-} \cdot \text{-}$

mp: *Null mp*

allI (*P*): $\lambda p.\ p\ \boldsymbol{\lambda}(c\colon \text{-})\ P\ (d\colon \text{-})\ p.\ allI \cdot \text{-} \bullet d$

allI: *Null allI*

spec (*P*): $\lambda x\ p.\ p\ x\ \boldsymbol{\lambda}(c\colon \text{-})\ P\ x\ (d\colon \text{-})\ p.\ spec \cdot \text{-} \cdot x \bullet d$

spec: *Null spec*

exI (*P*): $\lambda x\ p.\ (x,\ p)\ \boldsymbol{\lambda}(c\colon \text{-})\ P\ x\ (d\colon \text{-})\ p.\ exI\text{-}realizer \cdot P \cdot p \cdot x \bullet c \bullet d$

exI: $\lambda x.\ x\ \boldsymbol{\lambda}P\ x\ (c\colon \text{-})\ (h\colon \text{-}).\ h$

exE (*P*, *Q*): $\lambda p\ pq.\ let\ (x,\ y) = p\ in\ pq\ x\ y$
$\boldsymbol{\lambda}(c\colon \text{-})\ (d\colon \text{-})\ P\ Q\ (e\colon \text{-})\ p\ (h\colon \text{-})\ pq.\ exE\text{-}realizer \cdot P \cdot p \cdot Q \cdot pq \bullet c \bullet e \bullet d \bullet h$

exE (*P*): *Null*
$\boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ (d\colon \text{-})\ p.\ exE\text{-}realizer' \cdot \text{-} \cdot \text{-} \cdot \text{-} \bullet c \bullet d$

exE (*Q*): $\lambda x\ pq.\ pq\ x$
$\boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ (d\colon \text{-})\ x\ (h1\colon \text{-})\ pq\ (h2\colon \text{-}).\ h2 \cdot x \bullet h1$

exE: *Null*
$\boldsymbol{\lambda}P\ Q\ (c\colon \text{-})\ x\ (h1\colon \text{-})\ (h2\colon \text{-}).\ h2 \cdot x \bullet h1$

conjI (*P*, *Q*): *Pair*
$\boldsymbol{\lambda}(c\colon \text{-})\ (d\colon \text{-})\ P\ Q\ p\ (h\colon \text{-})\ q.\ conjI\text{-}realizer \cdot P \cdot p \cdot Q \cdot q \bullet c \bullet d \bullet h$

conjI (*P*): $\lambda p.\ p$
$\boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ p.\ conjI \cdot \text{-} \cdot \text{-}$

conjI (*Q*): $\lambda q.\ q$
$\boldsymbol{\lambda}(c\colon \text{-})\ P\ Q\ (h\colon \text{-})\ q.\ conjI \cdot \text{-} \cdot \text{-} \bullet h$

conjI: *Null conjI*

conjunct1 (*P*, *Q*): *fst*
λ(*c*: -) (*d*: -) *P Q pq*. *conjunct1* · - - -

conjunct1 (*P*): *λp*. *p*
λ(*c*: -) *P Q p*. *conjunct1* · - - -

conjunct1 (*Q*): *Null*
λ(*c*: -) *P Q q*. *conjunct1* · - - -

conjunct1: *Null conjunct1*

conjunct2 (*P*, *Q*): *snd*
λ(*c*: -) (*d*: -) *P Q pq*. *conjunct2* · - - -

conjunct2 (*P*): *Null*
λ(*c*: -) *P Q p*. *conjunct2* · - - -

conjunct2 (*Q*): *λp*. *p*
λ(*c*: -) *P Q p*. *conjunct2* · - - -

conjunct2: *Null conjunct2*

disjI1 (*P*, *Q*): *Inl*
λ(*c*: -) (*d*: -) *P Q p*. *iffD2* · - - - • (*sum.case-1* · *P* · - - *p* • *arity-type-bool* • *c* • *d*)

disjI1 (*P*): *Some*
λ(*c*: -) *P Q p*. *iffD2* · - - - • (*option.case-2* · - - *P* · *p* • *arity-type-bool* • *c*)

disjI1 (*Q*): *None*
λ(*c*: -) *P Q*. *iffD2* · - - - • (*option.case-1* · - - - • *arity-type-bool* • *c*)

disjI1: *Left*
λ*P Q*. *iffD2* · - - - • (*sumbool.case-1* · - - - • *arity-type-bool*)

disjI2 (*P*, *Q*): *Inr*
λ(*d*: -) (*c*: -) *Q P q*. *iffD2* · - - - • (*sum.case-2* · - - *Q* · *q* • *arity-type-bool* • *c* • *d*)

disjI2 (*P*): *None*
λ(*c*: -) *Q P*. *iffD2* · - - - • (*option.case-1* · - - - • *arity-type-bool* • *c*)

disjI2 (*Q*): *Some*
λ(*c*: -) *Q P q*. *iffD2* · - - - • (*option.case-2* · - - *Q* · *q* • *arity-type-bool* • *c*)

disjI2: *Right*

λ*Q P. iffD2* · - - - • (*sumbool.case-2* · - - - • *arity-type-bool*)

disjE (*P*, *Q*, *R*): λ*pq pr qr*.
  (*case pq of Inl p* ⇒ *pr p* | *Inr q* ⇒ *qr q*)
  λ(*c*: -) (*d*: -) (*e*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
    disjE-realizer · - - - - *pq* · *R* · *pr* · *qr* • *c* • *d* • *e* • *h1* • *h2*

disjE (*Q*, *R*): λ*pq pr qr*.
  (*case pq of None* ⇒ *pr* | *Some q* ⇒ *qr q*)
  λ(*c*: -) (*d*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
    disjE-realizer2 · - - - - *pq* · *R* · *pr* · *qr* • *c* • *d* • *h1* • *h2*

disjE (*P*, *R*): λ*pq pr qr*.
  (*case pq of None* ⇒ *qr* | *Some p* ⇒ *pr p*)
  λ(*c*: -) (*d*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr* (*h3*: -).
    disjE-realizer2 · - - - - *pq* · *R* · *qr* · *pr* • *c* • *d* • *h1* • *h3* • *h2*

disjE (*R*): λ*pq pr qr*.
  (*case pq of Left* ⇒ *pr* | *Right* ⇒ *qr*)
  λ(*c*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
    disjE-realizer3 · - - - - *pq* · *R* · *pr* · *qr* • *c* • *h1* • *h2*

disjE (*P*, *Q*): *Null*
  λ(*c*: -) (*d*: -) *P Q R pq*. *disjE-realizer* · - - - - *pq* · (λ*x*. *R*) · - - - • *c* • *d* • *arity-type-bool*

disjE (*Q*): *Null*
  λ(*c*: -) *P Q R pq*. *disjE-realizer2* · - - - - *pq* · (λ*x*. *R*) · - - - • *c* • *arity-type-bool*

disjE (*P*): *Null*
  λ(*c*: -) *P Q R pq* (*h1*: -) (*h2*: -) (*h3*: -).
    disjE-realizer2 · - - - - *pq* · (λ*x*. *R*) · - - - • *c* • *arity-type-bool* • *h1* • *h3* • *h2*

disjE: *Null*
  λ*P Q R pq*. *disjE-realizer3* · - - - - *pq* · (λ*x*. *R*) · - - - • *arity-type-bool*

FalseE (*P*): *default*
  λ(*c*: -) *P*. *FalseE* · -

FalseE: *Null FalseE*

notI (*P*): *Null*
  λ(*c*: -) *P* (*h*: -). *allI* · - • *c* • (**λ***x*. *notI* · - • (*h* · *x*))

notI: *Null notI*

notE (*P*, *R*): λ*p*. *default*
  λ(*c*: -) (*d*: -) *P R* (*h*: -) *p*. *notE* · - - - • (*spec* · - - *p* • *c* • *h*)

notE (*P*): *Null*
$\boldsymbol{\lambda}(c\text{: -})\ P\ R\ (h\text{: -})\ p.\ notE \cdot \cdot \cdot \cdot \bullet (spec \cdot \cdot \cdot p \bullet c \bullet h)$

notE (*R*): *default*
$\boldsymbol{\lambda}(c\text{: -})\ P\ R.\ notE \cdot \cdot \cdot \cdot$

notE: *Null notE*

subst (*P*): $\lambda s\ t\ ps.\ ps$
$\boldsymbol{\lambda}(c\text{: -})\ s\ t\ P\ (d\text{: -})\ (h\text{: -})\ ps.\ subst \cdot s \cdot t \cdot P\ ps \bullet d \bullet h$

subst: *Null subst*

iffD1 (*P*, *Q*): *fst*
$\boldsymbol{\lambda}(d\text{: -})\ (c\text{: -})\ Q\ P\ pq\ (h\text{: -})\ p.$
$mp \cdot \cdot \cdot \cdot \bullet (spec \cdot \cdot \cdot p \bullet d \bullet (conjunct1 \cdot \cdot \cdot \cdot \bullet h))$

iffD1 (*P*): $\lambda p.\ p$
$\boldsymbol{\lambda}(c\text{: -})\ Q\ P\ p\ (h\text{: -}).\ mp \cdot \cdot \cdot \cdot \bullet (conjunct1 \cdot \cdot \cdot \cdot \bullet h)$

iffD1 (*Q*): *Null*
$\boldsymbol{\lambda}(c\text{: -})\ Q\ P\ q1\ (h\text{: -})\ q2.$
$mp \cdot \cdot \cdot \cdot \bullet (spec \cdot \cdot \cdot q2 \bullet c \bullet (conjunct1 \cdot \cdot \cdot \cdot \bullet h))$

iffD1: *Null iffD1*

iffD2 (*P*, *Q*): *snd*
$\boldsymbol{\lambda}(c\text{: -})\ (d\text{: -})\ P\ Q\ pq\ (h\text{: -})\ q.$
$mp \cdot \cdot \cdot \cdot \bullet (spec \cdot \cdot \cdot q \bullet d \bullet (conjunct2 \cdot \cdot \cdot \cdot \bullet h))$

iffD2 (*P*): $\lambda p.\ p$
$\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ p\ (h\text{: -}).\ mp \cdot \cdot \cdot \cdot \bullet (conjunct2 \cdot \cdot \cdot \cdot \bullet h)$

iffD2 (*Q*): *Null*
$\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ q1\ (h\text{: -})\ q2.$
$mp \cdot \cdot \cdot \cdot \bullet (spec \cdot \cdot \cdot q2 \bullet c \bullet (conjunct2 \cdot \cdot \cdot \cdot \bullet h))$

iffD2: *Null iffD2*

iffI (*P*, *Q*): *Pair*
$\boldsymbol{\lambda}(c\text{: -})\ (d\text{: -})\ P\ Q\ pq\ (h1\ \text{: -})\ qp\ (h2\ \text{: -}).\ conjI\text{-}realizer\ \cdot$
$(\lambda pq.\ \forall x.\ P\ x \longrightarrow Q\ (pq\ x)) \cdot pq \cdot$
$(\lambda qp.\ \forall x.\ Q\ x \longrightarrow P\ (qp\ x)) \cdot qp \bullet$
$(arity\text{-}type\text{-}fun \bullet c \bullet d) \bullet$
$(arity\text{-}type\text{-}fun \bullet d \bullet c) \bullet$
$(allI \cdot \cdot \bullet c \bullet (\boldsymbol{\lambda} x.\ impI \cdot \cdot \cdot \cdot \bullet (h1 \cdot x))) \bullet$
$(allI \cdot \cdot \bullet d \bullet (\boldsymbol{\lambda} x.\ impI \cdot \cdot \cdot \cdot \bullet (h2 \cdot x)))$

iffI (*P*): $\lambda p.\ p$

λ(*c*: -) *P Q* (*h1* : -) *p* (*h2* : -). *conjI* · - · - •
  (*allI* · - • *c* • (*λx*. *impI* · - · - • (*h1* · *x*))) •
  (*impI* · - · - • *h2*)

iffI (*Q*): *λq*. *q*
 λ(*c*: -) *P Q q* (*h1* : -) (*h2* : -). *conjI* · - · - •
  (*impI* · - · - • *h1*) •
  (*allI* · - • *c* • (*λx*. *impI* · - · - • (*h2* · *x*)))

iffI: *Null iffI*

end

86 Extensible records with structural subtyping

theory *Record*
imports *Quickcheck-Exhaustive*
keywords
 record :: *thy-defn* **and**
 print-record :: *diag*
begin

86.1 Introduction

Records are isomorphic to compound tuple types. To implement efficient records, we make this isomorphism explicit. Consider the record access/update simplification *alpha* (*beta-update f rec*) = *alpha rec* for distinct fields alpha and beta of some record rec with n fields. There are *n* ^ *2* such theorems, which prohibits storage of all of them for large n. The rules can be proved on the fly by case decomposition and simplification in O(n) time. By creating O(n) isomorphic-tuple types while defining the record, however, we can prove the access/update simplification in *O*(*log*(*n*)^*2*) time.

The O(n) cost of case decomposition is not because O(n) steps are taken, but rather because the resulting rule must contain O(n) new variables and an O(n) size concrete record construction. To sidestep this cost, we would like to avoid case decomposition in proving access/update theorems.

Record types are defined as isomorphic to tuple types. For instance, a record type with fields *'a*, *'b*, *'c* and *'d* might be introduced as isomorphic to $'a \times ('b \times ('c \times 'd))$. If we balance the tuple tree to $('a \times 'b) \times ('c \times 'd)$ then accessors can be defined by converting to the underlying type then using O(log(n)) fst or snd operations. Updators can be defined similarly, if we introduce a *fst-update* and *snd-update* function. Furthermore, we can prove the access/update theorem in O(log(n)) steps by using simple rewrites on fst, snd, *fst-update* and *snd-update*.

The catch is that, although O(log(n)) steps were taken, the underlying type we converted to is a tuple tree of size O(n). Processing this term type wastes performance. We avoid this for large n by taking each subtree of size K and defining a new type isomorphic to that tuple subtree. A record can now be defined as isomorphic to a tuple tree of these O(n/K) new types, or, if $n > K*K$, we can repeat the process, until the record can be defined in terms of a tuple tree of complexity less than the constant K.

If we prove the access/update theorem on this type with the analogous steps to the tuple tree, we consume $O(log(n)\hat{\ }2)$ time as the intermediate terms are $O(log(n))$ in size and the types needed have size bounded by K. To enable this analogous traversal, we define the functions seen below: *iso-tuple-fst*, *iso-tuple-snd*, *iso-tuple-fst-update* and *iso-tuple-snd-update*. These functions generalise tuple operations by taking a parameter that encapsulates a tuple isomorphism. The rewrites needed on these functions now need an additional assumption which is that the isomorphism works.

These rewrites are typically used in a structured way. They are here presented as the introduction rule *isomorphic-tuple.intros* rather than as a rewrite rule set. The introduction form is an optimisation, as net matching can be performed at one term location for each step rather than the simplifier searching the term for possible pattern matches. The rule set is used as it is viewed outside the locale, with the locale assumption (that the isomorphism is valid) left as a rule assumption. All rules are structured to aid net matching, using either a point-free form or an encapsulating predicate.

86.2 Operators and lemmas for types isomorphic to tuples

datatype (*dead 'a*, *dead 'b*, *dead 'c*) *tuple-isomorphism* =
  Tuple-Isomorphism 'a ⇒ 'b × 'c 'b × 'c ⇒ 'a

primrec
  repr :: (*'a*, *'b*, *'c*) *tuple-isomorphism ⇒ 'a ⇒ 'b × 'c* **where**
  repr (*Tuple-Isomorphism r a*) = *r*

primrec
  abst :: (*'a*, *'b*, *'c*) *tuple-isomorphism ⇒ 'b × 'c ⇒ 'a* **where**
  abst (*Tuple-Isomorphism r a*) = *a*

definition
  iso-tuple-fst :: (*'a*, *'b*, *'c*) *tuple-isomorphism ⇒ 'a ⇒ 'b* **where**
  iso-tuple-fst isom = *fst ∘ repr isom*

definition
  iso-tuple-snd :: (*'a*, *'b*, *'c*) *tuple-isomorphism ⇒ 'a ⇒ 'c* **where**
  iso-tuple-snd isom = *snd ∘ repr isom*

definition
  iso-tuple-fst-update ::

$('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow ('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)$ **where**
iso-tuple-fst-update isom f = *abst isom* ∘ *apfst f* ∘ *repr isom*

definition
iso-tuple-snd-update ::
$('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow ('c \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'a)$ **where**
iso-tuple-snd-update isom f = *abst isom* ∘ *apsnd f* ∘ *repr isom*

definition
iso-tuple-cons ::
$('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'b \Rightarrow 'c \Rightarrow 'a$ **where**
iso-tuple-cons isom = *curry* (*abst isom*)

86.3 Logical infrastructure for records

definition
iso-tuple-surjective-proof-assist :: $'a \Rightarrow 'b \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool* **where**
iso-tuple-surjective-proof-assist x y f $\longleftrightarrow f\ x = y$

definition
iso-tuple-update-accessor-cong-assist ::
$(('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)) \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool* **where**
iso-tuple-update-accessor-cong-assist upd ac $\longleftrightarrow$
$(\forall f\ v.\ upd\ (\lambda x.\ f\ (ac\ v))\ v = upd\ f\ v) \wedge (\forall v.\ upd\ id\ v = v)$

definition
iso-tuple-update-accessor-eq-assist ::
$(('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow ('b \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b \Rightarrow$ *bool*
where
iso-tuple-update-accessor-eq-assist upd ac v f v' x $\longleftrightarrow$
$upd\ f\ v = v' \wedge ac\ v = x \wedge$ *iso-tuple-update-accessor-cong-assist upd ac*

lemma *update-accessor-congruence-foldE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd ac*
and r: $r = r'$ **and** v: $ac\ r' = v'$
and f: $\bigwedge v.\ v' = v \Longrightarrow f\ v = f'\ v$
shows $upd\ f\ r = upd\ f'\ r'$
using *uac r v* [*symmetric*]
apply (*subgoal-tac* $upd\ (\lambda x.\ f\ (ac\ r'))\ r' = upd\ (\lambda x.\ f'\ (ac\ r'))\ r'$)
apply (*simp add*: *iso-tuple-update-accessor-cong-assist-def*)
apply (*simp add*: *f*)
done

lemma *update-accessor-congruence-unfoldE*:
iso-tuple-update-accessor-cong-assist upd ac $\Longrightarrow$
$r = r' \Longrightarrow ac\ r' = v' \Longrightarrow (\bigwedge v.\ v = v' \Longrightarrow f\ v = f'\ v) \Longrightarrow$
$upd\ f\ r = upd\ f'\ r'$
apply (*erule*(*2*) *update-accessor-congruence-foldE*)
apply *simp*

done

lemma *iso-tuple-update-accessor-cong-assist-id*:
iso-tuple-update-accessor-cong-assist upd ac $\Longrightarrow$ *upd id* = *id*
by *rule* (*simp add*: *iso-tuple-update-accessor-cong-assist-def*)

lemma *update-accessor-noopE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd ac*
and *ac*: *f* (*ac x*) = *ac x*
shows *upd f x* = *x*
using *uac*
by (*simp add*: *ac iso-tuple-update-accessor-cong-assist-id* [*OF uac*, *unfolded id-def*]
cong: *update-accessor-congruence-unfoldE* [*OF uac*])

lemma *update-accessor-noop-compE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd ac*
and *ac*: *f* (*ac x*) = *ac x*
shows *upd* ($g \circ f$) *x* = *upd g x*
by (*simp add*: *ac cong*: *update-accessor-congruence-unfoldE*[*OF uac*])

lemma *update-accessor-cong-assist-idI*:
iso-tuple-update-accessor-cong-assist id id
by (*simp add*: *iso-tuple-update-accessor-cong-assist-def*)

lemma *update-accessor-cong-assist-triv*:
iso-tuple-update-accessor-cong-assist upd ac $\Longrightarrow$
iso-tuple-update-accessor-cong-assist upd ac
by *assumption*

lemma *update-accessor-accessor-eqE*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x $\Longrightarrow$ *ac v* = *x*
by (*simp add*: *iso-tuple-update-accessor-eq-assist-def*)

lemma *update-accessor-updator-eqE*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x $\Longrightarrow$ *upd f v* = *v'*
by (*simp add*: *iso-tuple-update-accessor-eq-assist-def*)

lemma *iso-tuple-update-accessor-eq-assist-idI*:
v' = *f v* $\Longrightarrow$ *iso-tuple-update-accessor-eq-assist id id v f v' v*
by (*simp add*: *iso-tuple-update-accessor-eq-assist-def update-accessor-cong-assist-idI*)

lemma *iso-tuple-update-accessor-eq-assist-triv*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x $\Longrightarrow$
iso-tuple-update-accessor-eq-assist upd ac v f v' x
by *assumption*

lemma *iso-tuple-update-accessor-cong-from-eq*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x $\Longrightarrow$
iso-tuple-update-accessor-cong-assist upd ac

by (*simp add*: *iso-tuple-update-accessor-eq-assist-def*)

lemma *iso-tuple-surjective-proof-assistI*:
f x = *y* ⟹ *iso-tuple-surjective-proof-assist x y f*
by (*simp add*: *iso-tuple-surjective-proof-assist-def*)

lemma *iso-tuple-surjective-proof-assist-idE*:
iso-tuple-surjective-proof-assist x y id ⟹ *x* = *y*
by (*simp add*: *iso-tuple-surjective-proof-assist-def*)

locale *isomorphic-tuple* =
fixes *isom* :: ('*a*, '*b*, '*c*) *tuple-isomorphism*
assumes *repr-inv*: ⋀*x*. *abst isom* (*repr isom x*) = *x*
and *abst-inv*: ⋀*y*. *repr isom* (*abst isom y*) = *y*
begin

lemma *repr-inj*: *repr isom x* = *repr isom y* ⟷ *x* = *y*
by (*auto dest*: *arg-cong* [*of repr isom x repr isom y abst isom*]
simp add: *repr-inv*)

lemma *abst-inj*: *abst isom x* = *abst isom y* ⟷ *x* = *y*
by (*auto dest*: *arg-cong* [*of abst isom x abst isom y repr isom*]
simp add: *abst-inv*)

lemmas *simps* = *Let-def repr-inv abst-inv repr-inj abst-inj*

lemma *iso-tuple-access-update-fst-fst*:
f ∘ *h g* = *j* ∘ *f* ⟹
(*f* ∘ *iso-tuple-fst isom*) ∘ (*iso-tuple-fst-update isom* ∘ *h*) *g* =
j ∘ (*f* ∘ *iso-tuple-fst isom*)
by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-fst-def simps*
fun-eq-iff)

lemma *iso-tuple-access-update-snd-snd*:
f ∘ *h g* = *j* ∘ *f* ⟹
(*f* ∘ *iso-tuple-snd isom*) ∘ (*iso-tuple-snd-update isom* ∘ *h*) *g* =
j ∘ (*f* ∘ *iso-tuple-snd isom*)
by (*clarsimp simp*: *iso-tuple-snd-update-def iso-tuple-snd-def simps*
fun-eq-iff)

lemma *iso-tuple-access-update-fst-snd*:
(*f* ∘ *iso-tuple-fst isom*) ∘ (*iso-tuple-snd-update isom* ∘ *h*) *g* =
id ∘ (*f* ∘ *iso-tuple-fst isom*)
by (*clarsimp simp*: *iso-tuple-snd-update-def iso-tuple-fst-def simps*
fun-eq-iff)

lemma *iso-tuple-access-update-snd-fst*:
(*f* ∘ *iso-tuple-snd isom*) ∘ (*iso-tuple-fst-update isom* ∘ *h*) *g* =
id ∘ (*f* ∘ *iso-tuple-snd isom*)

by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-snd-def simps fun-eq-iff*)

lemma *iso-tuple-update-swap-fst-fst*:
h f ∘ *j g* = *j g* ∘ *h f* ⟹
(*iso-tuple-fst-update isom* ∘ *h*) *f* ∘ (*iso-tuple-fst-update isom* ∘ *j*) *g* =
(*iso-tuple-fst-update isom* ∘ *j*) *g* ∘ (*iso-tuple-fst-update isom* ∘ *h*) *f*
by (*clarsimp simp*: *iso-tuple-fst-update-def simps apfst-compose fun-eq-iff*)

lemma *iso-tuple-update-swap-snd-snd*:
h f ∘ *j g* = *j g* ∘ *h f* ⟹
(*iso-tuple-snd-update isom* ∘ *h*) *f* ∘ (*iso-tuple-snd-update isom* ∘ *j*) *g* =
(*iso-tuple-snd-update isom* ∘ *j*) *g* ∘ (*iso-tuple-snd-update isom* ∘ *h*) *f*
by (*clarsimp simp*: *iso-tuple-snd-update-def simps apsnd-compose fun-eq-iff*)

lemma *iso-tuple-update-swap-fst-snd*:
(*iso-tuple-snd-update isom* ∘ *h*) *f* ∘ (*iso-tuple-fst-update isom* ∘ *j*) *g* =
(*iso-tuple-fst-update isom* ∘ *j*) *g* ∘ (*iso-tuple-snd-update isom* ∘ *h*) *f*
by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-snd-update-def simps fun-eq-iff*)

lemma *iso-tuple-update-swap-snd-fst*:
(*iso-tuple-fst-update isom* ∘ *h*) *f* ∘ (*iso-tuple-snd-update isom* ∘ *j*) *g* =
(*iso-tuple-snd-update isom* ∘ *j*) *g* ∘ (*iso-tuple-fst-update isom* ∘ *h*) *f*
by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-snd-update-def simps fun-eq-iff*)

lemma *iso-tuple-update-compose-fst-fst*:
h f ∘ *j g* = *k* (*f* ∘ *g*) ⟹
(*iso-tuple-fst-update isom* ∘ *h*) *f* ∘ (*iso-tuple-fst-update isom* ∘ *j*) *g* =
(*iso-tuple-fst-update isom* ∘ *k*) (*f* ∘ *g*)
by (*clarsimp simp*: *iso-tuple-fst-update-def simps apfst-compose fun-eq-iff*)

lemma *iso-tuple-update-compose-snd-snd*:
h f ∘ *j g* = *k* (*f* ∘ *g*) ⟹
(*iso-tuple-snd-update isom* ∘ *h*) *f* ∘ (*iso-tuple-snd-update isom* ∘ *j*) *g* =
(*iso-tuple-snd-update isom* ∘ *k*) (*f* ∘ *g*)
by (*clarsimp simp*: *iso-tuple-snd-update-def simps apsnd-compose fun-eq-iff*)

lemma *iso-tuple-surjective-proof-assist-step*:
iso-tuple-surjective-proof-assist v a (*iso-tuple-fst isom* ∘ *f*) ⟹
iso-tuple-surjective-proof-assist v b (*iso-tuple-snd isom* ∘ *f*) ⟹
iso-tuple-surjective-proof-assist v (*iso-tuple-cons isom a b*) *f*
by (*clarsimp simp*: *iso-tuple-surjective-proof-assist-def simps iso-tuple-fst-def iso-tuple-snd-def iso-tuple-cons-def*)

lemma *iso-tuple-fst-update-accessor-cong-assist*:
assumes *iso-tuple-update-accessor-cong-assist f g*
shows *iso-tuple-update-accessor-cong-assist*

(*iso-tuple-fst-update isom* ∘ *f*) (*g* ∘ *iso-tuple-fst isom*)
proof −
from *assms* **have** *f id* = *id*
by (*rule iso-tuple-update-accessor-cong-assist-id*)
with *assms* **show** *?thesis*
by (*clarsimp simp*: *iso-tuple-update-accessor-cong-assist-def simps iso-tuple-fst-update-def iso-tuple-fst-def*)
qed

lemma *iso-tuple-snd-update-accessor-cong-assist*:
assumes *iso-tuple-update-accessor-cong-assist f g*
shows *iso-tuple-update-accessor-cong-assist*
(*iso-tuple-snd-update isom* ∘ *f*) (*g* ∘ *iso-tuple-snd isom*)
proof −
from *assms* **have** *f id* = *id*
by (*rule iso-tuple-update-accessor-cong-assist-id*)
with *assms* **show** *?thesis*
by (*clarsimp simp*: *iso-tuple-update-accessor-cong-assist-def simps iso-tuple-snd-update-def iso-tuple-snd-def*)
qed

lemma *iso-tuple-fst-update-accessor-eq-assist*:
assumes *iso-tuple-update-accessor-eq-assist f g a u a′ v*
shows *iso-tuple-update-accessor-eq-assist*
(*iso-tuple-fst-update isom* ∘ *f*) (*g* ∘ *iso-tuple-fst isom*)
(*iso-tuple-cons isom a b*) *u* (*iso-tuple-cons isom a′ b*) *v*
proof −
from *assms* **have** *f id* = *id*
by (*auto simp add*: *iso-tuple-update-accessor-eq-assist-def intro*: *iso-tuple-update-accessor-cong-assist-id*)
with *assms* **show** *?thesis*
by (*clarsimp simp*: *iso-tuple-update-accessor-eq-assist-def iso-tuple-fst-update-def iso-tuple-fst-def iso-tuple-update-accessor-cong-assist-def iso-tuple-cons-def simps*)
qed

lemma *iso-tuple-snd-update-accessor-eq-assist*:
assumes *iso-tuple-update-accessor-eq-assist f g b u b′ v*
shows *iso-tuple-update-accessor-eq-assist*
(*iso-tuple-snd-update isom* ∘ *f*) (*g* ∘ *iso-tuple-snd isom*)
(*iso-tuple-cons isom a b*) *u* (*iso-tuple-cons isom a b′*) *v*
proof −
from *assms* **have** *f id* = *id*
by (*auto simp add*: *iso-tuple-update-accessor-eq-assist-def intro*: *iso-tuple-update-accessor-cong-assist-id*)
with *assms* **show** *?thesis*
by (*clarsimp simp*: *iso-tuple-update-accessor-eq-assist-def iso-tuple-snd-update-def iso-tuple-snd-def iso-tuple-update-accessor-cong-assist-def iso-tuple-cons-def simps*)

qed

lemma *iso-tuple-cons-conj-eqI*:
$a = c \wedge b = d \wedge P \longleftrightarrow Q \Longrightarrow$
iso-tuple-cons isom a b = *iso-tuple-cons isom c d* $\wedge$ *P* $\longleftrightarrow$ *Q*
by (*clarsimp simp*: *iso-tuple-cons-def simps*)

lemmas *intros* =
iso-tuple-access-update-fst-fst
iso-tuple-access-update-snd-snd
iso-tuple-access-update-fst-snd
iso-tuple-access-update-snd-fst
iso-tuple-update-swap-fst-fst
iso-tuple-update-swap-snd-snd
iso-tuple-update-swap-fst-snd
iso-tuple-update-swap-snd-fst
iso-tuple-update-compose-fst-fst
iso-tuple-update-compose-snd-snd
iso-tuple-surjective-proof-assist-step
iso-tuple-fst-update-accessor-eq-assist
iso-tuple-snd-update-accessor-eq-assist
iso-tuple-fst-update-accessor-cong-assist
iso-tuple-snd-update-accessor-cong-assist
iso-tuple-cons-conj-eqI

end

lemma *isomorphic-tuple-intro*:
fixes *repr abst*
assumes *repr-inj*: $\bigwedge x\ y.$ *repr x* = *repr y* $\longleftrightarrow x = y$
and *abst-inv*: $\bigwedge z.$ *repr* (*abst z*) = *z*
and *v*: *v* $\equiv$ *Tuple-Isomorphism repr abst*
shows *isomorphic-tuple v*
proof
fix *x* **have** *repr* (*abst* (*repr x*)) = *repr x*
by (*simp add*: *abst-inv*)
then show *Record.abst v* (*Record.repr v x*) = *x*
by (*simp add*: *v repr-inj*)
next
fix *y*
show *Record.repr v* (*Record.abst v y*) = *y*
by (*simp add*: *v*) (*fact abst-inv*)
qed

definition
tuple-iso-tuple $\equiv$ *Tuple-Isomorphism id id*

lemma *tuple-iso-tuple*:
isomorphic-tuple tuple-iso-tuple

by (*simp add*: *isomorphic-tuple-intro* [*OF - - reflexive*] *tuple-iso-tuple-def*)

lemma *refl-conj-eq*: $Q = R \Longrightarrow P \wedge Q \longleftrightarrow P \wedge R$
by *simp*

lemma *iso-tuple-UNIV-I*: $x \in UNIV \equiv True$
by *simp*

lemma *iso-tuple-True-simp*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
by *simp*

lemma *prop-subst*: $s = t \Longrightarrow PROP\ P\ t \Longrightarrow PROP\ P\ s$
by *simp*

lemma *K-record-comp*: $(\lambda x.\ c) \circ f = (\lambda x.\ c)$
by (*simp add*: *comp-def*)

86.4 Concrete record syntax

nonterminal
ident **and**
field-type **and**
field-types **and**
field **and**
fields **and**
field-update **and**
field-updates

open-bundle *record-syntax*
begin

syntax
-constify :: *id* => *ident* (‹-›)
-constify :: *longid* => *ident* (‹-›)

-field-type :: *ident* => *type* => *field-type* (‹(‹indent=2 notation=‹infix field type››- ::/ -)›)
:: *field-type* => *field-types* (‹-›)
-field-types :: *field-type* => *field-types* => *field-types* (‹-,/ -›)
-record-type :: *field-types* => *type* (‹(‹indent=3 notation=‹mixfix record type››⦇-⦈)›)
-record-type-scheme :: *field-types* => *type* => *type* (‹(‹indent=3 notation=‹mixfix record type››⦇-,/ (‹indent=2 notation=‹infix more type››. . . ::/ -)⦈)›)

-field :: *ident* => *'a* => *field* (‹(‹indent=2 notation=‹infix field value››(‹open-block markup=‹const››-) =/ -)›)
:: *field* => *fields* (‹-›)
-fields :: *field* => *fields* => *fields* (‹-,/ -›)
-record :: *fields* => *'a* (‹(‹indent=3 notation=‹mixfix

record value›› (|-|) ›)
-record-scheme :: *fields* => *'a* => *'a* (‹(‹*indent=3 notation=*‹*mixfix record value*›› (|-,/ (‹*indent=2 notation=*‹*infix more value*››... =/ -)|) ›)

-field-update :: *ident* => *'a* => *field-update* (‹(‹*indent=2 notation=*‹*infix field update*››(‹*open-block markup=*‹*const*››-) :=/ -)›)
:: *field-update* => *field-updates* (‹-›)
-field-updates :: *field-update* => *field-updates* => *field-updates* (‹-,/ -›)
-record-update :: *'a* => *field-updates* => *'b* (‹(‹*open-block notation=*‹*mixfix record update*››-/(*3*(|-|)))› [*900*, *0*] *900*)

syntax (*ASCII*)
-record-type :: *field-types* => *type* (‹(‹*indent=3 notation=*‹*mixfix record type*››'(| - |'))›)
-record-type-scheme :: *field-types* => *type* => *type* (‹(‹*indent=3 notation=*‹*mixfix record type*››'(| -,/ (‹*indent=2 notation=*‹*infix more type*››... ::/ -) |'))›)
-record :: *fields* => *'a* (‹(‹*indent=3 notation=*‹*mixfix record value*››'(| - |'))›)
-record-scheme :: *fields* => *'a* => *'a* (‹(‹*indent=3 notation=*‹*mixfix record value*››'(| -,/ (‹*indent=2 notation=*‹*infix more value*››... =/ -) |'))›)
-record-update :: *'a* => *field-updates* => *'b* (‹(‹*open-block notation=*‹*mixfix record update*››-/(*3*'(| - |')))› [*900*, *0*] *900*)

end

86.5 Record package

ML-file ‹*Tools/record.ML*›

hide-const (**open**) *Tuple-Isomorphism repr abst iso-tuple-fst iso-tuple-snd iso-tuple-fst-update iso-tuple-snd-update iso-tuple-cons iso-tuple-surjective-proof-assist iso-tuple-update-accessor-cong-assist iso-tuple-update-accessor-eq-assist tuple-iso-tuple*

end

87 Greatest common divisor and least common multiple

theory *GCD*
imports *Groups-List Code-Numeral*
begin

87.1 Abstract bounded quasi semilattices as common foundation

locale *bounded-quasi-semilattice* = *abel-semigroup* +

fixes *top* :: *'a* (‹⊤›) **and** *bot* :: *'a* (‹⊥›)
and *normalize* :: *'a* ⇒ *'a*
assumes *idem-normalize* [*simp*]: *a* ∗ *a* = *normalize a*
and *normalize-left-idem* [*simp*]: *normalize a* ∗ *b* = *a* ∗ *b*
and *normalize-idem* [*simp*]: *normalize* (*a* ∗ *b*) = *a* ∗ *b*
and *normalize-top* [*simp*]: *normalize* ⊤ = ⊤
and *normalize-bottom* [*simp*]: *normalize* ⊥ = ⊥
and *top-left-normalize* [*simp*]: ⊤ ∗ *a* = *normalize a*
and *bottom-left-bottom* [*simp*]: ⊥ ∗ *a* = ⊥
begin

lemma *left-idem* [*simp*]:
a ∗ (*a* ∗ *b*) = *a* ∗ *b*
using *assoc* [*of a a b*, *symmetric*] **by** *simp*

lemma *right-idem* [*simp*]:
(*a* ∗ *b*) ∗ *b* = *a* ∗ *b*
using *left-idem* [*of b a*] **by** (*simp add*: *ac-simps*)

lemma *comp-fun-idem*: *comp-fun-idem f*
by *standard* (*simp-all add*: *fun-eq-iff ac-simps*)

interpretation *comp-fun-idem f*
by (*fact comp-fun-idem*)

lemma *top-right-normalize* [*simp*]:
a ∗ ⊤ = *normalize a*
using *top-left-normalize* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *bottom-right-bottom* [*simp*]:
a ∗ ⊥ = ⊥
using *bottom-left-bottom* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *normalize-right-idem* [*simp*]:
a ∗ *normalize b* = *a* ∗ *b*
using *normalize-left-idem* [*of b a*] **by** (*simp add*: *ac-simps*)

end

locale *bounded-quasi-semilattice-set* = *bounded-quasi-semilattice*
begin

interpretation *comp-fun-idem f*
by (*fact comp-fun-idem*)

definition *F* :: *'a set* ⇒ *'a*
where
eq-fold: *F A* = (*if finite A then Finite-Set.fold f* ⊤ *A else* ⊥)

lemma *infinite* [*simp*]:
infinite $A \Longrightarrow F\ A = \bot$
by (*simp add*: *eq-fold*)

lemma *set-eq-fold* [*code*]:
F (*set xs*) = *fold f xs* $\top$
by (*simp add*: *eq-fold fold-set-fold*)

lemma *empty* [*simp*]:
$F\ \{\} = \top$
by (*simp add*: *eq-fold*)

lemma *insert* [*simp*]:
F (*insert a A*) $= a * F\ A$
by (*cases finite A*) (*simp-all add*: *eq-fold*)

lemma *normalize* [*simp*]:
normalize $(F\ A) = F\ A$
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *in-idem*:
assumes $a \in A$
shows $a * F\ A = F\ A$
using *assms* **by** (*induct A rule*: *infinite-finite-induct*)
(*auto simp*: *left-commute* [*of a*])

lemma *union*:
$F\ (A \cup B) = F\ A * F\ B$
by (*induct A rule*: *infinite-finite-induct*)
(*simp-all add*: *ac-simps*)

lemma *remove*:
assumes $a \in A$
shows $F\ A = a * F\ (A - \{a\})$
proof −
from *assms* **obtain** B **where** $A =$ *insert a B* **and** $a \notin B$
by (*blast dest*: *mk-disjoint-insert*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *insert-remove*:
F (*insert a A*) $= a * F\ (A - \{a\})$
by (*cases* $a \in A$) (*simp-all add*: *insert-absorb remove*)

lemma *subset*:
assumes $B \subseteq A$
shows $F\ B * F\ A = F\ A$
using *assms* **by** (*simp add*: *union* [*symmetric*] *Un-absorb1*)

end

87.2 Abstract GCD and LCM

class *gcd* = *zero* + *one* + *dvd* +
  fixes *gcd* :: *'a* ⇒ *'a* ⇒ *'a*
    and *lcm* :: *'a* ⇒ *'a* ⇒ *'a*

class *Gcd* = *gcd* +
  fixes *Gcd* :: *'a set* ⇒ *'a*
    and *Lcm* :: *'a set* ⇒ *'a*

syntax
  -GCD1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder GCD››GCD -./ -)› [*0*, *10*] *10*)
  -GCD :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder GCD››GCD -∈-./ -)› [*0*, *0*, *10*] *10*)
  -LCM1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder LCM››LCM -./ -)› [*0*, *10*] *10*)
  -LCM :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder LCM››LCM -∈-./ -)› [*0*, *0*, *10*] *10*)

syntax-consts
  -GCD1 -GCD ⇌ *Gcd* **and**
  -LCM1 -LCM ⇌ *Lcm*

translations
  GCD x y. f ⇌ *GCD x. GCD y. f*
  GCD x. f ⇌ *CONST Gcd* (*CONST range* (λ*x. f*))
  GCD x∈*A. f* ⇌ *CONST Gcd* ((λ*x. f*) ‘ *A*)
  LCM x y. f ⇌ *LCM x. LCM y. f*
  LCM x. f ⇌ *CONST Lcm* (*CONST range* (λ*x. f*))
  LCM x∈*A. f* ⇌ *CONST Lcm* ((λ*x. f*) ‘ *A*)

class *semiring-gcd* = *normalization-semidom* + *gcd* +
  assumes *gcd-dvd1* [*iff*]: *gcd a b dvd a*
    and *gcd-dvd2* [*iff*]: *gcd a b dvd b*
    and *gcd-greatest*: *c dvd a* ⟹ *c dvd b* ⟹ *c dvd gcd a b*
    and *normalize-gcd* [*simp*]: *normalize* (*gcd a b*) = *gcd a b*
    and *lcm-gcd*: *lcm a b* = *normalize* (*a* ∗ *b div gcd a b*)
begin

lemma *gcd-greatest-iff* [*simp*]: *a dvd gcd b c* ⟷ *a dvd b* ∧ *a dvd c*
  by (*blast intro*!: *gcd-greatest intro*: *dvd-trans*)

lemma *gcd-dvdI1*: *a dvd c* ⟹ *gcd a b dvd c*
  by (*rule dvd-trans*) (*rule gcd-dvd1*)

lemma *gcd-dvdI2*: *b dvd c* ⟹ *gcd a b dvd c*

by (*rule dvd-trans*) (*rule gcd-dvd2*)

lemma *dvd-gcdD1*: *a dvd gcd b c* $\Longrightarrow$ *a dvd b*
using *gcd-dvd1* [*of b c*] **by** (*blast intro*: *dvd-trans*)

lemma *dvd-gcdD2*: *a dvd gcd b c* $\Longrightarrow$ *a dvd c*
using *gcd-dvd2* [*of b c*] **by** (*blast intro*: *dvd-trans*)

lemma *gcd-0-left* [*simp*]: *gcd 0 a = normalize a*
by (*rule associated-eqI*) *simp-all*

lemma *gcd-0-right* [*simp*]: *gcd a 0 = normalize a*
by (*rule associated-eqI*) *simp-all*

lemma *gcd-eq-0-iff* [*simp*]: *gcd a b = 0* $\longleftrightarrow$ *a = 0* $\wedge$ *b = 0*
(**is** *?P* $\longleftrightarrow$ *?Q*)
proof
assume *?P*
then have *0 dvd gcd a b*
by *simp*
then have *0 dvd a* **and** *0 dvd b*
by (*blast intro*: *dvd-trans*)+
then show *?Q*
by *simp*
next
assume *?Q*
then show *?P*
by *simp*
qed

lemma *unit-factor-gcd*: *unit-factor* (*gcd a b*) = (*if a = 0* $\wedge$ *b = 0 then 0 else 1*)
proof (*cases gcd a b = 0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have *unit-factor* (*gcd a b*) $*$ *normalize* (*gcd a b*) = *gcd a b*
by (*rule unit-factor-mult-normalize*)
then have *unit-factor* (*gcd a b*) $*$ *gcd a b = gcd a b*
by *simp*
then have *unit-factor* (*gcd a b*) $*$ *gcd a b div gcd a b = gcd a b div gcd a b*
by *simp*
with *False* **show** *?thesis*
by *simp*
qed

lemma *is-unit-gcd-iff* [*simp*]:
is-unit (*gcd a b*) $\longleftrightarrow$ *gcd a b = 1*
by (*cases a = 0* $\wedge$ *b = 0*) (*auto simp*: *unit-factor-gcd dest*: *is-unit-unit-factor*)

sublocale *gcd*: *abel-semigroup gcd*
proof
  fix *a b c*
  show *gcd a b* = *gcd b a*
    by (*rule associated-eqI*) *simp-all*
  from *gcd-dvd1* **have** *gcd* (*gcd a b*) *c dvd a*
    by (*rule dvd-trans*) *simp*
  moreover from *gcd-dvd1* **have** *gcd* (*gcd a b*) *c dvd b*
    by (*rule dvd-trans*) *simp*
  ultimately have *P1*: *gcd* (*gcd a b*) *c dvd gcd a* (*gcd b c*)
    by (*auto intro*!: *gcd-greatest*)
  from *gcd-dvd2* **have** *gcd a* (*gcd b c*) *dvd b*
    by (*rule dvd-trans*) *simp*
  moreover from *gcd-dvd2* **have** *gcd a* (*gcd b c*) *dvd c*
    by (*rule dvd-trans*) *simp*
  ultimately have *P2*: *gcd a* (*gcd b c*) *dvd gcd* (*gcd a b*) *c*
    by (*auto intro*!: *gcd-greatest*)
  from *P1 P2* **show** *gcd* (*gcd a b*) *c* = *gcd a* (*gcd b c*)
    by (*rule associated-eqI*) *simp-all*
qed

sublocale *gcd*: *bounded-quasi-semilattice gcd 0 1 normalize*
proof
  show *gcd a a* = *normalize a* **for** *a*
  proof −
    have *a dvd gcd a a*
      by (*rule gcd-greatest*) *simp-all*
    then show *?thesis*
      by (*auto intro*: *associated-eqI*)
  qed
  show *gcd* (*normalize a*) *b* = *gcd a b* **for** *a b*
    using *gcd-dvd1* [*of normalize a b*]
    by (*auto intro*: *associated-eqI*)
  show *gcd 1 a* = *1* **for** *a*
    by (*rule associated-eqI*) *simp-all*
qed *simp-all*

lemma *gcd-self*: *gcd a a* = *normalize a*
  by (*fact gcd.idem-normalize*)

lemma *gcd-left-idem*: *gcd a* (*gcd a b*) = *gcd a b*
  by (*fact gcd.left-idem*)

lemma *gcd-right-idem*: *gcd* (*gcd a b*) *b* = *gcd a b*
  by (*fact gcd.right-idem*)

lemma *gcdI*:
  assumes *c dvd a* **and** *c dvd b*

and *greatest*: $\bigwedge d.\ d\ dvd\ a \Longrightarrow d\ dvd\ b \Longrightarrow d\ dvd\ c$
and *normalize* $c = c$
shows $c = gcd\ a\ b$
by (*rule associated-eqI*) (*auto simp*: *assms intro*: *gcd-greatest*)

lemma *gcd-unique*:
$d\ dvd\ a \wedge d\ dvd\ b \wedge normalize\ d = d \wedge (\forall e.\ e\ dvd\ a \wedge e\ dvd\ b \longrightarrow e\ dvd\ d) \longleftrightarrow d = gcd\ a\ b$
by *rule* (*auto intro*: *gcdI simp*: *gcd-greatest*)

lemma *gcd-dvd-prod*: $gcd\ a\ b\ dvd\ k * b$
using *mult-dvd-mono* [*of 1*] **by** *auto*

lemma *gcd-proj2-if-dvd*: $b\ dvd\ a \Longrightarrow gcd\ a\ b = normalize\ b$
by (*rule gcdI* [*symmetric*]) *simp-all*

lemma *gcd-proj1-if-dvd*: $a\ dvd\ b \Longrightarrow gcd\ a\ b = normalize\ a$
by (*rule gcdI* [*symmetric*]) *simp-all*

lemma *gcd-proj1-iff*: $gcd\ m\ n = normalize\ m \longleftrightarrow m\ dvd\ n$
proof
assume $*$: $gcd\ m\ n = normalize\ m$
show $m\ dvd\ n$
proof (*cases* $m = 0$)
case *True*
with $*$ **show** *?thesis* **by** *simp*
next
case [*simp*]: *False*
from $*$ **have** $**$: $m = gcd\ m\ n * unit\text{-}factor\ m$
by (*simp add*: *unit-eq-div2*)
show *?thesis*
by (*subst* $**$) (*simp add*: *mult-unit-dvd-iff*)
qed
next
assume $m\ dvd\ n$
then show $gcd\ m\ n = normalize\ m$
by (*rule gcd-proj1-if-dvd*)
qed

lemma *gcd-proj2-iff*: $gcd\ m\ n = normalize\ n \longleftrightarrow n\ dvd\ m$
using *gcd-proj1-iff* [*of n m*] **by** (*simp add*: *ac-simps*)

lemma *gcd-mult-left*: $gcd\ (c * a)\ (c * b) = normalize\ (c * gcd\ a\ b)$
proof (*cases* $c = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have $*$: $c * gcd\ a\ b\ dvd\ gcd\ (c * a)\ (c * b)$

by (*auto intro*: *gcd-greatest*)
moreover from *False* ∗ **have** *gcd* (*c* ∗ *a*) (*c* ∗ *b*) *dvd c* ∗ *gcd a b*
by (*metis div-dvd-iff-mult dvd-mult-left gcd-dvd1 gcd-dvd2 gcd-greatest mult-commute*)
ultimately have *normalize* (*gcd* (*c* ∗ *a*) (*c* ∗ *b*)) = *normalize* (*c* ∗ *gcd a b*)
by (*auto intro*: *associated-eqI*)
then show *?thesis*
by (*simp add*: *normalize-mult*)
qed

lemma *gcd-mult-right*: *gcd* (*a* ∗ *c*) (*b* ∗ *c*) = *normalize* (*gcd b a* ∗ *c*)
using *gcd-mult-left* [*of c a b*] **by** (*simp add*: *ac-simps*)

lemma *dvd-lcm1* [*iff*]: *a dvd lcm a b*
by (*metis div-mult-swap dvd-mult2 dvd-normalize-iff dvd-refl gcd-dvd2 lcm-gcd*)

lemma *dvd-lcm2* [*iff*]: *b dvd lcm a b*
by (*metis dvd-div-mult dvd-mult dvd-normalize-iff dvd-refl gcd-dvd1 lcm-gcd*)

lemma *dvd-lcmI1*: *a dvd b* ⟹ *a dvd lcm b c*
by (*rule dvd-trans*) (*assumption*, *blast*)

lemma *dvd-lcmI2*: *a dvd c* ⟹ *a dvd lcm b c*
by (*rule dvd-trans*) (*assumption*, *blast*)

lemma *lcm-dvdD1*: *lcm a b dvd c* ⟹ *a dvd c*
using *dvd-lcm1* [*of a b*] **by** (*blast intro*: *dvd-trans*)

lemma *lcm-dvdD2*: *lcm a b dvd c* ⟹ *b dvd c*
using *dvd-lcm2* [*of a b*] **by** (*blast intro*: *dvd-trans*)

lemma *lcm-least*:
assumes *a dvd c* **and** *b dvd c*
shows *lcm a b dvd c*
proof (*cases c* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have ∗: *is-unit* (*unit-factor c*)
by *simp*
show *?thesis*
proof (*cases gcd a b* = *0*)
case *True*
with *assms* **show** *?thesis* **by** *simp*
next
case *False*
have *a* ∗ *b dvd normalize* (*c* ∗ *gcd a b*)
using *assms* **by** (*subst gcd-mult-left* [*symmetric*]) (*auto intro*!: *gcd-greatest simp*: *mult-ac*)

```
    with False have (a * b div gcd a b) dvd c
      by (subst div-dvd-iff-mult) auto
    thus ?thesis by (simp add: lcm-gcd)
  qed
qed

lemma lcm-least-iff [simp]: lcm a b dvd c ⟷ a dvd c ∧ b dvd c
  by (blast intro!: lcm-least intro: dvd-trans)

lemma normalize-lcm [simp]: normalize (lcm a b) = lcm a b
  by (simp add: lcm-gcd dvd-normalize-div)

lemma lcm-0-left [simp]: lcm 0 a = 0
  by (simp add: lcm-gcd)

lemma lcm-0-right [simp]: lcm a 0 = 0
  by (simp add: lcm-gcd)

lemma lcm-eq-0-iff: lcm a b = 0 ⟷ a = 0 ∨ b = 0
  (is ?P ⟷ ?Q)
proof
  assume ?P
  then have 0 dvd lcm a b
    by simp
  also have lcm a b dvd (a * b)
    by simp
  finally show ?Q
    by auto
next
  assume ?Q
  then show ?P
    by auto
qed

lemma zero-eq-lcm-iff: 0 = lcm a b ⟷ a = 0 ∨ b = 0
  using lcm-eq-0-iff[of a b] by auto

lemma lcm-eq-1-iff [simp]: lcm a b = 1 ⟷ is-unit a ∧ is-unit b
  by (auto intro: associated-eqI)

lemma unit-factor-lcm: unit-factor (lcm a b) = (if a = 0 ∨ b = 0 then 0 else 1)
  using lcm-eq-0-iff[of a b] by (cases lcm a b = 0) (auto simp: lcm-gcd)

sublocale lcm: abel-semigroup lcm
proof
  fix a b c
  show lcm a b = lcm b a
    by (simp add: lcm-gcd ac-simps normalize-mult dvd-normalize-div)
  have lcm (lcm a b) c dvd lcm a (lcm b c)
```

and *lcm a* (*lcm b c*) *dvd lcm* (*lcm a b*) *c*
by (*auto intro*: *lcm-least*
dvd-trans [*of b lcm b c lcm a* (*lcm b c*)]
dvd-trans [*of c lcm b c lcm a* (*lcm b c*)]
dvd-trans [*of a lcm a b lcm* (*lcm a b*) *c*]
dvd-trans [*of b lcm a b lcm* (*lcm a b*) *c*])
then show *lcm* (*lcm a b*) *c* = *lcm a* (*lcm b c*)
by (*rule associated-eqI*) *simp-all*
qed

sublocale *lcm*: *bounded-quasi-semilattice lcm 1 0 normalize*
proof
show *lcm a a* = *normalize a* **for** *a*
proof −
have *lcm a a dvd a*
by (*rule lcm-least*) *simp-all*
then show *?thesis*
by (*auto intro*: *associated-eqI*)
qed
show *lcm* (*normalize a*) *b* = *lcm a b* **for** *a b*
using *dvd-lcm1* [*of normalize a b*] **unfolding** *normalize-dvd-iff*
by (*auto intro*: *associated-eqI*)
show *lcm 1 a* = *normalize a* **for** *a*
by (*rule associated-eqI*) *simp-all*
qed *simp-all*

lemma *lcm-self*: *lcm a a* = *normalize a*
by (*fact lcm.idem-normalize*)

lemma *lcm-left-idem*: *lcm a* (*lcm a b*) = *lcm a b*
by (*fact lcm.left-idem*)

lemma *lcm-right-idem*: *lcm* (*lcm a b*) *b* = *lcm a b*
by (*fact lcm.right-idem*)

lemma *gcd-lcm*:
assumes *a* ≠ *0* **and** *b* ≠ *0*
shows *gcd a b* = *normalize* (*a* ∗ *b div lcm a b*)
proof −
from *assms* **have** [*simp*]: *a* ∗ *b div gcd a b* ≠ *0*
by (*subst dvd-div-eq-0-iff*) *auto*
let *?u* = *unit-factor* (*a* ∗ *b div gcd a b*)
have *gcd a b* ∗ *normalize* (*a* ∗ *b div gcd a b*) =
gcd a b ∗ ((*a* ∗ *b div gcd a b*) ∗ (*1 div ?u*))
by *simp*
also have . . . = *a* ∗ *b div ?u*
by (*subst mult.assoc* [*symmetric*]) *auto*
also have . . . *dvd a* ∗ *b*
by (*subst div-unit-dvd-iff*) *auto*

finally have *gcd a b dvd* ((*a* ∗ *b*) *div lcm a b*)
by (*intro dvd-mult-imp-div*) (*auto simp*: *lcm-gcd*)
moreover have *a* ∗ *b div lcm a b dvd a* **and** *a* ∗ *b div lcm a b dvd b*
using *assms* **by** (*subst div-dvd-iff-mult*; *simp add*: *lcm-eq-0-iff mult.commute*[*of b lcm a b*])+
ultimately have *normalize* (*gcd a b*) = *normalize* (*a* ∗ *b div lcm a b*)
apply −
apply (*rule associated-eqI*)
using *assms*
apply (*auto simp*: *div-dvd-iff-mult zero-eq-lcm-iff*)
done
thus *?thesis* **by** *simp*
qed

lemma *lcm-1-left*: *lcm 1 a* = *normalize a*
by (*fact lcm.top-left-normalize*)

lemma *lcm-1-right*: *lcm a 1* = *normalize a*
by (*fact lcm.top-right-normalize*)

lemma *lcm-mult-left*: *lcm* (*c* ∗ *a*) (*c* ∗ *b*) = *normalize* (*c* ∗ *lcm a b*)
proof (*cases c* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have ∗: *lcm* (*c* ∗ *a*) (*c* ∗ *b*) *dvd c* ∗ *lcm a b*
by (*auto intro*: *lcm-least*)
moreover have *lcm a b dvd lcm* (*c* ∗ *a*) (*c* ∗ *b*) *div c*
by (*intro lcm-least*) (*auto intro*!: *dvd-mult-imp-div simp*: *mult-ac*)
hence *c* ∗ *lcm a b dvd lcm* (*c* ∗ *a*) (*c* ∗ *b*)
using *False* **by** (*subst* (*asm*) *dvd-div-iff-mult*) (*auto simp*: *mult-ac intro*: *dvd-lcmI1*)
ultimately have *normalize* (*lcm* (*c* ∗ *a*) (*c* ∗ *b*)) = *normalize* (*c* ∗ *lcm a b*)
by (*auto intro*: *associated-eqI*)
then show *?thesis*
by (*simp add*: *normalize-mult*)
qed

lemma *lcm-mult-right*: *lcm* (*a* ∗ *c*) (*b* ∗ *c*) = *normalize* (*lcm b a* ∗ *c*)
using *lcm-mult-left* [*of c a b*] **by** (*simp add*: *ac-simps*)

lemma *lcm-mult-unit1*: *is-unit a* ⟹ *lcm* (*b* ∗ *a*) *c* = *lcm b c*
by (*rule associated-eqI*) (*simp-all add*: *mult-unit-dvd-iff dvd-lcmI1*)

lemma *lcm-mult-unit2*: *is-unit a* ⟹ *lcm b* (*c* ∗ *a*) = *lcm b c*
using *lcm-mult-unit1* [*of a c b*] **by** (*simp add*: *ac-simps*)

lemma *lcm-div-unit1*:
is-unit a ⟹ *lcm* (*b div a*) *c* = *lcm b c*

by (*erule is-unitE* [*of* - *b*]) (*simp add*: *lcm-mult-unit1*)

lemma *lcm-div-unit2*: *is-unit a* ⟹ *lcm b* (*c div a*) = *lcm b c*
by (*erule is-unitE* [*of* - *c*]) (*simp add*: *lcm-mult-unit2*)

lemma *normalize-lcm-left*: *lcm* (*normalize a*) *b* = *lcm a b*
by (*fact lcm.normalize-left-idem*)

lemma *normalize-lcm-right*: *lcm a* (*normalize b*) = *lcm a b*
by (*fact lcm.normalize-right-idem*)

lemma *comp-fun-idem-gcd*: *comp-fun-idem gcd*
by *standard* (*simp-all add*: *fun-eq-iff ac-simps*)

lemma *comp-fun-idem-lcm*: *comp-fun-idem lcm*
by *standard* (*simp-all add*: *fun-eq-iff ac-simps*)

lemma *gcd-dvd-antisym*: *gcd a b dvd gcd c d* ⟹ *gcd c d dvd gcd a b* ⟹ *gcd a b* = *gcd c d*
proof (*rule gcdI*)
assume ∗: *gcd a b dvd gcd c d*
and ∗∗: *gcd c d dvd gcd a b*
have *gcd c d dvd c*
by *simp*
with ∗ **show** *gcd a b dvd c*
by (*rule dvd-trans*)
have *gcd c d dvd d*
by *simp*
with ∗ **show** *gcd a b dvd d*
by (*rule dvd-trans*)
show *normalize* (*gcd a b*) = *gcd a b*
by *simp*
fix *l* **assume** *l dvd c* **and** *l dvd d*
then have *l dvd gcd c d*
by (*rule gcd-greatest*)
from *this* **and** ∗∗ **show** *l dvd gcd a b*
by (*rule dvd-trans*)
qed

declare *unit-factor-lcm* [*simp*]

lemma *lcmI*:
assumes *a dvd c* **and** *b dvd c* **and** ⋀*d*. *a dvd d* ⟹ *b dvd d* ⟹ *c dvd d*
and *normalize c* = *c*
shows *c* = *lcm a b*
by (*rule associated-eqI*) (*auto simp*: *assms intro*: *lcm-least*)

lemma *gcd-dvd-lcm* [*simp*]: *gcd a b dvd lcm a b*
using *gcd-dvd2* **by** (*rule dvd-lcmI2*)

lemmas *lcm-0* = *lcm-0-right*

lemma *lcm-unique*:
a dvd d ∧ *b dvd d* ∧ *normalize d* = *d* ∧ (∀ *e*. *a dvd e* ∧ *b dvd e* ⟶ *d dvd e*) ⟷ *d* = *lcm a b*
by *rule* (*auto intro*: *lcmI simp*: *lcm-least lcm-eq-0-iff*)

lemma *lcm-proj1-if-dvd*:
assumes *b dvd a* **shows** *lcm a b* = *normalize a*
proof −
have *normalize* (*lcm a b*) = *normalize a*
by (*rule associatedI*) (*use assms* **in** *auto*)
thus *?thesis* **by** *simp*
qed

lemma *lcm-proj2-if-dvd*: *a dvd b* ⟹ *lcm a b* = *normalize b*
using *lcm-proj1-if-dvd* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *lcm-proj1-iff*: *lcm m n* = *normalize m* ⟷ *n dvd m*
proof
assume ∗: *lcm m n* = *normalize m*
show *n dvd m*
proof (*cases m* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case [*simp*]: *False*
from ∗ **have** ∗∗: *m* = *lcm m n* ∗ *unit-factor m*
by (*simp add*: *unit-eq-div2*)
show *?thesis* **by** (*subst* ∗∗) *simp*
qed
next
assume *n dvd m*
then show *lcm m n* = *normalize m*
by (*rule lcm-proj1-if-dvd*)
qed

lemma *lcm-proj2-iff*: *lcm m n* = *normalize n* ⟷ *m dvd n*
using *lcm-proj1-iff* [*of n m*] **by** (*simp add*: *ac-simps*)

lemma *gcd-mono*: *a dvd c* ⟹ *b dvd d* ⟹ *gcd a b dvd gcd c d*
by (*simp add*: *gcd-dvdI1 gcd-dvdI2*)

lemma *lcm-mono*: *a dvd c* ⟹ *b dvd d* ⟹ *lcm a b dvd lcm c d*
by (*simp add*: *dvd-lcmI1 dvd-lcmI2*)

lemma *dvd-productE*:
assumes *p dvd a* ∗ *b*

obtains *x y* **where** *p* = *x* ∗ *y x dvd a y dvd b*
proof (*cases a* = *0*)
case *True*
thus *?thesis* **by** (*intro that*[*of p 1*]) *simp-all*
next
case *False*
define *x y* **where** *x* = *gcd a p* **and** *y* = *p div x*
have *p* = *x* ∗ *y* **by** (*simp add*: *x-def y-def*)
moreover have *x dvd a* **by** (*simp add*: *x-def*)
moreover from *assms* **have** *p dvd gcd* (*b* ∗ *a*) (*b* ∗ *p*)
by (*intro gcd-greatest*) (*simp-all add*: *mult.commute*)
hence *p dvd b* ∗ *gcd a p* **by** (*subst* (*asm*) *gcd-mult-left*) *auto*
with *False* **have** *y dvd b*
by (*simp add*: *x-def y-def div-dvd-iff-mult assms*)
ultimately show *?thesis* **by** (*rule that*)
qed

lemma *gcd-mult-unit1*:
assumes *is-unit a* **shows** *gcd* (*b* ∗ *a*) *c* = *gcd b c*
proof −
have *gcd* (*b* ∗ *a*) *c dvd b*
using *assms dvd-mult-unit-iff* **by** *blast*
then show *?thesis*
by (*rule gcdI*) *simp-all*
qed

lemma *gcd-mult-unit2*: *is-unit a* ⟹ *gcd b* (*c* ∗ *a*) = *gcd b c*
using *gcd.commute gcd-mult-unit1* **by** *auto*

lemma *gcd-div-unit1*: *is-unit a* ⟹ *gcd* (*b div a*) *c* = *gcd b c*
by (*erule is-unitE* [*of - b*]) (*simp add*: *gcd-mult-unit1*)

lemma *gcd-div-unit2*: *is-unit a* ⟹ *gcd b* (*c div a*) = *gcd b c*
by (*erule is-unitE* [*of - c*]) (*simp add*: *gcd-mult-unit2*)

lemma *normalize-gcd-left*: *gcd* (*normalize a*) *b* = *gcd a b*
by (*fact gcd.normalize-left-idem*)

lemma *normalize-gcd-right*: *gcd a* (*normalize b*) = *gcd a b*
by (*fact gcd.normalize-right-idem*)

lemma *gcd-add1* [*simp*]: *gcd* (*m* + *n*) *n* = *gcd m n*
by (*rule gcdI* [*symmetric*]) (*simp-all add*: *dvd-add-left-iff*)

lemma *gcd-add2* [*simp*]: *gcd m* (*m* + *n*) = *gcd m n*
using *gcd-add1* [*of n m*] **by** (*simp add*: *ac-simps*)

lemma *gcd-add-mult*: *gcd m* (*k* ∗ *m* + *n*) = *gcd m n*
by (*rule gcdI* [*symmetric*]) (*simp-all add*: *dvd-add-right-iff*)

end

class *ring-gcd* = *comm-ring-1* + *semiring-gcd*
begin

lemma *gcd-neg1* [*simp*]: *gcd* (−*a*) *b* = *gcd a b*
by (*rule sym*, *rule gcdI*) (*simp-all add*: *gcd-greatest*)

lemma *gcd-neg2* [*simp*]: *gcd a* (−*b*) = *gcd a b*
by (*rule sym*, *rule gcdI*) (*simp-all add*: *gcd-greatest*)

lemma *gcd-neg-numeral-1* [*simp*]: *gcd* (− *numeral n*) *a* = *gcd* (*numeral n*) *a*
by (*fact gcd-neg1*)

lemma *gcd-neg-numeral-2* [*simp*]: *gcd a* (− *numeral n*) = *gcd a* (*numeral n*)
by (*fact gcd-neg2*)

lemma *gcd-diff1*: *gcd* (*m* − *n*) *n* = *gcd m n*
by (*subst diff-conv-add-uminus*, *subst gcd-neg2*[*symmetric*], *subst gcd-add1*, *simp*)

lemma *gcd-diff2*: *gcd* (*n* − *m*) *n* = *gcd m n*
by (*subst gcd-neg1*[*symmetric*]) (*simp only*: *minus-diff-eq gcd-diff1*)

lemma *lcm-neg1* [*simp*]: *lcm* (−*a*) *b* = *lcm a b*
by (*rule sym*, *rule lcmI*) (*simp-all add*: *lcm-least lcm-eq-0-iff*)

lemma *lcm-neg2* [*simp*]: *lcm a* (−*b*) = *lcm a b*
by (*rule sym*, *rule lcmI*) (*simp-all add*: *lcm-least lcm-eq-0-iff*)

lemma *lcm-neg-numeral-1* [*simp*]: *lcm* (− *numeral n*) *a* = *lcm* (*numeral n*) *a*
by (*fact lcm-neg1*)

lemma *lcm-neg-numeral-2* [*simp*]: *lcm a* (− *numeral n*) = *lcm a* (*numeral n*)
by (*fact lcm-neg2*)

end

class *semiring-Gcd* = *semiring-gcd* + *Gcd* +
assumes *Gcd-dvd*: *a* ∈ *A* ⟹ *Gcd A dvd a*
and *Gcd-greatest*: (⋀*b*. *b* ∈ *A* ⟹ *a dvd b*) ⟹ *a dvd Gcd A*
and *normalize-Gcd* [*simp*]: *normalize* (*Gcd A*) = *Gcd A*
assumes *dvd-Lcm*: *a* ∈ *A* ⟹ *a dvd Lcm A*
and *Lcm-least*: (⋀*b*. *b* ∈ *A* ⟹ *b dvd a*) ⟹ *Lcm A dvd a*
and *normalize-Lcm* [*simp*]: *normalize* (*Lcm A*) = *Lcm A*
begin

lemma *Lcm-Gcd*: *Lcm A* = *Gcd* {*b*. ∀ *a*∈*A*. *a dvd b*}
by (*rule associated-eqI*) (*auto intro*: *Gcd-dvd dvd-Lcm Gcd-greatest Lcm-least*)

```
lemma Gcd-Lcm: Gcd A = Lcm {b. ∀ a∈A. b dvd a}
  by (rule associated-eqI) (auto intro: Gcd-dvd dvd-Lcm Gcd-greatest Lcm-least)

lemma Gcd-empty [simp]: Gcd {} = 0
  by (rule dvd-0-left, rule Gcd-greatest) simp

lemma Lcm-empty [simp]: Lcm {} = 1
  by (auto intro: associated-eqI Lcm-least)

lemma Gcd-insert [simp]: Gcd (insert a A) = gcd a (Gcd A)
proof −
  have Gcd (insert a A) dvd gcd a (Gcd A)
    by (auto intro: Gcd-dvd Gcd-greatest)
  moreover have gcd a (Gcd A) dvd Gcd (insert a A)
  proof (rule Gcd-greatest)
    fix b
    assume b ∈ insert a A
    then show gcd a (Gcd A) dvd b
    proof
      assume b = a
      then show ?thesis
        by simp
    next
      assume b ∈ A
      then have Gcd A dvd b
        by (rule Gcd-dvd)
      moreover have gcd a (Gcd A) dvd Gcd A
        by simp
      ultimately show ?thesis
        by (blast intro: dvd-trans)
    qed
  qed
  ultimately show ?thesis
    by (auto intro: associated-eqI)
qed

lemma Lcm-insert [simp]: Lcm (insert a A) = lcm a (Lcm A)
proof (rule sym)
  have lcm a (Lcm A) dvd Lcm (insert a A)
    by (auto intro: dvd-Lcm Lcm-least)
  moreover have Lcm (insert a A) dvd lcm a (Lcm A)
  proof (rule Lcm-least)
    fix b
    assume b ∈ insert a A
    then show b dvd lcm a (Lcm A)
    proof
      assume b = a
      then show ?thesis by simp
```

next
assume *b* ∈ *A*
then have *b dvd Lcm A*
by (*rule dvd-Lcm*)
moreover have *Lcm A dvd lcm a* (*Lcm A*)
by *simp*
ultimately show *?thesis*
by (*blast intro*: *dvd-trans*)
qed
qed
ultimately show *lcm a* (*Lcm A*) = *Lcm* (*insert a A*)
by (*rule associated-eqI*) (*simp-all add*: *lcm-eq-0-iff*)
qed

lemma *LcmI*:
assumes ⋀*a*. *a* ∈ *A* ⟹ *a dvd b*
and ⋀*c*. (⋀*a*. *a* ∈ *A* ⟹ *a dvd c*) ⟹ *b dvd c*
and *normalize b* = *b*
shows *b* = *Lcm A*
by (*rule associated-eqI*) (*auto simp*: *assms dvd-Lcm intro*: *Lcm-least*)

lemma *Lcm-subset*: *A* ⊆ *B* ⟹ *Lcm A dvd Lcm B*
by (*blast intro*: *Lcm-least dvd-Lcm*)

lemma *Lcm-Un*: *Lcm* (*A* ∪ *B*) = *lcm* (*Lcm A*) (*Lcm B*)
proof −
have ⋀*d*. ⟦*Lcm A dvd d*; *Lcm B dvd d*⟧ ⟹ *Lcm* (*A* ∪ *B*) *dvd d*
by (*meson UnE Lcm-least dvd-Lcm dvd-trans*)
then show *?thesis*
by (*meson Lcm-subset lcm-unique normalize-Lcm sup.cobounded1 sup.cobounded2*)
qed

lemma *Gcd-0-iff* [*simp*]: *Gcd A* = *0* ⟷ *A* ⊆ {*0*}
(**is** *?P* ⟷ *?Q*)
proof
assume *?P*
show *?Q*
proof
fix *a*
assume *a* ∈ *A*
then have *Gcd A dvd a*
by (*rule Gcd-dvd*)
with ‹*?P*› **have** *a* = *0*
by *simp*
then show *a* ∈ {*0*}
by *simp*
qed
next
assume *?Q*

```
  have 0 dvd Gcd A
  proof (rule Gcd-greatest)
    fix a
    assume a ∈ A
    with ‹?Q› have a = 0
      by auto
    then show 0 dvd a
      by simp
  qed
  then show ?P
    by simp
qed

lemma Lcm-1-iff [simp]: Lcm A = 1 ⟷ (∀ a∈A. is-unit a)
  (is ?P ⟷ ?Q)
proof
  assume ?P
  show ?Q
  proof
    fix a
    assume a ∈ A
    then have a dvd Lcm A
      by (rule dvd-Lcm)
    with ‹?P› show is-unit a
      by simp
  qed
next
  assume ?Q
  then have is-unit (Lcm A)
    by (blast intro: Lcm-least)
  then have normalize (Lcm A) = 1
    by (rule is-unit-normalize)
  then show ?P
    by simp
qed

lemma unit-factor-Lcm: unit-factor (Lcm A) = (if Lcm A = 0 then 0 else 1)
proof (cases Lcm A = 0)
  case True
  then show ?thesis
    by simp
next
  case False
  with unit-factor-normalize have unit-factor (normalize (Lcm A)) = 1
    by blast
  with False show ?thesis
    by simp
qed
```

lemma *unit-factor-Gcd*: *unit-factor* (*Gcd A*) = (*if Gcd A* = *0 then 0 else 1*)
  by (*simp add*: *Gcd-Lcm unit-factor-Lcm*)

lemma *GcdI*:
  assumes $\bigwedge a.\ a \in A \Longrightarrow b\ dvd\ a$
    and $\bigwedge c.\ (\bigwedge a.\ a \in A \Longrightarrow c\ dvd\ a) \Longrightarrow c\ dvd\ b$
    and *normalize b* = *b*
  shows *b* = *Gcd A*
  by (*rule associated-eqI*) (*auto simp*: *assms Gcd-dvd intro*: *Gcd-greatest*)

lemma *Gcd-eq-1-I*:
  assumes *is-unit a* **and** $a \in A$
  shows *Gcd A* = *1*
proof −
  from *assms* **have** *is-unit* (*Gcd A*)
    by (*blast intro*: *Gcd-dvd dvd-unit-imp-unit*)
  then have *normalize* (*Gcd A*) = *1*
    by (*rule is-unit-normalize*)
  then show *?thesis*
    by *simp*
qed

lemma *Lcm-eq-0-I*:
  assumes $0 \in A$
  shows *Lcm A* = *0*
proof −
  from *assms* **have** *0 dvd Lcm A*
    by (*rule dvd-Lcm*)
  then show *?thesis*
    by *simp*
qed

lemma *Gcd-UNIV* [*simp*]: *Gcd UNIV* = *1*
  using *dvd-refl* **by** (*rule Gcd-eq-1-I*) *simp*

lemma *Lcm-UNIV* [*simp*]: *Lcm UNIV* = *0*
  by (*rule Lcm-eq-0-I*) *simp*

lemma *Lcm-0-iff*:
  assumes *finite A*
  shows $Lcm\ A = 0 \longleftrightarrow 0 \in A$
proof (*cases A* = {})
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  with *assms* **show** *?thesis*
    by (*induct A rule*: *finite-ne-induct*) (*auto simp*: *lcm-eq-0-iff*)
qed

lemma *Gcd-image-normalize* [*simp*]: *Gcd* (*normalize* ‘ *A*) = *Gcd A*
proof −
  have *Gcd* (*normalize* ‘ *A*) *dvd a* **if** *a* ∈ *A* **for** *a*
  proof −
    from *that* **obtain** *B* **where** *A* = *insert a B*
      by *blast*
    moreover have *gcd* (*normalize a*) (*Gcd* (*normalize* ‘ *B*)) *dvd normalize a*
      by (*rule gcd-dvd1*)
    ultimately show *Gcd* (*normalize* ‘ *A*) *dvd a*
      by *simp*
  qed
  then have *Gcd* (*normalize* ‘ *A*) *dvd Gcd A* **and** *Gcd A dvd Gcd* (*normalize* ‘ *A*)
    by (*auto intro*!: *Gcd-greatest intro*: *Gcd-dvd*)
  then show *?thesis*
    by (*auto intro*: *associated-eqI*)
qed

lemma *Gcd-eqI*:
  assumes *normalize a* = *a*
  assumes ⋀*b*. *b* ∈ *A* ⟹ *a dvd b*
    and ⋀*c*. (⋀*b*. *b* ∈ *A* ⟹ *c dvd b*) ⟹ *c dvd a*
  shows *Gcd A* = *a*
  using *assms* **by** (*blast intro*: *associated-eqI Gcd-greatest Gcd-dvd normalize-Gcd*)

lemma *dvd-GcdD*: *x dvd Gcd A* ⟹ *y* ∈ *A* ⟹ *x dvd y*
  using *Gcd-dvd dvd-trans* **by** *blast*

lemma *dvd-Gcd-iff*: *x dvd Gcd A* ⟷ (∀ *y*∈*A*. *x dvd y*)
  by (*blast dest*: *dvd-GcdD intro*: *Gcd-greatest*)

lemma *Gcd-mult*: *Gcd* ((∗) *c* ‘ *A*) = *normalize* (*c* ∗ *Gcd A*)
proof (*cases c* = *0*)
  case *True*
  then show *?thesis* **by** *auto*
next
  case [*simp*]: *False*
  have *Gcd* ((∗) *c* ‘ *A*) *div c dvd Gcd A*
    by (*intro Gcd-greatest*, *subst div-dvd-iff-mult*)
      (*auto intro*!: *Gcd-greatest Gcd-dvd simp*: *mult.commute*[*of - c*])
  then have *Gcd* ((∗) *c* ‘ *A*) *dvd c* ∗ *Gcd A*
    by (*subst* (*asm*) *div-dvd-iff-mult*) (*auto intro*: *Gcd-greatest simp*: *mult-ac*)
  moreover have *c* ∗ *Gcd A dvd Gcd* ((∗) *c* ‘ *A*)
    by (*intro Gcd-greatest*) (*auto intro*: *mult-dvd-mono Gcd-dvd*)
  ultimately have *normalize* (*Gcd* ((∗) *c* ‘ *A*)) = *normalize* (*c* ∗ *Gcd A*)
    by (*rule associatedI*)
  then show *?thesis* **by** *simp*
qed

lemma *Lcm-eqI*:
assumes *normalize a* = *a*
and $\bigwedge b$. *b* ∈ *A* ⟹ *b dvd a*
and $\bigwedge c$. ($\bigwedge b$. *b* ∈ *A* ⟹ *b dvd c*) ⟹ *a dvd c*
shows *Lcm A* = *a*
using *assms* **by** (*blast intro*: *associated-eqI Lcm-least dvd-Lcm normalize-Lcm*)

lemma *Lcm-dvdD*: *Lcm A dvd x* ⟹ *y* ∈ *A* ⟹ *y dvd x*
using *dvd-Lcm dvd-trans* **by** *blast*

lemma *Lcm-dvd-iff*: *Lcm A dvd x* ⟷ (∀ *y*∈*A*. *y dvd x*)
by (*blast dest*: *Lcm-dvdD intro*: *Lcm-least*)

lemma *Lcm-mult*:
assumes *A* ≠ {}
shows *Lcm* ((∗) *c* ' *A*) = *normalize* (*c* ∗ *Lcm A*)
proof (*cases c* = *0*)
case *True*
with *assms* **have** (∗) *c* ' *A* = {*0*}
by *auto*
with *True* **show** *?thesis* **by** *auto*
next
case [*simp*]: *False*
from *assms* **obtain** *x* **where** *x*: *x* ∈ *A*
by *blast*
have *c dvd c* ∗ *x*
by *simp*
also from *x* **have** *c* ∗ *x dvd Lcm* ((∗) *c* ' *A*)
by (*intro dvd-Lcm*) *auto*
finally have *dvd*: *c dvd Lcm* ((∗) *c* ' *A*) **.**
moreover have *Lcm A dvd Lcm* ((∗) *c* ' *A*) *div c*
by (*intro Lcm-least dvd-mult-imp-div*)
(*auto intro*!: *Lcm-least dvd-Lcm simp*: *mult.commute*[*of* - *c*])
ultimately have *c* ∗ *Lcm A dvd Lcm* ((∗) *c* ' *A*)
by *auto*
moreover have *Lcm* ((∗) *c* ' *A*) *dvd c* ∗ *Lcm A*
by (*intro Lcm-least*) (*auto intro*: *mult-dvd-mono dvd-Lcm*)
ultimately have *normalize* (*c* ∗ *Lcm A*) = *normalize* (*Lcm* ((∗) *c* ' *A*))
by (*rule associatedI*)
then show *?thesis* **by** *simp*
qed

lemma *Lcm-no-units*: *Lcm A* = *Lcm* (*A* − {*a*. *is-unit a*})
proof −
have (*A* − {*a*. *is-unit a*}) ∪ {*a*∈*A*. *is-unit a*} = *A*
by *blast*
then have *Lcm A* = *lcm* (*Lcm* (*A* − {*a*. *is-unit a*})) (*Lcm* {*a*∈*A*. *is-unit a*})
by (*simp add*: *Lcm-Un* [*symmetric*])
also have *Lcm* {*a*∈*A*. *is-unit a*} = *1*

by *simp*
finally show *?thesis*
by *simp*
qed

lemma *Lcm-0-iff'*: *Lcm A = 0* ⟷ (∄ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*))
by (*metis Lcm-least dvd-0-left dvd-Lcm*)

lemma *Lcm-no-multiple*: (∀ *m*. *m* ≠ *0* ⟶ (∃ *a*∈*A*. ¬ *a dvd m*)) ⟹ *Lcm A = 0*
by (*auto simp*: *Lcm-0-iff'*)

lemma *Lcm-singleton* [*simp*]: *Lcm* {*a*} = *normalize a*
by *simp*

lemma *Lcm-2* [*simp*]: *Lcm* {*a*, *b*} = *lcm a b*
by *simp*

lemma *Gcd-1*: *1* ∈ *A* ⟹ *Gcd A = 1*
by (*auto intro*!: *Gcd-eq-1-I*)

lemma *Gcd-singleton* [*simp*]: *Gcd* {*a*} = *normalize a*
by *simp*

lemma *Gcd-2* [*simp*]: *Gcd* {*a*, *b*} = *gcd a b*
by *simp*

lemma *Gcd-mono*:
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x dvd g x*
shows (*GCD x*∈*A*. *f x*) *dvd* (*GCD x*∈*A*. *g x*)
proof (*intro Gcd-greatest*, *safe*)
fix *x* **assume** *x* ∈ *A*
hence (*GCD x*∈*A*. *f x*) *dvd f x*
by (*intro Gcd-dvd*) *auto*
also have *f x dvd g x*
using ‹*x* ∈ *A*› *assms* **by** *blast*
finally show (*GCD x*∈*A*. *f x*) *dvd* . . . **.**
qed

lemma *Lcm-mono*:
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x dvd g x*
shows (*LCM x*∈*A*. *f x*) *dvd* (*LCM x*∈*A*. *g x*)
proof (*intro Lcm-least*, *safe*)
fix *x* **assume** *x* ∈ *A*
hence *f x dvd g x* **by** (*rule assms*)
also have *g x dvd* (*LCM x*∈*A*. *g x*)
using ‹*x* ∈ *A*› **by** (*intro dvd-Lcm*) *auto*
finally show *f x dvd* . . . **.**
qed

end

87.3 An aside: GCD and LCM on finite sets for incomplete gcd rings

context *semiring-gcd*
begin

sublocale *Gcd-fin*: *bounded-quasi-semilattice-set gcd 0 1 normalize*
defines
Gcd-fin (‹Gcd_{fin}›) = *Gcd-fin.F* :: *'a set* ⇒ *'a* **..**

abbreviation *gcd-list* :: *'a list* ⇒ *'a*
where *gcd-list xs* ≡ Gcd_{fin} (*set xs*)

sublocale *Lcm-fin*: *bounded-quasi-semilattice-set lcm 1 0 normalize*
defines
Lcm-fin (‹Lcm_{fin}›) = *Lcm-fin.F* **..**

abbreviation *lcm-list* :: *'a list* ⇒ *'a*
where *lcm-list xs* ≡ Lcm_{fin} (*set xs*)

lemma *Gcd-fin-dvd*:
a ∈ *A* ⟹ Gcd_{fin} *A dvd a*
by (*induct A rule*: *infinite-finite-induct*)
(*auto intro*: *dvd-trans*)

lemma *dvd-Lcm-fin*:
a ∈ *A* ⟹ *a dvd* Lcm_{fin} *A*
by (*induct A rule*: *infinite-finite-induct*)
(*auto intro*: *dvd-trans*)

lemma *Gcd-fin-greatest*:
a dvd Gcd_{fin} *A* **if** *finite A* **and** ⋀*b*. *b* ∈ *A* ⟹ *a dvd b*
using *that* **by** (*induct A*) *simp-all*

lemma *Lcm-fin-least*:
Lcm_{fin} *A dvd a* **if** *finite A* **and** ⋀*b*. *b* ∈ *A* ⟹ *b dvd a*
using *that* **by** (*induct A*) *simp-all*

lemma *gcd-list-greatest*:
a dvd gcd-list bs **if** ⋀*b*. *b* ∈ *set bs* ⟹ *a dvd b*
by (*rule Gcd-fin-greatest*) (*simp-all add*: *that*)

lemma *lcm-list-least*:
lcm-list bs dvd a **if** ⋀*b*. *b* ∈ *set bs* ⟹ *b dvd a*
by (*rule Lcm-fin-least*) (*simp-all add*: *that*)

lemma *dvd-Gcd-fin-iff*:

b dvd Gcd_{fin} *A* ⟷ (∀ *a*∈*A*. *b dvd a*) **if** *finite A*
using *that* **by** (*auto intro*: *Gcd-fin-greatest Gcd-fin-dvd dvd-trans* [*of b* Gcd_{fin} *A*])

lemma *dvd-gcd-list-iff*:
b dvd gcd-list xs ⟷ (∀ *a*∈*set xs*. *b dvd a*)
by (*simp add*: *dvd-Gcd-fin-iff*)

lemma *Lcm-fin-dvd-iff*:
Lcm_{fin} *A dvd b* ⟷ (∀ *a*∈*A*. *a dvd b*) **if** *finite A*
using *that* **by** (*auto intro*: *Lcm-fin-least dvd-Lcm-fin dvd-trans* [*of* - Lcm_{fin} *A b*])

lemma *lcm-list-dvd-iff*:
lcm-list xs dvd b ⟷ (∀ *a*∈*set xs*. *a dvd b*)
by (*simp add*: *Lcm-fin-dvd-iff*)

lemma *Gcd-fin-mult*:
Gcd_{fin} (*image* (*times b*) *A*) = *normalize* (*b* ∗ Gcd_{fin} *A*) **if** *finite A*
using *that* **by** *induction* (*auto simp*: *gcd-mult-left*)

lemma *Lcm-fin-mult*:
Lcm_{fin} (*image* (*times b*) *A*) = *normalize* (*b* ∗ Lcm_{fin} *A*) **if** *A* ≠ {}
proof (*cases b* = *0*)
  case *True*
  moreover from *that* **have** *times 0* ' *A* = {*0*}
    by *auto*
  ultimately show *?thesis*
    by *simp*
next
  case *False*
  show *?thesis* **proof** (*cases finite A*)
    case *False*
    moreover have *inj-on* (*times b*) *A*
      using ‹*b* ≠ *0*› **by** (*rule inj-on-mult*)
    ultimately have *infinite* (*times b* ' *A*)
      by (*simp add*: *finite-image-iff*)
    with *False* **show** *?thesis*
      by *simp*
  next
    case *True*
    then show *?thesis* **using** *that*
      by (*induct A rule*: *finite-ne-induct*) (*auto simp*: *lcm-mult-left*)
  qed
qed

lemma *unit-factor-Gcd-fin*:
unit-factor (Gcd_{fin} *A*) = *of-bool* (Gcd_{fin} *A* ≠ *0*)
by (*rule normalize-idem-imp-unit-factor-eq*) *simp*

lemma *unit-factor-Lcm-fin*:
unit-factor (Lcm_{fin} *A*) = *of-bool* (Lcm_{fin} *A* ≠ *0*)
by (*rule normalize-idem-imp-unit-factor-eq*) *simp*

lemma *is-unit-Gcd-fin-iff* [*simp*]:
is-unit (Gcd_{fin} *A*) ⟷ Gcd_{fin} *A* = *1*
by (*rule normalize-idem-imp-is-unit-iff*) *simp*

lemma *is-unit-Lcm-fin-iff* [*simp*]:
is-unit (Lcm_{fin} *A*) ⟷ Lcm_{fin} *A* = *1*
by (*rule normalize-idem-imp-is-unit-iff*) *simp*

lemma *Gcd-fin-0-iff*:
Gcd_{fin} *A* = *0* ⟷ *A* ⊆ {*0*} ∧ *finite A*
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *Lcm-fin-0-iff*:
Lcm_{fin} *A* = *0* ⟷ *0* ∈ *A* **if** *finite A*
using *that* **by** (*induct A*) (*auto simp*: *lcm-eq-0-iff*)

lemma *Lcm-fin-1-iff*:
Lcm_{fin} *A* = *1* ⟷ (∀ *a*∈*A*. *is-unit a*) ∧ *finite A*
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

end

context *semiring-Gcd*
begin

lemma *Gcd-fin-eq-Gcd* [*simp*]:
Gcd_{fin} *A* = *Gcd A* **if** *finite A* **for** *A* :: *'a set*
using *that* **by** *induct simp-all*

lemma *Gcd-set-eq-fold* [*code-unfold*]:
Gcd (*set xs*) = *fold gcd xs 0*
by (*simp add*: *Gcd-fin.set-eq-fold* [*symmetric*])

lemma *Lcm-fin-eq-Lcm* [*simp*]:
Lcm_{fin} *A* = *Lcm A* **if** *finite A* **for** *A* :: *'a set*
using *that* **by** *induct simp-all*

lemma *Lcm-set-eq-fold* [*code-unfold*]:
Lcm (*set xs*) = *fold lcm xs 1*
by (*simp add*: *Lcm-fin.set-eq-fold* [*symmetric*])

end

87.4 Coprimality

context *semiring-gcd*
begin

lemma *coprime-imp-gcd-eq-1* [*simp*]:
  gcd a b = *1* **if** *coprime a b*
proof –
  define *t r s* **where** *t* = *gcd a b* **and** *r* = *a div t* **and** *s* = *b div t*
  then have *a* = *t* ∗ *r* **and** *b* = *t* ∗ *s*
    by *simp-all*
  with *that* **have** *coprime* (*t* ∗ *r*) (*t* ∗ *s*)
    by *simp*
  then show *?thesis*
    by (*simp add*: *t-def*)
qed

lemma *gcd-eq-1-imp-coprime* [*dest*!]:
  coprime a b **if** *gcd a b* = *1*
proof (*rule coprimeI*)
  fix *c*
  assume *c dvd a* **and** *c dvd b*
  then have *c dvd gcd a b*
    by (*rule gcd-greatest*)
  with *that* **show** *is-unit c*
    by *simp*
qed

lemma *coprime-iff-gcd-eq-1* [*presburger*, *code*]:
  coprime a b ⟷ *gcd a b* = *1*
  by *rule* (*simp-all add*: *gcd-eq-1-imp-coprime*)

lemma *is-unit-gcd* [*simp*]:
  is-unit (*gcd a b*) ⟷ *coprime a b*
  by (*simp add*: *coprime-iff-gcd-eq-1*)

lemma *coprime-add-one-left* [*simp*]: *coprime* (*a* + *1*) *a*
  by (*simp add*: *gcd-eq-1-imp-coprime ac-simps*)

lemma *coprime-add-one-right* [*simp*]: *coprime a* (*a* + *1*)
  using *coprime-add-one-left* [*of a*] **by** (*simp add*: *ac-simps*)

lemma *coprime-mult-left-iff* [*simp*]:
  coprime (*a* ∗ *b*) *c* ⟷ *coprime a c* ∧ *coprime b c*
proof
  assume *coprime* (*a* ∗ *b*) *c*
  with *coprime-common-divisor* [*of a* ∗ *b c*]
  have ∗: *is-unit d* **if** *d dvd a* ∗ *b* **and** *d dvd c* **for** *d*
    using *that* **by** *blast*
  have *coprime a c*

```
    by (rule coprimeI, rule *) simp-all
  moreover have coprime b c
    by (rule coprimeI, rule *) simp-all
  ultimately show coprime a c ∧ coprime b c ..
next
  assume coprime a c ∧ coprime b c
  then have coprime a c coprime b c
    by simp-all
  show coprime (a * b) c
  proof (rule coprimeI)
    fix d
    assume d dvd a * b
    then obtain r s where d: d = r * s r dvd a s dvd b
      by (rule dvd-productE)
    assume d dvd c
    with d have r * s dvd c
      by simp
    then have r dvd c s dvd c
      by (auto intro: dvd-mult-left dvd-mult-right)
    from ‹coprime a c› ‹r dvd a› ‹r dvd c›
    have is-unit r
      by (rule coprime-common-divisor)
    moreover from ‹coprime b c› ‹s dvd b› ‹s dvd c›
    have is-unit s
      by (rule coprime-common-divisor)
    ultimately show is-unit d
      by (simp add: d is-unit-mult-iff)
  qed
qed

lemma coprime-mult-right-iff [simp]:
  coprime c (a * b) ⟷ coprime c a ∧ coprime c b
  using coprime-mult-left-iff [of a b c] by (simp add: ac-simps)

lemma coprime-power-left-iff [simp]:
  coprime (a ^ n) b ⟷ coprime a b ∨ n = 0
proof (cases n = 0)
  case True
  then show ?thesis
    by simp
next
  case False
  then have n > 0
    by simp
  then show ?thesis
    by (induction n rule: nat-induct-non-zero) simp-all
qed

lemma coprime-power-right-iff [simp]:
```

coprime a $(b \hat{} n) \longleftrightarrow$ *coprime a b* $\vee$ $n = 0$
using *coprime-power-left-iff* [*of b n a*] **by** (*simp add*: *ac-simps*)

lemma *prod-coprime-left*:
coprime ($\prod i{\in}A.$ *f i*) *a* **if** $\bigwedge i.\ i \in A \Longrightarrow$ *coprime* (*f i*) *a*
using *that* **by** (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *prod-coprime-right*:
coprime a ($\prod i{\in}A.$ *f i*) **if** $\bigwedge i.\ i \in A \Longrightarrow$ *coprime a* (*f i*)
using *that prod-coprime-left* [*of A f a*] **by** (*simp add*: *ac-simps*)

lemma *prod-list-coprime-left*:
coprime (*prod-list xs*) *a* **if** $\bigwedge x.\ x \in$ *set xs* $\Longrightarrow$ *coprime x a*
using *that* **by** (*induct xs*) *simp-all*

lemma *prod-list-coprime-right*:
coprime a (*prod-list xs*) **if** $\bigwedge x.\ x \in$ *set xs* $\Longrightarrow$ *coprime a x*
using *that prod-list-coprime-left* [*of xs a*] **by** (*simp add*: *ac-simps*)

lemma *coprime-dvd-mult-left-iff*:
a dvd b $*$ *c* $\longleftrightarrow$ *a dvd b* **if** *coprime a c*
proof
assume *a dvd b*
then show *a dvd b* $*$ *c*
by *simp*
next
assume *a dvd b* $*$ *c*
show *a dvd b*
proof (*cases b* = *0*)
case *True*
then show *?thesis*
by *simp*
next
case *False*
then have *unit*: *is-unit* (*unit-factor b*)
by *simp*
from ‹*coprime a c*›
have *gcd* (*b* $*$ *a*) (*b* $*$ *c*) $*$ *unit-factor b* = *b*
by (*subst gcd-mult-left*) (*simp add*: *ac-simps*)
moreover from ‹*a dvd b* $*$ *c*›
have *a dvd gcd* (*b* $*$ *a*) (*b* $*$ *c*) $*$ *unit-factor b*
by (*simp add*: *dvd-mult-unit-iff unit*)
ultimately show *?thesis*
by *simp*
qed
qed

lemma *coprime-dvd-mult-right-iff*:
a dvd c $*$ *b* $\longleftrightarrow$ *a dvd b* **if** *coprime a c*

using *that coprime-dvd-mult-left-iff* [*of a c b*] **by** (*simp add*: *ac-simps*)

lemma *divides-mult*:
a ∗ *b dvd c* **if** *a dvd c* **and** *b dvd c* **and** *coprime a b*
proof −
from ‹*b dvd c*› **obtain** *b'* **where** *c* = *b* ∗ *b'* **..**
with ‹*a dvd c*› **have** *a dvd b'* ∗ *b*
by (*simp add*: *ac-simps*)
with ‹*coprime a b*› **have** *a dvd b'*
by (*simp add*: *coprime-dvd-mult-left-iff*)
then obtain *a'* **where** *b'* = *a* ∗ *a'* **..**
with ‹*c* = *b* ∗ *b'*› **have** *c* = (*a* ∗ *b*) ∗ *a'*
by (*simp add*: *ac-simps*)
then show *?thesis* **..**
qed

lemma *div-gcd-coprime*:
assumes *a* ≠ *0* ∨ *b* ≠ *0*
shows *coprime* (*a div gcd a b*) (*b div gcd a b*)
proof −
let *?g* = *gcd a b*
let *?a'* = *a div ?g*
let *?b'* = *b div ?g*
let *?g'* = *gcd ?a' ?b'*
have *dvdg*: *?g dvd a ?g dvd b*
by *simp-all*
have *dvdg'*: *?g' dvd ?a' ?g' dvd ?b'*
by *simp-all*
from *dvdg dvdg'* **obtain** *ka kb ka' kb'* **where**
kab: *a* = *?g* ∗ *ka b* = *?g* ∗ *kb ?a'* = *?g'* ∗ *ka' ?b'* = *?g'* ∗ *kb'*
unfolding *dvd-def* **by** *blast*
from *this* [*symmetric*] **have** *?g* ∗ *?a'* = (*?g* ∗ *?g'*) ∗ *ka' ?g* ∗ *?b'* = (*?g* ∗ *?g'*) ∗ *kb'*
by (*simp-all add*: *mult.assoc mult.left-commute* [*of gcd a b*])
then have *dvdgg'*:*?g* ∗ *?g' dvd a ?g*∗ *?g' dvd b*
by (*auto simp*: *dvd-mult-div-cancel* [*OF dvdg*(*1*)] *dvd-mult-div-cancel* [*OF dvdg*(*2*)] *dvd-def*)
have *?g* ≠ *0*
using *assms* **by** *simp*
moreover from *gcd-greatest* [*OF dvdgg'*] **have** *?g* ∗ *?g' dvd ?g* **.**
ultimately show *?thesis*
using *dvd-times-left-cancel-iff* [*of gcd a b - 1*]
by *simp* (*simp only*: *coprime-iff-gcd-eq-1*)
qed

lemma *gcd-coprime*:
assumes *c*: *gcd a b* ≠ *0*
and *a*: *a* = *a'* ∗ *gcd a b*
and *b*: *b* = *b'* ∗ *gcd a b*

shows *coprime* a' b'
proof −
from *c* **have** $a \neq 0 \lor b \neq 0$
by *simp*
with *div-gcd-coprime* **have** *coprime* (*a div gcd a b*) (*b div gcd a b*) **.**
also from *assms* **have** *a div gcd a b* $= a'$
using *dvd-div-eq-mult gcd-dvd1* **by** *blast*
also from *assms* **have** *b div gcd a b* $= b'$
using *dvd-div-eq-mult gcd-dvd1* **by** *blast*
finally show *?thesis* **.**
qed

lemma *gcd-coprime-exists*:
assumes *gcd a b* $\neq 0$
shows $\exists\, a'\ b'.\ a = a' * gcd\ a\ b \land b = b' * gcd\ a\ b \land coprime\ a'\ b'$
proof −
have *coprime* (*a div gcd a b*) (*b div gcd a b*)
using *assms div-gcd-coprime* **by** *auto*
then show *?thesis*
by *force*
qed

lemma *pow-divides-pow-iff* [*simp*]:
$a \ \hat{}\ n$ *dvd* $b \ \hat{}\ n \longleftrightarrow a$ *dvd* b **if** $n > 0$
proof (*cases gcd a b* $= 0$)
case *True*
then show *?thesis*
by *simp*
next
case *False*
show *?thesis*
proof
let $?d = gcd\ a\ b$
from ‹$n > 0$› **obtain** *m* **where** *m*: $n = Suc\ m$
by (*cases n*) *simp-all*
from *False* **have** *zn*: $?d \ \hat{}\ n \neq 0$
by (*rule power-not-zero*)
from *gcd-coprime-exists* [*OF False*]
obtain a' b' **where** ab': $a = a' * ?d$ $b = b' * ?d$ *coprime* a' b'
by *blast*
assume $a \ \hat{}\ n$ *dvd* $b \ \hat{}\ n$
then have $(a' * ?d) \ \hat{}\ n$ *dvd* $(b' * ?d) \ \hat{}\ n$
by (*simp add*: ab'(*1*,*2*)[*symmetric*])
then have $?d\hat{}n * a'\hat{}n$ *dvd* $?d\hat{}n * b'\hat{}n$
by (*simp only*: *power-mult-distrib ac-simps*)
with *zn* **have** $a' \ \hat{}\ n$ *dvd* $b' \ \hat{}\ n$
by *simp*
then have a' *dvd* $b' \ \hat{}\ n$
using *dvd-trans*[*of* a' $a'\hat{}n$ $b'\hat{}n$] **by** (*simp add*: *m*)

then have *a′ dvd b′ ^ m ∗ b′*
by (*simp add*: *m ac-simps*)
moreover have *coprime a′* (*b′ ^ n*)
using ‹*coprime a′ b′*› **by** *simp*
then have *a′ dvd b′*
using ‹*a′ dvd b′ ^ n*› *coprime-dvd-mult-left-iff dvd-mult* **by** *blast*
then have *a′ ∗ ?d dvd b′ ∗ ?d*
by (*rule mult-dvd-mono*) *simp*
with *ab′*(*1*,*2*) **show** *a dvd b*
by *simp*
next
assume *a dvd b*
with ‹*n* > *0*› **show** *a ^ n dvd b ^ n*
by (*induction rule*: *nat-induct-non-zero*)
(*simp-all add*: *mult-dvd-mono*)
qed
qed

lemma *coprime-crossproduct*:
fixes *a b c d* :: *′a*
assumes *coprime a d* **and** *coprime b c*
shows *normalize a ∗ normalize c = normalize b ∗ normalize d ⟷*
normalize a = normalize b ∧ normalize c = normalize d
(**is** *?lhs ⟷ ?rhs*)
proof
assume *?rhs*
then show *?lhs* **by** *simp*
next
assume *?lhs*
from ‹*?lhs*› **have** *normalize a dvd normalize b ∗ normalize d*
by (*auto intro*: *dvdI dest*: *sym*)
with ‹*coprime a d*› **have** *a dvd b*
by (*simp add*: *coprime-dvd-mult-left-iff normalize-mult* [*symmetric*])
from ‹*?lhs*› **have** *normalize b dvd normalize a ∗ normalize c*
by (*auto intro*: *dvdI dest*: *sym*)
with ‹*coprime b c*› **have** *b dvd a*
by (*simp add*: *coprime-dvd-mult-left-iff normalize-mult* [*symmetric*])
from ‹*?lhs*› **have** *normalize c dvd normalize d ∗ normalize b*
by (*auto intro*: *dvdI dest*: *sym simp add*: *mult.commute*)
with ‹*coprime b c*› **have** *c dvd d*
by (*simp add*: *coprime-dvd-mult-left-iff coprime-commute normalize-mult* [*symmetric*])
from ‹*?lhs*› **have** *normalize d dvd normalize c ∗ normalize a*
by (*auto intro*: *dvdI dest*: *sym simp add*: *mult.commute*)
with ‹*coprime a d*› **have** *d dvd c*
by (*simp add*: *coprime-dvd-mult-left-iff coprime-commute normalize-mult* [*symmetric*])
from ‹*a dvd b*› ‹*b dvd a*› **have** *normalize a = normalize b*
by (*rule associatedI*)
moreover from ‹*c dvd d*› ‹*d dvd c*› **have** *normalize c = normalize d*
by (*rule associatedI*)

ultimately show *?rhs* **..**
qed

lemma *gcd-mult-left-left-cancel*:
gcd (*c* ∗ *a*) *b* = *gcd a b* **if** *coprime b c*
proof −
have *coprime* (*gcd b* (*a* ∗ *c*)) *c*
by (*rule coprimeI*) (*auto intro*: *that coprime-common-divisor*)
then have *gcd b* (*a* ∗ *c*) *dvd a*
using *coprime-dvd-mult-left-iff* [*of gcd b* (*a* ∗ *c*) *c a*]
by *simp*
then show *?thesis*
by (*auto intro*: *associated-eqI simp add*: *ac-simps*)
qed

lemma *gcd-mult-left-right-cancel*:
gcd (*a* ∗ *c*) *b* = *gcd a b* **if** *coprime b c*
using *that gcd-mult-left-left-cancel* [*of b c a*]
by (*simp add*: *ac-simps*)

lemma *gcd-mult-right-left-cancel*:
gcd a (*c* ∗ *b*) = *gcd a b* **if** *coprime a c*
using *that gcd-mult-left-left-cancel* [*of a c b*]
by (*simp add*: *ac-simps*)

lemma *gcd-mult-right-right-cancel*:
gcd a (*b* ∗ *c*) = *gcd a b* **if** *coprime a c*
using *that gcd-mult-right-left-cancel* [*of a c b*]
by (*simp add*: *ac-simps*)

lemma *gcd-exp-weak*:
gcd (*a* ^ *n*) (*b* ^ *n*) = *normalize* (*gcd a b* ^ *n*)
proof (*cases a* = *0* ∧ *b* = *0* ∨ *n* = *0*)
case *True*
then show *?thesis*
by (*cases n*) *simp-all*
next
case *False*
then have *coprime* (*a div gcd a b*) (*b div gcd a b*) **and** *n* > *0*
by (*auto intro*: *div-gcd-coprime*)
then have *coprime* ((*a div gcd a b*) ^ *n*) ((*b div gcd a b*) ^ *n*)
by *simp*
then have *1* = *gcd* ((*a div gcd a b*) ^ *n*) ((*b div gcd a b*) ^ *n*)
by *simp*
then have *normalize* (*gcd a b* ^ *n*) = *normalize* (*gcd a b* ^ *n* ∗ ...)
by *simp*
also have ... = *gcd* (*gcd a b* ^ *n* ∗ (*a div gcd a b*) ^ *n*) (*gcd a b* ^ *n* ∗ (*b div gcd a b*) ^ *n*)
by (*rule gcd-mult-left* [*symmetric*])

also have $(gcd\ a\ b)$ ^ $n * (a\ div\ gcd\ a\ b)$ ^ $n = a$ ^ n
by (*simp add*: *ac-simps div-power dvd-power-same*)
also have $(gcd\ a\ b)$ ^ $n * (b\ div\ gcd\ a\ b)$ ^ $n = b$ ^ n
by (*simp add*: *ac-simps div-power dvd-power-same*)
finally show *?thesis* **by** *simp*
qed

lemma *division-decomp*:
assumes $a\ dvd\ b * c$
shows $\exists\, b'\ c'.\ a = b' * c' \wedge b'\ dvd\ b \wedge c'\ dvd\ c$
proof (*cases gcd a b = 0*)
case *True*
then have $a = 0 \wedge b = 0$
by *simp*
then have $a = 0 * c \wedge 0\ dvd\ b \wedge c\ dvd\ c$
by *simp*
then show *?thesis* **by** *blast*
next
case *False*
let *?d = gcd a b*
from *gcd-coprime-exists* [*OF False*]
obtain a' b' **where** ab': $a = a' * ?d\ b = b' * ?d\ coprime\ a'\ b'$
by *blast*
from $ab'(1)$ **have** $a'\ dvd\ a$ **..**
with *assms* **have** $a'\ dvd\ b * c$
using *dvd-trans* [*of* $a'\ a\ b * c$] **by** *simp*
from *assms* $ab'(1,2)$ **have** $a' * ?d\ dvd\ (b' * ?d) * c$
by *simp*
then have $?d * a'\ dvd\ ?d * (b' * c)$
by (*simp add*: *mult-ac*)
with ‹$?d \neq 0$› **have** $a'\ dvd\ b' * c$
by *simp*
then have $a'\ dvd\ c$
using ‹*coprime* $a'\ b'$› **by** (*simp add*: *coprime-dvd-mult-right-iff*)
with $ab'(1)$ **have** $a = ?d * a' \wedge ?d\ dvd\ b \wedge a'\ dvd\ c$
by (*simp add*: *ac-simps*)
then show *?thesis* **by** *blast*
qed

lemma *lcm-coprime*: $coprime\ a\ b \Longrightarrow lcm\ a\ b = normalize\ (a * b)$
by (*subst lcm-gcd*) *simp*

end

context *ring-gcd*
begin

lemma *coprime-minus-left-iff* [*simp*]:
$coprime\ (-\ a)\ b \longleftrightarrow coprime\ a\ b$

by (*rule*; *rule coprimeI*) (*auto intro*: *coprime-common-divisor*)

lemma *coprime-minus-right-iff* [*simp*]:
coprime a (− *b*) ⟷ *coprime a b*
using *coprime-minus-left-iff* [*of b a*] **by** (*simp add*: *ac-simps*)

lemma *coprime-diff-one-left* [*simp*]: *coprime* (*a* − *1*) *a*
using *coprime-add-one-right* [*of a* − *1*] **by** *simp*

lemma *coprime-doff-one-right* [*simp*]: *coprime a* (*a* − *1*)
using *coprime-diff-one-left* [*of a*] **by** (*simp add*: *ac-simps*)

end

context *semiring-Gcd*
begin

lemma *Lcm-coprime*:
assumes *finite A*
and $A \neq \{\}$
and $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \neq b \Longrightarrow$ *coprime a b*
shows *Lcm A* = *normalize* ($\prod A$)
using *assms*
proof (*induct rule*: *finite-ne-induct*)
case *singleton*
then show *?case* **by** *simp*
next
case (*insert a A*)
have *Lcm* (*insert a A*) = *lcm a* (*Lcm A*)
by *simp*
also from *insert* **have** *Lcm A* = *normalize* ($\prod A$)
by *blast*
also have *lcm a* . . . = *lcm a* ($\prod A$)
by (*cases* $\prod A = 0$) (*simp-all add*: *lcm-div-unit2*)
also from *insert* **have** *coprime a* ($\prod A$)
by (*subst coprime-commute*, *intro prod-coprime-left*) *auto*
with *insert* **have** *lcm a* ($\prod A$) = *normalize* ($\prod$ (*insert a A*))
by (*simp add*: *lcm-coprime*)
finally show *?case* **.**
qed

lemma *Lcm-coprime′*:
card $A \neq 0 \Longrightarrow$
($\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \neq b \Longrightarrow$ *coprime a b*) $\Longrightarrow$
Lcm A = *normalize* ($\prod A$)
by (*rule Lcm-coprime*) (*simp-all add*: *card-eq-0-iff*)

end

And some consequences: cancellation modulo m

lemma *mult-mod-cancel-right*:
fixes *m* :: *'a*::{*euclidean-ring-cancel*,*semiring-gcd*}
assumes *eq*: (*a* ∗ *n*) *mod m* = (*b* ∗ *n*) *mod m* **and** *coprime m n*
shows *a mod m* = *b mod m*
proof −
have *m dvd* (*a*∗*n* − *b*∗*n*)
using *eq mod-eq-dvd-iff* **by** *blast*
then have *m dvd a*−*b*
by (*metis* ‹*coprime m n*› *coprime-dvd-mult-left-iff left-diff-distrib'*)
then show *?thesis*
using *mod-eq-dvd-iff* **by** *blast*
qed

lemma *mult-mod-cancel-left*:
fixes *m* :: *'a*::{*euclidean-ring-cancel*,*semiring-gcd*}
assumes (*n* ∗ *a*) *mod m* = (*n* ∗ *b*) *mod m* **and** *coprime m n*
shows *a mod m* = *b mod m*
by (*metis assms mult.commute mult-mod-cancel-right*)

87.5 GCD and LCM for multiplicative normalisation functions

class *semiring-gcd-mult-normalize* = *semiring-gcd* + *normalization-semidom-multiplicative*
begin

lemma *mult-gcd-left*: *c* ∗ *gcd a b* = *unit-factor c* ∗ *gcd* (*c* ∗ *a*) (*c* ∗ *b*)
by (*simp add*: *gcd-mult-left normalize-mult mult.assoc* [*symmetric*])

lemma *mult-gcd-right*: *gcd a b* ∗ *c* = *gcd* (*a* ∗ *c*) (*b* ∗ *c*) ∗ *unit-factor c*
using *mult-gcd-left* [*of c a b*] **by** (*simp add*: *ac-simps*)

lemma *gcd-mult-distrib'*: *normalize c* ∗ *gcd a b* = *gcd* (*c* ∗ *a*) (*c* ∗ *b*)
by (*subst gcd-mult-left*) (*simp-all add*: *normalize-mult*)

lemma *gcd-mult-distrib*: *k* ∗ *gcd a b* = *gcd* (*k* ∗ *a*) (*k* ∗ *b*) ∗ *unit-factor k*
proof−
have *normalize k* ∗ *gcd a b* = *gcd* (*k* ∗ *a*) (*k* ∗ *b*)
by (*simp add*: *gcd-mult-distrib'*)
then have *normalize k* ∗ *gcd a b* ∗ *unit-factor k* = *gcd* (*k* ∗ *a*) (*k* ∗ *b*) ∗ *unit-factor k*
by *simp*
then have *normalize k* ∗ *unit-factor k* ∗ *gcd a b* = *gcd* (*k* ∗ *a*) (*k* ∗ *b*) ∗ *unit-factor k*
by (*simp only*: *ac-simps*)
then show *?thesis*
by *simp*
qed

lemma *gcd-mult-lcm* [*simp*]: *gcd a b* ∗ *lcm a b* = *normalize a* ∗ *normalize b*

by (*simp add*: *lcm-gcd normalize-mult dvd-normalize-div*)

lemma *lcm-mult-gcd* [*simp*]: *lcm a b* ∗ *gcd a b* = *normalize a* ∗ *normalize b*
using *gcd-mult-lcm* [*of a b*] **by** (*simp add*: *ac-simps*)

lemma *mult-lcm-left*: *c* ∗ *lcm a b* = *unit-factor c* ∗ *lcm* (*c* ∗ *a*) (*c* ∗ *b*)
by (*simp add*: *lcm-mult-left mult.assoc* [*symmetric*] *normalize-mult*)

lemma *mult-lcm-right*: *lcm a b* ∗ *c* = *lcm* (*a* ∗ *c*) (*b* ∗ *c*) ∗ *unit-factor c*
using *mult-lcm-left* [*of c a b*] **by** (*simp add*: *ac-simps*)

lemma *lcm-gcd-prod*: *lcm a b* ∗ *gcd a b* = *normalize* (*a* ∗ *b*)
by (*simp add*: *lcm-gcd dvd-normalize-div normalize-mult*)

lemma *lcm-mult-distrib′*: *normalize c* ∗ *lcm a b* = *lcm* (*c* ∗ *a*) (*c* ∗ *b*)
by (*subst lcm-mult-left*) (*simp add*: *normalize-mult*)

lemma *lcm-mult-distrib*: *k* ∗ *lcm a b* = *lcm* (*k* ∗ *a*) (*k* ∗ *b*) ∗ *unit-factor k*
proof−
have *normalize k* ∗ *lcm a b* = *lcm* (*k* ∗ *a*) (*k* ∗ *b*)
by (*simp add*: *lcm-mult-distrib′*)
then have *normalize k* ∗ *lcm a b* ∗ *unit-factor k* = *lcm* (*k* ∗ *a*) (*k* ∗ *b*) ∗ *unit-factor k*
by *simp*
then have *normalize k* ∗ *unit-factor k* ∗ *lcm a b* = *lcm* (*k* ∗ *a*) (*k* ∗ *b*) ∗ *unit-factor k*
by (*simp only*: *ac-simps*)
then show *?thesis*
by *simp*
qed

lemma *coprime-crossproduct′*:
fixes *a b c d*
assumes *b* ≠ *0*
assumes *unit-factors*: *unit-factor b* = *unit-factor d*
assumes *coprime*: *coprime a b coprime c d*
shows *a* ∗ *d* = *b* ∗ *c* ⟷ *a* = *c* ∧ *b* = *d*
proof *safe*
assume *eq*: *a* ∗ *d* = *b* ∗ *c*
hence *normalize a* ∗ *normalize d* = *normalize c* ∗ *normalize b*
by (*simp only*: *normalize-mult* [*symmetric*] *mult-ac*)
with *coprime* **have** *normalize b* = *normalize d*
by (*subst* (*asm*) *coprime-crossproduct*) *simp-all*
from *this* **and** *unit-factors* **show** *b* = *d*
by (*rule normalize-unit-factor-eqI*)
from *eq* **have** *a* ∗ *d* = *c* ∗ *d* **by** (*simp only*: ‹*b* = *d*› *mult-ac*)
with ‹*b* ≠ *0*› ‹*b* = *d*› **show** *a* = *c* **by** *simp*
qed (*simp-all add*: *mult-ac*)

lemma *gcd-exp* [*simp*]:
  gcd (*a* ^ *n*) (*b* ^ *n*) = *gcd a b* ^ *n*
  using *gcd-exp-weak*[*of a n b*] **by** (*simp add*: *normalize-power*)

end

87.6 GCD and LCM on *nat* and *int*

instantiation *nat* :: *gcd*
begin

fun *gcd-nat* :: *nat* ⇒ *nat* ⇒ *nat*
  where *gcd-nat x y* = (*if y* = *0 then x else gcd y* (*x mod y*))

definition *lcm-nat* :: *nat* ⇒ *nat* ⇒ *nat*
  where *lcm-nat x y* = *x* ∗ *y div* (*gcd x y*)

instance ..

end

instantiation *int* :: *gcd*
begin

definition *gcd-int* :: *int* ⇒ *int* ⇒ *int*
  where *gcd-int x y* = *int* (*gcd* (*nat* |*x*|) (*nat* |*y*|))

definition *lcm-int* :: *int* ⇒ *int* ⇒ *int*
  where *lcm-int x y* = *int* (*lcm* (*nat* |*x*|) (*nat* |*y*|))

instance ..

end

lemma *gcd-int-int-eq* [*simp*]:
  gcd (*int m*) (*int n*) = *int* (*gcd m n*)
  by (*simp add*: *gcd-int-def*)

lemma *gcd-nat-abs-left-eq* [*simp*]:
  gcd (*nat* |*k*|) *n* = *nat* (*gcd k* (*int n*))
  by (*simp add*: *gcd-int-def*)

lemma *gcd-nat-abs-right-eq* [*simp*]:
  gcd n (*nat* |*k*|) = *nat* (*gcd* (*int n*) *k*)
  by (*simp add*: *gcd-int-def*)

lemma *abs-gcd-int* [*simp*]:
  |*gcd x y*| = *gcd x y*
  for *x y* :: *int*

by (*simp only*: *gcd-int-def*)

lemma *gcd-abs1-int* [*simp*]:
gcd $|x|$ *y* = *gcd x y*
for *x y* :: *int*
by (*simp only*: *gcd-int-def*) *simp*

lemma *gcd-abs2-int* [*simp*]:
gcd x $|y|$ = *gcd x y*
for *x y* :: *int*
by (*simp only*: *gcd-int-def*) *simp*

lemma *lcm-int-int-eq* [*simp*]:
lcm (*int m*) (*int n*) = *int* (*lcm m n*)
by (*simp add*: *lcm-int-def*)

lemma *lcm-nat-abs-left-eq* [*simp*]:
lcm (*nat* $|k|$) *n* = *nat* (*lcm k* (*int n*))
by (*simp add*: *lcm-int-def*)

lemma *lcm-nat-abs-right-eq* [*simp*]:
lcm n (*nat* $|k|$) = *nat* (*lcm* (*int n*) *k*)
by (*simp add*: *lcm-int-def*)

lemma *lcm-abs1-int* [*simp*]:
lcm $|x|$ *y* = *lcm x y*
for *x y* :: *int*
by (*simp only*: *lcm-int-def*) *simp*

lemma *lcm-abs2-int* [*simp*]:
lcm x $|y|$ = *lcm x y*
for *x y* :: *int*
by (*simp only*: *lcm-int-def*) *simp*

lemma *abs-lcm-int* [*simp*]: $|$*lcm i j*$|$ = *lcm i j*
for *i j* :: *int*
by (*simp only*: *lcm-int-def*)

lemma *gcd-nat-induct* [*case-names base step*]:
fixes *m n* :: *nat*
assumes $\bigwedge m.\ P\ m\ 0$
and $\bigwedge m\ n.\ 0 < n \Longrightarrow P\ n\ (m\ mod\ n) \Longrightarrow P\ m\ n$
shows *P m n*
proof (*induction m n rule*: *gcd-nat.induct*)
case (*1 x y*)
then show *?case*
using *assms neq0-conv* **by** *blast*
qed

lemma *gcd-neg1-int* [*simp*]: *gcd* (− *x*) *y* = *gcd x y*
  for *x y* :: *int*
  by (*simp only*: *gcd-int-def*) *simp*

lemma *gcd-neg2-int* [*simp*]: *gcd x* (− *y*) = *gcd x y*
  for *x y* :: *int*
  by (*simp only*: *gcd-int-def*) *simp*

lemma *gcd-cases-int*:
  fixes *x y* :: *int*
  assumes $x \geq 0 \Longrightarrow y \geq 0 \Longrightarrow P\ (gcd\ x\ y)$
    and $x \geq 0 \Longrightarrow y \leq 0 \Longrightarrow P\ (gcd\ x\ (-\ y))$
    and $x \leq 0 \Longrightarrow y \geq 0 \Longrightarrow P\ (gcd\ (-\ x)\ y)$
    and $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow P\ (gcd\ (-\ x)\ (-\ y))$
  shows *P* (*gcd x y*)
  using *assms* **by** *auto arith*

lemma *gcd-ge-0-int* [*simp*]: *gcd* (*x*::*int*) *y* >= *0*
  for *x y* :: *int*
  by (*simp add*: *gcd-int-def*)

lemma *lcm-neg1-int*: *lcm* (− *x*) *y* = *lcm x y*
  for *x y* :: *int*
  by (*simp only*: *lcm-int-def*) *simp*

lemma *lcm-neg2-int*: *lcm x* (− *y*) = *lcm x y*
  for *x y* :: *int*
  by (*simp only*: *lcm-int-def*) *simp*

lemma *lcm-cases-int*:
  fixes *x y* :: *int*
  assumes $x \geq 0 \Longrightarrow y \geq 0 \Longrightarrow P\ (lcm\ x\ y)$
    and $x \geq 0 \Longrightarrow y \leq 0 \Longrightarrow P\ (lcm\ x\ (-\ y))$
    and $x \leq 0 \Longrightarrow y \geq 0 \Longrightarrow P\ (lcm\ (-\ x)\ y)$
    and $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow P\ (lcm\ (-\ x)\ (-\ y))$
  shows *P* (*lcm x y*)
  using *assms* **by** (*auto simp*: *lcm-neg1-int lcm-neg2-int*) *arith*

lemma *lcm-ge-0-int* [*simp*]: $lcm\ x\ y \geq 0$
  for *x y* :: *int*
  by (*simp only*: *lcm-int-def*)

lemma *gcd-0-nat*: *gcd x 0* = *x*
  for *x* :: *nat*
  by *simp*

lemma *gcd-0-int* [*simp*]: *gcd x 0* = $|x|$
  for *x* :: *int*
  by (*auto simp*: *gcd-int-def*)

lemma *gcd-0-left-nat*: *gcd 0 x = x*
for *x* :: *nat*
by *simp*

lemma *gcd-0-left-int* [*simp*]: *gcd 0 x* = |*x*|
for *x* :: *int*
by (*auto simp*: *gcd-int-def*)

lemma *gcd-red-nat*: *gcd x y = gcd y* (*x mod y*)
for *x y* :: *nat*
by (*cases y = 0*) *auto*

Weaker, but useful for the simplifier.

lemma *gcd-non-0-nat*: *y* ≠ *0* ⟹ *gcd x y = gcd y* (*x mod y*)
for *x y* :: *nat*
by *simp*

lemma *gcd-1-nat* [*simp*]: *gcd m 1 = 1*
for *m* :: *nat*
by *simp*

lemma *gcd-Suc-0* [*simp*]: *gcd m* (*Suc 0*) = *Suc 0*
for *m* :: *nat*
by *simp*

lemma *gcd-1-int* [*simp*]: *gcd m 1 = 1*
for *m* :: *int*
by (*simp add*: *gcd-int-def*)

lemma *gcd-idem-nat*: *gcd x x = x*
for *x* :: *nat*
by *simp*

lemma *gcd-idem-int*: *gcd x x* = |*x*|
for *x* :: *int*
by (*auto simp*: *gcd-int-def*)

declare *gcd-nat.simps* [*simp del*]

gcd m n divides *m* and *n*. The conjunctions don't seem provable separately.

instance *nat* :: *semiring-gcd*
proof
fix *m n* :: *nat*
show *gcd m n dvd m* **and** *gcd m n dvd n*
proof (*induct m n rule*: *gcd-nat-induct*)
case (*step m n*)
then have *gcd n* (*m mod n*) *dvd m*
by (*metis dvd-mod-imp-dvd*)

with *step* **show** *gcd m n dvd m*
by (*simp add*: *gcd-non-0-nat*)
qed (*simp-all add*: *gcd-0-nat gcd-non-0-nat*)
next
fix *m n k* :: *nat*
assume *k dvd m* **and** *k dvd n*
then show *k dvd gcd m n*
by (*induct m n rule*: *gcd-nat-induct*) (*simp-all add*: *gcd-non-0-nat dvd-mod gcd-0-nat*)
qed (*simp-all add*: *lcm-nat-def*)

instance *int* :: *ring-gcd*
proof
fix *k l r* :: *int*
show [*simp*]: *gcd k l dvd k gcd k l dvd l*
using *gcd-dvd1* [*of nat* |*k*| *nat* |*l*|]
gcd-dvd2 [*of nat* |*k*| *nat* |*l*|]
by *simp-all*
show *lcm k l* = *normalize* (*k* ∗ *l div gcd k l*)
using *lcm-gcd* [*of nat* |*k*| *nat* |*l*|]
by (*simp add*: *nat-eq-iff of-nat-div abs-mult abs-div*)
assume *r dvd k r dvd l*
then show *r dvd gcd k l*
using *gcd-greatest* [*of nat* |*r*| *nat* |*k*| *nat* |*l*|]
by *simp*
qed *simp*

lemma *gcd-le1-nat* [*simp*]: *a* ≠ *0* ⟹ *gcd a b* ≤ *a*
for *a b* :: *nat*
by (*rule dvd-imp-le*) *auto*

lemma *gcd-le2-nat* [*simp*]: *b* ≠ *0* ⟹ *gcd a b* ≤ *b*
for *a b* :: *nat*
by (*rule dvd-imp-le*) *auto*

lemma *gcd-le1-int* [*simp*]: *a* > *0* ⟹ *gcd a b* ≤ *a*
for *a b* :: *int*
by (*rule zdvd-imp-le*) *auto*

lemma *gcd-le2-int* [*simp*]: *b* > *0* ⟹ *gcd a b* ≤ *b*
for *a b* :: *int*
by (*rule zdvd-imp-le*) *auto*

lemma *gcd-pos-nat* [*simp*]: *gcd m n* > *0* ⟷ *m* ≠ *0* ∨ *n* ≠ *0*
for *m n* :: *nat*
using *gcd-eq-0-iff* [*of m n*] **by** *arith*

lemma *gcd-pos-int* [*simp*]: *gcd m n* > *0* ⟷ *m* ≠ *0* ∨ *n* ≠ *0*
for *m n* :: *int*

using *gcd-eq-0-iff* [*of m n*] *gcd-ge-0-int* [*of m n*] **by** *arith*

lemma *gcd-unique-nat*: d *dvd* $a \wedge d$ *dvd* $b \wedge (\forall e.\ e$ *dvd* $a \wedge e$ *dvd* $b \longrightarrow e$ *dvd* $d)$ $\longleftrightarrow d = gcd\ a\ b$
for $d\ a$ :: *nat*
using *gcd-unique* **by** *fastforce*

lemma *gcd-unique-int*:
$d \geq 0 \wedge d$ *dvd* $a \wedge d$ *dvd* $b \wedge (\forall e.\ e$ *dvd* $a \wedge e$ *dvd* $b \longrightarrow e$ *dvd* $d) \longleftrightarrow d = gcd$ $a\ b$
for $d\ a$ :: *int*
using *zdvd-antisym-nonneg* **by** *auto*

interpretation *gcd-nat*:
semilattice-neutr-order gcd 0::nat Rings.dvd $\lambda m\ n.\ m$ *dvd* $n \wedge m \neq n$
by *standard* (*auto simp*: *gcd-unique-nat* [*symmetric*] *intro*: *dvd-antisym dvd-trans*)

lemma *gcd-proj1-if-dvd-int* [*simp*]: x *dvd* $y \Longrightarrow gcd\ x\ y = |x|$
for $x\ y$ :: *int*
by (*metis abs-dvd-iff gcd-0-left-int gcd-unique-int*)

lemma *gcd-proj2-if-dvd-int* [*simp*]: y *dvd* $x \Longrightarrow gcd\ x\ y = |y|$
for $x\ y$ :: *int*
by (*metis gcd-proj1-if-dvd-int gcd.commute*)

Multiplication laws.

lemma *gcd-mult-distrib-nat*: $k * gcd\ m\ n = gcd\ (k * m)\ (k * n)$
for $k\ m\ n$ :: *nat*
— [1, page 27]
by (*simp add*: *gcd-mult-left*)

lemma *gcd-mult-distrib-int*: $|k| * gcd\ m\ n = gcd\ (k * m)\ (k * n)$
for $k\ m\ n$ :: *int*
by (*simp add*: *gcd-mult-left abs-mult*)

Addition laws.

lemma *gcd-diff1-nat*: $m \geq n \Longrightarrow gcd\ (m - n)\ n = gcd\ m\ n$
for $m\ n$ :: *nat*
by (*subst gcd-add1* [*symmetric*]) *auto*

lemma *gcd-diff2-nat*: $n \geq m \Longrightarrow gcd\ (n - m)\ n = gcd\ m\ n$
for $m\ n$ :: *nat*
by (*metis gcd.commute gcd-add2 gcd-diff1-nat le-add-diff-inverse2*)

lemma *gcd-non-0-int*:
fixes $x\ y$ :: *int*
assumes $y > 0$ **shows** $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$
proof (*cases* $x\ mod\ y = 0$)

case *False*
then have *neg*: *x mod y* = *y* − (− *x*) *mod y*
by (*simp add*: *zmod-zminus1-eq-if*)
have *xy*: *0* ≤ *x mod y*
by (*simp add*: *assms*)
show *?thesis*
proof (*cases x* < *0*)
case *True*
have *nat* (− *x mod y*) ≤ *nat y*
by (*simp add*: *assms dual-order.order-iff-strict*)
moreover have *gcd* (*nat* (− *x*)) (*nat y*) = *gcd* (*nat* (− *x mod y*)) (*nat y*)
using *True assms gcd-non-0-nat nat-mod-distrib* **by** *auto*
ultimately have *gcd* (*nat* (− *x*)) (*nat y*) = *gcd* (*nat y*) (*nat* (*x mod y*))
using *assms*
by (*simp add*: *neg nat-diff-distrib′*) (*metis gcd.commute gcd-diff2-nat*)
with *assms* ‹*0* ≤ *x mod y*› **show** *?thesis*
by (*simp add*: *True dual-order.order-iff-strict gcd-int-def*)
next
case *False*
with *assms xy* **have** *gcd* (*nat x*) (*nat y*) = *gcd* (*nat y*) (*nat x mod nat y*)
using *gcd-red-nat* **by** *blast*
with *False assms* **show** *?thesis*
by (*simp add*: *gcd-int-def nat-mod-distrib*)
qed
qed (*use assms* **in** *auto*)

lemma *gcd-red-int*: *gcd x y* = *gcd y* (*x mod y*)
for *x y* :: *int*
proof (*cases y 0*::*int rule*: *linorder-cases*)
case *less*
with *gcd-non-0-int* [*of* − *y* − *x*] **show** *?thesis*
by *auto*
next
case *greater*
with *gcd-non-0-int* [*of y x*] **show** *?thesis*
by *auto*
qed *auto*

lemma *finite-divisors-nat* [*simp*]:
fixes *m* :: *nat*
assumes *m* > *0*
shows *finite* {*d*. *d dvd m*}
proof−
from *assms* **have** {*d*. *d dvd m*} ⊆ {*d*. *d* ≤ *m*}
by (*auto dest*: *dvd-imp-le*)

then show *?thesis*
using *finite-Collect-le-nat* **by** (*rule finite-subset*)
qed

lemma *finite-divisors-int* [*simp*]:
fixes *i* :: *int*
assumes $i \neq 0$
shows *finite* {*d. d dvd i*}
proof –
have $\{d.\ |d| \leq |i|\} = \{-\ |i|..|i|\}$
by (*auto simp*: *abs-if*)
then have *finite* $\{d.\ |d| \leq |i|\}$
by *simp*
from *finite-subset* [*OF - this*] **show** *?thesis*
using *assms* **by** (*simp add*: *dvd-imp-le-int subset-iff*)
qed

lemma *Max-divisors-self-nat* [*simp*]: $n \neq 0 \Longrightarrow$ *Max* {*d*::*nat. d dvd n*} = *n*
by (*fastforce intro*: *antisym Max-le-iff*[*THEN iffD2*] *simp*: *dvd-imp-le*)

lemma *Max-divisors-self-int* [*simp*]:
assumes $n \neq 0$ **shows** *Max* {*d*::*int. d dvd n*} = $|n|$
proof (*rule antisym*)
show *Max* {*d. d dvd n*} $\leq |n|$
using *assms* **by** (*auto intro*: *abs-le-D1 dvd-imp-le-int intro*!: *Max-le-iff* [*THEN iffD2*])
qed (*simp add*: *assms*)

lemma *gcd-is-Max-divisors-nat*:
fixes *m n* :: *nat*
assumes $n > 0$ **shows** *gcd m n* = *Max* {*d. d dvd m* ∧ *d dvd n*}
proof (*rule Max-eqI*[*THEN sym*], *simp-all*)
show *finite* {*d. d dvd m* ∧ *d dvd n*}
by (*simp add*: ‹$n > 0$›)
show ⋀*y. y dvd m* ∧ *y dvd n* ⟹ *y* ≤ *gcd m n*
by (*simp add*: ‹$n > 0$› *dvd-imp-le*)
qed

lemma *gcd-is-Max-divisors-int*:
fixes *m n* :: *int*
assumes $n \neq 0$ **shows** *gcd m n* = *Max* {*d. d dvd m* ∧ *d dvd n*}
proof (*rule Max-eqI*[*THEN sym*], *simp-all*)
show *finite* {*d. d dvd m* ∧ *d dvd n*}
by (*simp add*: ‹$n \neq 0$›)
show ⋀*y. y dvd m* ∧ *y dvd n* ⟹ *y* ≤ *gcd m n*
by (*simp add*: ‹$n \neq 0$› *zdvd-imp-le*)
qed

lemma *gcd-code-int* [*code*]: *gcd k l* = |*if l* = *0 then k else gcd l* (|*k*| *mod* |*l*|)|

for *k l* :: *int*
using *gcd-red-int* [*of* |*k*| |*l*|] **by** *simp*

lemma *coprime-Suc-left-nat* [*simp*]:
coprime (*Suc n*) *n*
using *coprime-add-one-left* [*of n*] **by** *simp*

lemma *coprime-Suc-right-nat* [*simp*]:
coprime n (*Suc n*)
using *coprime-Suc-left-nat* [*of n*] **by** (*simp add*: *ac-simps*)

lemma *coprime-diff-one-left-nat* [*simp*]:
coprime (*n* − *1*) *n* **if** *n* > *0* **for** *n* :: *nat*
using *that coprime-Suc-right-nat* [*of n* − *1*] **by** *simp*

lemma *coprime-diff-one-right-nat* [*simp*]:
coprime n (*n* − *1*) **if** *n* > *0* **for** *n* :: *nat*
using *that coprime-diff-one-left-nat* [*of n*] **by** (*simp add*: *ac-simps*)

lemma *coprime-crossproduct-nat*:
fixes *a b c d* :: *nat*
assumes *coprime a d* **and** *coprime b c*
shows $a * c = b * d \longleftrightarrow a = b \wedge c = d$
using *assms coprime-crossproduct* [*of a d b c*] **by** *simp*

lemma *coprime-crossproduct-int*:
fixes *a b c d* :: *int*
assumes *coprime a d* **and** *coprime b c*
shows $|a| * |c| = |b| * |d| \longleftrightarrow |a| = |b| \wedge |c| = |d|$
using *assms coprime-crossproduct* [*of a d b c*] **by** *simp*

87.7 Bezout's theorem

Function *bezw* returns a pair of witnesses to Bezout's theorem – see the theorems that follow the definition.

fun *bezw* :: *nat* ⇒ *nat* ⇒ *int* ∗ *int*
where *bezw x y* =
(*if y* = *0 then* (*1*, *0*)
else
(*snd* (*bezw y* (*x mod y*)),
fst (*bezw y* (*x mod y*)) − *snd* (*bezw y* (*x mod y*)) ∗ *int*(*x div y*)))

lemma *bezw-0* [*simp*]: *bezw x 0* = (*1*, *0*)
by *simp*

lemma *bezw-non-0*:
y > *0* ⟹ *bezw x y* =
(*snd* (*bezw y* (*x mod y*)), *fst* (*bezw y* (*x mod y*)) − *snd* (*bezw y* (*x mod y*)) ∗ *int*(*x div y*))

by *simp*

declare *bezw.simps* [*simp del*]

lemma *bezw-aux*: *int* (*gcd x y*) = *fst* (*bezw x y*) ∗ *int x* + *snd* (*bezw x y*) ∗ *int y*
proof (*induct x y rule*: *gcd-nat-induct*)
  case (*step m n*)
  then have *fst* (*bezw m n*) ∗ *int m* + *snd* (*bezw m n*) ∗ *int n* − *int* (*gcd m n*)
          = *int m* ∗ *snd* (*bezw n* (*m mod n*)) −
            (*int* (*m mod n*) ∗ *snd* (*bezw n* (*m mod n*)) + *int n* ∗ (*int* (*m div n*) ∗ *snd* (*bezw n* (*m mod n*))))
    by (*simp add*: *bezw-non-0 gcd-non-0-nat field-simps*)
  also have ... = *int m* ∗ *snd* (*bezw n* (*m mod n*)) − (*int* (*m mod n*) + *int* (*n* ∗ (*m div n*))) ∗ *snd* (*bezw n* (*m mod n*))
    by (*simp add*: *distrib-right*)
  also have ... = *0*
   by (*metis cancel-comm-monoid-add-class.diff-cancel mod-mult-div-eq of-nat-add*)
  finally show *?case*
    by *simp*
qed *auto*

lemma *bezout-int*: ∃ *u v*. *u* ∗ *x* + *v* ∗ *y* = *gcd x y*
  for *x y* :: *int*
proof −
  have *aux*: *x* ≥ *0* ⟹ *y* ≥ *0* ⟹ ∃ *u v*. *u* ∗ *x* + *v* ∗ *y* = *gcd x y* **for** *x y* :: *int*
    apply (*rule-tac x* = *fst* (*bezw* (*nat x*) (*nat y*)) **in** *exI*)
    apply (*rule-tac x* = *snd* (*bezw* (*nat x*) (*nat y*)) **in** *exI*)
    by (*simp add*: *bezw-aux gcd-int-def*)
  consider *x* ≥ *0 y* ≥ *0* | *x* ≥ *0 y* ≤ *0* | *x* ≤ *0 y* ≥ *0* | *x* ≤ *0 y* ≤ *0*
    using *linear* **by** *blast*
  then show *?thesis*
  proof *cases*
    case *1*
    then show *?thesis* **by** (*rule aux*)
  next
    case *2*
    then show *?thesis*
      using *aux* [*of x* −*y*]
      by (*metis gcd-neg2-int mult.commute mult-minus-right neg-0-le-iff-le*)
  next
    case *3*
    then show *?thesis*
      using *aux* [*of* −*x y*]
     by (*metis gcd.commute gcd-neg2-int mult.commute mult-minus-right neg-0-le-iff-le*)
  next
    case *4*
    then show *?thesis*

```
      using aux [of −x −y]
    by (metis diff-0 diff-ge-0-iff-ge gcd-neg1-int gcd-neg2-int mult.commute mult-minus-right)
  qed
qed
```

Versions of Bezout for *nat*, by Amine Chaieb.

```
lemma Euclid-induct [case-names swap zero add]:
  fixes P :: nat ⇒ nat ⇒ bool
  assumes c: ⋀a b. P a b ⟷ P b a
    and z: ⋀a. P a 0
    and add: ⋀a b. P a b ⟶ P a (a + b)
  shows P a b
proof (induct a + b arbitrary: a b rule: less-induct)
  case less
  consider (eq) a = b | (lt) a < b a + b − a < a + b | b = 0 | b + a − b < a + b
    by arith
  show ?case
  proof (cases a b rule: linorder-cases)
    case equal
    with add [rule-format, OF z [rule-format, of a]] show ?thesis by simp
  next
    case lt: less
    then consider a = 0 | a + b − a < a + b by arith
    then show ?thesis
    proof cases
      case 1
      with z c show ?thesis by blast
    next
      case 2
      also have ∗: a + b − a = a + (b − a) using lt by arith
      finally have a + (b − a) < a + b .
      then have P a (a + (b − a)) by (rule add [rule-format, OF less])
      then show ?thesis by (simp add: ∗[symmetric])
    qed
  next
    case gt: greater
    then consider b = 0 | b + a − b < a + b by arith
    then show ?thesis
    proof cases
      case 1
      with z c show ?thesis by blast
    next
      case 2
      also have ∗: b + a − b = b + (a − b) using gt by arith
      finally have b + (a − b) < a + b .
      then have P b (b + (a − b)) by (rule add [rule-format, OF less])
      then have P b a by (simp add: ∗[symmetric])
      with c show ?thesis by blast
    qed
```

  qed
qed

lemma *bezout-lemma-nat*:
  fixes *d*::*nat*
  shows $[\![d\ dvd\ a;\ d\ dvd\ b;\ a * x = b * y + d \vee b * x = a * y + d]\!]$
    $\Longrightarrow \exists\, x\ y.\ d\ dvd\ a \wedge d\ dvd\ a + b \wedge (a * x = (a + b) * y + d \vee (a + b) * x = a * y + d)$
  apply *auto*
  apply (*metis add-mult-distrib2 left-add-mult-distrib*)
  apply (*rule-tac x=x* **in** *exI*)
  by (*metis add-mult-distrib2 mult.commute add.assoc*)

lemma *bezout-add-nat*:
  $\exists\,(d{::}nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x = b * y + d \vee b * x = a * y + d)$
proof (*induct a b rule*: *Euclid-induct*)
  case (*swap a b*)
  then show *?case*
    by *blast*
next
  case (*zero a*)
  then show *?case*
    by *fastforce*
next
  case (*add a b*)
  then show *?case*
    by (*meson bezout-lemma-nat*)
qed

lemma *bezout1-nat*: $\exists\,(d{::}nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x - b * y = d \vee b * x - a * y = d)$
  using *bezout-add-nat*[*of a b*] **by** (*metis add-diff-cancel-left'*)

lemma *bezout-add-strong-nat*:
  fixes *a b* :: *nat*
  assumes *a*: $a \neq 0$
  shows $\exists\, d\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge a * x = b * y + d$
proof −
  consider *d x y* **where** $d\ dvd\ a\ d\ dvd\ b\ a * x = b * y + d$
        | *d x y* **where** $d\ dvd\ a\ d\ dvd\ b\ b * x = a * y + d$
    using *bezout-add-nat* [*of a b*] **by** *blast*
  then show *?thesis*
  proof *cases*
    case *1*
    then show *?thesis* **by** *blast*
  next
    case *H*: *2*
    show *?thesis*
    proof (*cases* $b = 0$)

```
      case True
      with H show ?thesis by simp
    next
      case False
      then have bp: b > 0 by simp
      with dvd-imp-le [OF H(2)] consider d = b | d < b
        by atomize-elim auto
      then show ?thesis
      proof cases
        case 1
        with a H show ?thesis
          by (metis Suc-pred add.commute mult.commute mult-Suc-right neq0-conv)
      next
        case 2
        show ?thesis
        proof (cases x = 0)
          case True
          with a H show ?thesis by simp
        next
          case x0: False
          then have xp: x > 0 by simp
          from ‹d < b› have d ≤ b − 1 by simp
          then have d ∗ b ≤ b ∗ (b − 1) by simp
          with xp mult-mono[of 1 x d ∗ b b ∗ (b − 1)]
          have dble: d ∗ b ≤ x ∗ b ∗ (b − 1) using bp by simp
          from H(3) have d + (b − 1) ∗ (b ∗ x) = d + (b − 1) ∗ (a ∗ y + d)
            by simp
          then have d + (b − 1) ∗ a ∗ y + (b − 1) ∗ d = d + (b − 1) ∗ b ∗ x
            by (simp only: mult.assoc distrib-left)
          then have a ∗ ((b − 1) ∗ y) + d ∗ (b − 1 + 1) = d + x ∗ b ∗ (b − 1)
            by algebra
          then have a ∗ ((b − 1) ∗ y) = d + x ∗ b ∗ (b − 1) − d ∗ b
            using bp by simp
          then have a ∗ ((b − 1) ∗ y) = d + (x ∗ b ∗ (b − 1) − d ∗ b)
            by (simp only: diff-add-assoc[OF dble, of d, symmetric])
          then have a ∗ ((b − 1) ∗ y) = b ∗ (x ∗ (b − 1) − d) + d
            by (simp only: diff-mult-distrib2 ac-simps)
          with H(1,2) show ?thesis
            by blast
        qed
      qed
    qed
  qed
qed

lemma bezout-nat:
  fixes a :: nat
  assumes a: a ≠ 0
  shows ∃x y. a ∗ x = b ∗ y + gcd a b
```

proof −
obtain *d x y* **where** *d*: *d dvd a d dvd b* **and** *eq*: $a * x = b * y + d$
using *bezout-add-strong-nat* [*OF a*, *of b*] **by** *blast*
from *d* **have** *d dvd gcd a b*
by *simp*
then obtain *k* **where** *k*: $gcd\ a\ b = d * k$
unfolding *dvd-def* **by** *blast*
from *eq* **have** $a * x * k = (b * y + d) * k$
by *auto*
then have $a * (x * k) = b * (y * k) + gcd\ a\ b$
by (*algebra add*: *k*)
then show *?thesis*
by *blast*
qed

87.8 LCM properties on *nat* and *int*

lemma *lcm-altdef-int* [*code*]: $lcm\ a\ b = |a| * |b|\ div\ gcd\ a\ b$
for *a b* :: *int*
by (*simp add*: *abs-mult lcm-gcd abs-div*)

lemma *prod-gcd-lcm-nat*: $m * n = gcd\ m\ n * lcm\ m\ n$
for *m n* :: *nat*
by (*simp add*: *lcm-gcd*)

lemma *prod-gcd-lcm-int*: $|m| * |n| = gcd\ m\ n * lcm\ m\ n$
for *m n* :: *int*
by (*simp add*: *lcm-gcd abs-div abs-mult*)

lemma *lcm-pos-nat*: $m > 0 \Longrightarrow n > 0 \Longrightarrow lcm\ m\ n > 0$
for *m n* :: *nat*
using *lcm-eq-0-iff* [*of m n*] **by** *auto*

lemma *lcm-pos-int*: $m \neq 0 \Longrightarrow n \neq 0 \Longrightarrow lcm\ m\ n > 0$
for *m n* :: *int*
by (*simp add*: *less-le lcm-eq-0-iff*)

lemma *dvd-pos-nat*: $n > 0 \Longrightarrow m\ dvd\ n \Longrightarrow m > 0$
for *m n* :: *nat*
by *auto*

lemma *lcm-unique-nat*:
$a\ dvd\ d \wedge b\ dvd\ d \wedge (\forall e.\ a\ dvd\ e \wedge b\ dvd\ e \longrightarrow d\ dvd\ e) \longleftrightarrow d = lcm\ a\ b$
for *a b d* :: *nat*
by (*auto intro*: *dvd-antisym lcm-least*)

lemma *lcm-unique-int*:
$d \geq 0 \wedge a\ dvd\ d \wedge b\ dvd\ d \wedge (\forall e.\ a\ dvd\ e \wedge b\ dvd\ e \longrightarrow d\ dvd\ e) \longleftrightarrow d = lcm\ a\ b$

for *a b d* :: *int*
using *lcm-least zdvd-antisym-nonneg* **by** *auto*

lemma *lcm-proj2-if-dvd-nat* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ x\ y = y$
for *x y* :: *nat*
by (*simp add*: *lcm-proj2-if-dvd*)

lemma *lcm-proj2-if-dvd-int* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ x\ y = |y|$
for *x y* :: *int*
by (*simp add*: *lcm-proj2-if-dvd*)

lemma *lcm-proj1-if-dvd-nat* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ y\ x = y$
for *x y* :: *nat*
by (*subst lcm.commute*) (*erule lcm-proj2-if-dvd-nat*)

lemma *lcm-proj1-if-dvd-int* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ y\ x = |y|$
for *x y* :: *int*
by (*subst lcm.commute*) (*erule lcm-proj2-if-dvd-int*)

lemma *lcm-proj1-iff-nat* [*simp*]: $lcm\ m\ n = m \longleftrightarrow n\ dvd\ m$
for *m n* :: *nat*
by (*metis lcm-proj1-if-dvd-nat lcm-unique-nat*)

lemma *lcm-proj2-iff-nat* [*simp*]: $lcm\ m\ n = n \longleftrightarrow m\ dvd\ n$
for *m n* :: *nat*
by (*metis lcm-proj2-if-dvd-nat lcm-unique-nat*)

lemma *lcm-proj1-iff-int* [*simp*]: $lcm\ m\ n = |m| \longleftrightarrow n\ dvd\ m$
for *m n* :: *int*
by (*metis dvd-abs-iff lcm-proj1-if-dvd-int lcm-unique-int*)

lemma *lcm-proj2-iff-int* [*simp*]: $lcm\ m\ n = |n| \longleftrightarrow m\ dvd\ n$
for *m n* :: *int*
by (*metis dvd-abs-iff lcm-proj2-if-dvd-int lcm-unique-int*)

lemma *lcm-1-iff-nat* [*simp*]: $lcm\ m\ n = Suc\ 0 \longleftrightarrow m = Suc\ 0 \wedge n = Suc\ 0$
for *m n* :: *nat*
using *lcm-eq-1-iff* [*of m n*] **by** *simp*

lemma *lcm-1-iff-int* [*simp*]: $lcm\ m\ n = 1 \longleftrightarrow (m = 1 \vee m = -1) \wedge (n = 1 \vee n = -1)$
for *m n* :: *int*
by *auto*

87.9 The complete divisibility lattice on *nat* and *int*

Lifting *gcd* and *lcm* to sets (*Gcd* / *Lcm*). *Gcd* is defined via *Lcm* to facilitate the proof that we have a complete lattice.

instantiation *nat* :: *semiring-Gcd*

begin

interpretation *semilattice-neutr-set lcm 1::nat*
  by *standard simp-all*

definition *Lcm M* = (*if finite M then F M else 0*) **for** *M* :: *nat set*

lemma *Lcm-nat-empty*: *Lcm* {} = (*1::nat*)
  by (*simp add*: *Lcm-nat-def del*: *One-nat-def*)

lemma *Lcm-nat-insert*: *Lcm* (*insert n M*) = *lcm n* (*Lcm M*) **for** *n* :: *nat*
  by (*cases finite M*) (*auto simp*: *Lcm-nat-def simp del*: *One-nat-def*)

lemma *Lcm-nat-infinite*: *infinite M* ⟹ *Lcm M* = *0* **for** *M* :: *nat set*
  by (*simp add*: *Lcm-nat-def*)

lemma *dvd-Lcm-nat* [*simp*]:
  fixes *M* :: *nat set*
  assumes *m* ∈ *M*
  shows *m dvd Lcm M*
proof −
  from *assms* **have** *insert m M* = *M*
    by *auto*
  moreover have *m dvd Lcm* (*insert m M*)
    by (*simp add*: *Lcm-nat-insert*)
  ultimately show *?thesis*
    by *simp*
qed

lemma *Lcm-dvd-nat* [*simp*]:
  fixes *M* :: *nat set*
  assumes ∀ *m*∈*M*. *m dvd n*
  shows *Lcm M dvd n*
proof (*cases n* > *0*)
  case *False*
  then show *?thesis* **by** *simp*
next
  case *True*
  then have *finite* {*d*. *d dvd n*}
    by (*rule finite-divisors-nat*)
  moreover have *M* ⊆ {*d*. *d dvd n*}
    using *assms* **by** *fast*
  ultimately have *finite M*
    by (*rule rev-finite-subset*)
  then show *?thesis*
    using *assms* **by** (*induct M*) (*simp-all add*: *Lcm-nat-empty Lcm-nat-insert*)
qed

definition *Gcd M* = *Lcm* {*d*. ∀ *m*∈*M*. *d dvd m*} **for** *M* :: *nat set*

```
instance
proof
  fix N :: nat set
  fix n :: nat
  show Gcd N dvd n if n ∈ N
    using that by (induct N rule: infinite-finite-induct) (auto simp: Gcd-nat-def)
  show n dvd Gcd N if ⋀m. m ∈ N ⟹ n dvd m
    using that by (induct N rule: infinite-finite-induct) (auto simp: Gcd-nat-def)
  show n dvd Lcm N if n ∈ N
    using that by (induct N rule: infinite-finite-induct) auto
  show Lcm N dvd n if ⋀m. m ∈ N ⟹ m dvd n
    using that by (induct N rule: infinite-finite-induct) auto
  show normalize (Gcd N) = Gcd N and normalize (Lcm N) = Lcm N
    by simp-all
qed

end

lemma Gcd-nat-eq-one: 1 ∈ N ⟹ Gcd N = 1
  for N :: nat set
  by (rule Gcd-eq-1-I) auto

instance nat :: semiring-gcd-mult-normalize
  by intro-classes (auto simp: unit-factor-nat-def)
```

Alternative characterizations of Gcd:

```
lemma Gcd-eq-Max:
  fixes M :: nat set
  assumes finite (M::nat set) and M ≠ {} and 0 ∉ M
  shows Gcd M = Max (⋂m∈M. {d. d dvd m})
proof (rule antisym)
  from assms obtain m where m ∈ M and m > 0
    by auto
  from ‹m > 0› have finite {d. d dvd m}
    by (blast intro: finite-divisors-nat)
  with ‹m ∈ M› have fin: finite (⋂m∈M. {d. d dvd m})
    by blast
  from fin show Gcd M ≤ Max (⋂m∈M. {d. d dvd m})
    by (auto intro: Max-ge Gcd-dvd)
  from fin show Max (⋂m∈M. {d. d dvd m}) ≤ Gcd M
  proof (rule Max.boundedI, simp-all)
    show (⋂m∈M. {d. d dvd m}) ≠ {}
      by auto
    show ⋀a. ∀x∈M. a dvd x ⟹ a ≤ Gcd M
      by (meson Gcd-dvd Gcd-greatest ‹0 < m› ‹m ∈ M› dvd-imp-le dvd-pos-nat)
  qed
qed
```

lemma *Gcd-remove0-nat*: *finite M* ⟹ *Gcd M* = *Gcd* (*M* − {*0*})
  for *M* :: *nat set*
proof (*induct pred*: *finite*)
  case (*insert x M*)
  then show *?case*
    by (*simp add*: *insert-Diff-if*)
qed *auto*

lemma *Lcm-in-lcm-closed-set-nat*:
  fixes *M* :: *nat set*
  assumes *finite M M* ≠ {} ⋀*m n*. ⟦*m* ∈ *M*; *n* ∈ *M*⟧ ⟹ *lcm m n* ∈ *M*
  shows *Lcm M* ∈ *M*
  using *assms*
proof (*induction M rule*: *finite-linorder-min-induct*)
  case (*insert x M*)
  then have ⋀*m n*. *m* ∈ *M* ⟹ *n* ∈ *M* ⟹ *lcm m n* ∈ *M*
    by (*metis dvd-lcm1 gr0I insert-iff lcm-pos-nat nat-dvd-not-less*)
  with *insert* **show** *?case*
    by *simp* (*metis Lcm-nat-empty One-nat-def dvd-1-left dvd-lcm2*)
qed *auto*

lemma *Lcm-eq-Max-nat*:
  fixes *M* :: *nat set*
  assumes *M*: *finite M M* ≠ {} *0* ∉ *M* **and** *lcm*: ⋀*m n*. ⟦*m* ∈ *M*; *n* ∈ *M*⟧ ⟹
lcm m n ∈ *M*
  shows *Lcm M* = *Max M*
proof (*rule antisym*)
  show *Lcm M* ≤ *Max M*
    by (*simp add*: *Lcm-in-lcm-closed-set-nat* ‹*finite M*› ‹*M* ≠ {}› *lcm*)
  show *Max M* ≤ *Lcm M*
    by (*meson Lcm-0-iff Max-in M dvd-Lcm dvd-imp-le le-0-eq not-le*)
qed

lemma *mult-inj-if-coprime-nat*:
  inj-on f A ⟹ *inj-on g B* ⟹ (⋀*a b*. ⟦*a*∈*A*; *b*∈*B*⟧ ⟹ *coprime* (*f a*) (*g b*)) ⟹
    inj-on (λ(*a*, *b*). *f a* ∗ *g b*) (*A* × *B*)
  for *f* :: ′*a* ⇒ *nat* **and** *g* :: ′*b* ⇒ *nat*
  by (*auto simp*: *inj-on-def coprime-crossproduct-nat simp del*: *One-nat-def*)

87.9.1 Setwise GCD and LCM for integers

instantiation *int* :: *Gcd*
begin

definition *Gcd-int* :: *int set* ⇒ *int*
  where *Gcd K* = *int* (*GCD k*∈*K*. (*nat* ∘ *abs*) *k*)

definition *Lcm-int* :: *int set* ⇒ *int*
  where *Lcm K* = *int* (*LCM k*∈*K*. (*nat* ∘ *abs*) *k*)

instance ..

end

lemma *Gcd-int-eq* [*simp*]:
($GCD\ n{\in}N.\ int\ n) = int\ (Gcd\ N)$
by (*simp add*: *Gcd-int-def image-image*)

lemma *Gcd-nat-abs-eq* [*simp*]:
$(GCD\ k{\in}K.\ nat\ |k|) = nat\ (Gcd\ K)$
by (*simp add*: *Gcd-int-def*)

lemma *abs-Gcd-eq* [*simp*]:
$|Gcd\ K| = Gcd\ K$ **for** K :: *int set*
by (*simp only*: *Gcd-int-def*)

lemma *Gcd-int-greater-eq-0* [*simp*]:
$Gcd\ K \geq 0$
for K :: *int set*
using *abs-ge-zero* [*of Gcd K*] **by** *simp*

lemma *Gcd-abs-eq* [*simp*]:
$(GCD\ k{\in}K.\ |k|) = Gcd\ K$
for K :: *int set*
by (*simp only*: *Gcd-int-def image-image*) *simp*

lemma *Lcm-int-eq* [*simp*]:
$(LCM\ n{\in}N.\ int\ n) = int\ (Lcm\ N)$
by (*simp add*: *Lcm-int-def image-image*)

lemma *Lcm-nat-abs-eq* [*simp*]:
$(LCM\ k{\in}K.\ nat\ |k|) = nat\ (Lcm\ K)$
by (*simp add*: *Lcm-int-def*)

lemma *abs-Lcm-eq* [*simp*]:
$|Lcm\ K| = Lcm\ K$ **for** K :: *int set*
by (*simp only*: *Lcm-int-def*)

lemma *Lcm-int-greater-eq-0* [*simp*]:
$Lcm\ K \geq 0$
for K :: *int set*
using *abs-ge-zero* [*of Lcm K*] **by** *simp*

lemma *Lcm-abs-eq* [*simp*]:
$(LCM\ k{\in}K.\ |k|) = Lcm\ K$
for K :: *int set*
by (*simp only*: *Lcm-int-def image-image*) *simp*

instance *int* :: *semiring-Gcd*
proof
  fix *K* :: *int set* **and** *k* :: *int*
  show *Gcd K dvd k* **and** *k dvd Lcm K* **if** $k \in K$
    using *that Gcd-dvd* [*of nat* |*k*| (*nat* ∘ *abs*) ' *K*]
      dvd-Lcm [*of nat* |*k*| (*nat* ∘ *abs*) ' *K*]
    by (*simp-all add*: *comp-def*)
  show *k dvd Gcd K* **if** $\bigwedge l.\ l \in K \Longrightarrow k\ dvd\ l$
  proof −
    have *nat* |*k*| *dvd* (*GCD k*∈*K*. *nat* |*k*|)
      by (*rule Gcd-greatest*) (*use that* **in** *auto*)
    then show *?thesis* **by** *simp*
  qed
  show *Lcm K dvd k* **if** $\bigwedge l.\ l \in K \Longrightarrow l\ dvd\ k$
  proof −
    have (*LCM k*∈*K*. *nat* |*k*|) *dvd nat* |*k*|
      by (*rule Lcm-least*) (*use that* **in** *auto*)
    then show *?thesis* **by** *simp*
  qed
qed (*simp-all add*: *sgn-mult*)

instance *int* :: *semiring-gcd-mult-normalize*
  by *intro-classes* (*auto simp*: *sgn-mult*)

87.10 GCD and LCM on *integer*

instantiation *integer* :: *gcd*
begin

context
  includes *integer.lifting*
begin

lift-definition *gcd-integer* :: *integer* ⇒ *integer* ⇒ *integer* **is** *gcd* **.**

lift-definition *lcm-integer* :: *integer* ⇒ *integer* ⇒ *integer* **is** *lcm* **.**

end

instance ..

end

lifting-update *integer.lifting*
lifting-forget *integer.lifting*

context
  includes *integer.lifting*
begin

lemma *gcd-code-integer* [*code*]: *gcd k l* = |*if l* = (*0*::*integer*) *then k else gcd l* (|*k*| *mod* |*l*|)|
by *transfer* (*fact gcd-code-int*)

lemma *lcm-code-integer* [*code*]: *lcm a b* = |*a*| ∗ |*b*| *div gcd a b*
for *a b* :: *integer*
by *transfer* (*fact lcm-altdef-int*)

end

code-printing
constant *gcd* :: *integer* ⇒ - ⇀
(*OCaml*) !(*fun k l* −> *if Z.equal k Z.zero then*/ *Z.abs l else if Z.equal*/ *l Z.zero then Z.abs k else Z.gcd k l*)
and (*Haskell*) *Prelude.gcd*
and (*Scala*) -.*gcd*′((-)′)
— There is no gcd operation in the SML standard library, so no code setup for SML

Some code equations

lemmas *Gcd-nat-set-eq-fold* [*code*] = *Gcd-set-eq-fold* [**where** *?′a* = *nat*]
lemmas *Lcm-nat-set-eq-fold* [*code*] = *Lcm-set-eq-fold* [**where** *?′a* = *nat*]
lemmas *Gcd-int-set-eq-fold* [*code*] = *Gcd-set-eq-fold* [**where** *?′a* = *int*]
lemmas *Lcm-int-set-eq-fold* [*code*] = *Lcm-set-eq-fold* [**where** *?′a* = *int*]

Fact aliases.

lemma *lcm-0-iff-nat* [*simp*]: *lcm m n* = *0* ⟷ *m* = *0* ∨ *n* = *0*
for *m n* :: *nat*
by (*fact lcm-eq-0-iff*)

lemma *lcm-0-iff-int* [*simp*]: *lcm m n* = *0* ⟷ *m* = *0* ∨ *n* = *0*
for *m n* :: *int*
by (*fact lcm-eq-0-iff*)

lemma *dvd-lcm-I1-nat* [*simp*]: *k dvd m* ⟹ *k dvd lcm m n*
for *k m n* :: *nat*
by (*fact dvd-lcmI1*)

lemma *dvd-lcm-I2-nat* [*simp*]: *k dvd n* ⟹ *k dvd lcm m n*
for *k m n* :: *nat*
by (*fact dvd-lcmI2*)

lemma *dvd-lcm-I1-int* [*simp*]: *i dvd m* ⟹ *i dvd lcm m n*
for *i m n* :: *int*
by (*fact dvd-lcmI1*)

lemma *dvd-lcm-I2-int* [*simp*]: *i dvd n* ⟹ *i dvd lcm m n*
for *i m n* :: *int*

by (*fact dvd-lcmI2*)

lemmas *Gcd-dvd-nat* [*simp*] = *Gcd-dvd* [**where** *?'a* = *nat*]
lemmas *Gcd-dvd-int* [*simp*] = *Gcd-dvd* [**where** *?'a* = *int*]
lemmas *Gcd-greatest-nat* [*simp*] = *Gcd-greatest* [**where** *?'a* = *nat*]
lemmas *Gcd-greatest-int* [*simp*] = *Gcd-greatest* [**where** *?'a* = *int*]

lemma *dvd-Lcm-int* [*simp*]: $m \in M \Longrightarrow m$ *dvd Lcm M*
for *M* :: *int set*
by (*fact dvd-Lcm*)

lemma *gcd-neg-numeral-1-int* [*simp*]: *gcd* (− *numeral n* :: *int*) *x* = *gcd* (*numeral n*) *x*
by (*fact gcd-neg1-int*)

lemma *gcd-neg-numeral-2-int* [*simp*]: *gcd x* (− *numeral n* :: *int*) = *gcd x* (*numeral n*)
by (*fact gcd-neg2-int*)

lemma *gcd-proj1-if-dvd-nat* [*simp*]: *x dvd y* $\Longrightarrow$ *gcd x y* = *x*
for *x y* :: *nat*
by (*fact gcd-nat.absorb1*)

lemma *gcd-proj2-if-dvd-nat* [*simp*]: *y dvd x* $\Longrightarrow$ *gcd x y* = *y*
for *x y* :: *nat*
by (*fact gcd-nat.absorb2*)

lemma *Gcd-in*:
fixes *A* :: *nat set*
assumes $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow$ *gcd a b* $\in A$
assumes $A \neq \{\}$
shows *Gcd A* $\in A$
proof (*cases A* = {*0*})
case *False*
with *assms* **obtain** *x* **where** $x \in A$ $x > 0$
by *auto*
thus *Gcd A* $\in A$
proof (*induction x rule*: *less-induct*)
case (*less x*)
show *?case*
proof (*cases x* = *Gcd A*)
case *False*
have $\exists y \in A.\ \neg x$ *dvd y*
using *False less.prems* **by** (*metis Gcd-dvd Gcd-greatest-nat gcd-nat.asym*)
then obtain *y* **where** *y*: $y \in A$ $\neg x$ *dvd y*
by *blast*
have *gcd x y* $\in A$
by (*rule assms*(*1*)) (*use* ‹$x \in A$› *y* **in** *auto*)
moreover have *gcd x y* $< x$

using ‹*x* > *0*› *y* **by** (*metis gcd-dvd1 gcd-dvd2 nat-dvd-not-less nat-neq-iff*)
moreover have *gcd x y* > *0*
using ‹*x* > *0*› **by** *auto*
ultimately show *?thesis* **using** *less.IH* **by** *blast*
qed (*use less* **in** *auto*)
qed
qed *auto*

lemma *bezout-gcd-nat'*:
fixes *a b* :: *nat*
shows $\exists x\ y.\ b * y \leq a * x \wedge a * x - b * y = gcd\ a\ b \vee a * y \leq b * x \wedge b * x - a * y = gcd\ a\ b$
using *bezout-nat*[*of a b*]
by (*metis add-diff-cancel-left' diff-zero gcd.commute gcd-0-nat le-add-same-cancel1 mult.right-neutral zero-le*)

lemmas *Lcm-eq-0-I-nat* [*simp*] = *Lcm-eq-0-I* [**where** *?'a* = *nat*]
lemmas *Lcm-0-iff-nat* [*simp*] = *Lcm-0-iff* [**where** *?'a* = *nat*]
lemmas *Lcm-least-int* [*simp*] = *Lcm-least* [**where** *?'a* = *int*]

87.11 Characteristic of a semiring

definition (**in** *semiring-1*) *semiring-char* :: *'a itself* ⇒ *nat*
where *semiring-char* - = *Gcd* {*n*. *of-nat n* = (*0* :: *'a*)}

syntax *-type-char* :: *type* => *nat* (‹(‹*indent=1 notation*=‹*mixfix CHAR*››*CHAR*/(*1*'(-')))›)
syntax-consts *-type-char* ⇌ *semiring-char*
translations *CHAR*('*t*) ⇀ *CONST semiring-char* (*CONST Pure.type* :: '*t itself*)
print-translation ‹
let
fun char-type-tr' ctxt [*Const* (***const-syntax***‹*Pure.type*›, *Type* (-, [*T*]))] =
Syntax.const ***syntax-const***‹*-type-char*› $ *Syntax-Phases.term-of-typ ctxt T*
in [(***const-syntax***‹*semiring-char*›, *char-type-tr'*)] *end*
›

context *semiring-1*
begin

lemma *of-nat-CHAR* [*simp*]: *of-nat CHAR*('*a*) = (*0* :: '*a*)
proof −
have *CHAR*('*a*) ∈ {*n*. *of-nat n* = (*0*::'*a*)}
unfolding *semiring-char-def*
proof (*rule Gcd-in*, *clarify*)
fix *a b* :: *nat*
assume ∗: *of-nat a* = (*0* :: '*a*) *of-nat b* = (*0* :: '*a*)
show *of-nat* (*gcd a b*) = (*0* :: '*a*)
proof (*cases a* = *0*)
case *False*
with *bezout-nat* **obtain** *x y* **where** $a * x = b * y + gcd\ a\ b$

```
      by blast
    hence of-nat (a * x) = (of-nat (b * y + gcd a b) :: 'a)
      by (rule arg-cong)
    thus of-nat (gcd a b) = (0 :: 'a)
      using * by simp
  qed (use * in auto)
 next
  have of-nat 0 = (0 :: 'a)
    by simp
  thus {n. of-nat n = (0 :: 'a)} ≠ {}
    by blast
 qed
 thus ?thesis
  by simp
qed

lemma of-nat-eq-0-iff-char-dvd: of-nat n = (0 :: 'a) ⟷ CHAR('a) dvd n
proof
 assume of-nat n = (0 :: 'a)
 thus CHAR('a) dvd n
  unfolding semiring-char-def by (intro Gcd-dvd) auto
next
 assume CHAR('a) dvd n
 then obtain m where n = CHAR('a) * m
  by auto
 thus of-nat n = (0 :: 'a)
  by simp
qed

lemma CHAR-eqI:
 assumes of-nat c = (0 :: 'a)
 assumes ⋀x. of-nat x = (0 :: 'a) ⟹ c dvd x
 shows  CHAR('a) = c
 using assms by (intro dvd-antisym) (auto simp: of-nat-eq-0-iff-char-dvd)

lemma CHAR-eq0-iff: CHAR('a) = 0 ⟷ (∀ n>0. of-nat n ≠ (0::'a))
 by (auto simp: of-nat-eq-0-iff-char-dvd)

lemma CHAR-pos-iff: CHAR('a) > 0 ⟷ (∃ n>0. of-nat n = (0::'a))
 using CHAR-eq0-iff neq0-conv by blast

lemma CHAR-eq-posI:
 assumes c > 0 of-nat c = (0 :: 'a) ⋀x. x > 0 ⟹ x < c ⟹ of-nat x ≠ (0 ::
'a)
 shows  CHAR('a) = c
proof (rule antisym)
 from assms have CHAR('a) > 0
  by (auto simp: CHAR-pos-iff)
 from assms(3)[OF this] show CHAR('a) ≥ c
```

by *force*
next
have $CHAR('a)$ *dvd* c
using *assms* **by** (*auto simp*: *of-nat-eq-0-iff-char-dvd*)
thus $CHAR('a) \leq c$
using ‹$c > 0$› **by** (*intro dvd-imp-le*) *auto*
qed

end

lemma (**in** *semiring-char-0*) *CHAR-eq-0* [*simp*]: $CHAR('a) = 0$
by (*simp add*: *CHAR-eq0-iff*)

lemma *CHAR-not-1* [*simp*]: $CHAR('a :: \{semiring\text{-}1, zero\text{-}neq\text{-}one\}) \neq Suc\ 0$
by (*metis One-nat-def of-nat-1 of-nat-CHAR zero-neq-one*)

lemma (**in** *idom*) *CHAR-not-1'* [*simp*]: $CHAR('a) \neq Suc\ 0$
using *local.of-nat-CHAR* **by** *fastforce*

lemma (**in** *ring-1*) *uminus-CHAR-2*:
assumes $CHAR('a) = 2$
shows $-(x :: 'a) = x$
proof −
have $x + x = 2 * x$
by (*simp add*: *mult-2*)
also have $2 = (0 :: 'a)$
using *assms local.of-nat-CHAR* **by** *auto*
finally show *?thesis*
by (*simp add*: *add-eq-0-iff2*)
qed

lemma (**in** *ring-1*) *minus-CHAR-2*:
assumes $CHAR('a) = 2$
shows $(x - y :: 'a) = x + y$
proof −
have $x - y = x + (-y)$
by *simp*
also have $-y = y$
by (*rule uminus-CHAR-2*) *fact*
finally show *?thesis* **.**
qed

lemma (**in** *semiring-1-cancel*) *of-nat-eq-iff-char-dvd*:
assumes $m < n$
shows $of\text{-}nat\ m = (of\text{-}nat\ n :: 'a) \longleftrightarrow CHAR('a)\ dvd\ (n - m)$
proof
assume $*$: $of\text{-}nat\ m = (of\text{-}nat\ n :: 'a)$
have $of\text{-}nat\ n = (of\text{-}nat\ m + of\text{-}nat\ (n - m) :: 'a)$

```
    using assms by (metis le-add-diff-inverse local.of-nat-add nless-le)
  hence of-nat (n − m) = (0 :: 'a)
    by (simp add: ∗)
  thus CHAR('a) dvd (n − m)
    by (simp add: of-nat-eq-0-iff-char-dvd)
next
  assume CHAR('a) dvd (n − m)
  hence of-nat (n − m) = (0 :: 'a)
    by (simp add: of-nat-eq-0-iff-char-dvd)
  hence of-nat m = (of-nat m + of-nat (n − m) :: 'a)
    by simp
  also have ... = of-nat n
    using assms by (metis le-add-diff-inverse local.of-nat-add nless-le)
  finally show of-nat m = (of-nat n :: 'a) .
qed

lemma (in ring-1) of-int-eq-0-iff-char-dvd:
  (of-int n = (0 :: 'a)) = (int CHAR('a) dvd n)
proof (cases n ≥ 0)
  case True
  hence (of-int n = (0 :: 'a)) ⟷ (of-nat (nat n)) = (0 :: 'a)
    by auto
  also have ... ⟷ CHAR('a) dvd nat n
    by (subst of-nat-eq-0-iff-char-dvd) auto
  also have ... ⟷ int CHAR('a) dvd n
    using True by presburger
  finally show ?thesis .
next
  case False
  hence (of-int n = (0 :: 'a)) ⟷ −(of-nat (nat (−n))) = (0 :: 'a)
    by auto
  also have ... ⟷ CHAR('a) dvd nat (−n)
    by (auto simp: of-nat-eq-0-iff-char-dvd)
  also have ... ⟷ int CHAR('a) dvd n
    using False dvd-nat-abs-iff[of CHAR('a) n] by simp
  finally show ?thesis .
qed

lemma (in semiring-1-cancel) finite-imp-CHAR-pos:
  assumes finite (UNIV :: 'a set)
  shows   CHAR('a) > 0
proof −
  have ∃ n∈UNIV. infinite {m ∈ UNIV. of-nat m = (of-nat n :: 'a)}
  proof (rule pigeonhole-infinite)
    show infinite (UNIV :: nat set)
      by simp
    show finite (range (of-nat :: nat ⇒ 'a))
      by (rule finite-subset[OF - assms]) auto
  qed
```

then obtain *n* :: *nat* **where** *infinite* $\{m \in \mathit{UNIV}.\ \mathit{of\text{-}nat}\ m = (\mathit{of\text{-}nat}\ n :: 'a)\}$
 by *blast*
hence $\neg(\{m \in \mathit{UNIV}.\ \mathit{of\text{-}nat}\ m = (\mathit{of\text{-}nat}\ n :: 'a)\} \subseteq \{n\})$
 by (*intro notI*) (*use finite-subset* **in** *blast*)
then obtain *m* **where** $m \neq n$ *of-nat* $m = (\mathit{of\text{-}nat}\ n :: 'a)$
 by *blast*
thus *?thesis*
proof (*induction m n rule*: *linorder-wlog*)
 case (*le m n*)
 hence *CHAR*('a) *dvd* $(n - m)$
  using *of-nat-eq-iff-char-dvd*[*of m n*] **by** *auto*
 thus *?thesis*
  using *le* **by** (*intro Nat.gr0I*) *auto*
qed (*simp-all add*: *eq-commute*)
qed

end

88 Nitpick: Yet Another Counterexample Generator for Isabelle/HOL

theory *Nitpick*
imports *Record GCD*
keywords
 nitpick :: *diag* **and**
 nitpick-params :: *thy-decl*
begin

datatype (*plugins only*: *extraction*) (*dead* 'a, *dead* 'b) *fun-box* = *FunBox* $'a \Rightarrow 'b$
datatype (*plugins only*: *extraction*) (*dead* 'a, *dead* 'b) *pair-box* = *PairBox* 'a 'b
datatype (*plugins only*: *extraction*) (*dead* 'a) *word* = *Word* 'a *set*

typedecl *bisim-iterator*
typedecl *unsigned-bit*
typedecl *signed-bit*

consts
 unknown :: 'a
 is-unknown :: $'a \Rightarrow bool$
 bisim :: *bisim-iterator* $\Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
 bisim-iterator-max :: *bisim-iterator*
 Quot :: $'a \Rightarrow 'b$
 safe-The :: $('a \Rightarrow bool) \Rightarrow 'a$

Alternative definitions.

lemma *Ex1-unfold*[*nitpick-unfold*]: $\mathit{Ex1}\ P \equiv \exists x.\ \{x.\ P\ x\} = \{x\}$
 apply (*rule eq-reflection*)
 apply (*simp add*: *Ex1-def set-eq-iff*)

apply (*rule iffI*)
apply (*erule exE*)
apply (*erule conjE*)
apply (*rule-tac x = x* **in** *exI*)
apply (*rule allI*)
apply (*rename-tac y*)
apply (*erule-tac x = y* **in** *allE*)
by *auto*

lemma *rtrancl-unfold*[*nitpick-unfold*]: $r^* \equiv (r^+)^=$
by (*simp only*: *rtrancl-trancl-reflcl*)

lemma *rtranclp-unfold*[*nitpick-unfold*]: *rtranclp r a b* ≡ (*a* = *b* ∨ *tranclp r a b*)
by (*rule eq-reflection*) (*auto dest*: *rtranclpD*)

lemma *tranclp-unfold*[*nitpick-unfold*]:
tranclp r a b ≡ (*a*, *b*) ∈ *trancl* {(*x*, *y*). *r x y*}
by (*simp add*: *trancl-def*)

lemma [*nitpick-simp*]:
of-nat n = (*if n* = *0 then 0 else 1* + *of-nat* (*n* − *1*))
by (*cases n*) *auto*

definition *prod* :: *'a set* ⇒ *'b set* ⇒ (*'a* × *'b*) *set* **where**
prod A B = {(*a*, *b*). *a* ∈ *A* ∧ *b* ∈ *B*}

definition *refl'* :: (*'a* × *'a*) *set* ⇒ *bool* **where**
refl' r ≡ ∀ *x*. (*x*, *x*) ∈ *r*

definition *wf'* :: (*'a* × *'a*) *set* ⇒ *bool* **where**
wf' r ≡ *acyclic r* ∧ (*finite r* ∨ *unknown*)

definition *card'* :: *'a set* ⇒ *nat* **where**
card' A ≡ *if finite A then length* (*SOME xs*. *set xs* = *A* ∧ *distinct xs*) *else 0*

definition *sum'* :: (*'a* ⇒ *'b*::*comm-monoid-add*) ⇒ *'a set* ⇒ *'b* **where**
sum' f A ≡ *if finite A then sum-list* (*map f* (*SOME xs*. *set xs* = *A* ∧ *distinct xs*))
else 0

inductive *fold-graph'* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'b* ⇒ *'a set* ⇒ *'b* ⇒ *bool* **where**
fold-graph' f z {} *z* |
⟦*x* ∈ *A*; *fold-graph' f z* (*A* − {*x*}) *y*⟧ ⟹ *fold-graph' f z A* (*f x y*)

The following lemmas are not strictly necessary but they help the *specialize* optimization.

lemma *The-psimp*[*nitpick-psimp*]: *P* = (=) *x* ⟹ *The P* = *x*
by *auto*

lemma *Eps-psimp*[*nitpick-psimp*]:

$[\![P\ x;\ \neg\ P\ y;\ Eps\ P = y]\!] \Longrightarrow Eps\ P = x$
apply (*cases P* (*Eps P*))
apply *auto*
apply (*erule contrapos-np*)
by (*rule someI*)

lemma *case-unit-unfold*[*nitpick-unfold*]:
case-unit x u ≡ *x*
apply (*subgoal-tac u* = ())
apply (*simp only*: *unit.case*)
by *simp*

declare *unit.case*[*nitpick-simp del*]

lemma *case-nat-unfold*[*nitpick-unfold*]:
case-nat x f n ≡ *if n* = *0 then x else f* (*n* − *1*)
apply (*rule eq-reflection*)
by (*cases n*) *auto*

declare *nat.case*[*nitpick-simp del*]

lemma *size-list-simp*[*nitpick-simp*]:
size-list f xs = (*if xs* = [] *then 0 else Suc* (*f* (*hd xs*) + *size-list f* (*tl xs*)))
size xs = (*if xs* = [] *then 0 else Suc* (*size* (*tl xs*)))
by (*cases xs*) *auto*

Auxiliary definitions used to provide an alternative representation for *rat* and *real*.

fun *nat-gcd* :: *nat* ⇒ *nat* ⇒ *nat* **where**
nat-gcd x y = (*if y* = *0 then x else nat-gcd y* (*x mod y*))

declare *nat-gcd.simps* [*simp del*]

definition *nat-lcm* :: *nat* ⇒ *nat* ⇒ *nat* **where**
nat-lcm x y = *x* ∗ *y div* (*nat-gcd x y*)

lemma *gcd-eq-nitpick-gcd* [*nitpick-unfold*]:
gcd x y = *Nitpick.nat-gcd x y*
by (*induct x y rule*: *nat-gcd.induct*)
(*simp add*: *gcd-nat.simps Nitpick.nat-gcd.simps*)

lemma *lcm-eq-nitpick-lcm* [*nitpick-unfold*]:
lcm x y = *Nitpick.nat-lcm x y*
by (*simp only*: *lcm-nat-def Nitpick.nat-lcm-def gcd-eq-nitpick-gcd*)

definition *Frac* :: *int* × *int* ⇒ *bool* **where**
Frac ≡ λ(*a*, *b*). *b* > *0* ∧ *coprime a b*

consts

Abs-Frac :: *int* × *int* ⇒ *'a*
Rep-Frac :: *'a* ⇒ *int* × *int*

definition *zero-frac* :: *'a* **where**
zero-frac ≡ *Abs-Frac* (*0*, *1*)

definition *one-frac* :: *'a* **where**
one-frac ≡ *Abs-Frac* (*1*, *1*)

definition *num* :: *'a* ⇒ *int* **where**
num ≡ *fst* ∘ *Rep-Frac*

definition *denom* :: *'a* ⇒ *int* **where**
denom ≡ *snd* ∘ *Rep-Frac*

function *norm-frac* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
norm-frac a b =
(*if b* < *0 then norm-frac* (− *a*) (− *b*)
else if a = *0* ∨ *b* = *0 then* (*0*, *1*)
else let c = *gcd a b in* (*a div c*, *b div c*))
by *pat-completeness auto*
termination by (*relation measure* (λ(-, *b*). *if b* < *0 then 1 else 0*)) *auto*

declare *norm-frac.simps*[*simp del*]

definition *frac* :: *int* ⇒ *int* ⇒ *'a* **where**
frac a b ≡ *Abs-Frac* (*norm-frac a b*)

definition *plus-frac* :: *'a* ⇒ *'a* ⇒ *'a* **where**
[*nitpick-simp*]: *plus-frac q r* = (*let d* = *lcm* (*denom q*) (*denom r*) *in*
frac (*num q* ∗ (*d div denom q*) + *num r* ∗ (*d div denom r*)) *d*)

definition *times-frac* :: *'a* ⇒ *'a* ⇒ *'a* **where**
[*nitpick-simp*]: *times-frac q r* = *frac* (*num q* ∗ *num r*) (*denom q* ∗ *denom r*)

definition *uminus-frac* :: *'a* ⇒ *'a* **where**
uminus-frac q ≡ *Abs-Frac* (− *num q*, *denom q*)

definition *number-of-frac* :: *int* ⇒ *'a* **where**
number-of-frac n ≡ *Abs-Frac* (*n*, *1*)

definition *inverse-frac* :: *'a* ⇒ *'a* **where**
inverse-frac q ≡ *frac* (*denom q*) (*num q*)

definition *less-frac* :: *'a* ⇒ *'a* ⇒ *bool* **where**
[*nitpick-simp*]: *less-frac q r* ⟷ *num* (*plus-frac q* (*uminus-frac r*)) < *0*

definition *less-eq-frac* :: *'a* ⇒ *'a* ⇒ *bool* **where**
[*nitpick-simp*]: *less-eq-frac q r* ⟷ *num* (*plus-frac q* (*uminus-frac r*)) ≤ *0*

definition *of-frac* :: $'a \Rightarrow 'b$::{*inverse*,*ring-1*} **where**
of-frac q ≡ *of-int* (*num q*) / *of-int* (*denom q*)

axiomatization *wf-wfrec* :: $('a \times 'a)$ *set* $\Rightarrow (('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$

definition *wf-wfrec'* :: $('a \times 'a)$ *set* $\Rightarrow (('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ **where**
[*nitpick-simp*]: *wf-wfrec' R F x* = *F* (*cut* (*wf-wfrec R F*) *R x*) *x*

definition *wfrec'* :: $('a \times 'a)$ *set* $\Rightarrow (('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ **where**
wfrec' R F x ≡ *if wf R then wf-wfrec' R F x else THE y. wfrec-rel R* (λ*f x. F* (*cut f R x*) *x*) *x y*

ML-file ‹*Tools/Nitpick/kodkod.ML*›
ML-file ‹*Tools/Nitpick/kodkod-sat.ML*›
ML-file ‹*Tools/Nitpick/nitpick-util.ML*›
ML-file ‹*Tools/Nitpick/nitpick-hol.ML*›
ML-file ‹*Tools/Nitpick/nitpick-mono.ML*›
ML-file ‹*Tools/Nitpick/nitpick-preproc.ML*›
ML-file ‹*Tools/Nitpick/nitpick-scope.ML*›
ML-file ‹*Tools/Nitpick/nitpick-peephole.ML*›
ML-file ‹*Tools/Nitpick/nitpick-rep.ML*›
ML-file ‹*Tools/Nitpick/nitpick-nut.ML*›
ML-file ‹*Tools/Nitpick/nitpick-kodkod.ML*›
ML-file ‹*Tools/Nitpick/nitpick-model.ML*›
ML-file ‹*Tools/Nitpick/nitpick.ML*›
ML-file ‹*Tools/Nitpick/nitpick-commands.ML*›
ML-file ‹*Tools/Nitpick/nitpick-tests.ML*›

setup ‹
Nitpick-HOL.register-ersatz-global
[(***const-name*** ‹*card*›, ***const-name*** ‹*card'*›),
(***const-name*** ‹*sum*›, ***const-name*** ‹*sum'*›),
(***const-name*** ‹*fold-graph*›, ***const-name*** ‹*fold-graph'*›),
(***const-abbrev*** ‹*wf*›, ***const-name*** ‹*wf'*›),
(***const-name*** ‹*wf-wfrec*›, ***const-name*** ‹*wf-wfrec'*›),
(***const-name*** ‹*wfrec*›, ***const-name*** ‹*wfrec'*›)]
›

hide-const (**open**) *unknown is-unknown bisim bisim-iterator-max Quot safe-The FunBox PairBox Word prod*
refl' wf' card' sum' fold-graph' nat-gcd nat-lcm Frac Abs-Frac Rep-Frac
zero-frac one-frac num denom norm-frac frac plus-frac times-frac uminus-frac number-of-frac
inverse-frac less-frac less-eq-frac of-frac wf-wfrec wf-wfrec wfrec'

hide-type (**open**) *bisim-iterator fun-box pair-box unsigned-bit signed-bit word*

hide-fact (**open**) *Ex1-unfold rtrancl-unfold rtranclp-unfold tranclp-unfold prod-def*

refl′-def wf′-def
card′-def sum′-def The-psimp Eps-psimp case-unit-unfold case-nat-unfold
size-list-simp nat-lcm-def Frac-def zero-frac-def one-frac-def
num-def denom-def frac-def plus-frac-def times-frac-def uminus-frac-def
number-of-frac-def inverse-frac-def less-frac-def less-eq-frac-def of-frac-def wf-wfrec′-def
wfrec′-def

end

theory *Nunchaku*
imports *Nitpick*
keywords
nunchaku :: *diag* **and**
nunchaku-params :: *thy-decl*
begin

consts *unreachable* :: $'a$

definition *The-unsafe* :: $('a \Rightarrow bool) \Rightarrow 'a$ **where**
The-unsafe = *The*

definition *rmember* :: $'a\ set \Rightarrow 'a \Rightarrow bool$ **where**
rmember $A\ x \longleftrightarrow x \in A$

ML-file ‹*Tools/Nunchaku/nunchaku-util.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-collect.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-problem.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-translate.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-model.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-reconstruct.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-display.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-tool.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku.ML*›
ML-file ‹*Tools/Nunchaku/nunchaku-commands.ML*›

hide-const (**open**) *unreachable The-unsafe rmember*

end

89 Greatest Fixpoint (Codatatype) Operation on Bounded Natural Functors

theory *BNF-Greatest-Fixpoint*
imports *BNF-Fixpoint-Base String*
keywords
codatatype :: *thy-defn* **and**
primcorecursive :: *thy-goal-defn* **and**

primcorec :: *thy-defn*
begin

alias *proj* = *Equiv-Relations.proj*

lemma *one-pointE*: $[\![\bigwedge x.\ s = x \Longrightarrow P]\!] \Longrightarrow P$
  by *simp*

lemma *obj-sumE*: $[\![\forall x.\ s = Inl\ x \longrightarrow P; \forall x.\ s = Inr\ x \longrightarrow P]\!] \Longrightarrow P$
  by (*cases s*) *auto*

lemma *not-TrueE*: $\neg\ True \Longrightarrow P$
  by (*erule notE*, *rule TrueI*)

lemma *neq-eq-eq-contradict*: $[\![t \neq u;\ s = t;\ s = u]\!] \Longrightarrow P$
  by *fast*

lemma *converse-Times*: $(A \times B)^{-1} = B \times A$
  by *fast*

lemma *equiv-proj*:
  assumes *e*: *equiv A R* **and** *m*: $z \in R$
  shows (*proj R* ∘ *fst*) *z* = (*proj R* ∘ *snd*) *z*
proof −
  from *m* **have** *z*: (*fst z*, *snd z*) ∈ *R* **by** *auto*
  with *e* **have** $\bigwedge x.$ (*fst z*, *x*) ∈ *R* ⟹ (*snd z*, *x*) ∈ *R* $\bigwedge x.$ (*snd z*, *x*) ∈ *R* ⟹ (*fst z*, *x*) ∈ *R*
    unfolding *equiv-def sym-def trans-def* **by** *blast+*
  then show *?thesis* **unfolding** *proj-def*[*abs-def*] **by** *auto*
qed

definition *image2* **where** *image2 A f g* = {(*f a*, *g a*) | *a*. *a* ∈ *A*}

lemma *Id-on-Gr*: *Id-on A* = *Gr A id*
  unfolding *Id-on-def Gr-def* **by** *auto*

lemma *image2-eqI*: $[\![b = f\ x;\ c = g\ x;\ x \in A]\!] \Longrightarrow (b, c) \in image2\ A\ f\ g$
  unfolding *image2-def* **by** *auto*

lemma *IdD*: $(a, b) \in Id \Longrightarrow a = b$
  by *auto*

lemma *image2-Gr*: *image2 A f g* = $(Gr\ A\ f)^{-1}$ *O* (*Gr A g*)
  unfolding *image2-def Gr-def* **by** *auto*

lemma *GrD1*: $(x, fx) \in Gr\ A\ f \Longrightarrow x \in A$
  unfolding *Gr-def* **by** *simp*

lemma *GrD2*: $(x, fx) \in Gr\ A\ f \Longrightarrow f\ x = fx$
unfolding *Gr-def* **by** *simp*

lemma *Gr-incl*: $Gr\ A\ f \subseteq A \times B \longleftrightarrow f\ `\ A \subseteq B$
unfolding *Gr-def* **by** *auto*

lemma *subset-Collect-iff*: $B \subseteq A \Longrightarrow (B \subseteq \{x \in A.\ P\ x\}) = (\forall x \in B.\ P\ x)$
by *blast*

lemma *subset-CollectI*: $B \subseteq A \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow Q\ x \Longrightarrow P\ x) \Longrightarrow (\{x \in B.\ Q\ x\} \subseteq \{x \in A.\ P\ x\})$
by *blast*

lemma *in-rel-Collect-case-prod-eq*: *in-rel* (*Collect* (*case-prod* X)) = X
unfolding *fun-eq-iff* **by** *auto*

lemma *Collect-case-prod-in-rel-leI*: $X \subseteq Y \Longrightarrow X \subseteq$ *Collect* (*case-prod* (*in-rel* Y))
by *auto*

lemma *Collect-case-prod-in-rel-leE*: $X \subseteq$ *Collect* (*case-prod* (*in-rel* Y)) $\Longrightarrow (X \subseteq Y \Longrightarrow R) \Longrightarrow R$
by *force*

lemma *conversep-in-rel*: $(\textit{in-rel}\ R)^{-1-1} = \textit{in-rel}\ (R^{-1})$
unfolding *fun-eq-iff* **by** *auto*

lemma *relcompp-in-rel*: *in-rel* R *OO* *in-rel* S = *in-rel* (R O S)
unfolding *fun-eq-iff* **by** *auto*

lemma *in-rel-Gr*: *in-rel* ($Gr\ A\ f$) = $Grp\ A\ f$
unfolding *Gr-def Grp-def fun-eq-iff* **by** *auto*

definition *relImage* **where**
relImage $R\ f \equiv \{(f\ a1,\ f\ a2) \mid a1\ a2.\ (a1,a2) \in R\}$

definition *relInvImage* **where**
relInvImage $A\ R\ f \equiv \{(a1,\ a2) \mid a1\ a2.\ a1 \in A \land a2 \in A \land (f\ a1,\ f\ a2) \in R\}$

lemma *relImage-Gr*:
$[\![R \subseteq A \times A]\!] \Longrightarrow$ *relImage* $R\ f = (Gr\ A\ f)^{-1}\ O\ R\ O\ Gr\ A\ f$
unfolding *relImage-def Gr-def relcomp-def* **by** *auto*

lemma *relInvImage-Gr*: $[\![R \subseteq B \times B]\!] \Longrightarrow$ *relInvImage* $A\ R\ f = Gr\ A\ f\ O\ R\ O\ (Gr\ A\ f)^{-1}$
unfolding *Gr-def relcomp-def image-def relInvImage-def* **by** *auto*

lemma *relImage-mono*:
$R1 \subseteq R2 \Longrightarrow$ *relImage* $R1\ f \subseteq$ *relImage* $R2\ f$

unfolding *relImage-def* **by** *auto*

lemma *relInvImage-mono*:
$R1 \subseteq R2 \Longrightarrow relInvImage\ A\ R1\ f \subseteq relInvImage\ A\ R2\ f$
unfolding *relInvImage-def* **by** *auto*

lemma *relInvImage-Id-on*:
$(\bigwedge a1\ a2.\ f\ a1 = f\ a2 \longleftrightarrow a1 = a2) \Longrightarrow relInvImage\ A\ (Id\text{-}on\ B)\ f \subseteq Id$
unfolding *relInvImage-def Id-on-def* **by** *auto*

lemma *relInvImage-UNIV-relImage*:
$R \subseteq relInvImage\ UNIV\ (relImage\ R\ f)\ f$
unfolding *relInvImage-def relImage-def* **by** *auto*

lemma *relImage-proj*:
assumes *equiv A R*
shows $relImage\ R\ (proj\ R) \subseteq Id\text{-}on\ (A//R)$
unfolding *relImage-def Id-on-def*
using *proj-iff*[*OF assms*] *equiv-class-eq-iff*[*OF assms*]
by (*auto simp*: *proj-preserves*)

lemma *relImage-relInvImage*:
assumes $R \subseteq f\ `\ A \times f\ `\ A$
shows $relImage\ (relInvImage\ A\ R\ f)\ f = R$
using *assms* **unfolding** *relImage-def relInvImage-def* **by** *fast*

lemma *subst-Pair*: $P\ x\ y \Longrightarrow a = (x,\ y) \Longrightarrow P\ (fst\ a)\ (snd\ a)$
by *simp*

lemma *fst-diag-id*: $(fst \circ (\lambda x.\ (x,\ x)))\ z = id\ z$ **by** *simp*
lemma *snd-diag-id*: $(snd \circ (\lambda x.\ (x,\ x)))\ z = id\ z$ **by** *simp*

lemma *fst-diag-fst*: $fst \circ ((\lambda x.\ (x,\ x)) \circ fst) = fst$ **by** *auto*
lemma *snd-diag-fst*: $snd \circ ((\lambda x.\ (x,\ x)) \circ fst) = fst$ **by** *auto*
lemma *fst-diag-snd*: $fst \circ ((\lambda x.\ (x,\ x)) \circ snd) = snd$ **by** *auto*
lemma *snd-diag-snd*: $snd \circ ((\lambda x.\ (x,\ x)) \circ snd) = snd$ **by** *auto*

definition *Succ* **where** $Succ\ Kl\ kl = \{k\ .\ kl\ @\ [k] \in Kl\}$
definition *Shift* **where** $Shift\ Kl\ k = \{kl.\ k\ \#\ kl \in Kl\}$
definition *shift* **where** $shift\ lab\ k = (\lambda kl.\ lab\ (k\ \#\ kl))$

lemma *empty-Shift*: $[\![[] \in Kl;\ k \in Succ\ Kl\ []]\!] \Longrightarrow [] \in Shift\ Kl\ k$
unfolding *Shift-def Succ-def* **by** *simp*

lemma *SuccD*: $k \in Succ\ Kl\ kl \Longrightarrow kl\ @\ [k] \in Kl$
unfolding *Succ-def* **by** *simp*

lemmas *SuccE* = *SuccD*[*elim-format*]

lemma *SuccI*: *kl* @ [*k*] ∈ *Kl* ⟹ *k* ∈ *Succ Kl kl*
unfolding *Succ-def* **by** *simp*

lemma *ShiftD*: *kl* ∈ *Shift Kl k* ⟹ *k* # *kl* ∈ *Kl*
unfolding *Shift-def* **by** *simp*

lemma *Succ-Shift*: *Succ* (*Shift Kl k*) *kl* = *Succ Kl* (*k* # *kl*)
unfolding *Succ-def Shift-def* **by** *auto*

lemma *length-Cons*: *length* (*x* # *xs*) = *Suc* (*length xs*)
by *simp*

lemma *length-append-singleton*: *length* (*xs* @ [*x*]) = *Suc* (*length xs*)
by *simp*

definition *toCard-pred A r f* ≡ *inj-on f A* ∧ *f* ' *A* ⊆ *Field r* ∧ *Card-order r*
definition *toCard A r* ≡ *SOME f*. *toCard-pred A r f*

lemma *ex-toCard-pred*:
⟦|*A*| ≤*o r*; *Card-order r*⟧ ⟹ ∃ *f*. *toCard-pred A r f*
unfolding *toCard-pred-def*
using *card-of-ordLeq*[*of A Field r*]
ordLeq-ordIso-trans[*OF - card-of-unique*[*of Field r r*], *of* |*A*|]
by *blast*

lemma *toCard-pred-toCard*:
⟦|*A*| ≤*o r*; *Card-order r*⟧ ⟹ *toCard-pred A r* (*toCard A r*)
unfolding *toCard-def* **using** *someI-ex*[*OF ex-toCard-pred*] **.**

lemma *toCard-inj*: ⟦|*A*| ≤*o r*; *Card-order r*; *x* ∈ *A*; *y* ∈ *A*⟧ ⟹ *toCard A r x* = *toCard A r y* ⟷ *x* = *y*
using *toCard-pred-toCard* **unfolding** *inj-on-def toCard-pred-def* **by** *blast*

definition *fromCard A r k* ≡ *SOME b*. *b* ∈ *A* ∧ *toCard A r b* = *k*

lemma *fromCard-toCard*:
⟦|*A*| ≤*o r*; *Card-order r*; *b* ∈ *A*⟧ ⟹ *fromCard A r* (*toCard A r b*) = *b*
unfolding *fromCard-def* **by** (*rule some-equality*) (*auto simp add*: *toCard-inj*)

lemma *Inl-Field-csum*: *a* ∈ *Field r* ⟹ *Inl a* ∈ *Field* (*r* +*c s*)
unfolding *Field-card-of csum-def* **by** *auto*

lemma *Inr-Field-csum*: *a* ∈ *Field s* ⟹ *Inr a* ∈ *Field* (*r* +*c s*)
unfolding *Field-card-of csum-def* **by** *auto*

lemma *rec-nat-0-imp*: *f* = *rec-nat f1* (λ*n rec*. *f2 n rec*) ⟹ *f 0* = *f1*
by *auto*

lemma *rec-nat-Suc-imp*: $f = rec\text{-}nat\ f1\ (\lambda n\ rec.\ f2\ n\ rec) \Longrightarrow f\ (Suc\ n) = f2\ n\ (f\ n)$
  by *auto*

lemma *rec-list-Nil-imp*: $f = rec\text{-}list\ f1\ (\lambda x\ xs\ rec.\ f2\ x\ xs\ rec) \Longrightarrow f\ [] = f1$
  by *auto*

lemma *rec-list-Cons-imp*: $f = rec\text{-}list\ f1\ (\lambda x\ xs\ rec.\ f2\ x\ xs\ rec) \Longrightarrow f\ (x\ \#\ xs) = f2\ x\ xs\ (f\ xs)$
  by *auto*

lemma *not-arg-cong-Inr*: $x \neq y \Longrightarrow Inr\ x \neq Inr\ y$
  by *simp*

definition *image2p* **where**
  $image2p\ f\ g\ R = (\lambda x\ y.\ \exists\, x'\ y'.\ R\ x'\ y' \wedge f\ x' = x \wedge g\ y' = y)$

lemma *image2pI*: $R\ x\ y \Longrightarrow image2p\ f\ g\ R\ (f\ x)\ (g\ y)$
  unfolding *image2p-def* **by** *blast*

lemma *image2pE*: $[\![image2p\ f\ g\ R\ fx\ gy;\ (\bigwedge x\ y.\ fx = f\ x \Longrightarrow gy = g\ y \Longrightarrow R\ x\ y \Longrightarrow P)]\!] \Longrightarrow P$
  unfolding *image2p-def* **by** *blast*

lemma *rel-fun-iff-geq-image2p*: $rel\text{-}fun\ R\ S\ f\ g = (image2p\ f\ g\ R \leq S)$
  unfolding *rel-fun-def image2p-def* **by** *auto*

lemma *rel-fun-image2p*: $rel\text{-}fun\ R\ (image2p\ f\ g\ R)\ f\ g$
  unfolding *rel-fun-def image2p-def* **by** *auto*

89.1 Equivalence relations, quotients, and Hilbert's choice

lemma *equiv-Eps-in*:
$[\![equiv\ A\ r;\ X \in A//r]\!] \Longrightarrow Eps\ (\lambda x.\ x \in X) \in X$
  apply (*rule someI2-ex*)
  using *in-quotient-imp-non-empty* **by** *blast*

lemma *equiv-Eps-preserves*:
  assumes *ECH*: $equiv\ A\ r$ **and** *X*: $X \in A//r$
  shows $Eps\ (\lambda x.\ x \in X) \in A$
  apply (*rule in-mono*[*rule-format*])
   using *assms* **apply** (*rule in-quotient-imp-subset*)
  by (*rule equiv-Eps-in*) (*rule assms*)+

lemma *proj-Eps*:
  assumes $equiv\ A\ r$ **and** $X \in A//r$
  shows $proj\ r\ (Eps\ (\lambda x.\ x \in X)) = X$
unfolding *proj-def*
proof *auto*

fix *x* **assume** *x*: $x \in X$
thus (*Eps* ($\lambda x.\ x \in X$), *x*) $\in$ *r* **using** *assms equiv-Eps-in in-quotient-imp-in-rel* **by** *fast*
next
fix *x* **assume** (*Eps* ($\lambda x.\ x \in X$),*x*) $\in$ *r*
thus $x \in X$ **using** *in-quotient-imp-closed*[*OF assms equiv-Eps-in*[*OF assms*]] **by** *fast*
qed

definition *univ* **where** *univ f X* == *f* (*Eps* ($\lambda x.\ x \in X$))

lemma *univ-commute*:
assumes *ECH*: *equiv A r* **and** *RES*: *f respects r* **and** *x*: $x \in A$
shows (*univ f*) (*proj r x*) = *f x*
proof (*unfold univ-def*)
have *prj*: *proj r x* $\in$ *A*//*r* **using** *x proj-preserves* **by** *fast*
hence *Eps* ($\lambda y.\ y \in$ *proj r x*) $\in$ *A* **using** *ECH equiv-Eps-preserves* **by** *fast*
moreover have *proj r* (*Eps* ($\lambda y.\ y \in$ *proj r x*)) = *proj r x* **using** *ECH prj proj-Eps* **by** *fast*
ultimately have (*x*, *Eps* ($\lambda y.\ y \in$ *proj r x*)) $\in$ *r* **using** *x ECH proj-iff* **by** *fast*
thus *f* (*Eps* ($\lambda y.\ y \in$ *proj r x*)) = *f x* **using** *RES* **unfolding** *congruent-def* **by** *fastforce*
qed

lemma *univ-preserves*:
assumes *ECH*: *equiv A r* **and** *RES*: *f respects r* **and** *PRES*: $\forall x \in A.$ *f x* $\in$ *B*
shows $\forall X \in$ *A*//*r*. *univ f X* $\in$ *B*
proof
fix *X* **assume** $X \in$ *A*//*r*
then obtain *x* **where** *x*: $x \in A$ **and** *X*: *X* = *proj r x* **using** *ECH proj-image*[*of r A*] **by** *blast*
hence *univ f X* = *f x* **using** *ECH RES univ-commute* **by** *fastforce*
thus *univ f X* $\in$ *B* **using** *x PRES* **by** *simp*
qed

lemma *card-suc-ordLess-imp-ordLeq*:
assumes *ORD*: *Card-order r Card-order r′ card-order r′*
and *LESS*: *r* <*o card-suc r′*
shows *r* ≤*o r′*
proof –
have *Card-order* (*card-suc r′*) **by** (*rule Card-order-card-suc*[*OF ORD*(*3*)])
then have *cardSuc r* ≤*o card-suc r′* **using** *cardSuc-least ORD LESS* **by** *blast*
then have *cardSuc r* ≤*o cardSuc r′* **using** *cardSuc-ordIso-card-suc ordIso-symmetric ordLeq-ordIso-trans ORD*(*3*) **by** *blast*
then show *?thesis* **using** *cardSuc-mono-ordLeq ORD* **by** *blast*
qed

lemma *natLeq-ordLess-cinfinite*: ⟦*Cinfinite r*; *card-order r*⟧ ⟹ *natLeq* <*o card-suc r*

using *natLeq-ordLeq-cinfinite card-suc-greater ordLeq-ordLess-trans* **by** *blast*

corollary *natLeq-ordLess-cinfinite'*: $[\![$*Cinfinite r'*; *card-order r'*; *r* $\equiv$ *card-suc r'*$]\!]$
$\Longrightarrow$ *natLeq* $<$*o r*
using *natLeq-ordLess-cinfinite* **by** *blast*

ML-file ‹*Tools/BNF/bnf-gfp-util.ML*›
ML-file ‹*Tools/BNF/bnf-gfp-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-gfp.ML*›
ML-file ‹*Tools/BNF/bnf-gfp-rec-sugar-tactics.ML*›
ML-file ‹*Tools/BNF/bnf-gfp-rec-sugar.ML*›

end

90 Filters on predicates

theory *Filter*
imports *Set-Interval Lifting-Set*
begin

90.1 Filters

This definition also allows non-proper filters.

locale *is-filter* =
fixes *F* :: (*'a* ⇒ *bool*) ⇒ *bool*
assumes *True*: *F* (λ*x*. *True*)
assumes *conj*: *F* (λ*x*. *P x*) ⟹ *F* (λ*x*. *Q x*) ⟹ *F* (λ*x*. *P x* ∧ *Q x*)
assumes *mono*: ∀ *x*. *P x* ⟶ *Q x* ⟹ *F* (λ*x*. *P x*) ⟹ *F* (λ*x*. *Q x*)

typedef *'a filter* = {*F* :: (*'a* ⇒ *bool*) ⇒ *bool*. *is-filter F*}
proof
show (λ*x*. *True*) ∈ *?filter* **by** (*auto intro*: *is-filter.intro*)
qed

lemma *is-filter-Rep-filter*: *is-filter* (*Rep-filter F*)
using *Rep-filter* [*of F*] **by** *simp*

lemma *Abs-filter-inverse'*:
assumes *is-filter F* **shows** *Rep-filter* (*Abs-filter F*) = *F*
using *assms* **by** (*simp add*: *Abs-filter-inverse*)

90.1.1 Eventually

definition *eventually* :: (*'a* ⇒ *bool*) ⇒ *'a filter* ⇒ *bool*
where *eventually P F* ⟷ *Rep-filter F P*

syntax

-*eventually* :: *pttrn* => 'a *filter* => *bool* => *bool* (‹(‹*indent=3 notation=*‹*binder* $\forall_F$››$\forall_F$ - *in* -./ -)› [*0*, *0*, *10*] *10*)
syntax-consts
-*eventually* == *eventually*
translations
$\forall_F x$ *in F*. *P* == *CONST eventually* (λx. *P*) *F*

lemma *eventually-Abs-filter*:
assumes *is-filter F* **shows** *eventually P* (*Abs-filter F*) = *F P*
unfolding *eventually-def* **using** *assms* **by** (*simp add*: *Abs-filter-inverse*)

lemma *filter-eq-iff*:
shows $F = F' \longleftrightarrow (\forall P.$ *eventually P F* = *eventually P F'*)
unfolding *Rep-filter-inject* [*symmetric*] *fun-eq-iff eventually-def* **..**

lemma *eventually-True* [*simp*]: *eventually* (λx. *True*) *F*
unfolding *eventually-def*
by (*rule is-filter.True* [*OF is-filter-Rep-filter*])

lemma *always-eventually*: $\forall x.\ P\ x \Longrightarrow$ *eventually P F*
proof −
assume $\forall x.\ P\ x$ **hence** $P = (\lambda x.$ *True*) **by** (*simp add*: *ext*)
thus *eventually P F* **by** *simp*
qed

lemma *eventuallyI*: $(\bigwedge x.\ P\ x) \Longrightarrow$ *eventually P F*
by (*auto intro*: *always-eventually*)

lemma *filter-eqI*: $(\bigwedge P.$ *eventually P F* $\longleftrightarrow$ *eventually P G*) $\Longrightarrow F = G$
by (*auto simp*: *filter-eq-iff*)

lemma *eventually-mono*:
⟦*eventually P F*; $\bigwedge x.\ P\ x \Longrightarrow Q\ x$⟧ $\Longrightarrow$ *eventually Q F*
unfolding *eventually-def*
by (*blast intro*: *is-filter.mono* [*OF is-filter-Rep-filter*])

lemma *eventually-conj*:
assumes *P*: *eventually* ($\lambda x.\ P\ x$) *F*
assumes *Q*: *eventually* ($\lambda x.\ Q\ x$) *F*
shows *eventually* ($\lambda x.\ P\ x \wedge Q\ x$) *F*
using *assms* **unfolding** *eventually-def*
by (*rule is-filter.conj* [*OF is-filter-Rep-filter*])

lemma *eventually-mp*:
assumes *eventually* ($\lambda x.\ P\ x \longrightarrow Q\ x$) *F*
assumes *eventually* ($\lambda x.\ P\ x$) *F*
shows *eventually* ($\lambda x.\ Q\ x$) *F*
proof −
have *eventually* ($\lambda x.\ (P\ x \longrightarrow Q\ x) \wedge P\ x$) *F*

using *assms* **by** (*rule eventually-conj*)
then show *?thesis*
by (*blast intro*: *eventually-mono*)
qed

lemma *eventually-rev-mp*:
assumes *eventually* (λx. *P x*) *F*
assumes *eventually* (λx. *P x* $\longrightarrow$ *Q x*) *F*
shows *eventually* (λx. *Q x*) *F*
using *assms*(*2*) *assms*(*1*) **by** (*rule eventually-mp*)

lemma *eventually-conj-iff*:
eventually (λx. *P x* $\wedge$ *Q x*) *F* $\longleftrightarrow$ *eventually P F* $\wedge$ *eventually Q F*
by (*auto intro*: *eventually-conj elim*: *eventually-rev-mp*)

lemma *eventually-elim2*:
assumes *eventually* (λi. *P i*) *F*
assumes *eventually* (λi. *Q i*) *F*
assumes $\bigwedge i$. *P i* $\Longrightarrow$ *Q i* $\Longrightarrow$ *R i*
shows *eventually* (λi. *R i*) *F*
using *assms* **by** (*auto elim*!: *eventually-rev-mp*)

lemma *eventually-cong*:
assumes *eventually P F* **and** $\bigwedge x$. *P x* $\Longrightarrow$ *Q x* $\longleftrightarrow$ *R x*
shows *eventually Q F* $\longleftrightarrow$ *eventually R F*
using *assms eventually-elim2* **by** *blast*

lemma *eventually-ball-finite-distrib*:
finite A $\Longrightarrow$ (*eventually* (λx. $\forall y \in A$. *P x y*) *net*) $\longleftrightarrow$ ($\forall y \in A$. *eventually* (λx. *P x y*) *net*)
by (*induction A rule*: *finite-induct*) (*auto simp*: *eventually-conj-iff*)

lemma *eventually-ball-finite*:
finite A $\Longrightarrow$ $\forall y \in A$. *eventually* (λx. *P x y*) *net* $\Longrightarrow$ *eventually* (λx. $\forall y \in A$. *P x y*) *net*
by (*auto simp*: *eventually-ball-finite-distrib*)

lemma *eventually-all-finite*:
fixes *P* :: *'a* $\Rightarrow$ *'b*::*finite* $\Rightarrow$ *bool*
assumes $\bigwedge y$. *eventually* (λx. *P x y*) *net*
shows *eventually* (λx. $\forall y$. *P x y*) *net*
using *eventually-ball-finite* [*of UNIV P*] *assms* **by** *simp*

lemma *eventually-ex*: ($\forall_F x$ *in F*. $\exists y$. *P x y*) $\longleftrightarrow$ ($\exists Y$. $\forall_F x$ *in F*. *P x* (*Y x*))
proof
assume $\forall_F x$ *in F*. $\exists y$. *P x y*
then have $\forall_F x$ *in F*. *P x* (*SOME y*. *P x y*)
by (*auto intro*: *someI-ex eventually-mono*)
then show $\exists Y$. $\forall_F x$ *in F*. *P x* (*Y x*)

by *auto*
qed (*auto intro*: *eventually-mono*)

lemma *not-eventually-impI*: *eventually P F* $\Longrightarrow$ $\neg$ *eventually Q F* $\Longrightarrow$ $\neg$ *eventually* ($\lambda x.$ *P x* $\longrightarrow$ *Q x*) *F*
by (*auto intro*: *eventually-mp*)

lemma *not-eventuallyD*: $\neg$ *eventually P F* $\Longrightarrow$ $\exists\, x.$ $\neg$ *P x*
by (*metis always-eventually*)

lemma *eventually-subst*:
assumes *eventually* ($\lambda n.$ *P n* = *Q n*) *F*
shows *eventually P F* = *eventually Q F* (**is** *?L* = *?R*)
proof −
from *assms* **have** *eventually* ($\lambda x.$ *P x* $\longrightarrow$ *Q x*) *F*
and *eventually* ($\lambda x.$ *Q x* $\longrightarrow$ *P x*) *F*
by (*auto elim*: *eventually-mono*)
then show *?thesis* **by** (*auto elim*: *eventually-elim2*)
qed

90.2 Frequently as dual to eventually

definition *frequently* :: ('*a* $\Rightarrow$ *bool*) $\Rightarrow$ '*a filter* $\Rightarrow$ *bool*
where *frequently P F* $\longleftrightarrow$ $\neg$ *eventually* ($\lambda x.$ $\neg$ *P x*) *F*

syntax
-frequently :: *pttrn* $\Rightarrow$ '*a filter* $\Rightarrow$ *bool* $\Rightarrow$ *bool* (‹(‹*indent=3 notation=*‹*binder* $\exists_F$››$\exists_F$ *- in -./ -*)› [*0*, *0*, *10*] *10*)
syntax-consts
-frequently == *frequently*
translations
$\exists_F x$ *in F. P* == *CONST frequently* ($\lambda x.$ *P*) *F*

lemma *not-frequently-False* [*simp*]: $\neg$ ($\exists_F x$ *in F. False*)
by (*simp add*: *frequently-def*)

lemma *frequently-ex*: $\exists_F x$ *in F. P x* $\Longrightarrow$ $\exists\, x.$ *P x*
by (*auto simp*: *frequently-def dest*: *not-eventuallyD*)

lemma *frequentlyE*: **assumes** *frequently P F* **obtains** *x* **where** *P x*
using *frequently-ex*[*OF assms*] **by** *auto*

lemma *frequently-mp*:
assumes *ev*: $\forall_F x$ *in F. P x* $\longrightarrow$ *Q x* **and** *P*: $\exists_F x$ *in F. P x* **shows** $\exists_F x$ *in F. Q x*
proof −
from *ev* **have** *eventually* ($\lambda x.$ $\neg$ *Q x* $\longrightarrow$ $\neg$ *P x*) *F*
by (*rule eventually-rev-mp*) (*auto intro*!: *always-eventually*)
from *eventually-mp*[*OF this*] *P* **show** *?thesis*

by (*auto simp*: *frequently-def*)
qed

lemma *frequently-rev-mp*:
assumes $\exists_F x$ *in* F. $P\ x$
assumes $\forall_F x$ *in* F. $P\ x \longrightarrow Q\ x$
shows $\exists_F x$ *in* F. $Q\ x$
using *assms*(*2*) *assms*(*1*) **by** (*rule frequently-mp*)

lemma *frequently-mono*: $(\forall x.\ P\ x \longrightarrow Q\ x) \Longrightarrow$ *frequently* $P\ F \Longrightarrow$ *frequently* $Q\ F$
using *frequently-mp*[*of* $P\ Q$] **by** (*simp add*: *always-eventually*)

lemma *frequently-elim1*: $\exists_F x$ *in* F. $P\ x \Longrightarrow (\bigwedge i.\ P\ i \Longrightarrow Q\ i) \Longrightarrow \exists_F x$ *in* F. $Q\ x$
by (*metis frequently-mono*)

lemma *frequently-disj-iff*: $(\exists_F x$ *in* F. $P\ x \vee Q\ x) \longleftrightarrow (\exists_F x$ *in* F. $P\ x) \vee (\exists_F x$ *in* F. $Q\ x)$
by (*simp add*: *frequently-def eventually-conj-iff*)

lemma *frequently-disj*: $\exists_F x$ *in* F. $P\ x \Longrightarrow \exists_F x$ *in* F. $Q\ x \Longrightarrow \exists_F x$ *in* F. $P\ x \vee Q\ x$
by (*simp add*: *frequently-disj-iff*)

lemma *frequently-bex-finite-distrib*:
assumes *finite* A **shows** $(\exists_F x$ *in* F. $\exists y \in A.\ P\ x\ y) \longleftrightarrow (\exists y \in A.\ \exists_F x$ *in* F. $P\ x\ y)$
using *assms* **by** *induction* (*auto simp*: *frequently-disj-iff*)

lemma *frequently-bex-finite*: *finite* $A \Longrightarrow \exists_F x$ *in* F. $\exists y \in A.\ P\ x\ y \Longrightarrow \exists y \in A.\ \exists_F x$ *in* F. $P\ x\ y$
by (*simp add*: *frequently-bex-finite-distrib*)

lemma *frequently-all*: $(\exists_F x$ *in* F. $\forall y.\ P\ x\ y) \longleftrightarrow (\forall Y.\ \exists_F x$ *in* F. $P\ x\ (Y\ x))$
using *eventually-ex*[*of* $\lambda x\ y.\ \neg\ P\ x\ y\ F$] **by** (*simp add*: *frequently-def*)

lemma
shows *not-eventually*: $\neg$ *eventually* $P\ F \longleftrightarrow (\exists_F x$ *in* F. $\neg\ P\ x)$
and *not-frequently*: $\neg$ *frequently* $P\ F \longleftrightarrow (\forall_F x$ *in* F. $\neg\ P\ x)$
by (*auto simp*: *frequently-def*)

lemma *frequently-imp-iff*:
$(\exists_F x$ *in* F. $P\ x \longrightarrow Q\ x) \longleftrightarrow$ (*eventually* $P\ F \longrightarrow$ *frequently* $Q\ F$)
unfolding *imp-conv-disj frequently-disj-iff not-eventually*[*symmetric*] **..**

lemma *frequently-eventually-conj*:
assumes $\exists_F x$ *in* F. $P\ x$
assumes $\forall_F x$ *in* F. $Q\ x$

shows $\exists_F x$ *in* F. $Q\ x \wedge P\ x$
using *assms eventually-elim2* **by** (*force simp add*: *frequently-def*)

lemma *frequently-cong*:
assumes *ev*: *eventually* $P\ F$ **and** *QR*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x \longleftrightarrow R\ x$
shows *frequently* $Q\ F \longleftrightarrow$ *frequently* $R\ F$
unfolding *frequently-def*
using *QR* **by** (*auto intro*!: *eventually-cong* [*OF ev*])

lemma *frequently-eventually-frequently*:
frequently $P\ F \Longrightarrow$ *eventually* $Q\ F \Longrightarrow$ *frequently* $(\lambda x.\ P\ x \wedge Q\ x)\ F$
using *frequently-cong* [*of* $Q\ F\ P\ \lambda x.\ P\ x \wedge Q\ x$] **by** *meson*

lemma *eventually-frequently-const-simps* [*simp*]:
$(\exists_F x\ in\ F.\ P\ x \wedge C) \longleftrightarrow (\exists_F x\ in\ F.\ P\ x) \wedge C$
$(\exists_F x\ in\ F.\ C \wedge P\ x) \longleftrightarrow C \wedge (\exists_F x\ in\ F.\ P\ x)$
$(\forall_F x\ in\ F.\ P\ x \vee C) \longleftrightarrow (\forall_F x\ in\ F.\ P\ x) \vee C$
$(\forall_F x\ in\ F.\ C \vee P\ x) \longleftrightarrow C \vee (\forall_F x\ in\ F.\ P\ x)$
$(\forall_F x\ in\ F.\ P\ x \longrightarrow C) \longleftrightarrow ((\exists_F x\ in\ F.\ P\ x) \longrightarrow C)$
$(\forall_F x\ in\ F.\ C \longrightarrow P\ x) \longleftrightarrow (C \longrightarrow (\forall_F x\ in\ F.\ P\ x))$
by (*cases C*; *simp add*: *not-frequently*)+

lemmas *eventually-frequently-simps* =
eventually-frequently-const-simps
not-eventually
eventually-conj-iff
eventually-ball-finite-distrib
eventually-ex
not-frequently
frequently-disj-iff
frequently-bex-finite-distrib
frequently-all
frequently-imp-iff

ML ‹
```
fun eventually-elim-tac facts =
  CONTEXT-SUBGOAL (fn (goal, i) => fn (ctxt, st) =>
    let
      val mp-facts = facts RL @{thms eventually-rev-mp}
      val rule =
        @{thm eventuallyI}
        |> fold (fn mp-fact => fn th => th RS mp-fact) mp-facts
        |> funpow (length facts) (fn th => @{thm impI} RS th)
      val cases-prop =
     Thm.prop-of (Rule-Cases.internalize-params (rule RS Goal.init (Thm.cterm-of
ctxt goal)))
      val cases = Rule-Cases.make-common ctxt cases-prop [((elim, []), [])]
    in CONTEXT-CASES cases (resolve-tac ctxt [rule] i) (ctxt, st) end)
```
›

method-setup *eventually-elim* = ‹
Scan.succeed (*fn* - => *CONTEXT-METHOD* (*fn facts* => *eventually-elim-tac facts 1*))
› *elimination of eventually quantifiers*

90.2.1 Finer-than relation

$F \le F'$ means that filter F is finer than filter F'.

instantiation *filter* :: (*type*) *complete-lattice*
begin

definition *le-filter-def*:
$F \le F' \longleftrightarrow (\forall P.\ eventually\ P\ F' \longrightarrow eventually\ P\ F)$

definition
$(F :: 'a\ filter) < F' \longleftrightarrow F \le F' \wedge \neg\ F' \le F$

definition
$top = Abs\text{-}filter\ (\lambda P.\ \forall x.\ P\ x)$

definition
$bot = Abs\text{-}filter\ (\lambda P.\ True)$

definition
$sup\ F\ F' = Abs\text{-}filter\ (\lambda P.\ eventually\ P\ F \wedge eventually\ P\ F')$

definition
$inf\ F\ F' = Abs\text{-}filter$
$(\lambda P.\ \exists\ Q\ R.\ eventually\ Q\ F \wedge eventually\ R\ F' \wedge (\forall x.\ Q\ x \wedge R\ x \longrightarrow P\ x))$

definition
$Sup\ S = Abs\text{-}filter\ (\lambda P.\ \forall F \in S.\ eventually\ P\ F)$

definition
$Inf\ S = Sup\ \{F{::}'a\ filter.\ \forall F' \in S.\ F \le F'\}$

lemma *eventually-top* [*simp*]: $eventually\ P\ top \longleftrightarrow (\forall x.\ P\ x)$
unfolding *top-filter-def*
by (*rule eventually-Abs-filter*, *rule is-filter.intro*, *auto*)

lemma *eventually-bot* [*simp*]: *eventually P bot*
unfolding *bot-filter-def*
by (*subst eventually-Abs-filter*, *rule is-filter.intro*, *auto*)

lemma *eventually-sup*:
$eventually\ P\ (sup\ F\ F') \longleftrightarrow eventually\ P\ F \wedge eventually\ P\ F'$
unfolding *sup-filter-def*
by (*rule eventually-Abs-filter*, *rule is-filter.intro*)

(*auto elim*!: *eventually-rev-mp*)

lemma *eventually-inf*:
eventually P (*inf F F'*) $\longleftrightarrow$
($\exists$ *Q R*. *eventually Q F* $\wedge$ *eventually R F'* $\wedge$ ($\forall$ *x*. *Q x* $\wedge$ *R x* $\longrightarrow$ *P x*))
unfolding *inf-filter-def*
apply (*rule eventually-Abs-filter* [*OF is-filter.intro*])
apply (*blast intro*: *eventually-True*)
apply (*force elim*!: *eventually-conj*)+
done

lemma *eventually-Sup*:
eventually P (*Sup S*) $\longleftrightarrow$ ($\forall$ *F*$\in$*S*. *eventually P F*)
unfolding *Sup-filter-def*
apply (*rule eventually-Abs-filter* [*OF is-filter.intro*])
apply (*auto intro*: *eventually-conj elim*!: *eventually-rev-mp*)
done

instance proof
fix *F F' F''* :: *'a filter* **and** *S* :: *'a filter set*
{ **show** $F < F' \longleftrightarrow F \leq F' \wedge \neg F' \leq F$
by (*rule less-filter-def*) }
{ **show** $F \leq F$
unfolding *le-filter-def* **by** *simp* }
{ **assume** $F \leq F'$ **and** $F' \leq F''$ **thus** $F \leq F''$
unfolding *le-filter-def* **by** *simp* }
{ **assume** $F \leq F'$ **and** $F' \leq F$ **thus** $F = F'$
unfolding *le-filter-def filter-eq-iff* **by** *fast* }
{ **show** *inf F F'* $\leq$ *F* **and** *inf F F'* $\leq$ *F'*
unfolding *le-filter-def eventually-inf* **by** (*auto intro*: *eventually-True*) }
{ **assume** $F \leq F'$ **and** $F \leq F''$ **thus** *F* $\leq$ *inf F' F''*
unfolding *le-filter-def eventually-inf*
by (*auto intro*: *eventually-mono* [*OF eventually-conj*]) }
{ **show** *F* $\leq$ *sup F F'* **and** *F'* $\leq$ *sup F F'*
unfolding *le-filter-def eventually-sup* **by** *simp-all* }
{ **assume** $F \leq F''$ **and** $F' \leq F''$ **thus** *sup F F'* $\leq$ *F''*
unfolding *le-filter-def eventually-sup* **by** *simp* }
{ **assume** $F'' \in S$ **thus** *Inf S* $\leq$ *F''*
unfolding *le-filter-def Inf-filter-def eventually-Sup Ball-def* **by** *simp* }
{ **assume** $\bigwedge F'. F' \in S \Longrightarrow F \leq F'$ **thus** *F* $\leq$ *Inf S*
unfolding *le-filter-def Inf-filter-def eventually-Sup Ball-def* **by** *simp* }
{ **assume** $F \in S$ **thus** *F* $\leq$ *Sup S*
unfolding *le-filter-def eventually-Sup* **by** *simp* }
{ **assume** $\bigwedge F. F \in S \Longrightarrow F \leq F'$ **thus** *Sup S* $\leq$ *F'*
unfolding *le-filter-def eventually-Sup* **by** *simp* }
{ **show** *Inf* {} = (*top*::*'a filter*)
by (*auto simp*: *top-filter-def Inf-filter-def Sup-filter-def*)
(*metis* (*full-types*) *top-filter-def always-eventually eventually-top*) }
{ **show** *Sup* {} = (*bot*::*'a filter*)

by (*auto simp*: *bot-filter-def Sup-filter-def*) }
qed

end

instance *filter* :: (*type*) *distrib-lattice*
proof
fix $F\ G\ H :: 'a\ filter$
show $sup\ F\ (inf\ G\ H) = inf\ (sup\ F\ G)\ (sup\ F\ H)$
proof (*rule order.antisym*)
show $inf\ (sup\ F\ G)\ (sup\ F\ H) \leq sup\ F\ (inf\ G\ H)$
unfolding *le-filter-def eventually-sup*
proof *safe*
fix P **assume** *1*: *eventually* $P\ F$ **and** *2*: *eventually* $P\ (inf\ G\ H)$
from *2* **obtain** $Q\ R$
where QR: *eventually* $Q\ G$ *eventually* $R\ H$ $\bigwedge x.\ Q\ x \Longrightarrow R\ x \Longrightarrow P\ x$
by (*auto simp*: *eventually-inf*)
define Q' **where** $Q' = (\lambda x.\ Q\ x \vee P\ x)$
define R' **where** $R' = (\lambda x.\ R\ x \vee P\ x)$
from *1* **have** *eventually* $Q'\ F$
by (*elim eventually-mono*) (*auto simp*: Q'*-def*)
moreover from *1* **have** *eventually* $R'\ F$
by (*elim eventually-mono*) (*auto simp*: R'*-def*)
moreover from $QR(1)$ **have** *eventually* $Q'\ G$
by (*elim eventually-mono*) (*auto simp*: Q'*-def*)
moreover from $QR(2)$ **have** *eventually* $R'\ H$
by (*elim eventually-mono*)(*auto simp*: R'*-def*)
moreover from QR **have** $P\ x$ **if** $Q'\ x\ R'\ x$ **for** x
using *that* **by** (*auto simp*: Q'*-def* R'*-def*)
ultimately show *eventually* $P\ (inf\ (sup\ F\ G)\ (sup\ F\ H))$
by (*auto simp*: *eventually-inf eventually-sup*)
qed
qed (*auto intro*: *inf.coboundedI1 inf.coboundedI2*)
qed

lemma *filter-leD*:
$F \leq F' \Longrightarrow$ *eventually* $P\ F' \Longrightarrow$ *eventually* $P\ F$
unfolding *le-filter-def* **by** *simp*

lemma *filter-leI*:
$(\bigwedge P.$ *eventually* $P\ F' \Longrightarrow$ *eventually* $P\ F) \Longrightarrow F \leq F'$
unfolding *le-filter-def* **by** *simp*

lemma *eventually-False*:
eventually $(\lambda x.\ False)\ F \longleftrightarrow F = bot$
unfolding *filter-eq-iff* **by** (*auto elim*: *eventually-rev-mp*)

lemma *eventually-frequently*: $F \neq bot \Longrightarrow$ *eventually* $P\ F \Longrightarrow$ *frequently* $P\ F$

using *eventually-conj*[*of P F* $\lambda x.\ \neg\ P\ x$]
by (*auto simp add*: *frequently-def eventually-False*)

lemma *eventually-frequentlyE*:
assumes *eventually P F*
assumes *eventually* ($\lambda x.\ \neg\ P\ x \vee Q\ x$) *F* $F \neq bot$
shows *frequently Q F*
proof –
have *eventually Q F*
using *eventually-conj*[*OF assms*(*1*,*2*),*simplified*] **by** (*auto elim*:*eventually-mono*)
then show *?thesis* **using** *eventually-frequently*[*OF* ‹$F \neq bot$›] **by** *auto*
qed

lemma *eventually-const-iff*: *eventually* ($\lambda x.\ P$) $F \longleftrightarrow P \vee F = bot$
by (*cases P*) (*auto simp*: *eventually-False*)

lemma *eventually-const*[*simp*]: $F \neq bot \Longrightarrow$ *eventually* ($\lambda x.\ P$) $F \longleftrightarrow P$
by (*simp add*: *eventually-const-iff*)

lemma *frequently-const-iff*: *frequently* ($\lambda x.\ P$) $F \longleftrightarrow P \wedge F \neq bot$
by (*simp add*: *frequently-def eventually-const-iff*)

lemma *frequently-const*[*simp*]: $F \neq bot \Longrightarrow$ *frequently* ($\lambda x.\ P$) $F \longleftrightarrow P$
by (*simp add*: *frequently-const-iff*)

lemma *eventually-happens*: *eventually P net* $\Longrightarrow$ $net = bot \vee (\exists x.\ P\ x)$
by (*metis frequentlyE eventually-frequently*)

lemma *eventually-happens'*:
assumes $F \neq bot$ *eventually P F*
shows $\exists x.\ P\ x$
using *assms eventually-frequently frequentlyE* **by** *blast*

abbreviation (*input*) *trivial-limit* :: *'a filter* $\Rightarrow$ *bool*
where *trivial-limit F* $\equiv F = bot$

lemma *trivial-limit-def*: *trivial-limit F* $\longleftrightarrow$ *eventually* ($\lambda x.\ False$) *F*
by (*rule eventually-False* [*symmetric*])

lemma *False-imp-not-eventually*: ($\forall x.\ \neg\ P\ x$) $\Longrightarrow \neg$ *trivial-limit net* $\Longrightarrow \neg$ *eventually* ($\lambda x.\ P\ x$) *net*
by (*simp add*: *eventually-False*)

lemma *trivial-limit-eventually*: *trivial-limit net* $\Longrightarrow$ *eventually P net*
by *simp*

lemma *trivial-limit-eq*: *trivial-limit net* $\longleftrightarrow$ ($\forall P.$ *eventually P net*)
by (*simp add*: *filter-eq-iff*)

lemma *eventually-Inf*: *eventually P* (*Inf B*) $\longleftrightarrow$ ($\exists X \subseteq B$. *finite X* $\wedge$ *eventually P* (*Inf X*))
proof –
  let *?F* = λP. $\exists X \subseteq B$. *finite X* $\wedge$ *eventually P* (*Inf X*)

  have *eventually-F*: *eventually P* (*Abs-filter ?F*) $\longleftrightarrow$ *?F P* **for** *P*
  proof (*rule eventually-Abs-filter is-filter.intro*)+
    show *?F* (λx. *True*)
      by (*rule exI*[*of* - {}]) (*simp add*: *le-fun-def*)
  next
    fix *P Q*
    assume *?F P ?F Q*
    then obtain *X Y* **where**
      X $\subseteq$ *B finite X eventually P* ($\bigsqcap$ *X*)
      Y $\subseteq$ *B finite Y eventually Q* ($\bigsqcap$ *Y*) **by** *blast*
    then show *?F* (λx. *P x* $\wedge$ *Q x*)
      by (*intro exI*[*of* - *X* $\cup$ *Y*]) (*auto simp*: *Inf-union-distrib eventually-inf*)
  next
    fix *P Q*
    assume *?F P*
    then obtain *X* **where** *X* $\subseteq$ *B finite X eventually P* ($\bigsqcap$ *X*)
      by *blast*
    moreover assume $\forall x$. *P x* $\longrightarrow$ *Q x*
    ultimately show *?F Q*
      by (*intro exI*[*of* - *X*]) (*auto elim*: *eventually-mono*)
  qed

  have *Inf B* = *Abs-filter ?F*
  proof (*intro antisym Inf-greatest*)
    show *Inf B* $\leq$ *Abs-filter ?F*
      by (*auto simp*: *le-filter-def eventually-F dest*: *Inf-superset-mono*)
  next
    fix *F* **assume** *F* $\in$ *B* **then show** *Abs-filter ?F* $\leq$ *F*
      by (*auto simp add*: *le-filter-def eventually-F intro*!: *exI*[*of* - {*F*}])
  qed
  then show *?thesis*
    by (*simp add*: *eventually-F*)
qed

lemma *eventually-INF*: *eventually P* ($\bigsqcap b \in B$. *F b*) $\longleftrightarrow$ ($\exists X \subseteq B$. *finite X* $\wedge$ *eventually P* ($\bigsqcap b \in X$. *F b*))
  unfolding *eventually-Inf* [*of P F'B*]
  by (*metis finite-imageI image-mono finite-subset-image*)

lemma *Inf-filter-not-bot*:
  fixes *B* :: *'a filter set*
  shows ($\bigwedge X$. *X* $\subseteq$ *B* $\Longrightarrow$ *finite X* $\Longrightarrow$ *Inf X* $\neq$ *bot*) $\Longrightarrow$ *Inf B* $\neq$ *bot*
  unfolding *trivial-limit-def eventually-Inf*[*of* - *B*]
    bot-bool-def [*symmetric*] *bot-fun-def* [*symmetric*] *bot-unique* **by** *simp*

lemma *INF-filter-not-bot*:
fixes F :: $'i \Rightarrow 'a$ *filter*
shows $(\bigwedge X.\ X \subseteq B \Longrightarrow$ *finite* $X \Longrightarrow (\bigsqcap b{\in}X.\ F\ b) \neq bot) \Longrightarrow (\bigsqcap b{\in}B.\ F\ b) \neq$ *bot*
unfolding *trivial-limit-def eventually-INF* [*of - - B*]
bot-bool-def [*symmetric*] *bot-fun-def* [*symmetric*] *bot-unique* **by** *simp*

lemma *eventually-Inf-base*:
assumes $B \neq \{\}$ **and** *base*: $\bigwedge F\ G.\ F \in B \Longrightarrow G \in B \Longrightarrow \exists x{\in}B.\ x \leq inf\ F\ G$
shows *eventually* P (*Inf* B) $\longleftrightarrow$ ($\exists b{\in}B.$ *eventually* P b)
proof (*subst eventually-Inf*, *safe*)
fix X **assume** *finite* X $X \subseteq B$
then have $\exists b{\in}B.\ \forall x{\in}X.\ b \leq x$
proof *induct*
case *empty* **then show** *?case*
using ‹$B \neq \{\}$› **by** *auto*
next
case (*insert* x X)
then obtain b **where** $b \in B$ $\bigwedge x.\ x \in X \Longrightarrow b \leq x$
by *auto*
with ‹*insert* x $X \subseteq B$› *base*[*of* b x] **show** *?case*
by (*auto intro*: *order-trans*)
qed
then obtain b **where** $b \in B$ $b \leq$ *Inf* X
by (*auto simp*: *le-Inf-iff*)
then show *eventually* P (*Inf* X) $\Longrightarrow$ *Bex* B (*eventually* P)
by (*intro bexI*[*of* - b]) (*auto simp*: *le-filter-def*)
qed (*auto intro*!: *exI*[*of* - $\{x\}$ **for** x])

lemma *eventually-INF-base*:
$B \neq \{\} \Longrightarrow (\bigwedge a\ b.\ a \in B \Longrightarrow b \in B \Longrightarrow \exists x{\in}B.\ F\ x \leq inf\ (F\ a)\ (F\ b)) \Longrightarrow$
eventually P $(\bigsqcap b{\in}B.\ F\ b) \longleftrightarrow (\exists b{\in}B.$ *eventually* P $(F\ b))$
by (*subst eventually-Inf-base*) *auto*

lemma *eventually-INF1*: $i \in I \Longrightarrow$ *eventually* P $(F\ i) \Longrightarrow$ *eventually* P $(\bigsqcap i{\in}I.\ F\ i)$
using *filter-leD*[*OF INF-lower*] **.**

lemma *eventually-INF-finite*:
assumes *finite* A
shows *eventually* P $(\bigsqcap x{\in}A.\ F\ x) \longleftrightarrow$
$(\exists Q.\ (\forall x{\in}A.$ *eventually* $(Q\ x)\ (F\ x)) \wedge (\forall y.\ (\forall x{\in}A.\ Q\ x\ y) \longrightarrow P\ y))$
using *assms*
proof (*induction arbitrary*: P *rule*: *finite-induct*)
case (*insert* a A P)
from *insert.hyps* **have** [*simp*]: $x \neq a$ **if** $x \in A$ **for** x
using *that* **by** *auto*
have *eventually* P $(\bigsqcap x{\in}insert\ a\ A.\ F\ x) \longleftrightarrow$

(∃ Q R S. *eventually* Q (F a) ∧ (((∀ x∈A. *eventually* (S x) (F x)) ∧
(∀ y. (∀ x∈A. S x y) ⟶ R y)) ∧ (∀ x. Q x ∧ R x ⟶ P x)))
unfolding *ex-simps* **by** (*simp add*: *eventually-inf insert.IH*)
also have ... ⟷ (∃ Q. (∀ x∈*insert* a A. *eventually* (Q x) (F x)) ∧
(∀ y. (∀ x∈*insert* a A. Q x y) ⟶ P y))
proof (*safe*, *goal-cases*)
case (*1* Q R S)
thus *?case* **using** *1* **by** (*intro exI*[*of* - S(a := Q)]) *auto*
next
case (*2* Q)
show *?case*
by (*rule exI*[*of* - Q a], *rule exI*[*of* - λy. ∀ x∈A. Q x y],
rule exI[*of* - Q(a := (λ-. True))]) (*use 2* **in** *auto*)
qed
finally show *?case* **.**
qed *auto*

lemma *eventually-le-le*:
fixes P :: 'a ⇒ ('b :: *preorder*)
assumes *eventually* (λx. P x ≤ Q x) F
assumes *eventually* (λx. Q x ≤ R x) F
shows *eventually* (λx. P x ≤ R x) F
using *assms* **by** *eventually-elim* (*rule order-trans*)

90.2.2 Map function for filters

definition *filtermap* :: ('a ⇒ 'b) ⇒ 'a *filter* ⇒ 'b *filter*
where *filtermap f F* = *Abs-filter* (λP. *eventually* (λx. P (f x)) F)

lemma *eventually-filtermap*:
eventually P (*filtermap f F*) = *eventually* (λx. P (f x)) F
unfolding *filtermap-def*
apply (*rule eventually-Abs-filter* [*OF is-filter.intro*])
apply (*auto elim*!: *eventually-rev-mp*)
done

lemma *eventually-comp-filtermap*:
eventually (P ∘ f) F ⟷ *eventually* P (*filtermap f F*)
unfolding *comp-def* **using** *eventually-filtermap* **by** *auto*

lemma *filtermap-compose*: *filtermap* (f ∘ g) F = *filtermap f* (*filtermap g F*)
unfolding *filter-eq-iff* **by** (*simp add*: *eventually-filtermap*)

lemma *filtermap-ident*: *filtermap* (λx. x) F = F
by (*simp add*: *filter-eq-iff eventually-filtermap*)

lemma *filtermap-filtermap*:
filtermap f (*filtermap g F*) = *filtermap* (λx. f (g x)) F
by (*simp add*: *filter-eq-iff eventually-filtermap*)

lemma *filtermap-mono*: $F \le F' \Longrightarrow$ *filtermap f F* $\le$ *filtermap f F'*
 unfolding *le-filter-def eventually-filtermap* **by** *simp*

lemma *filtermap-bot* [*simp*]: *filtermap f bot* = *bot*
 by (*simp add*: *filter-eq-iff eventually-filtermap*)

lemma *filtermap-bot-iff*: *filtermap f F* = *bot* $\longleftrightarrow$ *F* = *bot*
 by (*simp add*: *trivial-limit-def eventually-filtermap*)

lemma *filtermap-sup*: *filtermap f* (*sup F1 F2*) = *sup* (*filtermap f F1*) (*filtermap f F2*)
 by (*simp add*: *filter-eq-iff eventually-filtermap eventually-sup*)

lemma *filtermap-SUP*: *filtermap f* ($\bigsqcup b \in B$. *F b*) = ($\bigsqcup b \in B$. *filtermap f* (*F b*))
 by (*simp add*: *filter-eq-iff eventually-Sup eventually-filtermap*)

lemma *filtermap-inf*: *filtermap f* (*inf F1 F2*) $\le$ *inf* (*filtermap f F1*) (*filtermap f F2*)
 by (*intro inf-greatest filtermap-mono inf-sup-ord*)

lemma *filtermap-INF*: *filtermap f* ($\bigsqcap b \in B$. *F b*) $\le$ ($\bigsqcap b \in B$. *filtermap f* (*F b*))
 by (*rule INF-greatest*, *rule filtermap-mono*, *erule INF-lower*)

lemma *frequently-filtermap*:
 frequently P (*filtermap f F*) = *frequently* (λx. *P* (*f x*)) *F*
 by (*simp add*: *frequently-def eventually-filtermap*)

90.2.3 Contravariant map function for filters

definition *filtercomap* :: ($'a \Rightarrow 'b$) $\Rightarrow$ $'b$ *filter* $\Rightarrow$ $'a$ *filter* **where**
 filtercomap f F = *Abs-filter* (λP. $\exists\, Q$. *eventually Q F* $\wedge$ ($\forall\, x$. *Q* (*f x*) $\longrightarrow$ *P x*))

lemma *eventually-filtercomap*:
 eventually P (*filtercomap f F*) $\longleftrightarrow$ ($\exists\, Q$. *eventually Q F* $\wedge$ ($\forall\, x$. *Q* (*f x*) $\longrightarrow$ *P x*))
 unfolding *filtercomap-def*
proof (*intro eventually-Abs-filter*, *unfold-locales*, *goal-cases*)
 case *1*
 show *?case* **by** (*auto intro*!: *exI*[*of* - λ-. *True*])
next
 case (*2 P Q*)
 then obtain *P' Q'* **where** *P'Q'*:
  eventually P' F $\forall\, x$. *P'* (*f x*) $\longrightarrow$ *P x*
  eventually Q' F $\forall\, x$. *Q'* (*f x*) $\longrightarrow$ *Q x*
  by (*elim exE conjE*)
 show *?case*
  by (*rule exI*[*of* - λx. *P' x* $\wedge$ *Q' x*]) (*use P'Q'* **in** ‹*auto intro*!: *eventually-conj*›)
next

case (*3 P Q*)
thus *?case* **by** *blast*
qed

lemma *filtercomap-ident*: *filtercomap* (*λx. x*) *F* = *F*
by (*auto simp*: *filter-eq-iff eventually-filtercomap elim*!: *eventually-mono*)

lemma *filtercomap-filtercomap*: *filtercomap f* (*filtercomap g F*) = *filtercomap* (*λx. g* (*f x*)) *F*
unfolding *filter-eq-iff* **by** (*auto simp*: *eventually-filtercomap*)

lemma *filtercomap-mono*: $F \leq F' \Longrightarrow$ *filtercomap f F* ≤ *filtercomap f F′*
by (*auto simp*: *eventually-filtercomap le-filter-def*)

lemma *filtercomap-bot* [*simp*]: *filtercomap f bot* = *bot*
by (*auto simp*: *filter-eq-iff eventually-filtercomap*)

lemma *filtercomap-top* [*simp*]: *filtercomap f top* = *top*
by (*auto simp*: *filter-eq-iff eventually-filtercomap*)

lemma *filtercomap-inf*: *filtercomap f* (*inf F1 F2*) = *inf* (*filtercomap f F1*) (*filtercomap f F2*)
unfolding *filter-eq-iff*
proof *safe*
fix *P*
assume *eventually P* (*filtercomap f* (*F1* ⊓ *F2*))
then obtain *Q R S* **where** ∗:
eventually Q F1 eventually R F2 ⋀*x. Q x* ⟹ *R x* ⟹ *S x* ⋀*x. S* (*f x*) ⟹ *P x*
unfolding *eventually-filtercomap eventually-inf* **by** *blast*
from ∗ **have** *eventually* (*λx. Q* (*f x*)) (*filtercomap f F1*)
eventually (*λx. R* (*f x*)) (*filtercomap f F2*)
by (*auto simp*: *eventually-filtercomap*)
with ∗ **show** *eventually P* (*filtercomap f F1* ⊓ *filtercomap f F2*)
unfolding *eventually-inf* **by** *blast*
next
fix *P*
assume *eventually P* (*inf* (*filtercomap f F1*) (*filtercomap f F2*))
then obtain *Q Q′ R R′* **where** ∗:
eventually Q F1 eventually R F2 ⋀*x. Q* (*f x*) ⟹ *Q′ x* ⋀*x. R* (*f x*) ⟹ *R′ x*
⋀*x. Q′ x* ⟹ *R′ x* ⟹ *P x*
unfolding *eventually-filtercomap eventually-inf* **by** *blast*
from ∗ **have** *eventually* (*λx. Q x* ∧ *R x*) (*F1* ⊓ *F2*) **by** (*auto simp*: *eventually-inf*)
with ∗ **show** *eventually P* (*filtercomap f* (*F1* ⊓ *F2*))
by (*auto simp*: *eventually-filtercomap*)
qed

lemma *filtercomap-sup*: *filtercomap f* (*sup F1 F2*) ≥ *sup* (*filtercomap f F1*) (*filtercomap*

f F2)
by (*intro sup-least filtercomap-mono inf-sup-ord*)

lemma *filtercomap-INF*: *filtercomap f* ($\bigsqcap$ *b*∈*B*. *F b*) = ($\bigsqcap$ *b*∈*B*. *filtercomap f* (*F b*))
proof −
have ∗: *filtercomap f* ($\bigsqcap$ *b*∈*B*. *F b*) = ($\bigsqcap$ *b*∈*B*. *filtercomap f* (*F b*)) **if** *finite B* **for** *B*
using *that* **by** *induction* (*simp-all add*: *filtercomap-inf*)
show *?thesis* **unfolding** *filter-eq-iff*
proof
fix *P*
have *eventually P* ($\bigsqcap$ *b*∈*B*. *filtercomap f* (*F b*)) ⟷
(∃ *X*. (*X* ⊆ *B* ∧ *finite X*) ∧ *eventually P* ($\bigsqcap$ *b*∈*X*. *filtercomap f* (*F b*)))
by (*subst eventually-INF*) *blast*
also have ... ⟷ (∃ *X*. (*X* ⊆ *B* ∧ *finite X*) ∧ *eventually P* (*filtercomap f* ($\bigsqcap$ *b*∈*X*. *F b*)))
by (*rule ex-cong*) (*simp add*: ∗)
also have ... ⟷ *eventually P* (*filtercomap f* ($\bigsqcap$(*F* ' *B*)))
unfolding *eventually-filtercomap* **by** (*subst eventually-INF*) *blast*
finally show *eventually P* (*filtercomap f* ($\bigsqcap$(*F* ' *B*))) =
eventually P ($\bigsqcap$ *b*∈*B*. *filtercomap f* (*F b*)) **..**
qed
qed

lemma *filtercomap-SUP*:
filtercomap f ($\bigsqcup$ *b*∈*B*. *F b*) ≥ ($\bigsqcup$ *b*∈*B*. *filtercomap f* (*F b*))
by (*intro SUP-least filtercomap-mono SUP-upper*)

lemma *filtermap-le-iff-le-filtercomap*: *filtermap f F* ≤ *G* ⟷ *F* ≤ *filtercomap f G*
unfolding *le-filter-def eventually-filtermap eventually-filtercomap*
using *eventually-mono* **by** *auto*

lemma *filtercomap-neq-bot*:
assumes ⋀*P*. *eventually P F* ⟹ ∃ *x*. *P* (*f x*)
shows *filtercomap f F* ≠ *bot*
using *assms* **by** (*auto simp*: *trivial-limit-def eventually-filtercomap*)

lemma *filtercomap-neq-bot-surj*:
assumes *F* ≠ *bot* **and** *surj f*
shows *filtercomap f F* ≠ *bot*
proof (*rule filtercomap-neq-bot*)
fix *P* **assume** ∗: *eventually P F*
show ∃ *x*. *P* (*f x*)
proof (*rule ccontr*)
assume ∗∗: ¬(∃ *x*. *P* (*f x*))
from ∗ **have** *eventually* (λ-. *False*) *F*
proof *eventually-elim*
case (*elim x*)

from ‹*surj f*› **obtain** *y* **where** $x = f\ y$ **by** *auto*
with *elim* **and** ∗∗ **show** *False* **by** *auto*
qed
with *assms* **show** *False* **by** (*simp add*: *trivial-limit-def*)
qed
qed

lemma *eventually-filtercomapI* [*intro*]:
assumes *eventually P F*
shows *eventually* (λ*x*. *P* (*f x*)) (*filtercomap f F*)
using *assms* **by** (*auto simp*: *eventually-filtercomap*)

lemma *filtermap-filtercomap*: *filtermap f* (*filtercomap f F*) ≤ *F*
by (*auto simp*: *le-filter-def eventually-filtermap eventually-filtercomap*)

lemma *filtercomap-filtermap*: *filtercomap f* (*filtermap f F*) ≥ *F*
unfolding *le-filter-def eventually-filtermap eventually-filtercomap*
by (*auto elim*!: *eventually-mono*)

90.2.4 Standard filters

definition *principal* :: ′*a set* ⇒ ′*a filter* **where**
principal S = *Abs-filter* (λ*P*. ∀ *x*∈*S*. *P x*)

lemma *eventually-principal*: *eventually P* (*principal S*) ⟷ (∀ *x*∈*S*. *P x*)
unfolding *principal-def*
by (*rule eventually-Abs-filter*, *rule is-filter.intro*) *auto*

lemma *eventually-inf-principal*: *eventually P* (*inf F* (*principal s*)) ⟷ *eventually* (λ*x*. *x* ∈ *s* ⟶ *P x*) *F*
unfolding *eventually-inf eventually-principal* **by** (*auto elim*: *eventually-mono*)

lemma *principal-UNIV*[*simp*]: *principal UNIV* = *top*
by (*auto simp*: *filter-eq-iff eventually-principal*)

lemma *principal-empty*[*simp*]: *principal* {} = *bot*
by (*auto simp*: *filter-eq-iff eventually-principal*)

lemma *principal-eq-bot-iff*: *principal X* = *bot* ⟷ *X* = {}
by (*auto simp add*: *filter-eq-iff eventually-principal*)

lemma *principal-le-iff*[*iff*]: *principal A* ≤ *principal B* ⟷ *A* ⊆ *B*
by (*auto simp*: *le-filter-def eventually-principal*)

lemma *le-principal*: *F* ≤ *principal A* ⟷ *eventually* (λ*x*. *x* ∈ *A*) *F*
unfolding *le-filter-def eventually-principal*
by (*force elim*: *eventually-mono*)

lemma *principal-inject*[*iff*]: *principal A* = *principal B* ⟷ *A* = *B*

unfolding *eq-iff* **by** *simp*

lemma *sup-principal*[*simp*]: *sup* (*principal* A) (*principal* B) = *principal* ($A \cup B$)
unfolding *filter-eq-iff eventually-sup eventually-principal* **by** *auto*

lemma *inf-principal*[*simp*]: *inf* (*principal* A) (*principal* B) = *principal* ($A \cap B$)
unfolding *filter-eq-iff eventually-inf eventually-principal*
by (*auto intro*: *exI*[*of - λx. x ∈ A*] *exI*[*of - λx. x ∈ B*])

lemma *SUP-principal*[*simp*]: ($\bigsqcup i \in I$. *principal* (A i)) = *principal* ($\bigcup i \in I$. A i)
unfolding *filter-eq-iff eventually-Sup* **by** (*auto simp*: *eventually-principal*)

lemma *INF-principal-finite*: *finite* X $\Longrightarrow$ ($\bigsqcap x \in X$. *principal* (f x)) = *principal* ($\bigcap x \in X$. f x)
by (*induct X rule*: *finite-induct*) *auto*

lemma *filtermap-principal*[*simp*]: *filtermap f* (*principal* A) = *principal* (f ‘ A)
unfolding *filter-eq-iff eventually-filtermap eventually-principal* **by** *simp*

lemma *filtercomap-principal*[*simp*]: *filtercomap f* (*principal* A) = *principal* (f −‘ A)
unfolding *filter-eq-iff eventually-filtercomap eventually-principal* **by** *fast*

90.2.5 Order filters

definition *at-top* :: (*'a*::*order*) *filter*
where *at-top* = ($\bigsqcap k$. *principal* {*k* ..})

lemma *at-top-sub*: *at-top* = ($\bigsqcap k \in${*c*::*'a*::*linorder*..}. *principal* {*k* ..})
by (*auto intro*!: *INF-eq max.cobounded1 max.cobounded2 simp*: *at-top-def*)

lemma *eventually-at-top-linorder*: *eventually P at-top* ⟷ (∃ *N*::*'a*::*linorder*. ∀ *n*≥*N*. *P n*)
unfolding *at-top-def*
by (*subst eventually-INF-base*) (*auto simp*: *eventually-principal intro*: *max.cobounded1 max.cobounded2*)

lemma *eventually-filtercomap-at-top-linorder*:
eventually P (*filtercomap f at-top*) ⟷ (∃ *N*::*'a*::*linorder*. ∀ *x*. *f x* ≥ *N* ⟶ *P x*)
by (*auto simp*: *eventually-filtercomap eventually-at-top-linorder*)

lemma *eventually-at-top-linorderI*:
fixes *c*::*'a*::*linorder*
assumes $\bigwedge x.\ c \le x \Longrightarrow P\ x$
shows *eventually P at-top*
using *assms* **by** (*auto simp*: *eventually-at-top-linorder*)

lemma *eventually-ge-at-top* [*simp*]:
eventually (*λx*. (*c*::-::*linorder*) ≤ *x*) *at-top*

unfolding *eventually-at-top-linorder* **by** *auto*

lemma *eventually-at-top-dense*: *eventually P at-top* $\longleftrightarrow$ ($\exists$ *N*::*'a*::{*no-top*, *linorder*}. $\forall$ *n*>*N*. *P n*)
proof −
have *eventually P* ($\bigsqcap$ *k*. *principal* {*k* <..}) $\longleftrightarrow$ ($\exists$ *N*::*'a*. $\forall$ *n*>*N*. *P n*)
by (*subst eventually-INF-base*) (*auto simp*: *eventually-principal intro*: *max.cobounded1 max.cobounded2*)
also have ($\bigsqcap$ *k*. *principal* {*k*::*'a* <..}) = *at-top*
unfolding *at-top-def*
by (*intro INF-eq*) (*auto intro*: *less-imp-le simp*: *Ici-subset-Ioi-iff gt-ex*)
finally show *?thesis* **.**
qed

lemma *eventually-filtercomap-at-top-dense*:
eventually P (*filtercomap f at-top*) $\longleftrightarrow$ ($\exists$ *N*::*'a*::{*no-top*, *linorder*}. $\forall$ *x*. *f x* > *N* $\longrightarrow$ *P x*)
by (*auto simp*: *eventually-filtercomap eventually-at-top-dense*)

lemma *eventually-at-top-not-equal* [*simp*]: *eventually* (λ*x*::*'a*::{*no-top*, *linorder*}. *x* $\neq$ *c*) *at-top*
unfolding *eventually-at-top-dense* **by** *auto*

lemma *eventually-gt-at-top* [*simp*]: *eventually* (λ*x*. (*c*::-::{*no-top*, *linorder*}) < *x*) *at-top*
unfolding *eventually-at-top-dense* **by** *auto*

lemma *eventually-all-ge-at-top*:
assumes *eventually P* (*at-top* :: (*'a* :: *linorder*) *filter*)
shows *eventually* (λ*x*. $\forall$ *y*$\geq$*x*. *P y*) *at-top*
proof −
from *assms* **obtain** *x* **where** $\bigwedge$*y*. *y* $\geq$ *x* $\Longrightarrow$ *P y* **by** (*auto simp*: *eventually-at-top-linorder*)
hence $\forall$ *z*$\geq$*y*. *P z* **if** *y* $\geq$ *x* **for** *y* **using** *that* **by** *simp*
thus *?thesis* **by** (*auto simp*: *eventually-at-top-linorder*)
qed

definition *at-bot* :: (*'a*::*order*) *filter*
where *at-bot* = ($\bigsqcap$ *k*. *principal* {.. *k*})

lemma *at-bot-sub*: *at-bot* = ($\bigsqcap$ *k*$\in${.. *c*::*'a*::*linorder*}. *principal* {.. *k*})
by (*auto intro*!: *INF-eq min.cobounded1 min.cobounded2 simp*: *at-bot-def*)

lemma *eventually-at-bot-linorder*:
fixes *P* :: *'a*::*linorder* $\Rightarrow$ *bool* **shows** *eventually P at-bot* $\longleftrightarrow$ ($\exists$ *N*. $\forall$ *n*$\leq$*N*. *P n*)
unfolding *at-bot-def*
by (*subst eventually-INF-base*) (*auto simp*: *eventually-principal intro*: *min.cobounded1 min.cobounded2*)

lemma *eventually-filtercomap-at-bot-linorder*:
eventually P (*filtercomap f at-bot*) ⟷ (∃ *N*::′*a*::*linorder*. ∀ *x*. *f x* ≤ *N* ⟶ *P x*)
by (*auto simp*: *eventually-filtercomap eventually-at-bot-linorder*)

lemma *eventually-le-at-bot* [*simp*]:
eventually (λ*x*. *x* ≤ (*c*::-::*linorder*)) *at-bot*
unfolding *eventually-at-bot-linorder* **by** *auto*

lemma *eventually-at-bot-dense*: *eventually P at-bot* ⟷ (∃ *N*::′*a*::{*no-bot*, *linorder*}. ∀ *n*<*N*. *P n*)
proof −
have *eventually P* (⨅ *k*. *principal* {..< *k*}) ⟷ (∃ *N*::′*a*. ∀ *n*<*N*. *P n*)
by (*subst eventually-INF-base*) (*auto simp*: *eventually-principal intro*: *min.cobounded1 min.cobounded2*)
also have (⨅ *k*. *principal* {..< *k*::′*a*}) = *at-bot*
unfolding *at-bot-def*
by (*intro INF-eq*) (*auto intro*: *less-imp-le simp*: *Iic-subset-Iio-iff lt-ex*)
finally show *?thesis* **.**
qed

lemma *eventually-filtercomap-at-bot-dense*:
eventually P (*filtercomap f at-bot*) ⟷ (∃ *N*::′*a*::{*no-bot*, *linorder*}. ∀ *x*. *f x* < *N* ⟶ *P x*)
by (*auto simp*: *eventually-filtercomap eventually-at-bot-dense*)

lemma *eventually-at-bot-not-equal* [*simp*]: *eventually* (λ*x*::′*a*::{*no-bot*, *linorder*}. *x* ≠ *c*) *at-bot*
unfolding *eventually-at-bot-dense* **by** *auto*

lemma *eventually-gt-at-bot* [*simp*]:
eventually (λ*x*. *x* < (*c*::-::*unbounded-dense-linorder*)) *at-bot*
unfolding *eventually-at-bot-dense* **by** *auto*

lemma *trivial-limit-at-bot-linorder* [*simp*]: ¬ *trivial-limit* (*at-bot* ::(′*a*::*linorder*) *filter*)
unfolding *trivial-limit-def*
by (*metis eventually-at-bot-linorder order-refl*)

lemma *trivial-limit-at-top-linorder* [*simp*]: ¬ *trivial-limit* (*at-top* ::(′*a*::*linorder*) *filter*)
unfolding *trivial-limit-def*
by (*metis eventually-at-top-linorder order-refl*)

90.3 Sequentially

abbreviation *sequentially* :: *nat filter*
where *sequentially* ≡ *at-top*

lemma *eventually-sequentially*:

eventually P sequentially $\longleftrightarrow$ ($\exists$ *N*. $\forall$ *n*$\geq$*N*. *P n*)
by (*rule eventually-at-top-linorder*)

lemma *frequently-sequentially*:
frequently P sequentially $\longleftrightarrow$ ($\forall$ *N*. $\exists$ *n*$\geq$*N*. *P n*)
by (*simp add*: *frequently-def eventually-sequentially*)

lemma *sequentially-bot* [*simp*, *intro*]: *sequentially* $\neq$ *bot*
unfolding *filter-eq-iff eventually-sequentially* **by** *auto*

lemmas *trivial-limit-sequentially* = *sequentially-bot*

lemma *eventually-False-sequentially* [*simp*]:
$\neg$ *eventually* (λ*n*. *False*) *sequentially*
by (*simp add*: *eventually-False*)

lemma *le-sequentially*:
F $\leq$ *sequentially* $\longleftrightarrow$ ($\forall$ *N*. *eventually* (λ*n*. *N* $\leq$ *n*) *F*)
by (*simp add*: *at-top-def le-INF-iff le-principal*)

lemma *eventually-sequentiallyI* [*intro?*]:
assumes $\bigwedge$*x*. *c* $\leq$ *x* $\Longrightarrow$ *P x*
shows *eventually P sequentially*
using *assms* **by** (*auto simp*: *eventually-sequentially*)

lemma *eventually-sequentially-Suc* [*simp*]: *eventually* (λ*i*. *P* (*Suc i*)) *sequentially* $\longleftrightarrow$ *eventually P sequentially*
unfolding *eventually-sequentially* **by** (*metis Suc-le-D Suc-le-mono le-Suc-eq*)

lemma *eventually-sequentially-seg* [*simp*]: *eventually* (λ*n*. *P* (*n* + *k*)) *sequentially* $\longleftrightarrow$ *eventually P sequentially*
using *eventually-sequentially-Suc*[*of* λ*n*. *P* (*n* + *k*) **for** *k*] **by** (*induction k*) *auto*

lemma *filtermap-sequentually-ne-bot*: *filtermap f sequentially* $\neq$ *bot*
by (*simp add*: *filtermap-bot-iff*)

90.4 Increasing finite subsets

definition *finite-subsets-at-top* **where**
finite-subsets-at-top A = ($\prod$ *X*$\in${*X*. *finite X* $\wedge$ *X* $\subseteq$ *A*}. *principal* {*Y*. *finite Y* $\wedge$ *X* $\subseteq$ *Y* $\wedge$ *Y* $\subseteq$ *A*})

lemma *eventually-finite-subsets-at-top*:
eventually P (*finite-subsets-at-top A*) $\longleftrightarrow$
($\exists$ *X*. *finite X* $\wedge$ *X* $\subseteq$ *A* $\wedge$ ($\forall$ *Y*. *finite Y* $\wedge$ *X* $\subseteq$ *Y* $\wedge$ *Y* $\subseteq$ *A* $\longrightarrow$ *P Y*))
unfolding *finite-subsets-at-top-def*
proof (*subst eventually-INF-base*, *goal-cases*)
show {*X*. *finite X* $\wedge$ *X* $\subseteq$ *A*} $\neq$ {} **by** *auto*
next

case (*2 B C*)
thus *?case* **by** (*intro bexI*[*of - B* ∪ *C*]) *auto*
qed (*simp-all add*: *eventually-principal*)

lemma *eventually-finite-subsets-at-top-weakI* [*intro*]:
assumes $\bigwedge X$. *finite X* ⟹ *X* ⊆ *A* ⟹ *P X*
shows *eventually P* (*finite-subsets-at-top A*)
proof −
have *eventually* (λ*X*. *finite X* ∧ *X* ⊆ *A*) (*finite-subsets-at-top A*)
by (*auto simp*: *eventually-finite-subsets-at-top*)
thus *?thesis* **by** *eventually-elim* (*use assms* **in** *auto*)
qed

lemma *finite-subsets-at-top-neq-bot* [*simp*]: *finite-subsets-at-top A* ≠ *bot*
proof −
have ¬*eventually* (λ*x*. *False*) (*finite-subsets-at-top A*)
by (*auto simp*: *eventually-finite-subsets-at-top*)
thus *?thesis* **by** *auto*
qed

lemma *filtermap-image-finite-subsets-at-top*:
assumes *inj-on f A*
shows *filtermap* ((') *f*) (*finite-subsets-at-top A*) = *finite-subsets-at-top* (*f* ' *A*)
unfolding *filter-eq-iff eventually-filtermap*
proof (*safe*, *goal-cases*)
case (*1 P*)
then obtain *X* **where** *X*: *finite X X* ⊆ *A* $\bigwedge Y$. *finite Y* ⟹ *X* ⊆ *Y* ⟹ *Y* ⊆ *A* ⟹ *P* (*f* ' *Y*)
unfolding *eventually-finite-subsets-at-top* **by** *force*
show *?case* **unfolding** *eventually-finite-subsets-at-top eventually-filtermap*
proof (*rule exI*[*of - f* ' *X*], *intro conjI allI impI*, *goal-cases*)
case (*3 Y*)
with *assms* **and** *X*(*1*,*2*) **have** *P* (*f* ' (*f* −' *Y* ∩ *A*)) **using** *X*(*1*,*2*)
by (*intro X*(*3*) *finite-vimage-IntI*) *auto*
also have *f* ' (*f* −' *Y* ∩ *A*) = *Y* **using** *assms 3* **by** *blast*
finally show *?case* **.**
qed (*insert assms X*(*1*,*2*), *auto intro*!: *finite-vimage-IntI*)
next
case (*2 P*)
then obtain *X* **where** *X*: *finite X X* ⊆ *f* ' *A* $\bigwedge Y$. *finite Y* ⟹ *X* ⊆ *Y* ⟹ *Y* ⊆ *f* ' *A* ⟹ *P Y*
unfolding *eventually-finite-subsets-at-top* **by** *force*
show *?case* **unfolding** *eventually-finite-subsets-at-top eventually-filtermap*
proof (*rule exI*[*of - f* −' *X* ∩ *A*], *intro conjI allI impI*, *goal-cases*)
case (*3 Y*)
with *X*(*1*,*2*) **and** *assms* **show** *?case* **by** (*intro X*(*3*)) *force+*
qed (*insert assms X*(*1*), *auto intro*!: *finite-vimage-IntI*)
qed

lemma *eventually-finite-subsets-at-top-finite*:
assumes *finite A*
shows *eventually P* (*finite-subsets-at-top A*) $\longleftrightarrow$ *P A*
unfolding *eventually-finite-subsets-at-top* **using** *assms* **by** *force*

lemma *finite-subsets-at-top-finite*: *finite A* $\Longrightarrow$ *finite-subsets-at-top A* = *principal* $\{A\}$
by (*auto simp*: *filter-eq-iff eventually-finite-subsets-at-top-finite eventually-principal*)

90.5 The cofinite filter

definition *cofinite* = *Abs-filter* (λP. *finite* $\{x. \neg P\ x\}$)

abbreviation *Inf-many* :: ($'a \Rightarrow bool$) $\Rightarrow$ *bool* (**binder** ‹$\exists_\infty$› *10*)
where *Inf-many P* $\equiv$ *frequently P cofinite*

abbreviation *Alm-all* :: ($'a \Rightarrow bool$) $\Rightarrow$ *bool* (**binder** ‹$\forall_\infty$› *10*)
where *Alm-all P* $\equiv$ *eventually P cofinite*

notation (*ASCII*)
Inf-many (**binder** ‹*INFM* › *10*) **and**
Alm-all (**binder** ‹*MOST* › *10*)

lemma *eventually-cofinite*: *eventually P cofinite* $\longleftrightarrow$ *finite* $\{x. \neg P\ x\}$
unfolding *cofinite-def*
proof (*rule eventually-Abs-filter*, *rule is-filter.intro*)
fix *P Q* :: $'a \Rightarrow bool$ **assume** *finite* $\{x. \neg P\ x\}$ *finite* $\{x. \neg Q\ x\}$
from *finite-UnI*[*OF this*] **show** *finite* $\{x. \neg (P\ x \wedge Q\ x)\}$
by (*rule rev-finite-subset*) *auto*
next
fix *P Q* :: $'a \Rightarrow bool$ **assume** *P*: *finite* $\{x. \neg P\ x\}$ **and** $*$: $\forall x.\ P\ x \longrightarrow Q\ x$
from $*$ **show** *finite* $\{x. \neg Q\ x\}$
by (*intro finite-subset*[*OF - P*]) *auto*
qed *simp*

lemma *frequently-cofinite*: *frequently P cofinite* $\longleftrightarrow$ $\neg$ *finite* $\{x. P\ x\}$
by (*simp add*: *frequently-def eventually-cofinite*)

lemma *cofinite-bot*[*simp*]: *cofinite* = (*bot*::$'a$ *filter*) $\longleftrightarrow$ *finite* (*UNIV* :: $'a$ *set*)
unfolding *trivial-limit-def eventually-cofinite* **by** *simp*

lemma *cofinite-eq-sequentially*: *cofinite* = *sequentially*
unfolding *filter-eq-iff eventually-sequentially eventually-cofinite*
proof *safe*
fix *P* :: *nat* $\Rightarrow$ *bool* **assume** [*simp*]: *finite* $\{x. \neg P\ x\}$
show $\exists N.\ \forall n \geq N.\ P\ n$
proof *cases*
assume $\{x. \neg P\ x\} \neq \{\}$ **then show** *?thesis*
by (*intro exI*[*of - Suc* (*Max* $\{x. \neg P\ x\}$)]) (*auto simp*: *Suc-le-eq*)

qed *auto*
next
fix P :: *nat* ⇒ *bool* **and** N :: *nat* **assume** $\forall n\geq N.\ P\ n$
then have $\{x.\ \neg\ P\ x\} \subseteq \{..< N\}$
by (*auto simp*: *not-le*)
then show *finite* $\{x.\ \neg\ P\ x\}$
by (*blast intro*: *finite-subset*)
qed

90.5.1 Product of filters

definition *prod-filter* :: *'a filter* ⇒ *'b filter* ⇒ (*'a* × *'b*) *filter* (**infixr** ‹$\times_F$› *80*)
where
prod-filter F G =
($\bigsqcap$(P, Q)∈{(P, Q). *eventually* P F ∧ *eventually* Q G}. *principal* {(x, y). P x ∧ Q y})

lemma *eventually-prod-filter*: *eventually* P (F $\times_F$ G) ⟷
(∃ Pf Pg. *eventually* Pf F ∧ *eventually* Pg G ∧ (∀ x y. Pf x ⟶ Pg y ⟶ P (x, y)))
unfolding *prod-filter-def*
proof (*subst eventually-INF-base*, *goal-cases*)
case *2*
moreover have *eventually* Pf F ⟹ *eventually* Qf F ⟹ *eventually* Pg G ⟹ *eventually* Qg G ⟹
∃ P Q. *eventually* P F ∧ *eventually* Q G ∧
Collect P × *Collect* Q ⊆ *Collect* Pf × *Collect* Pg ∩ *Collect* Qf × *Collect* Qg
for Pf Pg Qf Qg
by (*intro conjI exI*[*of* - *inf* Pf Qf] *exI*[*of* - *inf* Pg Qg])
(*auto simp*: *inf-fun-def eventually-conj*)
ultimately show *?case*
by *auto*
qed (*auto simp*: *eventually-principal intro*: *eventually-True*)

lemma *eventually-prod1*:
assumes B ≠ *bot*
shows ($\forall_F$ (x, y) *in* A $\times_F$ B. P x) ⟷ ($\forall_F$ x *in* A. P x)
unfolding *eventually-prod-filter*
proof *safe*
fix R Q
assume ∗: $\forall_F$ x *in* A. R x $\forall_F$ x *in* B. Q x ∀ x y. R x ⟶ Q y ⟶ P x
with ‹B ≠ *bot*› **obtain** y **where** Q y **by** (*auto dest*: *eventually-happens*)
with ∗ **show** *eventually* P A
by (*force elim*: *eventually-mono*)
next
assume *eventually* P A
then show ∃ Pf Pg. *eventually* Pf A ∧ *eventually* Pg B ∧ (∀ x y. Pf x ⟶ Pg y ⟶ P x)
by (*intro exI*[*of* - P] *exI*[*of* - λx. *True*]) *auto*

qed

lemma *eventually-prod2*:
 assumes $A \neq bot$
 shows $(\forall_F (x, y)$ *in* $A \times_F B.\ P\ y) \longleftrightarrow (\forall_F\ y$ *in* $B.\ P\ y)$
 unfolding *eventually-prod-filter*
proof *safe*
 fix $R\ Q$
 assume $*$: $\forall_F\ x$ *in* $A.\ R\ x\ \forall_F\ x$ *in* $B.\ Q\ x\ \forall x\ y.\ R\ x \longrightarrow Q\ y \longrightarrow P\ y$
 with ‹$A \neq bot$› **obtain** x **where** $R\ x$ **by** (*auto dest*: *eventually-happens*)
 with $*$ **show** *eventually P B*
  by (*force elim*: *eventually-mono*)
next
 assume *eventually P B*
 then show $\exists Pf\ Pg.$ *eventually Pf A* $\wedge$ *eventually Pg B* $\wedge$ $(\forall x\ y.\ Pf\ x \longrightarrow Pg\ y \longrightarrow P\ y)$
  by (*intro exI*[*of* - *P*] *exI*[*of* - $\lambda x.$ *True*]) *auto*
qed

lemma *INF-filter-bot-base*:
 fixes $F :: 'a \Rightarrow 'b$ *filter*
 assumes $*$: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow \exists k{\in}I.\ F\ k \leq F\ i \sqcap F\ j$
 shows $(\bigsqcap i{\in}I.\ F\ i) = bot \longleftrightarrow (\exists i{\in}I.\ F\ i = bot)$
proof (*cases* $\exists i{\in}I.\ F\ i = bot$)
 case *True*
 then have $(\bigsqcap i{\in}I.\ F\ i) \leq bot$
  by (*auto intro*: *INF-lower2*)
 with *True* **show** *?thesis*
  by (*auto simp*: *bot-unique*)
next
 case *False*
 moreover have $(\bigsqcap i{\in}I.\ F\ i) \neq bot$
 proof (*cases* $I = \{\}$)
  case *True*
  then show *?thesis*
   by (*auto simp add*: *filter-eq-iff*)
 next
  case *False'*: *False*
  show *?thesis*
  proof (*rule INF-filter-not-bot*)
   fix J
   assume *finite* $J\ J \subseteq I$
   then have $\exists k{\in}I.\ F\ k \leq (\bigsqcap i{\in}J.\ F\ i)$
   proof (*induct J*)
    case *empty*
    then show *?case*
     using ‹$I \neq \{\}$› **by** *auto*
   next
    case (*insert i J*)

then obtain k **where** $k \in I$ F $k \leq (\prod i{\in}J.\ F\ i)$ **by** *auto*
with *insert* $*[of\ i\ k]$ **show** *?case*
by *auto*
qed
with *False* **show** $(\prod i{\in}J.\ F\ i) \neq \bot$
by (*auto simp*: *bot-unique*)
qed
qed
ultimately show *?thesis*
by *auto*
qed

lemma *Collect-empty-eq-bot*: $Collect\ P = \{\} \longleftrightarrow P = \bot$
by *auto*

lemma *prod-filter-eq-bot*: $A \times_F B = bot \longleftrightarrow A = bot \lor B = bot$
unfolding *trivial-limit-def*
proof
assume $\forall_F\ x\ in\ A \times_F B.\ False$
then obtain *Pf Pg*
where *Pf*: *eventually* $(\lambda x.\ Pf\ x)$ A **and** *Pg*: *eventually* $(\lambda y.\ Pg\ y)$ B
and $*$: $\forall x\ y.\ Pf\ x \longrightarrow Pg\ y \longrightarrow False$
unfolding *eventually-prod-filter* **by** *fast*
from $*$ **have** $(\forall x.\ \neg\ Pf\ x) \lor (\forall y.\ \neg\ Pg\ y)$ **by** *fast*
with *Pf Pg* **show** $(\forall_F\ x\ in\ A.\ False) \lor (\forall_F\ x\ in\ B.\ False)$ **by** *auto*
next
assume $(\forall_F\ x\ in\ A.\ False) \lor (\forall_F\ x\ in\ B.\ False)$
then show $\forall_F\ x\ in\ A \times_F B.\ False$
unfolding *eventually-prod-filter* **by** (*force intro*: *eventually-True*)
qed

lemma *prod-filter-mono*: $F \leq F' \Longrightarrow G \leq G' \Longrightarrow F \times_F G \leq F' \times_F G'$
by (*auto simp*: *le-filter-def eventually-prod-filter*)

lemma *prod-filter-mono-iff*:
assumes *nAB*: $A \neq bot$ $B \neq bot$
shows $A \times_F B \leq C \times_F D \longleftrightarrow A \leq C \land B \leq D$
proof *safe*
assume $*$: $A \times_F B \leq C \times_F D$
with *assms* **have** $A \times_F B \neq bot$
by (*auto simp*: *bot-unique prod-filter-eq-bot*)
with $*$ **have** $C \times_F D \neq bot$
by (*auto simp*: *bot-unique*)
then have *nCD*: $C \neq bot$ $D \neq bot$
by (*auto simp*: *prod-filter-eq-bot*)

show $A \leq C$
proof (*rule filter-leI*)
fix P **assume** *eventually* P C **with** $*[THEN\ filter\text{-}leD,\ of\ \lambda(x,\ y).\ P\ x]$ **show**

eventually P A
 using *nAB nCD* **by** (*simp add*: *eventually-prod1 eventually-prod2*)
 qed

 show $B \leq D$
 proof (*rule filter-leI*)
 fix *P* **assume** *eventually P D* **with** ∗[*THEN filter-leD*, *of* λ(*x*, *y*). *P y*] **show** *eventually P B*
 using *nAB nCD* **by** (*simp add*: *eventually-prod1 eventually-prod2*)
 qed
qed (*intro prod-filter-mono*)

lemma *eventually-prod-same*: *eventually P* ($F \times_F F$) ⟷
 (∃ *Q*. *eventually Q F* ∧ (∀ *x y*. *Q x* ⟶ *Q y* ⟶ *P* (*x*, *y*)))
 unfolding *eventually-prod-filter* **by** (*blast intro*!: *eventually-conj*)

lemma *eventually-prod-sequentially*:
 eventually P (*sequentially* $\times_F$ *sequentially*) ⟷ (∃ *N*. ∀ $m \geq N$. ∀ $n \geq N$. *P* (*n*, *m*))
 unfolding *eventually-prod-same eventually-sequentially* **by** *auto*

lemma *principal-prod-principal*: *principal A* $\times_F$ *principal B* = *principal* ($A \times B$)
 unfolding *filter-eq-iff eventually-prod-filter eventually-principal*
 by (*fast intro*: *exI*[*of* - λ*x*. *x* ∈ *A*] *exI*[*of* - λ*x*. *x* ∈ *B*])

lemma *le-prod-filterI*:
 filtermap fst F ≤ *A* ⟹ *filtermap snd F* ≤ *B* ⟹ *F* ≤ *A* $\times_F$ *B*
 unfolding *le-filter-def eventually-filtermap eventually-prod-filter*
 by (*force elim*: *eventually-elim2*)

lemma *filtermap-fst-prod-filter*: *filtermap fst* (*A* $\times_F$ *B*) ≤ *A*
 unfolding *le-filter-def eventually-filtermap eventually-prod-filter*
 by (*force intro*: *eventually-True*)

lemma *filtermap-snd-prod-filter*: *filtermap snd* (*A* $\times_F$ *B*) ≤ *B*
 unfolding *le-filter-def eventually-filtermap eventually-prod-filter*
 by (*force intro*: *eventually-True*)

lemma *prod-filter-INF*:
 assumes $I \neq \{\}$ **and** $J \neq \{\}$
 shows (⨅*i*∈*I*. *A i*) $\times_F$ (⨅*j*∈*J*. *B j*) = (⨅*i*∈*I*. ⨅*j*∈*J*. *A i* $\times_F$ *B j*)
proof (*rule antisym*)
 from ‹$I \neq \{\}$› **obtain** *i* **where** *i* ∈ *I* **by** *auto*
 from ‹$J \neq \{\}$› **obtain** *j* **where** *j* ∈ *J* **by** *auto*

 show (⨅*i*∈*I*. ⨅*j*∈*J*. *A i* $\times_F$ *B j*) ≤ (⨅*i*∈*I*. *A i*) $\times_F$ (⨅*j*∈*J*. *B j*)
 by (*fast intro*: *le-prod-filterI INF-greatest INF-lower2*
 order-trans[*OF filtermap-INF*] ‹*i* ∈ *I*› ‹*j* ∈ *J*›
 filtermap-fst-prod-filter filtermap-snd-prod-filter)

show ($\bigsqcap i{\in}I.\ A\ i$) $\times_F$ ($\bigsqcap j{\in}J.\ B\ j$) $\leq$ ($\bigsqcap i{\in}I.\ \bigsqcap j{\in}J.\ A\ i \times_F B\ j$)
by (*intro INF-greatest prod-filter-mono INF-lower*)
qed

lemma *filtermap-Pair*: *filtermap* ($\lambda x.$ ($f\ x$, $g\ x$)) $F \leq$ *filtermap* $f\ F \times_F$ *filtermap* $g\ F$
by (*rule le-prod-filterI*, *simp-all add*: *filtermap-filtermap*)

lemma *eventually-prodI*: *eventually* $P\ F \Longrightarrow$ *eventually* $Q\ G \Longrightarrow$ *eventually* ($\lambda x.$ P (*fst* x) $\wedge$ Q (*snd* x)) ($F \times_F G$)
unfolding *eventually-prod-filter* **by** *auto*

lemma *prod-filter-INF1*: $I \neq \{\} \Longrightarrow$ ($\bigsqcap i{\in}I.\ A\ i$) $\times_F B$ = ($\bigsqcap i{\in}I.\ A\ i \times_F B$)
using *prod-filter-INF*[*of* I $\{B\}$ A $\lambda x.\ x$] **by** *simp*

lemma *prod-filter-INF2*: $J \neq \{\} \Longrightarrow A \times_F$ ($\bigsqcap i{\in}J.\ B\ i$) = ($\bigsqcap i{\in}J.\ A \times_F B\ i$)
using *prod-filter-INF*[*of* $\{A\}$ J $\lambda x.\ x\ B$] **by** *simp*

lemma *prod-filtermap1*: *prod-filter* (*filtermap* $f\ F$) G = *filtermap* (*apfst* f) (*prod-filter* $F\ G$)
unfolding *filter-eq-iff eventually-filtermap eventually-prod-filter*
apply *safe*
subgoal by *auto*
subgoal for $P\ Q\ R$ **by**(*rule exI*[**where** $x{=}\lambda y.\ \exists x.\ y = f\ x \wedge Q\ x$])(*auto intro*: *eventually-mono*)
done

lemma *prod-filtermap2*: *prod-filter* F (*filtermap* $g\ G$) = *filtermap* (*apsnd* g) (*prod-filter* $F\ G$)
unfolding *filter-eq-iff eventually-filtermap eventually-prod-filter*
apply *safe*
subgoal by *auto*
subgoal for $P\ Q\ R$ **by**(*auto intro*: *exI*[**where** $x{=}\lambda y.\ \exists x.\ y = g\ x \wedge R\ x$] *eventually-mono*)
done

lemma *prod-filter-assoc*:
prod-filter (*prod-filter* $F\ G$) H = *filtermap* ($\lambda(x, y, z).$ ((x, y), z)) (*prod-filter* F (*prod-filter* $G\ H$))
apply(*clarsimp simp add*: *filter-eq-iff eventually-filtermap eventually-prod-filter*; *safe*)
subgoal for $P\ Q\ R\ S\ T$ **by**(*auto 4 4 intro*: *exI*[**where** $x{=}\lambda(a, b).\ T\ a \wedge S\ b$])
subgoal for $P\ Q\ R\ S\ T$ **by**(*auto 4 3 intro*: *exI*[**where** $x{=}\lambda(a, b).\ Q\ a \wedge S\ b$])
done

lemma *prod-filter-principal-singleton*: *prod-filter* (*principal* $\{x\}$) F = *filtermap* (*Pair* x) F
by(*fastforce simp add*: *filter-eq-iff eventually-prod-filter eventually-principal eventually-filtermap elim*: *eventually-mono intro*: *exI*[**where** $x{=}\lambda a.\ a = x$])

lemma *prod-filter-principal-singleton2*: *prod-filter F* (*principal* {*x*}) = *filtermap* (λ*a*. (*a*, *x*)) *F*
by(*fastforce simp add*: *filter-eq-iff eventually-prod-filter eventually-principal eventually-filtermap elim*: *eventually-mono intro*: *exI*[**where** *x*=λ*a*. *a* = *x*])

lemma *prod-filter-commute*: *prod-filter F G* = *filtermap prod.swap* (*prod-filter G F*)
by(*auto simp add*: *filter-eq-iff eventually-prod-filter eventually-filtermap*)

90.6 Limits

definition *filterlim* :: ('*a* ⇒ '*b*) ⇒ '*b filter* ⇒ '*a filter* ⇒ *bool* **where**
filterlim f F2 F1 ⟷ *filtermap f F1* ≤ *F2*

syntax
-LIM :: *pttrns* ⇒ '*a* ⇒ '*b* ⇒ '*a* ⇒ *bool* (‹(‹*indent*=*3 notation*=‹*binder LIM*››*LIM* (-)/ (-)./ (-) :> (-))› [*1000*, *10*, *0*, *10*] *10*)
syntax-consts
-LIM == *filterlim*
translations
LIM x F1. *f* :> *F2* == *CONST filterlim* (λ*x*. *f*) *F2 F1*

lemma *filterlim-filtercomapI*: *filterlim f F G* ⟹ *filterlim* (λ*x*. *f* (*g x*)) *F* (*filtercomap g G*)
unfolding *filterlim-def*
by (*metis order-trans filtermap-filtercomap filtermap-filtermap filtermap-mono*)

lemma *filterlim-top* [*simp*]: *filterlim f top F*
by (*simp add*: *filterlim-def*)

lemma *filterlim-iff*:
(*LIM x F1*. *f x* :> *F2*) ⟷ (∀ *P*. *eventually P F2* ⟶ *eventually* (λ*x*. *P* (*f x*)) *F1*)
unfolding *filterlim-def le-filter-def eventually-filtermap* **..**

lemma *filterlim-compose*:
filterlim g F3 F2 ⟹ *filterlim f F2 F1* ⟹ *filterlim* (λ*x*. *g* (*f x*)) *F3 F1*
unfolding *filterlim-def filtermap-filtermap*[*symmetric*] **by** (*metis filtermap-mono order-trans*)

lemma *filterlim-mono*:
filterlim f F2 F1 ⟹ *F2* ≤ *F2'* ⟹ *F1'* ≤ *F1* ⟹ *filterlim f F2' F1'*
unfolding *filterlim-def* **by** (*metis filtermap-mono order-trans*)

lemma *filterlim-ident*: *LIM x F*. *x* :> *F*
by (*simp add*: *filterlim-def filtermap-ident*)

lemma *filterlim-cong*:

F1 = *F1'* ⟹ *F2* = *F2'* ⟹ *eventually* (λ*x*. *f x* = *g x*) *F2* ⟹ *filterlim f F1 F2* = *filterlim g F1' F2'*
by (*auto simp*: *filterlim-def le-filter-def eventually-filtermap elim*: *eventually-elim2*)

lemma *filterlim-mono-eventually*:
assumes *filterlim f F G* **and** *ord*: *F* ≤ *F' G'* ≤ *G*
assumes *eq*: *eventually* (λ*x*. *f x* = *f' x*) *G'*
shows *filterlim f' F' G'*
proof −
have *filterlim f F' G'*
by (*simp add*: *filterlim-mono*[*OF* - *ord*] *assms*)
then show *?thesis*
by (*rule filterlim-cong*[*OF refl refl eq*, *THEN iffD1*])
qed

lemma *filtermap-mono-strong*: *inj f* ⟹ *filtermap f F* ≤ *filtermap f G* ⟷ *F* ≤ *G*
apply (*safe intro*!: *filtermap-mono*)
apply (*auto simp*: *le-filter-def eventually-filtermap*)
apply (*erule-tac x*=λ*x*. *P* (*inv f x*) **in** *allE*)
apply *auto*
done

lemma *eventually-compose-filterlim*:
assumes *eventually P F filterlim f F G*
shows *eventually* (λ*x*. *P* (*f x*)) *G*
using *assms* **by** (*simp add*: *filterlim-iff*)

lemma *filtermap-eq-strong*: *inj f* ⟹ *filtermap f F* = *filtermap f G* ⟷ *F* = *G*
by (*simp add*: *filtermap-mono-strong eq-iff*)

lemma *filtermap-fun-inverse*:
assumes *g*: *filterlim g F G*
assumes *f*: *filterlim f G F*
assumes *ev*: *eventually* (λ*x*. *f* (*g x*) = *x*) *G*
shows *filtermap f F* = *G*
proof (*rule antisym*)
show *filtermap f F* ≤ *G*
using *f* **unfolding** *filterlim-def* **.**
have *G* = *filtermap f* (*filtermap g G*)
using *ev* **by** (*auto elim*: *eventually-elim2 simp*: *filter-eq-iff eventually-filtermap*)
also have ... ≤ *filtermap f F*
using *g* **by** (*intro filtermap-mono*) (*simp add*: *filterlim-def*)
finally show *G* ≤ *filtermap f F* **.**
qed

lemma *filterlim-principal*:
(*LIM x F*. *f x* :> *principal S*) ⟷ (*eventually* (λ*x*. *f x* ∈ *S*) *F*)
unfolding *filterlim-def eventually-filtermap le-principal* **..**

lemma *filterlim-filtercomap* [*intro*]: *filterlim f F* (*filtercomap f F*)
unfolding *filterlim-def* **by** (*rule filtermap-filtercomap*)

lemma *filterlim-inf*:
(*LIM x F1. f x :> inf F2 F3*) ⟷ ((*LIM x F1. f x :> F2*) ∧ (*LIM x F1. f x :> F3*))
unfolding *filterlim-def* **by** *simp*

lemma *filterlim-INF*:
(*LIM x F. f x :>* (⨅ *b*∈*B. G b*)) ⟷ (∀ *b*∈*B. LIM x F. f x :> G b*)
unfolding *filterlim-def le-INF-iff* **..**

lemma *filterlim-INF-INF*:
(⋀*m. m* ∈ *J* ⟹ ∃ *i*∈*I. filtermap f* (*F i*) ≤ *G m*) ⟹ *LIM x* (⨅ *i*∈*I. F i*). *f x* :> (⨅ *j*∈*J. G j*)
unfolding *filterlim-def* **by** (*rule order-trans*[*OF filtermap-INF INF-mono*])

lemma *filterlim-INF'*: *x* ∈ *A* ⟹ *filterlim f F* (*G x*) ⟹ *filterlim f F* (⨅ *x*∈*A. G x*)
unfolding *filterlim-def* **by** (*rule order.trans*[*OF filtermap-mono*[*OF INF-lower*]])

lemma *filterlim-filtercomap-iff*: *filterlim f* (*filtercomap g G*) *F* ⟷ *filterlim* (*g* ∘ *f*) *G F*
by (*simp add*: *filterlim-def filtermap-le-iff-le-filtercomap filtercomap-filtercomap o-def*)

lemma *filterlim-iff-le-filtercomap*: *filterlim f F G* ⟷ *G* ≤ *filtercomap f F*
by (*simp add*: *filterlim-def filtermap-le-iff-le-filtercomap*)

lemma *filterlim-base*:
(⋀*m x. m* ∈ *J* ⟹ *i m* ∈ *I*) ⟹ (⋀*m x. m* ∈ *J* ⟹ *x* ∈ *F* (*i m*) ⟹ *f x* ∈ *G m*) ⟹
LIM x (⨅ *i*∈*I. principal* (*F i*)). *f x* :> (⨅ *j*∈*J. principal* (*G j*))
by (*force intro*!: *filterlim-INF-INF simp*: *image-subset-iff*)

lemma *filterlim-base-iff*:
assumes *I* ≠ {} **and** *chain*: ⋀*i j. i* ∈ *I* ⟹ *j* ∈ *I* ⟹ *F i* ⊆ *F j* ∨ *F j* ⊆ *F i*
shows (*LIM x* (⨅ *i*∈*I. principal* (*F i*)). *f x* :> ⨅ *j*∈*J. principal* (*G j*)) ⟷
(∀ *j*∈*J*. ∃ *i*∈*I*. ∀ *x*∈*F i. f x* ∈ *G j*)
unfolding *filterlim-INF filterlim-principal*
proof (*subst eventually-INF-base*)
fix *i j* **assume** *i* ∈ *I j* ∈ *I*
with *chain*[*OF this*] **show** ∃ *x*∈*I. principal* (*F x*) ≤ *inf* (*principal* (*F i*)) (*principal* (*F j*))
by *auto*
qed (*auto simp*: *eventually-principal* ‹*I* ≠ {}›)

lemma *filterlim-filtermap*: *filterlim f F1* (*filtermap g F2*) = *filterlim* (λ*x. f* (*g x*))

F1 F2
unfolding *filterlim-def filtermap-filtermap* **..**

lemma *filterlim-sup*:
filterlim f F F1 $\Longrightarrow$ *filterlim f F F2* $\Longrightarrow$ *filterlim f F* (*sup F1 F2*)
unfolding *filterlim-def filtermap-sup* **by** *auto*

lemma *filterlim-sequentially-Suc*:
(*LIM x sequentially. f* (*Suc x*) :> *F*) $\longleftrightarrow$ (*LIM x sequentially. f x* :> *F*)
unfolding *filterlim-iff* **by** (*subst eventually-sequentially-Suc*) *simp*

lemma *filterlim-Suc*: *filterlim Suc sequentially sequentially*
by (*simp add*: *filterlim-iff eventually-sequentially*)

lemma *filterlim-If*:
LIM x inf F (*principal* {*x. P x*}). *f x* :> *G* $\Longrightarrow$
LIM x inf F (*principal* {*x.* ¬ *P x*}). *g x* :> *G* $\Longrightarrow$
LIM x F. if P x then f x else g x :> *G*
unfolding *filterlim-iff eventually-inf-principal* **by** (*auto simp*: *eventually-conj-iff*)

lemma *filterlim-Pair*:
LIM x F. f x :> *G* $\Longrightarrow$ *LIM x F. g x* :> *H* $\Longrightarrow$ *LIM x F.* (*f x*, *g x*) :> $G \times_F H$
unfolding *filterlim-def*
by (*rule order-trans*[*OF filtermap-Pair prod-filter-mono*])

90.7 Limits to *at-top* and *at-bot*

lemma *filterlim-at-top*:
fixes *f* :: *'a* ⇒ (*'b*::*linorder*)
shows (*LIM x F. f x* :> *at-top*) $\longleftrightarrow$ (∀ *Z. eventually* (λ*x. Z* ≤ *f x*) *F*)
by (*auto simp*: *filterlim-iff eventually-at-top-linorder elim*!: *eventually-mono*)

lemma *filterlim-at-top-mono*:
LIM x F. f x :> *at-top* $\Longrightarrow$ *eventually* (λ*x. f x* ≤ (*g x*::*'a*::*linorder*)) *F* $\Longrightarrow$
LIM x F. g x :> *at-top*
by (*auto simp*: *filterlim-at-top elim*: *eventually-elim2 intro*: *order-trans*)

lemma *filterlim-at-top-dense*:
fixes *f* :: *'a* ⇒ (*'b*::*unbounded-dense-linorder*)
shows (*LIM x F. f x* :> *at-top*) $\longleftrightarrow$ (∀ *Z. eventually* (λ*x. Z* < *f x*) *F*)
by (*metis eventually-mono*[*of - F*] *eventually-gt-at-top order-less-imp-le*
filterlim-at-top[*of f F*] *filterlim-iff*[*of f at-top F*])

lemma *filterlim-at-top-ge*:
fixes *f* :: *'a* ⇒ (*'b*::*linorder*) **and** *c* :: *'b*
shows (*LIM x F. f x* :> *at-top*) $\longleftrightarrow$ (∀ *Z*≥*c. eventually* (λ*x. Z* ≤ *f x*) *F*)
unfolding *at-top-sub*[*of c*] *filterlim-INF* **by** (*auto simp add*: *filterlim-principal*)

lemma *filterlim-at-top-at-top*:

fixes f :: $'a$::*linorder* $\Rightarrow$ $'b$::*linorder*
assumes *mono*: $\bigwedge x\ y.\ Q\ x \Longrightarrow Q\ y \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
assumes *bij*: $\bigwedge x.\ P\ x \Longrightarrow f\ (g\ x) = x\ \bigwedge x.\ P\ x \Longrightarrow Q\ (g\ x)$
assumes Q: *eventually* Q *at-top*
assumes P: *eventually* P *at-top*
shows *filterlim f at-top at-top*
proof −
from P **obtain** x **where** x: $\bigwedge y.\ x \leq y \Longrightarrow P\ y$
unfolding *eventually-at-top-linorder* **by** *auto*
show *?thesis*
proof (*intro filterlim-at-top-ge*[*THEN iffD2*] *allI impI*)
fix z **assume** $x \leq z$
with x **have** $P\ z$ **by** *auto*
have *eventually* $(\lambda x.\ g\ z \leq x)$ *at-top*
by (*rule eventually-ge-at-top*)
with Q **show** *eventually* $(\lambda x.\ z \leq f\ x)$ *at-top*
by *eventually-elim* (*metis mono bij* ‹$P\ z$›)
qed
qed

lemma *filterlim-at-top-gt*:
fixes f :: $'a \Rightarrow ('b$::*unbounded-dense-linorder*) **and** c :: $'b$
shows (*LIM* x F. $f\ x$:> *at-top*) $\longleftrightarrow$ ($\forall Z{>}c.$ *eventually* $(\lambda x.\ Z \leq f\ x)$ F)
by (*metis filterlim-at-top order-less-le-trans gt-ex filterlim-at-top-ge*)

lemma *filterlim-at-bot*:
fixes f :: $'a \Rightarrow ('b$::*linorder*)
shows (*LIM* x F. $f\ x$:> *at-bot*) $\longleftrightarrow$ ($\forall Z.$ *eventually* $(\lambda x.\ f\ x \leq Z)$ F)
by (*auto simp*: *filterlim-iff eventually-at-bot-linorder elim*!: *eventually-mono*)

lemma *filterlim-at-bot-dense*:
fixes f :: $'a \Rightarrow ('b$::{*dense-linorder*, *no-bot*})
shows (*LIM* x F. $f\ x$:> *at-bot*) $\longleftrightarrow$ ($\forall Z.$ *eventually* $(\lambda x.\ f\ x < Z)$ F)
proof (*auto simp add*: *filterlim-at-bot*[*of f F*])
fix Z :: $'b$
from *lt-ex* [*of Z*] **obtain** Z' **where** *1*: $Z' < Z$ **..**
assume $\forall Z.$ *eventually* $(\lambda x.\ f\ x \leq Z)$ F
hence *eventually* $(\lambda x.\ f\ x \leq Z')$ F **by** *auto*
thus *eventually* $(\lambda x.\ f\ x < Z)$ F
by (*rule eventually-mono*) (*use 1* **in** *auto*)
next
fix Z :: $'b$
show $\forall Z.$ *eventually* $(\lambda x.\ f\ x < Z)$ $F \Longrightarrow$ *eventually* $(\lambda x.\ f\ x \leq Z)$ F
by (*drule spec* [*of - Z*], *erule eventually-mono*, *auto simp add*: *less-imp-le*)
qed

lemma *filterlim-at-bot-le*:
fixes f :: $'a \Rightarrow ('b$::*linorder*) **and** c :: $'b$
shows (*LIM* x F. $f\ x$:> *at-bot*) $\longleftrightarrow$ ($\forall Z{\leq}c.$ *eventually* $(\lambda x.\ Z \geq f\ x)$ F)

unfolding *filterlim-at-bot*
proof *safe*
fix *Z* **assume** ∗: ∀ *Z*≤*c*. *eventually* (λ*x*. *Z* ≥ *f x*) *F*
with ∗[*THEN spec*, *of min Z c*] **show** *eventually* (λ*x*. *Z* ≥ *f x*) *F*
by (*auto elim*!: *eventually-mono*)
qed *simp*

lemma *filterlim-at-bot-lt*:
fixes *f* :: ′*a* ⇒ (′*b*::*unbounded-dense-linorder*) **and** *c* :: ′*b*
shows (*LIM x F*. *f x* :> *at-bot*) ⟷ (∀ *Z*<*c*. *eventually* (λ*x*. *Z* ≥ *f x*) *F*)
by (*metis filterlim-at-bot filterlim-at-bot-le lt-ex order-le-less-trans*)

lemma *filterlim-at-top-div-const-nat*:
assumes *c* > *0*
shows *filterlim* (λ*x*::*nat*. *x div c*) *at-top at-top*
unfolding *filterlim-at-top*
proof
fix *C* :: *nat*
have ∗: *n div c* ≥ *C* **if** *n* ≥ *C* ∗ *c* **for** *n*
using *assms that* **by** (*metis div-le-mono div-mult-self-is-m*)
have *eventually* (λ*n*. *n* ≥ *C* ∗ *c*) *at-top*
by (*rule eventually-ge-at-top*)
thus *eventually* (λ*n*. *n div c* ≥ *C*) *at-top*
by *eventually-elim* (*use* ∗ **in** *auto*)
qed

lemma *filterlim-finite-subsets-at-top*:
filterlim f (*finite-subsets-at-top A*) *F* ⟷
(∀ *X*. *finite X* ∧ *X* ⊆ *A* ⟶ *eventually* (λ*y*. *finite* (*f y*) ∧ *X* ⊆ *f y* ∧ *f y* ⊆ *A*) *F*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
thus *?rhs*
proof (*safe*, *goal-cases*)
case (*1 X*)
hence ∗: (∀$_F$ *x in F*. *P* (*f x*)) **if** *eventually P* (*finite-subsets-at-top A*) **for** *P*
using *that* **by** (*auto simp*: *filterlim-def le-filter-def eventually-filtermap*)
have ∀$_F$ *Y in finite-subsets-at-top A*. *finite Y* ∧ *X* ⊆ *Y* ∧ *Y* ⊆ *A*
using *1* **unfolding** *eventually-finite-subsets-at-top* **by** *force*
thus *?case* **by** (*intro* ∗) *auto*
qed
next
assume *rhs*: *?rhs*
show *?lhs* **unfolding** *filterlim-def le-filter-def eventually-finite-subsets-at-top*
proof (*safe*, *goal-cases*)
case (*1 P X*)
with *rhs* **have** ∀$_F$ *y in F*. *finite* (*f y*) ∧ *X* ⊆ *f y* ∧ *f y* ⊆ *A* **by** *auto*
thus *eventually P* (*filtermap f F*) **unfolding** *eventually-filtermap*

by *eventually-elim* (*insert 1*, *auto*)
qed
qed

lemma *filterlim-atMost-at-top*:
filterlim (λ*n*. {..*n*}) (*finite-subsets-at-top* (*UNIV* :: *nat set*)) *at-top*
unfolding *filterlim-finite-subsets-at-top*
proof (*safe*, *goal-cases*)
case (*1 X*)
then obtain *n* **where** *n*: *X* ⊆ {..*n*} **by** (*auto simp*: *finite-nat-set-iff-bounded-le*)
show *?case* **using** *eventually-ge-at-top*[*of n*]
by *eventually-elim* (*insert n*, *auto*)
qed

lemma *filterlim-lessThan-at-top*:
filterlim (λ*n*. {..<*n*}) (*finite-subsets-at-top* (*UNIV* :: *nat set*)) *at-top*
unfolding *filterlim-finite-subsets-at-top*
proof (*safe*, *goal-cases*)
case (*1 X*)
then obtain *n* **where** *n*: *X* ⊆ {..<*n*} **by** (*auto simp*: *finite-nat-set-iff-bounded*)
show *?case* **using** *eventually-ge-at-top*[*of n*]
by *eventually-elim* (*insert n*, *auto*)
qed

lemma *filterlim-minus-const-nat-at-top*:
filterlim (λ*n*. *n* − *c*) *sequentially sequentially*
unfolding *filterlim-at-top*
proof
fix *a* :: *nat*
show *eventually* (λ*n*. *n* − *c* ≥ *a*) *at-top*
using *eventually-ge-at-top*[*of a* + *c*] **by** *eventually-elim auto*
qed

lemma *filterlim-add-const-nat-at-top*:
filterlim (λ*n*. *n* + *c*) *sequentially sequentially*
unfolding *filterlim-at-top*
proof
fix *a* :: *nat*
show *eventually* (λ*n*. *n* + *c* ≥ *a*) *at-top*
using *eventually-ge-at-top*[*of a*] **by** *eventually-elim auto*
qed

90.8 Setup *'a filter* for lifting and transfer

lemma *filtermap-id* [*simp*, *id-simps*]: *filtermap id* = *id*
by(*simp add*: *fun-eq-iff id-def filtermap-ident*)

lemma *filtermap-id'* [*simp*]: *filtermap* (λ*x*. *x*) = (λ*F*. *F*)
using *filtermap-id* **unfolding** *id-def* **.**

context includes *lifting-syntax*
begin

definition *map-filter-on* :: *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ *'a filter* ⇒ *'b filter* **where**
map-filter-on X f F = *Abs-filter* (λ*P*. *eventually* (λ*x*. *P* (*f x*) ∧ *x* ∈ *X*) *F*)

lemma *is-filter-map-filter-on*:
is-filter (λ*P*. ∀$_F$ *x in F*. *P* (*f x*) ∧ *x* ∈ *X*) ⟷ *eventually* (λ*x*. *x* ∈ *X*) *F*
proof(*rule iffI*; *unfold-locales*)
show ∀$_F$ *x in F*. *True* ∧ *x* ∈ *X* **if** *eventually* (λ*x*. *x* ∈ *X*) *F* **using** *that* **by** *simp*
show ∀$_F$ *x in F*. (*P* (*f x*) ∧ *Q* (*f x*)) ∧ *x* ∈ *X* **if** ∀$_F$ *x in F*. *P* (*f x*) ∧ *x* ∈ *X*
∀$_F$ *x in F*. *Q* (*f x*) ∧ *x* ∈ *X* **for** *P Q*
using *eventually-conj*[*OF that*] **by**(*auto simp add*: *conj-ac cong*: *conj-cong*)
show ∀$_F$ *x in F*. *Q* (*f x*) ∧ *x* ∈ *X* **if** ∀*x*. *P x* ⟶ *Q x* ∀$_F$ *x in F*. *P* (*f x*) ∧ *x*
∈ *X* **for** *P Q*
using *that*(*2*) **by**(*rule eventually-mono*)(*use that*(*1*) **in** *auto*)
show *eventually* (λ*x*. *x* ∈ *X*) *F* **if** *is-filter* (λ*P*. ∀$_F$ *x in F*. *P* (*f x*) ∧ *x* ∈ *X*)
using *is-filter*.*True*[*OF that*] **by** *simp*
qed

lemma *eventually-map-filter-on*: *eventually P* (*map-filter-on X f F*) = (∀$_F$ *x in*
F. *P* (*f x*) ∧ *x* ∈ *X*)
if *eventually* (λ*x*. *x* ∈ *X*) *F*
by(*simp add*: *is-filter-map-filter-on map-filter-on-def eventually-Abs-filter that*)

lemma *map-filter-on-UNIV*: *map-filter-on UNIV* = *filtermap*
by(*simp add*: *map-filter-on-def filtermap-def fun-eq-iff*)

lemma *map-filter-on-comp*: *map-filter-on X f* (*map-filter-on Y g F*) = *map-filter-on*
Y (*f* ∘ *g*) *F*
if *g* ' *Y* ⊆ *X* **and** *eventually* (λ*x*. *x* ∈ *Y*) *F*
unfolding *map-filter-on-def* **using** *that*(*1*)
by(*auto simp add*: *eventually-Abs-filter that*(*2*) *is-filter-map-filter-on intro*!: *arg-cong*[**where**
f=*Abs-filter*] *arg-cong2*[**where** *f*=*eventually*])

inductive *rel-filter* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a filter* ⇒ *'b filter* ⇒ *bool* **for** *R F G*
where
rel-filter R F G **if** *eventually* (*case-prod R*) *Z map-filter-on* {(*x*, *y*). *R x y*} *fst Z*
= *F map-filter-on* {(*x*, *y*). *R x y*} *snd Z* = *G*

lemma *rel-filter-eq* [*relator-eq*]: *rel-filter* (=) = (=)
proof(*intro ext iffI*)+
show *F* = *G* **if** *rel-filter* (=) *F G* **for** *F G* **using** *that*
by *cases*(*clarsimp simp add*: *filter-eq-iff eventually-map-filter-on split-def cong*:
rev-conj-cong)
show *rel-filter* (=) *F G* **if** *F* = *G* **for** *F G* **unfolding** ‹*F* = *G*›
proof
let *?Z* = *map-filter-on UNIV* (λ*x*. (*x*, *x*)) *G*

have [*simp*]: *range* (λ*x*. (*x*, *x*)) ⊆ {(*x*, *y*). *x* = *y*} **by** *auto*
show *map-filter-on* {(*x*, *y*). *x* = *y*} *fst ?Z* = *G* **and** *map-filter-on* {(*x*, *y*). *x* = *y*} *snd ?Z* = *G*
by(*simp-all add*: *map-filter-on-comp*)(*simp-all add*: *map-filter-on-UNIV o-def*)
show $\forall_F$ (*x*, *y*) *in ?Z*. *x* = *y* **by**(*simp add*: *eventually-map-filter-on*)
qed
qed

lemma *rel-filter-mono* [*relator-mono*]: *rel-filter A* ≤ *rel-filter B* **if** *le*: *A* ≤ *B*
proof(*clarify elim*!: *rel-filter.cases*)
show *rel-filter B* (*map-filter-on* {(*x*, *y*). *A x y*} *fst Z*) (*map-filter-on* {(*x*, *y*). *A x y*} *snd Z*)
(**is** *rel-filter - ?X ?Y*) **if** $\forall_F$ (*x*, *y*) *in Z*. *A x y* **for** *Z*
proof
let *?Z* = *map-filter-on* {(*x*, *y*). *A x y*} *id Z*
show $\forall_F$ (*x*, *y*) *in ?Z*. *B x y* **using** *le that*
by(*simp add*: *eventually-map-filter-on le-fun-def split-def conj-commute cong*: *conj-cong*)
have [*simp*]: {(*x*, *y*). *A x y*} ⊆ {(*x*, *y*). *B x y*} **using** *le* **by** *auto*
show *map-filter-on* {(*x*, *y*). *B x y*} *fst ?Z* = *?X map-filter-on* {(*x*, *y*). *B x y*} *snd ?Z* = *?Y*
using *le that* **by**(*simp-all add*: *le-fun-def map-filter-on-comp*)
qed
qed

lemma *rel-filter-conversep*: *rel-filter* A^{-1-1} = (*rel-filter A*)$^{-1-1}$
proof(*safe intro*!: *ext elim*!: *rel-filter.cases*)
show ∗: *rel-filter A* (*map-filter-on* {(*x*, *y*). A^{-1-1} *x y*} *snd Z*) (*map-filter-on* {(*x*, *y*). A^{-1-1} *x y*} *fst Z*)
(**is** *rel-filter - ?X ?Y*) **if** $\forall_F$ (*x*, *y*) *in Z*. A^{-1-1} *x y* **for** *A Z*
proof
let *?Z* = *map-filter-on* {(*x*, *y*). *A y x*} *prod.swap Z*
show $\forall_F$ (*x*, *y*) *in ?Z*. *A x y* **using** *that* **by**(*simp add*: *eventually-map-filter-on*)
have [*simp*]: *prod.swap* ' {(*x*, *y*). *A y x*} ⊆ {(*x*, *y*). *A x y*} **by** *auto*
show *map-filter-on* {(*x*, *y*). *A x y*} *fst ?Z* = *?X map-filter-on* {(*x*, *y*). *A x y*} *snd ?Z* = *?Y*
using *that* **by**(*simp-all add*: *map-filter-on-comp o-def*)
qed
show *rel-filter* A^{-1-1} (*map-filter-on* {(*x*, *y*). *A x y*} *snd Z*) (*map-filter-on* {(*x*, *y*). *A x y*} *fst Z*)
if $\forall_F$ (*x*, *y*) *in Z*. *A x y* **for** *Z* **using** ∗[*of* A^{-1-1} *Z*] *that* **by** *simp*
qed

lemma *rel-filter-distr* [*relator-distr*]:
rel-filter A OO rel-filter B = *rel-filter* (*A OO B*)
proof(*safe intro*!: *ext elim*!: *rel-filter.cases*)
let *?AB* = {(*x*, *y*). (*A OO B*) *x y*}
show (*rel-filter A OO rel-filter B*)
(*map-filter-on* {(*x*, *y*). (*A OO B*) *x y*} *fst Z*) (*map-filter-on* {(*x*, *y*). (*A OO B*)

x y} *snd Z*)
(**is** (- *OO* -) *?F ?H*) **if** $\forall_F$ (*x*, *y*) *in Z*. (*A OO B*) *x y* **for** *Z*
proof
let *?G* = *map-filter-on ?AB* (λ(*x*, *y*). *SOME z*. *A x z* ∧ *B z y*) *Z*
show *rel-filter A ?F ?G*
proof
let *?Z* = *map-filter-on ?AB* (λ(*x*, *y*). (*x*, *SOME z*. *A x z* ∧ *B z y*)) *Z*
show $\forall_F$ (*x*, *y*) *in ?Z*. *A x y* **using** *that*
by(*auto simp add*: *eventually-map-filter-on split-def elim*!: *eventually-mono intro*: *someI2*)
have [*simp*]: (λ*p*. (*fst p*, *SOME z*. *A* (*fst p*) *z* ∧ *B z* (*snd p*))) ‘ {*p*. (*A OO B*) (*fst p*) (*snd p*)} ⊆ {*p*. *A* (*fst p*) (*snd p*)} **by**(*auto intro*: *someI2*)
show *map-filter-on* {(*x*, *y*). *A x y*} *fst ?Z* = *?F map-filter-on* {(*x*, *y*). *A x y*} *snd ?Z* = *?G*
using *that* **by**(*simp-all add*: *map-filter-on-comp split-def o-def*)
qed
show *rel-filter B ?G ?H*
proof
let *?Z* = *map-filter-on ?AB* (λ(*x*, *y*). (*SOME z*. *A x z* ∧ *B z y*, *y*)) *Z*
show $\forall_F$ (*x*, *y*) *in ?Z*. *B x y* **using** *that*
by(*auto simp add*: *eventually-map-filter-on split-def elim*!: *eventually-mono intro*: *someI2*)
have [*simp*]: (λ*p*. (*SOME z*. *A* (*fst p*) *z* ∧ *B z* (*snd p*), *snd p*)) ‘ {*p*. (*A OO B*) (*fst p*) (*snd p*)} ⊆ {*p*. *B* (*fst p*) (*snd p*)} **by**(*auto intro*: *someI2*)
show *map-filter-on* {(*x*, *y*). *B x y*} *fst ?Z* = *?G map-filter-on* {(*x*, *y*). *B x y*} *snd ?Z* = *?H*
using *that* **by**(*simp-all add*: *map-filter-on-comp split-def o-def*)
qed
qed

fix *F G*
assume *F*: $\forall_F$ (*x*, *y*) *in F*. *A x y* **and** *G*: $\forall_F$ (*x*, *y*) *in G*. *B x y*
and *eq*: *map-filter-on* {(*x*, *y*). *B x y*} *fst G* = *map-filter-on* {(*x*, *y*). *A x y*} *snd F* (**is** *?Y2* = *?Y1*)
let *?X* = *map-filter-on* {(*x*, *y*). *A x y*} *fst F*
and *?Z* = (*map-filter-on* {(*x*, *y*). *B x y*} *snd G*)
have *step*: ∃ *P'*≤*P*. ∃ *Q'* ≤ *Q*. *eventually P' F* ∧ *eventually Q' G* ∧ {*y*. ∃ *x*. *P'* (*x*, *y*)} = {*y*. ∃ *z*. *Q'* (*y*, *z*)}
if *P*: *eventually P F* **and** *Q*: *eventually Q G* **for** *P Q*
proof −
let *?P* = λ(*x*, *y*). *P* (*x*, *y*) ∧ *A x y* **and** *?Q* = λ(*y*, *z*). *Q* (*y*, *z*) ∧ *B y z*
define *P'* **where** *P'* ≡ λ(*x*, *y*). *?P* (*x*, *y*) ∧ (∃ *z*. *?Q* (*y*, *z*))
define *Q'* **where** *Q'* ≡ λ(*y*, *z*). *?Q* (*y*, *z*) ∧ (∃ *x*. *?P* (*x*, *y*))
have *P'* ≤ *P Q'* ≤ *Q* {*y*. ∃ *x*. *P'* (*x*, *y*)} = {*y*. ∃ *z*. *Q'* (*y*, *z*)}
by(*auto simp add*: *P'-def Q'-def*)
moreover
from *P Q F G* **have** *P'*: *eventually ?P F* **and** *Q'*: *eventually ?Q G*
by(*simp-all add*: *eventually-conj-iff split-def*)
from *P' F* **have** $\forall_F$ *y in ?Y1*. ∃ *x*. *P* (*x*, *y*) ∧ *A x y*

by(*auto simp add*: *eventually-map-filter-on elim*!: *eventually-mono*)
from *this*[*folded eq*] **obtain** Q'' **where** Q'': *eventually* Q'' *G*
and $Q''P$: $\{y. \exists z. Q'' (y, z)\} \subseteq \{y. \exists x. ?P (x, y)\}$
using *G* **by**(*fastforce simp add*: *eventually-map-filter-on*)
have *eventually* (*inf* Q'' *?Q*) *G* **using** Q'' Q' **by**(*auto intro*: *eventually-conj simp add*: *inf-fun-def*)
then have *eventually* Q' *G* **using** $Q''P$ **by**(*auto elim*!: *eventually-mono simp add*: Q'*-def*)
moreover
from Q' *G* **have** $\forall_F$ *y in ?Y2*. $\exists z$. *Q* (*y*, *z*) $\wedge$ *B y z*
by(*auto simp add*: *eventually-map-filter-on elim*!: *eventually-mono*)
from *this*[*unfolded eq*] **obtain** P'' **where** P'': *eventually* P'' *F*
and $P''Q$: $\{y. \exists x. P'' (x, y)\} \subseteq \{y. \exists z. ?Q (y, z)\}$
using *F* **by**(*fastforce simp add*: *eventually-map-filter-on*)
have *eventually* (*inf* P'' *?P*) *F* **using** P'' P' **by**(*auto intro*: *eventually-conj simp add*: *inf-fun-def*)
then have *eventually* P' *F* **using** $P''Q$ **by**(*auto elim*!: *eventually-mono simp add*: P'*-def*)
ultimately show *?thesis* **by** *blast*
qed

show *rel-filter* (*A OO B*) *?X ?Z*
proof
let *?Y* = λY. $\exists$ *X Z*. *eventually X ?X* $\wedge$ *eventually Z ?Z* $\wedge$ (λ(*x*, *z*). *X x* $\wedge$ *Z z* $\wedge$ (*A OO B*) *x z*) $\leq$ *Y*
have *Y*: *is-filter ?Y*
proof
show *?Y* (λ-. *True*) **by**(*auto simp add*: *le-fun-def intro*: *eventually-True*)
show *?Y* (λx. *P x* $\wedge$ *Q x*) **if** *?Y P ?Y Q* **for** *P Q* **using** *that*
apply *clarify*
apply(*intro exI conjI*; (*elim eventually-rev-mp*; *fold imp-conjL*; *intro always-eventually allI*; *rule imp-refl*)*?*)
apply *auto*
done
show *?Y Q* **if** *?Y P* $\forall x$. *P x* $\longrightarrow$ *Q x* **for** *P Q* **using** *that* **by** *blast*
qed
define *Y* **where** *Y* = *Abs-filter ?Y*
have *eventually-Y*: *eventually P Y* $\longleftrightarrow$ *?Y P* **for** *P*
using *eventually-Abs-filter*[*OF Y*, *of P*] **by**(*simp add*: *Y-def*)
show *YY*: $\forall_F$ (*x*, *y*) *in Y*. (*A OO B*) *x y* **using** *F G*
by(*auto simp add*: *eventually-Y eventually-map-filter-on eventually-conj-iff intro*!: *eventually-True*)
have *?Y* (λ(*x*, *z*). *P x* $\wedge$ (*A OO B*) *x z*) $\longleftrightarrow$ ($\forall_F$ (*x*, *y*) *in F*. *P x* $\wedge$ *A x y*) (**is** *?lhs* = *?rhs*) **for** *P*
proof
show *?lhs* **if** *?rhs* **using** *G F that*
by(*auto 4 3 intro*: *exI*[**where** *x*=λ-. *True*] *simp add*: *eventually-map-filter-on split-def*)
assume *?lhs*

then obtain *X Z* **where** $\forall_F$ (*x*, *y*) *in F*. *X x* ∧ *A x y*
and $\forall_F$ (*x*, *y*) *in G*. *Z y* ∧ *B x y*
and (λ(*x*, *z*). *X x* ∧ *Z z* ∧ (*A OO B*) *x z*) ≤ (λ(*x*, *z*). *P x* ∧ (*A OO B*) *x z*)
using *F G* **by**(*auto simp add*: *eventually-map-filter-on split-def*)
from *step*[*OF this*(*1*, *2*)] *this*(*3*)
show *?rhs* **by**(*clarsimp elim*!: *eventually-rev-mp simp add*: *le-fun-def*)(*fastforce intro*: *always-eventually*)
qed
then show *map-filter-on ?AB fst Y* = *?X*
by(*simp add*: *filter-eq-iff YY eventually-map-filter-on*)(*simp add*: *eventually-Y eventually-map-filter-on F G*; *simp add*: *split-def*)

have *?Y* (λ(*x*, *z*). *P z* ∧ (*A OO B*) *x z*) ⟷ ($\forall_F$ (*x*, *y*) *in G*. *P y* ∧ *B x y*) (**is** *?lhs* = *?rhs*) **for** *P*
proof
show *?lhs* **if** *?rhs* **using** *G F that*
by(*auto 4 3 intro*: *exI*[**where** *x*=λ-. *True*] *simp add*: *eventually-map-filter-on split-def*)
assume *?lhs*
then obtain *X Z* **where** $\forall_F$ (*x*, *y*) *in F*. *X x* ∧ *A x y*
and $\forall_F$ (*x*, *y*) *in G*. *Z y* ∧ *B x y*
and (λ(*x*, *z*). *X x* ∧ *Z z* ∧ (*A OO B*) *x z*) ≤ (λ(*x*, *z*). *P z* ∧ (*A OO B*) *x z*)
using *F G* **by**(*auto simp add*: *eventually-map-filter-on split-def*)
from *step*[*OF this*(*1*, *2*)] *this*(*3*)
show *?rhs* **by**(*clarsimp elim*!: *eventually-rev-mp simp add*: *le-fun-def*)(*fastforce intro*: *always-eventually*)
qed
then show *map-filter-on ?AB snd Y* = *?Z*
by(*simp add*: *filter-eq-iff YY eventually-map-filter-on*)(*simp add*: *eventually-Y eventually-map-filter-on F G*; *simp add*: *split-def*)
qed
qed

lemma *filtermap-parametric*: ((*A* ===> *B*) ===> *rel-filter A* ===> *rel-filter B*) *filtermap filtermap*
proof(*intro rel-funI*; *erule rel-filter.cases*; *hypsubst*)
fix *f g Z*
assume *fg*: (*A* ===> *B*) *f g* **and** *Z*: $\forall_F$ (*x*, *y*) *in Z*. *A x y*
have *rel-filter B* (*map-filter-on* {(*x*, *y*). *A x y*} (*f* ∘ *fst*) *Z*) (*map-filter-on* {(*x*, *y*). *A x y*} (*g* ∘ *snd*) *Z*)
(**is** *rel-filter - ?F ?G*)
proof
let *?Z* = *map-filter-on* {(*x*, *y*). *A x y*} (*map-prod f g*) *Z*
show $\forall_F$ (*x*, *y*) *in ?Z*. *B x y* **using** *fg Z*
by(*auto simp add*: *eventually-map-filter-on split-def elim*!: *eventually-mono rel-funD*)
have [*simp*]: *map-prod f g* ' {*p*. *A* (*fst p*) (*snd p*)} ⊆ {*p*. *B* (*fst p*) (*snd p*)}
using *fg* **by**(*auto dest*: *rel-funD*)
show *map-filter-on* {(*x*, *y*). *B x y*} *fst ?Z* = *?F map-filter-on* {(*x*, *y*). *B x y*}

snd ?Z = ?G
using *Z* **by**(*auto simp add*: *map-filter-on-comp split-def*)
qed
thus *rel-filter B* (*filtermap f* (*map-filter-on* {(*x*, *y*). *A x y*} *fst Z*)) (*filtermap g* (*map-filter-on* {(*x*, *y*). *A x y*} *snd Z*))
using *Z* **by**(*simp add*: *map-filter-on-UNIV*[*symmetric*] *map-filter-on-comp*)
qed

lemma *rel-filter-Grp*: *rel-filter* (*Grp UNIV f*) = *Grp UNIV* (*filtermap f*)
proof((*intro antisym predicate2I*; (*elim GrpE*; *hypsubst*)?), *rule GrpI*[*OF - UNIV-I*])
fix *F G*
assume *rel-filter* (*Grp UNIV f*) *F G*
hence *rel-filter* (=) (*filtermap f F*) (*filtermap id G*)
by(*rule filtermap-parametric*[*THEN rel-funD*, *THEN rel-funD*, *rotated*])(*simp add*: *Grp-def rel-fun-def*)
thus *filtermap f F* = *G* **by**(*simp add*: *rel-filter-eq*)
next
fix *F* :: *'a filter*
have *rel-filter* (=) *F F* **by**(*simp add*: *rel-filter-eq*)
hence *rel-filter* (*Grp UNIV f*) (*filtermap id F*) (*filtermap f F*)
by(*rule filtermap-parametric*[*THEN rel-funD*, *THEN rel-funD*, *rotated*])(*simp add*: *Grp-def rel-fun-def*)
thus *rel-filter* (*Grp UNIV f*) *F* (*filtermap f F*) **by** *simp*
qed

lemma *Quotient-filter* [*quot-map*]:
Quotient R Abs Rep T ⟹ *Quotient* (*rel-filter R*) (*filtermap Abs*) (*filtermap Rep*) (*rel-filter T*)
unfolding *Quotient-alt-def5 rel-filter-eq*[*symmetric*] *rel-filter-Grp*[*symmetric*]
by(*simp add*: *rel-filter-distr*[*symmetric*] *rel-filter-conversep*[*symmetric*] *rel-filter-mono*)

lemma *left-total-rel-filter* [*transfer-rule*]: *left-total A* ⟹ *left-total* (*rel-filter A*)
unfolding *left-total-alt-def rel-filter-eq*[*symmetric*] *rel-filter-conversep*[*symmetric*] *rel-filter-distr*
by(*rule rel-filter-mono*)

lemma *right-total-rel-filter* [*transfer-rule*]: *right-total A* ⟹ *right-total* (*rel-filter A*)
using *left-total-rel-filter*[*of* A^{-1-1}] **by**(*simp add*: *rel-filter-conversep*)

lemma *bi-total-rel-filter* [*transfer-rule*]: *bi-total A* ⟹ *bi-total* (*rel-filter A*)
unfolding *bi-total-alt-def* **by**(*simp add*: *left-total-rel-filter right-total-rel-filter*)

lemma *left-unique-rel-filter* [*transfer-rule*]: *left-unique A* ⟹ *left-unique* (*rel-filter A*)
unfolding *left-unique-alt-def rel-filter-eq*[*symmetric*] *rel-filter-conversep*[*symmetric*] *rel-filter-distr*
by(*rule rel-filter-mono*)

lemma *right-unique-rel-filter* [*transfer-rule*]:
right-unique A $\Longrightarrow$ *right-unique* (*rel-filter A*)
using *left-unique-rel-filter*[*of* A^{-1-1}] **by**(*simp add*: *rel-filter-conversep*)

lemma *bi-unique-rel-filter* [*transfer-rule*]: *bi-unique A* $\Longrightarrow$ *bi-unique* (*rel-filter A*)
by(*simp add*: *bi-unique-alt-def left-unique-rel-filter right-unique-rel-filter*)

lemma *eventually-parametric* [*transfer-rule*]:
((*A* ===> (=)) ===> *rel-filter A* ===> (=)) *eventually eventually*
by(*auto 4 4 intro*!: *rel-funI elim*!: *rel-filter.cases simp add*: *eventually-map-filter-on dest*: *rel-funD intro*: *always-eventually elim*!: *eventually-rev-mp*)

lemma *frequently-parametric* [*transfer-rule*]: ((*A* ===> (=)) ===> *rel-filter A* ===> (=)) *frequently frequently*
unfolding *frequently-def*[*abs-def*] **by** *transfer-prover*

lemma *is-filter-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A*
assumes [*transfer-rule*]: *bi-unique A*
shows (((*A* ===> (=)) ===> (=)) ===> (=)) *is-filter is-filter*
unfolding *is-filter-def* **by** *transfer-prover*

lemma *top-filter-parametric* [*transfer-rule*]: *rel-filter A top top* **if** *bi-total A*
proof
let *?Z* = *principal* {(*x*, *y*). *A x y*}
show $\forall_F$ (*x*, *y*) *in ?Z*. *A x y* **by**(*simp add*: *eventually-principal*)
show *map-filter-on* {(*x*, *y*). *A x y*} *fst ?Z* = *top map-filter-on* {(*x*, *y*). *A x y*} *snd ?Z* = *top*
using *that* **by**(*auto simp add*: *filter-eq-iff eventually-map-filter-on eventually-principal bi-total-def*)
qed

lemma *bot-filter-parametric* [*transfer-rule*]: *rel-filter A bot bot*
proof
show $\forall_F$ (*x*, *y*) *in bot*. *A x y* **by** *simp*
show *map-filter-on* {(*x*, *y*). *A x y*} *fst bot* = *bot map-filter-on* {(*x*, *y*). *A x y*} *snd bot* = *bot*
by(*simp-all add*: *filter-eq-iff eventually-map-filter-on*)
qed

lemma *principal-parametric* [*transfer-rule*]: (*rel-set A* ===> *rel-filter A*) *principal principal*
proof(*rule rel-funI rel-filter.intros*)+
fix *S S'*
assume ∗: *rel-set A S S'*
define *SS'* **where** *SS'* = *S* × *S'* ∩ {(*x*, *y*). *A x y*}
have *SS'*: *SS'* ⊆ {(*x*, *y*). *A x y*} **and** [*simp*]: *S* = *fst* ' *SS' S'* = *snd* ' *SS'*
using ∗ **by**(*auto 4 3 dest*: *rel-setD1 rel-setD2 intro*: *rev-image-eqI simp add*: *SS'-def*)

let *?Z = principal SS'*
show $\forall_F$ *(x, y) in ?Z. A x y* **using** *SS'* **by**(*auto simp add: eventually-principal*)
then show *map-filter-on {(x, y). A x y} fst ?Z = principal S*
and *map-filter-on {(x, y). A x y} snd ?Z = principal S'*
by(*auto simp add: filter-eq-iff eventually-map-filter-on eventually-principal*)
qed

lemma *sup-filter-parametric* [*transfer-rule*]:
(*rel-filter A ===> rel-filter A ===> rel-filter A*) *sup sup*
proof(*intro rel-funI*; *elim rel-filter.cases*; *hypsubst*)
show *rel-filter A*
(*map-filter-on {(x, y). A x y} fst FG* ⊔ *map-filter-on {(x, y). A x y} fst FG'*)
(*map-filter-on {(x, y). A x y} snd FG* ⊔ *map-filter-on {(x, y). A x y} snd FG'*)
(**is** *rel-filter - (sup ?F ?G) (sup ?F' ?G')*)
if $\forall_F$ *(x, y) in FG. A x y* $\forall_F$ *(x, y) in FG'. A x y* **for** *FG FG'*
proof
let *?Z = sup FG FG'*
show $\forall_F$ *(x, y) in ?Z. A x y* **by**(*simp add: eventually-sup that*)
then show *map-filter-on {(x, y). A x y} fst ?Z = sup ?F ?G*
and *map-filter-on {(x, y). A x y} snd ?Z = sup ?F' ?G'*
by(*simp-all add: filter-eq-iff eventually-map-filter-on eventually-sup*)
qed
qed

lemma *Sup-filter-parametric* [*transfer-rule*]: (*rel-set* (*rel-filter A*) *===> rel-filter A*) *Sup Sup*
proof(*rule rel-funI*)
fix *S S'*
define *SS'* **where** *SS' = S × S' ∩ {(F, G). rel-filter A F G}*
assume *rel-set (rel-filter A) S S'*
then have *SS'*: *SS' ⊆ {(F, G). rel-filter A F G}* **and** [*simp*]: *S = fst ' SS' S' = snd ' SS'*
by(*auto 4 3 dest: rel-setD1 rel-setD2 intro: rev-image-eqI simp add: SS'-def*)
from *SS'* **obtain** *Z* **where** *Z*: $\bigwedge$*F G. (F, G) ∈ SS'* ⟹
($\forall_F$ *(x, y) in Z F G. A x y*) ∧
id F = map-filter-on {(x, y). A x y} fst (Z F G) ∧
id G = map-filter-on {(x, y). A x y} snd (Z F G)
unfolding *rel-filter.simps* **by** *atomize-elim((rule choice allI)+; auto)*
have *id*: *eventually P F = eventually P (id F) eventually Q G = eventually Q (id G)*
if *(F, G) ∈ SS'* **for** *P Q F G* **by** *simp-all*
show *rel-filter A (Sup S) (Sup S')*
proof
let *?Z =* $\bigsqcup$*(F, G)∈SS'. Z F G*
show ∗: $\forall_F$ *(x, y) in ?Z. A x y* **using** *Z* **by**(*auto simp add: eventually-Sup*)
show *map-filter-on {(x, y). A x y} fst ?Z = Sup S map-filter-on {(x, y). A x y} snd ?Z = Sup S'*
unfolding *filter-eq-iff*
by(*auto 4 4 simp add: id eventually-Sup eventually-map-filter-on ∗[simplified*

eventually-Sup] *simp del*: *id-apply dest*: *Z*)
  qed
qed

context
  fixes *A* :: *'a* ⇒ *'b* ⇒ *bool*
  assumes [*transfer-rule*]: *bi-unique A*
begin

lemma *le-filter-parametric* [*transfer-rule*]:
  (*rel-filter A* ===> *rel-filter A* ===> (=)) (≤) (≤)
unfolding *le-filter-def*[*abs-def*] **by** *transfer-prover*

lemma *less-filter-parametric* [*transfer-rule*]:
  (*rel-filter A* ===> *rel-filter A* ===> (=)) (<) (<)
unfolding *less-filter-def*[*abs-def*] **by** *transfer-prover*

context
  assumes [*transfer-rule*]: *bi-total A*
begin

lemma *Inf-filter-parametric* [*transfer-rule*]:
  (*rel-set* (*rel-filter A*) ===> *rel-filter A*) *Inf Inf*
unfolding *Inf-filter-def*[*abs-def*] **by** *transfer-prover*

lemma *inf-filter-parametric* [*transfer-rule*]:
  (*rel-filter A* ===> *rel-filter A* ===> *rel-filter A*) *inf inf*
proof(*intro rel-funI*)+
  fix *F F' G G'*
  assume [*transfer-rule*]: *rel-filter A F F' rel-filter A G G'*
  have *rel-filter A* (*Inf* {*F*, *G*}) (*Inf* {*F'*, *G'*}) **by** *transfer-prover*
  thus *rel-filter A* (*inf F G*) (*inf F' G'*) **by** *simp*
qed

end

end

end

context
  includes *lifting-syntax*
begin

lemma *prod-filter-parametric* [*transfer-rule*]:
  (*rel-filter R* ===> *rel-filter S* ===> *rel-filter* (*rel-prod R S*)) *prod-filter prod-filter*
proof(*intro rel-funI*; *elim rel-filter.cases*; *hypsubst*)
  fix *F G*
  assume *F*: $\forall_F$ (*x*, *y*) *in F*. *R x y* **and** *G*: $\forall_F$ (*x*, *y*) *in G*. *S x y*

```
  show rel-filter (rel-prod R S)
    (map-filter-on {(x, y). R x y} fst F ×F map-filter-on {(x, y). S x y} fst G)
    (map-filter-on {(x, y). R x y} snd F ×F map-filter-on {(x, y). S x y} snd G)
    (is rel-filter ?RS ?F ?G)
  proof
    let ?Z = filtermap (λ((a, b), (a', b')). ((a, a'), (b, b'))) (prod-filter F G)
    show ∗: ∀F (x, y) in ?Z. rel-prod R S x y using F G
      by(auto simp add: eventually-filtermap split-beta eventually-prod-filter)
    show map-filter-on {(x, y). ?RS x y} fst ?Z = ?F
      using F G
      apply(clarsimp simp add: filter-eq-iff eventually-map-filter-on ∗)
      apply(simp add: eventually-filtermap split-beta eventually-prod-filter)
      apply(subst eventually-map-filter-on; simp)+
      apply(rule iffI; clarsimp)
      subgoal for P P' P''
        apply(rule exI[where x=λa. ∃ b. P' (a, b) ∧ R a b]; rule conjI)
        subgoal by(fastforce elim: eventually-rev-mp eventually-mono)
        subgoal
          by(rule exI[where x=λa. ∃ b. P'' (a, b) ∧ S a b])(fastforce elim: eventu-
ally-rev-mp eventually-mono)
        done
      subgoal by fastforce
      done
    show map-filter-on {(x, y). ?RS x y} snd ?Z = ?G
      using F G
      apply(clarsimp simp add: filter-eq-iff eventually-map-filter-on ∗)
      apply(simp add: eventually-filtermap split-beta eventually-prod-filter)
      apply(subst eventually-map-filter-on; simp)+
      apply(rule iffI; clarsimp)
      subgoal for P P' P''
        apply(rule exI[where x=λb. ∃ a. P' (a, b) ∧ R a b]; rule conjI)
        subgoal by(fastforce elim: eventually-rev-mp eventually-mono)
        subgoal
          by(rule exI[where x=λb. ∃ a. P'' (a, b) ∧ S a b])(fastforce elim: eventu-
ally-rev-mp eventually-mono)
        done
      subgoal by fastforce
      done
  qed
qed

end
```

Code generation for filters

```
definition abstract-filter :: (unit ⇒ 'a filter) ⇒ 'a filter
  where [simp]: abstract-filter f = f ()

code-datatype principal abstract-filter
```

hide-const (**open**) *abstract-filter*

declare [[*code drop*: *filterlim prod-filter filtermap eventually*
inf :: - *filter* ⇒ - *sup* :: - *filter* ⇒ - *less-eq* :: - *filter* ⇒ -
Abs-filter]]

declare *filterlim-principal* [*code*]
declare *principal-prod-principal* [*code*]
declare *filtermap-principal* [*code*]
declare *filtercomap-principal* [*code*]
declare *eventually-principal* [*code*]
declare *inf-principal* [*code*]
declare *sup-principal* [*code*]
declare *principal-le-iff* [*code*]

lemma *Rep-filter-iff-eventually* [*simp*, *code*]:
Rep-filter F P ⟷ *eventually P F*
by (*simp add*: *eventually-def*)

lemma *bot-eq-principal-empty* [*code*]:
bot = *principal* {}
by *simp*

lemma *top-eq-principal-UNIV* [*code*]:
top = *principal UNIV*
by *simp*

instantiation *filter* :: (*equal*) *equal*
begin

definition *equal-filter* :: *'a filter* ⇒ *'a filter* ⇒ *bool*
where *equal-filter F F'* ⟷ *F* = *F'*

lemma *equal-filter* [*code*]:
HOL.equal (*principal A*) (*principal B*) ⟷ *A* = *B*
by (*simp add*: *equal-filter-def*)

instance
by *standard* (*simp add*: *equal-filter-def*)

end

end

91 Conditionally-complete Lattices

theory *Conditionally-Complete-Lattices*
imports *Finite-Set Lattices-Big Set-Interval*
begin

locale *preordering-bdd* = *preordering*
begin

definition *bdd* :: ‹′*a set* ⇒ *bool*›
  where *unfold*: ‹*bdd A* ⟷ (∃ *M*. ∀ *x* ∈ *A*. *x* ≤ *M*)›

lemma *empty* [*simp*, *intro*]:
  ‹*bdd* {}›
  by (*simp add*: *unfold*)

lemma *I* [*intro*]:
  ‹*bdd A*› **if** ‹⋀*x*. *x* ∈ *A* ⟹ *x* ≤ *M*›
  using *that* **by** (*auto simp add*: *unfold*)

lemma *E*:
  assumes ‹*bdd A*›
  obtains *M* **where** ‹⋀*x*. *x* ∈ *A* ⟹ *x* ≤ *M*›
  using *assms that* **by** (*auto simp add*: *unfold*)

lemma *I2*:
  ‹*bdd* (*f* ‘ *A*)› **if** ‹⋀*x*. *x* ∈ *A* ⟹ *f x* ≤ *M*›
  using *that* **by** (*auto simp add*: *unfold*)

lemma *mono*:
  ‹*bdd A*› **if** ‹*bdd B*› ‹*A* ⊆ *B*›
  using *that* **by** (*auto simp add*: *unfold*)

lemma *Int1* [*simp*]:
  ‹*bdd* (*A* ∩ *B*)› **if** ‹*bdd A*›
  using *mono that* **by** *auto*

lemma *Int2* [*simp*]:
  ‹*bdd* (*A* ∩ *B*)› **if** ‹*bdd B*›
  using *mono that* **by** *auto*

end

91.1 Preorders

context *preorder*
begin

sublocale *bdd-above*: *preordering-bdd* ‹(≤)› ‹(<)›
  defines *bdd-above-primitive-def*: *bdd-above* = *bdd-above.bdd* **..**

sublocale *bdd-below*: *preordering-bdd* ‹(≥)› ‹(>)›
  defines *bdd-below-primitive-def*: *bdd-below* = *bdd-below.bdd* **..**

lemma *bdd-above-def*: ‹*bdd-above A* $\longleftrightarrow$ ($\exists$ *M*. $\forall$ *x* $\in$ *A*. *x* $\leq$ *M*)›
by (*fact bdd-above.unfold*)

lemma *bdd-below-def*: ‹*bdd-below A* $\longleftrightarrow$ ($\exists$ *M*. $\forall$ *x* $\in$ *A*. *M* $\leq$ *x*)›
by (*fact bdd-below.unfold*)

lemma *bdd-aboveI*: ($\bigwedge$*x*. *x* $\in$ *A* $\Longrightarrow$ *x* $\leq$ *M*) $\Longrightarrow$ *bdd-above A*
by (*fact bdd-above.I*)

lemma *bdd-belowI*: ($\bigwedge$*x*. *x* $\in$ *A* $\Longrightarrow$ *m* $\leq$ *x*) $\Longrightarrow$ *bdd-below A*
by (*fact bdd-below.I*)

lemma *bdd-aboveI2*: ($\bigwedge$*x*. *x* $\in$ *A* $\Longrightarrow$ *f x* $\leq$ *M*) $\Longrightarrow$ *bdd-above* (*f*‘*A*)
by (*fact bdd-above.I2*)

lemma *bdd-belowI2*: ($\bigwedge$*x*. *x* $\in$ *A* $\Longrightarrow$ *m* $\leq$ *f x*) $\Longrightarrow$ *bdd-below* (*f*‘*A*)
by (*fact bdd-below.I2*)

lemma *bdd-above-empty*: *bdd-above* {}
by (*fact bdd-above.empty*)

lemma *bdd-below-empty*: *bdd-below* {}
by (*fact bdd-below.empty*)

lemma *bdd-above-mono*: *bdd-above B* $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *bdd-above A*
by (*fact bdd-above.mono*)

lemma *bdd-below-mono*: *bdd-below B* $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *bdd-below A*
by (*fact bdd-below.mono*)

lemma *bdd-above-Int1*: *bdd-above A* $\Longrightarrow$ *bdd-above* (*A* $\cap$ *B*)
by (*fact bdd-above.Int1*)

lemma *bdd-above-Int2*: *bdd-above B* $\Longrightarrow$ *bdd-above* (*A* $\cap$ *B*)
by (*fact bdd-above.Int2*)

lemma *bdd-below-Int1*: *bdd-below A* $\Longrightarrow$ *bdd-below* (*A* $\cap$ *B*)
by (*fact bdd-below.Int1*)

lemma *bdd-below-Int2*: *bdd-below B* $\Longrightarrow$ *bdd-below* (*A* $\cap$ *B*)
by (*fact bdd-below.Int2*)

lemma *bdd-above-Ioo* [*simp*, *intro*]: *bdd-above* {*a* <..< *b*}
by (*auto simp add*: *bdd-above-def intro*!: *exI*[*of* - *b*] *less-imp-le*)

lemma *bdd-above-Ico* [*simp*, *intro*]: *bdd-above* {*a* ..< *b*}
by (*auto simp add*: *bdd-above-def intro*!: *exI*[*of* - *b*] *less-imp-le*)

lemma *bdd-above-Iio* [*simp*, *intro*]: *bdd-above* {..< *b*}

by (*auto simp add*: *bdd-above-def intro*: *exI*[*of* - *b*] *less-imp-le*)

lemma *bdd-above-Ioc* [*simp*, *intro*]: *bdd-above* {*a* <.. *b*}
by (*auto simp add*: *bdd-above-def intro*: *exI*[*of* - *b*] *less-imp-le*)

lemma *bdd-above-Icc* [*simp*, *intro*]: *bdd-above* {*a* .. *b*}
by (*auto simp add*: *bdd-above-def intro*: *exI*[*of* - *b*] *less-imp-le*)

lemma *bdd-above-Iic* [*simp*, *intro*]: *bdd-above* {.. *b*}
by (*auto simp add*: *bdd-above-def intro*: *exI*[*of* - *b*] *less-imp-le*)

lemma *bdd-below-Ioo* [*simp*, *intro*]: *bdd-below* {*a* <..< *b*}
by (*auto simp add*: *bdd-below-def intro*!: *exI*[*of* - *a*] *less-imp-le*)

lemma *bdd-below-Ioc* [*simp*, *intro*]: *bdd-below* {*a* <.. *b*}
by (*auto simp add*: *bdd-below-def intro*!: *exI*[*of* - *a*] *less-imp-le*)

lemma *bdd-below-Ioi* [*simp*, *intro*]: *bdd-below* {*a* <..}
by (*auto simp add*: *bdd-below-def intro*: *exI*[*of* - *a*] *less-imp-le*)

lemma *bdd-below-Ico* [*simp*, *intro*]: *bdd-below* {*a* ..< *b*}
by (*auto simp add*: *bdd-below-def intro*: *exI*[*of* - *a*] *less-imp-le*)

lemma *bdd-below-Icc* [*simp*, *intro*]: *bdd-below* {*a* .. *b*}
by (*auto simp add*: *bdd-below-def intro*: *exI*[*of* - *a*] *less-imp-le*)

lemma *bdd-below-Ici* [*simp*, *intro*]: *bdd-below* {*a* ..}
by (*auto simp add*: *bdd-below-def intro*: *exI*[*of* - *a*] *less-imp-le*)

end

context *order-top*
begin

lemma *bdd-above-top* [*simp*, *intro*!]: *bdd-above A*
by (*rule bdd-aboveI* [*of* - *top*]) *simp*

end

context *order-bot*
begin

lemma *bdd-below-bot* [*simp*, *intro*!]: *bdd-below A*
by (*rule bdd-belowI* [*of* - *bot*]) *simp*

end

lemma *bdd-above-image-mono*: *mono f* ⟹ *bdd-above A* ⟹ *bdd-above* (*f*'*A*)
by (*auto simp*: *bdd-above-def mono-def*)

lemma *bdd-below-image-mono*: *mono f* $\Longrightarrow$ *bdd-below A* $\Longrightarrow$ *bdd-below* (*f'A*)
by (*auto simp*: *bdd-below-def mono-def*)

lemma *bdd-above-image-antimono*: *antimono f* $\Longrightarrow$ *bdd-below A* $\Longrightarrow$ *bdd-above* (*f'A*)
by (*auto simp*: *bdd-above-def bdd-below-def antimono-def*)

lemma *bdd-below-image-antimono*: *antimono f* $\Longrightarrow$ *bdd-above A* $\Longrightarrow$ *bdd-below* (*f'A*)
by (*auto simp*: *bdd-above-def bdd-below-def antimono-def*)

lemma
fixes *X* :: *'a*::*ordered-ab-group-add set*
shows *bdd-above-uminus*[*simp*]: *bdd-above* (*uminus* ' *X*) $\longleftrightarrow$ *bdd-below X*
and *bdd-below-uminus*[*simp*]: *bdd-below* (*uminus* ' *X*) $\longleftrightarrow$ *bdd-above X*
using *bdd-above-image-antimono*[*of uminus X*] *bdd-below-image-antimono*[*of uminus uminus'X*]
using *bdd-below-image-antimono*[*of uminus X*] *bdd-above-image-antimono*[*of uminus uminus'X*]
by (*auto simp*: *antimono-def image-image*)

91.2 Lattices

context *lattice*
begin

lemma *bdd-above-insert* [*simp*]: *bdd-above* (*insert a A*) = *bdd-above A*
by (*auto simp*: *bdd-above-def intro*: *le-supI2 sup-ge1*)

lemma *bdd-below-insert* [*simp*]: *bdd-below* (*insert a A*) = *bdd-below A*
by (*auto simp*: *bdd-below-def intro*: *le-infI2 inf-le1*)

lemma *bdd-finite* [*simp*]:
assumes *finite A* **shows** *bdd-above-finite*: *bdd-above A* **and** *bdd-below-finite*: *bdd-below A*
using *assms* **by** (*induct rule*: *finite-induct*, *auto*)

lemma *bdd-above-Un* [*simp*]: *bdd-above* ($A \cup B$) = (*bdd-above A* $\wedge$ *bdd-above B*)
proof
assume *bdd-above* ($A \cup B$)
thus *bdd-above A* $\wedge$ *bdd-above B* **unfolding** *bdd-above-def* **by** *auto*
next
assume *bdd-above A* $\wedge$ *bdd-above B*
then obtain *a b* **where** $\forall x \in A.\ x \leq a$ $\forall x \in B.\ x \leq b$ **unfolding** *bdd-above-def* **by** *auto*
hence $\forall x \in A \cup B.\ x \leq$ *sup a b* **by** (*auto intro*: *Un-iff le-supI1 le-supI2*)
thus *bdd-above* ($A \cup B$) **unfolding** *bdd-above-def* **..**
qed

lemma *bdd-below-Un* [*simp*]: *bdd-below* ($A \cup B$) = (*bdd-below* $A \wedge$ *bdd-below* B)
proof
assume *bdd-below* ($A \cup B$)
thus *bdd-below* $A \wedge$ *bdd-below* B **unfolding** *bdd-below-def* **by** *auto*
next
assume *bdd-below* $A \wedge$ *bdd-below* B
then obtain a b **where** $\forall x \in A.\ a \leq x$ $\forall x \in B.\ b \leq x$ **unfolding** *bdd-below-def* **by** *auto*
hence $\forall x \in A \cup B.$ *inf* a $b \leq x$ **by** (*auto intro*: *Un-iff le-infI1 le-infI2*)
thus *bdd-below* ($A \cup B$) **unfolding** *bdd-below-def* **..**
qed

lemma *bdd-above-image-sup*[*simp*]:
bdd-above (($\lambda x.$ *sup* (f x) (g x)) ‘ A) $\longleftrightarrow$ *bdd-above* (f‘A) $\wedge$ *bdd-above* (g‘A)
by (*auto simp*: *bdd-above-def intro*: *le-supI1 le-supI2*)

lemma *bdd-below-image-inf*[*simp*]:
bdd-below (($\lambda x.$ *inf* (f x) (g x)) ‘ A) $\longleftrightarrow$ *bdd-below* (f‘A) $\wedge$ *bdd-below* (g‘A)
by (*auto simp*: *bdd-below-def intro*: *le-infI1 le-infI2*)

lemma *bdd-below-UN*[*simp*]: *finite* $I \Longrightarrow$ *bdd-below* ($\bigcup i \in I.\ A\ i$) = ($\forall i \in I.$ *bdd-below* (A i))
by (*induction I rule*: *finite.induct*) *auto*

lemma *bdd-above-UN*[*simp*]: *finite* $I \Longrightarrow$ *bdd-above* ($\bigcup i \in I.\ A\ i$) = ($\forall i \in I.$ *bdd-above* (A i))
by (*induction I rule*: *finite.induct*) *auto*

end

To avoid name classes with the *complete-lattice*-class we prefix *Sup* and *Inf* in theorem names with c.

91.3 Conditionally complete lattices

class *conditionally-complete-lattice* = *lattice* + *Sup* + *Inf* +
assumes *cInf-lower*: $x \in X \Longrightarrow$ *bdd-below* $X \Longrightarrow$ *Inf* $X \leq x$
and *cInf-greatest*: $X \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow z \leq$ *Inf* X
assumes *cSup-upper*: $x \in X \Longrightarrow$ *bdd-above* $X \Longrightarrow x \leq$ *Sup* X
and *cSup-least*: $X \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow$ *Sup* $X \leq z$
begin

lemma *cSup-upper2*: $x \in X \Longrightarrow y \leq x \Longrightarrow$ *bdd-above* $X \Longrightarrow y \leq$ *Sup* X
by (*metis cSup-upper order-trans*)

lemma *cInf-lower2*: $x \in X \Longrightarrow x \leq y \Longrightarrow$ *bdd-below* $X \Longrightarrow$ *Inf* $X \leq y$
by (*metis cInf-lower order-trans*)

lemma *cSup-mono*: $B \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow (\bigwedge b.\ b \in B \Longrightarrow \exists a{\in}A.\ b \leq a)$ $\Longrightarrow$ *Sup* $B \leq$ *Sup* A
by (*metis cSup-least cSup-upper2*)

lemma *cInf-mono*: $B \neq \{\} \Longrightarrow$ *bdd-below* $A \Longrightarrow (\bigwedge b.\ b \in B \Longrightarrow \exists a{\in}A.\ a \leq b)$ $\Longrightarrow$ *Inf* $A \leq$ *Inf* B
by (*metis cInf-greatest cInf-lower2*)

lemma *cSup-subset-mono*: $A \neq \{\} \Longrightarrow$ *bdd-above* $B \Longrightarrow A \subseteq B \Longrightarrow$ *Sup* $A \leq$ *Sup* B
by (*metis cSup-least cSup-upper subsetD*)

lemma *cInf-superset-mono*: $A \neq \{\} \Longrightarrow$ *bdd-below* $B \Longrightarrow A \subseteq B \Longrightarrow$ *Inf* $B \leq$ *Inf* A
by (*metis cInf-greatest cInf-lower subsetD*)

lemma *cSup-eq-maximum*: $z \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow$ *Sup* $X = z$
by (*intro order.antisym cSup-upper*[*of z X*] *cSup-least*[*of X z*]) *auto*

lemma *cInf-eq-minimum*: $z \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow$ *Inf* $X = z$
by (*intro order.antisym cInf-lower*[*of z X*] *cInf-greatest*[*of X z*]) *auto*

lemma *cSup-le-iff*: $S \neq \{\} \Longrightarrow$ *bdd-above* $S \Longrightarrow$ *Sup* $S \leq a \longleftrightarrow (\forall x{\in}S.\ x \leq a)$
by (*metis order-trans cSup-upper cSup-least*)

lemma *le-cInf-iff*: $S \neq \{\} \Longrightarrow$ *bdd-below* $S \Longrightarrow a \leq$ *Inf* $S \longleftrightarrow (\forall x{\in}S.\ a \leq x)$
by (*metis order-trans cInf-lower cInf-greatest*)

lemma *cSup-eq-non-empty*:
assumes *1*: $X \neq \{\}$
assumes *2*: $\bigwedge x.\ x \in X \Longrightarrow x \leq a$
assumes *3*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow x \leq y) \Longrightarrow a \leq y$
shows *Sup* $X = a$
by (*intro 3 1 order.antisym cSup-least*) (*auto intro*: *2 1 cSup-upper*)

lemma *cInf-eq-non-empty*:
assumes *1*: $X \neq \{\}$
assumes *2*: $\bigwedge x.\ x \in X \Longrightarrow a \leq x$
assumes *3*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow y \leq x) \Longrightarrow y \leq a$
shows *Inf* $X = a$
by (*intro 3 1 order.antisym cInf-greatest*) (*auto intro*: *2 1 cInf-lower*)

lemma *cInf-cSup*: $S \neq \{\} \Longrightarrow$ *bdd-below* $S \Longrightarrow$ *Inf* $S =$ *Sup* $\{x.\ \forall s{\in}S.\ x \leq s\}$
by (*rule cInf-eq-non-empty*) (*auto intro*!: *cSup-upper cSup-least simp*: *bdd-below-def*)

lemma *cSup-cInf*: $S \neq \{\} \Longrightarrow$ *bdd-above* $S \Longrightarrow$ *Sup* $S =$ *Inf* $\{x.\ \forall s{\in}S.\ s \leq x\}$
by (*rule cSup-eq-non-empty*) (*auto intro*!: *cInf-lower cInf-greatest simp*: *bdd-above-def*)

lemma *cSup-insert*: $X \neq \{\} \Longrightarrow$ *bdd-above* $X \Longrightarrow$ *Sup* (*insert a X*) $=$ *sup a* (*Sup*

X)
  by (*intro cSup-eq-non-empty*) (*auto intro*: *le-supI2 cSup-upper cSup-least*)

lemma *cInf-insert*: $X \neq \{\} \Longrightarrow$ *bdd-below* $X \Longrightarrow$ *Inf* (*insert a X*) = *inf a* (*Inf X*)
  by (*intro cInf-eq-non-empty*) (*auto intro*: *le-infI2 cInf-lower cInf-greatest*)

lemma *cSup-singleton* [*simp*]: *Sup* $\{x\} = x$
  by (*intro cSup-eq-maximum*) *auto*

lemma *cInf-singleton* [*simp*]: *Inf* $\{x\} = x$
  by (*intro cInf-eq-minimum*) *auto*

lemma *cSup-insert-If*: *bdd-above* $X \Longrightarrow$ *Sup* (*insert a X*) = (*if* $X = \{\}$ *then a else sup a* (*Sup X*))
  using *cSup-insert*[*of X*] **by** *simp*

lemma *cInf-insert-If*: *bdd-below* $X \Longrightarrow$ *Inf* (*insert a X*) = (*if* $X = \{\}$ *then a else inf a* (*Inf X*))
  using *cInf-insert*[*of X*] **by** *simp*

lemma *le-cSup-finite*: *finite* $X \Longrightarrow x \in X \Longrightarrow x \leq$ *Sup X*
proof (*induct X arbitrary*: *x rule*: *finite-induct*)
  case (*insert x X y*) **then show** *?case*
    by (*cases* $X = \{\}$) (*auto simp*: *cSup-insert intro*: *le-supI2*)
qed *simp*

lemma *cInf-le-finite*: *finite* $X \Longrightarrow x \in X \Longrightarrow$ *Inf* $X \leq x$
proof (*induct X arbitrary*: *x rule*: *finite-induct*)
  case (*insert x X y*) **then show** *?case*
    by (*cases* $X = \{\}$) (*auto simp*: *cInf-insert intro*: *le-infI2*)
qed *simp*

lemma *cSup-eq-Sup-fin*: *finite* $X \Longrightarrow X \neq \{\} \Longrightarrow$ *Sup X* = *Sup-fin X*
  by (*induct X rule*: *finite-ne-induct*) (*simp-all add*: *cSup-insert*)

lemma *cInf-eq-Inf-fin*: *finite* $X \Longrightarrow X \neq \{\} \Longrightarrow$ *Inf X* = *Inf-fin X*
  by (*induct X rule*: *finite-ne-induct*) (*simp-all add*: *cInf-insert*)

lemma *cSup-atMost*[*simp*]: *Sup* $\{..x\} = x$
  by (*auto intro*!: *cSup-eq-maximum*)

lemma *cSup-greaterThanAtMost*[*simp*]: $y < x \Longrightarrow$ *Sup* $\{y<..x\} = x$
  by (*auto intro*!: *cSup-eq-maximum*)

lemma *cSup-atLeastAtMost*[*simp*]: $y \leq x \Longrightarrow$ *Sup* $\{y..x\} = x$
  by (*auto intro*!: *cSup-eq-maximum*)

lemma *cInf-atLeast*[*simp*]: *Inf* $\{x..\} = x$
  by (*auto intro*!: *cInf-eq-minimum*)

lemma *cInf-atLeastLessThan*[*simp*]: $y < x \Longrightarrow Inf\ \{y..<x\} = y$
by (*auto intro*!: *cInf-eq-minimum*)

lemma *cInf-atLeastAtMost*[*simp*]: $y \le x \Longrightarrow Inf\ \{y..x\} = y$
by (*auto intro*!: *cInf-eq-minimum*)

lemma *cINF-lower*: $bdd\text{-}below\ (f\ `\ A) \Longrightarrow x \in A \Longrightarrow \bigsqcap(f\ `\ A) \le f\ x$
using *cInf-lower* [*of - f ' A*] **by** *simp*

lemma *cINF-greatest*: $A \ne \{\} \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow m \le f\ x) \Longrightarrow m \le \bigsqcap(f\ `\ A)$
using *cInf-greatest* [*of f ' A*] **by** *auto*

lemma *cSUP-upper*: $x \in A \Longrightarrow bdd\text{-}above\ (f\ `\ A) \Longrightarrow f\ x \le \bigsqcup(f\ `\ A)$
using *cSup-upper* [*of - f ' A*] **by** *simp*

lemma *cSUP-least*: $A \ne \{\} \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \le M) \Longrightarrow \bigsqcup(f\ `\ A) \le M$
using *cSup-least* [*of f ' A*] **by** *auto*

lemma *cINF-lower2*: $bdd\text{-}below\ (f\ `\ A) \Longrightarrow x \in A \Longrightarrow f\ x \le u \Longrightarrow \bigsqcap(f\ `\ A) \le u$
by (*auto intro*: *cINF-lower order-trans*)

lemma *cSUP-upper2*: $bdd\text{-}above\ (f\ `\ A) \Longrightarrow x \in A \Longrightarrow u \le f\ x \Longrightarrow u \le \bigsqcup(f\ `\ A)$
by (*auto intro*: *cSUP-upper order-trans*)

lemma *cSUP-const* [*simp*]: $A \ne \{\} \Longrightarrow (\bigsqcup x\in A.\ c) = c$
by (*intro order.antisym cSUP-least*) (*auto intro*: *cSUP-upper*)

lemma *cINF-const* [*simp*]: $A \ne \{\} \Longrightarrow (\bigsqcap x\in A.\ c) = c$
by (*intro order.antisym cINF-greatest*) (*auto intro*: *cINF-lower*)

lemma *le-cINF-iff*: $A \ne \{\} \Longrightarrow bdd\text{-}below\ (f\ `\ A) \Longrightarrow u \le \bigsqcap(f\ `\ A) \longleftrightarrow (\forall x\in A.\ u \le f\ x)$
by (*metis cINF-greatest cINF-lower order-trans*)

lemma *cSUP-le-iff*: $A \ne \{\} \Longrightarrow bdd\text{-}above\ (f\ `\ A) \Longrightarrow \bigsqcup(f\ `\ A) \le u \longleftrightarrow (\forall x\in A.\ f\ x \le u)$
by (*metis cSUP-least cSUP-upper order-trans*)

lemma *less-cINF-D*: $bdd\text{-}below\ (f`A) \Longrightarrow y < (\bigsqcap i\in A.\ f\ i) \Longrightarrow i \in A \Longrightarrow y < f\ i$
by (*metis cINF-lower less-le-trans*)

lemma *cSUP-lessD*: $bdd\text{-}above\ (f`A) \Longrightarrow (\bigsqcup i\in A.\ f\ i) < y \Longrightarrow i \in A \Longrightarrow f\ i < y$
by (*metis cSUP-upper le-less-trans*)

lemma *cINF-insert*: $A \ne \{\} \Longrightarrow bdd\text{-}below\ (f\ `\ A) \Longrightarrow \bigsqcap(f\ `\ insert\ a\ A) = inf\ (f\ a)\ (\bigsqcap(f\ `\ A))$
by (*simp add*: *cInf-insert*)

lemma *cSUP-insert*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f \text{ ` } A) \Longrightarrow \bigsqcup(f \text{ ` } insert\ a\ A) = sup\ (f\ a)\ (\bigsqcup(f \text{ ` } A))$
by (*simp add*: *cSup-insert*)

lemma *cINF-mono*: $B \neq \{\} \Longrightarrow$ *bdd-below* $(f \text{ ` } A) \Longrightarrow (\bigwedge m.\ m \in B \Longrightarrow \exists n{\in}A.\ f\ n \leq g\ m) \Longrightarrow \bigsqcap(f \text{ ` } A) \leq \bigsqcap(g \text{ ` } B)$
using *cInf-mono* [*of g ` B f ` A*] **by** *auto*

lemma *cSUP-mono*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(g \text{ ` } B) \Longrightarrow (\bigwedge n.\ n \in A \Longrightarrow \exists m{\in}B.\ f\ n \leq g\ m) \Longrightarrow \bigsqcup(f \text{ ` } A) \leq \bigsqcup(g \text{ ` } B)$
using *cSup-mono* [*of f ` A g ` B*] **by** *auto*

lemma *cINF-superset-mono*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(g \text{ ` } B) \Longrightarrow A \subseteq B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow g\ x \leq f\ x) \Longrightarrow \bigsqcap(g \text{ ` } B) \leq \bigsqcap(f \text{ ` } A)$
by (*rule cINF-mono*) *auto*

lemma *cSUP-subset-mono*:
$[\![A \neq \{\};$ *bdd-above* $(g \text{ ` } B);\ A \subseteq B;\ \bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x]\!] \Longrightarrow \bigsqcup\ (f \text{ ` } A) \leq \bigsqcup\ (g \text{ ` } B)$
by (*rule cSUP-mono*) *auto*

lemma *less-eq-cInf-inter*: *bdd-below* $A \Longrightarrow$ *bdd-below* $B \Longrightarrow A \cap B \neq \{\} \Longrightarrow inf\ (Inf\ A)\ (Inf\ B) \leq Inf\ (A \cap B)$
by (*metis cInf-superset-mono lattice-class.inf-sup-ord*(*1*) *le-infI1*)

lemma *cSup-inter-less-eq*: *bdd-above* $A \Longrightarrow$ *bdd-above* $B \Longrightarrow A \cap B \neq \{\} \Longrightarrow Sup\ (A \cap B) \leq sup\ (Sup\ A)\ (Sup\ B)$
by (*metis cSup-subset-mono lattice-class.inf-sup-ord*(*1*) *le-supI1*)

lemma *cInf-union-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-below* $A \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-below* $B \Longrightarrow Inf\ (A \cup B) = inf\ (Inf\ A)\ (Inf\ B)$
by (*intro order.antisym le-infI cInf-greatest cInf-lower*) (*auto intro*: *le-infI1 le-infI2 cInf-lower*)

lemma *cINF-union*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f \text{ ` } A) \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-below* $(f \text{ ` } B) \Longrightarrow \bigsqcap\ (f \text{ ` } (A \cup B)) = \bigsqcap\ (f \text{ ` } A) \sqcap \bigsqcap\ (f \text{ ` } B)$
using *cInf-union-distrib* [*of f ` A f ` B*] **by** (*simp add*: *image-Un*)

lemma *cSup-union-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-above* $B \Longrightarrow Sup\ (A \cup B) = sup\ (Sup\ A)\ (Sup\ B)$
by (*intro order.antisym le-supI cSup-least cSup-upper*) (*auto intro*: *le-supI1 le-supI2 cSup-upper*)

lemma *cSUP-union*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f \text{ ` } A) \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-above* $(f \text{ ` } B) \Longrightarrow \bigsqcup\ (f \text{ ` } (A \cup B)) = \bigsqcup\ (f \text{ ` } A) \sqcup \bigsqcup\ (f \text{ ` } B)$
using *cSup-union-distrib* [*of f ` A f ` B*] **by** (*simp add*: *image-Un*)

lemma *cINF-inf-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f\text{`}A) \Longrightarrow$ *bdd-below* $(g\text{`}A) \Longrightarrow \bigsqcap\ (f \text{ ` } A) \sqcap \bigsqcap\ (g \text{ ` } A) = (\bigsqcap a{\in}A.\ inf\ (f\ a)\ (g\ a))$

by (*intro order.antisym le-infI cINF-greatest cINF-lower2*)
(*auto intro*: *le-infI1 le-infI2 cINF-greatest cINF-lower le-infI*)

lemma *SUP-sup-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f`A) \Longrightarrow$ *bdd-above* $(g`A) \Longrightarrow \bigsqcup (f ` A) \sqcup \bigsqcup (g ` A) = (\bigsqcup a{\in}A.\ sup\ (f\ a)\ (g\ a))$
by (*intro order.antisym le-supI cSUP-least cSUP-upper2*)
(*auto intro*: *le-supI1 le-supI2 cSUP-least cSUP-upper le-supI*)

lemma *cInf-le-cSup*:
$A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow$ *bdd-below* $A \Longrightarrow$ *Inf* $A \leq$ *Sup* A
by (*auto intro*!: *cSup-upper2*[*of SOME a. a* $\in$ *A*] *intro*: *someI cInf-lower*)

context
fixes f :: $'a \Rightarrow 'b$::*conditionally-complete-lattice*
assumes *mono f*
begin

lemma *mono-cInf*: ⟦*bdd-below A*; $A{\neq}\{\}$⟧ $\Longrightarrow f$ (*Inf A*) $\leq$ (*INF* $x{\in}A.$ *f x*)
by (*simp add*: ‹*mono f*› *conditionally-complete-lattice-class.cINF-greatest cInf-lower monoD*)

lemma *mono-cSup*: ⟦*bdd-above A*; $A{\neq}\{\}$⟧ $\Longrightarrow$ (*SUP* $x{\in}A.$ *f x*) $\leq f$ (*Sup A*)
by (*simp add*: ‹*mono f*› *conditionally-complete-lattice-class.cSUP-least cSup-upper monoD*)

lemma *mono-cINF*: ⟦*bdd-below* $(A`I)$; $I{\neq}\{\}$⟧ $\Longrightarrow f$ (*INF* $i{\in}I.$ *A i*) $\leq$ (*INF* $x{\in}I.$ *f* (*A x*))
by (*simp add*: ‹*mono f*› *conditionally-complete-lattice-class.cINF-greatest cINF-lower monoD*)

lemma *mono-cSUP*: ⟦*bdd-above* $(A`I)$; $I{\neq}\{\}$⟧ $\Longrightarrow$ (*SUP* $x{\in}I.$ *f* (*A x*)) $\leq f$ (*SUP* $i{\in}I.$ *A i*)
by (*simp add*: ‹*mono f*› *conditionally-complete-lattice-class.cSUP-least cSUP-upper monoD*)

end

end

The special case of well-orderings

lemma *wellorder-InfI*:
fixes k :: $'a$::{*wellorder*,*conditionally-complete-lattice*}
assumes $k \in A$ **shows** *Inf* $A \in A$
using *wellorder-class.LeastI* [*of* $\lambda x.\ x \in A\ k$]
by (*simp add*: *Least-le assms cInf-eq-minimum*)

lemma *wellorder-Inf-le1*:
fixes k :: $'a$::{*wellorder*,*conditionally-complete-lattice*}
assumes $k \in A$ **shows** *Inf* $A \leq k$

by (*meson Least-le assms bdd-below.I cInf-lower*)

91.4 Complete lattices

instance *complete-lattice* ⊆ *conditionally-complete-lattice*
by *standard* (*auto intro*: *Sup-upper Sup-least Inf-lower Inf-greatest*)

lemma *cSup-eq*:
fixes $a :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}bot\}$
assumes *upper*: $\bigwedge x.\ x \in X \Longrightarrow x \le a$
assumes *least*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow x \le y) \Longrightarrow a \le y$
shows $Sup\ X = a$
proof *cases*
assume $X = \{\}$ **with** *lt-ex*[*of a*] *least* **show** *?thesis* **by** (*auto simp*: *less-le-not-le*)
qed (*intro cSup-eq-non-empty assms*)

lemma *cSup-unique*:
fixes $b :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}bot\}$
assumes $\bigwedge c.\ (\forall x \in s.\ x \le c) \longleftrightarrow b \le c$
shows $Sup\ s = b$
by (*metis assms cSup-eq order.refl*)

lemma *cInf-eq*:
fixes $a :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}top\}$
assumes *upper*: $\bigwedge x.\ x \in X \Longrightarrow a \le x$
assumes *least*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow y \le x) \Longrightarrow y \le a$
shows $Inf\ X = a$
proof *cases*
assume $X = \{\}$ **with** *gt-ex*[*of a*] *least* **show** *?thesis* **by** (*auto simp*: *less-le-not-le*)
qed (*intro cInf-eq-non-empty assms*)

lemma *cInf-unique*:
fixes $b :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}top\}$
assumes $\bigwedge c.\ (\forall x \in s.\ x \ge c) \longleftrightarrow b \ge c$
shows $Inf\ s = b$
by (*meson assms cInf-eq order.refl*)

class *conditionally-complete-linorder* = *conditionally-complete-lattice* + *linorder*
begin

lemma *less-cSup-iff*:
$X \ne \{\} \Longrightarrow bdd\text{-}above\ X \Longrightarrow y < Sup\ X \longleftrightarrow (\exists x \in X.\ y < x)$
by (*rule iffI*) (*metis cSup-least not-less*, *metis cSup-upper less-le-trans*)

lemma *cInf-less-iff*: $X \ne \{\} \Longrightarrow bdd\text{-}below\ X \Longrightarrow Inf\ X < y \longleftrightarrow (\exists x \in X.\ x < y)$
by (*rule iffI*) (*metis cInf-greatest not-less*, *metis cInf-lower le-less-trans*)

lemma *cINF-less-iff*: $A \ne \{\} \Longrightarrow bdd\text{-}below\ (f`A) \Longrightarrow (\prod i \in A.\ f\ i) < a \longleftrightarrow (\exists x \in A.\ f\ x < a)$

using *cInf-less-iff* [*of f'A*] **by** *auto*

lemma *less-cSUP-iff*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f`A) \Longrightarrow a < (\bigsqcup i{\in}A.\ f\ i) \longleftrightarrow (\exists x{\in}A.\ a < f\ x)$
using *less-cSup-iff* [*of f'A*] **by** *auto*

lemma *less-cSupE*:
assumes $y <$ *Sup* X $X \neq \{\}$ **obtains** x **where** $x \in X$ $y < x$
by (*metis cSup-least assms not-le that*)

lemma *less-cSupD*:
$X \neq \{\} \Longrightarrow z <$ *Sup* $X \Longrightarrow \exists x{\in}X.\ z < x$
by (*metis less-cSup-iff not-le-imp-less bdd-above-def*)

lemma *cInf-lessD*:
$X \neq \{\} \Longrightarrow$ *Inf* $X < z \Longrightarrow \exists x{\in}X.\ x < z$
by (*metis cInf-less-iff not-le-imp-less bdd-below-def*)

lemma *complete-interval*:
assumes $a < b$ **and** $P\ a$ **and** $\neg\ P\ b$
shows $\exists\, c.\ a \leq c \wedge c \leq b \wedge (\forall\, x.\ a \leq x \wedge x < c \longrightarrow P\ x) \wedge$
$(\forall\, d.\ (\forall\, x.\ a \leq x \wedge x < d \longrightarrow P\ x) \longrightarrow d \leq c)$
proof (*rule exI* [**where** $x =$ *Sup* $\{d.\ \forall\, x.\ a \leq x \wedge x < d \longrightarrow P\ x\}$], *safe*)
show $a \leq$ *Sup* $\{d.\ \forall\, c.\ a \leq c \wedge c < d \longrightarrow P\ c\}$
by (*rule cSup-upper*, *auto simp*: *bdd-above-def*)
(*metis* ‹$a < b$› ‹$\neg\ P\ b$› *linear less-le*)
next
show *Sup* $\{d.\ \forall\, c.\ a \leq c \wedge c < d \longrightarrow P\ c\} \leq b$
by (*rule cSup-least*)
(*use* ‹$a<b$› ‹$\neg\ P\ b$› **in** ‹*auto simp add*: *less-le-not-le*›)
next
fix x
assume x: $a \leq x$ **and** *lt*: $x <$ *Sup* $\{d.\ \forall\, c.\ a \leq c \wedge c < d \longrightarrow P\ c\}$
show $P\ x$
by (*rule less-cSupE* [*OF lt*]) (*use less-le-not-le x* **in** ‹*auto*›)
next
fix d
assume *0*: $\forall\, x.\ a \leq x \wedge x < d \longrightarrow P\ x$
then have $d \in \{d.\ \forall\, c.\ a \leq c \wedge c < d \longrightarrow P\ c\}$
by *auto*
moreover have *bdd-above* $\{d.\ \forall\, c.\ a \leq c \wedge c < d \longrightarrow P\ c\}$
unfolding *bdd-above-def* **using** ‹$a<b$› ‹$\neg\ P\ b$› *linear*
by (*simp add*: *less-le*) *blast*
ultimately show $d \leq$ *Sup* $\{d.\ \forall\, c.\ a \leq c \wedge c < d \longrightarrow P\ c\}$
by (*auto simp*: *cSup-upper*)
qed

end

91.5 Instances

instance *complete-linorder* < *conditionally-complete-linorder*
..

lemma *cSup-eq-Max*: *finite* (*X*::′*a*::*conditionally-complete-linorder set*) ⟹ *X* ≠ {} ⟹ *Sup X* = *Max X*
using *cSup-eq-Sup-fin*[*of X*] **by** (*simp add*: *Sup-fin-Max*)

lemma *cInf-eq-Min*: *finite* (*X*::′*a*::*conditionally-complete-linorder set*) ⟹ *X* ≠ {} ⟹ *Inf X* = *Min X*
using *cInf-eq-Inf-fin*[*of X*] **by** (*simp add*: *Inf-fin-Min*)

lemma *cSup-lessThan*[*simp*]: *Sup* {..<*x*::′*a*::{*conditionally-complete-linorder*, *no-bot*, *dense-linorder*}} = *x*
by (*auto intro*!: *cSup-eq-non-empty intro*: *dense-le*)

lemma *cSup-greaterThanLessThan*[*simp*]: *y* < *x* ⟹ *Sup* {*y*<..<*x*::′*a*::{*conditionally-complete-linorder*, *dense-linorder*}} = *x*
by (*auto intro*!: *cSup-eq-non-empty intro*: *dense-le-bounded*)

lemma *cSup-atLeastLessThan*[*simp*]: *y* < *x* ⟹ *Sup* {*y*..<*x*::′*a*::{*conditionally-complete-linorder*, *dense-linorder*}} = *x*
by (*auto intro*!: *cSup-eq-non-empty intro*: *dense-le-bounded*)

lemma *cInf-greaterThan*[*simp*]: *Inf* {*x*::′*a*::{*conditionally-complete-linorder*, *no-top*, *dense-linorder*} <..} = *x*
by (*auto intro*!: *cInf-eq-non-empty intro*: *dense-ge*)

lemma *cInf-greaterThanAtMost*[*simp*]: *y* < *x* ⟹ *Inf* {*y*<..*x*::′*a*::{*conditionally-complete-linorder*, *dense-linorder*}} = *y*
by (*auto intro*!: *cInf-eq-non-empty intro*: *dense-ge-bounded*)

lemma *cInf-greaterThanLessThan*[*simp*]: *y* < *x* ⟹ *Inf* {*y*<..<*x*::′*a*::{*conditionally-complete-linorder*, *dense-linorder*}} = *y*
by (*auto intro*!: *cInf-eq-non-empty intro*: *dense-ge-bounded*)

lemma *Sup-inverse-eq-inverse-Inf*:
fixes *f*::′*b* ⇒ ′*a*::{*conditionally-complete-linorder*,*linordered-field*}
assumes *bdd-above* (*range f*) *L* > *0* **and** *geL*: ⋀*x*. *f x* ≥ *L*
shows (*SUP x*. *1* / *f x*) = *1* / (*INF x*. *f x*)
proof (*rule antisym*)
have *bdd-f*: *bdd-below* (*range f*)
by (*meson assms bdd-belowI2*)
have *Inf* (*range f*) ≥ *L*
by (*simp add*: *cINF-greatest geL*)
have *bdd-invf*: *bdd-above* (*range* (λ*x*. *1* / *f x*))
proof (*rule bdd-aboveI2*)
show ⋀*x*. *1* / *f x* ≤ *1*/*L*
using *assms* **by** (*auto simp*: *divide-simps*)

qed
moreover have *le-inverse-Inf*: *1 / f x* ≤ *1 / Inf (range f)* **for** *x*
proof −
have *Inf (range f)* ≤ *f x*
by (*simp add*: *bdd-f cInf-lower*)
then show *?thesis*
using *assms* ‹*L* ≤ *Inf (range f)*› **by** (*auto simp*: *divide-simps*)
qed
ultimately show ∗: (*SUP x. 1 / f x*) ≤ *1 / Inf (range f)*
by (*auto simp*: *cSup-le-iff cINF-lower*)
have *1 / (SUP x. 1 / f x)* ≤ *f y* **for** *y*
proof (*cases* (*SUP x. 1 / f x*) < *0*)
case *True*
with *assms* **show** *?thesis*
by (*meson less-asym′ order-trans linorder-not-le zero-le-divide-1-iff*)
next
case *False*
have *1 / f y* ≤ (*SUP x. 1 / f x*)
by (*simp add*: *bdd-invf cSup-upper*)
with *False assms* **show** *?thesis*
by (*metis (no-types) div-by-1 divide-divide-eq-right dual-order.strict-trans1 inverse-eq-divide*
inverse-le-imp-le mult.left-neutral)
qed
then have *1 / (SUP x. 1 / f x)* ≤ *Inf (range f)*
using *bdd-f* **by** (*simp add*: *le-cInf-iff*)
moreover have (*SUP x. 1 / f x*) > *0*
using *assms cSUP-upper [OF - bdd-invf]* **by** (*meson UNIV-I less-le-trans zero-less-divide-1-iff*)
ultimately show *1 / Inf (range f)* ≤ (*SUP t. 1 / f t*)
using ‹*L* ≤ *Inf (range f)*› ‹*L*>*0*› **by** (*auto simp*: *field-simps*)
qed

lemma *Inf-inverse-eq-inverse-Sup*:
fixes *f*::′*b* ⇒ ′*a*::{*conditionally-complete-linorder*,*linordered-field*}
assumes *bdd-above (range f) L* > *0* **and** *geL*: ⋀*x. f x* ≥ *L*
shows (*INF x. 1 / f x*) = *1 / (SUP x. f x)*
proof −
obtain *M* **where** *M*>*0* **and** *M*: ⋀*x. f x* ≤ *M*
by (*meson assms cSup-upper dual-order.strict-trans1 rangeI*)
have *bdd*: *bdd-above (range (inverse* ∘ *f))*
using *assms le-imp-inverse-le* **by** (*auto simp*: *bdd-above-def*)
have *f x* > *0* **for** *x*
using ‹*L*>*0*› *geL order-less-le-trans* **by** *blast*
then have [*simp*]: *1 / inverse(f x)* = *f x 1 / M* ≤ *1 / f x* **for** *x*
using *M* ‹*M*>*0*› **by** (*auto simp*: *divide-simps*)
show *?thesis*
using *Sup-inverse-eq-inverse-Inf [OF bdd, of inverse M]* ‹*M*>*0*›
by (*simp add*: *inverse-eq-divide*)

qed

lemma *Inf-insert-finite*:
fixes S :: $'a$::*conditionally-complete-linorder set*
shows *finite* $S \Longrightarrow$ *Inf* (*insert* x S) = (*if* $S = \{\}$ *then* x *else min* x (*Inf* S))
by (*simp add*: *cInf-eq-Min*)

lemma *Sup-insert-finite*:
fixes S :: $'a$::*conditionally-complete-linorder set*
shows *finite* $S \Longrightarrow$ *Sup* (*insert* x S) = (*if* $S = \{\}$ *then* x *else max* x (*Sup* S))
by (*simp add*: *cSup-insert sup-max*)

lemma *finite-imp-less-Inf*:
fixes a :: $'a$::*conditionally-complete-linorder*
shows $[\![$*finite* X; $x \in X$; $\bigwedge x.\ x{\in}X \Longrightarrow a < x]\!] \Longrightarrow a <$ *Inf* X
by (*induction X rule*: *finite-induct*) (*simp-all add*: *cInf-eq-Min Inf-insert-finite*)

lemma *finite-less-Inf-iff*:
fixes a :: $'a$ :: *conditionally-complete-linorder*
shows $[\![$*finite* X; $X \neq \{\}]\!] \Longrightarrow a <$ *Inf* $X \longleftrightarrow (\forall x \in X.\ a < x)$
by (*auto simp*: *cInf-eq-Min*)

lemma *finite-imp-Sup-less*:
fixes a :: $'a$::*conditionally-complete-linorder*
shows $[\![$*finite* X; $x \in X$; $\bigwedge x.\ x{\in}X \Longrightarrow a > x]\!] \Longrightarrow a >$ *Sup* X
by (*induction X rule*: *finite-induct*) (*simp-all add*: *cSup-eq-Max Sup-insert-finite*)

lemma *finite-Sup-less-iff*:
fixes a :: $'a$ :: *conditionally-complete-linorder*
shows $[\![$*finite* X; $X \neq \{\}]\!] \Longrightarrow a >$ *Sup* $X \longleftrightarrow (\forall x \in X.\ a > x)$
by (*auto simp*: *cSup-eq-Max*)

class *linear-continuum* = *conditionally-complete-linorder* + *dense-linorder* +
assumes *UNIV-not-singleton*: $\exists\, a\ b$::$'a.\ a \neq b$
begin

lemma *ex-gt-or-lt*: $\exists\, b.\ a < b \lor b < a$
by (*metis UNIV-not-singleton neq-iff*)

end

context
fixes f::$'a \Rightarrow$ $'b$::{*conditionally-complete-linorder*,*ordered-ab-group-add*}
begin

lemma *bdd-above-uminus-image*: *bdd-above* (($\lambda x.\ - f\ x$) ' A) $\longleftrightarrow$ *bdd-below* (f ' A)
by (*metis bdd-above-uminus image-image*)

lemma *bdd-below-uminus-image*: *bdd-below* $((\lambda x. - f\ x)$ ‘ $A) \longleftrightarrow$ *bdd-above* $(f$ ‘ $A)$
by (*metis bdd-below-uminus image-image*)

lemma *uminus-cSUP*:
assumes *bdd-above* $(f$ ‘ $A)$ $A \neq \{\}$
shows $-$ (*SUP* $x{\in}A.\ f\ x$) = (*INF* $x{\in}A.\ -\ f\ x$)
proof (*rule antisym*)
show (*INF* $x{\in}A.\ -\ f\ x$) $\leq -$ *Sup* $(f$ ‘ $A)$
by (*metis cINF-lower cSUP-least bdd-below-uminus-image assms le-minus-iff*)
have $*$: $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq$ *Sup* $(f$ ‘ $A)$
by (*simp add*: *assms cSup-upper*)
then show $-$ *Sup* $(f$ ‘ $A) \leq$ (*INF* $x{\in}A.\ -\ f\ x$)
by (*simp add*: *assms cINF-greatest*)
qed

end

context
fixes $f{::}'a \Rightarrow 'b{::}\{$*conditionally-complete-linorder*,*ordered-ab-group-add*$\}$
begin

lemma *uminus-cINF*:
assumes *bdd-below* $(f$ ‘ $A)$ $A \neq \{\}$
shows $-$ (*INF* $x{\in}A.\ f\ x$) = (*SUP* $x{\in}A.\ -\ f\ x$)
by (*metis* (*mono-tags*, *lifting*) *INF-cong uminus-cSUP assms bdd-above-uminus-image minus-equation-iff*)

lemma *Sup-add-eq*:
assumes *bdd-above* $(f$ ‘ $A)$ $A \neq \{\}$
shows (*SUP* $x{\in}A.\ a + f\ x$) = a + (*SUP* $x{\in}A.\ f\ x$) (**is** *?L=?R*)
proof (*rule antisym*)
have *bdd*: *bdd-above* $((\lambda x.\ a + f\ x)$ ‘ $A)$
by (*metis assms bdd-above-image-mono image-image mono-add*)
with *assms* **show** *?L* $\leq$ *?R*
by (*simp add*: *assms cSup-le-iff cSUP-upper*)
have $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq$ (*SUP* $x{\in}A.\ a + f\ x$) $-\ a$
by (*simp add*: *bdd cSup-upper le-diff-eq*)
with ‹$A \neq \{\}$› **have** $\bigsqcup\ (f$ ‘ $A) \leq (\bigsqcup x{\in}A.\ a + f\ x) - a$
by (*simp add*: *cSUP-least*)
then show *?R* $\leq$ *?L*
by (*metis add.commute le-diff-eq*)
qed

lemma *Inf-add-eq*: — you don't get a shorter proof by duality
assumes *bdd-below* $(f$ ‘ $A)$ $A \neq \{\}$
shows (*INF* $x{\in}A.\ a + f\ x$) = a + (*INF* $x{\in}A.\ f\ x$) (**is** *?L=?R*)
proof (*rule antisym*)

show *?R* ≤ *?L*
  using *assms mono-add mono-cINF* **by** *blast*
have *bdd*: *bdd-below* ((λ*x*. *a* + *f x*) ‘ *A*)
  by (*metis add-left-mono assms*(*1*) *bdd-below.E bdd-below.I2 imageI*)
with *assms* **have** ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ (*INF x*∈*A*. *a* + *f x*) − *a*
  by (*simp add*: *cInf-lower diff-le-eq*)
with ‹*A* ≠ {}› **have** (⨅*x*∈*A*. *a* + *f x*) − *a* ≤ ⨅ (*f* ‘ *A*)
  by (*simp add*: *cINF-greatest*)
with *assms* **show** *?L* ≤ *?R*
  by (*metis add.commute diff-le-eq*)
qed

end

instantiation *nat* :: *conditionally-complete-linorder*
begin

definition *Sup* (*X*::*nat set*) = (*if X*={} *then 0 else Max X*)
definition *Inf* (*X*::*nat set*) = (*LEAST n*. *n* ∈ *X*)

lemma *bdd-above-nat*: *bdd-above X* ⟷ *finite* (*X*::*nat set*)
proof
  assume *bdd-above X*
  then obtain *z* **where** *X* ⊆ {.. *z*}
    by (*auto simp*: *bdd-above-def*)
  then show *finite X*
    by (*rule finite-subset*) *simp*
qed *simp*

instance
proof
  fix *x* :: *nat*
  fix *X* :: *nat set*
  show *Inf X* ≤ *x* **if** *x* ∈ *X bdd-below X*
    using *that* **by** (*simp add*: *Inf-nat-def Least-le*)
  show *x* ≤ *Inf X* **if** *X* ≠ {} ⋀*y*. *y* ∈ *X* ⟹ *x* ≤ *y*
    using *that* **unfolding** *Inf-nat-def ex-in-conv*[*symmetric*] **by** (*rule LeastI2-ex*)
  show *x* ≤ *Sup X* **if** *x* ∈ *X bdd-above X*
    using *that* **by** (*auto simp add*: *Sup-nat-def bdd-above-nat*)
  show *Sup X* ≤ *x* **if** *X* ≠ {} ⋀*y*. *y* ∈ *X* ⟹ *y* ≤ *x*
  proof −
    from *that* **have** *bdd-above X*
      by (*auto simp*: *bdd-above-def*)
    with *that* **show** *?thesis*
      by (*simp add*: *Sup-nat-def bdd-above-nat*)
  qed
qed

end

lemma *Inf-nat-def1*:
  fixes *K*::*nat set*
  assumes $K \neq \{\}$
  shows *Inf* $K \in K$
by (*auto simp add*: *Min-def Inf-nat-def*) (*meson LeastI assms bot.extremum-unique subsetI*)

lemma *Sup-nat-empty* [*simp*]: *Sup* {} = (*0*::*nat*)
  by (*auto simp add*: *Sup-nat-def*)

instantiation *int* :: *conditionally-complete-linorder*
begin

definition *Sup* (*X*::*int set*) = (*THE x*. $x \in X \wedge (\forall y{\in}X.\ y \leq x)$)
definition *Inf* (*X*::*int set*) = − (*Sup* (*uminus* ' *X*))

instance
proof
  { fix *x* :: *int* **and** *X* :: *int set* **assume** $X \neq \{\}$ *bdd-above X*
    then obtain *x y* **where** $X \subseteq \{..y\}$ $x \in X$
      by (*auto simp*: *bdd-above-def*)
    then have ∗: *finite* ($X \cap \{x..y\}$) $X \cap \{x..y\} \neq \{\}$ **and** $x \leq y$
      by (*auto simp*: *subset-eq*)
    have $\exists!x{\in}X.\ (\forall y{\in}X.\ y \leq x)$
    proof
      { fix *z* **assume** $z \in X$
        have $z \leq$ *Max* ($X \cap \{x..y\}$)
        proof *cases*
          assume $x \leq z$ **with** ‹$z \in X$› ‹$X \subseteq \{..y\}$› ∗(*1*) **show** *?thesis*
            by (*auto intro*!: *Max-ge*)
        next
          assume ¬ $x \leq z$
          then have $z < x$ **by** *simp*
          also have $x \leq$ *Max* ($X \cap \{x..y\}$)
            using ‹$x \in X$› ∗(*1*) ‹$x \leq y$› **by** (*intro Max-ge*) *auto*
          finally show *?thesis* **by** *simp*
        qed }
      note *le* = *this*
      with *Max-in*[*OF* ∗] **show** *ex*: *Max* ($X \cap \{x..y\}$) $\in X \wedge (\forall z{\in}X.\ z \leq$ *Max* ($X \cap \{x..y\}$)) **by** *auto*

      fix *z* **assume** ∗: $z \in X \wedge (\forall y{\in}X.\ y \leq z)$
      with *le* **have** $z \leq$ *Max* ($X \cap \{x..y\}$)
        by *auto*
      moreover have *Max* ($X \cap \{x..y\}$) $\leq z$
        using ∗ *ex* **by** *auto*

ultimately show $z = Max\ (X \cap \{x..y\})$
by *auto*
qed
then have $Sup\ X \in X \wedge (\forall y \in X.\ y \le Sup\ X)$
unfolding *Sup-int-def* **by** (*rule theI'*) **}**
note *Sup-int* = *this*

{ fix x :: *int* **and** X :: *int set* **assume** $x \in X$ *bdd-above* X **then show** $x \le Sup$ X
using *Sup-int*[*of X*] **by** *auto* **}**
note *le-Sup* = *this*
{ fix x :: *int* **and** X :: *int set* **assume** $X \neq \{\}$ $\bigwedge y.\ y \in X \Longrightarrow y \le x$ **then show** $Sup\ X \le x$
using *Sup-int*[*of X*] **by** (*auto simp*: *bdd-above-def*) **}**
note *Sup-le* = *this*

{ fix x :: *int* **and** X :: *int set* **assume** $x \in X$ *bdd-below* X **then show** $Inf\ X \le x$
using *le-Sup*[*of* $-x$ *uminus* ' X] **by** (*auto simp*: *Inf-int-def*) **}**
{ fix x :: *int* **and** X :: *int set* **assume** $X \neq \{\}$ $\bigwedge y.\ y \in X \Longrightarrow x \le y$ **then show** $x \le Inf\ X$
using *Sup-le*[*of uminus* ' X $-x$] **by** (*force simp*: *Inf-int-def*) **}**
qed
end

lemma *interval-cases*:
fixes S :: *'a* :: *conditionally-complete-linorder set*
assumes *ivl*: $\bigwedge a\ b\ x.\ a \in S \Longrightarrow b \in S \Longrightarrow a \le x \Longrightarrow x \le b \Longrightarrow x \in S$
shows $\exists\, a\ b.\ S = \{\} \vee$
$S = UNIV \vee$
$S = \{..<b\} \vee$
$S = \{..b\} \vee$
$S = \{a<..\} \vee$
$S = \{a..\} \vee$
$S = \{a<..<b\} \vee$
$S = \{a<..b\} \vee$
$S = \{a..<b\} \vee$
$S = \{a..b\}$
proof −
define *lower upper* **where** $lower = \{x.\ \exists\, s \in S.\ s \le x\}$ **and** $upper = \{x.\ \exists\, s \in S.\ x \le s\}$
with *ivl* **have** $S = lower \cap upper$
by *auto*
moreover
have $\exists\, a.\ upper = UNIV \vee upper = \{\} \vee upper = \{..\ a\} \vee upper = \{..<\ a\}$
proof *cases*
assume $*$: *bdd-above* $S \wedge S \neq \{\}$
from $*$ **have** $upper \subseteq \{..\ Sup\ S\}$
by (*auto simp*: *upper-def intro*: *cSup-upper2*)
moreover from $*$ **have** $\{..<\ Sup\ S\} \subseteq upper$

by (*force simp add*: *less-cSup-iff upper-def subset-eq Ball-def*)
ultimately have *upper* = {.. *Sup S*} ∨ *upper* = {..< *Sup S*}
unfolding *ivl-disj-un*(*2*)[*symmetric*] **by** *auto*
then show *?thesis* **by** *auto*
next
assume ¬ (*bdd-above S* ∧ *S* ≠ {})
then have *upper* = *UNIV* ∨ *upper* = {}
by (*auto simp*: *upper-def bdd-above-def not-le dest*: *less-imp-le*)
then show *?thesis*
by *auto*
qed
moreover
have ∃ *b*. *lower* = *UNIV* ∨ *lower* = {} ∨ *lower* = {*b* ..} ∨ *lower* = {*b* <..}
proof *cases*
assume ∗: *bdd-below S* ∧ *S* ≠ {}
from ∗ **have** *lower* ⊆ {*Inf S* ..}
by (*auto simp*: *lower-def intro*: *cInf-lower2*)
moreover from ∗ **have** {*Inf S* <..} ⊆ *lower*
by (*force simp add*: *cInf-less-iff lower-def subset-eq Ball-def*)
ultimately have *lower* = {*Inf S* ..} ∨ *lower* = {*Inf S* <..}
unfolding *ivl-disj-un*(*1*)[*symmetric*] **by** *auto*
then show *?thesis* **by** *auto*
next
assume ¬ (*bdd-below S* ∧ *S* ≠ {})
then have *lower* = *UNIV* ∨ *lower* = {}
by (*auto simp*: *lower-def bdd-below-def not-le dest*: *less-imp-le*)
then show *?thesis*
by *auto*
qed
ultimately show *?thesis*
unfolding *greaterThanAtMost-def greaterThanLessThan-def atLeastAtMost-def atLeastLessThan-def*
by (*metis inf-bot-left inf-bot-right inf-top.left-neutral inf-top.right-neutral*)
qed

lemma *cSUP-eq-cINF-D*:
fixes *f* :: - ⇒ *'b*::*conditionally-complete-lattice*
assumes *eq*: ($\bigsqcup$*x*∈*A*. *f x*) = ($\bigsqcap$*x*∈*A*. *f x*)
and *bdd*: *bdd-above* (*f* ' *A*) *bdd-below* (*f* ' *A*)
and *a*: *a* ∈ *A*
shows *f a* = ($\bigsqcap$*x*∈*A*. *f x*)
proof (*rule antisym*)
show *f a* ≤ $\bigsqcap$ (*f* ' *A*)
by (*metis a bdd*(*1*) *eq cSUP-upper*)
show $\bigsqcap$ (*f* ' *A*) ≤ *f a*
using *a bdd* **by** (*auto simp*: *cINF-lower*)
qed

lemma *cSUP-UNION*:

fixes *f* :: - ⇒ *'b::conditionally-complete-lattice*
assumes *ne*: $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow B(x) \neq \{\}$
and *bdd-UN*: *bdd-above* $(\bigcup x{\in}A.\ f\ `\ B\ x)$
shows $(\bigsqcup z \in \bigcup x{\in}A.\ B\ x.\ f\ z) = (\bigsqcup x{\in}A.\ \bigsqcup z{\in}B\ x.\ f\ z)$
proof –
have *bdd*: $\bigwedge x.\ x \in A \Longrightarrow$ *bdd-above* $(f\ `\ B\ x)$
using *bdd-UN* **by** (*meson UN-upper bdd-above-mono*)
obtain *M* **where** $\bigwedge x\ y.\ x \in A \Longrightarrow y \in B(x) \Longrightarrow f\ y \leq M$
using *bdd-UN* **by** (*auto simp*: *bdd-above-def*)
then have *bdd2*: *bdd-above* $((\lambda x.\ \bigsqcup z{\in}B\ x.\ f\ z)\ `\ A)$
unfolding *bdd-above-def* **by** (*force simp*: *bdd cSUP-le-iff ne*(*2*))
have $(\bigsqcup z \in \bigcup x{\in}A.\ B\ x.\ f\ z) \leq (\bigsqcup x{\in}A.\ \bigsqcup z{\in}B\ x.\ f\ z)$
using *assms* **by** (*fastforce simp add*: *intro*!: *cSUP-least intro*: *cSUP-upper2 simp*: *bdd2 bdd*)
moreover have $(\bigsqcup x{\in}A.\ \bigsqcup z{\in}B\ x.\ f\ z) \leq (\bigsqcup z \in \bigcup x{\in}A.\ B\ x.\ f\ z)$
using *assms* **by** (*fastforce simp add*: *intro*!: *cSUP-least intro*: *cSUP-upper simp*: *image-UN bdd-UN*)
ultimately show *?thesis*
by (*rule order-antisym*)
qed

lemma *cINF-UNION*:
fixes *f* :: - ⇒ *'b::conditionally-complete-lattice*
assumes *ne*: $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow B(x) \neq \{\}$
and *bdd-UN*: *bdd-below* $(\bigcup x{\in}A.\ f\ `\ B\ x)$
shows $(\bigsqcap z \in \bigcup x{\in}A.\ B\ x.\ f\ z) = (\bigsqcap x{\in}A.\ \bigsqcap z{\in}B\ x.\ f\ z)$
proof –
have *bdd*: $\bigwedge x.\ x \in A \Longrightarrow$ *bdd-below* $(f\ `\ B\ x)$
using *bdd-UN* **by** (*meson UN-upper bdd-below-mono*)
obtain *M* **where** $\bigwedge x\ y.\ x \in A \Longrightarrow y \in B(x) \Longrightarrow f\ y \geq M$
using *bdd-UN* **by** (*auto simp*: *bdd-below-def*)
then have *bdd2*: *bdd-below* $((\lambda x.\ \bigsqcap z{\in}B\ x.\ f\ z)\ `\ A)$
unfolding *bdd-below-def* **by** (*force simp*: *bdd le-cINF-iff ne*(*2*))
have $(\bigsqcap z \in \bigcup x{\in}A.\ B\ x.\ f\ z) \leq (\bigsqcap x{\in}A.\ \bigsqcap z{\in}B\ x.\ f\ z)$
using *assms* **by** (*fastforce simp add*: *intro*!: *cINF-greatest intro*: *cINF-lower simp*: *bdd2 bdd*)
moreover have $(\bigsqcap x{\in}A.\ \bigsqcap z{\in}B\ x.\ f\ z) \leq (\bigsqcap z \in \bigcup x{\in}A.\ B\ x.\ f\ z)$
using *assms* **by** (*fastforce simp add*: *intro*!: *cINF-greatest intro*: *cINF-lower2 simp*: *bdd bdd-UN bdd2*)
ultimately show *?thesis*
by (*rule order-antisym*)
qed

lemma *cSup-abs-le*:
fixes *S* :: (*'a*::{*linordered-idom*,*conditionally-complete-linorder*}) *set*
shows $S \neq \{\} \Longrightarrow (\bigwedge x.\ x{\in}S \Longrightarrow |x| \leq a) \Longrightarrow |Sup\ S| \leq a$
apply (*auto simp add*: *abs-le-iff intro*: *cSup-least*)
by (*metis bdd-aboveI cSup-upper neg-le-iff-le order-trans*)

end

92 Factorial Function, Rising Factorials

theory *Factorial*
imports *Groups-List*
begin

92.1 Factorial Function

context *semiring-char-0*
begin

definition *fact* :: *nat* ⇒ *'a*
where *fact-prod*: *fact n* = *of-nat* (∏{*1..n*})

lemma *fact-prod-Suc*: *fact n* = *of-nat* (*prod Suc* {*0..<n*})
unfolding *fact-prod* **using** *atLeast0LessThan prod.atLeast1-atMost-eq* **by** *auto*

lemma *fact-prod-rev*: *fact n* = *of-nat* (∏ *i* = *0..<n. n* − *i*)
proof −
have *prod Suc* {*0..<n*} = ∏{*1..n*}
by (*simp add*: *atLeast0LessThan prod.atLeast1-atMost-eq*)
then have *prod Suc* {*0..<n*} = *prod* ((−) (*n* + *1*)) {*1..n*}
using *prod.atLeastAtMost-rev* [*of* λ*i. i 1 n*] **by** *presburger*
then show *?thesis*
unfolding *fact-prod-Suc* **by** (*simp add*: *atLeast0LessThan prod.atLeast1-atMost-eq*)
qed

lemma *fact-0* [*simp*]: *fact 0* = *1*
by (*simp add*: *fact-prod*)

lemma *fact-1* [*simp*]: *fact 1* = *1*
by (*simp add*: *fact-prod*)

lemma *fact-Suc-0* [*simp*]: *fact* (*Suc 0*) = *1*
by (*simp add*: *fact-prod*)

lemma *fact-Suc* [*simp*]: *fact* (*Suc n*) = *of-nat* (*Suc n*) ∗ *fact n*
by (*simp add*: *fact-prod atLeastAtMostSuc-conv algebra-simps*)

lemma *fact-2* [*simp*]: *fact 2* = *2*
by (*simp add*: *numeral-2-eq-2*)

lemma *fact-split*: *k* ≤ *n* ⟹ *fact n* = *of-nat* (*prod Suc* {*n* − *k..<n*}) ∗ *fact* (*n* − *k*)
by (*simp add*: *fact-prod-Suc prod.union-disjoint* [*symmetric*]
ivl-disj-un ac-simps of-nat-mult [*symmetric*])

end

lemma *of-nat-fact* [*simp*]: *of-nat* (*fact n*) = *fact n*
  by (*simp add*: *fact-prod*)

lemma *of-int-fact* [*simp*]: *of-int* (*fact n*) = *fact n*
  by (*simp only*: *fact-prod of-int-of-nat-eq*)

lemma *fact-reduce*: $n > 0 \Longrightarrow$ *fact* $n =$ *of-nat* $n *$ *fact* $(n - 1)$
  by (*cases n*) *auto*

lemma *fact-nonzero* [*simp*]: *fact* $n \neq$ (*0*::*'a*::{*semiring-char-0*,*semiring-no-zero-divisors*})
  using *of-nat-0-neq* **by** (*induct n*) *auto*

lemma *fact-mono-nat*: $m \leq n \Longrightarrow$ *fact* $m \leq$ (*fact n* :: *nat*)
  by (*induct n*) (*auto simp*: *le-Suc-eq*)

lemma *fact-in-Nats*: *fact* $n \in \mathbb{N}$
  by (*induct n*) *auto*

lemma *fact-in-Ints*: *fact* $n \in \mathbb{Z}$
  by (*induct n*) *auto*

context
  assumes *SORT-CONSTRAINT*(*'a*::*linordered-semidom*)
begin

lemma *fact-mono*: $m \leq n \Longrightarrow$ *fact* $m \leq$ (*fact n* :: *'a*)
  by (*metis of-nat-fact of-nat-le-iff fact-mono-nat*)

lemma *fact-ge-1* [*simp*]: *fact* $n \geq$ (*1* :: *'a*)
  by (*metis le0 fact-0 fact-mono*)

lemma *fact-gt-zero* [*simp*]: *fact* $n >$ (*0* :: *'a*)
  using *fact-ge-1 less-le-trans zero-less-one* **by** *blast*

lemma *fact-ge-zero* [*simp*]: *fact* $n \geq$ (*0* :: *'a*)
  by (*simp add*: *less-imp-le*)

lemma *fact-not-neg* [*simp*]: ¬ *fact* $n <$ (*0* :: *'a*)
  by (*simp add*: *not-less-iff-gr-or-eq*)

lemma *fact-le-power*: *fact* $n \leq$ (*of-nat* (*n*^*n*) :: *'a*)
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  then have ∗: *fact* $n \leq$ (*of-nat* (*Suc n* ^ *n*) ::*'a*)

by (*rule order-trans*) (*simp add*: *power-mono del*: *of-nat-power*)
have *fact* (*Suc n*) = (*of-nat* (*Suc n*) ∗ *fact n* ::'*a*)
by (*simp add*: *algebra-simps*)
also have ... ≤ *of-nat* (*Suc n*) ∗ *of-nat* (*Suc n* ^ *n*)
by (*simp add*: ∗ *ordered-comm-semiring-class.comm-mult-left-mono del*: *of-nat-power*)
also have ... ≤ *of-nat* (*Suc n* ^ *Suc n*)
by (*metis of-nat-mult order-refl power-Suc*)
finally show *?case* **.**
qed

end

lemma *fact-less-mono-nat*: *0* < *m* ⟹ *m* < *n* ⟹ *fact m* < (*fact n* :: *nat*)
by (*induct n*) (*auto simp*: *less-Suc-eq*)

lemma *fact-less-mono*: *0* < *m* ⟹ *m* < *n* ⟹ *fact m* < (*fact n* :: '*a*::*linordered-semidom*)
by (*metis of-nat-fact of-nat-less-iff fact-less-mono-nat*)

lemma *fact-ge-Suc-0-nat* [*simp*]: *fact n* ≥ *Suc 0*
by (*metis One-nat-def fact-ge-1*)

lemma *dvd-fact*: *1* ≤ *m* ⟹ *m* ≤ *n* ⟹ *m dvd fact n*
by (*induct n*) (*auto simp*: *dvdI le-Suc-eq*)

lemma *fact-ge-self*: *fact n* ≥ *n*
by (*cases n* = *0*) (*simp-all add*: *dvd-imp-le dvd-fact*)

lemma *fact-dvd*: *n* ≤ *m* ⟹ *fact n dvd* (*fact m* :: '*a*::*linordered-semidom*)
by (*induct m*) (*auto simp*: *le-Suc-eq*)

lemma *fact-mod*: *m* ≤ *n* ⟹ *fact n mod* (*fact m* :: '*a*::{*semidom-modulo*, *linordered-semidom*}) = *0*
by (*simp add*: *mod-eq-0-iff-dvd fact-dvd*)

lemma *fact-eq-fact-times*:
assumes *m* ≥ *n*
shows *fact m* = *fact n* ∗ ∏{*Suc n..m*}
unfolding *fact-prod*
by (*metis add.commute assms le-add1 le-add-diff-inverse of-nat-id plus-1-eq-Suc prod.ub-add-nat*)

lemma *fact-div-fact*:
assumes *m* ≥ *n*
shows *fact m div fact n* = ∏{*n* + *1..m*}
by (*simp add*: *fact-eq-fact-times* [*OF assms*])

lemma *fact-num-eq-if*: *fact m* = (*if m* = *0 then 1 else of-nat m* ∗ *fact* (*m* − *1*))
by (*cases m*) *auto*

lemma *fact-div-fact-le-pow*:
  assumes $r \leq n$
  shows *fact n div fact* $(n - r) \leq n \hat{} r$
proof –
  have $r \leq n \Longrightarrow \prod\{n - r..n\} = (n - r) * \prod\{Suc\ (n - r)..n\}$ **for** r
    by (*subst prod.insert*[*symmetric*]) (*auto simp*: *atLeastAtMost-insertL*)
  with *assms* **show** *?thesis*
   by (*induct r rule*: *nat.induct*) (*auto simp add*: *fact-div-fact Suc-diff-Suc mult-le-mono*)
qed

lemma *prod-Suc-fact*: *prod Suc* $\{0..<n\}$ = *fact n*
  by (*simp add*: *fact-prod-Suc*)

lemma *prod-Suc-Suc-fact*: *prod Suc* $\{Suc\ 0..<n\}$ = *fact n*
proof (*cases* $n = 0$)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  have *prod Suc* $\{Suc\ 0..<n\} = Suc\ 0 *$ *prod Suc* $\{Suc\ 0..<n\}$
    by *simp*
  also have ... = *prod Suc* (*insert 0* $\{Suc\ 0..<n\}$)
    by *simp*
  also have *insert 0* $\{Suc\ 0..<n\} = \{0..<n\}$
    using *False* **by** *auto*
  finally show *?thesis*
    by (*simp add*: *fact-prod-Suc*)
qed

lemma *fact-numeral*: *fact* (*numeral k*) = *numeral k* * *fact* (*pred-numeral k*)
  — Evaluation for specific numerals
  by (*metis fact-Suc numeral-eq-Suc of-nat-numeral*)

92.2 Pochhammer's symbol: generalized rising factorial

See https://en.wikipedia.org/wiki/Pochhammer_symbol.

context *comm-semiring-1*
begin

definition *pochhammer* :: $'a \Rightarrow nat \Rightarrow 'a$
  where *pochhammer-prod*: *pochhammer a n* = *prod* ($\lambda i.\ a +$ *of-nat i*) $\{0..<n\}$

lemma *pochhammer-prod-rev*: *pochhammer a n* = *prod* ($\lambda i.\ a +$ *of-nat* $(n - i)$) $\{1..n\}$
  using *prod.atLeastLessThan-rev-at-least-Suc-atMost* [*of* $\lambda i.\ a +$ *of-nat i 0 n*]
  by (*simp add*: *pochhammer-prod*)

lemma *pochhammer-Suc-prod*: *pochhammer a* (*Suc n*) = *prod* ($\lambda i.\ a +$ *of-nat i*) $\{0..n\}$

by (*simp add*: *pochhammer-prod atLeastLessThanSuc-atLeastAtMost*)

lemma *pochhammer-Suc-prod-rev*: *pochhammer a* (*Suc n*) = *prod* (*λi*. *a* + *of-nat* (*n* − *i*)) {*0*..*n*}
using *prod.atLeast-Suc-atMost-Suc-shift*
by (*simp add*: *pochhammer-prod-rev prod.atLeast-Suc-atMost-Suc-shift del*: *prod.cl-ivl-Suc*)

lemma *pochhammer-0* [*simp*]: *pochhammer a 0* = *1*
by (*simp add*: *pochhammer-prod*)

lemma *pochhammer-1* [*simp*]: *pochhammer a 1* = *a*
by (*simp add*: *pochhammer-prod lessThan-Suc*)

lemma *pochhammer-Suc0* [*simp*]: *pochhammer a* (*Suc 0*) = *a*
by (*simp add*: *pochhammer-prod lessThan-Suc*)

lemma *pochhammer-Suc*: *pochhammer a* (*Suc n*) = *pochhammer a n* ∗ (*a* + *of-nat* *n*)
by (*simp add*: *pochhammer-prod atLeast0-lessThan-Suc ac-simps*)

end

lemma *pochhammer-nonneg*:
fixes *x* :: *'a* :: *linordered-semidom*
shows *x* > *0* ⟹ *pochhammer x n* ≥ *0*
by (*induction n*) (*auto simp*: *pochhammer-Suc intro*!: *mult-nonneg-nonneg add-nonneg-nonneg*)

lemma *pochhammer-pos*:
fixes *x* :: *'a* :: *linordered-semidom*
shows *x* > *0* ⟹ *pochhammer x n* > *0*
by (*induction n*) (*auto simp*: *pochhammer-Suc intro*!: *mult-pos-pos add-pos-nonneg*)

context *comm-semiring-1*
begin

lemma *pochhammer-of-nat*: *pochhammer* (*of-nat x*) *n* = *of-nat* (*pochhammer x n*)
by (*simp add*: *pochhammer-prod Factorial.pochhammer-prod*)

end

context *comm-ring-1*
begin

lemma *pochhammer-of-int*: *pochhammer* (*of-int x*) *n* = *of-int* (*pochhammer x n*)
by (*simp add*: *pochhammer-prod Factorial.pochhammer-prod*)

end

lemma *pochhammer-rec*: *pochhammer a* (*Suc n*) = *a* ∗ *pochhammer* (*a* + *1*) *n*

by (*simp add*: *pochhammer-prod prod.atLeast0-lessThan-Suc-shift ac-simps del*: *prod.op-ivl-Suc*)

lemma *pochhammer-rec'*: *pochhammer z* (*Suc n*) = (*z* + *of-nat n*) ∗ *pochhammer z n*
by (*simp add*: *pochhammer-prod prod.atLeast0-lessThan-Suc ac-simps*)

lemma *pochhammer-fact*: *fact n* = *pochhammer 1 n*
by (*simp add*: *pochhammer-prod fact-prod-Suc*)

lemma *pochhammer-of-nat-eq-0-lemma*: *k* > *n* ⟹ *pochhammer* (− (*of-nat n* :: *'a*:: *idom*)) *k* = *0*
by (*auto simp add*: *pochhammer-prod*)

lemma *pochhammer-of-nat-eq-0-lemma'*:
assumes *kn*: *k* ≤ *n*
shows *pochhammer* (− (*of-nat n* :: *'a*::{*idom*,*ring-char-0*})) *k* ≠ *0*
proof (*cases k*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc h*)
then show *?thesis*
apply (*simp add*: *pochhammer-Suc-prod*)
using *Suc kn*
apply (*auto simp add*: *algebra-simps*)
done
qed

lemma *pochhammer-of-nat-eq-0-iff*:
pochhammer (− (*of-nat n* :: *'a*::{*idom*,*ring-char-0*})) *k* = *0* ⟷ *k* > *n*
(**is** *?l* = *?r*)
using *pochhammer-of-nat-eq-0-lemma*[*of n k*, **where** *?'a*=*'a*]
pochhammer-of-nat-eq-0-lemma'[*of k n*, **where** *?'a* = *'a*]
by (*auto simp add*: *not-le*[*symmetric*])

lemma *pochhammer-0-left*:
pochhammer 0 n = (*if n* = *0 then 1 else 0*)
by (*induction n*) (*simp-all add*: *pochhammer-rec*)

lemma *pochhammer-eq-0-iff*: *pochhammer a n* = (*0*::*'a*::*field-char-0*) ⟷ (∃ *k* < *n*. *a* = − *of-nat k*)
by (*auto simp add*: *pochhammer-prod eq-neg-iff-add-eq-0*)

lemma *pochhammer-eq-0-mono*:
pochhammer a n = (*0*::*'a*::*field-char-0*) ⟹ *m* ≥ *n* ⟹ *pochhammer a m* = *0*
unfolding *pochhammer-eq-0-iff* **by** *auto*

lemma *pochhammer-neq-0-mono*:

pochhammer a m ≠ (*0*::′*a*::*field-char-0*) ⟹ *m* ≥ *n* ⟹ *pochhammer a n* ≠ *0*
unfolding *pochhammer-eq-0-iff* **by** *auto*

lemma *pochhammer-minus*:
pochhammer (− *b*) *k* = ((− *1*) ^ *k* :: ′*a*::*comm-ring-1*) ∗ *pochhammer* (*b* − *of-nat* *k* + *1*) *k*
proof (*cases k*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc h*)
have *eq*: ((− *1*) ^ *Suc h* :: ′*a*) = (∏ *i* = *0*..*h*. − *1*)
using *prod-constant* [**where** *A*={*0*.. *h*} **and** *y*=− *1* :: ′*a*]
by *auto*
with *Suc* **show** *?thesis*
using *pochhammer-Suc-prod-rev* [*of b* − *of-nat k* + *1*]
by (*auto simp add*: *pochhammer-Suc-prod prod.distrib* [*symmetric*] *eq of-nat-diff simp del*: *prod-constant*)
qed

lemma *pochhammer-minus′*:
pochhammer (*b* − *of-nat k* + *1*) *k* = ((− *1*) ^ *k* :: ′*a*::*comm-ring-1*) ∗ *pochhammer* (− *b*) *k*
by (*simp add*: *pochhammer-minus*)

lemma *pochhammer-same*: *pochhammer* (− *of-nat n*) *n* =
((− *1*) ^ *n* :: ′*a*::{*semiring-char-0*,*comm-ring-1*,*semiring-no-zero-divisors*}) ∗ *fact n*
unfolding *pochhammer-minus*
by (*simp add*: *of-nat-diff pochhammer-fact*)

lemma *pochhammer-product′*: *pochhammer z* (*n* + *m*) = *pochhammer z n* ∗ *pochhammer* (*z* + *of-nat n*) *m*
proof (*induct n arbitrary*: *z*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n z*)
have *pochhammer z* (*Suc n*) ∗ *pochhammer* (*z* + *of-nat* (*Suc n*)) *m* =
z ∗ (*pochhammer* (*z* + *1*) *n* ∗ *pochhammer* (*z* + *1* + *of-nat n*) *m*)
by (*simp add*: *pochhammer-rec ac-simps*)
also note *Suc*[*symmetric*]
also have *z* ∗ *pochhammer* (*z* + *1*) (*n* + *m*) = *pochhammer z* (*Suc* (*n* + *m*))
by (*subst pochhammer-rec*) *simp*
finally show *?case*
by *simp*
qed

lemma *pochhammer-product*:

$m \leq n \Longrightarrow$ *pochhammer z n* = *pochhammer z m* ∗ *pochhammer* (*z* + *of-nat m*) (*n* − *m*)
using *pochhammer-product′*[*of z m n* − *m*] **by** *simp*

lemma *pochhammer-times-pochhammer-half*:
fixes *z* :: *′a::field-char-0*
shows *pochhammer z* (*Suc n*) ∗ *pochhammer* (*z* + *1/2*) (*Suc n*) = (∏ *k*=*0*..*2*∗*n*+*1*. *z* + *of-nat k* / *2*)
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *atLeast0-atMost-Suc*)
next
case (*Suc n*)
define *n′* **where** *n′* = *Suc n*
have *pochhammer z* (*Suc n′*) ∗ *pochhammer* (*z* + *1* / *2*) (*Suc n′*) =
(*pochhammer z n′* ∗ *pochhammer* (*z* + *1* / *2*) *n′*) ∗ ((*z* + *of-nat n′*) ∗ (*z* + *1/2* + *of-nat n′*))
(**is** - = - ∗ *?A*)
by (*simp-all add*: *pochhammer-rec′ mult-ac*)
also have *?A* = (*z* + *of-nat* (*Suc* (*2* ∗ *n* + *1*)) / *2*) ∗ (*z* + *of-nat* (*Suc* (*Suc* (*2* ∗ *n* + *1*))) / *2*)
(**is** - = *?B*)
by (*simp add*: *field-simps n′-def*)
also note *Suc*[*folded n′-def*]
also have (∏ *k*=*0*..*2* ∗ *n* + *1*. *z* + *of-nat k* / *2*) ∗ *?B* = (∏ *k*=*0*..*2* ∗ *Suc n* + *1*. *z* + *of-nat k* / *2*)
by (*simp add*: *atLeast0-atMost-Suc*)
finally show *?case*
by (*simp add*: *n′-def*)
qed

lemma *pochhammer-double*:
fixes *z* :: *′a::field-char-0*
shows *pochhammer* (*2* ∗ *z*) (*2* ∗ *n*) = *of-nat* (*2*^(*2*∗*n*)) ∗ *pochhammer z n* ∗ *pochhammer* (*z*+*1/2*) *n*
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
have *pochhammer* (*2* ∗ *z*) (*2* ∗ (*Suc n*)) = *pochhammer* (*2* ∗ *z*) (*2* ∗ *n*) ∗
(*2* ∗ (*z* + *of-nat n*)) ∗ (*2* ∗ (*z* + *of-nat n*) + *1*)
by (*simp add*: *pochhammer-rec′ ac-simps*)
also note *Suc*
also have *of-nat* (*2* ^ (*2* ∗ *n*)) ∗ *pochhammer z n* ∗ *pochhammer* (*z* + *1/2*) *n* ∗
(*2* ∗ (*z* + *of-nat n*)) ∗ (*2* ∗ (*z* + *of-nat n*) + *1*) =
of-nat (*2* ^ (*2* ∗ (*Suc n*))) ∗ *pochhammer z* (*Suc n*) ∗ *pochhammer* (*z* + *1/2*) (*Suc n*)

by (*simp add*: *field-simps pochhammer-rec'*)
finally show *?case* **.**
qed

lemma *fact-double*:
fact (*2* ∗ *n*) = (*2* ^ (*2* ∗ *n*) ∗ *pochhammer* (*1* / *2*) *n* ∗ *fact n* :: *'a*::*field-char-0*)
using *pochhammer-double*[*of 1/2*::*'a n*] **by** (*simp add*: *pochhammer-fact*)

lemma *pochhammer-absorb-comp*: (*r* − *of-nat k*) ∗ *pochhammer* (− *r*) *k* = *r* ∗ *pochhammer* (−*r* + *1*) *k*
(**is** *?lhs* = *?rhs*)
for *r* :: *'a*::*comm-ring-1*
proof −
have *?lhs* = − *pochhammer* (− *r*) (*Suc k*)
by (*subst pochhammer-rec'*) (*simp add*: *algebra-simps*)
also have ... = *?rhs*
by (*subst pochhammer-rec*) *simp*
finally show *?thesis* **.**
qed

92.3 Misc

lemma *fact-code* [*code*]:
fact n = (*of-nat* (*fold-atLeastAtMost-nat* ((∗)) *2 n 1*) :: *'a*::*semiring-char-0*)
proof −
have *fact n* = (*of-nat* (∏{*1*..*n*}) :: *'a*)
by (*simp add*: *fact-prod*)
also have ∏{*1*..*n*} = ∏{*2*..*n*}
by (*intro prod.mono-neutral-right*) *auto*
also have ... = *fold-atLeastAtMost-nat* ((∗)) *2 n 1*
by (*simp add*: *prod-atLeastAtMost-code*)
finally show *?thesis* **.**
qed

lemma *pochhammer-code* [*code*]:
pochhammer a n =
(*if n* = *0 then 1*
else fold-atLeastAtMost-nat (λ*n acc*. (*a* + *of-nat n*) ∗ *acc*) *0* (*n* − *1*) *1*)
by (*cases n*)
(*simp-all add*: *pochhammer-prod prod-atLeastAtMost-code* [*symmetric*]
atLeastLessThanSuc-atLeastAtMost)

end

93 Binomial Coefficients, Binomial Theorem, Inclusion-exclusion Principle

theory *Binomial*

imports *Presburger Factorial*
begin

93.1 Binomial coefficients

This development is based on the work of Andy Gordon and Florian Kammueller.

Combinatorial definition

```
definition binomial :: nat ⇒ nat ⇒ nat
  where binomial n k = card {K∈Pow {0..<n}. card K = k}

open-bundle binomial-syntax
begin
notation binomial (infix ‹choose› 64)
end

lemma binomial-right-mono:
  assumes m ≤ n shows m choose k ≤ n choose k
proof −
  have {K. K ⊆ {0..<m} ∧ card K = k} ⊆ {K. K ⊆ {0..<n} ∧ card K = k}
    using assms by auto
  then show ?thesis
    by (simp add: binomial-def card-mono)
qed

theorem n-subsets:
  assumes finite A
  shows card {B. B ⊆ A ∧ card B = k} = card A choose k
proof −
  from assms obtain f where bij: bij-betw f {0..<card A} A
    by (blast dest: ex-bij-betw-nat-finite)
  then have [simp]: card (f ‘ C) = card C if C ⊆ {0..<card A} for C
    by (meson bij-betw-imp-inj-on bij-betw-subset card-image that)
  from bij have bij-betw (image f) (Pow {0..<card A}) (Pow A)
    by (rule bij-betw-Pow)
  then have inj-on (image f) (Pow {0..<card A})
    by (rule bij-betw-imp-inj-on)
  moreover have {K. K ⊆ {0..<card A} ∧ card K = k} ⊆ Pow {0..<card A}
    by auto
  ultimately have inj-on (image f) {K. K ⊆ {0..<card A} ∧ card K = k}
    by (rule inj-on-subset)
  then have card {K. K ⊆ {0..<card A} ∧ card K = k} =
      card (image f ‘ {K. K ⊆ {0..<card A} ∧ card K = k}) (is - = card ?C)
    by (simp add: card-image)
  also have ?C = {K. K ⊆ f ‘ {0..<card A} ∧ card K = k}
    by (auto elim!: subset-imageE)
  also have f ‘ {0..<card A} = A
    by (meson bij bij-betw-def)
```

finally show *?thesis*
by (*simp add*: *binomial-def*)
qed

Recursive characterization

lemma *binomial-n-0* [*simp*]: *n choose 0 = 1*
proof −
have $\{K \in Pow\ \{0..<n\}.\ card\ K = 0\} = \{\{\}\}$
by (*auto dest*: *finite-subset*)
then show *?thesis*
by (*simp add*: *binomial-def*)
qed

lemma *binomial-0-Suc* [*simp*]: *0 choose Suc k = 0*
by (*simp add*: *binomial-def*)

lemma *binomial-Suc-Suc* [*simp*]: *Suc n choose Suc k* = (*n choose k*) + (*n choose Suc k*)
proof −
let $?P = \lambda n\ k.\ \{K.\ K \subseteq \{0..<n\} \wedge card\ K = k\}$
let $?Q = ?P\ (Suc\ n)\ (Suc\ k)$
have *inj*: *inj-on* (*insert n*) (*?P n k*)
by *rule* (*auto*; *metis atLeastLessThan-iff insert-iff less-irrefl subsetCE*)
have *disjoint*: $insert\ n\ `\ ?P\ n\ k \cap ?P\ n\ (Suc\ k) = \{\}$
by *auto*
have $?Q = \{K{\in}?Q.\ n \in K\} \cup \{K{\in}?Q.\ n \notin K\}$
by *auto*
also have $\{K{\in}?Q.\ n \in K\} = insert\ n\ `\ ?P\ n\ k$ (**is** *?A = ?B*)
proof (*rule set-eqI*)
fix *K*
have *K-finite*: *finite K* **if** $K \subseteq insert\ n\ \{0..<n\}$
using *that* **by** (*rule finite-subset*) *simp-all*
have *Suc-card-K*: *Suc* (*card K* − *Suc 0*) = *card K* **if** $n \in K$
and *finite K*
proof −
from ‹$n \in K$› **obtain** *L* **where** *K = insert n L* **and** $n \notin L$
by (*blast elim*: *Set.set-insert*)
with *that* **show** *?thesis* **by** (*simp add*: *card.insert-remove*)
qed
show $K \in ?A \longleftrightarrow K \in ?B$
by (*subst in-image-insert-iff*)
(*auto simp add*: *card.insert-remove subset-eq-atLeast0-lessThan-finite Diff-subset-conv K-finite Suc-card-K*)
qed
also have $\{K{\in}?Q.\ n \notin K\} = ?P\ n\ (Suc\ k)$
by (*auto simp add*: *atLeast0-lessThan-Suc*)
finally show *?thesis* **using** *inj disjoint*
by (*simp add*: *binomial-def card-Un-disjoint card-image*)
qed

lemma *binomial-eq-0*: $n < k \Longrightarrow n\ choose\ k = 0$
by (*auto simp add*: *binomial-def dest*: *subset-eq-atLeast0-lessThan-card*)

lemma *zero-less-binomial*: $k \leq n \Longrightarrow n\ choose\ k > 0$
by (*induct n k rule*: *diff-induct*) *simp-all*

lemma *binomial-eq-0-iff* [*simp*]: $n\ choose\ k = 0 \longleftrightarrow n < k$
by (*metis binomial-eq-0 less-numeral-extra*(*3*) *not-less zero-less-binomial*)

lemma *zero-less-binomial-iff* [*simp*]: $n\ choose\ k > 0 \longleftrightarrow k \leq n$
by (*metis binomial-eq-0-iff not-less0 not-less zero-less-binomial*)

lemma *binomial-n-n* [*simp*]: $n\ choose\ n = 1$
by (*induct n*) (*simp-all add*: *binomial-eq-0*)

lemma *binomial-Suc-n* [*simp*]: $Suc\ n\ choose\ n = Suc\ n$
by (*induct n*) *simp-all*

lemma *binomial-1* [*simp*]: $n\ choose\ Suc\ 0 = n$
by (*induct n*) *simp-all*

lemma *choose-one*: $n\ choose\ 1 = n$ **for** $n :: nat$
by *simp*

lemma *choose-reduce-nat*:
$0 < n \Longrightarrow 0 < k \Longrightarrow$
$n\ choose\ k = ((n - 1)\ choose\ (k - 1)) + ((n - 1)\ choose\ k)$
using *binomial-Suc-Suc* [*of* $n - 1$ $k - 1$] **by** *simp*

lemma *Suc-times-binomial-eq*: $Suc\ n * (n\ choose\ k) = (Suc\ n\ choose\ Suc\ k) * Suc\ k$
proof (*induction n arbitrary*: *k*)
case *0*
then show *?case*
by *auto*
next
case (*Suc n*)
show *?case*
proof (*cases k*)
case (*Suc k'*)
then show *?thesis*
using *Suc.IH*
by (*auto simp add*: *add-mult-distrib add-mult-distrib2 le-Suc-eq binomial-eq-0*)
qed *auto*
qed

lemma *binomial-le-pow2*: $n\ choose\ k \leq 2\hat{}n$
proof (*induction n arbitrary*: *k*)

```
  case 0
  then show ?case
    using le-less less-le-trans by fastforce
next
  case (Suc n)
  show ?case
  proof (cases k)
    case (Suc k')
    then show ?thesis
      using Suc.IH by (simp add: add-le-mono mult-2)
  qed auto
qed
```

The absorption property.

```
lemma Suc-times-binomial: Suc k * (Suc n choose Suc k) = Suc n * (n choose k)
  using Suc-times-binomial-eq by auto
```

This is the well-known version of absorption, but it's harder to use because of the need to reason about division.

```
lemma binomial-Suc-Suc-eq-times: (Suc n choose Suc k) = (Suc n * (n choose k))
div Suc k
  by (simp add: Suc-times-binomial-eq del: mult-Suc mult-Suc-right)
```

Another version of absorption, with -1 instead of *Suc*.

```
lemma times-binomial-minus1-eq: 0 < k ⟹ k * (n choose k) = n * ((n − 1)
choose (k − 1))
  using Suc-times-binomial-eq [where n = n − 1 and k = k − 1]
  by (auto split: nat-diff-split)
```

93.2 The binomial theorem (courtesy of Tobias Nipkow):

Avigad's version, generalized to any commutative ring

```
theorem (in comm-semiring-1) binomial-ring:
  (a + b :: 'a)^n = (∑k≤n. (of-nat (n choose k)) * a^k * b^(n−k))
proof (induct n)
  case 0
  then show ?case by simp
next
  case (Suc n)
   have decomp: {0..n+1} = {0} ∪ {n + 1} ∪ {1..n} and decomp2: {0..n} =
{0} ∪ {1..n}
    by auto
  have (a + b)^(n+1) = (a + b) * (∑k≤n. of-nat (n choose k) * a^k * b^(n −
k))
    using Suc.hyps by simp
  also have ... = a * (∑k≤n. of-nat (n choose k) * a^k * b^(n−k)) +
      b * (∑k≤n. of-nat (n choose k) * a^k * b^(n−k))
    by (rule distrib-right)
```

also have ... = ($\sum k \le n$. *of-nat* (*n choose k*) $*$ a^(k+1) $*$ b^(n−k)) +
($\sum k \le n$. *of-nat* (*n choose k*) $*$ a^k $*$ b^(n − k + 1))
by (*auto simp add*: *sum-distrib-left ac-simps*)
also have ... = ($\sum k \le n$. *of-nat* (*n choose k*) $*$ a^k $*$ b^(n + 1 − k)) +
($\sum k$=1..n+1. *of-nat* (*n choose* (k − 1)) $*$ a^k $*$ b^(n + 1 − k))
by (*simp add*: *atMost-atLeast0 sum.shift-bounds-cl-Suc-ivl Suc-diff-le field-simps del*: *sum.cl-ivl-Suc*)
also have ... = b^(n + 1) +
($\sum k$=1..n. *of-nat* (*n choose k*) $*$ a^k $*$ b^(n + 1 − k)) + (a^(n + 1) +
($\sum k$=1..n. *of-nat* (*n choose* (k − 1)) $*$ a^k $*$ b^(n + 1 − k)))
using *sum.nat-ivl-Suc′* [*of* 1 *n* λk. *of-nat* (*n choose* (k−1)) $*$ a ^ k $*$ b ^ (n + 1 − k)]
by (*simp add*: *sum.atLeast-Suc-atMost atMost-atLeast0*)
also have ... = a^(n + 1) + b^(n + 1) +
($\sum k$=1..n. *of-nat* (n + 1 *choose k*) $*$ a^k $*$ b^(n + 1 − k))
by (*auto simp add*: *field-simps sum.distrib* [*symmetric*] *choose-reduce-nat*)
also have ... = ($\sum k \le n$+1. *of-nat* (n + 1 *choose k*) $*$ a^k $*$ b^(n + 1 − k))
using *decomp* **by** (*simp add*: *atMost-atLeast0 field-simps*)
finally show *?case*
by *simp*
qed

Original version for the naturals.

corollary *binomial*: (a + b :: *nat*)^n = ($\sum k \le n$. (*of-nat* (*n choose k*)) $*$ a^k $*$ b^(n − k))
using *binomial-ring* [*of int a int b n*]
by (*simp only*: *of-nat-add* [*symmetric*] *of-nat-mult* [*symmetric*] *of-nat-power* [*symmetric*]
of-nat-sum [*symmetric*] *of-nat-eq-iff of-nat-id*)

lemma *binomial-fact-lemma*: $k \le n \Longrightarrow$ *fact k* $*$ *fact* (n − k) $*$ (*n choose k*) = *fact n*
proof (*induct n arbitrary*: *k rule*: *nat-less-induct*)
fix *n k*
assume *H*: $\forall m<n. \forall x \le m$. *fact x* $*$ *fact* (m − x) $*$ (*m choose x*) = *fact m*
assume *kn*: $k \le n$
let *?ths* = *fact k* $*$ *fact* (n − k) $*$ (*n choose k*) = *fact n*
consider $n = 0 \lor k = 0 \lor n = k$ | *m h* **where** n = *Suc m* k = *Suc h* $h < m$
using *kn* **by** *atomize-elim presburger*
then show *fact k* $*$ *fact* (n − k) $*$ (*n choose k*) = *fact n*
proof *cases*
case *1*
with *kn* **show** *?thesis* **by** *auto*
next
case *2*
note n = ‹n = *Suc m*›
note k = ‹k = *Suc h*›
note *hm* = ‹$h < m$›
have *mn*: $m < n$
using *n* **by** *arith*

```
    have hm': h ≤ m
      using hm by arith
    have km: k ≤ m
      using hm k n kn by arith
    have m − h = Suc (m − Suc h)
      using  k km hm by arith
    with km k have fact (m − h) = (m − h) ∗ fact (m − k)
      by simp
    with n k have fact k ∗ fact (n − k) ∗ (n choose k) =
        k ∗ (fact h ∗ fact (m − h) ∗ (m choose h)) +
        (m − h) ∗ (fact k ∗ fact (m − k) ∗ (m choose k))
      by (simp add: field-simps)
    also have ... = (k + (m − h)) ∗ fact m
      using H[rule-format, OF mn hm'] H[rule-format, OF mn km]
      by (simp add: field-simps)
    finally show ?thesis
      using k n km by simp
  qed
qed

lemma binomial-fact':
  assumes k ≤ n
  shows n choose k = fact n div (fact k ∗ fact (n − k))
  using binomial-fact-lemma [OF assms]
  by (metis fact-nonzero mult-eq-0-iff nonzero-mult-div-cancel-left)

lemma binomial-fact:
  assumes kn: k ≤ n
  shows (of-nat (n choose k) :: 'a::field-char-0) = fact n / (fact k ∗ fact (n − k))
  using binomial-fact-lemma[OF kn]
 by (metis (mono-tags, lifting) fact-nonzero mult-eq-0-iff nonzero-mult-div-cancel-left
of-nat-fact of-nat-mult)

lemma fact-binomial:
  assumes k ≤ n
  shows fact k ∗ of-nat (n choose k) = (fact n / fact (n − k) :: 'a::field-char-0)
  unfolding binomial-fact [OF assms] by (simp add: field-simps)

lemma binomial-fact-pow: (n choose s) ∗ fact s ≤ n^s
proof (cases s ≤ n)
  case True
  then show ?thesis
   by (smt (verit) binomial-fact-lemma mult.assoc mult.commute fact-div-fact-le-pow
fact-nonzero nonzero-mult-div-cancel-right)
qed (simp add: binomial-eq-0)

lemma choose-two: n choose 2 = n ∗ (n − 1) div 2
proof (cases n ≥ 2)
  case False
```

then have *n* = *0* ∨ *n* = *1*
  by *auto*
then show *?thesis* **by** *auto*
next
  case *True*
  define *m* **where** *m* = *n* − *2*
  with *True* **have** *n* = *m* + *2*
    by *simp*
  then have *fact n* = *n* ∗ (*n* − *1*) ∗ *fact* (*n* − *2*)
    by (*simp add*: *fact-prod-Suc atLeast0-lessThan-Suc algebra-simps*)
  with *True* **show** *?thesis*
    by (*simp add*: *binomial-fact′*)
qed

lemma *choose-row-sum*: (∑ *k*≤*n*. *n choose k*) = *2*^*n*
  using *binomial* [*of 1 1 n*] **by** (*simp add*: *numeral-2-eq-2*)

lemma *sum-choose-lower*: (∑ *k*≤*n*. (*r*+*k*) *choose k*) = *Suc* (*r*+*n*) *choose n*
  by (*induct n*) *auto*

lemma *sum-choose-upper*: (∑ *k*≤*n*. *k choose m*) = *Suc n choose Suc m*
  by (*induct n*) *auto*

lemma *choose-alternating-sum*:
  n > *0* ⟹ (∑ *i*≤*n*. (−*1*)^*i* ∗ *of-nat* (*n choose i*)) = (*0* :: ′*a*::*comm-ring-1*)
  using *binomial-ring*[*of* −*1* :: ′*a 1 n*]
  by (*simp add*: *atLeast0AtMost mult-of-nat-commute zero-power*)

lemma *choose-even-sum*:
  assumes *n* > *0*
  shows *2* ∗ (∑ *i*≤*n*. *if even i then of-nat* (*n choose i*) *else 0*) = (*2* ^ *n* :: ′*a*::*comm-ring-1*)
proof −
  have *2* ^ *n* = (∑ *i*≤*n*. *of-nat* (*n choose i*)) + (∑ *i*≤*n*. (−*1*) ^ *i* ∗ *of-nat* (*n choose i*) :: ′*a*)
    using *choose-row-sum*[*of n*]
    by (*simp add*: *choose-alternating-sum assms atLeast0AtMost of-nat-sum*[*symmetric*])
  also have ... = (∑ *i*≤*n*. *of-nat* (*n choose i*) + (−*1*) ^ *i* ∗ *of-nat* (*n choose i*))
    by (*simp add*: *sum.distrib*)
  also have ... = *2* ∗ (∑ *i*≤*n*. *if even i then of-nat* (*n choose i*) *else 0*)
    by (*subst sum-distrib-left*, *intro sum.cong*) *simp-all*
  finally show *?thesis* **..**
qed

lemma *choose-odd-sum*:
  assumes *n* > *0*
  shows *2* ∗ (∑ *i*≤*n*. *if odd i then of-nat* (*n choose i*) *else 0*) = (*2* ^ *n* :: ′*a*::*comm-ring-1*)
proof −

have *2 ^ n* = ($\sum$ *i*≤*n*. *of-nat* (*n choose i*)) − ($\sum$ *i*≤*n*. (−*1*) ^ *i* ∗ *of-nat* (*n choose i*) :: ′*a*)
using *choose-row-sum*[*of n*]
by (*simp add*: *choose-alternating-sum assms atLeast0AtMost of-nat-sum*[*symmetric*])
also have ... = ($\sum$ *i*≤*n*. *of-nat* (*n choose i*) − (−*1*) ^ *i* ∗ *of-nat* (*n choose i*))
by (*simp add*: *sum-subtractf*)
also have ... = *2* ∗ ($\sum$ *i*≤*n*. *if odd i then of-nat* (*n choose i*) *else 0*)
by (*subst sum-distrib-left*, *intro sum.cong*) *simp-all*
finally show *?thesis* **..**
qed

NW diagonal sum property

lemma *sum-choose-diagonal*:
assumes *m* ≤ *n*
shows ($\sum$ *k*≤*m*. (*n* − *k*) *choose* (*m* − *k*)) = *Suc n choose m*
proof −
have ($\sum$ *k*≤*m*. (*n*−*k*) *choose* (*m* − *k*)) = ($\sum$ *k*≤*m*. (*n* − *m* + *k*) *choose k*)
using *sum.atLeastAtMost-rev* [*of* λ*k*. (*n* − *k*) *choose* (*m* − *k*) *0 m*] *assms*
by (*simp add*: *atMost-atLeast0*)
also have ... = *Suc* (*n* − *m* + *m*) *choose m*
by (*rule sum-choose-lower*)
also have ... = *Suc n choose m*
using *assms* **by** *simp*
finally show *?thesis* **.**
qed

93.3 Generalized binomial coefficients

definition *gbinomial* :: ′*a*::{*semidom-divide*,*semiring-char-0*} ⇒ *nat* ⇒ ′*a* (**infix** ‹*gchoose*› *64*)
where *gbinomial-prod-rev*: *a gchoose k* = *prod* (λ*i*. *a* − *of-nat i*) {*0*..<*k*} *div fact k*

lemma *gbinomial-0* [*simp*]:
a gchoose 0 = *1*
0 gchoose (*Suc k*) = *0*
by (*simp-all add*: *gbinomial-prod-rev prod.atLeast0-lessThan-Suc-shift del*: *prod.op-ivl-Suc*)

lemma *gbinomial-Suc*: *a gchoose* (*Suc k*) = *prod* (λ*i*. *a* − *of-nat i*) {*0*..*k*} *div fact* (*Suc k*)
by (*simp add*: *gbinomial-prod-rev atLeastLessThanSuc-atLeastAtMost*)

lemma *gbinomial-1* [*simp*]: *a gchoose 1* = *a*
by (*simp add*: *gbinomial-prod-rev lessThan-Suc*)

lemma *gbinomial-Suc0* [*simp*]: *a gchoose Suc 0* = *a*
by (*simp add*: *gbinomial-prod-rev lessThan-Suc*)

lemma *gbinomial-0-left*: *0 gchoose k* = (*if k* = *0 then 1 else 0*)

by (*cases k*) *simp-all*

lemma *gbinomial-mult-fact*: *fact k* ∗ (*a gchoose k*) = (∏ *i* = *0*..<*k*. *a* − *of-nat i*)
for *a* :: ′*a*::*field-char-0*
by (*simp-all add*: *gbinomial-prod-rev field-simps*)

lemma *gbinomial-mult-fact′*: (*a gchoose k*) ∗ *fact k* = (∏ *i* = *0*..<*k*. *a* − *of-nat i*)
for *a* :: ′*a*::*field-char-0*
using *gbinomial-mult-fact* [*of k a*] **by** (*simp add*: *ac-simps*)

lemma *gbinomial-pochhammer*: *a gchoose k* = (− *1*) ^ *k* ∗ *pochhammer* (− *a*) *k* / *fact k*
for *a* :: ′*a*::*field-char-0*
proof (*cases k*)
case (*Suc k′*)
then have *a gchoose k* = *pochhammer* (*a* − *of-nat k′*) (*Suc k′*) / ((*1* + *of-nat k′*) ∗ *fact k′*)
by (*simp add*: *gbinomial-prod-rev pochhammer-prod-rev atLeastLessThanSuc-atLeastAtMost prod.atLeast-Suc-atMost-Suc-shift of-nat-diff flip*: *power-mult-distrib prod.cl-ivl-Suc*)
then show *?thesis*
by (*simp add*: *pochhammer-minus Suc*)
qed *auto*

lemma *gbinomial-pochhammer′*: *a gchoose k* = *pochhammer* (*a* − *of-nat k* + *1*) *k* / *fact k*
for *a* :: ′*a*::*field-char-0*
proof −
have *a gchoose k* = ((−*1*)^*k* ∗ (−*1*)^*k*) ∗ *pochhammer* (*a* − *of-nat k* + *1*) *k* / *fact k*
by (*simp add*: *gbinomial-pochhammer pochhammer-minus mult-ac*)
also have (−*1* :: ′*a*)^*k* ∗ (−*1*)^*k* = *1*
by (*subst power-add* [*symmetric*]) *simp*
finally show *?thesis*
by *simp*
qed

lemma *gbinomial-binomial*: *n gchoose k* = *n choose k*
proof (*cases k* ≤ *n*)
case *False*
then have *n* < *k*
by (*simp add*: *not-le*)
then have *0* ∈ ((−) *n*) ‘ {*0*..<*k*}
by *auto*
then have *prod* ((−) *n*) {*0*..<*k*} = *0*
by (*auto intro*: *prod-zero*)
with ‹*n* < *k*› **show** *?thesis*
by (*simp add*: *binomial-eq-0 gbinomial-prod-rev prod-zero*)
next
case *True*

from *True* **have** ∗: *prod* ((−) *n*) {*0*..<*k*} = ∏{*Suc* (*n* − *k*)..*n*}
by (*intro prod.reindex-bij-witness*[*of* - λ*i*. *n* − *i* λ*i*. *n* − *i*]) *auto*
from *True* **have** *n choose k* = *fact n div* (*fact k* ∗ *fact* (*n* − *k*))
by (*rule binomial-fact′*)
with ∗ **show** *?thesis*
by (*simp add*: *gbinomial-prod-rev mult.commute* [*of fact k*] *div-mult2-eq fact-div-fact*)
qed

lemma *of-nat-gbinomial*: *of-nat* (*n gchoose k*) = (*of-nat n gchoose k* :: ′*a*::*field-char-0*)
proof (*cases k* ≤ *n*)
case *False*
then show *?thesis*
by (*simp add*: *not-le gbinomial-binomial binomial-eq-0 gbinomial-prod-rev*)
next
case *True*
define *m* **where** *m* = *n* − *k*
with *True* **have** *n*: *n* = *m* + *k*
by *arith*
from *n* **have** *fact n* = ((∏ *i* = *0*..<*m* + *k*. *of-nat* (*m* + *k* − *i*)):: ′*a*)
by (*simp add*: *fact-prod-rev*)
also have ... = ((∏ *i*∈{*0*..<*k*} ∪ {*k*..<*m* + *k*}. *of-nat* (*m* + *k* − *i*)) :: ′*a*)
by (*simp add*: *ivl-disj-un*)
finally have *fact n* = (*fact m* ∗ (∏ *i* = *0*..<*k*. *of-nat m* + *of-nat k* − *of-nat i*)
:: ′*a*)
using *prod.shift-bounds-nat-ivl* [*of* λ*i*. *of-nat* (*m* + *k* − *i*) :: ′*a 0 k m*]
by (*simp add*: *fact-prod-rev* [*of m*] *prod.union-disjoint of-nat-diff*)
then have *fact n* / *fact* (*n* − *k*) = ((∏ *i* = *0*..<*k*. *of-nat n* − *of-nat i*) :: ′*a*)
by (*simp add*: *n*)
with *True* **have** *fact k* ∗ *of-nat* (*n gchoose k*) = (*fact k* ∗ (*of-nat n gchoose k*) ::
′*a*)
by (*simp only*: *gbinomial-mult-fact* [*of k of-nat n*] *gbinomial-binomial* [*of n k*]
fact-binomial)
then show *?thesis*
by *simp*
qed

lemma *binomial-gbinomial*: *of-nat* (*n choose k*) = (*of-nat n gchoose k* :: ′*a*::*field-char-0*)
by (*simp add*: *gbinomial-binomial* [*symmetric*] *of-nat-gbinomial*)

setup
‹*Sign.add-const-constraint* (***const-name*** ‹*gbinomial*›, *SOME* ***typ*** ‹′*a*::*field-char-0*
⇒ *nat* ⇒ ′*a*›)›

lemma *gbinomial-mult-1*:
fixes *a* :: ′*a*::*field-char-0*
shows *a* ∗ (*a gchoose k*) = *of-nat k* ∗ (*a gchoose k*) + *of-nat* (*Suc k*) ∗ (*a gchoose*
(*Suc k*))
(**is** *?l* = *?r*)
proof −

have *?r* = ((− *1*) ^*k* ∗ *pochhammer* (− *a*) *k* / *fact k*) ∗ (*of-nat k* − (− *a* + *of-nat k*))
unfolding *gbinomial-pochhammer pochhammer-Suc right-diff-distrib power-Suc*
by (*auto simp add*: *field-simps simp del*: *of-nat-Suc*)
also have . . . = *?l*
by (*simp add*: *field-simps gbinomial-pochhammer*)
finally show *?thesis* **..**
qed

lemma *gbinomial-mult-1′*:
(*a gchoose k*) ∗ *a* = *of-nat k* ∗ (*a gchoose k*) + *of-nat* (*Suc k*) ∗ (*a gchoose* (*Suc k*))
for *a* :: ′*a*::*field-char-0*
by (*simp add*: *mult.commute gbinomial-mult-1*)

lemma *gbinomial-Suc-Suc*: (*a* + *1*) *gchoose* (*Suc k*) = (*a gchoose k*) + (*a gchoose* (*Suc k*))
for *a* :: ′*a*::*field-char-0*
proof (*cases k*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc h*)
have *eq0*: (∏ *i*∈{*1*..*k*}. (*a* + *1*) − *of-nat i*) = (∏ *i*∈{*0*..*h*}. *a* − *of-nat i*)
proof (*rule prod.reindex-cong*)
show {*1*..*k*} = *Suc* ‘ {*0*..*h*}
using *Suc* **by** (*auto simp add*: *image-Suc-atMost*)
qed *auto*
have *fact* (*Suc k*) ∗ ((*a gchoose k*) + (*a gchoose* (*Suc k*))) =
(*a gchoose Suc h*) ∗ (*fact* (*Suc* (*Suc h*))) +
(*a gchoose Suc* (*Suc h*)) ∗ (*fact* (*Suc* (*Suc h*)))
by (*simp add*: *Suc field-simps del*: *fact-Suc*)
also have . . . =
(*a gchoose Suc h*) ∗ *of-nat* (*Suc* (*Suc h*) ∗ *fact* (*Suc h*)) + (∏ *i*=*0*..*Suc h*. *a* − *of-nat i*)
by (*metis atLeastLessThanSuc-atLeastAtMost fact-Suc gbinomial-mult-fact mult.commute of-nat-fact of-nat-mult*)
also have . . . =
(*fact* (*Suc h*) ∗ (*a gchoose Suc h*)) ∗ *of-nat* (*Suc* (*Suc h*)) + (∏ *i*=*0*..*Suc h*. *a* − *of-nat i*)
by (*simp only*: *fact-Suc mult.commute mult.left-commute of-nat-fact of-nat-id of-nat-mult*)
also have . . . =
of-nat (*Suc* (*Suc h*)) ∗ (∏ *i*=*0*..*h*. *a* − *of-nat i*) + (∏ *i*=*0*..*Suc h*. *a* − *of-nat i*)
unfolding *gbinomial-mult-fact atLeastLessThanSuc-atLeastAtMost* **by** *auto*
also have . . . =
(∏ *i*=*0*..*Suc h*. *a* − *of-nat i*) + (*of-nat h* ∗ (∏ *i*=*0*..*h*. *a* − *of-nat i*) + *2* ∗ (∏ *i*=*0*..*h*. *a* − *of-nat i*))
by (*simp add*: *field-simps*)

also have ... =
((*a gchoose Suc h*) ∗ (*fact* (*Suc h*)) ∗ *of-nat* (*Suc k*)) + (∏ *i*∈{*0..Suc h*}. *a* − *of-nat i*)
unfolding *gbinomial-mult-fact′*
by (*simp add*: *comm-semiring-class.distrib field-simps Suc atLeastLessThanSuc-atLeastAtMost*)
also have ... = (∏ *i*∈{*0..h*}. *a* − *of-nat i*) ∗ (*a* + *1*)
unfolding *gbinomial-mult-fact′ atLeast0-atMost-Suc*
by (*simp add*: *field-simps Suc atLeastLessThanSuc-atLeastAtMost*)
also have ... = (∏ *i*∈{*0..k*}. (*a* + *1*) − *of-nat i*)
using *eq0*
by (*simp add*: *Suc prod.atLeast0-atMost-Suc-shift del*: *prod.cl-ivl-Suc*)
also have ... = (*fact* (*Suc k*)) ∗ ((*a* + *1*) *gchoose* (*Suc k*))
by (*simp only*: *gbinomial-mult-fact atLeastLessThanSuc-atLeastAtMost*)
finally show *?thesis*
using *fact-nonzero* [*of Suc k*] **by** *auto*
qed

lemma *gbinomial-reduce-nat*: *0* < *k* ⟹ *a gchoose k* = (*a*−*1 gchoose k*−*1*) + (*a*−*1 gchoose k*)
for *a* :: ′*a*::*field-char-0*
by (*metis Suc-pred′ diff-add-cancel gbinomial-Suc-Suc*)

lemma *gchoose-row-sum-weighted*:
(∑ *k* = *0..m*. (*r gchoose k*) ∗ (*r*/*2* − *of-nat k*)) = *of-nat*(*Suc m*) / *2* ∗ (*r gchoose* (*Suc m*))
for *r* :: ′*a*::*field-char-0*
by (*induct m*) (*simp-all add*: *field-simps distrib gbinomial-mult-1*)

lemma *binomial-symmetric*:
assumes *kn*: *k* ≤ *n*
shows *n choose k* = *n choose* (*n* − *k*)
proof −
have *kn′*: *n* − *k* ≤ *n*
using *kn* **by** *arith*
from *binomial-fact-lemma*[*OF kn*] *binomial-fact-lemma*[*OF kn′*]
have *fact k* ∗ *fact* (*n* − *k*) ∗ (*n choose k*) = *fact* (*n* − *k*) ∗ *fact* (*n* − (*n* − *k*)) ∗ (*n choose* (*n* − *k*))
by *simp*
then show *?thesis*
using *kn* **by** *simp*
qed

lemma *choose-rising-sum*:
(∑ *j*≤*m*. ((*n* + *j*) *choose n*)) = ((*n* + *m* + *1*) *choose* (*n* + *1*))
(∑ *j*≤*m*. ((*n* + *j*) *choose n*)) = ((*n* + *m* + *1*) *choose m*)
proof −
show (∑ *j*≤*m*. ((*n* + *j*) *choose n*)) = ((*n* + *m* + *1*) *choose* (*n* + *1*))
by (*induct m*) *simp-all*
also have ... = (*n* + *m* + *1*) *choose m*

  by (*subst binomial-symmetric*) *simp-all*
 finally show ($\sum j \le m.$ ((*n* + *j*) *choose n*)) = (*n* + *m* + *1*) *choose m* **.**
qed

lemma *choose-linear-sum*: ($\sum i \le n.$ *i* ∗ (*n choose i*)) = *n* ∗ *2* ^ (*n* − *1*)
proof (*cases n*)
 case *0*
 then show *?thesis* **by** *simp*
next
 case (*Suc m*)
 have ($\sum i \le n.$ *i* ∗ (*n choose i*)) = ($\sum i \le Suc\ m.$ *i* ∗ (*Suc m choose i*))
  by (*simp add*: *Suc*)
 also have ... = *Suc m* ∗ *2* ^ *m*
  unfolding *sum.atMost-Suc-shift Suc-times-binomial sum-distrib-left*[*symmetric*]
  by (*simp add*: *choose-row-sum*)
 finally show *?thesis*
  using *Suc* **by** *simp*
qed

lemma *choose-alternating-linear-sum*:
 assumes *n* ≠ *1*
 shows ($\sum i \le n.$ (−*1*)^*i* ∗ *of-nat i* ∗ *of-nat* (*n choose i*) :: '*a*::*comm-ring-1*) = *0*
proof (*cases n*)
 case *0*
 then show *?thesis* **by** *simp*
next
 case (*Suc m*)
 with *assms* **have** *m* > *0*
  by *simp*
 have ($\sum i \le n.$ (−*1*) ^ *i* ∗ *of-nat i* ∗ *of-nat* (*n choose i*) :: '*a*) =
   ($\sum i \le Suc\ m.$ (−*1*) ^ *i* ∗ *of-nat i* ∗ *of-nat* (*Suc m choose i*))
  by (*simp add*: *Suc*)
 also have ... = ($\sum i \le m.$ (−*1*) ^ (*Suc i*) ∗ *of-nat* (*Suc i* ∗ (*Suc m choose Suc i*)))
  by (*simp only*: *sum.atMost-Suc-shift sum-distrib-left*[*symmetric*] *mult-ac of-nat-mult*) *simp*
 also have ... = − *of-nat* (*Suc m*) ∗ ($\sum i \le m.$ (−*1*) ^ *i* ∗ *of-nat* (*m choose i*))
  by (*subst sum-distrib-left*, *rule sum.cong*[*OF refl*], *subst Suc-times-binomial*)
   (*simp add*: *algebra-simps*)
 also have ($\sum i \le m.$ (−*1* :: '*a*) ^ *i* ∗ *of-nat* ((*m choose i*))) = *0*
  using *choose-alternating-sum*[*OF* ‹*m* > *0*›] **by** *simp*
 finally show *?thesis*
  by *simp*
qed

lemma *vandermonde*: ($\sum k \le r.$ (*m choose k*) ∗ (*n choose* (*r* − *k*))) = (*m* + *n*) *choose r*
proof (*induct n arbitrary*: *r*)
 case *0*

have ($\sum k\le r.$ (*m choose k*) $*$ (*0 choose* (*r* $-$ *k*))) = ($\sum k\le r.$ *if k* = *r then* (*m choose k*) *else 0*)
by (*intro sum.cong*) *simp-all*
also have ... = *m choose r*
by *simp*
finally show *?case*
by *simp*
next
case (*Suc n r*)
show *?case*
by (*cases r*) (*simp-all add*: *Suc* [*symmetric*] *algebra-simps sum.distrib Suc-diff-le*)
qed

lemma *choose-square-sum*: ($\sum k\le n.$ (*n choose k*)^*2*) = ((*2*$*$*n*) *choose n*)
using *vandermonde*[*of n n n*]
by (*simp add*: *power2-eq-square mult-2 binomial-symmetric* [*symmetric*])

lemma *pochhammer-binomial-sum*:
fixes *a b* :: *'a*::*comm-ring-1*
shows *pochhammer* (*a* + *b*) *n* = ($\sum k\le n.$ *of-nat* (*n choose k*) $*$ *pochhammer a k* $*$ *pochhammer b* (*n* $-$ *k*))
proof (*induction n arbitrary*: *a b*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n a b*)
have ($\sum k\le Suc\ n.$ *of-nat* (*Suc n choose k*) $*$ *pochhammer a k* $*$ *pochhammer b* (*Suc n* $-$ *k*)) =
($\sum i\le n.$ *of-nat* (*n choose i*) $*$ *pochhammer a* (*Suc i*) $*$ *pochhammer b* (*n* $-$ *i*)) +
(($\sum i\le n.$ *of-nat* (*n choose Suc i*) $*$ *pochhammer a* (*Suc i*) $*$ *pochhammer b* (*n* $-$ *i*)) +
pochhammer b (*Suc n*))
by (*subst sum.atMost-Suc-shift*) (*simp add*: *ring-distribs sum.distrib*)
also have ($\sum i\le n.$ *of-nat* (*n choose i*) $*$ *pochhammer a* (*Suc i*) $*$ *pochhammer b* (*n* $-$ *i*)) =
a $*$ *pochhammer* ((*a* + *1*) + *b*) *n*
by (*subst Suc*) (*simp add*: *sum-distrib-left pochhammer-rec mult-ac*)
also have ($\sum i\le n.$ *of-nat* (*n choose Suc i*) $*$ *pochhammer a* (*Suc i*) $*$ *pochhammer b* (*n* $-$ *i*)) +
pochhammer b (*Suc n*) =
($\sum i=0..Suc\ n.$ *of-nat* (*n choose i*) $*$ *pochhammer a i* $*$ *pochhammer b* (*Suc n* $-$ *i*))
apply (*subst sum.atLeast-Suc-atMost*, *simp*)
apply (*simp add*: *sum.shift-bounds-cl-Suc-ivl atLeast0AtMost del*: *sum.cl-ivl-Suc*)
done
also have ... = ($\sum i\le n.$ *of-nat* (*n choose i*) $*$ *pochhammer a i* $*$ *pochhammer b* (*Suc n* $-$ *i*))
using *Suc* **by** (*intro sum.mono-neutral-right*) (*auto simp*: *not-le binomial-eq-0*)

also have ... = ($\sum i \le n$. *of-nat* (*n choose i*) ∗ *pochhammer a i* ∗ *pochhammer b* (*Suc* (*n* − *i*)))
  by (*intro sum.cong*) (*simp-all add*: *Suc-diff-le*)
 also have ... = *b* ∗ *pochhammer* (*a* + (*b* + *1*)) *n*
  by (*subst Suc*) (*simp add*: *sum-distrib-left mult-ac pochhammer-rec*)
 also have *a* ∗ *pochhammer* ((*a* + *1*) + *b*) *n* + *b* ∗ *pochhammer* (*a* + (*b* + *1*)) *n* =
   pochhammer (*a* + *b*) (*Suc n*)
  by (*simp add*: *pochhammer-rec algebra-simps*)
 finally show *?case* **..**
qed

Contributed by Manuel Eberl, generalised by LCP. Alternative definition of the binomial coefficient as $\prod i<k.\ (n - i) / (k - i)$.

lemma *gbinomial-altdef-of-nat*: *a gchoose k* = ($\prod$ *i* = *0..<k*. (*a* − *of-nat i*) / *of-nat* (*k* − *i*) :: ′*a*)
 for *k* :: *nat* **and** *a* :: ′*a*::*field-char-0*
 by (*simp add*: *prod-dividef gbinomial-prod-rev fact-prod-rev*)

lemma *gbinomial-ge-n-over-k-pow-k*:
 fixes *k* :: *nat*
  and *a* :: ′*a*::*linordered-field*
 assumes *of-nat k* ≤ *a*
 shows (*a* / *of-nat k* :: ′*a*) ^ *k* ≤ *a gchoose k*
proof −
 have *x*: *0* ≤ *a*
  using *assms of-nat-0-le-iff order-trans* **by** *blast*
 have (*a* / *of-nat k* :: ′*a*) ^ *k* = ($\prod$ *i* = *0..<k*. *a* / *of-nat k* :: ′*a*)
  by *simp*
 also have ... ≤ *a gchoose k*
 proof −
  have ⋀*i*. *i* < *k* ⟹ *0* ≤ *a* / *of-nat k*
   by (*simp add*: *x zero-le-divide-iff*)
  moreover have *a* / *of-nat k* ≤ (*a* − *of-nat i*) / *of-nat* (*k* − *i*) **if** *i* < *k* **for** *i*
  proof −
   from *assms* **have** *a* ∗ *of-nat i* ≥ *of-nat* (*i* ∗ *k*)
    by (*metis mult.commute mult-le-cancel-right of-nat-less-0-iff of-nat-mult*)
   then have *a* ∗ *of-nat k* − *a* ∗ *of-nat i* ≤ *a* ∗ *of-nat k* − *of-nat* (*i* ∗ *k*)
    by *arith*
   then have *a* ∗ *of-nat* (*k* − *i*) ≤ (*a* − *of-nat i*) ∗ *of-nat k*
    using ‹*i* < *k*› **by** (*simp add*: *algebra-simps zero-less-mult-iff of-nat-diff*)
   then have *a* ∗ *of-nat* (*k* − *i*) ≤ (*a* − *of-nat i*) ∗ (*of-nat k* :: ′*a*)
    by *blast*
   with *assms* **show** *?thesis*
    using ‹*i* < *k*› **by** (*simp add*: *field-simps*)
  qed
  ultimately show *?thesis*
   unfolding *gbinomial-altdef-of-nat*
   by (*intro prod-mono*) *auto*

qed
finally show *?thesis* **.**
qed

lemma *gbinomial-negated-upper*: (*a gchoose k*) = (−*1*) ^ *k* ∗ ((*of-nat k* − *a* − *1*) *gchoose k*)
by (*simp add*: *gbinomial-pochhammer pochhammer-minus algebra-simps*)

lemma *gbinomial-minus*: ((−*a*) *gchoose k*) = (−*1*) ^ *k* ∗ ((*a* + *of-nat k* − *1*) *gchoose k*)
by (*subst gbinomial-negated-upper*) (*simp add*: *add-ac*)

lemma *Suc-times-gbinomial*: *of-nat* (*Suc k*) ∗ ((*a* + *1*) *gchoose* (*Suc k*)) = (*a* + *1*) ∗ (*a gchoose k*)
proof (*cases k*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc b*)
then have ((*a* + *1*) *gchoose* (*Suc* (*Suc b*))) = (∏ *i* = *0*..*Suc b*. *a* + (*1* − *of-nat i*)) / *fact* (*b* + *2*)
by (*simp add*: *field-simps gbinomial-prod-rev atLeastLessThanSuc-atLeastAtMost*)
also have (∏ *i* = *0*..*Suc b*. *a* + (*1* − *of-nat i*)) = (*a* + *1*) ∗ (∏ *i* = *0*..*b*. *a* − *of-nat i*)
by (*simp add*: *prod.atLeast0-atMost-Suc-shift del*: *prod.cl-ivl-Suc*)
also have ... / *fact* (*b* + *2*) = (*a* + *1*) / *of-nat* (*Suc* (*Suc b*)) ∗ (*a gchoose Suc b*)
by (*simp-all add*: *gbinomial-prod-rev atLeastLessThanSuc-atLeastAtMost*)
finally show *?thesis* **by** (*simp add*: *Suc field-simps del*: *of-nat-Suc*)
qed

lemma *gbinomial-factors*: ((*a* + *1*) *gchoose* (*Suc k*)) = (*a* + *1*) / *of-nat* (*Suc k*) ∗ (*a gchoose k*)
proof (*cases k*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc b*)
then have ((*a* + *1*) *gchoose* (*Suc* (*Suc b*))) = (∏ *i* = *0* .. *Suc b*. *a* + (*1* − *of-nat i*)) / *fact* (*b* + *2*)
by (*simp add*: *field-simps gbinomial-prod-rev atLeastLessThanSuc-atLeastAtMost*)
also have (∏ *i* = *0* .. *Suc b*. *a* + (*1* − *of-nat i*)) = (*a* + *1*) ∗ (∏ *i* = *0*..*b*. *a* − *of-nat i*)
by (*simp add*: *prod.atLeast0-atMost-Suc-shift del*: *prod.cl-ivl-Suc*)
also have ... / *fact* (*b* + *2*) = (*a* + *1*) / *of-nat* (*Suc* (*Suc b*)) ∗ (*a gchoose Suc b*)
by (*simp-all add*: *gbinomial-prod-rev atLeastLessThanSuc-atLeastAtMost atLeast0AtMost*)
finally show *?thesis*

by (*simp add*: *Suc*)
qed

lemma *gbinomial-rec*: ((*a* + *1*) *gchoose* (*Suc k*)) = (*a gchoose k*) ∗ ((*a* + *1*) / *of-nat* (*Suc k*))
using *gbinomial-mult-1*[*of a k*]
by (*subst gbinomial-Suc-Suc*) (*simp add*: *field-simps del*: *of-nat-Suc*, *simp add*: *algebra-simps*)

lemma *gbinomial-of-nat-symmetric*: *k* ≤ *n* ⟹ (*of-nat n*) *gchoose k* = (*of-nat n*) *gchoose* (*n* − *k*)
using *binomial-symmetric*[*of k n*] **by** (*simp add*: *binomial-gbinomial* [*symmetric*])

The absorption identity (equation 5.5 [3, p. 157]):

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \quad \text{integer } k \neq 0.$$

lemma *gbinomial-absorption′*: *k* > *0* ⟹ *a gchoose k* = (*a* / *of-nat k*) ∗ (*a* − *1* *gchoose* (*k* − *1*))
using *gbinomial-rec*[*of a* − *1 k* − *1*]
by (*simp-all add*: *gbinomial-rec field-simps del*: *of-nat-Suc*)

The absorption identity is written in the following form to avoid division by k (the lower index) and therefore remove the $k \neq 0$ restriction [3, p. 157]:

$$k\binom{r}{k} = r\binom{r-1}{k-1}, \quad \text{integer } k.$$

lemma *gbinomial-absorption*: *of-nat* (*Suc k*) ∗ (*a gchoose Suc k*) = *a* ∗ ((*a* − *1*) *gchoose k*)
using *gbinomial-absorption′*[*of Suc k a*] **by** (*simp add*: *field-simps del*: *of-nat-Suc*)

The absorption identity for natural number binomial coefficients:

lemma *binomial-absorption*: *Suc k* ∗ (*n choose Suc k*) = *n* ∗ ((*n* − *1*) *choose k*)
by (*cases n*) (*simp-all add*: *binomial-eq-0 Suc-times-binomial del*: *binomial-Suc-Suc mult-Suc*)

The absorption companion identity for natural number coefficients, following the proof by GKP [3, p. 157]:

lemma *binomial-absorb-comp*: (*n* − *k*) ∗ (*n choose k*) = *n* ∗ ((*n* − *1*) *choose k*)
(**is** *?lhs* = *?rhs*)
proof (*cases n* ≤ *k*)
case *True*
then show *?thesis* **by** *auto*
next
case *False*
then have *?rhs* = *Suc* ((*n* − *1*) − *k*) ∗ (*n choose Suc* ((*n* − *1*) − *k*))

using *binomial-symmetric*[*of k n* − *1*] *binomial-absorption*[*of* (*n* − *1*) − *k n*]
by *simp*
also have *Suc* ((*n* − *1*) − *k*) = *n* − *k*
using *False* **by** *simp*
also have *n choose* ... = *n choose k*
using *False* **by** (*intro binomial-symmetric* [*symmetric*]) *simp-all*
finally show *?thesis* **..**
qed

The generalised absorption companion identity:

lemma *gbinomial-absorb-comp*: (*a* − *of-nat k*) ∗ (*a gchoose k*) = *a* ∗ ((*a* − *1*) *gchoose k*)
using *pochhammer-absorb-comp*[*of a k*] **by** (*simp add*: *gbinomial-pochhammer*)

lemma *gbinomial-addition-formula*:
a gchoose (*Suc k*) = ((*a* − *1*) *gchoose* (*Suc k*)) + ((*a* − *1*) *gchoose k*)
using *gbinomial-Suc-Suc*[*of a* − *1 k*] **by** (*simp add*: *algebra-simps*)

lemma *binomial-addition-formula*:
0 < *n* ⟹ *n choose* (*Suc k*) = ((*n* − *1*) *choose* (*Suc k*)) + ((*n* − *1*) *choose k*)
by (*subst choose-reduce-nat*) *simp-all*

Equation 5.9 of the reference material [3, p. 159] is a useful summation formula, operating on both indices:

$$\sum_{k \le n} \binom{r+k}{k} = \binom{r+n+1}{n}, \quad \text{integer } n.$$

lemma *gbinomial-parallel-sum*: (∑ *k*≤*n*. (*a* + *of-nat k*) *gchoose k*) = (*a* + *of-nat n* + *1*) *gchoose n*
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc m*)
then show *?case*
using *gbinomial-Suc-Suc*[*of* (*a* + *of-nat m* + *1*) *m*]
by (*simp add*: *add-ac*)
qed

93.4 Summation on the upper index

Another summation formula is equation 5.10 of the reference material [3, p. 160], aptly named *summation on the upper index*:

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}, \quad \text{integers } m, n \ge 0.$$

lemma *gbinomial-sum-up-index*:
 ($\sum$ *j* = *0..n*. (*of-nat j gchoose k*) :: *'a::field-char-0*) = (*of-nat n* + *1*) *gchoose* (*k* + *1*)
proof (*induct n*)
 case *0*
 show *?case*
  using *gbinomial-Suc-Suc*[*of 0 k*]
  by (*cases k*) *auto*
next
 case (*Suc n*)
 then show *?case*
  using *gbinomial-Suc-Suc*[*of of-nat* (*Suc n*) :: *'a k*]
  by (*simp add*: *add-ac*)
qed

lemma *gbinomial-index-swap*:
 ((−*1*) ^ *k*) ∗ ((− (*of-nat n*) − *1*) *gchoose k*) = ((−*1*) ^ *n*) ∗ ((− (*of-nat k*) − *1*) *gchoose n*)
 (**is** *?lhs* = *?rhs*)
proof −
 have *?lhs* = (*of-nat* (*k* + *n*) *gchoose k*)
  by (*subst gbinomial-negated-upper*) (*simp add*: *power-mult-distrib* [*symmetric*])
 also have ... = (*of-nat* (*k* + *n*) *gchoose n*)
  by (*subst gbinomial-of-nat-symmetric*) *simp-all*
 also have ... = *?rhs*
  by (*subst gbinomial-negated-upper*) *simp*
 finally show *?thesis* **.**
qed

lemma *gbinomial-sum-lower-neg*: ($\sum$ *k*≤*m*. (*a gchoose k*) ∗ (− *1*) ^ *k*) = (− *1*) ^ *m* ∗ (*a* − *1 gchoose m*)
 (**is** *?lhs* = *?rhs*)
proof −
 have *?lhs* = ($\sum$ *k*≤*m*. −(*a* + *1*) + *of-nat k gchoose k*)
  by (*intro sum.cong*[*OF refl*]) (*subst gbinomial-negated-upper*, *simp add*: *power-mult-distrib*)
 also have ... = − *a* + *of-nat m gchoose m*
  by (*subst gbinomial-parallel-sum*) *simp*
 also have ... = *?rhs*
  by (*subst gbinomial-negated-upper*) (*simp add*: *power-mult-distrib*)
 finally show *?thesis* **.**
qed

lemma *gbinomial-partial-row-sum*:
 ($\sum$ *k*≤*m*. (*a gchoose k*) ∗ ((*a* / *2*) − *of-nat k*)) = ((*of-nat m* + *1*)/*2*) ∗ (*a gchoose* (*m* + *1*))
proof (*induct m*)
 case *0*
 then show *?case* **by** *simp*
next

case (*Suc mm*)
then have ($\sum$ *k*≤*Suc mm*. (*a gchoose k*) ∗ (*a* / *2* − *of-nat k*)) =
(*a* − *of-nat* (*Suc mm*)) ∗ (*a gchoose Suc mm*) / *2*
by (*simp add*: *field-simps*)
also have . . . = *a* ∗ (*a* − *1 gchoose Suc mm*) / *2*
by (*subst gbinomial-absorb-comp*) (*rule refl*)
also have . . . = (*of-nat* (*Suc mm*) + *1*) / *2* ∗ (*a gchoose* (*Suc mm* + *1*))
by (*subst gbinomial-absorption* [*symmetric*]) *simp*
finally show *?case* **.**
qed

lemma *sum-bounds-lt-plus1*: ($\sum$ *k*<*mm*. *f* (*Suc k*)) = ($\sum$ *k*=*1*..*mm*. *f k*)
by (*induct mm*) *simp-all*

lemma *gbinomial-partial-sum-poly*:
($\sum$ *k*≤*m*. (*of-nat m* + *a gchoose k*) ∗ *x*^*k* ∗ *y*^(*m*−*k*)) =
($\sum$ *k*≤*m*. (−*a gchoose k*) ∗ (−*x*)^*k* ∗ (*x* + *y*)^(*m*−*k*))
(**is** *?lhs m* = *?rhs m*)
proof (*induction m*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc mm*)
define *G* **where** *G i k* = (*of-nat i* + *a gchoose k*) ∗ *x*^*k* ∗ *y*^(*i* − *k*) **for** *i k*
define *S* **where** *S* = *?lhs*
have *SG-def*: *S* = (λ*i*. ($\sum$ *k*≤*i*. (*G i k*)))
unfolding *S-def G-def* **..**

have *S* (*Suc mm*) = *G* (*Suc mm*) *0* + ($\sum$ *k*=*Suc 0*..*Suc mm*. *G* (*Suc mm*) *k*)
using *SG-def* **by** (*simp add*: *sum.atLeast-Suc-atMost atLeast0AtMost* [*symmetric*])
also have ($\sum$ *k*=*Suc 0*..*Suc mm*. *G* (*Suc mm*) *k*) = ($\sum$ *k*=*0*..*mm*. *G* (*Suc mm*) (*Suc k*))
by (*subst sum.shift-bounds-cl-Suc-ivl*) *simp*
also have . . . = ($\sum$ *k*=*0*..*mm*. ((*of-nat mm* + *a gchoose* (*Suc k*)) +
(*of-nat mm* + *a gchoose k*)) ∗ *x*^(*Suc k*) ∗ *y*^(*mm* − *k*))
unfolding *G-def* **by** (*subst gbinomial-addition-formula*) *simp*
also have . . . = ($\sum$ *k*=*0*..*mm*. (*of-nat mm* + *a gchoose* (*Suc k*)) ∗ *x*^(*Suc k*) ∗ *y*^(*mm* − *k*)) +
($\sum$ *k*=*0*..*mm*. (*of-nat mm* + *a gchoose k*) ∗ *x*^(*Suc k*) ∗ *y*^(*mm* − *k*))
by (*subst sum.distrib* [*symmetric*]) (*simp add*: *algebra-simps*)
also have ($\sum$ *k*=*0*..*mm*. (*of-nat mm* + *a gchoose* (*Suc k*)) ∗ *x*^(*Suc k*) ∗ *y*^(*mm* − *k*)) =
($\sum$ *k*<*Suc mm*. (*of-nat mm* + *a gchoose* (*Suc k*)) ∗ *x*^(*Suc k*) ∗ *y*^(*mm* − *k*))
by (*simp only*: *atLeast0AtMost lessThan-Suc-atMost*)
also have . . . = ($\sum$ *k*<*mm*. (*of-nat mm* + *a gchoose Suc k*) ∗ *x*^(*Suc k*) ∗ *y*^(*mm*−*k*)) +
(*of-nat mm* + *a gchoose* (*Suc mm*)) ∗ *x*^(*Suc mm*)
(**is** - = *?A* + *?B*)
by (*subst sum.lessThan-Suc*) *simp*

also have *?A* = (∑ *k*=*1*..*mm*. (*of-nat mm* + *a gchoose k*) ∗ *x*^*k* ∗ *y*^(*mm* − *k* + *1*))
proof (*subst sum-bounds-lt-plus1* [*symmetric*], *intro sum.cong*[*OF refl*], *clarify*)
fix *k*
assume *k* < *mm*
then have *mm* − *k* = *mm* − *Suc k* + *1*
by *linarith*
then show (*of-nat mm* + *a gchoose Suc k*) ∗ *x* ^ *Suc k* ∗ *y* ^ (*mm* − *k*) =
(*of-nat mm* + *a gchoose Suc k*) ∗ *x* ^ *Suc k* ∗ *y* ^ (*mm* − *Suc k* + *1*)
by (*simp only*:)
qed
also have . . . + *?B* = *y* ∗ (∑ *k*=*1*..*mm*. (*G mm k*)) + (*of-nat mm* + *a gchoose* (*Suc mm*)) ∗ *x*^(*Suc mm*)
unfolding *G-def* **by** (*subst sum-distrib-left*) (*simp add*: *algebra-simps*)
also have (∑ *k*=*0*..*mm*. (*of-nat mm* + *a gchoose k*) ∗ *x*^(*Suc k*) ∗ *y*^(*mm* − *k*)) = *x* ∗ (*S mm*)
unfolding *S-def* **by** (*subst sum-distrib-left*) (*simp add*: *atLeast0AtMost algebra-simps*)
also have (*G* (*Suc mm*) *0*) = *y* ∗ (*G mm 0*)
by (*simp add*: *G-def*)
finally have *S* (*Suc mm*) =
y ∗ (*G mm 0* + (∑ *k*=*1*..*mm*. (*G mm k*))) + (*of-nat mm* + *a gchoose* (*Suc mm*)) ∗ *x*^(*Suc mm*) + *x* ∗ (*S mm*)
by (*simp add*: *ring-distribs*)
also have *G mm 0* + (∑ *k*=*1*..*mm*. (*G mm k*)) = *S mm*
by (*simp add*: *sum.atLeast-Suc-atMost*[*symmetric*] *SG-def atLeast0AtMost*)
finally have *S* (*Suc mm*) = (*x* + *y*) ∗ (*S mm*) + (*of-nat mm* + *a gchoose* (*Suc mm*)) ∗ *x*^(*Suc mm*)
by (*simp add*: *algebra-simps*)
also have (*of-nat mm* + *a gchoose* (*Suc mm*)) = (−*1*) ^ (*Suc mm*) ∗ (− *a gchoose* (*Suc mm*))
by (*subst gbinomial-negated-upper*) *simp*
also have (−*1*) ^ *Suc mm* ∗ (− *a gchoose Suc mm*) ∗ *x* ^ *Suc mm* =
(− *a gchoose* (*Suc mm*)) ∗ (−*x*) ^ *Suc mm*
by (*simp add*: *power-minus*[*of x*])
also have (*x* + *y*) ∗ *S mm* + . . . = (*x* + *y*) ∗ *?rhs mm* + (− *a gchoose* (*Suc mm*)) ∗ (− *x*)^*Suc mm*
unfolding *S-def* **by** (*subst Suc.IH*) *simp*
also have (*x* + *y*) ∗ *?rhs mm* = (∑ *n*≤*mm*. ((− *a gchoose n*) ∗ (− *x*) ^ *n* ∗ (*x* + *y*) ^ (*Suc mm* − *n*)))
by (*subst sum-distrib-left*, *rule sum.cong*) (*simp-all add*: *Suc-diff-le*)
also have . . . + (−*a gchoose* (*Suc mm*)) ∗ (−*x*)^*Suc mm* =
(∑ *n*≤*Suc mm*. (− *a gchoose n*) ∗ (− *x*) ^ *n* ∗ (*x* + *y*) ^ (*Suc mm* − *n*))
by *simp*
finally show *?case*
by (*simp only*: *S-def*)
qed

lemma *gbinomial-partial-sum-poly-xpos*:

($\sum k \leq m.$ (*of-nat m* + *a gchoose k*) ∗ *x*^*k* ∗ *y*^(*m*−*k*)) =
($\sum k \leq m.$ (*of-nat k* + *a* − *1 gchoose k*) ∗ *x*^*k* ∗ (*x* + *y*)^(*m*−*k*)) (**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = ($\sum k \leq m.$ (− *a gchoose k*) ∗ (− *x*) ^ *k* ∗ (*x* + *y*) ^ (*m* − *k*))
by (*simp add*: *gbinomial-partial-sum-poly*)
also have ... = ($\sum k \leq m.$ (−*1*) ^ *k* ∗ (*of-nat k* − − *a* − *1 gchoose k*) ∗ (− *x*) ^ *k* ∗ (*x* + *y*) ^ (*m* − *k*))
by (*metis* (*no-types*, *opaque-lifting*) *gbinomial-negated-upper*)
also have ... = *?rhs*
by (*intro sum.cong*) (*auto simp flip*: *power-mult-distrib*)
finally show *?thesis* **.**
qed

lemma *binomial-r-part-sum*: ($\sum k \leq m.$ (*2* ∗ *m* + *1 choose k*)) = *2* ^ (*2* ∗ *m*)
proof −
have *2* ∗ *2*^(*2*∗*m*) = ($\sum k = 0..(2 * m + 1).$ (*2* ∗ *m* + *1 choose k*))
using *choose-row-sum*[**where** *n*=*2* ∗ *m* + *1*] **by** (*simp add*: *atMost-atLeast0*)
also have ($\sum k = 0..(2 * m + 1).$ (*2* ∗ *m* + *1 choose k*)) =
($\sum k = 0..m.$ (*2* ∗ *m* + *1 choose k*)) +
($\sum k = m+1..2*m+1.$ (*2* ∗ *m* + *1 choose k*))
using *sum.ub-add-nat*[*of 0 m* λ*k*. *2* ∗ *m* + *1 choose k m*+*1*]
by (*simp add*: *mult-2*)
also have ($\sum k = m+1..2*m+1.$ (*2* ∗ *m* + *1 choose k*)) =
($\sum k = 0..m.$ (*2* ∗ *m* + *1 choose* (*k* + (*m* + *1*))))
by (*subst sum.shift-bounds-cl-nat-ivl* [*symmetric*]) (*simp add*: *mult-2*)
also have . . . = ($\sum k = 0..m.$ (*2* ∗ *m* + *1 choose* (*m* − *k*)))
by (*intro sum.cong*[*OF refl*], *subst binomial-symmetric*) *simp-all*
also have . . . = ($\sum k = 0..m.$ (*2* ∗ *m* + *1 choose k*))
using *sum.atLeastAtMost-rev* [*of* λ*k*. *2* ∗ *m* + *1 choose* (*m* − *k*) *0 m*]
by *simp*
also have . . . + . . . = *2* ∗ . . .
by *simp*
finally show *?thesis*
by (*subst* (*asm*) *mult-cancel1*) (*simp add*: *atLeast0AtMost*)
qed

lemma *gbinomial-r-part-sum*: ($\sum k \leq m.$ (*2* ∗ (*of-nat m*) + *1 gchoose k*)) = *2* ^ (*2* ∗ *m*)
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = *of-nat* ($\sum k \leq m.$ (*2* ∗ *m* + *1*) *choose k*)
by (*simp add*: *binomial-gbinomial add-ac*)
also have . . . = *of-nat* (*2* ^ (*2* ∗ *m*))
by (*subst binomial-r-part-sum*) (*rule refl*)
finally show *?thesis* **by** *simp*
qed

lemma *gbinomial-sum-nat-pow2*:

($\sum k \le m.$ (*of-nat* ($m + k$) *gchoose* k :: *'a::field-char-0*) / $2 \mathbin{\hat{}} k$) = $2 \mathbin{\hat{}} m$
(**is** *?lhs* = *?rhs*)
proof −
have $2 \mathbin{\hat{}} m * 2 \mathbin{\hat{}} m = (2 \mathbin{\hat{}} (2*m)$ :: *'a*)
by (*induct m*) *simp-all*
also have ... = ($\sum k \le m.$ ($2 *$ (*of-nat m*) + *1 gchoose k*))
using *gbinomial-r-part-sum* **..**
also have ... = ($\sum k \le m.$ (*of-nat* ($m + k$) *gchoose k*) $* 2 \mathbin{\hat{}} (m - k)$)
using *gbinomial-partial-sum-poly-xpos*[**where** *x=1* **and** *y=1* **and** *a=of-nat m* + *1* **and** *m=m*]
by (*simp add*: *add-ac*)
also have ... = $2 \mathbin{\hat{}} m *$ ($\sum k \le m.$ (*of-nat* ($m + k$) *gchoose k*) / $2 \mathbin{\hat{}} k$)
by (*subst sum-distrib-left*) (*simp add*: *algebra-simps power-diff*)
finally show *?thesis*
by (*subst* (*asm*) *mult-left-cancel*) *simp-all*
qed

lemma *gbinomial-trinomial-revision*:
assumes $k \le m$
shows (*a gchoose m*) $*$ (*of-nat m gchoose k*) = (*a gchoose k*) $*$ (*a* − *of-nat k gchoose* ($m - k$))
proof −
have (*a gchoose m*) $*$ (*of-nat m gchoose k*) = (*a gchoose m*) $*$ *fact m* / (*fact k* $*$ *fact* ($m - k$))
using *assms* **by** (*simp add*: *binomial-gbinomial* [*symmetric*] *binomial-fact*)
also have ... = (*a gchoose k*) $*$ (*a* − *of-nat k gchoose* ($m - k$))
using *assms* **by** (*simp add*: *gbinomial-pochhammer power-diff pochhammer-product*)
finally show *?thesis* **.**
qed

Versions of the theorems above for the natural-number version of "choose"

lemma *binomial-altdef-of-nat*:
$k \le n \Longrightarrow$ *of-nat* (*n choose k*) = ($\prod i = 0..<k.$ *of-nat* ($n - i$) / *of-nat* ($k - i$) :: *'a*)
for *n k* :: *nat* **and** *x* :: *'a::field-char-0*
by (*simp add*: *gbinomial-altdef-of-nat binomial-gbinomial of-nat-diff*)

lemma *binomial-ge-n-over-k-pow-k*: $k \le n \Longrightarrow$ (*of-nat n* / *of-nat k* :: *'a*) $\mathbin{\hat{}} k \le$ *of-nat* (*n choose k*)
for *k n* :: *nat* **and** *x* :: *'a::linordered-field*
by (*simp add*: *gbinomial-ge-n-over-k-pow-k binomial-gbinomial of-nat-diff*)

lemma *binomial-le-pow*:
assumes $r \le n$
shows *n choose r* $\le n \mathbin{\hat{}} r$
proof −
have *n choose r* $\le$ *fact n div fact* ($n - r$)
using *assms* **by** (*subst binomial-fact-lemma*[*symmetric*]) *auto*
with *fact-div-fact-le-pow* [*OF assms*] **show** *?thesis*

by *auto*
qed

lemma *binomial-altdef-nat*: $k \leq n \Longrightarrow n$ *choose* $k =$ *fact* n *div* (*fact* $k *$ *fact* ($n - k$))
for k n :: *nat*
by (*subst binomial-fact-lemma* [*symmetric*]) *auto*

lemma *choose-dvd*:
assumes $k \leq n$ **shows** *fact* $k *$ *fact* ($n - k$) *dvd* (*fact* n :: *'a::linordered-semidom*)
unfolding *dvd-def*
proof
show *fact* $n =$ *fact* $k *$ *fact* ($n - k$) $*$ *of-nat* (n *choose* k)
by (*metis assms binomial-fact-lemma of-nat-fact of-nat-mult*)
qed

lemma *fact-fact-dvd-fact*:
fact $k *$ *fact* n *dvd* (*fact* ($k + n$) :: *'a::linordered-semidom*)
by (*metis add.commute add-diff-cancel-left' choose-dvd le-add2*)

lemma *choose-mult-lemma*:
(($m + r + k$) *choose* ($m + k$)) $*$ (($m + k$) *choose* k) = (($m + r + k$) *choose* k) $*$ (($m + r$) *choose* m)
(**is** *?lhs* = -)
proof −
have *?lhs* =
fact ($m + r + k$) *div* (*fact* ($m + k$) $*$ *fact* ($m + r - m$)) $*$ (*fact* ($m + k$) *div* (*fact* $k *$ *fact* m))
by (*simp add*: *binomial-altdef-nat*)
also have ... = *fact* ($m + r + k$) $*$ *fact* ($m + k$) *div*
(*fact* ($m + k$) $*$ *fact* ($m + r - m$) $*$ (*fact* $k *$ *fact* m))
by (*metis add-implies-diff add-le-mono1 choose-dvd diff-cancel2 div-mult-div-if-dvd le-add1 le-add2*)
also have ... = *fact* ($m + r + k$) *div* (*fact* $r *$ (*fact* $k *$ *fact* m))
by (*auto simp*: *algebra-simps fact-fact-dvd-fact*)
also have ... = (*fact* ($m + r + k$) $*$ *fact* ($m + r$)) *div* (*fact* $r *$ (*fact* $k *$ *fact* m) $*$ *fact* ($m + r$))
by *simp*
also have ... =
(*fact* ($m + r + k$) *div* (*fact* $k *$ *fact* ($m + r$)) $*$ (*fact* ($m + r$) *div* (*fact* $r *$ *fact* m)))
by (*auto simp*: *div-mult-div-if-dvd fact-fact-dvd-fact algebra-simps*)
finally show *?thesis*
by (*simp add*: *binomial-altdef-nat mult.commute*)
qed

The "Subset of a Subset" identity.

lemma *choose-mult*:
$k \leq m \Longrightarrow m \leq n \Longrightarrow$ (n *choose* m) $*$ (m *choose* k) = (n *choose* k) $*$ (($n - k$)

choose (*m* − *k*))
  using *choose-mult-lemma* [*of m−k n−m k*] **by** *simp*

lemma *of-nat-binomial-eq-mult-binomial-Suc*:
  assumes $k \leq n$
  shows (*of-nat* :: (*nat* ⇒ ('*a* :: *field-char-0*))) (*n choose k*) = *of-nat* (*n* + *1* − *k*) / *of-nat* (*n* + *1*) ∗ *of-nat* (*Suc n choose k*)
proof (*cases k*)
  case *0* **then show** *?thesis*
    using *of-nat-neq-0* **by** *auto*
next
  case (*Suc l*)
  have *of-nat* (*n* + *1*) ∗ (∏ *i=0..<k. of-nat* (*n* − *i*)) = (*of-nat* :: (*nat* ⇒ '*a*)) (*n* + *1* − *k*) ∗ (∏ *i=0..<k. of-nat* (*Suc n* − *i*))
    using *prod.atLeast0-lessThan-Suc* [**where** *?'a* = '*a*, *symmetric*, *of* λ*i. of-nat* (*Suc n* − *i*) *k*]
    by (*simp add*: *ac-simps prod.atLeast0-lessThan-Suc-shift del*: *prod.op-ivl-Suc*)
  also have ... = (*of-nat* :: (*nat* ⇒ '*a*)) (*Suc n* − *k*) ∗ (∏ *i=0..<k. of-nat* (*Suc n* − *i*))
    by (*simp add*: *Suc atLeast0-atMost-Suc atLeastLessThanSuc-atLeastAtMost*)
  also have ... = (*of-nat* :: (*nat* ⇒ '*a*)) (*n* + *1* − *k*) ∗ (∏ *i=0..<k. of-nat* (*Suc n* − *i*))
    by (*simp only*: *Suc-eq-plus1*)
  finally have (∏ *i=0..<k. of-nat* (*n* − *i*)) = (*of-nat* :: (*nat* ⇒ '*a*)) (*n* + *1* − *k*) / *of-nat* (*n* + *1*) ∗ (∏ *i=0..<k. of-nat* (*Suc n* − *i*))
    using *of-nat-neq-0* **by** (*auto simp*: *mult.commute divide-simps*)
  with *assms* **show** *?thesis*
    by (*simp add*: *binomial-altdef-of-nat prod-dividef*)
qed

93.5 More on Binomial Coefficients

The number of nat lists of length *m* summing to *N* is *N* + *m* − *1 choose N*:

lemma *card-length-sum-list-rec*:
  assumes $m \geq 1$
  shows *card* {*l*::*nat list. length l* = *m* ∧ *sum-list l* = *N*} =
      card {*l. length l* = (*m* − *1*) ∧ *sum-list l* = *N*} +
      card {*l. length l* = *m* ∧ *sum-list l* + *1* = *N*}
    (**is** *card ?C* = *card ?A* + *card ?B*)
proof −
  let *?A'* = {*l. length l* = *m* ∧ *sum-list l* = *N* ∧ *hd l* = *0*}
  let *?B'* = {*l. length l* = *m* ∧ *sum-list l* = *N* ∧ *hd l* ≠ *0*}
  let *?f* = λ*l. 0* # *l*
  let *?g* = λ*l.* (*hd l* + *1*) # *tl l*
  have *1*: *xs* ≠ [] ⟹ *x* = *hd xs* ⟹ *x* # *tl xs* = *xs* **for** *x* :: *nat* **and** *xs*
    by *simp*
  have *2*: *xs* ≠ [] ⟹ *sum-list*(*tl xs*) = *sum-list xs* − *hd xs* **for** *xs* :: *nat list*
    by (*auto simp add*: *neq-Nil-conv*)

have *f*: *bij-betw ?f ?A ?A′*
by (*rule bij-betw-byWitness*[**where** *f′* = *tl*]) (*use assms* **in** ‹*auto simp*: *2 1 simp flip*: *length-0-conv*›)
have *3*: $xs \neq [] \Longrightarrow hd\ xs + (sum\text{-}list\ xs - hd\ xs) = sum\text{-}list\ xs$ **for** *xs* :: *nat list*
by (*metis 1 sum-list-simps*(*2*) *2*)
have *g*: *bij-betw ?g ?B ?B′*
apply (*rule bij-betw-byWitness*[**where** $f' = \lambda l.\ (hd\ l - 1)\ \#\ tl\ l$])
using *assms*
by (*auto simp*: *2 simp flip*: *length-0-conv intro*!: *3*)
have *fin*: *finite* $\{xs.\ size\ xs = M \wedge set\ xs \subseteq \{0..<N\}\}$ **for** *M N* :: *nat*
using *finite-lists-length-eq*[*OF finite-atLeastLessThan*] *conj-commute* **by** *auto*
have *fin-A*: *finite ?A* **using** *fin*[*of - N+1*]
by (*intro finite-subset*[**where** *?A* = *?A* **and** $?B = \{xs.\ size\ xs = m - 1 \wedge set\ xs \subseteq \{0..<N+1\}\}$])
(*auto simp*: *member-le-sum-list less-Suc-eq-le*)
have *fin-B*: *finite ?B*
by (*intro finite-subset*[**where** *?A* = *?B* **and** $?B = \{xs.\ size\ xs = m \wedge set\ xs \subseteq \{0..<N\}\}$])
(*auto simp*: *member-le-sum-list less-Suc-eq-le fin*)
have *uni*: $?C = ?A' \cup ?B'$
by *auto*
have *disj*: $?A' \cap ?B' = \{\}$ **by** *blast*
have *card* $?C = card(?A' \cup ?B')$
using *uni* **by** *simp*
also have ... = *card ?A* + *card ?B*
using *card-Un-disjoint*[*OF - - disj*] *bij-betw-finite*[*OF f*] *bij-betw-finite*[*OF g*]
bij-betw-same-card[*OF f*] *bij-betw-same-card*[*OF g*] *fin-A fin-B*
by *presburger*
finally show *?thesis* **.**
qed

lemma *card-length-sum-list*: *card* $\{l::nat\ list.\ size\ l = m \wedge sum\text{-}list\ l = N\} = (N + m - 1)$ *choose N*
— by Holden Lee, tidied by Tobias Nipkow
proof (*cases m*)
case *0*
then show *?thesis*
by (*cases N*) (*auto cong*: *conj-cong*)
next
case (*Suc m′*)
have *m*: $m \geq 1$
by (*simp add*: *Suc*)
then show *?thesis*
proof (*induct N + m − 1 arbitrary*: *N m*)
case *0* — In the base case, the only solution is [0].
have [*simp*]: $\{l::nat\ list.\ length\ l = Suc\ 0 \wedge (\forall n \in set\ l.\ n = 0)\} = \{[0]\}$
by (*auto simp*: *length-Suc-conv*)
have $m = 1 \wedge N = 0$
using *0* **by** *linarith*

then show *?case*
by *simp*
next
case (*Suc k*)
have *c1*: *card* {*l*::*nat list*. *size l* = (*m* − *1*) ∧ *sum-list l* = *N*} = (*N* + (*m* − *1*) − *1*) *choose N*
proof (*cases m* = *1*)
case *True*
with *Suc.hyps* **have** *N* ≥ *1*
by *auto*
with *True* **show** *?thesis*
by (*simp add*: *binomial-eq-0*)
next
case *False*
then show *?thesis*
using *Suc* **by** *fastforce*
qed
from *Suc* **have** *c2*: *card* {*l*::*nat list*. *size l* = *m* ∧ *sum-list l* + *1* = *N*} =
(*if N* > *0 then* ((*N* − *1*) + *m* − *1*) *choose* (*N* − *1*) *else 0*)
proof −
have ∗: *n* > *0* ⟹ *Suc m* = *n* ⟷ *m* = *n* − *1* **for** *m n*
by *arith*
from *Suc* **have** *N* > *0* ⟹
card {*l*::*nat list*. *size l* = *m* ∧ *sum-list l* + *1* = *N*} =
((*N* − *1*) + *m* − *1*) *choose* (*N* − *1*)
by (*simp add*: ∗)
then show *?thesis*
by *auto*
qed
from *Suc.prems* **have** (*card* {*l*::*nat list*. *size l* = (*m* − *1*) ∧ *sum-list l* = *N*} +
card {*l*::*nat list*. *size l* = *m* ∧ *sum-list l* + *1* = *N*}) = (*N* + *m* − *1*) *choose N*
by (*auto simp*: *c1 c2 choose-reduce-nat*[*of N* + *m* − *1 N*] *simp del*: *One-nat-def*)
then show *?case*
using *card-length-sum-list-rec*[*OF Suc.prems*] **by** *auto*
qed
qed

lemma *card-disjoint-shuffles*:
assumes *set xs* ∩ *set ys* = {}
shows *card* (*shuffles xs ys*) = (*length xs* + *length ys*) *choose length xs*
using *assms*
proof (*induction xs ys rule*: *shuffles.induct*)
case (*3 x xs y ys*)
have *shuffles* (*x* # *xs*) (*y* # *ys*) = (#) *x* ‘ *shuffles xs* (*y* # *ys*) ∪ (#) *y* ‘ *shuffles* (*x* # *xs*) *ys*
by (*rule shuffles.simps*)
also have *card* ... = *card* ((#) *x* ‘ *shuffles xs* (*y* # *ys*)) + *card* ((#) *y* ‘ *shuffles* (*x* # *xs*) *ys*)

by (*rule card-Un-disjoint*) (*insert 3.prems*, *auto*)
also have *card* ((#) *x* ' *shuffles xs* (*y* # *ys*)) = *card* (*shuffles xs* (*y* # *ys*))
by (*rule card-image*) *auto*
also have ... = (*length xs* + *length* (*y* # *ys*)) *choose length xs*
using *3.prems* **by** (*intro 3.IH*) *auto*
also have *card* ((#) *y* ' *shuffles* (*x* # *xs*) *ys*) = *card* (*shuffles* (*x* # *xs*) *ys*)
by (*rule card-image*) *auto*
also have ... = (*length* (*x* # *xs*) + *length ys*) *choose length* (*x* # *xs*)
using *3.prems* **by** (*intro 3.IH*) *auto*
also have (*length xs* + *length* (*y* # *ys*) *choose length xs*) + ... =
(*length* (*x* # *xs*) + *length* (*y* # *ys*)) *choose length* (*x* # *xs*) **by** *simp*
finally show *?case* **.**
qed *auto*

lemma *Suc-times-binomial-add*: *Suc a* ∗ (*Suc* (*a* + *b*) *choose Suc a*) = *Suc b* ∗ (*Suc* (*a* + *b*) *choose a*)
— by Lukas Bulwahn
proof −
have *dvd*: *Suc a* ∗ (*fact a* ∗ *fact b*) *dvd fact* (*Suc* (*a* + *b*)) **for** *a b*
using *fact-fact-dvd-fact*[*of Suc a b*, **where** ′*a*=*nat*]
by (*simp only*: *fact-Suc add-Suc*[*symmetric*] *of-nat-id mult.assoc*)
have *Suc a* ∗ (*fact* (*Suc* (*a* + *b*)) *div* (*Suc a* ∗ *fact a* ∗ *fact b*)) =
Suc a ∗ *fact* (*Suc* (*a* + *b*)) *div* (*Suc a* ∗ (*fact a* ∗ *fact b*))
by (*subst div-mult-swap*[*symmetric*]; *simp only*: *mult.assoc dvd*)
also have ... = *Suc b* ∗ *fact* (*Suc* (*a* + *b*)) *div* (*Suc b* ∗ (*fact a* ∗ *fact b*))
by (*simp only*: *div-mult-mult1*)
also have ... = *Suc b* ∗ (*fact* (*Suc* (*a* + *b*)) *div* (*Suc b* ∗ (*fact a* ∗ *fact b*)))
using *dvd*[*of b a*] **by** (*subst div-mult-swap*[*symmetric*]; *simp only*: *ac-simps dvd*)
finally show *?thesis*
by (*subst* (*1 2*) *binomial-altdef-nat*)
(*simp-all only*: *ac-simps diff-Suc-Suc Suc-diff-le diff-add-inverse fact-Suc of-nat-id*)
qed

93.6 Inclusion-exclusion principle

Ported from HOL Light by lcp

lemma *Inter-over-Union*:
$\bigcap$ {$\bigcup$ ($\mathcal{F}$ *x*) |*x*. *x* ∈ *S*} = $\bigcup$ {$\bigcap$ (*G* ' *S*) |*G*. ∀ *x*∈*S*. *G x* ∈ $\mathcal{F}$ *x*}
proof −
have ⋀*x*. ∀ *s*∈*S*. ∃ *X* ∈ $\mathcal{F}$ *s*. *x* ∈ *X* ⟹ ∃ *G*. (∀ *x*∈*S*. *G x* ∈ $\mathcal{F}$ *x*) ∧ (∀ *s*∈*S*. *x* ∈ *G s*)
by *metis*
then show *?thesis*
by (*auto simp flip*: *all-simps ex-simps*)
qed

lemma *subset-insert-lemma*:
{*T*. *T* ⊆ (*insert a S*) ∧ *P T*} = {*T*. *T* ⊆ *S* ∧ *P T*} ∪ {*insert a T* |*T*. *T* ⊆ *S*

$\wedge$ *P*(*insert a T*)} (**is** *?L=?R*)
proof
  show *?L* $\subseteq$ *?R*
    by (*smt* (*verit*) *UnI1 UnI2 insert-Diff mem-Collect-eq subsetI subset-insert-iff*)
qed *blast*

Versions for additive real functions, where the additivity applies only to some specific subsets (e.g. cardinality of finite sets, measurable sets with bounded measure. (From HOL Light)

locale *Incl-Excl* =
  fixes *P* :: *'a set* $\Rightarrow$ *bool* **and** *f* :: *'a set* $\Rightarrow$ *'b::ring-1*
  assumes *disj-add*: $[\![P\ S;\ P\ T;\ disjnt\ S\ T]\!] \Longrightarrow f(S \cup T) = f\ S + f\ T$
    and *empty*: *P*{}
    and *Int*: $[\![P\ S;\ P\ T]\!] \Longrightarrow P(S \cap T)$
    and *Un*: $[\![P\ S;\ P\ T]\!] \Longrightarrow P(S \cup T)$
    and *Diff*: $[\![P\ S;\ P\ T]\!] \Longrightarrow P(S - T)$

begin

lemma *f-empty* [*simp*]: *f*{} = *0*
  using *disj-add empty* **by** *fastforce*

lemma *f-Un-Int*: $[\![P\ S;\ P\ T]\!] \Longrightarrow f(S \cup T) + f(S \cap T) = f\ S + f\ T$
  by (*smt* (*verit*, *ccfv-threshold*) *Groups.add-ac*(*2*) *Incl-Excl.Diff Incl-Excl.Int Incl-Excl-axioms Int-Diff-Un Int-Diff-disjoint Int-absorb Un-Diff Un-Int-eq*(*2*) *disj-add disjnt-def group-cancel.add2 sup-bot.right-neutral*)

lemma *restricted-indexed*:
  assumes *finite A* **and** *X*: $\bigwedge a.\ a \in A \Longrightarrow P(X\ a)$
  shows $f(\bigcup(X\ `\ A)) = (\sum B\ |\ B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap (X\ `\ B)))$
proof −
  have $[\![finite\ A;\ card\ A = n;\ \forall a \in A.\ P\ (X\ a)]\!]$
    $\Longrightarrow f(\bigcup(X\ `\ A)) = (\sum B\ |\ B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap\ (X\ `\ B)))$ **for** *n X* **and** *A* :: *'c set*
  proof (*induction n arbitrary*: *A X rule*: *less-induct*)
    case (*less n0 A0 X*)
    show *?case*
    proof (*cases n0=0*)
      case *True*
      with *less* **show** *?thesis*
        by *fastforce*
    next
      case *False*
      with *less.prems* **obtain** *A n a* **where** *: *n0* = *Suc n A0* = *insert a A a* $\notin$ *A card A* = *n finite A*
        by (*metis card-Suc-eq-finite not0-implies-Suc*)
      with *less* **have** *P* (*X a*) **by** *blast*
      have *APX*: $\forall a \in A.\ P\ (X\ a)$

by (*simp add*: ∗ *less.prems*)
have *PUXA*: $P\ (\bigcup\ (X\ `\ A))$
using ‹*finite A*› *APX*
by (*induction*) (*auto simp*: *empty Un*)
have $f\ (\bigcup\ (X\ `\ A0)) = f\ (X\ a \cup \bigcup\ (X\ `\ A))$
by (*simp add*: ∗)
also have ... $= f\ (X\ a) + f\ (\bigcup\ (X\ `\ A)) - f\ (X\ a \cap \bigcup\ (X\ `\ A))$
using *f-Un-Int add-diff-cancel PUXA* ‹*P* (*X a*)› **by** *metis*
also have ... $= f\ (X\ a) - (\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}\ card\ B * f\ (\bigcap$
$(X\ `\ B))) +$
$(\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}\ card\ B * f\ (X\ a \cap \bigcap\ (X\ `\ B)))$
proof −
have *1*: $f\ (\bigcup i{\in}A.\ X\ a \cap X\ i) = (\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}(card$
$B + 1) * f\ (\bigcap b{\in}B.\ X\ a \cap X\ b))$
using *less.IH* [*of n A* λ*i*. *X a* ∩ *X i*] *APX Int* ‹*P* (*X a*)› **by** (*simp add*: ∗)
have *2*: $X\ a \cap \bigcup\ (X\ `\ A) = (\bigcup i{\in}A.\ X\ a \cap X\ i)$
by *auto*
have *3*: $f\ (\bigcup\ (X\ `\ A)) = (\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}(card\ B + 1)$
$* f\ (\bigcap\ (X\ `\ B)))$
using *less.IH* [*of n A X*] *APX Int* ‹*P* (*X a*)› **by** (*simp add*: ∗)
show *?thesis*
unfolding *3 2 1*
by (*simp add*: *sum-negf*)
qed
also have ... $= (\sum B \mid B \subseteq A0 \wedge B \neq \{\}.\ (-\ 1)\ \hat{}(card\ B + 1) * f\ (\bigcap\ (X$
$`\ B)))$
proof −
have *F*: $\{insert\ a\ B \mid B.\ B \subseteq A\} = insert\ a\ `\ Pow\ A \wedge \{B.\ B \subseteq A \wedge B \neq$
$\{\}\} = Pow\ A - \{\{\}\}$
by *auto*
have *G*: $(\sum B{\in}Pow\ A.\ (-\ 1)\ \hat{}\ card\ (insert\ a\ B) * f\ (X\ a \cap \bigcap\ (X\ `\ B)))$
$= (\sum B{\in}Pow\ A.\ -\ ((-\ 1)\ \hat{}\ card\ B * f\ (X\ a \cap \bigcap\ (X\ `\ B))))$
proof (*rule sum.cong* [*OF refl*])
fix *B*
assume *B*: $B \in Pow\ A$
then have *finite B*
using ‹*finite A*› *finite-subset* **by** *auto*
show $(-\ 1)\ \hat{}\ card\ (insert\ a\ B) * f\ (X\ a \cap \bigcap\ (X\ `\ B)) = -\ ((-\ 1)\ \hat{}\ card$
$B * f\ (X\ a \cap \bigcap\ (X\ `\ B)))$
using *B* ∗ **by** (*auto simp add*: *card-insert-if* ‹*finite B*›)
qed
have *disj*: $\{B.\ B \subseteq A \wedge B \neq \{\}\} \cap \{insert\ a\ B \mid B.\ B \subseteq A\} = \{\}$
using ∗ **by** *blast*
have *inj*: *inj-on* (*insert a*) (*Pow A*)
using ∗ *inj-on-def* **by** *fastforce*
show *?thesis*
apply (*simp add*: ∗ *subset-insert-lemma sum.union-disjoint disj sum-negf*)
apply (*simp add*: *F G sum-negf sum.reindex* [*OF inj*] *o-def sum-diff* ∗)
done

qed
finally show *?thesis* **.**
qed
qed
then show *?thesis*
by (*meson assms*)
qed

lemma *restricted*:
assumes *finite A* $\bigwedge a.\ a \in A \Longrightarrow P\ a$
shows $f(\bigcup A) = (\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap B))$
using *restricted-indexed* [*of A* $\lambda x.\ x$] *assms* **by** *auto*

end

93.7 Versions for unrestrictedly additive functions

lemma *Incl-Excl-UN*:
fixes $f :: 'a\ set \Rightarrow 'b::ring\text{-}1$
assumes $\bigwedge S\ T.\ disjnt\ S\ T \Longrightarrow f(S \cup T) = f\ S + f\ T$ *finite A*
shows $f(\bigcup(G\ `\ A)) = (\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap (G\ `\ B)))$
proof −
interpret *Incl-Excl* $\lambda x.$ *True f*
by (*simp add*: *Incl-Excl.intro assms*(*1*))
show *?thesis*
using *restricted-indexed assms* **by** *blast*
qed

lemma *Incl-Excl-Union*:
fixes $f :: 'a\ set \Rightarrow 'b::ring\text{-}1$
assumes $\bigwedge S\ T.\ disjnt\ S\ T \Longrightarrow f(S \cup T) = f\ S + f\ T$ *finite A*
shows $f(\bigcup A) = (\sum B \mid B \subseteq A \wedge B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap B))$
using *Incl-Excl-UN*[*of f A* $\lambda X.\ X$] *assms* **by** *simp*

The famous inclusion-exclusion formula for the cardinality of a union

lemma *int-card-UNION*:
assumes *finite A* $\bigwedge K.\ K \in A \Longrightarrow$ *finite K*
shows $int\ (card\ (\bigcup A)) = (\sum I \mid I \subseteq A \wedge I \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ I + 1) * int\ (card\ (\bigcap I)))$
proof −
interpret *Incl-Excl finite int o card*
proof qed (*auto simp add*: *card-Un-disjnt*)
show *?thesis*
using *restricted assms* **by** *auto*
qed

A more conventional form

lemma *inclusion-exclusion*:

assumes *finite A* $\bigwedge K.\ K \in A \Longrightarrow$ *finite K*
shows $int(card(\bigcup\ A)) =$
$(\sum n=1..card\ A.\ (-1)\ \hat{}\ (Suc\ n) * (\sum B \mid B \subseteq A \wedge card\ B = n.\ int\ (card\ (\bigcap\ B))))$ (**is** -=?R)
proof −
have *fin*: *finite* $\{I.\ I \subseteq A \wedge I \neq \{\}\}$
by (*simp add*: *assms*)
have $\bigwedge k.\ [\![Suc\ 0 \leq k;\ k \leq card\ A]\!] \Longrightarrow \exists B \subseteq A.\ B \neq \{\} \wedge k = card\ B$
by (*metis* (*mono-tags*, *lifting*) *Suc-le-D Zero-neq-Suc card-eq-0-iff obtain-subset-with-card-n*)
with ‹*finite A*› *finite-subset*
have *card-eq*: *card* ‘ $\{I.\ I \subseteq A \wedge I \neq \{\}\} = \{1..card\ A\}$
using *not-less-eq-eq card-mono* **by** (*fastforce simp*: *image-iff*)
have $int(card(\bigcup\ A))$
$= (\sum y = 1..card\ A.\ \sum I \in \{x.\ x \subseteq A \wedge x \neq \{\} \wedge card\ x = y\}.\ -\ ((-\ 1)\ \hat{}\ y * int\ (card\ (\bigcap\ I))))$
by (*simp add*: *int-card-UNION assms sum.image-gen* [*OF fin*, **where** *g=card*] *card-eq*)
also have ... = *?R*
proof −
have $\{B.\ B \subseteq A \wedge B \neq \{\} \wedge card\ B = k\} = \{B.\ B \subseteq A \wedge card\ B = k\}$
if *Suc 0* $\leq k$ **and** $k \leq card\ A$ **for** *k*
using *that* **by** *auto*
then show *?thesis*
by (*clarsimp simp add*: *sum-negf simp flip*: *sum-distrib-left*)
qed
finally show *?thesis* **.**
qed

lemma *card-UNION*:
assumes *finite A* **and** $\bigwedge K.\ K \in A \Longrightarrow$ *finite K*
shows $card\ (\bigcup A) = nat\ (\sum I \mid I \subseteq A \wedge I \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ I + 1) * int\ (card\ (\bigcap I)))$
by (*simp only*: *flip*: *int-card-UNION* [*OF assms*])

lemma *card-UNION-nonneg*:
assumes *finite A* **and** $\bigwedge K.\ K \in A \Longrightarrow$ *finite K*
shows $(\sum I \mid I \subseteq A \wedge I \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ I + 1) * int\ (card\ (\bigcap I))) \geq 0$
using *int-card-UNION* [*OF assms*] **by** *presburger*

93.8 General "Moebius inversion" inclusion-exclusion principle

This "symmetric" form is from Ira Gessel: "Symmetric Inclusion-Exclusion"

lemma *sum-Un-eq*:
$[\![S \cap T = \{\};\ S \cup T = U;\ finite\ U]\!]$
$\Longrightarrow (sum\ f\ S + sum\ f\ T = sum\ f\ U)$
by (*metis finite-Un sum.union-disjoint*)

lemma *card-adjust-lemma*: $[\![inj\text{-}on\ f\ S;\ x = y + card\ (f\ `\ S)]\!] \Longrightarrow x = y + card\ S$

by (*simp add*: *card-image*)

lemma *card-subsets-step*:
assumes *finite S* $x \notin S$ $U \subseteq S$
shows *card* $\{T.\ T \subseteq (insert\ x\ S) \land U \subseteq T \land odd(card\ T)\}$
$= card\ \{T.\ T \subseteq S \land U \subseteq T \land odd(card\ T)\} + card\ \{T.\ T \subseteq S \land U \subseteq T \land even(card\ T)\} \land$
card $\{T.\ T \subseteq (insert\ x\ S) \land U \subseteq T \land even(card\ T)\}$
$= card\ \{T.\ T \subseteq S \land U \subseteq T \land even(card\ T)\} + card\ \{T.\ T \subseteq S \land U \subseteq T \land odd(card\ T)\}$
proof −
have *inj*: *inj-on* (*insert x*) $\{T.\ T \subseteq S \land P\ T\}$ **for** P
using *assms* **by** (*auto simp*: *inj-on-def*)
have [*simp*]: *finite* $\{T.\ T \subseteq S \land P\ T\}$ *finite* (*insert x* ' $\{T.\ T \subseteq S \land P\ T\}$) **for** P
using ‹*finite S*› **by** *auto*
have [*simp*]: *disjnt* $\{T.\ T \subseteq S \land P\ T\}$ (*insert x* ' $\{T.\ T \subseteq S \land Q\ T\}$) **for** $P\ Q$
using *assms* **by** (*auto simp*: *disjnt-iff*)
have *eq*: $\{T.\ T \subseteq S \land U \subseteq T \land P\ T\} \cup insert\ x$ ' $\{T.\ T \subseteq S \land U \subseteq T \land Q\ T\}$
$= \{T.\ T \subseteq insert\ x\ S \land U \subseteq T \land P\ T\}$ (**is** *?L* = *?R*)
if $\bigwedge A.\ A \subseteq S \Longrightarrow Q\ (insert\ x\ A) \longleftrightarrow P\ A$ $\bigwedge A.\ \neg\ Q\ A \longleftrightarrow P\ A$ **for** $P\ Q$
proof
show $?L \subseteq ?R$
by (*clarsimp simp*: *image-iff subset-iff*) (*meson subsetI that*)
show $?R \subseteq ?L$
using ‹$U \subseteq S$›
by (*clarsimp simp*: *image-iff*) (*smt* (*verit*) *insert-iff mk-disjoint-insert subset-iff that*)
qed
have [*simp*]: $\bigwedge A.\ A \subseteq S \Longrightarrow even\ (card\ (insert\ x\ A)) \longleftrightarrow odd\ (card\ A)$
by (*metis* ‹*finite S*› ‹$x \notin S$› *card-insert-disjoint even-Suc finite-subset subsetD*)
show *?thesis*
by (*intro conjI card-adjust-lemma* [*OF inj*]; *simp add*: *eq flip*: *card-Un-disjnt*)
qed

lemma *card-subsupersets-even-odd*:
assumes *finite S* $U \subset S$
shows *card* $\{T.\ T \subseteq S \land U \subseteq T \land even(card\ T)\}$
$= card\ \{T.\ T \subseteq S \land U \subseteq T \land odd(card\ T)\}$
using *assms*
proof (*induction card S arbitrary*: *S rule*: *less-induct*)
case (*less S*)
then obtain x **where** $x \notin U$ $x \in S$
by *blast*
then have U: $U \subseteq S - \{x\}$
using *less.prems*(*2*) **by** *blast*
let $?V = S - \{x\}$
show *?case*

using *card-subsets-step* [*of ?V x U*] *less.prems U*
by (*simp add*: *insert-absorb* ‹$x \in S$›)
qed

lemma *sum-alternating-cancels*:
assumes *finite S card* $\{x.\ x \in S \wedge even(f\ x)\} = card\ \{x.\ x \in S \wedge odd(f\ x)\}$
shows $(\sum x{\in}S.\ (-1)\ \hat{}\ f\ x) = (0{::}'b{::}ring\text{-}1)$
proof −
have $(\sum x{\in}S.\ (-1)\ \hat{}\ f\ x)$
$= (\sum x \mid x \in S \wedge even\ (f\ x).\ (-1)\ \hat{}\ f\ x) + (\sum x \mid x \in S \wedge odd\ (f\ x).\ (-1)\ \hat{}\ f\ x)$
by (*rule sum-Un-eq* [*symmetric*]; *force simp*: ‹*finite S*›)
also have ... $= (0{::}'b{::}ring\text{-}1)$
by (*simp add*: *minus-one-power-iff assms cong*: *conj-cong*)
finally show *?thesis* **.**
qed

lemma *inclusion-exclusion-symmetric*:
fixes $f :: 'a\ set \Rightarrow 'b{::}ring\text{-}1$
assumes §: $\bigwedge S.\ finite\ S \Longrightarrow g\ S = (\sum T \in Pow\ S.\ (-1)\ \hat{}\ card\ T * f\ T)$
and *finite S*
shows $f\ S = (\sum T \in Pow\ S.\ (-1)\ \hat{}\ card\ T * g\ T)$
proof −
have $(-1)\ \hat{}\ card\ T * g\ T = (-1)\ \hat{}\ card\ T * (\sum U \mid U \subseteq S \wedge U \subseteq T.\ (-1)\ \hat{}\ card\ U * f\ U)$
if $T \subseteq S$ **for** T
proof −
have [*simp*]: $\{U.\ U \subseteq S \wedge U \subseteq T\} = Pow\ T$
using *that* **by** *auto*
show *?thesis*
using *that* **by** (*simp add*: ‹*finite S*› *finite-subset* §)
qed
then have $(\sum T \in Pow\ S.\ (-1)\ \hat{}\ card\ T * g\ T)$
$= (\sum T{\in}Pow\ S.\ (-1)\ \hat{}\ card\ T * (\sum U \mid U \in \{U.\ U \subseteq S\} \wedge U \subseteq T.\ (-1)\ \hat{}\ card\ U * f\ U))$
by *simp*
also have ... $= (\sum U{\in}Pow\ S.\ (\sum T \mid T \subseteq S \wedge U \subseteq T.\ (-1)\ \hat{}\ card\ T) * (-1)\ \hat{}\ card\ U * f\ U)$
unfolding *sum-distrib-left*
by (*subst sum.swap-restrict*; *simp add*: ‹*finite S*› *algebra-simps sum-distrib-right Pow-def*)
also have ... $= (\sum U{\in}Pow\ S.\ if\ U{=}S\ then\ f\ S\ else\ 0)$
proof −
have [*simp*]: $\{T.\ T \subseteq S \wedge S \subseteq T\} = \{S\}$
by *auto*
show *?thesis*
apply (*rule sum.cong* [*OF refl*])
by (*simp add*: *sum-alternating-cancels card-subsupersets-even-odd* ‹*finite S*› *flip*: *power-add*)

qed
also have ... = *f S*
by (*simp add*: ‹*finite S*›)
finally show *?thesis*
by *presburger*
qed

The more typical non-symmetric version.

lemma *inclusion-exclusion-mobius*:
fixes *f* :: *'a set* ⇒ *'b*::*ring-1*
assumes §: $\bigwedge S$. *finite S* ⟹ *g S* = *sum f* (*Pow S*) **and** *finite S*
shows *f S* = ($\sum$ *T* ∈ *Pow S*. (−*1*) ^ (*card S* − *card T*) ∗ *g T*) (**is** - = *?rhs*)
proof −
have (− *1*) ^ *card S* ∗ *f S* = ($\sum$ *T*∈*Pow S*. (− *1*) ^ *card T* ∗ *g T*)
by (*rule inclusion-exclusion-symmetric*; *simp add*: *assms flip*: *power-add mult.assoc*)
then have ((− *1*) ^ *card S* ∗ (− *1*) ^ *card S*) ∗ *f S* = ((− *1*) ^ *card S*) ∗ ($\sum$ *T*∈*Pow S*. (− *1*) ^ *card T* ∗ *g T*)
by (*simp add*: *mult-ac*)
then have *f S* = ($\sum$ *T*∈*Pow S*. (− *1*) ^ (*card S* + *card T*) ∗ *g T*)
by (*simp add*: *sum-distrib-left flip*: *power-add mult.assoc*)
also have ... = *?rhs*
by (*simp add*: ‹*finite S*› *card-mono neg-one-power-add-eq-neg-one-power-diff*)
finally show *?thesis* **.**
qed

93.9 Executable code

lemma *gbinomial-code* [*code*]:
a gchoose k =
(*if k* = *0 then 1*
else fold-atLeastAtMost-nat (λ*k acc*. (*a* − *of-nat k*) ∗ *acc*) *0* (*k* − *1*) *1* / *fact k*)
by (*cases k*)
(*simp-all add*: *gbinomial-prod-rev prod-atLeastAtMost-code* [*symmetric*]
atLeastLessThanSuc-atLeastAtMost)

lemma *binomial-code* [*code*]:
n choose k =
(*if k* > *n then 0*
else if 2 ∗ *k* > *n then n choose* (*n* − *k*)
else (*fold-atLeastAtMost-nat* (∗) (*n* − *k* + *1*) *n 1 div fact k*))
proof −
{
assume *k* ≤ *n*
then have {*1*..*n*} = {*1*..*n*−*k*} ∪ {*n*−*k*+*1*..*n*} **by** *auto*
then have (*fact n* :: *nat*) = *fact* (*n*−*k*) ∗ $\prod$ {*n*−*k*+*1*..*n*}
by (*simp add*: *prod.union-disjoint fact-prod*)
}
then show *?thesis* **by** (*auto simp*: *binomial-altdef-nat mult-ac prod-atLeastAtMost-code*)

qed

end

94 Main HOL

Classical Higher-order Logic – only "Main", excluding real and complex numbers etc.

theory *Main*
  imports
    Predicate-Compile
    Quickcheck-Narrowing
    Mirabelle
    Extraction
    Nunchaku
    BNF-Greatest-Fixpoint
    Filter
    Conditionally-Complete-Lattices
    Binomial
    GCD
begin

94.1 Namespace cleanup

hide-const (**open**)
  czero cinfinite cfinite csum cone ctwo Csum cprod cexp image2 image2p vimage2p Gr Grp collect
  fsts snds setl setr convol pick-middlep fstOp sndOp csquare relImage relInvImage Succ Shift
  shift proj id-bnf

hide-fact (**open**) *id-bnf-def type-definition-id-bnf-UNIV*

94.2 Syntax cleanup

no-notation
  ordLeq2 (**infix** ‹<=o› *50*) **and**
  ordLeq3 (**infix** ‹≤o› *50*) **and**
  ordLess2 (**infix** ‹<o› *50*) **and**
  ordIso2 (**infix** ‹=o› *50*) **and**
  card-of (‹(‹open-block notation=‹mixfix card-of››|-|)›) **and**
  BNF-Cardinal-Arithmetic.csum (**infixr** ‹+c› *65*) **and**
  BNF-Cardinal-Arithmetic.cprod (**infixr** ‹∗c› *80*) **and**
  BNF-Cardinal-Arithmetic.cexp (**infixr** ‹^c› *90*) **and**
  BNF-Def.convol (‹(‹indent=1 notation=‹mixfix convol››⟨-,/ -⟩)›)

bundle *cardinal-syntax*
begin

notation
ordLeq2 (**infix** ‹<=o› *50*) **and**
ordLeq3 (**infix** ‹≤o› *50*) **and**
ordLess2 (**infix** ‹<o› *50*) **and**
ordIso2 (**infix** ‹=o› *50*) **and**
card-of (‹(‹open-block notation=‹mixfix card-of››|-|)›) **and**
BNF-Cardinal-Arithmetic.csum (**infixr** ‹+c› *65*) **and**
BNF-Cardinal-Arithmetic.cprod (**infixr** ‹*c› *80*) **and**
BNF-Cardinal-Arithmetic.cexp (**infixr** ‹^c› *90*)

alias *cinfinite* = *BNF-Cardinal-Arithmetic.cinfinite*
alias *czero* = *BNF-Cardinal-Arithmetic.czero*
alias *cone* = *BNF-Cardinal-Arithmetic.cone*
alias *ctwo* = *BNF-Cardinal-Arithmetic.ctwo*

end

94.3 Lattice syntax

bundle *lattice-syntax*
begin

notation
bot (‹⊥›) **and**
top (‹⊤›) **and**
inf (**infixl** ‹⊓› *70*) **and**
sup (**infixl** ‹⊔› *65*) **and**
Inf (‹(‹open-block notation=‹prefix ⨅››⨅ -)› [*900*] *900*) **and**
Sup (‹(‹open-block notation=‹prefix ⨆››⨆ -)› [*900*] *900*)

syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder ⨅››⨅-./
-)› [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder ⨅››⨅-∈-./
-)› [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder ⨆››⨆-./
-)› [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹indent=3 notation=‹binder ⨆››⨆-∈-./
-)› [*0*, *0*, *10*] *10*)

end

unbundle *no lattice-syntax*

end

95 Archimedean Fields, Floor and Ceiling Functions

theory *Archimedean-Field*
imports *Main*
begin

lemma *cInf-abs-ge*:
  fixes S :: $'a$::{*linordered-idom*,*conditionally-complete-linorder*} *set*
  assumes $S \neq \{\}$
    and *bdd*: $\bigwedge x.\ x{\in}S \Longrightarrow |x| \leq a$
  shows $|Inf\ S| \leq a$
proof −
  have *Sup* (*uminus* ‘ *S*) = − (*Inf S*)
  proof (*rule antisym*)
    have $\bigwedge x.\ x \in S \Longrightarrow$ *bdd-above* (*uminus* ‘ *S*)
      using *bdd* **by** (*force simp*: *abs-le-iff bdd-above-def*)
  then show − (*Inf S*) ≤ *Sup* (*uminus* ‘ *S*)
    by (*meson cInf-greatest* [*OF* ‹$S \neq \{\}$›] *cSUP-upper minus-le-iff*)
  next
    have ∗: $\bigwedge x.\ x \in S \Longrightarrow$ *Inf S* ≤ *x*
      by (*meson abs-le-iff bdd bdd-below-def cInf-lower minus-le-iff*)
    show *Sup* (*uminus* ‘ *S*) ≤ − *Inf S*
      using ‹$S \neq \{\}$› **by** (*force intro*: ∗ *cSup-least*)
  qed
  with *cSup-abs-le* [*of uminus* ‘ *S*] *assms* **show** *?thesis*
    by *fastforce*
qed

lemma *cSup-asclose*:
  fixes S :: $'a$::{*linordered-idom*,*conditionally-complete-linorder*} *set*
  assumes *S*: $S \neq \{\}$
    and *b*: $\forall x{\in}S.\ |x - l| \leq e$
  shows $|Sup\ S - l| \leq e$
proof −
  have ∗: $|x - l| \leq e \longleftrightarrow l - e \leq x \wedge x \leq l + e$ **for** $x\ l\ e$ :: $'a$
    by *arith*
  have *bdd-above S*
    using *b* **by** (*auto intro*!: *bdd-aboveI*[*of* - *l* + *e*])
  with *S b* **show** *?thesis*
    unfolding ∗ **by** (*auto intro*!: *cSup-upper2 cSup-least*)
qed

lemma *cInf-asclose*:
  fixes S :: $'a$::{*linordered-idom*,*conditionally-complete-linorder*} *set*
  assumes *S*: $S \neq \{\}$
    and *b*: $\forall x{\in}S.\ |x - l| \leq e$
  shows $|Inf\ S - l| \leq e$
proof −

have $*$: $|x - l| \le e \longleftrightarrow l - e \le x \wedge x \le l + e$ **for** $x\ l\ e :: 'a$
  by *arith*
have *bdd-below S*
  using *b* **by** (*auto intro*!: *bdd-belowI*[*of* - *l* − *e*])
with *S b* **show** *?thesis*
  unfolding $*$ **by** (*auto intro*!: *cInf-lower2 cInf-greatest*)
qed

95.1 Class of Archimedean fields

Archimedean fields have no infinite elements.

class *archimedean-field* = *linordered-field* +
  assumes *ex-le-of-int*: $\exists z.\ x \le$ *of-int* z

lemma *ex-less-of-int*: $\exists z.\ x <$ *of-int* z
  for $x :: 'a::archimedean\text{-}field$
proof −
  from *ex-le-of-int* **obtain** z **where** $x \le$ *of-int* z **..**
  then have $x <$ *of-int* $(z + 1)$ **by** *simp*
  then show *?thesis* **..**
qed

lemma *ex-of-int-less*: $\exists z.$ *of-int* $z < x$
  for $x :: 'a::archimedean\text{-}field$
proof −
  from *ex-less-of-int* **obtain** z **where** $- x <$ *of-int* z **..**
  then have *of-int* $(- z) < x$ **by** *simp*
  then show *?thesis* **..**
qed

lemma *reals-Archimedean2*: $\exists n.\ x <$ *of-nat* n
  for $x :: 'a::archimedean\text{-}field$
proof −
  obtain z **where** $x <$ *of-int* z
    using *ex-less-of-int* **..**
  also have $\ldots \le$ *of-int* (*int* (*nat* z))
    by *simp*
  also have $\ldots =$ *of-nat* (*nat* z)
    by (*simp only*: *of-int-of-nat-eq*)
  finally show *?thesis* **..**
qed

lemma *real-arch-simple*: $\exists n.\ x \le$ *of-nat* n
  for $x :: 'a::archimedean\text{-}field$
proof −
  obtain n **where** $x <$ *of-nat* n
    using *reals-Archimedean2* **..**
  then have $x \le$ *of-nat* n
    by *simp*

then show *?thesis* **..**
qed

Archimedean fields have no infinitesimal elements.

lemma *reals-Archimedean*:
fixes x :: $'a$::*archimedean-field*
assumes $0 < x$
shows $\exists n.$ *inverse* (*of-nat* (*Suc* n)) $< x$
proof −
from ‹$0 < x$› **have** $0 <$ *inverse* x
by (*rule positive-imp-inverse-positive*)
obtain n **where** *inverse* $x <$ *of-nat* n
using *reals-Archimedean2* **..**
then obtain m **where** *inverse* $x <$ *of-nat* (*Suc* m)
using ‹$0 <$ *inverse* x› **by** (*cases* n) (*simp-all del*: *of-nat-Suc*)
then have *inverse* (*of-nat* (*Suc* m)) $<$ *inverse* (*inverse* x)
using ‹$0 <$ *inverse* x› **by** (*rule less-imp-inverse-less*)
then have *inverse* (*of-nat* (*Suc* m)) $< x$
using ‹$0 < x$› **by** (*simp add*: *nonzero-inverse-inverse-eq*)
then show *?thesis* **..**
qed

lemma *ex-inverse-of-nat-less*:
fixes x :: $'a$::*archimedean-field*
assumes $0 < x$
shows $\exists n>0.$ *inverse* (*of-nat* n) $< x$
using *reals-Archimedean* [*OF* ‹$0 < x$›] **by** *auto*

lemma *ex-less-of-nat-mult*:
fixes x :: $'a$::*archimedean-field*
assumes $0 < x$
shows $\exists n.$ $y <$ *of-nat* $n * x$
proof −
obtain n **where** $y / x <$ *of-nat* n
using *reals-Archimedean2* **..**
with ‹$0 < x$› **have** $y <$ *of-nat* $n * x$
by (*simp add*: *pos-divide-less-eq*)
then show *?thesis* **..**
qed

95.2 Existence and uniqueness of floor function

lemma *exists-least-lemma*:
assumes $\neg$ P 0 **and** $\exists n.$ P n
shows $\exists n.$ $\neg$ P $n \wedge P$ (*Suc* n)
proof −
from ‹$\exists n.$ P n› **have** P (*Least* P)
by (*rule LeastI-ex*)
with ‹$\neg$ P 0› **obtain** n **where** *Least* P = *Suc* n

```
    by (cases Least P) auto
  then have n < Least P
    by simp
  then have ¬ P n
    by (rule not-less-Least)
  then have ¬ P n ∧ P (Suc n)
    using ‹P (Least P)› ‹Least P = Suc n› by simp
  then show ?thesis ..
qed

lemma floor-exists:
  fixes x :: 'a::archimedean-field
  shows ∃ z. of-int z ≤ x ∧ x < of-int (z + 1)
proof (cases 0 ≤ x)
  case True
  then have ¬ x < of-nat 0
    by simp
  then have ∃ n. ¬ x < of-nat n ∧ x < of-nat (Suc n)
    using reals-Archimedean2 by (rule exists-least-lemma)
  then obtain n where ¬ x < of-nat n ∧ x < of-nat (Suc n) ..
  then have of-int (int n) ≤ x ∧ x < of-int (int n + 1)
    by simp
  then show ?thesis ..
next
  case False
  then have ¬ − x ≤ of-nat 0
    by simp
  then have ∃ n. ¬ − x ≤ of-nat n ∧ − x ≤ of-nat (Suc n)
    using real-arch-simple by (rule exists-least-lemma)
  then obtain n where ¬ − x ≤ of-nat n ∧ − x ≤ of-nat (Suc n) ..
  then have of-int (− int n − 1) ≤ x ∧ x < of-int (− int n − 1 + 1)
    by simp
  then show ?thesis ..
qed

lemma floor-exists1: ∃!z. of-int z ≤ x ∧ x < of-int (z + 1)
  for x :: 'a::archimedean-field
proof (rule ex-ex1I)
  show ∃ z. of-int z ≤ x ∧ x < of-int (z + 1)
    by (rule floor-exists)
next
  fix y z
  assume of-int y ≤ x ∧ x < of-int (y + 1)
    and of-int z ≤ x ∧ x < of-int (z + 1)
  with le-less-trans [of of-int y x of-int (z + 1)]
       le-less-trans [of of-int z x of-int (y + 1)] show y = z
    by (simp del: of-int-add)
qed
```

95.3 Floor function

class *floor-ceiling* = *archimedean-field* +
fixes *floor* :: *'a* ⇒ *int* (‹(‹*open-block notation*=‹*mixfix floor*››⌊-⌋)›)
assumes *floor-correct*: *of-int* ⌊*x*⌋ ≤ *x* ∧ *x* < *of-int* (⌊*x*⌋ + *1*)

lemma *floor-unique*: *of-int z* ≤ *x* ⟹ *x* < *of-int z* + *1* ⟹ ⌊*x*⌋ = *z*
using *floor-correct* [*of x*] *floor-exists1* [*of x*] **by** *auto*

lemma *floor-eq-iff*: ⌊*x*⌋ = *a* ⟷ *of-int a* ≤ *x* ∧ *x* < *of-int a* + *1*
using *floor-correct floor-unique* **by** *auto*

lemma *of-int-floor-le* [*simp*]: *of-int* ⌊*x*⌋ ≤ *x*
using *floor-correct* **..**

lemma *le-floor-iff*: *z* ≤ ⌊*x*⌋ ⟷ *of-int z* ≤ *x*
proof
assume *z* ≤ ⌊*x*⌋
then have (*of-int z* :: *'a*) ≤ *of-int* ⌊*x*⌋ **by** *simp*
also have *of-int* ⌊*x*⌋ ≤ *x* **by** (*rule of-int-floor-le*)
finally show *of-int z* ≤ *x* **.**
next
assume *of-int z* ≤ *x*
also have *x* < *of-int* (⌊*x*⌋ + *1*) **using** *floor-correct* **..**
finally show *z* ≤ ⌊*x*⌋ **by** (*simp del*: *of-int-add*)
qed

lemma *floor-less-iff*: ⌊*x*⌋ < *z* ⟷ *x* < *of-int z*
by (*simp add*: *not-le* [*symmetric*] *le-floor-iff*)

lemma *less-floor-iff*: *z* < ⌊*x*⌋ ⟷ *of-int z* + *1* ≤ *x*
using *le-floor-iff* [*of z* + *1 x*] **by** *auto*

lemma *floor-le-iff*: ⌊*x*⌋ ≤ *z* ⟷ *x* < *of-int z* + *1*
by (*simp add*: *not-less* [*symmetric*] *less-floor-iff*)

lemma *floor-split*[*linarith-split*]: *P* ⌊*t*⌋ ⟷ (∀ *i*. *of-int i* ≤ *t* ∧ *t* < *of-int i* + *1* ⟶ *P i*)
by (*metis floor-correct floor-unique less-floor-iff not-le order-refl*)

lemma *floor-mono*:
assumes *x* ≤ *y*
shows ⌊*x*⌋ ≤ ⌊*y*⌋
proof −
have *of-int* ⌊*x*⌋ ≤ *x* **by** (*rule of-int-floor-le*)
also note ‹*x* ≤ *y*›
finally show *?thesis* **by** (*simp add*: *le-floor-iff*)
qed

lemma *floor-less-cancel*: ⌊*x*⌋ < ⌊*y*⌋ ⟹ *x* < *y*

by (*auto simp add*: *not-le* [*symmetric*] *floor-mono*)

lemma *floor-of-int* [*simp*]: $\lfloor$*of-int z*$\rfloor$ = *z*
by (*rule floor-unique*) *simp-all*

lemma *floor-of-nat* [*simp*]: $\lfloor$*of-nat n*$\rfloor$ = *int n*
using *floor-of-int* [*of of-nat n*] **by** *simp*

lemma *le-floor-add*: $\lfloor x\rfloor + \lfloor y\rfloor \leq \lfloor x + y\rfloor$
by (*simp only*: *le-floor-iff of-int-add add-mono of-int-floor-le*)

Floor with numerals.

lemma *floor-zero* [*simp*]: $\lfloor 0\rfloor = 0$
using *floor-of-int* [*of 0*] **by** *simp*

lemma *floor-one* [*simp*]: $\lfloor 1\rfloor = 1$
using *floor-of-int* [*of 1*] **by** *simp*

lemma *floor-numeral* [*simp*]: $\lfloor$*numeral v*$\rfloor$ = *numeral v*
using *floor-of-int* [*of numeral v*] **by** *simp*

lemma *floor-neg-numeral* [*simp*]: $\lfloor$− *numeral v*$\rfloor$ = − *numeral v*
using *floor-of-int* [*of* − *numeral v*] **by** *simp*

lemma *zero-le-floor* [*simp*]: $0 \leq \lfloor x\rfloor \longleftrightarrow 0 \leq x$
by (*simp add*: *le-floor-iff*)

lemma *one-le-floor* [*simp*]: $1 \leq \lfloor x\rfloor \longleftrightarrow 1 \leq x$
by (*simp add*: *le-floor-iff*)

lemma *numeral-le-floor* [*simp*]: *numeral v* $\leq \lfloor x\rfloor \longleftrightarrow$ *numeral v* $\leq x$
by (*simp add*: *le-floor-iff*)

lemma *neg-numeral-le-floor* [*simp*]: − *numeral v* $\leq \lfloor x\rfloor \longleftrightarrow$ − *numeral v* $\leq x$
by (*simp add*: *le-floor-iff*)

lemma *zero-less-floor* [*simp*]: $0 < \lfloor x\rfloor \longleftrightarrow 1 \leq x$
by (*simp add*: *less-floor-iff*)

lemma *one-less-floor* [*simp*]: $1 < \lfloor x\rfloor \longleftrightarrow 2 \leq x$
by (*simp add*: *less-floor-iff*)

lemma *numeral-less-floor* [*simp*]: *numeral v* $< \lfloor x\rfloor \longleftrightarrow$ *numeral v* + 1 $\leq x$
by (*simp add*: *less-floor-iff*)

lemma *neg-numeral-less-floor* [*simp*]: − *numeral v* $< \lfloor x\rfloor \longleftrightarrow$ − *numeral v* + 1
$\leq x$
by (*simp add*: *less-floor-iff*)

lemma *floor-le-zero* [*simp*]: $\lfloor x \rfloor \leq 0 \longleftrightarrow x < 1$
by (*simp add*: *floor-le-iff*)

lemma *floor-le-one* [*simp*]: $\lfloor x \rfloor \leq 1 \longleftrightarrow x < 2$
by (*simp add*: *floor-le-iff*)

lemma *floor-le-numeral* [*simp*]: $\lfloor x \rfloor \leq numeral\ v \longleftrightarrow x < numeral\ v + 1$
by (*simp add*: *floor-le-iff*)

lemma *floor-le-neg-numeral* [*simp*]: $\lfloor x \rfloor \leq -\ numeral\ v \longleftrightarrow x < -\ numeral\ v + 1$
by (*simp add*: *floor-le-iff*)

lemma *floor-less-zero* [*simp*]: $\lfloor x \rfloor < 0 \longleftrightarrow x < 0$
by (*simp add*: *floor-less-iff*)

lemma *floor-less-one* [*simp*]: $\lfloor x \rfloor < 1 \longleftrightarrow x < 1$
by (*simp add*: *floor-less-iff*)

lemma *floor-less-numeral* [*simp*]: $\lfloor x \rfloor < numeral\ v \longleftrightarrow x < numeral\ v$
by (*simp add*: *floor-less-iff*)

lemma *floor-less-neg-numeral* [*simp*]: $\lfloor x \rfloor < -\ numeral\ v \longleftrightarrow x < -\ numeral\ v$
by (*simp add*: *floor-less-iff*)

lemma *le-mult-floor-Ints*:
assumes $0 \leq a$ $a \in Ints$
shows $of\text{-}int\ (\lfloor a \rfloor * \lfloor b \rfloor) \leq (of\text{-}int \lfloor a * b \rfloor :: 'a :: linordered\text{-}idom)$
by (*metis Ints-cases assms floor-less-iff floor-of-int linorder-not-less mult-left-mono of-int-floor-le of-int-less-iff of-int-mult*)

Addition and subtraction of integers.

lemma *floor-add-int*: $\lfloor x \rfloor + z = \lfloor x + of\text{-}int\ z \rfloor$
using *floor-correct* [*of x*] **by** (*simp add*: *floor-unique*[*symmetric*])

lemma *int-add-floor*: $z + \lfloor x \rfloor = \lfloor of\text{-}int\ z + x \rfloor$
using *floor-correct* [*of x*] **by** (*simp add*: *floor-unique*[*symmetric*])

lemma *one-add-floor*: $\lfloor x \rfloor + 1 = \lfloor x + 1 \rfloor$
using *floor-add-int* [*of x 1*] **by** *simp*

lemma *floor-diff-of-int* [*simp*]: $\lfloor x - of\text{-}int\ z \rfloor = \lfloor x \rfloor - z$
using *floor-add-int* [*of x − z*] **by** (*simp add*: *algebra-simps*)

lemma *floor-uminus-of-int* [*simp*]: $\lfloor - (of\text{-}int\ z) \rfloor = -\ z$
by (*metis floor-diff-of-int* [*of 0*] *diff-0 floor-zero*)

lemma *floor-diff-numeral* [*simp*]: $\lfloor x - numeral\ v \rfloor = \lfloor x \rfloor - numeral\ v$
using *floor-diff-of-int* [*of x numeral v*] **by** *simp*

lemma *floor-diff-one* [*simp*]: ⌊*x* − *1*⌋ = ⌊*x*⌋ − *1*
  using *floor-diff-of-int* [*of x 1*] **by** *simp*

lemma *le-mult-floor*:
  assumes *0* ≤ *a* **and** *0* ≤ *b*
  shows ⌊*a*⌋ ∗ ⌊*b*⌋ ≤ ⌊*a* ∗ *b*⌋
proof −
  have *of-int* ⌊*a*⌋ ≤ *a* **and** *of-int* ⌊*b*⌋ ≤ *b*
    by (*auto intro*: *of-int-floor-le*)
  then have *of-int* (⌊*a*⌋ ∗ ⌊*b*⌋) ≤ *a* ∗ *b*
    using *assms* **by** (*auto intro*!: *mult-mono*)
  also have *a* ∗ *b* < *of-int* (⌊*a* ∗ *b*⌋ + *1*)
    using *floor-correct*[*of a* ∗ *b*] **by** *auto*
  finally show *?thesis*
    unfolding *of-int-less-iff* **by** *simp*
qed

lemma *floor-divide-of-int-eq*: ⌊*of-int k* / *of-int l*⌋ = *k div l*
  for *k l* :: *int*
proof (*cases l* = *0*)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  have ∗: ⌊*of-int* (*k mod l*) / *of-int l* :: ′*a*⌋ = *0*
  proof (*cases l* > *0*)
    case *True*
    then show *?thesis*
      by (*auto intro*: *floor-unique*)
  next
    case *False*
    obtain *r* **where** *r* = − *l*
      by *blast*
    then have *l*: *l* = − *r*
      by *simp*
    with ‹*l* ≠ *0*› *False* **have** *r* > *0*
      by *simp*
    with *l* **show** *?thesis*
      using *pos-mod-bound* [*of r*]
      by (*auto simp add*: *zmod-zminus2-eq-if less-le field-simps intro*: *floor-unique*)
  qed
  have (*of-int k* :: ′*a*) = *of-int* (*k div l* ∗ *l* + *k mod l*)
    by *simp*
  also have . . . = (*of-int* (*k div l*) + *of-int* (*k mod l*) / *of-int l*) ∗ *of-int l*
    using *False* **by** (*simp only*: *of-int-add*) (*simp add*: *field-simps*)
  finally have (*of-int k* / *of-int l* :: ′*a*) = . . . / *of-int l*
    by *simp*
  then have (*of-int k* / *of-int l* :: ′*a*) = *of-int* (*k div l*) + *of-int* (*k mod l*) / *of-int l*

using *False* **by** (*simp only*:) (*simp add*: *field-simps*)
then have ⌊*of-int k* / *of-int l* :: ′*a*⌋ = ⌊*of-int* (*k div l*) + *of-int* (*k mod l*) / *of-int l* :: ′*a*⌋
by *simp*
then have ⌊*of-int k* / *of-int l* :: ′*a*⌋ = ⌊*of-int* (*k mod l*) / *of-int l* + *of-int* (*k div l*) :: ′*a*⌋
by (*simp add*: *ac-simps*)
then have ⌊*of-int k* / *of-int l* :: ′*a*⌋ = ⌊*of-int* (*k mod l*) / *of-int l* :: ′*a*⌋ + *k div l*
by (*simp add*: *floor-add-int*)
with ∗ **show** *?thesis*
by *simp*
qed

lemma *floor-divide-of-nat-eq*: ⌊*of-nat m* / *of-nat n*⌋ = *of-nat* (*m div n*)
for *m n* :: *nat*
by (*metis floor-divide-of-int-eq of-int-of-nat-eq linordered-euclidean-semiring-class.of-nat-div*)

lemma *floor-divide-lower*:
fixes *q* :: ′*a*::*floor-ceiling*
shows *q* > *0* ⟹ *of-int* ⌊*p* / *q*⌋ ∗ *q* ≤ *p*
using *of-int-floor-le pos-le-divide-eq* **by** *blast*

lemma *floor-divide-upper*:
fixes *q* :: ′*a*::*floor-ceiling*
shows *q* > *0* ⟹ *p* < (*of-int* ⌊*p* / *q*⌋ + *1*) ∗ *q*
by (*meson floor-eq-iff pos-divide-less-eq*)

95.4 Ceiling function

definition *ceiling* :: ′*a*::*floor-ceiling* ⇒ *int* (‹(‹*open-block notation*=‹*mixfix ceiling*››⌈-⌉)›)
where ⌈*x*⌉ = − ⌊− *x*⌋

lemma *ceiling-correct*: *of-int* ⌈*x*⌉ − *1* < *x* ∧ *x* ≤ *of-int* ⌈*x*⌉
unfolding *ceiling-def* **using** *floor-correct* [*of* − *x*]
by (*simp add*: *le-minus-iff*)

lemma *ceiling-unique*: *of-int z* − *1* < *x* ⟹ *x* ≤ *of-int z* ⟹ ⌈*x*⌉ = *z*
unfolding *ceiling-def* **using** *floor-unique* [*of* − *z* − *x*] **by** *simp*

lemma *ceiling-eq-iff*: ⌈*x*⌉ = *a* ⟷ *of-int a* − *1* < *x* ∧ *x* ≤ *of-int a*
using *ceiling-correct ceiling-unique* **by** *auto*

lemma *le-of-int-ceiling* [*simp*]: *x* ≤ *of-int* ⌈*x*⌉
using *ceiling-correct* **..**

lemma *ceiling-le-iff*: ⌈*x*⌉ ≤ *z* ⟷ *x* ≤ *of-int z*
unfolding *ceiling-def* **using** *le-floor-iff* [*of* − *z* − *x*] **by** *auto*

lemma *less-ceiling-iff*: $z < \lceil x \rceil \longleftrightarrow$ *of-int* $z < x$
by (*simp add*: *not-le* [*symmetric*] *ceiling-le-iff*)

lemma *ceiling-less-iff*: $\lceil x \rceil < z \longleftrightarrow x \leq$ *of-int* $z - 1$
using *ceiling-le-iff* [*of x z* $- 1$] **by** *simp*

lemma *le-ceiling-iff*: $z \leq \lceil x \rceil \longleftrightarrow$ *of-int* $z - 1 < x$
by (*simp add*: *not-less* [*symmetric*] *ceiling-less-iff*)

lemma *ceiling-mono*: $x \geq y \Longrightarrow \lceil x \rceil \geq \lceil y \rceil$
unfolding *ceiling-def* **by** (*simp add*: *floor-mono*)

lemma *ceiling-less-cancel*: $\lceil x \rceil < \lceil y \rceil \Longrightarrow x < y$
by (*auto simp add*: *not-le* [*symmetric*] *ceiling-mono*)

lemma *ceiling-of-int* [*simp*]: $\lceil$*of-int* $z\rceil = z$
by (*rule ceiling-unique*) *simp-all*

lemma *ceiling-of-nat* [*simp*]: $\lceil$*of-nat* $n\rceil =$ *int* n
using *ceiling-of-int* [*of of-nat n*] **by** *simp*

lemma *ceiling-add-le*: $\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$
by (*simp only*: *ceiling-le-iff of-int-add add-mono le-of-int-ceiling*)

lemma *mult-ceiling-le*:
assumes $0 \leq a$ **and** $0 \leq b$
shows $\lceil a * b \rceil \leq \lceil a \rceil * \lceil b \rceil$
by (*metis assms ceiling-le-iff ceiling-mono le-of-int-ceiling mult-mono of-int-mult*)

lemma *mult-ceiling-le-Ints*:
assumes $0 \leq a$ $a \in$ *Ints*
shows (*of-int* $\lceil a * b \rceil$ :: $'a$ :: *linordered-idom*) $\leq$ *of-int*($\lceil a \rceil * \lceil b \rceil$)
by (*metis Ints-cases assms ceiling-le-iff ceiling-of-int le-of-int-ceiling mult-left-mono of-int-le-iff of-int-mult*)

lemma *finite-int-segment*:
fixes a :: $'a$::*floor-ceiling*
shows *finite* $\{x \in \mathbb{Z}.\ a \leq x \land x \leq b\}$
proof −
have *finite* {*ceiling a..floor b*}
by *simp*
moreover have $\{x \in \mathbb{Z}.\ a \leq x \land x \leq b\} =$ *of-int* ‘ {*ceiling a..floor b*}
by (*auto simp*: *le-floor-iff ceiling-le-iff elim*!: *Ints-cases*)
ultimately show *?thesis*
by *simp*
qed

corollary *finite-abs-int-segment*:
fixes a :: $'a$::*floor-ceiling*

shows *finite* $\{k \in \mathbb{Z}.\ |k| \le a\}$
using *finite-int-segment* [*of* $-a$ *a*] **by** (*auto simp add*: *abs-le-iff conj-commute minus-le-iff*)

95.4.1 Ceiling with numerals.

lemma *ceiling-zero* [*simp*]: $\lceil 0 \rceil = 0$
using *ceiling-of-int* [*of 0*] **by** *simp*

lemma *ceiling-one* [*simp*]: $\lceil 1 \rceil = 1$
using *ceiling-of-int* [*of 1*] **by** *simp*

lemma *ceiling-numeral* [*simp*]: $\lceil numeral\ v \rceil = numeral\ v$
using *ceiling-of-int* [*of numeral v*] **by** *simp*

lemma *ceiling-neg-numeral* [*simp*]: $\lceil -\ numeral\ v \rceil = -\ numeral\ v$
using *ceiling-of-int* [*of* $-$ *numeral v*] **by** *simp*

lemma *ceiling-le-zero* [*simp*]: $\lceil x \rceil \le 0 \longleftrightarrow x \le 0$
by (*simp add*: *ceiling-le-iff*)

lemma *ceiling-le-one* [*simp*]: $\lceil x \rceil \le 1 \longleftrightarrow x \le 1$
by (*simp add*: *ceiling-le-iff*)

lemma *ceiling-le-numeral* [*simp*]: $\lceil x \rceil \le numeral\ v \longleftrightarrow x \le numeral\ v$
by (*simp add*: *ceiling-le-iff*)

lemma *ceiling-le-neg-numeral* [*simp*]: $\lceil x \rceil \le -\ numeral\ v \longleftrightarrow x \le -\ numeral\ v$
by (*simp add*: *ceiling-le-iff*)

lemma *ceiling-less-zero* [*simp*]: $\lceil x \rceil < 0 \longleftrightarrow x \le -1$
by (*simp add*: *ceiling-less-iff*)

lemma *ceiling-less-one* [*simp*]: $\lceil x \rceil < 1 \longleftrightarrow x \le 0$
by (*simp add*: *ceiling-less-iff*)

lemma *ceiling-less-numeral* [*simp*]: $\lceil x \rceil < numeral\ v \longleftrightarrow x \le numeral\ v - 1$
by (*simp add*: *ceiling-less-iff*)

lemma *ceiling-less-neg-numeral* [*simp*]: $\lceil x \rceil < -\ numeral\ v \longleftrightarrow x \le -\ numeral\ v - 1$
by (*simp add*: *ceiling-less-iff*)

lemma *zero-le-ceiling* [*simp*]: $0 \le \lceil x \rceil \longleftrightarrow -1 < x$
by (*simp add*: *le-ceiling-iff*)

lemma *one-le-ceiling* [*simp*]: $1 \le \lceil x \rceil \longleftrightarrow 0 < x$
by (*simp add*: *le-ceiling-iff*)

lemma *numeral-le-ceiling* [*simp*]: *numeral v* $\leq \lceil x \rceil \longleftrightarrow$ *numeral v* $-$ *1* $< x$
by (*simp add*: *le-ceiling-iff*)

lemma *neg-numeral-le-ceiling* [*simp*]: $-$ *numeral v* $\leq \lceil x \rceil \longleftrightarrow -$ *numeral v* $-$ *1* $< x$
by (*simp add*: *le-ceiling-iff*)

lemma *zero-less-ceiling* [*simp*]: *0* $< \lceil x \rceil \longleftrightarrow$ *0* $< x$
by (*simp add*: *less-ceiling-iff*)

lemma *one-less-ceiling* [*simp*]: *1* $< \lceil x \rceil \longleftrightarrow$ *1* $< x$
by (*simp add*: *less-ceiling-iff*)

lemma *numeral-less-ceiling* [*simp*]: *numeral v* $< \lceil x \rceil \longleftrightarrow$ *numeral v* $< x$
by (*simp add*: *less-ceiling-iff*)

lemma *neg-numeral-less-ceiling* [*simp*]: $-$ *numeral v* $< \lceil x \rceil \longleftrightarrow -$ *numeral v* $< x$
by (*simp add*: *less-ceiling-iff*)

lemma *ceiling-altdef*: $\lceil x \rceil$ = (*if x* = *of-int* $\lfloor x \rfloor$ *then* $\lfloor x \rfloor$ *else* $\lfloor x \rfloor$ + *1*)
by (*intro ceiling-unique*; *simp*, *linarith?*)

lemma *floor-le-ceiling* [*simp*]: $\lfloor x \rfloor \leq \lceil x \rceil$
by (*simp add*: *ceiling-altdef*)

95.4.2 Addition and subtraction of integers.

lemma *ceiling-add-of-int* [*simp*]: $\lceil x$ + *of-int z*$\rceil$ = $\lceil x \rceil$ + *z*
using *ceiling-correct* [*of x*] **by** (*simp add*: *ceiling-def*)

lemma *ceiling-add-numeral* [*simp*]: $\lceil x$ + *numeral v*$\rceil$ = $\lceil x \rceil$ + *numeral v*
using *ceiling-add-of-int* [*of x numeral v*] **by** *simp*

lemma *ceiling-add-one* [*simp*]: $\lceil x$ + *1*$\rceil$ = $\lceil x \rceil$ + *1*
using *ceiling-add-of-int* [*of x 1*] **by** *simp*

lemma *ceiling-diff-of-int* [*simp*]: $\lceil x$ $-$ *of-int z*$\rceil$ = $\lceil x \rceil$ $-$ *z*
using *ceiling-add-of-int* [*of x* $-$ *z*] **by** (*simp add*: *algebra-simps*)

lemma *ceiling-diff-numeral* [*simp*]: $\lceil x$ $-$ *numeral v*$\rceil$ = $\lceil x \rceil$ $-$ *numeral v*
using *ceiling-diff-of-int* [*of x numeral v*] **by** *simp*

lemma *ceiling-diff-one* [*simp*]: $\lceil x$ $-$ *1*$\rceil$ = $\lceil x \rceil$ $-$ *1*
using *ceiling-diff-of-int* [*of x 1*] **by** *simp*

lemma *ceiling-split*[*linarith-split*]: *P* $\lceil t \rceil \longleftrightarrow (\forall\, i.$ *of-int i* $-$ *1* $< t \wedge t \leq$ *of-int i* $\longrightarrow$ *P i*)
by (*auto simp add*: *ceiling-unique ceiling-correct*)

lemma *ceiling-diff-floor-le-1*: $\lceil x\rceil - \lfloor x\rfloor \leq 1$
proof −
  have *of-int* $\lceil x\rceil - 1 < x$
    using *ceiling-correct*[*of x*] **by** *simp*
  also have $x <$ *of-int* $\lfloor x\rfloor + 1$
    using *floor-correct*[*of x*] **by** *simp-all*
  finally have *of-int* $(\lceil x\rceil - \lfloor x\rfloor) <$ (*of-int 2*::'*a*)
    by *simp*
  then show *?thesis*
    unfolding *of-int-less-iff* **by** *simp*
qed

lemma *nat-approx-posE*:
  fixes *e*:: '*a*::{*archimedean-field*,*floor-ceiling*}
  assumes $0 < e$
  obtains *n* :: *nat* **where** *1* / *of-nat*(*Suc n*) < *e*
proof
  have (*1*::'*a*) / *of-nat* (*Suc* (*nat* ⌈*1*/*e*⌉)) < *1* / *of-int* (⌈*1*/*e*⌉)
  proof (*rule divide-strict-left-mono*)
    show (*of-int* ⌈*1* / *e*⌉::'*a*) < *of-nat* (*Suc* (*nat* ⌈*1* / *e*⌉))
      using *assms* **by** (*simp add*: *field-simps*)
    show (*0*::'*a*) < *of-nat* (*Suc* (*nat* ⌈*1* / *e*⌉)) ∗ *of-int* ⌈*1* / *e*⌉
      using *assms* **by** (*auto simp*: *zero-less-mult-iff pos-add-strict*)
  qed *auto*
  also have *1* / *of-int* (⌈*1*/*e*⌉) ≤ *1* / (*1*/*e*)
    by (*rule divide-left-mono*) (*auto simp*: ‹*0* < *e*› *ceiling-correct*)
  also have . . . = *e* **by** *simp*
  finally show *1* / *of-nat* (*Suc* (*nat* ⌈*1* / *e*⌉)) < *e*
    by *metis*
qed

lemma *ceiling-divide-upper*:
  fixes *q* :: '*a*::*floor-ceiling*
  shows $q > 0 \Longrightarrow p \leq$ *of-int* (*ceiling* (*p* / *q*)) ∗ *q*
  by (*meson divide-le-eq le-of-int-ceiling*)

lemma *ceiling-divide-lower*:
  fixes *q* :: '*a*::*floor-ceiling*
  shows $q > 0 \Longrightarrow$ (*of-int* ⌈*p* / *q*⌉ − *1*) ∗ *q* < *p*
  by (*meson ceiling-eq-iff pos-less-divide-eq*)

95.5 Negation

lemma *floor-minus*: $\lfloor - x\rfloor = - \lceil x\rceil$
  unfolding *ceiling-def* **by** *simp*

lemma *ceiling-minus*: $\lceil - x\rceil = - \lfloor x\rfloor$
  unfolding *ceiling-def* **by** *simp*

95.6 Natural numbers

lemma *of-nat-floor*: $r \geq 0 \Longrightarrow$ *of-nat* (*nat* $\lfloor r \rfloor$) $\leq r$
  by *simp*

lemma *of-nat-ceiling*: *of-nat* (*nat* $\lceil r \rceil$) $\geq r$
  by (*cases* $r \geq 0$) *auto*

lemma *of-nat-int-floor* [*simp*]: $x \geq 0 \Longrightarrow$ *of-nat* (*nat*$\lfloor x \rfloor$) = *of-int* $\lfloor x \rfloor$
  by *auto*

lemma *of-nat-int-ceiling* [*simp*]: $x \geq 0 \Longrightarrow$ *of-nat* (*nat* $\lceil x \rceil$) = *of-int* $\lceil x \rceil$
  by *auto*

95.7 Frac Function

definition *frac* :: $'a \Rightarrow 'a$::*floor-ceiling*
  where *frac* $x \equiv x -$ *of-int* $\lfloor x \rfloor$

lemma *frac-lt-1*: *frac* $x < 1$
  by (*simp add*: *frac-def*) *linarith*

lemma *frac-eq-0-iff* [*simp*]: *frac* $x = 0 \longleftrightarrow x \in \mathbb{Z}$
  by (*simp add*: *frac-def*) (*metis Ints-cases Ints-of-int floor-of-int*)

lemma *frac-ge-0* [*simp*]: *frac* $x \geq 0$
  unfolding *frac-def* **by** *linarith*

lemma *frac-gt-0-iff* [*simp*]: *frac* $x > 0 \longleftrightarrow x \notin \mathbb{Z}$
  by (*metis frac-eq-0-iff frac-ge-0 le-less less-irrefl*)

lemma *frac-of-int* [*simp*]: *frac* (*of-int* z) = 0
  by (*simp add*: *frac-def*)

lemma *frac-frac* [*simp*]: *frac* (*frac* x) = *frac* x
  by (*simp add*: *frac-def*)

lemma *floor-add*: $\lfloor x + y \rfloor$ = (*if frac* x + *frac* $y < 1$ *then* $\lfloor x \rfloor + \lfloor y \rfloor$ *else* ($\lfloor x \rfloor$ + $\lfloor y \rfloor$) + 1)
proof −
  have $x + y < 1$ + (*of-int* $\lfloor x \rfloor$ + *of-int* $\lfloor y \rfloor$) $\Longrightarrow \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$
    by (*metis add.commute floor-unique le-floor-add le-floor-iff of-int-add*)
  moreover
  have $\neg$ $x + y < 1$ + (*of-int* $\lfloor x \rfloor$ + *of-int* $\lfloor y \rfloor$) $\Longrightarrow \lfloor x + y \rfloor = 1 + (\lfloor x \rfloor + \lfloor y \rfloor)$
    apply (*simp add*: *floor-eq-iff*)
    apply (*auto simp add*: *algebra-simps*)
    apply *linarith*
    done
  ultimately show *?thesis* **by** (*auto simp add*: *frac-def algebra-simps*)
qed

lemma *floor-add2*[*simp*]: $x \in \mathbb{Z} \vee y \in \mathbb{Z} \Longrightarrow \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$
by (*metis add.commute add.left-neutral frac-lt-1 floor-add frac-eq-0-iff*)

lemma *frac-add*:
frac $(x + y) =$ (*if frac* x + *frac* $y < 1$ *then frac* x + *frac* y *else* (*frac* x + *frac* y) $- 1$)
by (*simp add*: *frac-def floor-add*)

lemma *frac-unique-iff*: *frac* $x = a \longleftrightarrow x - a \in \mathbb{Z} \wedge 0 \leq a \wedge a < 1$
for x :: *'a::floor-ceiling*
apply (*auto simp*: *Ints-def frac-def algebra-simps floor-unique*)
apply *linarith+*
done

lemma *frac-eq*: *frac* $x = x \longleftrightarrow 0 \leq x \wedge x < 1$
by (*simp add*: *frac-unique-iff*)

lemma *frac-neg*: *frac* $(- x) =$ (*if* $x \in \mathbb{Z}$ *then* 0 *else* $1 -$ *frac* x)
for x :: *'a::floor-ceiling*
apply (*auto simp add*: *frac-unique-iff*)
apply (*simp add*: *frac-def*)
apply (*meson frac-lt-1 less-iff-diff-less-0 not-le not-less-iff-gr-or-eq*)
done

lemma *frac-in-Ints-iff* [*simp*]: *frac* $x \in \mathbb{Z} \longleftrightarrow x \in \mathbb{Z}$
proof *safe*
assume *frac* $x \in \mathbb{Z}$
hence *of-int* $\lfloor x \rfloor$ + *frac* $x \in \mathbb{Z}$ **by** *auto*
also have *of-int* $\lfloor x \rfloor$ + *frac* $x = x$ **by** (*simp add*: *frac-def*)
finally show $x \in \mathbb{Z}$ **.**
qed (*auto simp*: *frac-def*)

lemma *frac-1-eq*: *frac* $(x+1)$ = *frac* x
by (*simp add*: *frac-def*)

95.8 Fractional part arithmetic

Many thanks to Stepan Holub

lemma *frac-non-zero*: *frac* $x \neq 0 \Longrightarrow$ *frac* $(-x) = 1 -$ *frac* x
using *frac-eq-0-iff frac-neg* **by** *metis*

lemma *frac-add-simps* [*simp*]:
frac (*frac* $a + b$) = *frac* $(a + b)$
frac (a + *frac* b) = *frac* $(a + b)$
by (*simp-all add*: *frac-add*)

lemma *frac-neg-frac*: *frac* $(-$ *frac* $x)$ = *frac* $(-x)$
unfolding *frac-neg frac-frac* **by** *force*

lemma *frac-diff-simp*: *frac* (*y* − *frac x*) = *frac* (*y* − *x*)
unfolding *diff-conv-add-uminus frac-add frac-neg-frac*..

lemma *frac-diff*: *frac* (*a* − *b*) = *frac* (*frac a* + (− *frac b*))
unfolding *frac-add-simps*(*1*)
unfolding *ab-group-add-class.ab-diff-conv-add-uminus*[*symmetric*] *frac-diff-simp*..

lemma *frac-diff-pos*: *frac x* ≤ *frac y* ⟹ *frac* (*y* − *x*) = *frac y* − *frac x*
unfolding *diff-conv-add-uminus frac-add frac-neg*
using *frac-lt-1* **by** *force*

lemma *frac-diff-neg*: **assumes** *frac y* < *frac x*
shows *frac* (*y* − *x*) = *frac y* + *1* − *frac x*
proof−
have *x* ∉ ℤ
unfolding *frac-gt-0-iff*[*symmetric*]
using *assms frac-ge-0*[*of y*] **by** *order*
have *frac y* + (*1* + − *frac x*) < *1*
using *frac-lt-1*[*of x*] *assms* **by** *fastforce*
show *?thesis*
unfolding *diff-conv-add-uminus frac-add frac-neg*
if-not-P[*OF* ‹*x* ∉ ℤ›] *if-P*[*OF* ‹*frac y* + (*1* + − *frac x*) < *1*›]
by *simp*
qed

lemma *frac-diff-eq*: **assumes** *frac y* = *frac x*
shows *frac* (*y* − *x*) = *0*
by (*simp add*: *assms frac-diff-pos*)

lemma *frac-diff-zero*: **assumes** *frac* (*x* − *y*) = *0*
shows *frac x* = *frac y*
using *frac-add-simps*(*1*)[*of x* − *y y*, *symmetric*]
unfolding *assms add.group-left-neutral diff-add-cancel*.

lemma *frac-neg-eq-iff*: *frac* (−*x*) = *frac* (−*y*) ⟷ *frac x* = *frac y*
using *add.inverse-inverse frac-neg-frac* **by** *metis*

95.9 Rounding to the nearest integer

definition *round* :: ′*a*::*floor-ceiling* ⇒ *int*
where *round x* = ⌊*x* + *1*/*2*⌋

lemma *of-int-round-ge*: *of-int* (*round x*) ≥ *x* − *1*/*2*
and *of-int-round-le*: *of-int* (*round x*) ≤ *x* + *1*/*2*
and *of-int-round-abs-le*: |*of-int* (*round x*) − *x*| ≤ *1*/*2*
and *of-int-round-gt*: *of-int* (*round x*) > *x* − *1*/*2*
proof −
from *floor-correct*[*of x* + *1*/*2*] **have** *x* + *1*/*2* < *of-int* (*round x*) + *1*

by (*simp add*: *round-def*)
from *add-strict-right-mono*[*OF this*, *of* −*1*] **show** *A*: *of-int* (*round x*) > *x* − *1/2*
by *simp*
then show *of-int* (*round x*) ≥ *x* − *1/2*
by *simp*
from *floor-correct*[*of x* + *1/2*] **show** *of-int* (*round x*) ≤ *x* + *1/2*
by (*simp add*: *round-def*)
with *A* **show** |*of-int* (*round x*) − *x*| ≤ *1/2*
by *linarith*
qed

lemma *round-of-int* [*simp*]: *round* (*of-int n*) = *n*
unfolding *round-def* **by** (*subst floor-eq-iff*) *force*

lemma *round-0* [*simp*]: *round 0* = *0*
using *round-of-int*[*of 0*] **by** *simp*

lemma *round-1* [*simp*]: *round 1* = *1*
using *round-of-int*[*of 1*] **by** *simp*

lemma *round-numeral* [*simp*]: *round* (*numeral n*) = *numeral n*
using *round-of-int*[*of numeral n*] **by** *simp*

lemma *round-neg-numeral* [*simp*]: *round* (−*numeral n*) = −*numeral n*
using *round-of-int*[*of* −*numeral n*] **by** *simp*

lemma *round-of-nat* [*simp*]: *round* (*of-nat n*) = *of-nat n*
using *round-of-int*[*of int n*] **by** *simp*

lemma *round-mono*: *x* ≤ *y* ⟹ *round x* ≤ *round y*
unfolding *round-def* **by** (*intro floor-mono*) *simp*

lemma *round-unique*: *of-int y* > *x* − *1/2* ⟹ *of-int y* ≤ *x* + *1/2* ⟹ *round x* = *y*
unfolding *round-def*
proof (*rule floor-unique*)
assume *x* − *1* / *2* < *of-int y*
from *add-strict-left-mono*[*OF this*, *of 1*] **show** *x* + *1* / *2* < *of-int y* + *1*
by *simp*
qed

lemma *round-unique'*: |*x* − *of-int n*| < *1/2* ⟹ *round x* = *n*
by (*subst* (*asm*) *abs-less-iff*, *rule round-unique*) (*simp-all add*: *field-simps*)

lemma *round-altdef*: *round x* = (*if frac x* ≥ *1/2 then* ⌈*x*⌉ *else* ⌊*x*⌋)
by (*cases frac x* ≥ *1/2*)
(*rule round-unique*, ((*simp add*: *frac-def field-simps ceiling-altdef*; *linarith*)+)[*2*])+

lemma *floor-le-round*: ⌊*x*⌋ ≤ *round x*

unfolding *round-def* **by** (*intro floor-mono*) *simp*

lemma *ceiling-ge-round*: $\lceil x \rceil \geq$ *round x*
unfolding *round-altdef* **by** *simp*

lemma *round-diff-minimal*: $|z -$ *of-int* (*round z*)$| \leq |z -$ *of-int m*$|$
for *z* :: *'a::floor-ceiling*
proof (*cases of-int m* $\geq$ *z*)
case *True*
then have $|z -$ *of-int* (*round z*)$| \leq |$*of-int* $\lceil z \rceil - z|$
unfolding *round-altdef* **by** (*simp add*: *field-simps ceiling-altdef frac-def*) *linarith*
also have *of-int* $\lceil z \rceil - z \geq 0$
by *linarith*
with *True* **have** $|$*of-int* $\lceil z \rceil - z| \leq |z -$ *of-int m*$|$
by (*simp add*: *ceiling-le-iff*)
finally show *?thesis* **.**
next
case *False*
then have $|z -$ *of-int* (*round z*)$| \leq |$*of-int* $\lfloor z \rfloor - z|$
unfolding *round-altdef* **by** (*simp add*: *field-simps ceiling-altdef frac-def*) *linarith*
also have $z -$ *of-int* $\lfloor z \rfloor \geq 0$
by *linarith*
with *False* **have** $|$*of-int* $\lfloor z \rfloor - z| \leq |z -$ *of-int m*$|$
by (*simp add*: *le-floor-iff*)
finally show *?thesis* **.**
qed

bundle *floor-ceiling-syntax*
begin
notation *floor* (‹(‹*open-block notation*=‹*mixfix floor*››⌊-⌋)›)
and *ceiling* (‹(‹*open-block notation*=‹*mixfix ceiling*››⌈-⌉)›)
end

end

96 Rational numbers

theory *Rat*
imports *Archimedean-Field*
begin

96.1 Rational numbers as quotient

96.1.1 Construction of the type of rational numbers

definition *ratrel* :: (*int* × *int*) ⇒ (*int* × *int*) ⇒ *bool*
where *ratrel* = (λ*x y*. *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*)

lemma *ratrel-iff* [*simp*]: *ratrel x y* ⟷ *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* =

fst y ∗ *snd x*
  by (*simp add*: *ratrel-def*)

lemma *exists-ratrel-refl*: $\exists x.\ ratrel\ x\ x$
  by (*auto intro*!: *one-neq-zero*)

lemma *symp-ratrel*: *symp ratrel*
  by (*simp add*: *ratrel-def symp-def*)

lemma *transp-ratrel*: *transp ratrel*
proof (*rule transpI*, *unfold split-paired-all*)
  fix $a\ b\ a'\ b'\ a''\ b''$ :: *int*
  assume ∗: *ratrel* $(a, b)\ (a', b')$
  assume ∗∗: *ratrel* $(a', b')\ (a'', b'')$
  have $b' * (a * b'') = b'' * (a * b')$ **by** *simp*
  also from ∗ **have** $a * b' = a' * b$ **by** *auto*
  also have $b'' * (a' * b) = b * (a' * b'')$ **by** *simp*
  also from ∗∗ **have** $a' * b'' = a'' * b'$ **by** *auto*
  also have $b * (a'' * b') = b' * (a'' * b)$ **by** *simp*
  finally have $b' * (a * b'') = b' * (a'' * b)$ **.**
  moreover from ∗∗ **have** $b' \neq 0$ **by** *auto*
  ultimately have $a * b'' = a'' * b$ **by** *simp*
  with ∗ ∗∗ **show** *ratrel* $(a, b)\ (a'', b'')$ **by** *auto*
qed

lemma *part-equivp-ratrel*: *part-equivp ratrel*
  by (*rule part-equivpI* [*OF exists-ratrel-refl symp-ratrel transp-ratrel*])

quotient-type *rat* = *int* × *int* / *partial*: *ratrel*
  morphisms *Rep-Rat Abs-Rat*
  by (*rule part-equivp-ratrel*)

lemma *Domainp-cr-rat* [*transfer-domain-rule*]: *Domainp pcr-rat* = $(\lambda x.\ snd\ x \neq 0)$
  by (*simp add*: *rat.domain-eq*)

96.1.2 Representation and basic operations

lift-definition *Fract* :: *int* ⇒ *int* ⇒ *rat*
  is $\lambda a\ b.\ if\ b = 0\ then\ (0, 1)\ else\ (a, b)$
  by *simp*

lemma *eq-rat*:
  $\bigwedge a\ b\ c\ d.\ b \neq 0 \Longrightarrow d \neq 0 \Longrightarrow Fract\ a\ b = Fract\ c\ d \longleftrightarrow a * d = c * b$
  $\bigwedge a.\ Fract\ a\ 0 = Fract\ 0\ 1$
  $\bigwedge a\ c.\ Fract\ 0\ a = Fract\ 0\ c$
  by (*transfer*, *simp*)+

lemma *Rat-cases* [*case-names Fract*, *cases type*: *rat*]:

assumes *that*: $\bigwedge a\ b.\ q = Fract\ a\ b \Longrightarrow b > 0 \Longrightarrow coprime\ a\ b \Longrightarrow C$
shows *C*
proof −
obtain *a b* :: *int* **where** *q*: $q = Fract\ a\ b$ **and** *b*: $b \neq 0$
by *transfer simp*
let $?a = a\ div\ gcd\ a\ b$
let $?b = b\ div\ gcd\ a\ b$
from *b* **have** $?b * gcd\ a\ b = b$
by *simp*
with *b* **have** $?b \neq 0$
by *fastforce*
with *q b* **have** *q2*: $q = Fract\ ?a\ ?b$
by (*simp add*: *eq-rat dvd-div-mult mult.commute* [*of a*])
from *b* **have** *coprime*: *coprime ?a ?b*
by (*auto intro*: *div-gcd-coprime*)
show *C*
proof (*cases* $b > 0$)
case *True*
then have $?b > 0$
by (*simp add*: *nonneg1-imp-zdiv-pos-iff*)
from *q2 this coprime* **show** *C* **by** (*rule that*)
next
case *False*
have $q = Fract\ (-\ ?a)\ (-\ ?b)$
unfolding *q2* **by** *transfer simp*
moreover from *False b* **have** $-\ ?b > 0$
by (*simp add*: *pos-imp-zdiv-neg-iff*)
moreover from *coprime* **have** *coprime* $(-\ ?a)\ (-\ ?b)$
by *simp*
ultimately show *C*
by (*rule that*)
qed
qed

lemma *Rat-induct* [*case-names Fract*, *induct type*: *rat*]:
assumes $\bigwedge a\ b.\ b > 0 \Longrightarrow coprime\ a\ b \Longrightarrow P\ (Fract\ a\ b)$
shows *P q*
using *assms* **by** (*cases q*) *simp*

instantiation *rat* :: *field*
begin

lift-definition *zero-rat* :: *rat* **is** (*0*, *1*)
by *simp*

lift-definition *one-rat* :: *rat* **is** (*1*, *1*)
by *simp*

lemma *Zero-rat-def*: $0 = Fract\ 0\ 1$

by *transfer simp*

lemma *One-rat-def*: *1* = *Fract 1 1*
by *transfer simp*

lift-definition *plus-rat* :: *rat* ⇒ *rat* ⇒ *rat*
is λ*x y*. (*fst x* ∗ *snd y* + *fst y* ∗ *snd x*, *snd x* ∗ *snd y*)
by (*auto simp*: *distrib-right*) (*simp add*: *ac-simps*)

lemma *add-rat* [*simp*]:
assumes *b* ≠ *0* **and** *d* ≠ *0*
shows *Fract a b* + *Fract c d* = *Fract* (*a* ∗ *d* + *c* ∗ *b*) (*b* ∗ *d*)
using *assms* **by** *transfer simp*

lift-definition *uminus-rat* :: *rat* ⇒ *rat* **is** λ*x*. (− *fst x*, *snd x*)
by *simp*

lemma *minus-rat* [*simp*]: − *Fract a b* = *Fract* (− *a*) *b*
by *transfer simp*

lemma *minus-rat-cancel* [*simp*]: *Fract* (− *a*) (− *b*) = *Fract a b*
by (*cases b* = *0*) (*simp-all add*: *eq-rat*)

definition *diff-rat-def*: *q* − *r* = *q* + − *r* **for** *q r* :: *rat*

lemma *diff-rat* [*simp*]:
b ≠ *0* ⟹ *d* ≠ *0* ⟹ *Fract a b* − *Fract c d* = *Fract* (*a* ∗ *d* − *c* ∗ *b*) (*b* ∗ *d*)
by (*simp add*: *diff-rat-def*)

lift-definition *times-rat* :: *rat* ⇒ *rat* ⇒ *rat*
is λ*x y*. (*fst x* ∗ *fst y*, *snd x* ∗ *snd y*)
by (*simp add*: *ac-simps*)

lemma *mult-rat* [*simp*]: *Fract a b* ∗ *Fract c d* = *Fract* (*a* ∗ *c*) (*b* ∗ *d*)
by *transfer simp*

lemma *mult-rat-cancel*: *c* ≠ *0* ⟹ *Fract* (*c* ∗ *a*) (*c* ∗ *b*) = *Fract a b*
by *transfer simp*

lift-definition *inverse-rat* :: *rat* ⇒ *rat*
is λ*x*. *if fst x* = *0 then* (*0*, *1*) *else* (*snd x*, *fst x*)
by (*auto simp add*: *mult.commute*)

lemma *inverse-rat* [*simp*]: *inverse* (*Fract a b*) = *Fract b a*
by *transfer simp*

definition *divide-rat-def*: *q div r* = *q* ∗ *inverse r* **for** *q r* :: *rat*

lemma *divide-rat* [*simp*]: *Fract a b div Fract c d* = *Fract* (*a* ∗ *d*) (*b* ∗ *c*)

by (*simp add*: *divide-rat-def*)

instance
proof
fix *q r s* :: *rat*
show (*q* ∗ *r*) ∗ *s* = *q* ∗ (*r* ∗ *s*)
by *transfer simp*
show *q* ∗ *r* = *r* ∗ *q*
by *transfer simp*
show *1* ∗ *q* = *q*
by *transfer simp*
show (*q* + *r*) + *s* = *q* + (*r* + *s*)
by *transfer* (*simp add*: *algebra-simps*)
show *q* + *r* = *r* + *q*
by *transfer simp*
show *0* + *q* = *q*
by *transfer simp*
show − *q* + *q* = *0*
by *transfer simp*
show *q* − *r* = *q* + − *r*
by (*fact diff-rat-def*)
show (*q* + *r*) ∗ *s* = *q* ∗ *s* + *r* ∗ *s*
by *transfer* (*simp add*: *algebra-simps*)
show (*0*::*rat*) ≠ *1*
by *transfer simp*
show *inverse q* ∗ *q* = *1* **if** *q* ≠ *0*
using *that* **by** *transfer simp*
show *q div r* = *q* ∗ *inverse r*
by (*fact divide-rat-def*)
show *inverse 0* = (*0*::*rat*)
by *transfer simp*
qed

end

lemma *div-add-self1-no-field* [*simp*]:
assumes *NO-MATCH* (*x* :: *'b* :: *field*) *b* (*b* :: *'a* :: *euclidean-semiring-cancel*) ≠ *0*
shows (*b* + *a*) *div b* = *a div b* + *1*
using *assms*(*2*) **by** (*fact div-add-self1*)

lemma *div-add-self2-no-field* [*simp*]:
assumes *NO-MATCH* (*x* :: *'b* :: *field*) *b* (*b* :: *'a* :: *euclidean-semiring-cancel*) ≠ *0*
shows (*a* + *b*) *div b* = *a div b* + *1*
using *assms*(*2*) **by** (*fact div-add-self2*)

lemma *of-nat-rat*: *of-nat k* = *Fract* (*of-nat k*) *1*

by (*induct k*) (*simp-all add*: *Zero-rat-def One-rat-def*)

lemma *of-int-rat*: *of-int k* = *Fract k 1*
by (*cases k rule*: *int-diff-cases*) (*simp add*: *of-nat-rat*)

lemma *Fract-of-nat-eq*: *Fract* (*of-nat k*) *1* = *of-nat k*
by (*rule of-nat-rat* [*symmetric*])

lemma *Fract-of-int-eq*: *Fract k 1* = *of-int k*
by (*rule of-int-rat* [*symmetric*])

lemma *rat-number-collapse*:
Fract 0 k = *0*
Fract 1 1 = *1*
Fract (*numeral w*) *1* = *numeral w*
Fract (− *numeral w*) *1* = − *numeral w*
Fract (− *1*) *1* = − *1*
Fract k 0 = *0*
using *Fract-of-int-eq* [*of numeral w*]
and *Fract-of-int-eq* [*of* − *numeral w*]
by (*simp-all add*: *Zero-rat-def One-rat-def eq-rat*)

lemma *rat-number-expand*:
0 = *Fract 0 1*
1 = *Fract 1 1*
numeral k = *Fract* (*numeral k*) *1*
− *1* = *Fract* (− *1*) *1*
− *numeral k* = *Fract* (− *numeral k*) *1*
by (*simp-all add*: *rat-number-collapse*)

lemma *Rat-cases-nonzero* [*case-names Fract 0*]:
assumes *Fract*: $\bigwedge a\ b.\ q = Fract\ a\ b \Longrightarrow b > 0 \Longrightarrow a \neq 0 \Longrightarrow coprime\ a\ b \Longrightarrow$
C
and *0*: $q = 0 \Longrightarrow C$
shows *C*
proof (*cases q* = *0*)
case *True*
then show *C* **using** *0* **by** *auto*
next
case *False*
then obtain *a b* **where** ∗: *q* = *Fract a b b* > *0 coprime a b*
by (*cases q*) *auto*
with *False* **have** *0* ≠ *Fract a b*
by *simp*
with ‹*b* > *0*› **have** *a* ≠ *0*
by (*simp add*: *Zero-rat-def eq-rat*)
with *Fract* ∗ **show** *C* **by** *blast*
qed

96.1.3 Function *normalize*

lemma *Fract-coprime*: *Fract* (*a div gcd a b*) (*b div gcd a b*) = *Fract a b*
proof (*cases b* = *0*)
  case *True*
  then show *?thesis*
    by (*simp add*: *eq-rat*)
next
  case *False*
  moreover have *b div gcd a b* ∗ *gcd a b* = *b*
    by (*rule dvd-div-mult-self*) *simp*
  ultimately have *b div gcd a b* ∗ *gcd a b* ≠ *0*
    by *simp*
  then have *b div gcd a b* ≠ *0*
    by *fastforce*
  with *False* **show** *?thesis*
    by (*simp add*: *eq-rat dvd-div-mult mult.commute* [*of a*])
qed

definition *normalize* :: *int* × *int* ⇒ *int* × *int*
  where *normalize p* =
   (*if snd p* > *0 then* (*let a* = *gcd* (*fst p*) (*snd p*) *in* (*fst p div a*, *snd p div a*))
    else if snd p = *0 then* (*0*, *1*)
    else (*let a* = − *gcd* (*fst p*) (*snd p*) *in* (*fst p div a*, *snd p div a*)))

lemma *normalize-crossproduct*:
  assumes *q* ≠ *0 s* ≠ *0*
  assumes *normalize* (*p*, *q*) = *normalize* (*r*, *s*)
  shows *p* ∗ *s* = *r* ∗ *q*
proof −
  have ∗: *p* ∗ *s* = *q* ∗ *r*
    if *p* ∗ *gcd r s* = *sgn* (*q* ∗ *s*) ∗ *r* ∗ *gcd p q* **and** *q* ∗ *gcd r s* = *sgn* (*q* ∗ *s*) ∗ *s* ∗ *gcd p q*
  proof −
    from *that* **have** (*p* ∗ *gcd r s*) ∗ (*sgn* (*q* ∗ *s*) ∗ *s* ∗ *gcd p q*) =
        (*q* ∗ *gcd r s*) ∗ (*sgn* (*q* ∗ *s*) ∗ *r* ∗ *gcd p q*)
      by *simp*
    with *assms* **show** *?thesis*
      by (*auto simp add*: *ac-simps sgn-mult sgn-0-0*)
  qed
  from *assms* **show** *?thesis*
   by (*auto simp*: *normalize-def Let-def dvd-div-div-eq-mult mult.commute sgn-mult*
        split: *if-splits intro*: ∗)
qed

lemma *normalize-eq*: *normalize* (*a*, *b*) = (*p*, *q*) ⟹ *Fract p q* = *Fract a b*
 by (*auto simp*: *normalize-def Let-def Fract-coprime dvd-div-neg rat-number-collapse*
      split: *if-split-asm*)

lemma *normalize-denom-pos*: *normalize r* = (*p*, *q*) ⟹ *q* > *0*

by (*auto simp*: *normalize-def Let-def dvd-div-neg pos-imp-zdiv-neg-iff nonneg1-imp-zdiv-pos-iff split*: *if-split-asm*)

lemma *normalize-coprime*: *normalize r* = (*p*, *q*) $\Longrightarrow$ *coprime p q*
by (*auto simp*: *normalize-def Let-def dvd-div-neg div-gcd-coprime split*: *if-split-asm*)

lemma *normalize-stable* [*simp*]: *q* > *0* $\Longrightarrow$ *coprime p q* $\Longrightarrow$ *normalize* (*p*, *q*) = (*p*, *q*)
by (*simp add*: *normalize-def*)

lemma *normalize-denom-zero* [*simp*]: *normalize* (*p*, *0*) = (*0*, *1*)
by (*simp add*: *normalize-def*)

lemma *normalize-negative* [*simp*]: *q* < *0* $\Longrightarrow$ *normalize* (*p*, *q*) = *normalize* (− *p*, − *q*)
by (*simp add*: *normalize-def Let-def dvd-div-neg dvd-neg-div*)

Decompose a fraction into normalized, i.e. coprime numerator and denominator:

definition *quotient-of* :: *rat* ⇒ *int* × *int*
where *quotient-of x* =
(*THE pair*. *x* = *Fract* (*fst pair*) (*snd pair*) ∧ *snd pair* > *0* ∧ *coprime* (*fst pair*) (*snd pair*))

lemma *quotient-of-unique*: ∃!*p*. *r* = *Fract* (*fst p*) (*snd p*) ∧ *snd p* > *0* ∧ *coprime* (*fst p*) (*snd p*)
proof (*cases r*)
case (*Fract a b*)
then have *r* = *Fract* (*fst* (*a*, *b*)) (*snd* (*a*, *b*)) ∧
snd (*a*, *b*) > *0* ∧ *coprime* (*fst* (*a*, *b*)) (*snd* (*a*, *b*))
by *auto*
then show *?thesis*
proof (*rule ex1I*)
fix *p*
assume *r*: *r* = *Fract* (*fst p*) (*snd p*) ∧ *snd p* > *0* ∧ *coprime* (*fst p*) (*snd p*)
obtain *c d* **where** *p*: *p* = (*c*, *d*) **by** (*cases p*)
with *r* **have** *Fract'*: *r* = *Fract c d d* > *0 coprime c d*
by *simp-all*
have (*c*, *d*) = (*a*, *b*)
proof (*cases a* = *0*)
case *True*
with *Fract Fract'* **show** *?thesis*
by (*simp add*: *eq-rat*)
next
case *False*
with *Fract Fract'* **have** ∗: *c* ∗ *b* = *a* ∗ *d* **and** *c* ≠ *0*
by (*auto simp add*: *eq-rat*)
then have *c* ∗ *b* > *0* ⟷ *a* ∗ *d* > *0*
by *auto*

with ‹$b > 0$› ‹$d > 0$› **have** $a > 0 \longleftrightarrow c > 0$
by (*simp add*: *zero-less-mult-iff*)
with ‹$a \neq 0$› ‹$c \neq 0$› **have** *sgn*: *sgn a* = *sgn c*
by (*auto simp add*: *not-less*)
from ‹*coprime a b*› ‹*coprime c d*› **have** $|a| * |d| = |c| * |b| \longleftrightarrow |a| = |c| \wedge |d| = |b|$
by (*simp add*: *coprime-crossproduct-int*)
with ‹$b > 0$› ‹$d > 0$› **have** $|a| * d = |c| * b \longleftrightarrow |a| = |c| \wedge d = b$
by *simp*
then have $a * sgn\ a * d = c * sgn\ c * b \longleftrightarrow a * sgn\ a = c * sgn\ c \wedge d = b$
by (*simp add*: *abs-sgn*)
with *sgn* ∗ **show** *?thesis*
by (*auto simp add*: *sgn-0-0*)
qed
with *p* **show** $p = (a, b)$
by *simp*
qed
qed

lemma *quotient-of-Fract* [*code*]: *quotient-of* (*Fract a b*) = *normalize* (*a*, *b*)
proof −
have *Fract a b* = *Fract* (*fst* (*normalize* (*a*, *b*))) (*snd* (*normalize* (*a*, *b*))) (**is** *?Fract*)
by (*rule sym*) (*auto intro*: *normalize-eq*)
moreover have $0 <$ *snd* (*normalize* (*a*, *b*)) (**is** *?denom-pos*)
by (*cases normalize* (*a*, *b*)) (*rule normalize-denom-pos*, *simp*)
moreover have *coprime* (*fst* (*normalize* (*a*, *b*))) (*snd* (*normalize* (*a*, *b*))) (**is** *?coprime*)
by (*rule normalize-coprime*) *simp*
ultimately have *?Fract* ∧ *?denom-pos* ∧ *?coprime* **by** *blast*
then have (*THE p*. *Fract a b* = *Fract* (*fst p*) (*snd p*) ∧ $0 <$ *snd p* ∧
coprime (*fst p*) (*snd p*)) = *normalize* (*a*, *b*)
by (*rule the1-equality* [*OF quotient-of-unique*])
then show *?thesis* **by** (*simp add*: *quotient-of-def*)
qed

lemma *quotient-of-number* [*simp*]:
quotient-of $0 = (0, 1)$
quotient-of $1 = (1, 1)$
quotient-of (*numeral k*) = (*numeral k*, *1*)
quotient-of $(- 1) = (- 1, 1)$
quotient-of (− *numeral k*) = (− *numeral k*, *1*)
by (*simp-all add*: *rat-number-expand quotient-of-Fract*)

lemma *quotient-of-eq*: *quotient-of* (*Fract a b*) = (*p*, *q*) ⟹ *Fract p q* = *Fract a b*
by (*simp add*: *quotient-of-Fract normalize-eq*)

lemma *quotient-of-denom-pos*: *quotient-of r* = (*p*, *q*) ⟹ $q > 0$
by (*cases r*) (*simp add*: *quotient-of-Fract normalize-denom-pos*)

lemma *quotient-of-denom-pos'*: *snd* (*quotient-of r*) > *0*
using *quotient-of-denom-pos* [*of r*] **by** (*simp add*: *prod-eq-iff*)

lemma *quotient-of-coprime*: *quotient-of r* = (*p*, *q*) ⟹ *coprime p q*
by (*cases r*) (*simp add*: *quotient-of-Fract normalize-coprime*)

lemma *quotient-of-inject*:
assumes *quotient-of a* = *quotient-of b*
shows *a* = *b*
proof −
obtain *p q r s* **where** *a*: *a* = *Fract p q* **and** *b*: *b* = *Fract r s* **and** *q* > *0* **and** *s* > *0*
by (*cases a*, *cases b*)
with *assms* **show** *?thesis*
by (*simp add*: *eq-rat quotient-of-Fract normalize-crossproduct*)
qed

lemma *quotient-of-inject-eq*: *quotient-of a* = *quotient-of b* ⟷ *a* = *b*
by (*auto simp add*: *quotient-of-inject*)

96.1.4 Various

lemma *Fract-of-int-quotient*: *Fract k l* = *of-int k* / *of-int l*
by (*simp add*: *Fract-of-int-eq* [*symmetric*])

lemma *Fract-add-one*: *n* ≠ *0* ⟹ *Fract* (*m* + *n*) *n* = *Fract m n* + *1*
by (*simp add*: *rat-number-expand*)

lemma *quotient-of-div*:
assumes *r*: *quotient-of r* = (*n*,*d*)
shows *r* = *of-int n* / *of-int d*
proof −
from *theI'*[*OF quotient-of-unique*[*of r*], *unfolded r*[*unfolded quotient-of-def*]]
have *r* = *Fract n d* **by** *simp*
then show *?thesis* **using** *Fract-of-int-quotient*
by *simp*
qed

96.1.5 The ordered field of rational numbers

lift-definition *positive* :: *rat* ⇒ *bool*
is λ*x*. *0* < *fst x* ∗ *snd x*
proof *clarsimp*
fix *a b c d* :: *int*
assume *b* ≠ *0* **and** *d* ≠ *0* **and** *a* ∗ *d* = *c* ∗ *b*
then have *a* ∗ *d* ∗ *b* ∗ *d* = *c* ∗ *b* ∗ *b* ∗ *d*
by *simp*
then have $a * b * d^2 = c * d * b^2$
unfolding *power2-eq-square* **by** (*simp add*: *ac-simps*)

then have $0 < a * b * d^2 \longleftrightarrow 0 < c * d * b^2$
  by *simp*
then show $0 < a * b \longleftrightarrow 0 < c * d$
  using ‹$b \neq 0$› **and** ‹$d \neq 0$›
  by (*simp add*: *zero-less-mult-iff*)
qed

lemma *positive-zero*: ¬ *positive 0*
  by *transfer simp*

lemma *positive-add*: *positive x* ⟹ *positive y* ⟹ *positive* (*x* + *y*)
  apply *transfer*
   apply (*auto simp add*: *zero-less-mult-iff add-pos-pos add-neg-neg mult-pos-neg mult-neg-pos mult-neg-neg*)
  done

lemma *positive-mult*: *positive x* ⟹ *positive y* ⟹ *positive* (*x* ∗ *y*)
  apply *transfer*
  by (*metis fst-conv mult.commute mult-pos-neg2 snd-conv zero-less-mult-iff*)

lemma *positive-minus*: ¬ *positive x* ⟹ $x \neq 0$ ⟹ *positive* (− *x*)
  by *transfer* (*auto simp*: *neq-iff zero-less-mult-iff mult-less-0-iff*)

instantiation *rat* :: *linordered-field*
begin

definition $x < y \longleftrightarrow$ *positive* $(y - x)$

definition $x \leq y \longleftrightarrow x < y \lor x = y$ **for** *x y* :: *rat*

definition $|a| =$ (*if* $a < 0$ *then* $- a$ *else* a) **for** *a* :: *rat*

definition *sgn a* = (*if* $a = 0$ *then 0 else if* $0 < a$ *then 1 else* $- 1$) **for** *a* :: *rat*

instance
proof
  fix *a b c* :: *rat*
  show $|a| =$ (*if* $a < 0$ *then* $- a$ *else* a)
    by (*rule abs-rat-def*)
  show $a < b \longleftrightarrow a \leq b \land \neg b \leq a$
    unfolding *less-eq-rat-def less-rat-def*
    using *positive-add positive-zero* **by** *force*
  show $a \leq a$
    unfolding *less-eq-rat-def* **by** *simp*
  show $a \leq b \Longrightarrow b \leq c \Longrightarrow a \leq c$
    unfolding *less-eq-rat-def less-rat-def*
    using *positive-add* **by** *fastforce*
  show $a \leq b \Longrightarrow b \leq a \Longrightarrow a = b$
    unfolding *less-eq-rat-def less-rat-def*

using *positive-add positive-zero* **by** *fastforce*
show $a \le b \Longrightarrow c + a \le c + b$
unfolding *less-eq-rat-def less-rat-def* **by** *auto*
show $sgn\ a = (if\ a = 0\ then\ 0\ else\ if\ 0 < a\ then\ 1\ else - 1)$
by (*rule sgn-rat-def*)
show $a \le b \vee b \le a$
unfolding *less-eq-rat-def less-rat-def*
by (*auto dest*!: *positive-minus*)
show $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
unfolding *less-rat-def*
by (*metis diff-zero positive-mult right-diff-distrib'*)
qed

end

instantiation *rat* :: *distrib-lattice*
begin

definition $(inf :: rat \Rightarrow rat \Rightarrow rat) = min$

definition $(sup :: rat \Rightarrow rat \Rightarrow rat) = max$

instance
by *standard* (*auto simp add*: *inf-rat-def sup-rat-def max-min-distrib2*)

end

lemma *positive-rat*: $positive\ (Fract\ a\ b) \longleftrightarrow 0 < a * b$
by *transfer simp*

lemma *less-rat* [*simp*]:
$b \ne 0 \Longrightarrow d \ne 0 \Longrightarrow Fract\ a\ b < Fract\ c\ d \longleftrightarrow (a * d) * (b * d) < (c * b) * (b * d)$
by (*simp add*: *less-rat-def positive-rat algebra-simps*)

lemma *le-rat* [*simp*]:
$b \ne 0 \Longrightarrow d \ne 0 \Longrightarrow Fract\ a\ b \le Fract\ c\ d \longleftrightarrow (a * d) * (b * d) \le (c * b) * (b * d)$
by (*simp add*: *le-less eq-rat*)

lemma *abs-rat* [*simp*, *code*]: $|Fract\ a\ b| = Fract\ |a|\ |b|$
by (*auto simp add*: *abs-rat-def zabs-def Zero-rat-def not-less le-less eq-rat zero-less-mult-iff*)

lemma *sgn-rat* [*simp*, *code*]: $sgn\ (Fract\ a\ b) = of\text{-}int\ (sgn\ a * sgn\ b)$
unfolding *Fract-of-int-eq*
by (*auto simp*: *zsgn-def sgn-rat-def Zero-rat-def eq-rat*)
(*auto simp*: *rat-number-collapse not-less le-less zero-less-mult-iff*)

lemma *Rat-induct-pos* [*case-names Fract*, *induct type*: *rat*]:

assumes *step*: $\bigwedge a\ b.\ 0 < b \Longrightarrow P\ (Fract\ a\ b)$
shows $P\ q$
proof (*cases q*)
case (*Fract a b*)
have *step'*: $P\ (Fract\ a\ b)$ **if** *b*: $b < 0$ **for** $a\ b :: int$
proof −
from *b* **have** $0 < -\ b$
by *simp*
then have $P\ (Fract\ (-\ a)\ (-\ b))$
by (*rule step*)
then show $P\ (Fract\ a\ b)$
by (*simp add*: *order-less-imp-not-eq* [*OF b*])
qed
from *Fract* **show** $P\ q$
by (*auto simp add*: *linorder-neq-iff step step'*)
qed

lemma *zero-less-Fract-iff*: $0 < b \Longrightarrow 0 < Fract\ a\ b \longleftrightarrow 0 < a$
by (*simp add*: *Zero-rat-def zero-less-mult-iff*)

lemma *Fract-less-zero-iff*: $0 < b \Longrightarrow Fract\ a\ b < 0 \longleftrightarrow a < 0$
by (*simp add*: *Zero-rat-def mult-less-0-iff*)

lemma *zero-le-Fract-iff*: $0 < b \Longrightarrow 0 \leq Fract\ a\ b \longleftrightarrow 0 \leq a$
by (*simp add*: *Zero-rat-def zero-le-mult-iff*)

lemma *Fract-le-zero-iff*: $0 < b \Longrightarrow Fract\ a\ b \leq 0 \longleftrightarrow a \leq 0$
by (*simp add*: *Zero-rat-def mult-le-0-iff*)

lemma *one-less-Fract-iff*: $0 < b \Longrightarrow 1 < Fract\ a\ b \longleftrightarrow b < a$
by (*simp add*: *One-rat-def mult-less-cancel-right-disj*)

lemma *Fract-less-one-iff*: $0 < b \Longrightarrow Fract\ a\ b < 1 \longleftrightarrow a < b$
by (*simp add*: *One-rat-def mult-less-cancel-right-disj*)

lemma *one-le-Fract-iff*: $0 < b \Longrightarrow 1 \leq Fract\ a\ b \longleftrightarrow b \leq a$
by (*simp add*: *One-rat-def mult-le-cancel-right*)

lemma *Fract-le-one-iff*: $0 < b \Longrightarrow Fract\ a\ b \leq 1 \longleftrightarrow a \leq b$
by (*simp add*: *One-rat-def mult-le-cancel-right*)

96.1.6 Rationals are an Archimedean field

lemma *rat-floor-lemma*: $of\text{-}int\ (a\ div\ b) \leq Fract\ a\ b \wedge Fract\ a\ b < of\text{-}int\ (a\ div\ b + 1)$
proof −
have $Fract\ a\ b = of\text{-}int\ (a\ div\ b) + Fract\ (a\ mod\ b)\ b$
by (*cases* $b = 0$) (*simp*, *simp add*: *of-int-rat*)
moreover have $0 \leq Fract\ (a\ mod\ b)\ b \wedge Fract\ (a\ mod\ b)\ b < 1$

unfolding *Fract-of-int-quotient*
by (*rule linorder-cases* [*of b 0*]) (*simp-all add*: *divide-nonpos-neg*)
ultimately show *?thesis* **by** *simp*
qed

instance *rat* :: *archimedean-field*
proof
show $\exists z.\ r \leq$ *of-int* z **for** r :: *rat*
proof (*induct r*)
case (*Fract a b*)
have *Fract a b* $\leq$ *of-int* (a *div* $b + 1$)
using *rat-floor-lemma* [*of a b*] **by** *simp*
then show $\exists z.$ *Fract a b* $\leq$ *of-int* z **..**
qed
qed

instantiation *rat* :: *floor-ceiling*
begin

definition *floor-rat* :: *rat* $\Rightarrow$ *int*
where$\lfloor x \rfloor = (THE\ z.\ of\text{-}int\ z \leq x \wedge x < of\text{-}int\ (z + 1))$ **for** x :: *rat*

instance
proof
show *of-int* $\lfloor x \rfloor \leq x \wedge x <$ *of-int* ($\lfloor x \rfloor + 1$) **for** x :: *rat*
unfolding *floor-rat-def* **using** *floor-exists1* **by** (*rule theI′*)
qed

end

lemma *floor-Fract* [*simp*]: $\lfloor Fract\ a\ b \rfloor = a\ div\ b$
by (*simp add*: *Fract-of-int-quotient floor-divide-of-int-eq*)

96.2 Linear arithmetic setup

declaration ‹
K (*Lin-Arith.add-inj-thms* @{*thms of-int-le-iff* [*THEN iffD2*] *of-int-eq-iff* [*THEN iffD2*]}
(∗ *not needed because* $x <$ (y::*int*) *can be rewritten as* $x + 1$ <= y: *of-int-less-iff RS iffD2* ∗)
#> *Lin-Arith.add-inj-const* (***const-name*** ‹*of-nat*›, ***typ*** ‹*nat* ⇒ *rat*›)
#> *Lin-Arith.add-inj-const* (***const-name*** ‹*of-int*›, ***typ*** ‹*int* ⇒ *rat*›))
›

96.3 Embedding from Rationals to other Fields

context *field-char-0*
begin

lift-definition *of-rat* :: *rat* ⇒ *′a*

is $\lambda x.$ *of-int* (*fst x*) / *of-int* (*snd x*)
by (*auto simp*: *nonzero-divide-eq-eq nonzero-eq-divide-eq*) (*simp only*: *of-int-mult* [*symmetric*])

end

lemma *of-rat-rat*: $b \neq 0 \Longrightarrow$ *of-rat* (*Fract a b*) = *of-int a* / *of-int b*
by *transfer simp*

lemma *of-rat-0* [*simp*]: *of-rat 0* = *0*
by *transfer simp*

lemma *of-rat-1* [*simp*]: *of-rat 1* = *1*
by *transfer simp*

lemma *of-rat-add*: *of-rat* (*a* + *b*) = *of-rat a* + *of-rat b*
by *transfer* (*simp add*: *add-frac-eq*)

lemma *of-rat-minus*: *of-rat* (− *a*) = − *of-rat a*
by *transfer simp*

lemma *of-rat-neg-one* [*simp*]: *of-rat* (− *1*) = − *1*
by (*simp add*: *of-rat-minus*)

lemma *of-rat-diff*: *of-rat* (*a* − *b*) = *of-rat a* − *of-rat b*
using *of-rat-add* [*of a* − *b*] **by** (*simp add*: *of-rat-minus*)

lemma *of-rat-mult*: *of-rat* (*a* ∗ *b*) = *of-rat a* ∗ *of-rat b*
by *transfer* (*simp add*: *divide-inverse nonzero-inverse-mult-distrib ac-simps*)

lemma *of-rat-sum*: *of-rat* ($\sum a \in A.$ *f a*) = ($\sum a \in A.$ *of-rat* (*f a*))
by (*induct rule*: *infinite-finite-induct*) (*auto simp*: *of-rat-add*)

lemma *of-rat-prod*: *of-rat* ($\prod a \in A.$ *f a*) = ($\prod a \in A.$ *of-rat* (*f a*))
by (*induct rule*: *infinite-finite-induct*) (*auto simp*: *of-rat-mult*)

lemma *nonzero-of-rat-inverse*: $a \neq 0 \Longrightarrow$ *of-rat* (*inverse a*) = *inverse* (*of-rat a*)
by (*rule inverse-unique* [*symmetric*]) (*simp add*: *of-rat-mult* [*symmetric*])

lemma *of-rat-inverse*: (*of-rat* (*inverse a*) :: *'a::field-char-0*) = *inverse* (*of-rat a*)
by (*cases a* = *0*) (*simp-all add*: *nonzero-of-rat-inverse*)

lemma *nonzero-of-rat-divide*: $b \neq 0 \Longrightarrow$ *of-rat* (*a* / *b*) = *of-rat a* / *of-rat b*
by (*simp add*: *divide-inverse of-rat-mult nonzero-of-rat-inverse*)

lemma *of-rat-divide*: (*of-rat* (*a* / *b*) :: *'a::field-char-0*) = *of-rat a* / *of-rat b*
by (*cases b* = *0*) (*simp-all add*: *nonzero-of-rat-divide*)

lemma *of-rat-power*: (*of-rat* (*a* ^ *n*) :: *'a::field-char-0*) = *of-rat a* ^ *n*

by (*induct n*) (*simp-all add*: *of-rat-mult*)

lemma *of-rat-eq-iff* [*simp*]: *of-rat a* = *of-rat b* ⟷ *a* = *b*
apply *transfer*
apply (*simp add*: *nonzero-divide-eq-eq nonzero-eq-divide-eq flip*: *of-int-mult*)
done

lemma *of-rat-eq-0-iff* [*simp*]: *of-rat a* = *0* ⟷ *a* = *0*
using *of-rat-eq-iff* [*of - 0*] **by** *simp*

lemma *zero-eq-of-rat-iff* [*simp*]: *0* = *of-rat a* ⟷ *0* = *a*
by *simp*

lemma *of-rat-eq-1-iff* [*simp*]: *of-rat a* = *1* ⟷ *a* = *1*
using *of-rat-eq-iff* [*of - 1*] **by** *simp*

lemma *one-eq-of-rat-iff* [*simp*]: *1* = *of-rat a* ⟷ *1* = *a*
by *simp*

lemma *of-rat-less*: (*of-rat r* :: *'a*::*linordered-field*) < *of-rat s* ⟷ *r* < *s*
proof (*induct r*, *induct s*)
fix *a b c d* :: *int*
assume *not-zero*: *b* > *0 d* > *0*
then have *b* ∗ *d* > *0* **by** *simp*
have *of-int-divide-less-eq*:
(*of-int a* :: *'a*) / *of-int b* < *of-int c* / *of-int d* ⟷
(*of-int a* :: *'a*) ∗ *of-int d* < *of-int c* ∗ *of-int b*
using *not-zero* **by** (*simp add*: *pos-less-divide-eq pos-divide-less-eq*)
show (*of-rat* (*Fract a b*) :: *'a*::*linordered-field*) < *of-rat* (*Fract c d*) ⟷
Fract a b < *Fract c d*
using *not-zero* ‹*b* ∗ *d* > *0*›
by (*simp add*: *of-rat-rat of-int-divide-less-eq of-int-mult* [*symmetric*] *del*: *of-int-mult*)
qed

lemma *of-rat-less-eq*: (*of-rat r* :: *'a*::*linordered-field*) ≤ *of-rat s* ⟷ *r* ≤ *s*
unfolding *le-less* **by** (*auto simp add*: *of-rat-less*)

lemma *of-rat-le-0-iff* [*simp*]: (*of-rat r* :: *'a*::*linordered-field*) ≤ *0* ⟷ *r* ≤ *0*
using *of-rat-less-eq* [*of r 0*, **where** *'a* = *'a*] **by** *simp*

lemma *zero-le-of-rat-iff* [*simp*]: *0* ≤ (*of-rat r* :: *'a*::*linordered-field*) ⟷ *0* ≤ *r*
using *of-rat-less-eq* [*of 0 r*, **where** *'a* = *'a*] **by** *simp*

lemma *of-rat-le-1-iff* [*simp*]: (*of-rat r* :: *'a*::*linordered-field*) ≤ *1* ⟷ *r* ≤ *1*
using *of-rat-less-eq* [*of r 1*] **by** *simp*

lemma *one-le-of-rat-iff* [*simp*]: *1* ≤ (*of-rat r* :: *'a*::*linordered-field*) ⟷ *1* ≤ *r*
using *of-rat-less-eq* [*of 1 r*] **by** *simp*

lemma *of-rat-less-0-iff* [*simp*]: (*of-rat r* :: *'a::linordered-field*) $<$ *0* $\longleftrightarrow$ *r* $<$ *0*
using *of-rat-less* [*of r 0*, **where** *'a* = *'a*] **by** *simp*

lemma *zero-less-of-rat-iff* [*simp*]: *0* $<$ (*of-rat r* :: *'a::linordered-field*) $\longleftrightarrow$ *0* $<$ *r*
using *of-rat-less* [*of 0 r*, **where** *'a* = *'a*] **by** *simp*

lemma *of-rat-less-1-iff* [*simp*]: (*of-rat r* :: *'a::linordered-field*) $<$ *1* $\longleftrightarrow$ *r* $<$ *1*
using *of-rat-less* [*of r 1*] **by** *simp*

lemma *one-less-of-rat-iff* [*simp*]: *1* $<$ (*of-rat r* :: *'a::linordered-field*) $\longleftrightarrow$ *1* $<$ *r*
using *of-rat-less* [*of 1 r*] **by** *simp*

lemma *of-rat-eq-id* [*simp*]: *of-rat* = *id*
proof
show *of-rat a* = *id a* **for** *a*
by (*induct a*) (*simp add*: *of-rat-rat Fract-of-int-eq* [*symmetric*])
qed

lemma *abs-of-rat* [*simp*]:
|*of-rat r*| = (*of-rat* |*r*| :: *'a* :: *linordered-field*)
by (*cases r* $\geq$ *0*) (*simp-all add*: *not-le of-rat-minus*)

Collapse nested embeddings.

lemma *of-rat-of-nat-eq* [*simp*]: *of-rat* (*of-nat n*) = *of-nat n*
by (*induct n*) (*simp-all add*: *of-rat-add*)

lemma *of-rat-of-int-eq* [*simp*]: *of-rat* (*of-int z*) = *of-int z*
by (*cases z rule*: *int-diff-cases*) (*simp add*: *of-rat-diff*)

lemma *of-rat-numeral-eq* [*simp*]: *of-rat* (*numeral w*) = *numeral w*
using *of-rat-of-int-eq* [*of numeral w*] **by** *simp*

lemma *of-rat-neg-numeral-eq* [*simp*]: *of-rat* (− *numeral w*) = − *numeral w*
using *of-rat-of-int-eq* [*of* − *numeral w*] **by** *simp*

lemma *of-rat-floor* [*simp*]:
$\lfloor$*of-rat r*$\rfloor$ = $\lfloor$*r*$\rfloor$
by (*cases r*) (*simp add*: *Fract-of-int-quotient of-rat-divide floor-divide-of-int-eq*)

lemma *of-rat-ceiling* [*simp*]:
$\lceil$*of-rat r*$\rceil$ = $\lceil$*r*$\rceil$
using *of-rat-floor* [*of* − *r*] **by** (*simp add*: *of-rat-minus ceiling-def*)

lemmas *zero-rat* = *Zero-rat-def*
lemmas *one-rat* = *One-rat-def*

abbreviation *rat-of-nat* :: *nat* ⇒ *rat*
where *rat-of-nat* ≡ *of-nat*

abbreviation *rat-of-int* :: *int* ⇒ *rat*
  where *rat-of-int* ≡ *of-int*

96.4 The Set of Rational Numbers

context *field-char-0*
begin

definition *Rats* :: *'a set* (‹ℚ›)
  where ℚ = *range of-rat*

end

lemma *Rats-cases* [*cases set*: *Rats*]:
  assumes *q* ∈ ℚ
  obtains (*of-rat*) *r* **where** *q* = *of-rat r*
proof −
  from ‹*q* ∈ ℚ› **have** *q* ∈ *range of-rat*
    by (*simp only*: *Rats-def*)
  then obtain *r* **where** *q* = *of-rat r* **..**
  then show *thesis* **..**
qed

lemma *Rats-cases'*:
  assumes (*x* :: *'a* :: *field-char-0*) ∈ ℚ
  obtains *a b* **where** *b* > *0* *coprime a b* *x* = *of-int a* / *of-int b*
proof −
  from *assms* **obtain** *r* **where** *x* = *of-rat r*
    by (*auto simp*: *Rats-def*)
  obtain *a b* **where** *quot*: *quotient-of r* = (*a*,*b*) **by** *force*
  have *b* > *0* **using** *quotient-of-denom-pos*[*OF quot*] **.**
  moreover have *coprime a b* **using** *quotient-of-coprime*[*OF quot*] **.**
  moreover have *x* = *of-int a* / *of-int b* **unfolding** ‹*x* = *of-rat r*›
      quotient-of-div[*OF quot*] **by** (*simp add*: *of-rat-divide*)
  ultimately show *?thesis* **using** *that* **by** *blast*
qed

lemma *Rats-of-rat* [*simp*]: *of-rat r* ∈ ℚ
  by (*simp add*: *Rats-def*)

lemma *Rats-of-int* [*simp*]: *of-int z* ∈ ℚ
  by (*subst of-rat-of-int-eq* [*symmetric*]) (*rule Rats-of-rat*)

lemma *Ints-subset-Rats*: ℤ ⊆ ℚ
  using *Ints-cases Rats-of-int* **by** *blast*

lemma *Rats-of-nat* [*simp*]: *of-nat n* ∈ ℚ
  by (*subst of-rat-of-nat-eq* [*symmetric*]) (*rule Rats-of-rat*)

lemma *Nats-subset-Rats*: $\mathbb{N} \subseteq \mathbb{Q}$
using *Ints-subset-Rats Nats-subset-Ints* **by** *blast*

lemma *Rats-number-of* [*simp*]: *numeral w* $\in \mathbb{Q}$
by (*subst of-rat-numeral-eq* [*symmetric*]) (*rule Rats-of-rat*)

lemma *Rats-0* [*simp*]: *0* $\in \mathbb{Q}$
unfolding *Rats-def* **by** (*rule range-eqI*) (*rule of-rat-0* [*symmetric*])

lemma *Rats-1* [*simp*]: *1* $\in \mathbb{Q}$
unfolding *Rats-def* **by** (*rule range-eqI*) (*rule of-rat-1* [*symmetric*])

lemma *Rats-add* [*simp*]: $a \in \mathbb{Q} \Longrightarrow b \in \mathbb{Q} \Longrightarrow a + b \in \mathbb{Q}$
by (*metis Rats-cases Rats-of-rat of-rat-add*)

lemma *Rats-minus-iff* [*simp*]: $- a \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q}$
by (*metis Rats-cases Rats-of-rat add.inverse-inverse of-rat-minus*)

lemma *Rats-diff* [*simp*]: $a \in \mathbb{Q} \Longrightarrow b \in \mathbb{Q} \Longrightarrow a - b \in \mathbb{Q}$
by (*metis Rats-add Rats-minus-iff diff-conv-add-uminus*)

lemma *Rats-mult* [*simp*]: $a \in \mathbb{Q} \Longrightarrow b \in \mathbb{Q} \Longrightarrow a * b \in \mathbb{Q}$
by (*metis Rats-cases Rats-of-rat of-rat-mult*)

lemma *Rats-inverse* [*simp*]: $a \in \mathbb{Q} \Longrightarrow$ *inverse a* $\in \mathbb{Q}$
for *a* :: *'a::field-char-0*
by (*metis Rats-cases Rats-of-rat of-rat-inverse*)

lemma *Rats-divide* [*simp*]: $a \in \mathbb{Q} \Longrightarrow b \in \mathbb{Q} \Longrightarrow a / b \in \mathbb{Q}$
for *a b* :: *'a::field-char-0*
by (*simp add*: *divide-inverse*)

lemma *Rats-power* [*simp*]: $a \in \mathbb{Q} \Longrightarrow a \,\widehat{}\, n \in \mathbb{Q}$
for *a* :: *'a::field-char-0*
by (*metis Rats-cases Rats-of-rat of-rat-power*)

lemma *Rats-sum* [*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow f\,x \in \mathbb{Q}) \Longrightarrow$ *sum f A* $\in \mathbb{Q}$
by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *Rats-prod* [*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow f\,x \in \mathbb{Q}) \Longrightarrow$ *prod f A* $\in \mathbb{Q}$
by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *Rats-induct* [*case-names of-rat*, *induct set*: *Rats*]: $q \in \mathbb{Q} \Longrightarrow (\bigwedge r.\ P$ (*of-rat r*)) $\Longrightarrow P\ q$
by (*rule Rats-cases*) *auto*

lemma *Rats-infinite*: $\neg$ *finite* $\mathbb{Q}$
by (*auto dest*!: *finite-imageD simp*: *inj-on-def infinite-UNIV-char-0 Rats-def*)

lemma *Rats-add-iff*: $a \in \mathbb{Q} \vee b \in \mathbb{Q} \Longrightarrow a+b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
by (*metis Rats-add Rats-diff add-diff-cancel add-diff-cancel-left'*)

lemma *Rats-diff-iff*: $a \in \mathbb{Q} \vee b \in \mathbb{Q} \Longrightarrow a-b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
by (*metis Rats-add-iff diff-add-cancel*)

lemma *Rats-mult-iff*: $a \in \mathbb{Q}-\{0\} \vee b \in \mathbb{Q}-\{0\} \Longrightarrow a*b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
by (*metis Diff-iff Rats-divide Rats-mult insertI1 mult.commute nonzero-divide-eq-eq*)

lemma *Rats-inverse-iff* [*simp*]: *inverse* $a \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q}$
using *Rats-inverse* **by** *force*

lemma *Rats-divide-iff*: $a \in \mathbb{Q}-\{0\} \vee b \in \mathbb{Q}-\{0\} \Longrightarrow a/b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in$
$\mathbb{Q}$
by (*metis Rats-divide Rats-mult-iff divide-eq-0-iff divide-inverse nonzero-mult-div-cancel-right*)

96.5 Implementation of rational numbers as pairs of integers

Formal constructor

definition *Frct* :: *int* × *int* ⇒ *rat*
where [*simp*]: *Frct p* = *Fract* (*fst p*) (*snd p*)

lemma [*code abstype*]: *Frct* (*quotient-of q*) = *q*
by (*cases q*) (*auto intro*: *quotient-of-eq*)

Numerals

declare *quotient-of-Fract* [*code abstract*]

definition *of-int* :: *int* ⇒ *rat*
where [*code-abbrev*]: *of-int* = *Int.of-int*

hide-const (**open**) *of-int*

lemma *quotient-of-int* [*code abstract*]: *quotient-of* (*Rat.of-int a*) = (*a*, *1*)
by (*simp add*: *of-int-def of-int-rat quotient-of-Fract*)

lemma [*code-unfold*]: *numeral k* = *Rat.of-int* (*numeral k*)
by (*simp add*: *Rat.of-int-def*)

lemma [*code-unfold*]: − *numeral k* = *Rat.of-int* (− *numeral k*)
by (*simp add*: *Rat.of-int-def*)

lemma *Frct-code-post* [*code-post*]:
Frct (*0*, *a*) = *0*
Frct (*a*, *0*) = *0*
Frct (*1*, *1*) = *1*
Frct (*numeral k*, *1*) = *numeral k*
Frct (*1*, *numeral k*) = *1* / *numeral k*
Frct (*numeral k*, *numeral l*) = *numeral k* / *numeral l*

```
  Frct (− a, b) = − Frct (a, b)
  Frct (a, − b) = − Frct (a, b)
  − (− Frct q) = Frct q
  by (simp-all add: Fract-of-int-quotient)
```

Operations

```
lemma rat-zero-code [code abstract]: quotient-of 0 = (0, 1)
  by (simp add: Zero-rat-def quotient-of-Fract normalize-def)

lemma rat-one-code [code abstract]: quotient-of 1 = (1, 1)
  by (simp add: One-rat-def quotient-of-Fract normalize-def)

lemma rat-plus-code [code abstract]:
  quotient-of (p + q) = (let (a, c) = quotient-of p; (b, d) = quotient-of q
     in normalize (a ∗ d + b ∗ c, c ∗ d))
  by (cases p, cases q) (simp add: quotient-of-Fract)

lemma rat-uminus-code [code abstract]:
  quotient-of (− p) = (let (a, b) = quotient-of p in (− a, b))
  by (cases p) (simp add: quotient-of-Fract)

lemma rat-minus-code [code abstract]:
  quotient-of (p − q) =
    (let (a, c) = quotient-of p; (b, d) = quotient-of q
     in normalize (a ∗ d − b ∗ c, c ∗ d))
  by (cases p, cases q) (simp add: quotient-of-Fract)

lemma rat-times-code [code abstract]:
  quotient-of (p ∗ q) =
    (let (a, c) = quotient-of p; (b, d) = quotient-of q
     in normalize (a ∗ b, c ∗ d))
  by (cases p, cases q) (simp add: quotient-of-Fract)

lemma rat-inverse-code [code abstract]:
  quotient-of (inverse p) =
    (let (a, b) = quotient-of p
     in if a = 0 then (0, 1) else (sgn a ∗ b, |a|))
proof (cases p)
  case (Fract a b)
  then show ?thesis
    by (cases 0::int a rule: linorder-cases) (simp-all add: quotient-of-Fract ac-simps)
qed

lemma rat-divide-code [code abstract]:
  quotient-of (p / q) =
    (let (a, c) = quotient-of p; (b, d) = quotient-of q
     in normalize (a ∗ d, c ∗ b))
  by (cases p, cases q) (simp add: quotient-of-Fract)
```

lemma *rat-abs-code* [*code abstract*]:
quotient-of $|p|$ = (*let* (*a*, *b*) = *quotient-of p in* ($|a|$, *b*))
by (*cases p*) (*simp add*: *quotient-of-Fract*)

lemma *rat-sgn-code* [*code abstract*]: *quotient-of* (*sgn p*) = (*sgn* (*fst* (*quotient-of p*)), *1*)
proof (*cases p*)
case (*Fract a b*)
then show *?thesis*
by (*cases 0::int a rule*: *linorder-cases*) (*simp-all add*: *quotient-of-Fract*)
qed

lemma *rat-floor-code* [*code*]: $\lfloor p \rfloor$ = (*let* (*a*, *b*) = *quotient-of p in a div b*)
by (*cases p*) (*simp add*: *quotient-of-Fract floor-Fract*)

instantiation *rat* :: *equal*
begin

definition [*code*]: *HOL.equal a b* ⟷ *quotient-of a* = *quotient-of b*

instance
by *standard* (*simp add*: *equal-rat-def quotient-of-inject-eq*)

lemma *rat-eq-refl* [*code nbe*]: *HOL.equal* (*r*::*rat*) *r* ⟷ *True*
by (*rule equal-refl*)

end

lemma *rat-less-eq-code* [*code*]:
$p \le q$ ⟷ (*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q in* $a * d \le c * b$)
by (*cases p*, *cases q*) (*simp add*: *quotient-of-Fract mult.commute*)

lemma *rat-less-code* [*code*]:
$p < q$ ⟷ (*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q in* $a * d < c * b$)
by (*cases p*, *cases q*) (*simp add*: *quotient-of-Fract mult.commute*)

lemma [*code*]: *of-rat p* = (*let* (*a*, *b*) = *quotient-of p in of-int a* / *of-int b*)
by (*cases p*) (*simp add*: *quotient-of-Fract of-rat-rat*)

Quickcheck

context
includes *term-syntax*
begin

definition
valterm-fract :: *int* × (*unit* ⇒ *Code-Evaluation.term*) ⇒
int × (*unit* ⇒ *Code-Evaluation.term*) ⇒
rat × (*unit* ⇒ *Code-Evaluation.term*)
where [*code-unfold*]: *valterm-fract k l* = *Code-Evaluation.valtermify Fract* {·} *k*

{·} *l*

end

instantiation *rat* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
Quickcheck-Random.random i =
Quickcheck-Random.random i ∘→ (λ*num. Random.range i* ∘→ (λ*denom. Pair*
(*let j* = *int-of-integer* (*integer-of-natural* (*denom* + *1*))
in valterm-fract num (*j*, λ*u. Code-Evaluation.term-of j*))))

instance ..

end

end

instantiation *rat* :: *exhaustive*
begin

definition
exhaustive-rat f d =
Quickcheck-Exhaustive.exhaustive
(λ*l. Quickcheck-Exhaustive.exhaustive*
(λ*k. f* (*Fract k* (*int-of-integer* (*integer-of-natural l*) + *1*))) *d*) *d*

instance ..

end

instantiation *rat* :: *full-exhaustive*
begin

definition
full-exhaustive-rat f d =
Quickcheck-Exhaustive.full-exhaustive
(λ(*l*, -). *Quickcheck-Exhaustive.full-exhaustive*
(λ*k. f*
(*let j* = *int-of-integer* (*integer-of-natural l*) + *1*
in valterm-fract k (*j*, λ-. *Code-Evaluation.term-of j*))) *d*) *d*

instance ..

end

instance *rat* :: *partial-term-of* **..**

lemma [*code*]:
partial-term-of (*ty* :: *rat itself*) (*Quickcheck-Narrowing.Narrowing-variable p tt*) ≡
Code-Evaluation.Free (*STR ''-''*) (*Typerep.Typerep* (*STR ''Rat.rat''*) [])
partial-term-of (*ty* :: *rat itself*) (*Quickcheck-Narrowing.Narrowing-constructor 0* [*l*, *k*]) ≡
Code-Evaluation.App
(*Code-Evaluation.Const* (*STR ''Rat.Frct''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.Typerep* (*STR ''Product-Type.prod''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [], *Typerep.Typerep* (*STR ''Int.int''*) []],
Typerep.Typerep (*STR ''Rat.rat''*) []]))
(*Code-Evaluation.App*
(*Code-Evaluation.App*
(*Code-Evaluation.Const* (*STR ''Product-Type.Pair''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [],
Typerep.Typerep (*STR ''fun''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [],
Typerep.Typerep (*STR ''Product-Type.prod''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [], *Typerep.Typerep* (*STR ''Int.int''*) []]]]]))
(*partial-term-of* (*TYPE*(*int*)) *l*)) (*partial-term-of* (*TYPE*(*int*)) *k*))
by (*rule partial-term-of-anything*)+

instantiation *rat* :: *narrowing*
begin

definition
narrowing =
Quickcheck-Narrowing.apply
(*Quickcheck-Narrowing.apply*
(*Quickcheck-Narrowing.cons* (λ*nom denom. Fract nom denom*)) *narrowing*)
narrowing

instance ..

end

96.6 Setup for Nitpick

declaration ‹
Nitpick-HOL.register-frac-type ***type-name*** ‹*rat*›
[(***const-name*** ‹*Abs-Rat*›, ***const-name*** ‹*Nitpick.Abs-Frac*›),
(***const-name*** ‹*zero-rat-inst.zero-rat*›, ***const-name*** ‹*Nitpick.zero-frac*›),

(**const-name** ‹*one-rat-inst.one-rat*›, **const-name** ‹*Nitpick.one-frac*›),
(**const-name** ‹*plus-rat-inst.plus-rat*›, **const-name** ‹*Nitpick.plus-frac*›),
(**const-name** ‹*times-rat-inst.times-rat*›, **const-name** ‹*Nitpick.times-frac*›),
(**const-name** ‹*uminus-rat-inst.uminus-rat*›, **const-name** ‹*Nitpick.uminus-frac*›),
(**const-name** ‹*inverse-rat-inst.inverse-rat*›, **const-name** ‹*Nitpick.inverse-frac*›),
(**const-name** ‹*ord-rat-inst.less-rat*›, **const-name** ‹*Nitpick.less-frac*›),
(**const-name** ‹*ord-rat-inst.less-eq-rat*›, **const-name** ‹*Nitpick.less-eq-frac*›),
(**const-name** ‹*field-char-0-class.of-rat*›, **const-name** ‹*Nitpick.of-frac*›)]
›

lemmas [*nitpick-unfold*] =
inverse-rat-inst.inverse-rat
one-rat-inst.one-rat ord-rat-inst.less-rat
ord-rat-inst.less-eq-rat plus-rat-inst.plus-rat times-rat-inst.times-rat
uminus-rat-inst.uminus-rat zero-rat-inst.zero-rat

96.7 Float syntax

syntax *-Float* :: *float-const* ⇒ *'a* (‹(‹*open-block notation*=‹*literal number*››-)›)

parse-translation ‹
let
fun mk-frac str =
let
val {*mant* = *i*, *exp* = *n*} = *Lexicon.read-float str*;
val exp = *Syntax.const* **const-syntax** ‹*Power.power*›;
val ten = *Numeral.mk-number-syntax 10*;
val exp10 = *if n* = *1 then ten else exp* $ *ten* $ *Numeral.mk-number-syntax n*;
in Syntax.const **const-syntax** ‹*Fields.inverse-divide*› $ *Numeral.mk-number-syntax i* $ *exp10 end*;

fun float-tr [(*c as Const* (**syntax-const** ‹*-constrain*›, -)) $ *t* $ *u*] = *c* $ *float-tr* [*t*] $ *u*
| *float-tr* [*t as Const* (*str*, -)] = *mk-frac str*
| *float-tr ts* = *raise TERM* (*float-tr*, *ts*);
in [(**syntax-const** ‹*-Float*›, *K float-tr*)] *end*
›

Test:

lemma $123.456 = -111.111 + 200 + 30 + 4 + 5/10 + 6/100 + (7/1000::rat)$
by *simp*

96.8 Hiding implementation details

hide-const (**open**) *normalize positive*

lifting-update *rat.lifting*
lifting-forget *rat.lifting*

end

97 Development of the Reals using Cauchy Sequences

theory *Real*
imports *Rat*
begin

This theory contains a formalization of the real numbers as equivalence classes of Cauchy sequences of rationals. See the AFP entry *Dedekind-Real* for an alternative construction using Dedekind cuts.

97.1 Preliminary lemmas

Useful in convergence arguments

lemma *inverse-of-nat-le*:
fixes $n::nat$ **shows** $[\![n \leq m;\ n\neq 0]\!] \Longrightarrow 1\ /\ of\text{-}nat\ m \leq (1::'a::linordered\text{-}field)\ /\ of\text{-}nat\ n$
by (*simp add*: *frac-le*)

lemma *add-diff-add*: $(a + c) - (b + d) = (a - b) + (c - d)$
for $a\ b\ c\ d :: 'a::ab\text{-}group\text{-}add$
by *simp*

lemma *minus-diff-minus*: $-\ a - -\ b = -\ (a - b)$
for $a\ b :: 'a::ab\text{-}group\text{-}add$
by *simp*

lemma *mult-diff-mult*: $(x * y - a * b) = x * (y - b) + (x - a) * b$
for $x\ y\ a\ b :: 'a::ring$
by (*simp add*: *algebra-simps*)

lemma *inverse-diff-inverse*:
fixes $a\ b :: 'a::division\text{-}ring$
assumes $a \neq 0$ **and** $b \neq 0$
shows $inverse\ a - inverse\ b = -\ (inverse\ a * (a - b) * inverse\ b)$
using *assms* **by** (*simp add*: *algebra-simps*)

lemma *obtain-pos-sum*:
fixes $r :: rat$ **assumes** r: $0 < r$
obtains $s\ t$ **where** $0 < s$ **and** $0 < t$ **and** $r = s + t$
proof
from r **show** $0 < r/2$ **by** *simp*
from r **show** $0 < r/2$ **by** *simp*
show $r = r/2 + r/2$ **by** *simp*

qed

97.2 Sequences that converge to zero

definition *vanishes* :: (*nat* ⇒ *rat*) ⇒ *bool*
 where *vanishes X* ⟷ (∀ *r*>*0*. ∃ *k*. ∀ *n*≥*k*. |*X n*| < *r*)

lemma *vanishesI*: (⋀*r*. *0* < *r* ⟹ ∃ *k*. ∀ *n*≥*k*. |*X n*| < *r*) ⟹ *vanishes X*
 unfolding *vanishes-def* **by** *simp*

lemma *vanishesD*: *vanishes X* ⟹ *0* < *r* ⟹ ∃ *k*. ∀ *n*≥*k*. |*X n*| < *r*
 unfolding *vanishes-def* **by** *simp*

lemma *vanishes-const* [*simp*]: *vanishes* (λ*n*. *c*) ⟷ *c* = *0*
proof (*cases c* = *0*)
 case *True*
 then show *?thesis*
  by (*simp add*: *vanishesI*)
next
 case *False*
 then show *?thesis*
  unfolding *vanishes-def*
  using *zero-less-abs-iff* **by** *blast*
qed

lemma *vanishes-minus*: *vanishes X* ⟹ *vanishes* (λ*n*. − *X n*)
 unfolding *vanishes-def* **by** *simp*

lemma *vanishes-add*:
 assumes *X*: *vanishes X*
  and *Y*: *vanishes Y*
 shows *vanishes* (λ*n*. *X n* + *Y n*)
proof (*rule vanishesI*)
 fix *r* :: *rat*
 assume *0* < *r*
 then obtain *s t* **where** *s*: *0* < *s* **and** *t*: *0* < *t* **and** *r*: *r* = *s* + *t*
  by (*rule obtain-pos-sum*)
 obtain *i* **where** *i*: ∀ *n*≥*i*. |*X n*| < *s*
  using *vanishesD* [*OF X s*] **..**
 obtain *j* **where** *j*: ∀ *n*≥*j*. |*Y n*| < *t*
  using *vanishesD* [*OF Y t*] **..**
 have ∀ *n*≥*max i j*. |*X n* + *Y n*| < *r*
 proof *clarsimp*
  fix *n*
  assume *n*: *i* ≤ *n j* ≤ *n*
  have |*X n* + *Y n*| ≤ |*X n*| + |*Y n*|
   by (*rule abs-triangle-ineq*)
  also have . . . < *s* + *t*
   by (*simp add*: *add-strict-mono i j n*)

finally show $|X\ n\ +\ Y\ n|\ <\ r$
by (*simp only*: *r*)
qed
then show $\exists\, k.\ \forall\, n{\ge}k.\ |X\ n\ +\ Y\ n|\ <\ r$ **..**
qed

lemma *vanishes-diff*:
assumes *vanishes X vanishes Y*
shows *vanishes* $(\lambda n.\ X\ n\ -\ Y\ n)$
unfolding *diff-conv-add-uminus* **by** (*intro vanishes-add vanishes-minus assms*)

lemma *vanishes-mult-bounded*:
assumes *X*: $\exists\, a{>}0.\ \forall\, n.\ |X\ n|\ <\ a$
assumes *Y*: *vanishes* $(\lambda n.\ Y\ n)$
shows *vanishes* $(\lambda n.\ X\ n\ *\ Y\ n)$
proof (*rule vanishesI*)
fix *r* :: *rat*
assume *r*: $0\ <\ r$
obtain *a* **where** *a*: $0\ <\ a\ \forall\, n.\ |X\ n|\ <\ a$
using *X* **by** *blast*
obtain *b* **where** *b*: $0\ <\ b\ r\ =\ a\ *\ b$
proof
show $0\ <\ r\ /\ a$ **using** *r a* **by** *simp*
show $r\ =\ a\ *\ (r\ /\ a)$ **using** *a* **by** *simp*
qed
obtain *k* **where** *k*: $\forall\, n{\ge}k.\ |Y\ n|\ <\ b$
using *vanishesD* [*OF Y b*(*1*)] **..**
have $\forall\, n{\ge}k.\ |X\ n\ *\ Y\ n|\ <\ r$
by (*simp add*: *b*(*2*) *abs-mult mult-strict-mono′ a k*)
then show $\exists\, k.\ \forall\, n{\ge}k.\ |X\ n\ *\ Y\ n|\ <\ r$ **..**
qed

97.3 Cauchy sequences

definition *cauchy* :: $(nat \Rightarrow rat) \Rightarrow bool$
where *cauchy X* $\longleftrightarrow (\forall\, r{>}0.\ \exists\, k.\ \forall\, m{\ge}k.\ \forall\, n{\ge}k.\ |X\ m\ -\ X\ n|\ <\ r)$

lemma *cauchyI*: $(\bigwedge r.\ 0\ <\ r \Longrightarrow \exists\, k.\ \forall\, m{\ge}k.\ \forall\, n{\ge}k.\ |X\ m\ -\ X\ n|\ <\ r) \Longrightarrow$ *cauchy X*
unfolding *cauchy-def* **by** *simp*

lemma *cauchyD*: *cauchy X* $\Longrightarrow 0\ <\ r \Longrightarrow \exists\, k.\ \forall\, m{\ge}k.\ \forall\, n{\ge}k.\ |X\ m\ -\ X\ n|\ <\ r$
unfolding *cauchy-def* **by** *simp*

lemma *cauchy-const* [*simp*]: *cauchy* $(\lambda n.\ x)$
unfolding *cauchy-def* **by** *simp*

lemma *cauchy-add* [*simp*]:
assumes *X*: *cauchy X* **and** *Y*: *cauchy Y*

```
  shows cauchy (λn. X n + Y n)
proof (rule cauchyI)
  fix r :: rat
  assume 0 < r
  then obtain s t where s: 0 < s and t: 0 < t and r: r = s + t
    by (rule obtain-pos-sum)
  obtain i where i: ∀m≥i. ∀n≥i. |X m − X n| < s
    using cauchyD [OF X s] ..
  obtain j where j: ∀m≥j. ∀n≥j. |Y m − Y n| < t
    using cauchyD [OF Y t] ..
  have ∀m≥max i j. ∀n≥max i j. |(X m + Y m) − (X n + Y n)| < r
  proof clarsimp
    fix m n
    assume *: i ≤ m j ≤ m i ≤ n j ≤ n
    have |(X m + Y m) − (X n + Y n)| ≤ |X m − X n| + |Y m − Y n|
      unfolding add-diff-add by (rule abs-triangle-ineq)
    also have ... < s + t
      by (rule add-strict-mono) (simp-all add: i j *)
    finally show |(X m + Y m) − (X n + Y n)| < r by (simp only: r)
  qed
  then show ∃k. ∀m≥k. ∀n≥k. |(X m + Y m) − (X n + Y n)| < r ..
qed

lemma cauchy-minus [simp]:
  assumes X: cauchy X
  shows cauchy (λn. − X n)
  using assms unfolding cauchy-def
  unfolding minus-diff-minus abs-minus-cancel .

lemma cauchy-diff [simp]:
  assumes cauchy X cauchy Y
  shows cauchy (λn. X n − Y n)
 using assms unfolding diff-conv-add-uminus by (simp del: add-uminus-conv-diff)

lemma cauchy-imp-bounded:
  assumes cauchy X
  shows ∃b>0. ∀n. |X n| < b
proof −
  obtain k where k: ∀m≥k. ∀n≥k. |X m − X n| < 1
    using cauchyD [OF assms zero-less-one] ..
  show ∃b>0. ∀n. |X n| < b
  proof (intro exI conjI allI)
    have 0 ≤ |X 0| by simp
    also have |X 0| ≤ Max (abs ‘ X ‘ {..k}) by simp
    finally have 0 ≤ Max (abs ‘ X ‘ {..k}) .
    then show 0 < Max (abs ‘ X ‘ {..k}) + 1 by simp
  next
    fix n :: nat
    show |X n| < Max (abs ‘ X ‘ {..k}) + 1
```

```
  proof (rule linorder-le-cases)
    assume n ≤ k
    then have |X n| ≤ Max (abs ` X ` {..k}) by simp
    then show |X n| < Max (abs ` X ` {..k}) + 1 by simp
  next
    assume k ≤ n
    have |X n| = |X k + (X n − X k)| by simp
    also have |X k + (X n − X k)| ≤ |X k| + |X n − X k|
      by (rule abs-triangle-ineq)
    also have ... < Max (abs ` X ` {..k}) + 1
      by (rule add-le-less-mono) (simp-all add: k ‹k ≤ n›)
    finally show |X n| < Max (abs ` X ` {..k}) + 1 .
  qed
 qed
qed

lemma cauchy-mult [simp]:
 assumes X: cauchy X and Y: cauchy Y
 shows cauchy (λn. X n ∗ Y n)
proof (rule cauchyI)
 fix r :: rat assume 0 < r
 then obtain u v where u: 0 < u and v: 0 < v and r = u + v
  by (rule obtain-pos-sum)
 obtain a where a: 0 < a ∀ n. |X n| < a
  using cauchy-imp-bounded [OF X] by blast
 obtain b where b: 0 < b ∀ n. |Y n| < b
  using cauchy-imp-bounded [OF Y] by blast
 obtain s t where s: 0 < s and t: 0 < t and r: r = a ∗ t + s ∗ b
 proof
  show 0 < v/b using v b(1) by simp
  show 0 < u/a using u a(1) by simp
  show r = a ∗ (u/a) + (v/b) ∗ b
   using a(1) b(1) ‹r = u + v› by simp
 qed
 obtain i where i: ∀ m≥i. ∀ n≥i. |X m − X n| < s
  using cauchyD [OF X s] ..
 obtain j where j: ∀ m≥j. ∀ n≥j. |Y m − Y n| < t
  using cauchyD [OF Y t] ..
 have ∀ m≥max i j. ∀ n≥max i j. |X m ∗ Y m − X n ∗ Y n| < r
 proof clarsimp
  fix m n
  assume ∗: i ≤ m j ≤ m i ≤ n j ≤ n
  have |X m ∗ Y m − X n ∗ Y n| = |X m ∗ (Y m − Y n) + (X m − X n) ∗ Y
n|
    unfolding mult-diff-mult ..
  also have ... ≤ |X m ∗ (Y m − Y n)| + |(X m − X n) ∗ Y n|
    by (rule abs-triangle-ineq)
  also have ... = |X m| ∗ |Y m − Y n| + |X m − X n| ∗ |Y n|
    unfolding abs-mult ..
```

```
    also have ... < a ∗ t + s ∗ b
      by (simp-all add: add-strict-mono mult-strict-mono' a b i j ∗)
    finally show |X m ∗ Y m − X n ∗ Y n| < r
      by (simp only: r)
  qed
  then show ∃ k. ∀ m≥k. ∀ n≥k. |X m ∗ Y m − X n ∗ Y n| < r ..
qed

lemma cauchy-not-vanishes-cases:
  assumes X: cauchy X
  assumes nz: ¬ vanishes X
  shows ∃ b>0. ∃ k. (∀ n≥k. b < − X n) ∨ (∀ n≥k. b < X n)
proof −
  obtain r where 0 < r and r: ∀ k. ∃ n≥k. r ≤ |X n|
    using nz unfolding vanishes-def by (auto simp add: not-less)
  obtain s t where s: 0 < s and t: 0 < t and r = s + t
    using ‹0 < r› by (rule obtain-pos-sum)
  obtain i where i: ∀ m≥i. ∀ n≥i. |X m − X n| < s
    using cauchyD [OF X s] ..
  obtain k where i ≤ k and r ≤ |X k|
    using r by blast
  have k: ∀ n≥k. |X n − X k| < s
    using i ‹i ≤ k› by auto
  have X k ≤ − r ∨ r ≤ X k
    using ‹r ≤ |X k|› by auto
  then have (∀ n≥k. t < − X n) ∨ (∀ n≥k. t < X n)
    unfolding ‹r = s + t› using k by auto
  then have ∃ k. (∀ n≥k. t < − X n) ∨ (∀ n≥k. t < X n) ..
  then show ∃ t>0. ∃ k. (∀ n≥k. t < − X n) ∨ (∀ n≥k. t < X n)
    using t by auto
qed

lemma cauchy-not-vanishes:
  assumes X: cauchy X
    and nz: ¬ vanishes X
  shows ∃ b>0. ∃ k. ∀ n≥k. b < |X n|
  using cauchy-not-vanishes-cases [OF assms]
  by (elim ex-forward conj-forward asm-rl) auto

lemma cauchy-inverse [simp]:
  assumes X: cauchy X
    and nz: ¬ vanishes X
  shows cauchy (λn. inverse (X n))
proof (rule cauchyI)
  fix r :: rat
  assume 0 < r
  obtain b i where b: 0 < b and i: ∀ n≥i. b < |X n|
    using cauchy-not-vanishes [OF X nz] by blast
  from b i have nz: ∀ n≥i. X n ≠ 0 by auto
```

```
  obtain s where s: 0 < s and r: r = inverse b * s * inverse b
  proof
    show 0 < b * r * b by (simp add: ‹0 < r› b)
    show r = inverse b * (b * r * b) * inverse b
      using b by simp
  qed
  obtain j where j: ∀m≥j. ∀n≥j. |X m − X n| < s
    using cauchyD [OF X s] ..
  have ∀m≥max i j. ∀n≥max i j. |inverse (X m) − inverse (X n)| < r
  proof clarsimp
    fix m n
    assume *: i ≤ m j ≤ m i ≤ n j ≤ n
    have |inverse (X m) − inverse (X n)| = inverse |X m| * |X m − X n| * inverse
|X n|
      by (simp add: inverse-diff-inverse nz * abs-mult)
    also have ... < inverse b * s * inverse b
      by (simp add: mult-strict-mono less-imp-inverse-less i j b * s)
    finally show |inverse (X m) − inverse (X n)| < r by (simp only: r)
  qed
  then show ∃k. ∀m≥k. ∀n≥k. |inverse (X m) − inverse (X n)| < r ..
qed

lemma vanishes-diff-inverse:
  assumes X: cauchy X ¬ vanishes X
    and Y: cauchy Y ¬ vanishes Y
    and XY: vanishes (λn. X n − Y n)
  shows vanishes (λn. inverse (X n) − inverse (Y n))
proof (rule vanishesI)
  fix r :: rat
  assume r: 0 < r
  obtain a i where a: 0 < a and i: ∀n≥i. a < |X n|
    using cauchy-not-vanishes [OF X] by blast
  obtain b j where b: 0 < b and j: ∀n≥j. b < |Y n|
    using cauchy-not-vanishes [OF Y] by blast
  obtain s where s: 0 < s and inverse a * s * inverse b = r
  proof
    show 0 < a * r * b
      using a r b by simp
    show inverse a * (a * r * b) * inverse b = r
      using a r b by simp
  qed
  obtain k where k: ∀n≥k. |X n − Y n| < s
    using vanishesD [OF XY s] ..
  have ∀n≥max (max i j) k. |inverse (X n) − inverse (Y n)| < r
  proof clarsimp
    fix n
    assume n: i ≤ n j ≤ n k ≤ n
    with i j a b have X n ≠ 0 and Y n ≠ 0
      by auto
```

then have $|inverse\ (X\ n) - inverse\ (Y\ n)| = inverse\ |X\ n| * |X\ n - Y\ n| * inverse\ |Y\ n|$
by (*simp add*: *inverse-diff-inverse abs-mult*)
also have ... $< inverse\ a * s * inverse\ b$
by (*intro mult-strict-mono′ less-imp-inverse-less*) (*simp-all add*: *a b i j k n*)
also note ‹$inverse\ a * s * inverse\ b = r$›
finally show $|inverse\ (X\ n) - inverse\ (Y\ n)| < r$ **.**
qed
then show $\exists k.\ \forall n \geq k.\ |inverse\ (X\ n) - inverse\ (Y\ n)| < r$ **..**
qed

97.4 Equivalence relation on Cauchy sequences

definition *realrel* :: $(nat \Rightarrow rat) \Rightarrow (nat \Rightarrow rat) \Rightarrow bool$
where $realrel = (\lambda X\ Y.\ cauchy\ X \wedge cauchy\ Y \wedge vanishes\ (\lambda n.\ X\ n - Y\ n))$

lemma *realrelI* [*intro?*]: $cauchy\ X \Longrightarrow cauchy\ Y \Longrightarrow vanishes\ (\lambda n.\ X\ n - Y\ n) \Longrightarrow realrel\ X\ Y$
by (*simp add*: *realrel-def*)

lemma *realrel-refl*: $cauchy\ X \Longrightarrow realrel\ X\ X$
by (*simp add*: *realrel-def*)

lemma *symp-realrel*: *symp realrel*
by (*simp add*: *abs-minus-commute realrel-def symp-def vanishes-def*)

lemma *transp-realrel*: *transp realrel*
unfolding *realrel-def*
by (*rule transpI*) (*force simp add*: *dest*: *vanishes-add*)

lemma *part-equivp-realrel*: *part-equivp realrel*
by (*blast intro*: *part-equivpI symp-realrel transp-realrel realrel-refl cauchy-const*)

97.5 The field of real numbers

quotient-type $real = nat \Rightarrow rat$ / *partial*: *realrel*
morphisms *rep-real Real*
by (*rule part-equivp-realrel*)

lemma *cr-real-eq*: $pcr\text{-}real = (\lambda x\ y.\ cauchy\ x \wedge Real\ x = y)$
unfolding *real.pcr-cr-eq cr-real-def realrel-def* **by** *auto*

lemma *Real-induct* [*induct type*: *real*]:
assumes $\bigwedge X.\ cauchy\ X \Longrightarrow P\ (Real\ X)$
shows $P\ x$
proof (*induct x*)
case (*1 X*)
then have *cauchy X* **by** (*simp add*: *realrel-def*)
then show $P\ (Real\ X)$ **by** (*rule assms*)
qed

lemma *eq-Real*: *cauchy X* $\Longrightarrow$ *cauchy Y* $\Longrightarrow$ *Real X* = *Real Y* $\longleftrightarrow$ *vanishes* (λ*n*. *X n* − *Y n*)
using *real.rel-eq-transfer*
unfolding *real.pcr-cr-eq cr-real-def rel-fun-def realrel-def* **by** *simp*

lemma *Domainp-pcr-real* [*transfer-domain-rule*]: *Domainp pcr-real* = *cauchy*
by (*simp add*: *real.domain-eq realrel-def*)

instantiation *real* :: *field*
begin

lift-definition *zero-real* :: *real* **is** λ*n*. *0*
by (*simp add*: *realrel-refl*)

lift-definition *one-real* :: *real* **is** λ*n*. *1*
by (*simp add*: *realrel-refl*)

lift-definition *plus-real* :: *real* ⇒ *real* ⇒ *real* **is** λ*X Y n*. *X n* + *Y n*
unfolding *realrel-def add-diff-add*
by (*simp only*: *cauchy-add vanishes-add simp-thms*)

lift-definition *uminus-real* :: *real* ⇒ *real* **is** λ*X n*. − *X n*
unfolding *realrel-def minus-diff-minus*
by (*simp only*: *cauchy-minus vanishes-minus simp-thms*)

lift-definition *times-real* :: *real* ⇒ *real* ⇒ *real* **is** λ*X Y n*. *X n* ∗ *Y n*
proof −
fix *f1 f2 f3 f4*
have ⟦*cauchy f1*; *cauchy f4*; *vanishes* (λ*n*. *f1 n* − *f2 n*); *vanishes* (λ*n*. *f3 n* − *f4 n*)⟧
$\Longrightarrow$ *vanishes* (λ*n*. *f1 n* ∗ (*f3 n* − *f4 n*) + *f4 n* ∗ (*f1 n* − *f2 n*))
by (*simp add*: *vanishes-add vanishes-mult-bounded cauchy-imp-bounded*)
then show ⟦*realrel f1 f2*; *realrel f3 f4*⟧ $\Longrightarrow$ *realrel* (λ*n*. *f1 n* ∗ *f3 n*) (λ*n*. *f2 n* ∗ *f4 n*)
by (*simp add*: *mult.commute realrel-def mult-diff-mult*)
qed

lift-definition *inverse-real* :: *real* ⇒ *real*
is λ*X*. *if vanishes X then* (λ*n*. *0*) *else* (λ*n*. *inverse* (*X n*))
proof −
fix *X Y*
assume *realrel X Y*
then have *X*: *cauchy X* **and** *Y*: *cauchy Y* **and** *XY*: *vanishes* (λ*n*. *X n* − *Y n*)
by (*simp-all add*: *realrel-def*)
have *vanishes X* $\longleftrightarrow$ *vanishes Y*
proof
assume *vanishes X*
from *vanishes-diff* [*OF this XY*] **show** *vanishes Y*

by *simp*
next
assume *vanishes Y*
from *vanishes-add* [*OF this XY*] **show** *vanishes X*
by *simp*
qed
then show *?thesis X Y*
by (*simp add*: *vanishes-diff-inverse X Y XY realrel-def*)
qed

definition $x - y = x + - y$ **for** $x\ y$ :: *real*

definition *x div y* = $x * $ *inverse y* **for** $x\ y$ :: *real*

lemma *add-Real*: *cauchy X* ⟹ *cauchy Y* ⟹ *Real X* + *Real Y* = *Real* (λ*n*. *X n* + *Y n*)
using *plus-real.transfer* **by** (*simp add*: *cr-real-eq rel-fun-def*)

lemma *minus-Real*: *cauchy X* ⟹ − *Real X* = *Real* (λ*n*. − *X n*)
using *uminus-real.transfer* **by** (*simp add*: *cr-real-eq rel-fun-def*)

lemma *diff-Real*: *cauchy X* ⟹ *cauchy Y* ⟹ *Real X* − *Real Y* = *Real* (λ*n*. *X n* − *Y n*)
by (*simp add*: *minus-Real add-Real minus-real-def*)

lemma *mult-Real*: *cauchy X* ⟹ *cauchy Y* ⟹ *Real X* ∗ *Real Y* = *Real* (λ*n*. *X n* ∗ *Y n*)
using *times-real.transfer* **by** (*simp add*: *cr-real-eq rel-fun-def*)

lemma *inverse-Real*:
cauchy X ⟹ *inverse* (*Real X*) = (*if vanishes X then 0 else Real* (λ*n*. *inverse* (*X n*)))
using *inverse-real.transfer zero-real.transfer*
unfolding *cr-real-eq rel-fun-def* **by** (*simp split*: *if-split-asm*, *metis*)

instance
proof
fix $a\ b\ c$ :: *real*
show $a + b = b + a$
by *transfer* (*simp add*: *ac-simps realrel-def*)
show $(a + b) + c = a + (b + c)$
by *transfer* (*simp add*: *ac-simps realrel-def*)
show $0 + a = a$
by *transfer* (*simp add*: *realrel-def*)
show $- a + a = 0$
by *transfer* (*simp add*: *realrel-def*)
show $a - b = a + - b$
by (*rule minus-real-def*)
show $(a * b) * c = a * (b * c)$

by *transfer* (*simp add*: *ac-simps realrel-def*)
show $a * b = b * a$
by *transfer* (*simp add*: *ac-simps realrel-def*)
show $1 * a = a$
by *transfer* (*simp add*: *ac-simps realrel-def*)
show $(a + b) * c = a * c + b * c$
by *transfer* (*simp add*: *distrib-right realrel-def*)
show $(0::real) \neq (1::real)$
by *transfer* (*simp add*: *realrel-def*)
have *vanishes* $(\lambda n.\ inverse\ (X\ n) * X\ n - 1)$ **if** *X*: *cauchy X* ¬ *vanishes X* **for** *X*
proof (*rule vanishesI*)
fix *r*::*rat*
assume $0 < r$
obtain *b k* **where** $b>0$ $\forall n \geq k.\ b < |X\ n|$
using *X cauchy-not-vanishes* **by** *blast*
then show $\exists k.\ \forall n \geq k.\ |inverse\ (X\ n) * X\ n - 1| < r$
using ‹$0 < r$› **by** *force*
qed
then show $a \neq 0 \Longrightarrow inverse\ a * a = 1$
by *transfer* (*simp add*: *realrel-def*)
show *a div b* = *a* ∗ *inverse b*
by (*rule divide-real-def*)
show *inverse* $(0::real) = 0$
by *transfer* (*simp add*: *realrel-def*)
qed

end

97.6 Positive reals

lift-definition *positive* :: *real* ⇒ *bool*
is $\lambda X.\ \exists r>0.\ \exists k.\ \forall n\geq k.\ r < X\ n$
proof −
have *1*: $\exists r>0.\ \exists k.\ \forall n\geq k.\ r < Y\ n$
if ∗: *realrel X Y* **and** ∗∗: $\exists r>0.\ \exists k.\ \forall n\geq k.\ r < X\ n$ **for** *X Y*
proof −
from ∗ **have** *XY*: *vanishes* $(\lambda n.\ X\ n - Y\ n)$
by (*simp-all add*: *realrel-def*)
from ∗∗ **obtain** *r i* **where** $0 < r$ **and** *i*: $\forall n\geq i.\ r < X\ n$
by *blast*
obtain *s t* **where** *s*: $0 < s$ **and** *t*: $0 < t$ **and** *r*: $r = s + t$
using ‹$0 < r$› **by** (*rule obtain-pos-sum*)
obtain *j* **where** *j*: $\forall n\geq j.\ |X\ n - Y\ n| < s$
using *vanishesD* [*OF XY s*] **..**
have $\forall n\geq max\ i\ j.\ t < Y\ n$
proof *clarsimp*
fix *n*
assume *n*: $i \leq n$ $j \leq n$

have $|X\ n - Y\ n| < s$ **and** $r < X\ n$
using *i j n* **by** *simp-all*
then show $t < Y\ n$ **by** (*simp add*: *r*)
qed
then show *?thesis* **using** *t* **by** *blast*
qed
fix *X Y* **assume** *realrel X Y*
then have *realrel X Y* **and** *realrel Y X*
using *symp-realrel* **by** (*auto simp*: *symp-def*)
then show *?thesis X Y*
by (*safe elim*!: *1*)
qed

lemma *positive-Real*: *cauchy X* $\Longrightarrow$ *positive* (*Real X*) $\longleftrightarrow$ ($\exists\, r>0.\ \exists\, k.\ \forall\, n\geq k.\ r < X\ n$)
using *positive.transfer* **by** (*simp add*: *cr-real-eq rel-fun-def*)

lemma *positive-zero*: $\neg$ *positive 0*
by *transfer auto*

lemma *positive-add*:
assumes *positive x positive y* **shows** *positive* $(x + y)$
proof $-$
have $*$: $[\![\forall\, n\geq i.\ a < x\ n;\ \forall\, n\geq j.\ b < y\ n;\ 0 < a;\ 0 < b;\ n \geq max\ i\ j]\!]$
$\Longrightarrow a+b < x\ n + y\ n$ **for** *x y* **and** *a b*::*rat* **and** *i j n*::*nat*
by (*simp add*: *add-strict-mono*)
show *?thesis*
using *assms*
by *transfer* (*blast intro*: $*$ *pos-add-strict*)
qed

lemma *positive-mult*:
assumes *positive x positive y* **shows** *positive* $(x * y)$
proof $-$
have $*$: $[\![\forall\, n\geq i.\ a < x\ n;\ \forall\, n\geq j.\ b < y\ n;\ 0 < a;\ 0 < b;\ n \geq max\ i\ j]\!]$
$\Longrightarrow a*b < x\ n * y\ n$ **for** *x y* **and** *a b*::*rat* **and** *i j n*::*nat*
by (*simp add*: *mult-strict-mono'*)
show *?thesis*
using *assms*
by *transfer* (*blast intro*: $*$ *mult-pos-pos*)
qed

lemma *positive-minus*: $\neg$ *positive x* $\Longrightarrow x \neq 0 \Longrightarrow$ *positive* $(- x)$
apply *transfer*
apply (*simp add*: *realrel-def*)
apply (*blast dest*: *cauchy-not-vanishes-cases*)
done

instantiation *real* :: *linordered-field*

begin

definition $x < y \longleftrightarrow$ *positive* $(y - x)$

definition $x \leq y \longleftrightarrow x < y \vee x = y$ **for** $x\ y$ :: *real*

definition $|a| = ($*if* $a < 0$ *then* $- a$ *else* $a)$ **for** a :: *real*

definition *sgn* $a = ($*if* $a = 0$ *then* 0 *else if* $0 < a$ *then* 1 *else* $- 1)$ **for** a :: *real*

instance
proof
  fix $a\ b\ c$ :: *real*
  show $|a| = ($*if* $a < 0$ *then* $- a$ *else* $a)$
    by (*rule abs-real-def*)
  show $a < b \longleftrightarrow a \leq b \wedge \neg\ b \leq a$
      $a \leq b \Longrightarrow b \leq c \Longrightarrow a \leq c \quad a \leq a$
      $a \leq b \Longrightarrow b \leq a \Longrightarrow a = b$
      $a \leq b \Longrightarrow c + a \leq c + b$
    unfolding *less-eq-real-def less-real-def*
    by (*force simp add*: *positive-zero dest*: *positive-add*)+
  show *sgn* $a = ($*if* $a = 0$ *then* 0 *else if* $0 < a$ *then* 1 *else* $- 1)$
    by (*rule sgn-real-def*)
  show $a \leq b \vee b \leq a$
    by (*auto dest*!: *positive-minus simp*: *less-eq-real-def less-real-def*)
  show $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
    unfolding *less-real-def*
    by (*force simp add*: *algebra-simps dest*: *positive-mult*)
qed

end

instantiation *real* :: *distrib-lattice*
begin

definition (*inf* :: *real* ⇒ *real* ⇒ *real*) = *min*

definition (*sup* :: *real* ⇒ *real* ⇒ *real*) = *max*

instance
  by *standard* (*auto simp add*: *inf-real-def sup-real-def max-min-distrib2*)

end

lemma *of-nat-Real*: *of-nat* x = *Real* ($\lambda n.$ *of-nat* x)
  by (*induct* x) (*simp-all add*: *zero-real-def one-real-def add-Real*)

lemma *of-int-Real*: *of-int* x = *Real* ($\lambda n.$ *of-int* x)
  by (*cases* x *rule*: *int-diff-cases*) (*simp add*: *of-nat-Real diff-Real*)

lemma *of-rat-Real*: *of-rat x* = *Real* (*λn. x*)
proof (*induct x*)
  case (*Fract a b*)
  then show *?case*
  apply (*simp add*: *Fract-of-int-quotient of-rat-divide*)
  apply (*simp add*: *of-int-Real divide-inverse inverse-Real mult-Real*)
  done
qed

instance *real* :: *archimedean-field*
proof
  show $\exists z.\ x \leq$ *of-int z* **for** *x* :: *real*
  proof (*induct x*)
    case (*1 X*)
    then obtain *b* **where** $0 < b$ **and** *b*: $\bigwedge n.\ |X\ n| < b$
      by (*blast dest*: *cauchy-imp-bounded*)
    then have *Real X* < *of-int* ($\lceil b \rceil + 1$)
      using *1*
      apply (*simp add*: *of-int-Real less-real-def diff-Real positive-Real*)
      apply (*rule-tac x=1* **in** *exI*)
      apply (*simp add*: *algebra-simps*)
      by (*metis abs-ge-self le-less-trans le-of-int-ceiling less-le*)
    then show *?case*
      using *less-eq-real-def* **by** *blast*
  qed
qed

instantiation *real* :: *floor-ceiling*
begin

definition [*code del*]: $\lfloor x::real \rfloor$ = (*THE z. of-int z* $\leq x \wedge x <$ *of-int* (*z* + *1*))

instance
proof
  show *of-int* $\lfloor x \rfloor \leq x \wedge x <$ *of-int* ($\lfloor x \rfloor + 1$) **for** *x* :: *real*
    unfolding *floor-real-def* **using** *floor-exists1* **by** (*rule theI'*)
qed

end

97.7 Completeness

lemma *not-positive-Real*:
  assumes *cauchy X* **shows** ¬ *positive* (*Real X*) ⟷ ($\forall r>0.\ \exists k.\ \forall n\geq k.\ X\ n \leq r$) (**is** *?lhs* = *?rhs*)
  unfolding *positive-Real* [*OF assms*]
proof (*intro iffI allI notI impI*)
  show $\exists k.\ \forall n\geq k.\ X\ n \leq r$ **if** *r*: ¬ ($\exists r>0.\ \exists k.\ \forall n\geq k.\ r < X\ n$) **and** $0 < r$ **for** *r*

```
  proof -
    obtain s t where s > 0 t > 0 r = s+t
      using ‹r > 0› obtain-pos-sum by blast
    obtain k where k: ⋀m n. ⟦m≥k; n≥k⟧ ⟹ |X m − X n| < t
      using cauchyD [OF assms ‹t > 0›] by blast
    obtain n where n ≥ k X n ≤ s
      by (meson r ‹0 < s› not-less)
    then have X l ≤ r if l ≥ n for l
      using k [OF ‹n ≥ k›, of l] that ‹r = s+t› by linarith
    then show ?thesis
      by blast
    qed
qed (meson le-cases not-le)

lemma le-Real:
  assumes cauchy X cauchy Y
  shows Real X ≤ Real Y = (∀r>0. ∃k. ∀n≥k. X n ≤ Y n + r)
  unfolding not-less [symmetric, where 'a=real] less-real-def
  apply (simp add: diff-Real not-positive-Real assms)
  apply (simp add: diff-le-eq ac-simps)
  done

lemma le-RealI:
  assumes Y: cauchy Y
  shows ∀n. x ≤ of-rat (Y n) ⟹ x ≤ Real Y
proof (induct x)
  fix X
  assume X: cauchy X and ∀n. Real X ≤ of-rat (Y n)
  then have le: ⋀m r. 0 < r ⟹ ∃k. ∀n≥k. X n ≤ Y m + r
    by (simp add: of-rat-Real le-Real)
  then have ∃k. ∀n≥k. X n ≤ Y n + r if 0 < r for r :: rat
  proof -
    from that obtain s t where s: 0 < s and t: 0 < t and r: r = s + t
      by (rule obtain-pos-sum)
    obtain i where i: ∀m≥i. ∀n≥i. |Y m − Y n| < s
      using cauchyD [OF Y s] ..
    obtain j where j: ∀n≥j. X n ≤ Y i + t
      using le [OF t] ..
    have ∀n≥max i j. X n ≤ Y n + r
    proof clarsimp
      fix n
      assume n: i ≤ n j ≤ n
      have X n ≤ Y i + t
        using n j by simp
      moreover have |Y i − Y n| < s
        using n i by simp
      ultimately show X n ≤ Y n + r
        unfolding r by simp
    qed
```

```
    then show ?thesis ..
  qed
  then show Real X ≤ Real Y
    by (simp add: of-rat-Real le-Real X Y)
qed

lemma Real-leI:
  assumes X: cauchy X
  assumes le: ∀ n. of-rat (X n) ≤ y
  shows Real X ≤ y
proof −
  have − y ≤ − Real X
    by (simp add: minus-Real X le-RealI of-rat-minus le)
  then show ?thesis by simp
qed

lemma less-RealD:
  assumes cauchy Y
  shows x < Real Y ⟹ ∃ n. x < of-rat (Y n)
  by (meson Real-leI assms leD leI)

lemma of-nat-less-two-power [simp]: of-nat n < (2::'a::linordered-idom) ^ n
  by auto

lemma complete-real:
  fixes S :: real set
  assumes ∃ x. x ∈ S and ∃ z. ∀ x∈S. x ≤ z
  shows ∃ y. (∀ x∈S. x ≤ y) ∧ (∀ z. (∀ x∈S. x ≤ z) ⟶ y ≤ z)
proof −
  obtain x where x: x ∈ S using assms(1) ..
  obtain z where z: ∀ x∈S. x ≤ z using assms(2) ..

  define P where P x ⟷ (∀ y∈S. y ≤ of-rat x) for x
  obtain a where a: ¬ P a
  proof
    have of-int ⌊x − 1⌋ ≤ x − 1 by (rule of-int-floor-le)
    also have x − 1 < x by simp
    finally have of-int ⌊x − 1⌋ < x .
    then have ¬ x ≤ of-int ⌊x − 1⌋ by (simp only: not-le)
    then show ¬ P (of-int ⌊x − 1⌋)
      unfolding P-def of-rat-of-int-eq using x by blast
  qed
  obtain b where b: P b
  proof
    show P (of-int ⌈z⌉)
    unfolding P-def of-rat-of-int-eq
    proof
      fix y assume y ∈ S
      then have y ≤ z using z by simp
```

```
      also have z ≤ of-int ⌈z⌉ by (rule le-of-int-ceiling)
      finally show y ≤ of-int ⌈z⌉ .
    qed
  qed

  define avg where avg x y = x/2 + y/2 for x y :: rat
  define bisect where bisect = (λ(x, y). if P (avg x y) then (x, avg x y) else (avg
x y, y))
  define A where A n = fst ((bisect ^^ n) (a, b)) for n
  define B where B n = snd ((bisect ^^ n) (a, b)) for n
  define C where C n = avg (A n) (B n) for n
  have A-0 [simp]: A 0 = a unfolding A-def by simp
  have B-0 [simp]: B 0 = b unfolding B-def by simp
  have A-Suc [simp]: ⋀n. A (Suc n) = (if P (C n) then A n else C n)
    unfolding A-def B-def C-def bisect-def split-def by simp
  have B-Suc [simp]: ⋀n. B (Suc n) = (if P (C n) then C n else B n)
    unfolding A-def B-def C-def bisect-def split-def by simp

  have width: B n − A n = (b − a) / 2^n for n
  proof (induct n)
    case (Suc n)
    then show ?case
      by (simp add: C-def eq-divide-eq avg-def algebra-simps)
  qed simp
  have twos: ∃ n. y / 2 ^ n < r if 0 < r for y r :: rat
  proof −
    obtain n where y / r < rat-of-nat n
      using ‹0 < r› reals-Archimedean2 by blast
    then have ∃ n. y < r ∗ 2 ^ n
      by (metis divide-less-eq less-trans mult.commute of-nat-less-two-power that)
    then show ?thesis
      by (simp add: field-split-simps)
  qed
  have PA: ¬ P (A n) for n
    by (induct n) (simp-all add: a)
  have PB: P (B n) for n
    by (induct n) (simp-all add: b)
  have ab: a < b
    using a b unfolding P-def
    by (meson leI less-le-trans of-rat-less)
  have AB: A n < B n for n
    by (induct n) (simp-all add: ab C-def avg-def)
  have A i ≤ A j ∧ B j ≤ B i if i < j for i j
    using that
  proof (induction rule: less-Suc-induct)
    case (1 i)
    then show ?case
      apply (clarsimp simp add: C-def avg-def add-divide-distrib [symmetric])
      apply (rule AB [THEN less-imp-le])
```

```
    done
  qed simp
  then have A-mono: A i ≤ A j and B-mono: B j ≤ B i if i ≤ j for i j
    by (metis eq-refl le-neq-implies-less that)+
  have cauchy-lemma: cauchy X if *: ⋀n i. i≥n ⟹ A n ≤ X i ∧ X i ≤ B n for
X
  proof (rule cauchyI)
    fix r::rat
    assume 0 < r
    then obtain k where k: (b − a) / 2 ^ k < r
      using twos by blast
    have |X m − X n| < r if m≥k n≥k for m n
    proof −
      have |X m − X n| ≤ B k − A k
        by (simp add: * abs-rat-def diff-mono that)
      also have ... < r
        by (simp add: k width)
      finally show ?thesis .
    qed
    then show ∃k. ∀m≥k. ∀n≥k. |X m − X n| < r
      by blast
  qed
  have cauchy A
   by (rule cauchy-lemma) (meson AB A-mono B-mono dual-order.strict-implies-order
less-le-trans)
  have cauchy B
   by (rule cauchy-lemma) (meson AB A-mono B-mono dual-order.strict-implies-order
le-less-trans)
  have ∀x∈S. x ≤ Real B
  proof
    fix x
    assume x ∈ S
    then show x ≤ Real B
      using PB [unfolded P-def] ‹cauchy B›
      by (simp add: le-RealI)
  qed
  moreover have ∀z. (∀x∈S. x ≤ z) ⟶ Real A ≤ z
    by (meson PA Real-leI P-def ‹cauchy A› le-cases order.trans)
  moreover have vanishes (λn. (b − a) / 2 ^ n)
  proof (rule vanishesI)
    fix r :: rat
    assume 0 < r
    then obtain k where k: |b − a| / 2 ^ k < r
      using twos by blast
    have ∀n≥k. |(b − a) / 2 ^ n| < r
    proof clarify
      fix n
      assume n: k ≤ n
      have |(b − a) / 2 ^ n| = |b − a| / 2 ^ n
```

by *simp*
also have ... $\leq |b - a| / 2 \hat{} k$
using *n* **by** (*simp add*: *divide-left-mono*)
also note *k*
finally show $|(b - a) / 2 \hat{} n| < r$ **.**
qed
then show $\exists k. \forall n \geq k. |(b - a) / 2 \hat{} n| < r$ **..**
qed
then have *Real B = Real A*
by (*simp add*: *eq-Real* ‹*cauchy A*› ‹*cauchy B*› *width*)
ultimately show $\exists y. (\forall x \in S. x \leq y) \wedge (\forall z. (\forall x \in S. x \leq z) \longrightarrow y \leq z)$
by *force*
qed

instantiation *real* :: *linear-continuum*
begin

97.8 Supremum of a set of reals

definition $Sup\ X = (LEAST\ z::real. \forall x \in X. x \leq z)$
definition $Inf\ X = - Sup\ (uminus\ \text{‘}\ X)$ **for** *X* :: *real set*

instance
proof
show *Sup-upper*: $x \leq Sup\ X$
if $x \in X$ *bdd-above X*
for *x* :: *real* **and** *X* :: *real set*
proof −
from *that* **obtain** *s* **where** *s*: $\forall y \in X. y \leq s \bigwedge z. \forall y \in X. y \leq z \Longrightarrow s \leq z$
using *complete-real*[*of X*] **unfolding** *bdd-above-def* **by** *blast*
then show *?thesis*
unfolding *Sup-real-def* **by** (*rule LeastI2-order*) (*auto simp*: *that*)
qed
show *Sup-least*: $Sup\ X \leq z$
if $X \neq \{\}$ **and** *z*: $\bigwedge x. x \in X \Longrightarrow x \leq z$
for *z* :: *real* **and** *X* :: *real set*
proof −
from *that* **obtain** *s* **where** *s*: $\forall y \in X. y \leq s \bigwedge z. \forall y \in X. y \leq z \Longrightarrow s \leq z$
using *complete-real* [*of X*] **by** *blast*
then have *Sup X = s*
unfolding *Sup-real-def* **by** (*best intro*: *Least-equality*)
also from *s z* **have** ... $\leq z$
by *blast*
finally show *?thesis* **.**
qed
show $Inf\ X \leq x$ **if** $x \in X$ *bdd-below X*
for *x* :: *real* **and** *X* :: *real set*
using *Sup-upper* [*of* $-x$ *uminus* ‘ *X*] **by** (*auto simp*: *Inf-real-def that*)
show $z \leq Inf\ X$ **if** $X \neq \{\}$ $\bigwedge x. x \in X \Longrightarrow z \leq x$

for *z* :: *real* **and** *X* :: *real set*
using *Sup-least* [*of uminus* ' *X* − *z*] **by** (*force simp*: *Inf-real-def that*)
show ∃ *a b*::*real*. *a* ≠ *b*
using *zero-neq-one* **by** *blast*
qed

end

97.9 Hiding implementation details

hide-const (**open**) *vanishes cauchy positive Real*

declare *Real-induct* [*induct del*]
declare *Abs-real-induct* [*induct del*]
declare *Abs-real-cases* [*cases del*]

lifting-update *real.lifting*
lifting-forget *real.lifting*

97.10 Embedding numbers into the Reals

abbreviation *real-of-nat* :: *nat* ⇒ *real*
where *real-of-nat* ≡ *of-nat*

abbreviation *real* :: *nat* ⇒ *real*
where *real* ≡ *of-nat*

abbreviation *real-of-int* :: *int* ⇒ *real*
where *real-of-int* ≡ *of-int*

abbreviation *real-of-rat* :: *rat* ⇒ *real*
where *real-of-rat* ≡ *of-rat*

declare [[*coercion-enabled*]]

declare [[*coercion of-nat* :: *nat* ⇒ *int*]]
declare [[*coercion of-nat* :: *nat* ⇒ *real*]]
declare [[*coercion of-int* :: *int* ⇒ *real*]]

declare [[*coercion-map map*]]
declare [[*coercion-map* λ*f g h x*. *g* (*h* (*f x*))]]
declare [[*coercion-map* λ*f g* (*x*,*y*). (*f x*, *g y*)]]

declare *of-int-eq-0-iff* [*algebra*, *presburger*]
declare *of-int-eq-1-iff* [*algebra*, *presburger*]
declare *of-int-eq-iff* [*algebra*, *presburger*]
declare *of-int-less-0-iff* [*algebra*, *presburger*]
declare *of-int-less-1-iff* [*algebra*, *presburger*]

declare *of-int-less-iff* [*algebra*, *presburger*]
declare *of-int-le-0-iff* [*algebra*, *presburger*]
declare *of-int-le-1-iff* [*algebra*, *presburger*]
declare *of-int-le-iff* [*algebra*, *presburger*]
declare *of-int-0-less-iff* [*algebra*, *presburger*]
declare *of-int-0-le-iff* [*algebra*, *presburger*]
declare *of-int-1-less-iff* [*algebra*, *presburger*]
declare *of-int-1-le-iff* [*algebra*, *presburger*]

lemma *int-less-real-le*: $n < m \longleftrightarrow$ *real-of-int* $n + 1 \leq$ *real-of-int* m
proof −
  have (*0*::*real*) $\leq$ *1*
    by (*metis less-eq-real-def zero-less-one*)
  then show *?thesis*
    by (*metis floor-of-int less-floor-iff*)
qed

lemma *int-le-real-less*: $n \leq m \longleftrightarrow$ *real-of-int* $n <$ *real-of-int* $m + 1$
  by (*meson int-less-real-le not-le*)

lemma (**in** *field-char-0*) *of-int-div-aux*:
  (*of-int* x) / (*of-int* d) =
    of-int (x *div* d) + (*of-int* (x *mod* d)) / (*of-int* d)
proof −
  have $x = (x$ *div* $d) * d + x$ *mod* d
    by *auto*
  then have *of-int* x = *of-int* (x *div* d) ∗ *of-int* d + *of-int*(x *mod* d)
    by (*metis local.of-int-add local.of-int-mult*)
  then show *?thesis*
    by (*simp add*: *divide-simps*)
qed

lemma *real-of-int-div*:
  d *dvd* $n \Longrightarrow$ *real-of-int* (n *div* d) = *real-of-int* n / *real-of-int* d **for** d n :: *int*
  by *auto*

lemma *real-of-int-div2*: $0 \leq$ *real-of-int* n / *real-of-int* x − *real-of-int* (n *div* x)
proof (*cases* $x = 0$)
  case *False*
  then show *?thesis*
    by (*metis diff-ge-0-iff-ge floor-divide-of-int-eq of-int-floor-le*)
qed *simp*

lemma *real-of-int-div3*: *real-of-int* n / *real-of-int* x − *real-of-int* (n *div* x) ≤ 1
  apply (*simp add*: *algebra-simps*)
  by (*metis add.commute floor-correct floor-divide-of-int-eq less-eq-real-def of-int-1 of-int-add*)

lemma *real-of-int-div4*: *real-of-int* (n *div* x) $\leq$ *real-of-int* n / *real-of-int* x

using *real-of-int-div2* [*of n x*] **by** *simp*

97.11 Embedding the Naturals into the Reals

lemma (**in** *field-char-0*) *of-nat-of-nat-div-aux*:
of-nat x / of-nat d = of-nat (x div d) + of-nat (x mod d) / of-nat d
by (*metis add-divide-distrib diff-add-cancel of-nat-div*)

lemma(**in** *field-char-0*) *of-nat-of-nat-div*: *d dvd n* $\Longrightarrow$ *of-nat*(*n div d*) = *of-nat n / of-nat d*
by *auto*

lemma (**in** *linordered-field*) *of-nat-div-le-of-nat*: *of-nat* (*n div x*) $\leq$ *of-nat n / of-nat x*
by (*metis le-add-same-cancel1 of-nat-0-le-iff of-nat-of-nat-div-aux zero-le-divide-iff*)

lemma *real-of-card*: *real* (*card A*) = *sum* (λx. *1*) *A*
by *simp*

lemma *nat-less-real-le*: $n < m \longleftrightarrow$ *real n* + *1* $\leq$ *real m*
by (*metis less-iff-succ-less-eq of-nat-1 of-nat-add of-nat-le-iff*)

lemma *nat-le-real-less*: $n \leq m \longleftrightarrow$ *real n* < *real m* + *1*
by (*meson nat-less-real-le not-le*)

lemma *real-of-nat-div*: *d dvd n* $\Longrightarrow$ *real*(*n div d*) = *real n / real d*
by *auto*

lemma *real-binomial-eq-mult-binomial-Suc*:
assumes $k \leq n$
shows *real*(*n choose k*) = (*n* + *1* − *k*) / (*n* + *1*) ∗ (*Suc n choose k*)
using *assms*
by (*simp add*: *of-nat-binomial-eq-mult-binomial-Suc* [*of k n*] *add.commute*)

97.12 The Archimedean Property of the Reals

lemma *real-arch-inverse*: *0* < *e* $\longleftrightarrow$ ($\exists$ *n*::*nat*. *n* $\neq$ *0* $\wedge$ *0* < *inverse* (*real n*) $\wedge$ *inverse* (*real n*) < *e*)
using *reals-Archimedean*[*of e*] *less-trans*[*of 0 1 / real n e* **for** *n*::*nat*]
by (*auto simp add*: *field-simps cong*: *conj-cong simp del*: *of-nat-Suc*)

lemma *reals-Archimedean3*: *0* < *x* $\Longrightarrow$ $\forall$ *y*. $\exists$ *n*. *y* < *real n* ∗ *x*
by (*auto intro*: *ex-less-of-nat-mult*)

lemma *real-archimedian-rdiv-eq-0*:
assumes *x0*: *x* $\geq$ *0*
and *c*: *c* $\geq$ *0*
and *xc*: $\bigwedge$*m*::*nat*. *m* > *0* $\Longrightarrow$ *real m* ∗ *x* $\leq$ *c*
shows *x* = *0*

by (*metis reals-Archimedean3 dual-order.order-iff-strict le0 le-less-trans not-le x0 xc*)

lemma *inverse-Suc*: *inverse* (*Suc n*) > *0*
using *of-nat-0-less-iff positive-imp-inverse-positive zero-less-Suc* **by** *blast*

lemma *Archimedean-eventually-inverse*:
fixes *ε*::*real* **shows** ($\forall_F$ *n in sequentially. inverse* (*real* (*Suc n*)) < *ε*) ⟷ *0* < *ε*
(**is** *?lhs*=*?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *eventually-at-top-dense* **using** *inverse-Suc order-less-trans* **by** *blast*
next
assume *?rhs*
then obtain *N* **where** *inverse* (*Suc N*) < *ε*
using *reals-Archimedean* **by** *blast*
moreover have *inverse* (*Suc n*) ≤ *inverse* (*Suc N*) **if** *n* ≥ *N* **for** *n*
using *inverse-Suc that* **by** *fastforce*
ultimately show *?lhs*
unfolding *eventually-sequentially*
using *order-le-less-trans* **by** *blast*
qed

On the relationship between two different ways of converting to 0

lemma *Inter-eq-Inter-inverse-Suc*:
assumes ⋀*r' r. r'* < *r* ⟹ *A r'* ⊆ *A r*
shows ⋂ (*A* ' {*0*<..}) = (⋂ *n. A*(*inverse*(*Suc n*)))
proof
have *x* ∈ *A ε*
if *x*: ∀ *n. x* ∈ *A* (*inverse* (*Suc n*)) **and** *ε*>*0* **for** *x* **and** *ε* :: *real*
proof −
obtain *n* **where** *inverse* (*Suc n*) < *ε*
using ‹*ε*>*0*› *reals-Archimedean* **by** *blast*
with *assms x* **show** *?thesis*
by *blast*
qed
then show (⋂ *n. A*(*inverse*(*Suc n*))) ⊆ (⋂*ε*∈{*0*<..}. *A ε*)
by *auto*
qed (*use inverse-Suc* **in** *fastforce*)

97.13 Rationals

lemma *Rats-abs-iff* [*simp*]:
|(*x*::*real*)| ∈ ℚ ⟷ *x* ∈ ℚ
by(*simp add*: *abs-real-def split*: *if-splits*)

lemma *Rats-eq-int-div-int*: ℚ = {*real-of-int i* / *real-of-int j* | *i j. j* ≠ *0*} (**is** - =

?S)
proof
  show $\mathbb{Q} \subseteq$ *?S*
  proof
    fix x :: *real*
    assume $x \in \mathbb{Q}$
    then obtain r **where** x = *of-rat r*
      unfolding *Rats-def* **..**
    have *of-rat r* ∈ *?S*
      by (*cases r*) (*auto simp add*: *of-rat-rat*)
    then show $x \in$ *?S*
      using ‹x = *of-rat r*› **by** *simp*
  qed
next
  show *?S* $\subseteq \mathbb{Q}$
  proof (*auto simp*: *Rats-def*)
    fix *i j* :: *int*
    assume $j \neq 0$
    then have *real-of-int i* / *real-of-int j* = *of-rat* (*Fract i j*)
      by (*simp add*: *of-rat-rat*)
    then show *real-of-int i* / *real-of-int j* ∈ *range of-rat*
      by *blast*
  qed
qed

lemma *Rats-eq-int-div-nat*: $\mathbb{Q}$ = { *real-of-int i* / *real n* | *i n*. $n \neq 0$}
proof (*auto simp*: *Rats-eq-int-div-int*)
  fix *i j* :: *int*
  assume $j \neq 0$
  show ∃(*i'*::*int*) (*n*::*nat*). *real-of-int i* / *real-of-int j* = *real-of-int i'* / *real n* ∧ 0 < *n*
  proof (*cases* $j > 0$)
    case *True*
    then have *real-of-int i* / *real-of-int j* = *real-of-int i* / *real* (*nat j*) ∧ 0 < *nat j*
      by *simp*
    then show *?thesis* **by** *blast*
  next
    case *False*
    with ‹$j \neq 0$›
    have *real-of-int i* / *real-of-int j* = *real-of-int* (− *i*) / *real* (*nat* (− *j*)) ∧ 0 < *nat* (− *j*)
      by *simp*
    then show *?thesis* **by** *blast*
  qed
next
  fix *i* :: *int* **and** *n* :: *nat*
  assume $0 < n$
  then have *real-of-int i* / *real n* = *real-of-int i* / *real-of-int*(*int n*) ∧ *int n* ≠ 0
    by *simp*

then show $\exists\, i'\ j.\ \mathit{real\text{-}of\text{-}int}\ i\ /\ \mathit{real}\ n = \mathit{real\text{-}of\text{-}int}\ i'\ /\ \mathit{real\text{-}of\text{-}int}\ j \wedge j \neq 0$
  by *blast*
qed

lemma *Rats-abs-nat-div-natE*:
  assumes $x \in \mathbb{Q}$
  obtains $m\ n :: \mathit{nat}$ **where** $n \neq 0$ **and** $|x| = \mathit{real}\ m\ /\ \mathit{real}\ n$ **and** *coprime m n*
proof −
  from ‹$x \in \mathbb{Q}$› **obtain** $i :: \mathit{int}$ **and** $n :: \mathit{nat}$ **where** $n \neq 0$ **and** $x = \mathit{real\text{-}of\text{-}int}\ i$ $/\ \mathit{real}\ n$
    by (*auto simp add*: *Rats-eq-int-div-nat*)
  then have $|x| = \mathit{real}\ (\mathit{nat}\ |i|)\ /\ \mathit{real}\ n$ **by** *simp*
  then obtain $m :: \mathit{nat}$ **where** *x-rat*: $|x| = \mathit{real}\ m\ /\ \mathit{real}\ n$ **by** *blast*
  let *?gcd* = *gcd m n*
  from ‹$n \neq 0$› **have** *gcd*: *?gcd* $\neq 0$ **by** *simp*
  let *?k* = *m div ?gcd*
  let *?l* = *n div ?gcd*
  let *?gcd'* = *gcd ?k ?l*
  have *?gcd dvd m* **..**
  then have *gcd-k*: *?gcd* ∗ *?k* = *m*
    by (*rule dvd-mult-div-cancel*)
  have *?gcd dvd n* **..**
  then have *gcd-l*: *?gcd* ∗ *?l* = *n*
    by (*rule dvd-mult-div-cancel*)
  from ‹$n \neq 0$› **and** *gcd-l* **have** *?gcd* ∗ *?l* $\neq 0$ **by** *simp*
  then have *?l* $\neq 0$ **by** (*blast dest!*: *mult-not-zero*)
  moreover
  have $|x| = \mathit{real}$ *?k* / *real ?l*
  proof −
    from *gcd* **have** *real ?k* / *real ?l* = *real* (*?gcd* ∗ *?k*) / *real* (*?gcd* ∗ *?l*)
      by (*simp add*: *real-of-nat-div*)
    also from *gcd-k* **and** *gcd-l* **have** ... = *real m* / *real n* **by** *simp*
    also from *x-rat* **have** ... = $|x|$ **..**
    finally show *?thesis* **..**
  qed
  moreover
  have *?gcd'* = *1*
  proof −
    have *?gcd* ∗ *?gcd'* = *gcd* (*?gcd* ∗ *?k*) (*?gcd* ∗ *?l*)
      by (*rule gcd-mult-distrib-nat*)
    with *gcd-k gcd-l* **have** *?gcd* ∗ *?gcd'* = *?gcd* **by** *simp*
    with *gcd* **show** *?thesis* **by** *auto*
  qed
  then have *coprime ?k ?l*
    by (*simp only*: *coprime-iff-gcd-eq-1*)
  ultimately show *?thesis* **..**
qed

97.14 Density of the Rational Reals in the Reals

This density proof is due to Stefan Richter and was ported by TN. The original source is *Real Analysis* by H.L. Royden. It employs the Archimedean property of the reals.

lemma *Rats-dense-in-real*:
  fixes $x :: real$
  assumes $x < y$
  shows $\exists r \in \mathbb{Q}.\ x < r \wedge r < y$
proof −
  from ‹$x < y$› **have** $0 < y - x$ **by** *simp*
  with *reals-Archimedean* **obtain** $q :: nat$ **where** q: *inverse* (*real* q) $< y - x$ **and** $0 < q$
    by *blast*
  define p **where** $p = \lceil y * real\ q \rceil - 1$
  define r **where** $r = of\text{-}int\ p\ /\ real\ q$
  from q **have** $x < y - inverse\ (real\ q)$
    by *simp*
  also from ‹$0 < q$› **have** $y - inverse\ (real\ q) \leq r$
    by (*simp add*: *r-def p-def le-divide-eq left-diff-distrib*)
  finally have $x < r$ **.**
  moreover from ‹$0 < q$› **have** $r < y$
    by (*simp add*: *r-def p-def divide-less-eq diff-less-eq less-ceiling-iff* [*symmetric*])
  moreover have $r \in \mathbb{Q}$
    by (*simp add*: *r-def*)
  ultimately show *?thesis* **by** *blast*
qed

lemma *of-rat-dense*:
  fixes $x\ y :: real$
  assumes $x < y$
  shows $\exists q :: rat.\ x < of\text{-}rat\ q \wedge of\text{-}rat\ q < y$
  using *Rats-dense-in-real* [*OF* ‹$x < y$›]
  by (*auto elim*: *Rats-cases*)

97.15 Numerals and Arithmetic

declaration ‹
  K (*Lin-Arith.add-inj-const* (***const-name*** ‹*of-nat*›, ***typ*** ‹*nat* ⇒ *real*›)
  #> *Lin-Arith.add-inj-const* (***const-name*** ‹*of-int*›, ***typ*** ‹*int* ⇒ *real*›))
›

97.16 Simprules combining $x + y$ and 0

lemma *real-add-minus-iff* [*simp*]: $x + - a = 0 \longleftrightarrow x = a$
  for $x\ a :: real$
  by *arith*

lemma *real-add-less-0-iff*: $x + y < 0 \longleftrightarrow y < - x$

for *x y* :: *real*
by *auto*

lemma *real-0-less-add-iff*: $0 < x + y \longleftrightarrow - x < y$
for *x y* :: *real*
by *auto*

lemma *real-add-le-0-iff*: $x + y \leq 0 \longleftrightarrow y \leq - x$
for *x y* :: *real*
by *auto*

lemma *real-0-le-add-iff*: $0 \leq x + y \longleftrightarrow - x \leq y$
for *x y* :: *real*
by *auto*

lemma *mult-ge1-I*: $[\![x \geq 1;\ y \geq 1]\!] \Longrightarrow x*y \geq (1::real)$
using *mult-mono* **by** *fastforce*

97.17 Lemmas about powers

lemma *two-realpow-ge-one*: $(1::real) \leq 2 \mathbin{\hat{}} n$
by *simp*

declare *sum-squares-eq-zero-iff* [*simp*] *sum-power2-eq-zero-iff* [*simp*]

lemma *real-minus-mult-self-le* [*simp*]: $- (u * u) \leq x * x$
for *u x* :: *real*
by (*rule order-trans* [**where** $y = 0$]) *auto*

lemma *realpow-square-minus-le* [*simp*]: $- u^2 \leq x^2$
for *u x* :: *real*
by (*auto simp add*: *power2-eq-square*)

97.18 Density of the Reals

lemma *field-lbound-gt-zero*: $0 < d1 \Longrightarrow 0 < d2 \Longrightarrow \exists e.\ 0 < e \wedge e < d1 \wedge e < d2$
for *d1 d2* :: *'a::linordered-field*
by (*rule exI* [**where** $x = min\ d1\ d2\ /\ 2$]) (*simp add*: *min-def*)

lemma *field-less-half-sum*: $x < y \Longrightarrow x < (x + y) / 2$
for *x y* :: *'a::linordered-field*
by *auto*

lemma *field-sum-of-halves*: $x / 2 + x / 2 = x$
for *x* :: *'a::linordered-field*
by *simp*

97.19 Archimedean properties and useful consequences

Bernoulli's inequality

proposition *Bernoulli-inequality*:
fixes x :: *real*
assumes $-1 \leq x$
shows $1 + n * x \leq (1 + x)$ ^ n
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
have $1 + Suc\ n * x \leq 1 + (Suc\ n)*x + n * x$^$2$
by (*simp add*: *algebra-simps*)
also have ... $= (1 + x) * (1 + n*x)$
by (*auto simp*: *power2-eq-square algebra-simps*)
also have ... $\leq (1 + x)$ ^ *Suc n*
using *Suc.hyps assms mult-left-mono* **by** *fastforce*
finally show *?case* **.**
qed

corollary *Bernoulli-inequality-even*:
fixes x :: *real*
assumes *even n*
shows $1 + n * x \leq (1 + x)$ ^ n
proof (*cases* $-1 \leq x \vee n=0$)
case *True*
then show *?thesis*
by (*auto simp*: *Bernoulli-inequality*)
next
case *False*
then have *real* $n \geq 1$
by *simp*
with *False* **have** $n * x \leq -1$
by (*metis linear minus-zero mult.commute mult.left-neutral mult-left-mono-neg neg-le-iff-le order-trans zero-le-one*)
then have $1 + n * x \leq 0$
by *auto*
also have ... $\leq (1 + x)$ ^ n
using *assms*
using *zero-le-even-power* **by** *blast*
finally show *?thesis* **.**
qed

corollary *real-arch-pow*:
fixes x :: *real*
assumes x: $1 < x$
shows $\exists n.\ y < x$^n
proof –

from *x* **have** *x0*: $x - 1 > 0$
  by *arith*
from *reals-Archimedean3*[*OF x0*, *rule-format*, *of y*]
obtain *n* :: *nat* **where** *n*: $y <$ *real* $n * (x - 1)$ **by** *metis*
from *x0* **have** *x00*: $x - 1 \geq -1$ **by** *arith*
from *Bernoulli-inequality*[*OF x00*, *of n*] *n*
have $y < x\hat{}n$ **by** *auto*
then show *?thesis* **by** *metis*
qed

corollary *real-arch-pow-inv*:
fixes *x y* :: *real*
assumes *y*: $y > 0$
  and *x1*: $x < 1$
shows $\exists n.\ x\hat{}n < y$
proof (*cases* $x > 0$)
case *True*
with *x1* **have** *ix*: $1 < 1/x$ **by** (*simp add*: *field-simps*)
from *real-arch-pow*[*OF ix*, *of 1/y*]
obtain *n* **where** *n*: $1/y < (1/x)\hat{}n$ **by** *blast*
then show *?thesis* **using** *y* ‹$x > 0$›
  by (*auto simp add*: *field-simps*)
next
case *False*
with *y x1* **show** *?thesis*
  by (*metis less-le-trans not-less power-one-right*)
qed

lemma *forall-pos-mono*:
$(\bigwedge d\ e{::}real.\ d < e \Longrightarrow P\ d \Longrightarrow P\ e) \Longrightarrow$
  $(\bigwedge n{::}nat.\ n \neq 0 \Longrightarrow P\ (inverse\ (real\ n))) \Longrightarrow (\bigwedge e.\ 0 < e \Longrightarrow P\ e)$
by (*metis real-arch-inverse*)

lemma *forall-pos-mono-1*:
$(\bigwedge d\ e{::}real.\ d < e \Longrightarrow P\ d \Longrightarrow P\ e) \Longrightarrow$
  $(\bigwedge n.\ P\ (inverse\ (real\ (Suc\ n)))) \Longrightarrow 0 < e \Longrightarrow P\ e$
using *reals-Archimedean* **by** *blast*

lemma *Archimedean-eventually-pow*:
fixes *x*::*real*
assumes $1 < x$
shows $\forall_F\ n$ *in sequentially*. $b < x\ \hat{}\ n$
proof −
obtain *N* **where** $\bigwedge n.\ n{\geq}N \Longrightarrow b < x\ \hat{}\ n$
  by (*metis assms le-less order-less-trans power-strict-increasing-iff real-arch-pow*)
then show *?thesis*
  using *eventually-sequentially* **by** *blast*
qed

lemma *Archimedean-eventually-pow-inverse*:
fixes *x*::*real*
assumes $|x| < 1$ $\varepsilon > 0$
shows $\forall_F$ *n in sequentially*. $|x\hat{}n| < \varepsilon$
proof (*cases* $x = 0$)
case *True*
then show *?thesis*
by (*simp add*: *assms eventually-at-top-dense zero-power*)
next
case *False*
then have $\forall_F$ *n in sequentially*. *inverse* $\varepsilon <$ *inverse* $|x|$ ^ *n*
by (*simp add*: *Archimedean-eventually-pow assms*(*1*) *one-less-inverse*)
then show *?thesis*
by *eventually-elim* (*metis* ‹$\varepsilon > 0$› *inverse-less-imp-less power-abs power-inverse*)
qed

97.20 Floor and Ceiling Functions from the Reals to the Integers

lemma *real-of-nat-less-numeral-iff* [*simp*]: *real n* < *numeral w* ⟷ *n* < *numeral w*
for *n* :: *nat*
by (*metis of-nat-less-iff of-nat-numeral*)

lemma *numeral-less-real-of-nat-iff* [*simp*]: *numeral w* < *real n* ⟷ *numeral w* < *n*
for *n* :: *nat*
by (*metis of-nat-less-iff of-nat-numeral*)

lemma *numeral-le-real-of-nat-iff* [*simp*]: *numeral n* ≤ *real m* ⟷ *numeral n* ≤ *m*
for *m* :: *nat*
by (*metis not-le real-of-nat-less-numeral-iff*)

lemma *of-int-floor-cancel* [*simp*]: *of-int* $\lfloor x \rfloor = x$ ⟷ (∃ *n*::*int*. *x* = *of-int n*)
by (*metis floor-of-int*)

lemma *of-int-floor* [*simp*]: $a \in \mathbb{Z} \Longrightarrow$ *of-int* (*floor a*) = *a*
by (*metis Ints-cases of-int-floor-cancel*)

lemma *floor-frac* [*simp*]: ⌊*frac r*⌋ = *0*
by (*simp add*: *frac-def*)

lemma *frac-1* [*simp*]: *frac 1* = *0*
by (*simp add*: *frac-def*)

lemma *frac-in-Rats-iff* [*simp*]:
fixes *r*::*'a*::{*floor-ceiling*,*field-char-0*}
shows *frac r* $\in \mathbb{Q}$ ⟷ *r* $\in \mathbb{Q}$
by (*metis Rats-add Rats-diff Rats-of-int diff-add-cancel frac-def*)

lemma *floor-eq*: *real-of-int n* $<$ *x* $\Longrightarrow$ *x* $<$ *real-of-int n* $+$ *1* $\Longrightarrow$ $\lfloor x \rfloor = n$
  by *linarith*

lemma *floor-eq2*: *real-of-int n* $\leq$ *x* $\Longrightarrow$ *x* $<$ *real-of-int n* $+$ *1* $\Longrightarrow$ $\lfloor x \rfloor = n$
  by (*fact floor-unique*)

lemma *floor-eq3*: *real n* $<$ *x* $\Longrightarrow$ *x* $<$ *real* (*Suc n*) $\Longrightarrow$ *nat* $\lfloor x \rfloor = n$
  by *linarith*

lemma *floor-eq4*: *real n* $\leq$ *x* $\Longrightarrow$ *x* $<$ *real* (*Suc n*) $\Longrightarrow$ *nat* $\lfloor x \rfloor = n$
  by *linarith*

lemma *real-of-int-floor-ge-diff-one* [*simp*]: *r* $-$ *1* $\leq$ *real-of-int* $\lfloor r \rfloor$
  by *linarith*

lemma *real-of-int-floor-gt-diff-one* [*simp*]: *r* $-$ *1* $<$ *real-of-int* $\lfloor r \rfloor$
  by *linarith*

lemma *real-of-int-floor-add-one-ge* [*simp*]: *r* $\leq$ *real-of-int* $\lfloor r \rfloor$ $+$ *1*
  by *linarith*

lemma *real-of-int-floor-add-one-gt* [*simp*]: *r* $<$ *real-of-int* $\lfloor r \rfloor$ $+$ *1*
  by *linarith*

lemma *floor-divide-real-eq-div*:
  assumes *0* $\leq$ *b*
  shows $\lfloor$*a* / *real-of-int b*$\rfloor$ = $\lfloor a \rfloor$ *div b*
proof (*cases b* = *0*)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  with *assms* **have** *b*: *b* $>$ *0* **by** *simp*
  have *j* = *i div b*
    if *real-of-int i* $\leq$ *a a* $<$ *1* $+$ *real-of-int i*
      real-of-int j $*$ *real-of-int b* $\leq$ *a a* $<$ *real-of-int b* $+$ *real-of-int j* $*$ *real-of-int b*
    for *i j* :: *int*
  proof $-$
    from *that* **have** *i* $<$ *b* $+$ *j* $*$ *b*
      by (*metis le-less-trans of-int-add of-int-less-iff of-int-mult*)
    moreover have *j* $*$ *b* $<$ *1* $+$ *i*
    proof $-$
      have *real-of-int* (*j* $*$ *b*) $<$ *real-of-int i* $+$ *1*
        using ‹*a* $<$ *1* $+$ *real-of-int i*› ‹*real-of-int j* $*$ *real-of-int b* $\leq$ *a*› **by** *force*
      then show *j* $*$ *b* $<$ *1* $+$ *i* **by** *linarith*
    qed
    ultimately have (*j* $-$ *i div b*) $*$ *b* $\leq$ *i mod b i mod b* $<$ ((*j* $-$ *i div b*) $+$ *1*) $*$ *b*
      by (*auto simp*: *field-simps*)

then have $(j - i\ div\ b) * b < 1 * b\ 0 * b < ((j - i\ div\ b) + 1) * b$
using *pos-mod-bound* [*OF b*, *of i*] *pos-mod-sign* [*OF b*, *of i*]
by *linarith*+
then show *?thesis* **using** *b* **unfolding** *mult-less-cancel-right* **by** *auto*
qed
with *b* **show** *?thesis* **by** (*auto split*: *floor-split simp*: *field-simps*)
qed

lemma *floor-one-divide-eq-div-numeral* [*simp*]:
$\lfloor 1\ /\ numeral\ b::real \rfloor = 1\ div\ numeral\ b$
by (*metis floor-divide-of-int-eq of-int-1 of-int-numeral*)

lemma *floor-minus-one-divide-eq-div-numeral* [*simp*]:
$\lfloor - (1\ /\ numeral\ b)::real \rfloor = - 1\ div\ numeral\ b$
by (*metis* (*mono-tags*, *opaque-lifting*) *div-minus-right minus-divide-right floor-divide-of-int-eq of-int-neg-numeral of-int-1*)

lemma *floor-divide-eq-div-numeral* [*simp*]:
$\lfloor numeral\ a\ /\ numeral\ b::real \rfloor = numeral\ a\ div\ numeral\ b$
by (*metis floor-divide-of-int-eq of-int-numeral*)

lemma *floor-minus-divide-eq-div-numeral* [*simp*]:
$\lfloor - (numeral\ a\ /\ numeral\ b)::real \rfloor = - numeral\ a\ div\ numeral\ b$
by (*metis divide-minus-left floor-divide-of-int-eq of-int-neg-numeral of-int-numeral*)

lemma *of-int-ceiling-cancel* [*simp*]: $of\text{-}int\ \lceil x \rceil = x \longleftrightarrow (\exists\ n::int.\ x = of\text{-}int\ n)$
using *ceiling-of-int* **by** *metis*

lemma *of-int-ceiling* [*simp*]: $a \in \mathbb{Z} \Longrightarrow of\text{-}int\ (ceiling\ a) = a$
by (*metis Ints-cases of-int-ceiling-cancel*)

lemma *ceiling-eq*: $of\text{-}int\ n < x \Longrightarrow x \leq of\text{-}int\ n + 1 \Longrightarrow \lceil x \rceil = n + 1$
by (*simp add*: *ceiling-unique*)

lemma *of-int-ceiling-diff-one-le* [*simp*]: $of\text{-}int\ \lceil r \rceil - 1 \leq r$
by *linarith*

lemma *of-int-ceiling-le-add-one* [*simp*]: $of\text{-}int\ \lceil r \rceil \leq r + 1$
by *linarith*

lemma *ceiling-le*: $x \leq of\text{-}int\ a \Longrightarrow \lceil x \rceil \leq a$
by (*simp add*: *ceiling-le-iff*)

lemma *ceiling-divide-eq-div*: $\lceil of\text{-}int\ a\ /\ of\text{-}int\ b \rceil = - (- a\ div\ b)$
by (*metis ceiling-def floor-divide-of-int-eq minus-divide-left of-int-minus*)

lemma *ceiling-divide-eq-div-numeral* [*simp*]:
$\lceil numeral\ a\ /\ numeral\ b :: real \rceil = - (- numeral\ a\ div\ numeral\ b)$
using *ceiling-divide-eq-div*[*of numeral a numeral b*] **by** *simp*

lemma *ceiling-minus-divide-eq-div-numeral* [*simp*]:
$\lceil - (numeral\ a\ /\ numeral\ b :: real) \rceil = - (numeral\ a\ div\ numeral\ b)$
using *ceiling-divide-eq-div*[*of* − *numeral a numeral b*] **by** *simp*

The following lemmas are remnants of the erstwhile functions natfloor and natceiling.

lemma *nat-floor-neg*: $x \leq 0 \Longrightarrow nat\ \lfloor x \rfloor = 0$
for $x :: real$
by *linarith*

lemma *le-nat-floor*: $real\ x \leq a \Longrightarrow x \leq nat\ \lfloor a \rfloor$
by *linarith*

lemma *le-mult-nat-floor*: $nat\ \lfloor a \rfloor * nat\ \lfloor b \rfloor \leq nat\ \lfloor a * b \rfloor$
by ($cases\ 0 \leq a \wedge 0 \leq b$)
(*auto simp add*: *nat-mult-distrib*[*symmetric*] *nat-mono le-mult-floor*)

lemma *nat-ceiling-le-eq* [*simp*]: $nat\ \lceil x \rceil \leq a \longleftrightarrow x \leq real\ a$
by *linarith*

lemma *real-nat-ceiling-ge*: $x \leq real\ (nat\ \lceil x \rceil)$
by *linarith*

lemma *Rats-no-top-le*: $\exists\, q \in \mathbb{Q}.\ x \leq q$
for $x :: real$
by (*auto intro*!: *bexI*[*of - of-nat* $(nat\ \lceil x \rceil)$]) *linarith*

lemma *Rats-no-bot-less*: $\exists\, q \in \mathbb{Q}.\ q < x$ **for** $x :: real$
by (*auto intro*!: *bexI*[*of - of-int* $(\lfloor x \rfloor - 1)$]) *linarith*

lemma *floor-ceiling-diff-le*: $0 \leq r \Longrightarrow nat\lfloor real\ k - r \rfloor \leq k - nat\lceil r \rceil$
by *linarith*

lemma *floor-ceiling-diff-le'*: $nat\lfloor r - real\ k \rfloor \leq nat\lceil r \rceil - k$
by *linarith*

lemma *ceiling-floor-diff-ge*: $nat\lceil r - real\ k \rceil \geq nat\lfloor r \rfloor - k$
by *linarith*

lemma *ceiling-floor-diff-ge'*: $r \leq k \Longrightarrow nat\lceil r - real\ k \rceil \leq k - nat\lfloor r \rfloor$
by *linarith*

97.21 Exponentiation with floor

lemma *floor-power*:
assumes $x = of\text{-}int\ \lfloor x \rfloor$
shows $\lfloor x \mathbin{\char`\^} n \rfloor = \lfloor x \rfloor \mathbin{\char`\^} n$
proof −

have *x* ^ *n* = *of-int* (⌊*x*⌋ ^ *n*)
  using *assms* **by** (*induct n arbitrary*: *x*) *simp-all*
 then show *?thesis* **by** (*metis floor-of-int*)
qed

lemma *floor-numeral-power* [*simp*]: ⌊*numeral x* ^ *n*⌋ = *numeral x* ^ *n*
 by (*metis floor-of-int of-int-numeral of-int-power*)

lemma *ceiling-numeral-power* [*simp*]: ⌈*numeral x* ^ *n*⌉ = *numeral x* ^ *n*
 by (*metis ceiling-of-int of-int-numeral of-int-power*)

97.22 Implementation of rational real numbers

Formal constructor

definition *Ratreal* :: *rat* ⇒ *real*
 where [*code-abbrev*, *simp*]: *Ratreal* = *real-of-rat*

code-datatype *Ratreal*

Quasi-Numerals

lemma [*code-abbrev*]:
 real-of-rat (*numeral k*) = *numeral k*
 real-of-rat (− *numeral k*) = − *numeral k*
 real-of-rat (*rat-of-int a*) = *real-of-int a*
 by *simp-all*

lemma [*code-post*]:
 real-of-rat 0 = *0*
 real-of-rat 1 = *1*
 real-of-rat (− *1*) = − *1*
 real-of-rat (*1* / *numeral k*) = *1* / *numeral k*
 real-of-rat (*numeral k* / *numeral l*) = *numeral k* / *numeral l*
 real-of-rat (− (*1* / *numeral k*)) = − (*1* / *numeral k*)
 real-of-rat (− (*numeral k* / *numeral l*)) = − (*numeral k* / *numeral l*)
 by (*simp-all add*: *of-rat-divide of-rat-minus*)

Operations

lemma *zero-real-code* [*code*]: *0* = *Ratreal 0*
 by *simp*

lemma *one-real-code* [*code*]: *1* = *Ratreal 1*
 by *simp*

instantiation *real* :: *equal*
begin

definition *HOL.equal x y* ⟷ *x* − *y* = *0* **for** *x* :: *real*

instance by *standard* (*simp add*: *equal-real-def*)

lemma *real-equal-code* [*code*]: *HOL.equal* (*Ratreal x*) (*Ratreal y*) $\longleftrightarrow$ *HOL.equal x y*
by (*simp add*: *equal-real-def equal*)

lemma [*code nbe*]: *HOL.equal x x* $\longleftrightarrow$ *True*
for *x* :: *real*
by (*rule equal-refl*)

end

lemma *real-less-eq-code* [*code*]: *Ratreal x* $\leq$ *Ratreal y* $\longleftrightarrow$ $x \leq y$
by (*simp add*: *of-rat-less-eq*)

lemma *real-less-code* [*code*]: *Ratreal x* $<$ *Ratreal y* $\longleftrightarrow$ $x < y$
by (*simp add*: *of-rat-less*)

lemma *real-plus-code* [*code*]: *Ratreal x* + *Ratreal y* = *Ratreal* ($x + y$)
by (*simp add*: *of-rat-add*)

lemma *real-times-code* [*code*]: *Ratreal x* $*$ *Ratreal y* = *Ratreal* ($x * y$)
by (*simp add*: *of-rat-mult*)

lemma *real-uminus-code* [*code*]: $-$ *Ratreal x* = *Ratreal* ($- x$)
by (*simp add*: *of-rat-minus*)

lemma *real-minus-code* [*code*]: *Ratreal x* $-$ *Ratreal y* = *Ratreal* ($x - y$)
by (*simp add*: *of-rat-diff*)

lemma *real-inverse-code* [*code*]: *inverse* (*Ratreal x*) = *Ratreal* (*inverse x*)
by (*simp add*: *of-rat-inverse*)

lemma *real-divide-code* [*code*]: *Ratreal x* / *Ratreal y* = *Ratreal* (x / y)
by (*simp add*: *of-rat-divide*)

lemma *real-floor-code* [*code*]: $\lfloor$*Ratreal x*$\rfloor$ = $\lfloor x \rfloor$
by (*metis Ratreal-def floor-le-iff floor-unique le-floor-iff of-int-floor-le of-rat-of-int-eq real-less-eq-code*)

Quickcheck

context
includes *term-syntax*
begin

definition
valterm-ratreal :: *rat* $\times$ (*unit* $\Rightarrow$ *Code-Evaluation.term*) $\Rightarrow$ *real* $\times$ (*unit* $\Rightarrow$ *Code-Evaluation.term*)
where [*code-unfold*]: *valterm-ratreal k* = *Code-Evaluation.valtermify Ratreal* $\{\cdot\}$

k

end

instantiation *real* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
Quickcheck-Random.random i = *Quickcheck-Random.random i* ∘→ (λ*r*. *Pair* (*valterm-ratreal r*))

instance ..

end

end

instantiation *real* :: *exhaustive*
begin

definition
exhaustive-real f d = *Quickcheck-Exhaustive.exhaustive* (λ*r*. *f* (*Ratreal r*)) *d*

instance ..

end

instantiation *real* :: *full-exhaustive*
begin

definition
full-exhaustive-real f d = *Quickcheck-Exhaustive.full-exhaustive* (λ*r*. *f* (*valterm-ratreal r*)) *d*

instance ..

end

instantiation *real* :: *narrowing*
begin

definition
narrowing-real = *Quickcheck-Narrowing.apply* (*Quickcheck-Narrowing.cons Ratreal*) *narrowing*

instance ..

end

97.23 Setup for Nitpick

declaration ‹
Nitpick-HOL.register-frac-type ***type-name*** ‹*real*›
[(***const-name*** ‹*zero-real-inst.zero-real*›, ***const-name*** ‹*Nitpick.zero-frac*›),
(***const-name*** ‹*one-real-inst.one-real*›, ***const-name*** ‹*Nitpick.one-frac*›),
(***const-name*** ‹*plus-real-inst.plus-real*›, ***const-name*** ‹*Nitpick.plus-frac*›),
(***const-name*** ‹*times-real-inst.times-real*›, ***const-name*** ‹*Nitpick.times-frac*›),
(***const-name*** ‹*uminus-real-inst.uminus-real*›, ***const-name*** ‹*Nitpick.uminus-frac*›),
(***const-name*** ‹*inverse-real-inst.inverse-real*›, ***const-name*** ‹*Nitpick.inverse-frac*›),
(***const-name*** ‹*ord-real-inst.less-real*›, ***const-name*** ‹*Nitpick.less-frac*›),
(***const-name*** ‹*ord-real-inst.less-eq-real*›, ***const-name*** ‹*Nitpick.less-eq-frac*›)]
›

lemmas [*nitpick-unfold*] = *inverse-real-inst.inverse-real one-real-inst.one-real*
ord-real-inst.less-real ord-real-inst.less-eq-real plus-real-inst.plus-real
times-real-inst.times-real uminus-real-inst.uminus-real
zero-real-inst.zero-real

97.24 Setup for SMT

ML-file ‹*Tools/SMT/smt-real.ML*›
ML-file ‹*Tools/SMT/z3-real.ML*›

lemma [*z3-rule*]:
$0 + x = x$
$x + 0 = x$
$0 * x = 0$
$1 * x = x$
$-x = -1 * x$
$x + y = y + x$
for *x y* :: *real*
by *auto*

lemma [*smt-arith-multiplication*]:
fixes *A B* :: *real* **and** *p n* :: *real*
assumes $A \leq B$ $0 < n$ $p > 0$
shows $(A / n) * p \leq (B / n) * p$
using *assms* **by** (*auto simp*: *field-simps*)

lemma [*smt-arith-multiplication*]:
fixes *A B* :: *real* **and** *p n* :: *real*
assumes $A < B$ $0 < n$ $p > 0$
shows $(A / n) * p < (B / n) * p$
using *assms* **by** (*auto simp*: *field-simps*)

lemma [*smt-arith-multiplication*]:
fixes *A B* :: *real* **and** *p n* :: *int*
assumes $A \leq B$ $0 < n$ $p > 0$
shows $(A / n) * p \leq (B / n) * p$
using *assms* **by** (*auto simp*: *field-simps*)

lemma [*smt-arith-multiplication*]:
fixes *A B* :: *real* **and** *p n* :: *int*
assumes $A < B$ $0 < n$ $p > 0$
shows $(A / n) * p < (B / n) * p$
using *assms* **by** (*auto simp*: *field-simps*)

lemmas [*smt-arith-multiplication*] =
verit-le-mono-div[*THEN mult-left-mono*, *unfolded int-distrib*, *of - - ‹nat* (*floor* (- :: *real*))› ‹*nat* (*floor* (- :: *real*))›]
div-le-mono[*THEN mult-left-mono*, *unfolded int-distrib*, *of - - ‹nat* (*floor* (- :: *real*))› ‹*nat* (*floor* (- :: *real*))›]
verit-le-mono-div-int[*THEN mult-left-mono*, *unfolded int-distrib*, *of - - ‹floor* (- :: *real*)› ‹*floor* (- :: *real*)›]
zdiv-mono1[*THEN mult-left-mono*, *unfolded int-distrib*, *of - - ‹floor* (- :: *real*)› ‹*floor* (- :: *real*)›]
arg-cong[*of - - ‹λa* :: *real. a / real* (*n*::*nat*) ∗ *real* (*p*::*nat*)› **for** *n p* :: *nat*, *THEN sym*]
arg-cong[*of - - ‹λa* :: *real. a / real-of-int n* ∗ *real-of-int p*› **for** *n p* :: *int*, *THEN sym*]
arg-cong[*of - - ‹λa* :: *real. a / n* ∗ *p*› **for** *n p* :: *real*, *THEN sym*]

lemmas [*smt-arith-simplify*] =
floor-one floor-numeral div-by-1 times-divide-eq-right
nonzero-mult-div-cancel-left division-ring-divide-zero div-0
divide-minus-left zero-less-divide-iff

97.25 Setup for Argo

ML-file ‹*Tools/Argo/argo-real.ML*›

end

98 Topological Spaces

theory *Topological-Spaces*
imports *Main*
begin

named-theorems *continuous-intros structural introduction rules for continuity*

98.1 Topological space

class *open* =

fixes *open* :: *'a set* ⇒ *bool*

class *topological-space* = *open* +
assumes *open-UNIV* [*simp*, *intro*]: *open UNIV*
assumes *open-Int* [*intro*]: *open S* ⟹ *open T* ⟹ *open* (*S* ∩ *T*)
assumes *open-Union* [*intro*]: ∀ *S*∈*K*. *open S* ⟹ *open* (⋃ *K*)
begin

definition *closed* :: *'a set* ⇒ *bool*
where *closed S* ⟷ *open* (− *S*)

lemma *open-empty* [*continuous-intros*, *intro*, *simp*]: *open* {}
using *open-Union* [*of* {}] **by** *simp*

lemma *open-Un* [*continuous-intros*, *intro*]: *open S* ⟹ *open T* ⟹ *open* (*S* ∪ *T*)
using *open-Union* [*of* {*S*, *T*}] **by** *simp*

lemma *open-UN* [*continuous-intros*, *intro*]: ∀ *x*∈*A*. *open* (*B x*) ⟹ *open* (⋃ *x*∈*A*. *B x*)
using *open-Union* [*of B* ‘ *A*] **by** *simp*

lemma *open-Inter* [*continuous-intros*, *intro*]: *finite S* ⟹ ∀ *T*∈*S*. *open T* ⟹ *open* (⋂ *S*)
by (*induction set*: *finite*) *auto*

lemma *open-INT* [*continuous-intros*, *intro*]: *finite A* ⟹ ∀ *x*∈*A*. *open* (*B x*) ⟹ *open* (⋂ *x*∈*A*. *B x*)
using *open-Inter* [*of B* ‘ *A*] **by** *simp*

lemma *openI*:
assumes ⋀*x*. *x* ∈ *S* ⟹ ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*
shows *open S*
proof −
have *open* (⋃ {*T*. *open T* ∧ *T* ⊆ *S*}) **by** *auto*
moreover have ⋃ {*T*. *open T* ∧ *T* ⊆ *S*} = *S* **by** (*auto dest*!: *assms*)
ultimately show *open S* **by** *simp*
qed

lemma *open-subopen*: *open S* ⟷ (∀ *x*∈*S*. ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*)
by (*auto intro*: *openI*)

lemma *closed-empty* [*continuous-intros*, *intro*, *simp*]: *closed* {}
unfolding *closed-def* **by** *simp*

lemma *closed-Un* [*continuous-intros*, *intro*]: *closed S* ⟹ *closed T* ⟹ *closed* (*S* ∪ *T*)
unfolding *closed-def* **by** *auto*

lemma *closed-UNIV* [*continuous-intros*, *intro*, *simp*]: *closed UNIV*

unfolding *closed-def* **by** *simp*

lemma *closed-Int* [*continuous-intros*, *intro*]: *closed* $S \Longrightarrow$ *closed* $T \Longrightarrow$ *closed* $(S \cap T)$
unfolding *closed-def* **by** *auto*

lemma *closed-INT* [*continuous-intros*, *intro*]: $\forall x \in A$. *closed* $(B\ x) \Longrightarrow$ *closed* $(\bigcap x \in A.\ B\ x)$
unfolding *closed-def* **by** *auto*

lemma *closed-Inter* [*continuous-intros*, *intro*]: $\forall S \in K$. *closed* $S \Longrightarrow$ *closed* $(\bigcap K)$
unfolding *closed-def uminus-Inf* **by** *auto*

lemma *closed-Union* [*continuous-intros*, *intro*]: *finite* $S \Longrightarrow \forall T \in S$. *closed* $T \Longrightarrow$ *closed* $(\bigcup S)$
by (*induct set*: *finite*) *auto*

lemma *closed-UN* [*continuous-intros*, *intro*]:
finite $A \Longrightarrow \forall x \in A$. *closed* $(B\ x) \Longrightarrow$ *closed* $(\bigcup x \in A.\ B\ x)$
using *closed-Union* [*of* B ' A] **by** *simp*

lemma *open-closed*: *open* $S \longleftrightarrow$ *closed* $(-\ S)$
by (*simp add*: *closed-def*)

lemma *closed-open*: *closed* $S \longleftrightarrow$ *open* $(-\ S)$
by (*rule closed-def*)

lemma *open-Diff* [*continuous-intros*, *intro*]: *open* $S \Longrightarrow$ *closed* $T \Longrightarrow$ *open* $(S - T)$
by (*simp add*: *closed-open Diff-eq open-Int*)

lemma *closed-Diff* [*continuous-intros*, *intro*]: *closed* $S \Longrightarrow$ *open* $T \Longrightarrow$ *closed* $(S - T)$
by (*simp add*: *open-closed Diff-eq closed-Int*)

lemma *open-Compl* [*continuous-intros*, *intro*]: *closed* $S \Longrightarrow$ *open* $(-\ S)$
by (*simp add*: *closed-open*)

lemma *closed-Compl* [*continuous-intros*, *intro*]: *open* $S \Longrightarrow$ *closed* $(-\ S)$
by (*simp add*: *open-closed*)

lemma *open-Collect-neg*: *closed* $\{x.\ P\ x\} \Longrightarrow$ *open* $\{x.\ \neg\ P\ x\}$
unfolding *Collect-neg-eq* **by** (*rule open-Compl*)

lemma *open-Collect-conj*:
assumes *open* $\{x.\ P\ x\}$ *open* $\{x.\ Q\ x\}$
shows *open* $\{x.\ P\ x \wedge Q\ x\}$
using *open-Int*[*OF assms*] **by** (*simp add*: *Int-def*)

lemma *open-Collect-disj*:
assumes *open* $\{x.\ P\ x\}$ *open* $\{x.\ Q\ x\}$
shows *open* $\{x.\ P\ x \vee Q\ x\}$
using *open-Un*[*OF assms*] **by** (*simp add*: *Un-def*)

lemma *open-Collect-ex*: $(\bigwedge i.\ open\ \{x.\ P\ i\ x\}) \Longrightarrow open\ \{x.\ \exists\, i.\ P\ i\ x\}$
using *open-UN*[*of UNIV* $\lambda i.\ \{x.\ P\ i\ x\}$] **unfolding** *Collect-ex-eq* **by** *simp*

lemma *open-Collect-imp*: $closed\ \{x.\ P\ x\} \Longrightarrow open\ \{x.\ Q\ x\} \Longrightarrow open\ \{x.\ P\ x \longrightarrow Q\ x\}$
unfolding *imp-conv-disj* **by** (*intro open-Collect-disj open-Collect-neg*)

lemma *open-Collect-const*: *open* $\{x.\ P\}$
by (*cases P*) *auto*

lemma *closed-Collect-neg*: $open\ \{x.\ P\ x\} \Longrightarrow closed\ \{x.\ \neg\ P\ x\}$
unfolding *Collect-neg-eq* **by** (*rule closed-Compl*)

lemma *closed-Collect-conj*:
assumes *closed* $\{x.\ P\ x\}$ *closed* $\{x.\ Q\ x\}$
shows *closed* $\{x.\ P\ x \wedge Q\ x\}$
using *closed-Int*[*OF assms*] **by** (*simp add*: *Int-def*)

lemma *closed-Collect-disj*:
assumes *closed* $\{x.\ P\ x\}$ *closed* $\{x.\ Q\ x\}$
shows *closed* $\{x.\ P\ x \vee Q\ x\}$
using *closed-Un*[*OF assms*] **by** (*simp add*: *Un-def*)

lemma *closed-Collect-all*: $(\bigwedge i.\ closed\ \{x.\ P\ i\ x\}) \Longrightarrow closed\ \{x.\ \forall\, i.\ P\ i\ x\}$
using *closed-INT*[*of UNIV* $\lambda i.\ \{x.\ P\ i\ x\}$] **by** (*simp add*: *Collect-all-eq*)

lemma *closed-Collect-imp*: $open\ \{x.\ P\ x\} \Longrightarrow closed\ \{x.\ Q\ x\} \Longrightarrow closed\ \{x.\ P\ x \longrightarrow Q\ x\}$
unfolding *imp-conv-disj* **by** (*intro closed-Collect-disj closed-Collect-neg*)

lemma *closed-Collect-const*: *closed* $\{x.\ P\}$
by (*cases P*) *auto*

end

98.2 Hausdorff and other separation properties

class *t0-space* = *topological-space* +
assumes *t0-space*: $x \neq y \Longrightarrow \exists\, U.\ open\ U \wedge \neg\ (x \in U \longleftrightarrow y \in U)$

class *t1-space* = *topological-space* +
assumes *t1-space*: $x \neq y \Longrightarrow \exists\, U.\ open\ U \wedge x \in U \wedge y \notin U$

instance *t1-space* $\subseteq$ *t0-space*

by *standard* (*fast dest*: *t1-space*)

context *t1-space* **begin**

lemma *separation-t1*: $x \neq y \longleftrightarrow (\exists U. \; open \; U \land x \in U \land y \notin U)$
using *t1-space*[*of x y*] **by** *blast*

lemma *closed-singleton* [*iff*]: *closed* $\{a\}$
proof −
let $?T = \bigcup\{S. \; open \; S \land a \notin S\}$
have *open ?T*
by (*simp add*: *open-Union*)
also have $?T = - \{a\}$
by (*auto simp add*: *set-eq-iff separation-t1*)
finally show *closed* $\{a\}$
by (*simp only*: *closed-def*)
qed

lemma *closed-insert* [*continuous-intros*, *simp*]:
assumes *closed S*
shows *closed* (*insert a S*)
proof −
from *closed-singleton assms* **have** *closed* $(\{a\} \cup S)$
by (*rule closed-Un*)
then show *closed* (*insert a S*)
by *simp*
qed

lemma *finite-imp-closed*: *finite S* $\Longrightarrow$ *closed S*
by (*induct pred*: *finite*) *simp-all*

end

T2 spaces are also known as Hausdorff spaces.

class *t2-space* = *topological-space* +
assumes *hausdorff*: $x \neq y \Longrightarrow \exists U \; V. \; open \; U \land open \; V \land x \in U \land y \in V \land U \cap V = \{\}$

instance *t2-space* ⊆ *t1-space*
by *standard* (*fast dest*: *hausdorff*)

lemma (**in** *t2-space*) *separation-t2*: $x \neq y \longleftrightarrow (\exists U \; V. \; open \; U \land open \; V \land x \in U \land y \in V \land U \cap V = \{\})$
using *hausdorff* [*of x y*] **by** *blast*

lemma (**in** *t0-space*) *separation-t0*: $x \neq y \longleftrightarrow (\exists U. \; open \; U \land \neg (x \in U \longleftrightarrow y \in U))$
using *t0-space* [*of x y*] **by** *blast*

A classical separation axiom for topological space, the T3 axiom – also called regularity: if a point is not in a closed set, then there are open sets separating them.

class *t3-space* = *t2-space* +
assumes *t3-space*: *closed S* ⟹ $y \notin S$ ⟹ ∃ *U V*. *open U* ∧ *open V* ∧ $y \in U$ ∧ $S \subseteq V$ ∧ $U \cap V = \{\}$

A classical separation axiom for topological space, the T4 axiom – also called normality: if two closed sets are disjoint, then there are open sets separating them.

class *t4-space* = *t2-space* +
assumes *t4-space*: *closed S* ⟹ *closed T* ⟹ $S \cap T = \{\}$ ⟹ ∃ *U V*. *open U* ∧ *open V* ∧ $S \subseteq U$ ∧ $T \subseteq V$ ∧ $U \cap V = \{\}$

T4 is stronger than T3, and weaker than metric.

instance *t4-space* ⊆ *t3-space*
proof
fix *S* **and** *y*::'*a* **assume** *closed S* $y \notin S$
then show ∃ *U V*. *open U* ∧ *open V* ∧ $y \in U$ ∧ $S \subseteq V$ ∧ $U \cap V = \{\}$
using *t4-space*[*of* {*y*} *S*] **by** *auto*
qed

A perfect space is a topological space with no isolated points.

class *perfect-space* = *topological-space* +
assumes *not-open-singleton*: ¬ *open* {*x*}

lemma (**in** *perfect-space*) *UNIV-not-singleton*: *UNIV* ≠ {*x*}
for *x*::'*a*
by (*metis* (*no-types*) *open-UNIV not-open-singleton*)

98.3 Generators for toplogies

inductive *generate-topology* :: '*a set set* ⇒ '*a set* ⇒ *bool* **for** *S* :: '*a set set*
where
UNIV: *generate-topology S UNIV*
| *Int*: *generate-topology S* (*a* ∩ *b*) **if** *generate-topology S a* **and** *generate-topology S b*
| *UN*: *generate-topology S* (⋃*K*) **if** (⋀*k*. *k* ∈ *K* ⟹ *generate-topology S k*)
| *Basis*: *generate-topology S s* **if** *s* ∈ *S*

hide-fact (**open**) *UNIV Int UN Basis*

lemma *generate-topology-Union*:
(⋀*k*. *k* ∈ *I* ⟹ *generate-topology S* (*K k*)) ⟹ *generate-topology S* (⋃*k*∈*I*. *K k*)
using *generate-topology.UN* [*of K* ' *I*] **by** *auto*

lemma *topological-space-generate-topology*: *class.topological-space* (*generate-topology S*)
by *standard* (*auto intro*: *generate-topology.intros*)

98.4 Order topologies

class *order-topology* = *order* + *open* +
assumes *open-generated-order*: *open* = *generate-topology* (*range* (λa. {..< *a*}) ∪ *range* (λa. {*a* <..}))
begin

subclass *topological-space*
unfolding *open-generated-order*
by (*rule topological-space-generate-topology*)

lemma *open-greaterThan* [*continuous-intros*, *simp*]: *open* {*a* <..}
unfolding *open-generated-order* **by** (*auto intro*: *generate-topology.Basis*)

lemma *open-lessThan* [*continuous-intros*, *simp*]: *open* {..< *a*}
unfolding *open-generated-order* **by** (*auto intro*: *generate-topology.Basis*)

lemma *open-greaterThanLessThan* [*continuous-intros*, *simp*]: *open* {*a* <..< *b*}
unfolding *greaterThanLessThan-eq* **by** (*simp add*: *open-Int*)

end

class *linorder-topology* = *linorder* + *order-topology*

lemma *closed-atMost* [*continuous-intros*, *simp*]: *closed* {..*a*}
for *a* :: *'a*::*linorder-topology*
by (*simp add*: *closed-open*)

lemma *closed-atLeast* [*continuous-intros*, *simp*]: *closed* {*a*..}
for *a* :: *'a*::*linorder-topology*
by (*simp add*: *closed-open*)

lemma *closed-atLeastAtMost* [*continuous-intros*, *simp*]: *closed* {*a*..*b*}
for *a b* :: *'a*::*linorder-topology*
proof −
have {*a* .. *b*} = {*a* ..} ∩ {.. *b*}
by *auto*
then show *?thesis*
by (*simp add*: *closed-Int*)
qed

lemma (**in** *order*) *less-separate*:
assumes $x < y$
shows ∃ *a b*. $x \in$ {..< *a*} ∧ $y \in$ {*b* <..} ∧ {..< *a*} ∩ {*b* <..} = {}
proof (*cases* ∃ *z*. $x < z \wedge z < y$)
case *True*
then obtain *z* **where** $x < z \wedge z < y$ **..**
then have $x \in$ {..< *z*} ∧ $y \in$ {*z* <..} ∧ {*z* <..} ∩ {..< *z*} = {}
by *auto*
then show *?thesis* **by** *blast*

next
  case *False*
  with ‹$x < y$› **have** $x \in \{..< y\}$ $y \in \{x <..\}$ $\{x <..\} \cap \{..< y\} = \{\}$
    by *auto*
  then show *?thesis* **by** *blast*
qed

instance *linorder-topology* $\subseteq$ *t2-space*
proof
  fix x y :: $'a$
  show $x \neq y \Longrightarrow \exists\, U\ V.\ open\ U \wedge open\ V \wedge x \in U \wedge y \in V \wedge U \cap V = \{\}$
    using *less-separate* [*of x y*] *less-separate* [*of y x*]
    by (*elim neqE*; *metis open-lessThan open-greaterThan Int-commute*)
qed

lemma (**in** *linorder-topology*) *open-right*:
  assumes *open* S $x \in S$
    and *gt-ex*: $x < y$
  shows $\exists\, b{>}x.\ \{x\ ..< b\} \subseteq S$
  using *assms* **unfolding** *open-generated-order*
proof *induct*
  case *UNIV*
  then show *?case* **by** *blast*
next
  case (*Int A B*)
  then obtain a b **where** $a > x$ $\{x\ ..< a\} \subseteq A$ $b > x$ $\{x\ ..< b\} \subseteq B$
    by *auto*
  then show *?case*
    by (*auto intro*!: *exI*[*of - min a b*])
next
  case *UN*
  then show *?case* **by** *blast*
next
  case *Basis*
  then show *?case*
    by (*fastforce intro*: *exI*[*of - y*] *gt-ex*)
qed

lemma (**in** *linorder-topology*) *open-left*:
  assumes *open* S $x \in S$
    and *lt-ex*: $y < x$
  shows $\exists\, b{<}x.\ \{b <..\ x\} \subseteq S$
  using *assms* **unfolding** *open-generated-order*
proof *induction*
  case *UNIV*
  then show *?case* **by** *blast*
next
  case (*Int A B*)
  then obtain a b **where** $a < x$ $\{a <..\ x\} \subseteq A$ $b < x$ $\{b <..\ x\} \subseteq B$

```
    by auto
  then show ?case
    by (auto intro!: exI[of - max a b])
next
  case UN
  then show ?case by blast
next
  case Basis
  then show ?case
    by (fastforce intro: exI[of - y] lt-ex)
qed
```

98.5 Setup some topologies

98.5.1 Boolean is an order topology

```
class discrete-topology = topological-space +
  assumes open-discrete: ⋀A. open A

instance discrete-topology < t2-space
proof
  fix x y :: 'a
  assume x ≠ y
  then show ∃ U V. open U ∧ open V ∧ x ∈ U ∧ y ∈ V ∧ U ∩ V = {}
    by (intro exI[of - {-}]) (auto intro!: open-discrete)
qed

instantiation bool :: linorder-topology
begin

definition open-bool :: bool set ⇒ bool
  where open-bool = generate-topology (range (λa. {..< a}) ∪ range (λa. {a <..}))

instance
  by standard (rule open-bool-def)

end

instance bool :: discrete-topology
proof
  fix A :: bool set
  have *: {False <..} = {True} {..< True} = {False}
    by auto
  have A = UNIV ∨ A = {} ∨ A = {False <..} ∨ A = {..< True}
    using subset-UNIV[of A] unfolding UNIV-bool * by blast
  then show open A
    by auto
qed

instantiation nat :: linorder-topology
```

begin

definition *open-nat* :: *nat set* ⇒ *bool*
  where *open-nat* = *generate-topology* (*range* (λ*a*. {..< *a*}) ∪ *range* (λ*a*. {*a* <..}))

instance
  by *standard* (*rule open-nat-def*)

end

instance *nat* :: *discrete-topology*
proof
  fix *A* :: *nat set*
  have *open* {*n*} **for** *n* :: *nat*
  proof (*cases n*)
    case *0*
    moreover have {*0*} = {..<*1*::*nat*}
      by *auto*
    ultimately show *?thesis*
       by *auto*
  next
    case (*Suc n'*)
    then have {*n*} = {..<*Suc n*} ∩ {*n'* <..}
      by *auto*
    with *Suc* **show** *?thesis*
      by (*auto intro*: *open-lessThan open-greaterThan*)
  qed
  then have *open* (⋃*a*∈*A*. {*a*})
    by (*intro open-UN*) *auto*
  then show *open A*
    by *simp*
qed

instantiation *int* :: *linorder-topology*
begin

definition *open-int* :: *int set* ⇒ *bool*
  where *open-int* = *generate-topology* (*range* (λ*a*. {..< *a*}) ∪ *range* (λ*a*. {*a* <..}))

instance
  by *standard* (*rule open-int-def*)

end

instance *int* :: *discrete-topology*
proof
  fix *A* :: *int set*
  have {..<*i* + *1*} ∩ {*i*−*1* <..} = {*i*} **for** *i* :: *int*
    by *auto*

then have *open* $\{i\}$ **for** i :: *int*
using *open-Int*[*OF open-lessThan*[*of* $i + 1$] *open-greaterThan*[*of* $i - 1$]] **by** *auto*
then have *open* $(\bigcup a{\in}A.\ \{a\})$
by (*intro open-UN*) *auto*
then show *open A*
by *simp*
qed

98.5.2 Topological filters

definition (**in** *topological-space*) *nhds* :: *'a ⇒ 'a filter*
where *nhds a* = (*INF S*∈{*S. open S* ∧ *a* ∈ *S*}. *principal S*)

definition (**in** *topological-space*) *at-within* :: *'a ⇒ 'a set ⇒ 'a filter*
(‹*at* (-)/ *within* (-)› [*1000*, *60*] *60*)
where *at a within s* = *inf* (*nhds a*) (*principal* (*s* − {*a*}))

abbreviation (**in** *topological-space*) *at* :: *'a ⇒ 'a filter* (‹*at*›)
where *at x* ≡ *at x within* (*CONST UNIV*)

abbreviation (**in** *order-topology*) *at-right* :: *'a ⇒ 'a filter*
where *at-right x* ≡ *at x within* {*x* <..}

abbreviation (**in** *order-topology*) *at-left* :: *'a ⇒ 'a filter*
where *at-left x* ≡ *at x within* {..< *x*}

lemma (**in** *topological-space*) *nhds-generated-topology*:
open = *generate-topology T* ⟹ *nhds x* = (*INF S*∈{*S*∈*T*. *x* ∈ *S*}. *principal S*)
unfolding *nhds-def*
proof (*safe intro*!: *antisym INF-greatest*)
fix *S*
assume *generate-topology T S x* ∈ *S*
then show (*INF S*∈{*S* ∈ *T*. *x* ∈ *S*}. *principal S*) ≤ *principal S*
by *induct*
(*auto intro*: *INF-lower order-trans simp*: *inf-principal*[*symmetric*] *simp del*: *inf-principal*)
qed (*auto intro*!: *INF-lower intro*: *generate-topology.intros*)

lemma (**in** *topological-space*) *eventually-nhds*:
eventually P (*nhds a*) ⟷ (∃ *S*. *open S* ∧ *a* ∈ *S* ∧ (∀ *x*∈*S*. *P x*))
unfolding *nhds-def* **by** (*subst eventually-INF-base*) (*auto simp*: *eventually-principal*)

lemma *eventually-eventually*:
eventually (λ*y*. *eventually P* (*nhds y*)) (*nhds x*) = *eventually P* (*nhds x*)
by (*auto simp*: *eventually-nhds*)

lemma (**in** *topological-space*) *eventually-nhds-in-open*:
open s ⟹ *x* ∈ *s* ⟹ *eventually* (λ*y*. *y* ∈ *s*) (*nhds x*)

by (*subst eventually-nhds*) *blast*

lemma (**in** *topological-space*) *eventually-nhds-x-imp-x*: *eventually* P (*nhds* x) $\Longrightarrow$ $P\ x$
by (*subst* (*asm*) *eventually-nhds*) *blast*

lemma (**in** *topological-space*) *nhds-neq-bot* [*simp*]: *nhds* $a \neq$ *bot*
by (*simp add*: *trivial-limit-def eventually-nhds*)

lemma (**in** *t1-space*) *t1-space-nhds*: $x \neq y \Longrightarrow (\forall_F\ x$ *in nhds* $x.\ x \neq y)$
by (*drule t1-space*) (*auto simp*: *eventually-nhds*)

lemma (**in** *topological-space*) *nhds-discrete-open*: *open* $\{x\} \Longrightarrow$ *nhds* x = *principal* $\{x\}$
by (*auto simp*: *nhds-def intro*!: *antisym INF-greatest INF-lower2*[*of* $\{x\}$])

lemma (**in** *discrete-topology*) *nhds-discrete*: *nhds* x = *principal* $\{x\}$
by (*simp add*: *nhds-discrete-open open-discrete*)

lemma (**in** *discrete-topology*) *at-discrete*: *at* x *within* S = *bot*
unfolding *at-within-def nhds-discrete* **by** *simp*

lemma (**in** *discrete-topology*) *tendsto-discrete*:
filterlim (f :: $'b \Rightarrow 'a$) (*nhds* y) $F \longleftrightarrow$ *eventually* ($\lambda x.\ f\ x = y$) F
by (*auto simp*: *nhds-discrete filterlim-principal*)

lemma (**in** *topological-space*) *at-within-eq*:
at x *within* s = (*INF* $S \in \{S.$ *open* $S \wedge x \in S\}$. *principal* $(S \cap s - \{x\})$)
unfolding *nhds-def at-within-def*
by (*subst INF-inf-const2*[*symmetric*]) (*auto simp*: *Diff-Int-distrib*)

lemma (**in** *topological-space*) *eventually-at-filter*:
eventually P (*at* a *within* s) $\longleftrightarrow$ *eventually* ($\lambda x.\ x \neq a \longrightarrow x \in s \longrightarrow P\ x$) (*nhds* a)
by (*simp add*: *at-within-def eventually-inf-principal imp-conjL*[*symmetric*] *conj-commute*)

lemma (**in** *topological-space*) *at-le*: $s \subseteq t \Longrightarrow$ *at* x *within* $s \leq$ *at* x *within* t
unfolding *at-within-def* **by** (*intro inf-mono*) *auto*

lemma (**in** *topological-space*) *eventually-at-topological*:
eventually P (*at* a *within* s) $\longleftrightarrow$ ($\exists S.$ *open* $S \wedge a \in S \wedge (\forall x \in S.\ x \neq a \longrightarrow x \in s \longrightarrow P\ x)$)
by (*simp add*: *eventually-nhds eventually-at-filter*)

lemma *eventually-nhds-conv-at*:
eventually P (*nhds* x) $\longleftrightarrow$ *eventually* P (*at* x) $\wedge P\ x$
unfolding *eventually-at-topological eventually-nhds* **by** *fast*

lemma *eventually-at-in-open*:

assumes *open A x* ∈ *A*
shows *eventually* (λ*y*. *y* ∈ *A* − {*x*}) (*at x*)
using *assms eventually-at-topological* **by** *blast*

lemma *eventually-at-in-open′*:
assumes *open A x* ∈ *A*
shows *eventually* (λ*y*. *y* ∈ *A*) (*at x*)
using *assms eventually-at-topological* **by** *blast*

lemma (**in** *topological-space*) *at-within-open*: *a* ∈ *S* ⟹ *open S* ⟹ *at a within S* = *at a*
unfolding *filter-eq-iff eventually-at-topological* **by** (*metis open-Int Int-iff UNIV-I*)

lemma (**in** *topological-space*) *at-within-open-NO-MATCH*:
a ∈ *s* ⟹ *open s* ⟹ *NO-MATCH UNIV s* ⟹ *at a within s* = *at a*
by (*simp only*: *at-within-open*)

lemma (**in** *topological-space*) *at-within-open-subset*:
a ∈ *S* ⟹ *open S* ⟹ *S* ⊆ *T* ⟹ *at a within T* = *at a*
by (*metis at-le at-within-open dual-order.antisym subset-UNIV*)

lemma (**in** *topological-space*) *at-within-nhd*:
assumes *x* ∈ *S open S T* ∩ *S* − {*x*} = *U* ∩ *S* − {*x*}
shows *at x within T* = *at x within U*
unfolding *filter-eq-iff eventually-at-filter*
proof (*intro allI eventually-subst*)
have *eventually* (λ*x*. *x* ∈ *S*) (*nhds x*)
using ‹*x* ∈ *S*› ‹*open S*› **by** (*auto simp*: *eventually-nhds*)
then show $\forall_F$ *n in nhds x*. (*n* ≠ *x* ⟶ *n* ∈ *T* ⟶ *P n*) = (*n* ≠ *x* ⟶ *n* ∈ *U* ⟶ *P n*) **for** *P*
by *eventually-elim* (*insert* ‹*T* ∩ *S* − {*x*} = *U* ∩ *S* − {*x*}›, *blast*)
qed

lemma (**in** *topological-space*) *at-within-empty* [*simp*]: *at a within* {} = *bot*
unfolding *at-within-def* **by** *simp*

lemma (**in** *topological-space*) *at-within-union*:
at x within (*S* ∪ *T*) = *sup* (*at x within S*) (*at x within T*)
unfolding *filter-eq-iff eventually-sup eventually-at-filter*
by (*auto elim*!: *eventually-rev-mp*)

lemma (**in** *topological-space*) *at-eq-bot-iff*: *at a* = *bot* ⟷ *open* {*a*}
unfolding *trivial-limit-def eventually-at-topological*
by (*metis UNIV-I empty-iff is-singletonE is-singletonI′ singleton-iff*)

lemma (**in** *t1-space*) *eventually-neq-at-within*:
eventually (λ*w*. *w* ≠ *x*) (*at z within A*)
by (*smt* (*verit*, *ccfv-threshold*) *eventually-True eventually-at-topological separation-t1*)

lemma (**in** *perfect-space*) *at-neq-bot* [*simp*]: *at a* ≠ *bot*
by (*simp add*: *at-eq-bot-iff not-open-singleton*)

lemma (**in** *order-topology*) *nhds-order*:
nhds x = *inf* (*INF a*∈{*x* <..}. *principal* {..< *a*}) (*INF a*∈{..< *x*}. *principal* {*a* <..})
proof −
have *1*: {*S* ∈ *range lessThan* ∪ *range greaterThan*. *x* ∈ *S*} =
(λ*a*. {..< *a*}) ‘ {*x* <..} ∪ (λ*a*. {*a* <..}) ‘ {..< *x*}
by *auto*
show *?thesis*
by (*simp only*: *nhds-generated-topology*[*OF open-generated-order*] *INF-union 1 INF-image comp-def*)
qed

lemma (**in** *topological-space*) *filterlim-at-within-If*:
assumes *filterlim f G* (*at x within* (*A* ∩ {*x*. *P x*}))
and *filterlim g G* (*at x within* (*A* ∩ {*x*. ¬*P x*}))
shows *filterlim* (λ*x*. *if P x then f x else g x*) *G* (*at x within A*)
proof (*rule filterlim-If*)
note *assms*(*1*)
also have *at x within* (*A* ∩ {*x*. *P x*}) = *inf* (*nhds x*) (*principal* (*A* ∩ *Collect P* − {*x*}))
by (*simp add*: *at-within-def*)
also have *A* ∩ *Collect P* − {*x*} = (*A* − {*x*}) ∩ *Collect P*
by *blast*
also have *inf* (*nhds x*) (*principal* . . .) = *inf* (*at x within A*) (*principal* (*Collect P*))
by (*simp add*: *at-within-def inf-assoc*)
finally show *filterlim f G* (*inf* (*at x within A*) (*principal* (*Collect P*))) **.**
next
note *assms*(*2*)
also have *at x within* (*A* ∩ {*x*. ¬ *P x*}) = *inf* (*nhds x*) (*principal* (*A* ∩ {*x*. ¬ *P x*} − {*x*}))
by (*simp add*: *at-within-def*)
also have *A* ∩ {*x*. ¬ *P x*} − {*x*} = (*A* − {*x*}) ∩ {*x*. ¬ *P x*}
by *blast*
also have *inf* (*nhds x*) (*principal* . . .) = *inf* (*at x within A*) (*principal* {*x*. ¬ *P x*})
by (*simp add*: *at-within-def inf-assoc*)
finally show *filterlim g G* (*inf* (*at x within A*) (*principal* {*x*. ¬ *P x*})) **.**
qed

lemma (**in** *topological-space*) *filterlim-at-If*:
assumes *filterlim f G* (*at x within* {*x*. *P x*})
and *filterlim g G* (*at x within* {*x*. ¬*P x*})
shows *filterlim* (λ*x*. *if P x then f x else g x*) *G* (*at x*)
using *assms* **by** (*intro filterlim-at-within-If*) *simp-all*

lemma (**in** *linorder-topology*) *at-within-order*:
assumes $UNIV \neq \{x\}$
shows *at x within s* =
inf (*INF* $a\in\{x <..\}$. *principal* ($\{..< a\} \cap s - \{x\}$))
(*INF* $a\in\{..< x\}$. *principal* ($\{a <..\} \cap s - \{x\}$))
proof (*cases* $\{x <..\} = \{\}$ $\{..< x\} = \{\}$ *rule*: *case-split* [*case-product case-split*])
case *True-True*
have $UNIV = \{..< x\} \cup \{x\} \cup \{x <..\}$
by *auto*
with *assms True-True* **show** *?thesis*
by *auto*
qed (*auto simp del*: *inf-principal simp*: *at-within-def nhds-order Int-Diff inf-principal*[*symmetric*] *INF-inf-const2 inf-sup-aci*[**where** *'a='a filter*])

lemma (**in** *linorder-topology*) *at-left-eq*:
$y < x \Longrightarrow$ *at-left* x = (*INF* $a\in\{..< x\}$. *principal* $\{a <..< x\}$)
by (*subst at-within-order*)
(*auto simp*: *greaterThan-Int-greaterThan greaterThanLessThan-eq*[*symmetric*]
min.absorb2 INF-constant
intro!: *INF-lower2 inf-absorb2*)

lemma (**in** *linorder-topology*) *eventually-at-left*:
$y < x \Longrightarrow$ *eventually P* (*at-left x*) $\longleftrightarrow$ ($\exists\, b<x.\ \forall\, y>b.\ y < x \longrightarrow P\ y$)
unfolding *at-left-eq*
by (*subst eventually-INF-base*) (*auto simp*: *eventually-principal Ball-def*)

lemma (**in** *linorder-topology*) *at-right-eq*:
$x < y \Longrightarrow$ *at-right* x = (*INF* $a\in\{x <..\}$. *principal* $\{x <..< a\}$)
by (*subst at-within-order*)
(*auto simp*: *lessThan-Int-lessThan greaterThanLessThan-eq*[*symmetric*] *max.absorb2*
INF-constant Int-commute
intro!: *INF-lower2 inf-absorb1*)

lemma (**in** *linorder-topology*) *eventually-at-right*:
$x < y \Longrightarrow$ *eventually P* (*at-right x*) $\longleftrightarrow$ ($\exists\, b>x.\ \forall\, y>x.\ y < b \longrightarrow P\ y$)
unfolding *at-right-eq*
by (*subst eventually-INF-base*) (*auto simp*: *eventually-principal Ball-def*)

lemma *eventually-at-right-less*: $\forall_F$ *y in at-right* (*x*::*'a*::{*linorder-topology*, *no-top*}).
$x < y$
using *gt-ex*[*of x*] *eventually-at-right*[*of x*] **by** *auto*

lemma *trivial-limit-at-right-top*: *at-right* (*top*::-::{*order-top*,*linorder-topology*}) =
bot
by (*auto simp*: *filter-eq-iff eventually-at-topological*)

lemma *trivial-limit-at-left-bot*: *at-left* (*bot*::-::{*order-bot*,*linorder-topology*}) = *bot*
by (*auto simp*: *filter-eq-iff eventually-at-topological*)

lemma *trivial-limit-at-left-real* [*simp*]: ¬ *trivial-limit* (*at-left x*)
for *x* :: *'a*::{*no-bot*,*dense-order*,*linorder-topology*}
using *lt-ex* [*of x*]
by *safe* (*auto simp add*: *trivial-limit-def eventually-at-left dest*: *dense*)

lemma *trivial-limit-at-right-real* [*simp*]: ¬ *trivial-limit* (*at-right x*)
for *x* :: *'a*::{*no-top*,*dense-order*,*linorder-topology*}
using *gt-ex*[*of x*]
by *safe* (*auto simp add*: *trivial-limit-def eventually-at-right dest*: *dense*)

lemma (**in** *linorder-topology*) *at-eq-sup-left-right*: *at x* = *sup* (*at-left x*) (*at-right x*)
by (*auto simp*: *eventually-at-filter filter-eq-iff eventually-sup*
elim: *eventually-elim2 eventually-mono*)

lemma (**in** *linorder-topology*) *eventually-at-split*:
eventually P (*at x*) ⟷ *eventually P* (*at-left x*) ∧ *eventually P* (*at-right x*)
by (*subst at-eq-sup-left-right*) (*simp add*: *eventually-sup*)

lemma (**in** *order-topology*) *eventually-at-leftI*:
assumes ⋀*x*. *x* ∈ {*a*<..<*b*} ⟹ *P x a* < *b*
shows *eventually P* (*at-left b*)
using *assms* **unfolding** *eventually-at-topological* **by** (*intro exI*[*of* - {*a*<..}]) *auto*

lemma (**in** *order-topology*) *eventually-at-rightI*:
assumes ⋀*x*. *x* ∈ {*a*<..<*b*} ⟹ *P x a* < *b*
shows *eventually P* (*at-right a*)
using *assms* **unfolding** *eventually-at-topological* **by** (*intro exI*[*of* - {..<*b*}]) *auto*

lemma *eventually-filtercomap-nhds*:
eventually P (*filtercomap f* (*nhds x*)) ⟷ (∃ *S*. *open S* ∧ *x* ∈ *S* ∧ (∀ *x*. *f x* ∈ *S* ⟶ *P x*))
unfolding *eventually-filtercomap eventually-nhds* **by** *auto*

lemma *eventually-filtercomap-at-topological*:
eventually P (*filtercomap f* (*at A within B*)) ⟷
(∃ *S*. *open S* ∧ *A* ∈ *S* ∧ (∀ *x*. *f x* ∈ *S* ∩ *B* − {*A*} ⟶ *P x*)) (**is** *?lhs* = *?rhs*)
unfolding *at-within-def filtercomap-inf eventually-inf-principal filtercomap-principal*

eventually-filtercomap-nhds eventually-principal **by** *blast*

lemma *eventually-at-right-field*:
eventually P (*at-right x*) ⟷ (∃ *b*>*x*. ∀ *y*>*x*. *y* < *b* ⟶ *P y*)
for *x* :: *'a*::{*linordered-field*, *linorder-topology*}
using *linordered-field-no-ub*[*rule-format*, *of x*]
by (*auto simp*: *eventually-at-right*)

lemma *eventually-at-left-field*:
eventually P (*at-left x*) ⟷ (∃ *b*<*x*. ∀ *y*>*b*. *y* < *x* ⟶ *P y*)

for *x* :: *'a*::{*linordered-field*, *linorder-topology*}
using *linordered-field-no-lb*[*rule-format*, *of x*]
by (*auto simp*: *eventually-at-left*)

lemma *filtermap-nhds-eq-imp-filtermap-at-eq*:
assumes *filtermap f* (*nhds z*) = *nhds* (*f z*)
assumes *eventually* (λ*x*. *f x* = *f z* ⟶ *x* = *z*) (*at z*)
shows *filtermap f* (*at z*) = *at* (*f z*)
proof (*rule filter-eqI*)
fix *P* :: *'a* ⇒ *bool*
have *eventually P* (*filtermap f* (*at z*)) ⟷ (∀$_F$ *x in nhds z*. *x* ≠ *z* ⟶ *P* (*f x*))
by (*simp add*: *eventually-filtermap eventually-at-filter*)
also have ... ⟷ (∀$_F$ *x in nhds z*. *f x* ≠ *f z* ⟶ *P* (*f x*))
by (*rule eventually-cong* [*OF assms*(*2*)[*unfolded eventually-at-filter*]]) *auto*
also have ... ⟷ (∀$_F$ *x in filtermap f* (*nhds z*). *x* ≠ *f z* ⟶ *P x*)
by (*simp add*: *eventually-filtermap*)
also have *filtermap f* (*nhds z*) = *nhds* (*f z*)
by (*rule assms*)
also have (∀$_F$ *x in nhds* (*f z*). *x* ≠ *f z* ⟶ *P x*) ⟷ (∀$_F$ *x in at* (*f z*). *P x*)
by (*simp add*: *eventually-at-filter*)
finally show *eventually P* (*filtermap f* (*at z*)) = *eventually P* (*at* (*f z*)) **.**
qed

98.5.3 Tendsto

abbreviation (**in** *topological-space*)
tendsto :: (*'b* ⇒ *'a*) ⇒ *'a* ⇒ *'b filter* ⇒ *bool* (**infixr** ‹⟶› *55*)
where (*f* ⟶ *l*) *F* ≡ *filterlim f* (*nhds l*) *F*

definition (**in** *t2-space*) *Lim* :: *'f filter* ⇒ (*'f* ⇒ *'a*) ⇒ *'a*
where *Lim A f* = (*THE l*. (*f* ⟶ *l*) *A*)

lemma (**in** *topological-space*) *tendsto-eq-rhs*: (*f* ⟶ *x*) *F* ⟹ *x* = *y* ⟹ (*f* ⟶ *y*) *F*
by *simp*

named-theorems *tendsto-intros introduction rules for tendsto*
setup ‹
Global-Theory.add-thms-dynamic (***binding*** ‹*tendsto-eq-intros*›,
fn context =>
Named-Theorems.get (*Context.proof-of context*) ***named-theorems*** ‹*tendsto-intros*›
|> *map-filter* (*try* (*fn thm* => @{*thm tendsto-eq-rhs*} *OF* [*thm*])))
›

context *topological-space* **begin**

lemma *tendsto-def*:
(*f* ⟶ *l*) *F* ⟷ (∀ *S*. *open S* ⟶ *l* ∈ *S* ⟶ *eventually* (λ*x*. *f x* ∈ *S*) *F*)
unfolding *nhds-def filterlim-INF filterlim-principal* **by** *auto*

lemma *tendsto-cong*: $(f \longrightarrow c)\ F \longleftrightarrow (g \longrightarrow c)\ F$ **if** *eventually* $(\lambda x.\ f\ x = g\ x)\ F$
by (*rule filterlim-cong* [*OF refl refl that*])

lemma *tendsto-mono*: $F \le F' \Longrightarrow (f \longrightarrow l)\ F' \Longrightarrow (f \longrightarrow l)\ F$
unfolding *tendsto-def le-filter-def* **by** *fast*

lemma *tendsto-ident-at* [*tendsto-intros*, *simp*, *intro*]: $((\lambda x.\ x) \longrightarrow a)$ (*at a within s*)
by (*auto simp*: *tendsto-def eventually-at-topological*)

lemma *tendsto-const* [*tendsto-intros*, *simp*, *intro*]: $((\lambda x.\ k) \longrightarrow k)\ F$
by (*simp add*: *tendsto-def*)

lemma *filterlim-at*:
(*LIM x F. f x :> at b within s*) $\longleftrightarrow$ *eventually* $(\lambda x.\ f\ x \in s \land f\ x \neq b)\ F \land (f \longrightarrow b)\ F$
by (*simp add*: *at-within-def filterlim-inf filterlim-principal conj-commute*)

lemma (**in** −)
assumes *filterlim f* (*nhds L*) *F*
shows *tendsto-imp-filterlim-at-right*:
eventually $(\lambda x.\ f\ x > L)\ F \Longrightarrow$ *filterlim f* (*at-right L*) *F*
and *tendsto-imp-filterlim-at-left*:
eventually $(\lambda x.\ f\ x < L)\ F \Longrightarrow$ *filterlim f* (*at-left L*) *F*
using *assms* **by** (*auto simp*: *filterlim-at elim*: *eventually-mono*)

lemma *filterlim-at-withinI*:
assumes *filterlim f* (*nhds c*) *F*
assumes *eventually* $(\lambda x.\ f\ x \in A - \{c\})\ F$
shows *filterlim f* (*at c within A*) *F*
using *assms* **by** (*simp add*: *filterlim-at*)

lemma *filterlim-atI*:
assumes *filterlim f* (*nhds c*) *F*
assumes *eventually* $(\lambda x.\ f\ x \neq c)\ F$
shows *filterlim f* (*at c*) *F*
using *assms* **by** (*intro filterlim-at-withinI*) *simp-all*

lemma *topological-tendstoI*:
$(\bigwedge S.\ open\ S \Longrightarrow l \in S \Longrightarrow eventually\ (\lambda x.\ f\ x \in S)\ F) \Longrightarrow (f \longrightarrow l)\ F$
by (*auto simp*: *tendsto-def*)

lemma *topological-tendstoD*:
$(f \longrightarrow l)\ F \Longrightarrow open\ S \Longrightarrow l \in S \Longrightarrow eventually\ (\lambda x.\ f\ x \in S)\ F$
by (*auto simp*: *tendsto-def*)

lemma *tendsto-bot* [*simp*]: $(f \longrightarrow a)$ *bot*

by (*simp add*: *tendsto-def*)

lemma *tendsto-eventually*: *eventually* ($\lambda x.\ f\ x = l$) *net* $\Longrightarrow$ (($\lambda x.\ f\ x$) $\longrightarrow$ l) *net*
by (*rule topological-tendstoI*) (*auto elim*: *eventually-mono*)

lemma *tendsto-principal-singleton*[*simp*]:
shows ($f \longrightarrow f\ x$) (*principal* $\{x\}$)
unfolding *tendsto-def eventually-principal* **by** *simp*

end

lemma (**in** *topological-space*) *filterlim-within-subset*:
filterlim f l (*at x within S*) $\Longrightarrow T \subseteq S \Longrightarrow$ *filterlim f l* (*at x within T*)
by (*blast intro*: *filterlim-mono at-le*)

lemmas *tendsto-within-subset* = *filterlim-within-subset*

lemma (**in** *order-topology*) *order-tendsto-iff*:
($f \longrightarrow x$) $F \longleftrightarrow$ ($\forall\, l<x.$ *eventually* ($\lambda x.\ l < f\ x$) F) $\wedge$ ($\forall\, u>x.$ *eventually* ($\lambda x.$ $f\ x < u$) F)
by (*auto simp*: *nhds-order filterlim-inf filterlim-INF filterlim-principal*)

lemma (**in** *order-topology*) *order-tendstoI*:
($\bigwedge a.\ a < y \Longrightarrow$ *eventually* ($\lambda x.\ a < f\ x$) F) $\Longrightarrow$ ($\bigwedge a.\ y < a \Longrightarrow$ *eventually* ($\lambda x.$ $f\ x < a$) F) $\Longrightarrow$
($f \longrightarrow y$) F
by (*auto simp*: *order-tendsto-iff*)

lemma (**in** *order-topology*) *order-tendstoD*:
assumes ($f \longrightarrow y$) F
shows $a < y \Longrightarrow$ *eventually* ($\lambda x.\ a < f\ x$) F
and $y < a \Longrightarrow$ *eventually* ($\lambda x.\ f\ x < a$) F
using *assms* **by** (*auto simp*: *order-tendsto-iff*)

lemma (**in** *linorder-topology*) *tendsto-max*[*tendsto-intros*]:
assumes X: ($X \longrightarrow x$) *net*
and Y: ($Y \longrightarrow y$) *net*
shows (($\lambda x.$ *max* ($X\ x$) ($Y\ x$)) $\longrightarrow$ *max* $x\ y$) *net*
proof (*rule order-tendstoI*)
fix a
assume $a <$ *max* $x\ y$
then show *eventually* ($\lambda x.\ a <$ *max* ($X\ x$) ($Y\ x$)) *net*
using *order-tendstoD*(*1*)[*OF X*, *of a*] *order-tendstoD*(*1*)[*OF Y*, *of a*]
by (*auto simp*: *less-max-iff-disj elim*: *eventually-mono*)
next
fix a
assume *max* $x\ y < a$
then show *eventually* ($\lambda x.$ *max* ($X\ x$) ($Y\ x$) $< a$) *net*

using *order-tendstoD(2)[OF X, of a] order-tendstoD(2)[OF Y, of a]*
by (*auto simp*: *eventually-conj-iff*)
qed

lemma (**in** *linorder-topology*) *tendsto-min[tendsto-intros]*:
assumes *X*: $(X \longrightarrow x)$ *net*
and *Y*: $(Y \longrightarrow y)$ *net*
shows $((\lambda x.\ min\ (X\ x)\ (Y\ x)) \longrightarrow min\ x\ y)$ *net*
proof (*rule order-tendstoI*)
fix *a*
assume $a < min\ x\ y$
then show *eventually* $(\lambda x.\ a < min\ (X\ x)\ (Y\ x))$ *net*
using *order-tendstoD(1)[OF X, of a] order-tendstoD(1)[OF Y, of a]*
by (*auto simp*: *eventually-conj-iff*)
next
fix *a*
assume $min\ x\ y < a$
then show *eventually* $(\lambda x.\ min\ (X\ x)\ (Y\ x) < a)$ *net*
using *order-tendstoD(2)[OF X, of a] order-tendstoD(2)[OF Y, of a]*
by (*auto simp*: *min-less-iff-disj elim*: *eventually-mono*)
qed

lemma (**in** *order-topology*)
assumes $a < b$
shows *at-within-Icc-at-right*: *at a within* $\{a..b\}$ = *at-right a*
and *at-within-Icc-at-left*: *at b within* $\{a..b\}$ = *at-left b*
using *order-tendstoD(2)[OF tendsto-ident-at assms, of* $\{a<..\}$*]*
using *order-tendstoD(1)[OF tendsto-ident-at assms, of* $\{..<b\}$*]*
by (*auto intro!*: *order-class.order-antisym filter-leI*
simp: *eventually-at-filter less-le*
elim: *eventually-elim2*)

lemma (**in** *order-topology*)
shows *at-within-Ici-at-right*: *at a within* $\{a..\}$ = *at-right a*
and *at-within-Iic-at-left*: *at a within* $\{..a\}$ = *at-left a*
using *order-tendstoD(2)[OF tendsto-ident-at* [**where** $s = \{a<..\}$]]
using *order-tendstoD(1)[OF tendsto-ident-at*[**where** $s = \{..<a\}$]]
by (*auto intro!*: *order-class.order-antisym filter-leI*
simp: *eventually-at-filter less-le*
elim: *eventually-elim2*)

lemma (**in** *order-topology*) *at-within-Icc-at*: $a < x \Longrightarrow x < b \Longrightarrow$ *at x within* $\{a..b\}$ = *at x*
by (*rule at-within-open-subset*[**where** $S=\{a<..<b\}$]) *auto*

lemma (**in** *t2-space*) *tendsto-unique*:
assumes $F \neq bot$
and $(f \longrightarrow a)$ *F*
and $(f \longrightarrow b)$ *F*

shows $a = b$
proof (*rule ccontr*)
assume $a \neq b$
obtain U V **where** *open* U *open* V $a \in U$ $b \in V$ $U \cap V = \{\}$
using *hausdorff* [*OF* ‹$a \neq b$›] **by** *fast*
have *eventually* ($\lambda x.\ f\ x \in U$) F
using ‹($f \longrightarrow a$) F› ‹*open* U› ‹$a \in U$› **by** (*rule topological-tendstoD*)
moreover
have *eventually* ($\lambda x.\ f\ x \in V$) F
using ‹($f \longrightarrow b$) F› ‹*open* V› ‹$b \in V$› **by** (*rule topological-tendstoD*)
ultimately
have *eventually* ($\lambda x.$ *False*) F
proof *eventually-elim*
case (*elim x*)
then have $f\ x \in U \cap V$ **by** *simp*
with ‹$U \cap V = \{\}$› **show** *?case* **by** *simp*
qed
with ‹$\neg$ *trivial-limit* F› **show** *False*
by (*simp add*: *trivial-limit-def*)
qed

lemma (**in** *t2-space*) *tendsto-const-iff*:
fixes a b :: $'a$
assumes $\neg$ *trivial-limit* F
shows (($\lambda x.\ a$) $\longrightarrow b$) $F \longleftrightarrow a = b$
by (*auto intro*!: *tendsto-unique* [*OF assms tendsto-const*])

lemma (**in** *t2-space*) *tendsto-unique'*:
assumes $F \neq bot$
shows $\exists_{\leq 1} l.$ ($f \longrightarrow l$) F
using *Uniq-def assms local.tendsto-unique* **by** *fastforce*

lemma *Lim-in-closed-set*:
assumes *closed* S *eventually* ($\lambda x.\ f(x) \in S$) F $F \neq bot$ ($f \longrightarrow l$) F
shows $l \in S$
proof (*rule ccontr*)
assume $l \notin S$
with ‹*closed* S› **have** *open* ($-$ S) $l \in -$ S
by (*simp-all add*: *open-Compl*)
with *assms*(*4*) **have** *eventually* ($\lambda x.\ f\ x \in -$ S) F
by (*rule topological-tendstoD*)
with *assms*(*2*) **have** *eventually* ($\lambda x.$ *False*) F
by (*rule eventually-elim2*) *simp*
with *assms*(*3*) **show** *False*
by (*simp add*: *eventually-False*)
qed

lemma (**in** *t3-space*) *nhds-closed*:
assumes $x \in A$ **and** *open* A

shows $\exists A'. x \in A' \wedge$ *closed* $A' \wedge A' \subseteq A \wedge$ *eventually* $(\lambda y. y \in A')$ (*nhds* x)
proof −
from *assms* **have** $\exists U V.$ *open* $U \wedge$ *open* $V \wedge x \in U \wedge - A \subseteq V \wedge U \cap V = \{\}$
by (*intro t3-space*) *auto*
then obtain U V **where** UV: *open* U *open* V $x \in U$ $-A \subseteq V$ $U \cap V = \{\}$
by *auto*
have *eventually* $(\lambda y. y \in U)$ (*nhds* x)
using ‹*open* U› **and** ‹$x \in U$› **by** (*intro eventually-nhds-in-open*)
hence *eventually* $(\lambda y. y \in -V)$ (*nhds* x)
by *eventually-elim* (*use* UV **in** *auto*)
with UV **show** *?thesis* **by** (*intro exI*[*of* - $-V$]) *auto*
qed

lemma (**in** *order-topology*) *increasing-tendsto*:
assumes *bdd*: *eventually* $(\lambda n. f\, n \leq l)$ F
and *en*: $\bigwedge x. x < l \Longrightarrow$ *eventually* $(\lambda n. x < f\, n)$ F
shows $(f \longrightarrow l)$ F
using *assms* **by** (*intro order-tendstoI*) (*auto elim*!: *eventually-mono*)

lemma (**in** *order-topology*) *decreasing-tendsto*:
assumes *bdd*: *eventually* $(\lambda n. l \leq f\, n)$ F
and *en*: $\bigwedge x. l < x \Longrightarrow$ *eventually* $(\lambda n. f\, n < x)$ F
shows $(f \longrightarrow l)$ F
using *assms* **by** (*intro order-tendstoI*) (*auto elim*!: *eventually-mono*)

lemma (**in** *order-topology*) *tendsto-sandwich*:
assumes *ev*: *eventually* $(\lambda n. f\, n \leq g\, n)$ *net* *eventually* $(\lambda n. g\, n \leq h\, n)$ *net*
assumes *lim*: $(f \longrightarrow c)$ *net* $(h \longrightarrow c)$ *net*
shows $(g \longrightarrow c)$ *net*
proof (*rule order-tendstoI*)
fix a
show $a < c \Longrightarrow$ *eventually* $(\lambda x. a < g\, x)$ *net*
using *order-tendstoD*[*OF lim*(*1*), *of a*] *ev* **by** (*auto elim*: *eventually-elim2*)
next
fix a
show $c < a \Longrightarrow$ *eventually* $(\lambda x. g\, x < a)$ *net*
using *order-tendstoD*[*OF lim*(*2*), *of a*] *ev* **by** (*auto elim*: *eventually-elim2*)
qed

lemma (**in** *t1-space*) *limit-frequently-eq*:
assumes $F \neq$ *bot*
and *frequently* $(\lambda x. f\, x = c)$ F
and $(f \longrightarrow d)$ F
shows $d = c$
proof (*rule ccontr*)
assume $d \neq c$
from *t1-space*[*OF this*] **obtain** U **where** *open* U $d \in U$ $c \notin U$
by *blast*

with *assms* **have** *eventually* $(\lambda x.\ f\ x \in U)\ F$
unfolding *tendsto-def* **by** *blast*
then have *eventually* $(\lambda x.\ f\ x \neq c)\ F$
by *eventually-elim* (*insert* ‹$c \notin U$›, *blast*)
with *assms*(*2*) **show** *False*
unfolding *frequently-def* **by** *contradiction*
qed

lemma (**in** *t1-space*) *tendsto-imp-eventually-ne*:
assumes $(f \longrightarrow c)\ F\ c \neq c'$
shows *eventually* $(\lambda z.\ f\ z \neq c')\ F$
proof (*cases* F=*bot*)
case *True*
thus *?thesis* **by** *auto*
next
case *False*
show *?thesis*
proof (*rule ccontr*)
assume $\neg$ *eventually* $(\lambda z.\ f\ z \neq c')\ F$
then have *frequently* $(\lambda z.\ f\ z = c')\ F$
by (*simp add*: *frequently-def*)
from *limit-frequently-eq*[*OF False this* ‹$(f \longrightarrow c)\ F$›] **and** ‹$c \neq c'$› **show** *False*
by *contradiction*
qed
qed

lemma (**in** *linorder-topology*) *tendsto-le*:
assumes F: $\neg$ *trivial-limit* F
and x: $(f \longrightarrow x)\ F$
and y: $(g \longrightarrow y)\ F$
and *ev*: *eventually* $(\lambda x.\ g\ x \leq f\ x)\ F$
shows $y \leq x$
proof (*rule ccontr*)
assume $\neg\ y \leq x$
with *less-separate*[*of x y*] **obtain** $a\ b$ **where** *xy*: $x < a\ b < y\ \{..<a\} \cap \{b<..\} = \{\}$
by (*auto simp*: *not-le*)
then have *eventually* $(\lambda x.\ f\ x < a)\ F$ *eventually* $(\lambda x.\ b < g\ x)\ F$
using $x\ y$ **by** (*auto intro*: *order-tendstoD*)
with *ev* **have** *eventually* $(\lambda x.$ *False*$)\ F$
by *eventually-elim* (*insert xy*, *fastforce*)
with F **show** *False*
by (*simp add*: *eventually-False*)
qed

lemma (**in** *linorder-topology*) *tendsto-lowerbound*:
assumes x: $(f \longrightarrow x)\ F$
and *ev*: *eventually* $(\lambda i.\ a \leq f\ i)\ F$

and *F*: ¬ *trivial-limit F*
shows $a \leq x$
using *F x tendsto-const ev* **by** (*rule tendsto-le*)

lemma (**in** *linorder-topology*) *tendsto-upperbound*:
assumes *x*: $(f \longrightarrow x)$ *F*
and *ev*: *eventually* $(\lambda i.\ a \geq f\ i)$ *F*
and *F*: ¬ *trivial-limit F*
shows $a \geq x$
by (*rule tendsto-le* [*OF F tendsto-const x ev*])

lemma *filterlim-at-within-not-equal*:
fixes $f::'a \Rightarrow 'b::t2\text{-}space$
assumes *filterlim f* (*at a within s*) *F*
shows *eventually* $(\lambda w.\ f\ w \in s \wedge f\ w \neq b)$ *F*
proof (*cases a=b*)
case *True*
then show *?thesis* **using** *assms* **by** (*simp add*: *filterlim-at*)
next
case *False*
from *hausdorff*[*OF this*] **obtain** *U V* **where** *UV*:*open U open V* $a \in U$ $b \in V$ $U \cap V = \{\}$
by *auto*
have $(f \longrightarrow a)$ *F* **using** *assms filterlim-at* **by** *auto*
then have $\forall_F$ *x in F*. $f\ x \in U$ **using** *UV* **unfolding** *tendsto-def* **by** *auto*
moreover have $\forall_F$ *x in F*. $f\ x \in s \wedge f\ x \neq a$ **using** *assms filterlim-at* **by** *auto*
ultimately show *?thesis*
apply *eventually-elim*
using *UV* **by** *auto*
qed

98.5.4 Rules about *Lim*

lemma *tendsto-Lim*: ¬ *trivial-limit net* $\Longrightarrow (f \longrightarrow l)$ *net* $\Longrightarrow$ *Lim net f* = *l*
unfolding *Lim-def* **using** *tendsto-unique* [*of net f*] **by** *auto*

lemma *Lim-ident-at*: ¬ *trivial-limit* (*at x within s*) $\Longrightarrow$ *Lim* (*at x within s*) $(\lambda x.\ x) = x$
by (*simp add*: *tendsto-Lim*)

lemma *Lim-cong*:
assumes $\forall_F$ *x in F*. $f\ x = g\ x$ $F = G$
shows *Lim F f* = *Lim F g*
unfolding *t2-space-class.Lim-def* **using** *tendsto-cong assms* **by** *fastforce*

lemma *eventually-Lim-ident-at*:
($\forall_F$ *y in at x within X*. *P* (*Lim* (*at x within X*) $(\lambda x.\ x)$) *y*) $\longleftrightarrow$
($\forall_F$ *y in at x within X*. *P x y*) **for** $x::'a::t2\text{-}space$
by (*cases at x within X = bot*) (*auto simp*: *Lim-ident-at*)

lemma *filterlim-at-bot-at-right*:
  fixes f :: $'a$::*linorder-topology* ⇒ $'b$::*linorder*
  assumes *mono*: $\bigwedge x\ y.\ Q\ x \Longrightarrow Q\ y \Longrightarrow x \le y \Longrightarrow f\ x \le f\ y$
    and *bij*: $\bigwedge x.\ P\ x \Longrightarrow f\ (g\ x) = x$ $\bigwedge x.\ P\ x \Longrightarrow Q\ (g\ x)$
    and Q: *eventually* Q (*at-right* a)
    and *bound*: $\bigwedge b.\ Q\ b \Longrightarrow a < b$
    and P: *eventually* P *at-bot*
  shows *filterlim* f *at-bot* (*at-right* a)
proof −
  from P **obtain** x **where** x: $\bigwedge y.\ y \le x \Longrightarrow P\ y$
    unfolding *eventually-at-bot-linorder* **by** *auto*
  show *?thesis*
  proof (*intro filterlim-at-bot-le*[*THEN iffD2*] *allI impI*)
    fix z
    assume $z \le x$
    with x **have** $P\ z$ **by** *auto*
    have *eventually* $(\lambda x.\ x \le g\ z)$ (*at-right* a)
      using *bound*[*OF bij*(*2*)[*OF* ‹$P\ z$›]]
      unfolding *eventually-at-right*[*OF bound*[*OF bij*(*2*)[*OF* ‹$P\ z$›]]]
      by (*auto intro*!: *exI*[*of - g z*])
    with Q **show** *eventually* $(\lambda x.\ f\ x \le z)$ (*at-right* a)
      by *eventually-elim* (*metis bij* ‹$P\ z$› *mono*)
  qed
qed

lemma *filterlim-at-top-at-left*:
  fixes f :: $'a$::*linorder-topology* ⇒ $'b$::*linorder*
  assumes *mono*: $\bigwedge x\ y.\ Q\ x \Longrightarrow Q\ y \Longrightarrow x \le y \Longrightarrow f\ x \le f\ y$
    and *bij*: $\bigwedge x.\ P\ x \Longrightarrow f\ (g\ x) = x$ $\bigwedge x.\ P\ x \Longrightarrow Q\ (g\ x)$
    and Q: *eventually* Q (*at-left* a)
    and *bound*: $\bigwedge b.\ Q\ b \Longrightarrow b < a$
    and P: *eventually* P *at-top*
  shows *filterlim* f *at-top* (*at-left* a)
proof −
  from P **obtain** x **where** x: $\bigwedge y.\ x \le y \Longrightarrow P\ y$
    unfolding *eventually-at-top-linorder* **by** *auto*
  show *?thesis*
  proof (*intro filterlim-at-top-ge*[*THEN iffD2*] *allI impI*)
    fix z
    assume $x \le z$
    with x **have** $P\ z$ **by** *auto*
    have *eventually* $(\lambda x.\ g\ z \le x)$ (*at-left* a)
      using *bound*[*OF bij*(*2*)[*OF* ‹$P\ z$›]]
      unfolding *eventually-at-left*[*OF bound*[*OF bij*(*2*)[*OF* ‹$P\ z$›]]]
      by (*auto intro*!: *exI*[*of - g z*])
    with Q **show** *eventually* $(\lambda x.\ z \le f\ x)$ (*at-left* a)
      by *eventually-elim* (*metis bij* ‹$P\ z$› *mono*)
  qed

qed

lemma *filterlim-split-at*:
filterlim f F (*at-left x*) ⟹ *filterlim f F* (*at-right x*) ⟹
filterlim f F (*at x*)
for *x* :: *'a::linorder-topology*
by (*subst at-eq-sup-left-right*) (*rule filterlim-sup*)

lemma *filterlim-at-split*:
filterlim f F (*at x*) ⟷ *filterlim f F* (*at-left x*) ∧ *filterlim f F* (*at-right x*)
for *x* :: *'a::linorder-topology*
by (*subst at-eq-sup-left-right*) (*simp add*: *filterlim-def filtermap-sup*)

lemma *eventually-nhds-top*:
fixes *P* :: *'a* :: {*order-top*,*linorder-topology*} ⇒ *bool*
and *b* :: *'a*
assumes *b* < *top*
shows *eventually P* (*nhds top*) ⟷ (∃ *b*<*top*. (∀ *z*. *b* < *z* ⟶ *P z*))
unfolding *eventually-nhds*
proof *safe*
fix *S* :: *'a set*
assume *open S top* ∈ *S*
note *open-left*[*OF this* ‹*b* < *top*›]
moreover assume ∀ *s*∈*S*. *P s*
ultimately show ∃ *b*<*top*. ∀ *z*>*b*. *P z*
by (*auto simp*: *subset-eq Ball-def*)
next
fix *b*
assume *b* < *top* ∀ *z*>*b*. *P z*
then show ∃ *S*. *open S* ∧ *top* ∈ *S* ∧ (∀ *xa*∈*S*. *P xa*)
by (*intro exI*[*of* - {*b* <..}]) *auto*
qed

lemma *tendsto-at-within-iff-tendsto-nhds*:
(*g* ⟶ *g l*) (*at l within S*) ⟷ (*g* ⟶ *g l*) (*inf* (*nhds l*) (*principal S*))
unfolding *tendsto-def eventually-at-filter eventually-inf-principal*
by (*intro ext all-cong imp-cong*) (*auto elim*!: *eventually-mono*)

98.6 Limits on sequences

abbreviation (**in** *topological-space*)
LIMSEQ :: [*nat* ⇒ *'a*, *'a*] ⇒ *bool* (‹(‹*notation*=‹*infix LIMSEQ*››(-)/ ⟶ (-))›
[*60*, *60*] *60*)
where *X* ⟶ *L* ≡ (*X* ⟶ *L*) *sequentially*

abbreviation (**in** *t2-space*) *lim* :: (*nat* ⇒ *'a*) ⇒ *'a*
where *lim X* ≡ *Lim sequentially X*

definition (**in** *topological-space*) *convergent* :: (*nat* ⇒ *'a*) ⇒ *bool*

where *convergent* $X = (\exists L.\ X \longrightarrow L)$

lemma *lim-def*: $lim\ X = (THE\ L.\ X \longrightarrow L)$
unfolding *Lim-def* **..**

lemma *lim-explicit*:
$f \longrightarrow f0 \longleftrightarrow (\forall S.\ open\ S \longrightarrow f0 \in S \longrightarrow (\exists N.\ \forall n\geq N.\ f\ n \in S))$
unfolding *tendsto-def eventually-sequentially* **by** *auto*

lemma *closed-sequentially*:
assumes *closed* S **and** $\bigwedge n.\ f\ n \in S$ **and** $f \longrightarrow l$
shows $l \in S$
by (*metis Lim-in-closed-set assms eventually-sequentially trivial-limit-sequentially*)

98.7 Monotone sequences and subsequences

Definition of monotonicity. The use of disjunction here complicates proofs considerably. One alternative is to add a Boolean argument to indicate the direction. Another is to develop the notions of increasing and decreasing first.

definition *monoseq* :: $(nat \Rightarrow 'a::order) \Rightarrow bool$
where $monoseq\ X \longleftrightarrow (\forall m.\ \forall n\geq m.\ X\ m \leq X\ n) \vee (\forall m.\ \forall n\geq m.\ X\ n \leq X\ m)$

abbreviation *incseq* :: $(nat \Rightarrow 'a::order) \Rightarrow bool$
where $incseq\ X \equiv mono\ X$

lemma *incseq-def*: $incseq\ X \longleftrightarrow (\forall m.\ \forall n\geq m.\ X\ n \geq X\ m)$
unfolding *mono-def* **..**

abbreviation *decseq* :: $(nat \Rightarrow 'a::order) \Rightarrow bool$
where $decseq\ X \equiv antimono\ X$

lemma *decseq-def*: $decseq\ X \longleftrightarrow (\forall m.\ \forall n\geq m.\ X\ n \leq X\ m)$
unfolding *antimono-def* **..**

98.7.1 Definition of subsequence.

lemma *strict-mono-leD*: $strict\text{-}mono\ r \Longrightarrow m \leq n \Longrightarrow r\ m \leq r\ n$
by (*erule* (*1*) *monoD* [*OF strict-mono-mono*])

lemma *strict-mono-id*: *strict-mono id*
by (*simp add*: *strict-mono-def*)

lemma *incseq-SucI*: $(\bigwedge n.\ X\ n \leq X\ (Suc\ n)) \Longrightarrow incseq\ X$
by (*simp add*: *mono-iff-le-Suc*)

lemma *incseqD*: $incseq\ f \Longrightarrow i \leq j \Longrightarrow f\ i \leq f\ j$
by (*auto simp*: *incseq-def*)

lemma *incseq-SucD*: *incseq* $A \Longrightarrow A\ i \leq A\ (Suc\ i)$
using *incseqD*[*of A i Suc i*] **by** *auto*

lemma *incseq-Suc-iff*: *incseq f* $\longleftrightarrow (\forall n.\ f\ n \leq f\ (Suc\ n))$
by (*auto intro*: *incseq-SucI dest*: *incseq-SucD*)

lemma *incseq-const*[*simp*, *intro*]: *incseq* $(\lambda x.\ k)$
unfolding *incseq-def* **by** *auto*

lemma *decseq-SucI*: $(\bigwedge n.\ X\ (Suc\ n) \leq X\ n) \Longrightarrow$ *decseq X*
by (*simp add*: *antimono-iff-le-Suc*)

lemma *decseqD*: *decseq f* $\Longrightarrow i \leq j \Longrightarrow f\ j \leq f\ i$
by (*auto simp*: *decseq-def*)

lemma *decseq-SucD*: *decseq* $A \Longrightarrow A\ (Suc\ i) \leq A\ i$
using *decseqD*[*of A i Suc i*] **by** *auto*

lemma *decseq-Suc-iff*: *decseq f* $\longleftrightarrow (\forall n.\ f\ (Suc\ n) \leq f\ n)$
by (*auto intro*: *decseq-SucI dest*: *decseq-SucD*)

lemma *decseq-const*[*simp*, *intro*]: *decseq* $(\lambda x.\ k)$
unfolding *decseq-def* **by** *auto*

lemma *monoseq-iff*: *monoseq X* $\longleftrightarrow$ *incseq X* $\vee$ *decseq X*
unfolding *monoseq-def incseq-def decseq-def* **..**

lemma *monoseq-Suc*: *monoseq X* $\longleftrightarrow (\forall n.\ X\ n \leq X\ (Suc\ n)) \vee (\forall n.\ X\ (Suc\ n)$
$\leq X\ n)$
unfolding *monoseq-iff incseq-Suc-iff decseq-Suc-iff* **..**

lemma *monoI1*: $\forall m.\ \forall n \geq m.\ X\ m \leq X\ n \Longrightarrow$ *monoseq X*
by (*simp add*: *monoseq-def*)

lemma *monoI2*: $\forall m.\ \forall n \geq m.\ X\ n \leq X\ m \Longrightarrow$ *monoseq X*
by (*simp add*: *monoseq-def*)

lemma *mono-SucI1*: $\forall n.\ X\ n \leq X\ (Suc\ n) \Longrightarrow$ *monoseq X*
by (*simp add*: *monoseq-Suc*)

lemma *mono-SucI2*: $\forall n.\ X\ (Suc\ n) \leq X\ n \Longrightarrow$ *monoseq X*
by (*simp add*: *monoseq-Suc*)

lemma *monoseq-minus*:
fixes $a :: nat \Rightarrow$ *'a::ordered-ab-group-add*
assumes *monoseq a*
shows *monoseq* $(\lambda\ n.\ -\ a\ n)$
proof (*cases* $\forall m.\ \forall n \geq m.\ a\ m \leq a\ n$)
case *True*

then have ∀ *m*. ∀ *n* ≥ *m*. − *a n* ≤ − *a m* **by** *auto*
then show *?thesis* **by** (*rule monoI2*)
next
case *False*
then have ∀ *m*. ∀ *n* ≥ *m*. − *a m* ≤ − *a n*
using ‹*monoseq a*›[*unfolded monoseq-def*] **by** *auto*
then show *?thesis* **by** (*rule monoI1*)
qed

98.7.2 Subsequence (alternative definition, (e.g. Hoskins)

For any sequence, there is a monotonic subsequence.

lemma *seq-monosub*:
fixes *s* :: *nat* ⇒ *'a*::*linorder*
shows ∃ *f*. *strict-mono f* ∧ *monoseq* (λ*n*. (*s* (*f n*)))
proof (*cases* ∀ *n*. ∃ *p*>*n*. ∀ *m*≥*p*. *s m* ≤ *s p*)
case *True*
then have ∃ *f*. ∀ *n*. (∀ *m*≥*f n*. *s m* ≤ *s* (*f n*)) ∧ *f n* < *f* (*Suc n*)
by (*intro dependent-nat-choice*) (*auto simp*: *conj-commute*)
then obtain *f* :: *nat* ⇒ *nat*
where *f*: *strict-mono f* **and** *mono*: ⋀*n m*. *f n* ≤ *m* ⟹ *s m* ≤ *s* (*f n*)
by (*auto simp*: *strict-mono-Suc-iff*)
then have *incseq f*
unfolding *strict-mono-Suc-iff incseq-Suc-iff* **by** (*auto intro*: *less-imp-le*)
then have *monoseq* (λ*n*. *s* (*f n*))
by (*auto simp add*: *incseq-def intro*!: *mono monoI2*)
with *f* **show** *?thesis*
by *auto*
next
case *False*
then obtain *N* **where** *N*: *p* > *N* ⟹ ∃ *m*>*p*. *s p* < *s m* **for** *p*
by (*force simp*: *not-le le-less*)
have ∃ *f*. ∀ *n*. *N* < *f n* ∧ *f n* < *f* (*Suc n*) ∧ *s* (*f n*) ≤ *s* (*f* (*Suc n*))
proof (*intro dependent-nat-choice*)
fix *x*
assume *N* < *x* **with** *N*[*of x*]
show ∃ *y*>*N*. *x* < *y* ∧ *s x* ≤ *s y*
by (*auto intro*: *less-trans*)
qed *auto*
then show *?thesis*
by (*auto simp*: *monoseq-iff incseq-Suc-iff strict-mono-Suc-iff*)
qed

lemma *seq-suble*:
assumes *sf*: *strict-mono* (*f* :: *nat* ⇒ *nat*)
shows *n* ≤ *f n*
proof (*induct n*)
case *0*
show *?case* **by** *simp*

next
  case (*Suc n*)
  with *sf* [*unfolded strict-mono-Suc-iff*, *rule-format*, *of n*] **have** $n < f$ (*Suc n*)
    by *arith*
  then show *?case* **by** *arith*
qed

lemma *eventually-subseq*:
  strict-mono r $\Longrightarrow$ *eventually P sequentially* $\Longrightarrow$ *eventually* ($\lambda n.$ *P* (*r n*)) *sequentially*
  unfolding *eventually-sequentially* **by** (*metis seq-suble le-trans*)

lemma *not-eventually-sequentiallyD*:
  assumes $\neg$ *eventually P sequentially*
  shows $\exists$ *r*::*nat*$\Rightarrow$*nat*. *strict-mono r* $\wedge$ ($\forall$ *n*. $\neg$ *P* (*r n*))
proof −
  from *assms* **have** $\forall n. \exists m \geq n. \neg P\ m$
    unfolding *eventually-sequentially* **by** (*simp add*: *not-less*)
  then obtain *r* **where** $\bigwedge n.\ r\ n \geq n$ $\bigwedge n.\ \neg P\ (r\ n)$
    by (*auto simp*: *choice-iff*)
  then show *?thesis*
    by (*auto intro*!: *exI*[*of* - $\lambda n.$ *r* (((*Suc* $\circ$ *r*) ^^ *Suc n*) *0*)]
          simp: *less-eq-Suc-le strict-mono-Suc-iff*)
qed

lemma *sequentially-offset*:
  assumes *eventually* ($\lambda i.$ *P i*) *sequentially*
  shows *eventually* ($\lambda i.$ *P* ($i + k$)) *sequentially*
  using *assms* **by** (*rule eventually-sequentially-seg* [*THEN iffD2*])

lemma *seq-offset-neg*:
  ($f \longrightarrow l$) *sequentially* $\Longrightarrow$ (($\lambda i.\ f(i - k)$) $\longrightarrow l$) *sequentially*
  apply (*erule filterlim-compose*)
  apply (*simp add*: *filterlim-def le-sequentially eventually-filtermap eventually-sequentially*, *arith*)
  done

lemma *filterlim-subseq*: *strict-mono f* $\Longrightarrow$ *filterlim f sequentially sequentially*
  unfolding *filterlim-iff* **by** (*metis eventually-subseq*)

lemma *strict-mono-o*: *strict-mono r* $\Longrightarrow$ *strict-mono s* $\Longrightarrow$ *strict-mono* ($r \circ s$)
  unfolding *strict-mono-def* **by** *simp*

lemma *strict-mono-compose*: *strict-mono r* $\Longrightarrow$ *strict-mono s* $\Longrightarrow$ *strict-mono* ($\lambda x.$ *r* (*s x*))
  using *strict-mono-o*[*of r s*] **by** (*simp add*: *o-def*)

lemma *incseq-imp-monoseq*: *incseq X* $\Longrightarrow$ *monoseq X*
  by (*simp add*: *incseq-def monoseq-def*)

lemma *decseq-imp-monoseq*: *decseq X* ⟹ *monoseq X*
 by (*simp add*: *decseq-def monoseq-def*)

lemma *decseq-eq-incseq*: *decseq X* = *incseq* (λ*n*. − *X n*)
 for *X* :: *nat* ⇒ ′*a*::*ordered-ab-group-add*
 by (*simp add*: *decseq-def incseq-def*)

lemma *INT-decseq-offset*:
 assumes *decseq F*
 shows (⋂ *i*. *F i*) = (⋂ *i*∈{*n*..}. *F i*)
proof *safe*
 fix *x i*
 assume *x*: *x* ∈ (⋂ *i*∈{*n*..}. *F i*)
 show *x* ∈ *F i*
 proof *cases*
  from *x* **have** *x* ∈ *F n* **by** *auto*
  also assume *i* ≤ *n* **with** ‹*decseq F*› **have** *F n* ⊆ *F i*
   unfolding *decseq-def* **by** *simp*
  finally show *?thesis* **.**
 qed (*insert x*, *simp*)
qed *auto*

lemma *LIMSEQ-const-iff*: (λ*n*. *k*) ⟶ *l* ⟷ *k* = *l*
 for *k l* :: ′*a*::*t2-space*
 using *trivial-limit-sequentially* **by** (*rule tendsto-const-iff*)

lemma *LIMSEQ-SUP*: *incseq X* ⟹ *X* ⟶ (*SUP i*. *X i* :: ′*a*::{*complete-linorder*,*linorder-topology*})
 by (*intro increasing-tendsto*)
  (*auto simp*: *SUP-upper less-SUP-iff incseq-def eventually-sequentially intro*:
less-le-trans)

lemma *LIMSEQ-INF*: *decseq X* ⟹ *X* ⟶ (*INF i*. *X i* :: ′*a*::{*complete-linorder*,*linorder-topology*})
 by (*intro decreasing-tendsto*)
  (*auto simp*: *INF-lower INF-less-iff decseq-def eventually-sequentially intro*:
le-less-trans)

lemma *LIMSEQ-ignore-initial-segment*: *f* ⟶ *a* ⟹ (λ*n*. *f* (*n* + *k*)) ⟶ *a*
 unfolding *tendsto-def* **by** (*subst eventually-sequentially-seg*[**where** *k*=*k*])

lemma *LIMSEQ-offset*: (λ*n*. *f* (*n* + *k*)) ⟶ *a* ⟹ *f* ⟶ *a*
 unfolding *tendsto-def*
 by (*subst* (*asm*) *eventually-sequentially-seg*[**where** *k*=*k*])

lemma *LIMSEQ-Suc*: *f* ⟶ *l* ⟹ (λ*n*. *f* (*Suc n*)) ⟶ *l*
 by (*drule LIMSEQ-ignore-initial-segment* [**where** *k*=*Suc 0*]) *simp*

lemma *LIMSEQ-imp-Suc*: (λ*n*. *f* (*Suc n*)) ⟶ *l* ⟹ *f* ⟶ *l*
 by (*rule LIMSEQ-offset* [**where** *k*=*Suc 0*]) *simp*

lemma *LIMSEQ-lessThan-iff-atMost*:
  shows $(\lambda n.\ f\ \{..<n\}) \longrightarrow x \longleftrightarrow (\lambda n.\ f\ \{..n\}) \longrightarrow x$
  apply (*subst filterlim-sequentially-Suc* [*symmetric*])
  apply (*simp only*: *lessThan-Suc-atMost*)
  done

lemma (**in** *t2-space*) *LIMSEQ-Uniq*: $\exists_{\leq 1} l.\ X \longrightarrow l$
 by (*simp add*: *tendsto-unique'*)

lemma (**in** *t2-space*) *LIMSEQ-unique*: $X \longrightarrow a \Longrightarrow X \longrightarrow b \Longrightarrow a = b$
  using *trivial-limit-sequentially* **by** (*rule tendsto-unique*)

lemma *LIMSEQ-le-const*: $X \longrightarrow x \Longrightarrow \exists N.\ \forall n \geq N.\ a \leq X\ n \Longrightarrow a \leq x$
  for $a\ x$ :: *'a::linorder-topology*
  by (*simp add*: *eventually-at-top-linorder tendsto-lowerbound*)

lemma *LIMSEQ-le*: $X \longrightarrow x \Longrightarrow Y \longrightarrow y \Longrightarrow \exists N.\ \forall n \geq N.\ X\ n \leq Y\ n \Longrightarrow x \leq y$
  for $x\ y$ :: *'a::linorder-topology*
  using *tendsto-le*[*of sequentially Y y X x*] **by** (*simp add*: *eventually-sequentially*)

lemma *LIMSEQ-le-const2*: $X \longrightarrow x \Longrightarrow \exists N.\ \forall n \geq N.\ X\ n \leq a \Longrightarrow x \leq a$
  for $a\ x$ :: *'a::linorder-topology*
  by (*rule LIMSEQ-le*[*of X x* $\lambda n.\ a$]) *auto*

lemma *Lim-bounded*: $f \longrightarrow l \Longrightarrow \forall n \geq M.\ f\ n \leq C \Longrightarrow l \leq C$
  for l :: *'a::linorder-topology*
  by (*intro LIMSEQ-le-const2*) *auto*

lemma *Lim-bounded2*:
  fixes f :: *nat* $\Rightarrow$ *'a::linorder-topology*
  assumes *lim*:$f \longrightarrow l$ **and** *ge*: $\forall n \geq N.\ f\ n \geq C$
  shows $l \geq C$
  using *ge*
  by (*intro tendsto-le*[*OF trivial-limit-sequentially lim tendsto-const*])
    (*auto simp*: *eventually-sequentially*)

lemma *lim-mono*:
  fixes $X\ Y$ :: *nat* $\Rightarrow$ *'a::linorder-topology*
  assumes $\bigwedge n.\ N \leq n \Longrightarrow X\ n \leq Y\ n$
    and $X \longrightarrow x$
    and $Y \longrightarrow y$
  shows $x \leq y$
  using *assms*(*1*) **by** (*intro LIMSEQ-le*[*OF assms*(*2*,*3*)]) *auto*

lemma *Sup-lim*:
  fixes a :: *'a::*{*complete-linorder*,*linorder-topology*}
  assumes $\bigwedge n.\ b\ n \in s$

and $b \longrightarrow a$
shows $a \le \mathit{Sup}\ s$
by (*metis Lim-bounded assms complete-lattice-class.Sup-upper*)

lemma *Inf-lim*:
fixes a :: *'a::{complete-linorder,linorder-topology}*
assumes $\bigwedge n.\ b\ n \in s$
and $b \longrightarrow a$
shows $\mathit{Inf}\ s \le a$
by (*metis Lim-bounded2 assms complete-lattice-class.Inf-lower*)

lemma *SUP-Lim*:
fixes X :: *nat* ⇒ *'a::{complete-linorder,linorder-topology}*
assumes *inc*: *incseq X*
and *l*: $X \longrightarrow l$
shows $(\mathit{SUP}\ n.\ X\ n) = l$
using *LIMSEQ-SUP*[*OF inc*] *tendsto-unique*[*OF trivial-limit-sequentially l*]
by *simp*

lemma *INF-Lim*:
fixes X :: *nat* ⇒ *'a::{complete-linorder,linorder-topology}*
assumes *dec*: *decseq X*
and *l*: $X \longrightarrow l$
shows $(\mathit{INF}\ n.\ X\ n) = l$
using *LIMSEQ-INF*[*OF dec*] *tendsto-unique*[*OF trivial-limit-sequentially l*]
by *simp*

lemma *convergentD*: *convergent X* $\Longrightarrow \exists L.\ X \longrightarrow L$
by (*simp add*: *convergent-def*)

lemma *convergentI*: $X \longrightarrow L \Longrightarrow$ *convergent X*
by (*auto simp add*: *convergent-def*)

lemma *convergent-LIMSEQ-iff*: *convergent X* $\longleftrightarrow X \longrightarrow \mathit{lim}\ X$
by (*auto intro*: *theI LIMSEQ-unique simp add*: *convergent-def lim-def*)

lemma *convergent-const*: *convergent* $(\lambda n.\ c)$
by (*rule convergentI*) (*rule tendsto-const*)

lemma *monoseq-le*:
monoseq a $\Longrightarrow a \longrightarrow x \Longrightarrow$
$(\forall n.\ a\ n \le x) \wedge (\forall m.\ \forall n{\ge}m.\ a\ m \le a\ n) \vee$
$(\forall n.\ x \le a\ n) \wedge (\forall m.\ \forall n{\ge}m.\ a\ n \le a\ m)$
for x :: *'a::linorder-topology*
by (*metis LIMSEQ-le-const LIMSEQ-le-const2 decseq-def incseq-def monoseq-iff*)

lemma *LIMSEQ-subseq-LIMSEQ*: $X \longrightarrow L \Longrightarrow$ *strict-mono f* $\Longrightarrow (X \circ f) \longrightarrow L$
unfolding *comp-def* **by** (*rule filterlim-compose* [*of X*, *OF - filterlim-subseq*])

lemma *convergent-subseq-convergent*: *convergent X* $\Longrightarrow$ *strict-mono f* $\Longrightarrow$ *convergent* $(X \circ f)$
by (*auto simp*: *convergent-def intro*: *LIMSEQ-subseq-LIMSEQ*)

lemma *limI*: $X \longrightarrow L \Longrightarrow$ *lim* $X = L$
by (*rule tendsto-Lim*) (*rule trivial-limit-sequentially*)

lemma *lim-le*: *convergent f* $\Longrightarrow$ $(\bigwedge n.\ f\ n \le x) \Longrightarrow$ *lim f* $\le x$
for x :: *'a::linorder-topology*
using *LIMSEQ-le-const2*[*of f lim f x*] **by** (*simp add*: *convergent-LIMSEQ-iff*)

lemma *lim-const* [*simp*]: *lim* $(\lambda m.\ a) = a$
by (*simp add*: *limI*)

98.7.3 Increasing and Decreasing Series

lemma *incseq-le*: *incseq X* $\Longrightarrow X \longrightarrow L \Longrightarrow X\ n \le L$
for L :: *'a::linorder-topology*
by (*metis incseq-def LIMSEQ-le-const*)

lemma *decseq-ge*: *decseq X* $\Longrightarrow X \longrightarrow L \Longrightarrow L \le X\ n$
for L :: *'a::linorder-topology*
by (*metis decseq-def LIMSEQ-le-const2*)

98.8 First countable topologies

class *first-countable-topology* = *topological-space* +
assumes *first-countable-basis*:
$\exists A$::*nat* $\Rightarrow$ *'a set*. $(\forall i.\ x \in A\ i \wedge$ *open* $(A\ i)) \wedge (\forall S.$ *open* $S \wedge x \in S \longrightarrow (\exists i.\ A\ i \subseteq S))$

lemma (**in** *first-countable-topology*) *countable-basis-at-decseq*:
obtains A :: *nat* $\Rightarrow$ *'a set* **where**
$\bigwedge i.$ *open* $(A\ i)$ $\bigwedge i.\ x \in (A\ i)$
$\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow$ *eventually* $(\lambda i.\ A\ i \subseteq S)$ *sequentially*
proof *atomize-elim*
from *first-countable-basis*[*of x*] **obtain** A :: *nat* $\Rightarrow$ *'a set*
where *nhds*: $\bigwedge i.$ *open* $(A\ i)$ $\bigwedge i.\ x \in A\ i$
and *incl*: $\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow \exists i.\ A\ i \subseteq S$
by *auto*
define F **where** $F\ n = (\bigcap i{\le}n.\ A\ i)$ **for** n
show $\exists A.\ (\forall i.$ *open* $(A\ i)) \wedge (\forall i.\ x \in A\ i) \wedge$
$(\forall S.$ *open* $S \longrightarrow x \in S \longrightarrow$ *eventually* $(\lambda i.\ A\ i \subseteq S)$ *sequentially*)
proof (*safe intro*!: *exI*[*of - F*])
fix i
show *open* $(F\ i)$
using *nhds*(*1*) **by** (*auto simp*: *F-def*)
show $x \in F\ i$
using *nhds*(*2*) **by** (*auto simp*: *F-def*)

next
fix *S*
assume *open S x* ∈ *S*
from *incl*[*OF this*] **obtain** *i* **where** *F i* ⊆ *S*
unfolding *F-def* **by** *auto*
moreover have ⋀*j*. *i* ≤ *j* ⟹ *F j* ⊆ *F i*
by (*simp add*: *Inf-superset-mono F-def image-mono*)
ultimately show *eventually* (λ*i*. *F i* ⊆ *S*) *sequentially*
by (*auto simp*: *eventually-sequentially*)
qed
qed

lemma (**in** *first-countable-topology*) *nhds-countable*:
obtains *X* :: *nat* ⇒ ′*a set*
where *decseq X* ⋀*n*. *open* (*X n*) ⋀*n*. *x* ∈ *X n nhds x* = (*INF n*. *principal* (*X n*))
proof −
from *first-countable-basis* **obtain** *A* :: *nat* ⇒ ′*a set*
where ∗: ⋀*n*. *x* ∈ *A n* ⋀*n*. *open* (*A n*) ⋀*S*. *open S* ⟹ *x* ∈ *S* ⟹ ∃ *i*. *A i* ⊆ *S*
by *metis*
show *thesis*
proof
show *decseq* (λ*n*. ⋂ *i*≤*n*. *A i*)
by (*simp add*: *antimono-iff-le-Suc atMost-Suc*)
show *x* ∈ (⋂ *i*≤*n*. *A i*) ⋀*n*. *open* (⋂ *i*≤*n*. *A i*) **for** *n*
using ∗ **by** *auto*
with ∗ **show** *nhds x* = (*INF n*. *principal* (⋂ *i*≤*n*. *A i*))
unfolding *nhds-def*
apply (*intro INF-eq*)
apply *fastforce*
apply *blast*
done
qed
qed

lemma (**in** *first-countable-topology*) *countable-basis*:
obtains *A* :: *nat* ⇒ ′*a set* **where**
⋀*i*. *open* (*A i*) ⋀*i*. *x* ∈ *A i*
⋀*F*. (∀ *n*. *F n* ∈ *A n*) ⟹ *F* ⟶ *x*
proof *atomize-elim*
obtain *A* :: *nat* ⇒ ′*a set* **where** ∗:
⋀*i*. *open* (*A i*)
⋀*i*. *x* ∈ *A i*
⋀*S*. *open S* ⟹ *x* ∈ *S* ⟹ *eventually* (λ*i*. *A i* ⊆ *S*) *sequentially*
by (*rule countable-basis-at-decseq*) *blast*
have *eventually* (λ*n*. *F n* ∈ *S*) *sequentially*
if ∀ *n*. *F n* ∈ *A n open S x* ∈ *S* **for** *F S*
using ∗(*3*)[*of S*] *that* **by** (*auto elim*: *eventually-mono simp*: *subset-eq*)

with $*$ **show** $\exists A.\ (\forall i.\ open\ (A\ i)) \wedge (\forall i.\ x \in A\ i) \wedge (\forall F.\ (\forall n.\ F\ n \in A\ n) \longrightarrow F \longlonglongrightarrow x)$
by (*intro exI*[*of - A*]) (*auto simp*: *tendsto-def*)
qed

lemma (**in** *first-countable-topology*) *sequentially-imp-eventually-nhds-within*:
assumes $\forall f.\ (\forall n.\ f\,n \in s) \wedge f \longlonglongrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\,n))\ sequentially$
shows $eventually\ P\ (inf\ (nhds\ a)\ (principal\ s))$
proof (*rule ccontr*)
obtain $A :: nat \Rightarrow {}'a\ set$ **where** $*$:
$\bigwedge i.\ open\ (A\ i)$
$\bigwedge i.\ a \in A\ i$
$\bigwedge F.\ \forall n.\ F\ n \in A\ n \Longrightarrow F \longlonglongrightarrow a$
by (*rule countable-basis*) *blast*
assume $\neg$ *?thesis*
with $*$ **have** $\exists F.\ \forall n.\ F\ n \in s \wedge F\ n \in A\ n \wedge \neg\ P\ (F\ n)$
unfolding *eventually-inf-principal eventually-nhds*
by (*intro choice*) *fastforce*
then obtain F **where** F: $\forall n.\ F\ n \in s$ **and** $\forall n.\ F\ n \in A\ n$ **and** F': $\forall n.\ \neg\ P\ (F\ n)$
by *blast*
with $*$ **have** $F \longlonglongrightarrow a$
by *auto*
then have $eventually\ (\lambda n.\ P\ (F\ n))\ sequentially$
using *assms F* **by** *simp*
then show *False*
by (*simp add*: F')
qed

lemma (**in** *first-countable-topology*) *eventually-nhds-within-iff-sequentially*:
$eventually\ P\ (inf\ (nhds\ a)\ (principal\ s)) \longleftrightarrow$
$(\forall f.\ (\forall n.\ f\,n \in s) \wedge f \longlonglongrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\,n))\ sequentially)$
proof (*safe intro*!: *sequentially-imp-eventually-nhds-within*)
assume $eventually\ P\ (inf\ (nhds\ a)\ (principal\ s))$
then obtain S **where** $open\ S\ a \in S\ \forall x{\in}S.\ x \in s \longrightarrow P\ x$
by (*auto simp*: *eventually-inf-principal eventually-nhds*)
moreover
fix f
assume $\forall n.\ f\,n \in s\ f \longlonglongrightarrow a$
ultimately show $eventually\ (\lambda n.\ P\ (f\,n))\ sequentially$
by (*auto dest*!: *topological-tendstoD elim*: *eventually-mono*)
qed

lemma (**in** *first-countable-topology*) *eventually-nhds-iff-sequentially*:
$eventually\ P\ (nhds\ a) \longleftrightarrow (\forall f.\ f \longlonglongrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\,n))\ sequentially)$
using *eventually-nhds-within-iff-sequentially*[*of P a UNIV*] **by** *simp*

```
lemma Inf-as-limit:
  fixes A::'a::{linorder-topology, first-countable-topology, complete-linorder} set
  assumes A ≠ {}
  shows ∃u. (∀n. u n ∈ A) ∧ u ⟶ Inf A
proof (cases Inf A ∈ A)
  case True
  show ?thesis
    by (rule exI[of - λn. Inf A], auto simp add: True)
next
  case False
  obtain y where y ∈ A using assms by auto
  then have Inf A < y using False Inf-lower less-le by auto
  obtain F :: nat ⇒ 'a set where F: ⋀i. open (F i) ⋀i. Inf A ∈ F i
                                    ⋀u. (∀n. u n ∈ F n) ⟹ u ⟶ Inf A
    by (metis first-countable-topology-class.countable-basis)
  define u where u = (λn. SOME z. z ∈ F n ∧ z ∈ A)
  have ∃z. z ∈ U ∧ z ∈ A if Inf A ∈ U open U for U
  proof -
    obtain b where b > Inf A {Inf A ..<b} ⊆ U
      using open-right[OF ‹open U› ‹Inf A ∈ U› ‹Inf A < y›] by auto
    obtain z where z < b z ∈ A
      using ‹Inf A < b› Inf-less-iff by auto
    then have z ∈ {Inf A ..<b}
      by (simp add: Inf-lower)
    then show ?thesis using ‹z ∈ A› ‹{Inf A ..<b} ⊆ U› by auto
  qed
  then have *: u n ∈ F n ∧ u n ∈ A for n
   using ‹Inf A ∈ F n› ‹open (F n)› unfolding u-def by (metis (no-types, lifting)
someI-ex)
  then have u ⟶ Inf A using F(3) by simp
  then show ?thesis using * by auto
qed

lemma tendsto-at-iff-sequentially:
  (f ⟶ a) (at x within s) ⟷ (∀X. (∀i. X i ∈ s − {x}) ⟶ X ⟶ x ⟶
((f ∘ X) ⟶ a))
  for f :: 'a::first-countable-topology ⇒ -
  unfolding filterlim-def[of - nhds a] le-filter-def eventually-filtermap
    at-within-def eventually-nhds-within-iff-sequentially comp-def
  by metis

lemma approx-from-above-dense-linorder:
  fixes x::'a::{dense-linorder, linorder-topology, first-countable-topology}
  assumes x < y
  shows ∃u. (∀n. u n > x) ∧ (u ⟶ x)
proof -
  obtain A :: nat ⇒ 'a set where A: ⋀i. open (A i) ⋀i. x ∈ A i
                                    ⋀F. (∀n. F n ∈ A n) ⟹ F ⟶ x
    by (metis first-countable-topology-class.countable-basis)
```

define *u* **where** $u = (\lambda n.\ SOME\ z.\ z \in A\ n \wedge z > x)$
have $\exists z.\ z \in U \wedge x < z$ **if** $x \in U$ *open* U **for** U
using *open-right*[*OF* ‹*open U*› ‹$x \in U$› ‹$x < y$›]
by (*meson atLeastLessThan-iff dense less-imp-le subset-eq*)
then have ∗: $u\ n \in A\ n \wedge x < u\ n$ **for** n
using ‹$x \in A\ n$› ‹*open* (*A n*)› **unfolding** *u-def* **by** (*metis* (*no-types*, *lifting*) *someI-ex*)
then have $u \longrightarrow x$ **using** *A*(*3*) **by** *simp*
then show *?thesis* **using** ∗ **by** *auto*
qed

lemma *approx-from-below-dense-linorder*:
fixes *x*::*'a*::{*dense-linorder*, *linorder-topology*, *first-countable-topology*}
assumes $x > y$
shows $\exists u.\ (\forall n.\ u\ n < x) \wedge (u \longrightarrow x)$
proof −
obtain $A :: nat \Rightarrow 'a\ set$ **where** A: $\bigwedge i.\ open\ (A\ i)$ $\bigwedge i.\ x \in A\ i$
$\bigwedge F.\ (\forall n.\ F\ n \in A\ n) \Longrightarrow F \longrightarrow x$
by (*metis first-countable-topology-class.countable-basis*)
define *u* **where** $u = (\lambda n.\ SOME\ z.\ z \in A\ n \wedge z < x)$
have $\exists z.\ z \in U \wedge z < x$ **if** $x \in U$ *open* U **for** U
using *open-left*[*OF* ‹*open U*› ‹$x \in U$› ‹$x > y$›]
by (*meson dense greaterThanAtMost-iff less-imp-le subset-eq*)
then have ∗: $u\ n \in A\ n \wedge u\ n < x$ **for** n
using ‹$x \in A\ n$› ‹*open* (*A n*)› **unfolding** *u-def* **by** (*metis* (*no-types*, *lifting*) *someI-ex*)
then have $u \longrightarrow x$ **using** *A*(*3*) **by** *simp*
then show *?thesis* **using** ∗ **by** *auto*
qed

98.9 Function limit at a point

abbreviation *LIM* :: (*'a*::*topological-space* ⇒ *'b*::*topological-space*) ⇒ *'a* ⇒ *'b* ⇒ *bool*
(‹(‹*notation*=‹*infix LIM*››(-)/ −(-)/→ (-))› [*60*, *0*, *60*] *60*)
where $f\ {-a\rightarrow}\ L \equiv (f \longrightarrow L)\ (at\ a)$

lemma *tendsto-within-open*: $a \in S \Longrightarrow open\ S \Longrightarrow (f \longrightarrow l)\ (at\ a\ within\ S) \longleftrightarrow (f\ {-a\rightarrow}\ l)$
by (*simp add*: *tendsto-def at-within-open*[**where** $S = S$])

lemma *tendsto-within-open-NO-MATCH*:
$a \in S \Longrightarrow NO\text{-}MATCH\ UNIV\ S \Longrightarrow open\ S \Longrightarrow (f \longrightarrow l)(at\ a\ within\ S) \longleftrightarrow (f \longrightarrow l)(at\ a)$
for *f* :: *'a*::*topological-space* ⇒ *'b*::*topological-space*
using *tendsto-within-open* **by** *blast*

lemma *LIM-const-not-eq*[*tendsto-intros*]: $k \neq L \Longrightarrow \neg\ (\lambda x.\ k)\ {-a\rightarrow}\ L$
for *a* :: *'a*::*perfect-space* **and** *k L* :: *'b*::*t2-space*

by (*simp add*: *tendsto-const-iff*)

lemmas *LIM-not-zero* = *LIM-const-not-eq* [**where** $L = 0$]

lemma *LIM-const-eq*: $(\lambda x.\ k)\ -a\rightarrow\ L \Longrightarrow k = L$
for a :: $'a$::*perfect-space* **and** $k\ L$ :: $'b$::*t2-space*
by (*simp add*: *tendsto-const-iff*)

lemma *LIM-unique*: $f\ -a\rightarrow\ L \Longrightarrow f\ -a\rightarrow\ M \Longrightarrow L = M$
for a :: $'a$::*perfect-space* **and** $L\ M$ :: $'b$::*t2-space*
using *at-neq-bot* **by** (*rule tendsto-unique*)

lemma *LIM-Uniq*: $\exists_{\leq 1} L$::$'b$::*t2-space*. $f\ -a\rightarrow\ L$
for a :: $'a$::*perfect-space*
by (*auto simp add*: *Uniq-def LIM-unique*)

Limits are equal for functions equal except at limit point.

lemma *LIM-equal*: $\forall x.\ x \neq a \longrightarrow f\,x = g\,x \Longrightarrow (f\ -a\rightarrow\ l) \longleftrightarrow (g\ -a\rightarrow\ l)$
by (*simp add*: *tendsto-def eventually-at-topological*)

lemma *LIM-cong*: $a = b \Longrightarrow (\bigwedge x.\ x \neq b \Longrightarrow f\,x = g\,x) \Longrightarrow l = m \Longrightarrow (f\ -a\rightarrow\ l) \longleftrightarrow (g\ -b\rightarrow\ m)$
by (*simp add*: *LIM-equal*)

lemma *tendsto-cong-limit*: $(f \longrightarrow l)\ F \Longrightarrow k = l \Longrightarrow (f \longrightarrow k)\ F$
by *simp*

lemma *tendsto-at-iff-tendsto-nhds*: $g\ -l\rightarrow\ g\ l \longleftrightarrow (g \longrightarrow g\ l)\ (nhds\ l)$
unfolding *tendsto-def eventually-at-filter*
by (*intro ext all-cong imp-cong*) (*auto elim*!: *eventually-mono*)

lemma *tendsto-compose*: $g\ -l\rightarrow\ g\ l \Longrightarrow (f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ g\ (f\ x)) \longrightarrow g\ l)\ F$
unfolding *tendsto-at-iff-tendsto-nhds* **by** (*rule filterlim-compose*[*of g*])

lemma *tendsto-compose-eventually*:
$g\ -l\rightarrow\ m \Longrightarrow (f \longrightarrow l)\ F \Longrightarrow$ *eventually* $(\lambda x.\ f\ x \neq l)\ F \Longrightarrow ((\lambda x.\ g\ (f\ x)) \longrightarrow m)\ F$
by (*rule filterlim-compose*[*of g - at l*]) (*auto simp add*: *filterlim-at*)

lemma *LIM-compose-eventually*:
assumes $f\ -a\rightarrow\ b$
and $g\ -b\rightarrow\ c$
and *eventually* $(\lambda x.\ f\ x \neq b)\ (at\ a)$
shows $(\lambda x.\ g\ (f\ x))\ -a\rightarrow\ c$
using *assms*(*2*,*1*,*3*) **by** (*rule tendsto-compose-eventually*)

lemma *tendsto-compose-filtermap*: $((g \circ f) \longrightarrow T)\ F \longleftrightarrow (g \longrightarrow T)$ (*filtermap* $f\ F$)

by (*simp add*: *filterlim-def filtermap-filtermap comp-def*)

lemma *tendsto-compose-at*:
assumes *f*: $(f \longrightarrow y)$ F **and** *g*: $(g \longrightarrow z)$ $(at\ y)$ **and** *fg*: *eventually* $(\lambda w.\ f\ w = y \longrightarrow g\ y = z)$ F
shows $((g \circ f) \longrightarrow z)$ F
proof −
have $(\forall_F\ a\ in\ F.\ f\ a \neq y) \vee g\ y = z$
using *fg* **by** *force*
moreover have $(g \longrightarrow z)$ (*filtermap f F*) $\vee \neg (\forall_F\ a\ in\ F.\ f\ a \neq y)$
by (*metis* (*no-types*) *filterlim-atI filterlim-def tendsto-mono f g*)
ultimately show *?thesis*
by (*metis* (*no-types*) *f filterlim-compose filterlim-filtermap g tendsto-at-iff-tendsto-nhds tendsto-compose-filtermap*)
qed

lemma *tendsto-nhds-iff*: $(f \longrightarrow (c :: 'a :: t1\text{-}space))$ $(nhds\ x) \longleftrightarrow f\ -x\rightarrow\ c \wedge f\ x = c$
proof *safe*
assume *lim*: $(f \longrightarrow c)$ $(nhds\ x)$
show $f\ x = c$
proof (*rule ccontr*)
assume $f\ x \neq c$
hence $c \neq f\ x$
by *auto*
then obtain A **where** A: *open* A $c \in A$ $f\ x \notin A$
by (*subst* (*asm*) *separation-t1*) *auto*
with *lim* **obtain** B **where** *open* B $x \in B$ $\bigwedge x.\ x \in B \Longrightarrow f\ x \in A$
unfolding *tendsto-def eventually-nhds* **by** *metis*
with ‹$f\ x \notin A$› **show** *False*
by *blast*
qed
show $(f \longrightarrow c)$ $(at\ x)$
using *lim* **by** (*rule filterlim-mono*) (*auto simp*: *at-within-def*)
next
assume $f\ -x\rightarrow\ f\ x$ $c = f\ x$
thus $(f \longrightarrow f\ x)$ $(nhds\ x)$
unfolding *tendsto-def eventually-at-filter* **by** (*fast elim*: *eventually-mono*)
qed

98.9.1 Relation of *LIM* and *LIMSEQ*

lemma (**in** *first-countable-topology*) *sequentially-imp-eventually-within*:
$(\forall f.\ (\forall n.\ f\ n \in s \wedge f\ n \neq a) \wedge f \longrightarrow a \longrightarrow$ *eventually* $(\lambda n.\ P\ (f\ n))$ *sequentially*$) \Longrightarrow$
eventually P $(at\ a\ within\ s)$
unfolding *at-within-def*
by (*intro sequentially-imp-eventually-nhds-within*) *auto*

lemma (**in** *first-countable-topology*) *sequentially-imp-eventually-at*:
(∀ *f*. (∀ *n*. *f n* ≠ *a*) ∧ *f* ⟶ *a* ⟶ *eventually* (λ*n*. *P* (*f n*)) *sequentially*) ⟹ *eventually P* (*at a*)
using *sequentially-imp-eventually-within* [**where** *s*=*UNIV*] **by** *simp*

lemma *LIMSEQ-SEQ-conv*:
(∀ *S*. (∀ *n*. *S n* ≠ *a*) ∧ *S* ⟶ *a* ⟶ (λ*n*. *X* (*S n*)) ⟶ *L*) ⟷ *X* −*a*→ *L* (**is** *?lhs*=*?rhs*)
for *a* :: ′*a*::*first-countable-topology* **and** *L* :: ′*b*::*topological-space*
proof
assume *?lhs* **then show** *?rhs*
by (*simp add*: *sequentially-imp-eventually-within tendsto-def*)
next
assume *?rhs* **then show** *?lhs*
using *tendsto-compose-eventually eventuallyI* **by** *blast*
qed

lemma *sequentially-imp-eventually-at-left*:
fixes *a* :: ′*a*::{*linorder-topology*,*first-countable-topology*}
assumes *b*[*simp*]: *b* < *a*
and ∗: ⋀*f*. (⋀*n*. *b* < *f n*) ⟹ (⋀*n*. *f n* < *a*) ⟹ *incseq f* ⟹ *f* ⟶ *a* ⟹ *eventually* (λ*n*. *P* (*f n*)) *sequentially*
shows *eventually P* (*at-left a*)
proof (*safe intro*!: *sequentially-imp-eventually-within*)
fix *X*
assume *X*: ∀ *n*. *X n* ∈ {..< *a*} ∧ *X n* ≠ *a X* ⟶ *a*
show *eventually* (λ*n*. *P* (*X n*)) *sequentially*
proof (*rule ccontr*)
assume *neg*: ¬ *?thesis*
have ∃ *s*. ∀ *n*. (¬ *P* (*X* (*s n*)) ∧ *b* < *X* (*s n*)) ∧ (*X* (*s n*) ≤ *X* (*s* (*Suc n*)) ∧ *Suc* (*s n*) ≤ *s* (*Suc n*))
(**is** ∃ *s*. *?P s*)
proof (*rule dependent-nat-choice*)
have ¬ *eventually* (λ*n*. *b* < *X n* ⟶ *P* (*X n*)) *sequentially*
by (*intro not-eventually-impI neg order-tendstoD*(*1*) [*OF X*(*2*) *b*])
then show ∃ *x*. ¬ *P* (*X x*) ∧ *b* < *X x*
by (*auto dest*!: *not-eventuallyD*)
next
fix *x n*
have ¬ *eventually* (λ*n*. *Suc x* ≤ *n* ⟶ *b* < *X n* ⟶ *X x* < *X n* ⟶ *P* (*X n*)) *sequentially*
using *X*
by (*intro not-eventually-impI order-tendstoD*(*1*)[*OF X*(*2*)] *eventually-ge-at-top neg*) *auto*
then show ∃ *n*. (¬ *P* (*X n*) ∧ *b* < *X n*) ∧ (*X x* ≤ *X n* ∧ *Suc x* ≤ *n*)
by (*auto dest*!: *not-eventuallyD*)
qed
then obtain *s* **where** *?P s* **..**
with *X* **have** *b* < *X* (*s n*)

and $X\ (s\ n) < a$
and *incseq* $(\lambda n.\ X\ (s\ n))$
and $(\lambda n.\ X\ (s\ n)) \longrightarrow a$
and $\neg\ P\ (X\ (s\ n))$
for n
by (*auto simp*: *strict-mono-Suc-iff Suc-le-eq incseq-Suc-iff*
intro!: *LIMSEQ-subseq-LIMSEQ*[*OF* ‹$X \longrightarrow a$›, *unfolded comp-def*])
from $*$[*OF this*(*1*,*2*,*3*,*4*)] *this*(*5*) **show** *False*
by *auto*
qed
qed

lemma *tendsto-at-left-sequentially*:
fixes $a\ b$ :: $'b$::{*linorder-topology*,*first-countable-topology*}
assumes $b < a$
assumes $*$: $\bigwedge S.\ (\bigwedge n.\ S\ n < a) \Longrightarrow (\bigwedge n.\ b < S\ n) \Longrightarrow incseq\ S \Longrightarrow S \longrightarrow a \Longrightarrow$
$(\lambda n.\ X\ (S\ n)) \longrightarrow L$
shows $(X \longrightarrow L)$ (*at-left* a)
using *assms* **by** (*simp add*: *tendsto-def* [**where** $l=L$] *sequentially-imp-eventually-at-left*)

lemma *sequentially-imp-eventually-at-right*:
fixes $a\ b$ :: $'a$::{*linorder-topology*,*first-countable-topology*}
assumes b[*simp*]: $a < b$
assumes $*$: $\bigwedge f.\ (\bigwedge n.\ a < f\ n) \Longrightarrow (\bigwedge n.\ f\ n < b) \Longrightarrow decseq\ f \Longrightarrow f \longrightarrow a \Longrightarrow$
eventually $(\lambda n.\ P\ (f\ n))$ *sequentially*
shows *eventually* P (*at-right* a)
proof (*safe intro*!: *sequentially-imp-eventually-within*)
fix X
assume X: $\forall n.\ X\ n \in \{a <..\} \wedge X\ n \neq a\ X \longrightarrow a$
show *eventually* $(\lambda n.\ P\ (X\ n))$ *sequentially*
proof (*rule ccontr*)
assume *neg*: $\neg$ *?thesis*
have $\exists s.\ \forall n.\ (\neg\ P\ (X\ (s\ n)) \wedge X\ (s\ n) < b) \wedge (X\ (s\ (Suc\ n)) \le X\ (s\ n) \wedge Suc\ (s\ n) \le s\ (Suc\ n))$
(**is** $\exists s.\ ?P\ s$)
proof (*rule dependent-nat-choice*)
have $\neg$ *eventually* $(\lambda n.\ X\ n < b \longrightarrow P\ (X\ n))$ *sequentially*
by (*intro not-eventually-impI neg order-tendstoD*(*2*) [*OF X*(*2*) *b*])
then show $\exists x.\ \neg\ P\ (X\ x) \wedge X\ x < b$
by (*auto dest*!: *not-eventuallyD*)
next
fix $x\ n$
have $\neg$ *eventually* $(\lambda n.\ Suc\ x \le n \longrightarrow X\ n < b \longrightarrow X\ n < X\ x \longrightarrow P\ (X\ n))$ *sequentially*
using X
by (*intro not-eventually-impI order-tendstoD*(*2*)[*OF X*(*2*)] *eventually-ge-at-top neg*) *auto*

then show $\exists n. (\neg P (X n) \wedge X n < b) \wedge (X n \leq X x \wedge Suc x \leq n)$
by (*auto dest*!: *not-eventuallyD*)
qed
then obtain *s* **where** *?P s* **..**
with *X* **have** $a < X (s n)$
and $X (s n) < b$
and *decseq* ($\lambda n. X (s n)$)
and $(\lambda n. X (s n)) \longrightarrow a$
and $\neg P (X (s n))$
for *n*
by (*auto simp*: *strict-mono-Suc-iff Suc-le-eq decseq-Suc-iff*
intro!: *LIMSEQ-subseq-LIMSEQ*[*OF* ‹$X \longrightarrow a$›, *unfolded comp-def*])
from ∗[*OF this*(*1*,*2*,*3*,*4*)] *this*(*5*) **show** *False*
by *auto*
qed
qed

lemma *tendsto-at-right-sequentially*:
fixes *a* :: - :: {*linorder-topology*, *first-countable-topology*}
assumes $a < b$
and ∗: $\bigwedge S. (\bigwedge n. a < S n) \Longrightarrow (\bigwedge n. S n < b) \Longrightarrow decseq\ S \Longrightarrow S \longrightarrow a \Longrightarrow$
$(\lambda n. X (S n)) \longrightarrow L$
shows $(X \longrightarrow L)$ (*at-right a*)
using *assms* **by** (*simp add*: *tendsto-def* [**where** *l*=*L*] *sequentially-imp-eventually-at-right*)

98.10 Continuity

98.10.1 Continuity on a set

definition *continuous-on* :: *'a set* ⇒ (*'a*::*topological-space* ⇒ *'b*::*topological-space*) ⇒ *bool*
where *continuous-on s f* ⟷ (∀ *x*∈*s*. ($f \longrightarrow f\ x$) (*at x within s*))

lemma *continuous-on-cong* [*cong*]:
$s = t \Longrightarrow (\bigwedge x. x \in t \Longrightarrow f\ x = g\ x) \Longrightarrow$ *continuous-on s f* ⟷ *continuous-on t g*
unfolding *continuous-on-def*
by (*intro ball-cong filterlim-cong*) (*auto simp*: *eventually-at-filter*)

lemma *continuous-on-cong-simp*:
$s = t \Longrightarrow (\bigwedge x. x \in t$ =*simp*=> $f\ x = g\ x) \Longrightarrow$ *continuous-on s f* ⟷ *continuous-on t g*
unfolding *simp-implies-def* **by** (*rule continuous-on-cong*)

lemma *continuous-on-topological*:
continuous-on s f ⟷
(∀ *x*∈*s*. ∀ *B*. *open B* ⟶ *f x* ∈ *B* ⟶ (∃ *A*. *open A* ∧ *x* ∈ *A* ∧ (∀ *y*∈*s*. *y* ∈ *A* ⟶ *f y* ∈ *B*)))
unfolding *continuous-on-def tendsto-def eventually-at-topological* **by** *metis*

lemma *continuous-on-open-invariant*:
  continuous-on s f $\longleftrightarrow$ ($\forall$ *B*. *open B* $\longrightarrow$ ($\exists$ *A*. *open A* $\wedge$ *A* $\cap$ *s* = *f* $-$‘ *B* $\cap$ *s*))
proof *safe*
  fix *B* :: ′*b set*
  assume *continuous-on s f open B*
  then have $\forall$ *x*$\in$*f* $-$‘ *B* $\cap$ *s*. ($\exists$ *A*. *open A* $\wedge$ *x* $\in$ *A* $\wedge$ *s* $\cap$ *A* $\subseteq$ *f* $-$‘ *B*)
    by (*auto simp*: *continuous-on-topological subset-eq Ball-def imp-conjL*)
  then obtain *A* **where** $\forall$ *x*$\in$*f* $-$‘ *B* $\cap$ *s*. *open* (*A x*) $\wedge$ *x* $\in$ *A x* $\wedge$ *s* $\cap$ *A x* $\subseteq$ *f* $-$‘ *B*
    unfolding *bchoice-iff* **..**
  then show $\exists$ *A*. *open A* $\wedge$ *A* $\cap$ *s* = *f* $-$‘ *B* $\cap$ *s*
    by (*intro exI*[*of* - $\bigcup$*x*$\in$*f* $-$‘ *B* $\cap$ *s*. *A x*]) *auto*
next
  assume *B*: $\forall$ *B*. *open B* $\longrightarrow$ ($\exists$ *A*. *open A* $\wedge$ *A* $\cap$ *s* = *f* $-$‘ *B* $\cap$ *s*)
  show *continuous-on s f*
    unfolding *continuous-on-topological*
  proof *safe*
    fix *x B*
    assume *x* $\in$ *s open B f x* $\in$ *B*
    with *B* **obtain** *A* **where** *A*: *open A A* $\cap$ *s* = *f* $-$‘ *B* $\cap$ *s*
      by *auto*
    with ‹*x* $\in$ *s*› ‹*f x* $\in$ *B*› **show** $\exists$ *A*. *open A* $\wedge$ *x* $\in$ *A* $\wedge$ ($\forall$ *y*$\in$*s*. *y* $\in$ *A* $\longrightarrow$ *f y* $\in$ *B*)
      by (*intro exI*[*of* - *A*]) *auto*
  qed
qed

lemma *continuous-on-open-vimage*:
  open s $\Longrightarrow$ *continuous-on s f* $\longleftrightarrow$ ($\forall$ *B*. *open B* $\longrightarrow$ *open* (*f* $-$‘ *B* $\cap$ *s*))
  unfolding *continuous-on-open-invariant*
  by (*metis open-Int Int-absorb Int-commute*[*of s*] *Int-assoc*[*of* - - *s*])

corollary *continuous-imp-open-vimage*:
  assumes *continuous-on s f open s open B f* $-$‘ *B* $\subseteq$ *s*
  shows *open* (*f* $-$‘ *B*)
  by (*metis assms continuous-on-open-vimage le-iff-inf*)

corollary *open-vimage*[*continuous-intros*]:
  assumes *open s*
    and *continuous-on UNIV f*
  shows *open* (*f* $-$‘ *s*)
  using *assms* **by** (*simp add*: *continuous-on-open-vimage* [*OF open-UNIV*])

lemma *continuous-on-closed-invariant*:
  continuous-on s f $\longleftrightarrow$ ($\forall$ *B*. *closed B* $\longrightarrow$ ($\exists$ *A*. *closed A* $\wedge$ *A* $\cap$ *s* = *f* $-$‘ *B* $\cap$ *s*))
proof $-$
  have $*$: ($\bigwedge$*A*. *P A* $\longleftrightarrow$ *Q* ($-$ *A*)) $\Longrightarrow$ ($\forall$ *A*. *P A*) $\longleftrightarrow$ ($\forall$ *A*. *Q A*)
    for *P Q* :: ′*b set* $\Rightarrow$ *bool*

by (*metis double-compl*)
show *?thesis*
unfolding *continuous-on-open-invariant*
by (*intro* ∗) (*auto simp*: *open-closed*[*symmetric*])
qed

lemma *continuous-on-closed-vimage*:
closed s ⟹ *continuous-on s f* ⟷ (∀ *B*. *closed B* ⟶ *closed* (*f* −' *B* ∩ *s*))
unfolding *continuous-on-closed-invariant*
by (*metis closed-Int Int-absorb Int-commute*[*of s*] *Int-assoc*[*of - - s*])

corollary *closed-vimage-Int*[*continuous-intros*]:
assumes *closed s*
and *continuous-on t f*
and *t*: *closed t*
shows *closed* (*f* −' *s* ∩ *t*)
using *assms* **by** (*simp add*: *continuous-on-closed-vimage* [*OF t*])

corollary *closed-vimage*[*continuous-intros*]:
assumes *closed s*
and *continuous-on UNIV f*
shows *closed* (*f* −' *s*)
using *closed-vimage-Int* [*OF assms*] **by** *simp*

lemma *continuous-on-empty* [*simp*]: *continuous-on* {} *f*
by (*simp add*: *continuous-on-def*)

lemma *continuous-on-sing* [*simp*]: *continuous-on* {*x*} *f*
by (*simp add*: *continuous-on-def at-within-def*)

lemma *continuous-on-open-Union*:
(⋀*s*. *s* ∈ *S* ⟹ *open s*) ⟹ (⋀*s*. *s* ∈ *S* ⟹ *continuous-on s f*) ⟹ *continuous-on* (⋃*S*) *f*
unfolding *continuous-on-def*
by *safe* (*metis open-Union at-within-open UnionI*)

lemma *continuous-on-open-UN*:
(⋀*s*. *s* ∈ *S* ⟹ *open* (*A s*)) ⟹ (⋀*s*. *s* ∈ *S* ⟹ *continuous-on* (*A s*) *f*) ⟹
continuous-on (⋃*s*∈*S*. *A s*) *f*
by (*rule continuous-on-open-Union*) *auto*

lemma *continuous-on-open-Un*:
open s ⟹ *open t* ⟹ *continuous-on s f* ⟹ *continuous-on t f* ⟹ *continuous-on* (*s* ∪ *t*) *f*
using *continuous-on-open-Union* [*of* {*s*,*t*}] **by** *auto*

lemma *continuous-on-closed-Un*:
closed s ⟹ *closed t* ⟹ *continuous-on s f* ⟹ *continuous-on t f* ⟹ *continuous-on* (*s* ∪ *t*) *f*

by (*auto simp add*: *continuous-on-closed-vimage closed-Un Int-Un-distrib*)

lemma *continuous-on-closed-Union*:
assumes *finite I*
$\bigwedge i.\ i \in I \Longrightarrow$ *closed* $(U\ i)$
$\bigwedge i.\ i \in I \Longrightarrow$ *continuous-on* $(U\ i)\ f$
shows *continuous-on* $(\bigcup\ i \in I.\ U\ i)\ f$
using *assms*
by (*induction I*) (*auto intro*!: *continuous-on-closed-Un*)

lemma *continuous-on-If*:
assumes *closed*: *closed s closed t*
and *cont*: *continuous-on s f continuous-on t g*
and *P*: $\bigwedge x.\ x \in s \Longrightarrow \neg\ P\ x \Longrightarrow f\ x = g\ x$ $\bigwedge x.\ x \in t \Longrightarrow P\ x \Longrightarrow f\ x = g\ x$
shows *continuous-on* $(s \cup t)$ $(\lambda x.$ *if P x then f x else g x*)
(**is** *continuous-on - ?h*)
proof−
from *P* **have** $\forall x{\in}s.\ f\ x = ?h\ x$ $\forall x{\in}t.\ g\ x = ?h\ x$
by *auto*
with *cont* **have** *continuous-on s ?h continuous-on t ?h*
by *simp-all*
with *closed* **show** *?thesis*
by (*rule continuous-on-closed-Un*)
qed

lemma *continuous-on-cases*:
closed s $\Longrightarrow$ *closed t* $\Longrightarrow$ *continuous-on s f* $\Longrightarrow$ *continuous-on t g* $\Longrightarrow$
$\forall x.\ (x{\in}s \wedge \neg\ P\ x) \vee (x \in t \wedge P\ x) \longrightarrow f\ x = g\ x \Longrightarrow$
continuous-on $(s \cup t)$ $(\lambda x.$ *if P x then f x else g x*)
by (*rule continuous-on-If*) *auto*

lemma *continuous-on-id*[*continuous-intros*,*simp*]: *continuous-on s* $(\lambda x.\ x)$
unfolding *continuous-on-def* **by** *fast*

lemma *continuous-on-id'*[*continuous-intros*,*simp*]: *continuous-on s id*
unfolding *continuous-on-def id-def* **by** *fast*

lemma *continuous-on-const*[*continuous-intros*,*simp*]: *continuous-on s* $(\lambda x.\ c)$
unfolding *continuous-on-def* **by** *auto*

lemma *continuous-on-subset*: *continuous-on s f* $\Longrightarrow t \subseteq s \Longrightarrow$ *continuous-on t f*
unfolding *continuous-on-def*
by (*metis subset-eq tendsto-within-subset*)

lemma *continuous-on-compose*[*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on* $(f\ `\ s)$ $g \Longrightarrow$ *continuous-on s* $(g \circ f)$
unfolding *continuous-on-topological* **by** *simp metis*

lemma *continuous-on-compose2*:

continuous-on t g ⟹ *continuous-on s f* ⟹ *f ‘ s* ⊆ *t* ⟹ *continuous-on s* (λ*x*. *g* (*f x*))
using *continuous-on-compose*[*of s f g*] *continuous-on-subset* **by** (*force simp add*: *comp-def*)

lemma *continuous-on-generate-topology*:
assumes ∗: *open* = *generate-topology X*
and ∗∗: ⋀*B*. *B* ∈ *X* ⟹ ∃ *C*. *open C* ∧ *C* ∩ *A* = *f* −‘ *B* ∩ *A*
shows *continuous-on A f*
unfolding *continuous-on-open-invariant*
proof *safe*
fix *B* :: ′*a set*
assume *open B*
then show ∃ *C*. *open C* ∧ *C* ∩ *A* = *f* −‘ *B* ∩ *A*
unfolding ∗
proof *induct*
case (*UN K*)
then obtain *C* **where** ⋀*k*. *k* ∈ *K* ⟹ *open* (*C k*) ⋀*k*. *k* ∈ *K* ⟹ *C k* ∩ *A* = *f* −‘ *k* ∩ *A*
by *metis*
then show *?case*
by (*intro exI*[*of* - ⋃*k*∈*K*. *C k*]) *blast*
qed (*auto intro*: ∗∗)
qed

lemma *continuous-onI-mono*:
fixes *f* :: ′*a*::*linorder-topology* ⇒ ′*b*::{*dense-order*,*linorder-topology*}
assumes *open* (*f*‘*A*)
and *mono*: ⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≤ *y* ⟹ *f x* ≤ *f y*
shows *continuous-on A f*
proof (*rule continuous-on-generate-topology*[*OF open-generated-order*], *safe*)
have *monoD*: ⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *f x* < *f y* ⟹ *x* < *y*
by (*auto simp*: *not-le*[*symmetric*] *mono*)
have ∃ *x*. *x* ∈ *A* ∧ *f x* < *b* ∧ *a* < *x* **if** *a*: *a* ∈ *A* **and** *fa*: *f a* < *b* **for** *a b*
proof −
obtain *y* **where** *f a* < *y* {*f a* ..< *y*} ⊆ *f*‘*A*
using *open-right*[*OF* ‹*open* (*f*‘*A*)›, *of f a b*] *a fa*
by *auto*
obtain *z* **where** *z*: *f a* < *z z* < *min b y*
using *dense*[*of f a min b y*] ‹*f a* < *y*› ‹*f a* < *b*› **by** *auto*
then obtain *c* **where** *z* = *f c c* ∈ *A*
using ‹{*f a* ..< *y*} ⊆ *f*‘*A*›[*THEN subsetD*, *of z*] **by** (*auto simp*: *less-imp-le*)
with *a z* **show** *?thesis*
by (*auto intro*!: *exI*[*of* - *c*] *simp*: *monoD*)
qed
then show ∃ *C*. *open C* ∧ *C* ∩ *A* = *f* −‘ {..<*b*} ∩ *A* **for** *b*
by (*intro exI*[*of* - (⋃*x*∈{*x*∈*A*. *f x* < *b*}. {..< *x*})])
(*auto intro*: *le-less-trans*[*OF mono*] *less-imp-le*)

have $\exists x.\ x \in A \wedge b < f\ x \wedge x < a$ **if** *a*: $a \in A$ **and** *fa*: $b < f\ a$ **for** *a b*
proof −
note *a fa*
moreover
obtain *y* **where** $y < f\ a$ $\{y <..\ f\ a\} \subseteq f`A$
using *open-left*[*OF* ‹*open* (*f`A*)›, *of f a b*] *a fa*
by *auto*
then obtain *z* **where** *z*: $max\ b\ y < z$ $z < f\ a$
using *dense*[*of max b y f a*] ‹$y < f\ a$› ‹$b < f\ a$› **by** *auto*
then obtain *c* **where** $z = f\ c$ $c \in A$
using ‹$\{y <..\ f\ a\} \subseteq f`A$›[*THEN subsetD*, *of z*] **by** (*auto simp*: *less-imp-le*)
with *a z* **show** *?thesis*
by (*auto intro*!: *exI*[*of* - *c*] *simp*: *monoD*)
qed
then show $\exists C.\ open\ C \wedge C \cap A = f\ -`\ \{b <..\} \cap A$ **for** *b*
by (*intro exI*[*of* - ($\bigcup x \in \{x \in A.\ b < f\ x\}.\ \{x <..\}$)])
(*auto intro*: *less-le-trans*[*OF* - *mono*] *less-imp-le*)
qed

lemma *continuous-on-IccI*:
$[\![(f \longrightarrow f\ a)\ (at\text{-}right\ a);$
$(f \longrightarrow f\ b)\ (at\text{-}left\ b);$
$(\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow f\ -x\rightarrow\ f\ x);\ a < b]\!] \Longrightarrow$
continuous-on $\{a\ ..\ b\}$ *f*
for *a*::*'a*::*linorder-topology*
using *at-within-open*[*of* - $\{a<..<b\}$]
by (*auto simp*: *continuous-on-def at-within-Icc-at-right at-within-Icc-at-left le-less at-within-Icc-at*)

lemma
fixes *a b*::*'a*::*linorder-topology*
assumes *continuous-on* $\{a\ ..\ b\}$ *f* $a < b$
shows *continuous-on-Icc-at-rightD*: $(f \longrightarrow f\ a)$ (*at-right a*)
and *continuous-on-Icc-at-leftD*: $(f \longrightarrow f\ b)$ (*at-left b*)
using *assms*
by (*auto simp*: *at-within-Icc-at-right at-within-Icc-at-left continuous-on-def dest*: *bspec*[**where** *x*=*a*] *bspec*[**where** *x*=*b*])

lemma *continuous-on-discrete* [*simp*]:
continuous-on A (*f* :: *'a* :: *discrete-topology* ⇒ -)
by (*auto simp*: *continuous-on-def at-discrete*)

lemma *continuous-on-of-nat* [*continuous-intros*]:
assumes *continuous-on A f*
shows *continuous-on A* (λ*n*. *of-nat* (*f n*))
using *continuous-on-compose*[*OF assms continuous-on-discrete*[*of* - *of-nat*]]
by (*simp add*: *o-def*)

lemma *continuous-on-of-int* [*continuous-intros*]:

assumes *continuous-on A f*
shows *continuous-on A* ($\lambda n.$ *of-int* (f n))
using *continuous-on-compose*[*OF assms continuous-on-discrete*[*of - of-int*]]
by (*simp add*: *o-def*)

98.10.2 Continuity at a point

definition *continuous* :: *'a::t2-space filter* $\Rightarrow$ (*'a* $\Rightarrow$ *'b::topological-space*) $\Rightarrow$ *bool*
where *continuous F f* $\longleftrightarrow$ (f $\longrightarrow$ f (*Lim F* ($\lambda x.$ x))) F

lemma *continuous-bot*[*continuous-intros*, *simp*]: *continuous bot f*
unfolding *continuous-def* **by** *auto*

lemma *continuous-trivial-limit*: *trivial-limit net* $\Longrightarrow$ *continuous net f*
by *simp*

lemma *continuous-within*: *continuous* (*at x within s*) f $\longleftrightarrow$ (f $\longrightarrow$ f x) (*at x within s*)
by (*cases trivial-limit* (*at x within s*)) (*auto simp add*: *Lim-ident-at continuous-def*)

lemma *continuous-within-topological*:
continuous (*at x within s*) f $\longleftrightarrow$
($\forall B.$ *open B* $\longrightarrow$ $f\,x \in B$ $\longrightarrow$ ($\exists A.$ *open A* $\wedge$ $x \in A$ $\wedge$ ($\forall y{\in}s.$ $y \in A$ $\longrightarrow$ $f\,y \in B$)))
unfolding *continuous-within tendsto-def eventually-at-topological* **by** *metis*

lemma *continuous-within-compose*[*continuous-intros*]:
continuous (*at x within s*) f $\Longrightarrow$ *continuous* (*at* (f x) *within* f ‘ s) g $\Longrightarrow$
continuous (*at x within s*) ($g \circ f$)
by (*simp add*: *continuous-within-topological*) *metis*

lemma *continuous-within-compose2*:
continuous (*at x within s*) f $\Longrightarrow$ *continuous* (*at* (f x) *within* f ‘ s) g $\Longrightarrow$
continuous (*at x within s*) ($\lambda x.$ g (f x))
using *continuous-within-compose*[*of x s f g*] **by** (*simp add*: *comp-def*)

lemma *continuous-at*: *continuous* (*at x*) f $\longleftrightarrow$ f $-x\rightarrow$ f x
using *continuous-within*[*of x UNIV f*] **by** *simp*

lemma *continuous-ident*[*continuous-intros*, *simp*]: *continuous* (*at x within S*) ($\lambda x.$ x)
unfolding *continuous-within* **by** (*rule tendsto-ident-at*)

lemma *continuous-id*[*continuous-intros*, *simp*]: *continuous* (*at x within S*) *id*
by (*simp add*: *id-def*)

lemma *continuous-const*[*continuous-intros*, *simp*]: *continuous F* ($\lambda x.$ c)
unfolding *continuous-def* **by** (*rule tendsto-const*)

lemma *continuous-on-eq-continuous-within*:
continuous-on s f ⟷ (∀ *x*∈*s*. *continuous* (*at x within s*) *f*)
unfolding *continuous-on-def continuous-within* **..**

lemma *continuous-discrete* [*simp*]:
continuous (*at x within A*) (*f* :: *'a* :: *discrete-topology* ⇒ -)
by (*auto simp*: *continuous-def at-discrete*)

abbreviation *isCont* :: (*'a*::*t2-space* ⇒ *'b*::*topological-space*) ⇒ *'a* ⇒ *bool*
where *isCont f a* ≡ *continuous* (*at a*) *f*

lemma *isCont-def*: *isCont f a* ⟷ *f* −*a*→ *f a*
by (*rule continuous-at*)

lemma *isContD*: *isCont f x* ⟹ *f* −*x*→ *f x*
by (*simp add*: *isCont-def*)

lemma *isCont-cong*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*nhds x*)
shows *isCont f x* ⟷ *isCont g x*
proof −
from *assms* **have** [*simp*]: *f x* = *g x*
by (*rule eventually-nhds-x-imp-x*)
from *assms* **have** *eventually* (λ*x*. *f x* = *g x*) (*at x*)
by (*auto simp*: *eventually-at-filter elim*!: *eventually-mono*)
with *assms* **have** *isCont f x* ⟷ *isCont g x* **unfolding** *isCont-def*
by (*intro filterlim-cong*) (*auto elim*!: *eventually-mono*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *continuous-at-imp-continuous-at-within*: *isCont f x* ⟹ *continuous* (*at x within s*) *f*
by (*auto intro*: *tendsto-mono at-le simp*: *continuous-at continuous-within*)

lemma *continuous-on-eq-continuous-at*: *open s* ⟹ *continuous-on s f* ⟷ (∀ *x*∈*s*. *isCont f x*)
by (*simp add*: *continuous-on-def continuous-at at-within-open*[*of - s*])

lemma *continuous-within-open*: *a* ∈ *A* ⟹ *open A* ⟹ *continuous* (*at a within A*) *f* ⟷ *isCont f a*
by (*simp add*: *at-within-open-NO-MATCH*)

lemma *continuous-at-imp-continuous-on*: ∀ *x*∈*s*. *isCont f x* ⟹ *continuous-on s f*
by (*auto intro*: *continuous-at-imp-continuous-at-within simp*: *continuous-on-eq-continuous-within*)

lemma *isCont-o2*: *isCont f a* ⟹ *isCont g* (*f a*) ⟹ *isCont* (λ*x*. *g* (*f x*)) *a*
unfolding *isCont-def* **by** (*rule tendsto-compose*)

lemma *continuous-at-compose*[*continuous-intros*]: *isCont f a* $\Longrightarrow$ *isCont g* (*f a*) $\Longrightarrow$ *isCont* (*g* $\circ$ *f*) *a*
  unfolding *o-def* **by** (*rule isCont-o2*)

lemma *isCont-tendsto-compose*: *isCont g l* $\Longrightarrow$ (*f* $\longrightarrow$ *l*) *F* $\Longrightarrow$ ((λx. *g* (*f x*)) $\longrightarrow$ *g l*) *F*
  unfolding *isCont-def* **by** (*rule tendsto-compose*)

lemma *continuous-on-tendsto-compose*:
  assumes *f-cont*: *continuous-on s f*
    and *g*: (*g* $\longrightarrow$ *l*) *F*
    and *l*: *l* $\in$ *s*
    and *ev*: $\forall_F$*x in F*. *g x* $\in$ *s*
  shows ((λx. *f* (*g x*)) $\longrightarrow$ *f l*) *F*
proof −
  from *f-cont l* **have** *f*: (*f* $\longrightarrow$ *f l*) (*at l within s*)
    by (*simp add*: *continuous-on-def*)
  have *i*: ((λx. *if g x* = *l then f l else f* (*g x*)) $\longrightarrow$ *f l*) *F*
    by (*rule filterlim-If*)
      (*auto intro*!: *filterlim-compose*[*OF f*] *eventually-conj tendsto-mono*[*OF - g*]
          simp: *filterlim-at eventually-inf-principal eventually-mono*[*OF ev*])
  show *?thesis*
    by (*rule filterlim-cong*[*THEN iffD1*[*OF - i*]]) *auto*
qed

lemma *continuous-within-compose3*:
  isCont g (*f x*) $\Longrightarrow$ *continuous* (*at x within s*) *f* $\Longrightarrow$ *continuous* (*at x within s*) (λx. *g* (*f x*))
  using *continuous-at-imp-continuous-at-within continuous-within-compose2* **by** *blast*

lemma *at-within-isCont-imp-nhds*:
  fixes *f*:: *'a*:: {*t2-space*,*perfect-space*} $\Rightarrow$ *'b*:: *t2-space*
  assumes $\forall_F$ *w in at z*. *f w* = *g w isCont f z isCont g z*
  shows $\forall_F$ *w in nhds z*. *f w* = *g w*
proof −
  have *g* −*z*→ *f z*
    using *assms isContD tendsto-cong* **by** *blast*
  moreover have *g* −*z*→ *g z* **using** ‹*isCont g z*› **using** *isCont-def* **by** *blast*
  ultimately have *f z*=*g z* **using** *LIM-unique* **by** *auto*
  moreover have $\forall_F$ *x in nhds z*. *x* $\neq$ *z* $\longrightarrow$ *f x* = *g x*
    using *assms* **unfolding** *eventually-at-filter* **by** *auto*
  ultimately show *?thesis*
    by (*auto elim*:*eventually-mono*)
qed

lemma *filtermap-nhds-open-map'*:
  assumes *cont*: *isCont f a*
    and *open A a* $\in$ *A*

and *open-map*: $\bigwedge S$. *open* $S \Longrightarrow S \subseteq A \Longrightarrow$ *open* (f ‘ S)
shows *filtermap f* (*nhds a*) = *nhds* (*f a*)
unfolding *filter-eq-iff*
proof *safe*
fix *P*
assume *eventually P* (*filtermap f* (*nhds a*))
then obtain *S* **where** *S*: *open S a* ∈ *S* ∀ *x*∈*S*. *P* (*f x*)
by (*auto simp*: *eventually-filtermap eventually-nhds*)
show *eventually P* (*nhds* (*f a*))
unfolding *eventually-nhds*
proof (*rule exI* [*of* - *f* ‘ (*A* ∩ *S*)], *safe*)
show *open* (*f* ‘ (*A* ∩ *S*))
using *S* **by** (*intro open-Int assms*) *auto*
show *f a* ∈ *f* ‘ (*A* ∩ *S*)
using *assms S* **by** *auto*
show *P* (*f x*) **if** *x* ∈ *A x* ∈ *S* **for** *x*
using *S that* **by** *auto*
qed
qed (*metis filterlim-iff tendsto-at-iff-tendsto-nhds isCont-def eventually-filtermap cont*)

lemma *filtermap-nhds-open-map*:
assumes *cont*: *isCont f a*
and *open-map*: $\bigwedge S$. *open* $S \Longrightarrow$ *open* (*f*‘*S*)
shows *filtermap f* (*nhds a*) = *nhds* (*f a*)
using *cont filtermap-nhds-open-map′ open-map* **by** *blast*

lemma *continuous-at-split*:
continuous (*at x*) *f* ⟷ *continuous* (*at-left x*) *f* ∧ *continuous* (*at-right x*) *f*
for *x* :: ′*a*::*linorder-topology*
by (*simp add*: *continuous-within filterlim-at-split*)

lemma *continuous-on-max* [*continuous-intros*]:
fixes *f g* :: ′*a*::*topological-space* ⇒ ′*b*::*linorder-topology*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (λ*x*. *max* (*f x*) (*g x*))
by (*auto simp*: *continuous-on-def intro*!: *tendsto-max*)

lemma *continuous-on-min* [*continuous-intros*]:
fixes *f g* :: ′*a*::*topological-space* ⇒ ′*b*::*linorder-topology*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (λ*x*. *min* (*f x*) (*g x*))
by (*auto simp*: *continuous-on-def intro*!: *tendsto-min*)

lemma *continuous-max* [*continuous-intros*]:
fixes *f* :: ′*a*::*t2-space* ⇒ ′*b*::*linorder-topology*
shows ⟦*continuous F f*; *continuous F g*⟧ ⟹ *continuous F* (λ*x*. (*max* (*f x*) (*g x*)))
by (*simp add*: *tendsto-max continuous-def*)

lemma *continuous-min* [*continuous-intros*]:
  fixes *f* :: *'a::t2-space* ⇒ *'b::linorder-topology*
  shows ⟦*continuous F f*; *continuous F g*⟧ ⟹ *continuous F* (*λx.* (*min* (*f x*) (*g x*)))
  by (*simp add*: *tendsto-min continuous-def*)

The following open/closed Collect lemmas are ported from Sébastien Gouëzel's *Ergodic-Theory*.

lemma *open-Collect-neq*:
  fixes *f g* :: *'a::topological-space* ⇒ *'b::t2-space*
  assumes *f*: *continuous-on UNIV f* **and** *g*: *continuous-on UNIV g*
  shows *open* {*x*. *f x* ≠ *g x*}
proof (*rule openI*)
  fix *t*
  assume *t* ∈ {*x*. *f x* ≠ *g x*}
  then obtain *U V* **where** ∗: *open U open V f t* ∈ *U g t* ∈ *V U* ∩ *V* = {}
    by (*auto simp add*: *separation-t2*)
  with *open-vimage*[*OF* ‹*open U*› *f*] *open-vimage*[*OF* ‹*open V*› *g*]
  show ∃ *T*. *open T* ∧ *t* ∈ *T* ∧ *T* ⊆ {*x*. *f x* ≠ *g x*}
    by (*intro exI*[*of* - *f* −' *U* ∩ *g* −' *V*]) *auto*
qed

lemma *closed-Collect-eq*:
  fixes *f g* :: *'a::topological-space* ⇒ *'b::t2-space*
  assumes *f*: *continuous-on UNIV f* **and** *g*: *continuous-on UNIV g*
  shows *closed* {*x*. *f x* = *g x*}
  using *open-Collect-neq*[*OF f g*] **by** (*simp add*: *closed-def Collect-neg-eq*)

lemma *open-Collect-less*:
  fixes *f g* :: *'a::topological-space* ⇒ *'b::linorder-topology*
  assumes *f*: *continuous-on UNIV f* **and** *g*: *continuous-on UNIV g*
  shows *open* {*x*. *f x* < *g x*}
proof (*rule openI*)
  fix *t*
  assume *t*: *t* ∈ {*x*. *f x* < *g x*}
  show ∃ *T*. *open T* ∧ *t* ∈ *T* ∧ *T* ⊆ {*x*. *f x* < *g x*}
  proof (*cases* ∃ *z*. *f t* < *z* ∧ *z* < *g t*)
    case *True*
    then obtain *z* **where** *f t* < *z* ∧ *z* < *g t* **by** *blast*
    then show *?thesis*
      using *open-vimage*[*OF* - *f*, *of* {..< *z*}] *open-vimage*[*OF* - *g*, *of* {*z* <..}]
      by (*intro exI*[*of* - *f* −' {..<*z*} ∩ *g* −' {*z*<..}]) *auto*
  next
    case *False*
    then have ∗: {*g t* ..} = {*f t* <..} {..< *g t*} = {.. *f t*}
      using *t* **by** (*auto intro*: *leI*)
    show *?thesis*
      using *open-vimage*[*OF* - *f*, *of* {..< *g t*}] *open-vimage*[*OF* - *g*, *of* {*f t* <..}] *t*
      apply (*intro exI*[*of* - *f* −' {..< *g t*} ∩ *g* −' {*f t*<..}])

apply (*simp add*: *open-Int*)
apply (*auto simp add*: $*$)
done
qed
qed

lemma *closed-Collect-le*:
fixes *f g* :: *'a* :: *topological-space* ⇒ *'b*::*linorder-topology*
assumes *f*: *continuous-on UNIV f*
and *g*: *continuous-on UNIV g*
shows *closed* $\{x.\ f\ x \le g\ x\}$
using *open-Collect-less* [*OF g f*]
by (*simp add*: *closed-def Collect-neg-eq*[*symmetric*] *not-le*)

98.10.3 Open-cover compactness

context *topological-space*
begin

definition *compact* :: *'a set* ⇒ *bool* **where**
compact-eq-Heine-Borel:
compact S $\longleftrightarrow$ ($\forall$ *C*. ($\forall$ *c*∈*C*. *open c*) $\wedge$ *S* $\subseteq \bigcup$ *C* $\longrightarrow$ ($\exists$ *D*⊆*C*. *finite D* $\wedge$ *S* $\subseteq \bigcup$ *D*))

lemma *compactI*:
assumes $\bigwedge C.\ \forall t{\in}C.\ open\ t \Longrightarrow s \subseteq \bigcup C \Longrightarrow \exists C'.\ C' \subseteq C \wedge finite\ C' \wedge s \subseteq \bigcup C'$
shows *compact s*
unfolding *compact-eq-Heine-Borel* **using** *assms* **by** *metis*

lemma *compact-empty*[*simp*]: *compact* {}
by (*auto intro*!: *compactI*)

lemma *compactE*:
assumes *compact S* $S \subseteq \bigcup \mathcal{T}$ $\bigwedge B.\ B \in \mathcal{T} \Longrightarrow open\ B$
obtains $\mathcal{T}'$ **where** $\mathcal{T}' \subseteq \mathcal{T}$ *finite* $\mathcal{T}'$ $S \subseteq \bigcup \mathcal{T}'$
by (*meson assms compact-eq-Heine-Borel*)

lemma *compactE-image*:
assumes *compact S*
and *opn*: $\bigwedge T.\ T \in C \Longrightarrow open\ (f\ T)$
and *S*: $S \subseteq (\bigcup c{\in}C.\ f\ c)$
obtains C' **where** $C' \subseteq C$ **and** *finite* C' **and** $S \subseteq (\bigcup c{\in}C'.\ f\ c)$
apply (*rule compactE*[*OF* ‹*compact S*› *S*])
using *opn* **apply** *force*
by (*metis finite-subset-image*)

lemma *compact-Int-closed* [*intro*]:
assumes *compact S*

and *closed T*
shows *compact* $(S \cap T)$
proof (*rule compactI*)
fix *C*
assume *C*: $\forall c \in C$. *open c*
assume *cover*: $S \cap T \subseteq \bigcup C$
from *C* ‹*closed T*› **have** $\forall c \in C \cup \{- T\}$. *open c*
by *auto*
moreover from *cover* **have** $S \subseteq \bigcup(C \cup \{- T\})$
by *auto*
ultimately have $\exists D \subseteq C \cup \{- T\}$. *finite D* $\wedge S \subseteq \bigcup D$
using ‹*compact S*› **unfolding** *compact-eq-Heine-Borel* **by** *auto*
then obtain *D* **where** $D \subseteq C \cup \{- T\} \wedge$ *finite D* $\wedge S \subseteq \bigcup D$ **..**
then show $\exists D \subseteq C$. *finite D* $\wedge S \cap T \subseteq \bigcup D$
by (*intro exI*[*of - D* $-$ $\{-T\}$]) *auto*
qed

lemma *compact-diff*: $[\![$*compact S*; *open T*$]\!] \Longrightarrow$ *compact*$(S - T)$
by (*simp add*: *Diff-eq compact-Int-closed open-closed*)

lemma *inj-setminus*: *inj-on uminus* (*A*::'*a set set*)
by (*auto simp*: *inj-on-def*)

98.11 Finite intersection property

lemma *compact-fip*:
compact U $\longleftrightarrow$
$(\forall A. (\forall a \in A.$ *closed a*$) \longrightarrow (\forall B \subseteq A.$ *finite B* $\longrightarrow U \cap \bigcap B \neq \{\}) \longrightarrow U \cap \bigcap A \neq \{\})$
(**is** - $\longleftrightarrow$ *?R*)
proof (*safe intro*!: *compact-eq-Heine-Borel*[*THEN iffD2*])
fix *A*
assume *compact U*
assume *A*: $\forall a \in A$. *closed a* $U \cap \bigcap A = \{\}$
assume *fin*: $\forall B \subseteq A$. *finite B* $\longrightarrow U \cap \bigcap B \neq \{\}$
from *A* **have** ($\forall a \in$*uminus*'*A*. *open a*) $\wedge U \subseteq \bigcup($*uminus*'*A*$)$
by *auto*
with ‹*compact U*› **obtain** *B* **where** $B \subseteq A$ *finite* (*uminus*'*B*) $U \subseteq \bigcup($*uminus*'*B*$)$
unfolding *compact-eq-Heine-Borel* **by** (*metis subset-image-iff*)
with *fin*[*THEN spec*, *of B*] **show** *False*
by (*auto dest*: *finite-imageD intro*: *inj-setminus*)
next
fix *A*
assume *?R*
assume $\forall a \in A$. *open a* $U \subseteq \bigcup A$
then have $U \cap \bigcap($*uminus*'*A*$) = \{\}$ $\forall a \in$*uminus*'*A*. *closed a*
by *auto*
with ‹*?R*› **obtain** *B* **where** $B \subseteq A$ *finite* (*uminus*'*B*) $U \cap \bigcap($*uminus*'*B*$) = \{\}$
by (*metis subset-image-iff*)

then show $\exists\, T \subseteq A.\ \mathit{finite}\ T \wedge U \subseteq \bigcup T$
  by (*auto intro*!: *exI*[*of* - *B*] *inj-setminus dest*: *finite-imageD*)
qed

lemma *compact-imp-fip*:
  assumes *compact S*
    and $\bigwedge T.\ T \in F \Longrightarrow \mathit{closed}\ T$
    and $\bigwedge F'.\ \mathit{finite}\ F' \Longrightarrow F' \subseteq F \Longrightarrow S \cap (\bigcap F') \neq \{\}$
  shows $S \cap (\bigcap F) \neq \{\}$
  using *assms* **unfolding** *compact-fip* **by** *auto*

lemma *compact-imp-fip-image*:
  assumes *compact s*
    and P: $\bigwedge i.\ i \in I \Longrightarrow \mathit{closed}\ (f\ i)$
    and Q: $\bigwedge I'.\ \mathit{finite}\ I' \Longrightarrow I' \subseteq I \Longrightarrow (s \cap (\bigcap i{\in}I'.\ f\ i) \neq \{\})$
  shows $s \cap (\bigcap i{\in}I.\ f\ i) \neq \{\}$
proof −
  from P **have** $\forall\, i \in f\ \text{‘}\ I.\ \mathit{closed}\ i$
    by *blast*
  moreover have $\forall\, A.\ \mathit{finite}\ A \wedge A \subseteq f\ \text{‘}\ I \longrightarrow (s \cap (\bigcap A) \neq \{\})$
    by (*metis Q finite-subset-image*)
  ultimately show $s \cap (\bigcap (f\ \text{‘}\ I)) \neq \{\}$
    by (*metis* ‹*compact s*› *compact-imp-fip*)
qed

end

lemma (**in** *t2-space*) *compact-imp-closed*:
  assumes *compact s*
  shows *closed s*
  unfolding *closed-def*
proof (*rule openI*)
  fix *y*
  assume $y \in -\ s$
  let $?C = \bigcup x{\in}s.\ \{u.\ \mathit{open}\ u \wedge x \in u \wedge \mathit{eventually}\ (\lambda y.\ y \notin u)\ (\mathit{nhds}\ y)\}$
  have $s \subseteq \bigcup ?C$
  proof
    fix *x*
    assume $x \in s$
    with ‹$y \in -\ s$› **have** $x \neq y$ **by** *clarsimp*
    then have $\exists\, u\ v.\ \mathit{open}\ u \wedge \mathit{open}\ v \wedge x \in u \wedge y \in v \wedge u \cap v = \{\}$
      by (*rule hausdorff*)
    with ‹$x \in s$› **show** $x \in \bigcup ?C$
      unfolding *eventually-nhds* **by** *auto*
  qed
  then obtain D **where** $D \subseteq ?C$ **and** *finite D* **and** $s \subseteq \bigcup D$
    by (*rule compactE* [*OF* ‹*compact s*›]) *auto*
  from ‹$D \subseteq ?C$› **have** $\forall\, x{\in}D.\ \mathit{eventually}\ (\lambda y.\ y \notin x)\ (\mathit{nhds}\ y)$
    by *auto*

with ‹*finite D*› **have** *eventually* ($\lambda y.\ y \notin \bigcup D$) (*nhds y*)
  by (*simp add*: *eventually-ball-finite*)
with ‹$s \subseteq \bigcup D$› **have** *eventually* ($\lambda y.\ y \notin s$) (*nhds y*)
  by (*auto elim*!: *eventually-mono*)
then show $\exists t.$ *open* $t \wedge y \in t \wedge t \subseteq - s$
  by (*simp add*: *eventually-nhds subset-eq*)
qed

lemma *compact-continuous-image*:
  assumes *f*: *continuous-on s f*
    and *s*: *compact s*
  shows *compact* (*f* ‘ *s*)
proof (*rule compactI*)
  fix *C*
  assume $\forall c \in C.$ *open c* **and** *cover*: $f`s \subseteq \bigcup C$
  with *f* **have** $\forall c \in C.\ \exists A.$ *open* $A \wedge A \cap s = f -` c \cap s$
    unfolding *continuous-on-open-invariant* **by** *blast*
  then obtain *A* **where** *A*: $\forall c \in C.$ *open* $(A\ c) \wedge A\ c \cap s = f -` c \cap s$
    unfolding *bchoice-iff* **..**
  with *cover* **have** $\bigwedge c.\ c \in C \Longrightarrow$ *open* $(A\ c)$ $s \subseteq (\bigcup c \in C.\ A\ c)$
    by (*fastforce simp add*: *subset-eq set-eq-iff*)+
  from *compactE-image*[*OF s this*] **obtain** *D* **where** $D \subseteq C$ *finite D* $s \subseteq (\bigcup c \in D.\ A\ c)$ **.**
  with *A* **show** $\exists D \subseteq C.$ *finite* $D \wedge f`s \subseteq \bigcup D$
    by (*intro exI*[*of - D*]) (*fastforce simp add*: *subset-eq set-eq-iff*)+
qed

lemma *continuous-on-inv*:
  fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*t2-space*
  assumes *continuous-on s f*
    and *compact s*
    and $\forall x \in s.\ g\ (f\ x) = x$
  shows *continuous-on* (*f* ‘ *s*) *g*
  unfolding *continuous-on-topological*
proof (*clarsimp simp add*: *assms*(*3*))
  fix *x* :: *'a* **and** *B* :: *'a set*
  assume $x \in s$ **and** *open B* **and** $x \in B$
  have *1*: $\forall x \in s.\ f\ x \in f\ `\ (s - B) \longleftrightarrow x \in s - B$
    using *assms*(*3*) **by** (*auto*, *metis*)
  have *continuous-on* $(s - B)$ *f*
    using ‹*continuous-on s f*› *Diff-subset*
    by (*rule continuous-on-subset*)
  moreover have *compact* $(s - B)$
    using ‹*open B*› **and** ‹*compact s*›
    unfolding *Diff-eq* **by** (*intro compact-Int-closed closed-Compl*)
  ultimately have *compact* $(f\ `\ (s - B))$
    by (*rule compact-continuous-image*)
  then have *closed* $(f\ `\ (s - B))$
    by (*rule compact-imp-closed*)

then have *open* $(- f \text{ ` } (s - B))$
by (*rule open-Compl*)
moreover have $f\ x \in - f \text{ ` } (s - B)$
using ‹$x \in s$› **and** ‹$x \in B$› **by** (*simp add*: *1*)
moreover have $\forall y{\in}s.\ f\ y \in - f \text{ ` } (s - B) \longrightarrow y \in B$
by (*simp add*: *1*)
ultimately show $\exists A.$ *open* $A \wedge f\ x \in A \wedge (\forall y{\in}s.\ f\ y \in A \longrightarrow y \in B)$
by *fast*
qed

lemma *continuous-on-inv-into*:
fixes *f* :: *'a::topological-space* ⇒ *'b::t2-space*
assumes *s*: *continuous-on s f compact s*
and *f*: *inj-on f s*
shows *continuous-on* (*f* ‘ *s*) (*the-inv-into s f*)
by (*rule continuous-on-inv*[*OF s*]) (*auto simp*: *the-inv-into-f-f*[*OF f*])

lemma (**in** *linorder-topology*) *compact-attains-sup*:
assumes *compact* S $S \neq \{\}$
shows $\exists s{\in}S.\ \forall t{\in}S.\ t \leq s$
proof (*rule classical*)
assume $\neg (\exists s{\in}S.\ \forall t{\in}S.\ t \leq s)$
then obtain *t* **where** *t*: $\forall s{\in}S.\ t\ s \in S$ **and** $\forall s{\in}S.\ s < t\ s$
by (*metis not-le*)
then have $\bigwedge s.\ s{\in}S \Longrightarrow$ *open* $\{..< t\ s\}$ $S \subseteq (\bigcup s{\in}S.\ \{..< t\ s\})$
by *auto*
with ‹*compact S*› **obtain** *C* **where** $C \subseteq S$ *finite C* **and** *C*: $S \subseteq (\bigcup s{\in}C.\ \{..< t\ s\})$
by (*metis compactE-image*)
with ‹$S \neq \{\}$› **have** *Max*: *Max* $(t\text{`}C) \in t\text{`}C$ **and** $\forall s{\in}t\text{`}C.\ s \leq$ *Max* $(t\text{`}C)$
by (*auto intro*!: *Max-in*)
with *C* **have** $S \subseteq \{..<$ *Max* $(t\text{`}C)\}$
by (*auto intro*: *less-le-trans simp*: *subset-eq*)
with *t Max* ‹$C \subseteq S$› **show** *?thesis*
by *fastforce*
qed

lemma (**in** *linorder-topology*) *compact-attains-inf*:
assumes *compact* S $S \neq \{\}$
shows $\exists s{\in}S.\ \forall t{\in}S.\ s \leq t$
proof (*rule classical*)
assume $\neg (\exists s{\in}S.\ \forall t{\in}S.\ s \leq t)$
then obtain *t* **where** *t*: $\forall s{\in}S.\ t\ s \in S$ **and** $\forall s{\in}S.\ t\ s < s$
by (*metis not-le*)
then have $\bigwedge s.\ s{\in}S \Longrightarrow$ *open* $\{t\ s <..\}$ $S \subseteq (\bigcup s{\in}S.\ \{t\ s <..\})$
by *auto*
with ‹*compact S*› **obtain** *C* **where** $C \subseteq S$ *finite C* **and** *C*: $S \subseteq (\bigcup s{\in}C.\ \{t\ s <..\})$
by (*metis compactE-image*)

with ‹$S \neq \{\}$› **have** *Min*: $Min\ (t`C) \in t`C$ **and** $\forall\, s{\in}t`C.\ Min\ (t`C) \leq s$
by (*auto intro*!: *Min-in*)
with C **have** $S \subseteq \{Min\ (t`C) <..\}$
by (*auto intro*: *le-less-trans simp*: *subset-eq*)
with t *Min* ‹$C \subseteq S$› **show** *?thesis*
by *fastforce*
qed

lemma *continuous-attains-sup*:
fixes f :: $'a$::*topological-space* ⇒ $'b$::*linorder-topology*
shows *compact* $s \Longrightarrow s \neq \{\} \Longrightarrow$ *continuous-on* $s\ f \Longrightarrow (\exists\, x{\in}s.\ \forall\, y{\in}s.\ \ f\ y \leq f\ x)$
using *compact-attains-sup*[*of f ‘ s*] *compact-continuous-image*[*of s f*] **by** *auto*

lemma *continuous-attains-inf*:
fixes f :: $'a$::*topological-space* ⇒ $'b$::*linorder-topology*
shows *compact* $s \Longrightarrow s \neq \{\} \Longrightarrow$ *continuous-on* $s\ f \Longrightarrow (\exists\, x{\in}s.\ \forall\, y{\in}s.\ f\ x \leq f\ y)$
using *compact-attains-inf*[*of f ‘ s*] *compact-continuous-image*[*of s f*] **by** *auto*

98.12 Connectedness

context *topological-space*
begin

definition *connected* $S \longleftrightarrow$
$\neg\ (\exists\, A\ B.\ open\ A \wedge open\ B \wedge S \subseteq A \cup B \wedge A \cap B \cap S = \{\} \wedge A \cap S \neq \{\} \wedge B \cap S \neq \{\})$

lemma *connectedI*:
$(\bigwedge A\ B.\ open\ A \Longrightarrow open\ B \Longrightarrow A \cap U \neq \{\} \Longrightarrow B \cap U \neq \{\} \Longrightarrow A \cap B \cap U = \{\} \Longrightarrow U \subseteq A \cup B \Longrightarrow False)$
$\Longrightarrow$ *connected* U
by (*auto simp*: *connected-def*)

lemma *connected-empty* [*simp*]: *connected* $\{\}$
by (*auto intro*!: *connectedI*)

lemma *connected-sing* [*simp*]: *connected* $\{x\}$
by (*auto intro*!: *connectedI*)

lemma *connectedD*:
connected $A \Longrightarrow$ *open* $U \Longrightarrow$ *open* $V \Longrightarrow U \cap V \cap A = \{\} \Longrightarrow A \subseteq U \cup V \Longrightarrow U \cap A = \{\} \vee V \cap A = \{\}$
by (*auto simp*: *connected-def*)

end

lemma *connected-closed*:
connected $s \longleftrightarrow$

¬ (∃ *A B*. *closed A* ∧ *closed B* ∧ *s* ⊆ *A* ∪ *B* ∧ *A* ∩ *B* ∩ *s* = {} ∧ *A* ∩ *s* ≠ {} ∧ *B* ∩ *s* ≠ {})
apply (*simp add*: *connected-def del*: *ex-simps*, *safe*)
apply (*drule-tac x*=−*A* **in** *spec*)
apply (*drule-tac x*=−*B* **in** *spec*)
apply (*fastforce simp add*: *closed-def* [*symmetric*])
apply (*drule-tac x*=−*A* **in** *spec*)
apply (*drule-tac x*=−*B* **in** *spec*)
apply (*fastforce simp add*: *open-closed* [*symmetric*])
done

lemma *connected-closedD*:
⟦*connected s*; *A* ∩ *B* ∩ *s* = {}; *s* ⊆ *A* ∪ *B*; *closed A*; *closed B*⟧ ⟹ *A* ∩ *s* = {} ∨ *B* ∩ *s* = {}
by (*simp add*: *connected-closed*)

lemma *connected-Union*:
assumes *cs*: ⋀*s*. *s* ∈ *S* ⟹ *connected s*
and *ne*: ⋂ *S* ≠ {}
shows *connected*(⋃ *S*)
proof (*rule connectedI*)
fix *A B*
assume *A*: *open A* **and** *B*: *open B* **and** *Alap*: *A* ∩ ⋃ *S* ≠ {} **and** *Blap*: *B* ∩ ⋃ *S* ≠ {}
and *disj*: *A* ∩ *B* ∩ ⋃ *S* = {} **and** *cover*: ⋃ *S* ⊆ *A* ∪ *B*
have *disjs*:⋀*s*. *s* ∈ *S* ⟹ *A* ∩ *B* ∩ *s* = {}
using *disj* **by** *auto*
obtain *sa* **where** *sa*: *sa* ∈ *S A* ∩ *sa* ≠ {}
using *Alap* **by** *auto*
obtain *sb* **where** *sb*: *sb* ∈ *S B* ∩ *sb* ≠ {}
using *Blap* **by** *auto*
obtain *x* **where** *x*: ⋀*s*. *s* ∈ *S* ⟹ *x* ∈ *s*
using *ne* **by** *auto*
then have *x* ∈ ⋃ *S*
using ‹*sa* ∈ *S*› **by** *blast*
then have *x* ∈ *A* ∨ *x* ∈ *B*
using *cover* **by** *auto*
then show *False*
using *cs* [*unfolded connected-def*]
by (*metis A B IntI Sup-upper sa sb disjs x cover empty-iff subset-trans*)
qed

lemma *connected-Un*: *connected s* ⟹ *connected t* ⟹ *s* ∩ *t* ≠ {} ⟹ *connected* (*s* ∪ *t*)
using *connected-Union* [*of* {*s*,*t*}] **by** *auto*

lemma *connected-diff-open-from-closed*:
assumes *st*: *s* ⊆ *t*
and *tu*: *t* ⊆ *u*

and *s*: *open s*
and *t*: *closed t*
and *u*: *connected u*
and *ts*: *connected* $(t - s)$
shows *connected*$(u - s)$
proof (*rule connectedI*)
fix *A B*
assume *AB*: *open A open B* $A \cap (u - s) \neq \{\}$ $B \cap (u - s) \neq \{\}$
and *disj*: $A \cap B \cap (u - s) = \{\}$
and *cover*: $u - s \subseteq A \cup B$
then consider $A \cap (t - s) = \{\} \mid B \cap (t - s) = \{\}$
using *st ts tu connectedD* [*of t−s A B*] **by** *auto*
then show *False*
proof *cases*
case *1*
then have $(A - t) \cap (B \cup s) \cap u = \{\}$
using *disj st* **by** *auto*
moreover have $u \subseteq (A - t) \cup (B \cup s)$
using *1 cover* **by** *auto*
ultimately show *False*
using *connectedD* [*of u* $A - t$ $B \cup s$] *AB s t 1 u* **by** *auto*
next
case *2*
then have $(A \cup s) \cap (B - t) \cap u = \{\}$
using *disj st* **by** *auto*
moreover have $u \subseteq (A \cup s) \cup (B - t)$
using *2 cover* **by** *auto*
ultimately show *False*
using *connectedD* [*of u* $A \cup s$ $B - t$] *AB s t 2 u* **by** *auto*
qed
qed

lemma *connected-iff-const*:
fixes *S* :: *'a::topological-space set*
shows *connected S* $\longleftrightarrow$ ($\forall$ *P*::*'a* $\Rightarrow$ *bool. continuous-on S P* $\longrightarrow$ ($\exists$ *c.* $\forall$ *s*$\in$*S. P s* $= c$))
proof *safe*
fix *P* :: *'a* $\Rightarrow$ *bool*
assume *connected S continuous-on S P*
then have $\bigwedge b.\ \exists A.$ *open A* $\wedge$ $A \cap S = P -` \{b\} \cap S$
unfolding *continuous-on-open-invariant* **by** (*simp add*: *open-discrete*)
from *this*[*of True*] *this*[*of False*]
obtain *t f* **where** *open t open f* **and** $*$: $f \cap S = P -` \{False\} \cap S$ $t \cap S = P -` \{True\} \cap S$
by *meson*
then have $t \cap S = \{\} \vee f \cap S = \{\}$
by (*intro connectedD*[*OF* ‹*connected S*›]) *auto*
then show $\exists c.\ \forall s{\in}S.\ P\ s = c$
proof (*rule disjE*)

assume $t \cap S = \{\}$
then show *?thesis*
unfolding ∗ **by** (*intro exI*[*of - False*]) *auto*
next
assume $f \cap S = \{\}$
then show *?thesis*
unfolding ∗ **by** (*intro exI*[*of - True*]) *auto*
qed
next
assume *P*: $\forall P::'a \Rightarrow bool.$ *continuous-on S P* $\longrightarrow (\exists c. \forall s\in S. P\ s = c)$
show *connected S*
proof (*rule connectedI*)
fix *A B*
assume ∗: *open A open B* $A \cap S \neq \{\}$ $B \cap S \neq \{\}$ $A \cap B \cap S = \{\}$ $S \subseteq A \cup B$
have *continuous-on S* $(\lambda x. x \in A)$
unfolding *continuous-on-open-invariant*
proof *safe*
fix *C* :: *bool set*
have $C = UNIV \lor C = \{True\} \lor C = \{False\} \lor C = \{\}$
using *subset-UNIV*[*of C*] **unfolding** *UNIV-bool* **by** *auto*
with ∗ **show** $\exists T.$ *open* $T \land T \cap S = (\lambda x. x \in A) -` C \cap S$
by (*intro exI*[*of -* (*if True* $\in$ *C then A else* {}) $\cup$ (*if False* $\in$ *C then B else* {})]) *auto*
qed
from *P*[*rule-format*, *OF this*] **obtain** *c* **where** $\bigwedge s. s \in S \Longrightarrow (s \in A) = c$
by *blast*
with ∗ **show** *False*
by (*cases c*) *auto*
qed
qed

lemma *connectedD-const*: *connected S* $\Longrightarrow$ *continuous-on S P* $\Longrightarrow \exists c. \forall s\in S. P\ s = c$
for *P* :: *'a::topological-space* $\Rightarrow$ *bool*
by (*auto simp*: *connected-iff-const*)

lemma *connectedI-const*:
$(\bigwedge P::'a::topological\text{-}space \Rightarrow bool.$ *continuous-on S P* $\Longrightarrow \exists c. \forall s\in S. P\ s = c)$ $\Longrightarrow$ *connected S*
by (*auto simp*: *connected-iff-const*)

lemma *connected-local-const*:
assumes *connected A* $a \in A$ $b \in A$
and ∗: $\forall a\in A.$ *eventually* $(\lambda b. f\ a = f\ b)$ (*at a within A*)
shows $f\ a = f\ b$
proof −
obtain *S* **where** *S*: $\bigwedge a. a \in A \Longrightarrow a \in S\ a$ $\bigwedge a. a \in A \Longrightarrow$ *open* $(S\ a)$
$\bigwedge a\ x. a \in A \Longrightarrow x \in S\ a \Longrightarrow x \in A \Longrightarrow f\ a = f\ x$

using $*$ **unfolding** *eventually-at-topological* **by** *metis*
let $?P = \bigcup b \in \{b \in A.\ f\ a = f\ b\}.\ S\ b$ **and** $?N = \bigcup b \in \{b \in A.\ f\ a \neq f\ b\}.\ S\ b$
have $?P \cap A = \{\} \vee ?N \cap A = \{\}$
using ‹*connected A*› S ‹$a \in A$›
by (*intro connectedD*) (*auto*, *metis*)
then show $f\ a = f\ b$
proof
assume $?N \cap A = \{\}$
then have $\forall x \in A.\ f\ a = f\ x$
using $S(1)$ **by** *auto*
with ‹$b \in A$› **show** *?thesis* **by** *auto*
next
assume $?P \cap A = \{\}$ **then show** *?thesis*
using ‹$a \in A$› $S(1)[of\ a]$ **by** *auto*
qed
qed

lemma (**in** *linorder-topology*) *connectedD-interval*:
assumes *connected U*
and *xy*: $x \in U$ $y \in U$
and $x \leq z$ $z \leq y$
shows $z \in U$
proof −
have *eq*: $\{..<z\} \cup \{z<..\} = - \{z\}$
by *auto*
have $\neg$ *connected U* **if** $z \notin U$ $x < z$ $z < y$
using *xy that*
apply (*simp only*: *connected-def simp-thms*)
apply (*rule-tac exI*[*of* - $\{..< z\}$])
apply (*rule-tac exI*[*of* - $\{z <..\}$])
apply (*auto simp add*: *eq*)
done
with *assms* **show** $z \in U$
by (*metis less-le*)
qed

lemma (**in** *linorder-topology*) *not-in-connected-cases*:
assumes *conn*: *connected S*
assumes *nbdd*: $x \notin S$
assumes *ne*: $S \neq \{\}$
obtains *bdd-above S* $\bigwedge y.\ y \in S \Longrightarrow x \geq y$ | *bdd-below S* $\bigwedge y.\ y \in S \Longrightarrow x \leq y$
proof −
obtain s **where** $s \in S$ **using** *ne* **by** *blast*
{
assume $s \leq x$
have *False* **if** $x \leq y$ $y \in S$ **for** y
using *connectedD-interval*[*OF conn* ‹$s \in S$› ‹$y \in S$› ‹$s \leq x$› ‹$x \leq y$›] ‹$x \notin S$›
by *simp*
then have *wit*: $y \in S \Longrightarrow x \geq y$ **for** y

using *le-cases* **by** *blast*
then have *bdd-above S*
by (*rule local.bdd-aboveI*)
note *this wit*
} moreover {
assume $x \leq s$
have *False* **if** $x \geq y$ $y \in S$ **for** y
using *connectedD-interval*[*OF conn* ‹$y \in S$› ‹$s \in S$› ‹$x \geq y$› ‹$s \geq x$›] ‹$x \notin S$›
by *simp*
then have *wit*: $y \in S \Longrightarrow x \leq y$ **for** y
using *le-cases* **by** *blast*
then have *bdd-below S*
by (*rule bdd-belowI*)
note *this wit*
} ultimately show *?thesis*
by (*meson le-cases that*)
qed

lemma *connected-continuous-image*:
assumes $*$: *continuous-on s f*
and *connected s*
shows *connected* (f ‘ s)
proof (*rule connectedI-const*)
fix P :: $'b \Rightarrow$ *bool*
assume *continuous-on* (f ‘ s) P
then have *continuous-on s* ($P \circ f$)
by (*rule continuous-on-compose*[*OF* $*$])
from *connectedD-const*[*OF* ‹*connected s*› *this*] **show** $\exists c.\ \forall s \in f$ ‘ $s.\ P\ s = c$
by *auto*
qed

lemma *connected-Un-UN*:
assumes *connected A* $\bigwedge X.\ X \in B \Longrightarrow$ *connected X* $\bigwedge X.\ X \in B \Longrightarrow A \cap X \neq \{\}$
shows *connected* ($A \cup \bigcup B$)
proof (*rule connectedI-const*)
fix f :: $'a \Rightarrow$ *bool*
assume f: *continuous-on* ($A \cup \bigcup B$) f
have *connected A continuous-on A f*
by (*auto intro*: *assms continuous-on-subset*[*OF f*(*1*)])
from *connectedD-const*[*OF this*] **obtain** c **where** c: $\bigwedge x.\ x \in A \Longrightarrow f\,x = c$
by *metis*
have $f\,x = c$ **if** $x \in X$ $X \in B$ **for** x X
proof −
have *connected X continuous-on X f*
using *that* **by** (*auto intro*: *assms continuous-on-subset*[*OF f*])
from *connectedD-const*[*OF this*] **obtain** c' **where** c': $\bigwedge x.\ x \in X \Longrightarrow f\,x = c'$
by *metis*

from *assms*(*3*) **and** *that* **obtain** *y* **where** $y \in A \cap X$
by *auto*
with *c*[*of y*] *c'*[*of y*] *c'*[*of x*] *that* **show** *?thesis*
by *auto*
qed
with *c* **show** $\exists\, c.\ \forall x \in A \cup \bigcup B.\ f\ x = c$
by (*intro exI*[*of - c*]) *auto*
qed

99 Linear Continuum Topologies

class *linear-continuum-topology* = *linorder-topology* + *linear-continuum*
begin

lemma *Inf-notin-open*:
assumes *A*: *open A*
and *bnd*: $\forall a \in A.\ x < a$
shows $Inf\ A \notin A$
proof
assume $Inf\ A \in A$
then obtain *b* **where** $b < Inf\ A$ $\{b <..\ Inf\ A\} \subseteq A$
using *open-left*[*of A Inf A x*] *assms* **by** *auto*
with *dense*[*of b Inf A*] **obtain** *c* **where** $c < Inf\ A$ $c \in A$
by (*auto simp*: *subset-eq*)
then show *False*
using *cInf-lower*[*OF* ‹$c \in A$›] *bnd*
by (*metis not-le less-imp-le bdd-belowI*)
qed

lemma *Sup-notin-open*:
assumes *A*: *open A*
and *bnd*: $\forall a \in A.\ a < x$
shows $Sup\ A \notin A$
proof
assume $Sup\ A \in A$
with *assms* **obtain** *b* **where** $Sup\ A < b$ $\{Sup\ A\ ..< b\} \subseteq A$
using *open-right*[*of A Sup A x*] **by** *auto*
with *dense*[*of Sup A b*] **obtain** *c* **where** $Sup\ A < c$ $c \in A$
by (*auto simp*: *subset-eq*)
then show *False*
using *cSup-upper*[*OF* ‹$c \in A$›] *bnd*
by (*metis less-imp-le not-le bdd-aboveI*)
qed

end

instance *linear-continuum-topology* ⊆ *perfect-space*
proof
fix *x* :: *'a*

obtain y **where** $x < y \lor y < x$
  using *ex-gt-or-lt* [*of* x] **..**
with *Inf-notin-open*[*of* $\{x\}$ y] *Sup-notin-open*[*of* $\{x\}$ y] **show** $\neg$ *open* $\{x\}$
  by *auto*
qed

lemma *connectedI-interval*:
fixes U :: $'a$ :: *linear-continuum-topology set*
assumes $*$: $\bigwedge x\ y\ z.\ x \in U \Longrightarrow y \in U \Longrightarrow x \leq z \Longrightarrow z \leq y \Longrightarrow z \in U$
shows *connected* U
proof (*rule connectedI*)
{
  fix A B
  assume *open* A *open* B $A \cap B \cap U = \{\}$ $U \subseteq A \cup B$
  fix x y
  assume $x < y$ $x \in A$ $y \in B$ $x \in U$ $y \in U$

  let $?z = Inf\ (B \cap \{x <..\})$

  have $x \leq ?z$ $?z \leq y$
    using ‹$y \in B$› ‹$x < y$› **by** (*auto intro*: *cInf-lower cInf-greatest*)
  with ‹$x \in U$› ‹$y \in U$› **have** $?z \in U$
    by (*rule* $*$)
  moreover have $?z \notin B \cap \{x <..\}$
    using ‹*open* B› **by** (*intro Inf-notin-open*) *auto*
  ultimately have $?z \in A$
    using ‹$x \leq ?z$› ‹$A \cap B \cap U = \{\}$› ‹$x \in A$› ‹$U \subseteq A \cup B$› **by** *auto*
  have $\exists\, b \in B.\ b \in A \land b \in U$ **if** $?z < y$
  proof −
    obtain a **where** $?z < a$ $\{?z\ ..< a\} \subseteq A$
      using *open-right*[*OF* ‹*open* A› ‹$?z \in A$› ‹$?z < y$›] **by** *auto*
    moreover obtain b **where** $b \in B$ $x < b$ $b < min\ a\ y$
      using *cInf-less-iff*[*of* $B \cap \{x <..\}$ *min* a y] ‹$?z < a$› ‹$?z < y$› ‹$x < y$› ‹$y \in$
B›
      by *auto*
    moreover have $?z \leq b$
      using ‹$b \in B$› ‹$x < b$›
      by (*intro cInf-lower*) *auto*
    moreover have $b \in U$
      using ‹$x \leq ?z$› ‹$?z \leq b$› ‹$b < min\ a\ y$›
      by (*intro* $*$[*OF* ‹$x \in U$› ‹$y \in U$›]) (*auto simp*: *less-imp-le*)
    ultimately show *?thesis*
      by (*intro bexI*[*of* - b]) *auto*
  qed
  then have *False*
    using ‹$?z \leq y$› ‹$?z \in A$› ‹$y \in B$› ‹$y \in U$› ‹$A \cap B \cap U = \{\}$›
    unfolding *le-less* **by** *blast*
}
note *not-disjoint* = *this*

fix A B **assume** AB: *open* A *open* B $U \subseteq A \cup B$ $A \cap B \cap U = \{\}$
moreover assume $A \cap U \neq \{\}$ **then obtain** x **where** x: $x \in U$ $x \in A$ **by** *auto*
moreover assume $B \cap U \neq \{\}$ **then obtain** y **where** y: $y \in U$ $y \in B$ **by** *auto*
moreover note *not-disjoint*[*of* B A y x] *not-disjoint*[*of* A B x y]
ultimately show *False*
by (*cases* x y *rule*: *linorder-cases*) *auto*
qed

lemma *connected-iff-interval*: *connected* $U \longleftrightarrow (\forall x \in U.\ \forall y \in U.\ \forall z.\ x \leq z \longrightarrow z \leq y \longrightarrow z \in U)$
for U :: $'a$::*linear-continuum-topology set*
by (*auto intro*: *connectedI-interval dest*: *connectedD-interval*)

lemma *connected-UNIV*[*simp*]: *connected* (*UNIV*::$'a$::*linear-continuum-topology set*)
by (*simp add*: *connected-iff-interval*)

lemma *connected-Ioi*[*simp*]: *connected* $\{a<..\}$
for a :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-Ici*[*simp*]: *connected* $\{a..\}$
for a :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-Iio*[*simp*]: *connected* $\{..<a\}$
for a :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-Iic*[*simp*]: *connected* $\{..a\}$
for a :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-Ioo*[*simp*]: *connected* $\{a<..<b\}$
for a b :: $'a$::*linear-continuum-topology*
unfolding *connected-iff-interval* **by** *auto*

lemma *connected-Ioc*[*simp*]: *connected* $\{a<..b\}$
for a b :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-Ico*[*simp*]: *connected* $\{a..<b\}$
for a b :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-Icc*[*simp*]: *connected* $\{a..b\}$
for a b :: $'a$::*linear-continuum-topology*
by (*auto simp*: *connected-iff-interval*)

lemma *connected-contains-Ioo*:
fixes A :: $'a$ :: *linorder-topology set*
assumes *connected* A $a \in A$ $b \in A$ **shows** $\{a <..< b\} \subseteq A$
using *connectedD-interval*[*OF assms*] **by** (*simp add*: *subset-eq Ball-def less-imp-le*)

lemma *connected-contains-Icc*:
fixes A :: $'a$::*linorder-topology set*
assumes *connected* A $a \in A$ $b \in A$
shows $\{a..b\} \subseteq A$
proof
fix x **assume** $x \in \{a..b\}$
then have $x = a \lor x = b \lor x \in \{a<..<b\}$
by *auto*
then show $x \in A$
using *assms connected-contains-Ioo*[*of* A a b] **by** *auto*
qed

99.1 Intermediate Value Theorem

lemma *IVT'*:
fixes f :: $'a$::*linear-continuum-topology* $\Rightarrow$ $'b$::*linorder-topology*
assumes y: $f\ a \leq y$ $y \leq f\ b$ $a \leq b$
and $*$: *continuous-on* $\{a\ ..\ b\}$ f
shows $\exists x.\ a \leq x \land x \leq b \land f\ x = y$
proof −
have *connected* $\{a..b\}$
unfolding *connected-iff-interval* **by** *auto*
from *connected-continuous-image*[*OF* $*$ *this*, *THEN connectedD-interval*, *of* $f\ a$ $f\ b$ y] y
show *?thesis*
by (*auto simp add*: *atLeastAtMost-def atLeast-def atMost-def*)
qed

lemma *IVT2'*:
fixes f :: $'a$ :: *linear-continuum-topology* $\Rightarrow$ $'b$ :: *linorder-topology*
assumes y: $f\ b \leq y$ $y \leq f\ a$ $a \leq b$
and $*$: *continuous-on* $\{a\ ..\ b\}$ f
shows $\exists x.\ a \leq x \land x \leq b \land f\ x = y$
proof −
have *connected* $\{a..b\}$
unfolding *connected-iff-interval* **by** *auto*
from *connected-continuous-image*[*OF* $*$ *this*, *THEN connectedD-interval*, *of* $f\ b$ $f\ a$ y] y
show *?thesis*
by (*auto simp add*: *atLeastAtMost-def atLeast-def atMost-def*)
qed

lemma *IVT*:
fixes f :: $'a$::*linear-continuum-topology* $\Rightarrow$ $'b$::*linorder-topology*

shows $f\ a \leq y \Longrightarrow y \leq f\ b \Longrightarrow a \leq b \Longrightarrow (\forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x) \Longrightarrow$
$\exists x.\ a \leq x \wedge x \leq b \wedge f\ x = y$
by (*rule IVT'*) (*auto intro*: *continuous-at-imp-continuous-on*)

lemma *IVT2*:
fixes *f* :: *'a::linear-continuum-topology* ⇒ *'b::linorder-topology*
shows $f\ b \leq y \Longrightarrow y \leq f\ a \Longrightarrow a \leq b \Longrightarrow (\forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x) \Longrightarrow$
$\exists x.\ a \leq x \wedge x \leq b \wedge f\ x = y$
by (*rule IVT2'*) (*auto intro*: *continuous-at-imp-continuous-on*)

lemma *continuous-inj-imp-mono*:
fixes *f* :: *'a::linear-continuum-topology* ⇒ *'b::linorder-topology*
assumes *x*: $a < x$ $x < b$
and *cont*: *continuous-on* {*a..b*} *f*
and *inj*: *inj-on f* {*a..b*}
shows $(f\ a < f\ x \wedge f\ x < f\ b) \vee (f\ b < f\ x \wedge f\ x < f\ a)$
proof −
note *I* = *inj-on-eq-iff*[*OF inj*]
{
assume $f\ x < f\ a$ $f\ x < f\ b$
then obtain *s t* **where** $x \leq s$ $s \leq b$ $a \leq t$ $t \leq x$ $f\ s = f\ t$ $f\ x < f\ s$
using *IVT'*[*of f x min* (*f a*) (*f b*) *b*] *IVT2'*[*of f x min* (*f a*) (*f b*) *a*] *x*
by (*auto simp*: *continuous-on-subset*[*OF cont*] *less-imp-le*)
with *x I* **have** *False* **by** *auto*
}
moreover
{
assume $f\ a < f\ x$ $f\ b < f\ x$
then obtain *s t* **where** $x \leq s$ $s \leq b$ $a \leq t$ $t \leq x$ $f\ s = f\ t$ $f\ s < f\ x$
using *IVT'*[*of f a max* (*f a*) (*f b*) *x*] *IVT2'*[*of f b max* (*f a*) (*f b*) *x*] *x*
by (*auto simp*: *continuous-on-subset*[*OF cont*] *less-imp-le*)
with *x I* **have** *False* **by** *auto*
}
ultimately show *?thesis*
using *I*[*of a x*] *I*[*of x b*] *x less-trans*[*OF x*]
by (*auto simp add*: *le-less less-imp-neq neq-iff*)
qed

lemma *continuous-at-Sup-mono*:
fixes *f* :: *'a::*{*linorder-topology,conditionally-complete-linorder*} ⇒
'b::{*linorder-topology,conditionally-complete-linorder*}
assumes *mono f*
and *cont*: *continuous* (*at-left* (*Sup S*)) *f*
and *S*: $S \neq \{\}$ *bdd-above S*
shows *f* (*Sup S*) = (*SUP s*∈*S*. *f s*)
proof (*rule antisym*)
have *f*: (*f* ⟶ *f* (*Sup S*)) (*at-left* (*Sup S*))

using *cont* **unfolding** *continuous-within* **.**
show *f* (*Sup S*) ≤ (*SUP s*∈*S*. *f s*)
proof *cases*
assume *Sup S* ∈ *S*
then show *?thesis*
by (*rule cSUP-upper*) (*auto intro*: *bdd-above-image-mono S* ‹*mono f*›)
next
assume *Sup S* ∉ *S*
from ‹*S* ≠ {}› **obtain** *s* **where** *s* ∈ *S*
by *auto*
with ‹*Sup S* ∉ *S*› *S* **have** *s* < *Sup S*
unfolding *less-le* **by** (*blast intro*: *cSup-upper*)
show *?thesis*
proof (*rule ccontr*)
assume ¬ *?thesis*
with *order-tendstoD*(*1*)[*OF f*, *of SUP s*∈*S*. *f s*] **obtain** *b* **where** *b* < *Sup S*
and ∗: ⋀*y*. *b* < *y* ⟹ *y* < *Sup S* ⟹ (*SUP s*∈*S*. *f s*) < *f y*
by (*auto simp*: *not-le eventually-at-left*[*OF* ‹*s* < *Sup S*›])
with ‹*S* ≠ {}› **obtain** *c* **where** *c* ∈ *S b* < *c*
using *less-cSupD*[*of S b*] **by** *auto*
with ‹*Sup S* ∉ *S*› *S* **have** *c* < *Sup S*
unfolding *less-le* **by** (*blast intro*: *cSup-upper*)
from ∗[*OF* ‹*b* < *c*› ‹*c* < *Sup S*›] *cSUP-upper*[*OF* ‹*c* ∈ *S*› *bdd-above-image-mono*[*of f*]]
show *False*
by (*auto simp*: *assms*)
qed
qed
qed (*intro cSUP-least* ‹*mono f*›[*THEN monoD*] *cSup-upper S*)

lemma *continuous-at-Sup-antimono*:
fixes *f* :: '*a*::{*linorder-topology*,*conditionally-complete-linorder*} ⇒
'*b*::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *antimono f*
and *cont*: *continuous* (*at-left* (*Sup S*)) *f*
and *S*: *S* ≠ {} *bdd-above S*
shows *f* (*Sup S*) = (*INF s*∈*S*. *f s*)
proof (*rule antisym*)
have *f*: (*f* ⟶ *f* (*Sup S*)) (*at-left* (*Sup S*))
using *cont* **unfolding** *continuous-within* **.**
show (*INF s*∈*S*. *f s*) ≤ *f* (*Sup S*)
proof *cases*
assume *Sup S* ∈ *S*
then show *?thesis*
by (*intro cINF-lower*) (*auto intro*: *bdd-below-image-antimono S* ‹*antimono f*›)
next
assume *Sup S* ∉ *S*
from ‹*S* ≠ {}› **obtain** *s* **where** *s* ∈ *S*
by *auto*

with ‹*Sup S* ∉ *S*› *S* **have** *s* < *Sup S*
unfolding *less-le* **by** (*blast intro*: *cSup-upper*)
show *?thesis*
proof (*rule ccontr*)
assume ¬ *?thesis*
with *order-tendstoD*(*2*)[*OF f*, *of INF s*∈*S*. *f s*] **obtain** *b* **where** *b* < *Sup S*
and ∗: ⋀*y*. *b* < *y* ⟹ *y* < *Sup S* ⟹ *f y* < (*INF s*∈*S*. *f s*)
by (*auto simp*: *not-le eventually-at-left*[*OF* ‹*s* < *Sup S*›])
with ‹*S* ≠ {}› **obtain** *c* **where** *c* ∈ *S b* < *c*
using *less-cSupD*[*of S b*] **by** *auto*
with ‹*Sup S* ∉ *S*› *S* **have** *c* < *Sup S*
unfolding *less-le* **by** (*blast intro*: *cSup-upper*)
from ∗[*OF* ‹*b* < *c*› ‹*c* < *Sup S*›] *cINF-lower*[*OF bdd-below-image-antimono*, *of f S c*] ‹*c* ∈ *S*›
show *False*
by (*auto simp*: *assms*)
qed
qed
qed (*intro cINF-greatest* ‹*antimono f*›[*THEN antimonoD*] *cSup-upper S*)

lemma *continuous-at-Inf-mono*:
fixes *f* :: *'a*::{*linorder-topology*,*conditionally-complete-linorder*} ⇒ *'b*::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *mono f*
and *cont*: *continuous* (*at-right* (*Inf S*)) *f*
and *S*: *S* ≠ {} *bdd-below S*
shows *f* (*Inf S*) = (*INF s*∈*S*. *f s*)
proof (*rule antisym*)
have *f*: (*f* ⟶ *f* (*Inf S*)) (*at-right* (*Inf S*))
using *cont* **unfolding** *continuous-within* **.**
show (*INF s*∈*S*. *f s*) ≤ *f* (*Inf S*)
proof *cases*
assume *Inf S* ∈ *S*
then show *?thesis*
by (*rule cINF-lower*[*rotated*]) (*auto intro*: *bdd-below-image-mono S* ‹*mono f*›)
next
assume *Inf S* ∉ *S*
from ‹*S* ≠ {}› **obtain** *s* **where** *s* ∈ *S*
by *auto*
with ‹*Inf S* ∉ *S*› *S* **have** *Inf S* < *s*
unfolding *less-le* **by** (*blast intro*: *cInf-lower*)
show *?thesis*
proof (*rule ccontr*)
assume ¬ *?thesis*
with *order-tendstoD*(*2*)[*OF f*, *of INF s*∈*S*. *f s*] **obtain** *b* **where** *Inf S* < *b*
and ∗: ⋀*y*. *Inf S* < *y* ⟹ *y* < *b* ⟹ *f y* < (*INF s*∈*S*. *f s*)
by (*auto simp*: *not-le eventually-at-right*[*OF* ‹*Inf S* < *s*›])
with ‹*S* ≠ {}› **obtain** *c* **where** *c* ∈ *S c* < *b*
using *cInf-lessD*[*of S b*] **by** *auto*

with ‹*Inf S* ∉ *S*› *S* **have** *Inf S* < *c*
unfolding *less-le* **by** (*blast intro*: *cInf-lower*)
from ∗[*OF* ‹*Inf S* < *c*› ‹*c* < *b*›] *cINF-lower*[*OF bdd-below-image-mono*[*of f*] ‹*c* ∈ *S*›]
show *False*
by (*auto simp*: *assms*)
qed
qed
qed (*intro cINF-greatest* ‹*mono f*›[*THEN monoD*] *cInf-lower* ‹*bdd-below S*› ‹*S* ≠ {}›)

lemma *continuous-at-Inf-antimono*:
fixes *f* :: ′*a*::{*linorder-topology*,*conditionally-complete-linorder*} ⇒ ′*b*::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *antimono f*
and *cont*: *continuous* (*at-right* (*Inf S*)) *f*
and *S*: *S* ≠ {} *bdd-below S*
shows *f* (*Inf S*) = (*SUP s*∈*S*. *f s*)
proof (*rule antisym*)
have *f*: (*f* ⟶ *f* (*Inf S*)) (*at-right* (*Inf S*))
using *cont* **unfolding** *continuous-within* **.**
show *f* (*Inf S*) ≤ (*SUP s*∈*S*. *f s*)
proof *cases*
assume *Inf S* ∈ *S*
then show *?thesis*
by (*rule cSUP-upper*) (*auto intro*: *bdd-above-image-antimono S* ‹*antimono f*›)
next
assume *Inf S* ∉ *S*
from ‹*S* ≠ {}› **obtain** *s* **where** *s* ∈ *S*
by *auto*
with ‹*Inf S* ∉ *S*› *S* **have** *Inf S* < *s*
unfolding *less-le* **by** (*blast intro*: *cInf-lower*)
show *?thesis*
proof (*rule ccontr*)
assume ¬ *?thesis*
with *order-tendstoD*(*1*)[*OF f*, *of SUP s*∈*S*. *f s*] **obtain** *b* **where** *Inf S* < *b*
and ∗: ⋀*y*. *Inf S* < *y* ⟹ *y* < *b* ⟹ (*SUP s*∈*S*. *f s*) < *f y*
by (*auto simp*: *not-le eventually-at-right*[*OF* ‹*Inf S* < *s*›])
with ‹*S* ≠ {}› **obtain** *c* **where** *c* ∈ *S c* < *b*
using *cInf-lessD*[*of S b*] **by** *auto*
with ‹*Inf S* ∉ *S*› *S* **have** *Inf S* < *c*
unfolding *less-le* **by** (*blast intro*: *cInf-lower*)
from ∗[*OF* ‹*Inf S* < *c*› ‹*c* < *b*›] *cSUP-upper*[*OF* ‹*c* ∈ *S*› *bdd-above-image-antimono*[*of f*]]
show *False*
by (*auto simp*: *assms*)
qed
qed
qed (*intro cSUP-least* ‹*antimono f*›[*THEN antimonoD*] *cInf-lower S*)

99.2 Uniform spaces

class *uniformity* =
fixes *uniformity* :: ('*a* × '*a*) *filter*
begin

abbreviation *uniformity-on* :: '*a set* ⇒ ('*a* × '*a*) *filter*
where *uniformity-on s* ≡ *inf uniformity* (*principal* (*s*×*s*))

end

lemma *uniformity-Abort*:
uniformity =
Filter.abstract-filter (λ*u*. *Code.abort* (*STR* ''*uniformity is not executable*'') (λ*u*. *uniformity*))
by *simp*

class *open-uniformity* = *open* + *uniformity* +
assumes *open-uniformity*:
⋀*U*. *open U* ⟷ (∀ *x*∈*U*. *eventually* (λ(*x*', *y*). *x*' = *x* ⟶ *y* ∈ *U*) *uniformity*)
begin

subclass *topological-space*
by *standard* (*force elim*: *eventually-mono eventually-elim2 simp*: *split-beta*' *open-uniformity*)+

end

class *uniform-space* = *open-uniformity* +
assumes *uniformity-refl*: *eventually E uniformity* ⟹ *E* (*x*, *x*)
and *uniformity-sym*: *eventually E uniformity* ⟹ *eventually* (λ(*x*, *y*). *E* (*y*, *x*)) *uniformity*
and *uniformity-trans*:
eventually E uniformity ⟹
∃ *D*. *eventually D uniformity* ∧ (∀ *x y z*. *D* (*x*, *y*) ⟶ *D* (*y*, *z*) ⟶ *E* (*x*, *z*))
begin

lemma *uniformity-bot*: *uniformity* ≠ *bot*
using *uniformity-refl* **by** *auto*

lemma *uniformity-trans*':
eventually E uniformity ⟹
eventually (λ((*x*, *y*), (*y*', *z*)). *y* = *y*' ⟶ *E* (*x*, *z*)) (*uniformity* $\times_F$ *uniformity*)
by (*drule uniformity-trans*) (*auto simp add*: *eventually-prod-same*)

lemma *uniformity-transE*:
assumes *eventually E uniformity*
obtains *D* **where** *eventually D uniformity* ⋀*x y z*. *D* (*x*, *y*) ⟹ *D* (*y*, *z*) ⟹ *E* (*x*, *z*)
using *uniformity-trans* [*OF assms*] **by** *auto*

lemma *eventually-nhds-uniformity*:
eventually P (*nhds x*) $\longleftrightarrow$ *eventually* ($\lambda(x', y)$. $x' = x \longrightarrow P\ y$) *uniformity*
(**is** - $\longleftrightarrow$ *?N P x*)
unfolding *eventually-nhds*
proof *safe*
assume $*$: *?N P x*
have *?N* (*?N P*) *x* **if** *?N P x* **for** *x*
proof −
from *that* **obtain** *D* **where** *ev*: *eventually D uniformity*
and *D*: $D\ (a, b) \Longrightarrow D\ (b, c) \Longrightarrow$ *case* (*a*, *c*) *of* (x', *y*) $\Rightarrow x' = x \longrightarrow P\ y$
for *a b c*
by (*rule uniformity-transE*) *simp*
from *ev* **show** *?thesis*
by *eventually-elim* (*insert ev D*, *force elim*: *eventually-mono split*: *prod.split*)
qed
then have *open* {*x*. *?N P x*}
by (*simp add*: *open-uniformity*)
then show $\exists S$. *open* $S \land x \in S \land (\forall x{\in}S.\ P\ x)$
by (*intro exI*[*of* - {*x*. *?N P x*}]) (*auto dest*: *uniformity-refl simp*: $*$)
qed (*force simp add*: *open-uniformity elim*: *eventually-mono*)

99.2.1 Totally bounded sets

definition *totally-bounded* :: *'a set* $\Rightarrow$ *bool*
where *totally-bounded S* $\longleftrightarrow$
($\forall E$. *eventually E uniformity* $\longrightarrow$ ($\exists X$. *finite X* $\land$ ($\forall s{\in}S.\ \exists x{\in}X.\ E\ (x, s)$)))

lemma *totally-bounded-empty*[*iff*]: *totally-bounded* {}
by (*auto simp add*: *totally-bounded-def*)

lemma *totally-bounded-subset*: *totally-bounded S* $\Longrightarrow T \subseteq S \Longrightarrow$ *totally-bounded* *T*
by (*fastforce simp add*: *totally-bounded-def*)

lemma *totally-bounded-Union*[*intro*]:
assumes *M*: *finite M* $\bigwedge S.\ S \in M \Longrightarrow$ *totally-bounded S*
shows *totally-bounded* ($\bigcup M$)
unfolding *totally-bounded-def*
proof *safe*
fix *E*
assume *eventually E uniformity*
with *M* **obtain** *X* **where** $\forall S{\in}M$. *finite* (*X S*) $\land$ ($\forall s{\in}S.\ \exists x{\in}X\ S.\ E\ (x, s)$)
by (*metis totally-bounded-def*)
with ‹*finite M*› **show** $\exists X$. *finite X* $\land$ ($\forall s{\in}\bigcup M.\ \exists x{\in}X.\ E\ (x, s)$)
by (*intro exI*[*of* - $\bigcup S{\in}M.\ X\ S$]) *force*
qed

99.2.2 Cauchy filter

definition *cauchy-filter* :: *'a filter* $\Rightarrow$ *bool*

where *cauchy-filter* $F \longleftrightarrow F \times_F F \leq$ *uniformity*

definition *Cauchy* :: (*nat* ⇒ ′*a*) ⇒ *bool*
where *Cauchy-uniform*: *Cauchy X* = *cauchy-filter* (*filtermap X sequentially*)

lemma *Cauchy-uniform-iff*:
Cauchy X ⟷ (∀ *P*. *eventually P uniformity* ⟶ (∃ *N*. ∀ *n*≥*N*. ∀ *m*≥*N*. *P* (*X n*, *X m*)))
unfolding *Cauchy-uniform cauchy-filter-def le-filter-def eventually-prod-same eventually-filtermap eventually-sequentially*
proof *safe*
let *?U* = λ*P*. *eventually P uniformity*
{
fix *P*
assume *?U P* ∀ *P*. *?U P* ⟶ (∃ *Q*. (∃ *N*. ∀ *n*≥*N*. *Q* (*X n*)) ∧ (∀ *x y*. *Q x* ⟶ *Q y* ⟶ *P* (*x*, *y*)))
then obtain *Q N* **where** ⋀*n*. *n* ≥ *N* ⟹ *Q* (*X n*) ⋀*x y*. *Q x* ⟹ *Q y* ⟹ *P* (*x*, *y*)
by *metis*
then show ∃ *N*. ∀ *n*≥*N*. ∀ *m*≥*N*. *P* (*X n*, *X m*)
by *blast*
next
fix *P*
assume *?U P* **and** *P*: ∀ *P*. *?U P* ⟶ (∃ *N*. ∀ *n*≥*N*. ∀ *m*≥*N*. *P* (*X n*, *X m*))
then obtain *Q* **where** *?U Q* **and** *Q*: ⋀*x y z*. *Q* (*x*, *y*) ⟹ *Q* (*y*, *z*) ⟹ *P* (*x*, *z*)
by (*auto elim*: *uniformity-transE*)
then have *?U* (λ*x*. *Q x* ∧ (λ(*x*, *y*). *Q* (*y*, *x*)) *x*)
unfolding *eventually-conj-iff* **by** (*simp add*: *uniformity-sym*)
from *P*[*rule-format*, *OF this*]
obtain *N* **where** *N*: ⋀*n m*. *n* ≥ *N* ⟹ *m* ≥ *N* ⟹ *Q* (*X n*, *X m*) ∧ *Q* (*X m*, *X n*)
by *auto*
show ∃ *Q*. (∃ *N*. ∀ *n*≥*N*. *Q* (*X n*)) ∧ (∀ *x y*. *Q x* ⟶ *Q y* ⟶ *P* (*x*, *y*))
proof (*safe intro*!: *exI*[*of* - λ*x*. ∀ *n*≥*N*. *Q* (*x*, *X n*) ∧ *Q* (*X n*, *x*)] *exI*[*of* - *N*] *N*)
fix *x y*
assume ∀ *n*≥*N*. *Q* (*x*, *X n*) ∧ *Q* (*X n*, *x*) ∀ *n*≥*N*. *Q* (*y*, *X n*) ∧ *Q* (*X n*, *y*)
then have *Q* (*x*, *X N*) *Q* (*X N*, *y*) **by** *auto*
then show *P* (*x*, *y*)
by (*rule Q*)
qed
}
qed

lemma *nhds-imp-cauchy-filter*:
assumes ∗: *F* ≤ *nhds x*
shows *cauchy-filter F*
proof −

have $F \times_F F \leq$ *nhds x* $\times_F$ *nhds x*
by (*intro prod-filter-mono* ∗)
also have . . . ≤ *uniformity*
unfolding *le-filter-def eventually-nhds-uniformity eventually-prod-same*
proof *safe*
fix *P*
assume *eventually P uniformity*
then obtain *Ql* **where** *ev*: *eventually Ql uniformity*
and *Ql* (*x*, *y*) ⟹ *Ql* (*y*, *z*) ⟹ *P* (*x*, *z*) **for** *x y z*
by (*rule uniformity-transE*) *simp*
with *ev*[*THEN uniformity-sym*]
show ∃ *Q*. *eventually* (λ(*x'*, *y*). *x'* = *x* ⟶ *Q y*) *uniformity* ∧
(∀ *x y*. *Q x* ⟶ *Q y* ⟶ *P* (*x*, *y*))
by (*rule-tac exI*[*of* - λ*y*. *Ql* (*y*, *x*) ∧ *Ql* (*x*, *y*)]) (*fastforce elim*: *eventually-elim2*)
qed
finally show *?thesis*
by (*simp add*: *cauchy-filter-def*)
qed

lemma *LIMSEQ-imp-Cauchy*: *X* ⟶ *x* ⟹ *Cauchy X*
unfolding *Cauchy-uniform filterlim-def* **by** (*intro nhds-imp-cauchy-filter*)

lemma *Cauchy-subseq-Cauchy*:
assumes *Cauchy X strict-mono f*
shows *Cauchy* (*X* ∘ *f*)
unfolding *Cauchy-uniform comp-def filtermap-filtermap*[*symmetric*] *cauchy-filter-def*
by (*rule order-trans*[*OF* - ‹*Cauchy X*›[*unfolded Cauchy-uniform cauchy-filter-def*]])
(*intro prod-filter-mono filtermap-mono filterlim-subseq*[*OF* ‹*strict-mono f*›, *unfolded filterlim-def*])

lemma *convergent-Cauchy*: *convergent X* ⟹ *Cauchy X*
unfolding *convergent-def* **by** (*erule exE*, *erule LIMSEQ-imp-Cauchy*)

definition *complete* :: *'a set* ⇒ *bool*
where *complete-uniform*: *complete S* ⟷
(∀ *F* ≤ *principal S*. *F* ≠ *bot* ⟶ *cauchy-filter F* ⟶ (∃ *x*∈*S*. *F* ≤ *nhds x*))

lemma (**in** *uniform-space*) *cauchy-filter-complete-converges*:
assumes *cauchy-filter F complete A F* ≤ *principal A F* ≠ *bot*
shows ∃ *c*. *F* ≤ *nhds c*
using *assms* **unfolding** *complete-uniform* **by** *blast*

end

99.2.3 Uniformly continuous functions

definition *uniformly-continuous-on* :: *'a set* ⇒ (*'a*::*uniform-space* ⇒ *'b*::*uniform-space*) ⇒ *bool*
where *uniformly-continuous-on-uniformity*: *uniformly-continuous-on s f* ⟷

(*LIM* (*x*, *y*) (*uniformity-on s*). (*f x*, *f y*) :> *uniformity*)

lemma *uniformly-continuous-onD*:
uniformly-continuous-on s f ⟹ *eventually E uniformity* ⟹
eventually (λ(*x*, *y*). *x* ∈ *s* ⟶ *y* ∈ *s* ⟶ *E* (*f x*, *f y*)) *uniformity*
by (*simp add*: *uniformly-continuous-on-uniformity filterlim-iff*
eventually-inf-principal split-beta′ mem-Times-iff imp-conjL)

lemma *uniformly-continuous-on-const*[*continuous-intros*]: *uniformly-continuous-on s* (λ*x*. *c*)
by (*auto simp*: *uniformly-continuous-on-uniformity filterlim-iff uniformity-refl*)

lemma *uniformly-continuous-on-id*[*continuous-intros*]: *uniformly-continuous-on s* (λ*x*. *x*)
by (*auto simp*: *uniformly-continuous-on-uniformity filterlim-def*)

lemma *uniformly-continuous-on-compose*:
uniformly-continuous-on s g ⟹ *uniformly-continuous-on* (*g*‘*s*) *f* ⟹
uniformly-continuous-on s (λ*x*. *f* (*g x*))
using *filterlim-compose*[*of* λ(*x*, *y*). (*f x*, *f y*) *uniformity*
uniformity-on (*g*‘*s*) λ(*x*, *y*). (*g x*, *g y*) *uniformity-on s*]
by (*simp add*: *split-beta′ uniformly-continuous-on-uniformity*
filterlim-inf filterlim-principal eventually-inf-principal mem-Times-iff)

lemma *uniformly-continuous-imp-continuous*:
assumes *f*: *uniformly-continuous-on s f*
shows *continuous-on s f*
by (*auto simp*: *filterlim-iff eventually-at-filter eventually-nhds-uniformity continuous-on-def*
elim: *eventually-mono dest*!: *uniformly-continuous-onD*[*OF f*])

100 Product Topology

100.1 Product is a topological space

instantiation *prod* :: (*topological-space*, *topological-space*) *topological-space*
begin

definition *open-prod-def*[*code del*]:
open (*S* :: (′*a* × ′*b*) *set*) ⟷
(∀ *x*∈*S*. ∃ *A B*. *open A* ∧ *open B* ∧ *x* ∈ *A* × *B* ∧ *A* × *B* ⊆ *S*)

lemma *open-prod-elim*:
assumes *open S* **and** *x* ∈ *S*
obtains *A B* **where** *open A* **and** *open B* **and** *x* ∈ *A* × *B* **and** *A* × *B* ⊆ *S*
using *assms* **unfolding** *open-prod-def* **by** *fast*

lemma *open-prod-intro*:
assumes ⋀*x*. *x* ∈ *S* ⟹ ∃ *A B*. *open A* ∧ *open B* ∧ *x* ∈ *A* × *B* ∧ *A* × *B* ⊆ *S*

shows *open S*
using *assms* **unfolding** *open-prod-def* **by** *fast*

instance
proof
show *open* (*UNIV* :: ($'a \times 'b$) *set*)
unfolding *open-prod-def* **by** *auto*
next
fix *S T* :: ($'a \times 'b$) *set*
assume *open S open T*
show *open* ($S \cap T$)
proof (*rule open-prod-intro*)
fix *x*
assume *x*: $x \in S \cap T$
from *x* **have** $x \in S$ **by** *simp*
obtain *Sa Sb* **where** *A*: *open Sa open Sb* $x \in Sa \times Sb$ $Sa \times Sb \subseteq S$
using ‹*open S*› **and** ‹$x \in S$› **by** (*rule open-prod-elim*)
from *x* **have** $x \in T$ **by** *simp*
obtain *Ta Tb* **where** *B*: *open Ta open Tb* $x \in Ta \times Tb$ $Ta \times Tb \subseteq T$
using ‹*open T*› **and** ‹$x \in T$› **by** (*rule open-prod-elim*)
let *?A* = $Sa \cap Ta$ **and** *?B* = $Sb \cap Tb$
have *open ?A* ∧ *open ?B* ∧ $x \in$ *?A* × *?B* ∧ *?A* × *?B* ⊆ $S \cap T$
using *A B* **by** (*auto simp add*: *open-Int*)
then show ∃ *A B*. *open A* ∧ *open B* ∧ $x \in A \times B \wedge A \times B \subseteq S \cap T$
by *fast*
qed
next
fix *K* :: ($'a \times 'b$) *set set*
assume ∀ *S*∈*K*. *open S*
then show *open* ($\bigcup K$)
unfolding *open-prod-def* **by** *fast*
qed

end

declare [[*code abort*: *open* :: (*'a*::*topological-space* × *'b*::*topological-space*) *set* ⇒ *bool*]]

lemma *open-Times*: *open S* ⟹ *open T* ⟹ *open* ($S \times T$)
unfolding *open-prod-def* **by** *auto*

lemma *fst-vimage-eq-Times*: *fst* −‘ *S* = *S* × *UNIV*
by *auto*

lemma *snd-vimage-eq-Times*: *snd* −‘ *S* = *UNIV* × *S*
by *auto*

lemma *open-vimage-fst*: *open S* ⟹ *open* (*fst* −‘ *S*)
by (*simp add*: *fst-vimage-eq-Times open-Times*)

lemma *open-vimage-snd*: *open* $S \Longrightarrow$ *open* (*snd* −‘ S)
by (*simp add*: *snd-vimage-eq-Times open-Times*)

lemma *closed-vimage-fst*: *closed* $S \Longrightarrow$ *closed* (*fst* −‘ S)
unfolding *closed-open vimage-Compl* [*symmetric*]
by (*rule open-vimage-fst*)

lemma *closed-vimage-snd*: *closed* $S \Longrightarrow$ *closed* (*snd* −‘ S)
unfolding *closed-open vimage-Compl* [*symmetric*]
by (*rule open-vimage-snd*)

lemma *closed-Times*: *closed* $S \Longrightarrow$ *closed* $T \Longrightarrow$ *closed* ($S \times T$)
proof −
have $S \times T$ = (*fst* −‘ S) ∩ (*snd* −‘ T)
by *auto*
then show *closed* $S \Longrightarrow$ *closed* $T \Longrightarrow$ *closed* ($S \times T$)
by (*simp add*: *closed-vimage-fst closed-vimage-snd closed-Int*)
qed

lemma *subset-fst-imageI*: $A \times B \subseteq S \Longrightarrow y \in B \Longrightarrow A \subseteq$ *fst* ‘ S
unfolding *image-def subset-eq* **by** *force*

lemma *subset-snd-imageI*: $A \times B \subseteq S \Longrightarrow x \in A \Longrightarrow B \subseteq$ *snd* ‘ S
unfolding *image-def subset-eq* **by** *force*

lemma *open-image-fst*:
assumes *open* S
shows *open* (*fst* ‘ S)
proof (*rule openI*)
fix x
assume $x \in$ *fst* ‘ S
then obtain y **where** $(x, y) \in S$
by *auto*
then obtain A B **where** *open* A *open* B $x \in A$ $y \in B$ $A \times B \subseteq S$
using ‹*open* S› **unfolding** *open-prod-def* **by** *auto*
from ‹$A \times B \subseteq S$› ‹$y \in B$› **have** $A \subseteq$ *fst* ‘ S
by (*rule subset-fst-imageI*)
with ‹*open* A› ‹$x \in A$› **have** *open* $A \wedge x \in A \wedge A \subseteq$ *fst* ‘ S
by *simp*
then show $\exists T$. *open* $T \wedge x \in T \wedge T \subseteq$ *fst* ‘ S **..**
qed

lemma *open-image-snd*:
assumes *open* S
shows *open* (*snd* ‘ S)
proof (*rule openI*)
fix y
assume $y \in$ *snd* ‘ S

then obtain x **where** $(x, y) \in S$
by *auto*
then obtain A B **where** *open* A *open* B $x \in A$ $y \in B$ $A \times B \subseteq S$
using ‹*open* S› **unfolding** *open-prod-def* **by** *auto*
from ‹$A \times B \subseteq S$› ‹$x \in A$› **have** $B \subseteq$ *snd* ‘ S
by (*rule subset-snd-imageI*)
with ‹*open* B› ‹$y \in B$› **have** *open* $B \wedge y \in B \wedge B \subseteq$ *snd* ‘ S
by *simp*
then show $\exists T.$ *open* $T \wedge y \in T \wedge T \subseteq$ *snd* ‘ S **..**
qed

lemma *nhds-prod*: *nhds* $(a, b) =$ *nhds* $a \times_F$ *nhds* b
unfolding *nhds-def*
proof (*subst prod-filter-INF*, *auto intro*!: *antisym INF-greatest simp*: *principal-prod-principal*)
fix S T
assume *open* S $a \in S$ *open* T $b \in T$
then show (*INF* $x \in \{S.$ *open* $S \wedge (a, b) \in S\}$. *principal* x) $\leq$ *principal* ($S \times$
T)
by (*intro INF-lower*) (*auto intro*!: *open-Times*)
next
fix S'
assume *open* S' $(a, b) \in S'$
then obtain S T **where** *open* S $a \in S$ *open* T $b \in T$ $S \times T \subseteq S'$
by (*auto elim*: *open-prod-elim*)
then show (*INF* $x \in \{S.$ *open* $S \wedge a \in S\}$. *INF* $y \in \{S.$ *open* $S \wedge b \in S\}$.
principal $(x \times y)$) $\leq$ *principal* S'
by (*auto intro*!: *INF-lower2*)
qed

100.1.1 Continuity of operations

lemma *tendsto-fst* [*tendsto-intros*]:
assumes $(f \longrightarrow a)$ F
shows $((\lambda x.$ *fst* $(f\,x)) \longrightarrow$ *fst* $a)$ F
proof (*rule topological-tendstoI*)
fix S
assume *open* S **and** *fst* $a \in S$
then have *open* (*fst* −‘ S) **and** $a \in$ *fst* −‘ S
by (*simp-all add*: *open-vimage-fst*)
with *assms* **have** *eventually* ($\lambda x.$ $f\,x \in$ *fst* −‘ S) F
by (*rule topological-tendstoD*)
then show *eventually* ($\lambda x.$ *fst* $(f\,x) \in S$) F
by *simp*
qed

lemma *tendsto-snd* [*tendsto-intros*]:
assumes $(f \longrightarrow a)$ F
shows $((\lambda x.$ *snd* $(f\,x)) \longrightarrow$ *snd* $a)$ F
proof (*rule topological-tendstoI*)

fix *S*
assume *open S* **and** *snd a* ∈ *S*
then have *open* (*snd* −' *S*) **and** *a* ∈ *snd* −' *S*
by (*simp-all add*: *open-vimage-snd*)
with *assms* **have** *eventually* (λ*x*. *f x* ∈ *snd* −' *S*) *F*
by (*rule topological-tendstoD*)
then show *eventually* (λ*x*. *snd* (*f x*) ∈ *S*) *F*
by *simp*
qed

lemma *tendsto-Pair* [*tendsto-intros*]:
assumes (*f* ⟶ *a*) *F* **and** (*g* ⟶ *b*) *F*
shows ((λ*x*. (*f x*, *g x*)) ⟶ (*a*, *b*)) *F*
unfolding *nhds-prod* **using** *assms* **by** (*rule filterlim-Pair*)

lemma *continuous-fst*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *fst* (*f x*))
unfolding *continuous-def* **by** (*rule tendsto-fst*)

lemma *continuous-snd*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *snd* (*f x*))
unfolding *continuous-def* **by** (*rule tendsto-snd*)

lemma *continuous-Pair*[*continuous-intros*]:
continuous F f ⟹ *continuous F g* ⟹ *continuous F* (λ*x*. (*f x*, *g x*))
unfolding *continuous-def* **by** (*rule tendsto-Pair*)

lemma *continuous-on-fst*[*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s* (λ*x*. *fst* (*f x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-fst*)

lemma *continuous-on-snd*[*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s* (λ*x*. *snd* (*f x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-snd*)

lemma *continuous-on-Pair*[*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s g* ⟹ *continuous-on s* (λ*x*. (*f x*, *g x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-Pair*)

lemma *continuous-on-swap*[*continuous-intros*]: *continuous-on A prod.swap*
by (*simp add*: *prod.swap-def continuous-on-fst continuous-on-snd continuous-on-Pair continuous-on-id*)

lemma *continuous-on-swap-args*:
assumes *continuous-on* (*A*×*B*) (λ(*x*,*y*). *d x y*)
shows *continuous-on* (*B*×*A*) (λ(*x*,*y*). *d y x*)
proof −
have (λ(*x*,*y*). *d y x*) = (λ(*x*,*y*). *d x y*) ∘ *prod.swap*
by *force*

then show *?thesis*
by (*metis assms continuous-on-compose continuous-on-swap product-swap*)
qed

lemma *isCont-fst* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ *fst* (*f x*)) *a*
by (*fact continuous-fst*)

lemma *isCont-snd* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ *snd* (*f x*)) *a*
by (*fact continuous-snd*)

lemma *isCont-Pair* [*simp*]: $[\![$*isCont f a*; *isCont g a*$]\!]$ $\Longrightarrow$ *isCont* ($\lambda x.$ (*f x*, *g x*)) *a*
by (*fact continuous-Pair*)

lemma *continuous-on-compose-Pair*:
assumes *f*: *continuous-on* (*Sigma A B*) (λ(*a*, *b*). *f a b*)
assumes *g*: *continuous-on C g*
assumes *h*: *continuous-on C h*
assumes *subset*: $\bigwedge c.\ c \in C \Longrightarrow g\ c \in A$ $\bigwedge c.\ c \in C \Longrightarrow h\ c \in B\ (g\ c)$
shows *continuous-on C* ($\lambda c.$ *f* (*g c*) (*h c*))
using *continuous-on-compose2*[*OF f continuous-on-Pair*[*OF g h*]] *subset*
by *auto*

100.1.2 Connectedness of products

proposition *connected-Times*:
assumes *S*: *connected S* **and** *T*: *connected T*
shows *connected* ($S \times T$)
proof (*rule connectedI-const*)
fix P::$'a \times 'b \Rightarrow$ *bool*
assume *P*[*THEN continuous-on-compose2*, *continuous-intros*]: *continuous-on* ($S \times T$) *P*
have *continuous-on S* ($\lambda s.$ *P* (*s*, *t*)) **if** $t \in T$ **for** *t*
by (*auto intro*!: *continuous-intros that*)
from *connectedD-const*[*OF S this*]
obtain *c1* **where** *c1*: $\bigwedge s\ t.\ t \in T \Longrightarrow s \in S \Longrightarrow P\ (s, t) = c1\ t$
by *metis*
moreover
have *continuous-on T* ($\lambda t.$ *P* (*s*, *t*)) **if** $s \in S$ **for** *s*
by (*auto intro*!: *continuous-intros that*)
from *connectedD-const*[*OF T this*]
obtain *c2* **where** $\bigwedge s\ t.\ t \in T \Longrightarrow s \in S \Longrightarrow P\ (s, t) = c2\ s$
by *metis*
ultimately show $\exists c.\ \forall s \in S \times T.\ P\ s = c$
by *auto*
qed

corollary *connected-Times-eq* [*simp*]:
connected ($S \times T$) $\longleftrightarrow S = \{\} \vee T = \{\} \vee$ *connected S* $\wedge$ *connected T* (**is** *?lhs* = *?rhs*)

proof
  assume *L*: *?lhs*
  show *?rhs*
  proof *cases*
    assume $S \neq \{\} \wedge T \neq \{\}$
    moreover
    have *connected* (*fst* ‘ $(S \times T)$) *connected* (*snd* ‘ $(S \times T)$)
      using *continuous-on-fst continuous-on-snd continuous-on-id*
      by (*blast intro*: *connected-continuous-image* [*OF - L*])+
    ultimately show *?thesis*
      by *auto*
  qed *auto*
qed (*auto simp*: *connected-Times*)

100.1.3 Separation axioms

instance *prod* :: (*t0-space*, *t0-space*) *t0-space*
proof
  fix *x y* :: $'a \times 'b$
  assume $x \neq y$
  then have *fst* $x \neq$ *fst* $y \vee$ *snd* $x \neq$ *snd* y
    by (*simp add*: *prod-eq-iff*)
  then show $\exists U.$ *open* $U \wedge (x \in U) \neq (y \in U)$
    by (*fast dest*: *t0-space elim*: *open-vimage-fst open-vimage-snd*)
qed

instance *prod* :: (*t1-space*, *t1-space*) *t1-space*
proof
  fix *x y* :: $'a \times 'b$
  assume $x \neq y$
  then have *fst* $x \neq$ *fst* $y \vee$ *snd* $x \neq$ *snd* y
    by (*simp add*: *prod-eq-iff*)
  then show $\exists U.$ *open* $U \wedge x \in U \wedge y \notin U$
    by (*fast dest*: *t1-space elim*: *open-vimage-fst open-vimage-snd*)
qed

instance *prod* :: (*t2-space*, *t2-space*) *t2-space*
proof
  fix *x y* :: $'a \times 'b$
  assume $x \neq y$
  then have *fst* $x \neq$ *fst* $y \vee$ *snd* $x \neq$ *snd* y
    by (*simp add*: *prod-eq-iff*)
  then show $\exists U\ V.$ *open* $U \wedge$ *open* $V \wedge x \in U \wedge y \in V \wedge U \cap V = \{\}$
    by (*fast dest*: *hausdorff elim*: *open-vimage-fst open-vimage-snd*)
qed

lemma *isCont-swap*[*continuous-intros*]: *isCont prod.swap a*
  using *continuous-on-eq-continuous-within continuous-on-swap* **by** *blast*

lemma *open-diagonal-complement*:
  open {(*x*,*y*) |*x y*. *x* ≠ (*y*::('*a*::*t2-space*))}
proof –
  have *open* {(*x*, *y*). *x* ≠ (*y*::'*a*)}
    unfolding *split-def* **by** (*intro open-Collect-neq continuous-intros*)
  also have {(*x*, *y*). *x* ≠ (*y*::'*a*)} = {(*x*, *y*) |*x y*. *x* ≠ (*y*::'*a*)}
    by *auto*
  finally show *?thesis* **.**
qed

lemma *closed-diagonal*:
  closed {*y*. ∃ *x*::('*a*::*t2-space*). *y* = (*x*,*x*)}
proof –
  have {*y*. ∃ *x*::'*a*. *y* = (*x*,*x*)} = *UNIV* – {(*x*,*y*) | *x y*. *x* ≠ *y*} **by** *auto*
  then show *?thesis* **using** *open-diagonal-complement closed-Diff* **by** *auto*
qed

lemma *open-superdiagonal*:
  open {(*x*,*y*) | *x y*. *x* > (*y*::'*a*::{*linorder-topology*})}
proof –
  have *open* {(*x*, *y*). *x* > (*y*::'*a*)}
    unfolding *split-def* **by** (*intro open-Collect-less continuous-intros*)
  also have {(*x*, *y*). *x* > (*y*::'*a*)} = {(*x*, *y*) |*x y*. *x* > (*y*::'*a*)}
    by *auto*
  finally show *?thesis* **.**
qed

lemma *closed-subdiagonal*:
  closed {(*x*,*y*) | *x y*. *x* ≤ (*y*::'*a*::{*linorder-topology*})}
proof –
  have {(*x*,*y*) | *x y*. *x* ≤ (*y*::'*a*)} = *UNIV* – {(*x*,*y*) | *x y*. *x* > (*y*::'*a*)} **by** *auto*
  then show *?thesis* **using** *open-superdiagonal closed-Diff* **by** *auto*
qed

lemma *open-subdiagonal*:
  open {(*x*,*y*) | *x y*. *x* < (*y*::'*a*::{*linorder-topology*})}
proof –
  have *open* {(*x*, *y*). *x* < (*y*::'*a*)}
    unfolding *split-def* **by** (*intro open-Collect-less continuous-intros*)
  also have {(*x*, *y*). *x* < (*y*::'*a*)} = {(*x*, *y*) |*x y*. *x* < (*y*::'*a*)}
    by *auto*
  finally show *?thesis* **.**
qed

lemma *closed-superdiagonal*:
  closed {(*x*,*y*) | *x y*. *x* ≥ (*y*::('*a*::{*linorder-topology*}))}
proof –
  have {(*x*,*y*) | *x y*. *x* ≥ (*y*::'*a*)} = *UNIV* – {(*x*,*y*) | *x y*. *x* < *y*} **by** *auto*
  then show *?thesis* **using** *open-subdiagonal closed-Diff* **by** *auto*

qed

end

theory *Hull*
imports *Main*
begin

100.2 A generic notion of the convex, affine, conic hull, or closed "hull".

definition *hull* :: ($'a$ *set* ⇒ *bool*) ⇒ $'a$ *set* ⇒ $'a$ *set* (**infixl** ‹*hull*› *75*)
where *S hull s* = ⋂{*t*. *S t* ∧ *s* ⊆ *t*}

lemma *hull-same*: *S s* ⟹ *S hull s* = *s*
unfolding *hull-def* **by** *auto*

lemma *hull-in*: (⋀*T*. *Ball T S* ⟹ *S* (⋂ *T*)) ⟹ *S* (*S hull s*)
unfolding *hull-def Ball-def* **by** *auto*

lemma *hull-eq*: (⋀*T*. *Ball T S* ⟹ *S* (⋂ *T*)) ⟹ (*S hull s*) = *s* ⟷ *S s*
using *hull-same*[*of S s*] *hull-in*[*of S s*] **by** *metis*

lemma *hull-hull* [*simp*]: *S hull* (*S hull s*) = *S hull s*
unfolding *hull-def* **by** *blast*

lemma *hull-subset*[*intro*]: *s* ⊆ (*S hull s*)
unfolding *hull-def* **by** *blast*

lemma *hull-mono*: *s* ⊆ *t* ⟹ (*S hull s*) ⊆ (*S hull t*)
unfolding *hull-def* **by** *blast*

lemma *hull-antimono*: ∀ *x*. *S x* ⟶ *T x* ⟹ (*T hull s*) ⊆ (*S hull s*)
unfolding *hull-def* **by** *blast*

lemma *hull-minimal*: *s* ⊆ *t* ⟹ *S t* ⟹ (*S hull s*) ⊆ *t*
unfolding *hull-def* **by** *blast*

lemma *subset-hull*: *S t* ⟹ *S hull s* ⊆ *t* ⟷ *s* ⊆ *t*
unfolding *hull-def* **by** *blast*

lemma *hull-UNIV* [*simp*]: *S hull UNIV* = *UNIV*
unfolding *hull-def* **by** *auto*

lemma *hull-unique*: *s* ⊆ *t* ⟹ *S t* ⟹ (⋀*t'*. *s* ⊆ *t'* ⟹ *S t'* ⟹ *t* ⊆ *t'*) ⟹ (*S hull s* = *t*)
unfolding *hull-def* **by** *auto*

lemma *hull-induct*: $[\![a \in Q\ hull\ S;\ \bigwedge x.\ x\in S \Longrightarrow P\ x;\ Q\ \{x.\ P\ x\}]\!] \Longrightarrow P\ a$
using *hull-minimal*[*of S* {*x. P x*} *Q*]
by (*auto simp add*: *subset-eq*)

lemma *hull-inc*: $x \in S \Longrightarrow x \in P\ hull\ S$
by (*metis hull-subset subset-eq*)

lemma *hull-Un-subset*: $(S\ hull\ s) \cup (S\ hull\ t) \subseteq (S\ hull\ (s \cup t))$
unfolding *Un-subset-iff* **by** (*metis hull-mono Un-upper1 Un-upper2*)

lemma *hull-Un*:
assumes *T*: $\bigwedge T.\ Ball\ T\ S \Longrightarrow S\ (\bigcap T)$
shows $S\ hull\ (s \cup t) = S\ hull\ (S\ hull\ s \cup S\ hull\ t)$
apply (*rule equalityI*)
apply (*meson hull-mono hull-subset sup.mono*)
by (*metis hull-Un-subset hull-hull hull-mono*)

lemma *hull-Un-left*: $P\ hull\ (S \cup T) = P\ hull\ (P\ hull\ S \cup T)$
apply (*rule equalityI*)
apply (*simp add*: *Un-commute hull-mono hull-subset sup.coboundedI2*)
by (*metis Un-subset-iff hull-hull hull-mono hull-subset*)

lemma *hull-Un-right*: $P\ hull\ (S \cup T) = P\ hull\ (S \cup P\ hull\ T)$
by (*metis hull-Un-left sup.commute*)

lemma *hull-insert*:
$P\ hull\ (insert\ a\ S) = P\ hull\ (insert\ a\ (P\ hull\ S))$
by (*metis hull-Un-right insert-is-Un*)

lemma *hull-redundant-eq*: $a \in (S\ hull\ s) \longleftrightarrow S\ hull\ (insert\ a\ s) = S\ hull\ s$
unfolding *hull-def* **by** *blast*

lemma *hull-redundant*: $a \in (S\ hull\ s) \Longrightarrow S\ hull\ (insert\ a\ s) = S\ hull\ s$
by (*metis hull-redundant-eq*)

end

101 Modules

Bases of a linear algebra based on modules (i.e. vector spaces of rings).

theory *Modules*
imports *Hull*
begin

101.1 Locale for additive functions

locale *additive* =
fixes f :: *'a::ab-group-add* ⇒ *'b::ab-group-add*

assumes *add*: $f\ (x + y) = f\ x + f\ y$
begin

lemma *zero*: $f\ 0 = 0$
proof −
have $f\ 0 = f\ (0 + 0)$ **by** *simp*
also have $\ldots = f\ 0 + f\ 0$ **by** (*rule add*)
finally show $f\ 0 = 0$ **by** *simp*
qed

lemma *minus*: $f\ (-\ x) = -\ f\ x$
proof −
have $f\ (-\ x) + f\ x = f\ (-\ x + x)$ **by** (*rule add* [*symmetric*])
also have $\ldots = -\ f\ x + f\ x$ **by** (*simp add*: *zero*)
finally show $f\ (-\ x) = -\ f\ x$ **by** (*rule add-right-imp-eq*)
qed

lemma *diff*: $f\ (x - y) = f\ x - f\ y$
using *add* [*of* $x - y$] **by** (*simp add*: *minus*)

lemma *sum*: $f\ (sum\ g\ A) = (\sum x \in A.\ f\ (g\ x))$
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *zero add*)

end

Modules form the central spaces in linear algebra. They are a generalization from vector spaces by replacing the scalar field by a scalar ring.

locale *module* =
fixes *scale* :: *'a::comm-ring-1* ⇒ *'b::ab-group-add* ⇒ *'b* (**infixr** ‹∗s› *75*)
assumes *scale-right-distrib* [*algebra-simps*, *algebra-split-simps*]:
$a *s\ (x + y) = a *s\ x + a *s\ y$
and *scale-left-distrib* [*algebra-simps*, *algebra-split-simps*]:
$(a + b) *s\ x = a *s\ x + b *s\ x$
and *scale-scale* [*simp*]: $a *s\ (b *s\ x) = (a * b) *s\ x$
and *scale-one* [*simp*]: $1 *s\ x = x$
begin

lemma *scale-left-commute*: $a *s\ (b *s\ x) = b *s\ (a *s\ x)$
by (*simp add*: *mult.commute*)

lemma *scale-zero-left* [*simp*]: $0 *s\ x = 0$
and *scale-minus-left* [*simp*]: $(-\ a) *s\ x = -\ (a *s\ x)$
and *scale-left-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$(a - b) *s\ x = a *s\ x - b *s\ x$
and *scale-sum-left*: $(sum\ f\ A) *s\ x = (\sum a \in A.\ (f\ a) *s\ x)$
proof −
interpret *s*: *additive* $\lambda a.\ a *s\ x$
by *standard* (*rule scale-left-distrib*)
show $0 *s\ x = 0$ **by** (*rule s.zero*)

show (− *a*) ∗*s x* = − (*a* ∗*s x*) **by** (*rule s.minus*)
show (*a* − *b*) ∗*s x* = *a* ∗*s x* − *b* ∗*s x* **by** (*rule s.diff*)
show (*sum f A*) ∗*s x* = (∑ *a*∈*A*. (*f a*) ∗*s x*) **by** (*rule s.sum*)
qed

lemma *scale-zero-right* [*simp*]: *a* ∗*s 0* = *0*
and *scale-minus-right* [*simp*]: *a* ∗*s* (− *x*) = − (*a* ∗*s x*)
and *scale-right-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
a ∗*s* (*x* − *y*) = *a* ∗*s x* − *a* ∗*s y*
and *scale-sum-right*: *a* ∗*s* (*sum f A*) = (∑ *x*∈*A*. *a* ∗*s* (*f x*))
proof −
interpret *s*: *additive* λ*x*. *a* ∗*s x*
by *standard* (*rule scale-right-distrib*)
show *a* ∗*s 0* = *0* **by** (*rule s.zero*)
show *a* ∗*s* (− *x*) = − (*a* ∗*s x*) **by** (*rule s.minus*)
show *a* ∗*s* (*x* − *y*) = *a* ∗*s x* − *a* ∗*s y* **by** (*rule s.diff*)
show *a* ∗*s* (*sum f A*) = (∑ *x*∈*A*. *a* ∗*s* (*f x*)) **by** (*rule s.sum*)
qed

lemma *sum-constant-scale*: (∑ *x*∈*A*. *y*) = *scale* (*of-nat* (*card A*)) *y*
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *algebra-simps*)

end

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*divide*›, *SOME* ***typ*** ‹'*a* ⇒ '*a* ⇒ '*a*›)›

context *module*
begin

lemma [*field-simps*, *field-split-simps*]:
shows *scale-left-distrib-NO-MATCH*: *NO-MATCH* (*x div y*) *c* ⟹ (*a* + *b*) ∗*s x* = *a* ∗*s x* + *b* ∗*s x*
and *scale-right-distrib-NO-MATCH*: *NO-MATCH* (*x div y*) *a* ⟹ *a* ∗*s* (*x* + *y*) = *a* ∗*s x* + *a* ∗*s y*
and *scale-left-diff-distrib-NO-MATCH*: *NO-MATCH* (*x div y*) *c* ⟹ (*a* − *b*) ∗*s x* = *a* ∗*s x* − *b* ∗*s x*
and *scale-right-diff-distrib-NO-MATCH*: *NO-MATCH* (*x div y*) *a* ⟹ *a* ∗*s* (*x* − *y*) = *a* ∗*s x* − *a* ∗*s y*
by (*rule scale-left-distrib scale-right-distrib scale-left-diff-distrib scale-right-diff-distrib*)+

end

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*divide*›, *SOME* ***typ*** ‹'*a*::*divide* ⇒ '*a* ⇒ '*a*›)›

102 Subspace

context *module*

begin

definition *subspace* :: $'b$ *set* $\Rightarrow$ *bool*
where *subspace* $S \longleftrightarrow 0 \in S \wedge (\forall x{\in}S.\ \forall y{\in}S.\ x + y \in S) \wedge (\forall c.\ \forall x{\in}S.\ c *s\ x \in S)$

lemma *subspaceI*:
$0 \in S \Longrightarrow (\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow x + y \in S) \Longrightarrow (\bigwedge c\ x.\ x \in S \Longrightarrow c *s\ x \in S) \Longrightarrow subspace\ S$
by (*auto simp*: *subspace-def*)

lemma *subspace-UNIV*[*simp*]: *subspace UNIV*
by (*simp add*: *subspace-def*)

lemma *subspace-single-0*[*simp*]: *subspace* $\{0\}$
by (*simp add*: *subspace-def*)

lemma *subspace-0*: $subspace\ S \Longrightarrow 0 \in S$
by (*metis subspace-def*)

lemma *subspace-add*: $subspace\ S \Longrightarrow x \in S \Longrightarrow y \in S \Longrightarrow x + y \in S$
by (*metis subspace-def*)

lemma *subspace-scale*: $subspace\ S \Longrightarrow x \in S \Longrightarrow c *s\ x \in S$
by (*metis subspace-def*)

lemma *subspace-neg*: $subspace\ S \Longrightarrow x \in S \Longrightarrow -\ x \in S$
by (*metis scale-minus-left scale-one subspace-scale*)

lemma *subspace-diff*: $subspace\ S \Longrightarrow x \in S \Longrightarrow y \in S \Longrightarrow x - y \in S$
by (*metis diff-conv-add-uminus subspace-add subspace-neg*)

lemma *subspace-sum*: $subspace\ A \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow f\ x \in A) \Longrightarrow sum\ f\ B \in A$
by (*induct B rule*: *infinite-finite-induct*) (*auto simp add*: *subspace-add subspace-0*)

lemma *subspace-Int*: $(\bigwedge i.\ i \in I \Longrightarrow subspace\ (s\ i)) \Longrightarrow subspace\ (\bigcap i{\in}I.\ s\ i)$
by (*auto simp*: *subspace-def*)

lemma *subspace-Inter*: $\forall s \in f.\ subspace\ s \Longrightarrow subspace\ (\bigcap f)$
unfolding *subspace-def* **by** *auto*

lemma *subspace-inter*: $subspace\ A \Longrightarrow subspace\ B \Longrightarrow subspace\ (A \cap B)$
by (*simp add*: *subspace-def*)

103 Span: subspace generated by a set

definition *span* :: $'b$ *set* $\Rightarrow$ $'b$ *set*
where *span-explicit*: $span\ b = \{(\sum a{\in}t.\ r\ a *s\ \ a) \mid t\ r.\ finite\ t \wedge t \subseteq b\}$

lemma *span-explicit'*:
  span b = {(∑ *v* | *f v* ≠ *0*. *f v* ∗*s v*) | *f*. *finite* {*v*. *f v* ≠ *0*} ∧ (∀ *v*. *f v* ≠ *0* ⟶ *v* ∈ *b*)}
  unfolding *span-explicit*
proof *safe*
  fix *t r* **assume** *finite t t* ⊆ *b*
  then show ∃ *f*. (∑ *a*∈*t*. *r a* ∗*s a*) = (∑ *v* | *f v* ≠ *0*. *f v* ∗*s v*) ∧ *finite* {*v*. *f v* ≠ *0*} ∧ (∀ *v*. *f v* ≠ *0* ⟶ *v* ∈ *b*)
   by (*intro exI*[*of - λv. if v* ∈ *t then r v else 0*]) (*auto intro*!: *sum.mono-neutral-cong-right*)
next
  fix *f* :: *'b* ⇒ *'a* **assume** *finite* {*v*. *f v* ≠ *0*} (∀ *v*. *f v* ≠ *0* ⟶ *v* ∈ *b*)
  then show ∃ *t r*. (∑ *v* | *f v* ≠ *0*. *f v* ∗*s v*) = (∑ *a*∈*t*. *r a* ∗*s a*) ∧ *finite t* ∧ *t* ⊆ *b*
    by (*intro exI*[*of -* {*v*. *f v* ≠ *0*}] *exI*[*of - f*]) *auto*
qed

lemma *span-alt*:
  span B = {(∑ *x* | *f x* ≠ *0*. *f x* ∗*s x*) | *f*. {*x*. *f x* ≠ *0*} ⊆ *B* ∧ *finite* {*x*. *f x* ≠ *0*}}
  unfolding *span-explicit'* **by** *auto*

lemma *span-finite*:
  assumes *fS*: *finite S*
  shows *span S* = *range* (λ*u*. ∑ *v*∈*S*. *u v* ∗*s v*)
  unfolding *span-explicit*
proof *safe*
  fix *t r* **assume** *t* ⊆ *S* **then show** (∑ *a*∈*t*. *r a* ∗*s a*) ∈ *range* (λ*u*. ∑ *v*∈*S*. *u v* ∗*s v*)
    by (*intro image-eqI*[*of - - λa. if a* ∈ *t then r a else 0*])
      (*auto simp*: *if-distrib*[*of λr. r* ∗*s a* **for** *a*] *sum.If-cases fS Int-absorb1*)
next
  show ∃ *t r*. (∑ *v*∈*S*. *u v* ∗*s v*) = (∑ *a*∈*t*. *r a* ∗*s a*) ∧ *finite t* ∧ *t* ⊆ *S* **for** *u*
    by (*intro exI*[*of - u*] *exI*[*of - S*]) (*auto intro*: *fS*)
qed

lemma *span-induct-alt* [*consumes 1*, *case-names base step*, *induct set*: *span*]:
  assumes *x*: *x* ∈ *span S*
  assumes *h0*: *h 0* **and** *hS*: ⋀*c x y*. *x* ∈ *S* ⟹ *h y* ⟹ *h* (*c* ∗*s x* + *y*)
  shows *h x*
  using *x* **unfolding** *span-explicit*
proof *safe*
  fix *t r* **assume** *finite t t* ⊆ *S* **then show** *h* (∑ *a*∈*t*. *r a* ∗*s a*)
    by (*induction t*) (*auto intro*!: *hS h0*)
qed

lemma *span-mono*: *A* ⊆ *B* ⟹ *span A* ⊆ *span B*
  by (*auto simp*: *span-explicit*)

lemma *span-base*: *a* ∈ *S* ⟹ *a* ∈ *span S*
  by (*auto simp*: *span-explicit intro*!: *exI*[*of -* {*a*}] *exI*[*of - λ-. 1*])

lemma *span-superset*: $S \subseteq span\ S$
by (*auto simp*: *span-base*)

lemma *span-zero*: $0 \in span\ S$
by (*auto simp*: *span-explicit intro*!: *exI*[*of* - {}])

lemma *span-UNIV*[*simp*]: *span UNIV* = *UNIV*
by (*auto intro*: *span-base*)

lemma *span-add*: $x \in span\ S \Longrightarrow y \in span\ S \Longrightarrow x + y \in span\ S$
unfolding *span-explicit*
proof *safe*
fix *tx ty rx ry* **assume** $*$: *finite tx finite ty* $tx \subseteq S$ $ty \subseteq S$
have [*simp*]: $(tx \cup ty) \cap tx = tx$ $(tx \cup ty) \cap ty = ty$
by *auto*
show $\exists\, t\ r.\ (\sum a{\in}tx.\ rx\ a *s\ a) + (\sum a{\in}ty.\ ry\ a *s\ a) = (\sum a{\in}t.\ r\ a *s\ a) \wedge$ *finite* $t \wedge t \subseteq S$
apply (*intro exI*[*of* - $tx \cup ty$])
apply (*intro exI*[*of* - $\lambda a.$ (*if* $a \in tx$ *then rx a else* 0) + (*if* $a \in ty$ *then ry a else* 0)])
apply (*auto simp*: $*$ *scale-left-distrib sum.distrib if-distrib*[*of* $\lambda r.\ r *s\ a$ **for** *a*] *sum.If-cases*)
done
qed

lemma *span-scale*: $x \in span\ S \Longrightarrow c *s\ x \in span\ S$
unfolding *span-explicit*
proof *safe*
fix *t r* **assume** $*$: *finite t* $t \subseteq S$
show $\exists\, t'\ r'.\ c *s\ (\sum a{\in}t.\ r\ a *s\ a) = (\sum a{\in}t'.\ r'\ a *s\ a) \wedge$ *finite* $t' \wedge t' \subseteq S$
by (*intro exI*[*of* - *t*] *exI*[*of* - $\lambda a.\ c * r\ a$]) (*auto simp*: $*$ *scale-sum-right*)
qed

lemma *subspace-span* [*iff*]: *subspace* (*span S*)
by (*auto simp*: *subspace-def span-zero span-add span-scale*)

lemma *span-neg*: $x \in span\ S \Longrightarrow -\ x \in span\ S$
by (*metis subspace-neg subspace-span*)

lemma *span-diff*: $x \in span\ S \Longrightarrow y \in span\ S \Longrightarrow x - y \in span\ S$
by (*metis subspace-span subspace-diff*)

lemma *span-sum*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in span\ S) \Longrightarrow sum\ f\ A \in span\ S$
by (*rule subspace-sum*, *rule subspace-span*)

lemma *span-minimal*: $S \subseteq T \Longrightarrow subspace\ T \Longrightarrow span\ S \subseteq T$
by (*auto simp*: *span-explicit intro*!: *subspace-sum subspace-scale*)

lemma *span-def*: *span S* = *subspace hull S*

by (*intro hull-unique*[*symmetric*] *span-superset subspace-span span-minimal*)

lemma *span-unique*:
$S \subseteq T \Longrightarrow$ *subspace* $T \Longrightarrow (\bigwedge T'. S \subseteq T' \Longrightarrow$ *subspace* $T' \Longrightarrow T \subseteq T') \Longrightarrow$ *span* $S = T$
unfolding *span-def* **by** (*rule hull-unique*)

lemma *span-subspace-induct*[*consumes 2*]:
assumes *x*: $x \in$ *span* S
and *P*: *subspace* P
and *SP*: $\bigwedge x. x \in S \Longrightarrow x \in P$
shows $x \in P$
proof −
from *SP* **have** *SP'*: $S \subseteq P$
by (*simp add*: *subset-eq*)
from *x hull-minimal*[**where** *S*=*subspace*, *OF SP' P*, *unfolded span-def*[*symmetric*]]
show $x \in P$
by (*metis subset-eq*)
qed

lemma (**in** *module*) *span-induct*[*consumes 1*, *case-names base step*, *induct set*: *span*]:
assumes *x*: $x \in$ *span* S
and *P*: *subspace* (*Collect* P)
and *SP*: $\bigwedge x. x \in S \Longrightarrow P\ x$
shows $P\ x$
using *P SP span-subspace-induct x* **by** *fastforce*

lemma *span-empty*[*simp*]: *span* $\{\} = \{0\}$
by (*rule span-unique*) (*auto simp add*: *subspace-def*)

lemma *span-subspace*: $A \subseteq B \Longrightarrow B \subseteq$ *span* $A \Longrightarrow$ *subspace* $B \Longrightarrow$ *span* $A = B$
by (*metis order-antisym span-def hull-minimal*)

lemma *span-span*: *span* (*span* A) = *span* A
unfolding *span-def hull-hull* **..**

lemma *span-add-eq*: **assumes** *x*: $x \in$ *span* S **shows** $x + y \in$ *span* $S \longleftrightarrow y \in$ *span* S
proof
assume $*$: $x + y \in$ *span* S
have $(x + y) - x \in$ *span* S **using** $*$ x **by** (*rule span-diff*)
then show $y \in$ *span* S **by** *simp*
qed (*intro span-add x*)

lemma *span-add-eq2*: **assumes** *y*: $y \in$ *span* S **shows** $x + y \in$ *span* $S \longleftrightarrow x \in$ *span* S
using *span-add-eq*[*of y S x*] *y* **by** (*auto simp*: *ac-simps*)

lemma *span-singleton*: *span* $\{x\}$ = *range* ($\lambda k.\ k *s\ x$)
  by (*auto simp*: *span-finite*)

lemma *span-Un*: *span* ($S \cup T$) = $\{x + y \mid x\ y.\ x \in$ *span* $S \wedge y \in$ *span* $T\}$
proof *safe*
  fix x **assume** $x \in$ *span* ($S \cup T$)
  then obtain $t\ r$ **where** t: *finite* $t\ t \subseteq S \cup T$ **and** x: $x = (\sum a{\in}t.\ r\ a *s\ a)$
    by (*auto simp*: *span-explicit*)
  moreover have $t \cap S \cup (t - S) = t$ **by** *auto*
  ultimately show $\exists xa\ y.\ x = xa + y \wedge xa \in$ *span* $S \wedge y \in$ *span* T
    unfolding x
    apply (*rule-tac exI*[*of* - $\sum a{\in}t \cap S.\ r\ a *s\ a$])
    apply (*rule-tac exI*[*of* - $\sum a{\in}t - S.\ r\ a *s\ a$])
    apply (*subst sum.union-inter-neutral*[*symmetric*])
    apply (*auto intro*!: *span-sum span-scale intro*: *span-base*)
    done
next
  fix $x\ y$ **assume**$x \in$ *span* $S\ y \in$ *span* T **then show** $x + y \in$ *span* ($S \cup T$)
    using *span-mono*[*of* $S\ S \cup T$] *span-mono*[*of* $T\ S \cup T$]
    by (*auto intro*!: *span-add*)
qed

lemma *span-insert*: *span* (*insert* $a\ S$) = $\{x.\ \exists k.\ (x - k *s\ a) \in$ *span* $S\}$
proof −
  have *span* ($\{a\} \cup S$) = $\{x.\ \exists k.\ (x - k *s\ a) \in$ *span* $S\}$
    unfolding *span-Un span-singleton*
    apply (*auto simp add*: *set-eq-iff*)
    subgoal for $y\ k$ **by** (*auto intro*!: *exI*[*of* - k])
    subgoal for $y\ k$ **by** (*rule exI*[*of* - $k *s\ a$], *rule exI*[*of* - $y - k *s\ a$]) *auto*
    done
  then show *?thesis* **by** *simp*
qed

lemma *span-breakdown*:
  assumes *bS*: $b \in S$
    and *aS*: $a \in$ *span* S
  shows $\exists k.\ a - k *s\ b \in$ *span* ($S - \{b\}$)
  using *assms span-insert* [*of* $b\ S - \{b\}$]
  by (*simp add*: *insert-absorb*)

lemma *span-breakdown-eq*: $x \in$ *span* (*insert* $a\ S$) $\longleftrightarrow$ ($\exists k.\ x - k *s\ a \in$ *span* S)
  by (*simp add*: *span-insert*)

lemmas *span-clauses* = *span-base span-zero span-add span-scale*

lemma *span-eq-iff*[*simp*]: *span* $s = s \longleftrightarrow$ *subspace* s
  unfolding *span-def* **by** (*rule hull-eq*) (*rule subspace-Inter*)

lemma *span-eq*: *span S* = *span T* ⟷ *S* ⊆ *span T* ∧ *T* ⊆ *span S*
  by (*metis span-minimal span-subspace span-superset subspace-span*)

lemma *eq-span-insert-eq*:
  assumes (*x* − *y*) ∈ *span S*
    shows *span*(*insert x S*) = *span*(*insert y S*)
proof −
  have ∗: *span*(*insert x S*) ⊆ *span*(*insert y S*) **if** (*x* − *y*) ∈ *span S* **for** *x y*
  proof −
    have *1*: (*r* ∗*s x* − *r* ∗*s y*) ∈ *span S* **for** *r*
      by (*metis scale-right-diff-distrib span-scale that*)
    have *2*: (*z* − *k* ∗*s y*) − *k* ∗*s* (*x* − *y*) = *z* − *k* ∗*s x* **for** *z k*
      by (*simp add*: *scale-right-diff-distrib*)
  show *?thesis*
    apply (*clarsimp simp add*: *span-breakdown-eq*)
    by (*metis 1 2 diff-add-cancel scale-right-diff-distrib span-add-eq*)
  qed
  show *?thesis*
    apply (*intro subset-antisym* ∗ *assms*)
    using *assms subspace-neg subspace-span minus-diff-eq* **by** *force*
qed

104 Dependent and independent sets

definition *dependent* :: *'b set* ⇒ *bool*
  where *dependent-explicit*: *dependent s* ⟷ (∃ *t u*. *finite t* ∧ *t* ⊆ *s* ∧ (∑ *v*∈*t*. *u v* ∗*s v*) = *0* ∧ (∃ *v*∈*t*. *u v* ≠ *0*))

abbreviation *independent s* ≡ ¬ *dependent s*

lemma *dependent-mono*: *dependent B* ⟹ *B* ⊆ *A* ⟹ *dependent A*
  by (*auto simp add*: *dependent-explicit*)

lemma *independent-mono*: *independent A* ⟹ *B* ⊆ *A* ⟹ *independent B*
  by (*auto intro*: *dependent-mono*)

lemma *dependent-zero*: *0* ∈ *A* ⟹ *dependent A*
  by (*auto simp*: *dependent-explicit intro*!: *exI*[*of* - λ*i*. *1*] *exI*[*of* - {*0*}])

lemma *independent-empty*[*intro*]: *independent* {}
  by (*simp add*: *dependent-explicit*)

lemma *independent-explicit-module*:
  independent s ⟷ (∀ *t u v*. *finite t* ⟶ *t* ⊆ *s* ⟶ (∑ *v*∈*t*. *u v* ∗*s v*) = *0* ⟶ *v* ∈ *t* ⟶ *u v* = *0*)
  unfolding *dependent-explicit* **by** *auto*

lemma *independentD*: *independent s* ⟹ *finite t* ⟹ *t* ⊆ *s* ⟹ (∑ *v*∈*t*. *u v* ∗*s v*) = *0* ⟹ *v* ∈ *t* ⟹ *u v* = *0*

by (*simp add*: *independent-explicit-module*)

lemma *independent-Union-directed*:
assumes *directed*: $\bigwedge c\ d.\ c \in C \Longrightarrow d \in C \Longrightarrow c \subseteq d \vee d \subseteq c$
assumes *indep*: $\bigwedge c.\ c \in C \Longrightarrow$ *independent c*
shows *independent* $(\bigcup C)$
proof
assume *dependent* $(\bigcup C)$
then obtain *u v S* **where** *S*: *finite S* $S \subseteq \bigcup C$ $v \in S$ $u\ v \neq 0$ $(\sum v{\in}S.\ u\ v *s\ v) = 0$
by (*auto simp*: *dependent-explicit*)

have $S \neq \{\}$
using ‹$v \in S$› **by** *auto*
have $\exists c{\in}C.\ S \subseteq c$
using ‹*finite S*› ‹$S \neq \{\}$› ‹$S \subseteq \bigcup C$›
proof (*induction rule*: *finite-ne-induct*)
case (*insert i I*)
then obtain *c d* **where** *cd*: $c \in C$ $d \in C$ **and** *iI*: $I \subseteq c$ $i \in d$
by *blast*
from *directed*[*OF cd*] *cd* **have** $c \cup d \in C$
by (*auto simp*: *sup.absorb1 sup.absorb2*)
with *iI* **show** *?case*
by (*intro bexI*[*of - c* $\cup$ *d*]) *auto*
qed *auto*
then obtain *c* **where** $c \in C$ $S \subseteq c$
by *auto*
have *dependent c*
unfolding *dependent-explicit*
by (*intro exI*[*of - S*] *exI*[*of - u*] *bexI*[*of - v*] *conjI*) *fact+*
with *indep*[*OF* ‹$c \in C$›] **show** *False*
by *auto*
qed

lemma *dependent-finite*:
assumes *finite S*
shows *dependent S* $\longleftrightarrow (\exists u.\ (\exists v \in S.\ u\ v \neq 0) \wedge (\sum v{\in}S.\ u\ v *s\ v) = 0)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *T u v*
where *finite T* $T \subseteq S$ $v{\in}T$ $u\ v \neq 0$ $(\sum v{\in}T.\ u\ v *s\ v) = 0$
by (*force simp*: *dependent-explicit*)
with *assms* **show** *?rhs*
apply (*rule-tac x*=$\lambda v.$ *if* $v \in T$ *then u v else 0* **in** *exI*)
apply (*auto simp*: *sum.mono-neutral-right*)
done
next
assume *?rhs* **with** *assms* **show** *?lhs*

by (*fastforce simp add*: *dependent-explicit*)
qed

lemma *dependent-alt*:
dependent B $\longleftrightarrow$
($\exists X$. *finite* $\{x.\ X\ x \neq 0\} \wedge \{x.\ X\ x \neq 0\} \subseteq B \wedge (\sum x|X\ x \neq 0.\ X\ x *s\ x) = 0 \wedge (\exists x.\ X\ x \neq 0)$)
unfolding *dependent-explicit*
apply *safe*
subgoal for *S u v*
apply (*intro exI*[*of* - λx. *if* $x \in S$ *then u x else 0*])
apply (*subst sum.mono-neutral-cong-left*[**where** *T*=*S*])
apply (*auto intro*!: *sum.mono-neutral-cong-right cong*: *rev-conj-cong*)
done
apply *auto*
done

lemma *independent-alt*:
independent B $\longleftrightarrow$
($\forall X$. *finite* $\{x.\ X\ x \neq 0\} \longrightarrow \{x.\ X\ x \neq 0\} \subseteq B \longrightarrow (\sum x|X\ x \neq 0.\ X\ x *s\ x) = 0 \longrightarrow (\forall x.\ X\ x = 0)$)
unfolding *dependent-alt* **by** *auto*

lemma *independentD-alt*:
independent B $\Longrightarrow$ *finite* $\{x.\ X\ x \neq 0\} \Longrightarrow \{x.\ X\ x \neq 0\} \subseteq B \Longrightarrow (\sum x|X\ x \neq 0.\ X\ x *s\ x) = 0 \Longrightarrow X\ x = 0$
unfolding *independent-alt* **by** *blast*

lemma *independentD-unique*:
assumes *B*: *independent B*
and *X*: *finite* $\{x.\ X\ x \neq 0\}$ $\{x.\ X\ x \neq 0\} \subseteq B$
and *Y*: *finite* $\{x.\ Y\ x \neq 0\}$ $\{x.\ Y\ x \neq 0\} \subseteq B$
and $(\sum x \mid X\ x \neq 0.\ X\ x *s\ x) = (\sum x|\ Y\ x \neq 0.\ Y\ x *s\ x)$
shows $X = Y$
proof –
have $X\ x - Y\ x = 0$ **for** *x*
using *B*
proof (*rule independentD-alt*)
have $\{x.\ X\ x - Y\ x \neq 0\} \subseteq \{x.\ X\ x \neq 0\} \cup \{x.\ Y\ x \neq 0\}$
by *auto*
then show *finite* $\{x.\ X\ x - Y\ x \neq 0\}$ $\{x.\ X\ x - Y\ x \neq 0\} \subseteq B$
using *X Y* **by** (*auto dest*: *finite-subset*)
then have $(\sum x \mid X\ x - Y\ x \neq 0.\ (X\ x - Y\ x) *s\ x) = (\sum v \in \{S.\ X\ S \neq 0\} \cup \{S.\ Y\ S \neq 0\}.\ (X\ v - Y\ v) *s\ v)$
using *X Y* **by** (*intro sum.mono-neutral-cong-left*) *auto*
also have $\ldots = (\sum v \in \{S.\ X\ S \neq 0\} \cup \{S.\ Y\ S \neq 0\}.\ X\ v *s\ v) - (\sum v \in \{S.\ X\ S \neq 0\} \cup \{S.\ Y\ S \neq 0\}.\ Y\ v *s\ v)$
by (*simp add*: *scale-left-diff-distrib sum-subtractf assms*)
also have $(\sum v \in \{S.\ X\ S \neq 0\} \cup \{S.\ Y\ S \neq 0\}.\ X\ v *s\ v) = (\sum v \in \{S.\ X\ S$

$\neq$ *0*}. *X v ∗s v*)
using *X Y* **by** (*intro sum.mono-neutral-cong-right*) *auto*
also have ($\sum$ *v*∈{*S*. *X S* $\neq$ *0*} ∪ {*S*. *Y S* $\neq$ *0*}. *Y v ∗s v*) = ($\sum$ *v*∈{*S*. *Y S* $\neq$ *0*}. *Y v ∗s v*)
using *X Y* **by** (*intro sum.mono-neutral-cong-right*) *auto*
finally show ($\sum$ *x* | *X x* − *Y x* $\neq$ *0*. (*X x* − *Y x*) *∗s x*) = *0*
using *assms* **by** *simp*
qed
then show *?thesis*
by *auto*
qed

105 Representation of a vector on a specific basis

definition *representation* :: *'b set* ⇒ *'b* ⇒ *'b* ⇒ *'a*
where *representation basis v* =
(*if independent basis* ∧ *v* ∈ *span basis then*
SOME f. (∀ *v*. *f v* $\neq$ *0* ⟶ *v* ∈ *basis*) ∧ *finite* {*v*. *f v* $\neq$ *0*} ∧ ($\sum$ *v*∈{*v*. *f v* $\neq$ *0*}. *f v ∗s v*) = *v*
else (λ*b*. *0*))

lemma *unique-representation*:
assumes *basis*: *independent basis*
and *in-basis*: ⋀*v*. *f v* $\neq$ *0* ⟹ *v* ∈ *basis* ⋀*v*. *g v* $\neq$ *0* ⟹ *v* ∈ *basis*
and [*simp*]: *finite* {*v*. *f v* $\neq$ *0*} *finite* {*v*. *g v* $\neq$ *0*}
and *eq*: ($\sum$ *v*∈{*v*. *f v* $\neq$ *0*}. *f v ∗s v*) = ($\sum$ *v*∈{*v*. *g v* $\neq$ *0*}. *g v ∗s v*)
shows *f* = *g*
proof (*rule ext*, *rule ccontr*)
fix *v* **assume** *ne*: *f v* $\neq$ *g v*
have *dependent basis*
unfolding *dependent-explicit*
proof (*intro exI conjI*)
have ∗: {*v*. *f v* − *g v* $\neq$ *0*} ⊆ {*v*. *f v* $\neq$ *0*} ∪ {*v*. *g v* $\neq$ *0*}
by *auto*
show *finite* {*v*. *f v* − *g v* $\neq$ *0*}
by (*rule finite-subset*[*OF* ∗]) *simp*
show ∃ *v*∈{*v*. *f v* − *g v* $\neq$ *0*}. *f v* − *g v* $\neq$ *0*
by (*rule bexI*[*of - v*]) (*auto simp*: *ne*)
have ($\sum$ *v* | *f v* − *g v* $\neq$ *0*. (*f v* − *g v*) *∗s v*) =
($\sum$ *v*∈{*v*. *f v* $\neq$ *0*} ∪ {*v*. *g v* $\neq$ *0*}. (*f v* − *g v*) *∗s v*)
by (*intro sum.mono-neutral-cong-left* ∗) *auto*
also have ... =
($\sum$ *v*∈{*v*. *f v* $\neq$ *0*} ∪ {*v*. *g v* $\neq$ *0*}. *f v ∗s v*) − ($\sum$ *v*∈{*v*. *f v* $\neq$ *0*} ∪ {*v*. *g v* $\neq$ *0*}. *g v ∗s v*)
by (*simp add*: *algebra-simps sum-subtractf*)
also have ... = ($\sum$ *v* | *f v* $\neq$ *0*. *f v ∗s v*) − ($\sum$ *v* | *g v* $\neq$ *0*. *g v ∗s v*)
by (*intro arg-cong2*[**where** *f*= (−)] *sum.mono-neutral-cong-right*) *auto*
finally show ($\sum$ *v* | *f v* − *g v* $\neq$ *0*. (*f v* − *g v*) *∗s v*) = *0*
by (*simp add*: *eq*)

show {*v*. *f v* − *g v* ≠ *0*} ⊆ *basis*
using *in-basis* ∗ **by** *auto*
qed
with *basis* **show** *False* **by** *auto*
qed

lemma
shows *representation-ne-zero*: ⋀*b*. *representation basis v b* ≠ *0* ⟹ *b* ∈ *basis*
and *finite-representation*: *finite* {*b*. *representation basis v b* ≠ *0*}
and *sum-nonzero-representation-eq*:
independent basis ⟹ *v* ∈ *span basis* ⟹ (∑ *b* | *representation basis v b* ≠ *0*. *representation basis v b* ∗*s b*) = *v*
proof −
{ assume *basis*: *independent basis* **and** *v*: *v* ∈ *span basis*
define *p* **where** *p f* ⟷
(∀ *v*. *f v* ≠ *0* ⟶ *v* ∈ *basis*) ∧ *finite* {*v*. *f v* ≠ *0*} ∧ (∑ *v*∈{*v*. *f v* ≠ *0*}. *f v* ∗*s v*) = *v* **for** *f*
obtain *t r* **where** ∗: *finite t t* ⊆ *basis* (∑ *b*∈*t*. *r b* ∗*s b*) = *v*
using ‹*v* ∈ *span basis*› **by** (*auto simp*: *span-explicit*)
define *f* **where** *f b* = (*if b* ∈ *t then r b else 0*) **for** *b*
have *p f*
using ∗ **by** (*auto simp*: *p-def f-def intro*!: *sum.mono-neutral-cong-left*)
have ∗: *representation basis v* = *Eps p* **by** (*simp add*: *p-def*[*abs-def*] *representation-def basis v*)
from *someI*[*of p f*, *OF* ‹*p f*›] **have** *p* (*representation basis v*)
unfolding ∗ **. }**
note ∗ = *this*

show *representation basis v b* ≠ *0* ⟹ *b* ∈ *basis* **for** *b*
using ∗ **by** (*cases independent basis* ∧ *v* ∈ *span basis*) (*auto simp*: *representation-def*)

show *finite* {*b*. *representation basis v b* ≠ *0*}
using ∗ **by** (*cases independent basis* ∧ *v* ∈ *span basis*) (*auto simp*: *representation-def*)

show *independent basis* ⟹ *v* ∈ *span basis* ⟹ (∑ *b* | *representation basis v b* ≠ *0*. *representation basis v b* ∗*s b*) = *v*
using ∗ **by** *auto*
qed

lemma *sum-representation-eq*:
(∑ *b*∈*B*. *representation basis v b* ∗*s b*) = *v*
if *independent basis v* ∈ *span basis finite B basis* ⊆ *B*
proof −
have (∑ *b*∈*B*. *representation basis v b* ∗*s b*) =
(∑ *b* | *representation basis v b* ≠ *0*. *representation basis v b* ∗*s b*)
apply (*rule sum.mono-neutral-cong*)
apply (*rule finite-representation*)

apply *fact*
subgoal for *b*
using *that representation-ne-zero*[*of basis v b*]
by *auto*
subgoal by *auto*
subgoal by *simp*
done
also have ... $= v$
by (*rule sum-nonzero-representation-eq*; *fact*)
finally show *?thesis* **.**
qed

lemma *representation-eqI*:
assumes *basis*: *independent basis* **and** *b*: $v \in$ *span basis*
and *ne-zero*: $\bigwedge b.\ f\ b \neq 0 \Longrightarrow b \in basis$
and *finite*: *finite* $\{b.\ f\ b \neq 0\}$
and *eq*: $(\sum b \mid f\ b \neq 0.\ f\ b *s\ b) = v$
shows *representation basis* $v = f$
by (*rule unique-representation*[*OF basis*])
(*auto simp*: *representation-ne-zero finite-representation*
sum-nonzero-representation-eq[*OF basis b*] *ne-zero finite eq*)

lemma *representation-basis*:
assumes *basis*: *independent basis* **and** *b*: $b \in basis$
shows *representation basis* $b = (\lambda v.\ if\ v = b\ then\ 1\ else\ 0)$
proof (*rule unique-representation*[*OF basis*])
show *representation basis b v* $\neq 0 \Longrightarrow v \in basis$ **for** *v*
using *representation-ne-zero* **.**
show *finite* $\{v.\ representation\ basis\ b\ v \neq 0\}$
using *finite-representation* **.**
show $(if\ v = b\ then\ 1\ else\ 0) \neq 0 \Longrightarrow v \in basis$ **for** *v*
by (*cases* $v = b$) (*auto simp*: *b*)
have $*$: $\{v.\ (if\ v = b\ then\ 1\ else\ 0 :: 'a) \neq 0\} = \{b\}$
by *auto*
show *finite* $\{v.\ (if\ v = b\ then\ 1\ else\ 0) \neq 0\}$ **unfolding** $*$ **by** *auto*
show $(\sum v \mid representation\ basis\ b\ v \neq 0.\ representation\ basis\ b\ v *s\ v) =$
$(\sum v \mid (if\ v = b\ then\ 1\ else\ 0::'a) \neq 0.\ (if\ v = b\ then\ 1\ else\ 0) *s\ v)$
unfolding $*$ *sum-nonzero-representation-eq*[*OF basis span-base*[*OF b*]] **by** *auto*
qed

lemma *representation-zero*: *representation basis* $0 = (\lambda b.\ 0)$
proof *cases*
assume *basis*: *independent basis* **show** *?thesis*
by (*rule representation-eqI*[*OF basis span-zero*]) *auto*
qed (*simp add*: *representation-def*)

lemma *representation-diff*:
assumes *basis*: *independent basis* **and** *v*: $v \in$ *span basis* **and** *u*: $u \in$ *span basis*
shows *representation basis* $(u - v) = (\lambda b.\ representation\ basis\ u\ b - represen$-

tation basis v b)
proof (*rule representation-eqI*[*OF basis span-diff*[*OF u v*]])
let *?R* = *representation basis*
note *finite-representation*[*simp*] *u*[*simp*] *v*[*simp*]
have ∗: {*b*. *?R u b* − *?R v b* ≠ *0*} ⊆ {*b*. *?R u b* ≠ *0*} ∪ {*b*. *?R v b* ≠ *0*}
by *auto*
then show *?R u b* − *?R v b* ≠ *0* ⟹ *b* ∈ *basis* **for** *b*
by (*auto dest*: *representation-ne-zero*)
show *finite* {*b*. *?R u b* − *?R v b* ≠ *0*}
by (*intro finite-subset*[*OF* ∗]) *simp-all*
have (∑ *b* | *?R u b* − *?R v b* ≠ *0*. (*?R u b* − *?R v b*) ∗*s b*) =
(∑ *b*∈{*b*. *?R u b* ≠ *0*} ∪ {*b*. *?R v b* ≠ *0*}. (*?R u b* − *?R v b*) ∗*s b*)
by (*intro sum.mono-neutral-cong-left* ∗) *auto*
also have ... =
(∑ *b*∈{*b*. *?R u b* ≠ *0*} ∪ {*b*. *?R v b* ≠ *0*}. *?R u b* ∗*s b*) − (∑ *b*∈{*b*. *?R u b* ≠ *0*} ∪ {*b*. *?R v b* ≠ *0*}. *?R v b* ∗*s b*)
by (*simp add*: *algebra-simps sum-subtractf*)
also have ... = (∑ *b* | *?R u b* ≠ *0*. *?R u b* ∗*s b*) − (∑ *b* | *?R v b* ≠ *0*. *?R v b* ∗*s b*)
by (*intro arg-cong2*[**where** *f*= (−)] *sum.mono-neutral-cong-right*) *auto*
finally show (∑ *b* | *?R u b* − *?R v b* ≠ *0*. (*?R u b* − *?R v b*) ∗*s b*) = *u* − *v*
by (*simp add*: *sum-nonzero-representation-eq*[*OF basis*])
qed

lemma *representation-neg*:
independent basis ⟹ *v* ∈ *span basis* ⟹ *representation basis* (− *v*) = (λ*b*. − *representation basis v b*)
using *representation-diff*[*of basis v 0*] **by** (*simp add*: *representation-zero span-zero*)

lemma *representation-add*:
independent basis ⟹ *v* ∈ *span basis* ⟹ *u* ∈ *span basis* ⟹
representation basis (*u* + *v*) = (λ*b*. *representation basis u b* + *representation basis v b*)
using *representation-diff*[*of basis* −*v u*] **by** (*simp add*: *representation-neg representation-diff span-neg*)

lemma *representation-sum*:
independent basis ⟹ (⋀*i*. *i* ∈ *I* ⟹ *v i* ∈ *span basis*) ⟹
representation basis (*sum v I*) = (λ*b*. ∑ *i*∈*I*. *representation basis* (*v i*) *b*)
by (*induction I rule*: *infinite-finite-induct*)
(*auto simp*: *representation-zero representation-add span-sum*)

lemma *representation-scale*:
assumes *basis*: *independent basis* **and** *v*: *v* ∈ *span basis*
shows *representation basis* (*r* ∗*s v*) = (λ*b*. *r* ∗ *representation basis v b*)
proof (*rule representation-eqI*[*OF basis span-scale*[*OF v*]])
let *?R* = *representation basis*
note *finite-representation*[*simp*] *v*[*simp*]
have ∗: {*b*. *r* ∗ *?R v b* ≠ *0*} ⊆ {*b*. *?R v b* ≠ *0*}

by *auto*
then show $r * representation\ basis\ v\ b \neq 0 \Longrightarrow b \in basis$ **for** b
using *representation-ne-zero* **by** *auto*
show *finite* $\{b.\ r * ?R\ v\ b \neq 0\}$
by (*intro finite-subset*[*OF* ∗]) *simp-all*
have $(\sum b \mid r * ?R\ v\ b \neq 0.\ (r * ?R\ v\ b) *s\ b) = (\sum b \in \{b.\ ?R\ v\ b \neq 0\}.\ (r * ?R\ v\ b) *s\ b)$
by (*intro sum.mono-neutral-cong-left* ∗) *auto*
also have ... $= r *s\ (\sum b \mid ?R\ v\ b \neq 0.\ ?R\ v\ b *s\ b)$
by (*simp add*: *scale-scale*[*symmetric*] *scale-sum-right del*: *scale-scale*)
finally show $(\sum b \mid r * ?R\ v\ b \neq 0.\ (r * ?R\ v\ b) *s\ b) = r *s\ v$
by (*simp add*: *sum-nonzero-representation-eq*[*OF basis*])
qed

lemma *representation-extend*:
assumes *basis*: *independent basis* **and** *v*: $v \in span\ basis'$ **and** *basis'*: $basis' \subseteq basis$
shows $representation\ basis\ v = representation\ basis'\ v$
proof (*rule representation-eqI*[*OF basis*])
show *v'*: $v \in span\ basis$ **using** *span-mono*[*OF basis'*] *v* **by** *auto*
have ∗: *independent basis'* **using** *basis' basis* **by** (*auto intro*: *dependent-mono*)
show $representation\ basis'\ v\ b \neq 0 \Longrightarrow b \in basis$ **for** b
using *representation-ne-zero basis'* **by** *auto*
show *finite* $\{b.\ representation\ basis'\ v\ b \neq 0\}$
using *finite-representation* **.**
show $(\sum b \mid representation\ basis'\ v\ b \neq 0.\ representation\ basis'\ v\ b *s\ b) = v$
using *sum-nonzero-representation-eq*[*OF* ∗ *v*] **.**
qed

The set B is the maximal independent set for *span* B, or A is the minimal spanning set

lemma *spanning-subset-independent*:
assumes *BA*: $B \subseteq A$
and *iA*: *independent* A
and *AsB*: $A \subseteq span\ B$
shows $A = B$
proof (*intro antisym*[*OF* - *BA*] *subsetI*)
have *iB*: *independent* B **using** *independent-mono* [*OF iA BA*] **.**
fix v **assume** $v \in A$
with *AsB* **have** $v \in span\ B$ **by** *auto*
let $?RB = representation\ B\ v$ **and** $?RA = representation\ A\ v$
have $?RB\ v = 1$
unfolding *representation-extend*[*OF iA* ‹$v \in span\ B$› *BA*, *symmetric*] *representation-basis*[*OF iA* ‹$v \in A$›] **by** *simp*
then show $v \in B$
using *representation-ne-zero*[*of B v v*] **by** *auto*
qed

end

A linear function is a mapping between two modules over the same ring.

locale *module-hom* = *m1*: *module s1* + *m2*: *module s2*
  for *s1* :: *'a*::*comm-ring-1* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ‹∗*a*› *75*)
  and *s2* :: *'a*::*comm-ring-1* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ‹∗*b*› *75*) +
 fixes *f* :: *'b* ⇒ *'c*
 assumes *add*: *f* (*b1* + *b2*) = *f b1* + *f b2*
  and *scale*: *f* (*r* ∗*a b*) = *r* ∗*b f b*
begin

lemma *zero*[*simp*]: *f 0* = *0*
 using *scale*[*of 0 0*] **by** *simp*

lemma *neg*: *f* (− *x*) = − *f x*
 using *scale* [**where** *r*=−*1*] **by** (*metis add add-eq-0-iff zero*)

lemma *diff*: *f* (*x* − *y*) = *f x* − *f y*
 by (*metis diff-conv-add-uminus add neg*)

lemma *sum*: *f* (*sum g S*) = (∑ *a*∈*S*. *f* (*g a*))
proof (*induct S rule*: *infinite-finite-induct*)
 case (*insert x F*)
 have *f* (*sum g* (*insert x F*)) = *f* (*g x* + *sum g F*)
  using *insert.hyps* **by** *simp*
 also have . . . = *f* (*g x*) + *f* (*sum g F*)
  using *add* **by** *simp*
 also have . . . = (∑ *a*∈*insert x F*. *f* (*g a*))
  using *insert.hyps* **by** *simp*
 finally show *?case* **.**
qed *simp-all*

lemma *inj-on-iff-eq-0*:
 assumes *s*: *m1.subspace s*
 shows *inj-on f s* ⟷ (∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0*)
proof −
 have *inj-on f s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. *f x* − *f y* = *0* ⟶ *x* − *y* = *0*)
  by (*simp add*: *inj-on-def*)
 also have . . . ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. *f* (*x* − *y*) = *0* ⟶ *x* − *y* = *0*)
  by (*simp add*: *diff*)
 also have . . . ⟷ (∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0*) (**is** *?l* = *?r*)
 proof *safe*
  fix *x* **assume** *?l* **assume** *x* ∈ *s f x* = *0* **with** ‹*?l*›[*rule-format*, *of x 0*] *s* **show** *x* = *0*
   by (*auto simp*: *m1.subspace-0*)
 next
  fix *x y* **assume** *?r* **assume** *x* ∈ *s y* ∈ *s f* (*x* − *y*) = *0*
  with ‹*?r*›[*rule-format*, *of x* − *y*] *s*
  show *x* − *y* = *0*
   by (*auto simp*: *m1.subspace-diff*)
 qed

finally show *?thesis*
by *auto*
qed

lemma *inj-iff-eq-0*: *inj f* = (∀ *x*. *f x* = *0* ⟶ *x* = *0*)
by (*rule inj-on-iff-eq-0*[*OF m1.subspace-UNIV*, *unfolded ball-UNIV*])

lemma *subspace-image*: **assumes** *S*: *m1.subspace S* **shows** *m2.subspace* (*f* ' *S*)
unfolding *m2.subspace-def*
proof *safe*
show *0* ∈ *f* ' *S*
by (*rule image-eqI*[*of - - 0*]) (*auto simp*: *S m1.subspace-0*)
show *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *f x* + *f y* ∈ *f* ' *S* **for** *x y*
by (*rule image-eqI*[*of - - x* + *y*]) (*auto simp*: *S m1.subspace-add add*)
show *x* ∈ *S* ⟹ *r* ∗*b f x* ∈ *f* ' *S* **for** *r x*
by (*rule image-eqI*[*of - - r* ∗*a x*]) (*auto simp*: *S m1.subspace-scale scale*)
qed

lemma *subspace-vimage*: *m2.subspace S* ⟹ *m1.subspace* (*f* −' *S*)
by (*simp add*: *vimage-def add scale m1.subspace-def m2.subspace-0 m2.subspace-add m2.subspace-scale*)

lemma *subspace-kernel*: *m1.subspace* {*x*. *f x* = *0*}
using *subspace-vimage*[*OF m2.subspace-single-0*] **by** (*simp add*: *vimage-def*)

lemma *span-image*: *m2.span* (*f* ' *S*) = *f* ' (*m1.span S*)
proof (*rule m2.span-unique*)
show *f* ' *S* ⊆ *f* ' *m1.span S*
by (*rule image-mono*, *rule m1.span-superset*)
show *m2.subspace* (*f* ' *m1.span S*)
using *m1.subspace-span* **by** (*rule subspace-image*)
next
fix *T* **assume** *f* ' *S* ⊆ *T* **and** *m2.subspace T* **then show** *f* ' *m1.span S* ⊆ *T*
unfolding *image-subset-iff-subset-vimage* **by** (*metis subspace-vimage m1.span-minimal*)
qed

lemma *dependent-inj-imageD*:
assumes *d*: *m2.dependent* (*f* ' *s*) **and** *i*: *inj-on f* (*m1.span s*)
shows *m1.dependent s*
proof −
have [*intro*]: *inj-on f s*
using ‹*inj-on f* (*m1.span s*)› *m1.span-superset* **by** (*rule inj-on-subset*)
from *d* **obtain** *s'* *r v* **where** ∗: *finite s'* *s'* ⊆ *s* (∑ *v*∈*f* ' *s'*. *r v* ∗*b v*) = *0* *v* ∈ *s'* *r* (*f v*) ≠ *0*
by (*auto simp*: *m2.dependent-explicit subset-image-iff dest*!: *finite-imageD intro*: *inj-on-subset*)
have *f* (∑ *v*∈*s'*. *r* (*f v*) ∗*a v*) = (∑ *v*∈*s'*. *r* (*f v*) ∗*b f v*)
by (*simp add*: *sum scale*)
also have ... = (∑ *v*∈*f* ' *s'*. *r v* ∗*b v*)

using ‹$s' \subseteq s$› **by** (*subst sum.reindex*) (*auto dest*!: *finite-imageD intro*: *inj-on-subset*)
finally have f ($\sum v{\in}s'$. r (f v) $*a$ v) = 0
by (*simp add*: $*$)
with ‹$s' \subseteq s$› **have** ($\sum v{\in}s'$. r (f v) $*a$ v) = 0
by (*intro inj-onD*[*OF i*] *m1.span-zero m1.span-sum m1.span-scale*) (*auto intro*: *m1.span-base*)
then show *m1.dependent s*
using ‹*finite s'*› ‹$s' \subseteq s$› ‹$v \in s'$› ‹r (f v) $\neq 0$› **by** (*force simp add*: *m1.dependent-explicit*)
qed

lemma *eq-0-on-span*:
assumes *f0*: $\bigwedge x.\ x \in b \Longrightarrow f\ x = 0$ **and** *x*: $x \in$ *m1.span b* **shows** $f\ x = 0$
using *m1.span-induct*[*OF x subspace-kernel*] *f0* **by** *simp*

lemma *independent-injective-image*: *m1.independent s* $\Longrightarrow$ *inj-on f* (*m1.span s*) $\Longrightarrow$ *m2.independent* (f ‘ s)
using *dependent-inj-imageD*[*of s*] **by** *auto*

lemma *inj-on-span-independent-image*:
assumes *ifB*: *m2.independent* (f ‘ B) **and** *f*: *inj-on f B* **shows** *inj-on f* (*m1.span B*)
unfolding *inj-on-iff-eq-0*[*OF m1.subspace-span*] **unfolding** *m1.span-explicit'*
proof *safe*
fix *r* **assume** *fr*: *finite* $\{v.\ r\ v \neq 0\}$ **and** *r*: $\forall v.\ r\ v \neq 0 \longrightarrow v \in B$
and *eq0*: f ($\sum v \mid r\ v \neq 0.\ r\ v\ *a\ v$) = 0
have 0 = ($\sum v \mid r\ v \neq 0.\ r\ v\ *b\ f\ v$)
using *eq0* **by** (*simp add*: *sum scale*)
also have ... = ($\sum v{\in}f$ ‘ $\{v.\ r\ v \neq 0\}$. r (*the-inv-into B f v*) $*b$ v)
using *r* **by** (*subst sum.reindex*) (*auto simp*: *the-inv-into-f-f*[*OF f*] *intro*!: *inj-on-subset*[*OF f*] *sum.cong*)
finally have $r\ v \neq 0 \Longrightarrow r$ (*the-inv-into B f* (f v)) = 0 **for** v
using *fr r ifB*[*unfolded m2.independent-explicit-module*, *rule-format*,
of f ‘ $\{v.\ r\ v \neq 0\}$ $\lambda v.\ r$ (*the-inv-into B f v*)]
by *auto*
then have $r\ v = 0$ **for** v
using *the-inv-into-f-f*[*OF f*] *r* **by** *auto*
then show ($\sum v \mid r\ v \neq 0.\ r\ v\ *a\ v$) = 0 **by** *auto*
qed

lemma *inj-on-span-iff-independent-image*: *m2.independent* (f ‘ B) $\Longrightarrow$ *inj-on f* (*m1.span B*) $\longleftrightarrow$ *inj-on f B*
using *inj-on-span-independent-image*[*of B*] *inj-on-subset*[*OF - m1.span-superset*, *of f B*] **by** *auto*

lemma *subspace-linear-preimage*: *m2.subspace S* $\Longrightarrow$ *m1.subspace* $\{x.\ f\ x \in S\}$
by (*simp add*: *add scale m1.subspace-def m2.subspace-def*)

lemma *spans-image*: $V \subseteq$ *m1.span B* $\Longrightarrow$ f ‘ $V \subseteq$ *m2.span* (f ‘ B)
by (*metis image-mono span-image*)

Relation between bases and injectivity/surjectivity of map.

lemma *spanning-surjective-image*:
  assumes *us*: *UNIV* ⊆ *m1.span S*
    and *sf*: *surj f*
  shows *UNIV* ⊆ *m2.span* (*f* ' *S*)
proof −
  have *UNIV* ⊆ *f* ' *UNIV*
    using *sf* **by** (*auto simp add*: *surj-def*)
  also have ... ⊆ *m2.span* (*f* ' *S*)
    using *spans-image*[*OF us*] **.**
  finally show *?thesis* **.**
qed

lemmas *independent-inj-on-image* = *independent-injective-image*

lemma *independent-inj-image*:
  m1.independent S ⟹ *inj f* ⟹ *m2.independent* (*f* ' *S*)
  using *independent-inj-on-image*[*of S*] **by** (*auto simp*: *subset-inj-on*)

end

lemma *module-hom-iff*:
  module-hom s1 s2 f ⟷
    module s1 ∧ *module s2* ∧
    (∀ *x y*. *f* (*x* + *y*) = *f x* + *f y*) ∧ (∀ *c x*. *f* (*s1 c x*) = *s2 c* (*f x*))
  by (*simp add*: *module-hom-def module-hom-axioms-def*)

locale *module-pair* = *m1*: *module s1* + *m2*: *module s2*
  for *s1* :: *'a* :: *comm-ring-1* ⇒ *'b* ⇒ *'b* :: *ab-group-add*
  and *s2* :: *'a* :: *comm-ring-1* ⇒ *'c* ⇒ *'c* :: *ab-group-add*
begin

lemma *module-hom-zero*: *module-hom s1 s2* (λ*x*. *0*)
  by (*simp add*: *module-hom-iff m1.module-axioms m2.module-axioms*)

lemma *module-hom-add*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2 g* ⟹ *module-hom s1 s2* (λ*x*. *f x* + *g x*)
  by (*simp add*: *module-hom-iff module.scale-right-distrib*)

lemma *module-hom-sub*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2 g* ⟹ *module-hom s1 s2* (λ*x*. *f x* − *g x*)
  by (*simp add*: *module-hom-iff module.scale-right-diff-distrib*)

lemma *module-hom-neg*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2* (λ*x*. − *f x*)
  by (*simp add*: *module-hom-iff module.scale-minus-right*)

lemma *module-hom-scale*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2* (λ*x*. *s2 c* (*f x*))
  by (*simp add*: *module-hom-iff module.scale-scale module.scale-right-distrib ac-simps*)

lemma *module-hom-compose-scale*:
module-hom s1 s2 (λx. *s2* (*f x*) (*c*))
if *module-hom s1* (∗) *f*
proof −
interpret *mh*: *module-hom s1* (∗) *f* **by** *fact*
show *?thesis*
by *unfold-locales* (*simp-all add*: *mh.add mh.scale m2.scale-left-distrib*)
qed

lemma *bij-module-hom-imp-inv-module-hom*: *module-hom scale1 scale2 f* ⟹ *bij f* ⟹
module-hom scale2 scale1 (*inv f*)
by (*auto simp*: *module-hom-iff bij-is-surj bij-is-inj surj-f-inv-f*
intro!: *Hilbert-Choice.inv-f-eq*)

lemma *module-hom-sum*: ($\bigwedge i$. $i \in I$ ⟹ *module-hom s1 s2* (*f i*)) ⟹ ($I = \{\}$ ⟹ *module s1* ∧ *module s2*) ⟹ *module-hom s1 s2* (λx. $\sum i \in I$. *f i x*)
apply (*induction I rule*: *infinite-finite-induct*)
apply (*auto intro*!: *module-hom-zero module-hom-add*)
using *m1.module-axioms m2.module-axioms* **by** *blast*

lemma *module-hom-eq-on-span*: *f x* = *g x*
if *module-hom s1 s2 f module-hom s1 s2 g*
and ($\bigwedge x$. $x \in B$ ⟹ *f x* = *g x*) $x \in$ *m1.span B*
proof −
interpret *module-hom s1 s2* λx. *f x* − *g x*
by (*rule module-hom-sub that*)+
from *eq-0-on-span*[*OF* - *that*(*4*)] *that*(*3*) **show** *?thesis* **by** *auto*
qed

end

context *module* **begin**

lemma *module-hom-scale-self*[*simp*]:
module-hom scale scale (λx. *scale c x*)
using *module-axioms module-hom-iff scale-left-commute scale-right-distrib* **by** *blast*

lemma *module-hom-scale-left*[*simp*]:
module-hom (∗) *scale* (λr. *scale r x*)
by *unfold-locales* (*auto simp*: *algebra-simps*)

lemma *module-hom-id*: *module-hom scale scale id*
by (*simp add*: *module-hom-iff module-axioms*)

lemma *module-hom-ident*: *module-hom scale scale* (λx. *x*)
by (*simp add*: *module-hom-iff module-axioms*)

lemma *module-hom-uminus*: *module-hom scale scale uminus*
  by (*simp add*: *module-hom-iff module-axioms*)

end

lemma *module-hom-compose*: *module-hom s1 s2 f* $\Longrightarrow$ *module-hom s2 s3 g* $\Longrightarrow$ *module-hom s1 s3* (*g o f*)
  by (*auto simp*: *module-hom-iff*)

end

106 Vector Spaces

theory *Vector-Spaces*
  imports *Modules*
begin

lemma *isomorphism-expand*:
  $f \circ g = id \wedge g \circ f = id \longleftrightarrow (\forall x.\ f\ (g\ x) = x) \wedge (\forall x.\ g\ (f\ x) = x)$
  by (*simp add*: *fun-eq-iff o-def id-def*)

lemma *left-right-inverse-eq*:
  assumes *fg*: $f \circ g = id$
    and *gh*: $g \circ h = id$
  shows $f = h$
proof −
  have $f = f \circ (g \circ h)$
    unfolding *gh* **by** *simp*
  also have $\ldots = (f \circ g) \circ h$
    by (*simp add*: *o-assoc*)
  finally show $f = h$
    unfolding *fg* **by** *simp*
qed

lemma *ordLeq3-finite-infinite*:
  assumes *A*: *finite A* **and** *B*: *infinite B* **shows** *ordLeq3* (*card-of A*) (*card-of B*)
proof −
  have ‹*ordLeq3* (*card-of A*) (*card-of B*) ∨ *ordLeq3* (*card-of B*) (*card-of A*)›
    by (*intro ordLeq-total card-of-Well-order*)
  moreover have ¬ *ordLeq3* (*card-of B*) (*card-of A*)
    using *B A card-of-ordLeq-finite*[*of B A*] **by** *auto*
  ultimately show *?thesis* **by** *auto*
qed

locale *vector-space* =
  fixes *scale* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ‹*∗s*› *75*)
  assumes *vector-space-assms*:— re-stating the assumptions of *module* instead of extending *module* allows us to rewrite in the sublocale.

a ∗s (x + y) = a ∗s x + a ∗s y
(a + b) ∗s x = a ∗s x + b ∗s x
a ∗s (b ∗s x) = (a ∗ b) ∗s x
1 ∗s x = x

lemma *module-iff-vector-space*: *module s ⟷ vector-space s*
unfolding *module-def vector-space-def* **..**

locale *linear = vs1*: *vector-space s1 + vs2*: *vector-space s2 + module-hom s1 s2 f*
for *s1* :: *'a::field ⇒ 'b::ab-group-add ⇒ 'b* (**infixr** ‹∗a› *75*)
and *s2* :: *'a::field ⇒ 'c::ab-group-add ⇒ 'c* (**infixr** ‹∗b› *75*)
and *f* :: *'b ⇒ 'c*

lemma *module-hom-iff-linear*: *module-hom s1 s2 f ⟷ linear s1 s2 f*
unfolding *module-hom-def linear-def module-iff-vector-space* **by** *auto*
lemmas *module-hom-eq-linear = module-hom-iff-linear[abs-def, THEN meta-eq-to-obj-eq]*
lemmas *linear-iff-module-hom = module-hom-iff-linear[symmetric]*
lemmas *linear-module-homI = module-hom-iff-linear[THEN iffD1]*
and *module-hom-linearI = module-hom-iff-linear[THEN iffD2]*

context *vector-space* **begin**

sublocale *module scale* **rewrites** *module-hom = linear*
by *unfold-locales (fact vector-space-assms module-hom-eq-linear)+*

lemmas— from *module*
linear-id = module-hom-id
and *linear-ident = module-hom-ident*
and *linear-scale-self = module-hom-scale-self*
and *linear-scale-left = module-hom-scale-left*
and *linear-uminus = module-hom-uminus*

lemma *linear-imp-scale*:
fixes *D::'a ⇒ 'b*
assumes *linear (∗) scale D*
obtains *d* **where** *D = (λx. scale x d)*
proof −
interpret *linear (∗) scale D* **by** *fact*
show *?thesis*
by *(metis mult.commute mult.left-neutral scale that)*
qed

lemma *scale-eq-0-iff [simp]*: *scale a x = 0 ⟷ a = 0 ∨ x = 0*
by *(metis scale-left-commute right-inverse scale-one scale-scale scale-zero-left)*

lemma *scale-left-imp-eq*:
assumes *nonzero*: *a ≠ 0*
and *scale*: *scale a x = scale a y*
shows *x = y*

proof –
  from *scale* **have** *scale a* $(x - y) = 0$
    by (*simp add*: *scale-right-diff-distrib*)
  with *nonzero* **have** $x - y = 0$ **by** *simp*
  then show $x = y$ **by** (*simp only*: *right-minus-eq*)
qed

lemma *scale-right-imp-eq*:
  assumes *nonzero*: $x \neq 0$
    and *scale*: *scale a x* = *scale b x*
  shows $a = b$
proof –
  from *scale* **have** *scale* $(a - b)$ $x = 0$
    by (*simp add*: *scale-left-diff-distrib*)
  with *nonzero* **have** $a - b = 0$ **by** *simp*
  then show $a = b$ **by** (*simp only*: *right-minus-eq*)
qed

lemma *scale-cancel-left* [*simp*]: *scale a x* = *scale a y* $\longleftrightarrow x = y \lor a = 0$
  by (*auto intro*: *scale-left-imp-eq*)

lemma *scale-cancel-right* [*simp*]: *scale a x* = *scale b x* $\longleftrightarrow a = b \lor x = 0$
  by (*auto intro*: *scale-right-imp-eq*)

lemma *injective-scale*: $c \neq 0 \Longrightarrow$ *inj* ($\lambda x.$ *scale c x*)
  by (*simp add*: *inj-on-def*)

lemma *dependent-def*: *dependent P* $\longleftrightarrow$ ($\exists a \in P.$ $a \in$ *span* ($P - \{a\}$))
  unfolding *dependent-explicit*
proof *safe*
  fix *a* **assume** *aP*: $a \in P$ **and** $a \in$ *span* ($P - \{a\}$)
  then obtain *a S u*
    where *aP*: $a \in P$ **and** *fS*: *finite S* **and** *SP*: $S \subseteq P$ $a \notin S$ **and** *ua*: ($\sum v{\in}S.$ *u v* $*s$ *v*) = *a*
    unfolding *span-explicit* **by** *blast*
  let *?S* = *insert a S*
  let *?u* = $\lambda y.$ *if* $y = a$ *then* $- 1$ *else u y*
  from *fS SP* **have** ($\sum v{\in}?S.$ *?u v* $*s$ *v*) = *0*
   by (*simp add*: *if-distrib*[*of* $\lambda r.$ *r* $*s$ *a* **for** *a*] *sum.If-cases field-simps Diff-eq*[*symmetric*] *ua*)
  moreover have *finite ?S* *?S* $\subseteq$ *P* $a \in$ *?S* *?u a* $\neq 0$
    using *fS SP aP* **by** *auto*
  ultimately show $\exists t\ u.$ *finite t* $\land$ $t \subseteq P$ $\land$ ($\sum v{\in}t.$ *u v* $*s$ *v*) = *0* $\land$ ($\exists v{\in}t.$ *u v* $\neq 0$) **by** *fast*
next
  fix *S u v*
  assume *fS*: *finite S* **and** *SP*: $S \subseteq P$ **and** *vS*: $v \in S$
   and *uv*: *u v* $\neq 0$ **and** *u*: ($\sum v{\in}S.$ *u v* $*s$ *v*) = *0*
  let *?a* = *v*

let *?S* = *S* − {*v*}
let *?u* = λ*i*. (− *u i*) / *u v*
have *th0*: *?a* ∈ *P finite ?S ?S* ⊆ *P*
using *fS SP vS* **by** *auto*
have (∑ *v*∈*?S*. *?u v* ∗*s v*) = (∑ *v*∈*S*. (− (*inverse* (*u ?a*))) ∗*s* (*u v* ∗*s v*)) − *?u v* ∗*s v*
using *fS vS uv* **by** (*simp add*: *sum-diff1 field-simps*)
also have ... = *?a*
unfolding *scale-sum-right*[*symmetric*] *u* **using** *uv* **by** *simp*
finally have (∑ *v*∈*?S*. *?u v* ∗*s v*) = *?a* **.**
with *th0* **show** ∃ *a* ∈ *P*. *a* ∈ *span* (*P* − {*a*})
unfolding *span-explicit* **by** (*auto intro*!: *bexI*[**where** *x*=*?a*] *exI*[**where** *x*=*?S*] *exI*[**where** *x*=*?u*])
qed

lemma *dependent-single*[*simp*]: *dependent* {*x*} ⟷ *x* = *0*
unfolding *dependent-def* **by** *auto*

lemma *in-span-insert*:
assumes *a*: *a* ∈ *span* (*insert b S*)
and *na*: *a* ∉ *span S*
shows *b* ∈ *span* (*insert a S*)
proof −
from *span-breakdown*[*of b insert b S a*, *OF insertI1 a*]
obtain *k* **where** *k*: *a* − *k* ∗*s b* ∈ *span* (*S* − {*b*}) **by** *auto*
have *k* ≠ *0*
proof
assume *k* = *0*
with *k span-mono*[*of S* − {*b*} *S*] **have** *a* ∈ *span S* **by** *auto*
with *na* **show** *False* **by** *blast*
qed
then have *eq*: *b* = (*1*/*k*) ∗*s a* − (*1*/*k*) ∗*s* (*a* − *k* ∗*s b*)
by (*simp add*: *algebra-simps*)

from *k* **have** (*1*/*k*) ∗*s* (*a* − *k* ∗*s b*) ∈ *span* (*S* − {*b*})
by (*rule span-scale*)
also have ... ⊆ *span* (*insert a S*)
by (*rule span-mono*) *auto*
finally show *?thesis*
using *k* **by** (*subst eq*) (*blast intro*: *span-diff span-scale span-base*)
qed

lemma *dependent-insertD*: **assumes** *a*: *a* ∉ *span S* **and** *S*: *dependent* (*insert a S*)
shows *dependent S*
proof −
have *a* ∉ *S* **using** *a* **by** (*auto dest*: *span-base*)
obtain *b* **where** *b*: *b* = *a* ∨ *b* ∈ *S b* ∈ *span* (*insert a S* − {*b*})
using *S* **unfolding** *dependent-def* **by** *blast*
have *b* ≠ *a b* ∈ *S*

using *b* ‹*a* ∉ *S*› *a* **by** *auto*
with *b* **have** ∗: *b* ∈ *span* (*insert a* (*S* − {*b*}))
by (*auto simp*: *insert-Diff-if*)
show *dependent S*
proof *cases*
assume *b* ∈ *span* (*S* − {*b*}) **with** ‹*b* ∈ *S*› **show** *?thesis*
by (*auto simp add*: *dependent-def*)
next
assume *b* ∉ *span* (*S* − {*b*})
with ∗ **have** *a* ∈ *span* (*insert b* (*S* − {*b*})) **by** (*rule in-span-insert*)
with *a* **show** *?thesis*
using ‹*b* ∈ *S*› **by** (*auto simp*: *insert-absorb*)
qed
qed

lemma *independent-insertI*: *a* ∉ *span S* ⟹ *independent S* ⟹ *independent* (*insert a S*)
by (*auto dest*: *dependent-insertD*)

lemma *independent-insert*:
independent (*insert a S*) ⟷ (*if a* ∈ *S then independent S else independent S* ∧ *a* ∉ *span S*)
proof −
have *a* ∉ *S* ⟹ *a* ∈ *span S* ⟹ *dependent* (*insert a S*)
by (*auto simp*: *dependent-def*)
then show *?thesis*
by (*auto intro*: *dependent-mono simp*: *independent-insertI*)
qed

lemma *maximal-independent-subset-extend*:
assumes *S* ⊆ *V independent S*
obtains *B* **where** *S* ⊆ *B B* ⊆ *V independent B V* ⊆ *span B*
proof −
let *?C* = {*B*. *S* ⊆ *B* ∧ *independent B* ∧ *B* ⊆ *V*}
have ∃ *M*∈*?C*. ∀ *X*∈*?C*. *M* ⊆ *X* ⟶ *X* = *M*
proof (*rule subset-Zorn*)
fix *C* :: ′*b set set* **assume** *subset.chain ?C C*
then have *C*: ⋀*c*. *c* ∈ *C* ⟹ *c* ⊆ *V* ⋀*c*. *c* ∈ *C* ⟹ *S* ⊆ *c* ⋀*c*. *c* ∈ *C* ⟹ *independent c*
⋀*c d*. *c* ∈ *C* ⟹ *d* ∈ *C* ⟹ *c* ⊆ *d* ∨ *d* ⊆ *c*
unfolding *subset.chain-def* **by** *blast+*

show ∃ *U*∈*?C*. ∀ *X*∈*C*. *X* ⊆ *U*
proof *cases*
assume *C* = {} **with** *assms* **show** *?thesis*
by (*auto intro*!: *exI*[*of* - *S*])
next
assume *C* ≠ {}
with *C*(*2*) **have** *S* ⊆ ⋃ *C*

by *auto*
moreover have *independent* $(\bigcup C)$
by (*intro independent-Union-directed C*)
moreover have $\bigcup C \subseteq V$
using *C* **by** *auto*
ultimately show *?thesis*
by *auto*
qed
qed
then obtain *B* **where** *B*: *independent B* $B \subseteq V$ $S \subseteq B$
and *max*: $\bigwedge S.$ *independent S* $\Longrightarrow S \subseteq V \Longrightarrow B \subseteq S \Longrightarrow S = B$
by *auto*
moreover
{ assume $\neg$ $V \subseteq$ *span B*
then obtain *v* **where** $v \in V$ $v \notin$ *span B*
by *auto*
with *B* **have** *independent* (*insert v B*) **by** (*auto intro*: *dependent-insertD*)
from *max*[*OF this*] ‹$v \in V$› ‹$B \subseteq V$›
have $v \in B$
by *auto*
with ‹$v \notin$ *span B*› **have** *False*
by (*auto intro*: *span-base*) **}**
ultimately show *?thesis*
by (*meson that*)
qed

lemma *maximal-independent-subset*:
obtains *B* **where** $B \subseteq V$ *independent B* $V \subseteq$ *span B*
by (*metis maximal-independent-subset-extend*[*of* {}] *empty-subsetI independent-empty*)

Extends a basis from B to a basis of the entire space.

definition *extend-basis* :: *'b set* $\Rightarrow$ *'b set*
where *extend-basis B* = (*SOME B'*. $B \subseteq B'$ $\wedge$ *independent B'* $\wedge$ *span B'* = *UNIV*)

lemma
assumes *B*: *independent B*
shows *extend-basis-superset*: $B \subseteq$ *extend-basis B*
and *independent-extend-basis*: *independent* (*extend-basis B*)
and *span-extend-basis*[*simp*]: *span* (*extend-basis B*) = *UNIV*
proof −
define *p* **where** *p B'* $\equiv$ $B \subseteq B'$ $\wedge$ *independent B'* $\wedge$ *span B'* = *UNIV* **for** *B'*
obtain *B'* **where** *p B'*
using *maximal-independent-subset-extend*[*OF subset-UNIV B*]
by (*metis top.extremum-uniqueI p-def*)
then have *p* (*extend-basis B*)
unfolding *extend-basis-def p-def*[*symmetric*] **by** (*rule someI*)
then show $B \subseteq$ *extend-basis B independent* (*extend-basis B*) *span* (*extend-basis B*) = *UNIV*

```
    by (auto simp: p-def)
qed

lemma in-span-delete:
  assumes a: a ∈ span S and na: a ∉ span (S − {b})
  shows b ∈ span (insert a (S − {b}))
  by (metis Diff-empty Diff-insert0 a in-span-insert insert-Diff na)

lemma span-redundant: x ∈ span S ⟹ span (insert x S) = span S
  unfolding span-def by (rule hull-redundant)

lemma span-trans: x ∈ span S ⟹ y ∈ span (insert x S) ⟹ y ∈ span S
  by (simp only: span-redundant)

lemma span-insert-0[simp]: span (insert 0 S) = span S
  by (metis span-zero span-redundant)

lemma span-delete-0 [simp]: span(S − {0}) = span S
proof
  show span (S − {0}) ⊆ span S
    by (blast intro!: span-mono)
next
  have span S ⊆ span(insert 0 (S − {0}))
    by (blast intro!: span-mono)
  also have ... ⊆ span(S − {0})
    using span-insert-0 by blast
  finally show span S ⊆ span (S − {0}) .
qed

lemma span-image-scale:
  assumes finite S and nz: ⋀x. x ∈ S ⟹ c x ≠ 0
    shows span ((λx. c x *s x) ‘ S) = span S
using assms
proof (induction S arbitrary: c)
  case (empty c) show ?case by simp
next
  case (insert x F c)
  show ?case
  proof (intro set-eqI iffI)
    fix y
      assume y ∈ span ((λx. c x *s x) ‘ insert x F)
      then show y ∈ span (insert x F)
        using insert by (force simp: span-breakdown-eq)
  next
    fix y
      assume y ∈ span (insert x F)
      then show y ∈ span ((λx. c x *s x) ‘ insert x F)
        using insert
        apply (clarsimp simp: span-breakdown-eq)
```

```
      apply (rule-tac x=k / c x in exI)
      by simp
  qed
qed

lemma exchange-lemma:
  assumes f: finite T
    and i: independent S
    and sp: S ⊆ span T
  shows ∃ t'. card t' = card T ∧ finite t' ∧ S ⊆ t' ∧ t' ⊆ S ∪ T ∧ S ⊆ span t'
  using f i sp
proof (induct card (T − S) arbitrary: S T rule: less-induct)
  case less
  note ft = ‹finite T› and S = ‹independent S› and sp = ‹S ⊆ span T›
  let ?P = λt'. card t' = card T ∧ finite t' ∧ S ⊆ t' ∧ t' ⊆ S ∪ T ∧ S ⊆ span t'
  show ?case
  proof (cases S ⊆ T ∨ T ⊆ S)
    case True
    then show ?thesis
    proof
      assume S ⊆ T then show ?thesis
        by (metis ft Un-commute sp sup-ge1)
    next
      assume T ⊆ S then show ?thesis
        by (metis Un-absorb sp spanning-subset-independent[OF - S sp] ft)
    qed
  next
    case False
    then have st: ¬ S ⊆ T ¬ T ⊆ S
      by auto
    from st(2) obtain b where b: b ∈ T b ∉ S
      by blast
    from b have T − {b} − S ⊂ T − S
      by blast
    then have cardlt: card (T − {b} − S) < card (T − S)
      using ft by (auto intro: psubset-card-mono)
    from b ft have ct0: card T ≠ 0
      by auto
    show ?thesis
    proof (cases S ⊆ span (T − {b}))
      case True
      from ft have ftb: finite (T − {b})
        by auto
      from less(1)[OF cardlt ftb S True]
      obtain U where U: card U = card (T − {b}) S ⊆ U U ⊆ S ∪ (T − {b})
S ⊆ span U
        and fu: finite U by blast
      let ?w = insert b U
      have th0: S ⊆ insert b U
```

using *U* **by** *blast*
have *th1*: *insert b U* ⊆ *S* ∪ *T*
using *U b* **by** *blast*
have *bu*: *b* ∉ *U*
using *b U* **by** *blast*
from *U*(*1*) *ft b* **have** *card U* = (*card T* − *1*)
by *auto*
then have *th2*: *card* (*insert b U*) = *card T*
using *card-insert-disjoint*[*OF fu bu*] *ct0* **by** *auto*
from *U*(*4*) **have** *S* ⊆ *span U* **.**
also have ... ⊆ *span* (*insert b U*)
by (*rule span-mono*) *blast*
finally have *th3*: *S* ⊆ *span* (*insert b U*) **.**
from *th0 th1 th2 th3 fu* **have** *th*: *?P ?w*
by *blast*
from *th* **show** *?thesis* **by** *blast*
next
case *False*
then obtain *a* **where** *a*: *a* ∈ *S a* ∉ *span* (*T* − {*b*})
by *blast*
have *ab*: *a* ≠ *b*
using *a b* **by** *blast*
have *at*: *a* ∉ *T*
using *a ab span-base*[*of a T*− {*b*}] **by** *auto*
have *mlt*: *card* ((*insert a* (*T* − {*b*})) − *S*) < *card* (*T* − *S*)
using *cardlt ft a b* **by** *auto*
have *ft'*: *finite* (*insert a* (*T* − {*b*}))
using *ft* **by** *auto*
have *sp'*: *S* ⊆ *span* (*insert a* (*T* − {*b*}))
proof
fix *x*
assume *xs*: *x* ∈ *S*
have *T*: *T* ⊆ *insert b* (*insert a* (*T* − {*b*}))
using *b* **by** *auto*
have *bs*: *b* ∈ *span* (*insert a* (*T* − {*b*}))
by (*rule in-span-delete*) (*use a sp* **in** *auto*)
from *xs sp* **have** *x* ∈ *span T*
by *blast*
with *span-mono*[*OF T*] **have** *x*: *x* ∈ *span* (*insert b* (*insert a* (*T* − {*b*}))) **..**
from *span-trans*[*OF bs x*] **show** *x* ∈ *span* (*insert a* (*T* − {*b*})) **.**
qed
from *less*(*1*)[*OF mlt ft' S sp'*] **obtain** *U* **where** *U*:
card U = *card* (*insert a* (*T* − {*b*}))
finite U S ⊆ *U U* ⊆ *S* ∪ *insert a* (*T* − {*b*})
S ⊆ *span U* **by** *blast*
from *U a b ft at ct0* **have** *?P U*
by *auto*
then show *?thesis* **by** *blast*
qed

qed
qed

lemma *independent-span-bound*:
assumes *f*: *finite T*
and *i*: *independent S*
and *sp*: $S \subseteq span\ T$
shows $finite\ S \wedge card\ S \leq card\ T$
by (*metis exchange-lemma*[*OF f i sp*] *finite-subset card-mono*)

lemma *independent-explicit-finite-subsets*:
$independent\ A \longleftrightarrow (\forall S \subseteq A.\ finite\ S \longrightarrow (\forall u.\ (\sum v \in S.\ u\ v *s\ v) = 0 \longrightarrow (\forall v \in S.\ u\ v = 0)))$
unfolding *dependent-explicit* [*of A*] **by** (*simp add*: *disj-not2*)

lemma *independent-if-scalars-zero*:
assumes *fin-A*: *finite A*
and *sum*: $\bigwedge f\ x.\ (\sum x \in A.\ f\ x *s\ x) = 0 \Longrightarrow x \in A \Longrightarrow f\ x = 0$
shows *independent A*
proof (*unfold independent-explicit-finite-subsets*, *clarify*)
fix *S v* **and** $u :: 'b \Rightarrow 'a$
assume *S*: $S \subseteq A$ **and** *v*: $v \in S$
let $?g = \lambda x.\ if\ x \in S\ then\ u\ x\ else\ 0$
have $(\sum v \in A.\ ?g\ v *s\ v) = (\sum v \in S.\ u\ v *s\ v)$
using *S fin-A* **by** (*auto intro*!: *sum.mono-neutral-cong-right*)
also assume $(\sum v \in S.\ u\ v *s\ v) = 0$
finally have $?g\ v = 0$ **using** *v S sum* **by** *force*
thus $u\ v = 0$ **unfolding** *if-P*[*OF v*] **.**
qed

lemma *bij-if-span-eq-span-bases*:
assumes *B*: *independent B* **and** *C*: *independent C*
and *eq*: $span\ B = span\ C$
shows $\exists f.\ bij\text{-}betw\ f\ B\ C$
proof *cases*
assume $finite\ B \vee finite\ C$
then have $finite\ B \wedge finite\ C \wedge card\ C = card\ B$
using *independent-span-bound*[*of B C*] *independent-span-bound*[*of C B*] *assms span-superset*[*of B*] *span-superset*[*of C*]
by *auto*
then show *?thesis*
by (*auto intro*!: *finite-same-card-bij*)
next
assume $\neg (finite\ B \vee finite\ C)$
then have *infinite B infinite C* **by** *auto*
{ fix *B C* **assume** *B*: *independent B* **and** *C*: *independent C* **and** *infinite B infinite C* **and** *eq*: $span\ B = span\ C$
let $?R = representation\ B$ **and** $?R' = representation\ C$ **let** $?U = \lambda c.\ \{v.\ ?R\ c\ v \neq 0\}$

have *in-span-C* [*simp*, *intro*]: ‹$b \in B \Longrightarrow b \in span\ C$› **for** *b* **unfolding** *eq*[*symmetric*] **by** (*rule span-base*)
have *in-span-B* [*simp*, *intro*]: ‹$c \in C \Longrightarrow c \in span\ B$› **for** *c* **unfolding** *eq* **by** (*rule span-base*)
have ‹$B \subseteq (\bigcup c \in C.\ ?U\ c)$›
proof
fix *b* **assume** ‹$b \in B$›
have ‹$b \in span\ C$›
using ‹$b \in B$› **unfolding** *eq*[*symmetric*] **by** (*rule span-base*)
have ‹$(\sum v \mid ?R'\ b\ v \neq 0.\ \sum w \mid ?R\ v\ w \neq 0.\ (?R'\ b\ v * ?R\ v\ w) *s\ w) = (\sum v \mid ?R'\ b\ v \neq 0.\ ?R'\ b\ v *s\ (\sum w \mid ?R\ v\ w \neq 0.\ ?R\ v\ w *s\ w))$›
by (*simp add*: *scale-sum-right*)
also have ‹$\ldots = (\sum v \mid ?R'\ b\ v \neq 0.\ ?R'\ b\ v *s\ v)$›
by (*auto simp*: *sum-nonzero-representation-eq B eq span-base representation-ne-zero*)
also have ‹$\ldots = b$›
by (*rule sum-nonzero-representation-eq*[*OF C* ‹$b \in span\ C$›])
finally have $?R\ b\ b = ?R\ (\sum v \mid ?R'\ b\ v \neq 0.\ \sum w \mid ?R\ v\ w \neq 0.\ (?R'\ b\ v * ?R\ v\ w) *s\ w)\ b$
by *simp*
also have ... $= (\sum i \in \{v.\ ?R'\ b\ v \neq 0\}.\ ?R\ (\sum w \mid ?R\ i\ w \neq 0.\ (?R'\ b\ i * ?R\ i\ w) *s\ w)\ b)$
by (*subst representation-sum*[*OF B*]) (*auto intro*: *span-sum span-scale span-base representation-ne-zero*)
also have ... $= (\sum i \in \{v.\ ?R'\ b\ v \neq 0\}.\ \sum j \in \{w.\ ?R\ i\ w \neq 0\}.\ ?R\ ((?R'\ b\ i * ?R\ i\ j) *s\ j)\ b)$
by (*subst representation-sum*[*OF B*]) (*auto simp add*: *span-sum span-scale span-base representation-ne-zero*)
also have ‹$\ldots = (\sum v \mid ?R'\ b\ v \neq 0.\ \sum w \mid ?R\ v\ w \neq 0.\ ?R'\ b\ v * ?R\ v\ w * ?R\ w\ b)$›
using *B* ‹$b \in B$› **by** (*simp add*: *representation-scale*[*OF B*] *span-base representation-ne-zero*)
finally have $(\sum v \mid ?R'\ b\ v \neq 0.\ \sum w \mid ?R\ v\ w \neq 0.\ ?R'\ b\ v * ?R\ v\ w * ?R\ w\ b) \neq 0$
using *representation-basis*[*OF B* ‹$b \in B$›] **by** *auto*
then obtain *v w* **where** *bv*: $?R'\ b\ v \neq 0$ **and** *vw*: $?R\ v\ w \neq 0$ **and** $?R'\ b\ v * ?R\ v\ w * ?R\ w\ b \neq 0$
by (*blast elim*: *sum.not-neutral-contains-not-neutral*)
with *representation-basis*[*OF B*, *of w*] *vw*[*THEN representation-ne-zero*]
have ‹$?R'\ b\ v \neq 0$› ‹$?R\ v\ b \neq 0$› **by** (*auto split*: *if-splits*)
then show ‹$b \in (\bigcup c \in C.\ ?U\ c)$›
by (*auto dest*: *representation-ne-zero*)
qed
then have *B-eq*: ‹$B = (\bigcup c \in C.\ ?U\ c)$›
by (*auto intro*: *span-base representation-ne-zero eq*)
have *ordLeq3* (*card-of B*) (*card-of C*)
proof (*subst B-eq*, *rule card-of-UNION-ordLeq-infinite*[*OF* ‹*infinite C*›])
show *ordLeq3* (*card-of C*) (*card-of C*)
by (*intro ordLeq-refl card-of-Card-order*)

show ∀ c∈C. *ordLeq3* (*card-of* {v. ?R c v ≠ 0}) (*card-of* C)
by (*intro ballI ordLeq3-finite-infinite* ‹*infinite C*› *finite-representation*)
qed }
from *this*[*of B C*] *this*[*of C B*] *B C eq* ‹*infinite C*› ‹*infinite B*›
show *?thesis* **by** (*auto simp add*: *ordIso-iff-ordLeq card-of-ordIso*)
qed

definition *dim* :: 'b *set* ⇒ *nat*
where *dim V* = (*if* ∃ b. *independent b* ∧ *span b* = *span V then*
card (*SOME b. independent b* ∧ *span b* = *span V*) *else 0*)

lemma *dim-eq-card*:
assumes *BV*: *span B* = *span V* **and** *B*: *independent B*
shows *dim V* = *card B*
proof −
define *p* **where** *p b* ≡ *independent b* ∧ *span b* = *span V* **for** *b*
have *p* (*SOME B. p B*)
using *assms* **by** (*intro someI*[*of p B*]) (*auto simp*: *p-def*)
then have ∃ f. *bij-betw f B* (*SOME B. p B*)
by (*subst* (*asm*) *p-def*, *intro bij-if-span-eq-span-bases*[*OF B*]) (*simp-all add*: *BV*)
then have *card B* = *card* (*SOME B. p B*)
by (*auto intro*: *bij-betw-same-card*)
then show *?thesis*
using *BV B*
by (*auto simp add*: *dim-def p-def*)
qed

lemma *basis-card-eq-dim*: *B* ⊆ *V* ⟹ *V* ⊆ *span B* ⟹ *independent B* ⟹ *card B* = *dim V*
using *dim-eq-card*[*of B V*] *span-mono*[*of B V*] *span-minimal*[*OF - subspace-span*, *of V B*] **by** *auto*

lemma *basis-exists*:
obtains *B* **where** *B* ⊆ *V independent B V* ⊆ *span B card B* = *dim V*
by (*meson basis-card-eq-dim empty-subsetI independent-empty maximal-independent-subset-extend*)

lemma *dim-eq-card-independent*: *independent B* ⟹ *dim B* = *card B*
by (*rule dim-eq-card*[*OF refl*])

lemma *dim-span*[*simp*]: *dim* (*span S*) = *dim S*
by (*auto simp add*: *dim-def span-span*)

lemma *dim-span-eq-card-independent*: *independent B* ⟹ *dim* (*span B*) = *card B*
by (*simp add*: *dim-eq-card*)

lemma *dim-le-card*: **assumes** *V* ⊆ *span W finite W* **shows** *dim V* ≤ *card W*
proof −
obtain *A* **where** *independent A A* ⊆ *V V* ⊆ *span A*

using *maximal-independent-subset*[*of V*] **by** *force*
with *assms independent-span-bound*[*of W A*] *basis-card-eq-dim*[*of A V*]
show *?thesis* **by** *auto*
qed

lemma *span-eq-dim*: *span S* = *span T* ⟹ *dim S* = *dim T*
by (*metis dim-span*)

corollary *dim-le-card'*:
finite s ⟹ *dim s* ≤ *card s*
by (*metis basis-exists card-mono*)

lemma *span-card-ge-dim*:
B ⊆ *V* ⟹ *V* ⊆ *span B* ⟹ *finite B* ⟹ *dim V* ≤ *card B*
by (*simp add*: *dim-le-card*)

lemma *dim-unique*:
B ⊆ *V* ⟹ *V* ⊆ *span B* ⟹ *independent B* ⟹ *card B* = *n* ⟹ *dim V* = *n*
by (*metis basis-card-eq-dim*)

lemma *subspace-sums*: ⟦*subspace S*; *subspace T*⟧ ⟹ *subspace* {*x* + *y*|*x y*. *x* ∈ *S* ∧ *y* ∈ *T*}
apply (*simp add*: *subspace-def*)
apply (*intro conjI impI allI*; *clarsimp simp*: *algebra-simps*)
using *add.left-neutral* **apply** *blast*
apply (*metis add.assoc*)
using *scale-right-distrib* **by** *blast*

end

lemma *linear-iff*: *linear s1 s2 f* ⟷
(*vector-space s1* ∧ *vector-space s2* ∧ (∀ *x y*. *f* (*x* + *y*) = *f x* + *f y*) ∧ (∀ *c x*. *f* (*s1 c x*) = *s2 c* (*f x*)))
unfolding *linear-def module-hom-iff vector-space-def module-def* **by** *auto*

context begin
qualified lemma *linear-compose*: *linear s1 s2 f* ⟹ *linear s2 s3 g* ⟹ *linear s1 s3* (*g o f*)
unfolding *module-hom-iff-linear*[*symmetric*]
by (*rule module-hom-compose*)
end

locale *vector-space-pair* = *vs1*: *vector-space s1* + *vs2*: *vector-space s2*
for *s1* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ‹*∗a*› *75*)
and *s2* :: *'a*::*field* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ‹*∗b*› *75*)
begin

context fixes *f* **assumes** *linear s1 s2 f* **begin**
interpretation *linear s1 s2 f* **by** *fact*

lemmas— from locale *module-hom*
linear-0 = *zero*
and *linear-add* = *add*
and *linear-scale* = *scale*
and *linear-neg* = *neg*
and *linear-diff* = *diff*
and *linear-sum* = *sum*
and *linear-inj-on-iff-eq-0* = *inj-on-iff-eq-0*
and *linear-inj-iff-eq-0* = *inj-iff-eq-0*
and *linear-subspace-image* = *subspace-image*
and *linear-subspace-vimage* = *subspace-vimage*
and *linear-subspace-kernel* = *subspace-kernel*
and *linear-span-image* = *span-image*
and *linear-dependent-inj-imageD* = *dependent-inj-imageD*
and *linear-eq-0-on-span* = *eq-0-on-span*
and *linear-independent-injective-image* = *independent-injective-image*
and *linear-inj-on-span-independent-image* = *inj-on-span-independent-image*
and *linear-inj-on-span-iff-independent-image* = *inj-on-span-iff-independent-image*
and *linear-subspace-linear-preimage* = *subspace-linear-preimage*
and *linear-spans-image* = *spans-image*
and *linear-spanning-surjective-image* = *spanning-surjective-image*
end

sublocale *module-pair*
rewrites *module-hom* = *linear*
by *unfold-locales* (*fact module-hom-eq-linear*)

lemmas— from locale *module-pair*
linear-eq-on-span = *module-hom-eq-on-span*
and *linear-compose-scale-right* = *module-hom-scale*
and *linear-compose-add* = *module-hom-add*
and *linear-zero* = *module-hom-zero*
and *linear-compose-sub* = *module-hom-sub*
and *linear-compose-neg* = *module-hom-neg*
and *linear-compose-scale* = *module-hom-compose-scale*

lemma *linear-indep-image-lemma*:
assumes *lf*: *linear s1 s2 f*
and *fB*: *finite B*
and *ifB*: *vs2.independent* (*f* ` *B*)
and *fi*: *inj-on f B*
and *xsB*: *x* ∈ *vs1.span B*
and *fx*: *f x* = *0*
shows *x* = *0*
using *fB ifB fi xsB fx*
proof (*induction B arbitrary*: *x rule*: *finite-induct*)
case *empty*
then show *?case* **by** *auto*
next

case (*insert a b x*)
have *th0*: *f* ' *b* ⊆ *f* ' (*insert a b*)
by (*simp add*: *subset-insertI*)
have *ifb*: *vs2.independent* (*f* ' *b*)
using *vs2.independent-mono insert.prems*(*1*) *th0* **by** *blast*
have *fib*: *inj-on f b*
using *insert.prems*(*2*) **by** *blast*
from *vs1.span-breakdown*[*of a insert a b*, *simplified*, *OF insert.prems*(*3*)]
obtain *k* **where** *k*: *x* − *k* ∗*a a* ∈ *vs1.span* (*b* − {*a*})
by *blast*
have *f* (*x* − *k* ∗*a a*) ∈ *vs2.span* (*f* ' *b*)
unfolding *linear-span-image*[*OF lf*]
using *insert.hyps*(*2*) *k* **by** *auto*
then have *f x* − *k* ∗*b f a* ∈ *vs2.span* (*f* ' *b*)
by (*simp add*: *linear-diff linear-scale lf*)
then have *th*: −*k* ∗*b f a* ∈ *vs2.span* (*f* ' *b*)
using *insert.prems*(*4*) **by** *simp*
have *xsb*: *x* ∈ *vs1.span b*
proof (*cases k* = *0*)
case *True*
with *k* **have** *x* ∈ *vs1.span* (*b* − {*a*}) **by** *simp*
then show *?thesis* **using** *vs1.span-mono*[*of b* − {*a*} *b*]
by *blast*
next
case *False*
from *inj-on-image-set-diff*[*OF insert.prems*(*2*), *of insert a b* {*a*}, *symmetric*]
have *f* ' *insert a b* − *f* ' {*a*} = *f* ' (*insert a b* − {*a*}) **by** *blast*
then have *f a* ∉ *vs2.span* (*f* ' *b*)
using *vs2.dependent-def insert.hyps*(*2*) *insert.prems*(*1*) **by** *fastforce*
moreover have *f a* ∈ *vs2.span* (*f* ' *b*)
using *False vs2.span-scale*[*OF th*, *of* − *1*/ *k*] **by** *auto*
ultimately have *False*
by *blast*
then show *?thesis* **by** *blast*
qed
show *x* = *0*
using *ifb fib xsb insert.IH insert.prems*(*4*) **by** *blast*
qed

lemma *linear-eq-on*:
assumes *l*: *linear s1 s2 f linear s1 s2 g*
assumes *x*: *x* ∈ *vs1.span B* **and** *eq*: ⋀*b*. *b* ∈ *B* ⟹ *f b* = *g b*
shows *f x* = *g x*
proof −
interpret *d*: *linear s1 s2* λ*x*. *f x* − *g x*
using *l* **by** (*intro linear-compose-sub*) (*auto simp*: *module-hom-iff-linear*)
have *f x* − *g x* = *0*
by (*rule d.eq-0-on-span*[*OF - x*]) (*auto simp*: *eq*)
then show *?thesis* **by** *auto*

qed

definition *construct* :: $'b$ *set* $\Rightarrow$ $('b \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'c)$
where *construct B g v* = $(\sum b \mid$ *vs1.representation* (*vs1.extend-basis B*) *v b* $\neq$ *0*.
vs1.representation (*vs1.extend-basis B*) *v b* $*b$ (*if* $b \in B$ *then g b else 0*))

lemma *construct-cong*: $(\bigwedge b.\ b \in B \Longrightarrow f\ b = g\ b) \Longrightarrow$ *construct B f* = *construct B g*
unfolding *construct-def* **by** (*rule ext*, *auto intro*!: *sum.cong*)

lemma *linear-construct*:
assumes *B*[*simp*]: *vs1.independent B*
shows *linear s1 s2* (*construct B f*)
unfolding *module-hom-iff-linear linear-iff*
proof *safe*
have *eB*[*simp*]: *vs1.independent* (*vs1.extend-basis B*)
using *vs1.independent-extend-basis*[*OF B*] **.**
let *?R* = *vs1.representation* (*vs1.extend-basis B*)
fix *c x y*
have *construct B f* (*x* + *y*) =
$(\sum b \in \{b.$ *?R x b* $\neq$ *0*$\} \cup \{b.$ *?R y b* $\neq$ *0*$\}$. *?R* (*x* + *y*) *b* $*b$ (*if* $b \in B$ *then f b else 0*))
by (*auto intro*!: *sum.mono-neutral-cong-left simp*: *vs1.finite-representation vs1.representation-add construct-def*)
also have ... = *construct B f x* + *construct B f y*
by (*auto simp*: *construct-def vs1.representation-add vs2.scale-left-distrib sum.distrib intro*!: *arg-cong2*[**where** *f*=(+)] *sum.mono-neutral-cong-right vs1.finite-representation*)
finally show *construct B f* (*x* + *y*) = *construct B f x* + *construct B f y* **.**

show *construct B f* (*c* $*a$ *x*) = *c* $*b$ *construct B f x*
by (*auto simp del*: *vs2.scale-scale intro*!: *sum.mono-neutral-cong-left vs1.finite-representation simp add*: *construct-def vs2.scale-scale*[*symmetric*] *vs1.representation-scale vs2.scale-sum-right*)
qed *intro-locales*

lemma *construct-basis*:
assumes *B*[*simp*]: *vs1.independent B* **and** *b*: $b \in B$
shows *construct B f b* = *f b*
proof −
have $*$: *vs1.representation* (*vs1.extend-basis B*) *b* = ($\lambda v.$ *if v* = *b then 1 else 0*)
using *vs1.extend-basis-superset*[*OF B*] *b*
by (*intro vs1.representation-basis vs1.independent-extend-basis*) *auto*
then have $\{v.$ *vs1.representation* (*vs1.extend-basis B*) *b v* $\neq$ *0*$\} = \{b\}$
by *auto*
then show *?thesis*
unfolding *construct-def* **by** (*simp add*: $*$ *b*)
qed

lemma *construct-outside*:
assumes *B*: *vs1.independent B* **and** *v*: *v* ∈ *vs1.span* (*vs1.extend-basis B* − *B*)
shows *construct B f v* = *0*
unfolding *construct-def*
proof (*clarsimp intro*!: *sum.neutral simp del*: *vs2.scale-eq-0-iff*)
fix *b* **assume** *b* ∈ *B*
then have *vs1.representation* (*vs1.extend-basis B* − *B*) *v b* = *0*
using *vs1.representation-ne-zero*[*of vs1.extend-basis B* − *B v b*] **by** *auto*
moreover have *vs1.representation* (*vs1.extend-basis B*) *v* = *vs1.representation* (*vs1.extend-basis B* − *B*) *v*
using *vs1.representation-extend*[*OF vs1.independent-extend-basis*[*OF B*] *v*] **by** *auto*
ultimately show *vs1.representation* (*vs1.extend-basis B*) *v b* ∗*b f b* = *0*
by *simp*
qed

lemma *construct-add*:
assumes *B*[*simp*]: *vs1.independent B*
shows *construct B* (λ*x*. *f x* + *g x*) *v* = *construct B f v* + *construct B g v*
proof (*rule linear-eq-on*)
show *v* ∈ *vs1.span* (*vs1.extend-basis B*) **by** *simp*
show *b* ∈ *vs1.extend-basis B* ⟹ *construct B* (λ*x*. *f x* + *g x*) *b* = *construct B f b* + *construct B g b* **for** *b*
using *construct-outside*[*OF B vs1.span-base*, *of b*] **by** (*cases b* ∈ *B*) (*auto simp*: *construct-basis*)
qed (*intro linear-compose-add linear-construct B*)+

lemma *construct-scale*:
assumes *B*[*simp*]: *vs1.independent B*
shows *construct B* (λ*x*. *c* ∗*b f x*) *v* = *c* ∗*b construct B f v*
proof (*rule linear-eq-on*)
show *v* ∈ *vs1.span* (*vs1.extend-basis B*) **by** *simp*
show *b* ∈ *vs1.extend-basis B* ⟹ *construct B* (λ*x*. *c* ∗*b f x*) *b* = *c* ∗*b construct B f b* **for** *b*
using *construct-outside*[*OF B vs1.span-base*, *of b*] **by** (*cases b* ∈ *B*) (*auto simp*: *construct-basis*)
qed (*intro linear-construct module-hom-scale B*)+

lemma *construct-in-span*:
assumes *B*[*simp*]: *vs1.independent B*
shows *construct B f v* ∈ *vs2.span* (*f* ‘ *B*)
proof −
interpret *c*: *linear s1 s2 construct B f* **by** (*rule linear-construct*) *fact*
let *?R* = *vs1.representation B*
have *v* ∈ *vs1.span* ((*vs1.extend-basis B* − *B*) ∪ *B*)
by (*auto simp*: *Un-absorb2 vs1.extend-basis-superset*)
then obtain *x y* **where** *v* = *x* + *y x* ∈ *vs1.span* (*vs1.extend-basis B* − *B*) *y* ∈ *vs1.span B*
unfolding *vs1.span-Un* **by** *auto*

moreover have *construct B f* ($\sum b$ | *?R y b* ≠ *0*. *?R y b* ∗*a b*) ∈ *vs2.span* (*f* ' *B*)
  by (*auto simp add*: *c.sum c.scale construct-basis vs1.representation-ne-zero*
   intro!: *vs2.span-sum vs2.span-scale intro*: *vs2.span-base*)
ultimately show *construct B f v* ∈ *vs2.span* (*f* ' *B*)
  by (*auto simp add*: *c.add construct-outside vs1.sum-nonzero-representation-eq*)
qed

lemma *linear-compose-sum*:
 assumes *lS*: ∀ *a* ∈ *S*. *linear s1 s2* (*f a*)
 shows *linear s1 s2* (λ*x*. *sum* (λ*a*. *f a x*) *S*)
proof (*cases finite S*)
 case *True*
 then show *?thesis*
  using *lS* **by** *induct* (*simp-all add*: *linear-zero linear-compose-add*)
next
 case *False*
 then show *?thesis*
  by (*simp add*: *linear-zero*)
qed

lemma *in-span-in-range-construct*:
 x ∈ *range* (*construct B f*) **if** *i*: *vs1.independent B* **and** *x*: *x* ∈ *vs2.span* (*f* ' *B*)
proof −
 interpret *linear* (∗*a*) (∗*b*) *construct B f*
  using *i* **by** (*rule linear-construct*)
 obtain *bb* :: ('*b* ⇒ '*c*) ⇒ ('*b* ⇒ '*c*) ⇒ '*b set* ⇒ '*b* **where**
  ∀ *x0 x1 x2*. (∃ *v4*. *v4* ∈ *x2* ∧ *x1 v4* ≠ *x0 v4*) = (*bb x0 x1 x2* ∈ *x2* ∧ *x1* (*bb x0 x1 x2*) ≠ *x0* (*bb x0 x1 x2*))
  by *moura*
 then have *f2*: ∀ *B Ba f fa*. (*B* ≠ *Ba* ∨ *bb fa f Ba* ∈ *Ba* ∧ *f* (*bb fa f Ba*) ≠ *fa* (*bb fa f Ba*)) ∨ *f* ' *B* = *fa* ' *Ba*
  by (*meson image-cong*)
 have *vs1.span B* ⊆ *vs1.span* (*vs1.extend-basis B*)
  by (*simp add*: *vs1.extend-basis-superset*[*OF i*] *vs1.span-mono*)
 then show *x* ∈ *range* (*construct B f*)
  using *f2 x* **by** (*metis* (*no-types*) *construct-basis*[*OF i*, *of - f*]
   vs1.span-extend-basis[*OF i*] *subsetD span-image spans-image*)
qed

lemma *range-construct-eq-span*:
 range (*construct B f*) = *vs2.span* (*f* ' *B*)
 if *vs1.independent B*
 by (*auto simp*: *that construct-in-span in-span-in-range-construct*)

lemma *linear-independent-extend-subspace*:
 — legacy: use *construct* instead
 assumes *vs1.independent B*
 shows ∃ *g*. *linear s1 s2 g* ∧ (∀ *x*∈*B*. *g x* = *f x*) ∧ *range g* = *vs2.span* (*f*'*B*)

by (*rule exI*[**where** *x=construct B f*])
(*auto simp*: *linear-construct assms construct-basis range-construct-eq-span*)

lemma *linear-independent-extend*:
vs1.independent B ⟹ ∃ *g. linear s1 s2 g* ∧ (∀ *x*∈*B. g x = f x*)
using *linear-independent-extend-subspace*[*of B f*] **by** *auto*

lemma *linear-exists-left-inverse-on*:
assumes *lf*: *linear s1 s2 f*
assumes *V*: *vs1.subspace V* **and** *f*: *inj-on f V*
shows ∃ *g. g ' UNIV* ⊆ *V* ∧ *linear s2 s1 g* ∧ (∀ *v*∈*V. g* (*f v*) = *v*)
proof −
interpret *linear s1 s2 f* **by** *fact*
obtain *B* **where** *V-eq*: *V = vs1.span B* **and** *B*: *vs1.independent B*
using *vs1.maximal-independent-subset*[*of V*] *vs1.span-minimal*[*OF - ‹vs1.subspace V›*]
by (*metis antisym-conv*)
have *f*: *inj-on f* (*vs1.span B*)
using *f* **unfolding** *V-eq* **.**
show *?thesis*
proof (*intro exI ballI conjI*)
interpret *p*: *vector-space-pair s2 s1* **by** *unfold-locales*
have *fB*: *vs2.independent* (*f ' B*)
using *independent-injective-image*[*OF B f*] **.**
let *?g = p.construct* (*f ' B*) (*the-inv-into B f*)
show *linear* (∗*b*) (∗*a*) *?g*
by (*rule p.linear-construct*[*OF fB*])
have *?g b* ∈ *vs1.span* (*the-inv-into B f ' f ' B*) **for** *b*
by (*intro p.construct-in-span fB*)
moreover have *the-inv-into B f ' f ' B = B*
by (*auto simp*: *image-comp comp-def the-inv-into-f-f inj-on-subset*[*OF f vs1.span-superset*]
cong: *image-cong*)
ultimately show *?g ' UNIV* ⊆ *V*
by (*auto simp*: *V-eq*)
have (*?g* ∘ *f*) *v = id v* **if** *v* ∈ *vs1.span B* **for** *v*
proof (*rule vector-space-pair.linear-eq-on*[**where** *x=v*])
show *vector-space-pair* (∗*a*) (∗*a*) **by** *unfold-locales*
show *linear* (∗*a*) (∗*a*) (*?g* ∘ *f*)
proof (*rule Vector-Spaces.linear-compose*[*of -* (∗*b*)])
show *linear* (∗*a*) (∗*b*) *f*
by *unfold-locales*
qed *fact*
show *linear* (∗*a*) (∗*a*) *id* **by** (*rule vs1.linear-id*)
show *v* ∈ *vs1.span B* **by** *fact*
show *b* ∈ *B* ⟹ (*p.construct* (*f ' B*) (*the-inv-into B f*) ∘ *f*) *b = id b* **for** *b*
by (*simp add*: *p.construct-basis fB the-inv-into-f-f inj-on-subset*[*OF f vs1.span-superset*])
qed

then show *v* ∈ *V* ⟹ *?g* (*f v*) = *v* **for** *v* **by** (*auto simp*: *comp-def id-def V-eq*)
qed
qed

lemma *linear-exists-right-inverse-on*:
assumes *lf*: *linear s1 s2 f*
assumes *vs1.subspace V*
shows ∃ *g*. *g* ‘ *UNIV* ⊆ *V* ∧ *linear s2 s1 g* ∧ (∀ *v*∈*f* ‘ *V*. *f* (*g v*) = *v*)
proof −
obtain *B* **where** *V-eq*: *V* = *vs1.span B* **and** *B*: *vs1.independent B*
using *vs1.maximal-independent-subset*[*of V*] *vs1.span-minimal*[*OF* - ‹*vs1.subspace V*›]
by (*metis antisym-conv*)
obtain *C* **where** *C*: *vs2.independent C* **and** *fB-C*: *f* ‘ *B* ⊆ *vs2.span C C* ⊆ *f* ‘ *B*
using *vs2.maximal-independent-subset*[*of f* ‘ *B*] **by** *metis*
then have ∀ *v*∈*C*. ∃ *b*∈*B*. *v* = *f b* **by** *auto*
then obtain *g* **where** *g*: ⋀*v*. *v* ∈ *C* ⟹ *g v* ∈ *B* ⋀*v*. *v* ∈ *C* ⟹ *f* (*g v*) = *v* **by** *metis*
show *?thesis*
proof (*intro exI ballI conjI*)
interpret *p*: *vector-space-pair s2 s1* **by** *unfold-locales*
let *?g* = *p.construct C g*
show *linear* (*∗b*) (*∗a*) *?g*
by (*rule p.linear-construct*[*OF C*])
have *?g v* ∈ *vs1.span* (*g* ‘ *C*) **for** *v*
by (*rule p.construct-in-span*[*OF C*])
also have . . . ⊆ *V* **unfolding** *V-eq* **using** *g* **by** (*intro vs1.span-mono*) *auto*
finally show *?g* ‘ *UNIV* ⊆ *V* **by** *auto*
have (*f* ∘ *?g*) *v* = *id v* **if** *v*: *v* ∈ *f* ‘ *V* **for** *v*
proof (*rule vector-space-pair.linear-eq-on*[**where** *x*=*v*])
show *vector-space-pair* (*∗b*) (*∗b*) **by** *unfold-locales*
show *linear* (*∗b*) (*∗b*) (*f* ∘ *?g*)
by (*rule Vector-Spaces.linear-compose*[*of* - (*∗a*)]) *fact+*
show *linear* (*∗b*) (*∗b*) *id* **by** (*rule vs2.linear-id*)
have *vs2.span* (*f* ‘ *B*) = *vs2.span C*
using *fB-C vs2.span-mono*[*of C f* ‘ *B*] *vs2.span-minimal*[*of f*‘*B vs2.span C*]
by *auto*
then show *v* ∈ *vs2.span C*
using *v linear-span-image*[*OF lf*, *of B*] **by** (*simp add*: *V-eq*)
show (*f* ∘ *p.construct C g*) *b* = *id b* **if** *b*: *b* ∈ *C* **for** *b*
by (*auto simp*: *p.construct-basis g C b*)
qed
then show *v* ∈ *f* ‘ *V* ⟹ *f* (*?g v*) = *v* **for** *v* **by** (*auto simp*: *comp-def id-def*)
qed
qed

lemma *linear-inj-on-left-inverse*:
assumes *lf*: *linear s1 s2 f*

assumes *fi*: *inj-on f* (*vs1.span S*)
shows ∃ *g*. *range g* ⊆ *vs1.span S* ∧ *linear s2 s1 g* ∧ (∀ *x*∈*vs1.span S*. *g* (*f x*) = *x*)
using *linear-exists-left-inverse-on*[*OF lf vs1.subspace-span fi*]
by (*auto simp*: *linear-iff-module-hom*)

lemma *linear-injective-left-inverse*: *linear s1 s2 f* ⟹ *inj f* ⟹ ∃ *g*. *linear s2 s1 g* ∧ *g* ∘ *f* = *id*
using *linear-inj-on-left-inverse*[*of f UNIV*]
by *force*

lemma *linear-surj-right-inverse*:
assumes *lf*: *linear s1 s2 f*
assumes *sf*: *vs2.span T* ⊆ *f'vs1.span S*
shows ∃ *g*. *range g* ⊆ *vs1.span S* ∧ *linear s2 s1 g* ∧ (∀ *x*∈*vs2.span T*. *f* (*g x*) = *x*)
using *linear-exists-right-inverse-on*[*OF lf vs1.subspace-span*, *of S*] *sf*
by (*force simp*: *linear-iff-module-hom*)

lemma *linear-surjective-right-inverse*: *linear s1 s2 f* ⟹ *surj f* ⟹ ∃ *g*. *linear s2 s1 g* ∧ *f* ∘ *g* = *id*
using *linear-surj-right-inverse*[*of f UNIV UNIV*]
by (*auto simp*: *fun-eq-iff*)

lemma *finite-basis-to-basis-subspace-isomorphism*:
assumes *s*: *vs1.subspace S*
and *t*: *vs2.subspace T*
and *d*: *vs1.dim S* = *vs2.dim T*
and *fB*: *finite B*
and *B*: *B* ⊆ *S vs1.independent B S* ⊆ *vs1.span B card B* = *vs1.dim S*
and *fC*: *finite C*
and *C*: *C* ⊆ *T vs2.independent C T* ⊆ *vs2.span C card C* = *vs2.dim T*
shows ∃ *f*. *linear s1 s2 f* ∧ *f* ' *B* = *C* ∧ *f* ' *S* = *T* ∧ *inj-on f S*
proof −
from *B*(*4*) *C*(*4*) *card-le-inj*[*of B C*] *d* **obtain** *f* **where**
f: *f* ' *B* ⊆ *C inj-on f B* **using** ‹*finite B*› ‹*finite C*› **by** *auto*
from *linear-independent-extend*[*OF B*(*2*)] **obtain** *g* **where**
g: *linear s1 s2 g* ∀ *x* ∈ *B*. *g x* = *f x* **by** *blast*
interpret *g*: *linear s1 s2 g* **by** *fact*
from *inj-on-iff-eq-card*[*OF fB*, *of f*] *f*(*2*)
have *card* (*f* ' *B*) = *card B* **by** *simp*
with *B*(*4*) *C*(*4*) **have** *ceq*: *card* (*f* ' *B*) = *card C* **using** *d*
by *simp*
have *g* ' *B* = *f* ' *B* **using** *g*(*2*)
by (*auto simp add*: *image-iff*)
also have ... = *C* **using** *card-subset-eq*[*OF fC f*(*1*) *ceq*] **.**
finally have *gBC*: *g* ' *B* = *C* **.**
have *gi*: *inj-on g B* **using** *f*(*2*) *g*(*2*)
by (*auto simp add*: *inj-on-def*)

```
  note g0 = linear-indep-image-lemma[OF g(1) fB, unfolded gBC, OF C(2) gi]
  {
    fix x y
    assume x: x ∈ S and y: y ∈ S and gxy: g x = g y
    from B(3) x y have x': x ∈ vs1.span B and y': y ∈ vs1.span B
      by blast+
    from gxy have th0: g (x − y) = 0
      by (simp add: g.diff)
    have th1: x − y ∈ vs1.span B using x' y'
      by (metis vs1.span-diff)
    have x = y using g0[OF th1 th0] by simp
  }
  then have giS: inj-on g S unfolding inj-on-def by blast
  from vs1.span-subspace[OF B(1,3) s]
  have g ` S = vs2.span (g ` B)
    by (simp add: g.span-image)
  also have ... = vs2.span C
    unfolding gBC ..
  also have ... = T
    using vs2.span-subspace[OF C(1,3) t] .
  finally have gS: g ` S = T .
  from g(1) gS giS gBC show ?thesis
    by blast
qed

end


locale finite-dimensional-vector-space = vector-space +
  fixes Basis :: 'b set
  assumes finite-Basis: finite Basis
  and independent-Basis: independent Basis
  and span-Basis: span Basis = UNIV
begin

definition dimension = card Basis

lemma finiteI-independent: independent B ⟹ finite B
  using independent-span-bound[OF finite-Basis, of B] by (auto simp: span-Basis)

lemma dim-empty [simp]: dim {} = 0
  by (rule dim-unique[OF order-refl]) (auto simp: dependent-def)

lemma dim-insert:
  dim (insert x S) = (if x ∈ span S then dim S else dim S + 1)
proof −
  show ?thesis
  proof (cases x ∈ span S)
    case True then show ?thesis
```

by (*metis dim-span span-redundant*)
next
case *False*
obtain *B* **where** *B*: *B* ⊆ *span S independent B span S* ⊆ *span B card B* = *dim* (*span S*)
using *basis-exists* [*of span S*] **by** *blast*
have *dim* (*span* (*insert x S*)) = *Suc* (*dim S*)
proof (*rule dim-unique*)
show *insert x B* ⊆ *span* (*insert x S*)
by (*meson B*(*1*) *insertI1 insert-subset order-trans span-base span-mono subset-insertI*)
show *span* (*insert x S*) ⊆ *span* (*insert x B*)
by (*metis* ‹*B* ⊆ *span S*› ‹*span S* ⊆ *span B*› *span-breakdown-eq span-subspace subsetI subspace-span*)
show *independent* (*insert x B*)
by (*metis B*(*1*−*3*) *independent-insert span-subspace subspace-span False*)
show *card* (*insert x B*) = *Suc* (*dim S*)
using *B False finiteI-independent* **by** *force*
qed
then show *?thesis*
by (*metis False Suc-eq-plus1 dim-span*)
qed
qed

lemma *dim-singleton* [*simp*]: *dim*{*x*} = (*if x* = *0 then 0 else 1*)
by (*simp add*: *dim-insert*)

proposition *choose-subspace-of-subspace*:
assumes *n* ≤ *dim S*
obtains *T* **where** *subspace T T* ⊆ *span S dim T* = *n*
proof −
have ∃ *T*. *subspace T* ∧ *T* ⊆ *span S* ∧ *dim T* = *n*
using *assms*
proof (*induction n*)
case *0* **then show** *?case* **by** (*auto intro*!: *exI*[**where** *x*={*0*}] *span-zero*)
next
case (*Suc n*)
then obtain *T* **where** *subspace T T* ⊆ *span S dim T* = *n*
by *force*
then show *?case*
proof (*cases span S* ⊆ *span T*)
case *True*
have *span T* ⊆ *span S*
by (*simp add*: ‹*T* ⊆ *span S*› *span-minimal*)
then have *dim S* = *dim T*
by (*rule span-eq-dim* [*OF subset-antisym* [*OF True*]])
then show *?thesis*
using *Suc.prems* ‹*dim T* = *n*› **by** *linarith*
next

case *False*
then obtain *y* **where** *y*: *y* ∈ *S y* ∉ *T*
by (*meson span-mono subsetI*)
then have *span* (*insert y T*) ⊆ *span S*
by (*metis* (*no-types*) ‹*T* ⊆ *span S*› *subsetD insert-subset span-superset span-mono span-span*)
with ‹*dim T* = *n*› ‹*subspace T*› *y* **show** *?thesis*
apply (*rule-tac x=span*(*insert y T*) **in** *exI*)
using *span-eq-iff* **by** (*fastforce simp*: *dim-insert*)
qed
qed
with *that* **show** *?thesis* **by** *blast*
qed

lemma *basis-subspace-exists*:
assumes *subspace S*
obtains *B* **where** *finite B B* ⊆ *S independent B span B* = *S card B* = *dim S*
by (*metis assms span-subspace basis-exists finiteI-independent*)

lemma *dim-mono*: **assumes** *V* ⊆ *span W* **shows** *dim V* ≤ *dim W*
proof −
obtain *B* **where** *independent B B* ⊆ *W W* ⊆ *span B*
using *maximal-independent-subset*[*of W*] **by** *force*
with *dim-le-card*[*of V B*] *assms independent-span-bound*[*of Basis B*] *basis-card-eq-dim*[*of B W*]
span-mono[*of B W*] *span-minimal*[*OF - subspace-span*, *of W B*]
show *?thesis*
by (*auto simp*: *finite-Basis span-Basis*)
qed

lemma *dim-subset*: *S* ⊆ *T* ⟹ *dim S* ≤ *dim T*
using *dim-mono*[*of S T*] **by** (*auto intro*: *span-base*)

lemma *dim-eq-0* [*simp*]:
dim S = *0* ⟷ *S* ⊆ {*0*}
by (*metis basis-exists card-eq-0-iff dim-span finiteI-independent span-empty subset-empty subset-singletonD*)

lemma *dim-UNIV*[*simp*]: *dim UNIV* = *card Basis*
using *dim-eq-card*[*of Basis UNIV*] **by** (*simp add*: *independent-Basis span-Basis*)

lemma *independent-card-le-dim*: **assumes** *B* ⊆ *V* **and** *independent B* **shows** *card B* ≤ *dim V*
by (*subst dim-eq-card*[*symmetric*, *OF refl* ‹*independent B*›]) (*rule dim-subset*[*OF* ‹*B* ⊆ *V*›])

lemma *dim-subset-UNIV*: *dim S* ≤ *dimension*
by (*metis dim-subset subset-UNIV dim-UNIV dimension-def*)

lemma *card-ge-dim-independent*:
  assumes *BV*: $B \subseteq V$
    and *iB*: *independent B*
    and *dVB*: *dim* $V \leq$ *card B*
  shows $V \subseteq$ *span B*
proof
  fix *a*
  assume *aV*: $a \in V$
  {
    assume *aB*: $a \notin$ *span B*
    then have *iaB*: *independent* (*insert a B*)
      using *iB aV BV* **by** (*simp add*: *independent-insert*)
    from *aV BV* **have** *th0*: *insert a B* $\subseteq V$
      by *blast*
    from *aB* **have** $a \notin B$
      by (*auto simp add*: *span-base*)
    with *independent-card-le-dim*[*OF th0 iaB*] *dVB finiteI-independent*[*OF iB*]
    have *False* **by** *auto*
  }
  then show $a \in$ *span B* **by** *blast*
qed

lemma *card-le-dim-spanning*:
  assumes *BV*: $B \subseteq V$
    and *VB*: $V \subseteq$ *span B*
    and *fB*: *finite B*
    and *dVB*: *dim* $V \geq$ *card B*
  shows *independent B*
proof −
  {
    fix *a*
    assume *a*: $a \in B$ $a \in$ *span* $(B - \{a\})$
    from *a fB* **have** *c0*: *card* $B \neq 0$
      by *auto*
    from *a fB* **have** *cb*: *card* $(B - \{a\})$ = *card* $B - 1$
      by *auto*
    {
      fix *x*
      assume *x*: $x \in V$
      from *a* **have** *eq*: *insert a* $(B - \{a\}) = B$
        by *blast*
      from *x VB* **have** *x'*: $x \in$ *span B*
        by *blast*
      from *span-trans*[*OF a*(*2*), *unfolded eq*, *OF x'*]
      have $x \in$ *span* $(B - \{a\})$ **.**
    }
    then have *th1*: $V \subseteq$ *span* $(B - \{a\})$
      by *blast*
    have *th2*: *finite* $(B - \{a\})$

using *fB* **by** *auto*
from *dim-le-card*[*OF th1 th2*]
have *c*: $dim\ V \leq card\ (B - \{a\})$ **.**
from *c c0 dVB cb* **have** *False* **by** *simp*
}
then show *?thesis*
unfolding *dependent-def* **by** *blast*
qed

lemma *card-eq-dim*: $B \subseteq V \Longrightarrow card\ B = dim\ V \Longrightarrow finite\ B \Longrightarrow independent\ B \longleftrightarrow V \subseteq span\ B$
by (*metis order-eq-iff card-le-dim-spanning card-ge-dim-independent*)

lemma *subspace-dim-equal*:
assumes *subspace S*
and *subspace T*
and $S \subseteq T$
and $dim\ S \geq dim\ T$
shows $S = T$
proof −
obtain *B* **where** *B*: $B \leq S$ *independent* $B \wedge S \subseteq span\ B$ *card* $B = dim\ S$
using *basis-exists*[*of S*] **by** *metis*
then have $span\ B \subseteq S$
using *span-mono*[*of B S*] *span-eq-iff*[*of S*] *assms* **by** *metis*
then have $span\ B = S$
using *B* **by** *auto*
have $dim\ S = dim\ T$
using *assms dim-subset*[*of S T*] **by** *auto*
then have $T \subseteq span\ B$
using *card-eq-dim*[*of B T*] *B finiteI-independent assms* **by** *auto*
then show *?thesis*
using *assms* ‹$span\ B = S$› **by** *auto*
qed

corollary *dim-eq-span*:
shows $[\![S \subseteq T;\ dim\ T \leq dim\ S]\!] \Longrightarrow span\ S = span\ T$
by (*simp add*: *span-mono subspace-dim-equal*)

lemma *dim-psubset*:
$span\ S \subset span\ T \Longrightarrow dim\ S < dim\ T$
by (*metis* (*no-types*, *opaque-lifting*) *dim-span less-le not-le subspace-dim-equal subspace-span*)

lemma *dim-eq-full*:
shows $dim\ S = dimension \longleftrightarrow span\ S = UNIV$
by (*metis dim-eq-span dim-subset-UNIV span-Basis span-span subset-UNIV dim-UNIV dim-span dimension-def*)

lemma *indep-card-eq-dim-span*:

assumes *independent B*
shows *finite B* $\wedge$ *card B* = *dim* (*span B*)
using *dim-span-eq-card-independent*[*OF assms*] *finiteI-independent*[*OF assms*] **by** *auto*

More general size bound lemmas.

lemma *independent-bound-general*:
independent S $\Longrightarrow$ *finite S* $\wedge$ *card S* $\leq$ *dim S*
by (*simp add*: *dim-eq-card-independent finiteI-independent*)

lemma *independent-explicit*:
shows *independent B* $\longleftrightarrow$ *finite B* $\wedge$ ($\forall$ *c*. ($\sum$ *v*$\in$*B*. *c v* $*s$ *v*) = *0* $\longrightarrow$ ($\forall$ *v* $\in$ *B*. *c v* = *0*))
using *independent-bound-general*
by (*fastforce simp*: *dependent-finite*)

proposition *dim-sums-Int*:
assumes *subspace S subspace T*
shows *dim* {*x* + *y* |*x y*. *x* $\in$ *S* $\wedge$ *y* $\in$ *T*} + *dim*(*S* $\cap$ *T*) = *dim S* + *dim T* (**is** *dim ?ST* + - = -)
proof −
obtain *B* **where** *B*: *B* $\subseteq$ *S* $\cap$ *T S* $\cap$ *T* $\subseteq$ *span B*
and *indB*: *independent B*
and *cardB*: *card B* = *dim* (*S* $\cap$ *T*)
using *basis-exists* **by** *metis*
then obtain *C D* **where** *B* $\subseteq$ *C C* $\subseteq$ *S independent C S* $\subseteq$ *span C*
and *B* $\subseteq$ *D D* $\subseteq$ *T independent D T* $\subseteq$ *span D*
using *maximal-independent-subset-extend*
by (*metis Int-subset-iff* ‹*B* $\subseteq$ *S* $\cap$ *T*› *indB*)
then have *finite B finite C finite D*
by (*simp-all add*: *finiteI-independent indB independent-bound-general*)
have *Beq*: *B* = *C* $\cap$ *D*
proof (*rule spanning-subset-independent* [*symmetric*])
show *independent* (*C* $\cap$ *D*)
by (*meson* ‹*independent C*› *independent-mono inf.cobounded1*)
qed (*use B* ‹*C* $\subseteq$ *S*› ‹*D* $\subseteq$ *T*› ‹*B* $\subseteq$ *C*› ‹*B* $\subseteq$ *D*› **in** *auto*)
then have *Deq*: *D* = *B* $\cup$ (*D* − *C*)
by *blast*
have *CUD*: *C* $\cup$ *D* $\subseteq$ *?ST*
proof (*simp*, *intro conjI*)
show *C* $\subseteq$ *?ST*
using *span-zero span-minimal* [*OF* - ‹*subspace T*›] ‹*C* $\subseteq$ *S*› **by** *force*
show *D* $\subseteq$ *?ST*
using *span-zero span-minimal* [*OF* - ‹*subspace S*›] ‹*D* $\subseteq$ *T*› **by** *force*
qed
have *a v* = *0* **if** *0*: ($\sum$ *v*$\in$*C*. *a v* $*s$ *v*) + ($\sum$ *v*$\in$*D* − *C*. *a v* $*s$ *v*) = *0*
and *v*: *v* $\in$ *C* $\cup$ (*D*−*C*) **for** *a v*
proof −
have *CsS*: $\bigwedge$*x*. *x* $\in$ *C* $\Longrightarrow$ *a x* $*s$ *x* $\in$ *S*

using ‹$C \subseteq S$› ‹*subspace S*› *subspace-scale* **by** *auto*
have *eq*: $(\sum v{\in}D - C.\ a\ v *s\ v) = - (\sum v{\in}C.\ a\ v *s\ v)$
using *that add-eq-0-iff* **by** *blast*
have $(\sum v{\in}D - C.\ a\ v *s\ v) \in S$
by (*simp add*: *eq CsS* ‹*subspace S*› *subspace-neg subspace-sum*)
moreover have $(\sum v{\in}D - C.\ a\ v *s\ v) \in T$
apply (*rule subspace-sum* [*OF* ‹*subspace T*›])
by (*meson DiffD1* ‹$D \subseteq T$› ‹*subspace T*› *subset-eq subspace-def*)
ultimately have $(\sum v \in D{-}C.\ a\ v *s\ v) \in span\ B$
using *B* **by** *blast*
then obtain *e* **where** *e*: $(\sum v{\in}B.\ e\ v *s\ v) = (\sum v \in D{-}C.\ a\ v *s\ v)$
using *span-finite* [*OF* ‹*finite B*›] **by** *force*
have $\bigwedge c\ v.\ [\![(\sum v{\in}C.\ c\ v *s\ v) = 0;\ v \in C]\!] \Longrightarrow c\ v = 0$
using ‹*finite C*› ‹*independent C*› *independentD* **by** *blast*
define *cc* **where** $cc\ x = (if\ x \in B\ then\ a\ x + e\ x\ else\ a\ x)$ **for** *x*
have [*simp*]: $C \cap B = B\ D \cap B = B\ C \cap - B = C{-}D\ B \cap (D - C) = \{\}$
using ‹$B \subseteq C$› ‹$B \subseteq D$› *Beq* **by** *blast+*
have *f2*: $(\sum v{\in}C \cap D.\ e\ v *s\ v) = (\sum v{\in}D - C.\ a\ v *s\ v)$
using *Beq e* **by** *presburger*
have *f3*: $(\sum v{\in}C \cup D.\ a\ v *s\ v) = (\sum v{\in}C - D.\ a\ v *s\ v) + (\sum v{\in}D - C.\ a\ v *s\ v) + (\sum v{\in}C \cap D.\ a\ v *s\ v)$
using ‹*finite C*› ‹*finite D*› *sum.union-diff2* **by** *blast*
have *f4*: $(\sum v{\in}C \cup (D - C).\ a\ v *s\ v) = (\sum v{\in}C.\ a\ v *s\ v) + (\sum v{\in}D - C.\ a\ v *s\ v)$
by (*meson Diff-disjoint* ‹*finite C*› ‹*finite D*› *finite-Diff sum.union-disjoint*)
have $(\sum v{\in}C.\ cc\ v *s\ v) = 0$
using *0 f2 f3 f4*
apply (*simp add*: *cc-def Beq* ‹*finite C*› *sum.If-cases algebra-simps sum.distrib if-distrib if-distribR*)
apply (*simp add*: *add.commute add.left-commute diff-eq*)
done
then have $\bigwedge v.\ v \in C \Longrightarrow cc\ v = 0$
using *independent-explicit* ‹*independent C*› ‹*finite C*› **by** *blast*
then have *C0*: $\bigwedge v.\ v \in C - B \Longrightarrow a\ v = 0$
by (*simp add*: *cc-def Beq*) *meson*
then have [*simp*]: $(\sum x{\in}C - B.\ a\ x *s\ x) = 0$
by *simp*
have $(\sum x{\in}C.\ a\ x *s\ x) = (\sum x{\in}B.\ a\ x *s\ x)$
proof −
have $C - D = C - B$
using *Beq* **by** *blast*
then show *?thesis*
using *Beq* ‹$(\sum x{\in}C - B.\ a\ x *s\ x) = 0$› *f3 f4* **by** *auto*
qed
with *0* **have** *Dcc0*: $(\sum v{\in}D.\ a\ v *s\ v) = 0$
by (*subst Deq*) (*simp add*: ‹*finite B*› ‹*finite D*› *sum-Un*)
then have *D0*: $\bigwedge v.\ v \in D \Longrightarrow a\ v = 0$
using *independent-explicit* ‹*independent D*› ‹*finite D*› **by** *blast*
show *?thesis*

using *v C0 D0 Beq* **by** *blast*
qed
then have *independent* $(C \cup (D - C))$
unfolding *independent-explicit*
using *independent-explicit*
by (*simp add*: *independent-explicit* ‹*finite C*› ‹*finite D*› *sum-Un del*: *Un-Diff-cancel*)
then have *indCUD*: *independent* $(C \cup D)$ **by** *simp*
have *dim* $(S \cap T)$ = *card B*
by (*rule dim-unique* [*OF B indB refl*])
moreover have *dim S* = *card C*
by (*metis* ‹$C \subseteq S$› ‹*independent C*› ‹$S \subseteq$ *span C*› *basis-card-eq-dim*)
moreover have *dim T* = *card D*
by (*metis* ‹$D \subseteq T$› ‹*independent D*› ‹$T \subseteq$ *span D*› *basis-card-eq-dim*)
moreover have *dim ?ST* = *card*$(C \cup D)$
proof −
have ∗: $\bigwedge x\ y.\ [\![x \in S;\ y \in T]\!] \Longrightarrow x + y \in$ *span* $(C \cup D)$
by (*meson* ‹$S \subseteq$ *span C*› ‹$T \subseteq$ *span D*› *span-add span-mono subsetCE sup.cobounded1 sup.cobounded2*)
show *?thesis*
by (*auto intro*: ∗ *dim-unique* [*OF CUD - indCUD refl*])
qed
ultimately show *?thesis*
using ‹$B = C \cap D$› [*symmetric*]
by (*simp add*: ‹*independent C*› ‹*independent D*› *card-Un-Int finiteI-independent*)
qed

lemma *dependent-biggerset-general*:
(*finite S* ⟹ *card S* > *dim S*) ⟹ *dependent S*
using *independent-bound-general*[*of S*] **by** (*metis linorder-not-le*)

lemma *subset-le-dim*:
$S \subseteq$ *span T* ⟹ *dim S* ≤ *dim T*
by (*metis dim-span dim-subset*)

lemma *linear-inj-imp-surj*:
assumes *lf*: *linear scale scale f*
and *fi*: *inj f*
shows *surj f*
proof −
interpret *lf*: *linear scale scale f* **by** *fact*
from *basis-exists*[*of UNIV*] **obtain** *B*
where *B*: $B \subseteq$ *UNIV independent B UNIV* $\subseteq$ *span B card B* = *dim UNIV*
by *blast*
from *B*(*4*) **have** *d*: *dim UNIV* = *card B*
by *simp*
have *UNIV* $\subseteq$ *span* (*f* ‘ *B*)
proof (*rule card-ge-dim-independent*)
show *independent* (*f* ‘ *B*)
by (*simp add*: *B*(*2*) *fi lf.independent-inj-image*)

have *card* (*f* ' *B*) = *dim UNIV*
by (*metis B*(*1*) *card-image d fi inj-on-subset*)
then show *dim UNIV* ≤ *card* (*f* ' *B*)
by *simp*
qed *blast*
then show *?thesis*
unfolding *lf.span-image surj-def*
using *B*(*3*) **by** *blast*
qed

end

locale *finite-dimensional-vector-space-pair-1* =
vs1: *finite-dimensional-vector-space s1 B1* + *vs2*: *vector-space s2*
for *s1* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ‹∗*a*› *75*)
and *B1* :: *'b set*
and *s2* :: *'a*::*field* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ‹∗*b*› *75*)
begin

sublocale *vector-space-pair s1 s2* **by** *unfold-locales*

lemma *dim-image-eq*:
assumes *lf*: *linear s1 s2 f*
and *fi*: *inj-on f* (*vs1.span S*)
shows *vs2.dim* (*f* ' *S*) = *vs1.dim S*
proof −
interpret *lf*: *linear* **by** *fact*
obtain *B* **where** *B*: *B* ⊆ *S vs1.independent B S* ⊆ *vs1.span B card B* = *vs1.dim S*
using *vs1.basis-exists*[*of S*] **by** *auto*
then have *vs1.span S* = *vs1.span B*
using *vs1.span-mono*[*of B S*] *vs1.span-mono*[*of S vs1.span B*] *vs1.span-span*[*of B*] **by** *auto*
moreover have *card* (*f* ' *B*) = *card B*
using *assms card-image*[*of f B*] *subset-inj-on*[*of f vs1.span S B*] *B vs1.span-superset* **by** *auto*
moreover have (*f* ' *B*) ⊆ (*f* ' *S*)
using *B* **by** *auto*
ultimately show *?thesis*
by (*metis B*(*2*) *B*(*4*) *fi lf.dependent-inj-imageD lf.span-image vs2.dim-eq-card-independent vs2.dim-span*)
qed

lemma *dim-image-le*:
assumes *lf*: *linear s1 s2 f*
shows *vs2.dim* (*f* ' *S*) ≤ *vs1.dim* (*S*)
proof −
from *vs1.basis-exists*[*of S*] **obtain** *B* **where**
B: *B* ⊆ *S vs1.independent B S* ⊆ *vs1.span B card B* = *vs1.dim S* **by** *blast*

```
  from B have fB: finite B card B = vs1.dim S
    using vs1.independent-bound-general by blast+
  have vs2.dim (f ` S) ≤ card (f ` B)
    apply (rule vs2.span-card-ge-dim)
    using lf B fB
     apply (auto simp add: module-hom.span-image module-hom.spans-image sub-
set-image-iff
        linear-iff-module-hom)
    done
  also have ... ≤ vs1.dim S
    using card-image-le[OF fB(1)] fB by simp
  finally show ?thesis .
qed

end

locale finite-dimensional-vector-space-pair =
  vs1: finite-dimensional-vector-space s1 B1 + vs2: finite-dimensional-vector-space
s2 B2
  for s1 :: 'a::field ⇒ 'b::ab-group-add ⇒ 'b (infixr ‹*a› 75)
  and B1 :: 'b set
  and s2 :: 'a::field ⇒ 'c::ab-group-add ⇒ 'c (infixr ‹*b› 75)
  and B2 :: 'c set
begin

sublocale finite-dimensional-vector-space-pair-1 ..

lemma linear-surjective-imp-injective:
   assumes lf: linear s1 s2 f and sf: surj f and eq: vs2.dim UNIV = vs1.dim
UNIV
    shows inj f
proof -
  interpret linear s1 s2 f by fact
  have *: card (f ` B1) ≤ vs2.dim UNIV
    using vs1.finite-Basis vs1.dim-eq-card[of B1 UNIV] sf
    by (auto simp: vs1.span-Basis vs1.independent-Basis eq
        simp del: vs2.dim-UNIV
        intro!: card-image-le)
  have indep-fB: vs2.independent (f ` B1)
    using vs1.finite-Basis vs1.dim-eq-card[of B1 UNIV] sf *
     by (intro vs2.card-le-dim-spanning[of f ` B1 UNIV]) (auto simp: span-image
vs1.span-Basis )
  have vs2.dim UNIV ≤ card (f ` B1)
    unfolding eq sf[symmetric] vs2.dim-span-eq-card-independent[symmetric, OF
indep-fB]
      vs2.dim-span
    by (intro vs2.dim-mono) (auto simp: span-image vs1.span-Basis)
  with * have card (f ` B1) = vs2.dim UNIV by auto
  also have ... = card B1
```

unfolding *eq vs1.dim-UNIV* **..**
finally have *inj-on f B1*
by (*subst inj-on-iff-eq-card*[*OF vs1.finite-Basis*])
then show *inj f*
using *inj-on-span-iff-independent-image*[*OF indep-fB*] *vs1.span-Basis* **by** *auto*
qed

lemma *linear-injective-imp-surjective*:
assumes *lf*: *linear s1 s2 f* **and** *sf*: *inj f* **and** *eq*: *vs2.dim UNIV* = *vs1.dim UNIV*
shows *surj f*
proof −
interpret *linear s1 s2 f* **by** *fact*
have ∗: *False* **if** *b*: $b \notin$ *vs2.span* (*f* ‘ *B1*) **for** *b*
proof −
have ∗: *vs2.independent* (*f* ‘ *B1*)
using *vs1.independent-Basis* **by** (*intro independent-injective-image inj-on-subset*[*OF sf*]) *auto*
have ∗∗: *vs2.independent* (*insert b* (*f* ‘ *B1*))
using *b* ∗ **by** (*rule vs2.independent-insertI*)

have $b \notin f$ ‘ *B1* **using** *vs2.span-base*[*of b f* ‘ *B1*] *b* **by** *auto*
then have *Suc* (*card B1*) = *card* (*insert b* (*f* ‘ *B1*))
using *sf*[*THEN inj-on-subset*, *of B1*] **by** (*subst card.insert-remove*) (*auto intro*: *vs1.finite-Basis simp*: *card-image*)
also have ... = *vs2.dim* (*insert b* (*f* ‘ *B1*))
using *vs2.dim-eq-card-independent*[*OF* ∗∗] **by** *simp*
also have *vs2.dim* (*insert b* (*f* ‘ *B1*)) ≤ *vs2.dim B2*
by (*rule vs2.dim-mono*) (*auto simp*: *vs2.span-Basis*)
also have ... = *card B1*
using *vs1.dim-span*[*of B1*] *vs2.dim-span*[*of B2*] **unfolding** *vs1.span-Basis vs2.span-Basis eq*
vs1.dim-eq-card-independent[*OF vs1.independent-Basis*] **by** *simp*
finally show *False* **by** *simp*
qed
have *f* ‘ *UNIV* = *f* ‘ *vs1.span B1* **unfolding** *vs1.span-Basis* **..**
also have ... = *vs2.span* (*f* ‘ *B1*) **unfolding** *span-image* **..**
also have ... = *UNIV* **using** ∗ **by** *blast*
finally show *?thesis* **.**
qed

lemma *linear-injective-isomorphism*:
assumes *lf*: *linear s1 s2 f*
and *fi*: *inj f*
and *dims*: *vs2.dim UNIV* = *vs1.dim UNIV*
shows $\exists f'.$ *linear s2 s1* $f' \wedge (\forall x.\ f'\ (f\ x) = x) \wedge (\forall x.\ f\ (f'\ x) = x)$
unfolding *isomorphism-expand*[*symmetric*]
using *linear-injective-imp-surjective*[*OF lf fi dims*]
using *fi left-right-inverse-eq lf linear-injective-left-inverse linear-surjective-right-inverse*
by *blast*

lemma *linear-surjective-isomorphism*:
assumes *lf*: *linear s1 s2 f*
and *sf*: *surj f*
and *dims*: *vs2.dim UNIV* = *vs1.dim UNIV*
shows $\exists f'$. *linear s2 s1 f'* $\wedge$ ($\forall$ *x*. *f'* (*f x*) = *x*) $\wedge$ ($\forall$ *x*. *f* (*f' x*) = *x*)
using *linear-surjective-imp-injective*[*OF lf sf dims*] *sf*
linear-exists-right-inverse-on[*OF lf vs1.subspace-UNIV*]
linear-exists-left-inverse-on[*OF lf vs1.subspace-UNIV*]
dims lf linear-injective-isomorphism **by** *auto*

lemma *basis-to-basis-subspace-isomorphism*:
assumes *s*: *vs1.subspace S*
and *t*: *vs2.subspace T*
and *d*: *vs1.dim S* = *vs2.dim T*
and *B*: *B* $\subseteq$ *S vs1.independent B S* $\subseteq$ *vs1.span B card B* = *vs1.dim S*
and *C*: *C* $\subseteq$ *T vs2.independent C T* $\subseteq$ *vs2.span C card C* = *vs2.dim T*
shows $\exists f$. *linear s1 s2 f* $\wedge$ *f* ' *B* = *C* $\wedge$ *f* ' *S* = *T* $\wedge$ *inj-on f S*
proof –
from *B* **have** *fB*: *finite B*
by (*simp add*: *vs1.finiteI-independent*)
from *C* **have** *fC*: *finite C*
by (*simp add*: *vs2.finiteI-independent*)
from *finite-basis-to-basis-subspace-isomorphism*[*OF s t d fB B fC C*] **show** *?thesis*
.
qed

end

context *finite-dimensional-vector-space* **begin**

lemma *linear-surj-imp-inj*:
assumes *lf*: *linear scale scale f* **and** *sf*: *surj f*
shows *inj f*
proof –
interpret *finite-dimensional-vector-space-pair scale Basis scale Basis* **by** *unfold-locales*
let *?U* = *UNIV* :: *'b set*
from *basis-exists*[*of ?U*] **obtain** *B*
where *B*: *B* $\subseteq$ *?U independent B ?U* $\subseteq$ *span B* **and** *d*: *card B* = *dim ?U*
by *blast*
{
fix *x*
assume *x*: *x* $\in$ *span B* **and** *fx*: *f x* = *0*
from *B*(*2*) **have** *fB*: *finite B*
using *finiteI-independent* **by** *auto*
have *Uspan*: *UNIV* $\subseteq$ *span* (*f* ' *B*)
by (*simp add*: *B*(*3*) *lf linear-spanning-surjective-image sf*)
have *fBi*: *independent* (*f* ' *B*)

proof (*rule card-le-dim-spanning*)
show *card* (*f* ' *B*) ≤ *dim ?U*
using *card-image-le d fB* **by** *fastforce*
qed (*use fB Uspan* **in** *auto*)
have *th0*: *dim ?U* ≤ *card* (*f* ' *B*)
by (*rule span-card-ge-dim*) (*use Uspan fB* **in** *auto*)
moreover have *card* (*f* ' *B*) ≤ *card B*
by (*rule card-image-le*, *rule fB*)
ultimately have *th1*: *card B* = *card* (*f* ' *B*)
unfolding *d* **by** *arith*
have *fiB*: *inj-on f B*
by (*simp add*: *eq-card-imp-inj-on fB th1*)
from *linear-indep-image-lemma*[*OF lf fB fBi fiB x*] *fx*
have *x* = *0* **by** *blast*
}
then show *?thesis*
unfolding *linear-inj-iff-eq-0*[*OF lf*] **using** *B*(*3*) **by** *blast*
qed

lemma *linear-inverse-left*:
assumes *lf*: *linear scale scale f*
and *lf'*: *linear scale scale f'*
shows *f* ∘ *f'* = *id* ⟷ *f'* ∘ *f* = *id*
proof −
{
fix *f f'*:: *'b* ⇒ *'b*
assume *lf*: *linear scale scale f linear scale scale f'*
assume *f*: *f* ∘ *f'* = *id*
from *f* **have** *sf*: *surj f*
by (*auto simp add*: *o-def id-def surj-def*) *metis*
interpret *finite-dimensional-vector-space-pair scale Basis scale Basis* **by** *unfold-locales*
from *linear-surjective-isomorphism*[*OF lf*(*1*) *sf*] *lf f*
have *f'* ∘ *f* = *id*
unfolding *fun-eq-iff o-def id-def* **by** *metis*
}
then show *?thesis*
using *lf lf'* **by** *metis*
qed

lemma *left-inverse-linear*:
assumes *lf*: *linear scale scale f*
and *gf*: *g* ∘ *f* = *id*
shows *linear scale scale g*
proof −
from *gf* **have** *fi*: *inj f*
by (*auto simp add*: *inj-on-def o-def id-def fun-eq-iff*) *metis*
interpret *finite-dimensional-vector-space-pair scale Basis scale Basis* **by** *unfold-locales*

from *linear-injective-isomorphism*[*OF lf fi*]
obtain *h* :: *'b* ⇒ *'b* **where** *linear scale scale h* **and** *h*: ∀ *x*. *h* (*f x*) = *x* ∀ *x*. *f* (*h* *x*) = *x*
by *blast*
have *h* = *g*
by (*metis gf h isomorphism-expand left-right-inverse-eq*)
with ‹*linear scale scale h*› **show** *?thesis* **by** *blast*
qed

lemma *inj-linear-imp-inv-linear*:
assumes *linear scale scale f inj f* **shows** *linear scale scale* (*inv f*)
using *assms inj-iff left-inverse-linear* **by** *blast*

lemma *right-inverse-linear*:
assumes *lf*: *linear scale scale f*
and *gf*: *f* ∘ *g* = *id*
shows *linear scale scale g*
proof –
from *gf* **have** *fi*: *surj f*
by (*auto simp add*: *surj-def o-def id-def*) *metis*
interpret *finite-dimensional-vector-space-pair scale Basis scale Basis* **by** *unfold-locales*
from *linear-surjective-isomorphism*[*OF lf fi*]
obtain *h*:: *'b* ⇒ *'b* **where** *h*: *linear scale scale h* ∀ *x*. *h* (*f x*) = *x* ∀ *x*. *f* (*h x*) = *x*
by *blast*
then have *h* = *g*
by (*metis gf isomorphism-expand left-right-inverse-eq*)
with *h*(*1*) **show** *?thesis* **by** *blast*
qed

end

context *finite-dimensional-vector-space-pair* **begin**

lemma *subspace-isomorphism*:
assumes *s*: *vs1.subspace S*
and *t*: *vs2.subspace T*
and *d*: *vs1.dim S* = *vs2.dim T*
shows ∃ *f*. *linear s1 s2 f* ∧ *f* ‘ *S* = *T* ∧ *inj-on f S*
proof –
from *vs1.basis-exists*[*of S*] *vs1.finiteI-independent*
obtain *B* **where** *B*: *B* ⊆ *S vs1.independent B S* ⊆ *vs1.span B card B* = *vs1.dim S* **and** *fB*: *finite B*
by *metis*
from *vs2.basis-exists*[*of T*] *vs2.finiteI-independent*
obtain *C* **where** *C*: *C* ⊆ *T vs2.independent C T* ⊆ *vs2.span C card C* = *vs2.dim T* **and** *fC*: *finite C*
by *metis*
from *B*(*4*) *C*(*4*) *card-le-inj*[*of B C*] *d*

```
  obtain f where f: f ‘ B ⊆ C inj-on f B using ‹finite B› ‹finite C›
    by auto
  from linear-independent-extend[OF B(2)]
  obtain g where g: linear s1 s2 g ∀ x∈ B. g x = f x
    by blast
  interpret g: linear s1 s2 g by fact
  from inj-on-iff-eq-card[OF fB, of f] f(2) have card (f ‘ B) = card B
    by simp
  with B(4) C(4) have ceq: card (f ‘ B) = card C
    using d by simp
  have g ‘ B = f ‘ B
    using g(2) by (auto simp add: image-iff)
  also have . . . = C using card-subset-eq[OF fC f(1) ceq] .
  finally have gBC: g ‘ B = C .
  have gi: inj-on g B
    using f(2) g(2) by (auto simp add: inj-on-def)
  note g0 = linear-indep-image-lemma[OF g(1) fB, unfolded gBC, OF C(2) gi]
  {
    fix x y
    assume x: x ∈ S and y: y ∈ S and gxy: g x = g y
    from B(3) x y have x': x ∈ vs1.span B and y': y ∈ vs1.span B
      by blast+
    from gxy have th0: g (x − y) = 0
      by (simp add: linear-diff g)
    have th1: x − y ∈ vs1.span B
      using x' y' by (metis vs1.span-diff)
    have x = y
      using g0[OF th1 th0] by simp
  }
  then have giS: inj-on g S
    unfolding inj-on-def by blast
  from vs1.span-subspace[OF B(1,3) s] have g ‘ S = vs2.span (g ‘ B)
   by (simp add: module-hom.span-image[OF g(1)[unfolded linear-iff-module-hom]])
  also have . . . = vs2.span C unfolding gBC ..
  also have . . . = T using vs2.span-subspace[OF C(1,3) t] .
  finally have gS: g ‘ S = T .
  from g(1) gS giS show ?thesis
    by blast
qed

end

hide-const (open) linear

end
```

107 Vector Spaces and Algebras over the Reals

theory *Real-Vector-Spaces*

imports *Real Topological-Spaces Vector-Spaces*
begin

107.1 Real vector spaces

class *scaleR* =
  fixes *scaleR* :: *real* ⇒ *'a* ⇒ *'a* (**infixr** ‹$*_R$› *75*)
begin

abbreviation *divideR* :: *'a* ⇒ *real* ⇒ *'a* (**infixl** ‹$'/_R$› *70*)
  where $x /_R r \equiv$ *inverse* $r *_R x$

end

class *real-vector* = *scaleR* + *ab-group-add* +
  assumes *scaleR-add-right*: $a *_R (x + y) = a *_R x + a *_R y$
  and *scaleR-add-left*: $(a + b) *_R x = a *_R x + b *_R x$
  and *scaleR-scaleR*: $a *_R b *_R x = (a * b) *_R x$
  and *scaleR-one*: $1 *_R x = x$

class *real-algebra* = *real-vector* + *ring* +
  assumes *mult-scaleR-left* [*simp*]: $a *_R x * y = a *_R (x * y)$
    and *mult-scaleR-right* [*simp*]: $x * a *_R y = a *_R (x * y)$

class *real-algebra-1* = *real-algebra* + *ring-1*

class *real-div-algebra* = *real-algebra-1* + *division-ring*

class *real-field* = *real-div-algebra* + *field*

instantiation *real* :: *real-field*
begin

definition *real-scaleR-def* [*simp*]: *scaleR* $a\ x = a * x$

instance
  by *standard* (*simp-all add*: *algebra-simps*)

end

locale *linear* = *Vector-Spaces.linear scaleR*::-⇒-⇒*'a*::*real-vector scaleR*::-⇒-⇒*'b*::*real-vector*
begin

lemmas *scaleR* = *scale*

end

global-interpretation *real-vector?*: *vector-space scaleR* :: *real* ⇒ *'a* ⇒ *'a* :: *real-vector*
  rewrites *Vector-Spaces.linear* $(*_R)$ $(*_R)$ = *linear*

and *Vector-Spaces.linear* $(*)$ $(*_R)$ = *linear*
defines *dependent-raw-def*: *dependent* = *real-vector.dependent*
and *representation-raw-def*: *representation* = *real-vector.representation*
and *subspace-raw-def*: *subspace* = *real-vector.subspace*
and *span-raw-def*: *span* = *real-vector.span*
and *extend-basis-raw-def*: *extend-basis* = *real-vector.extend-basis*
and *dim-raw-def*: *dim* = *real-vector.dim*
proof *unfold-locales*
show *Vector-Spaces.linear* $(*_R)$ $(*_R)$ = *linear Vector-Spaces.linear* $(*)$ $(*_R)$ = *linear*
by (*force simp*: *linear-def real-scaleR-def*[*abs-def*])+
qed (*use scaleR-add-right scaleR-add-left scaleR-scaleR scaleR-one* **in** *auto*)

hide-const (**open**)— locale constants
real-vector.dependent
real-vector.independent
real-vector.representation
real-vector.subspace
real-vector.span
real-vector.extend-basis
real-vector.dim

abbreviation *independent x* ≡ ¬ *dependent x*

global-interpretation *real-vector?*: *vector-space-pair scaleR*::-⇒-⇒'*a*::*real-vector scaleR*::-⇒-⇒'*b*::*real-vector*
rewrites *Vector-Spaces.linear* $(*_R)$ $(*_R)$ = *linear*
and *Vector-Spaces.linear* $(*)$ $(*_R)$ = *linear*
defines *construct-raw-def*: *construct* = *real-vector.construct*
proof *unfold-locales*
show *Vector-Spaces.linear* $(*)$ $(*_R)$ = *linear*
unfolding *linear-def real-scaleR-def* **by** *auto*
qed (*auto simp*: *linear-def*)

hide-const (**open**)— locale constants
real-vector.construct

lemma *linear-compose*: *linear f* ⟹ *linear g* ⟹ *linear* $(g \circ f)$
unfolding *linear-def* **by** (*rule Vector-Spaces.linear-compose*)

Recover original theorem names

lemmas *scaleR-left-commute* = *real-vector.scale-left-commute*
lemmas *scaleR-zero-left* = *real-vector.scale-zero-left*
lemmas *scaleR-minus-left* = *real-vector.scale-minus-left*
lemmas *scaleR-diff-left* = *real-vector.scale-left-diff-distrib*
lemmas *scaleR-sum-left* = *real-vector.scale-sum-left*
lemmas *scaleR-zero-right* = *real-vector.scale-zero-right*
lemmas *scaleR-minus-right* = *real-vector.scale-minus-right*
lemmas *scaleR-diff-right* = *real-vector.scale-right-diff-distrib*

lemmas *scaleR-sum-right* = *real-vector.scale-sum-right*
lemmas *scaleR-eq-0-iff* = *real-vector.scale-eq-0-iff*
lemmas *scaleR-left-imp-eq* = *real-vector.scale-left-imp-eq*
lemmas *scaleR-right-imp-eq* = *real-vector.scale-right-imp-eq*
lemmas *scaleR-cancel-left* = *real-vector.scale-cancel-left*
lemmas *scaleR-cancel-right* = *real-vector.scale-cancel-right*

lemma [*field-simps*]:
$c \neq 0 \Longrightarrow a = b /_R\ c \longleftrightarrow c *_R\ a = b$
$c \neq 0 \Longrightarrow b /_R\ c = a \longleftrightarrow b = c *_R\ a$
$c \neq 0 \Longrightarrow a + b /_R\ c = (c *_R\ a + b) /_R\ c$
$c \neq 0 \Longrightarrow a /_R\ c + b = (a + c *_R\ b) /_R\ c$
$c \neq 0 \Longrightarrow a - b /_R\ c = (c *_R\ a - b) /_R\ c$
$c \neq 0 \Longrightarrow a /_R\ c - b = (a - c *_R\ b) /_R\ c$
$c \neq 0 \Longrightarrow - (a /_R\ c) + b = (- a + c *_R\ b) /_R\ c$
$c \neq 0 \Longrightarrow - (a /_R\ c) - b = (- a - c *_R\ b) /_R\ c$
for *a b* :: *'a* :: *real-vector*
by (*auto simp add*: *scaleR-add-right scaleR-add-left scaleR-diff-right scaleR-diff-left*)

Legacy names

lemmas *scaleR-left-distrib* = *scaleR-add-left*
lemmas *scaleR-right-distrib* = *scaleR-add-right*
lemmas *scaleR-left-diff-distrib* = *scaleR-diff-left*
lemmas *scaleR-right-diff-distrib* = *scaleR-diff-right*

lemmas *linear-injective-0* = *linear-inj-iff-eq-0*
and *linear-injective-on-subspace-0* = *linear-inj-on-iff-eq-0*
and *linear-cmul* = *linear-scale*
and *linear-scaleR* = *linear-scale-self*
and *subspace-mul* = *subspace-scale*
and *span-linear-image* = *linear-span-image*
and *span-0* = *span-zero*
and *span-mul* = *span-scale*
and *injective-scaleR* = *injective-scale*

lemma *scaleR-minus1-left* [*simp*]: *scaleR* (-1) $x = -\ x$
for *x* :: *'a*::*real-vector*
by *simp*

lemma *scaleR-2*:
fixes *x* :: *'a*::*real-vector*
shows *scaleR* $2\ x = x + x$
unfolding *one-add-one* [*symmetric*] *scaleR-left-distrib* **by** *simp*

lemma *scaleR-half-double* [*simp*]:
fixes *a* :: *'a*::*real-vector*
shows $(1\ /\ 2) *_R\ (a + a) = a$
proof −
have $\bigwedge r.\ r *_R\ (a + a) = (r * 2) *_R\ a$

by (*metis scaleR-2 scaleR-scaleR*)
then show *?thesis*
by *simp*
qed

lemma *shift-zero-ident* [*simp*]:
fixes f :: $'a \Rightarrow 'b$::*real-vector*
shows $(+)0 \circ f = f$
by *force*

lemma *linear-scale-real*:
fixes *r*::*real* **shows** *linear f* $\Longrightarrow f\ (r * b) = r * f\ b$
using *linear-scale* **by** *fastforce*

interpretation *scaleR-left*: *additive* ($\lambda a.$ *scaleR a x* :: $'a$::*real-vector*)
by *standard* (*rule scaleR-left-distrib*)

interpretation *scaleR-right*: *additive* ($\lambda x.$ *scaleR a x* :: $'a$::*real-vector*)
by *standard* (*rule scaleR-right-distrib*)

lemma *nonzero-inverse-scaleR-distrib*:
$a \neq 0 \Longrightarrow x \neq 0 \Longrightarrow$ *inverse* (*scaleR a x*) = *scaleR* (*inverse a*) (*inverse x*)
for x :: $'a$::*real-div-algebra*
by (*rule inverse-unique*) *simp*

lemma *inverse-scaleR-distrib*: *inverse* (*scaleR a x*) = *scaleR* (*inverse a*) (*inverse x*)
for x :: $'a$::{*real-div-algebra*,*division-ring*}
by (*metis inverse-zero nonzero-inverse-scaleR-distrib scale-eq-0-iff*)

lemmas *sum-constant-scaleR* = *real-vector.sum-constant-scale*— legacy name

named-theorems *vector-add-divide-simps to simplify sums of scaled vectors*

lemma [*vector-add-divide-simps*]:
$v + (b / z) *_R w = (\textit{if } z = 0 \textit{ then } v \textit{ else } (z *_R v + b *_R w) /_R z)$
$a *_R v + (b / z) *_R w = (\textit{if } z = 0 \textit{ then } a *_R v \textit{ else } ((a * z) *_R v + b *_R w) /_R z)$
$(a / z) *_R v + w = (\textit{if } z = 0 \textit{ then } w \textit{ else } (a *_R v + z *_R w) /_R z)$
$(a / z) *_R v + b *_R w = (\textit{if } z = 0 \textit{ then } b *_R w \textit{ else } (a *_R v + (b * z) *_R w) /_R z)$
$v - (b / z) *_R w = (\textit{if } z = 0 \textit{ then } v \textit{ else } (z *_R v - b *_R w) /_R z)$
$a *_R v - (b / z) *_R w = (\textit{if } z = 0 \textit{ then } a *_R v \textit{ else } ((a * z) *_R v - b *_R w) /_R z)$
$(a / z) *_R v - w = (\textit{if } z = 0 \textit{ then } -w \textit{ else } (a *_R v - z *_R w) /_R z)$
$(a / z) *_R v - b *_R w = (\textit{if } z = 0 \textit{ then } -b *_R w \textit{ else } (a *_R v - (b * z) *_R w) /_R z)$
for v :: $'a$ :: *real-vector*
by (*simp-all add*: *divide-inverse-commute scaleR-add-right scaleR-diff-right*)

lemma *eq-vector-fraction-iff* [*vector-add-divide-simps*]:
  fixes x :: $'a$ :: *real-vector*
  shows $(x = (u\ /\ v) *_R a) \longleftrightarrow (\mathit{if}\ v{=}0\ \mathit{then}\ x = 0\ \mathit{else}\ v *_R x = u *_R a)$
by *auto* (*metis* (*no-types*) *divide-eq-1-iff divide-inverse-commute scaleR-one scaleR-scaleR*)

lemma *vector-fraction-eq-iff* [*vector-add-divide-simps*]:
  fixes x :: $'a$ :: *real-vector*
  shows $((u\ /\ v) *_R a = x) \longleftrightarrow (\mathit{if}\ v{=}0\ \mathit{then}\ x = 0\ \mathit{else}\ u *_R a = v *_R x)$
by (*metis eq-vector-fraction-iff*)

lemma *real-vector-affinity-eq*:
  fixes x :: $'a$ :: *real-vector*
  assumes *m0*: $m \neq 0$
  shows $m *_R x + c = y \longleftrightarrow x = \mathit{inverse}\ m *_R y - (\mathit{inverse}\ m *_R c)$
    (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
  assume *?lhs*
  then have $m *_R x = y - c$ **by** (*simp add*: *field-simps*)
  then have $\mathit{inverse}\ m *_R (m *_R x) = \mathit{inverse}\ m *_R (y - c)$ **by** *simp*
  then show $x = \mathit{inverse}\ m *_R y - (\mathit{inverse}\ m *_R c)$
    using *m0*
  by (*simp add*: *scaleR-diff-right*)
next
  assume *?rhs*
  with *m0* **show** $m *_R x + c = y$
    by (*simp add*: *scaleR-diff-right*)
qed

lemma *real-vector-eq-affinity*: $m \neq 0 \Longrightarrow y = m *_R x + c \longleftrightarrow \mathit{inverse}\ m *_R y - (\mathit{inverse}\ m *_R c) = x$
  for x :: $'a$::*real-vector*
  using *real-vector-affinity-eq*[**where** $m{=}m$ **and** $x{=}x$ **and** $y{=}y$ **and** $c{=}c$]
  by *metis*

lemma *scaleR-eq-iff* [*simp*]: $b + u *_R a = a + u *_R b \longleftrightarrow a = b \lor u = 1$
  for a :: $'a$::*real-vector*
proof (*cases* $u = 1$)
  case *True*
  then show *?thesis* **by** *auto*
next
  case *False*
  have $a = b$ **if** $b + u *_R a = a + u *_R b$
  proof −
    from *that* **have** $(u - 1) *_R a = (u - 1) *_R b$
      by (*simp add*: *algebra-simps*)
    with *False* **show** *?thesis*
      by *auto*

qed
then show *?thesis* **by** *auto*
qed

lemma *scaleR-collapse* [*simp*]: $(1 - u) *_R a + u *_R a = a$
for *a* :: *'a::real-vector*
by (*simp add*: *algebra-simps*)

107.2 Embedding of the Reals into any *real-algebra-1*: *of-real*

definition *of-real* :: *real* ⇒ *'a::real-algebra-1*
where *of-real r* = *scaleR r 1*

lemma *scaleR-conv-of-real*: *scaleR r x* = *of-real r* ∗ *x*
by (*simp add*: *of-real-def*)

lemma *of-real-0* [*simp*]: *of-real 0* = *0*
by (*simp add*: *of-real-def*)

lemma *of-real-1* [*simp*]: *of-real 1* = *1*
by (*simp add*: *of-real-def*)

lemma *of-real-add* [*simp*]: *of-real* (*x* + *y*) = *of-real x* + *of-real y*
by (*simp add*: *of-real-def scaleR-left-distrib*)

lemma *of-real-minus* [*simp*]: *of-real* (− *x*) = − *of-real x*
by (*simp add*: *of-real-def*)

lemma *of-real-diff* [*simp*]: *of-real* (*x* − *y*) = *of-real x* − *of-real y*
by (*simp add*: *of-real-def scaleR-left-diff-distrib*)

lemma *of-real-mult* [*simp*]: *of-real* (*x* ∗ *y*) = *of-real x* ∗ *of-real y*
by (*simp add*: *of-real-def*)

lemma *of-real-sum*[*simp*]: *of-real* (*sum f s*) = (∑ *x*∈*s*. *of-real* (*f x*))
by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *of-real-prod*[*simp*]: *of-real* (*prod f s*) = (∏ *x*∈*s*. *of-real* (*f x*))
by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *sum-list-of-real*: *sum-list* (*map of-real xs*) = *of-real* (*sum-list xs*)
by (*induction xs*) *auto*

lemma *nonzero-of-real-inverse*:
x ≠ *0* ⟹ *of-real* (*inverse x*) = *inverse* (*of-real x* :: *'a::real-div-algebra*)
by (*simp add*: *of-real-def nonzero-inverse-scaleR-distrib*)

lemma *of-real-inverse* [*simp*]:
of-real (*inverse x*) = *inverse* (*of-real x* :: *'a::*{*real-div-algebra*,*division-ring*})

by (*simp add*: *of-real-def inverse-scaleR-distrib*)

lemma *nonzero-of-real-divide*:
$y \neq 0 \Longrightarrow$ *of-real* $(x\ /\ y) =$ (*of-real* x / *of-real* y :: *'a::real-field*)
by (*simp add*: *divide-inverse nonzero-of-real-inverse*)

lemma *of-real-divide* [*simp*]:
of-real $(x\ /\ y) =$ (*of-real* x / *of-real* y :: *'a::real-div-algebra*)
by (*simp add*: *divide-inverse*)

lemma *of-real-power* [*simp*]:
of-real $(x\ \hat{}\ n) =$ (*of-real* x :: *'a::{real-algebra-1}*) $\hat{}\ n$
by (*induct n*) *simp-all*

lemma *of-real-power-int* [*simp*]:
of-real (*power-int* x n) = *power-int* (*of-real* x :: *'a* :: {*real-div-algebra,division-ring*}) n
by (*auto simp*: *power-int-def*)

lemma *of-real-eq-iff* [*simp*]: *of-real* x = *of-real* $y \longleftrightarrow x = y$
by (*simp add*: *of-real-def*)

lemma *inj-of-real*: *inj of-real*
by (*auto intro*: *injI*)

lemmas *of-real-eq-0-iff* [*simp*] = *of-real-eq-iff* [*of - 0*, *simplified*]
lemmas *of-real-eq-1-iff* [*simp*] = *of-real-eq-iff* [*of - 1*, *simplified*]

lemma *minus-of-real-eq-of-real-iff* [*simp*]: $-$*of-real* x = *of-real* $y \longleftrightarrow -x = y$
using *of-real-eq-iff*[*of* $-x$ y] **by** (*simp only*: *of-real-minus*)

lemma *of-real-eq-minus-of-real-iff* [*simp*]: *of-real* $x = -$*of-real* $y \longleftrightarrow x = -y$
using *of-real-eq-iff*[*of* x $-y$] **by** (*simp only*: *of-real-minus*)

lemma *of-real-eq-id* [*simp*]: *of-real* = (*id* :: *real* ⇒ *real*)
by (*rule ext*) (*simp add*: *of-real-def*)

Collapse nested embeddings.

lemma *of-real-of-nat-eq* [*simp*]: *of-real* (*of-nat* n) = *of-nat* n
by (*induct n*) *auto*

lemma *of-real-of-int-eq* [*simp*]: *of-real* (*of-int* z) = *of-int* z
by (*cases z rule*: *int-diff-cases*) *simp*

lemma *of-real-numeral* [*simp*]: *of-real* (*numeral* w) = *numeral* w
using *of-real-of-int-eq* [*of numeral w*] **by** *simp*

lemma *of-real-neg-numeral* [*simp*]: *of-real* ($-$ *numeral* w) = $-$ *numeral* w
using *of-real-of-int-eq* [*of* $-$ *numeral w*] **by** *simp*

lemma *numeral-power-int-eq-of-real-cancel-iff* [*simp*]:
power-int (*numeral x*) *n* = (*of-real y* :: *'a* :: {*real-div-algebra*, *division-ring*}) ⟷
power-int (*numeral x*) *n* = *y*
proof −
have *power-int* (*numeral x*) *n* = (*of-real* (*power-int* (*numeral x*) *n*) :: *'a*)
by *simp*
also have ... = *of-real y* ⟷ *power-int* (*numeral x*) *n* = *y*
by (*subst of-real-eq-iff*) *auto*
finally show *?thesis* **.**
qed

lemma *of-real-eq-numeral-power-int-cancel-iff* [*simp*]:
(*of-real y* :: *'a* :: {*real-div-algebra*, *division-ring*}) = *power-int* (*numeral x*) *n* ⟷
y = *power-int* (*numeral x*) *n*
by (*subst* (*1 2*) *eq-commute*) *simp*

lemma *of-real-eq-of-real-power-int-cancel-iff* [*simp*]:
power-int (*of-real b* :: *'a* :: {*real-div-algebra*, *division-ring*}) *w* = *of-real x* ⟷
power-int b w = *x*
by (*metis of-real-power-int of-real-eq-iff*)

lemma *of-real-in-Ints-iff* [*simp*]: *of-real x* ∈ ℤ ⟷ *x* ∈ ℤ
proof *safe*
fix *x* **assume** (*of-real x* :: *'a*) ∈ ℤ
then obtain *n* **where** (*of-real x* :: *'a*) = *of-int n*
by (*auto simp*: *Ints-def*)
also have *of-int n* = *of-real* (*real-of-int n*)
by *simp*
finally have *x* = *real-of-int n*
by (*subst* (*asm*) *of-real-eq-iff*)
thus *x* ∈ ℤ
by *auto*
qed (*auto simp*: *Ints-def*)

lemma *Ints-of-real* [*intro*]: *x* ∈ ℤ ⟹ *of-real x* ∈ ℤ
by *simp*

Every real algebra has characteristic zero.

instance *real-algebra-1* < *ring-char-0*
proof
from *inj-of-real inj-of-nat* **have** *inj* (*of-real* ∘ *of-nat*)
by (*rule inj-compose*)
then show *inj* (*of-nat* :: *nat* ⇒ *'a*)
by (*simp add*: *comp-def*)
qed

lemma *fraction-scaleR-times* [*simp*]:
fixes *a* :: *'a*::*real-algebra-1*

shows (*numeral u* / *numeral v*) $*_R$ (*numeral w* $*$ *a*) = (*numeral u* $*$ *numeral w* / *numeral v*) $*_R$ *a*
by (*metis* (*no-types*, *lifting*) *of-real-numeral scaleR-conv-of-real scaleR-scaleR times-divide-eq-left*)

lemma *inverse-scaleR-times* [*simp*]:
fixes *a* :: *'a::real-algebra-1*
shows (*1* / *numeral v*) $*_R$ (*numeral w* $*$ *a*) = (*numeral w* / *numeral v*) $*_R$ *a*
by (*metis divide-inverse-commute inverse-eq-divide of-real-numeral scaleR-conv-of-real scaleR-scaleR*)

lemma *scaleR-times* [*simp*]:
fixes *a* :: *'a::real-algebra-1*
shows (*numeral u*) $*_R$ (*numeral w* $*$ *a*) = (*numeral u* $*$ *numeral w*) $*_R$ *a*
by (*simp add*: *scaleR-conv-of-real*)

instance *real-field* < *field-char-0* **..**

107.3 The Set of Real Numbers

definition *Reals* :: *'a::real-algebra-1 set* (‹ℝ›)
where ℝ = *range of-real*

lemma *Reals-of-real* [*simp*]: *of-real r* ∈ ℝ
by (*simp add*: *Reals-def*)

lemma *Reals-of-int* [*simp*]: *of-int z* ∈ ℝ
by (*subst of-real-of-int-eq* [*symmetric*], *rule Reals-of-real*)

lemma *Reals-of-nat* [*simp*]: *of-nat n* ∈ ℝ
by (*subst of-real-of-nat-eq* [*symmetric*], *rule Reals-of-real*)

lemma *Reals-numeral* [*simp*]: *numeral w* ∈ ℝ
by (*subst of-real-numeral* [*symmetric*], *rule Reals-of-real*)

lemma *Reals-0* [*simp*]: *0* ∈ ℝ **and** *Reals-1* [*simp*]: *1* ∈ ℝ
by (*simp-all add*: *Reals-def*)

lemma *Reals-add* [*simp*]: *a* ∈ ℝ ⟹ *b* ∈ ℝ ⟹ *a* + *b* ∈ ℝ
by (*metis* (*no-types*, *opaque-lifting*) *Reals-def Reals-of-real imageE of-real-add*)

lemma *Reals-minus* [*simp*]: *a* ∈ ℝ ⟹ − *a* ∈ ℝ
by (*auto simp*: *Reals-def*)

lemma *Reals-minus-iff* [*simp*]: − *a* ∈ ℝ ⟷ *a* ∈ ℝ
using *Reals-minus* **by** *fastforce*

lemma *Reals-diff* [*simp*]: *a* ∈ ℝ ⟹ *b* ∈ ℝ ⟹ *a* − *b* ∈ ℝ
by (*metis Reals-add Reals-minus-iff add-uminus-conv-diff*)

lemma *Reals-mult* [*simp*]: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow a * b \in \mathbb{R}$
by (*metis* (*no-types*, *lifting*) *Reals-def Reals-of-real imageE of-real-mult*)

lemma *nonzero-Reals-inverse*: $a \in \mathbb{R} \Longrightarrow a \neq 0 \Longrightarrow$ *inverse* $a \in \mathbb{R}$
for *a* :: *'a*::*real-div-algebra*
by (*metis Reals-def Reals-of-real imageE of-real-inverse*)

lemma *Reals-inverse*: $a \in \mathbb{R} \Longrightarrow$ *inverse* $a \in \mathbb{R}$
for *a* :: *'a*::{*real-div-algebra*,*division-ring*}
using *nonzero-Reals-inverse* **by** *fastforce*

lemma *Reals-inverse-iff* [*simp*]: *inverse* $x \in \mathbb{R} \longleftrightarrow x \in \mathbb{R}$
for *x* :: *'a*::{*real-div-algebra*,*division-ring*}
by (*metis Reals-inverse inverse-inverse-eq*)

lemma *nonzero-Reals-divide*: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow b \neq 0 \Longrightarrow a / b \in \mathbb{R}$
for *a b* :: *'a*::*real-field*
by (*simp add*: *divide-inverse*)

lemma *Reals-divide* [*simp*]: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow a / b \in \mathbb{R}$
for *a b* :: *'a*::{*real-field*,*field*}
using *nonzero-Reals-divide* **by** *fastforce*

lemma *Reals-power* [*simp*]: $a \in \mathbb{R} \Longrightarrow a \hat{} n \in \mathbb{R}$
for *a* :: *'a*::*real-algebra-1*
by (*metis Reals-def Reals-of-real imageE of-real-power*)

lemma *Reals-cases* [*cases set*: *Reals*]:
assumes $q \in \mathbb{R}$
obtains (*of-real*) *r* **where** $q =$ *of-real r*
unfolding *Reals-def*
proof −
from ‹$q \in \mathbb{R}$› **have** $q \in$ *range of-real* **unfolding** *Reals-def* **.**
then obtain *r* **where** $q =$ *of-real r* **..**
then show *thesis* **..**
qed

lemma *sum-in-Reals* [*intro*,*simp*]: $(\bigwedge i.\ i \in s \Longrightarrow f\ i \in \mathbb{R}) \Longrightarrow$ *sum f s* $\in \mathbb{R}$
proof (*induct s rule*: *infinite-finite-induct*)
case *infinite*
then show *?case* **by** (*metis Reals-0 sum.infinite*)
qed *simp-all*

lemma *prod-in-Reals* [*intro*,*simp*]: $(\bigwedge i.\ i \in s \Longrightarrow f\ i \in \mathbb{R}) \Longrightarrow$ *prod f s* $\in \mathbb{R}$
proof (*induct s rule*: *infinite-finite-induct*)
case *infinite*
then show *?case* **by** (*metis Reals-1 prod.infinite*)
qed *simp-all*

lemma *Reals-induct* [*case-names of-real*, *induct set*: *Reals*]:
$q \in \mathbb{R} \Longrightarrow (\bigwedge r.\ P\ (of\text{-}real\ r)) \Longrightarrow P\ q$
by (*rule Reals-cases*) *auto*

107.4 Ordered real vector spaces

class *ordered-real-vector* = *real-vector* + *ordered-ab-group-add* +
assumes *scaleR-left-mono*: $x \le y \Longrightarrow 0 \le a \Longrightarrow a *_R x \le a *_R y$
and *scaleR-right-mono*: $a \le b \Longrightarrow 0 \le x \Longrightarrow a *_R x \le b *_R x$
begin

lemma *scaleR-mono*:
$a \le b \Longrightarrow x \le y \Longrightarrow 0 \le b \Longrightarrow 0 \le x \Longrightarrow a *_R x \le b *_R y$
by (*meson order-trans scaleR-left-mono scaleR-right-mono*)

lemma *scaleR-mono'*:
$a \le b \Longrightarrow c \le d \Longrightarrow 0 \le a \Longrightarrow 0 \le c \Longrightarrow a *_R c \le b *_R d$
by (*rule scaleR-mono*) (*auto intro*: *order.trans*)

lemma *pos-le-divideR-eq* [*field-simps*]:
$a \le b\ /_R\ c \longleftrightarrow c *_R a \le b$ (**is** *?P* $\longleftrightarrow$ *?Q*) **if** $0 < c$
proof
assume *?P*
with *scaleR-left-mono that* **have** $c *_R a \le c *_R (b\ /_R\ c)$
by *simp*
with *that* **show** *?Q*
by (*simp add*: *scaleR-one scaleR-scaleR inverse-eq-divide*)
next
assume *?Q*
with *scaleR-left-mono that* **have** $c *_R a\ /_R\ c \le b\ /_R\ c$
by *simp*
with *that* **show** *?P*
by (*simp add*: *scaleR-one scaleR-scaleR inverse-eq-divide*)
qed

lemma *pos-less-divideR-eq* [*field-simps*]:
$a < b\ /_R\ c \longleftrightarrow c *_R a < b$ **if** $c > 0$
using *that pos-le-divideR-eq* [*of c a b*]
by (*auto simp add*: *le-less scaleR-scaleR scaleR-one*)

lemma *pos-divideR-le-eq* [*field-simps*]:
$b\ /_R\ c \le a \longleftrightarrow b \le c *_R a$ **if** $c > 0$
using *that pos-le-divideR-eq* [*of inverse c b a*] **by** *simp*

lemma *pos-divideR-less-eq* [*field-simps*]:
$b\ /_R\ c < a \longleftrightarrow b < c *_R a$ **if** $c > 0$
using *that pos-less-divideR-eq* [*of inverse c b a*] **by** *simp*

lemma *pos-le-minus-divideR-eq* [*field-simps*]:

$a \leq - (b /_R c) \longleftrightarrow c *_R a \leq - b$ **if** $c > 0$
using *that* **by** (*metis add-minus-cancel diff-0 left-minus minus-minus neg-le-iff-le scaleR-add-right uminus-add-conv-diff pos-le-divideR-eq*)

lemma *pos-less-minus-divideR-eq* [*field-simps*]:
$a < - (b /_R c) \longleftrightarrow c *_R a < - b$ **if** $c > 0$
using *that* **by** (*metis le-less less-le-not-le pos-divideR-le-eq pos-divideR-less-eq pos-le-minus-divideR-eq*)

lemma *pos-minus-divideR-le-eq* [*field-simps*]:
$- (b /_R c) \leq a \longleftrightarrow - b \leq c *_R a$ **if** $c > 0$
using *that* **by** (*metis pos-divideR-le-eq pos-le-minus-divideR-eq that inverse-positive-iff-positive le-imp-neg-le minus-minus*)

lemma *pos-minus-divideR-less-eq* [*field-simps*]:
$- (b /_R c) < a \longleftrightarrow - b < c *_R a$ **if** $c > 0$
using *that* **by** (*simp add*: *less-le-not-le pos-le-minus-divideR-eq pos-minus-divideR-le-eq*)

lemma *scaleR-image-atLeastAtMost*: $c > 0 \Longrightarrow scaleR\ c\ `\ \{x..y\} = \{c *_R x..c *_R y\}$
apply (*auto intro*!: *scaleR-left-mono simp*: *image-iff Bex-def*)
using *pos-divideR-le-eq* [*of c*] *pos-le-divideR-eq* [*of c*]
apply (*meson local.order-eq-iff*)
done

end

lemma *neg-le-divideR-eq* [*field-simps*]:
$a \leq b /_R c \longleftrightarrow b \leq c *_R a$ (**is** *?P* $\longleftrightarrow$ *?Q*) **if** $c < 0$
for $a\ b$ :: *'a* :: *ordered-real-vector*
using *that pos-le-divideR-eq* [*of* $-$ *c a* $-$ *b*] **by** *simp*

lemma *neg-less-divideR-eq* [*field-simps*]:
$a < b /_R c \longleftrightarrow b < c *_R a$ **if** $c < 0$
for $a\ b$ :: *'a* :: *ordered-real-vector*
using *that neg-le-divideR-eq* [*of c a b*] **by** (*auto simp add*: *le-less*)

lemma *neg-divideR-le-eq* [*field-simps*]:
$b /_R c \leq a \longleftrightarrow c *_R a \leq b$ **if** $c < 0$
for $a\ b$ :: *'a* :: *ordered-real-vector*
using *that pos-divideR-le-eq* [*of* $-$ *c* $-$ *b a*] **by** *simp*

lemma *neg-divideR-less-eq* [*field-simps*]:
$b /_R c < a \longleftrightarrow c *_R a < b$ **if** $c < 0$
for $a\ b$ :: *'a* :: *ordered-real-vector*
using *that neg-divideR-le-eq* [*of c b a*] **by** (*auto simp add*: *le-less*)

lemma *neg-le-minus-divideR-eq* [*field-simps*]:

$a \le -\ (b\ /_R\ c) \longleftrightarrow -\ b \le c *_R a$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
using *that pos-le-minus-divideR-eq* [*of* $-\ c\ a\ -\ b$] **by** (*simp add*: *minus-le-iff*)

lemma *neg-less-minus-divideR-eq* [*field-simps*]:
$a < -\ (b\ /_R\ c) \longleftrightarrow -\ b < c *_R a$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
proof −
have ∗: $-\ b = c *_R a \longleftrightarrow b = -\ (c *_R a)$
by (*metis add.inverse-inverse*)
from *that neg-le-minus-divideR-eq* [*of c a b*]
show *?thesis* **by** (*auto simp add*: *le-less* ∗)
qed

lemma *neg-minus-divideR-le-eq* [*field-simps*]:
$-\ (b\ /_R\ c) \le a \longleftrightarrow c *_R a \le -\ b$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
using *that pos-minus-divideR-le-eq* [*of* $-\ c\ -\ b\ a$] **by** (*simp add*: *le-minus-iff*)

lemma *neg-minus-divideR-less-eq* [*field-simps*]:
$-\ (b\ /_R\ c) < a \longleftrightarrow c *_R a < -\ b$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
using *that* **by** (*simp add*: *less-le-not-le neg-le-minus-divideR-eq neg-minus-divideR-le-eq*)

lemma [*field-split-simps*]:
$a = b\ /_R\ c \longleftrightarrow (if\ c = 0\ then\ a = 0\ else\ c *_R a = b)$
$b\ /_R\ c = a \longleftrightarrow (if\ c = 0\ then\ a = 0\ else\ b = c *_R a)$
$a + b\ /_R\ c = (if\ c = 0\ then\ a\ else\ (c *_R a + b)\ /_R\ c)$
$a\ /_R\ c + b = (if\ c = 0\ then\ b\ else\ (a + c *_R b)\ /_R\ c)$
$a - b\ /_R\ c = (if\ c = 0\ then\ a\ else\ (c *_R a - b)\ /_R\ c)$
$a\ /_R\ c - b = (if\ c = 0\ then\ -\ b\ else\ (a - c *_R b)\ /_R\ c)$
$-\ (a\ /_R\ c) + b = (if\ c = 0\ then\ b\ else\ (-\ a + c *_R b)\ /_R\ c)$
$-\ (a\ /_R\ c) - b = (if\ c = 0\ then\ -\ b\ else\ (-\ a - c *_R b)\ /_R\ c)$
for $a\ b :: 'a ::$ *real-vector*
by (*auto simp add*: *field-simps*)

lemma [*field-split-simps*]:
$0 < c \Longrightarrow a \le b\ /_R\ c \longleftrightarrow (if\ c > 0\ then\ c *_R a \le b\ else\ if\ c < 0\ then\ b \le c *_R a\ else\ a \le 0)$
$0 < c \Longrightarrow a < b\ /_R\ c \longleftrightarrow (if\ c > 0\ then\ c *_R a < b\ else\ if\ c < 0\ then\ b < c *_R a\ else\ a < 0)$
$0 < c \Longrightarrow b\ /_R\ c \le a \longleftrightarrow (if\ c > 0\ then\ b \le c *_R a\ else\ if\ c < 0\ then\ c *_R a \le b\ else\ a \ge 0)$
$0 < c \Longrightarrow b\ /_R\ c < a \longleftrightarrow (if\ c > 0\ then\ b < c *_R a\ else\ if\ c < 0\ then\ c *_R a < b\ else\ a > 0)$
$0 < c \Longrightarrow a \le -\ (b\ /_R\ c) \longleftrightarrow (if\ c > 0\ then\ c *_R a \le -\ b\ else\ if\ c < 0\ then\ -\ b \le c *_R a\ else\ a \le 0)$
$0 < c \Longrightarrow a < -\ (b\ /_R\ c) \longleftrightarrow (if\ c > 0\ then\ c *_R a < -\ b\ else\ if\ c < 0\ then\ -\ b < c *_R a\ else\ a < 0)$

$0 < c \Longrightarrow -(b /_R c) \le a \longleftrightarrow (\textit{if } c > 0 \textit{ then } - b \le c *_R a \textit{ else if } c < 0 \textit{ then } c *_R a \le - b \textit{ else } a \ge 0)$
$0 < c \Longrightarrow -(b /_R c) < a \longleftrightarrow (\textit{if } c > 0 \textit{ then } - b < c *_R a \textit{ else if } c < 0 \textit{ then } c *_R a < - b \textit{ else } a > 0)$
for *a b* :: *'a* :: *ordered-real-vector*
by (*clarsimp intro*!: *field-simps*)+

lemma *scaleR-nonneg-nonneg*: $0 \le a \Longrightarrow 0 \le x \Longrightarrow 0 \le a *_R x$
for *x* :: *'a*::*ordered-real-vector*
using *scaleR-left-mono* [*of 0 x a*] **by** *simp*

lemma *scaleR-nonneg-nonpos*: $0 \le a \Longrightarrow x \le 0 \Longrightarrow a *_R x \le 0$
for *x* :: *'a*::*ordered-real-vector*
using *scaleR-left-mono* [*of x 0 a*] **by** *simp*

lemma *scaleR-nonpos-nonneg*: $a \le 0 \Longrightarrow 0 \le x \Longrightarrow a *_R x \le 0$
for *x* :: *'a*::*ordered-real-vector*
using *scaleR-right-mono* [*of a 0 x*] **by** *simp*

lemma *split-scaleR-neg-le*: $(0 \le a \wedge x \le 0) \vee (a \le 0 \wedge 0 \le x) \Longrightarrow a *_R x \le 0$
for *x* :: *'a*::*ordered-real-vector*
by (*auto simp*: *scaleR-nonneg-nonpos scaleR-nonpos-nonneg*)

lemma *le-add-iff1*: $a *_R e + c \le b *_R e + d \longleftrightarrow (a - b) *_R e + c \le d$
for *c d e* :: *'a*::*ordered-real-vector*
by (*simp add*: *algebra-simps*)

lemma *le-add-iff2*: $a *_R e + c \le b *_R e + d \longleftrightarrow c \le (b - a) *_R e + d$
for *c d e* :: *'a*::*ordered-real-vector*
by (*simp add*: *algebra-simps*)

lemma *scaleR-left-mono-neg*: $b \le a \Longrightarrow c \le 0 \Longrightarrow c *_R a \le c *_R b$
for *a b* :: *'a*::*ordered-real-vector*
by (*drule scaleR-left-mono* [*of - - − c*], *simp-all*)

lemma *scaleR-right-mono-neg*: $b \le a \Longrightarrow c \le 0 \Longrightarrow a *_R c \le b *_R c$
for *c* :: *'a*::*ordered-real-vector*
by (*drule scaleR-right-mono* [*of - - − c*], *simp-all*)

lemma *scaleR-nonpos-nonpos*: $a \le 0 \Longrightarrow b \le 0 \Longrightarrow 0 \le a *_R b$
for *b* :: *'a*::*ordered-real-vector*
using *scaleR-right-mono-neg* [*of a 0 b*] **by** *simp*

lemma *split-scaleR-pos-le*: $(0 \le a \wedge 0 \le b) \vee (a \le 0 \wedge b \le 0) \Longrightarrow 0 \le a *_R b$
for *b* :: *'a*::*ordered-real-vector*
by (*auto simp*: *scaleR-nonneg-nonneg scaleR-nonpos-nonpos*)

lemma *zero-le-scaleR-iff*:
fixes *b* :: *'a*::*ordered-real-vector*

shows $0 \leq a *_R b \longleftrightarrow 0 < a \wedge 0 \leq b \vee a < 0 \wedge b \leq 0 \vee a = 0$
(**is** *?lhs* = *?rhs*)
proof (*cases a* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof
assume *?lhs*
from ‹$a \neq 0$› **consider** $a > 0 \mid a < 0$ **by** *arith*
then show *?rhs*
proof *cases*
case *1*
with ‹*?lhs*› **have** *inverse* $a *_R 0 \leq$ *inverse* $a *_R (a *_R b)$
by (*intro scaleR-mono*) *auto*
with *1* **show** *?thesis*
by *simp*
next
case *2*
with ‹*?lhs*› **have** $-$ *inverse* $a *_R 0 \leq -$ *inverse* $a *_R (a *_R b)$
by (*intro scaleR-mono*) *auto*
with *2* **show** *?thesis*
by *simp*
qed
next
assume *?rhs*
then show *?lhs*
by (*auto simp*: *not-le* ‹$a \neq 0$› *intro*!: *split-scaleR-pos-le*)
qed
qed

lemma *scaleR-le-0-iff*: $a *_R b \leq 0 \longleftrightarrow 0 < a \wedge b \leq 0 \vee a < 0 \wedge 0 \leq b \vee a = 0$
for *b*::*'a*::*ordered-real-vector*
by (*insert zero-le-scaleR-iff* [*of* $-a$ *b*]) *force*

lemma *scaleR-le-cancel-left*: $c *_R a \leq c *_R b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
for *b* :: *'a*::*ordered-real-vector*
by (*auto simp*: *neq-iff scaleR-left-mono scaleR-left-mono-neg*
dest: *scaleR-left-mono*[**where** *a*=*inverse c*] *scaleR-left-mono-neg*[**where** *c*=*inverse c*])

lemma *scaleR-le-cancel-left-pos*: $0 < c \Longrightarrow c *_R a \leq c *_R b \longleftrightarrow a \leq b$
for *b* :: *'a*::*ordered-real-vector*
by (*auto simp*: *scaleR-le-cancel-left*)

lemma *scaleR-le-cancel-left-neg*: $c < 0 \Longrightarrow c *_R a \leq c *_R b \longleftrightarrow b \leq a$
for *b* :: *'a*::*ordered-real-vector*

by (*auto simp*: *scaleR-le-cancel-left*)

lemma *scaleR-left-le-one-le*: $0 \leq x \Longrightarrow a \leq 1 \Longrightarrow a *_R x \leq x$
for x :: $'a$::*ordered-real-vector* **and** a :: *real*
using *scaleR-right-mono*[*of a 1 x*] **by** *simp*

107.5 Real normed vector spaces

class *dist* =
fixes *dist* :: $'a \Rightarrow 'a \Rightarrow$ *real*

class *norm* =
fixes *norm* :: $'a \Rightarrow$ *real*

class *sgn-div-norm* = *scaleR* + *norm* + *sgn* +
assumes *sgn-div-norm*: *sgn* $x = x /_R$ *norm* x

class *dist-norm* = *dist* + *norm* + *minus* +
assumes *dist-norm*: *dist* x y = *norm* $(x - y)$

class *uniformity-dist* = *dist* + *uniformity* +
assumes *uniformity-dist*: *uniformity* = (*INF* $e \in \{0 <..\}$. *principal* $\{(x, y)$. *dist* x $y < e\}$)
begin

lemma *eventually-uniformity-metric*:
eventually P uniformity $\longleftrightarrow$ ($\exists e>0. \forall x\ y.$ *dist* $x\ y < e \longrightarrow P\ (x, y)$)
unfolding *uniformity-dist*
by (*subst eventually-INF-base*)
(*auto simp*: *eventually-principal subset-eq intro*: *bexI*[*of - min - -*])

end

class *real-normed-vector* = *real-vector* + *sgn-div-norm* + *dist-norm* + *uniformity-dist* + *open-uniformity* +
assumes *norm-eq-zero* [*simp*]: *norm* $x = 0 \longleftrightarrow x = 0$
and *norm-triangle-ineq*: *norm* $(x + y) \leq$ *norm* x + *norm* y
and *norm-scaleR* [*simp*]: *norm* (*scaleR* a x) = $|a| *$ *norm* x
begin

lemma *norm-ge-zero* [*simp*]: $0 \leq$ *norm* x
proof −
have 0 = *norm* $(x + -1 *_R x)$
using *scaleR-add-left*[*of 1 −1 x*] *norm-scaleR*[*of 0 x*] **by** (*simp add*: *scaleR-one*)
also have ... $\leq$ *norm* x + *norm* $(-1 *_R x)$ **by** (*rule norm-triangle-ineq*)
finally show *?thesis* **by** *simp*
qed

lemma *bdd-below-norm-image*: *bdd-below* (*norm* ‘ *A*)

by (*meson bdd-belowI2 norm-ge-zero*)

end

class *real-normed-algebra* = *real-algebra* + *real-normed-vector* +
assumes *norm-mult-ineq*: *norm* (*x* ∗ *y*) ≤ *norm x* ∗ *norm y*

class *real-normed-algebra-1* = *real-algebra-1* + *real-normed-algebra* +
assumes *norm-one* [*simp*]: *norm 1* = *1*

lemma (**in** *real-normed-algebra-1*) *scaleR-power* [*simp*]: (*scaleR x y*) ^ *n* = *scaleR* (*x*^*n*) (*y*^*n*)
by (*induct n*) (*simp-all add*: *scaleR-one scaleR-scaleR mult-ac*)

class *real-normed-div-algebra* = *real-div-algebra* + *real-normed-vector* +
assumes *norm-mult*: *norm* (*x* ∗ *y*) = *norm x* ∗ *norm y*

lemma *divideR-right*:
fixes *x y* :: '*a*::*real-normed-vector*
shows $r \neq 0 \Longrightarrow y = x /_R r \longleftrightarrow r *_R y = x$
by *auto*

class *real-normed-field* = *real-field* + *real-normed-div-algebra*

instance *real-normed-div-algebra* < *real-normed-algebra-1*
proof
show *norm* (*x* ∗ *y*) ≤ *norm x* ∗ *norm y* **for** *x y* :: '*a*
by (*simp add*: *norm-mult*)
next
have *norm* (*1* ∗ *1*::'*a*) = *norm* (*1*::'*a*) ∗ *norm* (*1*::'*a*)
by (*rule norm-mult*)
then show *norm* (*1*::'*a*) = *1* **by** *simp*
qed

context *real-normed-vector* **begin**

lemma *norm-zero* [*simp*]: *norm* (*0*::'*a*) = *0*
by *simp*

lemma *zero-less-norm-iff* [*simp*]: *norm x* > *0* ⟷ *x* ≠ *0*
by (*simp add*: *order-less-le*)

lemma *norm-not-less-zero* [*simp*]: ¬ *norm x* < *0*
by (*simp add*: *linorder-not-less*)

lemma *norm-le-zero-iff* [*simp*]: *norm x* ≤ *0* ⟷ *x* = *0*
by (*simp add*: *order-le-less*)

lemma *norm-minus-cancel* [*simp*]: *norm* (− *x*) = *norm x*

proof −
  have − *1* $*_R$ *x* = − (*1* $*_R$ *x*)
    unfolding *add-eq-0-iff2*[*symmetric*] *scaleR-add-left*[*symmetric*]
    using *norm-eq-zero*
    by *fastforce*
  then have *norm* (− *x*) = *norm* (*scaleR* (− *1*) *x*)
    by (*simp only*: *scaleR-one*)
  also have ... = |− *1*| ∗ *norm x*
    by (*rule norm-scaleR*)
  finally show *?thesis* **by** *simp*
qed

lemma *norm-minus-commute*: *norm* (*a* − *b*) = *norm* (*b* − *a*)
proof −
  have *norm* (− (*b* − *a*)) = *norm* (*b* − *a*)
    by (*rule norm-minus-cancel*)
  then show *?thesis* **by** *simp*
qed

lemma *dist-add-cancel* [*simp*]: *dist* (*a* + *b*) (*a* + *c*) = *dist b c*
  by (*simp add*: *dist-norm*)

lemma *dist-add-cancel2* [*simp*]: *dist* (*b* + *a*) (*c* + *a*) = *dist b c*
  by (*simp add*: *dist-norm*)

lemma *norm-uminus-minus*: *norm* (− *x* − *y*) = *norm* (*x* + *y*)
  by (*subst* (*2*) *norm-minus-cancel*[*symmetric*], *subst minus-add-distrib*) *simp*

lemma *norm-triangle-ineq2*: *norm a* − *norm b* ≤ *norm* (*a* − *b*)
proof −
  have *norm* (*a* − *b* + *b*) ≤ *norm* (*a* − *b*) + *norm b*
    by (*rule norm-triangle-ineq*)
  then show *?thesis* **by** *simp*
qed

lemma *norm-triangle-ineq3*: |*norm a* − *norm b*| ≤ *norm* (*a* − *b*)
proof −
  have *norm a* − *norm b* ≤ *norm* (*a* − *b*)
    by (*simp add*: *norm-triangle-ineq2*)
  moreover have *norm b* − *norm a* ≤ *norm* (*a* − *b*)
    by (*metis norm-minus-commute norm-triangle-ineq2*)
  ultimately show *?thesis*
    by (*simp add*: *abs-le-iff*)
qed

lemma *norm-triangle-ineq4*: *norm* (*a* − *b*) ≤ *norm a* + *norm b*
proof −
  have *norm* (*a* + − *b*) ≤ *norm a* + *norm* (− *b*)
    by (*rule norm-triangle-ineq*)

then show *?thesis* **by** *simp*
qed

lemma *norm-triangle-le-diff*: *norm* x + *norm* $y \leq e \Longrightarrow$ *norm* $(x - y) \leq e$
by (*meson norm-triangle-ineq4 order-trans*)

lemma *norm-diff-ineq*: *norm* a − *norm* $b \leq$ *norm* $(a + b)$
proof −
have *norm* a − *norm* $(- b) \leq$ *norm* $(a - - b)$
by (*rule norm-triangle-ineq2*)
then show *?thesis* **by** *simp*
qed

lemma *norm-triangle-sub*: *norm* $x \leq$ *norm* y + *norm* $(x - y)$
using *norm-triangle-ineq*[*of y x − y*] **by** (*simp add*: *field-simps*)

lemma *norm-triangle-le*: *norm* x + *norm* $y \leq e \Longrightarrow$ *norm* $(x + y) \leq e$
by (*rule norm-triangle-ineq* [*THEN order-trans*])

lemma *norm-triangle-lt*: *norm* x + *norm* $y < e \Longrightarrow$ *norm* $(x + y) < e$
by (*rule norm-triangle-ineq* [*THEN le-less-trans*])

lemma *norm-add-leD*: *norm* $(a + b) \leq c \Longrightarrow$ *norm* $b \leq$ *norm* $a + c$
by (*metis ab-semigroup-add-class.add.commute add-commute diff-le-eq norm-diff-ineq order-trans*)

lemma *norm-diff-triangle-ineq*: *norm* $((a + b) - (c + d)) \leq$ *norm* $(a - c)$ + *norm* $(b - d)$
proof −
have *norm* $((a + b) - (c + d))$ = *norm* $((a - c) + (b - d))$
by (*simp add*: *algebra-simps*)
also have ... $\leq$ *norm* $(a - c)$ + *norm* $(b - d)$
by (*rule norm-triangle-ineq*)
finally show *?thesis* **.**
qed

lemma *norm-diff-triangle-le*: *norm* $(x - z) \leq e1 + e2$
if *norm* $(x - y) \leq e1$ *norm* $(y - z) \leq e2$
proof −
have *norm* $(x - (y + z - y)) \leq$ *norm* $(x - y)$ + *norm* $(y - z)$
using *norm-diff-triangle-ineq that diff-diff-eq2* **by** *presburger*
with *that* **show** *?thesis* **by** *simp*
qed

lemma *norm-diff-triangle-less*: *norm* $(x - z) < e1 + e2$
if *norm* $(x - y) < e1$ *norm* $(y - z) < e2$
proof −
have *norm* $(x - z) \leq$ *norm* $(x - y)$ + *norm* $(y - z)$
by (*metis norm-diff-triangle-ineq add-diff-cancel-left′ diff-diff-eq2*)

with *that* **show** *?thesis* **by** *auto*
qed

lemma *norm-triangle-mono*:
norm a $\leq$ *r* $\Longrightarrow$ *norm b* $\leq$ *s* $\Longrightarrow$ *norm* (*a* + *b*) $\leq$ *r* + *s*
by (*metis* (*mono-tags*) *add-mono-thms-linordered-semiring*(*1*) *norm-triangle-ineq order.trans*)

lemma *norm-sum*: *norm* (*sum f A*) $\leq$ ($\sum$ *i*$\in$*A*. *norm* (*f i*))
for *f*::'*b* $\Rightarrow$ '*a*
by (*induct A rule*: *infinite-finite-induct*) (*auto intro*: *norm-triangle-mono*)

lemma *sum-norm-le*: *norm* (*sum f S*) $\leq$ *sum g S*
if $\bigwedge$*x*. *x* $\in$ *S* $\Longrightarrow$ *norm* (*f x*) $\leq$ *g x*
for *f*::'*b* $\Rightarrow$ '*a*
by (*rule order-trans* [*OF norm-sum sum-mono*]) (*simp add*: *that*)

lemma *abs-norm-cancel* [*simp*]: |*norm a*| = *norm a*
by (*rule abs-of-nonneg* [*OF norm-ge-zero*])

lemma *sum-norm-bound*:
norm (*sum f S*) $\leq$ *of-nat* (*card S*)$*$*K*
if $\bigwedge$*x*. *x* $\in$ *S* $\Longrightarrow$ *norm* (*f x*) $\leq$ *K*
for *f* :: '*b* $\Rightarrow$ '*a*
using *sum-norm-le*[*OF that*] *sum-constant*[*symmetric*]
by *simp*

lemma *norm-add-less*: *norm x* < *r* $\Longrightarrow$ *norm y* < *s* $\Longrightarrow$ *norm* (*x* + *y*) < *r* + *s*
by (*rule order-le-less-trans* [*OF norm-triangle-ineq add-strict-mono*])

end

lemma *dist-scaleR* [*simp*]: *dist* (*x* $*_R$ *a*) (*y* $*_R$ *a*) = |*x* − *y*| $*$ *norm a*
for *a* :: '*a*::*real-normed-vector*
by (*metis dist-norm norm-scaleR scaleR-left.diff*)

lemma *norm-mult-less*: *norm x* < *r* $\Longrightarrow$ *norm y* < *s* $\Longrightarrow$ *norm* (*x* $*$ *y*) < *r* $*$ *s*
for *x y* :: '*a*::*real-normed-algebra*
by (*rule order-le-less-trans* [*OF norm-mult-ineq*]) (*simp add*: *mult-strict-mono'*)

lemma *norm-of-real* [*simp*]: *norm* (*of-real r* :: '*a*::*real-normed-algebra-1*) = |*r*|
by (*simp add*: *of-real-def*)

lemma *norm-numeral* [*simp*]: *norm* (*numeral w*::'*a*::*real-normed-algebra-1*) = *numeral w*
by (*subst of-real-numeral* [*symmetric*], *subst norm-of-real*, *simp*)

lemma *norm-neg-numeral* [*simp*]: *norm* (− *numeral w*::'*a*::*real-normed-algebra-1*) = *numeral w*

by (*subst of-real-neg-numeral* [*symmetric*], *subst norm-of-real*, *simp*)

lemma *norm-of-real-add1* [*simp*]: *norm* (*of-real x* + *1* :: *'a* :: *real-normed-div-algebra*) = |*x* + *1*|
by (*metis norm-of-real of-real-1 of-real-add*)

lemma *norm-of-real-addn* [*simp*]:
norm (*of-real x* + *numeral b* :: *'a* :: *real-normed-div-algebra*) = |*x* + *numeral b*|
by (*metis norm-of-real of-real-add of-real-numeral*)

lemma *norm-of-int* [*simp*]: *norm* (*of-int z*::*'a*::*real-normed-algebra-1*) = |*of-int z*|
by (*subst of-real-of-int-eq* [*symmetric*], *rule norm-of-real*)

lemma *norm-of-nat* [*simp*]: *norm* (*of-nat n*::*'a*::*real-normed-algebra-1*) = *of-nat n*
by (*metis abs-of-nat norm-of-real of-real-of-nat-eq*)

lemma *nonzero-norm-inverse*: *a* ≠ *0* ⟹ *norm* (*inverse a*) = *inverse* (*norm a*)
for *a* :: *'a*::*real-normed-div-algebra*
by (*metis inverse-unique norm-mult norm-one right-inverse*)

lemma *norm-inverse*: *norm* (*inverse a*) = *inverse* (*norm a*)
for *a* :: *'a*::{*real-normed-div-algebra*,*division-ring*}
by (*metis inverse-zero nonzero-norm-inverse norm-zero*)

lemma *nonzero-norm-divide*: *b* ≠ *0* ⟹ *norm* (*a* / *b*) = *norm a* / *norm b*
for *a b* :: *'a*::*real-normed-field*
by (*simp add*: *divide-inverse norm-mult nonzero-norm-inverse*)

lemma *norm-divide*: *norm* (*a* / *b*) = *norm a* / *norm b*
for *a b* :: *'a*::{*real-normed-field*,*field*}
by (*simp add*: *divide-inverse norm-mult norm-inverse*)

lemma *dist-divide-right*: *dist* (*a*/*c*) (*b*/*c*) = *dist a b* / *norm c* **for** *c* :: *'a* :: *real-normed-field*
by (*metis diff-divide-distrib dist-norm norm-divide*)

lemma *norm-inverse-le-norm*:
fixes *x* :: *'a*::*real-normed-div-algebra*
shows *r* ≤ *norm x* ⟹ *0* < *r* ⟹ *norm* (*inverse x*) ≤ *inverse r*
by (*simp add*: *le-imp-inverse-le norm-inverse*)

lemma *norm-power-ineq*: *norm* (*x* ^ *n*) ≤ *norm x* ^ *n*
for *x* :: *'a*::*real-normed-algebra-1*
proof (*induct n*)
case *0*
show *norm* (*x* ^ *0*) ≤ *norm x* ^ *0* **by** *simp*
next
case (*Suc n*)
have *norm* (*x* * *x* ^ *n*) ≤ *norm x* * *norm* (*x* ^ *n*)

by (*rule norm-mult-ineq*)
also from *Suc* **have** ... ≤ *norm x* ∗ *norm x* ^ *n*
using *norm-ge-zero* **by** (*rule mult-left-mono*)
finally show *norm* (*x* ^ *Suc n*) ≤ *norm x* ^ *Suc n*
by *simp*
qed

lemma *norm-power*: *norm* (*x* ^ *n*) = *norm x* ^ *n*
for *x* :: *'a::real-normed-div-algebra*
by (*induct n*) (*simp-all add*: *norm-mult*)

lemma *norm-power-int*: *norm* (*power-int x n*) = *power-int* (*norm x*) *n*
for *x* :: *'a::real-normed-div-algebra*
by (*cases n rule*: *int-cases4*) (*auto simp*: *norm-power power-int-minus norm-inverse*)

lemma *power-eq-imp-eq-norm*:
fixes *w* :: *'a::real-normed-div-algebra*
assumes *eq*: *w* ^ *n* = *z* ^ *n* **and** *n* > *0*
shows *norm w* = *norm z*
proof −
have *norm w* ^ *n* = *norm z* ^ *n*
by (*metis* (*no-types*) *eq norm-power*)
then show *?thesis*
using *assms* **by** (*force intro*: *power-eq-imp-eq-base*)
qed

lemma *power-eq-1-iff*:
fixes *w* :: *'a::real-normed-div-algebra*
shows *w* ^ *n* = *1* ⟹ *norm w* = *1* ∨ *n* = *0*
by (*metis norm-one power-0-left power-eq-0-iff power-eq-imp-eq-norm power-one*)

lemma *norm-mult-numeral1* [*simp*]: *norm* (*numeral w* ∗ *a*) = *numeral w* ∗ *norm a*
for *a b* :: *'a::{real-normed-field,field}*
by (*simp add*: *norm-mult*)

lemma *norm-mult-numeral2* [*simp*]: *norm* (*a* ∗ *numeral w*) = *norm a* ∗ *numeral w*
for *a b* :: *'a::{real-normed-field,field}*
by (*simp add*: *norm-mult*)

lemma *norm-divide-numeral* [*simp*]: *norm* (*a* / *numeral w*) = *norm a* / *numeral w*
for *a b* :: *'a::{real-normed-field,field}*
by (*simp add*: *norm-divide*)

lemma *norm-of-real-diff* [*simp*]:
norm (*of-real b* − *of-real a* :: *'a::real-normed-algebra-1*) ≤ |*b* − *a*|
by (*metis norm-of-real of-real-diff order-refl*)

Despite a superficial resemblance, *norm-eq-1* is not relevant.

lemma *square-norm-one*:
  fixes x :: *'a::real-normed-div-algebra*
  assumes $x^2 = 1$
  shows *norm* $x = 1$
  by (*metis assms norm-minus-cancel norm-one power2-eq-1-iff*)

lemma *norm-less-p1*: *norm* $x <$ *norm* (*of-real* (*norm* x) + *1* :: *'a*)
  for x :: *'a::real-normed-algebra-1*
proof −
  have *norm* $x <$ *norm* (*of-real* (*norm* x + *1*) :: *'a*)
    by (*simp add*: *of-real-def*)
  then show *?thesis*
    by *simp*
qed

lemma *prod-norm*: *prod* (λx. *norm* (f x)) A = *norm* (*prod* f A)
  for f :: *'a* ⇒ *'b*::{*comm-semiring-1*,*real-normed-div-algebra*}
  by (*induct A rule*: *infinite-finite-induct*) (*auto simp*: *norm-mult*)

lemma *norm-prod-le*:
  norm (*prod* f A) $\le$ ($\prod a \in A$. *norm* (f a :: *'a* :: {*real-normed-algebra-1*,*comm-monoid-mult*}))
proof (*induct A rule*: *infinite-finite-induct*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert a A*)
  then have *norm* (*prod* f (*insert a A*)) $\le$ *norm* (f a) $*$ *norm* (*prod* f A)
    by (*simp add*: *norm-mult-ineq*)
  also have *norm* (*prod* f A) $\le$ ($\prod a \in A$. *norm* (f a))
    by (*rule insert*)
  finally show *?case*
    by (*simp add*: *insert mult-left-mono*)
next
  case *infinite*
  then show *?case* **by** *simp*
qed

lemma *norm-prod-diff*:
  fixes z w :: *'i* ⇒ *'a*::{*real-normed-algebra-1*, *comm-monoid-mult*}
  shows ($\bigwedge i$. $i \in I \Longrightarrow$ *norm* (z i) ≤ 1) $\Longrightarrow$ ($\bigwedge i$. $i \in I \Longrightarrow$ *norm* (w i) ≤ 1) $\Longrightarrow$
    norm (($\prod i \in I$. z i) − ($\prod i \in I$. w i)) $\le$ ($\sum i \in I$. *norm* (z i − w i))
proof (*induction I rule*: *infinite-finite-induct*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert i I*)
  note *insert.hyps*[*simp*]

have *norm* $((\prod i{\in}insert\ i\ I.\ z\ i) - (\prod i{\in}insert\ i\ I.\ w\ i)) =$
norm $((\prod i{\in}I.\ z\ i) * (z\ i - w\ i) + ((\prod i{\in}I.\ z\ i) - (\prod i{\in}I.\ w\ i)) * w\ i)$
(**is** - = *norm* (*?t1* + *?t2*))
by (*auto simp*: *field-simps*)
also have ... $\leq$ *norm ?t1* + *norm ?t2*
by (*rule norm-triangle-ineq*)
also have *norm ?t1* $\leq$ *norm* $(\prod i{\in}I.\ z\ i) *$ *norm* $(z\ i - w\ i)$
by (*rule norm-mult-ineq*)
also have ... $\leq (\prod i{\in}I.\ norm\ (z\ i)) * norm(z\ i - w\ i)$
by (*rule mult-right-mono*) (*auto intro*: *norm-prod-le*)
also have $(\prod i{\in}I.\ norm\ (z\ i)) \leq (\prod i{\in}I.\ 1)$
by (*intro prod-mono*) (*auto intro*!: *insert*)
also have *norm ?t2* $\leq$ *norm* $((\prod i{\in}I.\ z\ i) - (\prod i{\in}I.\ w\ i)) *$ *norm* $(w\ i)$
by (*rule norm-mult-ineq*)
also have *norm* $(w\ i) \leq 1$
by (*auto intro*: *insert*)
also have *norm* $((\prod i{\in}I.\ z\ i) - (\prod i{\in}I.\ w\ i)) \leq (\sum i{\in}I.\ norm\ (z\ i - w\ i))$
using *insert* **by** *auto*
finally show *?case*
by (*auto simp*: *ac-simps mult-right-mono mult-left-mono*)
next
case *infinite*
then show *?case* **by** *simp*
qed

lemma *norm-power-diff*:
fixes *z w* :: *'a*::{*real-normed-algebra-1*, *comm-monoid-mult*}
assumes *norm z* $\leq$ *1 norm w* $\leq$ *1*
shows *norm* $(z\hat{}m - w\hat{}m) \leq m *$ *norm* $(z - w)$
proof −
have *norm* $(z\hat{}m - w\hat{}m) =$ *norm* $((\prod\ i < m.\ z) - (\prod\ i < m.\ w))$
by *simp*
also have ... $\leq (\sum i{<}m.\ norm\ (z - w))$
by (*intro norm-prod-diff*) (*auto simp*: *assms*)
also have ... $= m *$ *norm* $(z - w)$
by *simp*
finally show *?thesis* **.**
qed

107.6 Metric spaces

class *metric-space* = *uniformity-dist* + *open-uniformity* +
assumes *dist-eq-0-iff* [*simp*]: *dist x y* = *0* $\longleftrightarrow$ *x* = *y*
and *dist-triangle2*: *dist x y* $\leq$ *dist x z* + *dist y z*
begin

lemma *dist-self* [*simp*]: *dist x x* = *0*
by *simp*

lemma *zero-le-dist* [*simp*]: *0* ≤ *dist x y*
  using *dist-triangle2* [*of x x y*] **by** *simp*

lemma *zero-less-dist-iff*: *0* < *dist x y* ⟷ *x* ≠ *y*
  by (*simp add*: *less-le*)

lemma *dist-not-less-zero* [*simp*]: ¬ *dist x y* < *0*
  by (*simp add*: *not-less*)

lemma *dist-le-zero-iff* [*simp*]: *dist x y* ≤ *0* ⟷ *x* = *y*
  by (*simp add*: *le-less*)

lemma *dist-commute*: *dist x y* = *dist y x*
proof (*rule order-antisym*)
  show *dist x y* ≤ *dist y x*
    using *dist-triangle2* [*of x y x*] **by** *simp*
  show *dist y x* ≤ *dist x y*
    using *dist-triangle2* [*of y x y*] **by** *simp*
qed

lemma *dist-commute-lessI*: *dist y x* < *e* ⟹ *dist x y* < *e*
  by (*simp add*: *dist-commute*)

lemma *dist-triangle*: *dist x z* ≤ *dist x y* + *dist y z*
  using *dist-triangle2* [*of x z y*] **by** (*simp add*: *dist-commute*)

lemma *dist-triangle3*: *dist x y* ≤ *dist a x* + *dist a y*
  using *dist-triangle2* [*of x y a*] **by** (*simp add*: *dist-commute*)

lemma *abs-dist-diff-le*: |*dist a b* − *dist b c*| ≤ *dist a c*
  using *dist-triangle3*[*of b c a*] *dist-triangle2*[*of a b c*] **by** *simp*

lemma *dist-pos-lt*: *x* ≠ *y* ⟹ *0* < *dist x y*
  by (*simp add*: *zero-less-dist-iff*)

lemma *dist-nz*: *x* ≠ *y* ⟷ *0* < *dist x y*
  by (*simp add*: *zero-less-dist-iff*)

declare *dist-nz* [*symmetric*, *simp*]

lemma *dist-triangle-le*: *dist x z* + *dist y z* ≤ *e* ⟹ *dist x y* ≤ *e*
  by (*rule order-trans* [*OF dist-triangle2*])

lemma *dist-triangle-lt*: *dist x z* + *dist y z* < *e* ⟹ *dist x y* < *e*
  by (*rule le-less-trans* [*OF dist-triangle2*])

lemma *dist-triangle-less-add*: *dist x1 y* < *e1* ⟹ *dist x2 y* < *e2* ⟹ *dist x1 x2* < *e1* + *e2*
  by (*rule dist-triangle-lt* [**where** *z*=*y*]) *simp*

lemma *dist-triangle-half-l*: *dist x1 y* < *e* / *2* ⟹ *dist x2 y* < *e* / *2* ⟹ *dist x1 x2* < *e*
  by (*rule dist-triangle-lt* [**where** *z*=*y*]) *simp*

lemma *dist-triangle-half-r*: *dist y x1* < *e* / *2* ⟹ *dist y x2* < *e* / *2* ⟹ *dist x1 x2* < *e*
  by (*rule dist-triangle-half-l*) (*simp-all add*: *dist-commute*)

lemma *dist-triangle-third*:
  assumes *dist x1 x2* < *e*/*3* *dist x2 x3* < *e*/*3* *dist x3 x4* < *e*/*3*
  shows *dist x1 x4* < *e*
proof −
  have *dist x1 x3* < *e*/*3* + *e*/*3*
    by (*metis assms*(*1*) *assms*(*2*) *dist-commute dist-triangle-less-add*)
  then have *dist x1 x4* < (*e*/*3* + *e*/*3*) + *e*/*3*
    by (*metis assms*(*3*) *dist-commute dist-triangle-less-add*)
  then show *?thesis*
    by *simp*
qed

subclass *uniform-space*
proof
  fix *E x*
  assume *eventually E uniformity*
  then obtain *e* **where** *E*: *0* < *e* ⋀*x y*. *dist x y* < *e* ⟹ *E* (*x*, *y*)
    by (*auto simp*: *eventually-uniformity-metric*)
  then show *E* (*x*, *x*) $\forall_F$ (*x*, *y*) *in uniformity*. *E* (*y*, *x*)
    by (*auto simp*: *eventually-uniformity-metric dist-commute*)
  show ∃ *D*. *eventually D uniformity* ∧ (∀ *x y z*. *D* (*x*, *y*) ⟶ *D* (*y*, *z*) ⟶ *E* (*x*, *z*))
    using *E dist-triangle-half-l*[**where** *e*=*e*]
    unfolding *eventually-uniformity-metric*
    by (*intro exI*[*of* - λ(*x*, *y*). *dist x y* < *e* / *2*] *exI*[*of* - *e*/*2*] *conjI*)
      (*auto simp*: *dist-commute*)
qed

lemma *open-dist*: *open S* ⟷ (∀ *x*∈*S*. ∃ *e*>*0*. ∀ *y*. *dist y x* < *e* ⟶ *y* ∈ *S*)
  by (*simp add*: *dist-commute open-uniformity eventually-uniformity-metric*)

lemma *open-ball*: *open* {*y*. *dist x y* < *d*}
  unfolding *open-dist*
proof (*intro ballI*)
  fix *y*
  assume ∗: *y* ∈ {*y*. *dist x y* < *d*}
  then show ∃ *e*>*0*. ∀ *z*. *dist z y* < *e* ⟶ *z* ∈ {*y*. *dist x y* < *d*}
    by (*auto intro*!: *exI*[*of* - *d* − *dist x y*] *simp*: *field-simps dist-triangle-lt*)
qed

subclass *first-countable-topology*
proof
  fix x
  show $\exists A::nat \Rightarrow 'a\ set.\ (\forall i.\ x \in A\ i \wedge open\ (A\ i)) \wedge (\forall S.\ open\ S \wedge x \in S \longrightarrow (\exists i.\ A\ i \subseteq S))$
  proof (*safe intro*!: *exI*[*of* - $\lambda n.\ \{y.\ dist\ x\ y < inverse\ (Suc\ n)\}$])
    fix S
    assume *open* S $x \in S$
    then obtain e **where** e: $0 < e$ **and** $\{y.\ dist\ x\ y < e\} \subseteq S$
      by (*auto simp*: *open-dist subset-eq dist-commute*)
    moreover
    from e **obtain** i **where** $inverse\ (Suc\ i) < e$
      by (*auto dest*!: *reals-Archimedean*)
    then have $\{y.\ dist\ x\ y < inverse\ (Suc\ i)\} \subseteq \{y.\ dist\ x\ y < e\}$
      by *auto*
    ultimately show $\exists i.\ \{y.\ dist\ x\ y < inverse\ (Suc\ i)\} \subseteq S$
      by *blast*
  qed (*auto intro*: *open-ball*)
qed

end

instance *metric-space* $\subseteq$ *t2-space*
proof
  fix $x\ y :: 'a::metric\text{-}space$
  assume xy: $x \neq y$
  let $?U = \{y'.\ dist\ x\ y' < dist\ x\ y\ /\ 2\}$
  let $?V = \{x'.\ dist\ y\ x' < dist\ x\ y\ /\ 2\}$
  have $*$: $d\ x\ z \leq d\ x\ y + d\ y\ z \Longrightarrow d\ y\ z = d\ z\ y \Longrightarrow \neg\ (d\ x\ y * 2 < d\ x\ z \wedge d\ z\ y * 2 < d\ x\ z)$
    for $d :: 'a \Rightarrow 'a \Rightarrow real$ **and** $x\ y\ z :: 'a$
    by *arith*
  have $open\ ?U \wedge open\ ?V \wedge x \in ?U \wedge y \in ?V \wedge ?U \cap ?V = \{\}$
    using *dist-pos-lt*[*OF xy*] $*$[*of dist*, *OF dist-triangle dist-commute*]
    using *open-ball*[*of* - $dist\ x\ y\ /\ 2$] **by** *auto*
  then show $\exists U\ V.\ open\ U \wedge open\ V \wedge x \in U \wedge y \in V \wedge U \cap V = \{\}$
    by *blast*
qed

Every normed vector space is a metric space.

instance *real-normed-vector* < *metric-space*
proof
  fix $x\ y\ z :: 'a$
  show $dist\ x\ y = 0 \longleftrightarrow x = y$
    by (*simp add*: *dist-norm*)
  show $dist\ x\ y \leq dist\ x\ z + dist\ y\ z$
    using *norm-triangle-ineq4* [*of* $x - z$ $y - z$] **by** (*simp add*: *dist-norm*)
qed

107.7 Class instances for real numbers

instantiation *real* :: *real-normed-field*
begin

definition *dist-real-def*: *dist x y* = |*x* − *y*|

definition *uniformity-real-def* [*code del*]:
(*uniformity* :: (*real* × *real*) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-real-def* [*code del*]:
open (*U* :: *real set*) ⟷ (∀ *x*∈*U*. *eventually* (λ(*x'*, *y*). *x'* = *x* ⟶ *y* ∈ *U*) *uniformity*)

definition *real-norm-def* [*simp*]: *norm r* = |*r*|

instance
by *intro-classes* (*auto simp*: *abs-mult open-real-def dist-real-def sgn-real-def uniformity-real-def*)

end

declare *uniformity-Abort*[**where** *'a*=*real*, *code*]

lemma *dist-of-real* [*simp*]: *dist* (*of-real x* :: *'a*) (*of-real y*) = *dist x y*
for *a* :: *'a*::*real-normed-div-algebra*
by (*metis dist-norm norm-of-real of-real-diff real-norm-def*)

declare [[*code abort*: *open* :: *real set* ⇒ *bool*]]

instance *real* :: *linorder-topology*
proof
show (*open* :: *real set* ⇒ *bool*) = *generate-topology* (*range lessThan* ∪ *range greaterThan*)
proof (*rule ext*, *safe*)
fix *S* :: *real set*
assume *open S*
then obtain *f* **where** ∀ *x*∈*S*. *0* < *f x* ∧ (∀ *y*. *dist y x* < *f x* ⟶ *y* ∈ *S*)
unfolding *open-dist bchoice-iff* **..**
then have ∗: (⋃*x*∈*S*. {*x* − *f x* <..} ∩ {..< *x* + *f x*}) = *S* (**is** *?S* = *S*)
by (*fastforce simp*: *dist-real-def*)
moreover have *generate-topology* (*range lessThan* ∪ *range greaterThan*) *?S*
by (*force intro*: *generate-topology.Basis generate-topology-Union generate-topology.Int*)
ultimately show *generate-topology* (*range lessThan* ∪ *range greaterThan*) *S*
by *simp*
next
fix *S* :: *real set*
assume *generate-topology* (*range lessThan* ∪ *range greaterThan*) *S*
moreover have ⋀*a*::*real*. *open* {..<*a*}

unfolding *open-dist dist-real-def*
proof *clarify*
fix *x a* :: *real*
assume $x < a$
then have $0 < a - x \wedge (\forall y.\ |y - x| < a - x \longrightarrow y \in \{..<a\})$ **by** *auto*
then show $\exists e>0.\ \forall y.\ |y - x| < e \longrightarrow y \in \{..<a\}$ **..**
qed
moreover have $\bigwedge a$::*real*. *open* $\{a <..\}$
unfolding *open-dist dist-real-def*
proof *clarify*
fix *x a* :: *real*
assume $a < x$
then have $0 < x - a \wedge (\forall y.\ |y - x| < x - a \longrightarrow y \in \{a<..\})$ **by** *auto*
then show $\exists e>0.\ \forall y.\ |y - x| < e \longrightarrow y \in \{a<..\}$ **..**
qed
ultimately show *open S*
by *induct auto*
qed
qed

instance *real* :: *linear-continuum-topology* **..**

lemmas *open-real-greaterThan* = *open-greaterThan*[**where** *'a=real*]
lemmas *open-real-lessThan* = *open-lessThan*[**where** *'a=real*]
lemmas *open-real-greaterThanLessThan* = *open-greaterThanLessThan*[**where** *'a=real*]
lemmas *closed-real-atMost* = *closed-atMost*[**where** *'a=real*]
lemmas *closed-real-atLeast* = *closed-atLeast*[**where** *'a=real*]
lemmas *closed-real-atLeastAtMost* = *closed-atLeastAtMost*[**where** *'a=real*]

instance *real* :: *ordered-real-vector*
by *standard* (*auto intro*: *mult-left-mono mult-right-mono*)

107.8 Extra type constraints

Only allow *open* in class *topological-space*.

setup ‹*Sign.add-const-constraint*
(***const-name*** ‹*open*›, *SOME* ***typ*** ‹*'a::topological-space set ⇒ bool*›)›

Only allow *uniformity* in class *uniform-space*.

setup ‹*Sign.add-const-constraint*
(***const-name*** ‹*uniformity*›, *SOME* ***typ*** ‹*('a::uniformity × 'a) filter*›)›

Only allow *dist* in class *metric-space*.

setup ‹*Sign.add-const-constraint*
(***const-name*** ‹*dist*›, *SOME* ***typ*** ‹*'a::metric-space ⇒ 'a ⇒ real*›)›

Only allow *norm* in class *real-normed-vector*.

setup ‹*Sign.add-const-constraint*
(***const-name*** ‹*norm*›, *SOME* ***typ*** ‹*'a::real-normed-vector ⇒ real*›)›

107.9 Sign function

lemma *norm-sgn*: *norm* (*sgn x*) = (*if x* = *0 then 0 else 1*)
  for *x* :: *'a*::*real-normed-vector*
  by (*simp add*: *sgn-div-norm*)

lemma *sgn-zero* [*simp*]: *sgn* (*0*::*'a*::*real-normed-vector*) = *0*
  by (*simp add*: *sgn-div-norm*)

lemma *sgn-zero-iff*: *sgn x* = *0* ⟷ *x* = *0*
  for *x* :: *'a*::*real-normed-vector*
  by (*simp add*: *sgn-div-norm*)

lemma *sgn-minus*: *sgn* (− *x*) = − *sgn x*
  for *x* :: *'a*::*real-normed-vector*
  by (*simp add*: *sgn-div-norm*)

lemma *sgn-scaleR*: *sgn* (*scaleR r x*) = *scaleR* (*sgn r*) (*sgn x*)
  for *x* :: *'a*::*real-normed-vector*
  by (*simp add*: *sgn-div-norm ac-simps*)

lemma *sgn-one* [*simp*]: *sgn* (*1*::*'a*::*real-normed-algebra-1*) = *1*
  by (*simp add*: *sgn-div-norm*)

lemma *sgn-of-real*: *sgn* (*of-real r* :: *'a*::*real-normed-algebra-1*) = *of-real* (*sgn r*)
  unfolding *of-real-def* **by** (*simp only*: *sgn-scaleR sgn-one*)

lemma *sgn-mult*: *sgn* (*x* ∗ *y*) = *sgn x* ∗ *sgn y*
  for *x y* :: *'a*::*real-normed-div-algebra*
  by (*simp add*: *sgn-div-norm norm-mult*)

hide-fact (**open**) *sgn-mult*

lemma *real-sgn-eq*: *sgn x* = *x* / |*x*|
  for *x* :: *real*
  by (*simp add*: *sgn-div-norm divide-inverse*)

lemma *zero-le-sgn-iff* [*simp*]: *0* ≤ *sgn x* ⟷ *0* ≤ *x*
  for *x* :: *real*
  by (*cases 0*::*real x rule*: *linorder-cases*) *simp-all*

lemma *sgn-le-0-iff* [*simp*]: *sgn x* ≤ *0* ⟷ *x* ≤ *0*
  for *x* :: *real*
  by (*cases 0*::*real x rule*: *linorder-cases*) *simp-all*

lemma *norm-conv-dist*: *norm x* = *dist x 0*
  unfolding *dist-norm* **by** *simp*

declare *norm-conv-dist* [*symmetric*, *simp*]

lemma *dist-0-norm* [*simp*]: *dist 0 x = norm x*
  for *x* :: *'a::real-normed-vector*
  by (*simp add*: *dist-norm*)

lemma *dist-diff* [*simp*]: *dist a* (*a* − *b*) = *norm b* *dist* (*a* − *b*) *a* = *norm b*
  by (*simp-all add*: *dist-norm*)

lemma *dist-of-int*: *dist* (*of-int m*) (*of-int n* :: *'a* :: *real-normed-algebra-1*) = *of-int* |*m* − *n*|
proof −
  have *dist* (*of-int m*) (*of-int n* :: *'a*) = *dist* (*of-int m* :: *'a*) (*of-int m* − (*of-int* (*m* − *n*)))
    by *simp*
  also have . . . = *of-int* |*m* − *n*| **by** (*subst dist-diff*, *subst norm-of-int*) *simp*
  finally show *?thesis* **.**
qed

lemma *dist-of-nat*:
  dist (*of-nat m*) (*of-nat n* :: *'a* :: *real-normed-algebra-1*) = *of-int* |*int m* − *int n*|
  by (*subst* (*1 2*) *of-int-of-nat-eq* [*symmetric*]) (*rule dist-of-int*)

107.10 Bounded Linear and Bilinear Operators

lemma *linearI*: *linear f*
  if $\bigwedge$*b1 b2*. *f* (*b1* + *b2*) = *f b1* + *f b2*
    $\bigwedge$*r b*. *f* (*r* $*_R$ *b*) = *r* $*_R$ *f b*
  using *that*
  by *unfold-locales* (*auto simp*: *algebra-simps*)

lemma *linear-iff*:
  linear f ⟷ (∀ *x y*. *f* (*x* + *y*) = *f x* + *f y*) ∧ (∀ *c x*. *f* (*c* $*_R$ *x*) = *c* $*_R$ *f x*)
  (**is** *linear f* ⟷ *?rhs*)
proof
  assume *linear f*
  then interpret *f*: *linear f* **.**
  show *?rhs* **by** (*simp add*: *f.add f.scale*)
next
  assume *?rhs*
  then show *linear f* **by** (*intro linearI*) *auto*
qed

lemma *linear-of-real* [*simp*]: *linear of-real*
  by (*simp add*: *linear-iff scaleR-conv-of-real*)

lemmas *linear-scaleR-left* = *linear-scale-left*
lemmas *linear-imp-scaleR* = *linear-imp-scale*

corollary *real-linearD*:
  fixes *f* :: *real* ⇒ *real*

assumes *linear f* **obtains** *c* **where** $f = (*)\ c$
by (*rule linear-imp-scaleR* [*OF assms*]) (*force simp*: *scaleR-conv-of-real*)

lemma *linear-times-of-real*: *linear* ($\lambda x.\ a * of\text{-}real\ x$)
by (*auto intro*!: *linearI simp*: *distrib-left*)
(*metis mult-scaleR-right scaleR-conv-of-real*)

locale *bounded-linear* = *linear f* **for** *f* :: *'a::real-normed-vector* ⇒ *'b::real-normed-vector*
+
assumes *bounded*: $\exists K.\ \forall x.\ norm\ (f\ x) \le norm\ x * K$
begin

lemma *pos-bounded*: $\exists K>0.\ \forall x.\ norm\ (f\ x) \le norm\ x * K$
proof −
obtain *K* **where** *K*: $\bigwedge x.\ norm\ (f\ x) \le norm\ x * K$
using *bounded* **by** *blast*
show *?thesis*
proof (*intro exI impI conjI allI*)
show $0 < max\ 1\ K$
by (*rule order-less-le-trans* [*OF zero-less-one max.cobounded1*])
next
fix *x*
have $norm\ (f\ x) \le norm\ x * K$ **using** *K* **.**
also have $\ldots \le norm\ x * max\ 1\ K$
by (*rule mult-left-mono* [*OF max.cobounded2 norm-ge-zero*])
finally show $norm\ (f\ x) \le norm\ x * max\ 1\ K$ **.**
qed
qed

lemma *nonneg-bounded*: $\exists K\ge 0.\ \forall x.\ norm\ (f\ x) \le norm\ x * K$
using *pos-bounded* **by** (*auto intro*: *order-less-imp-le*)

lemma *linear*: *linear f*
by (*fact local.linear-axioms*)

end

lemma *bounded-linear-intro*:
assumes $\bigwedge x\ y.\ f\ (x + y) = f\ x + f\ y$
and $\bigwedge r\ x.\ f\ (scaleR\ r\ x) = scaleR\ r\ (f\ x)$
and $\bigwedge x.\ norm\ (f\ x) \le norm\ x * K$
shows *bounded-linear f*
by *standard* (*blast intro*: *assms*)+

locale *bounded-bilinear* =
fixes *prod* :: *'a::real-normed-vector* ⇒ *'b::real-normed-vector* ⇒ *'c::real-normed-vector*
(**infixl** ‹**› *70*)
assumes *add-left*: $prod\ (a + a')\ b = prod\ a\ b + prod\ a'\ b$
and *add-right*: $prod\ a\ (b + b') = prod\ a\ b + prod\ a\ b'$

and *scaleR-left*: *prod* (*scaleR r a*) *b* = *scaleR r* (*prod a b*)
and *scaleR-right*: *prod a* (*scaleR r b*) = *scaleR r* (*prod a b*)
and *bounded*: $\exists K. \forall a\ b.\ norm\ (prod\ a\ b) \leq norm\ a * norm\ b * K$
begin

lemma *pos-bounded*: $\exists K>0. \forall a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
proof −
obtain *K* **where** $\bigwedge a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
using *bounded* **by** *blast*
then have $norm\ (a ** b) \leq norm\ a * norm\ b * (max\ 1\ K)$ **for** *a b*
by (*rule order.trans*) (*simp add*: *mult-left-mono*)
then show *?thesis*
by *force*
qed

lemma *nonneg-bounded*: $\exists K\geq 0. \forall a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
using *pos-bounded* **by** (*auto intro*: *order-less-imp-le*)

lemma *additive-right*: *additive* ($\lambda b.$ *prod a b*)
by (*rule additive.intro*, *rule add-right*)

lemma *additive-left*: *additive* ($\lambda a.$ *prod a b*)
by (*rule additive.intro*, *rule add-left*)

lemma *zero-left*: *prod 0 b* = *0*
by (*rule additive.zero* [*OF additive-left*])

lemma *zero-right*: *prod a 0* = *0*
by (*rule additive.zero* [*OF additive-right*])

lemma *minus-left*: *prod* (− *a*) *b* = − *prod a b*
by (*rule additive.minus* [*OF additive-left*])

lemma *minus-right*: *prod a* (− *b*) = − *prod a b*
by (*rule additive.minus* [*OF additive-right*])

lemma *diff-left*: *prod* ($a - a'$) *b* = *prod a b* − *prod* a' *b*
by (*rule additive.diff* [*OF additive-left*])

lemma *diff-right*: *prod a* ($b - b'$) = *prod a b* − *prod a* b'
by (*rule additive.diff* [*OF additive-right*])

lemma *sum-left*: *prod* (*sum g S*) *x* = *sum* (($\lambda i.$ *prod* (*g i*) *x*)) *S*
by (*rule additive.sum* [*OF additive-left*])

lemma *sum-right*: *prod x* (*sum g S*) = *sum* (($\lambda i.$ (*prod x* (*g i*)))) *S*
by (*rule additive.sum* [*OF additive-right*])

lemma *bounded-linear-left*: *bounded-linear* ($\lambda a.\ a ** b$)
proof –
  obtain K **where** $\bigwedge a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
    using *pos-bounded* **by** *blast*
  then show *?thesis*
    by (*rule-tac K=norm b * K* **in** *bounded-linear-intro*) (*auto simp*: *algebra-simps scaleR-left add-left*)
qed

lemma *bounded-linear-right*: *bounded-linear* ($\lambda b.\ a ** b$)
proof –
  obtain K **where** $\bigwedge a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
    using *pos-bounded* **by** *blast*
  then show *?thesis*
    by (*rule-tac K=norm a * K* **in** *bounded-linear-intro*) (*auto simp*: *algebra-simps scaleR-right add-right*)
qed

lemma *prod-diff-prod*: $(x ** y - a ** b) = (x - a) ** (y - b) + (x - a) ** b + a ** (y - b)$
  by (*simp add*: *diff-left diff-right*)

lemma *flip*: *bounded-bilinear* ($\lambda x\ y.\ y ** x$)
proof
  show $\exists K.\ \forall a\ b.\ norm\ (b ** a) \leq norm\ a * norm\ b * K$
    by (*metis bounded mult.commute*)
qed (*simp-all add*: *add-right add-left scaleR-right scaleR-left*)

lemma *comp1*:
  assumes *bounded-linear g*
  shows *bounded-bilinear* ($\lambda x.\ (**)\ (g\ x)$)
proof *unfold-locales*
  interpret *g*: *bounded-linear g* **by** *fact*
  show $\bigwedge a\ a'\ b.\ g\ (a + a')\ **\ b = g\ a ** b + g\ a' ** b$
    $\bigwedge a\ b\ b'.\ g\ a ** (b + b') = g\ a ** b + g\ a ** b'$
    $\bigwedge r\ a\ b.\ g\ (r *_R a) ** b = r *_R (g\ a ** b)$
    $\bigwedge a\ r\ b.\ g\ a ** (r *_R b) = r *_R (g\ a ** b)$
    by (*auto simp*: *g.add add-left add-right g.scaleR scaleR-left scaleR-right*)
  from *g.nonneg-bounded nonneg-bounded* **obtain** $K\ L$
    where *nn*: $0 \leq K\ 0 \leq L$
      and *K*: $\bigwedge x.\ norm\ (g\ x) \leq norm\ x * K$
      and *L*: $\bigwedge a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * L$
    by *auto*
  have $norm\ (g\ a ** b) \leq norm\ a * K * norm\ b * L$ **for** $a\ b$
    by (*auto intro*!: *order-trans*[*OF K*] *order-trans*[*OF L*] *mult-mono simp*: *nn*)
  then show $\exists K.\ \forall a\ b.\ norm\ (g\ a ** b) \leq norm\ a * norm\ b * K$
    by (*auto intro*!: *exI*[**where** *x=K * L*] *simp*: *ac-simps*)
qed

lemma *comp*: *bounded-linear f* ⟹ *bounded-linear g* ⟹ *bounded-bilinear* (*λx y. f x* ** *g y*)
by (*rule bounded-bilinear.flip*[*OF bounded-bilinear.comp1*[*OF bounded-bilinear.flip*[*OF comp1*]]])

end

lemma *bounded-linear-ident*[*simp*]: *bounded-linear* (*λx. x*)
by *standard* (*auto intro*!: *exI*[*of - 1*])

lemma *bounded-linear-zero*[*simp*]: *bounded-linear* (*λx. 0*)
by *standard* (*auto intro*!: *exI*[*of - 1*])

lemma *bounded-linear-add*:
assumes *bounded-linear f*
and *bounded-linear g*
shows *bounded-linear* (*λx. f x* + *g x*)
proof −
interpret *f*: *bounded-linear f* **by** *fact*
interpret *g*: *bounded-linear g* **by** *fact*
show *?thesis*
proof
from *f.bounded* **obtain** *Kf* **where** *Kf*: *norm* (*f x*) ≤ *norm x* ∗ *Kf* **for** *x*
by *blast*
from *g.bounded* **obtain** *Kg* **where** *Kg*: *norm* (*g x*) ≤ *norm x* ∗ *Kg* **for** *x*
by *blast*
show ∃ *K*. ∀ *x*. *norm* (*f x* + *g x*) ≤ *norm x* ∗ *K*
using *add-mono*[*OF Kf Kg*]
by (*intro exI*[*of - Kf* + *Kg*]) (*auto simp*: *field-simps intro*: *norm-triangle-ineq order-trans*)
qed (*simp-all add*: *f.add g.add f.scaleR g.scaleR scaleR-right-distrib*)
qed

lemma *bounded-linear-minus*:
assumes *bounded-linear f*
shows *bounded-linear* (*λx.* − *f x*)
proof −
interpret *f*: *bounded-linear f* **by** *fact*
show *?thesis*
by *unfold-locales* (*simp-all add*: *f.add f.scaleR f.bounded*)
qed

lemma *bounded-linear-sub*: *bounded-linear f* ⟹ *bounded-linear g* ⟹ *bounded-linear* (*λx. f x* − *g x*)
using *bounded-linear-add*[*of f λx.* − *g x*] *bounded-linear-minus*[*of g*]
by (*auto simp*: *algebra-simps*)

lemma *bounded-linear-sum*:
fixes *f* :: *'i* ⇒ *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*

shows ($\bigwedge i.\ i \in I \Longrightarrow$ *bounded-linear* $(f\ i)) \Longrightarrow$ *bounded-linear* $(\lambda x.\ \sum i{\in}I.\ f\ i\ x)$
by (*induct I rule*: *infinite-finite-induct*) (*auto intro*!: *bounded-linear-add*)

lemma *bounded-linear-compose*:
assumes *bounded-linear f*
and *bounded-linear g*
shows *bounded-linear* $(\lambda x.\ f\ (g\ x))$
proof −
interpret *f*: *bounded-linear f* **by** *fact*
interpret *g*: *bounded-linear g* **by** *fact*
show *?thesis*
proof *unfold-locales*
show $f\ (g\ (x + y)) = f\ (g\ x) + f\ (g\ y)$ **for** *x y*
by (*simp only*: *f.add g.add*)
show $f\ (g\ (scaleR\ r\ x)) = scaleR\ r\ (f\ (g\ x))$ **for** *r x*
by (*simp only*: *f.scaleR g.scaleR*)
from *f.pos-bounded* **obtain** *Kf* **where** *f*: $\bigwedge x.\ norm\ (f\ x) \leq norm\ x * Kf$ **and** *Kf*: $0 < Kf$
by *blast*
from *g.pos-bounded* **obtain** *Kg* **where** *g*: $\bigwedge x.\ norm\ (g\ x) \leq norm\ x * Kg$
by *blast*
show $\exists K.\ \forall x.\ norm\ (f\ (g\ x)) \leq norm\ x * K$
proof (*intro exI allI*)
fix *x*
have $norm\ (f\ (g\ x)) \leq norm\ (g\ x) * Kf$
using *f* **.**
also have $\ldots \leq (norm\ x * Kg) * Kf$
using *g Kf* [*THEN order-less-imp-le*] **by** (*rule mult-right-mono*)
also have $(norm\ x * Kg) * Kf = norm\ x * (Kg * Kf)$
by (*rule mult.assoc*)
finally show $norm\ (f\ (g\ x)) \leq norm\ x * (Kg * Kf)$ **.**
qed
qed
qed

lemma *bounded-bilinear-mult*: *bounded-bilinear* $((*) :: 'a \Rightarrow 'a \Rightarrow 'a::real\text{-}normed\text{-}algebra)$
proof (*rule bounded-bilinear.intro*)
show $\exists K.\ \forall a\ b::'a.\ norm\ (a * b) \leq norm\ a * norm\ b * K$
by (*rule-tac x=1* **in** *exI*) (*simp add*: *norm-mult-ineq*)
qed (*auto simp*: *algebra-simps*)

lemma *bounded-linear-mult-left*: *bounded-linear* $(\lambda x::'a::real\text{-}normed\text{-}algebra.\ x * y)$
using *bounded-bilinear-mult*
by (*rule bounded-bilinear.bounded-linear-left*)

lemma *bounded-linear-mult-right*: *bounded-linear* $(\lambda y::'a::real\text{-}normed\text{-}algebra.\ x * y)$

using *bounded-bilinear-mult*
by (*rule bounded-bilinear.bounded-linear-right*)

lemmas *bounded-linear-mult-const* =
bounded-linear-mult-left [*THEN bounded-linear-compose*]

lemmas *bounded-linear-const-mult* =
bounded-linear-mult-right [*THEN bounded-linear-compose*]

lemma *bounded-linear-divide*: *bounded-linear* (λ*x*. *x* / *y*)
for *y* :: *'a*::*real-normed-field*
unfolding *divide-inverse* **by** (*rule bounded-linear-mult-left*)

lemma *bounded-bilinear-scaleR*: *bounded-bilinear scaleR*
proof (*rule bounded-bilinear.intro*)
show ∃ *K*. ∀ *a b*. *norm* ($a *_R b$) ≤ *norm a* ∗ *norm b* ∗ *K*
using *less-eq-real-def* **by** *auto*
qed (*auto simp*: *algebra-simps*)

lemma *bounded-linear-scaleR-left*: *bounded-linear* (λ*r*. *scaleR r x*)
using *bounded-bilinear-scaleR*
by (*rule bounded-bilinear.bounded-linear-left*)

lemma *bounded-linear-scaleR-right*: *bounded-linear* (λ*x*. *scaleR r x*)
using *bounded-bilinear-scaleR*
by (*rule bounded-bilinear.bounded-linear-right*)

lemmas *bounded-linear-scaleR-const* =
bounded-linear-scaleR-left[*THEN bounded-linear-compose*]

lemmas *bounded-linear-const-scaleR* =
bounded-linear-scaleR-right[*THEN bounded-linear-compose*]

lemma *bounded-linear-of-real*: *bounded-linear* (λ*r*. *of-real r*)
unfolding *of-real-def* **by** (*rule bounded-linear-scaleR-left*)

lemma *real-bounded-linear*: *bounded-linear f* ⟷ (∃ *c*::*real*. *f* = (λ*x*. *x* ∗ *c*))
for *f* :: *real* ⇒ *real*
proof −
{
fix *x*
assume *bounded-linear f*
then interpret *bounded-linear f* **.**
from *scaleR*[*of x 1*] **have** *f x* = *x* ∗ *f 1*
by *simp*
}
then show *?thesis*
by (*auto intro*: *exI*[*of - f 1*] *bounded-linear-mult-left*)
qed

instance *real-normed-algebra-1* ⊆ *perfect-space*
proof
fix x::'a
have $\bigwedge e.\ 0 < e \Longrightarrow \exists y.\ norm\ (y - x) < e \wedge y \neq x$
by (*rule-tac* $x = x + $ *of-real* $(e/2)$ **in** *exI*) *auto*
then show ¬ *open* {x}
by (*clarsimp simp*: *open-dist dist-norm*)
qed

107.11 Filters and Limits on Metric Space

lemma (**in** *metric-space*) *nhds-metric*: *nhds* x = (*INF* e∈{*0* <..}. *principal* {y. *dist* y x < e})
unfolding *nhds-def*
proof (*safe intro*!: *INF-eq*)
fix S
assume *open* S $x \in S$
then obtain e **where** {y. *dist* y x < e} ⊆ S *0* < e
by (*auto simp*: *open-dist subset-eq*)
then show ∃ e∈{*0*<..}. *principal* {y. *dist* y x < e} ≤ *principal* S
by *auto*
qed (*auto intro*!: *exI*[*of* - {y. *dist* x y < e} **for** e] *open-ball simp*: *dist-commute*)

lemma *tendsto-iff-uniformity*:
— More general analogus of *tendsto-iff* below. Applies to all uniform spaces, not just metric ones.
fixes l :: ‹'b :: *uniform-space*›
shows ‹(f ⟶ l) F ⟷ (∀ E. *eventually* E *uniformity* ⟶ ($\forall_F$ x *in* F. E (f x, l)))›
proof (*intro iffI allI impI*)
fix E :: ‹('b × 'b) ⇒ *bool*›
assume ‹(f ⟶ l) F› **and** ‹*eventually* E *uniformity*›
from ‹*eventually* E *uniformity*›
have ‹*eventually* (λ(x, y). E (y, x)) *uniformity*›
by (*simp add*: *uniformity-sym*)
then have ‹$\forall_F$ (y, x) *in uniformity*. $y = l$ ⟶ E (x, y)›
using *eventually-mono* **by** *fastforce*
with ‹(f ⟶ l) F› **have** ‹*eventually* (λx. E (x ,l)) (*filtermap* f F)›
by (*simp add*: *filterlim-def le-filter-def eventually-nhds-uniformity*)
then show ‹$\forall_F$ x *in* F. E (f x, l)›
by (*simp add*: *eventually-filtermap*)
next
assume *assm*: ‹∀ E. *eventually* E *uniformity* ⟶ ($\forall_F$ x *in* F. E (f x, l))›
have ‹*eventually* P (*filtermap* f F)› **if** ‹$\forall_F$ (x, y) *in uniformity*. $x = l$ ⟶ P y› **for** P
proof −
from *that* **have** ‹$\forall_F$ (y, x) *in uniformity*. $x = l$ ⟶ P y›

using *uniformity-sym*[**where** *E*=‹λ(*x*,*y*). *x*=*l* ⟶ *P y*›] **by** *auto*
with *assm* **have** ‹∀$_F$ *x in F*. *P* (*f x*)›
by *auto*
then show *?thesis*
by (*auto simp*: *eventually-filtermap*)
qed
then show ‹(*f* ⟶ *l*) *F*›
by (*simp add*: *filterlim-def le-filter-def eventually-nhds-uniformity*)
qed

lemma (**in** *metric-space*) *tendsto-iff*: (*f* ⟶ *l*) *F* ⟷ (∀ *e*>*0*. *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *F*)
unfolding *nhds-metric filterlim-INF filterlim-principal* **by** *auto*

lemma *tendsto-dist-iff*:
((*f* ⟶ *l*) *F*) ⟷ (((λ*x*. *dist* (*f x*) *l*) ⟶ *0*) *F*)
unfolding *tendsto-iff* **by** *simp*

lemma (**in** *metric-space*) *tendstoI* [*intro?*]:
(⋀*e*. *0* < *e* ⟹ *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *F*) ⟹ (*f* ⟶ *l*) *F*
by (*auto simp*: *tendsto-iff*)

lemma (**in** *metric-space*) *tendstoD*: (*f* ⟶ *l*) *F* ⟹ *0* < *e* ⟹ *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *F*
by (*auto simp*: *tendsto-iff*)

lemma (**in** *metric-space*) *eventually-nhds-metric*:
eventually P (*nhds a*) ⟷ (∃ *d*>*0*. ∀ *x*. *dist x a* < *d* ⟶ *P x*)
unfolding *nhds-metric*
by (*subst eventually-INF-base*)
(*auto simp*: *eventually-principal Bex-def subset-eq intro*: *exI*[*of - min a b* **for** *a b*])

lemma *eventually-at*: *eventually P* (*at a within S*) ⟷ (∃ *d*>*0*. ∀ *x*∈*S*. *x* ≠ *a* ∧ *dist x a* < *d* ⟶ *P x*)
for *a* :: *'a* :: *metric-space*
by (*auto simp*: *eventually-at-filter eventually-nhds-metric*)

lemma *frequently-at*: *frequently P* (*at a within S*) ⟷ (∀ *d*>*0*. ∃ *x*∈*S*. *x* ≠ *a* ∧ *dist x a* < *d* ∧ *P x*)
for *a* :: *'a* :: *metric-space*
unfolding *frequently-def eventually-at* **by** *auto*

lemma *eventually-at-le*: *eventually P* (*at a within S*) ⟷ (∃ *d*>*0*. ∀ *x*∈*S*. *x* ≠ *a* ∧ *dist x a* ≤ *d* ⟶ *P x*)
for *a* :: *'a*::*metric-space*
unfolding *eventually-at-filter eventually-nhds-metric*
apply *safe*
apply (*rule-tac x*=*d* / *2* **in** *exI*, *auto*)

done

lemma *eventually-at-left-real*: $a > (b :: real) \Longrightarrow$ *eventually* $(\lambda x.\ x \in \{b<..<a\})$ (*at-left* a)
by (*subst eventually-at*, *rule exI*[*of* - $a - b$]) (*force simp*: *dist-real-def*)

lemma *eventually-at-right-real*: $a < (b :: real) \Longrightarrow$ *eventually* $(\lambda x.\ x \in \{a<..<b\})$ (*at-right* a)
by (*subst eventually-at*, *rule exI*[*of* - $b - a$]) (*force simp*: *dist-real-def*)

lemma *metric-tendsto-imp-tendsto*:
fixes a :: $'a$ :: *metric-space*
and b :: $'b$ :: *metric-space*
assumes f: $(f \longrightarrow a)$ F
and *le*: *eventually* $(\lambda x.\ dist\ (g\ x)\ b \le dist\ (f\ x)\ a)$ F
shows $(g \longrightarrow b)$ F
proof (*rule tendstoI*)
fix e :: *real*
assume $0 < e$
with f **have** *eventually* $(\lambda x.\ dist\ (f\ x)\ a < e)$ F **by** (*rule tendstoD*)
with *le* **show** *eventually* $(\lambda x.\ dist\ (g\ x)\ b < e)$ F
using *le-less-trans* **by** (*rule eventually-elim2*)
qed

lemma *filterlim-real-sequentially*: *LIM* x *sequentially*. *real* x :> *at-top*
proof (*clarsimp simp*: *filterlim-at-top*)
fix Z
show $\forall_F$ x *in sequentially*. $Z \le$ *real* x
by (*meson eventually-sequentiallyI nat-ceiling-le-eq*)
qed

lemma *filterlim-nat-sequentially*: *filterlim nat sequentially at-top*
proof −
have $\forall_F$ x *in at-top*. $Z \le$ *nat* x **for** Z
by (*auto intro*!: *eventually-at-top-linorderI*[**where** c=*int* Z])
then show *?thesis*
unfolding *filterlim-at-top* **..**
qed

lemma *filterlim-floor-sequentially*: *filterlim floor at-top at-top*
proof −
have $\forall_F$ x *in at-top*. $Z \le \lfloor x \rfloor$ **for** Z
by (*auto simp*: *le-floor-iff intro*!: *eventually-at-top-linorderI*[**where** c=*of-int* Z])
then show *?thesis*
unfolding *filterlim-at-top* **..**
qed

lemma *filterlim-sequentially-iff-filterlim-real*:
filterlim f *sequentially* $F \longleftrightarrow$ *filterlim* $(\lambda x.\ real\ (f\ x))$ *at-top* F (**is** *?lhs* = *?rhs*)

proof
  assume *?lhs* **then show** *?rhs*
    using *filterlim-compose filterlim-real-sequentially* **by** *blast*
next
  assume *R*: *?rhs*
  show *?lhs*
  proof −
    have *filterlim* ($\lambda x.$ *nat* (*floor* (*real* (*f x*)))) *sequentially F*
      by (*intro filterlim-compose*[*OF filterlim-nat-sequentially*]
        filterlim-compose[*OF filterlim-floor-sequentially*] *R*)
    then show *?thesis* **by** *simp*
  qed
qed

107.11.1 Limits of Sequences

lemma *lim-sequentially*: $X \longrightarrow L \longleftrightarrow (\forall r>0.\ \exists no.\ \forall n\geq no.\ dist\ (X\ n)\ L < r)$
  for $L :: 'a::metric\text{-}space$
  unfolding *tendsto-iff eventually-sequentially* **..**

lemmas *LIMSEQ-def* = *lim-sequentially*

lemma *LIMSEQ-iff-nz*: $X \longrightarrow L \longleftrightarrow (\forall r>0.\ \exists no>0.\ \forall n\geq no.\ dist\ (X\ n)\ L < r)$
  for $L :: 'a::metric\text{-}space$
  unfolding *lim-sequentially* **by** (*metis Suc-leD zero-less-Suc*)

lemma *metric-LIMSEQ-I*: $(\bigwedge r.\ 0 < r \Longrightarrow \exists no.\ \forall n\geq no.\ dist\ (X\ n)\ L < r) \Longrightarrow X \longrightarrow L$
  for $L :: 'a::metric\text{-}space$
  by (*simp add*: *lim-sequentially*)

lemma *metric-LIMSEQ-D*: $X \longrightarrow L \Longrightarrow 0 < r \Longrightarrow \exists no.\ \forall n\geq no.\ dist\ (X\ n)\ L < r$
  for $L :: 'a::metric\text{-}space$
  by (*simp add*: *lim-sequentially*)

lemma *LIMSEQ-norm-0*:
  assumes $\bigwedge n::nat.\ norm\ (f\ n) < 1\ /\ real\ (Suc\ n)$
  shows $f \longrightarrow 0$
proof (*rule metric-LIMSEQ-I*)
  fix ε :: *real*
  assume $\varepsilon > 0$
  then obtain *N*::*nat* **where** $\varepsilon > inverse\ N\ N > 0$
    by (*metis neq0-conv real-arch-inverse*)
  then have *norm* (*f n*) $< \varepsilon$ **if** $n \geq N$ **for** *n*
  proof −
    have $1\ /\ (Suc\ n) \leq 1\ /\ N$

using ‹*0* < *N*› *inverse-of-nat-le le-SucI that* **by** *blast*
also have ... < *ε*
by (*metis* (*no-types*) ‹*inverse* (*real N*) < *ε*› *inverse-eq-divide*)
finally show *?thesis*
by (*meson assms less-eq-real-def not-le order-trans*)
qed
then show ∃ *no*. ∀ *n*≥*no*. *dist* (*f n*) *0* < *ε*
by *auto*
qed

107.11.2 Limits of Functions

lemma *LIM-def*: *f* −*a*→ *L* ⟷ (∀ *r* > *0*. ∃ *s* > *0*. ∀ *x*. *x* ≠ *a* ∧ *dist x a* < *s* ⟶ *dist* (*f x*) *L* < *r*)
for *a* :: *'a*::*metric-space* **and** *L* :: *'b*::*metric-space*
unfolding *tendsto-iff eventually-at* **by** *simp*

lemma *metric-LIM-I*:
(⋀*r*. *0* < *r* ⟹ ∃ *s*>*0*. ∀ *x*. *x* ≠ *a* ∧ *dist x a* < *s* ⟶ *dist* (*f x*) *L* < *r*) ⟹ *f* −*a*→ *L*
for *a* :: *'a*::*metric-space* **and** *L* :: *'b*::*metric-space*
by (*simp add*: *LIM-def*)

lemma *metric-LIM-D*: *f* −*a*→ *L* ⟹ *0* < *r* ⟹ ∃ *s*>*0*. ∀ *x*. *x* ≠ *a* ∧ *dist x a* < *s* ⟶ *dist* (*f x*) *L* < *r*
for *a* :: *'a*::*metric-space* **and** *L* :: *'b*::*metric-space*
by (*simp add*: *LIM-def*)

lemma *metric-LIM-imp-LIM*:
fixes *l* :: *'a*::*metric-space*
and *m* :: *'b*::*metric-space*
assumes *f*: *f* −*a*→ *l*
and *le*: ⋀*x*. *x* ≠ *a* ⟹ *dist* (*g x*) *m* ≤ *dist* (*f x*) *l*
shows *g* −*a*→ *m*
by (*rule metric-tendsto-imp-tendsto* [*OF f*]) (*auto simp*: *eventually-at-topological le*)

lemma *metric-LIM-equal2*:
fixes *a* :: *'a*::*metric-space*
assumes *g* −*a*→ *l* *0* < *R*
and ⋀*x*. *x* ≠ *a* ⟹ *dist x a* < *R* ⟹ *f x* = *g x*
shows *f* −*a*→ *l*
proof −
have ⋀*S*. ⟦*open S*; *l* ∈ *S*; ∀$_F$ *x in at a*. *g x* ∈ *S*⟧ ⟹ ∀$_F$ *x in at a*. *f x* ∈ *S*
apply (*simp add*: *eventually-at*)
by (*metis assms*(*2*) *assms*(*3*) *dual-order.strict-trans linorder-neqE-linordered-idom*)
then show *?thesis*
using *assms* **by** (*simp add*: *tendsto-def*)
qed

lemma *metric-LIM-compose2*:
 fixes *a* :: *'a::metric-space*
 assumes *f*: *f* $-a\rightarrow$ *b*
  and *g*: *g* $-b\rightarrow$ *c*
  and *inj*: $\exists\, d>0.\ \forall\, x.\ x \neq a \wedge dist\ x\ a < d \longrightarrow f\ x \neq b$
 shows $(\lambda x.\ g\ (f\ x))$ $-a\rightarrow$ *c*
 using *inj* **by** (*intro tendsto-compose-eventually*[*OF g f*]) (*auto simp*: *eventually-at*)

lemma *metric-isCont-LIM-compose2*:
 fixes *f* :: *'a* :: *metric-space* ⇒ -
 assumes *f* [*unfolded isCont-def*]: *isCont f a*
  and *g*: *g* $-f\ a\rightarrow$ *l*
  and *inj*: $\exists\, d>0.\ \forall\, x.\ x \neq a \wedge dist\ x\ a < d \longrightarrow f\ x \neq f\ a$
 shows $(\lambda x.\ g\ (f\ x))$ $-a\rightarrow$ *l*
 by (*rule metric-LIM-compose2* [*OF f g inj*])

107.12 Complete metric spaces

107.13 Cauchy sequences

lemma (**in** *metric-space*) *Cauchy-def*: *Cauchy X* = $(\forall\, e>0.\ \exists\, M.\ \forall\, m\geq M.\ \forall\, n\geq M.\ dist\ (X\ m)\ (X\ n) < e)$
proof −
 have ∗: *eventually P* (*INF M. principal* $\{(X\ m,\ X\ n) \mid n\ m.\ m \geq M \wedge n \geq M\}$) $\longleftrightarrow$
  $(\exists\, M.\ \forall\, m\geq M.\ \forall\, n\geq M.\ P\ (X\ m,\ X\ n))$ **for** *P*
  apply (*subst eventually-INF-base*)
  subgoal by *simp*
  subgoal for *a b*
   by (*intro bexI*[*of - max a b*]) (*auto simp*: *eventually-principal subset-eq*)
  subgoal by (*auto simp*: *eventually-principal*, *blast*)
  done
 have *Cauchy X* $\longleftrightarrow$ (*INF M. principal* $\{(X\ m,\ X\ n) \mid n\ m.\ m \geq M \wedge n \geq M\}$) $\leq$ *uniformity*
  unfolding *Cauchy-uniform-iff le-filter-def* ∗ **..**
 also have ... = $(\forall\, e>0.\ \exists\, M.\ \forall\, m\geq M.\ \forall\, n\geq M.\ dist\ (X\ m)\ (X\ n) < e)$
  unfolding *uniformity-dist le-INF-iff* **by** (*auto simp*: ∗ *le-principal*)
 finally show *?thesis* **.**
qed

lemma (**in** *metric-space*) *Cauchy-altdef*: *Cauchy f* $\longleftrightarrow$ $(\forall\, e>0.\ \exists\, M.\ \forall\, m\geq M.\ \forall\, n>m.\ dist\ (f\ m)\ (f\ n) < e)$
 (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
 assume *?rhs*
 show *?lhs*
  unfolding *Cauchy-def*
 proof (*intro allI impI*)

fix *e* :: *real* **assume** *e*: $e > 0$
with ‹*?rhs*› **obtain** *M* **where** *M*: $m \geq M \Longrightarrow n > m \Longrightarrow$ *dist* (*f m*) (*f n*) $<$ *e* **for** *m n*
by *blast*
have *dist* (*f m*) (*f n*) $< e$ **if** $m \geq M$ $n \geq M$ **for** *m n*
using *M*[*of m n*] *M*[*of n m*] *e that* **by** (*cases m n rule*: *linorder-cases*) (*auto simp*: *dist-commute*)
then show $\exists M. \forall m \geq M. \forall n \geq M.$ *dist* (*f m*) (*f n*) $< e$
by *blast*
qed
next
assume *?lhs*
show *?rhs*
proof (*intro allI impI*)
fix *e* :: *real*
assume *e*: $e > 0$
with ‹*Cauchy f*› **obtain** *M* **where** $\bigwedge m\ n.\ m \geq M \Longrightarrow n \geq M \Longrightarrow$ *dist* (*f m*) (*f n*) $< e$
unfolding *Cauchy-def* **by** *blast*
then show $\exists M. \forall m \geq M. \forall n > m.$ *dist* (*f m*) (*f n*) $< e$
by (*intro exI*[*of - M*]) *force*
qed
qed

lemma (**in** *metric-space*) *Cauchy-altdef2*: *Cauchy s* $\longleftrightarrow$ ($\forall e>0. \exists N::nat. \forall n \geq N.$ *dist*(*s n*)(*s N*) $< e$) (**is** *?lhs* = *?rhs*)
proof
assume *Cauchy s*
then show *?rhs* **by** (*force simp*: *Cauchy-def*)
next
assume *?rhs*
{
fix *e*::*real*
assume $e>0$
with ‹*?rhs*› **obtain** *N* **where** *N*: $\forall n \geq N.$ *dist* (*s n*) (*s N*) $< e/2$
by (*erule-tac* $x=e/2$ **in** *allE*) *auto*
{
fix *n m*
assume *nm*: $N \leq m \wedge N \leq n$
then have *dist* (*s m*) (*s n*) $< e$ **using** *N*
using *dist-triangle-half-l*[*of s m s N e s n*]
by *blast*
}
then have $\exists N. \forall m\ n.\ N \leq m \wedge N \leq n \longrightarrow$ *dist* (*s m*) (*s n*) $< e$
by *blast*
}
then have *?lhs*
unfolding *Cauchy-def* **by** *blast*
then show *?lhs*

by *blast*
qed

lemma (**in** *metric-space*) *metric-CauchyI*:
$(\bigwedge e.\ 0 < e \Longrightarrow \exists M.\ \forall m \geq M.\ \forall n \geq M.\ dist\ (X\ m)\ (X\ n) < e) \Longrightarrow Cauchy\ X$
by (*simp add*: *Cauchy-def*)

lemma (**in** *metric-space*) *CauchyI'*:
$(\bigwedge e.\ 0 < e \Longrightarrow \exists M.\ \forall m \geq M.\ \forall n > m.\ dist\ (X\ m)\ (X\ n) < e) \Longrightarrow Cauchy\ X$
unfolding *Cauchy-altdef* **by** *blast*

lemma (**in** *metric-space*) *metric-CauchyD*:
$Cauchy\ X \Longrightarrow 0 < e \Longrightarrow \exists M.\ \forall m \geq M.\ \forall n \geq M.\ dist\ (X\ m)\ (X\ n) < e$
by (*simp add*: *Cauchy-def*)

lemma (**in** *metric-space*) *metric-Cauchy-iff2*:
$Cauchy\ X = (\forall j.\ (\exists M.\ \forall m \geq M.\ \forall n \geq M.\ dist\ (X\ m)\ (X\ n) < inverse(real\ (Suc\ j))))$
apply (*auto simp add*: *Cauchy-def*)
by (*metis less-trans of-nat-Suc reals-Archimedean*)

lemma *Cauchy-iff2*: $Cauchy\ X \longleftrightarrow (\forall j.\ (\exists M.\ \forall m \geq M.\ \forall n \geq M.\ |X\ m - X\ n| < inverse\ (real\ (Suc\ j))))$
by (*simp only*: *metric-Cauchy-iff2 dist-real-def*)

lemma *lim-1-over-n* [*tendsto-intros*]: $((\lambda n.\ 1\ /\ of\text{-}nat\ n) \longlongrightarrow (0::'a::real\text{-}normed\text{-}field))$ *sequentially*
proof (*subst lim-sequentially*, *intro allI impI exI*)
fix *e*::*real* **and** *n*
assume *e*: $e > 0$
have $inverse\ e < of\text{-}nat\ (nat\ \lceil inverse\ e + 1\rceil)$ **by** *linarith*
also assume $n \geq nat\ \lceil inverse\ e + 1\rceil$
finally show $dist\ (1\ /\ of\text{-}nat\ n :: 'a)\ 0 < e$
using *e* **by** (*simp add*: *field-split-simps norm-divide*)
qed

lemma (**in** *metric-space*) *complete-def*:
shows $complete\ S = (\forall f.\ (\forall n.\ f\ n \in S) \wedge Cauchy\ f \longrightarrow (\exists l \in S.\ f \longlongrightarrow l))$
unfolding *complete-uniform*
proof *safe*
fix $f :: nat \Rightarrow 'a$
assume *f*: $\forall n.\ f\ n \in S\ Cauchy\ f$
and $*$: $\forall F \leq principal\ S.\ F \neq bot \longrightarrow cauchy\text{-}filter\ F \longrightarrow (\exists x \in S.\ F \leq nhds\ x)$
then show $\exists l \in S.\ f \longlongrightarrow l$
unfolding *filterlim-def* **using** *f*
by (*intro* $*$[*rule-format*])
(*auto simp*: *filtermap-sequentually-ne-bot le-principal eventually-filtermap Cauchy-uniform*)
next

fix F :: *'a filter*
assume $F \leq$ *principal S F* $\neq$ *bot cauchy-filter F*
assume *seq*: $\forall f.$ ($\forall n.$ $f\ n \in S$) $\wedge$ *Cauchy f* $\longrightarrow$ ($\exists l{\in}S.$ $f \longrightarrow l$)

from ‹$F \leq$ *principal S*› ‹*cauchy-filter F*›
have *FF-le*: $F \times_F F \leq$ *uniformity-on S*
by (*simp add*: *cauchy-filter-def principal-prod-principal*[*symmetric*] *prod-filter-mono*)

let *?P* = $\lambda P\ e.$ *eventually P F* $\wedge$ ($\forall x.$ $P\ x \longrightarrow x \in S$) $\wedge$ ($\forall x\ y.$ $P\ x \longrightarrow P\ y$ $\longrightarrow$ *dist x y* $< e$)
have *P*: $\exists P.$ *?P P* ε **if** $0 < \varepsilon$ **for** ε :: *real*
proof −
from *that* **have** *eventually* ($\lambda(x, y).$ $x \in S \wedge y \in S \wedge$ *dist x y* $< \varepsilon$) (*uniformity-on S*)
by (*auto simp*: *eventually-inf-principal eventually-uniformity-metric*)
from *filter-leD*[*OF FF-le this*] **show** *?thesis*
by (*auto simp*: *eventually-prod-same*)
qed

have $\exists P.$ $\forall n.$ *?P* ($P\ n$) (1 / *Suc n*) $\wedge$ P (*Suc n*) $\leq P\ n$
proof (*rule dependent-nat-choice*)
show $\exists P.$ *?P P* (1 / *Suc 0*)
using *P*[*of 1*] **by** *auto*
next
fix *P n* **assume** *?P P* (1/*Suc n*)
moreover obtain *Q* **where** *?P Q* (1 / *Suc* (*Suc n*))
using *P*[*of 1*/*Suc* (*Suc n*)] **by** *auto*
ultimately show $\exists Q.$ *?P Q* (1 / *Suc* (*Suc n*)) $\wedge$ $Q \leq P$
by (*intro exI*[*of* - $\lambda x.$ $P\ x \wedge Q\ x$]) (*auto simp*: *eventually-conj-iff*)
qed
then obtain *P* **where** *P*: *eventually* ($P\ n$) F $P\ n\ x \Longrightarrow x \in S$
$P\ n\ x \Longrightarrow P\ n\ y \Longrightarrow$ *dist x y* < 1 / *Suc n* P (*Suc n*) $\leq P\ n$
for *n x y*
by *metis*
have *antimono P*
using *P*(*4*) **by** (*rule decseq-SucI*)

obtain *X* **where** *X*: $P\ n$ ($X\ n$) **for** *n*
using *P*(*1*)[*THEN eventually-happens'*[*OF* ‹$F \neq$ *bot*›]] **by** *metis*
have *Cauchy X*
unfolding *metric-Cauchy-iff2 inverse-eq-divide*
proof (*intro exI allI impI*)
fix *j m n* :: *nat*
assume $j \leq m$ $j \leq n$
with ‹*antimono P*› *X* **have** $P\ j$ ($X\ m$) $P\ j$ ($X\ n$)
by (*auto simp*: *antimono-def*)
then show *dist* ($X\ m$) ($X\ n$) < 1 / *Suc j*
by (*rule P*)
qed

moreover have ∀ *n*. *X n* ∈ *S*
  using *P*(*2*) *X* **by** *auto*
ultimately obtain *x* **where** *X* ⟶ *x x* ∈ *S*
  using *seq* **by** *blast*

show ∃ *x*∈*S*. *F* ≤ *nhds x*
proof (*rule bexI*)
  have *eventually* (λ*y*. *dist y x* < *e*) *F* **if** *0* < *e* **for** *e* :: *real*
  proof −
    from *that* **have** (λ*n*. *1* / *Suc n* :: *real*) ⟶ *0* ∧ *0* < *e* / *2*
      by (*subst filterlim-sequentially-Suc*) (*auto intro*!: *lim-1-over-n*)
    then have ∀ *F* *n in sequentially*. *dist* (*X n*) *x* < *e* / *2* ∧ *1* / *Suc n* < *e* / *2*
      using ‹*X* ⟶ *x*›
      unfolding *tendsto-iff order-tendsto-iff*[**where** ′*a*=*real*] *eventually-conj-iff*
      by *blast*
    then obtain *n* **where** *dist x* (*X n*) < *e* / *2 1* / *Suc n* < *e* / *2*
      by (*auto simp*: *eventually-sequentially dist-commute*)
    show *?thesis*
      using ‹*eventually* (*P n*) *F*›
    proof *eventually-elim*
      case (*elim y*)
      then have *dist y* (*X n*) < *1* / *Suc n*
        by (*intro X P*)
      also have . . . < *e* / *2* **by** *fact*
      finally show *dist y x* < *e*
        by (*rule dist-triangle-half-l*) *fact*
    qed
  qed
  then show *F* ≤ *nhds x*
    unfolding *nhds-metric le-INF-iff le-principal* **by** *auto*
qed *fact*
qed

apparently unused

lemma (**in** *metric-space*) *totally-bounded-metric*:
  totally-bounded S ⟷ (∀ *e*>*0*. ∃ *k*. *finite k* ∧ *S* ⊆ (⋃ *x*∈*k*. {*y*. *dist x y* < *e*}))
  unfolding *totally-bounded-def eventually-uniformity-metric imp-ex*
  apply (*subst all-comm*)
  apply (*intro arg-cong*[**where** *f*=*All*] *ext*, *safe*)
  subgoal for *e*
    apply (*erule allE*[*of* - λ(*x*, *y*). *dist x y* < *e*])
    apply *auto*
    done
  subgoal for *e P k*
    apply (*intro exI*[*of* - *k*])
    apply (*force simp*: *subset-eq*)
    done
  done

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*dist*›, *SOME* ***typ*** ‹$'a$::*dist* ⇒ $'a$ ⇒ *real*›)›

lemma *cauchy-filter-metric*:
fixes F :: $'a$::{*uniformity-dist*,*uniform-space*} *filter*
shows *cauchy-filter* F ⟷ (∀ *e*. *e*>*0* ⟶ (∃ *P*. *eventually* P F ∧ (∀ *x y*. P x ∧ P y ⟶ *dist x y* < *e*)))
proof (*unfold cauchy-filter-def le-filter-def*, *auto*)
assume *assm*: ‹∀ *e*>*0*. ∃ *P*. *eventually* P F ∧ (∀ *x y*. P x ∧ P y ⟶ *dist x y* < *e*)›
then show ‹*eventually* P *uniformity* ⟹ *eventually* P (F $\times_F$ F)› **for** P
apply (*auto simp*: *eventually-uniformity-metric*)
using *eventually-prod-same* **by** *blast*
next
fix *e* :: *real*
assume ‹*e* > *0*›
assume *asm*: ‹∀ *P*. *eventually* P *uniformity* ⟶ *eventually* P (F $\times_F$ F)›

define P **where** ‹P ≡ λ(*x*,*y* :: $'a$). *dist x y* < *e*›
with *asm* ‹*e* > *0*› **have** ‹*eventually* P (F $\times_F$ F)›
by (*metis case-prod-conv eventually-uniformity-metric*)
then
show ‹∃ *P*. *eventually* P F ∧ (∀ *x y*. P x ∧ P y ⟶ *dist x y* < *e*)›
by (*auto simp add*: *eventually-prod-same P-def*)
qed

lemma *cauchy-filter-metric-filtermap*:
fixes *f* :: $'a$ ⇒ $'b$::{*uniformity-dist*,*uniform-space*}
shows *cauchy-filter* (*filtermap f F*) ⟷ (∀ *e*. *e*>*0* ⟶ (∃ *P*. *eventually* P F ∧ (∀ *x y*. P x ∧ P y ⟶ *dist* (*f x*) (*f y*) < *e*)))
proof (*subst cauchy-filter-metric*, *intro iffI allI impI*)
assume ‹∀ *e*>*0*. ∃ *P*. *eventually* P (*filtermap f F*) ∧ (∀ *x y*. P x ∧ P y ⟶ *dist x y* < *e*)›
then show ‹*e*>*0* ⟹ ∃ *P*. *eventually* P F ∧ (∀ *x y*. P x ∧ P y ⟶ *dist* (*f x*) (*f y*) < *e*)› **for** *e*
unfolding *eventually-filtermap* **by** *blast*
next
assume *asm*: ‹∀ *e*>*0*. ∃ *P*. *eventually* P F ∧ (∀ *x y*. P x ∧ P y ⟶ *dist* (*f x*) (*f y*) < *e*)›
fix *e*::*real* **assume** ‹*e* > *0*›
then obtain P **where** ‹*eventually* P F› **and** *PPe*: ‹P x ∧ P y ⟶ *dist* (*f x*) (*f y*) < *e*› **for** *x y*
using *asm* **by** *blast*

show ‹∃ *P*. *eventually* P (*filtermap f F*) ∧ (∀ *x y*. P x ∧ P y ⟶ *dist x y* < *e*)›
apply (*rule exI*[*of* - ‹λ*x*. ∃ *y*. P y ∧ x = *f y*›])

using *PPe* ‹*eventually P F*› **apply** (*auto simp*: *eventually-filtermap*)
by (*smt* (*verit*, *ccfv-SIG*) *eventually-elim2*)
qed

setup ‹*Sign.add-const-constraint* (***const-name*** ‹*dist*›, *SOME* ***typ*** ‹′*a*::*metric-space* ⇒ ′*a* ⇒ *real*›)›

107.13.1 Cauchy Sequences are Convergent

class *complete-space* = *metric-space* +
assumes *Cauchy-convergent*: *Cauchy X* ⟹ *convergent X*

lemma *Cauchy-convergent-iff*: *Cauchy X* ⟷ *convergent X*
for *X* :: *nat* ⇒ ′*a*::*complete-space*
by (*blast intro*: *Cauchy-convergent convergent-Cauchy*)

To prove that a Cauchy sequence converges, it suffices to show that a subsequence converges.

lemma *Cauchy-converges-subseq*:
fixes *u*::*nat* ⇒ ′*a*::*metric-space*
assumes *Cauchy u*
strict-mono r
(*u* ∘ *r*) ⟶ *l*
shows *u* ⟶ *l*
proof −
have ∗: *eventually* (λ*n*. *dist* (*u n*) *l* < *e*) *sequentially* **if** *e* > *0* **for** *e*
proof −
have *e*/*2* > *0* **using** *that* **by** *auto*
then obtain *N1* **where** *N1*: ⋀*m n*. *m* ≥ *N1* ⟹ *n* ≥ *N1* ⟹ *dist* (*u m*) (*u n*) < *e*/*2*
using ‹*Cauchy u*› **unfolding** *Cauchy-def* **by** *blast*
obtain *N2* **where** *N2*: ⋀*n*. *n* ≥ *N2* ⟹ *dist* ((*u* ∘ *r*) *n*) *l* < *e* / *2*
using *order-tendstoD*(*2*)[*OF iffD1*[*OF tendsto-dist-iff* ‹(*u* ∘ *r*) ⟶ *l*›] ‹*e*/*2* > *0*›]
unfolding *eventually-sequentially* **by** *auto*
have *dist* (*u n*) *l* < *e* **if** *n* ≥ *max N1 N2* **for** *n*
proof −
have *dist* (*u n*) *l* ≤ *dist* (*u n*) ((*u* ∘ *r*) *n*) + *dist* ((*u* ∘ *r*) *n*) *l*
by (*rule dist-triangle*)
also have ... < *e*/*2* + *e*/*2*
proof (*intro add-strict-mono*)
show *dist* (*u n*) ((*u* ∘ *r*) *n*) < *e* / *2*
using *N1*[*of n r n*] *N2*[*of n*] *that* **unfolding** *comp-def*
by (*meson assms*(*2*) *le-trans max.bounded-iff strict-mono-imp-increasing*)
show *dist* ((*u* ∘ *r*) *n*) *l* < *e* / *2*
using *N2 that* **by** *auto*
qed
finally show *?thesis* **by** *simp*
qed

then show *?thesis* **unfolding** *eventually-sequentially* **by** *blast*
qed
have $(\lambda n.\ dist\ (u\ n)\ l) \longrightarrow 0$
by (*simp add*: *less-le-trans* ∗ *order-tendstoI*)
then show *?thesis* **using** *tendsto-dist-iff* **by** *auto*
qed

107.14 The set of real numbers is a complete metric space

Proof that Cauchy sequences converge based on the one from http://pirate.shu.edu/~wachsmut/ira/numseq/proofs/cauconv.html

If sequence X is Cauchy, then its limit is the lub of $\{r.\ \exists N.\ \forall n\geq N.\ r < X\ n\}$

lemma *increasing-LIMSEQ*:
fixes $f :: nat \Rightarrow real$
assumes *inc*: $\bigwedge n.\ f\ n \leq f\ (Suc\ n)$
and *bdd*: $\bigwedge n.\ f\ n \leq l$
and *en*: $\bigwedge e.\ 0 < e \Longrightarrow \exists n.\ l \leq f\ n + e$
shows $f \longrightarrow l$
proof (*rule increasing-tendsto*)
fix x
assume $x < l$
with *dense*[*of* $0\ l - x$] **obtain** e **where** $0 < e\ e < l - x$
by *auto*
from *en*[*OF* ‹$0 < e$›] **obtain** n **where** $l - e \leq f\ n$
by (*auto simp*: *field-simps*)
with ‹$e < l - x$› ‹$0 < e$› **have** $x < f\ n$
by *simp*
with *incseq-SucI*[*of f*, *OF inc*] **show** *eventually* $(\lambda n.\ x < f\ n)$ *sequentially*
by (*auto simp*: *eventually-sequentially incseq-def intro*: *less-le-trans*)
qed (*use bdd* **in** *auto*)

lemma *real-Cauchy-convergent*:
fixes $X :: nat \Rightarrow real$
assumes X: *Cauchy* X
shows *convergent* X
proof −
define $S :: real\ set$ **where** $S = \{x.\ \exists N.\ \forall n\geq N.\ x < X\ n\}$
then have *mem-S*: $\bigwedge N\ x.\ \forall n\geq N.\ x < X\ n \Longrightarrow x \in S$
by *auto*

have *bound-isUb*: $y \leq x$ **if** N: $\forall n\geq N.\ X\ n < x$ **and** $y \in S$ **for** N **and** $x\ y :: real$
proof −
from *that* **have** $\exists M.\ \forall n\geq M.\ y < X\ n$
by (*simp add*: *S-def*)
then obtain M **where** $\forall n\geq M.\ y < X\ n$ **..**
then have $y < X\ (max\ M\ N)$ **by** *simp*
also have $\ldots < x$ **using** N **by** *simp*

```
  finally show ?thesis by (rule order-less-imp-le)
qed

obtain N where ∀ m≥N. ∀ n≥N. dist (X m) (X n) < 1
  using X[THEN metric-CauchyD, OF zero-less-one] by auto
then have N: ∀ n≥N. dist (X n) (X N) < 1 by simp
have [simp]: S ≠ {}
proof (intro exI ex-in-conv[THEN iffD1])
  from N have ∀ n≥N. X N − 1 < X n
    by (simp add: abs-diff-less-iff dist-real-def)
  then show X N − 1 ∈ S by (rule mem-S)
qed
have [simp]: bdd-above S
proof
  from N have ∀ n≥N. X n < X N + 1
    by (simp add: abs-diff-less-iff dist-real-def)
  then show ⋀s. s ∈ S ⟹ s ≤ X N + 1
    by (rule bound-isUb)
qed
have X ⟶ Sup S
proof (rule metric-LIMSEQ-I)
  fix r :: real
  assume 0 < r
  then have r: 0 < r/2 by simp
  obtain N where ∀ n≥N. ∀ m≥N. dist (X n) (X m) < r/2
    using metric-CauchyD [OF X r] by auto
  then have ∀ n≥N. dist (X n) (X N) < r/2 by simp
  then have N: ∀ n≥N. X N − r/2 < X n ∧ X n < X N + r/2
    by (simp only: dist-real-def abs-diff-less-iff)

  from N have ∀ n≥N. X N − r/2 < X n by blast
  then have X N − r/2 ∈ S by (rule mem-S)
  then have 1: X N − r/2 ≤ Sup S by (simp add: cSup-upper)

  from N have ∀ n≥N. X n < X N + r/2 by blast
  from bound-isUb[OF this]
  have 2: Sup S ≤ X N + r/2
    by (intro cSup-least) simp-all

  show ∃ N. ∀ n≥N. dist (X n) (Sup S) < r
  proof (intro exI allI impI)
    fix n
    assume n: N ≤ n
    from N n have X n < X N + r/2 and X N − r/2 < X n
      by simp-all
    then show dist (X n) (Sup S) < r using 1 2
      by (simp add: abs-diff-less-iff dist-real-def)
  qed
qed
```

then show *?thesis* **by** (*auto simp*: *convergent-def*)
qed

instance *real* :: *complete-space*
by *intro-classes* (*rule real-Cauchy-convergent*)

class *banach* = *real-normed-vector* + *complete-space*

instance *real* :: *banach* **..**

lemma *tendsto-at-topI-sequentially*:
fixes *f* :: *real* ⇒ *'b::first-countable-topology*
assumes ∗: $\bigwedge X$. *filterlim X at-top sequentially* ⟹ (λn. *f* (*X n*)) ⟶ *y*
shows (*f* ⟶ *y*) *at-top*
proof −
obtain *A* **where** *A*: *decseq A open* (*A n*) *y* ∈ *A n nhds y* = (*INF n. principal* (*A n*)) **for** *n*
by (*rule nhds-countable*[*of y*]) (*rule that*)

have ∀ *m*. ∃ *k*. ∀ *x*≥*k*. *f x* ∈ *A m*
proof (*rule ccontr*)
assume ¬ (∀ *m*. ∃ *k*. ∀ *x*≥*k*. *f x* ∈ *A m*)
then obtain *m* **where** $\bigwedge k$. ∃ *x*≥*k*. *f x* ∉ *A m*
by *auto*
then have ∃ *X*. ∀ *n*. (*f* (*X n*) ∉ *A m*) ∧ *max n* (*X n*) + *1* ≤ *X* (*Suc n*)
by (*intro dependent-nat-choice*) (*auto simp del*: *max.bounded-iff*)
then obtain *X* **where** *X*: $\bigwedge n$. *f* (*X n*) ∉ *A m* $\bigwedge n$. *max n* (*X n*) + *1* ≤ *X* (*Suc n*)
by *auto*
have *1* ≤ *n* ⟹ *real n* ≤ *X n* **for** *n*
using *X*[*of n* − *1*] **by** *auto*
then have *filterlim X at-top sequentially*
by (*force intro*!: *filterlim-at-top-mono*[*OF filterlim-real-sequentially*]
simp: *eventually-sequentially*)
from *topological-tendstoD*[*OF* ∗[*OF this*] *A*(*2*, *3*), *of m*] *X*(*1*) **show** *False*
by *auto*
qed
then obtain *k* **where** *k m* ≤ *x* ⟹ *f x* ∈ *A m* **for** *m x*
by *metis*
then show *?thesis*
unfolding *at-top-def A* **by** (*intro filterlim-base*[**where** *i*=*k*]) *auto*
qed

lemma *tendsto-at-topI-sequentially-real*:
fixes *f* :: *real* ⇒ *real*
assumes *mono*: *mono f*
and *limseq*: (λn. *f* (*real n*)) ⟶ *y*
shows (*f* ⟶ *y*) *at-top*
proof (*rule tendstoI*)

fix *e* :: *real*
assume $0 < e$
with *limseq* **obtain** *N* :: *nat* **where** *N*: $N \leq n \Longrightarrow |f\ (real\ n) - y| < e$ **for** *n*
by (*auto simp*: *lim-sequentially dist-real-def*)
have *le*: $f\ x \leq y$ **for** *x* :: *real*
proof −
obtain *n* **where** $x \leq$ *real-of-nat n*
using *real-arch-simple*[*of x*] **..**
note *monoD*[*OF mono this*]
also have *f* (*real-of-nat n*) $\leq y$
by (*rule LIMSEQ-le-const*[*OF limseq*]) (*auto intro*!: *exI*[*of - n*] *monoD*[*OF mono*])
finally show *?thesis* **.**
qed
have *eventually* (λx. *real N* $\leq x$) *at-top*
by (*rule eventually-ge-at-top*)
then show *eventually* (λx. *dist* (*f x*) $y < e$) *at-top*
proof *eventually-elim*
case (*elim x*)
with *N*[*of N*] *le* **have** $y - f$ (*real N*) $< e$ **by** *auto*
moreover note *monoD*[*OF mono elim*]
ultimately show *dist* (*f x*) $y < e$
using *le*[*of x*] **by** (*auto simp*: *dist-real-def field-simps*)
qed
qed

end

108 Limits on Real Vector Spaces

theory *Limits*
imports *Real-Vector-Spaces*
begin

lemma *range-mult* [*simp*]:
fixes *a*::*real* **shows** *range* ((∗) *a*) = (*if a=0 then* {*0*} *else UNIV*)
by (*simp add*: *surj-def*) (*meson dvdE dvd-field-iff*)

108.1 Filter going to infinity norm

definition *at-infinity* :: *'a*::*real-normed-vector filter*
where *at-infinity* = (*INF r. principal* {*x. r* ≤ *norm x*})

lemma *eventually-at-infinity*: *eventually P at-infinity* ⟷ (∃ *b*. ∀ *x*. *b* ≤ *norm x* ⟶ *P x*)
unfolding *at-infinity-def*
by (*subst eventually-INF-base*)
(*auto simp*: *subset-eq eventually-principal intro*!: *exI*[*of - max a b* **for** *a b*])

corollary *eventually-at-infinity-pos*:
eventually p at-infinity $\longleftrightarrow$ ($\exists$ *b*. *0* < *b* $\wedge$ ($\forall$ *x*. *norm x* $\geq$ *b* $\longrightarrow$ *p x*))
unfolding *eventually-at-infinity*
by (*meson le-less-trans norm-ge-zero not-le zero-less-one*)

lemma *at-infinity-eq-at-top-bot*: (*at-infinity* :: *real filter*) = *sup at-top at-bot*
proof −
have *1*: $[\![\forall n{\geq}u.\ A\ n;\ \forall n{\leq}v.\ A\ n]\!]$
$\Longrightarrow \exists b.\ \forall x.\ b \leq |x| \longrightarrow A\ x$ **for** *A* **and** *u v*::*real*
by (*rule-tac x*=*max* (− *v*) *u* **in** *exI*) (*auto simp*: *abs-real-def*)
have *2*: $\forall x.\ u \leq |x| \longrightarrow A\ x \Longrightarrow \exists N.\ \forall n{\geq}N.\ A\ n$ **for** *A* **and** *u*::*real*
by (*meson abs-less-iff le-cases less-le-not-le*)
have *3*: $\forall x.\ u \leq |x| \longrightarrow A\ x \Longrightarrow \exists N.\ \forall n{\leq}N.\ A\ n$ **for** *A* **and** *u*::*real*
by (*metis* (*full-types*) *abs-ge-self abs-minus-cancel le-minus-iff order-trans*)
show *?thesis*
by (*auto simp*: *filter-eq-iff eventually-sup eventually-at-infinity*
eventually-at-top-linorder eventually-at-bot-linorder intro: *1 2 3*)
qed

lemma *at-top-le-at-infinity*: *at-top* $\leq$ (*at-infinity* :: *real filter*)
unfolding *at-infinity-eq-at-top-bot* **by** *simp*

lemma *at-bot-le-at-infinity*: *at-bot* $\leq$ (*at-infinity* :: *real filter*)
unfolding *at-infinity-eq-at-top-bot* **by** *simp*

lemma *filterlim-at-top-imp-at-infinity*: *filterlim f at-top F* $\Longrightarrow$ *filterlim f at-infinity F*
for *f* :: - $\Rightarrow$ *real*
by (*rule filterlim-mono*[*OF* - *at-top-le-at-infinity order-refl*])

lemma *filterlim-real-at-infinity-sequentially*: *filterlim real at-infinity sequentially*
by (*simp add*: *filterlim-at-top-imp-at-infinity filterlim-real-sequentially*)

lemma *lim-infinity-imp-sequentially*: (*f* $\longrightarrow$ *l*) *at-infinity* $\Longrightarrow$ ((λ*n*. *f*(*n*)) $\longrightarrow$ *l*) *sequentially*
by (*simp add*: *filterlim-at-top-imp-at-infinity filterlim-compose filterlim-real-sequentially*)

108.1.1 Boundedness

definition *Bfun* :: ('*a* $\Rightarrow$ '*b*::*metric-space*) $\Rightarrow$ '*a filter* $\Rightarrow$ *bool*
where *Bfun-metric-def*: *Bfun f F* = ($\exists$ *y*. $\exists$ *K*>*0*. *eventually* (λ*x*. *dist* (*f x*) *y* $\leq$ *K*) *F*)

abbreviation *Bseq* :: (*nat* $\Rightarrow$ '*a*::*metric-space*) $\Rightarrow$ *bool*
where *Bseq X* $\equiv$ *Bfun X sequentially*

lemma *Bseq-conv-Bfun*: *Bseq X* $\longleftrightarrow$ *Bfun X sequentially* **..**

lemma *Bseq-ignore-initial-segment*: *Bseq X* $\Longrightarrow$ *Bseq* (λ*n*. *X* (*n* + *k*))

unfolding *Bfun-metric-def* **by** (*subst eventually-sequentially-seg*)

lemma *Bseq-offset*: *Bseq* ($\lambda n.$ *X* ($n + k$)) $\Longrightarrow$ *Bseq X*
unfolding *Bfun-metric-def* **by** (*subst* (*asm*) *eventually-sequentially-seg*)

lemma *Bfun-def*: *Bfun f F* $\longleftrightarrow$ ($\exists K>0.$ *eventually* ($\lambda x.$ *norm* (*f x*) $\leq K$) *F*)
unfolding *Bfun-metric-def norm-conv-dist*
proof *safe*
fix *y K*
assume *K*: $0 < K$ **and** $*$: *eventually* ($\lambda x.$ *dist* (*f x*) $y \leq K$) *F*
moreover have *eventually* ($\lambda x.$ *dist* (*f x*) $0 \leq$ *dist* (*f x*) $y +$ *dist* 0 y) *F*
by (*intro always-eventually*) (*metis dist-commute dist-triangle*)
with $*$ **have** *eventually* ($\lambda x.$ *dist* (*f x*) $0 \leq K +$ *dist* 0 y) *F*
by *eventually-elim auto*
with ‹$0 < K$› **show** $\exists K>0.$ *eventually* ($\lambda x.$ *dist* (*f x*) $0 \leq K$) *F*
by (*intro exI*[*of - K + dist 0 y*] *add-pos-nonneg conjI zero-le-dist*) *auto*
qed (*force simp del*: *norm-conv-dist* [*symmetric*])

lemma *BfunI*:
assumes *K*: *eventually* ($\lambda x.$ *norm* (*f x*) $\leq K$) *F*
shows *Bfun f F*
unfolding *Bfun-def*
proof (*intro exI conjI allI*)
show $0 <$ *max K 1* **by** *simp*
show *eventually* ($\lambda x.$ *norm* (*f x*) $\leq$ *max K 1*) *F*
using *K* **by** (*rule eventually-mono*) *simp*
qed

lemma *BfunE*:
assumes *Bfun f F*
obtains *B* **where** $0 < B$ **and** *eventually* ($\lambda x.$ *norm* (*f x*) $\leq B$) *F*
using *assms* **unfolding** *Bfun-def* **by** *blast*

lemma *Cauchy-Bseq*:
assumes *Cauchy X* **shows** *Bseq X*
proof −
have $\exists y\ K.\ 0 < K \wedge (\exists N.\ \forall n\geq N.$ *dist* (*X n*) $y \leq K$)
if $\bigwedge m\ n.\ [\![m \geq M;\ n \geq M]\!] \Longrightarrow$ *dist* (*X m*) (*X n*) < 1 **for** *M*
by (*meson order.order-iff-strict that zero-less-one*)
with *assms* **show** *?thesis*
by (*force simp*: *Cauchy-def Bfun-metric-def eventually-sequentially*)
qed

108.1.2 Bounded Sequences

lemma *BseqI'*: ($\bigwedge n.$ *norm* (*X n*) $\leq K$) $\Longrightarrow$ *Bseq X*
by (*intro BfunI*) (*auto simp*: *eventually-sequentially*)

lemma *Bseq-def*: *Bseq X* $\longleftrightarrow$ ($\exists K>0.\ \forall n.$ *norm* (*X n*) $\leq K$)

unfolding *Bfun-def eventually-sequentially*
proof *safe*
fix *N K*
assume $0 < K$ $\forall n \geq N$. *norm* (*X n*) $\leq K$
then show $\exists K>0$. $\forall n$. *norm* (*X n*) $\leq K$
by (*intro exI*[*of - max* (*Max* (*norm* ‘ *X* ‘ {..*N*})) *K*] *max.strict-coboundedI2*)
(*auto intro*!: *imageI not-less*[**where** *'a=nat*, *THEN iffD1*] *Max-ge simp*: *le-max-iff-disj*)
qed *auto*

lemma *BseqE*: *Bseq X* $\Longrightarrow$ ($\bigwedge K$. $0 < K \Longrightarrow \forall n$. *norm* (*X n*) $\leq K \Longrightarrow Q$) $\Longrightarrow$ *Q*
unfolding *Bseq-def* **by** *auto*

lemma *BseqD*: *Bseq X* $\Longrightarrow \exists K$. $0 < K \wedge$ ($\forall n$. *norm* (*X n*) $\leq K$)
by (*simp add*: *Bseq-def*)

lemma *BseqI*: $0 < K \Longrightarrow \forall n$. *norm* (*X n*) $\leq K \Longrightarrow$ *Bseq X*
by (*auto simp*: *Bseq-def*)

lemma *Bseq-bdd-above*: *Bseq X* $\Longrightarrow$ *bdd-above* (*range X*)
for *X* :: *nat* $\Rightarrow$ *real*
proof (*elim BseqE*, *intro bdd-aboveI2*)
fix *K n*
assume $0 < K$ $\forall n$. *norm* (*X n*) $\leq K$
then show *X n* $\leq K$
by (*auto elim*!: *allE*[*of - n*])
qed

lemma *Bseq-bdd-above'*: *Bseq X* $\Longrightarrow$ *bdd-above* (*range* (λn. *norm* (*X n*)))
for *X* :: *nat* $\Rightarrow$ *'a* :: *real-normed-vector*
proof (*elim BseqE*, *intro bdd-aboveI2*)
fix *K n*
assume $0 < K$ $\forall n$. *norm* (*X n*) $\leq K$
then show *norm* (*X n*) $\leq K$
by (*auto elim*!: *allE*[*of - n*])
qed

lemma *Bseq-bdd-below*: *Bseq X* $\Longrightarrow$ *bdd-below* (*range X*)
for *X* :: *nat* $\Rightarrow$ *real*
proof (*elim BseqE*, *intro bdd-belowI2*)
fix *K n*
assume $0 < K$ $\forall n$. *norm* (*X n*) $\leq K$
then show $- K \leq X\ n$
by (*auto elim*!: *allE*[*of - n*])
qed

lemma *Bseq-eventually-mono*:
assumes *eventually* (λn. *norm* (*f n*) $\leq$ *norm* (*g n*)) *sequentially Bseq g*

shows *Bseq f*
proof −
from *assms*(*2*) **obtain** *K* **where** $0 < K$ **and** *eventually* ($\lambda n.$ *norm* (*g n*) $\leq K$) *sequentially*
unfolding *Bfun-def* **by** *fast*
with *assms*(*1*) **have** *eventually* ($\lambda n.$ *norm* (*f n*) $\leq K$) *sequentially*
by (*fast elim*: *eventually-elim2 order-trans*)
with ‹$0 < K$› **show** *Bseq f*
unfolding *Bfun-def* **by** *fast*
qed

lemma *lemma-NBseq-def*: ($\exists K > 0.\ \forall n.$ *norm* (*X n*) $\leq K$) $\longleftrightarrow$ ($\exists N.\ \forall n.$ *norm* (*X n*) $\leq$ *real*(*Suc N*))
proof *safe*
fix *K* :: *real*
from *reals-Archimedean2* **obtain** *n* :: *nat* **where** $K <$ *real n* **..**
then have $K \leq$ *real* (*Suc n*) **by** *auto*
moreover assume $\forall m.$ *norm* (*X m*) $\leq K$
ultimately have $\forall m.$ *norm* (*X m*) $\leq$ *real* (*Suc n*)
by (*blast intro*: *order-trans*)
then show $\exists N.\ \forall n.$ *norm* (*X n*) $\leq$ *real* (*Suc N*) **..**
next
show $\bigwedge N.\ \forall n.$ *norm* (*X n*) $\leq$ *real* (*Suc N*) $\Longrightarrow \exists K>0.\ \forall n.$ *norm* (*X n*) $\leq K$
using *of-nat-0-less-iff* **by** *blast*
qed

Alternative definition for *Bseq*.

lemma *Bseq-iff*: *Bseq X* $\longleftrightarrow$ ($\exists N.\ \forall n.$ *norm* (*X n*) $\leq$ *real*(*Suc N*))
by (*simp add*: *Bseq-def*) (*simp add*: *lemma-NBseq-def*)

lemma *lemma-NBseq-def2*: ($\exists K > 0.\ \forall n.$ *norm* (*X n*) $\leq K$) = ($\exists N.\ \forall n.$ *norm* (*X n*) $<$ *real*(*Suc N*))
proof −
have ∗: $\bigwedge N.\ \forall n.$ *norm* (*X n*) $\leq 1 +$ *real N* $\Longrightarrow$
$\exists N.\ \forall n.$ *norm* (*X n*) $< 1 +$ *real N*
by (*metis add.commute le-less-trans less-add-one of-nat-Suc*)
then show *?thesis*
unfolding *lemma-NBseq-def*
by (*metis less-le-not-le not-less-iff-gr-or-eq of-nat-Suc*)
qed

Yet another definition for Bseq.

lemma *Bseq-iff1a*: *Bseq X* $\longleftrightarrow$ ($\exists N.\ \forall n.$ *norm* (*X n*) $<$ *real* (*Suc N*))
by (*simp add*: *Bseq-def lemma-NBseq-def2*)

108.1.3 A Few More Equivalence Theorems for Boundedness

Alternative formulation for boundedness.

lemma *Bseq-iff2*: *Bseq X* $\longleftrightarrow$ ($\exists k > 0.\ \exists x.\ \forall n.$ *norm* (*X n* + − *x*) $\leq k$)

by (*metis BseqE BseqI' add.commute add-cancel-right-left add-uminus-conv-diff norm-add-leD*
norm-minus-cancel norm-minus-commute)

Alternative formulation for boundedness.

lemma *Bseq-iff3*: *Bseq X* ⟷ (∃ *k*>*0*. ∃ *N*. ∀ *n*. *norm* (*X n* + − *X N*) ≤ *k*)
(**is** *?P* ⟷ *?Q*)
proof
assume *?P*
then obtain *K* **where** ∗: *0* < *K* **and** ∗∗: ⋀*n*. *norm* (*X n*) ≤ *K*
by (*auto simp*: *Bseq-def*)
from ∗ **have** *0* < *K* + *norm* (*X 0*) **by** (*rule order-less-le-trans*) *simp*
from ∗∗ **have** ∀ *n*. *norm* (*X n* − *X 0*) ≤ *K* + *norm* (*X 0*)
by (*auto intro*: *order-trans norm-triangle-ineq4*)
then have ∀ *n*. *norm* (*X n* + − *X 0*) ≤ *K* + *norm* (*X 0*)
by *simp*
with ‹*0* < *K* + *norm* (*X 0*)› **show** *?Q* **by** *blast*
next
assume *?Q*
then show *?P* **by** (*auto simp*: *Bseq-iff2*)
qed

108.1.4 Upper Bounds and Lubs of Bounded Sequences

lemma *Bseq-minus-iff*: *Bseq* (λ*n*. − (*X n*) :: '*a*::*real-normed-vector*) ⟷ *Bseq X*
by (*simp add*: *Bseq-def*)

lemma *Bseq-add*:
fixes *f* :: *nat* ⇒ '*a*::*real-normed-vector*
assumes *Bseq f*
shows *Bseq* (λ*x*. *f x* + *c*)
proof −
from *assms* **obtain** *K* **where** *K*: ⋀*x*. *norm* (*f x*) ≤ *K*
unfolding *Bseq-def* **by** *blast*
{
fix *x* :: *nat*
have *norm* (*f x* + *c*) ≤ *norm* (*f x*) + *norm c* **by** (*rule norm-triangle-ineq*)
also have *norm* (*f x*) ≤ *K* **by** (*rule K*)
finally have *norm* (*f x* + *c*) ≤ *K* + *norm c* **by** *simp*
}
then show *?thesis* **by** (*rule BseqI'*)
qed

lemma *Bseq-add-iff*: *Bseq* (λ*x*. *f x* + *c*) ⟷ *Bseq f*
for *f* :: *nat* ⇒ '*a*::*real-normed-vector*
using *Bseq-add*[*of f c*] *Bseq-add*[*of* λ*x*. *f x* + *c* −*c*] **by** *auto*

lemma *Bseq-mult*:
fixes *f g* :: *nat* ⇒ '*a*::*real-normed-field*

assumes *Bseq f* **and** *Bseq g*
shows *Bseq* (*λx. f x* ∗ *g x*)
proof −
from *assms* **obtain** *K1 K2* **where** *K*: *norm* (*f x*) ≤ *K1 K1* > *0 norm* (*g x*) ≤ *K2 K2* > *0*
for *x*
unfolding *Bseq-def* **by** *blast*
then have *norm* (*f x* ∗ *g x*) ≤ *K1* ∗ *K2* **for** *x*
by (*auto simp*: *norm-mult intro*!: *mult-mono*)
then show *?thesis* **by** (*rule BseqI′*)
qed

lemma *Bfun-const* [*simp*]: *Bfun* (*λ-. c*) *F*
unfolding *Bfun-metric-def* **by** (*auto intro*!: *exI*[*of - c*] *exI*[*of - 1::real*])

lemma *Bseq-cmult-iff*:
fixes *c* :: *′a::real-normed-field*
assumes *c* ≠ *0*
shows *Bseq* (*λx. c* ∗ *f x*) ⟷ *Bseq f*
proof
assume *Bseq* (*λx. c* ∗ *f x*)
with *Bfun-const* **have** *Bseq* (*λx. inverse c* ∗ (*c* ∗ *f x*))
by (*rule Bseq-mult*)
with ‹*c* ≠ *0*› **show** *Bseq f*
by (*simp add*: *field-split-simps*)
qed (*intro Bseq-mult Bfun-const*)

lemma *Bseq-subseq*: *Bseq f* ⟹ *Bseq* (*λx. f* (*g x*))
for *f* :: *nat* ⇒ *′a::real-normed-vector*
unfolding *Bseq-def* **by** *auto*

lemma *Bseq-Suc-iff*: *Bseq* (*λn. f* (*Suc n*)) ⟷ *Bseq f*
for *f* :: *nat* ⇒ *′a::real-normed-vector*
using *Bseq-offset*[*of f 1*] **by** (*auto intro*: *Bseq-subseq*)

lemma *increasing-Bseq-subseq-iff*:
assumes ⋀*x y. x* ≤ *y* ⟹ *norm* (*f x* :: *′a::real-normed-vector*) ≤ *norm* (*f y*) *strict-mono g*
shows *Bseq* (*λx. f* (*g x*)) ⟷ *Bseq f*
proof
assume *Bseq* (*λx. f* (*g x*))
then obtain *K* **where** *K*: ⋀*x. norm* (*f* (*g x*)) ≤ *K*
unfolding *Bseq-def* **by** *auto*
{
fix *x* :: *nat*
from *filterlim-subseq*[*OF assms*(*2*)] **obtain** *y* **where** *g y* ≥ *x*
by (*auto simp*: *filterlim-at-top eventually-at-top-linorder*)
then have *norm* (*f x*) ≤ *norm* (*f* (*g y*))
using *assms*(*1*) **by** *blast*

also have *norm* (*f* (*g y*)) ≤ *K* **by** (*rule K*)
finally have *norm* (*f x*) ≤ *K* **.**
}
then show *Bseq f* **by** (*rule BseqI'*)
qed (*use Bseq-subseq*[*of f g*] **in** *simp-all*)

lemma *nonneg-incseq-Bseq-subseq-iff*:
fixes *f* :: *nat* ⇒ *real*
and *g* :: *nat* ⇒ *nat*
assumes ⋀*x*. *f x* ≥ *0 incseq f strict-mono g*
shows *Bseq* (λ*x*. *f* (*g x*)) ⟷ *Bseq f*
using *assms* **by** (*intro increasing-Bseq-subseq-iff*) (*auto simp*: *incseq-def*)

lemma *Bseq-eq-bounded*: *range f* ⊆ {*a*..*b*} ⟹ *Bseq f*
for *a b* :: *real*
proof (*rule BseqI'*[**where** *K*=*max* (*norm a*) (*norm b*)])
fix *n* **assume** *range f* ⊆ {*a*..*b*}
then have *f n* ∈ {*a*..*b*}
by *blast*
then show *norm* (*f n*) ≤ *max* (*norm a*) (*norm b*)
by *auto*
qed

lemma *incseq-bounded*: *incseq X* ⟹ ∀ *i*. *X i* ≤ *B* ⟹ *Bseq X*
for *B* :: *real*
by (*intro Bseq-eq-bounded*[*of X X 0 B*]) (*auto simp*: *incseq-def*)

lemma *decseq-bounded*: *decseq X* ⟹ ∀ *i*. *B* ≤ *X i* ⟹ *Bseq X*
for *B* :: *real*
by (*intro Bseq-eq-bounded*[*of X B X 0*]) (*auto simp*: *decseq-def*)

108.1.5 Polynomal function extremal theorem, from HOL Light

lemma *polyfun-extremal-lemma*:
fixes *c* :: *nat* ⇒ *'a*::*real-normed-div-algebra*
assumes *0* < *e*
shows ∃ *M*. ∀ *z*. *M* ≤ *norm*(*z*) ⟶ *norm* (∑ *i*≤*n*. *c*(*i*) ∗ *z*^*i*) ≤ *e* ∗ *norm*(*z*) ^ (*Suc n*)
proof (*induct n*)
case *0* **with** *assms*
show *?case*
apply (*rule-tac x*=*norm* (*c 0*) / *e* **in** *exI*)
apply (*auto simp*: *field-simps*)
done
next
case (*Suc n*)
obtain *M* **where** *M*: ⋀*z*. *M* ≤ *norm z* ⟹ *norm* (∑ *i*≤*n*. *c i* ∗ *z*^*i*) ≤ *e* ∗ *norm z* ^ *Suc n*
using *Suc assms* **by** *blast*

show *?case*
proof (*rule exI* [**where** *x= max M* (*1* + *norm*(*c*(*Suc n*)) / *e*)], *clarsimp simp del*: *power-Suc*)
fix *z*::*'a*
assume *z1*: $M \leq$ *norm z* **and** *1* + *norm* (*c* (*Suc n*)) / $e \leq$ *norm z*
then have *z2*: *e* + *norm* (*c* (*Suc n*)) $\leq e *$ *norm z*
using *assms* **by** (*simp add*: *field-simps*)
have *norm* ($\sum i \leq n.\ c\ i * z\widehat{\ }i$) $\leq e *$ *norm z* ^ *Suc n*
using *M* [*OF z1*] **by** *simp*
then have *norm* ($\sum i \leq n.\ c\ i * z\widehat{\ }i$) + *norm* (*c* (*Suc n*) * *z* ^ *Suc n*) $\leq e *$ *norm z* ^ *Suc n* + *norm* (*c* (*Suc n*) * *z* ^ *Suc n*)
by *simp*
then have *norm* (($\sum i \leq n.\ c\ i * z\widehat{\ }i$) + *c* (*Suc n*) * *z* ^ *Suc n*) $\leq e *$ *norm z* ^ *Suc n* + *norm* (*c* (*Suc n*) * *z* ^ *Suc n*)
by (*blast intro*: *norm-triangle-le elim*:)
also have ... $\leq$ (*e* + *norm* (*c* (*Suc n*))) * *norm z* ^ *Suc n*
by (*simp add*: *norm-power norm-mult algebra-simps*)
also have ... $\leq$ (*e* * *norm z*) * *norm z* ^ *Suc n*
by (*metis z2 mult.commute mult-left-mono norm-ge-zero norm-power*)
finally show *norm* (($\sum i \leq n.\ c\ i * z\widehat{\ }i$) + *c* (*Suc n*) * *z* ^ *Suc n*) $\leq e *$ *norm z* ^ *Suc* (*Suc n*)
by *simp*
qed
qed

lemma *polyfun-extremal*:
fixes *c* :: *nat* ⇒ *'a*::*real-normed-div-algebra*
assumes *k*: *c k* ≠ *0* *1*≤*k* **and** *kn*: *k*≤*n*
shows *eventually* (λ*z*. *norm* ($\sum i \leq n.\ c(i) * z\widehat{\ }i$) ≥ *B*) *at-infinity*
using *kn*
proof (*induction n*)
case *0*
then show *?case*
using *k* **by** *simp*
next
case (*Suc m*)
show *?case*
proof (*cases c* (*Suc m*) = *0*)
case *True*
then show *?thesis* **using** *Suc k*
by *auto* (*metis antisym-conv less-eq-Suc-le not-le*)
next
case *False*
then obtain *M* **where** *M*:
$\bigwedge z.\ M \leq$ *norm z* ⟹ *norm* ($\sum i \leq m.\ c\ i * z\widehat{\ }i$) ≤ *norm* (*c* (*Suc m*)) / *2* * *norm z* ^ *Suc m*
using *polyfun-extremal-lemma* [*of norm*(*c* (*Suc m*)) / *2 c m*] *Suc*
by *auto*
have ∃ *b*. ∀ *z*. *b* ≤ *norm z* ⟶ *B* ≤ *norm* ($\sum i \leq Suc\ m.\ c\ i * z\widehat{\ }i$)

proof (*rule exI* [**where** *x=max M* (*max 1* (|*B*| / (*norm*(*c* (*Suc m*)) / *2*)))], *clarsimp simp del*: *power-Suc*)
fix *z*::′*a*
assume *z1*: *M* ≤ *norm z 1* ≤ *norm z*
and |*B*| ∗ *2* / *norm* (*c* (*Suc m*)) ≤ *norm z*
then have *z2*: |*B*| ≤ *norm* (*c* (*Suc m*)) ∗ *norm z* / *2*
using *False* **by** (*simp add*: *field-simps*)
have *nz*: *norm z* ≤ *norm z* ^ *Suc m*
by (*metis* ‹*1* ≤ *norm z*› *One-nat-def less-eq-Suc-le power-increasing power-one-right zero-less-Suc*)
have ∗: ⋀*y x*. *norm* (*c* (*Suc m*)) ∗ *norm z* / *2* ≤ *norm y* − *norm x* ⟹ *B* ≤ *norm* (*x* + *y*)
by (*metis abs-le-iff add.commute norm-diff-ineq order-trans z2*)
have *norm z* ∗ *norm* (*c* (*Suc m*)) + *2* ∗ *norm* (∑ *i*≤*m*. *c i* ∗ *z*^*i*)
≤ *norm* (*c* (*Suc m*)) ∗ *norm z* + *norm* (*c* (*Suc m*)) ∗ *norm z* ^ *Suc m*
using *M* [*of z*] *Suc z1* **by** *auto*
also have ... ≤ *2* ∗ (*norm* (*c* (*Suc m*)) ∗ *norm z* ^ *Suc m*)
using *nz* **by** (*simp add*: *mult-mono del*: *power-Suc*)
finally show *B* ≤ *norm* ((∑ *i*≤*m*. *c i* ∗ *z*^*i*) + *c* (*Suc m*) ∗ *z* ^ *Suc m*)
using *Suc.IH*
apply (*auto simp*: *eventually-at-infinity*)
apply (*rule* ∗)
apply (*simp add*: *field-simps norm-mult norm-power*)
done
qed
then show *?thesis*
by (*simp add*: *eventually-at-infinity*)
qed
qed

108.2 Convergence to Zero

definition *Zfun* :: (′*a* ⇒ ′*b*::*real-normed-vector*) ⇒ ′*a filter* ⇒ *bool*
where *Zfun f F* = (∀ *r*>*0*. *eventually* (λ*x*. *norm* (*f x*) < *r*) *F*)

lemma *ZfunI*: (⋀*r*. *0* < *r* ⟹ *eventually* (λ*x*. *norm* (*f x*) < *r*) *F*) ⟹ *Zfun f F*
by (*simp add*: *Zfun-def*)

lemma *ZfunD*: *Zfun f F* ⟹ *0* < *r* ⟹ *eventually* (λ*x*. *norm* (*f x*) < *r*) *F*
by (*simp add*: *Zfun-def*)

lemma *Zfun-ssubst*: *eventually* (λ*x*. *f x* = *g x*) *F* ⟹ *Zfun g F* ⟹ *Zfun f F*
unfolding *Zfun-def* **by** (*auto elim*!: *eventually-rev-mp*)

lemma *Zfun-zero*: *Zfun* (λ*x*. *0*) *F*
unfolding *Zfun-def* **by** *simp*

lemma *Zfun-norm-iff*: *Zfun* (λ*x*. *norm* (*f x*)) *F* = *Zfun* (λ*x*. *f x*) *F*
unfolding *Zfun-def* **by** *simp*

```
lemma Zfun-imp-Zfun:
  assumes f: Zfun f F
    and g: eventually (λx. norm (g x) ≤ norm (f x) ∗ K) F
  shows Zfun (λx. g x) F
proof (cases 0 < K)
  case K: True
  show ?thesis
  proof (rule ZfunI)
    fix r :: real
    assume 0 < r
    then have 0 < r / K using K by simp
    then have eventually (λx. norm (f x) < r / K) F
      using ZfunD [OF f] by blast
    with g show eventually (λx. norm (g x) < r) F
    proof eventually-elim
      case (elim x)
      then have norm (f x) ∗ K < r
        by (simp add: pos-less-divide-eq K)
      then show ?case
        by (simp add: order-le-less-trans [OF elim(1)])
    qed
  qed
next
  case False
  then have K: K ≤ 0 by (simp only: not-less)
  show ?thesis
  proof (rule ZfunI)
    fix r :: real
    assume 0 < r
    from g show eventually (λx. norm (g x) < r) F
    proof eventually-elim
      case (elim x)
      also have norm (f x) ∗ K ≤ norm (f x) ∗ 0
        using K norm-ge-zero by (rule mult-left-mono)
      finally show ?case
        using ‹0 < r› by simp
    qed
  qed
qed

lemma Zfun-le: Zfun g F =⇒ ∀ x. norm (f x) ≤ norm (g x) =⇒ Zfun f F
  by (erule Zfun-imp-Zfun [where K = 1]) simp

lemma Zfun-add:
  assumes f: Zfun f F
    and g: Zfun g F
  shows Zfun (λx. f x + g x) F
proof (rule ZfunI)
```

fix *r* :: *real*
assume *0* < *r*
then have *r*: *0* < *r* / *2* **by** *simp*
have *eventually* (λ*x*. *norm* (*f x*) < *r*/*2*) *F*
using *f r* **by** (*rule ZfunD*)
moreover
have *eventually* (λ*x*. *norm* (*g x*) < *r*/*2*) *F*
using *g r* **by** (*rule ZfunD*)
ultimately
show *eventually* (λ*x*. *norm* (*f x* + *g x*) < *r*) *F*
proof *eventually-elim*
case (*elim x*)
have *norm* (*f x* + *g x*) ≤ *norm* (*f x*) + *norm* (*g x*)
by (*rule norm-triangle-ineq*)
also have ... < *r*/*2* + *r*/*2*
using *elim* **by** (*rule add-strict-mono*)
finally show *?case*
by *simp*
qed
qed

lemma *Zfun-minus*: *Zfun f F* ⟹ *Zfun* (λ*x*. − *f x*) *F*
unfolding *Zfun-def* **by** *simp*

lemma *Zfun-diff*: *Zfun f F* ⟹ *Zfun g F* ⟹ *Zfun* (λ*x*. *f x* − *g x*) *F*
using *Zfun-add* [*of f F* λ*x*. − *g x*] **by** (*simp add*: *Zfun-minus*)

lemma (**in** *bounded-linear*) *Zfun*:
assumes *g*: *Zfun g F*
shows *Zfun* (λ*x*. *f* (*g x*)) *F*
proof −
obtain *K* **where** *norm* (*f x*) ≤ *norm x* ∗ *K* **for** *x*
using *bounded* **by** *blast*
then have *eventually* (λ*x*. *norm* (*f* (*g x*)) ≤ *norm* (*g x*) ∗ *K*) *F*
by *simp*
with *g* **show** *?thesis*
by (*rule Zfun-imp-Zfun*)
qed

lemma (**in** *bounded-bilinear*) *Zfun*:
assumes *f*: *Zfun f F*
and *g*: *Zfun g F*
shows *Zfun* (λ*x*. *f x* ∗∗ *g x*) *F*
proof (*rule ZfunI*)
fix *r* :: *real*
assume *r*: *0* < *r*
obtain *K* **where** *K*: *0* < *K*
and *norm-le*: *norm* (*x* ∗∗ *y*) ≤ *norm x* ∗ *norm y* ∗ *K* **for** *x y*
using *pos-bounded* **by** *blast*

from K **have** K': $0 < inverse\ K$
by (*rule positive-imp-inverse-positive*)
have *eventually* ($\lambda x.\ norm\ (f\ x) < r$) F
using $f\ r$ **by** (*rule ZfunD*)
moreover
have *eventually* ($\lambda x.\ norm\ (g\ x) < inverse\ K$) F
using $g\ K'$ **by** (*rule ZfunD*)
ultimately
show *eventually* ($\lambda x.\ norm\ (f\ x\ {**}\ g\ x) < r$) F
proof *eventually-elim*
case (*elim x*)
have $norm\ (f\ x\ {**}\ g\ x) \leq norm\ (f\ x) * norm\ (g\ x) * K$
by (*rule norm-le*)
also have $norm\ (f\ x) * norm\ (g\ x) * K < r * inverse\ K * K$
by (*intro mult-strict-right-mono mult-strict-mono' norm-ge-zero elim K*)
also from K **have** $r * inverse\ K * K = r$
by *simp*
finally show *?case* **.**
qed
qed

lemma (**in** *bounded-bilinear*) *Zfun-left*: *Zfun f F* $\Longrightarrow$ *Zfun* ($\lambda x.\ f\ x\ {**}\ a$) F
by (*rule bounded-linear-left* [*THEN bounded-linear.Zfun*])

lemma (**in** *bounded-bilinear*) *Zfun-right*: *Zfun f F* $\Longrightarrow$ *Zfun* ($\lambda x.\ a\ {**}\ f\ x$) F
by (*rule bounded-linear-right* [*THEN bounded-linear.Zfun*])

lemmas *Zfun-mult* = *bounded-bilinear.Zfun* [*OF bounded-bilinear-mult*]
lemmas *Zfun-mult-right* = *bounded-bilinear.Zfun-right* [*OF bounded-bilinear-mult*]
lemmas *Zfun-mult-left* = *bounded-bilinear.Zfun-left* [*OF bounded-bilinear-mult*]

lemma *tendsto-Zfun-iff*: ($f \longrightarrow a$) F = *Zfun* ($\lambda x.\ f\ x - a$) F
by (*simp only*: *tendsto-iff Zfun-def dist-norm*)

lemma *tendsto-0-le*:
($f \longrightarrow 0$) $F \Longrightarrow$ *eventually* ($\lambda x.\ norm\ (g\ x) \leq norm\ (f\ x) * K$) $F \Longrightarrow$ ($g \longrightarrow$ 0) F
by (*simp add*: *Zfun-imp-Zfun tendsto-Zfun-iff*)

108.2.1 Distance and norms

lemma *tendsto-dist* [*tendsto-intros*]:
fixes *l m* :: *'a::metric-space*
assumes *f*: ($f \longrightarrow l$) F
and *g*: ($g \longrightarrow m$) F
shows (($\lambda x.\ dist\ (f\ x)\ (g\ x)$) $\longrightarrow$ *dist l m*) F
proof (*rule tendstoI*)
fix *e* :: *real*
assume $0 < e$

then have *e2*: *0* < *e*/*2* **by** *simp*
from *tendstoD* [*OF f e2*] *tendstoD* [*OF g e2*]
show *eventually* (λ*x*. *dist* (*dist* (*f x*) (*g x*)) (*dist l m*) < *e*) *F*
proof (*eventually-elim*)
case (*elim x*)
then show *dist* (*dist* (*f x*) (*g x*)) (*dist l m*) < *e*
unfolding *dist-real-def*
using *dist-triangle2* [*of f x g x l*]
and *dist-triangle2* [*of g x l m*]
and *dist-triangle3* [*of l m f x*]
and *dist-triangle* [*of f x m g x*]
by *arith*
qed
qed

lemma *continuous-dist*[*continuous-intros*]:
fixes *f g* :: - ⇒ ′*a* :: *metric-space*
shows *continuous F f* ⟹ *continuous F g* ⟹ *continuous F* (λ*x*. *dist* (*f x*) (*g x*))
unfolding *continuous-def* **by** (*rule tendsto-dist*)

lemma *continuous-on-dist*[*continuous-intros*]:
fixes *f g* :: - ⇒ ′*a* :: *metric-space*
shows *continuous-on s f* ⟹ *continuous-on s g* ⟹ *continuous-on s* (λ*x*. *dist* (*f x*) (*g x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-dist*)

lemma *continuous-at-dist*: *isCont* (*dist a*) *b*
using *continuous-on-dist* [*OF continuous-on-const continuous-on-id*] *continuous-on-eq-continuous-within* **by** *blast*

lemma *tendsto-norm* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *norm* (*f x*)) ⟶ *norm a*) *F*
unfolding *norm-conv-dist* **by** (*intro tendsto-intros*)

lemma *continuous-norm* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *norm* (*f x*))
unfolding *continuous-def* **by** (*rule tendsto-norm*)

lemma *continuous-on-norm* [*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s* (λ*x*. *norm* (*f x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-norm*)

lemma *continuous-on-norm-id* [*continuous-intros*]: *continuous-on S norm*
by (*intro continuous-on-id continuous-on-norm*)

lemma *tendsto-norm-zero*: (*f* ⟶ *0*) *F* ⟹ ((λ*x*. *norm* (*f x*)) ⟶ *0*) *F*
by (*drule tendsto-norm*) *simp*

lemma *tendsto-norm-zero-cancel*: $((\lambda x.\ norm\ (f\ x)) \longrightarrow 0)\ F \Longrightarrow (f \longrightarrow 0)\ F$
unfolding *tendsto-iff dist-norm* **by** *simp*

lemma *tendsto-norm-zero-iff*: $((\lambda x.\ norm\ (f\ x)) \longrightarrow 0)\ F \longleftrightarrow (f \longrightarrow 0)\ F$
unfolding *tendsto-iff dist-norm* **by** *simp*

lemma *tendsto-rabs* [*tendsto-intros*]: $(f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ |f\ x|) \longrightarrow |l|)\ F$
for l :: *real*
by (*fold real-norm-def*) (*rule tendsto-norm*)

lemma *continuous-rabs* [*continuous-intros*]:
continuous F f $\Longrightarrow$ *continuous F* $(\lambda x.\ |f\ x :: real|)$
unfolding *real-norm-def*[*symmetric*] **by** (*rule continuous-norm*)

lemma *continuous-on-rabs* [*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on s* $(\lambda x.\ |f\ x :: real|)$
unfolding *real-norm-def*[*symmetric*] **by** (*rule continuous-on-norm*)

lemma *tendsto-rabs-zero*: $(f \longrightarrow (0::real))\ F \Longrightarrow ((\lambda x.\ |f\ x|) \longrightarrow 0)\ F$
by (*fold real-norm-def*) (*rule tendsto-norm-zero*)

lemma *tendsto-rabs-zero-cancel*: $((\lambda x.\ |f\ x|) \longrightarrow (0::real))\ F \Longrightarrow (f \longrightarrow 0)\ F$
by (*fold real-norm-def*) (*rule tendsto-norm-zero-cancel*)

lemma *tendsto-rabs-zero-iff*: $((\lambda x.\ |f\ x|) \longrightarrow (0::real))\ F \longleftrightarrow (f \longrightarrow 0)\ F$
by (*fold real-norm-def*) (*rule tendsto-norm-zero-iff*)

108.3 Topological Monoid

class *topological-monoid-add* = *topological-space* + *monoid-add* +
assumes *tendsto-add-Pair*: *LIM x* (*nhds a* $\times_F$ *nhds b*). *fst x* + *snd x* :> *nhds* (*a* + *b*)

class *topological-comm-monoid-add* = *topological-monoid-add* + *comm-monoid-add*

lemma *tendsto-add* [*tendsto-intros*]:
fixes *a b* :: *'a::topological-monoid-add*
shows $(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow ((\lambda x.\ f\ x + g\ x) \longrightarrow a + b)\ F$
using *filterlim-compose*[*OF tendsto-add-Pair*, *of* $\lambda x.$ (*f x*, *g x*) *a b F*]
by (*simp add*: *nhds-prod*[*symmetric*] *tendsto-Pair*)

lemma *continuous-add* [*continuous-intros*]:
fixes *f g* :: - ⇒ *'b::topological-monoid-add*
shows *continuous F f* $\Longrightarrow$ *continuous F g* $\Longrightarrow$ *continuous F* $(\lambda x.\ f\ x + g\ x)$
unfolding *continuous-def* **by** (*rule tendsto-add*)

lemma *continuous-on-add* [*continuous-intros*]:
fixes *f g* :: - ⇒ *'b::topological-monoid-add*

shows *continuous-on s f* $\Longrightarrow$ *continuous-on s g* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *f x* + *g x*)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-add*)

lemma *tendsto-add-zero*:
fixes *f g* :: - $\Rightarrow$ *'b::topological-monoid-add*
shows (*f* $\longrightarrow$ *0*) *F* $\Longrightarrow$ (*g* $\longrightarrow$ *0*) *F* $\Longrightarrow$ (($\lambda x.$ *f x* + *g x*) $\longrightarrow$ *0*) *F*
by (*drule* (*1*) *tendsto-add*) *simp*

lemma *tendsto-sum* [*tendsto-intros*]:
fixes *f* :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c::topological-comm-monoid-add*
shows ($\bigwedge i.$ $i \in I$ $\Longrightarrow$ (*f i* $\longrightarrow$ *a i*) *F*) $\Longrightarrow$ (($\lambda x.$ $\sum i{\in}I.$ *f i x*) $\longrightarrow$ ($\sum i{\in}I.$ *a i*)) *F*
by (*induct I rule*: *infinite-finite-induct*) (*simp-all add*: *tendsto-add*)

lemma *tendsto-null-sum*:
fixes *f* :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c::topological-comm-monoid-add*
assumes $\bigwedge i.$ $i \in I$ $\Longrightarrow$ (($\lambda x.$ *f x i*) $\longrightarrow$ *0*) *F*
shows (($\lambda i.$ *sum* (*f i*) *I*) $\longrightarrow$ *0*) *F*
using *tendsto-sum* [*of I* $\lambda x\ y.$ *f y x* $\lambda x.$ *0*] *assms* **by** *simp*

lemma *continuous-sum* [*continuous-intros*]:
fixes *f* :: *'a* $\Rightarrow$ *'b::t2-space* $\Rightarrow$ *'c::topological-comm-monoid-add*
shows ($\bigwedge i.$ $i \in I$ $\Longrightarrow$ *continuous F* (*f i*)) $\Longrightarrow$ *continuous F* ($\lambda x.$ $\sum i{\in}I.$ *f i x*)
unfolding *continuous-def* **by** (*rule tendsto-sum*)

lemma *continuous-on-sum* [*continuous-intros*]:
fixes *f* :: *'a* $\Rightarrow$ *'b::topological-space* $\Rightarrow$ *'c::topological-comm-monoid-add*
shows ($\bigwedge i.$ $i \in I$ $\Longrightarrow$ *continuous-on S* (*f i*)) $\Longrightarrow$ *continuous-on S* ($\lambda x.$ $\sum i{\in}I.$ *f i x*)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-sum*)

instance *nat* :: *topological-comm-monoid-add*
by *standard*
(*simp add*: *nhds-discrete principal-prod-principal filterlim-principal eventually-principal*)

instance *int* :: *topological-comm-monoid-add*
by *standard*
(*simp add*: *nhds-discrete principal-prod-principal filterlim-principal eventually-principal*)

108.3.1 Topological group

class *topological-group-add* = *topological-monoid-add* + *group-add* +
assumes *tendsto-uminus-nhds*: (*uminus* $\longrightarrow$ − *a*) (*nhds a*)
begin

lemma *tendsto-minus* [*tendsto-intros*]: (*f* $\longrightarrow$ *a*) *F* $\Longrightarrow$ (($\lambda x.$ − *f x*) $\longrightarrow$ − *a*) *F*
by (*rule filterlim-compose*[*OF tendsto-uminus-nhds*])

end

class *topological-ab-group-add* = *topological-group-add* + *ab-group-add*

instance *topological-ab-group-add* < *topological-comm-monoid-add* **..**

lemma *continuous-minus* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.$ $-$ *f x*)
  for *f* :: *'a*::*t2-space* $\Rightarrow$ *'b*::*topological-group-add*
  unfolding *continuous-def* **by** (*rule tendsto-minus*)

lemma *continuous-on-minus* [*continuous-intros*]: *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ $-$ *f x*)
  for *f* :: *-* $\Rightarrow$ *'b*::*topological-group-add*
  unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-minus*)

lemma *tendsto-minus-cancel*: (($\lambda x.$ $-$ *f x*) $\longrightarrow$ $-$ *a*) *F* $\Longrightarrow$ (*f* $\longrightarrow$ *a*) *F*
  for *a* :: *'a*::*topological-group-add*
  by (*drule tendsto-minus*) *simp*

lemma *tendsto-minus-cancel-left*:
  (*f* $\longrightarrow$ $-$ (*y*::*-*::*topological-group-add*)) *F* $\longleftrightarrow$ (($\lambda x.$ $-$ *f x*) $\longrightarrow$ *y*) *F*
  using *tendsto-minus-cancel*[*of f* $-$ *y F*] *tendsto-minus*[*of f* $-$ *y F*]
  by *auto*

lemma *tendsto-diff* [*tendsto-intros*]:
  fixes *a b* :: *'a*::*topological-group-add*
  shows (*f* $\longrightarrow$ *a*) *F* $\Longrightarrow$ (*g* $\longrightarrow$ *b*) *F* $\Longrightarrow$ (($\lambda x.$ *f x* $-$ *g x*) $\longrightarrow$ *a* $-$ *b*) *F*
  using *tendsto-add* [*of f a F* $\lambda x.$ $-$ *g x* $-$ *b*] **by** (*simp add*: *tendsto-minus*)

lemma *continuous-diff* [*continuous-intros*]:
  fixes *f g* :: *'a*::*t2-space* $\Rightarrow$ *'b*::*topological-group-add*
  shows *continuous F f* $\Longrightarrow$ *continuous F g* $\Longrightarrow$ *continuous F* ($\lambda x.$ *f x* $-$ *g x*)
  unfolding *continuous-def* **by** (*rule tendsto-diff*)

lemma *continuous-on-diff* [*continuous-intros*]:
  fixes *f g* :: *-* $\Rightarrow$ *'b*::*topological-group-add*
  shows *continuous-on s f* $\Longrightarrow$ *continuous-on s g* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *f x* $-$ *g x*)
  unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-diff*)

lemma *continuous-on-op-minus*: *continuous-on* (*s*::*'a*::*topological-group-add set*) (($-$) *x*)
  by (*rule continuous-intros* | *simp*)+

instance *real-normed-vector* < *topological-ab-group-add*
proof
  fix *a b* :: *'a*

show $((\lambda x.\ fst\ x + snd\ x) \longrightarrow a + b)\ (nhds\ a \times_F\ nhds\ b)$
unfolding *tendsto-Zfun-iff add-diff-add*
using *tendsto-fst*[*OF filterlim-ident*, *of* (*a*,*b*)] *tendsto-snd*[*OF filterlim-ident*, *of* (*a*,*b*)]
by (*intro Zfun-add*)
(*auto simp*: *tendsto-Zfun-iff*[*symmetric*] *nhds-prod*[*symmetric*] *intro*!: *tendsto-fst*)
show $(uminus \longrightarrow -\ a)\ (nhds\ a)$
unfolding *tendsto-Zfun-iff minus-diff-minus*
using *filterlim-ident*[*of nhds a*]
by (*intro Zfun-minus*) (*simp add*: *tendsto-Zfun-iff*)
qed

lemmas *real-tendsto-sandwich* = *tendsto-sandwich*[**where** *'a*=*real*]

108.3.2 Linear operators and multiplication

lemma *linear-times* [*simp*]: *linear* $(\lambda x.\ c * x)$
for *c* :: *'a*::*real-algebra*
by (*auto simp*: *linearI distrib-left*)

lemma (**in** *bounded-linear*) *tendsto*: $(g \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ f\ (g\ x)) \longrightarrow f\ a)\ F$
by (*simp only*: *tendsto-Zfun-iff diff* [*symmetric*] *Zfun*)

lemma (**in** *bounded-linear*) *continuous*: *continuous F g* $\Longrightarrow$ *continuous F* $(\lambda x.\ f\ (g\ x))$
using *tendsto*[*of g - F*] **by** (*auto simp*: *continuous-def*)

lemma (**in** *bounded-linear*) *continuous-on*: *continuous-on s g* $\Longrightarrow$ *continuous-on s* $(\lambda x.\ f\ (g\ x))$
using *tendsto*[*of g*] **by** (*auto simp*: *continuous-on-def*)

lemma (**in** *bounded-linear*) *tendsto-zero*: $(g \longrightarrow 0)\ F \Longrightarrow ((\lambda x.\ f\ (g\ x)) \longrightarrow 0)\ F$
by (*drule tendsto*) (*simp only*: *zero*)

lemma (**in** *bounded-bilinear*) *tendsto*:
$(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow ((\lambda x.\ f\ x ** g\ x) \longrightarrow a ** b)\ F$
by (*simp only*: *tendsto-Zfun-iff prod-diff-prod Zfun-add Zfun Zfun-left Zfun-right*)

lemma (**in** *bounded-bilinear*) *continuous*:
continuous F f $\Longrightarrow$ *continuous F g* $\Longrightarrow$ *continuous F* $(\lambda x.\ f\ x ** g\ x)$
using *tendsto*[*of f - F g*] **by** (*auto simp*: *continuous-def*)

lemma (**in** *bounded-bilinear*) *continuous-on*:
continuous-on s f $\Longrightarrow$ *continuous-on s g* $\Longrightarrow$ *continuous-on s* $(\lambda x.\ f\ x ** g\ x)$
using *tendsto*[*of f - - g*] **by** (*auto simp*: *continuous-on-def*)

lemma (**in** *bounded-bilinear*) *tendsto-zero*:

assumes *f*: (*f* ⟶ *0*) *F*
and *g*: (*g* ⟶ *0*) *F*
shows ((λ*x*. *f x* ** *g x*) ⟶ *0*) *F*
using *tendsto* [*OF f g*] **by** (*simp add*: *zero-left*)

lemma (**in** *bounded-bilinear*) *tendsto-left-zero*:
(*f* ⟶ *0*) *F* ⟹ ((λ*x*. *f x* ** *c*) ⟶ *0*) *F*
by (*rule bounded-linear.tendsto-zero* [*OF bounded-linear-left*])

lemma (**in** *bounded-bilinear*) *tendsto-right-zero*:
(*f* ⟶ *0*) *F* ⟹ ((λ*x*. *c* ** *f x*) ⟶ *0*) *F*
by (*rule bounded-linear.tendsto-zero* [*OF bounded-linear-right*])

lemmas *tendsto-of-real* [*tendsto-intros*] =
bounded-linear.tendsto [*OF bounded-linear-of-real*]

lemmas *tendsto-scaleR* [*tendsto-intros*] =
bounded-bilinear.tendsto [*OF bounded-bilinear-scaleR*]

Analogous type class for multiplication

class *topological-semigroup-mult* = *topological-space* + *semigroup-mult* +
assumes *tendsto-mult-Pair*: *LIM x* (*nhds a* $\times_F$ *nhds b*). *fst x* ∗ *snd x* :> *nhds* (*a* ∗ *b*)

instance *real-normed-algebra* < *topological-semigroup-mult*
proof
fix *a b* :: *'a*
show ((λ*x*. *fst x* ∗ *snd x*) ⟶ *a* ∗ *b*) (*nhds a* $\times_F$ *nhds b*)
unfolding *nhds-prod*[*symmetric*]
using *tendsto-fst*[*OF filterlim-ident*, *of* (*a*,*b*)] *tendsto-snd*[*OF filterlim-ident*, *of* (*a*,*b*)]
by (*simp add*: *bounded-bilinear.tendsto* [*OF bounded-bilinear-mult*])
qed

lemma *tendsto-mult* [*tendsto-intros*]:
fixes *a b* :: *'a*::*topological-semigroup-mult*
shows (*f* ⟶ *a*) *F* ⟹ (*g* ⟶ *b*) *F* ⟹ ((λ*x*. *f x* ∗ *g x*) ⟶ *a* ∗ *b*) *F*
using *filterlim-compose*[*OF tendsto-mult-Pair*, *of* λ*x*. (*f x*, *g x*) *a b F*]
by (*simp add*: *nhds-prod*[*symmetric*] *tendsto-Pair*)

lemma *tendsto-mult-left*: (*f* ⟶ *l*) *F* ⟹ ((λ*x*. *c* ∗ (*f x*)) ⟶ *c* ∗ *l*) *F*
for *c* :: *'a*::*topological-semigroup-mult*
by (*rule tendsto-mult* [*OF tendsto-const*])

lemma *tendsto-mult-right*: (*f* ⟶ *l*) *F* ⟹ ((λ*x*. (*f x*) ∗ *c*) ⟶ *l* ∗ *c*) *F*
for *c* :: *'a*::*topological-semigroup-mult*
by (*rule tendsto-mult* [*OF - tendsto-const*])

lemma *tendsto-mult-left-iff* [*simp*]:

$c \neq 0 \Longrightarrow$ *tendsto*$(\lambda x.\ c * f\,x)$ $(c * l)$ $F \longleftrightarrow$ *tendsto* $f\,l\,F$ **for** c :: $'a$::{*topological-semigroup-mult*,*field*}
by (*auto simp*: *tendsto-mult-left dest*: *tendsto-mult-left* [**where** $c = 1/c$])

lemma *tendsto-mult-right-iff* [*simp*]:
$c \neq 0 \Longrightarrow$ *tendsto*$(\lambda x.\ f\,x * c)$ $(l * c)$ $F \longleftrightarrow$ *tendsto* $f\,l\,F$ **for** c :: $'a$::{*topological-semigroup-mult*,*field*}
by (*auto simp*: *tendsto-mult-right dest*: *tendsto-mult-left* [**where** $c = 1/c$])

lemma *tendsto-zero-mult-left-iff* [*simp*]:
fixes c::$'a$::{*topological-semigroup-mult*,*field*} **assumes** $c \neq 0$ **shows** $(\lambda n.\ c * a\ n) \longrightarrow 0 \longleftrightarrow a \longrightarrow 0$
using *assms tendsto-mult-left tendsto-mult-left-iff* **by** *fastforce*

lemma *tendsto-zero-mult-right-iff* [*simp*]:
fixes c::$'a$::{*topological-semigroup-mult*,*field*} **assumes** $c \neq 0$ **shows** $(\lambda n.\ a\ n * c) \longrightarrow 0 \longleftrightarrow a \longrightarrow 0$
using *assms tendsto-mult-right tendsto-mult-right-iff* **by** *fastforce*

lemma *tendsto-zero-divide-iff* [*simp*]:
fixes c::$'a$::{*topological-semigroup-mult*,*field*} **assumes** $c \neq 0$ **shows** $(\lambda n.\ a\ n\ /\ c) \longrightarrow 0 \longleftrightarrow a \longrightarrow 0$
using *tendsto-zero-mult-right-iff* [*of* $1/c$ *a*] *assms* **by** (*simp add*: *field-simps*)

lemma *lim-const-over-n* [*tendsto-intros*]:
fixes a :: $'a$::*real-normed-field*
shows $(\lambda n.\ a\ /\ \textit{of-nat}\ n) \longrightarrow 0$
using *tendsto-mult* [*OF tendsto-const* [*of a*] *lim-1-over-n*] **by** *simp*

lemmas *continuous-of-real* [*continuous-intros*] =
bounded-linear.continuous [*OF bounded-linear-of-real*]

lemmas *continuous-scaleR* [*continuous-intros*] =
bounded-bilinear.continuous [*OF bounded-bilinear-scaleR*]

lemmas *continuous-mult* [*continuous-intros*] =
bounded-bilinear.continuous [*OF bounded-bilinear-mult*]

lemmas *continuous-on-of-real* [*continuous-intros*] =
bounded-linear.continuous-on [*OF bounded-linear-of-real*]

lemmas *continuous-on-scaleR* [*continuous-intros*] =
bounded-bilinear.continuous-on [*OF bounded-bilinear-scaleR*]

lemmas *continuous-on-mult* [*continuous-intros*] =
bounded-bilinear.continuous-on [*OF bounded-bilinear-mult*]

lemmas *tendsto-mult-zero* =
bounded-bilinear.tendsto-zero [*OF bounded-bilinear-mult*]

lemmas *tendsto-mult-left-zero* =

bounded-bilinear.tendsto-left-zero [*OF bounded-bilinear-mult*]

lemmas *tendsto-mult-right-zero* =
bounded-bilinear.tendsto-right-zero [*OF bounded-bilinear-mult*]

lemma *continuous-mult-left*:
fixes c::'a::*real-normed-algebra*
shows *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.\ c * f\ x$)
by (*rule continuous-mult* [*OF continuous-const*])

lemma *continuous-mult-right*:
fixes c::'a::*real-normed-algebra*
shows *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.\ f\ x * c$)
by (*rule continuous-mult* [*OF - continuous-const*])

lemma *continuous-on-mult-left*:
fixes c::'a::*real-normed-algebra*
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.\ c * f\ x$)
by (*rule continuous-on-mult* [*OF continuous-on-const*])

lemma *continuous-on-mult-right*:
fixes c::'a::*real-normed-algebra*
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.\ f\ x * c$)
by (*rule continuous-on-mult* [*OF - continuous-on-const*])

lemma *continuous-on-mult-const* [*simp*]:
fixes c::'a::*real-normed-algebra*
shows *continuous-on s* ((*) *c*)
by (*intro continuous-on-mult-left continuous-on-id*)

lemma *tendsto-divide-zero*:
fixes c :: 'a::*real-normed-field*
shows ($f \longrightarrow 0$) $F \Longrightarrow$ (($\lambda x.\ f\ x\ /\ c$) $\longrightarrow 0$) F
by (*cases c=0*) (*simp-all add*: *divide-inverse tendsto-mult-left-zero*)

lemma *tendsto-power* [*tendsto-intros*]: ($f \longrightarrow a$) $F \Longrightarrow$ (($\lambda x.\ f\ x\ \hat{}\ n$) $\longrightarrow a\ \hat{}\ n$) F
for f :: 'a $\Rightarrow$ 'b::{*power*,*real-normed-algebra*}
by (*induct n*) (*simp-all add*: *tendsto-mult*)

lemma *tendsto-null-power*: ⟦($f \longrightarrow 0$) F; $0 < n$⟧ $\Longrightarrow$ (($\lambda x.\ f\ x\ \hat{}\ n$) $\longrightarrow 0$) F
for f :: 'a $\Rightarrow$ 'b::{*power*,*real-normed-algebra-1*}
using *tendsto-power* [*of f 0 F n*] **by** (*simp add*: *power-0-left*)

lemma *continuous-power* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.\ (f\ x)\hat{}n$)
for f :: 'a::*t2-space* $\Rightarrow$ 'b::{*power*,*real-normed-algebra*}
unfolding *continuous-def* **by** (*rule tendsto-power*)

lemma *continuous-on-power* [*continuous-intros*]:
fixes *f* :: - ⇒ ′*b*::{*power*,*real-normed-algebra*}
shows *continuous-on s f* ⟹ *continuous-on s* (λ*x*. (*f x*)^*n*)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-power*)

lemma *tendsto-prod* [*tendsto-intros*]:
fixes *f* :: ′*a* ⇒ ′*b* ⇒ ′*c*::{*real-normed-algebra*,*comm-ring-1*}
shows (⋀*i*. *i* ∈ *S* ⟹ (*f i* ⟶ *L i*) *F*) ⟹ ((λ*x*. ∏ *i*∈*S*. *f i x*) ⟶ (∏ *i*∈*S*. *L i*)) *F*
by (*induct S rule*: *infinite-finite-induct*) (*simp-all add*: *tendsto-mult*)

lemma *continuous-prod* [*continuous-intros*]:
fixes *f* :: ′*a* ⇒ ′*b*::*t2-space* ⇒ ′*c*::{*real-normed-algebra*,*comm-ring-1*}
shows (⋀*i*. *i* ∈ *S* ⟹ *continuous F* (*f i*)) ⟹ *continuous F* (λ*x*. ∏ *i*∈*S*. *f i x*)
unfolding *continuous-def* **by** (*rule tendsto-prod*)

lemma *continuous-on-prod* [*continuous-intros*]:
fixes *f* :: ′*a* ⇒ - ⇒ ′*c*::{*real-normed-algebra*,*comm-ring-1*}
shows (⋀*i*. *i* ∈ *S* ⟹ *continuous-on s* (*f i*)) ⟹ *continuous-on s* (λ*x*. ∏ *i*∈*S*. *f i x*)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-prod*)

lemma *tendsto-of-real-iff*:
((λ*x*. *of-real* (*f x*) :: ′*a*::*real-normed-div-algebra*) ⟶ *of-real c*) *F* ⟷ (*f* ⟶ *c*) *F*
unfolding *tendsto-iff* **by** *simp*

lemma *tendsto-add-const-iff*:
((λ*x*. *c* + *f x* :: ′*a*::*topological-group-add*) ⟶ *c* + *d*) *F* ⟷ (*f* ⟶ *d*) *F*
using *tendsto-add*[*OF tendsto-const*[*of c*], *of f d*]
and *tendsto-add*[*OF tendsto-const*[*of* −*c*], *of* λ*x*. *c* + *f x c* + *d*] **by** *auto*

class *topological-monoid-mult* = *topological-semigroup-mult* + *monoid-mult*
class *topological-comm-monoid-mult* = *topological-monoid-mult* + *comm-monoid-mult*

lemma *tendsto-power-strong* [*tendsto-intros*]:
fixes *f* :: - ⇒ ′*b* :: *topological-monoid-mult*
assumes (*f* ⟶ *a*) *F* (*g* ⟶ *b*) *F*
shows ((λ*x*. *f x* ^ *g x*) ⟶ *a* ^ *b*) *F*
proof −
have ((λ*x*. *f x* ^ *b*) ⟶ *a* ^ *b*) *F*
by (*induction b*) (*auto intro*: *tendsto-intros assms*)
also from *assms*(*2*) **have** *eventually* (λ*x*. *g x* = *b*) *F*
by (*simp add*: *nhds-discrete filterlim-principal*)
hence *eventually* (λ*x*. *f x* ^ *b* = *f x* ^ *g x*) *F*
by *eventually-elim simp*
hence ((λ*x*. *f x* ^ *b*) ⟶ *a* ^ *b*) *F* ⟷ ((λ*x*. *f x* ^ *g x*) ⟶ *a* ^ *b*) *F*

by (*intro filterlim-cong refl*)
finally show *?thesis* **.**
qed

lemma *continuous-mult'* [*continuous-intros*]:
fixes *f g* :: *-* ⇒ *'b::topological-semigroup-mult*
shows *continuous F f* ⟹ *continuous F g* ⟹ *continuous F* (*λx. f x* ∗ *g x*)
unfolding *continuous-def* **by** (*rule tendsto-mult*)

lemma *continuous-power'* [*continuous-intros*]:
fixes *f* :: *-* ⇒ *'b::topological-monoid-mult*
shows *continuous F f* ⟹ *continuous F g* ⟹ *continuous F* (*λx. f x* ^ *g x*)
unfolding *continuous-def* **by** (*rule tendsto-power-strong*) *auto*

lemma *continuous-on-mult'* [*continuous-intros*]:
fixes *f g* :: *-* ⇒ *'b::topological-semigroup-mult*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (*λx. f x* ∗ *g x*)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-mult*)

lemma *continuous-on-power'* [*continuous-intros*]:
fixes *f* :: *-* ⇒ *'b::topological-monoid-mult*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (*λx. f x* ^ *g x*)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-power-strong*)

lemma *tendsto-mult-one*:
fixes *f g* :: *-* ⇒ *'b::topological-monoid-mult*
shows (*f* ⟶ *1*) *F* ⟹ (*g* ⟶ *1*) *F* ⟹ ((*λx. f x* ∗ *g x*) ⟶ *1*) *F*
by (*drule* (*1*) *tendsto-mult*) *simp*

lemma *tendsto-prod'* [*tendsto-intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c::topological-comm-monoid-mult*
shows (⋀*i. i* ∈ *I* ⟹ (*f i* ⟶ *a i*) *F*) ⟹ ((*λx.* ∏*i*∈*I. f i x*) ⟶ (∏*i*∈*I. a i*)) *F*
by (*induct I rule*: *infinite-finite-induct*) (*simp-all add*: *tendsto-mult*)

lemma *tendsto-one-prod'*:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c::topological-comm-monoid-mult*
assumes ⋀*i. i* ∈ *I* ⟹ ((*λx. f x i*) ⟶ *1*) *F*
shows ((*λi. prod* (*f i*) *I*) ⟶ *1*) *F*
using *tendsto-prod'* [*of I λx y. f y x λx. 1*] *assms* **by** *simp*

lemma *LIMSEQ-prod-0*:
fixes *f* :: *nat* ⇒ *'a::*{*semidom,topological-space*}
assumes *f i* = *0*
shows (*λn. prod f* {*..n*}) ⟶ *0*
proof (*subst tendsto-cong*)
show ∀$_F$ *n in sequentially. prod f* {*..n*} = *0*

using *assms eventually-at-top-linorder* **by** *auto*
qed *auto*

lemma *LIMSEQ-prod-nonneg*:
fixes *f* :: *nat* ⇒ *'a*::{*linordered-semidom*,*linorder-topology*}
assumes *0*: $\bigwedge n.\ 0 \le f\ n$ **and** *a*: ($\lambda n.\ prod\ f\ \{..n\}$) $\longrightarrow$ *a*
shows $a \ge 0$
by (*simp add*: *0 prod-nonneg LIMSEQ-le-const* [*OF a*])

lemma *continuous-prod'* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'b*::*t2-space* ⇒ *'c*::*topological-comm-monoid-mult*
shows ($\bigwedge i.\ i \in I \Longrightarrow$ *continuous F* (*f i*)) $\Longrightarrow$ *continuous F* ($\lambda x.\ \prod i{\in}I.\ f\ i\ x$)
unfolding *continuous-def* **by** (*rule tendsto-prod'*)

lemma *continuous-on-prod'* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'b*::*topological-space* ⇒ *'c*::*topological-comm-monoid-mult*
shows ($\bigwedge i.\ i \in I \Longrightarrow$ *continuous-on S* (*f i*)) $\Longrightarrow$ *continuous-on S* ($\lambda x.\ \prod i{\in}I.\ f$
i x)
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-prod'*)

instance *nat* :: *topological-comm-monoid-mult*
by *standard*
(*simp add*: *nhds-discrete principal-prod-principal filterlim-principal eventually-principal*)

instance *int* :: *topological-comm-monoid-mult*
by *standard*
(*simp add*: *nhds-discrete principal-prod-principal filterlim-principal eventually-principal*)

class *comm-real-normed-algebra-1* = *real-normed-algebra-1* + *comm-monoid-mult*

context *real-normed-field*
begin

subclass *comm-real-normed-algebra-1*
proof
from *norm-mult*[*of 1* :: *'a 1*] **show** *norm 1* = *1* **by** *simp*
qed (*simp-all add*: *norm-mult*)

end

108.3.3 Inverse and division

lemma (**in** *bounded-bilinear*) *Zfun-prod-Bfun*:
assumes *f*: *Zfun f F*
and *g*: *Bfun g F*
shows *Zfun* ($\lambda x.\ f\ x\ {**}\ g\ x$) *F*
proof −
obtain *K* **where** *K*: $0 \le K$
and *norm-le*: $\bigwedge x\ y.\ norm\ (x\ {**}\ y) \le norm\ x * norm\ y * K$

using *nonneg-bounded* **by** *blast*
obtain *B* **where** *B*: $0 < B$
and *norm-g*: *eventually* (λx. *norm* (*g x*) $\leq$ *B*) *F*
using *g* **by** (*rule BfunE*)
have *eventually* (λx. *norm* (*f x* $**$ *g x*) $\leq$ *norm* (*f x*) $*$ (*B* $*$ *K*)) *F*
using *norm-g* **proof** *eventually-elim*
case (*elim x*)
have *norm* (*f x* $**$ *g x*) $\leq$ *norm* (*f x*) $*$ *norm* (*g x*) $*$ *K*
by (*rule norm-le*)
also have ... $\leq$ *norm* (*f x*) $*$ *B* $*$ *K*
by (*intro mult-mono′ order-refl norm-g norm-ge-zero mult-nonneg-nonneg K elim*)
also have ... = *norm* (*f x*) $*$ (*B* $*$ *K*)
by (*rule mult.assoc*)
finally show *norm* (*f x* $**$ *g x*) $\leq$ *norm* (*f x*) $*$ (*B* $*$ *K*) **.**
qed
with *f* **show** *?thesis*
by (*rule Zfun-imp-Zfun*)
qed

lemma (**in** *bounded-bilinear*) *Bfun-prod-Zfun*:
assumes *f*: *Bfun f F*
and *g*: *Zfun g F*
shows *Zfun* (λx. *f x* $**$ *g x*) *F*
using *flip g f* **by** (*rule bounded-bilinear.Zfun-prod-Bfun*)

lemma *Bfun-inverse*:
fixes *a* :: *′a::real-normed-div-algebra*
assumes *f*: (*f* $\longrightarrow$ *a*) *F*
assumes *a*: $a \neq 0$
shows *Bfun* (λx. *inverse* (*f x*)) *F*
proof −
from *a* **have** $0 <$ *norm a* **by** *simp*
then have $\exists r>0$. $r <$ *norm a* **by** (*rule dense*)
then obtain *r* **where** *r1*: $0 < r$ **and** *r2*: $r <$ *norm a*
by *blast*
have *eventually* (λx. *dist* (*f x*) *a* $< r$) *F*
using *tendstoD* [*OF f r1*] **by** *blast*
then have *eventually* (λx. *norm* (*inverse* (*f x*)) $\leq$ *inverse* (*norm a* − *r*)) *F*
proof *eventually-elim*
case (*elim x*)
then have *1*: *norm* (*f x* − *a*) $< r$
by (*simp add*: *dist-norm*)
then have *2*: *f x* $\neq 0$ **using** *r2* **by** *auto*
then have *norm* (*inverse* (*f x*)) = *inverse* (*norm* (*f x*))
by (*rule nonzero-norm-inverse*)
also have ... $\leq$ *inverse* (*norm a* − *r*)
proof (*rule le-imp-inverse-le*)
show $0 <$ *norm a* − *r*

```
      using r2 by simp
    have norm a − norm (f x) ≤ norm (a − f x)
      by (rule norm-triangle-ineq2)
    also have ... = norm (f x − a)
      by (rule norm-minus-commute)
    also have ... < r using 1 .
    finally show norm a − r ≤ norm (f x)
      by simp
  qed
  finally show norm (inverse (f x)) ≤ inverse (norm a − r) .
 qed
 then show ?thesis by (rule BfunI)
qed

lemma tendsto-inverse [tendsto-intros]:
 fixes a :: 'a::real-normed-div-algebra
 assumes f: (f ⟶ a) F
  and a: a ≠ 0
 shows ((λx. inverse (f x)) ⟶ inverse a) F
proof −
 from a have 0 < norm a by simp
 with f have eventually (λx. dist (f x) a < norm a) F
  by (rule tendstoD)
 then have eventually (λx. f x ≠ 0) F
  unfolding dist-norm by (auto elim!: eventually-mono)
 with a have eventually (λx. inverse (f x) − inverse a =
  − (inverse (f x) ∗ (f x − a) ∗ inverse a)) F
  by (auto elim!: eventually-mono simp: inverse-diff-inverse)
 moreover have Zfun (λx. − (inverse (f x) ∗ (f x − a) ∗ inverse a)) F
  by (intro Zfun-minus Zfun-mult-left
   bounded-bilinear.Bfun-prod-Zfun [OF bounded-bilinear-mult]
   Bfun-inverse [OF f a] f [unfolded tendsto-Zfun-iff])
 ultimately show ?thesis
  unfolding tendsto-Zfun-iff by (rule Zfun-ssubst)
qed

lemma continuous-inverse:
 fixes f :: 'a::t2-space ⇒ 'b::real-normed-div-algebra
 assumes continuous F f
  and f (Lim F (λx. x)) ≠ 0
 shows continuous F (λx. inverse (f x))
 using assms unfolding continuous-def by (rule tendsto-inverse)

lemma continuous-at-within-inverse[continuous-intros]:
 fixes f :: 'a::t2-space ⇒ 'b::real-normed-div-algebra
 assumes continuous (at a within s) f
  and f a ≠ 0
 shows continuous (at a within s) (λx. inverse (f x))
 using assms unfolding continuous-within by (rule tendsto-inverse)
```

lemma *continuous-on-inverse*[*continuous-intros*]:
 fixes *f* :: *'a::topological-space* ⇒ *'b::real-normed-div-algebra*
 assumes *continuous-on s f*
  and ∀ *x*∈*s*. *f x* ≠ *0*
 shows *continuous-on s* (λ*x*. *inverse* (*f x*))
 using *assms* **unfolding** *continuous-on-def* **by** (*blast intro*: *tendsto-inverse*)

lemma *tendsto-divide* [*tendsto-intros*]:
 fixes *a b* :: *'a::real-normed-field*
 shows (*f* ⟶ *a*) *F* ⟹ (*g* ⟶ *b*) *F* ⟹ *b* ≠ *0* ⟹ ((λ*x*. *f x* / *g x*) ⟶ *a* / *b*) *F*
 by (*simp add*: *tendsto-mult tendsto-inverse divide-inverse*)

lemma *continuous-divide*:
 fixes *f g* :: *'a::t2-space* ⇒ *'b::real-normed-field*
 assumes *continuous F f*
  and *continuous F g*
  and *g* (*Lim F* (λ*x*. *x*)) ≠ *0*
 shows *continuous F* (λ*x*. (*f x*) / (*g x*))
 using *assms* **unfolding** *continuous-def* **by** (*rule tendsto-divide*)

lemma *continuous-at-within-divide*[*continuous-intros*]:
 fixes *f g* :: *'a::t2-space* ⇒ *'b::real-normed-field*
 assumes *continuous* (*at a within s*) *f continuous* (*at a within s*) *g*
  and *g a* ≠ *0*
 shows *continuous* (*at a within s*) (λ*x*. (*f x*) / (*g x*))
 using *assms* **unfolding** *continuous-within* **by** (*rule tendsto-divide*)

lemma *isCont-divide*[*continuous-intros*, *simp*]:
 fixes *f g* :: *'a::t2-space* ⇒ *'b::real-normed-field*
 assumes *isCont f a isCont g a g a* ≠ *0*
 shows *isCont* (λ*x*. (*f x*) / *g x*) *a*
 using *assms* **unfolding** *continuous-at* **by** (*rule tendsto-divide*)

lemma *continuous-on-divide*[*continuous-intros*]:
 fixes *f* :: *'a::topological-space* ⇒ *'b::real-normed-field*
 assumes *continuous-on s f continuous-on s g*
  and ∀ *x*∈*s*. *g x* ≠ *0*
 shows *continuous-on s* (λ*x*. (*f x*) / (*g x*))
 using *assms* **unfolding** *continuous-on-def* **by** (*blast intro*: *tendsto-divide*)

lemma *tendsto-power-int* [*tendsto-intros*]:
 fixes *a* :: *'a::real-normed-div-algebra*
 assumes *f*: (*f* ⟶ *a*) *F*
  and *a*: *a* ≠ *0*
 shows ((λ*x*. *power-int* (*f x*) *n*) ⟶ *power-int a n*) *F*
 using *assms* **by** (*cases n rule*: *int-cases4*) (*auto intro*!: *tendsto-intros simp*: *power-int-minus*)

lemma *continuous-power-int*:
fixes *f* :: *'a::t2-space* ⇒ *'b::real-normed-div-algebra*
assumes *continuous F f*
and *f* (*Lim F* (λ*x. x*)) ≠ *0*
shows *continuous F* (λ*x. power-int* (*f x*) *n*)
using *assms* **unfolding** *continuous-def* **by** (*rule tendsto-power-int*)

lemma *continuous-at-within-power-int*[*continuous-intros*]:
fixes *f* :: *'a::t2-space* ⇒ *'b::real-normed-div-algebra*
assumes *continuous* (*at a within s*) *f*
and *f a* ≠ *0*
shows *continuous* (*at a within s*) (λ*x. power-int* (*f x*) *n*)
using *assms* **unfolding** *continuous-within* **by** (*rule tendsto-power-int*)

lemma *continuous-on-power-int* [*continuous-intros*]:
fixes *f* :: *'a::topological-space* ⇒ *'b::real-normed-div-algebra*
assumes *continuous-on s f* **and** ∀ *x*∈*s. f x* ≠ *0*
shows *continuous-on s* (λ*x. power-int* (*f x*) *n*)
using *assms* **unfolding** *continuous-on-def* **by** (*blast intro*: *tendsto-power-int*)

lemma *tendsto-power-int'* [*tendsto-intros*]:
fixes *a* :: *'a::real-normed-div-algebra*
assumes *f*: (*f* ⟶ *a*) *F*
and *a*: *a* ≠ *0* ∨ *n* ≥ *0*
shows ((λ*x. power-int* (*f x*) *n*) ⟶ *power-int a n*) *F*
using *assms* **by** (*cases n rule*: *int-cases4*) (*auto intro*!: *tendsto-intros simp*: *power-int-minus*)

lemma *tendsto-sgn* [*tendsto-intros*]: (*f* ⟶ *l*) *F* ⟹ *l* ≠ *0* ⟹ ((λ*x. sgn* (*f x*)) ⟶ *sgn l*) *F*
for *l* :: *'a::real-normed-vector*
unfolding *sgn-div-norm* **by** (*simp add*: *tendsto-intros*)

lemma *continuous-sgn*:
fixes *f* :: *'a::t2-space* ⇒ *'b::real-normed-vector*
assumes *continuous F f*
and *f* (*Lim F* (λ*x. x*)) ≠ *0*
shows *continuous F* (λ*x. sgn* (*f x*))
using *assms* **unfolding** *continuous-def* **by** (*rule tendsto-sgn*)

lemma *continuous-at-within-sgn*[*continuous-intros*]:
fixes *f* :: *'a::t2-space* ⇒ *'b::real-normed-vector*
assumes *continuous* (*at a within s*) *f*
and *f a* ≠ *0*
shows *continuous* (*at a within s*) (λ*x. sgn* (*f x*))
using *assms* **unfolding** *continuous-within* **by** (*rule tendsto-sgn*)

lemma *isCont-sgn*[*continuous-intros*]:

fixes *f* :: *'a::t2-space* ⇒ *'b::real-normed-vector*
assumes *isCont f a*
and *f a* ≠ *0*
shows *isCont* (λ*x*. *sgn* (*f x*)) *a*
using *assms* **unfolding** *continuous-at* **by** (*rule tendsto-sgn*)

lemma *continuous-on-sgn*[*continuous-intros*]:
fixes *f* :: *'a::topological-space* ⇒ *'b::real-normed-vector*
assumes *continuous-on s f*
and ∀ *x*∈*s*. *f x* ≠ *0*
shows *continuous-on s* (λ*x*. *sgn* (*f x*))
using *assms* **unfolding** *continuous-on-def* **by** (*blast intro*: *tendsto-sgn*)

lemma *filterlim-at-infinity*:
fixes *f* :: *-* ⇒ *'a::real-normed-vector*
assumes *0* ≤ *c*
shows (*LIM x F*. *f x* :> *at-infinity*) ⟷ (∀ *r*>*c*. *eventually* (λ*x*. *r* ≤ *norm* (*f x*)) *F*)
unfolding *filterlim-iff eventually-at-infinity*
proof *safe*
fix *P* :: *'a* ⇒ *bool*
fix *b*
assume ∗: ∀ *r*>*c*. *eventually* (λ*x*. *r* ≤ *norm* (*f x*)) *F*
assume *P*: ∀ *x*. *b* ≤ *norm x* ⟶ *P x*
have *max b* (*c* + *1*) > *c* **by** *auto*
with ∗ **have** *eventually* (λ*x*. *max b* (*c* + *1*) ≤ *norm* (*f x*)) *F*
by *auto*
then show *eventually* (λ*x*. *P* (*f x*)) *F*
proof *eventually-elim*
case (*elim x*)
with *P* **show** *P* (*f x*) **by** *auto*
qed
qed *force*

lemma *filterlim-at-infinity-imp-norm-at-top*:
fixes *F*
assumes *filterlim f at-infinity F*
shows *filterlim* (λ*x*. *norm* (*f x*)) *at-top F*
proof −
{
fix *r* :: *real*
have $\forall_F$ *x in F*. *r* ≤ *norm* (*f x*) **using** *filterlim-at-infinity*[*of 0 f F*] *assms*
by (*cases r* > *0*)
(*auto simp*: *not-less intro*: *always-eventually order.trans*[*OF - norm-ge-zero*])
}
thus *?thesis* **by** (*auto simp*: *filterlim-at-top*)
qed

lemma *filterlim-norm-at-top-imp-at-infinity*:

fixes *F*
assumes *filterlim* ($\lambda x.$ *norm* (*f x*)) *at-top F*
shows *filterlim f at-infinity F*
using *filterlim-at-infinity*[*of 0 f F*] *assms* **by** (*auto simp*: *filterlim-at-top*)

lemma *filterlim-norm-at-top*: *filterlim norm at-top at-infinity*
by (*rule filterlim-at-infinity-imp-norm-at-top*) (*rule filterlim-ident*)

lemma *filterlim-at-infinity-conv-norm-at-top*:
filterlim f at-infinity G $\longleftrightarrow$ *filterlim* ($\lambda x.$ *norm* (*f x*)) *at-top G*
by (*auto simp*: *filterlim-at-infinity*[*OF order.refl*] *filterlim-at-top-gt*[*of - - 0*])

lemma *eventually-not-equal-at-infinity*:
eventually ($\lambda x.$ $x \neq$ (*a* :: *'a* :: {*real-normed-vector*})) *at-infinity*
proof −
from *filterlim-norm-at-top*[**where** *'a* = *'a*]
have $\forall_F$ *x in at-infinity. norm a* < *norm* (*x*::*'a*) **by** (*auto simp*: *filterlim-at-top-dense*)
thus *?thesis* **by** *eventually-elim auto*
qed

lemma *filterlim-int-of-nat-at-topD*:
fixes *F*
assumes *filterlim* ($\lambda x.$ *f* (*int x*)) *F at-top*
shows *filterlim f F at-top*
proof −
have *filterlim* ($\lambda x.$ *f* (*int* (*nat x*))) *F at-top*
by (*rule filterlim-compose*[*OF assms filterlim-nat-sequentially*])
also have *?this* $\longleftrightarrow$ *filterlim f F at-top*
by (*intro filterlim-cong refl eventually-mono* [*OF eventually-ge-at-top*[*of 0*::*int*]])
auto
finally show *?thesis* **.**
qed

lemma *filterlim-int-sequentially* [*tendsto-intros*]:
filterlim int at-top sequentially
unfolding *filterlim-at-top*
proof
fix *C* :: *int*
show *eventually* ($\lambda n.$ *int n* $\geq$ *C*) *at-top*
using *eventually-ge-at-top*[*of nat* $\lceil C \rceil$] **by** *eventually-elim linarith*
qed

lemma *filterlim-real-of-int-at-top* [*tendsto-intros*]:
filterlim real-of-int at-top at-top
unfolding *filterlim-at-top*
proof
fix *C* :: *real*
show *eventually* ($\lambda n.$ *real-of-int n* $\geq$ *C*) *at-top*
using *eventually-ge-at-top*[*of* $\lceil C \rceil$] **by** *eventually-elim linarith*

qed

lemma *filterlim-abs-real*: *filterlim* (*abs*::*real* ⇒ *real*) *at-top at-top*
proof (*subst filterlim-cong*[*OF refl refl*])
  from *eventually-ge-at-top*[*of 0*::*real*] **show** *eventually* (λ*x*::*real*. |*x*| = *x*) *at-top*
    by *eventually-elim simp*
qed (*simp-all add*: *filterlim-ident*)

lemma *filterlim-of-real-at-infinity* [*tendsto-intros*]:
  filterlim (*of-real* :: *real* ⇒ ′*a* :: *real-normed-algebra-1*) *at-infinity at-top*
  by (*intro filterlim-norm-at-top-imp-at-infinity*) (*auto simp*: *filterlim-abs-real*)

lemma *not-tendsto-and-filterlim-at-infinity*:
  fixes *c* :: ′*a*::*real-normed-vector*
  assumes *F* ≠ *bot*
    and (*f* ⟶ *c*) *F*
    and *filterlim f at-infinity F*
  shows *False*
proof −
  from *tendstoD*[*OF assms*(*2*), *of 1*/*2*]
  have *eventually* (λ*x*. *dist* (*f x*) *c* < *1*/*2*) *F*
    by *simp*
  moreover
  from *filterlim-at-infinity*[*of norm c f F*] *assms*(*3*)
  have *eventually* (λ*x*. *norm* (*f x*) ≥ *norm c* + *1*) *F* **by** *simp*
  ultimately have *eventually* (λ*x*. *False*) *F*
  proof *eventually-elim*
    fix *x*
    assume *A*: *dist* (*f x*) *c* < *1*/*2*
    assume *norm* (*f x*) ≥ *norm c* + *1*
    also have *norm* (*f x*) = *dist* (*f x*) *0* **by** *simp*
    also have ... ≤ *dist* (*f x*) *c* + *dist c 0* **by** (*rule dist-triangle*)
    finally show *False* **using** *A* **by** *simp*
  qed
  with *assms* **show** *False* **by** *simp*
qed

lemma *filterlim-at-infinity-imp-not-convergent*:
  assumes *filterlim f at-infinity sequentially*
  shows ¬ *convergent f*
  by (*rule notI*, *rule not-tendsto-and-filterlim-at-infinity*[*OF - - assms*])
    (*simp-all add*: *convergent-LIMSEQ-iff*)

lemma *filterlim-at-infinity-imp-eventually-ne*:
  assumes *filterlim f at-infinity F*
  shows *eventually* (λ*z*. *f z* ≠ *c*) *F*
proof −
  have *norm c* + *1* > *0*
    by (*intro add-nonneg-pos*) *simp-all*

with *filterlim-at-infinity*[*OF order.refl*, *of f F*] *assms*
have *eventually* (*λz. norm* (*f z*) ≥ *norm c* + *1*) *F*
by *blast*
then show *?thesis*
by *eventually-elim auto*
qed

lemma *tendsto-of-nat* [*tendsto-intros*]:
filterlim (*of-nat* :: *nat* ⇒ *′a::real-normed-algebra-1*) *at-infinity sequentially*
proof (*subst filterlim-at-infinity*[*OF order.refl*], *intro allI impI*)
fix *r* :: *real*
assume *r*: *r* > *0*
define *n* **where** *n* = *nat* ⌈*r*⌉
from *r* **have** *n*: ∀ *m*≥*n*. *of-nat m* ≥ *r*
unfolding *n-def* **by** *linarith*
from *eventually-ge-at-top*[*of n*] **show** *eventually* (*λm. norm* (*of-nat m* :: *′a*) ≥ *r*) *sequentially*
by *eventually-elim* (*use n* **in** *simp-all*)
qed

108.4 Relate *at*, *at-left* and *at-right*

This lemmas are useful for conversion between *at x* to *at-left x* and *at-right x* and also *at-right 0*.

lemmas *filterlim-split-at-real* = *filterlim-split-at*[**where** *′a=real*]

lemma *filtermap-nhds-shift*: *filtermap* (*λx. x* − *d*) (*nhds a*) = *nhds* (*a* − *d*)
for *a d* :: *′a::real-normed-vector*
by (*rule filtermap-fun-inverse*[**where** *g=λx. x* + *d*])
(*auto intro*!: *tendsto-eq-intros filterlim-ident*)

lemma *filtermap-nhds-minus*: *filtermap* (*λx.* − *x*) (*nhds a*) = *nhds* (− *a*)
for *a* :: *′a::real-normed-vector*
by (*rule filtermap-fun-inverse*[**where** *g=uminus*])
(*auto intro*!: *tendsto-eq-intros filterlim-ident*)

lemma *filtermap-at-shift*: *filtermap* (*λx. x* − *d*) (*at a*) = *at* (*a* − *d*)
for *a d* :: *′a::real-normed-vector*
by (*simp add*: *filter-eq-iff eventually-filtermap eventually-at-filter filtermap-nhds-shift*[*symmetric*])

lemma *filtermap-at-right-shift*: *filtermap* (*λx. x* − *d*) (*at-right a*) = *at-right* (*a* − *d*)
for *a d* :: *real*
by (*simp add*: *filter-eq-iff eventually-filtermap eventually-at-filter filtermap-nhds-shift*[*symmetric*])

lemma *filterlim-shift*:
fixes *d* :: *′a::real-normed-vector*
assumes *filterlim f F* (*at a*)
shows *filterlim* (*f* ∘ (+) *d*) *F* (*at* (*a* − *d*))

unfolding *filterlim-iff*
proof (*intro strip*)
fix *P*
assume *eventually P F*
then have ∀$_F$ *x in filtermap* (λ*y. y* − *d*) (*at a*). *P* (*f* (*d* + *x*))
using *assms* **by** (*force simp add*: *filterlim-iff eventually-filtermap*)
then show (∀$_F$ *x in at* (*a* − *d*). *P* ((*f* ∘ (+) *d*) *x*))
by (*force simp add*: *filtermap-at-shift*)
qed

lemma *filterlim-shift-iff*:
fixes *d* :: *'a::real-normed-vector*
shows *filterlim* (*f* ∘ (+) *d*) *F* (*at* (*a* − *d*)) = *filterlim f F* (*at a*) (**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs* **show** *?rhs*
using *filterlim-shift* [*OF L*, *of* −*d*] **by** (*simp add*: *filterlim-iff*)
qed (*metis filterlim-shift*)

lemma *at-right-to-0*: *at-right a* = *filtermap* (λ*x. x* + *a*) (*at-right 0*)
for *a* :: *real*
using *filtermap-at-right-shift*[*of* −*a 0*] **by** *simp*

lemma *filterlim-at-right-to-0*:
filterlim f F (*at-right a*) ⟷ *filterlim* (λ*x. f* (*x* + *a*)) *F* (*at-right 0*)
for *a* :: *real*
unfolding *filterlim-def filtermap-filtermap at-right-to-0*[*of a*] **..**

lemma *eventually-at-right-to-0*:
eventually P (*at-right a*) ⟷ *eventually* (λ*x. P* (*x* + *a*)) (*at-right 0*)
for *a* :: *real*
unfolding *at-right-to-0*[*of a*] **by** (*simp add*: *eventually-filtermap*)

lemma *at-to-0*: *at a* = *filtermap* (λ*x. x* + *a*) (*at 0*)
for *a* :: *'a::real-normed-vector*
using *filtermap-at-shift*[*of* −*a 0*] **by** *simp*

lemma *filterlim-at-to-0*:
filterlim f F (*at a*) ⟷ *filterlim* (λ*x. f* (*x* + *a*)) *F* (*at 0*)
for *a* :: *'a::real-normed-vector*
unfolding *filterlim-def filtermap-filtermap at-to-0*[*of a*] **..**

lemma *eventually-at-to-0*:
eventually P (*at a*) ⟷ *eventually* (λ*x. P* (*x* + *a*)) (*at 0*)
for *a* :: *'a::real-normed-vector*
unfolding *at-to-0*[*of a*] **by** (*simp add*: *eventually-filtermap*)

lemma *filtermap-at-minus*: *filtermap* (λ*x.* − *x*) (*at a*) = *at* (− *a*)
for *a* :: *'a::real-normed-vector*

by (*simp add*: *filter-eq-iff eventually-filtermap eventually-at-filter filtermap-nhds-minus*[*symmetric*])

lemma *at-left-minus*: *at-left a* = *filtermap* (λ*x*. − *x*) (*at-right* (− *a*))
for *a* :: *real*
by (*simp add*: *filter-eq-iff eventually-filtermap eventually-at-filter filtermap-nhds-minus*[*symmetric*])

lemma *at-right-minus*: *at-right a* = *filtermap* (λ*x*. − *x*) (*at-left* (− *a*))
for *a* :: *real*
by (*simp add*: *filter-eq-iff eventually-filtermap eventually-at-filter filtermap-nhds-minus*[*symmetric*])

lemma *filterlim-at-left-to-right*:
filterlim f F (*at-left a*) ⟷ *filterlim* (λ*x*. *f* (− *x*)) *F* (*at-right* (−*a*))
for *a* :: *real*
unfolding *filterlim-def filtermap-filtermap at-left-minus*[*of a*] **..**

lemma *eventually-at-left-to-right*:
eventually P (*at-left a*) ⟷ *eventually* (λ*x*. *P* (− *x*)) (*at-right* (−*a*))
for *a* :: *real*
unfolding *at-left-minus*[*of a*] **by** (*simp add*: *eventually-filtermap*)

lemma *filterlim-uminus-at-top-at-bot*: *LIM x at-bot*. − *x* :: *real* :> *at-top*
unfolding *filterlim-at-top eventually-at-bot-dense*
by (*metis leI minus-less-iff order-less-asym*)

lemma *filterlim-uminus-at-bot-at-top*: *LIM x at-top*. − *x* :: *real* :> *at-bot*
unfolding *filterlim-at-bot eventually-at-top-dense*
by (*metis leI less-minus-iff order-less-asym*)

lemma *at-bot-mirror* :
shows (*at-bot*::(′*a*::{*ordered-ab-group-add*,*linorder*} *filter*)) = *filtermap uminus at-top*
proof (*rule filtermap-fun-inverse*[*symmetric*])
show *filterlim uminus at-top* (*at-bot*::′*a filter*)
using *eventually-at-bot-linorder filterlim-at-top le-minus-iff* **by** *force*
show *filterlim uminus* (*at-bot*::′*a filter*) *at-top*
by (*simp add*: *filterlim-at-bot minus-le-iff*)
qed *auto*

lemma *at-top-mirror* :
shows (*at-top*::(′*a*::{*ordered-ab-group-add*,*linorder*} *filter*)) = *filtermap uminus at-bot*
apply (*subst at-bot-mirror*)
by (*auto simp*: *filtermap-filtermap*)

lemma *filterlim-at-top-mirror*: (*LIM x at-top*. *f x* :> *F*) ⟷ (*LIM x at-bot*. *f* (−*x*::*real*) :> *F*)
unfolding *filterlim-def at-top-mirror filtermap-filtermap* **..**

lemma *filterlim-at-bot-mirror*: (*LIM x at-bot. f x* :> *F*) ⟷ (*LIM x at-top. f* (−*x*::*real*) :> *F*)
unfolding *filterlim-def at-bot-mirror filtermap-filtermap* **..**

lemma *filterlim-uminus-at-top*: (*LIM x F. f x* :> *at-top*) ⟷ (*LIM x F.* − (*f x*) :: *real* :> *at-bot*)
using *filterlim-compose*[*OF filterlim-uminus-at-bot-at-top*, *of f F*]
and *filterlim-compose*[*OF filterlim-uminus-at-top-at-bot*, *of* λ*x.* − *f x F*]
by *auto*

lemma *tendsto-at-botI-sequentially*:
fixes *f* :: *real* ⇒ ′*b*::*first-countable-topology*
assumes ∗: ⋀*X. filterlim X at-bot sequentially* ⟹ (λ*n. f* (*X n*)) ⟶ *y*
shows (*f* ⟶ *y*) *at-bot*
unfolding *filterlim-at-bot-mirror*
proof (*rule tendsto-at-topI-sequentially*)
fix *X* :: *nat* ⇒ *real* **assume** *filterlim X at-top sequentially*
thus (λ*n. f* (−*X n*)) ⟶ *y* **by** (*intro* ∗) (*auto simp*: *filterlim-uminus-at-top*)
qed

lemma *filterlim-at-infinity-imp-filterlim-at-top*:
assumes *filterlim* (*f* :: ′*a* ⇒ *real*) *at-infinity F*
assumes *eventually* (λ*x. f x* > *0*) *F*
shows *filterlim f at-top F*
proof −
from *assms*(*2*) **have** ∗: *eventually* (λ*x. norm* (*f x*) = *f x*) *F* **by** *eventually-elim simp*
from *assms*(*1*) **show** *?thesis* **unfolding** *filterlim-at-infinity-conv-norm-at-top*
by (*subst* (*asm*) *filterlim-cong*[*OF refl refl* ∗])
qed

lemma *filterlim-at-infinity-imp-filterlim-at-bot*:
assumes *filterlim* (*f* :: ′*a* ⇒ *real*) *at-infinity F*
assumes *eventually* (λ*x. f x* < *0*) *F*
shows *filterlim f at-bot F*
proof −
from *assms*(*2*) **have** ∗: *eventually* (λ*x. norm* (*f x*) = −*f x*) *F* **by** *eventually-elim simp*
from *assms*(*1*) **have** *filterlim* (λ*x.* − *f x*) *at-top F*
unfolding *filterlim-at-infinity-conv-norm-at-top*
by (*subst* (*asm*) *filterlim-cong*[*OF refl refl* ∗])
thus *?thesis* **by** (*simp add*: *filterlim-uminus-at-top*)
qed

lemma *filterlim-uminus-at-bot*: (*LIM x F. f x* :> *at-bot*) ⟷ (*LIM x F.* − (*f x*) :: *real* :> *at-top*)
unfolding *filterlim-uminus-at-top* **by** *simp*

lemma *filterlim-inverse-at-top-right*: *LIM x at-right* (*0*::*real*). *inverse x* :> *at-top*

```
  unfolding filterlim-at-top-gt[where c=0] eventually-at-filter
proof safe
  fix Z :: real
  assume [arith]: 0 < Z
  then have eventually (λx. x < inverse Z) (nhds 0)
    by (auto simp: eventually-nhds-metric dist-real-def intro!: exI[of - |inverse Z|])
  then show eventually (λx. x ≠ 0 ⟶ x ∈ {0<..} ⟶ Z ≤ inverse x) (nhds 0)
    by (auto elim!: eventually-mono simp: inverse-eq-divide field-simps)
qed

lemma tendsto-inverse-0:
  fixes x :: - ⇒ 'a::real-normed-div-algebra
  shows (inverse ⟶ (0::'a)) at-infinity
  unfolding tendsto-Zfun-iff diff-0-right Zfun-def eventually-at-infinity
proof safe
  fix r :: real
  assume 0 < r
  show ∃ b. ∀ x. b ≤ norm x ⟶ norm (inverse x :: 'a) < r
  proof (intro exI[of - inverse (r / 2)] allI impI)
    fix x :: 'a
    from ‹0 < r› have 0 < inverse (r / 2) by simp
    also assume *: inverse (r / 2) ≤ norm x
    finally show norm (inverse x) < r
      using * ‹0 < r›
      by (subst nonzero-norm-inverse) (simp-all add: inverse-eq-divide field-simps)
  qed
qed

lemma tendsto-add-filterlim-at-infinity:
  fixes c :: 'b::real-normed-vector
    and F :: 'a filter
  assumes (f ⟶ c) F
    and filterlim g at-infinity F
  shows filterlim (λx. f x + g x) at-infinity F
proof (subst filterlim-at-infinity[OF order-refl], safe)
  fix r :: real
  assume r: r > 0
  from assms(1) have ((λx. norm (f x)) ⟶ norm c) F
    by (rule tendsto-norm)
  then have eventually (λx. norm (f x) < norm c + 1) F
    by (rule order-tendstoD) simp-all
  moreover from r have r + norm c + 1 > 0
    by (intro add-pos-nonneg) simp-all
  with assms(2) have eventually (λx. norm (g x) ≥ r + norm c + 1) F
    unfolding filterlim-at-infinity[OF order-refl]
    by (elim allE[of - r + norm c + 1]) simp-all
  ultimately show eventually (λx. norm (f x + g x) ≥ r) F
  proof eventually-elim
    fix x :: 'a
```

assume *A*: *norm* (*f x*) < *norm c* + *1* **and** *B*: *r* + *norm c* + *1* ≤ *norm* (*g x*)
from *A B* **have** *r* ≤ *norm* (*g x*) − *norm* (*f x*)
by *simp*
also have *norm* (*g x*) − *norm* (*f x*) ≤ *norm* (*g x* + *f x*)
by (*rule norm-diff-ineq*)
finally show *r* ≤ *norm* (*f x* + *g x*)
by (*simp add*: *add-ac*)
qed
qed

lemma *tendsto-add-filterlim-at-infinity'*:
fixes *c* :: *'b*::*real-normed-vector*
and *F* :: *'a filter*
assumes *filterlim f at-infinity F*
and (*g* ⟶ *c*) *F*
shows *filterlim* (*λx. f x* + *g x*) *at-infinity F*
by (*subst add.commute*) (*rule tendsto-add-filterlim-at-infinity assms*)+

lemma *filterlim-inverse-at-right-top*: *LIM x at-top. inverse x* :> *at-right* (*0*::*real*)
unfolding *filterlim-at*
by (*auto simp*: *eventually-at-top-dense*)
(*metis tendsto-inverse-0 filterlim-mono at-top-le-at-infinity order-refl*)

lemma *filterlim-inverse-at-top*:
(*f* ⟶ (*0* :: *real*)) *F* ⟹ *eventually* (*λx. 0* < *f x*) *F* ⟹ *LIM x F. inverse* (*f x*) :> *at-top*
by (*intro filterlim-compose*[*OF filterlim-inverse-at-top-right*])
(*simp add*: *filterlim-def eventually-filtermap eventually-mono at-within-def le-principal*)

lemma *filterlim-inverse-at-bot-neg*:
LIM x (*at-left* (*0*::*real*)). *inverse x* :> *at-bot*
by (*simp add*: *filterlim-inverse-at-top-right filterlim-uminus-at-bot filterlim-at-left-to-right*)

lemma *filterlim-inverse-at-bot*:
(*f* ⟶ (*0* :: *real*)) *F* ⟹ *eventually* (*λx. f x* < *0*) *F* ⟹ *LIM x F. inverse* (*f x*) :> *at-bot*
unfolding *filterlim-uminus-at-bot inverse-minus-eq*[*symmetric*]
by (*rule filterlim-inverse-at-top*) (*simp-all add*: *tendsto-minus-cancel-left*[*symmetric*])

lemma *at-right-to-top*: (*at-right* (*0*::*real*)) = *filtermap inverse at-top*
by (*intro filtermap-fun-inverse*[*symmetric*, **where** *g*=*inverse*])
(*auto intro*: *filterlim-inverse-at-top-right filterlim-inverse-at-right-top*)

lemma *eventually-at-right-to-top*:
eventually P (*at-right* (*0*::*real*)) ⟷ *eventually* (*λx. P* (*inverse x*)) *at-top*
unfolding *at-right-to-top eventually-filtermap* **..**

lemma *filterlim-at-right-to-top*:

filterlim f F (*at-right* (*0*::*real*)) ⟷ (*LIM x at-top*. *f* (*inverse x*) :> *F*)
unfolding *filterlim-def at-right-to-top filtermap-filtermap* **..**

lemma *at-top-to-right*: *at-top* = *filtermap inverse* (*at-right* (*0*::*real*))
unfolding *at-right-to-top filtermap-filtermap inverse-inverse-eq filtermap-ident* **..**

lemma *eventually-at-top-to-right*:
eventually P at-top ⟷ *eventually* (λ*x*. *P* (*inverse x*)) (*at-right* (*0*::*real*))
unfolding *at-top-to-right eventually-filtermap* **..**

lemma *filterlim-at-top-to-right*:
filterlim f F at-top ⟷ (*LIM x* (*at-right* (*0*::*real*)). *f* (*inverse x*) :> *F*)
unfolding *filterlim-def at-top-to-right filtermap-filtermap* **..**

lemma *filterlim-inverse-at-infinity*:
fixes *x* :: - ⇒ *'a*::{*real-normed-div-algebra*, *division-ring*}
shows *filterlim inverse at-infinity* (*at* (*0*::*'a*))
unfolding *filterlim-at-infinity*[*OF order-refl*]
proof *safe*
fix *r* :: *real*
assume *0* < *r*
then show *eventually* (λ*x*::*'a*. *r* ≤ *norm* (*inverse x*)) (*at 0*)
unfolding *eventually-at norm-inverse*
by (*intro exI*[*of - inverse r*])
(*auto simp*: *norm-conv-dist*[*symmetric*] *field-simps inverse-eq-divide*)
qed

lemma *filterlim-inverse-at-iff*:
fixes *g* :: *'a* ⇒ *'b*::{*real-normed-div-algebra*, *division-ring*}
shows (*LIM x F*. *inverse* (*g x*) :> *at 0*) ⟷ (*LIM x F*. *g x* :> *at-infinity*)
unfolding *filterlim-def filtermap-filtermap*[*symmetric*]
proof
assume *filtermap g F* ≤ *at-infinity*
then have *filtermap inverse* (*filtermap g F*) ≤ *filtermap inverse at-infinity*
by (*rule filtermap-mono*)
also have ... ≤ *at 0*
using *tendsto-inverse-0*[**where** *'a*=*'b*]
by (*auto intro*!: *exI*[*of - 1*]
simp: *le-principal eventually-filtermap filterlim-def at-within-def eventually-at-infinity*)
finally show *filtermap inverse* (*filtermap g F*) ≤ *at 0* **.**
next
assume *filtermap inverse* (*filtermap g F*) ≤ *at 0*
then have *filtermap inverse* (*filtermap inverse* (*filtermap g F*)) ≤ *filtermap inverse* (*at 0*)
by (*rule filtermap-mono*)
with *filterlim-inverse-at-infinity* **show** *filtermap g F* ≤ *at-infinity*
by (*auto intro*: *order-trans simp*: *filterlim-def filtermap-filtermap*)
qed

lemma *tendsto-mult-filterlim-at-infinity*:
fixes c :: $'a$::*real-normed-field*
assumes $(f \longrightarrow c)$ F $c \neq 0$
assumes *filterlim g at-infinity F*
shows *filterlim* $(\lambda x.\ f\ x * g\ x)$ *at-infinity F*
proof −
have $((\lambda x.\ inverse\ (f\ x) * inverse\ (g\ x)) \longrightarrow inverse\ c * 0)$ F
by (*intro tendsto-mult tendsto-inverse assms filterlim-compose*[*OF tendsto-inverse-0*])
then have *filterlim* $(\lambda x.\ inverse\ (f\ x) * inverse\ (g\ x))$ $(at\ (inverse\ c * 0))$ F
unfolding *filterlim-at*
using *assms*
by (*auto intro*: *filterlim-at-infinity-imp-eventually-ne tendsto-imp-eventually-ne eventually-conj*)
then show *?thesis*
by (*subst filterlim-inverse-at-iff*[*symmetric*]) *simp-all*
qed

lemma *filterlim-power-int-neg-at-infinity*:
fixes f :: - ⇒ $'a$::{*real-normed-div-algebra*, *division-ring*}
assumes $n < 0$ **and** *lim*: $(f \longrightarrow 0)$ F **and** *ev*: *eventually* $(\lambda x.\ f\ x \neq 0)$ F
shows *filterlim* $(\lambda x.\ f\ x\ powi\ n)$ *at-infinity F*
proof −
have *lim'*: $((\lambda x.\ f\ x\ \hat{}\ nat\ (-\ n)) \longrightarrow 0)$ F
by (*rule tendsto-eq-intros lim*)+ (*use* ‹$n < 0$› **in** *auto*)
have *ev'*: *eventually* $(\lambda x.\ f\ x\ \hat{}\ nat\ (-n) \neq 0)$ F
using *ev* **by** *eventually-elim* (*use* ‹$n < 0$› **in** *auto*)
have *filterlim* $(\lambda x.\ inverse\ (f\ x\ \hat{}\ nat\ (-n)))$ *at-infinity F*
by (*intro filterlim-compose*[*OF filterlim-inverse-at-infinity*])
(*use lim' ev'* **in** ‹*auto simp*: *filterlim-at*›)
thus *?thesis*
using ‹$n < 0$› **by** (*simp add*: *power-int-def power-inverse*)
qed

lemma *tendsto-inverse-0-at-top*: *LIM x F. f x :> at-top* ⟹ $((\lambda x.\ inverse\ (f\ x) :: real) \longrightarrow 0)$ F
by (*metis filterlim-at filterlim-mono*[*OF - at-top-le-at-infinity order-refl*] *filterlim-inverse-at-iff*)

lemma *filterlim-inverse-at-top-iff*:
eventually $(\lambda x.\ 0 < f\ x)$ F ⟹ (*LIM x F. inverse* (*f x*) :> *at-top*) ⟷ $(f \longrightarrow (0 :: real))$ F
by (*auto dest*: *tendsto-inverse-0-at-top filterlim-inverse-at-top*)

lemma *filterlim-at-top-iff-inverse-0*:
eventually $(\lambda x.\ 0 < f\ x)$ F ⟹ (*LIM x F. f x :> at-top*) ⟷ $((inverse \circ f) \longrightarrow (0 :: real))$ F
using *filterlim-inverse-at-top-iff* [*of inverse* ∘ *f*] **by** *auto*

lemma *real-tendsto-divide-at-top*:
  fixes *c*::*real*
  assumes (*f* ⟶ *c*) *F*
  assumes *filterlim g at-top F*
  shows ((λ*x*. *f x* / *g x*) ⟶ *0*) *F*
  by (*auto simp*: *divide-inverse-commute*
    intro!: *tendsto-mult*[*THEN tendsto-eq-rhs*] *tendsto-inverse-0-at-top assms*)

lemma *mult-nat-left-at-top*: *c* > *0* ⟹ *filterlim* (λ*x*. *c* ∗ *x*) *at-top sequentially*
  for *c* :: *nat*
  by (*rule filterlim-subseq*) (*auto simp*: *strict-mono-def*)

lemma *mult-nat-right-at-top*: *c* > *0* ⟹ *filterlim* (λ*x*. *x* ∗ *c*) *at-top sequentially*
  for *c* :: *nat*
  by (*rule filterlim-subseq*) (*auto simp*: *strict-mono-def*)

lemma *filterlim-times-pos*:
  LIM x F1. *c* ∗ *f x* :> *at-right l*
  if *filterlim f* (*at-right p*) *F1 0* < *c l* = *c* ∗ *p*
  for *c*::′*a*::{*linordered-field*, *linorder-topology*}
  unfolding *filterlim-iff*
proof *safe*
  fix *P*
  assume ∀$_F$ *x in at-right l*. *P x*
  then obtain *d* **where** *c* ∗ *p* < *d* ⋀*y*. *y* > *c* ∗ *p* ⟹ *y* < *d* ⟹ *P y*
    unfolding ‹*l* = - › *eventually-at-right-field*
    by *auto*
  then have ∀$_F$ *a in at-right p*. *P* (*c* ∗ *a*)
    by (*auto simp*: *eventually-at-right-field* ‹*0* < *c*› *field-simps intro*!: *exI*[**where** *x*=*d*/*c*])
  from *that*(*1*)[*unfolded filterlim-iff*, *rule-format*, *OF this*]
  show ∀$_F$ *x in F1*. *P* (*c* ∗ *f x*) **.**
qed

lemma *filtermap-nhds-times*: *c* ≠ *0* ⟹ *filtermap* (*times c*) (*nhds a*) = *nhds* (*c* ∗ *a*)
  for *a c* :: ′*a*::*real-normed-field*
  by (*rule filtermap-fun-inverse*[**where** *g*=λ*x*. *inverse c* ∗ *x*])
    (*auto intro*!: *tendsto-eq-intros filterlim-ident*)

lemma *filtermap-times-pos-at-right*:
  fixes *c*::′*a*::{*linordered-field*, *linorder-topology*}
  assumes *c* > *0*
  shows *filtermap* (*times c*) (*at-right p*) = *at-right* (*c* ∗ *p*)
  using *assms*
  by (*intro filtermap-fun-inverse*[**where** *g*=λ*x*. *inverse c* ∗ *x*])
    (*auto intro*!: *filterlim-ident filterlim-times-pos*)

lemma *at-to-infinity*: (*at* (*0*::′*a*::{*real-normed-field*,*field*})) = *filtermap inverse at-infinity*

proof (*rule antisym*)
  have (*inverse* ⟶ (*0*::'*a*)) *at-infinity*
    by (*fact tendsto-inverse-0*)
  then show *filtermap inverse at-infinity* ≤ *at* (*0*::'*a*)
    using *filterlim-def filterlim-ident filterlim-inverse-at-iff* **by** *fastforce*
next
  have *filtermap inverse* (*filtermap inverse* (*at* (*0*::'*a*))) ≤ *filtermap inverse at-infinity*
    using *filterlim-inverse-at-infinity* **unfolding** *filterlim-def*
    by (*rule filtermap-mono*)
  then show *at* (*0*::'*a*) ≤ *filtermap inverse at-infinity*
    by (*simp add*: *filtermap-ident filtermap-filtermap*)
qed

lemma *lim-at-infinity-0*:
  fixes *l* :: '*a*::{*real-normed-field*,*field*}
  shows (*f* ⟶ *l*) *at-infinity* ⟷ ((*f* ∘ *inverse*) ⟶ *l*) (*at* (*0*::'*a*))
  by (*simp add*: *tendsto-compose-filtermap at-to-infinity filtermap-filtermap*)

lemma *lim-zero-infinity*:
  fixes *l* :: '*a*::{*real-normed-field*,*field*}
  shows ((λ*x*. *f*(*1* / *x*)) ⟶ *l*) (*at* (*0*::'*a*)) ⟹ (*f* ⟶ *l*) *at-infinity*
  by (*simp add*: *inverse-eq-divide lim-at-infinity-0 comp-def*)

We only show rules for multiplication and addition when the functions are either against a real value or against infinity. Further rules are easy to derive by using *filterlim ?f at-top ?F* = (*LIM x ?F*. − *?f x* :> *at-bot*).

lemma *filterlim-tendsto-pos-mult-at-top*:
  assumes *f*: (*f* ⟶ *c*) *F*
    and *c*: *0* < *c*
    and *g*: *LIM x F*. *g x* :> *at-top*
  shows *LIM x F*. (*f x* ∗ *g x* :: *real*) :> *at-top*
  unfolding *filterlim-at-top-gt*[**where** *c*=*0*]
proof *safe*
  fix *Z* :: *real*
  assume *0* < *Z*
  from *f* ‹*0* < *c*› **have** *eventually* (λ*x*. *c* / *2* < *f x*) *F*
    by (*auto dest*!: *tendstoD*[**where** *e*=*c* / *2*] *elim*!: *eventually-mono*
        simp: *dist-real-def abs-real-def split*: *if-split-asm*)
  moreover from *g* **have** *eventually* (λ*x*. (*Z* / *c* ∗ *2*) ≤ *g x*) *F*
    unfolding *filterlim-at-top* **by** *auto*
  ultimately show *eventually* (λ*x*. *Z* ≤ *f x* ∗ *g x*) *F*
  proof *eventually-elim*
    case (*elim x*)
    with ‹*0* < *Z*› ‹*0* < *c*› **have** *c* / *2* ∗ (*Z* / *c* ∗ *2*) ≤ *f x* ∗ *g x*
      by (*intro mult-mono*) (*auto simp*: *zero-le-divide-iff*)
    with ‹*0* < *c*› **show** *Z* ≤ *f x* ∗ *g x*
       by *simp*
  qed
qed

lemma *filterlim-at-top-mult-at-top*:
assumes *f*: *LIM x F. f x :> at-top*
and *g*: *LIM x F. g x :> at-top*
shows *LIM x F.* (*f x* ∗ *g x* :: *real*) *:> at-top*
unfolding *filterlim-at-top-gt*[**where** *c=0*]
proof *safe*
fix *Z* :: *real*
assume *0* < *Z*
from *f* **have** *eventually* (λ*x. 1* ≤ *f x*) *F*
unfolding *filterlim-at-top* **by** *auto*
moreover from *g* **have** *eventually* (λ*x. Z* ≤ *g x*) *F*
unfolding *filterlim-at-top* **by** *auto*
ultimately show *eventually* (λ*x. Z* ≤ *f x* ∗ *g x*) *F*
proof *eventually-elim*
case (*elim x*)
with ‹*0* < *Z*› **have** *1* ∗ *Z* ≤ *f x* ∗ *g x*
by (*intro mult-mono*) (*auto simp*: *zero-le-divide-iff*)
then show *Z* ≤ *f x* ∗ *g x*
by *simp*
qed
qed

lemma *filterlim-at-top-mult-tendsto-pos*:
assumes *f*: (*f* ⟶ *c*) *F*
and *c*: *0* < *c*
and *g*: *LIM x F. g x :> at-top*
shows *LIM x F.* (*g x* ∗ *f x*:: *real*) *:> at-top*
by (*auto simp*: *mult.commute intro*!: *filterlim-tendsto-pos-mult-at-top f c g*)

lemma *filterlim-tendsto-pos-mult-at-bot*:
fixes *c* :: *real*
assumes (*f* ⟶ *c*) *F 0* < *c filterlim g at-bot F*
shows *LIM x F. f x* ∗ *g x :> at-bot*
using *filterlim-tendsto-pos-mult-at-top*[*OF assms*(*1*,*2*), *of* λ*x.* − *g x*] *assms*(*3*)
unfolding *filterlim-uminus-at-bot* **by** *simp*

lemma *filterlim-tendsto-neg-mult-at-bot*:
fixes *c* :: *real*
assumes *c*: (*f* ⟶ *c*) *F c* < *0* **and** *g*: *filterlim g at-top F*
shows *LIM x F. f x* ∗ *g x :> at-bot*
using *c filterlim-tendsto-pos-mult-at-top*[*of* λ*x.* − *f x* − *c F*, *OF - - g*]
unfolding *filterlim-uminus-at-bot tendsto-minus-cancel-left* **by** *simp*

lemma *filterlim-cmult-at-bot-at-top*:
assumes *filterlim* (*h* :: *-* ⇒ *real*) *at-top F c* ≠ *0 G* = (*if c* > *0 then at-top else at-bot*)
shows *filterlim* (λ*x. c* ∗ *h x*) *G F*

using *assms filterlim-tendsto-pos-mult-at-top*[*OF tendsto-const*[*of c*], *of h F*]
filterlim-tendsto-neg-mult-at-bot[*OF tendsto-const*[*of c*], *of h F*] **by** *simp*

lemma *filterlim-pow-at-top*:
fixes *f* :: *'a* ⇒ *real*
assumes *0* < *n*
and *f*: *LIM x F. f x* :> *at-top*
shows *LIM x F.* (*f x*)^*n* :: *real* :> *at-top*
using ‹*0* < *n*›
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*) **with** *f* **show** *?case*
by (*cases n* = *0*) (*auto intro*!: *filterlim-at-top-mult-at-top*)
qed

lemma *filterlim-pow-at-bot-even*:
fixes *f* :: *real* ⇒ *real*
shows *0* < *n* ⟹ *LIM x F. f x* :> *at-bot* ⟹ *even n* ⟹ *LIM x F.* (*f x*)^*n* :> *at-top*
using *filterlim-pow-at-top*[*of n* λ*x.* − *f x F*] **by** (*simp add*: *filterlim-uminus-at-top*)

lemma *filterlim-pow-at-bot-odd*:
fixes *f* :: *real* ⇒ *real*
shows *0* < *n* ⟹ *LIM x F. f x* :> *at-bot* ⟹ *odd n* ⟹ *LIM x F.* (*f x*)^*n* :> *at-bot*
using *filterlim-pow-at-top*[*of n* λ*x.* − *f x F*] **by** (*simp add*: *filterlim-uminus-at-bot*)

lemma *filterlim-power-at-infinity* [*tendsto-intros*]:
fixes *F* **and** *f* :: *'a* ⇒ *'b* :: *real-normed-div-algebra*
assumes *filterlim f at-infinity F n* > *0*
shows *filterlim* (λ*x. f x* ^ *n*) *at-infinity F*
by (*rule filterlim-norm-at-top-imp-at-infinity*)
(*auto simp*: *norm-power intro*!: *filterlim-pow-at-top assms*
intro: *filterlim-at-infinity-imp-norm-at-top*)

lemma *filterlim-tendsto-add-at-top*:
assumes *f*: (*f* ⟶ *c*) *F*
and *g*: *LIM x F. g x* :> *at-top*
shows *LIM x F.* (*f x* + *g x* :: *real*) :> *at-top*
unfolding *filterlim-at-top-gt*[**where** *c*=*0*]
proof *safe*
fix *Z* :: *real*
assume *0* < *Z*
from *f* **have** *eventually* (λ*x. c* − *1* < *f x*) *F*
by (*auto dest*!: *tendstoD*[**where** *e*=*1*] *elim*!: *eventually-mono simp*: *dist-real-def*)
moreover from *g* **have** *eventually* (λ*x. Z* − (*c* − *1*) ≤ *g x*) *F*
unfolding *filterlim-at-top* **by** *auto*

ultimately show *eventually* ($\lambda x.$ $Z \leq f\ x + g\ x$) F
  by *eventually-elim simp*
qed

lemma *LIM-at-top-divide*:
  fixes $f\ g$ :: $'a \Rightarrow real$
  assumes f: $(f \longrightarrow a)\ F\ 0 < a$
    and g: $(g \longrightarrow 0)\ F$ *eventually* ($\lambda x.$ $0 < g\ x$) F
  shows *LIM x F. f x / g x :> at-top*
  unfolding *divide-inverse*
  by (*rule filterlim-tendsto-pos-mult-at-top*[*OF f*]) (*rule filterlim-inverse-at-top*[*OF g*])

lemma *filterlim-at-top-add-at-top*:
  assumes f: *LIM x F. f x :> at-top*
    and g: *LIM x F. g x :> at-top*
  shows *LIM x F.* ($f\ x + g\ x$ :: *real*) :> *at-top*
  unfolding *filterlim-at-top-gt*[**where** $c=0$]
proof *safe*
  fix Z :: *real*
  assume $0 < Z$
  from f **have** *eventually* ($\lambda x.$ $0 \leq f\ x$) F
    unfolding *filterlim-at-top* **by** *auto*
  moreover from g **have** *eventually* ($\lambda x.$ $Z \leq g\ x$) F
    unfolding *filterlim-at-top* **by** *auto*
  ultimately show *eventually* ($\lambda x.$ $Z \leq f\ x + g\ x$) F
    by *eventually-elim simp*
qed

lemma *tendsto-divide-0*:
  fixes f :: - $\Rightarrow$ $'a$::{*real-normed-div-algebra*, *division-ring*}
  assumes f: $(f \longrightarrow c)\ F$
    and g: *LIM x F. g x :> at-infinity*
  shows (($\lambda x.$ $f\ x\ /\ g\ x$) $\longrightarrow 0$) F
  using *tendsto-mult*[*OF f filterlim-compose*[*OF tendsto-inverse-0 g*]]
  by (*simp add*: *divide-inverse*)

lemma *linear-plus-1-le-power*:
  fixes x :: *real*
  assumes x: $0 \leq x$
  shows *real* $n * x + 1 \leq (x + 1)$ ^ n
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  from x **have** *real* (*Suc n*) $* x + 1 \leq (x + 1) *$ (*real* $n * x + 1$)
    by (*simp add*: *field-simps*)
  also have ... $\leq (x + 1)$^*Suc n*

using *Suc x* **by** (*simp add*: *mult-left-mono*)
finally show *?case* **.**
qed

lemma *filterlim-realpow-sequentially-gt1*:
fixes *x* :: *'a* :: *real-normed-div-algebra*
assumes *x*[*arith*]: *1* < *norm x*
shows *LIM n sequentially. x ^ n :> at-infinity*
proof (*intro filterlim-at-infinity*[*THEN iffD2*] *allI impI*)
fix *y* :: *real*
assume *0* < *y*
obtain *N* :: *nat* **where** *y* < *real N* ∗ (*norm x* − *1*)
by (*meson diff-gt-0-iff-gt reals-Archimedean3 x*)
also have ... ≤ *real N* ∗ (*norm x* − *1*) + *1*
by *simp*
also have ... ≤ (*norm x* − *1* + *1*) ^ *N*
by (*rule linear-plus-1-le-power*) *simp*
also have ... = *norm x* ^ *N*
by *simp*
finally have ∀ *n*≥*N*. *y* ≤ *norm x* ^ *n*
by (*metis order-less-le-trans power-increasing order-less-imp-le x*)
then show *eventually* (λ*n*. *y* ≤ *norm* (*x* ^ *n*)) *sequentially*
unfolding *eventually-sequentially*
by (*auto simp*: *norm-power*)
qed *simp*

lemma *filterlim-divide-at-infinity*:
fixes *f g* :: *'a* ⇒ *'a* :: *real-normed-field*
assumes *filterlim f* (*nhds c*) *F filterlim g* (*at 0*) *F c* ≠ *0*
shows *filterlim* (λ*x*. *f x* / *g x*) *at-infinity F*
proof −
have *filterlim* (λ*x*. *f x* ∗ *inverse* (*g x*)) *at-infinity F*
by (*intro tendsto-mult-filterlim-at-infinity*[*OF assms*(*1*,*3*)]
filterlim-compose [*OF filterlim-inverse-at-infinity assms*(*2*)])
thus *?thesis* **by** (*simp add*: *field-simps*)
qed

108.5 Floor and Ceiling

lemma *eventually-floor-less*:
fixes *f* :: *'a* ⇒ *'b*::{*order-topology*,*floor-ceiling*}
assumes *f*: (*f* ⟶ *l*) *F*
and *l*: *l* ∉ ℤ
shows $\forall_F$ *x in F*. *of-int* (*floor l*) < *f x*
by (*intro order-tendstoD*[*OF f*]) (*metis Ints-of-int antisym-conv2 floor-correct l*)

lemma *eventually-less-ceiling*:
fixes *f* :: *'a* ⇒ *'b*::{*order-topology*,*floor-ceiling*}

assumes f: ($f \longrightarrow l$) F
and l: $l \notin \mathbb{Z}$
shows $\forall_F$ x *in* F. $f\,x <$ *of-int* (*ceiling* l)
by (*intro order-tendstoD*[*OF f*]) (*metis Ints-of-int l le-of-int-ceiling less-le*)

lemma *eventually-floor-eq*:
fixes f::'$a \Rightarrow$ 'b::{*order-topology,floor-ceiling*}
assumes f: ($f \longrightarrow l$) F
and l: $l \notin \mathbb{Z}$
shows $\forall_F$ x *in* F. *floor* ($f\,x$) = *floor* l
using *eventually-floor-less*[*OF assms*] *eventually-less-ceiling*[*OF assms*]
by *eventually-elim* (*meson floor-less-iff less-ceiling-iff not-less-iff-gr-or-eq*)

lemma *eventually-ceiling-eq*:
fixes f::'$a \Rightarrow$ 'b::{*order-topology,floor-ceiling*}
assumes f: ($f \longrightarrow l$) F
and l: $l \notin \mathbb{Z}$
shows $\forall_F$ x *in* F. *ceiling* ($f\,x$) = *ceiling* l
using *eventually-floor-less*[*OF assms*] *eventually-less-ceiling*[*OF assms*]
by *eventually-elim* (*meson floor-less-iff less-ceiling-iff not-less-iff-gr-or-eq*)

lemma *tendsto-of-int-floor*:
fixes f::'$a \Rightarrow$ 'b::{*order-topology,floor-ceiling*}
assumes ($f \longrightarrow l$) F
and $l \notin \mathbb{Z}$
shows ((λx. *of-int* (*floor* ($f\,x$)) :: 'c::{*ring-1*,*topological-space*}) $\longrightarrow$ *of-int* (*floor* l)) F
using *eventually-floor-eq*[*OF assms*]
by (*simp add*: *eventually-mono topological-tendstoI*)

lemma *tendsto-of-int-ceiling*:
fixes f::'$a \Rightarrow$ 'b::{*order-topology,floor-ceiling*}
assumes ($f \longrightarrow l$) F
and $l \notin \mathbb{Z}$
shows ((λx. *of-int* (*ceiling* (f x)):: 'c::{*ring-1*,*topological-space*}) $\longrightarrow$ *of-int* (*ceiling* l)) F
using *eventually-ceiling-eq*[*OF assms*]
by (*simp add*: *eventually-mono topological-tendstoI*)

lemma *continuous-on-of-int-floor*:
continuous-on (*UNIV* − $\mathbb{Z}$::'a::{*order-topology*, *floor-ceiling*} *set*)
(λx. *of-int* (*floor* x)::'b::{*ring-1*, *topological-space*})
unfolding *continuous-on-def*
by (*auto intro*!: *tendsto-of-int-floor*)

lemma *continuous-on-of-int-ceiling*:
continuous-on (*UNIV* − $\mathbb{Z}$::'a::{*order-topology*, *floor-ceiling*} *set*)
(λx. *of-int* (*ceiling* x)::'b::{*ring-1*, *topological-space*})
unfolding *continuous-on-def*

by (*auto intro*!: *tendsto-of-int-ceiling*)

108.6 Limits of Sequences

lemma [*trans*]: $X = Y \Longrightarrow Y \longrightarrow z \Longrightarrow X \longrightarrow z$
by *simp*

lemma *LIMSEQ-iff*:
fixes L :: *'a::real-normed-vector*
shows $(X \longrightarrow L) = (\forall r>0. \exists no. \forall n \geq no. norm (X n - L) < r)$
unfolding *lim-sequentially dist-norm* **..**

lemma *LIMSEQ-I*: $(\bigwedge r. 0 < r \Longrightarrow \exists no. \forall n\geq no. norm (X n - L) < r) \Longrightarrow X \longrightarrow L$
for L :: *'a::real-normed-vector*
by (*simp add*: *LIMSEQ-iff*)

lemma *LIMSEQ-D*: $X \longrightarrow L \Longrightarrow 0 < r \Longrightarrow \exists no. \forall n\geq no. norm (X n - L) < r$
for L :: *'a::real-normed-vector*
by (*simp add*: *LIMSEQ-iff*)

lemma *LIMSEQ-linear*: $X \longrightarrow x \Longrightarrow l > 0 \Longrightarrow (\lambda n. X (n * l)) \longrightarrow x$
unfolding *tendsto-def eventually-sequentially*
by (*metis div-le-dividend div-mult-self1-is-m le-trans mult.commute*)

Transformation of limit.

lemma *Lim-transform*: $(g \longrightarrow a) F \Longrightarrow ((\lambda x. f x - g x) \longrightarrow 0) F \Longrightarrow (f \longrightarrow a) F$
for $a\ b$ :: *'a::real-normed-vector*
using *tendsto-add* [*of g a F* $\lambda x. f x - g x$ *0*] **by** *simp*

lemma *Lim-transform2*: $(f \longrightarrow a) F \Longrightarrow ((\lambda x. f x - g x) \longrightarrow 0) F \Longrightarrow (g \longrightarrow a) F$
for $a\ b$ :: *'a::real-normed-vector*
by (*erule Lim-transform*) (*simp add*: *tendsto-minus-cancel*)

proposition *Lim-transform-eq*: $((\lambda x. f x - g x) \longrightarrow 0) F \Longrightarrow (f \longrightarrow a) F \longleftrightarrow (g \longrightarrow a) F$
for a :: *'a::real-normed-vector*
using *Lim-transform Lim-transform2* **by** *blast*

lemma *Lim-transform-eventually*:
$[\![(f \longrightarrow l) F; eventually (\lambda x. f x = g x) F]\!] \Longrightarrow (g \longrightarrow l) F$
using *eventually-elim2* **by** (*fastforce simp add*: *tendsto-def*)

lemma *Lim-transform-within*:
assumes $(f \longrightarrow l)$ (*at x within S*)
and $0 < d$

and $\bigwedge x'. x'{\in}S \Longrightarrow 0 < dist\ x'\ x \Longrightarrow dist\ x'\ x < d \Longrightarrow f\ x' = g\ x'$
shows $(g \longrightarrow l)$ (*at x within S*)
proof (*rule Lim-transform-eventually*)
show *eventually* $(\lambda x.\ f\ x = g\ x)$ (*at x within S*)
using *assms* **by** (*auto simp*: *eventually-at*)
show $(f \longrightarrow l)$ (*at x within S*)
by *fact*
qed

lemma *filterlim-transform-within*:
assumes *filterlim g G* (*at x within S*)
assumes $G \le F$ $0{<}d$ $(\bigwedge x'. x' \in S \Longrightarrow 0 < dist\ x'\ x \Longrightarrow dist\ x'\ x < d \Longrightarrow f\ x'$
$= g\ x')$
shows *filterlim f F* (*at x within S*)
using *assms*
apply (*elim filterlim-mono-eventually*)
unfolding *eventually-at* **by** *auto*

Common case assuming being away from some crucial point like 0.

lemma *Lim-transform-away-within*:
fixes *a b* :: *'a::t1-space*
assumes $a \neq b$
and $\forall x{\in}S.\ x \neq a \wedge x \neq b \longrightarrow f\ x = g\ x$
and $(f \longrightarrow l)$ (*at a within S*)
shows $(g \longrightarrow l)$ (*at a within S*)
proof (*rule Lim-transform-eventually*)
show $(f \longrightarrow l)$ (*at a within S*)
by *fact*
show *eventually* $(\lambda x.\ f\ x = g\ x)$ (*at a within S*)
unfolding *eventually-at-topological*
by (*rule exI* [**where** $x{=}-\ \{b\}$]) (*simp add*: *open-Compl assms*)
qed

lemma *Lim-transform-away-at*:
fixes *a b* :: *'a::t1-space*
assumes *ab*: $a \neq b$
and *fg*: $\forall x.\ x \neq a \wedge x \neq b \longrightarrow f\ x = g\ x$
and *fl*: $(f \longrightarrow l)$ (*at a*)
shows $(g \longrightarrow l)$ (*at a*)
using *Lim-transform-away-within*[*OF ab*, *of UNIV f g l*] *fg fl* **by** *simp*

Alternatively, within an open set.

lemma *Lim-transform-within-open*:
assumes $(f \longrightarrow l)$ (*at a within T*)
and *open s* **and** $a \in s$
and $\bigwedge x.\ x{\in}s \Longrightarrow x \neq a \Longrightarrow f\ x = g\ x$
shows $(g \longrightarrow l)$ (*at a within T*)
proof (*rule Lim-transform-eventually*)
show *eventually* $(\lambda x.\ f\ x = g\ x)$ (*at a within T*)

unfolding *eventually-at-topological*
using *assms* **by** *auto*
show $(f \longrightarrow l)$ (*at a within T*) **by** *fact*
qed

A congruence rule allowing us to transform limits assuming not at point.

lemma *Lim-cong-within*:
assumes $a = b$
and $x = y$
and $S = T$
and $\bigwedge x.\ x \neq b \Longrightarrow x \in T \Longrightarrow f\ x = g\ x$
shows $(f \longrightarrow x)$ (*at a within S*) $\longleftrightarrow$ $(g \longrightarrow y)$ (*at b within T*)
unfolding *tendsto-def eventually-at-topological*
using *assms* **by** *simp*

An unbounded sequence's inverse tends to 0.

lemma *LIMSEQ-inverse-zero*:
assumes $\bigwedge r$::*real*. $\exists N.\ \forall n \geq N.\ r < X\ n$
shows ($\lambda n.$ *inverse* $(X\ n)) \longrightarrow 0$
apply (*rule filterlim-compose*[*OF tendsto-inverse-0*])
by (*metis assms eventually-at-top-linorderI filterlim-at-top-dense filterlim-at-top-imp-at-infinity*)

The sequence *1* / *n* tends to 0 as *n* tends to infinity.

lemma *LIMSEQ-inverse-real-of-nat*: ($\lambda n.$ *inverse* (*real* (*Suc n*))) $\longrightarrow 0$
by (*metis filterlim-compose tendsto-inverse-0 filterlim-mono order-refl filterlim-Suc filterlim-compose*[*OF filterlim-real-sequentially*] *at-top-le-at-infinity*)

The sequence $r + 1 / n$ tends to r as n tends to infinity is now easily proved.

lemma *LIMSEQ-inverse-real-of-nat-add*: ($\lambda n.\ r +$ *inverse* (*real* (*Suc n*))) $\longrightarrow$ r
using *tendsto-add* [*OF tendsto-const LIMSEQ-inverse-real-of-nat*] **by** *auto*

lemma *LIMSEQ-inverse-real-of-nat-add-minus*: ($\lambda n.\ r + -$*inverse* (*real* (*Suc n*))) $\longrightarrow r$
using *tendsto-add* [*OF tendsto-const tendsto-minus* [*OF LIMSEQ-inverse-real-of-nat*]]
by *auto*

lemma *LIMSEQ-inverse-real-of-nat-add-minus-mult*: ($\lambda n.\ r * (1 + -$ *inverse* (*real* (*Suc n*)))) $\longrightarrow r$
using *tendsto-mult* [*OF tendsto-const LIMSEQ-inverse-real-of-nat-add-minus* [*of 1*]]
by *auto*

lemma *lim-inverse-n*: (($\lambda n.$ *inverse*(*of-nat n*)) $\longrightarrow$ (*0*::'*a*::*real-normed-field*)) *sequentially*
using *lim-1-over-n* **by** (*simp add*: *inverse-eq-divide*)

lemma *lim-inverse-n'*: (($\lambda n.\ 1 / n$) $\longrightarrow 0$) *sequentially*

using *lim-inverse-n*
by (*simp add*: *inverse-eq-divide*)

lemma *LIMSEQ-Suc-n-over-n*: (λ*n*. *of-nat* (*Suc n*) / *of-nat n* :: ′*a* :: *real-normed-field*) ⟶ *1*
proof (*rule Lim-transform-eventually*)
show *eventually* (λ*n*. *1* + *inverse* (*of-nat n* :: ′*a*) = *of-nat* (*Suc n*) / *of-nat n*) *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
by *eventually-elim* (*simp add*: *field-simps*)
have (λ*n*. *1* + *inverse* (*of-nat n*) :: ′*a*) ⟶ *1* + *0*
by (*intro tendsto-add tendsto-const lim-inverse-n*)
then show (λ*n*. *1* + *inverse* (*of-nat n*) :: ′*a*) ⟶ *1*
by *simp*
qed

lemma *LIMSEQ-n-over-Suc-n*: (λ*n*. *of-nat n* / *of-nat* (*Suc n*) :: ′*a* :: *real-normed-field*) ⟶ *1*
proof (*rule Lim-transform-eventually*)
show *eventually* (λ*n*. *inverse* (*of-nat* (*Suc n*) / *of-nat n* :: ′*a*) =
of-nat n / *of-nat* (*Suc n*)) *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
by *eventually-elim* (*simp add*: *field-simps del*: *of-nat-Suc*)
have (λ*n*. *inverse* (*of-nat* (*Suc n*) / *of-nat n* :: ′*a*)) ⟶ *inverse 1*
by (*intro tendsto-inverse LIMSEQ-Suc-n-over-n*) *simp-all*
then show (λ*n*. *inverse* (*of-nat* (*Suc n*) / *of-nat n* :: ′*a*)) ⟶ *1*
by *simp*
qed

108.7 Convergence on sequences

lemma *convergent-cong*:
assumes *eventually* (λ*x*. *f x* = *g x*) *sequentially*
shows *convergent f* ⟷ *convergent g*
unfolding *convergent-def*
by (*subst filterlim-cong*[*OF refl refl assms*]) (*rule refl*)

lemma *convergent-Suc-iff*: *convergent* (λ*n*. *f* (*Suc n*)) ⟷ *convergent f*
by (*auto simp*: *convergent-def filterlim-sequentially-Suc*)

lemma *convergent-ignore-initial-segment*: *convergent* (λ*n*. *f* (*n* + *m*)) = *convergent f*
proof (*induct m arbitrary*: *f*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc m*)
have *convergent* (λ*n*. *f* (*n* + *Suc m*)) ⟷ *convergent* (λ*n*. *f* (*Suc n* + *m*))
by *simp*

also have ... $\longleftrightarrow$ *convergent* ($\lambda n.\ f\ (n + m)$)
by (*rule convergent-Suc-iff*)
also have ... $\longleftrightarrow$ *convergent f*
by (*rule Suc*)
finally show *?case* **.**
qed

lemma *convergent-add*:
fixes $X\ Y$ :: *nat* $\Rightarrow$ *'a::topological-monoid-add*
assumes *convergent* ($\lambda n.\ X\ n$)
and *convergent* ($\lambda n.\ Y\ n$)
shows *convergent* ($\lambda n.\ X\ n + Y\ n$)
using *assms* **unfolding** *convergent-def* **by** (*blast intro*: *tendsto-add*)

lemma *convergent-sum*:
fixes X :: *'a* $\Rightarrow$ *nat* $\Rightarrow$ *'b::topological-comm-monoid-add*
shows ($\bigwedge i.\ i \in A \Longrightarrow$ *convergent* ($\lambda n.\ X\ i\ n$)) $\Longrightarrow$ *convergent* ($\lambda n.\ \sum i{\in}A.\ X\ i\ n$)
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *convergent-const convergent-add*)

lemma (**in** *bounded-linear*) *convergent*:
assumes *convergent* ($\lambda n.\ X\ n$)
shows *convergent* ($\lambda n.\ f\ (X\ n)$)
using *assms* **unfolding** *convergent-def* **by** (*blast intro*: *tendsto*)

lemma (**in** *bounded-bilinear*) *convergent*:
assumes *convergent* ($\lambda n.\ X\ n$)
and *convergent* ($\lambda n.\ Y\ n$)
shows *convergent* ($\lambda n.\ X\ n ** Y\ n$)
using *assms* **unfolding** *convergent-def* **by** (*blast intro*: *tendsto*)

lemma *convergent-minus-iff*:
fixes X :: *nat* $\Rightarrow$ *'a::topological-group-add*
shows *convergent X* $\longleftrightarrow$ *convergent* ($\lambda n.\ -\ X\ n$)
unfolding *convergent-def* **by** (*force dest*: *tendsto-minus*)

lemma *convergent-diff*:
fixes $X\ Y$ :: *nat* $\Rightarrow$ *'a::topological-group-add*
assumes *convergent* ($\lambda n.\ X\ n$)
assumes *convergent* ($\lambda n.\ Y\ n$)
shows *convergent* ($\lambda n.\ X\ n - Y\ n$)
using *assms* **unfolding** *convergent-def* **by** (*blast intro*: *tendsto-diff*)

lemma *convergent-norm*:
assumes *convergent f*
shows *convergent* ($\lambda n.$ *norm* ($f\ n$))
proof –
from *assms* **have** $f \longrightarrow$ *lim f*

by (*simp add*: *convergent-LIMSEQ-iff*)
then have ($\lambda n.$ *norm* (*f n*)) $\longlongrightarrow$ *norm* (*lim f*)
by (*rule tendsto-norm*)
then show *?thesis*
by (*auto simp*: *convergent-def*)
qed

lemma *convergent-of-real*:
convergent f $\Longrightarrow$ *convergent* ($\lambda n.$ *of-real* (*f n*) :: *'a::real-normed-algebra-1*)
unfolding *convergent-def* **by** (*blast intro*!: *tendsto-of-real*)

lemma *convergent-add-const-iff*:
convergent ($\lambda n.$ *c* + *f n* :: *'a::topological-ab-group-add*) $\longleftrightarrow$ *convergent f*
proof
assume *convergent* ($\lambda n.$ *c* + *f n*)
from *convergent-diff*[*OF this convergent-const*[*of c*]] **show** *convergent f*
by *simp*
next
assume *convergent f*
from *convergent-add*[*OF convergent-const*[*of c*] *this*] **show** *convergent* ($\lambda n.$ *c* + *f n*)
by *simp*
qed

lemma *convergent-add-const-right-iff*:
convergent ($\lambda n.$ *f n* + *c* :: *'a::topological-ab-group-add*) $\longleftrightarrow$ *convergent f*
using *convergent-add-const-iff*[*of c f*] **by** (*simp add*: *add-ac*)

lemma *convergent-diff-const-right-iff*:
convergent ($\lambda n.$ *f n* − *c* :: *'a::topological-ab-group-add*) $\longleftrightarrow$ *convergent f*
using *convergent-add-const-right-iff*[*of f* −*c*] **by** (*simp add*: *add-ac*)

lemma *convergent-mult*:
fixes *X Y* :: *nat* ⇒ *'a::topological-semigroup-mult*
assumes *convergent* ($\lambda n.$ *X n*)
and *convergent* ($\lambda n.$ *Y n*)
shows *convergent* ($\lambda n.$ *X n* ∗ *Y n*)
using *assms* **unfolding** *convergent-def* **by** (*blast intro*: *tendsto-mult*)

lemma *convergent-mult-const-iff*:
assumes $c \neq 0$
shows *convergent* ($\lambda n.$ *c* ∗ *f n* :: *'a::{field,topological-semigroup-mult}*) $\longleftrightarrow$ *convergent f*
proof
assume *convergent* ($\lambda n.$ *c* ∗ *f n*)
from *assms convergent-mult*[*OF this convergent-const*[*of inverse c*]]
show *convergent f* **by** (*simp add*: *field-simps*)
next
assume *convergent f*

from *convergent-mult*[*OF convergent-const*[*of c*] *this*] **show** *convergent* ($\lambda n.$ *c* ∗ *f n*)
by *simp*
qed

lemma *convergent-mult-const-right-iff*:
fixes *c* :: ′*a*::{*field*,*topological-semigroup-mult*}
assumes $c \neq 0$
shows *convergent* ($\lambda n.$ *f n* ∗ *c*) ⟷ *convergent f*
using *convergent-mult-const-iff*[*OF assms*, *of f*] **by** (*simp add*: *mult-ac*)

lemma *convergent-imp-Bseq*: *convergent f* ⟹ *Bseq f*
by (*simp add*: *Cauchy-Bseq convergent-Cauchy*)

A monotone sequence converges to its least upper bound.

lemma *LIMSEQ-incseq-SUP*:
fixes *X* :: *nat* ⇒ ′*a*::{*conditionally-complete-linorder*,*linorder-topology*}
assumes *u*: *bdd-above* (*range X*)
and *X*: *incseq X*
shows *X* ⟶ (*SUP i. X i*)
by (*rule order-tendstoI*)
(*auto simp*: *eventually-sequentially u less-cSUP-iff*
intro: *X*[*THEN incseqD*] *less-le-trans cSUP-lessD*[*OF u*])

lemma *LIMSEQ-decseq-INF*:
fixes *X* :: *nat* ⇒ ′*a*::{*conditionally-complete-linorder*, *linorder-topology*}
assumes *u*: *bdd-below* (*range X*)
and *X*: *decseq X*
shows *X* ⟶ (*INF i. X i*)
by (*rule order-tendstoI*)
(*auto simp*: *eventually-sequentially u cINF-less-iff*
intro: *X*[*THEN decseqD*] *le-less-trans less-cINF-D*[*OF u*])

Main monotonicity theorem.

lemma *Bseq-monoseq-convergent*: *Bseq X* ⟹ *monoseq X* ⟹ *convergent X*
for *X* :: *nat* ⇒ *real*
by (*auto simp*: *monoseq-iff convergent-def intro*: *LIMSEQ-decseq-INF LIMSEQ-incseq-SUP*
dest: *Bseq-bdd-above Bseq-bdd-below*)

lemma *Bseq-mono-convergent*: *Bseq X* ⟹ (∀ *m n*. $m \leq n \longrightarrow X\ m \leq X\ n$) ⟹ *convergent X*
for *X* :: *nat* ⇒ *real*
by (*auto intro*!: *Bseq-monoseq-convergent incseq-imp-monoseq simp*: *incseq-def*)

lemma *monoseq-imp-convergent-iff-Bseq*: *monoseq f* ⟹ *convergent f* ⟷ *Bseq f*
for *f* :: *nat* ⇒ *real*
using *Bseq-monoseq-convergent*[*of f*] *convergent-imp-Bseq*[*of f*] **by** *blast*

lemma *Bseq-monoseq-convergent*′*-inc*:

fixes *f* :: *nat* ⇒ *real*
shows *Bseq* (λ*n*. *f* (*n* + *M*)) ⟹ (⋀*m n*. *M* ≤ *m* ⟹ *m* ≤ *n* ⟹ *f m* ≤ *f n*) ⟹ *convergent f*
by (*subst convergent-ignore-initial-segment* [*symmetric*, *of* - *M*])
(*auto intro*!: *Bseq-monoseq-convergent simp*: *monoseq-def*)

lemma *Bseq-monoseq-convergent′-dec*:
fixes *f* :: *nat* ⇒ *real*
shows *Bseq* (λ*n*. *f* (*n* + *M*)) ⟹ (⋀*m n*. *M* ≤ *m* ⟹ *m* ≤ *n* ⟹ *f m* ≥ *f n*) ⟹ *convergent f*
by (*subst convergent-ignore-initial-segment* [*symmetric*, *of* - *M*])
(*auto intro*!: *Bseq-monoseq-convergent simp*: *monoseq-def*)

lemma *Cauchy-iff*: *Cauchy X* ⟷ (∀ *e*>*0*. ∃ *M*. ∀ *m*≥*M*. ∀ *n*≥*M*. *norm* (*X m* − *X n*) < *e*)
for *X* :: *nat* ⇒ ′*a*::*real-normed-vector*
unfolding *Cauchy-def dist-norm* **..**

lemma *CauchyI*: (⋀*e*. *0* < *e* ⟹ ∃ *M*. ∀ *m*≥*M*. ∀ *n*≥*M*. *norm* (*X m* − *X n*) < *e*) ⟹ *Cauchy X*
for *X* :: *nat* ⇒ ′*a*::*real-normed-vector*
by (*simp add*: *Cauchy-iff*)

lemma *CauchyD*: *Cauchy X* ⟹ *0* < *e* ⟹ ∃ *M*. ∀ *m*≥*M*. ∀ *n*≥*M*. *norm* (*X m* − *X n*) < *e*
for *X* :: *nat* ⇒ ′*a*::*real-normed-vector*
by (*simp add*: *Cauchy-iff*)

lemma *incseq-convergent*:
fixes *X* :: *nat* ⇒ *real*
assumes *incseq X*
and ∀ *i*. *X i* ≤ *B*
obtains *L* **where** *X* ⟶ *L* ∀ *i*. *X i* ≤ *L*
proof *atomize-elim*
from *incseq-bounded*[*OF assms*] ‹*incseq X*› *Bseq-monoseq-convergent*[*of X*]
obtain *L* **where** *X* ⟶ *L*
by (*auto simp*: *convergent-def monoseq-def incseq-def*)
with ‹*incseq X*› **show** ∃ *L*. *X* ⟶ *L* ∧ (∀ *i*. *X i* ≤ *L*)
by (*auto intro*!: *exI*[*of* - *L*] *incseq-le*)
qed

lemma *decseq-convergent*:
fixes *X* :: *nat* ⇒ *real*
assumes *decseq X*
and ∀ *i*. *B* ≤ *X i*
obtains *L* **where** *X* ⟶ *L* ∀ *i*. *L* ≤ *X i*
proof *atomize-elim*
from *decseq-bounded*[*OF assms*] ‹*decseq X*› *Bseq-monoseq-convergent*[*of X*]
obtain *L* **where** *X* ⟶ *L*

by (*auto simp*: *convergent-def monoseq-def decseq-def*)
with ‹*decseq X*› **show** $\exists L.\ X \longrightarrow L \wedge (\forall i.\ L \leq X\ i)$
by (*auto intro*!: *exI*[*of - L*] *decseq-ge*)
qed

lemma *monoseq-convergent*:
fixes X :: *nat* ⇒ *real*
assumes *X*: *monoseq X* **and** *B*: $\bigwedge i.\ |X\ i| \leq B$
obtains *L* **where** $X \longrightarrow L$
using *X* **unfolding** *monoseq-iff*
proof
assume *incseq X*
show *thesis*
using *abs-le-D1* [*OF B*] *incseq-convergent* [*OF* ‹*incseq X*›] *that* **by** *meson*
next
assume *decseq X*
show *thesis*
using *decseq-convergent* [*OF* ‹*decseq X*›] *that*
by (*metis B abs-le-iff add.inverse-inverse neg-le-iff-le*)
qed

108.8 More about *filterlim* (thanks to Wenda Li)

lemma *filterlim-at-infinity-times*:
fixes *f* :: *'a* ⇒ *'b*::*real-normed-field*
assumes *filterlim f at-infinity F filterlim g at-infinity F*
shows *filterlim* (λ*x*. *f x* ∗ *g x*) *at-infinity F*
proof −
have ((λ*x*. *inverse* (*f x*) ∗ *inverse* (*g x*)) ⟶ *0* ∗ *0*) *F*
by (*intro tendsto-mult tendsto-inverse assms filterlim-compose*[*OF tendsto-inverse-0*])
then have *filterlim* (λ*x*. *inverse* (*f x*) ∗ *inverse* (*g x*)) (*at 0*) *F*
unfolding *filterlim-at* **using** *assms*
by (*auto intro*: *filterlim-at-infinity-imp-eventually-ne tendsto-imp-eventually-ne eventually-conj*)
then show *?thesis*
by (*subst filterlim-inverse-at-iff*[*symmetric*]) *simp-all*
qed

lemma *filterlim-at-top-at-bot*[*elim*]:
fixes *f*::*'a* ⇒ *'b*::*unbounded-dense-linorder* **and** *F*::*'a filter*
assumes *top*:*filterlim f at-top F* **and** *bot*: *filterlim f at-bot F* **and** *F*≠*bot*
shows *False*
proof −
obtain *c*::*'b* **where** *True* **by** *auto*
have $\forall_F$ *x in F*. *c* < *f x*
using *top* **unfolding** *filterlim-at-top-dense* **by** *auto*
moreover have $\forall_F$ *x in F*. *f x* < *c*
using *bot* **unfolding** *filterlim-at-bot-dense* **by** *auto*
ultimately have $\forall_F$ *x in F*. *c* < *f x* ∧ *f x* < *c*

using *eventually-conj* **by** *auto*
then have $\forall_F$ *x in F. False* **by** (*auto elim*:*eventually-mono*)
then show *False* **using** ‹*F*≠*bot*› **by** *auto*
qed

lemma *filterlim-at-top-nhds*[*elim*]:
fixes *f*::'*a* ⇒ '*b*::{*unbounded-dense-linorder*,*order-topology*} **and** *F*::'*a filter*
assumes *top*:*filterlim f at-top F* **and** *tendsto*: (*f* ⟶ *c*) *F* **and** *F*≠*bot*
shows *False*
proof −
obtain *c*'::'*b* **where** *c*'>*c* **using** *gt-ex* **by** *blast*
have $\forall_F$ *x in F. c*' < *f x*
using *top* **unfolding** *filterlim-at-top-dense* **by** *auto*
moreover have $\forall_F$ *x in F. f x* < *c*'
using *order-tendstoD*[*OF tendsto*,*of c*'] ‹*c*'>*c*› **by** *auto*
ultimately have $\forall_F$ *x in F. c*' < *f x* ∧ *f x* < *c*'
using *eventually-conj* **by** *auto*
then have $\forall_F$ *x in F. False* **by** (*auto elim*:*eventually-mono*)
then show *False* **using** ‹*F*≠*bot*› **by** *auto*
qed

lemma *filterlim-at-bot-nhds*[*elim*]:
fixes *f*::'*a* ⇒ '*b*::{*unbounded-dense-linorder*,*order-topology*} **and** *F*::'*a filter*
assumes *top*:*filterlim f at-bot F* **and** *tendsto*: (*f* ⟶ *c*) *F* **and** *F*≠*bot*
shows *False*
proof −
obtain *c*'::'*b* **where** *c*'<*c* **using** *lt-ex* **by** *blast*
have $\forall_F$ *x in F. c*' > *f x*
using *top* **unfolding** *filterlim-at-bot-dense* **by** *auto*
moreover have $\forall_F$ *x in F. f x* > *c*'
using *order-tendstoD*[*OF tendsto*,*of c*'] ‹*c*'<*c*› **by** *auto*
ultimately have $\forall_F$ *x in F. c*' < *f x* ∧ *f x* < *c*'
using *eventually-conj* **by** *auto*
then have $\forall_F$ *x in F. False* **by** (*auto elim*:*eventually-mono*)
then show *False* **using** ‹*F*≠*bot*› **by** *auto*
qed

lemma *eventually-times-inverse-1*:
fixes *f*::'*a* ⇒ '*b*::{*field*,*t2-space*}
assumes (*f* ⟶ *c*) *F c*≠*0*
shows $\forall_F$ *x in F. inverse* (*f x*) ∗ *f x* = *1*
by (*smt* (*verit*) *assms eventually-mono mult.commute right-inverse tendsto-imp-eventually-ne*)

lemma *filterlim-at-infinity-divide-iff*:
fixes *f*::'*a* ⇒ '*b*::*real-normed-field*
assumes (*f* ⟶ *c*) *F c*≠*0*
shows (*LIM x F. f x* / *g x* :> *at-infinity*) ⟷ (*LIM x F. g x* :> *at 0*)
proof
assume *LIM x F. f x* / *g x* :> *at-infinity*

then have *LIM x F. inverse (f x) ∗ (f x / g x) :> at-infinity*
using *assms tendsto-inverse tendsto-mult-filterlim-at-infinity* **by** *fastforce*
then have *LIM x F. inverse (g x) :> at-infinity*
apply (*elim filterlim-mono-eventually*)
using *eventually-times-inverse-1*[*OF assms*]
by (*auto elim:eventually-mono simp add:field-simps*)
then show *filterlim g (at 0) F* **using** *filterlim-inverse-at-iff*[*symmetric*] **by** *force*

next
assume *filterlim g (at 0) F*
then have *filterlim (λx. inverse (g x)) at-infinity F*
using *filterlim-compose filterlim-inverse-at-infinity* **by** *blast*
then have *LIM x F. f x ∗ inverse (g x) :> at-infinity*
using *tendsto-mult-filterlim-at-infinity*[*OF assms, of λx. inverse(g x)*]
by *simp*
then show *LIM x F. f x / g x :> at-infinity* **by** (*simp add: divide-inverse*)
qed

lemma *filterlim-tendsto-pos-mult-at-top-iff*:
fixes *f::'a ⇒ real*
assumes (*f ⟶ c*) *F* **and** *0 < c*
shows (*LIM x F. (f x ∗ g x) :> at-top*) ⟷ (*LIM x F. g x :> at-top*)
proof
assume *filterlim g at-top F*
then show *LIM x F. f x ∗ g x :> at-top*
using *filterlim-tendsto-pos-mult-at-top*[*OF assms*] **by** *auto*
next
assume *asm:LIM x F. f x ∗ g x :> at-top*
have ((*λx. inverse (f x)*) ⟶ *inverse c*) *F*
using *tendsto-inverse*[*OF assms(1)*] ‹*0<c*› **by** *auto*
moreover have *inverse c >0* **using** *assms(2)* **by** *auto*
ultimately have *LIM x F. inverse (f x) ∗ (f x ∗ g x) :> at-top*
using *filterlim-tendsto-pos-mult-at-top*[*OF - - asm,of λx. inverse (f x) inverse c*] **by** *auto*
then show *LIM x F. g x :> at-top*
apply (*elim filterlim-mono-eventually*)
apply *simp-all*[*2*]
using *eventually-times-inverse-1*[*OF assms(1)*] ‹*c>0*› *eventually-mono* **by** *fastforce*
qed

lemma *filterlim-tendsto-pos-mult-at-bot-iff*:
fixes *c :: real*
assumes (*f ⟶ c*) *F 0 < c*
shows (*LIM x F. f x ∗ g x :> at-bot*) ⟷ *filterlim g at-bot F*
using *filterlim-tendsto-pos-mult-at-top-iff*[*OF assms(1,2), of λx. − g x*]
unfolding *filterlim-uminus-at-bot* **by** *simp*

lemma *filterlim-tendsto-neg-mult-at-top-iff*:

fixes $f::'a \Rightarrow real$
assumes $(f \longrightarrow c)$ F **and** $c < 0$
shows $(LIM\ x\ F.\ (f\ x * g\ x) :> at\text{-}top) \longleftrightarrow (LIM\ x\ F.\ g\ x :> at\text{-}bot)$
proof −
have $(LIM\ x\ F.\ f\ x * g\ x :> at\text{-}top) = (LIM\ x\ F.\ -\ g\ x :> at\text{-}top)$
apply (*rule filterlim-tendsto-pos-mult-at-top-iff*[*of* $\lambda x.\ -\ f\ x\ -c\ F\ \lambda x.\ -\ g\ x$, *simplified*])
using *assms* **by** (*auto intro*: *tendsto-intros*)
also have ... = $(LIM\ x\ F.\ g\ x :> at\text{-}bot)$
using *filterlim-uminus-at-bot*[*symmetric*] **by** *auto*
finally show *?thesis* **.**
qed

lemma *filterlim-tendsto-neg-mult-at-bot-iff*:
fixes c :: *real*
assumes $(f \longrightarrow c)$ F $0 > c$
shows $(LIM\ x\ F.\ f\ x * g\ x :> at\text{-}bot) \longleftrightarrow$ *filterlim g at-top F*
using *filterlim-tendsto-neg-mult-at-top-iff*[*OF assms*(*1*,*2*), *of* $\lambda x.\ -\ g\ x$]
unfolding *filterlim-uminus-at-top* **by** *simp*

108.9 Power Sequences

lemma *Bseq-realpow*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow Bseq\ (\lambda n.\ x\ \widehat{}\ n)$
for x :: *real*
by (*metis decseq-bounded decseq-def power-decreasing zero-le-power*)

lemma *monoseq-realpow*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow monoseq\ (\lambda n.\ x\ \widehat{}\ n)$
for x :: *real*
using *monoseq-def power-decreasing* **by** *blast*

lemma *convergent-realpow*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow convergent\ (\lambda n.\ x\ \widehat{}\ n)$
for x :: *real*
by (*blast intro*!: *Bseq-monoseq-convergent Bseq-realpow monoseq-realpow*)

lemma *LIMSEQ-inverse-realpow-zero*: $1 < x \Longrightarrow (\lambda n.\ inverse\ (x\ \widehat{}\ n)) \longrightarrow 0$
for x :: *real*
by (*rule filterlim-compose*[*OF tendsto-inverse-0 filterlim-realpow-sequentially-gt1*]) *simp*

lemma *LIMSEQ-realpow-zero*:
fixes x :: *real*
assumes $0 \leq x$ $x < 1$
shows $(\lambda n.\ x\ \widehat{}\ n) \longrightarrow 0$
proof (*cases* $x = 0$)
case *False*
with ‹$0 \leq x$› **have** $1 < inverse\ x$
using ‹$x < 1$› **by** (*simp add*: *one-less-inverse*)
then have $(\lambda n.\ inverse\ (inverse\ x\ \widehat{}\ n)) \longrightarrow 0$
by (*rule LIMSEQ-inverse-realpow-zero*)

then show *?thesis* **by** (*simp add*: *power-inverse*)
next
case *True*
show *?thesis*
by (*rule LIMSEQ-imp-Suc*) (*simp add*: *True*)
qed

lemma *LIMSEQ-power-zero* [*tendsto-intros*]: *norm* $x < 1 \Longrightarrow (\lambda n.\ x \,\hat{}\, n) \longrightarrow 0$
for *x* :: *'a*::*real-normed-algebra-1*
apply (*drule LIMSEQ-realpow-zero* [*OF norm-ge-zero*])
by (*simp add*: *Zfun-le norm-power-ineq tendsto-Zfun-iff*)

lemma *LIMSEQ-divide-realpow-zero*: $1 < x \Longrightarrow (\lambda n.\ a\ /\ (x \,\hat{}\, n) :: real) \longrightarrow 0$
by (*rule tendsto-divide-0* [*OF tendsto-const filterlim-realpow-sequentially-gt1*]) *simp*

lemma
tendsto-power-zero:
fixes *x*::*'a*::*real-normed-algebra-1*
assumes *filterlim f at-top F*
assumes *norm* $x < 1$
shows $((\lambda y.\ x \,\hat{}\, (f\ y)) \longrightarrow 0)\ F$
proof (*rule tendstoI*)
fix *e*::*real* **assume** $0 < e$
from *tendstoD*[*OF LIMSEQ-power-zero*[*OF* ‹*norm* $x < 1$›] ‹$0 < e$›]
have $\forall_F$ *xa in sequentially*. *norm* $(x \,\hat{}\, xa) < e$
by *simp*
then obtain *N* **where** *N*: *norm* $(x \,\hat{}\, n) < e$ **if** $n \geq N$ **for** *n*
by (*auto simp*: *eventually-sequentially*)
have $\forall_F$ *i in F*. $f\ i \geq N$
using ‹*filterlim f sequentially F*›
by (*simp add*: *filterlim-at-top*)
then show $\forall_F$ *i in F*. *dist* $(x \,\hat{}\, f\ i)\ 0 < e$
by *eventually-elim* (*auto simp*: *N*)
qed

Limit of c^n for $|c| < 1$.

lemma *LIMSEQ-abs-realpow-zero*: $|c| < 1 \Longrightarrow (\lambda n.\ |c| \,\hat{}\, n :: real) \longrightarrow 0$
by (*rule LIMSEQ-realpow-zero* [*OF abs-ge-zero*])

lemma *LIMSEQ-abs-realpow-zero2*: $|c| < 1 \Longrightarrow (\lambda n.\ c \,\hat{}\, n :: real) \longrightarrow 0$
by (*rule LIMSEQ-power-zero*) *simp*

108.10 Limits of Functions

lemma *LIM-eq*: $f\ {-a}\!\!\rightarrow L = (\forall r{>}0.\ \exists s{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (f\ x - L) < r)$
for *a* :: *'a*::*real-normed-vector* **and** *L* :: *'b*::*real-normed-vector*

by (*simp add*: *LIM-def dist-norm*)

lemma *LIM-I*:
($\bigwedge r.\ 0 < r \Longrightarrow \exists s{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (f\, x - L) < r) \Longrightarrow f\ {-a\to}\ L$
for a :: $'a$::*real-normed-vector* **and** L :: $'b$::*real-normed-vector*
by (*simp add*: *LIM-eq*)

lemma *LIM-D*: $f\ {-a\to}\ L \Longrightarrow 0 < r \Longrightarrow \exists s{>}0. \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (f\, x - L) < r$
for a :: $'a$::*real-normed-vector* **and** L :: $'b$::*real-normed-vector*
by (*simp add*: *LIM-eq*)

lemma *LIM-offset*: $f\ {-a\to}\ L \Longrightarrow (\lambda x.\ f\ (x + k))\ {-(a - k)\to}\ L$
for a :: $'a$::*real-normed-vector*
by (*simp add*: *filtermap-at-shift*[*symmetric*, *of a k*] *filterlim-def filtermap-filtermap*)

lemma *LIM-offset-zero*: $f\ {-a\to}\ L \Longrightarrow (\lambda h.\ f\ (a + h))\ {-0\to}\ L$
for a :: $'a$::*real-normed-vector*
by (*drule LIM-offset* [**where** $k = a$]) (*simp add*: *add.commute*)

lemma *LIM-offset-zero-cancel*: $(\lambda h.\ f\ (a + h))\ {-0\to}\ L \Longrightarrow f\ {-a\to}\ L$
for a :: $'a$::*real-normed-vector*
by (*drule LIM-offset* [**where** $k = -\ a$]) *simp*

lemma *LIM-offset-zero-iff*: *NO-MATCH 0 a* $\Longrightarrow f\ {-a\to}\ L \longleftrightarrow (\lambda h.\ f\ (a + h))\ {-0\to}\ L$
for f :: $'a$ :: *real-normed-vector* $\Rightarrow$ -
using *LIM-offset-zero-cancel*[*of f a L*] *LIM-offset-zero*[*of f L a*] **by** *auto*

lemma *tendsto-offset-zero-iff*:
fixes f :: $'a$ :: *real-normed-vector* $\Rightarrow$ -
assumes *NO-MATCH 0 a* $a \in S$ *open S*
shows $(f \longrightarrow L)$ (*at a within S*) $\longleftrightarrow ((\lambda h.\ f\ (a + h)) \longrightarrow L)$ (*at 0*)
using *assms* **by** (*simp add*: *tendsto-within-open-NO-MATCH LIM-offset-zero-iff*)

lemma *LIM-zero*: $(f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ f\, x - l) \longrightarrow 0)\ F$
for f :: $'a \Rightarrow 'b$::*real-normed-vector*
unfolding *tendsto-iff dist-norm* **by** *simp*

lemma *LIM-zero-cancel*:
fixes f :: $'a \Rightarrow 'b$::*real-normed-vector*
shows $((\lambda x.\ f\, x - l) \longrightarrow 0)\ F \Longrightarrow (f \longrightarrow l)\ F$
unfolding *tendsto-iff dist-norm* **by** *simp*

lemma *LIM-zero-iff*: $((\lambda x.\ f\, x - l) \longrightarrow 0)\ F = (f \longrightarrow l)\ F$
for f :: $'a \Rightarrow 'b$::*real-normed-vector*
unfolding *tendsto-iff dist-norm* **by** *simp*

lemma *LIM-imp-LIM*:
fixes f :: $'a$::*topological-space* ⇒ $'b$::*real-normed-vector*
fixes g :: $'a$::*topological-space* ⇒ $'c$::*real-normed-vector*
assumes f: f −a→ l
and *le*: $\bigwedge x.\ x \neq a \Longrightarrow norm\ (g\ x - m) \leq norm\ (f\ x - l)$
shows g −a→ m
by (*rule metric-LIM-imp-LIM* [*OF f*]) (*simp add*: *dist-norm le*)

lemma *LIM-equal2*:
fixes $f\ g$ :: $'a$::*real-normed-vector* ⇒ $'b$::*topological-space*
assumes $0 < R$
and $\bigwedge x.\ x \neq a \Longrightarrow norm\ (x - a) < R \Longrightarrow f\ x = g\ x$
shows g −a→ l ⟹ f −a→ l
by (*rule metric-LIM-equal2* [*OF - assms*]) (*simp-all add*: *dist-norm*)

lemma *LIM-compose2*:
fixes a :: $'a$::*real-normed-vector*
assumes f: f −a→ b
and g: g −b→ c
and *inj*: $\exists\, d>0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < d \longrightarrow f\ x \neq b$
shows $(\lambda x.\ g\ (f\ x))$ −a→ c
by (*rule metric-LIM-compose2* [*OF f g inj* [*folded dist-norm*]])

lemma *real-LIM-sandwich-zero*:
fixes $f\ g$ :: $'a$::*topological-space* ⇒ *real*
assumes f: f −a→ 0
and *1*: $\bigwedge x.\ x \neq a \Longrightarrow 0 \leq g\ x$
and *2*: $\bigwedge x.\ x \neq a \Longrightarrow g\ x \leq f\ x$
shows g −a→ 0
proof (*rule LIM-imp-LIM* [*OF f*])
fix x
assume x: $x \neq a$
with *1* **have** $norm\ (g\ x - 0) = g\ x$ **by** *simp*
also have $g\ x \leq f\ x$ **by** (*rule 2* [*OF x*])
also have $f\ x \leq |f\ x|$ **by** (*rule abs-ge-self*)
also have $|f\ x| = norm\ (f\ x - 0)$ **by** *simp*
finally show $norm\ (g\ x - 0) \leq norm\ (f\ x - 0)$ **.**
qed

108.11 Continuity

lemma *LIM-isCont-iff*: $(f$ −a→ $f\ a) = ((\lambda h.\ f\ (a + h))$ −0→ $f\ a)$
for f :: $'a$::*real-normed-vector* ⇒ $'b$::*topological-space*
by (*rule iffI* [*OF LIM-offset-zero LIM-offset-zero-cancel*])

lemma *isCont-iff*: *isCont* $f\ x = (\lambda h.\ f\ (x + h))$ −0→ $f\ x$
for f :: $'a$::*real-normed-vector* ⇒ $'b$::*topological-space*
by (*simp add*: *isCont-def LIM-isCont-iff*)

lemma *isCont-LIM-compose2*:
fixes *a* :: *'a::real-normed-vector*
assumes *f* [*unfolded isCont-def*]: *isCont f a*
and *g*: *g* −*f a*→ *l*
and *inj*: ∃ *d*>*0*. ∀ *x*. *x* ≠ *a* ∧ *norm* (*x* − *a*) < *d* ⟶ *f x* ≠ *f a*
shows (λ*x*. *g* (*f x*)) −*a*→ *l*
by (*rule LIM-compose2* [*OF f g inj*])

lemma *isCont-norm* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *norm* (*f x*)) *a*
for *f* :: *'a::t2-space* ⇒ *'b::real-normed-vector*
by (*fact continuous-norm*)

lemma *isCont-rabs* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. |*f x*|) *a*
for *f* :: *'a::t2-space* ⇒ *real*
by (*fact continuous-rabs*)

lemma *isCont-add* [*simp*]: *isCont f a* ⟹ *isCont g a* ⟹ *isCont* (λ*x*. *f x* + *g x*) *a*
for *f* :: *'a::t2-space* ⇒ *'b::topological-monoid-add*
by (*fact continuous-add*)

lemma *isCont-minus* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. − *f x*) *a*
for *f* :: *'a::t2-space* ⇒ *'b::real-normed-vector*
by (*fact continuous-minus*)

lemma *isCont-diff* [*simp*]: *isCont f a* ⟹ *isCont g a* ⟹ *isCont* (λ*x*. *f x* − *g x*) *a*
for *f* :: *'a::t2-space* ⇒ *'b::real-normed-vector*
by (*fact continuous-diff*)

lemma *isCont-mult* [*simp*]: *isCont f a* ⟹ *isCont g a* ⟹ *isCont* (λ*x*. *f x* ∗ *g x*) *a*
for *f g* :: *'a::t2-space* ⇒ *'b::real-normed-algebra*
by (*fact continuous-mult*)

lemma (**in** *bounded-linear*) *isCont*: *isCont g a* ⟹ *isCont* (λ*x*. *f* (*g x*)) *a*
by (*fact continuous*)

lemma (**in** *bounded-bilinear*) *isCont*: *isCont f a* ⟹ *isCont g a* ⟹ *isCont* (λ*x*. *f x* ∗∗ *g x*) *a*
by (*fact continuous*)

lemmas *isCont-scaleR* [*simp*] =
bounded-bilinear.isCont [*OF bounded-bilinear-scaleR*]

lemmas *isCont-of-real* [*simp*] =
bounded-linear.isCont [*OF bounded-linear-of-real*]

lemma *isCont-power* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *f x* ^ *n*) *a*
for *f* :: *'a::t2-space* ⇒ *'b::*{*power*,*real-normed-algebra*}
by (*fact continuous-power*)

lemma *isCont-sum* [*simp*]: ∀ *i*∈*A*. *isCont* (*f i*) *a* ⟹ *isCont* (λ*x*. ∑ *i*∈*A*. *f i x*) *a*
for *f* :: ′*a* ⇒ ′*b*::*t2-space* ⇒ ′*c*::*topological-comm-monoid-add*
by (*auto intro*: *continuous-sum*)

108.12 Uniform Continuity

lemma *uniformly-continuous-on-def*:
fixes *f* :: ′*a*::*metric-space* ⇒ ′*b*::*metric-space*
shows *uniformly-continuous-on s f* ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *x*∈*s*. ∀ *x*′∈*s*. *dist x*′ *x* < *d* ⟶ *dist* (*f x*′) (*f x*) < *e*)
unfolding *uniformly-continuous-on-uniformity*
uniformity-dist filterlim-INF filterlim-principal eventually-inf-principal
by (*force simp*: *Ball-def uniformity-dist*[*symmetric*] *eventually-uniformity-metric*)

abbreviation *isUCont* :: [′*a*::*metric-space* ⇒ ′*b*::*metric-space*] ⇒ *bool*
where *isUCont f* ≡ *uniformly-continuous-on UNIV f*

lemma *isUCont-def*: *isUCont f* ⟷ (∀ *r*>*0*. ∃ *s*>*0*. ∀ *x y*. *dist x y* < *s* ⟶ *dist* (*f x*) (*f y*) < *r*)
by (*auto simp*: *uniformly-continuous-on-def dist-commute*)

lemma *isUCont-isCont*: *isUCont f* ⟹ *isCont f x*
by (*drule uniformly-continuous-imp-continuous*) (*simp add*: *continuous-on-eq-continuous-at*)

lemma *uniformly-continuous-on-Cauchy*:
fixes *f* :: ′*a*::*metric-space* ⇒ ′*b*::*metric-space*
assumes *uniformly-continuous-on S f Cauchy X* ⋀*n*. *X n* ∈ *S*
shows *Cauchy* (λ*n*. *f* (*X n*))
using *assms*
unfolding *uniformly-continuous-on-def* **by** (*meson Cauchy-def*)

lemma *isUCont-Cauchy*: *isUCont f* ⟹ *Cauchy X* ⟹ *Cauchy* (λ*n*. *f* (*X n*))
by (*rule uniformly-continuous-on-Cauchy*[**where** *S*=*UNIV* **and** *f*=*f*]) *simp-all*

lemma (**in** *bounded-linear*) *isUCont*: *isUCont f*
unfolding *isUCont-def dist-norm*
proof (*intro allI impI*)
fix *r* :: *real*
assume *r*: *0* < *r*
obtain *K* **where** *K*: *0* < *K* **and** *norm-le*: *norm* (*f x*) ≤ *norm x* ∗ *K* **for** *x*
using *pos-bounded* **by** *blast*
show ∃ *s*>*0*. ∀ *x y*. *norm* (*x* − *y*) < *s* ⟶ *norm* (*f x* − *f y*) < *r*
proof (*rule exI*, *safe*)
from *r K* **show** *0* < *r* / *K* **by** *simp*
next
fix *x y* :: ′*a*
assume *xy*: *norm* (*x* − *y*) < *r* / *K*
have *norm* (*f x* − *f y*) = *norm* (*f* (*x* − *y*)) **by** (*simp only*: *diff*)
also have ... ≤ *norm* (*x* − *y*) ∗ *K* **by** (*rule norm-le*)

also from *K xy* **have** ... $<$ *r* **by** (*simp only*: *pos-less-divide-eq*)
finally show *norm* ($f\ x - f\ y$) $< r$ **.**
qed
qed

lemma (**in** *bounded-linear*) *Cauchy*: *Cauchy X* $\Longrightarrow$ *Cauchy* ($\lambda n.\ f\ (X\ n)$)
by (*rule isUCont* [*THEN isUCont-Cauchy*])

lemma *LIM-less-bound*:
fixes *f* :: *real* $\Rightarrow$ *real*
assumes *ev*: $b < x\ \forall\ x' \in \{\ b <..< x\}.\ 0 \leq f\ x'$ **and** *isCont f x*
shows $0 \leq f\ x$
proof (*rule tendsto-lowerbound*)
show ($f \longrightarrow f\ x$) (*at-left x*)
using ‹*isCont f x*› **by** (*simp add*: *filterlim-at-split isCont-def*)
show *eventually* ($\lambda x.\ 0 \leq f\ x$) (*at-left x*)
using *ev* **by** (*auto simp*: *eventually-at dist-real-def intro*!: *exI*[*of - x − b*])
qed *simp*

108.13 Nested Intervals and Bisection – Needed for Compactness

lemma *nested-sequence-unique*:
assumes $\forall n.\ f\ n \leq f\ (Suc\ n)$ $\forall n.\ g\ (Suc\ n) \leq g\ n$ $\forall n.\ f\ n \leq g\ n$ ($\lambda n.\ f\ n - g\ n$) $\longrightarrow 0$
shows $\exists\ l$::*real*. $((\forall n.\ f\ n \leq l) \wedge f \longrightarrow l) \wedge ((\forall n.\ l \leq g\ n) \wedge g \longrightarrow l)$
proof −
have *incseq f* **unfolding** *incseq-Suc-iff* **by** *fact*
have *decseq g* **unfolding** *decseq-Suc-iff* **by** *fact*
have $f\ n \leq g\ 0$ **for** *n*
proof −
from ‹*decseq g*› **have** $g\ n \leq g\ 0$
by (*rule decseqD*) *simp*
with ‹$\forall n.\ f\ n \leq g\ n$›[*THEN spec*, *of n*] **show** *?thesis*
by *auto*
qed
then obtain *u* **where** $f \longrightarrow u$ $\forall i.\ f\ i \leq u$
using *incseq-convergent*[*OF* ‹*incseq f*›] **by** *auto*
moreover have $f\ 0 \leq g\ n$ **for** *n*
proof −
from ‹*incseq f*› **have** $f\ 0 \leq f\ n$ **by** (*rule incseqD*) *simp*
with ‹$\forall n.\ f\ n \leq g\ n$›[*THEN spec*, *of n*] **show** *?thesis*
by *simp*
qed
then obtain *l* **where** $g \longrightarrow l$ $\forall i.\ l \leq g\ i$
using *decseq-convergent*[*OF* ‹*decseq g*›] **by** *auto*
moreover note *LIMSEQ-unique*[*OF assms*(*4*) *tendsto-diff*[*OF* ‹$f \longrightarrow u$› ‹$g \longrightarrow l$›]]
ultimately show *?thesis* **by** *auto*

qed

lemma *Bolzano*[*consumes 1*, *case-names trans local*]:
fixes *P* :: *real* ⇒ *real* ⇒ *bool*
assumes [*arith*]: $a \leq b$
and *trans*: $\bigwedge a\ b\ c.\ P\ a\ b \Longrightarrow P\ b\ c \Longrightarrow a \leq b \Longrightarrow b \leq c \Longrightarrow P\ a\ c$
and *local*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow \exists\, d{>}0.\ \forall\, a\ b.\ a \leq x \wedge x \leq b \wedge b - a < d \longrightarrow P\ a\ b$
shows *P a b*
proof −
define *bisect* **where** *bisect* ≡ λ(*x*,*y*). *if P x* ((*x*+*y*) / *2*) *then* ((*x*+*y*)/*2*, *y*) *else* (*x*, (*x*+*y*)/*2*)
define *l u* **where** *l n* ≡ *fst* ((*bisect*^^*n*)(*a*,*b*)) **and** *u n* ≡ *snd* ((*bisect*^^*n*)(*a*,*b*)) **for** *n*
have *l*[*simp*]: *l 0* = *a* ⋀*n*. *l* (*Suc n*) = (*if P* (*l n*) ((*l n* + *u n*) / *2*) *then* (*l n* + *u n*) / *2 else l n*)
and *u*[*simp*]: *u 0* = *b* ⋀*n*. *u* (*Suc n*) = (*if P* (*l n*) ((*l n* + *u n*) / *2*) *then u n else* (*l n* + *u n*) / *2*)
by (*simp-all add*: *l-def u-def bisect-def split*: *prod.split*)

have [*simp*]: *l n* ≤ *u n* **for** *n* **by** (*induct n*) *auto*

have ∃*x*. ((∀*n*. *l n* ≤ *x*) ∧ *l* ⟶ *x*) ∧ ((∀*n*. *x* ≤ *u n*) ∧ *u* ⟶ *x*)
proof (*safe intro*!: *nested-sequence-unique*)
show *l n* ≤ *l* (*Suc n*) *u* (*Suc n*) ≤ *u n* **for** *n*
by (*induct n*) *auto*
next
have *l n* − *u n* = (*a* − *b*) / *2*^*n* **for** *n*
by (*induct n*) (*auto simp*: *field-simps*)
then show (λ*n*. *l n* − *u n*) ⟶ *0*
by (*simp add*: *LIMSEQ-divide-realpow-zero*)
qed *fact*
then obtain *x* **where** *x*: ⋀*n*. *l n* ≤ *x* ⋀*n*. *x* ≤ *u n* **and** *l* ⟶ *x u* ⟶ *x*
by *auto*
obtain *d* **where** *0* < *d* **and** *d*: *a* ≤ *x* ⟹ *x* ≤ *b* ⟹ *b* − *a* < *d* ⟹ *P a b* **for** *a b*
using ‹*l 0* ≤ *x*› ‹*x* ≤ *u 0*› *local*[*of x*] **by** *auto*

show *P a b*
proof (*rule ccontr*)
assume ¬ *P a b*
have ¬ *P* (*l n*) (*u n*) **for** *n*
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: ‹¬ *P a b*›)
next
case (*Suc n*)
with *trans*[*of l n* (*l n* + *u n*) / *2 u n*] **show** *?case*

```
      by auto
    qed
    moreover
    {
      have eventually (λn. x − d / 2 < l n) sequentially
        using ‹0 < d› ‹l ⟶ x› by (intro order-tendstoD[of - x]) auto
      moreover have eventually (λn. u n < x + d / 2) sequentially
        using ‹0 < d› ‹u ⟶ x› by (intro order-tendstoD[of - x]) auto
      ultimately have eventually (λn. P (l n) (u n)) sequentially
      proof eventually-elim
        case (elim n)
        from add-strict-mono[OF this] have u n − l n < d by simp
        with x show P (l n) (u n) by (rule d)
      qed
    }
    ultimately show False by simp
  qed
qed

lemma compact-Icc[simp, intro]: compact {a .. b::real}
proof (cases a ≤ b, rule compactI)
  fix C
  assume C: a ≤ b ∀ t∈C. open t {a..b} ⊆ ⋃ C
  define T where T = {a .. b}
  from C(1,3) show ∃ C'⊆C. finite C' ∧ {a..b} ⊆ ⋃ C'
  proof (induct rule: Bolzano)
    case (trans a b c)
    then have ∗: {a..c} = {a..b} ∪ {b..c}
      by auto
    with trans obtain C1 C2
      where C1⊆C finite C1 {a..b} ⊆ ⋃ C1 C2⊆C finite C2 {b..c} ⊆ ⋃ C2
      by auto
    with trans show ?case
      unfolding ∗ by (intro exI[of - C1 ∪ C2]) auto
  next
    case (local x)
    with C have x ∈ ⋃ C by auto
    with C(2) obtain c where x ∈ c open c c ∈ C
      by auto
    then obtain e where 0 < e {x − e <..< x + e} ⊆ c
      by (auto simp: open-dist dist-real-def subset-eq Ball-def abs-less-iff)
    with ‹c ∈ C› show ?case
      by (safe intro!: exI[of - e/2] exI[of - {c}]) auto
  qed
qed simp


lemma continuous-image-closed-interval:
  fixes a b and f :: real ⇒ real
```

defines $S \equiv \{a..b\}$
assumes $a \leq b$ **and** f: *continuous-on S f*
shows $\exists\, c\ d.\ f`S = \{c..d\} \wedge c \leq d$
proof −
have S: *compact S* $S \neq \{\}$
using ‹$a \leq b$› **by** (*auto simp*: *S-def*)
obtain c **where** $c \in S\ \forall\, d{\in}S.\ f\ d \leq f\ c$
using *continuous-attains-sup*[*OF S f*] **by** *auto*
moreover obtain d **where** $d \in S\ \forall\, c{\in}S.\ f\ d \leq f\ c$
using *continuous-attains-inf*[*OF S f*] **by** *auto*
moreover have *connected* ($f`S$)
using *connected-continuous-image*[*OF f*] *connected-Icc* **by** (*auto simp*: *S-def*)
ultimately have $f\ `\ S = \{f\ d\ ..\ f\ c\} \wedge f\ d \leq f\ c$
by (*auto simp*: *connected-iff-interval*)
then show *?thesis*
by *auto*
qed

lemma *open-Collect-positive*:
fixes f :: *'a::topological-space* ⇒ *real*
assumes f: *continuous-on s f*
shows $\exists\, A.\ open\ A \wedge A \cap s = \{x{\in}s.\ 0 < f\ x\}$
using *continuous-on-open-invariant*[*THEN iffD1*, *OF f*, *rule-format*, *of* $\{0 <..\}$]
by (*auto simp*: *Int-def field-simps*)

lemma *open-Collect-less-Int*:
fixes $f\ g$ :: *'a::topological-space* ⇒ *real*
assumes f: *continuous-on s f*
and g: *continuous-on s g*
shows $\exists\, A.\ open\ A \wedge A \cap s = \{x{\in}s.\ f\ x < g\ x\}$
using *open-Collect-positive*[*OF continuous-on-diff*[*OF g f*]] **by** (*simp add*: *field-simps*)

108.14 Boundedness of continuous functions

By bisection, function continuous on closed interval is bounded above

lemma *isCont-eq-Ub*:
fixes f :: *real* ⇒ *'a::linorder-topology*
shows $a \leq b \Longrightarrow \forall\, x{::}real.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x \Longrightarrow$
$\exists\, M.\ (\forall\, x.\ a \leq x \wedge x \leq b \longrightarrow f\ x \leq M) \wedge (\exists\, x.\ a \leq x \wedge x \leq b \wedge f\ x = M)$
using *continuous-attains-sup*[*of* $\{a..b\}$ *f*]
by (*auto simp*: *continuous-at-imp-continuous-on Ball-def Bex-def*)

lemma *isCont-eq-Lb*:
fixes f :: *real* ⇒ *'a::linorder-topology*
shows $a \leq b \Longrightarrow \forall\, x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x \Longrightarrow$
$\exists\, M.\ (\forall\, x.\ a \leq x \wedge x \leq b \longrightarrow M \leq f\ x) \wedge (\exists\, x.\ a \leq x \wedge x \leq b \wedge f\ x = M)$
using *continuous-attains-inf*[*of* $\{a..b\}$ *f*]
by (*auto simp*: *continuous-at-imp-continuous-on Ball-def Bex-def*)

lemma *isCont-bounded*:
fixes f :: *real* ⇒ *'a::linorder-topology*
shows $a \leq b \Longrightarrow \forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x \Longrightarrow \exists M.\ \forall x.\ a \leq x \wedge x \leq b \longrightarrow f\ x \leq M$
using *isCont-eq-Ub*[*of a b f*] **by** *auto*

lemma *isCont-has-Ub*:
fixes f :: *real* ⇒ *'a::linorder-topology*
shows $a \leq b \Longrightarrow \forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x \Longrightarrow$
$\exists M.\ (\forall x.\ a \leq x \wedge x \leq b \longrightarrow f\ x \leq M) \wedge (\forall N.\ N < M \longrightarrow (\exists x.\ a \leq x \wedge x \leq b \wedge N < f\ x))$
using *isCont-eq-Ub*[*of a b f*] **by** *auto*

lemma *isCont-Lb-Ub*:
fixes f :: *real* ⇒ *real*
assumes $a \leq b\ \forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x$
shows $\exists L\ M.\ (\forall x.\ a \leq x \wedge x \leq b \longrightarrow L \leq f\ x \wedge f\ x \leq M) \wedge$
$(\forall y.\ L \leq y \wedge y \leq M \longrightarrow (\exists x.\ a \leq x \wedge x \leq b \wedge (f\ x = y)))$
proof −
obtain M **where** M: $a \leq M\ M \leq b\ \forall x.\ a \leq x \wedge x \leq b \longrightarrow f\ x \leq f\ M$
using *isCont-eq-Ub*[*OF assms*] **by** *auto*
obtain L **where** L: $a \leq L\ L \leq b\ \forall x.\ a \leq x \wedge x \leq b \longrightarrow f\ L \leq f\ x$
using *isCont-eq-Lb*[*OF assms*] **by** *auto*
have $(\forall x.\ a \leq x \wedge x \leq b \longrightarrow f\ L \leq f\ x \wedge f\ x \leq f\ M)$
using *M L* **by** *simp*
moreover
have $(\forall y.\ f\ L \leq y \wedge y \leq f\ M \longrightarrow (\exists x \geq a.\ x \leq b \wedge f\ x = y))$
proof (*cases* $L \leq M$)
case *True* **then show** *?thesis*
using *IVT*[*of f L - M*] *M L assms* **by** (*metis order.trans*)
next
case *False* **then show** *?thesis*
using *IVT2*[*of f L - M*]
by (*metis L*(*2*) *M*(*1*) *assms*(*2*) *le-cases order.trans*)
qed
ultimately show *?thesis*
by *blast*
qed

Continuity of inverse function.

lemma *isCont-inverse-function*:
fixes $f\ g$ :: *real* ⇒ *real*
assumes d: $0 < d$
and *inj*: $\bigwedge z.\ |z-x| \leq d \Longrightarrow g\ (f\ z) = z$
and *cont*: $\bigwedge z.\ |z-x| \leq d \Longrightarrow isCont\ f\ z$
shows $isCont\ g\ (f\ x)$
proof −
let $?A = f\ (x - d)$
let $?B = f\ (x + d)$

let *?D* = {*x* − *d*..*x* + *d*}

have *f*: *continuous-on ?D f*
using *cont* **by** (*intro continuous-at-imp-continuous-on ballI*) *auto*
then have *g*: *continuous-on* (*f*'*?D*) *g*
using *inj* **by** (*intro continuous-on-inv*) *auto*

from *d f* **have** {*min ?A ?B* <..< *max ?A ?B*} ⊆ *f* ' *?D*
by (*intro connected-contains-Ioo connected-continuous-image*) (*auto split*: *split-min split-max*)
with *g* **have** *continuous-on* {*min ?A ?B* <..< *max ?A ?B*} *g*
by (*rule continuous-on-subset*)
moreover
have (*?A* < *f x* ∧ *f x* < *?B*) ∨ (*?B* < *f x* ∧ *f x* < *?A*)
using *d inj* **by** (*intro continuous-inj-imp-mono*[*OF - - f*] *inj-on-imageI2*[*of g*, *OF inj-onI*]) *auto*
then have *f x* ∈ {*min ?A ?B* <..< *max ?A ?B*}
by *auto*
ultimately show *?thesis*
by (*simp add*: *continuous-on-eq-continuous-at*)
qed

lemma *isCont-inverse-function2*:
fixes *f g* :: *real* ⇒ *real*
shows
⟦*a* < *x*; *x* < *b*;
⋀*z*. ⟦*a* ≤ *z*; *z* ≤ *b*⟧ ⟹ *g* (*f z*) = *z*;
⋀*z*. ⟦*a* ≤ *z*; *z* ≤ *b*⟧ ⟹ *isCont f z*⟧ ⟹ *isCont g* (*f x*)
apply (*rule isCont-inverse-function* [**where** *f*=*f* **and** *d*=*min* (*x* − *a*) (*b* − *x*)])
apply (*simp-all add*: *abs-le-iff*)
done

Bartle/Sherbert: Introduction to Real Analysis, Theorem 4.2.9, p. 110.

lemma *LIM-fun-gt-zero*: *f* −*c*→ *l* ⟹ *0* < *l* ⟹ ∃ *r*. *0* < *r* ∧ (∀ *x*. *x* ≠ *c* ∧ |*c* − *x*| < *r* ⟶ *0* < *f x*)
for *f* :: *real* ⇒ *real*
by (*force simp*: *dest*: *LIM-D*)

lemma *LIM-fun-less-zero*: *f* −*c*→ *l* ⟹ *l* < *0* ⟹ ∃ *r*. *0* < *r* ∧ (∀ *x*. *x* ≠ *c* ∧ |*c* − *x*| < *r* ⟶ *f x* < *0*)
for *f* :: *real* ⇒ *real*
by (*drule LIM-D* [**where** *r*=−*l*]) *force*+

lemma *LIM-fun-not-zero*: *f* −*c*→ *l* ⟹ *l* ≠ *0* ⟹ ∃ *r*. *0* < *r* ∧ (∀ *x*. *x* ≠ *c* ∧ |*c* − *x*| < *r* ⟶ *f x* ≠ *0*)
for *f* :: *real* ⇒ *real*
using *LIM-fun-gt-zero*[*of f l c*] *LIM-fun-less-zero*[*of f l c*] **by** (*auto simp*: *neq-iff*)

lemma *Lim-topological*:

```
  (f ⟶ l) net ⟷
    trivial-limit net ∨ (∀ S. open S ⟶ l ∈ S ⟶ eventually (λx. f x ∈ S) net)
  unfolding tendsto-def trivial-limit-eq by auto

lemma eventually-within-Un:
  eventually P (at x within (s ∪ t)) ⟷
    eventually P (at x within s) ∧ eventually P (at x within t)
  unfolding eventually-at-filter
  by (auto elim!: eventually-rev-mp)

lemma Lim-within-Un:
 (f ⟶ l) (at x within (s ∪ t)) ⟷
  (f ⟶ l) (at x within s) ∧ (f ⟶ l) (at x within t)
  unfolding tendsto-def
  by (auto simp: eventually-within-Un)

end


theory Inequalities
  imports Real-Vector-Spaces
begin

lemma Chebyshev-sum-upper:
  fixes a b::nat ⇒ 'a::linordered-idom
  assumes ⋀i j. i ≤ j ⟹ j < n ⟹ a i ≤ a j
  assumes ⋀i j. i ≤ j ⟹ j < n ⟹ b i ≥ b j
  shows of-nat n ∗ (∑k=0..<n. a k ∗ b k) ≤ (∑k=0..<n. a k) ∗ (∑k=0..<n.
b k)
proof −
  let ?S = (∑j=0..<n. (∑k=0..<n. (a j − a k) ∗ (b j − b k)))
  have 2 ∗ (of-nat n ∗ (∑j=0..<n. (a j ∗ b j)) − (∑j=0..<n. b j) ∗ (∑k=0..<n.
a k)) = ?S
    by (simp only: one-add-one[symmetric] algebra-simps)
      (simp add: algebra-simps sum-subtractf sum.distrib sum.swap[of λi j. a i ∗ b
j] sum-distrib-left)
  also
  { fix i j::nat assume i<n j<n
    hence a i − a j ≤ 0 ∧ b i − b j ≥ 0 ∨ a i − a j ≥ 0 ∧ b i − b j ≤ 0
      using assms by (cases i ≤ j) (auto simp: algebra-simps)
  } then have ?S ≤ 0
    by (auto intro!: sum-nonpos simp: mult-le-0-iff)
  finally show ?thesis by (simp add: algebra-simps)
qed

lemma Chebyshev-sum-upper-nat:
  fixes a b :: nat ⇒ nat
  shows (⋀i j. ⟦ i≤j; j<n ⟧ ⟹ a i ≤ a j) ⟹
        (⋀i j. ⟦ i≤j; j<n ⟧ ⟹ b i ≥ b j) ⟹
```

$n * (\sum i=0..<n.\ a\ i * b\ i) \le (\sum i=0..<n.\ a\ i) * (\sum i=0..<n.\ b\ i)$
using *Chebyshev-sum-upper*[**where** *'a=real*, *of n a b*]
by (*simp del*: *of-nat-mult of-nat-sum add*: *of-nat-mult*[*symmetric*] *of-nat-sum*[*symmetric*])

end

109 Infinite Series

theory *Series*
imports *Limits Inequalities*
begin

109.1 Definition of infinite summability

definition *sums* :: (*nat* ⇒ *'a*::{*topological-space*, *comm-monoid-add*}) ⇒ *'a* ⇒ *bool*
(**infixr** ‹*sums*› *80*)
where *f sums s* ⟷ ($\lambda n. \sum i<n.\ f\ i$) ⟶ *s*

definition *summable* :: (*nat* ⇒ *'a*::{*topological-space*, *comm-monoid-add*}) ⇒ *bool*
where *summable f* ⟷ (∃ *s*. *f sums s*)

definition *suminf* :: (*nat* ⇒ *'a*::{*topological-space*, *comm-monoid-add*}) ⇒ *'a*
(**binder** ‹∑› *10*)
where *suminf f* = (*THE s*. *f sums s*)

Variants of the definition

lemma *sums-def'*: *f sums s* ⟷ ($\lambda n. \sum i = 0..n.\ f\ i$) ⟶ *s*
unfolding *sums-def*
apply (*subst filterlim-sequentially-Suc* [*symmetric*])
apply (*simp only*: *lessThan-Suc-atMost atLeast0AtMost*)
done

lemma *sums-def-le*: *f sums s* ⟷ ($\lambda n. \sum i\le n.\ f\ i$) ⟶ *s*
by (*simp add*: *sums-def' atMost-atLeast0*)

lemma *bounded-imp-summable*:
fixes *a* :: *nat* ⇒ *'a*::{*conditionally-complete-linorder*,*linorder-topology*,*linordered-comm-semiring-strict*}
assumes *0*: $\bigwedge n.\ a\ n \ge 0$ **and** *bounded*: $\bigwedge n.\ (\sum k\le n.\ a\ k) \le B$
shows *summable a*
proof −
have *bdd-above* (*range*($\lambda n. \sum k\le n.\ a\ k$))
by (*meson bdd-aboveI2 bounded*)
moreover have *incseq* ($\lambda n. \sum k\le n.\ a\ k$)
by (*simp add*: *mono-def 0 sum-mono2*)
ultimately obtain *s* **where** ($\lambda n. \sum k\le n.\ a\ k$) ⟶ *s*
using *LIMSEQ-incseq-SUP* **by** *blast*
then show *?thesis*
by (*auto simp*: *sums-def-le summable-def*)
qed

109.2 Infinite summability on topological monoids

lemma *sums-subst*[*trans*]: $f = g \Longrightarrow g\ sums\ z \Longrightarrow f\ sums\ z$
  by *simp*

lemma *sums-cong*: $(\bigwedge n.\ f\ n = g\ n) \Longrightarrow f\ sums\ c \longleftrightarrow g\ sums\ c$
  by *presburger*

lemma *sums-summable*: $f\ sums\ l \Longrightarrow summable\ f$
  by (*simp add*: *sums-def summable-def*, *blast*)

lemma *summable-iff-convergent*: $summable\ f \longleftrightarrow convergent\ (\lambda n.\ \sum i{<}n.\ f\ i)$
  by (*simp add*: *summable-def sums-def convergent-def*)

lemma *summable-iff-convergent'*: $summable\ f \longleftrightarrow convergent\ (\lambda n.\ sum\ f\ \{..n\})$
  by (*simp add*: *convergent-def summable-def sums-def-le*)

lemma *suminf-eq-lim*: $suminf\ f = lim\ (\lambda n.\ \sum i{<}n.\ f\ i)$
  by (*simp add*: *suminf-def sums-def lim-def*)

lemma *sums-zero*[*simp*, *intro*]: $(\lambda n.\ 0)\ sums\ 0$
  unfolding *sums-def* **by** *simp*

lemma *summable-zero*[*simp*, *intro*]: $summable\ (\lambda n.\ 0)$
  by (*rule sums-zero* [*THEN sums-summable*])

lemma *sums-group*: $f\ sums\ s \Longrightarrow 0 < k \Longrightarrow (\lambda n.\ sum\ f\ \{n * k\ ..{<}\ n * k + k\})\ sums\ s$
  apply (*simp only*: *sums-def sum.nat-group tendsto-def eventually-sequentially*)
  apply (*erule all-forward imp-forward exE*| *assumption*)+
  by (*metis le-square mult.commute mult.left-neutral mult-le-cancel2 mult-le-mono*)

lemma *suminf-cong*: $(\bigwedge n.\ f\ n = g\ n) \Longrightarrow suminf\ f = suminf\ g$
  by *presburger*

lemma *summable-cong*:
  fixes $f\ g :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector$
  assumes $eventually\ (\lambda x.\ f\ x = g\ x)\ sequentially$
  shows $summable\ f = summable\ g$
proof −
  from *assms* **obtain** N **where** N: $\forall n{\geq}N.\ f\ n = g\ n$
    by (*auto simp*: *eventually-at-top-linorder*)
  define C **where** $C = (\sum k{<}N.\ f\ k - g\ k)$
  from *eventually-ge-at-top*[*of N*]
  have $eventually\ (\lambda n.\ sum\ f\ \{..{<}n\} = C + sum\ g\ \{..{<}n\})\ sequentially$
  proof *eventually-elim*
    case (*elim n*)
    then have $\{..{<}n\} = \{..{<}N\} \cup \{N..{<}n\}$
      by *auto*
    also have $sum\ f\ ... = sum\ f\ \{..{<}N\} + sum\ f\ \{N..{<}n\}$

by (*intro sum.union-disjoint*) *auto*
also from *N* **have** *sum f* {*N..<n*} = *sum g* {*N..<n*}
by (*intro sum.cong*) *simp-all*
also have *sum f* {*..<N*} + *sum g* {*N..<n*} = *C* + (*sum g* {*..<N*} + *sum g* {*N..<n*})
unfolding *C-def* **by** (*simp add*: *algebra-simps sum-subtractf*)
also have *sum g* {*..<N*} + *sum g* {*N..<n*} = *sum g* ({*..<N*} ∪ {*N..<n*})
by (*intro sum.union-disjoint* [*symmetric*]) *auto*
also from *elim* **have** {*..<N*} ∪ {*N..<n*} = {*..<n*}
by *auto*
finally show *sum f* {*..<n*} = *C* + *sum g* {*..<n*} **.**
qed
from *convergent-cong*[*OF this*] **show** *?thesis*
by (*simp add*: *summable-iff-convergent convergent-add-const-iff*)
qed

lemma *sums-finite*:
assumes [*simp*]: *finite N*
and *f*: ⋀*n*. *n* ∉ *N* ⟹ *f n* = *0*
shows *f sums* (∑ *n*∈*N*. *f n*)
proof –
have *eq*: *sum f* {*..<n* + *Suc* (*Max N*)} = *sum f N* **for** *n*
by (*rule sum.mono-neutral-right*) (*auto simp*: *add-increasing less-Suc-eq-le f*)
show *?thesis*
unfolding *sums-def*
by (*rule LIMSEQ-offset*[*of - Suc* (*Max N*)])
(*simp add*: *eq atLeast0LessThan del*: *add-Suc-right*)
qed

corollary *sums-0*: (⋀*n*. *f n* = *0*) ⟹ (*f sums 0*)
by (*metis* (*no-types*) *finite.emptyI sum.empty sums-finite*)

lemma *summable-finite*: *finite N* ⟹ (⋀*n*. *n* ∉ *N* ⟹ *f n* = *0*) ⟹ *summable f*
by (*rule sums-summable*) (*rule sums-finite*)

lemma *sums-If-finite-set*: *finite A* ⟹ (λ*r*. *if r* ∈ *A then f r else 0*) *sums* (∑ *r*∈*A*. *f r*)
using *sums-finite*[*of A* (λ*r*. *if r* ∈ *A then f r else 0*)] **by** *simp*

lemma *summable-If-finite-set*[*simp*, *intro*]: *finite A* ⟹ *summable* (λ*r*. *if r* ∈ *A then f r else 0*)
by (*rule sums-summable*) (*rule sums-If-finite-set*)

lemma *sums-If-finite*: *finite* {*r*. *P r*} ⟹ (λ*r*. *if P r then f r else 0*) *sums* (∑ *r* | *P r*. *f r*)
using *sums-If-finite-set*[*of* {*r*. *P r*}] **by** *simp*

lemma *summable-If-finite*[*simp*, *intro*]: *finite* {*r*. *P r*} ⟹ *summable* (λ*r*. *if P r then f r else 0*)

by (*rule sums-summable*) (*rule sums-If-finite*)

lemma *sums-single*: ($\lambda r.$ *if* $r = i$ *then* $f\ r$ *else* 0) *sums* $f\ i$
using *sums-If-finite*[*of* $\lambda r.\ r = i$] **by** *simp*

lemma *summable-single*[*simp*, *intro*]: *summable* ($\lambda r.$ *if* $r = i$ *then* $f\ r$ *else* 0)
by (*rule sums-summable*) (*rule sums-single*)

context
fixes f :: *nat* $\Rightarrow$ $'a$::{*t2-space*,*comm-monoid-add*}
begin

lemma *summable-sums*[*intro*]: *summable* $f \Longrightarrow f$ *sums* (*suminf* f)
by (*simp add*: *summable-def sums-def suminf-def*)
(*metis convergent-LIMSEQ-iff convergent-def lim-def*)

lemma *summable-LIMSEQ*: *summable* $f \Longrightarrow (\lambda n. \sum i<n.\ f\ i) \longrightarrow$ *suminf* f
by (*rule summable-sums* [*unfolded sums-def*])

lemma *summable-LIMSEQ'*: *summable* $f \Longrightarrow (\lambda n. \sum i\leq n.\ f\ i) \longrightarrow$ *suminf* f
using *sums-def-le* **by** *blast*

lemma *sums-unique*: f *sums* $s \Longrightarrow s =$ *suminf* f
by (*metis limI suminf-eq-lim sums-def*)

lemma *sums-iff*: f *sums* $x \longleftrightarrow$ *summable* $f \wedge$ *suminf* $f = x$
by (*metis summable-sums sums-summable sums-unique*)

lemma *summable-sums-iff*: *summable* $f \longleftrightarrow f$ *sums suminf* f
by (*auto simp*: *sums-iff summable-sums*)

lemma *sums-unique2*: f *sums* $a \Longrightarrow f$ *sums* $b \Longrightarrow a = b$
for $a\ b$ :: $'a$
by (*simp add*: *sums-iff*)

lemma *sums-Uniq*: $\exists_{\leq 1} a.\ f$ *sums* a
for $a\ b$ :: $'a$
by (*simp add*: *sums-unique2 Uniq-def*)

lemma *suminf-finite*:
assumes N: *finite* N
and f: $\bigwedge n.\ n \notin N \Longrightarrow f\ n = 0$
shows *suminf* $f = (\sum n\in N.\ f\ n)$
using *sums-finite*[*OF assms*, *THEN sums-unique*] **by** *simp*

end

lemma *suminf-zero*[*simp*]: *suminf* ($\lambda n.\ 0$::$'a$::{*t2-space*, *comm-monoid-add*}) $= 0$
by (*rule sums-zero* [*THEN sums-unique*, *symmetric*])

109.3 Infinite summability on ordered, topological monoids

lemma *sums-le*: $(\bigwedge n.\ f\ n \leq g\ n) \Longrightarrow f\ sums\ s \Longrightarrow g\ sums\ t \Longrightarrow s \leq t$
for *f g* :: *nat* ⇒ *'a*::{*ordered-comm-monoid-add*,*linorder-topology*}
by (*rule LIMSEQ-le*) (*auto intro*: *sum-mono simp*: *sums-def*)

context
fixes *f* :: *nat* ⇒ *'a*::{*ordered-comm-monoid-add*,*linorder-topology*}
begin

lemma *suminf-le*: $(\bigwedge n.\ f\ n \leq g\ n) \Longrightarrow$ *summable f* $\Longrightarrow$ *summable g* $\Longrightarrow$ *suminf f* $\leq$ *suminf g*
using *sums-le* **by** *blast*

lemma *sum-le-suminf*:
shows *summable f* $\Longrightarrow$ *finite I* $\Longrightarrow (\bigwedge n.\ n \in- I \Longrightarrow 0 \leq f\ n) \Longrightarrow$ *sum f I* $\leq$ *suminf f*
by (*rule sums-le*[*OF - sums-If-finite-set summable-sums*]) *auto*

lemma *suminf-nonneg*: *summable f* $\Longrightarrow (\bigwedge n.\ 0 \leq f\ n) \Longrightarrow 0 \leq$ *suminf f*
using *sum-le-suminf* **by** *force*

lemma *suminf-le-const*: *summable f* $\Longrightarrow (\bigwedge n.$ *sum f* $\{..<n\} \leq x) \Longrightarrow$ *suminf f* $\leq x$
by (*metis LIMSEQ-le-const2 summable-LIMSEQ*)

lemma *suminf-eq-zero-iff*:
assumes *summable f* **and** *pos*: $\bigwedge n.\ 0 \leq f\ n$
shows *suminf f* $= 0 \longleftrightarrow (\forall n.\ f\ n = 0)$
proof
assume *L*: *suminf f* $= 0$
then have *f*: $(\lambda n.\ \sum i<n.\ f\ i) \longrightarrow 0$
using *summable-LIMSEQ*[*of f*] *assms* **by** *simp*
then have $\bigwedge i.\ (\sum n\in\{i\}.\ f\ n) \leq 0$
by (*metis L ‹summable f› order-refl pos sum.infinite sum-le-suminf*)
with *pos* **show** $\forall n.\ f\ n = 0$
by (*simp add*: *order.antisym*)
qed (*metis suminf-zero fun-eq-iff*)

lemma *suminf-pos-iff*: *summable f* $\Longrightarrow (\bigwedge n.\ 0 \leq f\ n) \Longrightarrow 0 <$ *suminf f* $\longleftrightarrow (\exists i.\ 0 < f\ i)$
using *sum-le-suminf*[*of* {}] *suminf-eq-zero-iff* **by** (*simp add*: *less-le*)

lemma *suminf-pos2*:
assumes *summable f* $\bigwedge n.\ 0 \leq f\ n\ 0 < f\ i$
shows $0 <$ *suminf f*
proof −
have $0 < (\sum n<Suc\ i.\ f\ n)$
using *assms* **by** (*intro sum-pos2*[**where** *i*=*i*]) *auto*
also have ... $\leq$ *suminf f*

  using *assms* **by** (*intro sum-le-suminf*) *auto*
 finally show *?thesis* **.**
qed

lemma *suminf-pos*: *summable f* ⟹ (⋀*n. 0* < *f n*) ⟹ *0* < *suminf f*
 by (*intro suminf-pos2*[**where** *i*=*0*]) (*auto intro*: *less-imp-le*)

end

context
 fixes *f* :: *nat* ⇒ *'a*::{*ordered-cancel-comm-monoid-add*,*linorder-topology*}
begin

lemma *sum-less-suminf2*:
 summable f ⟹ (⋀*m. m*≥*n* ⟹ *0* ≤ *f m*) ⟹ *n* ≤ *i* ⟹ *0* < *f i* ⟹ *sum f* {..<*n*} < *suminf f*
 using *sum-le-suminf*[*of f* {..< *Suc i*}]
  and *add-strict-increasing*[*of f i sum f* {..<*n*} *sum f* {..<*i*}]
  and *sum-mono2*[*of* {..<*i*} {..<*n*} *f*]
 by (*auto simp*: *less-imp-le ac-simps*)

lemma *sum-less-suminf*: *summable f* ⟹ (⋀*m. m*≥*n* ⟹ *0* < *f m*) ⟹ *sum f* {..<*n*} < *suminf f*
 using *sum-less-suminf2*[*of n n*] **by** (*simp add*: *less-imp-le*)

end

lemma *summableI-nonneg-bounded*:
 fixes *f* :: *nat* ⇒ *'a*::{*ordered-comm-monoid-add*,*linorder-topology*,*conditionally-complete-linorder*}
 assumes *pos*[*simp*]: ⋀*n. 0* ≤ *f n*
  and *le*: ⋀*n.* (∑ *i*<*n. f i*) ≤ *x*
 shows *summable f*
 unfolding *summable-def sums-def* [*abs-def*]
proof (*rule exI LIMSEQ-incseq-SUP*)+
 show *bdd-above* (*range* (λ*n. sum f* {..<*n*}))
  using *le* **by** (*auto simp*: *bdd-above-def*)
 show *incseq* (λ*n. sum f* {..<*n*})
  by (*auto simp*: *mono-def intro*!: *sum-mono2*)
qed

lemma *summableI*[*intro*, *simp*]: *summable f*
 for *f* :: *nat* ⇒ *'a*::{*canonically-ordered-monoid-add*,*linorder-topology*,*complete-linorder*}
 by (*intro summableI-nonneg-bounded*[**where** *x*=*top*] *zero-le top-greatest*)

lemma *suminf-eq-SUP-real*:
 assumes *X*: *summable X* ⋀*i. 0* ≤ *X i* **shows** *suminf X* = (*SUP i.* ∑ *n*<*i. X n*::*real*)
 by (*intro LIMSEQ-unique*[*OF summable-LIMSEQ*] *X LIMSEQ-incseq-SUP*)
  (*auto intro*!: *bdd-aboveI2*[**where** *M*=∑ *i. X i*] *sum-le-suminf X monoI sum-mono2*)

109.4 Infinite summability on topological monoids

context
fixes *f g* :: *nat* ⇒ *'a*::{*t2-space*,*topological-comm-monoid-add*}
begin

lemma *sums-Suc*:
assumes (*λn. f* (*Suc n*)) *sums l*
shows *f sums* (*l* + *f 0*)
proof −
have (*λn.* (∑ *i*<*n. f* (*Suc i*)) + *f 0*) ⟶ *l* + *f 0*
using *assms* **by** (*auto intro*!: *tendsto-add simp*: *sums-def*)
moreover have (∑ *i*<*n. f* (*Suc i*)) + *f 0* = (∑ *i*<*Suc n. f i*) **for** *n*
unfolding *lessThan-Suc-eq-insert-0*
by (*simp add*: *ac-simps sum.atLeast1-atMost-eq image-Suc-lessThan*)
ultimately show *?thesis*
by (*auto simp*: *sums-def simp del*: *sum.lessThan-Suc intro*: *filterlim-sequentially-Suc*[*THEN iffD1*])
qed

lemma *sums-add*: *f sums a* ⟹ *g sums b* ⟹ (*λn. f n* + *g n*) *sums* (*a* + *b*)
unfolding *sums-def* **by** (*simp add*: *sum.distrib tendsto-add*)

lemma *summable-add*: *summable f* ⟹ *summable g* ⟹ *summable* (*λn. f n* + *g n*)
unfolding *summable-def* **by** (*auto intro*: *sums-add*)

lemma *suminf-add*: *summable f* ⟹ *summable g* ⟹ *suminf f* + *suminf g* = (∑ *n. f n* + *g n*)
by (*intro sums-unique sums-add summable-sums*)

end

context
fixes *f* :: *'i* ⇒ *nat* ⇒ *'a*::{*t2-space*,*topological-comm-monoid-add*}
and *I* :: *'i set*
begin

lemma *sums-sum*: (⋀*i. i* ∈ *I* ⟹ (*f i*) *sums* (*x i*)) ⟹ (*λn.* ∑ *i*∈*I. f i n*) *sums* (∑ *i*∈*I. x i*)
by (*induct I rule*: *infinite-finite-induct*) (*auto intro*!: *sums-add*)

lemma *suminf-sum*: (⋀*i. i* ∈ *I* ⟹ *summable* (*f i*)) ⟹ (∑ *n.* ∑ *i*∈*I. f i n*) = (∑ *i*∈*I.* ∑ *n. f i n*)
using *sums-unique*[*OF sums-sum*, *OF summable-sums*] **by** *simp*

lemma *summable-sum*: (⋀*i. i* ∈ *I* ⟹ *summable* (*f i*)) ⟹ *summable* (*λn.* ∑ *i*∈*I. f i n*)
using *sums-summable*[*OF sums-sum*[*OF summable-sums*]] **.**

end

lemma *sums-If-finite-set'*:
fixes $f\ g$ :: *nat* ⇒ *'a*::{*t2-space*,*topological-ab-group-add*}
assumes g *sums* S **and** *finite* A **and** $S' = S + (\sum n{\in}A.\ f\ n - g\ n)$
shows $(\lambda n.$ *if* $n \in A$ *then* $f\ n$ *else* $g\ n)$ *sums* S'
proof −
have $(\lambda n.\ g\ n + ($*if* $n \in A$ *then* $f\ n - g\ n$ *else* $0))$ *sums* $(S + (\sum n{\in}A.\ f\ n - g\ n))$
by (*intro sums-add assms sums-If-finite-set*)
also have $(\lambda n.\ g\ n + ($*if* $n \in A$ *then* $f\ n - g\ n$ *else* $0)) = (\lambda n.$ *if* $n \in A$ *then* $f\ n$ *else* $g\ n)$
by (*simp add*: *fun-eq-iff*)
finally show *?thesis* **using** *assms* **by** *simp*
qed

109.5 Infinite summability on real normed vector spaces

context
fixes f :: *nat* ⇒ *'a*::*real-normed-vector*
begin

lemma *sums-Suc-iff*: $(\lambda n.\ f\ (Suc\ n))$ *sums* $s \longleftrightarrow f$ *sums* $(s + f\ 0)$
proof −
have f *sums* $(s + f\ 0) \longleftrightarrow (\lambda i.\ \sum j{<}Suc\ i.\ f\ j) \longrightarrow s + f\ 0$
by (*subst filterlim-sequentially-Suc*) (*simp add*: *sums-def*)
also have ... $\longleftrightarrow (\lambda i.\ (\sum j{<}i.\ f\ (Suc\ j)) + f\ 0) \longrightarrow s + f\ 0$
by (*simp add*: *ac-simps lessThan-Suc-eq-insert-0 image-Suc-lessThan sum.atLeast1-atMost-eq*)
also have ... $\longleftrightarrow (\lambda n.\ f\ (Suc\ n))$ *sums* s
proof
assume $(\lambda i.\ (\sum j{<}i.\ f\ (Suc\ j)) + f\ 0) \longrightarrow s + f\ 0$
with *tendsto-add*[*OF this tendsto-const*, *of* $- f\ 0$] **show** $(\lambda i.\ f\ (Suc\ i))$ *sums* s
by (*simp add*: *sums-def*)
qed (*auto intro*: *tendsto-add simp*: *sums-def*)
finally show *?thesis* **..**
qed

lemma *summable-Suc-iff*: *summable* $(\lambda n.\ f\ (Suc\ n))$ = *summable* f
proof
assume *summable* f
then have f *sums suminf* f
by (*rule summable-sums*)
then have $(\lambda n.\ f\ (Suc\ n))$ *sums* (*suminf* $f - f\ 0$)
by (*simp add*: *sums-Suc-iff*)
then show *summable* $(\lambda n.\ f\ (Suc\ n))$
unfolding *summable-def* **by** *blast*
qed (*auto simp*: *sums-Suc-iff summable-def*)

lemma *sums-Suc-imp*: $f\ 0 = 0 \Longrightarrow (\lambda n.\ f\ (Suc\ n))$ *sums* $s \Longrightarrow (\lambda n.\ f\ n)$ *sums* s

using *sums-Suc-iff* **by** *simp*

end

context
fixes f :: *nat* ⇒ *'a::real-normed-vector*
begin

lemma *sums-diff*: f *sums* $a \Longrightarrow g$ *sums* $b \Longrightarrow (\lambda n.\ f\ n - g\ n)$ *sums* $(a - b)$
unfolding *sums-def* **by** (*simp add*: *sum-subtractf tendsto-diff*)

lemma *summable-diff*: *summable* $f \Longrightarrow$ *summable* $g \Longrightarrow$ *summable* $(\lambda n.\ f\ n - g\ n)$
unfolding *summable-def* **by** (*auto intro*: *sums-diff*)

lemma *suminf-diff*: *summable* $f \Longrightarrow$ *summable* $g \Longrightarrow$ *suminf* f − *suminf* $g = (\sum n.\ f\ n - g\ n)$
by (*intro sums-unique sums-diff summable-sums*)

lemma *sums-minus*: f *sums* $a \Longrightarrow (\lambda n.\ - f\ n)$ *sums* $(- a)$
unfolding *sums-def* **by** (*simp add*: *sum-negf tendsto-minus*)

lemma *summable-minus*: *summable* $f \Longrightarrow$ *summable* $(\lambda n.\ - f\ n)$
unfolding *summable-def* **by** (*auto intro*: *sums-minus*)

lemma *suminf-minus*: *summable* $f \Longrightarrow (\sum n.\ - f\ n) = - (\sum n.\ f\ n)$
by (*intro sums-unique* [*symmetric*] *sums-minus summable-sums*)

lemma *sums-iff-shift*: $(\lambda i.\ f\ (i + n))$ *sums* $s \longleftrightarrow f$ *sums* $(s + (\sum i{<}n.\ f\ i))$
proof (*induct n arbitrary*: *s*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
then have $(\lambda i.\ f\ (Suc\ i + n))$ *sums* $s \longleftrightarrow (\lambda i.\ f\ (i + n))$ *sums* $(s + f\ n)$
by (*subst sums-Suc-iff*) *simp*
with *Suc* **show** *?case*
by (*simp add*: *ac-simps*)
qed

corollary *sums-iff-shift'*: $(\lambda i.\ f\ (i + n))$ *sums* $(s - (\sum i{<}n.\ f\ i)) \longleftrightarrow f$ *sums* s
by (*simp add*: *sums-iff-shift*)

lemma *sums-zero-iff-shift*:
assumes $\bigwedge i.\ i < n \Longrightarrow f\ i = 0$
shows $(\lambda i.\ f\ (i{+}n))$ *sums* $s \longleftrightarrow (\lambda i.\ f\ i)$ *sums* s
by (*simp add*: *assms sums-iff-shift*)

lemma *summable-iff-shift* [*simp*]: *summable* $(\lambda n.\ f\ (n + k)) \longleftrightarrow$ *summable* f

by (*metis diff-add-cancel summable-def sums-iff-shift* [*abs-def*])

lemma *sums-split-initial-segment*: *f sums s* $\Longrightarrow$ ($\lambda i.$ *f* (*i* + *n*)) *sums* (*s* − ($\sum i<n.$ *f i*))
by (*simp add*: *sums-iff-shift*)

lemma *summable-ignore-initial-segment*: *summable f* $\Longrightarrow$ *summable* ($\lambda n.$ *f*(*n* + *k*))
by (*simp add*: *summable-iff-shift*)

lemma *suminf-minus-initial-segment*: *summable f* $\Longrightarrow$ ($\sum n.$ *f* (*n* + *k*)) = ($\sum n.$ *f n*) − ($\sum i<k.$ *f i*)
by (*rule sums-unique*[*symmetric*]) (*auto simp*: *sums-iff-shift*)

lemma *suminf-split-initial-segment*: *summable f* $\Longrightarrow$ *suminf f* = ($\sum n.$ *f*(*n* + *k*)) + ($\sum i<k.$ *f i*)
by (*auto simp add*: *suminf-minus-initial-segment*)

lemma *suminf-split-head*: *summable f* $\Longrightarrow$ ($\sum n.$ *f* (*Suc n*)) = *suminf f* − *f 0*
using *suminf-split-initial-segment*[*of 1*] **by** *simp*

lemma *suminf-exist-split*:
fixes *r* :: *real*
assumes *0* < *r* **and** *summable f*
shows $\exists N. \forall n\geq N.$ *norm* ($\sum i.$ *f* (*i* + *n*)) < *r*
proof −
from *LIMSEQ-D*[*OF summable-LIMSEQ*[*OF* ‹*summable f*›] ‹*0* < *r*›]
obtain *N* :: *nat* **where** ∀ *n* ≥ *N*. *norm* (*sum f* {..<*n*} − *suminf f*) < *r*
by *auto*
then show *?thesis*
by (*auto simp*: *norm-minus-commute suminf-minus-initial-segment*[*OF* ‹*summable f*›])
qed

lemma *summable-LIMSEQ-zero*:
assumes *summable f* **shows** *f* ⟶ *0*
proof −
have *Cauchy* ($\lambda n.$ *sum f* {..<*n*})
using *LIMSEQ-imp-Cauchy assms summable-LIMSEQ* **by** *blast*
then show *?thesis*
unfolding *Cauchy-iff LIMSEQ-iff*
by (*metis add.commute add-diff-cancel-right′ diff-zero le-SucI sum.lessThan-Suc*)
qed

lemma *summable-imp-convergent*: *summable f* $\Longrightarrow$ *convergent f*
by (*force dest*!: *summable-LIMSEQ-zero simp*: *convergent-def*)

lemma *summable-imp-Bseq*: *summable f* $\Longrightarrow$ *Bseq f*
by (*simp add*: *convergent-imp-Bseq summable-imp-convergent*)

end

lemma *summable-minus-iff*: *summable* (λ*n*. − *f n*) ⟷ *summable f*
for *f* :: *nat* ⇒ '*a*::*real-normed-vector*
by (*auto dest*: *summable-minus*)

lemma (**in** *bounded-linear*) *sums*: (λ*n*. *X n*) *sums a* ⟹ (λ*n*. *f* (*X n*)) *sums* (*f a*)
unfolding *sums-def* **by** (*drule tendsto*) (*simp only*: *sum*)

lemma (**in** *bounded-linear*) *summable*: *summable* (λ*n*. *X n*) ⟹ *summable* (λ*n*. *f* (*X n*))
unfolding *summable-def* **by** (*auto intro*: *sums*)

lemma (**in** *bounded-linear*) *suminf*: *summable* (λ*n*. *X n*) ⟹ *f* (∑*n*. *X n*) = (∑*n*. *f* (*X n*))
by (*intro sums-unique sums summable-sums*)

lemmas *sums-of-real* = *bounded-linear*.*sums* [*OF bounded-linear-of-real*]
lemmas *summable-of-real* = *bounded-linear*.*summable* [*OF bounded-linear-of-real*]
lemmas *suminf-of-real* = *bounded-linear*.*suminf* [*OF bounded-linear-of-real*]

lemmas *sums-scaleR-left* = *bounded-linear*.*sums*[*OF bounded-linear-scaleR-left*]
lemmas *summable-scaleR-left* = *bounded-linear*.*summable*[*OF bounded-linear-scaleR-left*]
lemmas *suminf-scaleR-left* = *bounded-linear*.*suminf*[*OF bounded-linear-scaleR-left*]

lemmas *sums-scaleR-right* = *bounded-linear*.*sums*[*OF bounded-linear-scaleR-right*]
lemmas *summable-scaleR-right* = *bounded-linear*.*summable*[*OF bounded-linear-scaleR-right*]
lemmas *suminf-scaleR-right* = *bounded-linear*.*suminf*[*OF bounded-linear-scaleR-right*]

lemma *summable-const-iff*: *summable* (λ-. *c*) ⟷ *c* = *0*
for *c* :: '*a*::*real-normed-vector*
proof −
have ¬ *summable* (λ-. *c*) **if** *c* ≠ *0*
proof −
from *that* **have** *filterlim* (λ*n*. *of-nat n* ∗ *norm c*) *at-top sequentially*
by (*subst mult*.*commute*)
(*auto intro*!: *filterlim-tendsto-pos-mult-at-top filterlim-real-sequentially*)
then have ¬ *convergent* (λ*n*. *norm* (∑*k*<*n*. *c*))
by (*intro filterlim-at-infinity-imp-not-convergent filterlim-at-top-imp-at-infinity*)
(*simp-all add*: *sum-constant-scaleR*)
then show *?thesis*
unfolding *summable-iff-convergent* **using** *convergent-norm* **by** *blast*
qed
then show *?thesis* **by** *auto*
qed

109.6 Infinite summability on real normed algebras

context
fixes *f* :: *nat* ⇒ *'a*::*real-normed-algebra*
begin

lemma *sums-mult*: *f sums a* ⟹ (*λn. c* ∗ *f n*) *sums* (*c* ∗ *a*)
by (*rule bounded-linear.sums* [*OF bounded-linear-mult-right*])

lemma *summable-mult*: *summable f* ⟹ *summable* (*λn. c* ∗ *f n*)
by (*rule bounded-linear.summable* [*OF bounded-linear-mult-right*])

lemma *suminf-mult*: *summable f* ⟹ *suminf* (*λn. c* ∗ *f n*) = *c* ∗ *suminf f*
by (*rule bounded-linear.suminf* [*OF bounded-linear-mult-right*, *symmetric*])

lemma *sums-mult2*: *f sums a* ⟹ (*λn. f n* ∗ *c*) *sums* (*a* ∗ *c*)
by (*rule bounded-linear.sums* [*OF bounded-linear-mult-left*])

lemma *summable-mult2*: *summable f* ⟹ *summable* (*λn. f n* ∗ *c*)
by (*rule bounded-linear.summable* [*OF bounded-linear-mult-left*])

lemma *suminf-mult2*: *summable f* ⟹ *suminf f* ∗ *c* = (∑ *n. f n* ∗ *c*)
by (*rule bounded-linear.suminf* [*OF bounded-linear-mult-left*])

end

lemma *sums-mult-iff*:
fixes *f* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*field*}
assumes *c* ≠ *0*
shows (*λn. c* ∗ *f n*) *sums* (*c* ∗ *d*) ⟷ *f sums d*
using *sums-mult*[*of f d c*] *sums-mult*[*of λn. c* ∗ *f n c* ∗ *d inverse c*]
by (*force simp*: *field-simps assms*)

lemma *sums-mult2-iff*:
fixes *f* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*field*}
assumes *c* ≠ *0*
shows (*λn. f n* ∗ *c*) *sums* (*d* ∗ *c*) ⟷ *f sums d*
using *sums-mult-iff*[*OF assms*, *of f d*] **by** (*simp add*: *mult.commute*)

lemma *sums-of-real-iff*:
(*λn. of-real* (*f n*) :: *'a*::*real-normed-div-algebra*) *sums of-real c* ⟷ *f sums c*
by (*simp add*: *sums-def of-real-sum*[*symmetric*] *tendsto-of-real-iff del*: *of-real-sum*)

109.7 Infinite summability on real normed fields

context
fixes *c* :: *'a*::*real-normed-field*
begin

lemma *sums-divide*: *f sums a* ⟹ (*λn. f n* / *c*) *sums* (*a* / *c*)

by (*rule bounded-linear.sums* [*OF bounded-linear-divide*])

lemma *summable-divide*: *summable f* $\Longrightarrow$ *summable* ($\lambda n.$ *f n* / *c*)
by (*rule bounded-linear.summable* [*OF bounded-linear-divide*])

lemma *suminf-divide*: *summable f* $\Longrightarrow$ *suminf* ($\lambda n.$ *f n* / *c*) = *suminf f* / *c*
by (*rule bounded-linear.suminf* [*OF bounded-linear-divide*, *symmetric*])

lemma *summable-inverse-divide*: *summable* (*inverse* $\circ$ *f*) $\Longrightarrow$ *summable* ($\lambda n.$ *c* / *f n*)
by (*auto dest*: *summable-mult* [*of - c*] *simp*: *field-simps*)

lemma *sums-mult-D*: ($\lambda n.$ *c* $*$ *f n*) *sums a* $\Longrightarrow$ *c* $\neq$ *0* $\Longrightarrow$ *f sums* (*a*/*c*)
using *sums-mult-iff* **by** *fastforce*

lemma *summable-mult-D*: *summable* ($\lambda n.$ *c* $*$ *f n*) $\Longrightarrow$ *c* $\neq$ *0* $\Longrightarrow$ *summable f*
by (*auto dest*: *summable-divide*)

Sum of a geometric progression.

lemma *geometric-sums*:
assumes *norm c* < *1*
shows ($\lambda n.$ *c*^*n*) *sums* (*1* / (*1* − *c*))
proof −
have *neq-0*: *c* − *1* $\neq$ *0*
using *assms* **by** *auto*
then have ($\lambda n.$ *c* ^ *n* / (*c* − *1*) − *1* / (*c* − *1*)) $\longrightarrow$ *0* / (*c* − *1*) − *1* / (*c* − *1*)
by (*intro tendsto-intros assms*)
then have ($\lambda n.$ (*c* ^ *n* − *1*) / (*c* − *1*)) $\longrightarrow$ *1* / (*1* − *c*)
by (*simp add*: *nonzero-minus-divide-right* [*OF neq-0*] *diff-divide-distrib*)
with *neq-0* **show** ($\lambda n.$ *c* ^ *n*) *sums* (*1* / (*1* − *c*))
by (*simp add*: *sums-def geometric-sum*)
qed

lemma *summable-geometric*: *norm c* < *1* $\Longrightarrow$ *summable* ($\lambda n.$ *c*^*n*)
by (*rule geometric-sums* [*THEN sums-summable*])

lemma *suminf-geometric*: *norm c* < *1* $\Longrightarrow$ *suminf* ($\lambda n.$ *c*^*n*) = *1* / (*1* − *c*)
by (*rule sums-unique*[*symmetric*]) (*rule geometric-sums*)

lemma *summable-geometric-iff* [*simp*]: *summable* ($\lambda n.$ *c* ^ *n*) $\longleftrightarrow$ *norm c* < *1*
proof
assume *summable* ($\lambda n.$ *c* ^ *n* :: *'a* :: *real-normed-field*)
then have ($\lambda n.$ *norm c* ^ *n*) $\longrightarrow$ *0*
by (*simp add*: *norm-power* [*symmetric*] *tendsto-norm-zero-iff summable-LIMSEQ-zero*)
from *order-tendstoD*(*2*)[*OF this zero-less-one*] **obtain** *n* **where** *norm c* ^ *n* < *1*
by (*auto simp*: *eventually-at-top-linorder*)
then show *norm c* < *1* **using** *one-le-power*[*of norm c n*]
by (*cases norm c* $\geq$ *1*) (*linarith*, *simp*)

qed (*rule summable-geometric*)

end

Biconditional versions for constant factors

context
fixes *c* :: *'a::real-normed-field*
begin

lemma *summable-cmult-iff* [*simp*]: *summable* ($\lambda n.\ c * f\ n$) $\longleftrightarrow$ *c=0* $\vee$ *summable f*
proof −
have ⟦*summable* ($\lambda n.\ c * f\ n$); $c \neq 0$⟧ $\Longrightarrow$ *summable f*
using *summable-mult-D* **by** *blast*
then show *?thesis*
by (*auto simp*: *summable-mult*)
qed

lemma *summable-divide-iff* [*simp*]: *summable* ($\lambda n.\ f\ n\ /\ c$) $\longleftrightarrow$ *c=0* $\vee$ *summable f*
proof −
have ⟦*summable* ($\lambda n.\ f\ n\ /\ c$); $c \neq 0$⟧ $\Longrightarrow$ *summable f*
by (*auto dest*: *summable-divide* [**where** $c = 1/c$])
then show *?thesis*
by (*auto simp*: *summable-divide*)
qed

end

lemma *power-half-series*: ($\lambda n.\ (1/2::real)$^*Suc n*) *sums 1*
proof −
have *2*: ($\lambda n.\ (1/2::real)$^*n*) *sums 2*
using *geometric-sums* [*of 1/2::real*] **by** *auto*
have ($\lambda n.\ (1/2::real)$^*Suc n*) = ($\lambda n.\ (1\ /\ 2)$ ^ $n\ /\ 2$)
by (*simp add*: *mult.commute*)
then show *?thesis*
using *sums-divide* [*OF 2*, *of 2*] **by** *simp*
qed

109.8 Telescoping

lemma *telescope-sums*:
fixes *c* :: *'a::real-normed-vector*
assumes $f \longrightarrow c$
shows ($\lambda n.\ f\ (Suc\ n) - f\ n$) *sums* ($c - f\ 0$)
unfolding *sums-def*
proof (*subst filterlim-sequentially-Suc* [*symmetric*])
have ($\lambda n.\ \sum k<Suc\ n.\ f\ (Suc\ k) - f\ k$) = ($\lambda n.\ f\ (Suc\ n) - f\ 0$)
by (*simp add*: *lessThan-Suc-atMost atLeast0AtMost* [*symmetric*] *sum-Suc-diff*)

also have ... $\longrightarrow$ *c* − *f 0*
by (*intro tendsto-diff LIMSEQ-Suc*[*OF assms*] *tendsto-const*)
finally show ($\lambda n. \sum n{<}Suc\ n.\ f\ (Suc\ n) - f\ n$) $\longrightarrow$ *c* − *f 0* **.**
qed

lemma *telescope-sums′*:
fixes *c* :: *′a*::*real-normed-vector*
assumes *f* $\longrightarrow$ *c*
shows ($\lambda n.\ f\ n - f\ (Suc\ n)$) *sums* (*f 0* − *c*)
using *sums-minus*[*OF telescope-sums*[*OF assms*]] **by** (*simp add*: *algebra-simps*)

lemma *telescope-summable*:
fixes *c* :: *′a*::*real-normed-vector*
assumes *f* $\longrightarrow$ *c*
shows *summable* ($\lambda n.\ f\ (Suc\ n) - f\ n$)
using *telescope-sums*[*OF assms*] **by** (*simp add*: *sums-iff*)

lemma *telescope-summable′*:
fixes *c* :: *′a*::*real-normed-vector*
assumes *f* $\longrightarrow$ *c*
shows *summable* ($\lambda n.\ f\ n - f\ (Suc\ n)$)
using *summable-minus*[*OF telescope-summable*[*OF assms*]] **by** (*simp add*: *algebra-simps*)

109.9 Infinite summability on Banach spaces

Cauchy-type criterion for convergence of series (c.f. Harrison).

lemma *summable-Cauchy*: *summable f* $\longleftrightarrow$ ($\forall e{>}0.\ \exists N.\ \forall m{\ge}N.\ \forall n.\ norm\ (sum\ f\ \{m..{<}n\}) < e$) (**is** - = *?rhs*)
for *f* :: *nat* ⇒ *′a*::*banach*
proof
assume *f*: *summable f*
show *?rhs*
proof *clarify*
fix *e* :: *real*
assume *0* < *e*
then obtain *M* **where** *M*: $\bigwedge m\ n.\ [\![m{\ge}M;\ n{\ge}M]\!] \Longrightarrow norm\ (sum\ f\ \{..{<}m\} - sum\ f\ \{..{<}n\}) < e$
using *f* **by** (*force simp add*: *summable-iff-convergent Cauchy-convergent-iff* [*symmetric*] *Cauchy-iff*)
have *norm* (*sum f* {*m..<n*}) < *e* **if** *m* ≥ *M* **for** *m n*
proof (*cases m n rule*: *linorder-class.le-cases*)
assume *m* ≤ *n*
then show *?thesis*
by (*metis* (*mono-tags*, *opaque-lifting*) *M atLeast0LessThan order-trans sum-diff-nat-ivl that zero-le*)
next
assume *n* ≤ *m*
then show *?thesis*

```
      by (simp add: ‹0 < e›)
    qed
    then show ∃N. ∀m≥N. ∀n. norm (sum f {m..<n}) < e
      by blast
  qed
next
  assume r: ?rhs
  then show summable f
   unfolding summable-iff-convergent Cauchy-convergent-iff [symmetric] Cauchy-iff
  proof clarify
    fix e :: real
    assume 0 < e
    with r obtain N where N: ⋀m n. m ≥ N ⟹ norm (sum f {m..<n}) < e
      by blast
    have norm (sum f {..<m} − sum f {..<n}) < e if m≥N n≥N for m n
    proof (cases m n rule: linorder-class.le-cases)
      assume m ≤ n
      then show ?thesis
          by (metis N finite-lessThan lessThan-minus-lessThan lessThan-subset-iff
norm-minus-commute sum-diff ‹m≥N›)
    next
      assume n ≤ m
      then show ?thesis
          by (metis N finite-lessThan lessThan-minus-lessThan lessThan-subset-iff
sum-diff ‹n≥N›)
    qed
    then show ∃M. ∀m≥M. ∀n≥M. norm (sum f {..<m} − sum f {..<n}) < e
      by blast
  qed
qed

lemma summable-Cauchy':
  fixes f :: nat ⇒ 'a :: banach
  assumes ev: eventually (λm. ∀n≥m. norm (sum f {m..<n}) ≤ g m) sequentially
  assumes g0: g ⟶ 0
  shows summable f
proof (subst summable-Cauchy, intro allI impI, goal-cases)
  case (1 e)
  then have ∀F x in sequentially. g x < e
    using g0 order-tendstoD(2) by blast
  with ev have eventually (λm. ∀n. norm (sum f {m..<n}) < e) at-top
  proof eventually-elim
    case (elim m)
    show ?case
    proof
      fix n
      from elim show norm (sum f {m..<n}) < e
        by (cases n ≥ m) auto
    qed
```

qed
thus *?case* **by** (*auto simp*: *eventually-at-top-linorder*)
qed

context
fixes *f* :: *nat* ⇒ *'a*::*banach*
begin

Absolute convergence imples normal convergence.

lemma *summable-norm-cancel*: *summable* (*λn*. *norm* (*f n*)) ⟹ *summable f*
unfolding *summable-Cauchy*
apply (*erule all-forward imp-forward ex-forward* | *assumption*)+
apply (*fastforce simp add*: *order-le-less-trans* [*OF norm-sum*] *order-le-less-trans* [*OF abs-ge-self*])
done

lemma *summable-norm*: *summable* (*λn*. *norm* (*f n*)) ⟹ *norm* (*suminf f*) ≤ (∑ *n*. *norm* (*f n*))
by (*auto intro*: *LIMSEQ-le tendsto-norm summable-norm-cancel summable-LIMSEQ norm-sum*)

Comparison tests.

lemma *summable-comparison-test*:
assumes *fg*: ∃ *N*. ∀ *n*≥*N*. *norm* (*f n*) ≤ *g n* **and** *g*: *summable g*
shows *summable f*
proof −
obtain *N* **where** *N*: ⋀*n*. *n*≥*N* ⟹ *norm* (*f n*) ≤ *g n*
using *assms* **by** *blast*
show *?thesis*
proof (*clarsimp simp add*: *summable-Cauchy*)
fix *e* :: *real*
assume *0* < *e*
then obtain *Ng* **where** *Ng*: ⋀*m n*. *m* ≥ *Ng* ⟹ *norm* (*sum g* {*m*..<*n*}) < *e*
using *g* **by** (*fastforce simp*: *summable-Cauchy*)
with *N* **have** *norm* (*sum f* {*m*..<*n*}) < *e* **if** *m*≥*max N Ng* **for** *m n*
proof −
have *norm* (*sum f* {*m*..<*n*}) ≤ *sum g* {*m*..<*n*}
using *N that* **by** (*force intro*: *sum-norm-le*)
also have ... ≤ *norm* (*sum g* {*m*..<*n*})
by *simp*
also have ... < *e*
using *Ng that* **by** *auto*
finally show *?thesis* **.**
qed
then show ∃ *N*. ∀ *m*≥*N*. ∀ *n*. *norm* (*sum f* {*m*..<*n*}) < *e*
by *blast*
qed
qed

lemma *summable-comparison-test-ev*:
eventually ($\lambda n.$ *norm* (*f n*) $\leq$ *g n*) *sequentially* $\Longrightarrow$ *summable g* $\Longrightarrow$ *summable f*
by (*rule summable-comparison-test*) (*auto simp*: *eventually-at-top-linorder*)

A better argument order.

lemma *summable-comparison-test'*: *summable g* $\Longrightarrow$ ($\bigwedge n.$ $n \geq N$ $\Longrightarrow$ *norm* (*f n*) $\leq$ *g n*) $\Longrightarrow$ *summable f*
by (*rule summable-comparison-test*) *auto*

109.10 The Ratio Test

```
lemma summable-ratio-test:
  assumes c < 1 ⋀n. n ≥ N ⟹ norm (f (Suc n)) ≤ c ∗ norm (f n)
  shows summable f
proof (cases 0 < c)
  case True
  show summable f
  proof (rule summable-comparison-test)
    show ∃N'. ∀n≥N'. norm (f n) ≤ (norm (f N) / (c ^ N)) ∗ c ^ n
    proof (intro exI allI impI)
      fix n
      assume N ≤ n
      then show norm (f n) ≤ (norm (f N) / (c ^ N)) ∗ c ^ n
      proof (induct rule: inc-induct)
        case base
        with True show ?case by simp
      next
        case (step m)
        have norm (f (Suc m)) / c ^ Suc m ∗ c ^ n ≤ norm (f m) / c ^ m ∗ c ^ n
          using ‹0 < c› ‹c < 1› assms(2)[OF ‹N ≤ m›] by (simp add: field-simps)
        with step show ?case by simp
      qed
    qed
    show summable (λn. norm (f N) / c ^ N ∗ c ^ n)
      using ‹0 < c› ‹c < 1› by (intro summable-mult summable-geometric) simp
  qed
next
  case False
  have f (Suc n) = 0 if n ≥ N for n
  proof −
    from that have norm (f (Suc n)) ≤ c ∗ norm (f n)
      by (rule assms(2))
    also have ... ≤ 0
      using False by (simp add: not-less mult-nonpos-nonneg)
    finally show ?thesis
      by auto
  qed
  then show summable f
    by (intro sums-summable[OF sums-finite, of {.. Suc N}]) (auto simp: not-le
```

Suc-less-eq2)
qed

end

Application to convergence of the log function

lemma *norm-summable-ln-series*:
fixes *z* :: *'a* :: {*real-normed-field*, *banach*}
assumes *norm z* < *1*
shows *summable* (λ*n*. *norm* (*z* ^ *n* / *of-nat n*))
proof (*rule summable-comparison-test*)
show *summable* (λ*n*. *norm* (*z* ^ *n*))
using *assms* **unfolding** *norm-power* **by** (*intro summable-geometric*) *auto*
have *norm z* ^ *n* / *real n* ≤ *norm z* ^ *n* **for** *n*
proof (*cases n* = *0*)
case *False*
hence *norm z* ^ *n* ∗ *1* ≤ *norm z* ^ *n* ∗ *real n*
by (*intro mult-left-mono*) *auto*
thus *?thesis*
using *False* **by** (*simp add*: *field-simps*)
qed *auto*
thus ∃*N*. ∀*n*≥*N*. *norm* (*norm* (*z* ^ *n* / *of-nat n*)) ≤ *norm* (*z* ^ *n*)
by (*intro exI*[*of - 0*]) (*auto simp*: *norm-power norm-divide*)
qed

Relations among convergence and absolute convergence for power series.

lemma *Abel-lemma*:
fixes *a* :: *nat* ⇒ *'a*::*real-normed-vector*
assumes *r*: *0* ≤ *r*
and *r0*: *r* < *r0*
and *M*: ⋀*n*. *norm* (*a n*) ∗ *r0*^*n* ≤ *M*
shows *summable* (λ*n*. *norm* (*a n*) ∗ *r*^*n*)
proof (*rule summable-comparison-test'*)
show *summable* (λ*n*. *M* ∗ (*r* / *r0*) ^ *n*)
using *assms* **by** (*auto simp add*: *summable-mult summable-geometric*)
show *norm* (*norm* (*a n*) ∗ *r* ^ *n*) ≤ *M* ∗ (*r* / *r0*) ^ *n* **for** *n*
using *r r0 M* [*of n*] *dual-order.order-iff-strict*
by (*fastforce simp add*: *abs-mult field-simps*)
qed

Summability of geometric series for real algebras.

lemma *complete-algebra-summable-geometric*:
fixes *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
assumes *norm x* < *1*
shows *summable* (λ*n*. *x* ^ *n*)
proof (*rule summable-comparison-test*)
show ∃*N*. ∀*n*≥*N*. *norm* (*x* ^ *n*) ≤ *norm x* ^ *n*
by (*simp add*: *norm-power-ineq*)
from *assms* **show** *summable* (λ*n*. *norm x* ^ *n*)

by (*simp add*: *summable-geometric*)
qed

109.11 Cauchy Product Formula

Proof based on Analysis WebNotes: Chapter 07, Class 41 http://www.math.unl.edu/~webnotes/classes/class41/prp77.htm

lemma *Cauchy-product-sums*:
fixes *a b* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*banach*}
assumes *a*: *summable* (λ*k*. *norm* (*a k*))
and *b*: *summable* (λ*k*. *norm* (*b k*))
shows (λ*k*. ∑ *i*≤*k*. *a i* ∗ *b* (*k* − *i*)) *sums* ((∑ *k*. *a k*) ∗ (∑ *k*. *b k*))
proof −
let *?S1* = λ*n*::*nat*. {..<*n*} × {..<*n*}
let *?S2* = λ*n*::*nat*. {(*i*,*j*). *i* + *j* < *n*}
have *S1-mono*: ⋀*m n*. *m* ≤ *n* ⟹ *?S1 m* ⊆ *?S1 n* **by** *auto*
have *S2-le-S1*: ⋀*n*. *?S2 n* ⊆ *?S1 n* **by** *auto*
have *S1-le-S2*: ⋀*n*. *?S1* (*n div 2*) ⊆ *?S2 n* **by** *auto*
have *finite-S1*: ⋀*n*. *finite* (*?S1 n*) **by** *simp*
with *S2-le-S1* **have** *finite-S2*: ⋀*n*. *finite* (*?S2 n*) **by** (*rule finite-subset*)

let *?g* = λ(*i*,*j*). *a i* ∗ *b j*
let *?f* = λ(*i*,*j*). *norm* (*a i*) ∗ *norm* (*b j*)
have *f-nonneg*: ⋀*x*. *0* ≤ *?f x* **by** *auto*
then have *norm-sum-f*: ⋀*A*. *norm* (*sum ?f A*) = *sum ?f A*
unfolding *real-norm-def*
by (*simp only*: *abs-of-nonneg sum-nonneg* [*rule-format*])

have (λ*n*. (∑ *k*<*n*. *a k*) ∗ (∑ *k*<*n*. *b k*)) ⟶ (∑ *k*. *a k*) ∗ (∑ *k*. *b k*)
by (*intro tendsto-mult summable-LIMSEQ summable-norm-cancel* [*OF a*] *summable-norm-cancel* [*OF b*])
then have *1*: (λ*n*. *sum ?g* (*?S1 n*)) ⟶ (∑ *k*. *a k*) ∗ (∑ *k*. *b k*)
by (*simp only*: *sum-product sum.Sigma* [*rule-format*] *finite-lessThan*)

have (λ*n*. (∑ *k*<*n*. *norm* (*a k*)) ∗ (∑ *k*<*n*. *norm* (*b k*))) ⟶ (∑ *k*. *norm* (*a k*)) ∗ (∑ *k*. *norm* (*b k*))
using *a b* **by** (*intro tendsto-mult summable-LIMSEQ*)
then have (λ*n*. *sum ?f* (*?S1 n*)) ⟶ (∑ *k*. *norm* (*a k*)) ∗ (∑ *k*. *norm* (*b k*))
by (*simp only*: *sum-product sum.Sigma* [*rule-format*] *finite-lessThan*)
then have *convergent* (λ*n*. *sum ?f* (*?S1 n*))
by (*rule convergentI*)
then have *Cauchy*: *Cauchy* (λ*n*. *sum ?f* (*?S1 n*))
by (*rule convergent-Cauchy*)
have *Zfun* (λ*n*. *sum ?f* (*?S1 n* − *?S2 n*)) *sequentially*
proof (*rule ZfunI*, *simp only*: *eventually-sequentially norm-sum-f*)
fix *r* :: *real*
assume *r*: *0* < *r*
from *CauchyD* [*OF Cauchy r*] **obtain** *N*
where ∀ *m*≥*N*. ∀ *n*≥*N*. *norm* (*sum ?f* (*?S1 m*) − *sum ?f* (*?S1 n*)) < *r* **..**

then have $\bigwedge m\ n.\ N \le n \Longrightarrow n \le m \Longrightarrow$ *norm* (*sum ?f* (*?S1 m* − *?S1 n*)) < *r*
  by (*simp only*: *sum-diff finite-S1 S1-mono*)
 then have *N*: $\bigwedge m\ n.\ N \le n \Longrightarrow n \le m \Longrightarrow$ *sum ?f* (*?S1 m* − *?S1 n*) < *r*
  by (*simp only*: *norm-sum-f*)
 show ∃ *N*. ∀ *n*≥*N*. *sum ?f* (*?S1 n* − *?S2 n*) < *r*
 proof (*intro exI allI impI*)
  fix *n*
  assume *2* ∗ *N* ≤ *n*
  then have *n*: *N* ≤ *n div 2* **by** *simp*
  have *sum ?f* (*?S1 n* − *?S2 n*) ≤ *sum ?f* (*?S1 n* − *?S1* (*n div 2*))
   by (*intro sum-mono2 finite-Diff finite-S1 f-nonneg Diff-mono subset-refl S1-le-S2*)
  also have . . . < *r*
   using *n div-le-dividend* **by** (*rule N*)
  finally show *sum ?f* (*?S1 n* − *?S2 n*) < *r* **.**
 qed
qed
then have *Zfun* (λ*n*. *sum ?g* (*?S1 n* − *?S2 n*)) *sequentially*
 apply (*rule Zfun-le* [*rule-format*])
 apply (*simp only*: *norm-sum-f*)
 apply (*rule order-trans* [*OF norm-sum sum-mono*])
 apply (*auto simp add*: *norm-mult-ineq*)
 done
then have *2*: (λ*n*. *sum ?g* (*?S1 n*) − *sum ?g* (*?S2 n*)) ⟶ *0*
 unfolding *tendsto-Zfun-iff diff-0-right*
 by (*simp only*: *sum-diff finite-S1 S2-le-S1*)
with *1* **have** (λ*n*. *sum ?g* (*?S2 n*)) ⟶ ($\sum$ *k*. *a k*) ∗ ($\sum$ *k*. *b k*)
 by (*rule Lim-transform2*)
then show *?thesis*
 by (*simp only*: *sums-def sum.triangle-reindex*)
qed

lemma *Cauchy-product*:
 fixes *a b* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*banach*}
 assumes *summable* (λ*k*. *norm* (*a k*))
  and *summable* (λ*k*. *norm* (*b k*))
 shows ($\sum$ *k*. *a k*) ∗ ($\sum$ *k*. *b k*) = ($\sum$ *k*. $\sum$ *i*≤*k*. *a i* ∗ *b* (*k* − *i*))
 using *assms* **by** (*rule Cauchy-product-sums* [*THEN sums-unique*])

lemma *summable-Cauchy-product*:
 fixes *a b* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*banach*}
 assumes *summable* (λ*k*. *norm* (*a k*))
  and *summable* (λ*k*. *norm* (*b k*))
 shows *summable* (λ*k*. $\sum$ *i*≤*k*. *a i* ∗ *b* (*k* − *i*))
 using *Cauchy-product-sums*[*OF assms*] **by** (*simp add*: *sums-iff*)

109.12 Series on *real*s

lemma *summable-norm-comparison-test*:
$\exists N. \forall n \geq N.$ *norm* $(f\ n) \leq g\ n \Longrightarrow$ *summable* $g \Longrightarrow$ *summable* $(\lambda n.$ *norm* $(f\ n))$
by (*rule summable-comparison-test*) *auto*

lemma *summable-rabs-comparison-test*: $\exists N. \forall n \geq N. |f\ n| \leq g\ n \Longrightarrow$ *summable* g
$\Longrightarrow$ *summable* $(\lambda n. |f\ n|)$
for f :: *nat* $\Rightarrow$ *real*
by (*rule summable-comparison-test*) *auto*

lemma *summable-rabs-cancel*: *summable* $(\lambda n. |f\ n|) \Longrightarrow$ *summable* f
for f :: *nat* $\Rightarrow$ *real*
by (*rule summable-norm-cancel*) *simp*

lemma *summable-rabs*: *summable* $(\lambda n. |f\ n|) \Longrightarrow |$*suminf* $f| \leq (\sum n. |f\ n|)$
for f :: *nat* $\Rightarrow$ *real*
by (*fold real-norm-def*) (*rule summable-norm*)

lemma *norm-suminf-le*:
assumes $\bigwedge n.$ *norm* $(f\ n$ :: $'a$ :: *banach*$) \leq g\ n$ *summable* g
shows *norm* (*suminf* f) $\leq$ *suminf* g
proof −
have ∗: *summable* $(\lambda n.$ *norm* $(f\ n))$
using *assms summable-norm-comparison-test* **by** *blast*
hence *norm* (*suminf* f) $\leq (\sum n.$ *norm* $(f\ n))$ **by** (*intro summable-norm*) *auto*
also have ... $\leq$ *suminf* g **by** (*intro suminf-le* ∗ *assms allI*)
finally show *?thesis* **.**
qed

lemma *summable-zero-power* [*simp*]: *summable* $(\lambda n.\ 0$ ^ n :: $'a$::{*comm-ring-1*,*topological-space*})
proof −
have $(\lambda n.\ 0$ ^ n :: $'a) = (\lambda n.$ *if* $n = 0$ *then* 0^0 *else* $0)$
by (*intro ext*) (*simp add*: *zero-power*)
moreover have *summable* ... **by** *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *summable-zero-power'* [*simp*]: *summable* $(\lambda n.\ f\ n * 0$ ^ n :: $'a$::{*ring-1*,*topological-space*})
proof −
have $(\lambda n.\ f\ n * 0$ ^ n :: $'a) = (\lambda n.$ *if* $n = 0$ *then* $f\ 0 * 0$^0 *else* $0)$
by (*intro ext*) (*simp add*: *zero-power*)
moreover have *summable* ... **by** *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *summable-power-series*:
fixes z :: *real*
assumes *le-1*: $\bigwedge i.\ f\ i \leq 1$
and *nonneg*: $\bigwedge i.\ 0 \leq f\ i$

and *z*: *0* ≤ *z z* < *1*
shows *summable* (λ*i*. *f i* ∗ *z*^*i*)
proof (*rule summable-comparison-test*[*OF* - *summable-geometric*])
show *norm z* < *1*
using *z* **by** (*auto simp*: *less-imp-le*)
show ⋀*n*. ∃ *N*. ∀ *na*≥*N*. *norm* (*f na* ∗ *z* ^ *na*) ≤ *z* ^ *na*
using *z*
by (*auto intro*!: *exI*[*of* - *0*] *mult-left-le-one-le simp*: *abs-mult nonneg power-abs less-imp-le le-1*)
qed

lemma *summable-0-powser*: *summable* (λ*n*. *f n* ∗ *0* ^ *n* :: *'a*::*real-normed-div-algebra*)
by *simp*

lemma *summable-powser-split-head*:
summable (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n* :: *'a*::*real-normed-div-algebra*) = *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
proof −
have *summable* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*) ⟷ *summable* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *Suc n*)
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?rhs* **if** *?lhs*
using *summable-mult2*[*OF that*, *of z*]
by (*simp add*: *power-commutes algebra-simps*)
show *?lhs* **if** *?rhs*
using *summable-mult2*[*OF that*, *of inverse z*]
by (*cases z* ≠ *0*, *subst* (*asm*) *power-Suc2*) (*simp-all add*: *algebra-simps*)
qed
also have ... ⟷ *summable* (λ*n*. *f n* ∗ *z* ^ *n*) **by** (*rule summable-Suc-iff*)
finally show *?thesis* **.**
qed

lemma *summable-powser-ignore-initial-segment*:
fixes *f* :: *nat* ⇒ *'a* :: *real-normed-div-algebra*
shows *summable* (λ*n*. *f* (*n* + *m*) ∗ *z* ^ *n*) ⟷ *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
proof (*induction m*)
case (*Suc m*)
have *summable* (λ*n*. *f* (*n* + *Suc m*) ∗ *z* ^ *n*) = *summable* (λ*n*. *f* (*Suc n* + *m*) ∗ *z* ^ *n*)
by *simp*
also have ... = *summable* (λ*n*. *f* (*n* + *m*) ∗ *z* ^ *n*)
by (*rule summable-powser-split-head*)
also have ... = *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
by (*rule Suc.IH*)
finally show *?case* **.**
qed *simp-all*

lemma *powser-split-head*:

fixes *f* :: *nat* ⇒ *'a*::{*real-normed-div-algebra*,*banach*}
assumes *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
shows *suminf* (λ*n*. *f n* ∗ *z* ^ *n*) = *f 0* + *suminf* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*) ∗ *z*
and *suminf* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*) ∗ *z* = *suminf* (λ*n*. *f n* ∗ *z* ^ *n*) − *f 0*
and *summable* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*)
proof −
from *assms* **show** *summable* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*)
by (*subst summable-powser-split-head*)
from *suminf-mult2*[*OF this*, *of z*]
have (∑*n*. *f* (*Suc n*) ∗ *z* ^ *n*) ∗ *z* = (∑*n*. *f* (*Suc n*) ∗ *z* ^ *Suc n*)
by (*simp add*: *power-commutes algebra-simps*)
also from *assms* **have** ... = *suminf* (λ*n*. *f n* ∗ *z* ^ *n*) − *f 0*
by (*subst suminf-split-head*) *simp-all*
finally show *suminf* (λ*n*. *f n* ∗ *z* ^ *n*) = *f 0* + *suminf* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*) ∗ *z*
by *simp*
then show *suminf* (λ*n*. *f* (*Suc n*) ∗ *z* ^ *n*) ∗ *z* = *suminf* (λ*n*. *f n* ∗ *z* ^ *n*) − *f 0*
by *simp*
qed

lemma *summable-partial-sum-bound*:
fixes *f* :: *nat* ⇒ *'a* :: *banach*
and *e* :: *real*
assumes *summable*: *summable f*
and *e*: *e* > *0*
obtains *N* **where** ⋀*m n*. *m* ≥ *N* ⟹ *norm* (∑*k*=*m*..*n*. *f k*) < *e*
proof −
from *summable* **have** *Cauchy* (λ*n*. ∑*k*<*n*. *f k*)
by (*simp add*: *Cauchy-convergent-iff summable-iff-convergent*)
from *CauchyD* [*OF this e*] **obtain** *N*
where *N*: ⋀*m n*. *m* ≥ *N* ⟹ *n* ≥ *N* ⟹ *norm* ((∑*k*<*m*. *f k*) − (∑*k*<*n*. *f k*)) < *e*
by *blast*
have *norm* (∑*k*=*m*..*n*. *f k*) < *e* **if** *m*: *m* ≥ *N* **for** *m n*
proof (*cases n* ≥ *m*)
case *True*
with *m* **have** *norm* ((∑*k*<*Suc n*. *f k*) − (∑*k*<*m*. *f k*)) < *e*
by (*intro N*) *simp-all*
also from *True* **have** (∑*k*<*Suc n*. *f k*) − (∑*k*<*m*. *f k*) = (∑*k*=*m*..*n*. *f k*)
by (*subst sum-diff* [*symmetric*]) (*simp-all add*: *sum.last-plus*)
finally show *?thesis* **.**
next
case *False*
with *e* **show** *?thesis* **by** *simp-all*
qed
then show *?thesis* **by** (*rule that*)
qed

lemma *powser-sums-if*:

$(\lambda n.\ (\mathit{if}\ n = m\ \mathit{then}\ (\mathit{1} :: {'}a{::}\{\mathit{ring\text{-}1}, \mathit{topological\text{-}space}\})\ \mathit{else}\ \mathit{0}) * z\hat{\ }n)\ \mathit{sums}\ z\hat{\ }m$
proof −
  have $(\lambda n.\ (\mathit{if}\ n = m\ \mathit{then}\ \mathit{1}\ \mathit{else}\ \mathit{0}) * z\hat{\ }n) = (\lambda n.\ \mathit{if}\ n = m\ \mathit{then}\ z\hat{\ }n\ \mathit{else}\ \mathit{0})$
    by (*intro ext*) *auto*
  then show *?thesis*
    by (*simp add*: *sums-single*)
qed

lemma
  fixes $f :: \mathit{nat} \Rightarrow \mathit{real}$
  assumes *summable f*
    and *inj g*
    and *pos*: $\bigwedge x.\ \mathit{0} \le f\ x$
  shows *summable-reindex*: *summable* $(f \circ g)$
    and *suminf-reindex-mono*: *suminf* $(f \circ g) \le$ *suminf f*
    and *suminf-reindex*: $(\bigwedge x.\ x \notin \mathit{range}\ g \Longrightarrow f\ x = \mathit{0}) \Longrightarrow \mathit{suminf}\ (f \circ g) =$ *suminf f*
proof −
  from ‹*inj g*› **have** [*simp*]: $\bigwedge A.\ \mathit{inj\text{-}on}\ g\ A$
    by (*rule subset-inj-on*) *simp*

  have *smaller*: $\forall n.\ (\sum i<n.\ (f \circ g)\ i) \le \mathit{suminf}\ f$
  proof
    fix *n*
    have $\forall\ n' \in (g\ `\ \{..<n\}).\ n' < \mathit{Suc}\ (\mathit{Max}\ (g\ `\ \{..<n\}))$
      by (*metis Max-ge finite-imageI finite-lessThan not-le not-less-eq*)
    then obtain *m* **where** *n*: $\bigwedge n'.\ n' < n \Longrightarrow g\ n' < m$
      by *blast*

    have $(\sum i<n.\ f\ (g\ i)) = \mathit{sum}\ f\ (g\ `\ \{..<n\})$
      by (*simp add*: *sum.reindex*)
    also have $\ldots \le (\sum i<m.\ f\ i)$
      by (*rule sum-mono2*) (*auto simp add*: *pos n*[*rule-format*])
    also have $\ldots \le$ *suminf f*
      using ‹*summable f*›
      by (*rule sum-le-suminf*) (*simp-all add*: *pos*)
    finally show $(\sum i<n.\ (f \circ g)\ i) \le \mathit{suminf}\ f$
      by *simp*
  qed

  have *incseq* $(\lambda n.\ \sum i<n.\ (f \circ g)\ i)$
    by (*rule incseq-SucI*) (*auto simp add*: *pos*)
  then obtain *L* **where** *L*: $(\lambda\ n.\ \sum i<n.\ (f \circ g)\ i) \longrightarrow L$
    using *smaller* **by**(*rule incseq-convergent*)
  then have $(f \circ g)$ *sums L*
    by (*simp add*: *sums-def*)
  then show *summable* $(f \circ g)$
    by (*auto simp add*: *sums-iff*)

then have $(\lambda n.\ \sum i{<}n.\ (f \circ g)\ i) \longrightarrow suminf\ (f \circ g)$
by (*rule summable-LIMSEQ*)
then show *le*: $suminf\ (f \circ g) \le suminf\ f$
by(*rule LIMSEQ-le-const2*)(*blast intro*: *smaller*[*rule-format*])

assume *f*: $\bigwedge x.\ x \notin range\ g \Longrightarrow f\ x = 0$

from ‹*summable f*› **have** $suminf\ f \le suminf\ (f \circ g)$
proof (*rule suminf-le-const*)
fix *n*
have $\forall\ n' \in (g -`\{..<n\}).\ n' < Suc\ (Max\ (g -`\{..<n\}))$
by(*auto intro*: *Max-ge simp add*: *finite-vimageI less-Suc-eq-le*)
then obtain *m* **where** *n*: $\bigwedge n'.\ g\ n' < n \Longrightarrow n' < m$
by *blast*
have $(\sum i{<}n.\ f\ i) = (\sum i{\in}\{..<n\} \cap range\ g.\ f\ i)$
using *f* **by**(*auto intro*: *sum.mono-neutral-cong-right*)
also have $\ldots = (\sum i{\in}g -`\{..<n\}.\ (f \circ g)\ i)$
by (*rule sum.reindex-cong*[**where** *l*=*g*])(*auto*)
also have $\ldots \le (\sum i{<}m.\ (f \circ g)\ i)$
by (*rule sum-mono2*)(*auto simp add*: *pos n*)
also have $\ldots \le suminf\ (f \circ g)$
using ‹*summable* $(f \circ g)$› **by** (*rule sum-le-suminf*) (*simp-all add*: *pos*)
finally show $sum\ f\ \{..<n\} \le suminf\ (f \circ g)$ **.**
qed
with *le* **show** $suminf\ (f \circ g) = suminf\ f$
by (*rule antisym*)
qed

lemma *sums-mono-reindex*:
assumes *subseq*: *strict-mono g*
and *zero*: $\bigwedge n.\ n \notin range\ g \Longrightarrow f\ n = 0$
shows $(\lambda n.\ f\ (g\ n))\ sums\ c \longleftrightarrow f\ sums\ c$
unfolding *sums-def*
proof
assume *lim*: $(\lambda n.\ \sum k{<}n.\ f\ k) \longrightarrow c$
have $(\lambda n.\ \sum k{<}n.\ f\ (g\ k)) = (\lambda n.\ \sum k{<}g\ n.\ f\ k)$
proof
fix *n* :: *nat*
from *subseq* **have** $(\sum k{<}n.\ f\ (g\ k)) = (\sum k{\in}g`\{..<n\}.\ f\ k)$
by (*subst sum.reindex*) (*auto intro*: *strict-mono-imp-inj-on*)
also from *subseq* **have** $\ldots = (\sum k{<}g\ n.\ f\ k)$
by (*intro sum.mono-neutral-left ballI zero*)
(*auto simp*: *strict-mono-less strict-mono-less-eq*)
finally show $(\sum k{<}n.\ f\ (g\ k)) = (\sum k{<}g\ n.\ f\ k)$ **.**
qed
also from *LIMSEQ-subseq-LIMSEQ*[*OF lim subseq*] **have** $\ldots \longrightarrow c$
by (*simp only*: *o-def*)
finally show $(\lambda n.\ \sum k{<}n.\ f\ (g\ k)) \longrightarrow c$ **.**
next

assume *lim*: $(\lambda n. \sum k{<}n.\ f\ (g\ k)) \longrightarrow c$
define *g-inv* **where** *g-inv* $n = (LEAST\ m.\ g\ m \geq n)$ **for** n
from *filterlim-subseq*[*OF subseq*] **have** *g-inv-ex*: $\exists m.\ g\ m \geq n$ **for** n
  by (*auto simp*: *filterlim-at-top eventually-at-top-linorder*)
then have *g-inv*: $g\ (g\text{-}inv\ n) \geq n$ **for** n
  unfolding *g-inv-def* **by** (*rule LeastI-ex*)
have *g-inv-least*: $m \geq g\text{-}inv\ n$ **if** $g\ m \geq n$ **for** $m\ n$
  using *that* **unfolding** *g-inv-def* **by** (*rule Least-le*)
have *g-inv-least'*: $g\ m < n$ **if** $m < g\text{-}inv\ n$ **for** $m\ n$
  using *that g-inv-least*[*of n m*] **by** *linarith*
have $(\lambda n. \sum k{<}n.\ f\ k) = (\lambda n. \sum k{<}g\text{-}inv\ n.\ f\ (g\ k))$
proof
  fix n :: *nat*
  {
    fix k
    assume k: $k \in \{..{<}n\} - g`\{..{<}g\text{-}inv\ n\}$
    have $k \notin range\ g$
    proof (*rule notI*, *elim imageE*)
      fix l
      assume l: $k = g\ l$
      have $g\ l < g\ (g\text{-}inv\ n)$
        by (*rule less-le-trans*[*OF - g-inv*]) (*use k l* **in** *simp-all*)
      with *subseq* **have** $l < g\text{-}inv\ n$
        by (*simp add*: *strict-mono-less*)
      with *k l* **show** *False*
        by *simp*
    qed
    then have $f\ k = 0$
      by (*rule zero*)
  }
  with *g-inv-least' g-inv* **have** $(\sum k{<}n.\ f\ k) = (\sum k{\in}g`\{..{<}g\text{-}inv\ n\}.\ f\ k)$
    by (*intro sum.mono-neutral-right*) *auto*
  also from *subseq* **have** $\ldots = (\sum k{<}g\text{-}inv\ n.\ f\ (g\ k))$
    using *strict-mono-imp-inj-on* **by** (*subst sum.reindex*) *simp-all*
  finally show $(\sum k{<}n.\ f\ k) = (\sum k{<}g\text{-}inv\ n.\ f\ (g\ k))$ **.**
qed
also {
  fix $K\ n$ :: *nat*
  assume $g\ K \leq n$
  also have $n \leq g\ (g\text{-}inv\ n)$
    by (*rule g-inv*)
  finally have $K \leq g\text{-}inv\ n$
    using *subseq* **by** (*simp add*: *strict-mono-less-eq*)
}
then have *filterlim g-inv at-top sequentially*
  by (*auto simp*: *filterlim-at-top eventually-at-top-linorder*)
with *lim* **have** $(\lambda n. \sum k{<}g\text{-}inv\ n.\ f\ (g\ k)) \longrightarrow c$
  by (*rule filterlim-compose*)
finally show $(\lambda n. \sum k{<}n.\ f\ k) \longrightarrow c$ **.**

qed

lemma *summable-mono-reindex*:
  assumes *subseq*: *strict-mono g*
    and *zero*: $\bigwedge n.\ n \notin range\ g \Longrightarrow f\ n = 0$
  shows *summable* $(\lambda n.\ f\ (g\ n)) \longleftrightarrow$ *summable f*
  using *sums-mono-reindex*[*of g f*, *OF assms*] **by** (*simp add*: *summable-def*)

lemma *suminf-mono-reindex*:
  fixes $f :: nat \Rightarrow 'a::\{t2\text{-}space, comm\text{-}monoid\text{-}add\}$
  assumes *strict-mono g* $\bigwedge n.\ n \notin range\ g \Longrightarrow f\ n = 0$
  shows *suminf* $(\lambda n.\ f\ (g\ n)) =$ *suminf f*
proof (*cases summable f*)
  case *True*
  with *sums-mono-reindex* [*of g f*, *OF assms*]
    and *summable-mono-reindex* [*of g f*, *OF assms*]
  show *?thesis*
    by (*simp add*: *sums-iff*)
next
  case *False*
  then have $\neg(\exists\, c.\ f\ sums\ c)$
    unfolding *summable-def* **by** *blast*
  then have *suminf f* = *The* (λ-. *False*)
    by (*simp add*: *suminf-def*)
  moreover from *False* **have** ¬ *summable* $(\lambda n.\ f\ (g\ n))$
    using *summable-mono-reindex*[*of g f*, *OF assms*] **by** *simp*
  then have $\neg(\exists\, c.\ (\lambda n.\ f\ (g\ n))\ sums\ c)$
    unfolding *summable-def* **by** *blast*
  then have *suminf* $(\lambda n.\ f\ (g\ n)) =$ *The* (λ-. *False*)
    by (*simp add*: *suminf-def*)
  ultimately show *?thesis* **by** *simp*
qed

lemma *summable-bounded-partials*:
  fixes $f :: nat \Rightarrow 'a :: \{real\text{-}normed\text{-}vector, complete\text{-}space\}$
  assumes *bound*: *eventually* $(\lambda x0.\ \forall\, a \geq x0.\ \forall\, b > a.\ norm\ (sum\ f\ \{a<..b\}) \leq g\ a)$ *sequentially*
  assumes *g*: $g \longrightarrow 0$
  shows *summable f* **unfolding** *summable-iff-convergent'*
proof (*intro Cauchy-convergent CauchyI'*, *goal-cases*)
  case (*1* ε)
  with *g* **have** *eventually* $(\lambda x.\ |g\ x| < \varepsilon)$ *sequentially*
    by (*auto simp*: *tendsto-iff*)
  from *eventually-conj*[*OF this bound*] **obtain** *x0* **where** *x0*:
    $\bigwedge x.\ x \geq x0 \Longrightarrow |g\ x| < \varepsilon\ \bigwedge a\ b.\ x0 \leq a \Longrightarrow a < b \Longrightarrow norm\ (sum\ f\ \{a<..b\}) \leq g\ a$
    unfolding *eventually-at-top-linorder* **by** *auto*
  show *?case*
  proof (*intro exI*[*of - x0*] *allI impI*)

fix *m n* **assume** *mn*: $x0 \le m\ m < n$
have *dist* (*sum f* {..*m*}) (*sum f* {..*n*}) = *norm* (*sum f* {..*n*} − *sum f* {..*m*})
by (*simp add*: *dist-norm norm-minus-commute*)
also have *sum f* {..*n*} − *sum f* {..*m*} = *sum f* ({..*n*} − {..*m*})
using *mn* **by** (*intro Groups-Big.sum-diff* [*symmetric*]) *auto*
also have {..*n*} − {..*m*} = {*m*<..*n*} **using** *mn* **by** *auto*
also have *norm* (*sum f* {*m*<..*n*}) ≤ *g m* **using** *mn* **by** (*intro x0*) *auto*
also have ... ≤ |*g m*| **by** *simp*
also have ... < ε **using** *mn* **by** (*intro x0*) *auto*
finally show *dist* (*sum f* {..*m*}) (*sum f* {..*n*}) < ε **.**
qed
qed

end

110 Differentiation

theory *Deriv*
imports *Limits*
begin

110.1 Frechet derivative

definition *has-derivative* :: (*'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*) ⇒
(*'a* ⇒ *'b*) ⇒ *'a filter* ⇒ *bool* (**infix** ‹(*has'-derivative*)› *50*)
where (*f has-derivative f'*) *F* ⟷
bounded-linear f' ∧
((λ*y*. ((*f y* − *f* (*Lim F* (λ*x*. *x*))) − *f'* (*y* − *Lim F* (λ*x*. *x*))) $/_R$ *norm* (*y* − *Lim F* (λ*x*. *x*))) ⟶ *0*) *F*

Usually the filter *F* is *at x within s*. (*f has-derivative D*) (*at x within s*) means: *D* is the derivative of function *f* at point *x* within the set *s*. Where *s* is used to express left or right sided derivatives. In most cases *s* is either a variable or *UNIV*.

These are the only cases we'll care about, probably.

lemma *has-derivative-within*: (*f has-derivative f'*) (*at x within s*) ⟷
bounded-linear f' ∧ ((λ*y*. (*1* / *norm*(*y* − *x*)) $*_R$ (*f y* − (*f x* + *f'* (*y* − *x*)))) ⟶ *0*) (*at x within s*)
unfolding *has-derivative-def tendsto-iff*
by (*subst eventually-Lim-ident-at*) (*auto simp add*: *field-simps*)

lemma *has-derivative-eq-rhs*: (*f has-derivative f'*) *F* ⟹ *f'* = *g'* ⟹ (*f has-derivative g'*) *F*
by *simp*

definition *has-field-derivative* :: (*'a*::*real-normed-field* ⇒ *'a*) ⇒ *'a* ⇒ *'a filter* ⇒ *bool*
(**infix** ‹(*has'-field'-derivative*)› *50*)

where (*f has-field-derivative D*) *F* ⟷ (*f has-derivative* (∗) *D*) *F*

lemma *DERIV-cong*: (*f has-field-derivative X*) *F* ⟹ *X* = *Y* ⟹ (*f has-field-derivative Y*) *F*
by *simp*

definition *has-vector-derivative* :: (*real* ⇒ ′*b*::*real-normed-vector*) ⇒ ′*b* ⇒ *real filter* ⇒ *bool*
(**infix** ‹*has*′-*vector*′-*derivative*› *50*)
where (*f has-vector-derivative f*′) *net* ⟷ (*f has-derivative* (λ*x*. *x* $*_R$ *f*′)) *net*

lemma *has-vector-derivative-eq-rhs*:
(*f has-vector-derivative X*) *F* ⟹ *X* = *Y* ⟹ (*f has-vector-derivative Y*) *F*
by *simp*

named-theorems *derivative-intros structural introduction rules for derivatives*
setup ‹
```
let
  val eq-thms = @{thms has-derivative-eq-rhs DERIV-cong has-vector-derivative-eq-rhs}
  fun eq-rule thm = get-first (try (fn eq-thm => eq-thm OF [thm])) eq-thms
in
  Global-Theory.add-thms-dynamic
    (binding ‹derivative-eq-intros›,
      fn context =>
      Named-Theorems.get (Context.proof-of context) named-theorems ‹derivative-intros›
        |> map-filter eq-rule)
end
```
›

The following syntax is only used as a legacy syntax.

abbreviation (*input*)
FDERIV :: (′*a*::*real-normed-vector* ⇒ ′*b*::*real-normed-vector*) ⇒ ′*a* ⇒ (′*a* ⇒ ′*b*) ⇒ *bool*
(‹(‹*notation*=‹*mixfix FDERIV*››*FDERIV* (-)/ (-)/ :> (-))› [*1000*, *1000*, *60*] *60*)
where *FDERIV f x* :> *f*′ ≡ (*f has-derivative f*′) (*at x*)

lemma *has-derivative-bounded-linear*: (*f has-derivative f*′) *F* ⟹ *bounded-linear f*′
by (*simp add*: *has-derivative-def*)

lemma *has-derivative-linear*: (*f has-derivative f*′) *F* ⟹ *linear f*′
using *bounded-linear*.*linear*[*OF has-derivative-bounded-linear*] **.**

lemma *has-derivative-ident*[*derivative-intros*, *simp*]: ((λ*x*. *x*) *has-derivative* (λ*x*. *x*)) *F*
by (*simp add*: *has-derivative-def*)

lemma *has-derivative-id* [*derivative-intros*, *simp*]: (*id has-derivative id*) *F*
by (*metis eq-id-iff has-derivative-ident*)

lemma *shift-has-derivative-id*: ((+) *d has-derivative* (λ*x*. *x*)) *F*
using *has-derivative-def* **by** *fastforce*

lemma *has-derivative-const*[*derivative-intros*, *simp*]: ((λ*x*. *c*) *has-derivative* (λ*x*. *0*)) *F*
by (*simp add*: *has-derivative-def*)

lemma (**in** *bounded-linear*) *bounded-linear*: *bounded-linear f* **..**

lemma (**in** *bounded-linear*) *has-derivative*:
(*g has-derivative g'*) *F* ⟹ ((λ*x*. *f* (*g x*)) *has-derivative* (λ*x*. *f* (*g' x*))) *F*
unfolding *has-derivative-def*
by (*auto simp add*: *bounded-linear-compose* [*OF bounded-linear*] *scaleR diff dest*: *tendsto*)

lemma *has-derivative-bot* [*intro*]: *bounded-linear f'* ⟹ (*f has-derivative f'*) *bot*
by (*auto simp*: *has-derivative-def*)

lemma *has-field-derivative-bot* [*simp*, *intro*]: (*f has-field-derivative f'*) *bot*
by (*auto simp*: *has-field-derivative-def intro*!: *has-derivative-bot bounded-linear-mult-right*)

lemmas *has-derivative-scaleR-right* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-scaleR-right*]

lemmas *has-derivative-scaleR-left* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-scaleR-left*]

lemmas *has-derivative-mult-right* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-mult-right*]

lemmas *has-derivative-mult-left* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-mult-left*]

lemmas *has-derivative-of-real*[*derivative-intros*, *simp*] =
bounded-linear.has-derivative[*OF bounded-linear-of-real*]

lemma *has-derivative-add*[*simp*, *derivative-intros*]:
assumes *f*: (*f has-derivative f'*) *F*
and *g*: (*g has-derivative g'*) *F*
shows ((λ*x*. *f x* + *g x*) *has-derivative* (λ*x*. *f' x* + *g' x*)) *F*
unfolding *has-derivative-def*
proof *safe*
let *?x* = *Lim F* (λ*x*. *x*)
let *?D* = λ*f f' y*. ((*f y* − *f ?x*) − *f'* (*y* − *?x*)) $/_R$ *norm* (*y* − *?x*)
have ((λ*x*. *?D f f' x* + *?D g g' x*) ⟶ (*0* + *0*)) *F*
using *f g* **by** (*intro tendsto-add*) (*auto simp*: *has-derivative-def*)
then show (*?D* (λ*x*. *f x* + *g x*) (λ*x*. *f' x* + *g' x*) ⟶ *0*) *F*
by (*simp add*: *field-simps scaleR-add-right scaleR-diff-right*)
qed (*blast intro*: *bounded-linear-add f g has-derivative-bounded-linear*)

lemma *has-derivative-sum*[*simp*, *derivative-intros*]:
($\bigwedge i.\ i \in I \Longrightarrow (f\ i$ *has-derivative* $f'\ i)\ F) \Longrightarrow$
$((\lambda x. \sum i{\in}I.\ f\ i\ x)$ *has-derivative* $(\lambda x. \sum i{\in}I.\ f'\ i\ x))\ F$
by (*induct I rule*: *infinite-finite-induct*) *simp-all*

lemma *has-derivative-minus*[*simp*, *derivative-intros*]:
(f *has-derivative* f') $F \Longrightarrow ((\lambda x. - f\ x)$ *has-derivative* $(\lambda x. - f'\ x))\ F$
using *has-derivative-scaleR-right*[*of f f' F −1*] **by** *simp*

lemma *has-derivative-diff*[*simp*, *derivative-intros*]:
(f *has-derivative* f') $F \Longrightarrow$ (g *has-derivative* g') $F \Longrightarrow$
$((\lambda x.\ f\ x - g\ x)$ *has-derivative* $(\lambda x.\ f'\ x - g'\ x))\ F$
by (*simp only*: *diff-conv-add-uminus has-derivative-add has-derivative-minus*)

lemma *has-derivative-at-within*:
(f *has-derivative* f') (*at x within s*) $\longleftrightarrow$
(*bounded-linear* $f' \wedge ((\lambda y.\ ((f\ y - f\ x) - f'\ (y - x))\ /_R$ *norm* $(y - x)) \longrightarrow$
0) (*at x within s*))
proof (*cases at x within s = bot*)
case *True*
then show *?thesis*
by (*metis* (*no-types*, *lifting*) *has-derivative-within tendsto-bot*)
next
case *False*
then show *?thesis*
by (*simp add*: *Lim-ident-at has-derivative-def*)
qed

lemma *has-derivative-iff-norm*:
(f *has-derivative* f') (*at x within s*) $\longleftrightarrow$
bounded-linear $f' \wedge ((\lambda y.$ *norm* $((f\ y - f\ x) - f'\ (y - x))$ / *norm* $(y - x))$
$\longrightarrow$ *0*) (*at x within s*)
using *tendsto-norm-zero-iff*[*of - at x within s*, **where** *'b='b*, *symmetric*]
by (*simp add*: *has-derivative-at-within divide-inverse ac-simps*)

lemma *has-derivative-at*:
(f *has-derivative* D) (*at x*) $\longleftrightarrow$
(*bounded-linear* $D \wedge (\lambda h.$ *norm* $(f\ (x + h) - f\ x - D\ h)$ / *norm* $h)$ −*0*→ *0*)
by (*simp add*: *has-derivative-iff-norm LIM-offset-zero-iff*)

lemma *field-has-derivative-at*:
fixes x :: *'a::real-normed-field*
shows (f *has-derivative* $(*)\ D$) (*at x*) $\longleftrightarrow (\lambda h.\ (f\ (x + h) - f\ x) / h)$ −*0*→ D
(**is** *?lhs = ?rhs*)
proof −
have *?lhs* = $(\lambda h.$ *norm* $(f\ (x + h) - f\ x - D * h)$ / *norm* $h)$ −*0* → *0*
by (*simp add*: *bounded-linear-mult-right has-derivative-at*)
also have ... = $(\lambda y.$ *norm* $((f\ (x + y) - f\ x - D * y) / y))$ −*0*→ *0*

by (*simp cong*: *LIM-cong flip*: *nonzero-norm-divide*)
also have ... = (λ*y*. *norm* ((*f* (*x* + *y*) − *f x*) / *y* − *D* / *y* ∗ *y*)) −*0*→ *0*
by (*simp only*: *diff-divide-distrib times-divide-eq-left* [*symmetric*])
also have ... = *?rhs*
by (*simp add*: *tendsto-norm-zero-iff LIM-zero-iff cong*: *LIM-cong*)
finally show *?thesis* **.**
qed

lemma *has-derivative-iff-Ex*:
(*f has-derivative f′*) (*at x*) ⟷
bounded-linear f′ ∧ (∃ *e*. (∀ *h*. *f* (*x*+*h*) = *f x* + *f′ h* + *e h*) ∧ ((λ*h*. *norm* (*e h*) / *norm h*) ⟶ *0*) (*at 0*))
unfolding *has-derivative-at* **by** *force*

lemma *has-derivative-at-within-iff-Ex*:
assumes *x* ∈ *S open S*
shows (*f has-derivative f′*) (*at x within S*) ⟷
bounded-linear f′ ∧ (∃ *e*. (∀ *h*. *x*+*h* ∈ *S* ⟶ *f* (*x*+*h*) = *f x* + *f′ h* + *e h*) ∧ ((λ*h*. *norm* (*e h*) / *norm h*) ⟶ *0*) (*at 0*))
(**is** *?lhs* = *?rhs*)
proof *safe*
show *bounded-linear f′*
if (*f has-derivative f′*) (*at x within S*)
using *has-derivative-bounded-linear that* **by** *blast*
show ∃ *e*. (∀ *h*. *x* + *h* ∈ *S* ⟶ *f* (*x* + *h*) = *f x* + *f′ h* + *e h*) ∧ (λ*h*. *norm* (*e h*) / *norm h*) −*0*→ *0*
if (*f has-derivative f′*) (*at x within S*)
by (*metis* (*full-types*) *assms that has-derivative-iff-Ex at-within-open*)
show (*f has-derivative f′*) (*at x within S*)
if *bounded-linear f′*
and *eq* [*rule-format*]: ∀ *h*. *x* + *h* ∈ *S* ⟶ *f* (*x* + *h*) = *f x* + *f′ h* + *e h*
and *0*: (λ*h*. *norm* (*e* (*h*::′*a*)::′*b*) / *norm h*) −*0*→ *0*
for *e*
proof −
have *1*: *f y* − *f x* = *f′* (*y*−*x*) + *e* (*y*−*x*) **if** *y* ∈ *S* **for** *y*
using *eq* [*of y*−*x*] *that* **by** *simp*
have *2*: ((λ*y*. *norm* (*e* (*y*−*x*)) / *norm* (*y* − *x*)) ⟶ *0*) (*at x within S*)
by (*simp add*: *0 assms tendsto-offset-zero-iff*)
have ((λ*y*. *norm* (*f y* − *f x* − *f′* (*y* − *x*)) / *norm* (*y* − *x*)) ⟶ *0*) (*at x within S*)
by (*simp add*: *Lim-cong-within 1 2*)
then show *?thesis*
by (*simp add*: *has-derivative-iff-norm* ‹*bounded-linear f′*›)
qed
qed

lemma *has-derivativeI*:
bounded-linear f′ ⟹
((λ*y*. ((*f y* − *f x*) − *f′* (*y* − *x*)) $/_R$ *norm* (*y* − *x*)) ⟶ *0*) (*at x within s*) ⟹

(*f has-derivative f′*) (*at x within s*)
by (*simp add*: *has-derivative-at-within*)

lemma *has-derivativeI-sandwich*:
assumes *e*: $0 < e$
and *bounded*: *bounded-linear f′*
and *sandwich*: ($\bigwedge y.\ y \in s \Longrightarrow y \neq x \Longrightarrow dist\ y\ x < e \Longrightarrow$
$norm\ ((f\ y - f\ x) - f'\ (y - x))\ /\ norm\ (y - x) \leq H\ y$)
and ($H \longrightarrow 0$) (*at x within s*)
shows (*f has-derivative f′*) (*at x within s*)
unfolding *has-derivative-iff-norm*
proof *safe*
show (($\lambda y.\ norm\ (f\ y - f\ x - f'\ (y - x))\ /\ norm\ (y - x)) \longrightarrow 0$) (*at x within s*)
proof (*rule tendsto-sandwich*[**where** $f=\lambda x.\ 0$])
show ($H \longrightarrow 0$) (*at x within s*) **by** *fact*
show *eventually* ($\lambda n.\ norm\ (f\ n - f\ x - f'\ (n - x))\ /\ norm\ (n - x) \leq H\ n$) (*at x within s*)
unfolding *eventually-at* **using** *e sandwich* **by** *auto*
qed (*auto simp*: *le-divide-eq*)
qed *fact*

lemma *has-derivative-subset*:
(*f has-derivative f′*) (*at x within s*) $\Longrightarrow t \subseteq s \Longrightarrow$ (*f has-derivative f′*) (*at x within t*)
by (*auto simp add*: *has-derivative-iff-norm intro*: *tendsto-within-subset*)

lemma *has-derivative-within-singleton-iff*:
(*f has-derivative g*) (*at x within* $\{x\}$) $\longleftrightarrow$ *bounded-linear g*
by (*auto intro*!: *has-derivativeI-sandwich*[**where** $e=1$] *has-derivative-bounded-linear*)

110.1.1 Limit transformation for derivatives

lemma *has-derivative-transform-within*:
assumes (*f has-derivative f′*) (*at x within s*)
and $0 < d$
and $x \in s$
and $\bigwedge x'.\ [\![x' \in s;\ dist\ x'\ x < d]\!] \Longrightarrow f\ x' = g\ x'$
shows (*g has-derivative f′*) (*at x within s*)
using *assms*
unfolding *has-derivative-within*
by (*force simp add*: *intro*: *Lim-transform-within*)

lemma *has-derivative-transform-within-open*:
assumes (*f has-derivative f′*) (*at x within t*)
and *open s*
and $x \in s$
and $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
shows (*g has-derivative f′*) (*at x within t*)

using *assms* **unfolding** *has-derivative-within*
by (*force simp add*: *intro*: *Lim-transform-within-open*)

lemma *has-derivative-transform*:
assumes $x \in s \bigwedge x.\ x \in s \Longrightarrow g\ x = f\ x$
assumes (*f has-derivative f′*) (*at x within s*)
shows (*g has-derivative f′*) (*at x within s*)
using *assms*
by (*intro has-derivative-transform-within*[*OF - zero-less-one*, **where** *g*=*g*]) *auto*

lemma *has-derivative-transform-eventually*:
assumes (*f has-derivative f′*) (*at x within s*)
($\forall_F$ *x′ in at x within s. f x′ = g x′*)
assumes $f\ x = g\ x\ x \in s$
shows (*g has-derivative f′*) (*at x within s*)
using *assms*
proof −
from *assms*(*2*,*3*) **obtain** *d* **where** $d > 0 \bigwedge x'.\ x' \in s \Longrightarrow dist\ x'\ x < d \Longrightarrow f\ x' = g\ x'$
by (*force simp*: *eventually-at*)
from *has-derivative-transform-within*[*OF assms*(*1*) *this*(*1*) *assms*(*4*) *this*(*2*)]
show *?thesis* **.**
qed

lemma *has-field-derivative-transform-within*:
assumes (*f has-field-derivative f′*) (*at a within S*)
and $0 < d$
and $a \in S$
and $\bigwedge x.\ [\![x \in S;\ dist\ x\ a < d]\!] \Longrightarrow f\ x = g\ x$
shows (*g has-field-derivative f′*) (*at a within S*)
using *assms* **unfolding** *has-field-derivative-def*
by (*metis has-derivative-transform-within*)

lemma *has-field-derivative-transform-within-open*:
assumes (*f has-field-derivative f′*) (*at a*)
and *open S* $a \in S$
and $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows (*g has-field-derivative f′*) (*at a*)
using *assms* **unfolding** *has-field-derivative-def*
by (*metis has-derivative-transform-within-open*)

110.2 Continuity

lemma *has-derivative-continuous*:
assumes *f*: (*f has-derivative f′*) (*at x within s*)
shows *continuous* (*at x within s*) *f*
proof −
from *f* **interpret** *F*: *bounded-linear f′*
by (*rule has-derivative-bounded-linear*)

note *F.tendsto*[*tendsto-intros*]
let *?L* = $\lambda f.$ (*f* $\longrightarrow$ *0*) (*at x within s*)
have *?L* ($\lambda y.$ *norm* ((*f y* − *f x*) − *f′* (*y* − *x*)) / *norm* (*y* − *x*))
using *f* **unfolding** *has-derivative-iff-norm* **by** *blast*
then have *?L* ($\lambda y.$ *norm* ((*f y* − *f x*) − *f′* (*y* − *x*)) / *norm* (*y* − *x*) ∗ *norm* (*y* − *x*)) (**is** *?m*)
by (*rule tendsto-mult-zero*) (*auto intro*!: *tendsto-eq-intros*)
also have *?m* $\longleftrightarrow$ *?L* ($\lambda y.$ *norm* ((*f y* − *f x*) − *f′* (*y* − *x*)))
by (*intro filterlim-cong*) (*simp-all add*: *eventually-at-filter*)
finally have *?L* ($\lambda y.$ (*f y* − *f x*) − *f′* (*y* − *x*))
by (*rule tendsto-norm-zero-cancel*)
then have *?L* ($\lambda y.$ ((*f y* − *f x*) − *f′* (*y* − *x*)) + *f′* (*y* − *x*))
by (*rule tendsto-eq-intros*) (*auto intro*!: *tendsto-eq-intros simp*: *F.zero*)
then have *?L* ($\lambda y.$ *f y* − *f x*)
by *simp*
from *tendsto-add*[*OF this tendsto-const*, *of f x*] **show** *?thesis*
by (*simp add*: *continuous-within*)
qed

110.3 Composition

lemma *tendsto-at-iff-tendsto-nhds-within*:
f x = *y* $\Longrightarrow$ (*f* $\longrightarrow$ *y*) (*at x within s*) $\longleftrightarrow$ (*f* $\longrightarrow$ *y*) (*inf* (*nhds x*) (*principal s*))
unfolding *tendsto-def eventually-inf-principal eventually-at-filter*
by (*intro ext all-cong imp-cong*) (*auto elim*!: *eventually-mono*)

lemma *has-derivative-in-compose*:
assumes *f*: (*f has-derivative f′*) (*at x within s*)
and *g*: (*g has-derivative g′*) (*at* (*f x*) *within* (*f‘s*))
shows (($\lambda x.$ *g* (*f x*)) *has-derivative* ($\lambda x.$ *g′* (*f′ x*))) (*at x within s*)
proof −
from *f* **interpret** *F*: *bounded-linear f′*
by (*rule has-derivative-bounded-linear*)
from *g* **interpret** *G*: *bounded-linear g′*
by (*rule has-derivative-bounded-linear*)
from *F.bounded* **obtain** *kF* **where** *kF*: $\bigwedge x.$ *norm* (*f′ x*) ≤ *norm x* ∗ *kF*
by *fast*
from *G.bounded* **obtain** *kG* **where** *kG*: $\bigwedge x.$ *norm* (*g′ x*) ≤ *norm x* ∗ *kG*
by *fast*
note *G.tendsto*[*tendsto-intros*]

let *?L* = $\lambda f.$ (*f* $\longrightarrow$ *0*) (*at x within s*)
let *?D* = $\lambda f\, f'\, x\, y.$ (*f y* − *f x*) − *f′* (*y* − *x*)
let *?N* = $\lambda f\, f'\, x\, y.$ *norm* (*?D f f′ x y*) / *norm* (*y* − *x*)
let *?gf* = $\lambda x.$ *g* (*f x*) **and** *?gf′* = $\lambda x.$ *g′* (*f′ x*)
define *Nf* **where** *Nf* = *?N f f′ x*
define *Ng* **where** [*abs-def*]: *Ng y* = *?N g g′* (*f x*) (*f y*) **for** *y*

show *?thesis*
proof (*rule has-derivativeI-sandwich*[*of 1*])
show *bounded-linear* (*λx. g′* (*f′ x*))
using *f g* **by** (*blast intro*: *bounded-linear-compose has-derivative-bounded-linear*)
next
fix *y* :: *′a*
assume *neq*: *y* ≠ *x*
have *?N ?gf ?gf′ x y* = *norm* (*g′* (*?D f f′ x y*) + *?D g g′* (*f x*) (*f y*)) / *norm* (*y* − *x*)
by (*simp add*: *G.diff G.add field-simps*)
also have ... ≤ *norm* (*g′* (*?D f f′ x y*)) / *norm* (*y* − *x*) + *Ng y* ∗ (*norm* (*f y* − *f x*) / *norm* (*y* − *x*))
by (*simp add*: *add-divide-distrib*[*symmetric*] *divide-right-mono norm-triangle-ineq G.zero Ng-def*)
also have ... ≤ *Nf y* ∗ *kG* + *Ng y* ∗ (*Nf y* + *kF*)
proof (*intro add-mono mult-left-mono*)
have *norm* (*f y* − *f x*) = *norm* (*?D f f′ x y* + *f′* (*y* − *x*))
by *simp*
also have ... ≤ *norm* (*?D f f′ x y*) + *norm* (*f′* (*y* − *x*))
by (*rule norm-triangle-ineq*)
also have ... ≤ *norm* (*?D f f′ x y*) + *norm* (*y* − *x*) ∗ *kF*
using *kF* **by** (*intro add-mono*) *simp*
finally show *norm* (*f y* − *f x*) / *norm* (*y* − *x*) ≤ *Nf y* + *kF*
by (*simp add*: *neq Nf-def field-simps*)
qed (*use kG* **in** ‹*simp-all add*: *Ng-def Nf-def neq zero-le-divide-iff field-simps*›)
finally show *?N ?gf ?gf′ x y* ≤ *Nf y* ∗ *kG* + *Ng y* ∗ (*Nf y* + *kF*) **.**
next
have [*tendsto-intros*]: *?L Nf*
using *f* **unfolding** *has-derivative-iff-norm Nf-def* **..**
from *f* **have** (*f* ⟶ *f x*) (*at x within s*)
by (*blast intro*: *has-derivative-continuous continuous-within*[*THEN iffD1*])
then have *f′*: *LIM x at x within s. f x* :> *inf* (*nhds* (*f x*)) (*principal* (*f‘s*))
unfolding *filterlim-def*
by (*simp add*: *eventually-filtermap eventually-at-filter le-principal*)

have ((*?N g g′* (*f x*)) ⟶ *0*) (*at* (*f x*) *within f‘s*)
using *g* **unfolding** *has-derivative-iff-norm* **..**
then have *g′*: ((*?N g g′* (*f x*)) ⟶ *0*) (*inf* (*nhds* (*f x*)) (*principal* (*f‘s*)))
by (*rule tendsto-at-iff-tendsto-nhds-within*[*THEN iffD1*, *rotated*]) *simp*

have [*tendsto-intros*]: *?L Ng*
unfolding *Ng-def* **by** (*rule filterlim-compose*[*OF g′ f′*])
show ((*λy. Nf y* ∗ *kG* + *Ng y* ∗ (*Nf y* + *kF*)) ⟶ *0*) (*at x within s*)
by (*intro tendsto-eq-intros*) *auto*
qed *simp*
qed

lemma *has-derivative-compose*:
(*f has-derivative f′*) (*at x within s*) ⟹ (*g has-derivative g′*) (*at* (*f x*)) ⟹

((λx. g (f x)) has-derivative (λx. g′ (f′ x))) (at x within s)
by (blast intro: has-derivative-in-compose has-derivative-subset)

lemma has-derivative-in-compose2:
assumes ⋀x. x ∈ t ⟹ (g has-derivative g′ x) (at x within t)
assumes f ‘ s ⊆ t x ∈ s
assumes (f has-derivative f′) (at x within s)
shows ((λx. g (f x)) has-derivative (λy. g′ (f x) (f′ y))) (at x within s)
using assms
by (auto intro: has-derivative-subset intro!: has-derivative-in-compose[of f f′ x s g])

lemma (**in** bounded-bilinear) FDERIV:
assumes f: (f has-derivative f′) (at x within s) **and** g: (g has-derivative g′) (at x within s)
shows ((λx. f x ** g x) has-derivative (λh. f x ** g′ h + f′ h ** g x)) (at x within s)
proof −
from bounded-linear.bounded [OF has-derivative-bounded-linear [OF f]]
obtain KF **where** norm-F: ⋀x. norm (f′ x) ≤ norm x * KF **by** fast

from pos-bounded **obtain** K
where K: 0 < K **and** norm-prod: ⋀a b. norm (a ** b) ≤ norm a * norm b * K
by fast
let ?D = λf f′ y. f y − f x − f′ (y − x)
let ?N = λf f′ y. norm (?D f f′ y) / norm (y − x)
define Ng **where** Ng = ?N g g′
define Nf **where** Nf = ?N f f′

let ?fun1 = λy. norm (f y ** g y − f x ** g x − (f x ** g′ (y − x) + f′ (y − x) ** g x)) / norm (y − x)
let ?fun2 = λy. norm (f x) * Ng y * K + Nf y * norm (g y) * K + KF * norm (g y − g x) * K
let ?F = at x within s

show ?thesis
proof (rule has-derivativeI-sandwich[of 1])
show bounded-linear (λh. f x ** g′ h + f′ h ** g x)
by (intro bounded-linear-add
bounded-linear-compose [OF bounded-linear-right] bounded-linear-compose [OF bounded-linear-left]
has-derivative-bounded-linear [OF g] has-derivative-bounded-linear [OF f])
next
from g **have** (g ⟶ g x) ?F
by (intro continuous-within[THEN iffD1] has-derivative-continuous)
moreover from f g **have** (Nf ⟶ 0) ?F (Ng ⟶ 0) ?F
by (simp-all add: has-derivative-iff-norm Ng-def Nf-def)
ultimately have (?fun2 ⟶ norm (f x) * 0 * K + 0 * norm (g x) * K +

$KF * norm\ (0::'b) * K)\ ?F$
  by (*intro tendsto-intros*) (*simp-all add*: *LIM-zero-iff*)
 then show $(?fun2 \longrightarrow 0)\ ?F$
  by *simp*
 next
 fix $y :: 'd$
 assume $y \neq x$
 have $?fun1\ y =$
  $norm\ (f\ x ** ?D\ g\ g'\ y + ?D\ f\ f'\ y ** g\ y + f'\ (y - x) ** (g\ y - g\ x))\ /$
$norm\ (y - x)$
  by (*simp add*: *diff-left diff-right add-left add-right field-simps*)
 also have $\ldots \leq (norm\ (f\ x) * norm\ (?D\ g\ g'\ y) * K + norm\ (?D\ f\ f'\ y) *$
$norm\ (g\ y) * K +$
  $norm\ (y - x) * KF * norm\ (g\ y - g\ x) * K)\ /\ norm\ (y - x)$
  by (*intro divide-right-mono mult-mono'*
   order-trans [*OF norm-triangle-ineq add-mono*]
   order-trans [*OF norm-prod mult-right-mono*]
   mult-nonneg-nonneg order-refl norm-ge-zero norm-F
   K [*THEN order-less-imp-le*])
 also have $\ldots = ?fun2\ y$
  by (*simp add*: *add-divide-distrib Ng-def Nf-def*)
 finally show $?fun1\ y \leq ?fun2\ y$ **.**
 qed *simp*
qed

lemmas *has-derivative-mult*[*simp*, *derivative-intros*] = *bounded-bilinear.FDERIV*[*OF bounded-bilinear-mult*]
lemmas *has-derivative-scaleR*[*simp*, *derivative-intros*] = *bounded-bilinear.FDERIV*[*OF bounded-bilinear-scaleR*]

lemma *has-derivative-prod*[*simp*, *derivative-intros*]:
 fixes $f :: 'i \Rightarrow 'a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}field$
 shows $(\bigwedge i.\ i \in I \Longrightarrow (f\ i\ has\text{-}derivative\ f'\ i)\ (at\ x\ within\ S)) \Longrightarrow$
  $((\lambda x.\ \prod i{\in}I.\ f\ i\ x)\ has\text{-}derivative\ (\lambda y.\ \sum i{\in}I.\ f'\ i\ y * (\prod j{\in}I - \{i\}.\ f\ j\ x)))$
$(at\ x\ within\ S)$
proof (*induct I rule*: *infinite-finite-induct*)
 case *infinite*
 then show *?case* **by** *simp*
next
 case *empty*
 then show *?case* **by** *simp*
next
 case (*insert i I*)
 let $?P = \lambda y.\ f\ i\ x * (\sum i{\in}I.\ f'\ i\ y * (\prod j{\in}I - \{i\}.\ f\ j\ x)) + (f'\ i\ y) * (\prod i{\in}I.\ f$
$i\ x)$
 have $((\lambda x.\ f\ i\ x * (\prod i{\in}I.\ f\ i\ x))\ has\text{-}derivative\ ?P)\ (at\ x\ within\ S)$
  using *insert* **by** (*intro has-derivative-mult*) *auto*
 also have $?P = (\lambda y.\ \sum i'{\in}insert\ i\ I.\ f'\ i'\ y * (\prod j{\in}insert\ i\ I - \{i'\}.\ f\ j\ x))$
  using *insert*(*1*,*2*)

by (*auto simp add*: *sum-distrib-left insert-Diff-if intro*!: *ext sum.cong*)
finally show *?case*
using *insert* **by** *simp*
qed

lemma *has-derivative-power*[*simp*, *derivative-intros*]:
fixes f :: *'a* :: *real-normed-vector* ⇒ *'b* :: *real-normed-field*
assumes *f*: (*f has-derivative f'*) (*at x within S*)
shows ((λx. *f x*^*n*) *has-derivative* (λy. *of-nat n* ∗ *f' y* ∗ *f x*^(*n* − *1*))) (*at x within S*)
using *has-derivative-prod*[*OF f*, *of* {..< *n*}] **by** (*simp add*: *prod-constant ac-simps*)

lemma *has-derivative-inverse'*:
fixes *x* :: *'a*::*real-normed-div-algebra*
assumes *x*: $x \neq 0$
shows (*inverse has-derivative* (λh. − (*inverse x* ∗ *h* ∗ *inverse x*))) (*at x within S*)
(**is** (- *has-derivative ?f*) -)
proof (*rule has-derivativeI-sandwich*)
show *bounded-linear* (λh. − (*inverse x* ∗ *h* ∗ *inverse x*))
by (*simp add*: *bounded-linear-minus bounded-linear-mult-const bounded-linear-mult-right*)
show $0 <$ *norm x* **using** *x* **by** *simp*
have (*inverse* ⟶ *inverse x*) (*at x within S*)
using *tendsto-inverse tendsto-ident-at x* **by** *auto*
then show ((λy. *norm* (*inverse y* − *inverse x*) ∗ *norm* (*inverse x*)) ⟶ *0*) (*at x within S*)
by (*simp add*: *LIM-zero-iff tendsto-mult-left-zero tendsto-norm-zero*)
next
fix *y* :: *'a*
assume *h*: $y \neq x$ *dist y x* < *norm x*
then have $y \neq 0$ **by** *auto*
have *norm* (*inverse y* − *inverse x* − *?f* (*y* −*x*)) / *norm* (*y* − *x*)
= *norm* (− (*inverse y* ∗ (*y* − *x*) ∗ *inverse x* − *inverse x* ∗ (*y* − *x*) ∗ *inverse x*)) /
norm (*y* − *x*)
by (*simp add*: ‹$y \neq 0$› *inverse-diff-inverse x*)
also have ... = *norm* ((*inverse y* − *inverse x*) ∗ (*y* − *x*) ∗ *inverse x*) / *norm* (*y* − *x*)
by (*simp add*: *left-diff-distrib norm-minus-commute*)
also have ... ≤ *norm* (*inverse y* − *inverse x*) ∗ *norm* (*y* − *x*) ∗ *norm* (*inverse x*) / *norm* (*y* − *x*)
by (*simp add*: *norm-mult*)
also have ... = *norm* (*inverse y* − *inverse x*) ∗ *norm* (*inverse x*)
by *simp*
finally show *norm* (*inverse y* − *inverse x* − *?f* (*y* −*x*)) / *norm* (*y* − *x*) ≤
norm (*inverse y* − *inverse x*) ∗ *norm* (*inverse x*) **.**
qed

lemma *has-derivative-inverse*[*simp*, *derivative-intros*]:

fixes *f* :: - ⇒ *'a::real-normed-div-algebra*
assumes *x*: *f x* ≠ *0*
and *f*: (*f has-derivative f'*) (*at x within S*)
shows ((λ*x*. *inverse* (*f x*)) *has-derivative* (λ*h*. − (*inverse* (*f x*) ∗ *f' h* ∗ *inverse* (*f x*))))
(*at x within S*)
using *has-derivative-compose*[*OF f has-derivative-inverse'*, *OF x*] **.**

lemma *has-derivative-divide*[*simp*, *derivative-intros*]:
fixes *f* :: - ⇒ *'a::real-normed-div-algebra*
assumes *f*: (*f has-derivative f'*) (*at x within S*)
and *g*: (*g has-derivative g'*) (*at x within S*)
assumes *x*: *g x* ≠ *0*
shows ((λ*x*. *f x* / *g x*) *has-derivative*
(λ*h*. − *f x* ∗ (*inverse* (*g x*) ∗ *g' h* ∗ *inverse* (*g x*)) + *f' h* / *g x*)) (*at x within S*)
using *has-derivative-mult*[*OF f has-derivative-inverse*[*OF x g*]]
by (*simp add*: *field-simps*)

lemma *has-derivative-power-int'*:
fixes *x* :: *'a::real-normed-field*
assumes *x*: *x* ≠ *0*
shows ((λ*x*. *power-int x n*) *has-derivative* (λ*y*. *y* ∗ (*of-int n* ∗ *power-int x* (*n* − *1*)))) (*at x within S*)
proof (*cases n rule*: *int-cases4*)
case (*nonneg n*)
thus *?thesis* **using** *x*
by (*cases n* = *0*) (*auto intro*!: *derivative-eq-intros simp*: *field-simps power-int-diff fun-eq-iff*
simp flip: *power-Suc*)
next
case (*neg n*)
thus *?thesis* **using** *x*
by (*auto intro*!: *derivative-eq-intros simp*: *field-simps power-int-diff power-int-minus*
simp flip: *power-Suc power-Suc2 power-add*)
qed

lemma *has-derivative-power-int*[*simp*, *derivative-intros*]:
fixes *f* :: - ⇒ *'a::real-normed-field*
assumes *x*: *f x* ≠ *0*
and *f*: (*f has-derivative f'*) (*at x within S*)
shows ((λ*x*. *power-int* (*f x*) *n*) *has-derivative* (λ*h*. *f' h* ∗ (*of-int n* ∗ *power-int* (*f x*) (*n* − *1*))))
(*at x within S*)
using *has-derivative-compose*[*OF f has-derivative-power-int'*, *OF x*] **.**

Conventional form requires mult-AC laws. Types real and complex only.

lemma *has-derivative-divide'*[*derivative-intros*]:
fixes *f* :: - ⇒ *'a::real-normed-field*

assumes f: (f *has-derivative* f') (*at* x *within* S)
and g: (g *has-derivative* g') (*at* x *within* S)
and x: $g\ x \neq 0$
shows (($\lambda x.\ f\ x\ /\ g\ x$) *has-derivative* ($\lambda h.\ (f'\ h * g\ x - f\ x * g'\ h)\ /\ (g\ x * g\ x)$)) (*at* x *within* S)
proof −
have $f'\ h\ /\ g\ x - f\ x * (inverse\ (g\ x) * g'\ h * inverse\ (g\ x)) = (f'\ h * g\ x - f\ x * g'\ h)\ /\ (g\ x * g\ x)$ **for** h
by (*simp add*: *field-simps* x)
then show *?thesis*
using *has-derivative-divide* [*OF* f g] x
by *simp*
qed

110.4 Uniqueness

This can not generally shown for (*has-derivative*), as we need to approach the point from all directions. There is a proof in *Analysis* for *euclidean-space*.

lemma *has-derivative-at2*: (f *has-derivative* f') (*at* x) $\longleftrightarrow$
bounded-linear $f' \wedge ((\lambda y.\ (1\ /\ (norm(y - x))) *_R (f\ y - (f\ x + f'\ (y - x)))) \longrightarrow 0)$ (*at* x)
using *has-derivative-within* [*of* f f' x *UNIV*]
by *simp*

lemma *has-derivative-zero-unique*:
assumes (($\lambda x.\ 0$) *has-derivative* F) (*at* x)
shows $F = (\lambda h.\ 0)$
proof −
interpret F: *bounded-linear* F
using *assms* **by** (*rule has-derivative-bounded-linear*)
let $?r = \lambda h.\ norm\ (F\ h)\ /\ norm\ h$
have $*$: $?r\ {-0\rightarrow}\ 0$
using *assms* **unfolding** *has-derivative-at* **by** *simp*
show $F = (\lambda h.\ 0)$
proof
show $F\ h = 0$ **for** h
proof (*rule ccontr*)
assume $**$: $\neg$ *?thesis*
then have h: $h \neq 0$
by (*auto simp add*: *F.zero*)
with $**$ **have** $0 < ?r\ h$
by *simp*
from *LIM-D* [*OF* $*$ *this*] **obtain** S
where S: $0 < S$ **and** r: $\bigwedge x.\ x \neq 0 \Longrightarrow norm\ x < S \Longrightarrow ?r\ x < ?r\ h$
by *auto*
from *dense* [*OF* S] **obtain** t **where** t: $0 < t \wedge t < S$ **..**
let $?x = scaleR\ (t\ /\ norm\ h)\ h$
have $?x \neq 0$ **and** $norm\ ?x < S$
using t h **by** *simp-all*

then have *?r ?x* < *?r h*
by (*rule r*)
then show *False*
using *t h* **by** (*simp add*: *F.scaleR*)
qed
qed
qed

lemma *has-derivative-unique*:
assumes (*f has-derivative F*) (*at x*)
and (*f has-derivative F′*) (*at x*)
shows *F* = *F′*
proof −
have ((λ*x*. *0*) *has-derivative* (λ*h*. *F h* − *F′ h*)) (*at x*)
using *has-derivative-diff* [*OF assms*] **by** *simp*
then have (λ*h*. *F h* − *F′ h*) = (λ*h*. *0*)
by (*rule has-derivative-zero-unique*)
then show *F* = *F′*
unfolding *fun-eq-iff right-minus-eq* **.**
qed

lemma *has-derivative-Uniq*: $\exists_{\leq 1}$*F*. (*f has-derivative F*) (*at x*)
by (*simp add*: *Uniq-def has-derivative-unique*)

110.5 Differentiability predicate

definition *differentiable* :: (*′a*::*real-normed-vector* ⇒ *′b*::*real-normed-vector*) ⇒ *′a filter* ⇒ *bool*
(**infix** ‹*differentiable*› *50*)
where *f differentiable F* ⟷ (∃ *D*. (*f has-derivative D*) *F*)

lemma *differentiable-subset*:
f differentiable (*at x within s*) ⟹ *t* ⊆ *s* ⟹ *f differentiable* (*at x within t*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-subset*)

lemmas *differentiable-within-subset* = *differentiable-subset*

lemma *differentiable-ident* [*simp*, *derivative-intros*]: (λ*x*. *x*) *differentiable F*
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-ident*)

lemma *differentiable-const* [*simp*, *derivative-intros*]: (λ*z*. *a*) *differentiable F*
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-const*)

lemma *differentiable-in-compose*:
f differentiable (*at* (*g x*) *within* (*g‘s*)) ⟹ *g differentiable* (*at x within s*) ⟹
(λ*x*. *f* (*g x*)) *differentiable* (*at x within s*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-in-compose*)

lemma *differentiable-compose*:

f differentiable (*at* (*g x*)) $\Longrightarrow$ *g differentiable* (*at x within s*) $\Longrightarrow$
(λ*x*. *f* (*g x*)) *differentiable* (*at x within s*)
by (*blast intro*: *differentiable-in-compose differentiable-subset*)

lemma *differentiable-add* [*simp*, *derivative-intros*]:
f differentiable F $\Longrightarrow$ *g differentiable F* $\Longrightarrow$ (λ*x*. *f x* + *g x*) *differentiable F*
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-add*)

lemma *differentiable-sum*[*simp*, *derivative-intros*]:
assumes *finite s* ∀ *a*∈*s*. (*f a*) *differentiable net*
shows (λ*x*. *sum* (λ*a*. *f a x*) *s*) *differentiable net*
proof −
from *bchoice*[*OF assms*(*2*)[*unfolded differentiable-def*]]
show *?thesis*
by (*auto intro*!: *has-derivative-sum simp*: *differentiable-def*)
qed

lemma *differentiable-minus* [*simp*, *derivative-intros*]:
f differentiable F $\Longrightarrow$ (λ*x*. − *f x*) *differentiable F*
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-minus*)

lemma *differentiable-diff* [*simp*, *derivative-intros*]:
f differentiable F $\Longrightarrow$ *g differentiable F* $\Longrightarrow$ (λ*x*. *f x* − *g x*) *differentiable F*
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-diff*)

lemma *differentiable-mult* [*simp*, *derivative-intros*]:
fixes *f g* :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-algebra*
shows *f differentiable* (*at x within s*) $\Longrightarrow$ *g differentiable* (*at x within s*) $\Longrightarrow$
(λ*x*. *f x* ∗ *g x*) *differentiable* (*at x within s*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-mult*)

lemma *differentiable-cmult-left-iff* [*simp*]:
fixes *c*::*'a*::*real-normed-field*
shows (λ*t*. *c* ∗ *q t*) *differentiable at t* ⟷ *c* = *0* ∨ (λ*t*. *q t*) *differentiable at t*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
{**assume** *c* ≠ *0*
then have *q differentiable at t*
using *differentiable-mult* [*OF differentiable-const L*, *of concl*: *1*/*c*] **by** *auto*
} **then show** *?rhs*
by *auto*
qed *auto*

lemma *differentiable-cmult-right-iff* [*simp*]:
fixes *c*::*'a*::*real-normed-field*
shows (λ*t*. *q t* ∗ *c*) *differentiable at t* ⟷ *c* = *0* ∨ (λ*t*. *q t*) *differentiable at t*
(**is** *?lhs* = *?rhs*)
by (*simp add*: *mult.commute flip*: *differentiable-cmult-left-iff*)

lemma *differentiable-inverse* [*simp*, *derivative-intros*]:
fixes *f* :: *'a::real-normed-vector* ⇒ *'b::real-normed-field*
shows *f differentiable* (*at x within s*) ⟹ *f x* ≠ *0* ⟹
(λ*x*. *inverse* (*f x*)) *differentiable* (*at x within s*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-inverse*)

lemma *differentiable-divide* [*simp*, *derivative-intros*]:
fixes *f g* :: *'a::real-normed-vector* ⇒ *'b::real-normed-field*
shows *f differentiable* (*at x within s*) ⟹ *g differentiable* (*at x within s*) ⟹
g x ≠ *0* ⟹ (λ*x*. *f x* / *g x*) *differentiable* (*at x within s*)
unfolding *divide-inverse* **by** *simp*

lemma *differentiable-power* [*simp*, *derivative-intros*]:
fixes *f g* :: *'a::real-normed-vector* ⇒ *'b::real-normed-field*
shows *f differentiable* (*at x within s*) ⟹ (λ*x*. *f x* ^ *n*) *differentiable* (*at x within s*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-power*)

lemma *differentiable-power-int* [*simp*, *derivative-intros*]:
fixes *f* :: *'a::real-normed-vector* ⇒ *'b::real-normed-field*
shows *f differentiable* (*at x within s*) ⟹ *f x* ≠ *0* ⟹
(λ*x*. *power-int* (*f x*) *n*) *differentiable* (*at x within s*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-power-int*)

lemma *differentiable-scaleR* [*simp*, *derivative-intros*]:
f differentiable (*at x within s*) ⟹ *g differentiable* (*at x within s*) ⟹
(λ*x*. *f x* $*_R$ *g x*) *differentiable* (*at x within s*)
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-scaleR*)

lemma *has-derivative-imp-has-field-derivative*:
(*f has-derivative D*) *F* ⟹ (⋀*x*. *x* * *D'* = *D x*) ⟹ (*f has-field-derivative D'*) *F*
unfolding *has-field-derivative-def*
by (*rule has-derivative-eq-rhs*[*of f D*]) (*simp-all add*: *fun-eq-iff mult.commute*)

lemma *has-field-derivative-imp-has-derivative*:
(*f has-field-derivative D*) *F* ⟹ (*f has-derivative* (*) *D*) *F*
by (*simp add*: *has-field-derivative-def*)

lemma *DERIV-subset*:
(*f has-field-derivative f'*) (*at x within s*) ⟹ *t* ⊆ *s* ⟹
(*f has-field-derivative f'*) (*at x within t*)
by (*simp add*: *has-field-derivative-def has-derivative-subset*)

lemma *has-field-derivative-at-within*:
(*f has-field-derivative f'*) (*at x*) ⟹ (*f has-field-derivative f'*) (*at x within s*)
using *DERIV-subset* **by** *blast*

abbreviation (*input*)

DERIV :: ('*a*::*real-normed-field* ⇒ '*a*) ⇒ '*a* ⇒ '*a* ⇒ *bool*
(‹(‹*notation*=‹*mixfix DERIV*››*DERIV* (-)/ (-)/ :> (-))› [*1000*, *1000*, *60*] *60*)
where *DERIV f x* :> *D* ≡ (*f has-field-derivative D*) (*at x*)

abbreviation *has-real-derivative* :: (*real* ⇒ *real*) ⇒ *real* ⇒ *real filter* ⇒ *bool*
(**infix** ‹(*has*′-*real*′-*derivative*)› *50*)
where (*f has-real-derivative D*) *F* ≡ (*f has-field-derivative D*) *F*

lemma *real-differentiable-def*:
f differentiable at x within s ⟷ (∃ *D*. (*f has-real-derivative D*) (*at x within s*))
proof *safe*
assume *f differentiable at x within s*
then obtain *f*′ **where** ∗: (*f has-derivative f*′) (*at x within s*)
unfolding *differentiable-def* **by** *auto*
then obtain *c* **where** *f*′ = ((∗) *c*)
by (*metis real-bounded-linear has-derivative-bounded-linear mult.commute fun-eq-iff*)
with ∗ **show** ∃ *D*. (*f has-real-derivative D*) (*at x within s*)
unfolding *has-field-derivative-def* **by** *auto*
qed (*auto simp*: *differentiable-def has-field-derivative-def*)

lemma *real-differentiableE* [*elim?*]:
assumes *f*: *f differentiable* (*at x within s*)
obtains *df* **where** (*f has-real-derivative df*) (*at x within s*)
using *assms* **by** (*auto simp*: *real-differentiable-def*)

lemma *has-field-derivative-iff*:
(*f has-field-derivative D*) (*at x within S*) ⟷
((λ*y*. (*f y* − *f x*) / (*y* − *x*)) ⟶ *D*) (*at x within S*)
proof −
have ((λ*y*. *norm* (*f y* − *f x* − *D* ∗ (*y* − *x*)) / *norm* (*y* − *x*)) ⟶ *0*) (*at x within S*)
= ((λ*y*. (*f y* − *f x*) / (*y* − *x*) − *D*) ⟶ *0*) (*at x within S*)
by (*smt* (*verit*, *best*) *Lim-cong-within divide-diff-eq-iff norm-divide right-minus-eq tendsto-norm-zero-iff*)
then show *?thesis*
by (*simp add*: *has-field-derivative-def has-derivative-iff-norm bounded-linear-mult-right LIM-zero-iff*)
qed

lemma *DERIV-def*: *DERIV f x* :> *D* ⟷ (λ*h*. (*f* (*x* + *h*) − *f x*) / *h*) −*0*→ *D*
unfolding *field-has-derivative-at has-field-derivative-def has-field-derivative-iff* **..**

lemma *has-field-derivative-unique*:
assumes (*f has-field-derivative f*′*1*) (*at x within A*)
assumes (*f has-field-derivative f*′*2*) (*at x within A*)
assumes *at x within A* ≠ *bot*
shows *f*′*1* = *f*′*2*
using *assms* **unfolding** *has-field-derivative-iff* **using** *tendsto-unique* **by** *blast*

due to Christian Pardillo Laursen, replacing a proper epsilon-delta horror

lemma *field-derivative-lim-unique*:
assumes *f*: (*f has-field-derivative df*) (*at z*)
and *s*: $s \longrightarrow 0$ $\bigwedge n.\ s\ n \neq 0$
and *a*: $(\lambda n.\ (f\ (z + s\ n) - f\ z)\ /\ s\ n) \longrightarrow a$
shows $df = a$
proof −
have $((\lambda k.\ (f\ (z + k) - f\ z)\ /\ k) \longrightarrow df)$ (*at 0*)
using *f* **by** (*simp add*: *DERIV-def*)
with *s* **have** $((\lambda n.\ (f\ (z + s\ n) - f\ z)\ /\ s\ n) \longrightarrow df)$
by (*simp flip*: *LIMSEQ-SEQ-conv*)
then show *?thesis*
using *a* **by** (*rule LIMSEQ-unique*)
qed

lemma *mult-commute-abs*: $(\lambda x.\ x * c) = (*)\ c$
for *c* :: *'a::ab-semigroup-mult*
by (*simp add*: *fun-eq-iff mult.commute*)

lemma *DERIV-compose-FDERIV*:
fixes *f::real⇒real*
assumes *DERIV f* (*g x*) :> *f'*
assumes (*g has-derivative g'*) (*at x within s*)
shows $((\lambda x.\ f\ (g\ x))$ *has-derivative* $(\lambda x.\ g'\ x * f'))$ (*at x within s*)
using *assms has-derivative-compose*[*of g g' x s f* (*) *f'*]
by (*auto simp*: *has-field-derivative-def ac-simps*)

110.6 Vector derivative

It's for real derivatives only, and not obviously generalisable to field derivatives

lemma *has-real-derivative-iff-has-vector-derivative*:
(*f has-real-derivative y*) *F* ⟷ (*f has-vector-derivative y*) *F*
unfolding *has-vector-derivative-def has-field-derivative-def real-scaleR-def mult-commute-abs*
..

lemma *has-field-derivative-subset*:
(*f has-field-derivative y*) (*at x within s*) ⟹ $t \subseteq s$ ⟹
(*f has-field-derivative y*) (*at x within t*)
by (*fact DERIV-subset*)

lemma *has-vector-derivative-const*[*simp*, *derivative-intros*]: $((\lambda x.\ c)$ *has-vector-derivative* *0*) *net*
by (*auto simp*: *has-vector-derivative-def*)

lemma *has-vector-derivative-id*[*simp*, *derivative-intros*]: $((\lambda x.\ x)$ *has-vector-derivative* *1*) *net*
by (*auto simp*: *has-vector-derivative-def*)

lemma *has-vector-derivative-minus*[*derivative-intros*]:

(*f has-vector-derivative f'*) *net* $\Longrightarrow$ ((*λx.* − *f x*) *has-vector-derivative* (− *f'*)) *net*
by (*auto simp*: *has-vector-derivative-def*)

lemma *has-vector-derivative-add*[*derivative-intros*]:
(*f has-vector-derivative f'*) *net* $\Longrightarrow$ (*g has-vector-derivative g'*) *net* $\Longrightarrow$
((*λx. f x* + *g x*) *has-vector-derivative* (*f'* + *g'*)) *net*
by (*auto simp*: *has-vector-derivative-def scaleR-right-distrib*)

lemma *has-vector-derivative-sum*[*derivative-intros*]:
($\bigwedge$*i. i* ∈ *I* $\Longrightarrow$ (*f i has-vector-derivative f' i*) *net*) $\Longrightarrow$
((*λx.* $\sum$ *i*∈*I. f i x*) *has-vector-derivative* ($\sum$ *i*∈*I. f' i*)) *net*
by (*auto simp*: *has-vector-derivative-def fun-eq-iff scaleR-sum-right intro*!: *derivative-eq-intros*)

lemma *has-vector-derivative-diff*[*derivative-intros*]:
(*f has-vector-derivative f'*) *net* $\Longrightarrow$ (*g has-vector-derivative g'*) *net* $\Longrightarrow$
((*λx. f x* − *g x*) *has-vector-derivative* (*f'* − *g'*)) *net*
by (*auto simp*: *has-vector-derivative-def scaleR-diff-right*)

lemma *has-vector-derivative-add-const*:
((*λt. g t* + *z*) *has-vector-derivative f'*) *net* = ((*λt. g t*) *has-vector-derivative f'*) *net*
apply (*intro iffI*)
apply (*force dest*: *has-vector-derivative-diff* [**where** *g* = *λt. z*, *OF* - *has-vector-derivative-const*])
apply (*force dest*: *has-vector-derivative-add* [*OF* - *has-vector-derivative-const*])
done

lemma *has-vector-derivative-diff-const*:
((*λt. g t* − *z*) *has-vector-derivative f'*) *net* = ((*λt. g t*) *has-vector-derivative f'*) *net*
using *has-vector-derivative-add-const* [**where** *z* = −*z*]
by *simp*

lemma (**in** *bounded-linear*) *has-vector-derivative*:
assumes (*g has-vector-derivative g'*) *F*
shows ((*λx. f* (*g x*)) *has-vector-derivative f g'*) *F*
using *has-derivative*[*OF assms*[*unfolded has-vector-derivative-def*]]
by (*simp add*: *has-vector-derivative-def scaleR*)

lemma (**in** *bounded-bilinear*) *has-vector-derivative*:
assumes (*f has-vector-derivative f'*) (*at x within s*)
and (*g has-vector-derivative g'*) (*at x within s*)
shows ((*λx. f x* ** *g x*) *has-vector-derivative* (*f x* ** *g'* + *f'* ** *g x*)) (*at x within s*)
using *FDERIV*[*OF assms*(*1*−*2*)[*unfolded has-vector-derivative-def*]]
by (*simp add*: *has-vector-derivative-def scaleR-right scaleR-left scaleR-right-distrib*)

lemma *has-vector-derivative-scaleR*[*derivative-intros*]:
(*f has-field-derivative f'*) (*at x within s*) $\Longrightarrow$ (*g has-vector-derivative g'*) (*at x*

within s) $\Longrightarrow$
((λ*x*. *f x* $*_R$ *g x*) *has-vector-derivative* (*f x* $*_R$ *g'* + *f'* $*_R$ *g x*)) (*at x within s*)
unfolding *has-real-derivative-iff-has-vector-derivative*
by (*rule bounded-bilinear.has-vector-derivative*[*OF bounded-bilinear-scaleR*])

lemma *has-vector-derivative-mult*[*derivative-intros*]:
(*f has-vector-derivative f'*) (*at x within s*) $\Longrightarrow$ (*g has-vector-derivative g'*) (*at x within s*) $\Longrightarrow$
((λ*x*. *f x* $*$ *g x*) *has-vector-derivative* (*f x* $*$ *g'* + *f'* $*$ *g x*)) (*at x within s*)
for *f g* :: *real* ⇒ *'a::real-normed-algebra*
by (*rule bounded-bilinear.has-vector-derivative*[*OF bounded-bilinear-mult*])

lemma *has-vector-derivative-of-real*[*derivative-intros*]:
(*f has-field-derivative D*) *F* $\Longrightarrow$ ((λ*x*. *of-real* (*f x*)) *has-vector-derivative* (*of-real D*)) *F*
by (*rule bounded-linear.has-vector-derivative*[*OF bounded-linear-of-real*])
(*simp add*: *has-real-derivative-iff-has-vector-derivative*)

lemma *has-vector-derivative-real-field*:
(*f has-field-derivative f'*) (*at* (*of-real a*)) $\Longrightarrow$ ((λ*x*. *f* (*of-real x*)) *has-vector-derivative f'*) (*at a within s*)
using *has-derivative-compose*[*of of-real of-real a - f* (*) *f'*]
by (*simp add*: *scaleR-conv-of-real ac-simps has-vector-derivative-def has-field-derivative-def*)

lemma *has-vector-derivative-continuous*:
(*f has-vector-derivative D*) (*at x within s*) $\Longrightarrow$ *continuous* (*at x within s*) *f*
by (*auto intro*: *has-derivative-continuous simp*: *has-vector-derivative-def*)

lemma *continuous-on-vector-derivative*:
(⋀*x*. *x* ∈ *S* $\Longrightarrow$ (*f has-vector-derivative f' x*) (*at x within S*)) $\Longrightarrow$ *continuous-on S f*
by (*auto simp*: *continuous-on-eq-continuous-within intro*!: *has-vector-derivative-continuous*)

lemma *has-vector-derivative-mult-right*[*derivative-intros*]:
fixes *a* :: *'a::real-normed-algebra*
shows (*f has-vector-derivative x*) *F* $\Longrightarrow$ ((λ*x*. *a* $*$ *f x*) *has-vector-derivative* (*a* $*$ *x*)) *F*
by (*rule bounded-linear.has-vector-derivative*[*OF bounded-linear-mult-right*])

lemma *has-vector-derivative-mult-left*[*derivative-intros*]:
fixes *a* :: *'a::real-normed-algebra*
shows (*f has-vector-derivative x*) *F* $\Longrightarrow$ ((λ*x*. *f x* $*$ *a*) *has-vector-derivative* (*x* $*$ *a*)) *F*
by (*rule bounded-linear.has-vector-derivative*[*OF bounded-linear-mult-left*])

lemma *has-vector-derivative-divide*[*derivative-intros*]:
fixes *a* :: *'a::real-normed-field*
shows (*f has-vector-derivative x*) *F* $\Longrightarrow$ ((λ*x*. *f x* / *a*) *has-vector-derivative* (*x* / *a*)) *F*

using *has-vector-derivative-mult-left* [*of f x F inverse a*]
by (*simp add*: *field-class.field-divide-inverse*)

110.7 Derivatives

lemma *DERIV-D*: *DERIV f x :> D* $\Longrightarrow$ (λh. (*f* (*x* + *h*) − *f x*) / *h*) −*0*→ *D*
by (*simp add*: *DERIV-def*)

lemma *has-field-derivativeD*:
(*f has-field-derivative D*) (*at x within S*) $\Longrightarrow$
((λy. (*f y* − *f x*) / (*y* − *x*)) $\longrightarrow$ *D*) (*at x within S*)
by (*simp add*: *has-field-derivative-iff*)

lemma *DERIV-const* [*simp*, *derivative-intros*]: ((λx. *k*) *has-field-derivative 0*) *F*
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-const*]) *auto*

lemma *DERIV-ident* [*simp*, *derivative-intros*]: ((λx. *x*) *has-field-derivative 1*) *F*
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-ident*]) *auto*

lemma *field-differentiable-add*[*derivative-intros*]:
(*f has-field-derivative f'*) *F* $\Longrightarrow$ (*g has-field-derivative g'*) *F* $\Longrightarrow$
((λz. *f z* + *g z*) *has-field-derivative f'* + *g'*) *F*
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-add*])
(*auto simp*: *has-field-derivative-def field-simps mult-commute-abs*)

corollary *DERIV-add*:
(*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$ (*g has-field-derivative E*) (*at x within s*) $\Longrightarrow$
((λx. *f x* + *g x*) *has-field-derivative D* + *E*) (*at x within s*)
by (*rule field-differentiable-add*)

lemma *field-differentiable-minus*[*derivative-intros*]:
(*f has-field-derivative f'*) *F* $\Longrightarrow$ ((λz. − (*f z*)) *has-field-derivative* −*f'*) *F*
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-minus*])
(*auto simp*: *has-field-derivative-def field-simps mult-commute-abs*)

corollary *DERIV-minus*:
(*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$
((λx. − *f x*) *has-field-derivative* −*D*) (*at x within s*)
by (*rule field-differentiable-minus*)

lemma *field-differentiable-diff*[*derivative-intros*]:
(*f has-field-derivative f'*) *F* $\Longrightarrow$
(*g has-field-derivative g'*) *F* $\Longrightarrow$ ((λz. *f z* − *g z*) *has-field-derivative f'* − *g'*) *F*
by (*simp only*: *diff-conv-add-uminus field-differentiable-add field-differentiable-minus*)

corollary *DERIV-diff*:
(*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$
(*g has-field-derivative E*) (*at x within s*) $\Longrightarrow$

((λx. f x − g x) has-field-derivative D − E) (at x within s)
by (rule field-differentiable-diff)

lemma DERIV-continuous: (f has-field-derivative D) (at x within s) ⟹ continuous (at x within s) f
by (drule has-derivative-continuous[OF has-field-derivative-imp-has-derivative]) simp

corollary DERIV-isCont: DERIV f x :> D ⟹ isCont f x
by (rule DERIV-continuous)

lemma DERIV-atLeastAtMost-imp-continuous-on:
assumes ⋀x. ⟦a ≤ x; x ≤ b⟧ ⟹ ∃ y. DERIV f x :> y
shows continuous-on {a..b} f
by (meson DERIV-isCont assms atLeastAtMost-iff continuous-at-imp-continuous-at-within continuous-on-eq-continuous-within)

lemma DERIV-continuous-on:
(⋀x. x ∈ s ⟹ (f has-field-derivative (D x)) (at x within s)) ⟹ continuous-on s f
unfolding continuous-on-eq-continuous-within
by (intro continuous-at-imp-continuous-on ballI DERIV-continuous)

lemma DERIV-mult′:
(f has-field-derivative D) (at x within s) ⟹ (g has-field-derivative E) (at x within s) ⟹
((λx. f x ∗ g x) has-field-derivative f x ∗ E + D ∗ g x) (at x within s)
by (rule has-derivative-imp-has-field-derivative[OF has-derivative-mult])
(auto simp: field-simps mult-commute-abs dest: has-field-derivative-imp-has-derivative)

lemma DERIV-mult[derivative-intros]:
(f has-field-derivative Da) (at x within s) ⟹ (g has-field-derivative Db) (at x within s) ⟹
((λx. f x ∗ g x) has-field-derivative Da ∗ g x + Db ∗ f x) (at x within s)
by (rule has-derivative-imp-has-field-derivative[OF has-derivative-mult])
(auto simp: field-simps dest: has-field-derivative-imp-has-derivative)

Derivative of linear multiplication

lemma DERIV-cmult:
(f has-field-derivative D) (at x within s) ⟹
((λx. c ∗ f x) has-field-derivative c ∗ D) (at x within s)
by (drule DERIV-mult′ [OF DERIV-const]) simp

lemma DERIV-cmult-right:
(f has-field-derivative D) (at x within s) ⟹
((λx. f x ∗ c) has-field-derivative D ∗ c) (at x within s)
using DERIV-cmult **by** (auto simp add: ac-simps)

lemma DERIV-cmult-Id [simp]: ((∗) c has-field-derivative c) (at x within s)

using *DERIV-ident* [*THEN DERIV-cmult*, **where** $c = c$ **and** $x = x$] **by** *simp*

lemma *DERIV-cdivide*:
(*f has-field-derivative D*) (*at x within s*) ⟹
((λ*x*. *f x* / *c*) *has-field-derivative D* / *c*) (*at x within s*)
using *DERIV-cmult-right*[*of f D x s 1* / *c*] **by** *simp*

lemma *DERIV-unique*: *DERIV f x* :> *D* ⟹ *DERIV f x* :> *E* ⟹ *D* = *E*
unfolding *DERIV-def* **by** (*rule LIM-unique*)

lemma *DERIV-Uniq*: $\exists_{\leq 1}D$. *DERIV f x* :> *D*
by (*simp add*: *DERIV-unique Uniq-def*)

lemma *DERIV-sum*[*derivative-intros*]:
(⋀ *n*. *n* ∈ *S* ⟹ ((λ*x*. *f x n*) *has-field-derivative* (*f′ x n*)) *F*) ⟹
((λ*x*. *sum* (*f x*) *S*) *has-field-derivative sum* (*f′ x*) *S*) *F*
by (*rule has-derivative-imp-has-field-derivative* [*OF has-derivative-sum*])
(*auto simp*: *sum-distrib-left mult-commute-abs dest*: *has-field-derivative-imp-has-derivative*)

lemma *DERIV-inverse′*[*derivative-intros*]:
assumes (*f has-field-derivative D*) (*at x within s*)
and *f x* ≠ *0*
shows ((λ*x*. *inverse* (*f x*)) *has-field-derivative* − (*inverse* (*f x*) ∗ *D* ∗ *inverse* (*f x*)))
(*at x within s*)
proof −
have (*f has-derivative* (λ*x*. *x* ∗ *D*)) = (*f has-derivative* (∗) *D*)
by (*rule arg-cong* [*of* λ*x*. *x* ∗ *D*]) (*simp add*: *fun-eq-iff*)
with *assms* **have** (*f has-derivative* (λ*x*. *x* ∗ *D*)) (*at x within s*)
by (*auto dest*!: *has-field-derivative-imp-has-derivative*)
then show *?thesis* **using** ‹*f x* ≠ *0*›
by (*auto intro*: *has-derivative-imp-has-field-derivative has-derivative-inverse*)
qed

Power of −*1*

lemma *DERIV-inverse*:
x ≠ *0* ⟹ ((λ*x*. *inverse*(*x*)) *has-field-derivative* − (*inverse x* ^ *Suc* (*Suc 0*))) (*at x within s*)
by (*drule DERIV-inverse′* [*OF DERIV-ident*]) *simp*

Derivative of inverse

lemma *DERIV-inverse-fun*:
(*f has-field-derivative d*) (*at x within s*) ⟹ *f x* ≠ *0* ⟹
((λ*x*. *inverse* (*f x*)) *has-field-derivative* (− (*d* ∗ *inverse*(*f x* ^ *Suc* (*Suc 0*)))))
(*at x within s*)
by (*drule* (*1*) *DERIV-inverse′*) (*simp add*: *ac-simps nonzero-inverse-mult-distrib*)

Derivative of quotient

lemma *DERIV-divide*[*derivative-intros*]:

(*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$
(*g has-field-derivative E*) (*at x within s*) $\Longrightarrow$ *g x* $\neq$ *0* $\Longrightarrow$
((*λx. f x / g x*) *has-field-derivative* (*D* ∗ *g x* − *f x* ∗ *E*) / (*g x* ∗ *g x*)) (*at x within s*)
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-divide*])
(*auto dest*: *has-field-derivative-imp-has-derivative simp*: *field-simps*)

lemma *DERIV-quotient*:
(*f has-field-derivative d*) (*at x within s*) $\Longrightarrow$
(*g has-field-derivative e*) (*at x within s*)$\Longrightarrow$ *g x* $\neq$ *0* $\Longrightarrow$
((*λy. f y / g y*) *has-field-derivative* (*d* ∗ *g x* − (*e* ∗ *f x*)) / (*g x* ^ *Suc* (*Suc 0*)))
(*at x within s*)
by (*drule* (*2*) *DERIV-divide*) (*simp add*: *mult.commute*)

lemma *DERIV-power-Suc*:
(*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$
((*λx. f x* ^ *Suc n*) *has-field-derivative* (*1* + *of-nat n*) ∗ (*D* ∗ *f x* ^ *n*)) (*at x within s*)
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-power*])
(*auto simp*: *has-field-derivative-def*)

lemma *DERIV-power*[*derivative-intros*]:
(*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$
((*λx. f x* ^ *n*) *has-field-derivative of-nat n* ∗ (*D* ∗ *f x* ^ (*n* − *Suc 0*))) (*at x within s*)
by (*rule has-derivative-imp-has-field-derivative*[*OF has-derivative-power*])
(*auto simp*: *has-field-derivative-def*)

lemma *DERIV-pow*: ((*λx. x* ^ *n*) *has-field-derivative real n* ∗ (*x* ^ (*n* − *Suc 0*)))
(*at x within s*)
using *DERIV-power* [*OF DERIV-ident*] **by** *simp*

lemma *DERIV-power-int* [*derivative-intros*]:
assumes [*derivative-intros*]: (*f has-field-derivative d*) (*at x within s*) **and** [*simp*]: *f x* $\neq$ *0*
shows ((*λx. power-int* (*f x*) *n*) *has-field-derivative*
(*of-int n* ∗ *power-int* (*f x*) (*n* − *1*) ∗ *d*)) (*at x within s*)
proof (*cases n rule*: *int-cases4*)
case (*nonneg n*)
thus *?thesis*
by (*cases n* = *0*)
(*auto intro*!: *derivative-eq-intros simp*: *field-simps power-int-diff*
simp flip: *power-Suc power-Suc2 power-add*)
next
case (*neg n*)
thus *?thesis*
by (*auto intro*!: *derivative-eq-intros simp*: *field-simps power-int-diff power-int-minus*
simp flip: *power-Suc power-Suc2 power-add*)
qed

lemma *DERIV-chain'*: (*f has-field-derivative D*) (*at x within s*) $\Longrightarrow$ *DERIV g* (*f x*) :> *E* $\Longrightarrow$
((*λx. g* (*f x*)) *has-field-derivative E* ∗ *D*) (*at x within s*)
using *has-derivative-compose*[*of f* (∗) *D x s g* (∗) *E*]
by (*simp only*: *has-field-derivative-def mult-commute-abs ac-simps*)

corollary *DERIV-chain2*: *DERIV f* (*g x*) :> *Da* $\Longrightarrow$ (*g has-field-derivative Db*) (*at x within s*) $\Longrightarrow$
((*λx. f* (*g x*)) *has-field-derivative Da* ∗ *Db*) (*at x within s*)
by (*rule DERIV-chain'*)

Standard version

lemma *DERIV-chain*:
DERIV f (*g x*) :> *Da* $\Longrightarrow$ (*g has-field-derivative Db*) (*at x within s*) $\Longrightarrow$
(*f* ∘ *g has-field-derivative Da* ∗ *Db*) (*at x within s*)
by (*drule* (*1*) *DERIV-chain'*, *simp add*: *o-def mult.commute*)

lemma *DERIV-image-chain*:
(*f has-field-derivative Da*) (*at* (*g x*) *within* (*g ‘ s*)) $\Longrightarrow$
(*g has-field-derivative Db*) (*at x within s*) $\Longrightarrow$
(*f* ∘ *g has-field-derivative Da* ∗ *Db*) (*at x within s*)
using *has-derivative-in-compose* [*of g* (∗) *Db x s f* (∗) *Da*]
by (*simp add*: *has-field-derivative-def o-def mult-commute-abs ac-simps*)

lemma *DERIV-chain-s*:
assumes (⋀*x. x* ∈ *s* $\Longrightarrow$ *DERIV g x* :> *g'*(*x*))
and *DERIV f x* :> *f'*
and *f x* ∈ *s*
shows *DERIV* (*λx. g*(*f x*)) *x* :> *f'* ∗ *g'*(*f x*)
by (*metis* (*full-types*) *DERIV-chain' mult.commute assms*)

lemma *DERIV-chain3*:
assumes (⋀*x. DERIV g x* :> *g'*(*x*))
and *DERIV f x* :> *f'*
shows *DERIV* (*λx. g*(*f x*)) *x* :> *f'* ∗ *g'*(*f x*)
by (*metis UNIV-I DERIV-chain-s* [*of UNIV*] *assms*)

Alternative definition for differentiability

lemma *DERIV-LIM-iff*:
fixes *f* :: *'a*::{*real-normed-vector*,*inverse*} ⇒ *'a*
shows ((*λh.* (*f* (*a* + *h*) − *f a*) / *h*) −*0*→ *D*) = ((*λx.* (*f x* − *f a*) / (*x* − *a*)) −*a*→ *D*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have (*λx.* (*f* (*a* + (*x* + − *a*)) − *f a*) / (*x* + − *a*)) −*0* − − *a*→ *D*
by (*rule LIM-offset*)
then show *?rhs*

by *simp*
next
assume *?rhs*
then have $(\lambda x.\ (f\ (x{+}a) - f\ a)\ /\ ((x{+}a) - a))\ {-}a{-}a{\rightarrow}\ D$
by (*rule LIM-offset*)
then show *?lhs*
by (*simp add*: *add.commute*)
qed

lemma *has-field-derivative-cong-ev*:
assumes $x = y$
and $*$: *eventually* $(\lambda x.\ x \in S \longrightarrow f\ x = g\ x)$ (*nhds* x)
and $u = v\ S = t\ x \in S$
shows (f *has-field-derivative* u) (*at* x *within* S) = (g *has-field-derivative* v) (*at* y *within* t)
unfolding *has-field-derivative-iff*
proof (*rule filterlim-cong*)
from *assms* **have** $f\ y = g\ y$
by (*auto simp*: *eventually-nhds*)
with $*$ **show** $\forall_F$ z *in at* x *within* S. $(f\ z - f\ x)\ /\ (z - x) = (g\ z - g\ y)\ /\ (z - y)$
unfolding *eventually-at-filter*
by *eventually-elim* (*auto simp*: *assms* ‹$f\ y = g\ y$›)
qed (*simp-all add*: *assms*)

lemma *has-field-derivative-cong-eventually*:
assumes *eventually* $(\lambda x.\ f\ x = g\ x)$ (*at* x *within* S) $f\ x = g\ x$
shows (f *has-field-derivative* u) (*at* x *within* S) = (g *has-field-derivative* u) (*at* x *within* S)
unfolding *has-field-derivative-iff*
proof (*rule tendsto-cong*)
show $\forall_F$ y *in at* x *within* S. $(f\ y - f\ x)\ /\ (y - x) = (g\ y - g\ x)\ /\ (y - x)$
using *assms* **by** (*auto elim*: *eventually-mono*)
qed

lemma *DERIV-cong-ev*:
$x = y \Longrightarrow$ *eventually* $(\lambda x.\ f\ x = g\ x)$ (*nhds* x) $\Longrightarrow u = v \Longrightarrow$
DERIV $f\ x$:> $u \longleftrightarrow$ *DERIV* $g\ y$:> v
by (*rule has-field-derivative-cong-ev*) *simp-all*

lemma *DERIV-mirror*: (*DERIV* $f\ (-\ x)$:> y) $\longleftrightarrow$ (*DERIV* $(\lambda x.\ f\ (-\ x))\ x$:> $-\ y$)
for f :: *real* $\Rightarrow$ *real* **and** $x\ y$ :: *real*
by (*simp add*: *DERIV-def filterlim-at-split filterlim-at-left-to-right tendsto-minus-cancel-left field-simps conj-commute*)

lemma *DERIV-shift*:
(f *has-field-derivative* y) (*at* $(x + z)$) = $((\lambda x.\ f\ (x + z))$ *has-field-derivative* y) (*at* x)

by (*simp add*: *DERIV-def field-simps*)

lemma *DERIV-at-within-shift-lemma*:
assumes (*f has-field-derivative y*) (*at* (*z*+*x*) *within* (+) *z* ' *S*)
shows (*f* ∘ (+)*z has-field-derivative y*) (*at x within S*)
proof −
have ((+)*z has-field-derivative 1*) (*at x within S*)
by (*rule derivative-eq-intros* | *simp*)+
with *assms DERIV-image-chain* **show** *?thesis*
by (*metis mult.right-neutral*)
qed

lemma *DERIV-at-within-shift*:
(*f has-field-derivative y*) (*at* (*z*+*x*) *within* (+) *z* ' *S*) ⟷
((λ*x*. *f* (*z*+*x*)) *has-field-derivative y*) (*at x within S*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *DERIV-at-within-shift-lemma* **unfolding** *o-def* **by** *blast*
next
have [*simp*]: (λ*x*. *x* − *z*) ' (+) *z* ' *S* = *S*
by *force*
assume *R*: *?rhs*
have (*f* ∘ (+) *z* ∘ (+) (− *z*) *has-field-derivative y*) (*at* (*z* + *x*) *within* (+) *z* ' *S*)
by (*rule DERIV-at-within-shift-lemma*) (*use R* **in** ‹*simp add*: *o-def*›)
then show *?lhs*
by (*simp add*: *o-def*)
qed

lemma *floor-has-real-derivative*:
fixes *f* :: *real* ⇒ '*a*::{*floor-ceiling*,*order-topology*}
assumes *isCont f x*
and *f x* ∉ ℤ
shows ((λ*x*. *floor* (*f x*)) *has-real-derivative 0*) (*at x*)
proof (*subst DERIV-cong-ev*[*OF refl - refl*])
show ((λ-. *floor* (*f x*)) *has-real-derivative 0*) (*at x*)
by *simp*
have $\forall_F$ *y in at x*. ⌊*f y*⌋ = ⌊*f x*⌋
by (*rule eventually-floor-eq*[*OF assms*[*unfolded continuous-at*]])
then show $\forall_F$ *y in nhds x*. *real-of-int* ⌊*f y*⌋ = *real-of-int* ⌊*f x*⌋
unfolding *eventually-at-filter*
by *eventually-elim auto*
qed

lemmas *has-derivative-floor*[*derivative-intros*] =
floor-has-real-derivative[*THEN DERIV-compose-FDERIV*]

lemma *continuous-floor*:
fixes *x*::*real*
shows *x* ∉ ℤ ⟹ *continuous* (*at x*) (*real-of-int* ∘ *floor*)

using *floor-has-real-derivative* [**where** *f=id*]
by (*auto simp*: *o-def has-field-derivative-def intro*: *has-derivative-continuous*)

lemma *continuous-frac*:
fixes *x*::*real*
assumes $x \notin \mathbb{Z}$
shows *continuous* (*at x*) *frac*
proof −
have *isCont* (λx. *real-of-int* $\lfloor x \rfloor$) *x*
using *continuous-floor* [*OF assms*] **by** (*simp add*: *o-def*)
then have ∗: *continuous* (*at x*) (λx. $x -$ *real-of-int* $\lfloor x \rfloor$)
by (*intro continuous-intros*)
moreover have $\forall_F$ *x in nhds x*. *frac* $x = x -$ *real-of-int* $\lfloor x \rfloor$
by (*simp add*: *frac-def*)
ultimately show *?thesis*
by (*simp add*: *LIM-imp-LIM frac-def isCont-def*)
qed

Caratheodory formulation of derivative at a point

lemma *CARAT-DERIV*:
(*DERIV f x* :> *l*) ⟷ (∃ *g*. (∀ *z*. *f z* − *f x* = *g z* ∗ (*z* − *x*)) ∧ *isCont g x* ∧ *g x* = *l*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
show ∃ *g*. (∀ *z*. *f z* − *f x* = *g z* ∗ (*z* − *x*)) ∧ *isCont g x* ∧ *g x* = *l*
proof (*intro exI conjI*)
let *?g* = (λ*z*. *if z* = *x then l else* (*f z* − *f x*) / (*z*−*x*))
show ∀ *z*. *f z* − *f x* = *?g z* ∗ (*z* − *x*)
by *simp*
show *isCont ?g x*
using ‹*?lhs*› **by** (*simp add*: *isCont-iff DERIV-def cong*: *LIM-equal* [*rule-format*])
show *?g x* = *l*
by *simp*
qed
next
assume *?rhs*
then show *?lhs*
by (*auto simp add*: *isCont-iff DERIV-def cong*: *LIM-cong*)
qed

110.8 Local extrema

If $0 < f'\ x$ then x is Locally Strictly Increasing At The Right.

lemma *has-real-derivative-pos-inc-right*:
fixes *f* :: *real* ⇒ *real*
assumes *der*: (*f has-real-derivative l*) (*at x within S*)
and *l*: $0 < l$
shows $\exists\, d > 0.\ \forall\, h > 0.\ x + h \in S \longrightarrow h < d \longrightarrow f\ x < f\ (x + h)$

using *assms*
proof −
from *der* [*THEN has-field-derivativeD*, *THEN tendstoD*, *OF l*, *unfolded eventually-at*]
obtain *s* **where** *s*: *0* < *s*
and *all*: $\bigwedge$*xa*. *xa*∈*S* ⟹ *xa* ≠ *x* ∧ *dist xa x* < *s* ⟶ |(*f xa* − *f x*) / (*xa* − *x*) − *l*| < *l*
by (*auto simp*: *dist-real-def*)
then show *?thesis*
proof (*intro exI conjI strip*)
show *0* < *s* **by** (*rule s*)
next
fix *h* :: *real*
assume *0* < *h h* < *s x* + *h* ∈ *S*
with *all* [*of x* + *h*] **show** *f x* < *f* (*x*+*h*)
proof (*simp add*: *abs-if dist-real-def pos-less-divide-eq split*: *if-split-asm*)
assume ¬ (*f* (*x* + *h*) − *f x*) / *h* < *l* **and** *h*: *0* < *h*
with *l* **have** *0* < (*f* (*x* + *h*) − *f x*) / *h*
by *arith*
then show *f x* < *f* (*x* + *h*)
by (*simp add*: *pos-less-divide-eq h*)
qed
qed
qed

lemma *DERIV-pos-inc-right*:
fixes *f* :: *real* ⇒ *real*
assumes *der*: *DERIV f x* :> *l*
and *l*: *0* < *l*
shows ∃ *d* > *0*. ∀ *h* > *0*. *h* < *d* ⟶ *f x* < *f* (*x* + *h*)
using *has-real-derivative-pos-inc-right*[*OF assms*]
by *auto*

lemma *has-real-derivative-neg-dec-left*:
fixes *f* :: *real* ⇒ *real*
assumes *der*: (*f has-real-derivative l*) (*at x within S*)
and *l* < *0*
shows ∃ *d* > *0*. ∀ *h* > *0*. *x* − *h* ∈ *S* ⟶ *h* < *d* ⟶ *f x* < *f* (*x* − *h*)
proof −
from ‹*l* < *0*› **have** *l*: − *l* > *0*
by *simp*
from *der* [*THEN has-field-derivativeD*, *THEN tendstoD*, *OF l*, *unfolded eventually-at*]
obtain *s* **where** *s*: *0* < *s*
and *all*: $\bigwedge$*xa*. *xa*∈*S* ⟹ *xa* ≠ *x* ∧ *dist xa x* < *s* ⟶ |(*f xa* − *f x*) / (*xa* − *x*) − *l*| < − *l*
by (*auto simp*: *dist-real-def*)
then show *?thesis*
proof (*intro exI conjI strip*)

show *0 < s* **by** (*rule s*)
next
fix *h* :: *real*
assume *0 < h h < s x − h ∈ S*
with *all* [*of x − h*] **show** *f x < f (x−h)*
proof (*simp add*: *abs-if pos-less-divide-eq dist-real-def split*: *if-split-asm*)
assume *− ((f (x−h) − f x) / h) < l* **and** *h*: *0 < h*
with *l* **have** *0 < (f (x−h) − f x) / h*
by *arith*
then show *f x < f (x − h)*
by (*simp add*: *pos-less-divide-eq h*)
qed
qed
qed

lemma *DERIV-neg-dec-left*:
fixes *f* :: *real ⇒ real*
assumes *der*: *DERIV f x :> l*
and *l*: *l < 0*
shows *∃ d > 0. ∀ h > 0. h < d ⟶ f x < f (x − h)*
using *has-real-derivative-neg-dec-left*[*OF assms*]
by *auto*

lemma *has-real-derivative-pos-inc-left*:
fixes *f* :: *real ⇒ real*
shows (*f has-real-derivative l*) (*at x within S*) *⟹ 0 < l ⟹*
∃ d>0. ∀ h>0. x − h ∈ S ⟶ h < d ⟶ f (x − h) < f x
by (*rule has-real-derivative-neg-dec-left* [*of λx. − f x −l x S*, *simplified*])
(*auto simp add*: *DERIV-minus*)

lemma *DERIV-pos-inc-left*:
fixes *f* :: *real ⇒ real*
shows *DERIV f x :> l ⟹ 0 < l ⟹ ∃ d > 0. ∀ h > 0. h < d ⟶ f (x − h) < f x*
using *has-real-derivative-pos-inc-left*
by *blast*

lemma *has-real-derivative-neg-dec-right*:
fixes *f* :: *real ⇒ real*
shows (*f has-real-derivative l*) (*at x within S*) *⟹ l < 0 ⟹*
∃ d > 0. ∀ h > 0. x + h ∈ S ⟶ h < d ⟶ f x > f (x + h)
by (*rule has-real-derivative-pos-inc-right* [*of λx. − f x −l x S*, *simplified*])
(*auto simp add*: *DERIV-minus*)

lemma *DERIV-neg-dec-right*:
fixes *f* :: *real ⇒ real*
shows *DERIV f x :> l ⟹ l < 0 ⟹ ∃ d > 0. ∀ h > 0. h < d ⟶ f x > f (x + h)*
using *has-real-derivative-neg-dec-right* **by** *blast*

lemma *DERIV-local-max*:
  fixes f :: *real* ⇒ *real*
  assumes *der*: *DERIV f x* :> l
    and d: $0 < d$
    and *le*: $\forall y.\ |x - y| < d \longrightarrow f\ y \leq f\ x$
  shows $l = 0$
proof (*cases rule*: *linorder-cases* [*of l 0*])
  case *equal*
  then show *?thesis* **.**
next
  case *less*
  from *DERIV-neg-dec-left* [*OF der less*]
  obtain d' **where** d': $0 < d'$ **and** *lt*: $\forall h > 0.\ h < d' \longrightarrow f\ x < f\ (x - h)$
    by *blast*
  obtain e **where** $0 < e \land e < d \land e < d'$
    using *field-lbound-gt-zero* [*OF d d'*] **..**
  with *lt le* [*THEN spec* [**where** $x=x - e$]] **show** *?thesis*
    by (*auto simp add*: *abs-if*)
next
  case *greater*
  from *DERIV-pos-inc-right* [*OF der greater*]
  obtain d' **where** d': $0 < d'$ **and** *lt*: $\forall h > 0.\ h < d' \longrightarrow f\ x < f\ (x + h)$
    by *blast*
  obtain e **where** $0 < e \land e < d \land e < d'$
    using *field-lbound-gt-zero* [*OF d d'*] **..**
  with *lt le* [*THEN spec* [**where** $x=x + e$]] **show** *?thesis*
    by (*auto simp add*: *abs-if*)
qed

Similar theorem for a local minimum

lemma *DERIV-local-min*:
  fixes f :: *real* ⇒ *real*
  shows *DERIV f x* :> $l \Longrightarrow 0 < d \Longrightarrow \forall y.\ |x - y| < d \longrightarrow f\ x \leq f\ y \Longrightarrow l = 0$
  by (*drule DERIV-minus* [*THEN DERIV-local-max*]) *auto*

In particular, if a function is locally flat

lemma *DERIV-local-const*:
  fixes f :: *real* ⇒ *real*
  shows *DERIV f x* :> $l \Longrightarrow 0 < d \Longrightarrow \forall y.\ |x - y| < d \longrightarrow f\ x = f\ y \Longrightarrow l = 0$
  by (*auto dest*!: *DERIV-local-max*)

110.9 Rolle's Theorem

Lemma about introducing open ball in open interval

lemma *lemma-interval-lt*:
  fixes $a\ b\ x$ :: *real*
  assumes $a < x\ x < b$

shows $\exists d.\ 0 < d \wedge (\forall y.\ |x - y| < d \longrightarrow a < y \wedge y < b)$
using *linorder-linear* [*of* $x - a$ $b - x$]
proof
assume $x - a \le b - x$
with *assms* **show** *?thesis*
by (*rule-tac* $x = x - a$ **in** *exI*) *auto*
next
assume $b - x \le x - a$
with *assms* **show** *?thesis*
by (*rule-tac* $x = b - x$ **in** *exI*) *auto*
qed

lemma *lemma-interval*: $a < x \Longrightarrow x < b \Longrightarrow \exists d.\ 0 < d \wedge (\forall y.\ |x - y| < d \longrightarrow a \le y \wedge y \le b)$
for *a b x* :: *real*
by (*force dest*: *lemma-interval-lt*)

Rolle's Theorem. If f is defined and continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and $f\ a = f\ b$, then there exists $x0 \in (a,b)$ such that $f'\ x0 = 0$

theorem *Rolle-deriv*:
fixes f :: *real* $\Rightarrow$ *real*
assumes $a < b$
and *fab*: $f\ a = f\ b$
and *contf*: *continuous-on* $\{a..b\}$ f
and *derf*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow (f$ *has-derivative* $f'\ x)$ (*at* x)
shows $\exists z.\ a < z \wedge z < b \wedge f'\ z = (\lambda v.\ 0)$
proof −
have *le*: $a \le b$
using ‹$a < b$› **by** *simp*
have $(a + b) / 2 \in \{a..b\}$
using *assms*(*1*) **by** *auto*
then have ∗: $\{a..b\} \ne \{\}$
by *auto*
obtain x **where** *x-max*: $\forall z.\ a \le z \wedge z \le b \longrightarrow f\ z \le f\ x$ **and** $a \le x$ $x \le b$
using *continuous-attains-sup*[*OF compact-Icc* ∗ *contf*]
by (*meson atLeastAtMost-iff*)
obtain x' **where** *x'-min*: $\forall z.\ a \le z \wedge z \le b \longrightarrow f\ x' \le f\ z$ **and** $a \le x'$ $x' \le b$
using *continuous-attains-inf*[*OF compact-Icc* ∗ *contf*] **by** (*meson atLeastAt-Most-iff*)
consider $a < x$ $x < b$ | $x = a \vee x = b$
using ‹$a \le x$› ‹$x \le b$› **by** *arith*
then show *?thesis*
proof *cases*
case *1*
— f attains its maximum within the interval
then obtain l **where** *der*: *DERIV f x* :> l
using *derf differentiable-def real-differentiable-def* **by** *blast*
obtain d **where** d: $0 < d$ **and** *bound*: $\forall y.\ |x - y| < d \longrightarrow a \le y \wedge y \le b$

using *lemma-interval* [*OF 1*] **by** *blast*
then have *bound'*: $\forall y. |x - y| < d \longrightarrow f\ y \le f\ x$
using *x-max* **by** *blast*
— the derivative at a local maximum is zero
have $l = 0$
by (*rule DERIV-local-max* [*OF der d bound'*])
with *1 der derf* [*of x*] **show** *?thesis*
by (*metis has-derivative-unique has-field-derivative-def mult-zero-left*)
next
case *2*
then have *fx*: $f\ b = f\ x$ **by** (*auto simp add*: *fab*)
consider $a < x'\ x' < b \mid x' = a \vee x' = b$
using ‹$a \le x'$› ‹$x' \le b$› **by** *arith*
then show *?thesis*
proof *cases*
case *1*
— *f* attains its minimum within the interval
then obtain *l* **where** *der*: *DERIV f x'* :> *l*
using *derf differentiable-def real-differentiable-def* **by** *blast*
from *lemma-interval* [*OF 1*]
obtain *d* **where** *d*: $0<d$ **and** *bound*: $\forall y. |x'-y| < d \longrightarrow a \le y \wedge y \le b$
by *blast*
then have *bound'*: $\forall y. |x' - y| < d \longrightarrow f\ x' \le f\ y$
using *x'-min* **by** *blast*
have $l = 0$ **by** (*rule DERIV-local-min* [*OF der d bound'*])
— the derivative at a local minimum is zero
then show *?thesis* **using** *1 der derf* [*of x'*]
by (*metis has-derivative-unique has-field-derivative-def mult-zero-left*)
next
case *2*
— *f* is constant throughout the interval
then have *fx'*: $f\ b = f\ x'$ **by** (*auto simp*: *fab*)
from *dense* [*OF* ‹$a < b$›] **obtain** *r* **where** *r*: $a < r\ r < b$ **by** *blast*
obtain *d* **where** *d*: $0 < d$ **and** *bound*: $\forall y. |r - y| < d \longrightarrow a \le y \wedge y \le b$
using *lemma-interval* [*OF r*] **by** *blast*
have *eq-fb*: $f\ z = f\ b$ **if** $a \le z$ **and** $z \le b$ **for** *z*
proof (*rule order-antisym*)
show $f\ z \le f\ b$ **by** (*simp add*: *fx x-max that*)
show $f\ b \le f\ z$ **by** (*simp add*: *fx' x'-min that*)
qed
have *bound'*: $\forall y. |r - y| < d \longrightarrow f\ r = f\ y$
proof (*intro strip*)
fix *y* :: *real*
assume *lt*: $|r - y| < d$
then have $f\ y = f\ b$ **by** (*simp add*: *eq-fb bound*)
then show $f\ r = f\ y$ **by** (*simp add*: *eq-fb r order-less-imp-le*)
qed
obtain *l* **where** *der*: *DERIV f r* :> *l*
using *derf differentiable-def r*(*1*) *r*(*2*) *real-differentiable-def* **by** *blast*

have $l = 0$
by (*rule DERIV-local-const* [*OF der d bound'*])
— the derivative of a constant function is zero
with *r der derf* [*of r*] **show** *?thesis*
by (*metis has-derivative-unique has-field-derivative-def mult-zero-left*)
qed
qed
qed

corollary *Rolle*:
fixes *a b* :: *real*
assumes *ab*: $a < b$ $f\ a = f\ b$ *continuous-on* $\{a..b\}$ f
and *dif* [*rule-format*]: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow f$ *differentiable* (*at* x)
shows $\exists z.\ a < z \wedge z < b \wedge$ *DERIV* $f\ z$:> *0*
proof −
obtain f' **where** f': $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow$ (f *has-derivative* $f'\ x$) (*at* x)
using *dif* **unfolding** *differentiable-def* **by** *metis*
then have $\exists z.\ a < z \wedge z < b \wedge f'\ z = (\lambda v.\ 0)$
by (*metis Rolle-deriv* [*OF ab*])
then show *?thesis*
using f' *has-derivative-imp-has-field-derivative* **by** *fastforce*
qed

110.10 Mean Value Theorem

theorem *mvt*:
fixes f :: *real* ⇒ *real*
assumes $a < b$
and *contf*: *continuous-on* $\{a..b\}$ f
and *derf*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow$ (f *has-derivative* $f'\ x$) (*at* x)
obtains ξ **where** $a < \xi$ $\xi < b$ $f\ b - f\ a = (f'\ \xi)\ (b - a)$
proof −
have $\exists \xi.\ a < \xi \wedge \xi < b \wedge (\lambda y.\ f'\ \xi\ y - (f\ b - f\ a) / (b - a) * y) = (\lambda v.\ 0)$
proof (*intro Rolle-deriv*[*OF* ‹$a < b$›])
fix x
assume x: $a < x$ $x < b$
show $((\lambda x.\ f\ x - (f\ b - f\ a) / (b - a) * x)$
has-derivative $(\lambda y.\ f'\ x\ y - (f\ b - f\ a) / (b - a) * y))$ (*at* x)
by (*intro derivative-intros derf*[*OF x*])
qed (*use assms* **in** ‹*auto intro*!: *continuous-intros simp*: *field-simps*›)
then show *?thesis*
by (*smt* (*verit*, *ccfv-SIG*) *pos-le-divide-eq pos-less-divide-eq that*)
qed

theorem *MVT*:
fixes *a b* :: *real*
assumes *lt*: $a < b$
and *contf*: *continuous-on* $\{a..b\}$ f
and *dif*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow f$ *differentiable* (*at* x)

shows $\exists\, l\ z.\ a < z \wedge z < b \wedge DERIV\ f\ z :> l \wedge f\ b - f\ a = (b - a) * l$
proof −
obtain $f' :: real \Rightarrow real \Rightarrow real$
where *derf*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (f\ has\text{-}derivative\ f'\ x)\ (at\ x)$
using *dif* **unfolding** *differentiable-def* **by** *metis*
then obtain z **where** $a < z\ z < b\ f\ b - f\ a = (f'\ z)\ (b - a)$
using *mvt* [*OF lt contf*] **by** *blast*
then show *?thesis*
by (*simp add*: *ac-simps*)
(*metis derf dif has-derivative-unique has-field-derivative-imp-has-derivative real-differentiable-def*)
qed

corollary *MVT2*:
assumes $a < b$ **and** *der*: $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow DERIV\ f\ x :> f'\ x$
shows $\exists\, z{::}real.\ a < z \wedge z < b \wedge (f\ b - f\ a = (b - a) * f'\ z)$
proof −
have $\exists\, l\ z.\ a < z \wedge$
$z < b \wedge$
$(f\ has\text{-}real\text{-}derivative\ l)\ (at\ z) \wedge$
$f\ b - f\ a = (b - a) * l$
proof (*rule MVT* [*OF* ‹$a < b$›])
show *continuous-on* $\{a..b\}$ f
by (*meson DERIV-continuous atLeastAtMost-iff continuous-at-imp-continuous-on der*)
show $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow f$ *differentiable* (*at* x)
using *assms* **by** (*force dest*: *order-less-imp-le simp add*: *real-differentiable-def*)
qed
with *assms* **show** *?thesis*
by (*blast dest*: *DERIV-unique order-less-imp-le*)
qed

110.10.1 A function is constant if its derivative is 0 over an interval.

lemma *DERIV-isconst-end*:
fixes $f :: real \Rightarrow real$
assumes $a < b$ **and** *contf*: *continuous-on* $\{a..b\}$ f
and *0*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow DERIV\ f\ x :> 0$
shows $f\ b = f\ a$
using *MVT* [*OF* ‹$a < b$›] *0 DERIV-unique contf real-differentiable-def*
by (*fastforce simp*: *algebra-simps*)

lemma *DERIV-isconst2*:
fixes $f :: real \Rightarrow real$
assumes $a < b$ **and** *contf*: *continuous-on* $\{a..b\}$ f **and** *derf*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow DERIV\ f\ x :> 0$
and $a \le x\ x \le b$
shows $f\ x = f\ a$

proof (*cases a* < *x*)
  case *True*
  have ∗: *continuous-on* {*a*..*x*} *f*
    using ‹*x* ≤ *b*› *contf continuous-on-subset* **by** *fastforce*
  show *?thesis*
    by (*rule DERIV-isconst-end* [*OF True* ∗]) (*use* ‹*x* ≤ *b*› *derf* **in** *auto*)
qed (*use* ‹*a* ≤ *x*› **in** *auto*)

lemma *DERIV-isconst3*:
  fixes *a b x y* :: *real*
  assumes *a* < *b*
    and *x* ∈ {*a* <..< *b*}
    and *y* ∈ {*a* <..< *b*}
    and *derivable*: ⋀*x*. *x* ∈ {*a* <..< *b*} ⟹ *DERIV f x* :> *0*
  shows *f x* = *f y*
proof (*cases x* = *y*)
  case *False*
  let *?a* = *min x y*
  let *?b* = *max x y*
  have ∗: *DERIV f z* :> *0* **if** *?a* ≤ *z z* ≤ *?b* **for** *z*
  proof −
    have *a* < *z* **and** *z* < *b*
      using *that* ‹*x* ∈ {*a* <..< *b*}› **and** ‹*y* ∈ {*a* <..< *b*}› **by** *auto*
    then have *z* ∈ {*a*<..<*b*} **by** *auto*
    then show *DERIV f z* :> *0* **by** (*rule derivable*)
  qed
  have *isCont*: *continuous-on* {*?a*..*?b*} *f*
   by (*meson* ∗ *DERIV-continuous-on atLeastAtMost-iff has-field-derivative-at-within*)
  have *DERIV*: ⋀*z*. ⟦*?a* < *z*; *z* < *?b*⟧ ⟹ *DERIV f z* :> *0*
    using ∗ **by** *auto*
  have *?a* < *?b* **using** ‹*x* ≠ *y*› **by** *auto*
  from *DERIV-isconst2*[*OF this isCont DERIV*, *of x*] **and** *DERIV-isconst2*[*OF this isCont DERIV*, *of y*]
  show *?thesis* **by** *auto*
qed *auto*

lemma *DERIV-isconst-all*:
  fixes *f* :: *real* ⇒ *real*
  shows ∀ *x*. *DERIV f x* :> *0* ⟹ *f x* = *f y*
  apply (*rule linorder-cases* [*of x y*])
  apply (*metis DERIV-continuous DERIV-isconst-end continuous-at-imp-continuous-on*)+
  done

lemma *DERIV-const-ratio-const*:
  fixes *f* :: *real* ⇒ *real*
  assumes *a* ≠ *b* **and** *df*: ⋀*x*. *DERIV f x* :> *k*
  shows *f b* − *f a* = (*b* − *a*) ∗ *k*
proof (*cases a b rule*: *linorder-cases*)
  case *less*

show *?thesis*
using *MVT* [*OF less*] *df*
by (*metis DERIV-continuous DERIV-unique continuous-at-imp-continuous-on real-differentiable-def*)
next
case *greater*
have $f\ a - f\ b = (a - b) * k$
using *MVT* [*OF greater*] *df*
by (*metis DERIV-continuous DERIV-unique continuous-at-imp-continuous-on real-differentiable-def*)
then show *?thesis*
by (*simp add*: *algebra-simps*)
qed *auto*

lemma *DERIV-const-ratio-const2*:
fixes f :: *real* ⇒ *real*
assumes $a \neq b$ **and** *df*: ⋀*x*. *DERIV f x* :> *k*
shows $(f\ b - f\ a) / (b - a) = k$
using *DERIV-const-ratio-const* [*OF assms*] ‹$a \neq b$› **by** *auto*

lemma *real-average-minus-first* [*simp*]: $(a + b) / 2 - a = (b - a) / 2$
for *a b* :: *real*
by *simp*

lemma *real-average-minus-second* [*simp*]: $(b + a) / 2 - a = (b - a) / 2$
for *a b* :: *real*
by *simp*

Gallileo's "trick": average velocity = av. of end velocities.

lemma *DERIV-const-average*:
fixes *v* :: *real* ⇒ *real*
and *a b* :: *real*
assumes *neq*: $a \neq b$
and *der*: ⋀*x*. *DERIV v x* :> *k*
shows $v\ ((a + b) / 2) = (v\ a + v\ b) / 2$
proof (*cases rule*: *linorder-cases* [*of a b*])
case *equal*
with *neq* **show** *?thesis* **by** *simp*
next
case *less*
have $(v\ b - v\ a) / (b - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF neq der*])
then have $(b - a) * ((v\ b - v\ a) / (b - a)) = (b - a) * k$
by *simp*
moreover have $(v\ ((a + b) / 2) - v\ a) / ((a + b) / 2 - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF - der*]) (*simp add*: *neq*)
ultimately show *?thesis*
using *neq* **by** *force*
next

case *greater*
have $(v\ b - v\ a)\ /\ (b - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF neq der*])
then have $(b - a) * ((v\ b - v\ a)\ /\ (b - a)) = (b - a) * k$
by *simp*
moreover have $(v\ ((b + a)\ /\ 2) - v\ a)\ /\ ((b + a)\ /\ 2 - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF - der*]) (*simp add*: *neq*)
ultimately show *?thesis*
using *neq* **by** (*force simp add*: *add.commute*)
qed

110.10.2 A function with positive derivative is increasing

A simple proof using the MVT, by Jeremy Avigad. And variants.

lemma *DERIV-pos-imp-increasing-open*:
fixes *a b* :: *real*
and *f* :: *real* ⇒ *real*
assumes $a < b$
and $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (\exists\, y.\ DERIV\ f\ x :> y \wedge y > 0)$
and *con*: *continuous-on* {*a..b*} *f*
shows $f\ a < f\ b$
proof (*rule ccontr*)
assume *f*: ¬ *?thesis*
have $\exists\, l\ z.\ a < z \wedge z < b \wedge DERIV\ f\ z :> l \wedge f\ b - f\ a = (b - a) * l$
by (*rule MVT*) (*use assms real-differentiable-def* **in** ‹*force+*›)
then obtain *l z* **where** *z*: $a < z$ $z < b$ *DERIV f z* :> *l* **and** $f\ b - f\ a = (b - a) * l$
by *auto*
with *assms f* **have** $\neg\ l > 0$
by (*metis linorder-not-le mult-le-0-iff diff-le-0-iff-le*)
with *assms z* **show** *False*
by (*metis DERIV-unique*)
qed

lemma *DERIV-pos-imp-increasing*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ *real*
assumes $a < b$
and *der*: $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow \exists\, y.\ DERIV\ f\ x :> y \wedge y > 0$
shows $f\ a < f\ b$
by (*metis less-le-not-le DERIV-atLeastAtMost-imp-continuous-on DERIV-pos-imp-increasing-open* [*OF* ‹$a < b$›] *der*)

lemma *DERIV-nonneg-imp-nondecreasing*:
fixes *a b* :: *real*
and *f* :: *real* ⇒ *real*
assumes $a \le b$
and $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow \exists\, y.\ DERIV\ f\ x :> y \wedge y \ge 0$
shows $f\ a \le f\ b$
proof (*rule ccontr*, *cases* $a = b$)

assume ¬ *?thesis* **and** *a* = *b*
then show *False* **by** *auto*
next
assume ∗: ¬ *?thesis*
assume *a* ≠ *b*
with ‹*a* ≤ *b*› **have** *a* < *b*
by *linarith*
moreover have *continuous-on* {*a..b*} *f*
by (*meson DERIV-isCont assms*(*2*) *atLeastAtMost-iff continuous-at-imp-continuous-on*)
ultimately have ∃ *l z*. *a* < *z* ∧ *z* < *b* ∧ *DERIV f z* :> *l* ∧ *f b* − *f a* = (*b* − *a*) ∗ *l*
using *assms MVT* [*OF* ‹*a* < *b*›, *of f*] *real-differentiable-def less-eq-real-def* **by** *blast*
then obtain *l z* **where** *lz*: *a* < *z z* < *b DERIV f z* :> *l* **and** ∗∗: *f b* − *f a* = (*b* − *a*) ∗ *l*
by *auto*
with ∗ **have** *a* < *b f b* < *f a* **by** *auto*
with ∗∗ **have** ¬ *l* ≥ *0* **by** (*auto simp add*: *not-le algebra-simps*)
(*metis* ∗ *add-le-cancel-right assms*(*1*) *less-eq-real-def mult-right-mono add-left-mono linear order-refl*)
with *assms lz* **show** *False*
by (*metis DERIV-unique order-less-imp-le*)
qed

lemma *DERIV-neg-imp-decreasing-open*:
fixes *a b* :: *real*
and *f* :: *real* ⇒ *real*
assumes *a* < *b*
and ⋀*x*. *a* < *x* ⟹ *x* < *b* ⟹ ∃ *y*. *DERIV f x* :> *y* ∧ *y* < *0*
and *con*: *continuous-on* {*a..b*} *f*
shows *f a* > *f b*
proof −
have (λ*x*. −*f x*) *a* < (λ*x*. −*f x*) *b*
proof (*rule DERIV-pos-imp-increasing-open* [*of a b*])
show ⋀*x*. ⟦*a* < *x*; *x* < *b*⟧ ⟹ ∃ *y*. ((λ*x*. − *f x*) *has-real-derivative y*) (*at x*) ∧ *0* < *y*
using *assms*
by *simp* (*metis field-differentiable-minus neg-0-less-iff-less*)
show *continuous-on* {*a..b*} (λ*x*. − *f x*)
using *con continuous-on-minus* **by** *blast*
qed (*use assms* **in** *auto*)
then show *?thesis*
by *simp*
qed

lemma *DERIV-neg-imp-decreasing*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ *real*
assumes *a* < *b*
and *der*: ⋀*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ ∃ *y*. *DERIV f x* :> *y* ∧ *y* < *0*

shows *f a* > *f b*
by (*metis less-le-not-le DERIV-atLeastAtMost-imp-continuous-on DERIV-neg-imp-decreasing-open* [*OF* ‹*a* < *b*›] *der*)

lemma *DERIV-nonpos-imp-nonincreasing*:
fixes *a b* :: *real*
and *f* :: *real* ⇒ *real*
assumes $a \le b$
and $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow \exists y.\ DERIV\ f\ x :> y \wedge y \le 0$
shows $f\ a \ge f\ b$
proof −
have $(\lambda x.\ -f\ x)\ a \le (\lambda x.\ -f\ x)\ b$
using *DERIV-nonneg-imp-nondecreasing* [*of a b* λ*x*. −*f x*] *assms DERIV-minus*
by *fastforce*
then show *?thesis*
by *simp*
qed

lemma *DERIV-pos-imp-increasing-at-bot*:
fixes *f* :: *real* ⇒ *real*
assumes $\bigwedge x.\ x \le b \Longrightarrow (\exists y.\ DERIV\ f\ x :> y \wedge y > 0)$
and *lim*: (*f* ⟶ *flim*) *at-bot*
shows *flim* < *f b*
proof −
have $\exists N.\ \forall n{\le}N.\ f\ n \le f\ (b - 1)$
by (*rule-tac x*=*b* − *2* **in** *exI*) (*force intro*: *order.strict-implies-order DERIV-pos-imp-increasing assms*)
then have $flim \le f\ (b - 1)$
by (*auto simp*: *eventually-at-bot-linorder tendsto-upperbound* [*OF lim*])
also have . . . < *f b*
by (*force intro*: *DERIV-pos-imp-increasing* [**where** *f*=*f*] *assms*)
finally show *?thesis* **.**
qed

lemma *DERIV-neg-imp-decreasing-at-top*:
fixes *f* :: *real* ⇒ *real*
assumes *der*: $\bigwedge x.\ x \ge b \Longrightarrow \exists y.\ DERIV\ f\ x :> y \wedge y < 0$
and *lim*: (*f* ⟶ *flim*) *at-top*
shows *flim* < *f b*
apply (*rule DERIV-pos-imp-increasing-at-bot* [**where** *f* = λ*i*. *f* (−*i*) **and** *b* = −*b*, *simplified*])
apply (*metis DERIV-mirror der le-minus-iff neg-0-less-iff-less*)
apply (*metis filterlim-at-top-mirror lim*)
done

proposition *deriv-nonpos-imp-antimono*:
assumes *deriv*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\text{-}real\text{-}derivative\ g'\ x)\ (at\ x)$
assumes *nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \le 0$
assumes $a \le b$

shows *g b* ≤ *g a*
proof −
have − *g a* ≤ − *g b*
proof (*intro DERIV-nonneg-imp-nondecreasing* [**where** *f* = λ*x*. − *g x*] *conjI exI*)
fix *x*
assume *x*: *a* ≤ *x x* ≤ *b*
show ((λ*x*. − *g x*) *has-real-derivative* − *g′ x*) (*at x*)
by (*simp add*: *DERIV-minus deriv x*)
show *0* ≤ − *g′ x*
by (*simp add*: *nonneg x*)
qed (*rule* ‹*a*≤*b*›)
then show *?thesis* **by** *simp*
qed

lemma *DERIV-nonneg-imp-increasing-open*:
fixes *a b* :: *real*
and *f* :: *real* ⇒ *real*
assumes *a* ≤ *b*
and ⋀*x*. *a* < *x* ⟹ *x* < *b* ⟹ (∃ *y*. *DERIV f x* :> *y* ∧ *y* ≥ *0*)
and *con*: *continuous-on* {*a*..*b*} *f*
shows *f a* ≤ *f b*
proof (*cases a*=*b*)
case *False*
with ‹*a*≤*b*› **have** *a*<*b* **by** *simp*
show *?thesis*
proof (*rule ccontr*)
assume *f*: ¬ *?thesis*
have ∃ *l z*. *a* < *z* ∧ *z* < *b* ∧ *DERIV f z* :> *l* ∧ *f b* − *f a* = (*b* − *a*) ∗ *l*
by (*rule MVT*) (*use assms* ‹*a*<*b*› *real-differentiable-def* **in** ‹*force*+›)
then obtain *l z* **where** *z*: *a* < *z z* < *b DERIV f z* :> *l* **and** *f b* − *f a* = (*b* − *a*) ∗ *l*
by *auto*
with *assms z f* **show** *False*
by (*metis DERIV-unique diff-ge-0-iff-ge zero-le-mult-iff*)
qed
qed *auto*

lemma *DERIV-nonpos-imp-decreasing-open*:
fixes *a b* :: *real*
and *f* :: *real* ⇒ *real*
assumes *a* ≤ *b*
and ⋀*x*. *a* < *x* ⟹ *x* < *b* ⟹ ∃ *y*. *DERIV f x* :> *y* ∧ *y* ≤ *0*
and *con*: *continuous-on* {*a*..*b*} *f*
shows *f a* ≥ *f b*
proof −
have (λ*x*. −*f x*) *a* ≤ (λ*x*. −*f x*) *b*
proof (*rule DERIV-nonneg-imp-increasing-open* [*of a b*])
show ⋀*x*. ⟦*a* < *x*; *x* < *b*⟧ ⟹ ∃ *y*. ((λ*x*. − *f x*) *has-real-derivative y*) (*at x*) ∧

```
0 ≤ y
      using assms
      by (metis Deriv.field-differentiable-minus neg-0-le-iff-le)
    show continuous-on {a..b} (λx. − f x)
      using con continuous-on-minus by blast
  qed (use assms in auto)
  then show ?thesis
    by simp
qed
```

```
proposition deriv-nonneg-imp-mono:
  assumes ⋀x. x ∈ {a..b} ⟹ (g has-real-derivative g′ x) (at x)
  assumes ⋀x. x ∈ {a..b} ⟹ g′ x ≥ 0
  assumes a ≤ b
  shows g a ≤ g b
  by (metis DERIV-nonneg-imp-nondecreasing atLeastAtMost-iff assms)
```

Derivative of inverse function

```
lemma DERIV-inverse-function:
  fixes f g :: real ⇒ real
  assumes der: DERIV f (g x) :> D
    and neq: D ≠ 0
    and x: a < x x < b
    and inj: ⋀y. ⟦a < y; y < b⟧ ⟹ f (g y) = y
    and cont: isCont g x
  shows DERIV g x :> inverse D
unfolding has-field-derivative-iff
proof (rule LIM-equal2)
  show 0 < min (x − a) (b − x)
    using x by arith
next
  fix y
  assume norm (y − x) < min (x − a) (b − x)
  then have a < y and y < b
    by (simp-all add: abs-less-iff)
  then show (g y − g x) / (y − x) = inverse ((f (g y) − x) / (g y − g x))
    by (simp add: inj)
next
  have (λz. (f z − f (g x)) / (z − g x)) −g x→ D
    by (rule der [unfolded has-field-derivative-iff])
  then have 1: (λz. (f z − x) / (z − g x)) −g x→ D
    using inj x by simp
  have 2: ∃ d>0. ∀ y. y ≠ x ∧ norm (y − x) < d ⟶ g y ≠ g x
  proof (rule exI, safe)
    show 0 < min (x − a) (b − x)
      using x by simp
  next
    fix y
```

assume *norm* $(y - x) < min\ (x - a)\ (b - x)$
then have *y*: $a < y\ y < b$
by (*simp-all add*: *abs-less-iff*)
assume $g\ y = g\ x$
then have $f\ (g\ y) = f\ (g\ x)$ **by** *simp*
then have $y = x$ **using** *inj y x* **by** *simp*
also assume $y \neq x$
finally show *False* **by** *simp*
qed
have $(\lambda y.\ (f\ (g\ y) - x)\ /\ (g\ y - g\ x))\ {-x\rightarrow}\ D$
using *cont 1 2* **by** (*rule isCont-LIM-compose2*)
then show $(\lambda y.\ inverse\ ((f\ (g\ y) - x)\ /\ (g\ y - g\ x)))\ {-x\rightarrow}\ inverse\ D$
using *neq* **by** (*rule tendsto-inverse*)
qed

110.11 Generalized Mean Value Theorem

theorem *GMVT*:
fixes *a b* :: *real*
assumes *alb*: $a < b$
and *fc*: $\forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ f\ x$
and *fd*: $\forall x.\ a < x \wedge x < b \longrightarrow f\ differentiable\ (at\ x)$
and *gc*: $\forall x.\ a \leq x \wedge x \leq b \longrightarrow isCont\ g\ x$
and *gd*: $\forall x.\ a < x \wedge x < b \longrightarrow g\ differentiable\ (at\ x)$
shows $\exists g'c\ f'c\ c.$
$DERIV\ g\ c :> g'c \wedge DERIV\ f\ c :> f'c \wedge a < c \wedge c < b \wedge (f\ b - f\ a) * g'c = (g\ b - g\ a) * f'c$
proof −
let $?h = \lambda x.\ (f\ b - f\ a) * g\ x - (g\ b - g\ a) * f\ x$
have $\exists l\ z.\ a < z \wedge z < b \wedge DERIV\ ?h\ z :> l \wedge ?h\ b - ?h\ a = (b - a) * l$
proof (*rule MVT*)
from *assms* **show** $a < b$ **by** *simp*
show *continuous-on* $\{a..b\}$ *?h*
by (*simp add*: *continuous-at-imp-continuous-on fc gc*)
show $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow\ ?h\ differentiable\ (at\ x)$
using *fd gd* **by** *simp*
qed
then obtain *l* **where** *l*: $\exists z.\ a < z \wedge z < b \wedge DERIV\ ?h\ z :> l \wedge ?h\ b - ?h\ a = (b - a) * l$ **..**
then obtain *c* **where** *c*: $a < c \wedge c < b \wedge DERIV\ ?h\ c :> l \wedge ?h\ b - ?h\ a = (b - a) * l$ **..**

from *c* **have** *cint*: $a < c \wedge c < b$ **by** *auto*
then obtain $g'c$ **where** $g'c$: $DERIV\ g\ c :> g'c$
using *gd real-differentiable-def* **by** *blast*
from *c* **have** $a < c \wedge c < b$ **by** *auto*
then obtain $f'c$ **where** $f'c$: $DERIV\ f\ c :> f'c$
using *fd real-differentiable-def* **by** *blast*

from c **have** *DERIV ?h c* :> l **by** *auto*
moreover have *DERIV ?h c* :> $g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a)$
using $g'c\ f'c$ **by** (*auto intro*!: *derivative-eq-intros*)
ultimately have *leq*: $l = g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a)$ **by** (*rule DERIV-unique*)

have $?h\ b - ?h\ a = (b - a) * (g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a))$
proof −
from c **have** $?h\ b - ?h\ a = (b - a) * l$ **by** *auto*
also from *leq* **have** $\ldots = (b - a) * (g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a))$ **by** *simp*
finally show *?thesis* **by** *simp*
qed
moreover have $?h\ b - ?h\ a = 0$
proof −
have $?h\ b - ?h\ a =$
$((f\ b)*(g\ b) - (f\ a)*(g\ b) - (g\ b)*(f\ b) + (g\ a)*(f\ b)) -$
$((f\ b)*(g\ a) - (f\ a)*(g\ a) - (g\ b)*(f\ a) + (g\ a)*(f\ a))$
by (*simp add*: *algebra-simps*)
then show *?thesis* **by** *auto*
qed
ultimately have $(b - a) * (g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a)) = 0$ **by** *auto*
with *alb* **have** $g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a) = 0$ **by** *simp*
then have $g'c * (f\ b - f\ a) = f'c * (g\ b - g\ a)$ **by** *simp*
then have $(f\ b - f\ a) * g'c = (g\ b - g\ a) * f'c$ **by** (*simp add*: *ac-simps*)
with $g'c\ f'c$ *cint* **show** *?thesis* **by** *auto*
qed

lemma *GMVT'*:
fixes $f\ g$ :: *real* ⇒ *real*
assumes $a < b$
and *isCont-f*: $\bigwedge z.\ a \le z \Longrightarrow z \le b \Longrightarrow$ *isCont f z*
and *isCont-g*: $\bigwedge z.\ a \le z \Longrightarrow z \le b \Longrightarrow$ *isCont g z*
and *DERIV-g*: $\bigwedge z.\ a < z \Longrightarrow z < b \Longrightarrow$ *DERIV g z* :> $(g'\ z)$
and *DERIV-f*: $\bigwedge z.\ a < z \Longrightarrow z < b \Longrightarrow$ *DERIV f z* :> $(f'\ z)$
shows $\exists\, c.\ a < c \land c < b \land (f\ b - f\ a) * g'\ c = (g\ b - g\ a) * f'\ c$
proof −
have $\exists\, g'c\ f'c\ c.$ *DERIV g c* :> $g'c \land$ *DERIV f c* :> $f'c \land$
$a < c \land c < b \land (f\ b - f\ a) * g'c = (g\ b - g\ a) * f'c$
using *assms* **by** (*intro GMVT*) (*force simp*: *real-differentiable-def*)+
then obtain c **where** $a < c\ c < b\ (f\ b - f\ a) * g'\ c = (g\ b - g\ a) * f'\ c$
using *DERIV-f DERIV-g* **by** (*force dest*: *DERIV-unique*)
then show *?thesis*
by *auto*
qed

110.12 L'Hopitals rule

lemma *isCont-If-ge*:

fixes *a* :: *'a* :: *linorder-topology*
assumes *continuous* (*at-left a*) *g* **and** *f*: (*f* ⟶ *g a*) (*at-right a*)
shows *isCont* (λ*x*. *if x* ≤ *a then g x else f x*) *a* (**is** *isCont ?gf a*)
proof −
have *g*: (*g* ⟶ *g a*) (*at-left a*)
using *assms continuous-within* **by** *blast*
show *?thesis*
unfolding *isCont-def continuous-within*
proof (*intro filterlim-split-at*; *simp*)
show (*?gf* ⟶ *g a*) (*at-left a*)
by (*subst filterlim-cong*[*OF refl refl*, **where** *g*=*g*]) (*simp-all add*: *eventually-at-filter less-le g*)
show (*?gf* ⟶ *g a*) (*at-right a*)
by (*subst filterlim-cong*[*OF refl refl*, **where** *g*=*f*]) (*simp-all add*: *eventually-at-filter less-le f*)
qed
qed

lemma *lhopital-right-0*:
fixes *f0 g0* :: *real* ⇒ *real*
assumes *f-0*: (*f0* ⟶ *0*) (*at-right 0*)
and *g-0*: (*g0* ⟶ *0*) (*at-right 0*)
and *ev*:
eventually (λ*x*. *g0 x* ≠ *0*) (*at-right 0*)
eventually (λ*x*. *g' x* ≠ *0*) (*at-right 0*)
eventually (λ*x*. *DERIV f0 x* :> *f' x*) (*at-right 0*)
eventually (λ*x*. *DERIV g0 x* :> *g' x*) (*at-right 0*)
and *lim*: *filterlim* (λ *x*. (*f' x* / *g' x*)) *F* (*at-right 0*)
shows *filterlim* (λ *x*. *f0 x* / *g0 x*) *F* (*at-right 0*)
proof −
define *f* **where** [*abs-def*]: *f x* = (*if x* ≤ *0 then 0 else f0 x*) **for** *x*
then have *f 0* = *0* **by** *simp*

define *g* **where** [*abs-def*]: *g x* = (*if x* ≤ *0 then 0 else g0 x*) **for** *x*
then have *g 0* = *0* **by** *simp*

have *eventually* (λ*x*. *g0 x* ≠ *0* ∧ *g' x* ≠ *0* ∧
DERIV f0 x :> (*f' x*) ∧ *DERIV g0 x* :> (*g' x*)) (*at-right 0*)
using *ev* **by** *eventually-elim auto*
then obtain *a* **where** [*arith*]: *0* < *a*
and *g0-neq-0*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *g0 x* ≠ *0*
and *g'-neq-0*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *g' x* ≠ *0*
and *f0*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *DERIV f0 x* :> (*f' x*)
and *g0*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *DERIV g0 x* :> (*g' x*)
unfolding *eventually-at* **by** (*auto simp*: *dist-real-def*)

have *g-neq-0*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *g x* ≠ *0*
using *g0-neq-0* **by** (*simp add*: *g-def*)

have *f*: *DERIV f x* :> (*f′ x*) **if** *x*: *0* < *x x* < *a* **for** *x*
using *that*
by (*intro DERIV-cong-ev*[*THEN iffD1*, *OF* - - - *f0*[*OF x*]])
(*auto simp*: *f-def eventually-nhds-metric dist-real-def intro*!: *exI*[*of* - *x*])

have *g*: *DERIV g x* :> (*g′ x*) **if** *x*: *0* < *x x* < *a* **for** *x*
using *that*
by (*intro DERIV-cong-ev*[*THEN iffD1*, *OF* - - - *g0*[*OF x*]])
(*auto simp*: *g-def eventually-nhds-metric dist-real-def intro*!: *exI*[*of* - *x*])

have *isCont f 0*
unfolding *f-def* **by** (*intro isCont-If-ge f-0 continuous-const*)

have *isCont g 0*
unfolding *g-def* **by** (*intro isCont-If-ge g-0 continuous-const*)

have ∃ζ. ∀*x*∈{*0* <..< *a*}. *0* < ζ *x* ∧ ζ *x* < *x* ∧ *f x* / *g x* = *f′* (ζ *x*) / *g′* (ζ *x*)
proof (*rule bchoice*, *rule ballI*)
fix *x*
assume *x* ∈ {*0* <..< *a*}
then have *x*[*arith*]: *0* < *x x* < *a* **by** *auto*
with *g′-neq-0 g-neq-0* ‹*g 0* = *0*› **have** *g′*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *0* ≠ *g′ x*
g 0 ≠ *g x*
by *auto*
have ⋀*x*. *0* ≤ *x* ⟹ *x* < *a* ⟹ *isCont f x*
using ‹*isCont f 0*› *f* **by** (*auto intro*: *DERIV-isCont simp*: *le-less*)
moreover have ⋀*x*. *0* ≤ *x* ⟹ *x* < *a* ⟹ *isCont g x*
using ‹*isCont g 0*› *g* **by** (*auto intro*: *DERIV-isCont simp*: *le-less*)
ultimately have ∃*c*. *0* < *c* ∧ *c* < *x* ∧ (*f x* − *f 0*) ∗ *g′ c* = (*g x* − *g 0*) ∗ *f′ c*
using *f g* ‹*x* < *a*› **by** (*intro GMVT′*) *auto*
then obtain *c* **where** ∗: *0* < *c c* < *x* (*f x* − *f 0*) ∗ *g′ c* = (*g x* − *g 0*) ∗ *f′ c*
by *blast*
moreover
from ∗ *g′*(*1*)[*of c*] *g′*(*2*) **have** (*f x* − *f 0*) / (*g x* − *g 0*) = *f′ c* / *g′ c*
by (*simp add*: *field-simps*)
ultimately show ∃*y*. *0* < *y* ∧ *y* < *x* ∧ *f x* / *g x* = *f′ y* / *g′ y*
using ‹*f 0* = *0*› ‹*g 0* = *0*› **by** (*auto intro*!: *exI*[*of* - *c*])
qed
then obtain ζ **where** ∀*x*∈{*0* <..< *a*}. *0* < ζ *x* ∧ ζ *x* < *x* ∧ *f x* / *g x* = *f′* (ζ
x) / *g′* (ζ *x*) **..**
then have ζ: *eventually* (λ*x*. *0* < ζ *x* ∧ ζ *x* < *x* ∧ *f x* / *g x* = *f′* (ζ *x*) / *g′* (ζ
x)) (*at-right 0*)
unfolding *eventually-at* **by** (*intro exI*[*of* - *a*]) (*auto simp*: *dist-real-def*)
moreover
from ζ **have** *eventually* (λ*x*. *norm* (ζ *x*) ≤ *x*) (*at-right 0*)
by *eventually-elim auto*
then have ((λ*x*. *norm* (ζ *x*)) ⟶ *0*) (*at-right 0*)
by (*rule-tac real-tendsto-sandwich*[**where** *f*=λ*x*. *0* **and** *h*=λ*x*. *x*]) *auto*
then have (ζ ⟶ *0*) (*at-right 0*)

by (*rule tendsto-norm-zero-cancel*)
with ζ **have** *filterlim* ζ (*at-right 0*) (*at-right 0*)
by (*auto elim*!: *eventually-mono simp*: *filterlim-at*)
from *this lim* **have** *filterlim* ($\lambda t.\ f'\ (\zeta\ t)\ /\ g'\ (\zeta\ t)$) *F* (*at-right 0*)
by (*rule-tac filterlim-compose*[*of* - - - ζ])
ultimately have *filterlim* ($\lambda t.\ f\ t\ /\ g\ t$) *F* (*at-right 0*) (**is** *?P*)
by (*rule-tac filterlim-cong*[*THEN iffD1*, *OF refl refl*])
(*auto elim*: *eventually-mono*)
also have *?P* $\longleftrightarrow$ *?thesis*
by (*rule filterlim-cong*) (*auto simp*: *f-def g-def eventually-at-filter*)
finally show *?thesis* **.**
qed

lemma *lhopital-right*:
($f \longrightarrow 0$) (*at-right x*) $\Longrightarrow$ ($g \longrightarrow 0$) (*at-right x*) $\Longrightarrow$
eventually ($\lambda x.\ g\ x \neq 0$) (*at-right x*) $\Longrightarrow$
eventually ($\lambda x.\ g'\ x \neq 0$) (*at-right x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV f x* :> $f'\ x$) (*at-right x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV g x* :> $g'\ x$) (*at-right x*) $\Longrightarrow$
filterlim ($\lambda\ x.\ (f'\ x\ /\ g'\ x)$) *F* (*at-right x*) $\Longrightarrow$
filterlim ($\lambda\ x.\ f\ x\ /\ g\ x$) *F* (*at-right x*)
for *x* :: *real*
unfolding *eventually-at-right-to-0*[*of* - *x*] *filterlim-at-right-to-0*[*of* - - *x*] *DERIV-shift*
by (*rule lhopital-right-0*)

lemma *lhopital-left*:
($f \longrightarrow 0$) (*at-left x*) $\Longrightarrow$ ($g \longrightarrow 0$) (*at-left x*) $\Longrightarrow$
eventually ($\lambda x.\ g\ x \neq 0$) (*at-left x*) $\Longrightarrow$
eventually ($\lambda x.\ g'\ x \neq 0$) (*at-left x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV f x* :> $f'\ x$) (*at-left x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV g x* :> $g'\ x$) (*at-left x*) $\Longrightarrow$
filterlim ($\lambda\ x.\ (f'\ x\ /\ g'\ x)$) *F* (*at-left x*) $\Longrightarrow$
filterlim ($\lambda\ x.\ f\ x\ /\ g\ x$) *F* (*at-left x*)
for *x* :: *real*
unfolding *eventually-at-left-to-right filterlim-at-left-to-right DERIV-mirror*
by (*rule lhopital-right*[**where** $f'=\lambda x.\ -\ f'\ (-\ x)$]) (*auto simp*: *DERIV-mirror*)

lemma *lhopital*:
($f \longrightarrow 0$) (*at x*) $\Longrightarrow$ ($g \longrightarrow 0$) (*at x*) $\Longrightarrow$
eventually ($\lambda x.\ g\ x \neq 0$) (*at x*) $\Longrightarrow$
eventually ($\lambda x.\ g'\ x \neq 0$) (*at x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV f x* :> $f'\ x$) (*at x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV g x* :> $g'\ x$) (*at x*) $\Longrightarrow$
filterlim ($\lambda\ x.\ (f'\ x\ /\ g'\ x)$) *F* (*at x*) $\Longrightarrow$
filterlim ($\lambda\ x.\ f\ x\ /\ g\ x$) *F* (*at x*)
for *x* :: *real*
unfolding *eventually-at-split filterlim-at-split*
by (*auto intro*!: *lhopital-right*[*of f x g g' f'*] *lhopital-left*[*of f x g g' f'*])

lemma *lhopital-right-0-at-top*:
fixes *f g* :: *real* ⇒ *real*
assumes *g-0*: *LIM x at-right 0. g x :> at-top*
and *ev*:
eventually (λ*x*. *g′ x* ≠ *0*) (*at-right 0*)
eventually (λ*x*. *DERIV f x* :> *f′ x*) (*at-right 0*)
eventually (λ*x*. *DERIV g x* :> *g′ x*) (*at-right 0*)
and *lim*: ((λ *x*. (*f′ x* / *g′ x*)) ⟶ *x*) (*at-right 0*)
shows ((λ *x*. *f x* / *g x*) ⟶ *x*) (*at-right 0*)
unfolding *tendsto-iff*
proof *safe*
fix *e* :: *real*
assume *0* < *e*
with *lim*[*unfolded tendsto-iff*, *rule-format*, *of e* / *4*]
have *eventually* (λ*t*. *dist* (*f′ t* / *g′ t*) *x* < *e* / *4*) (*at-right 0*)
by *simp*
from *eventually-conj*[*OF eventually-conj*[*OF ev*(*1*) *ev*(*2*)] *eventually-conj*[*OF ev*(*3*) *this*]]
obtain *a* **where** [*arith*]: *0* < *a*
and *g′-neq-0*: ⋀*x*. *0* < *x* ⟹ *x* < *a* ⟹ *g′ x* ≠ *0*
and *f0*: ⋀*x*. *0* < *x* ⟹ *x* ≤ *a* ⟹ *DERIV f x* :> (*f′ x*)
and *g0*: ⋀*x*. *0* < *x* ⟹ *x* ≤ *a* ⟹ *DERIV g x* :> (*g′ x*)
and *Df*: ⋀*t*. *0* < *t* ⟹ *t* < *a* ⟹ *dist* (*f′ t* / *g′ t*) *x* < *e* / *4*
unfolding *eventually-at-le* **by** (*auto simp*: *dist-real-def*)

from *Df* **have** *eventually* (λ*t*. *t* < *a*) (*at-right 0*) *eventually* (λ*t*::*real*. *0* < *t*) (*at-right 0*)
unfolding *eventually-at* **by** (*auto intro*!: *exI*[*of* - *a*] *simp*: *dist-real-def*)

moreover
have *eventually* (λ*t*. *0* < *g t*) (*at-right 0*) *eventually* (λ*t*. *g a* < *g t*) (*at-right 0*)
using *g-0* **by** (*auto elim*: *eventually-mono simp*: *filterlim-at-top-dense*)

moreover
have *inv-g*: ((λ*x*. *inverse* (*g x*)) ⟶ *0*) (*at-right 0*)
using *tendsto-inverse-0 filterlim-mono*[*OF g-0 at-top-le-at-infinity order-refl*]
by (*rule filterlim-compose*)
then have ((λ*x*. *norm* (*1* − *g a* ∗ *inverse* (*g x*))) ⟶ *norm* (*1* − *g a* ∗ *0*)) (*at-right 0*)
by (*intro tendsto-intros*)
then have ((λ*x*. *norm* (*1* − *g a* / *g x*)) ⟶ *1*) (*at-right 0*)
by (*simp add*: *inverse-eq-divide*)
from *this*[*unfolded tendsto-iff*, *rule-format*, *of 1*]
have *eventually* (λ*x*. *norm* (*1* − *g a* / *g x*) < *2*) (*at-right 0*)
by (*auto elim*!: *eventually-mono simp*: *dist-real-def*)

moreover

from *inv-g* **have** ((λ*t*. *norm* ((*f a* − *x* ∗ *g a*) ∗ *inverse* (*g t*))) ⟶ *norm* ((*f a* − *x* ∗ *g a*) ∗ *0*))
(*at-right 0*)
by (*intro tendsto-intros*)
then have ((λ*t*. *norm* (*f a* − *x* ∗ *g a*) / *norm* (*g t*)) ⟶ *0*) (*at-right 0*)
by (*simp add*: *inverse-eq-divide*)
from *this*[*unfolded tendsto-iff*, *rule-format*, *of e* / *2*] ‹*0* < *e*›
have *eventually* (λ*t*. *norm* (*f a* − *x* ∗ *g a*) / *norm* (*g t*) < *e* / *2*) (*at-right 0*)
by (*auto simp*: *dist-real-def*)

ultimately show *eventually* (λ*t*. *dist* (*f t* / *g t*) *x* < *e*) (*at-right 0*)
proof *eventually-elim*
fix *t* **assume** *t*[*arith*]: *0* < *t t* < *a g a* < *g t 0* < *g t*
assume *ineq*: *norm* (*1* − *g a* / *g t*) < *2 norm* (*f a* − *x* ∗ *g a*) / *norm* (*g t*) < *e* / *2*

have ∃ *y*. *t* < *y* ∧ *y* < *a* ∧ (*g a* − *g t*) ∗ *f′ y* = (*f a* − *f t*) ∗ *g′ y*
using *f0 g0 t*(*1*,*2*) **by** (*intro GMVT′*) (*force intro*!: *DERIV-isCont*)+
then obtain *y* **where** [*arith*]: *t* < *y y* < *a*
and *D-eq0*: (*g a* − *g t*) ∗ *f′ y* = (*f a* − *f t*) ∗ *g′ y*
by *blast*
from *D-eq0* **have** *D-eq*: (*f t* − *f a*) / (*g t* − *g a*) = *f′ y* / *g′ y*
using ‹*g a* < *g t*› *g′-neq-0*[*of y*] **by** (*auto simp add*: *field-simps*)

have ∗: *f t* / *g t* − *x* = ((*f t* − *f a*) / (*g t* − *g a*) − *x*) ∗ (*1* − *g a* / *g t*) + (*f a* − *x* ∗ *g a*) / *g t*
by (*simp add*: *field-simps*)
have *norm* (*f t* / *g t* − *x*) ≤
norm (((*f t* − *f a*) / (*g t* − *g a*) − *x*) ∗ (*1* − *g a* / *g t*)) + *norm* ((*f a* − *x* ∗ *g a*) / *g t*)
unfolding ∗ **by** (*rule norm-triangle-ineq*)
also have … = *dist* (*f′ y* / *g′ y*) *x* ∗ *norm* (*1* − *g a* / *g t*) + *norm* (*f a* − *x* ∗ *g a*) / *norm* (*g t*)
by (*simp add*: *abs-mult D-eq dist-real-def*)
also have … < (*e* / *4*) ∗ *2* + *e* / *2*
using *ineq Df*[*of y*] ‹*0* < *e*› **by** (*intro add-le-less-mono mult-mono*) *auto*
finally show *dist* (*f t* / *g t*) *x* < *e*
by (*simp add*: *dist-real-def*)
qed
qed

lemma *lhopital-right-at-top*:
LIM x at-right x. (*g*::*real* ⇒ *real*) *x* :> *at-top* ⟹
eventually (λ*x*. *g′ x* ≠ *0*) (*at-right x*) ⟹
eventually (λ*x*. *DERIV f x* :> *f′ x*) (*at-right x*) ⟹
eventually (λ*x*. *DERIV g x* :> *g′ x*) (*at-right x*) ⟹
((λ *x*. (*f′ x* / *g′ x*)) ⟶ *y*) (*at-right x*) ⟹
((λ *x*. *f x* / *g x*) ⟶ *y*) (*at-right x*)
unfolding *eventually-at-right-to-0*[*of - x*] *filterlim-at-right-to-0*[*of - - x*] *DE-*

RIV-shift
by (*rule lhopital-right-0-at-top*)

lemma *lhopital-left-at-top*:
LIM x at-left x. g x :> at-top $\Longrightarrow$
eventually ($\lambda x.$ g' $x \neq 0$) (*at-left x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV f x :>* f' x) (*at-left x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV g x :>* g' x) (*at-left x*) $\Longrightarrow$
((λ $x.$ (f' x / g' x)) $\longrightarrow$ y) (*at-left x*) $\Longrightarrow$
((λ $x.$ f x / g x) $\longrightarrow$ y) (*at-left x*)
for x :: *real*
unfolding *eventually-at-left-to-right filterlim-at-left-to-right DERIV-mirror*
by (*rule lhopital-right-at-top*[**where** f'=$\lambda x.$ $- f'$ ($- x$)]) (*auto simp*: *DERIV-mirror*)

lemma *lhopital-at-top*:
LIM x at x. (*g*::*real* $\Rightarrow$ *real*) *x :> at-top* $\Longrightarrow$
eventually ($\lambda x.$ g' $x \neq 0$) (*at x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV f x :>* f' x) (*at x*) $\Longrightarrow$
eventually ($\lambda x.$ *DERIV g x :>* g' x) (*at x*) $\Longrightarrow$
((λ $x.$ (f' x / g' x)) $\longrightarrow$ y) (*at x*) $\Longrightarrow$
((λ $x.$ f x / g x) $\longrightarrow$ y) (*at x*)
unfolding *eventually-at-split filterlim-at-split*
by (*auto intro*!: *lhopital-right-at-top*[*of g x* g' *f* f'] *lhopital-left-at-top*[*of g x* g' *f* f'])

lemma *lhospital-at-top-at-top*:
fixes *f g* :: *real* $\Rightarrow$ *real*
assumes *g-0*: *LIM x at-top. g x :> at-top*
and g': *eventually* ($\lambda x.$ g' $x \neq 0$) *at-top*
and *Df*: *eventually* ($\lambda x.$ *DERIV f x :>* f' x) *at-top*
and *Dg*: *eventually* ($\lambda x.$ *DERIV g x :>* g' x) *at-top*
and *lim*: ((λ $x.$ (f' x / g' x)) $\longrightarrow$ x) *at-top*
shows ((λ $x.$ f x / g x) $\longrightarrow$ x) *at-top*
unfolding *filterlim-at-top-to-right*
proof (*rule lhopital-right-0-at-top*)
let *?F* = $\lambda x.$ *f* (*inverse x*)
let *?G* = $\lambda x.$ *g* (*inverse x*)
let *?R* = *at-right* (*0*::*real*)
let *?D* = $\lambda f'$ $x.$ f' (*inverse x*) $*$ $-$ (*inverse x* ^ *Suc* (*Suc 0*))
show *LIM x ?R. ?G x :> at-top*
using *g-0* **unfolding** *filterlim-at-top-to-right* **.**
show *eventually* ($\lambda x.$ *DERIV ?G x :> ?D* g' x) *?R*
unfolding *eventually-at-right-to-top*
using *Dg eventually-ge-at-top*[**where** *c=1*]
by *eventually-elim* (*rule derivative-eq-intros DERIV-chain'*[**where** *f=inverse*] | *simp*)+
show *eventually* ($\lambda x.$ *DERIV ?F x :> ?D* f' x) *?R*
unfolding *eventually-at-right-to-top*
using *Df eventually-ge-at-top*[**where** *c=1*]

by *eventually-elim* (*rule derivative-eq-intros DERIV-chain'*[**where** *f=inverse*] | *simp*)+
show *eventually* (λ*x*. *?D g' x ≠ 0*) *?R*
unfolding *eventually-at-right-to-top*
using *g' eventually-ge-at-top*[**where** *c=1*]
by *eventually-elim auto*
show ((λ*x*. *?D f' x* / *?D g' x*) ⟶ *x*) *?R*
unfolding *filterlim-at-right-to-top*
apply (*intro filterlim-cong*[*THEN iffD2*, *OF refl refl - lim*])
using *eventually-ge-at-top*[**where** *c=1*]
by *eventually-elim simp*
qed

lemma *lhopital-right-at-top-at-top*:
fixes *f g* :: *real* ⇒ *real*
assumes *f-0*: *LIM x at-right a*. *f x* :> *at-top*
assumes *g-0*: *LIM x at-right a*. *g x* :> *at-top*
and *ev*:
eventually (λ*x*. *DERIV f x* :> *f' x*) (*at-right a*)
eventually (λ*x*. *DERIV g x* :> *g' x*) (*at-right a*)
and *lim*: *filterlim* (λ *x*. (*f' x* / *g' x*)) *at-top* (*at-right a*)
shows *filterlim* (λ *x*. *f x* / *g x*) *at-top* (*at-right a*)
proof −
from *lim* **have** *pos*: *eventually* (λ*x*. *f' x* / *g' x* > *0*) (*at-right a*)
unfolding *filterlim-at-top-dense* **by** *blast*
have ((λ*x*. *g x* / *f x*) ⟶ *0*) (*at-right a*)
proof (*rule lhopital-right-at-top*)
from *pos* **show** *eventually* (λ*x*. *f' x ≠ 0*) (*at-right a*) **by** *eventually-elim auto*
from *tendsto-inverse-0-at-top*[*OF lim*]
show ((λ*x*. *g' x* / *f' x*) ⟶ *0*) (*at-right a*) **by** *simp*
qed *fact+*
moreover from *f-0 g-0*
have *eventually* (λ*x*. *f x* > *0*) (*at-right a*) *eventually* (λ*x*. *g x* > *0*) (*at-right a*)
unfolding *filterlim-at-top-dense* **by** *blast+*
hence *eventually* (λ*x*. *g x* / *f x* > *0*) (*at-right a*) **by** *eventually-elim simp*
ultimately have *filterlim* (λ*x*. *inverse* (*g x* / *f x*)) *at-top* (*at-right a*)
by (*rule filterlim-inverse-at-top*)
thus *?thesis* **by** *simp*
qed

lemma *lhopital-right-at-top-at-bot*:
fixes *f g* :: *real* ⇒ *real*
assumes *f-0*: *LIM x at-right a*. *f x* :> *at-top*
assumes *g-0*: *LIM x at-right a*. *g x* :> *at-bot*
and *ev*:
eventually (λ*x*. *DERIV f x* :> *f' x*) (*at-right a*)
eventually (λ*x*. *DERIV g x* :> *g' x*) (*at-right a*)
and *lim*: *filterlim* (λ *x*. (*f' x* / *g' x*)) *at-bot* (*at-right a*)
shows *filterlim* (λ *x*. *f x* / *g x*) *at-bot* (*at-right a*)

proof –
  from *ev(2)* **have** *ev'*: *eventually* (λ*x*. *DERIV* (λ*x*. −*g x*) *x* :> −*g' x*) (*at-right a*)
    by *eventually-elim* (*auto intro*: *derivative-intros*)
  have *filterlim* (λ*x*. *f x* / (−*g x*)) *at-top* (*at-right a*)
    by (*rule lhopital-right-at-top-at-top*[**where** *f'* = *f'* **and** *g'* = λ*x*. −*g' x*])
      (*insert assms ev'*, *auto simp*: *filterlim-uminus-at-bot*)
  hence *filterlim* (λ*x*. −(*f x* / *g x*)) *at-top* (*at-right a*) **by** *simp*
  thus *?thesis* **by** (*simp add*: *filterlim-uminus-at-bot*)
qed

lemma *lhopital-left-at-top-at-top*:
  fixes *f g* :: *real* ⇒ *real*
  assumes *f-0*: *LIM x at-left a*. *f x* :> *at-top*
  assumes *g-0*: *LIM x at-left a*. *g x* :> *at-top*
    and *ev*:
      eventually (λ*x*. *DERIV f x* :> *f' x*) (*at-left a*)
      eventually (λ*x*. *DERIV g x* :> *g' x*) (*at-left a*)
    and *lim*: *filterlim* (λ *x*. (*f' x* / *g' x*)) *at-top* (*at-left a*)
  shows *filterlim* (λ *x*. *f x* / *g x*) *at-top* (*at-left a*)
  by (*insert assms*, *unfold eventually-at-left-to-right filterlim-at-left-to-right DERIV-mirror*,
      rule lhopital-right-at-top-at-top[**where** *f'*=λ*x*. − *f'* (− *x*)])
    (*insert assms*, *auto simp*: *DERIV-mirror*)

lemma *lhopital-left-at-top-at-bot*:
  fixes *f g* :: *real* ⇒ *real*
  assumes *f-0*: *LIM x at-left a*. *f x* :> *at-top*
  assumes *g-0*: *LIM x at-left a*. *g x* :> *at-bot*
    and *ev*:
      eventually (λ*x*. *DERIV f x* :> *f' x*) (*at-left a*)
      eventually (λ*x*. *DERIV g x* :> *g' x*) (*at-left a*)
    and *lim*: *filterlim* (λ *x*. (*f' x* / *g' x*)) *at-bot* (*at-left a*)
  shows *filterlim* (λ *x*. *f x* / *g x*) *at-bot* (*at-left a*)
  by (*insert assms*, *unfold eventually-at-left-to-right filterlim-at-left-to-right DERIV-mirror*,
      rule lhopital-right-at-top-at-bot[**where** *f'*=λ*x*. − *f'* (− *x*)])
    (*insert assms*, *auto simp*: *DERIV-mirror*)

lemma *lhopital-at-top-at-top*:
  fixes *f g* :: *real* ⇒ *real*
  assumes *f-0*: *LIM x at a*. *f x* :> *at-top*
  assumes *g-0*: *LIM x at a*. *g x* :> *at-top*
    and *ev*:
      eventually (λ*x*. *DERIV f x* :> *f' x*) (*at a*)
      eventually (λ*x*. *DERIV g x* :> *g' x*) (*at a*)
    and *lim*: *filterlim* (λ *x*. (*f' x* / *g' x*)) *at-top* (*at a*)
  shows *filterlim* (λ *x*. *f x* / *g x*) *at-top* (*at a*)
  using *assms* **unfolding** *eventually-at-split filterlim-at-split*

```
  by (auto intro!: lhopital-right-at-top-at-top[of f a g f' g']
               lhopital-left-at-top-at-top[of f a g f' g'])

lemma lhopital-at-top-at-bot:
  fixes f g :: real ⇒ real
  assumes f-0: LIM x at a. f x :> at-top
  assumes g-0: LIM x at a. g x :> at-bot
    and ev:
      eventually (λx. DERIV f x :> f' x) (at a)
      eventually (λx. DERIV g x :> g' x) (at a)
    and lim: filterlim (λ x. (f' x / g' x)) at-bot (at a)
  shows filterlim (λ x. f x / g x) at-bot (at a)
  using assms unfolding eventually-at-split filterlim-at-split
  by (auto intro!: lhopital-right-at-top-at-bot[of f a g f' g']
               lhopital-left-at-top-at-bot[of f a g f' g'])

end
```

111 Nth Roots of Real Numbers

```
theory NthRoot
  imports Deriv
begin
```

111.1 Existence of Nth Root

Existence follows from the Intermediate Value Theorem

```
lemma realpow-pos-nth:
  fixes a :: real
  assumes n: 0 < n
    and a: 0 < a
  shows ∃ r>0. r ^ n = a
proof −
  have ∃ r≥0. r ≤ (max 1 a) ∧ r ^ n = a
  proof (rule IVT)
    show 0 ^ n ≤ a
      using n a by (simp add: power-0-left)
    show 0 ≤ max 1 a
      by simp
    from n have n1: 1 ≤ n
      by simp
    have a ≤ max 1 a ^ 1
      by simp
    also have max 1 a ^ 1 ≤ max 1 a ^ n
      using n1 by (rule power-increasing) simp
    finally show a ≤ max 1 a ^ n .
    show ∀ r. 0 ≤ r ∧ r ≤ max 1 a ⟶ isCont (λx. x ^ n) r
      by simp
```

qed
then obtain *r* **where** *r*: $0 \leq r \wedge r \,\hat{}\, n = a$
by *fast*
with *n a* **have** $r \neq 0$
by (*auto simp add*: *power-0-left*)
with *r* **have** $0 < r \wedge r \,\hat{}\, n = a$
by *simp*
then show *?thesis* **..**
qed

lemma *realpow-pos-nth2*: $(0::real) < a \Longrightarrow \exists r>0.\ r \,\hat{}\, Suc\ n = a$
by (*blast intro*: *realpow-pos-nth*)

Uniqueness of nth positive root.

lemma *realpow-pos-nth-unique*: $0 < n \Longrightarrow 0 < a \Longrightarrow \exists!r.\ 0 < r \wedge r \,\hat{}\, n = a$ **for** *a* :: *real*
by (*auto intro*!: *realpow-pos-nth simp*: *power-eq-iff-eq-base*)

111.2 Nth Root

We define roots of negative reals such that *root n* $(-\ x) = -$ *root n x*. This allows us to omit side conditions from many theorems.

lemma *inj-sgn-power*:
assumes $0 < n$
shows *inj* $(\lambda y.\ sgn\ y * |y|\hat{}n :: real)$
(**is** *inj ?f*)
proof (*rule injI*)
have *x*: $(0 < a \wedge b < 0) \vee (a < 0 \wedge 0 < b) \Longrightarrow a \neq b$ **for** *a b* :: *real*
by *auto*
fix *x y*
assume *?f x* = *?f y*
with *power-eq-iff-eq-base*[*of n* |*x*| |*y*|] ‹$0 < n$› **show** $x = y$
by (*cases rule*: *linorder-cases*[*of 0 x*, *case-product linorder-cases*[*of 0 y*]])
(*simp-all add*: *x*)
qed

lemma *sgn-power-injE*:
$sgn\ a * |a| \,\hat{}\, n = x \Longrightarrow x = sgn\ b * |b| \,\hat{}\, n \Longrightarrow 0 < n \Longrightarrow a = b$
for *a b* :: *real*
using *inj-sgn-power*[*THEN injD*, *of n a b*] **by** *simp*

definition *root* :: *nat* ⇒ *real* ⇒ *real*
where *root n x* = (*if n* = *0 then 0 else the-inv* $(\lambda y.\ sgn\ y * |y|\hat{}n)$ *x*)

lemma *root-0* [*simp*]: *root 0 x* = *0*
by (*simp add*: *root-def*)

lemma *root-sgn-power*: $0 < n \Longrightarrow root\ n\ (sgn\ y * |y|\,\widehat{}\,n) = y$
using *the-inv-f-f*[*OF inj-sgn-power*] **by** (*simp add*: *root-def*)

lemma *sgn-power-root*:
assumes $0 < n$
shows $sgn\ (root\ n\ x) * |(root\ n\ x)|\,\widehat{}\,n = x$
(**is** $?f\ (root\ n\ x) = x$)
proof (*cases* $x = 0$)
case *True*
with *assms root-sgn-power*[*of n 0*] **show** *?thesis*
by *simp*
next
case *False*
with *realpow-pos-nth*[*OF* ‹$0 < n$›, *of* $|x|$]
obtain r **where** $0 < r\ r\ \widehat{}\ n = |x|$
by *auto*
with ‹$x \neq 0$› **have** S: $x \in range\ ?f$
by (*intro image-eqI*[*of - - sgn x * r*])
(*auto simp*: *abs-mult sgn-mult power-mult-distrib abs-sgn-eq mult-sgn-abs*)
from ‹$0 < n$› *f-the-inv-into-f*[*OF inj-sgn-power*[*OF* ‹$0 < n$›] *this*] **show** *?thesis*
by (*simp add*: *root-def*)
qed

lemma *split-root*: $P\ (root\ n\ x) \longleftrightarrow (n = 0 \longrightarrow P\ 0) \wedge (0 < n \longrightarrow (\forall\, y.\ sgn\ y * |y|\,\widehat{}\,n = x \longrightarrow P\ y))$
proof (*cases* $n = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then show *?thesis*
by *simp* (*metis root-sgn-power sgn-power-root*)
qed

lemma *real-root-zero* [*simp*]: $root\ n\ 0 = 0$
by (*simp split*: *split-root add*: *sgn-zero-iff*)

lemma *real-root-minus*: $root\ n\ (-\ x) = -\ root\ n\ x$
by (*clarsimp split*: *split-root elim!*: *sgn-power-injE simp*: *sgn-minus*)

lemma *real-root-less-mono*: $0 < n \Longrightarrow x < y \Longrightarrow root\ n\ x < root\ n\ y$
proof (*clarsimp split*: *split-root*)
have $*$: $0 < b \Longrightarrow a < 0 \Longrightarrow \neg\ a > b$ **for** $a\ b :: real$
by *auto*
fix $a\ b :: real$
assume $0 < n\ sgn\ a * |a|\ \widehat{}\ n < sgn\ b * |b|\ \widehat{}\ n$
then show $a < b$
using *power-less-imp-less-base*[*of a n b*]
power-less-imp-less-base[*of* $-\ b\ n\ -\ a$]

by (*simp add*: *sgn-real-def* $*$ [*of a* $\hat{}$ *n* $-$ (($-$ *b*) $\hat{}$ *n*)]
split: *if-split-asm*)
qed

lemma *real-root-gt-zero*: $0 < n \Longrightarrow 0 < x \Longrightarrow 0 < root\ n\ x$
using *real-root-less-mono*[*of n 0 x*] **by** *simp*

lemma *real-root-ge-zero*: $0 \leq x \Longrightarrow 0 \leq root\ n\ x$
using *real-root-gt-zero*[*of n x*]
by (*cases* $n = 0$) (*auto simp add*: *le-less*)

lemma *real-root-pow-pos*: $0 < n \Longrightarrow 0 < x \Longrightarrow root\ n\ x\ \hat{}\ n = x$
using *sgn-power-root*[*of n x*] *real-root-gt-zero*[*of n x*] **by** *simp*

lemma *real-root-pow-pos2* [*simp*]: $0 < n \Longrightarrow 0 \leq x \Longrightarrow root\ n\ x\ \hat{}\ n = x$
by (*auto simp add*: *order-le-less real-root-pow-pos*)

lemma *sgn-root*: $0 < n \Longrightarrow sgn\ (root\ n\ x) = sgn\ x$
by (*auto split*: *split-root simp*: *sgn-real-def*)

lemma *odd-real-root-pow*: $odd\ n \Longrightarrow root\ n\ x\ \hat{}\ n = x$
using *sgn-power-root*[*of n x*]
by (*simp add*: *odd-pos sgn-real-def split*: *if-split-asm*)

lemma *real-root-power-cancel*: $0 < n \Longrightarrow 0 \leq x \Longrightarrow root\ n\ (x\ \hat{}\ n) = x$
using *root-sgn-power*[*of n x*] **by** (*auto simp add*: *le-less power-0-left*)

lemma *odd-real-root-power-cancel*: $odd\ n \Longrightarrow root\ n\ (x\ \hat{}\ n) = x$
using *root-sgn-power*[*of n x*]
by (*simp add*: *odd-pos sgn-real-def power-0-left split*: *if-split-asm*)

lemma *real-root-pos-unique*: $0 < n \Longrightarrow 0 \leq y \Longrightarrow y\ \hat{}\ n = x \Longrightarrow root\ n\ x = y$
using *real-root-power-cancel* **by** *blast*

lemma *odd-real-root-unique*: $odd\ n \Longrightarrow y\ \hat{}\ n = x \Longrightarrow root\ n\ x = y$
using *odd-real-root-power-cancel* **by** *blast*

lemma *real-root-one* [*simp*]: $0 < n \Longrightarrow root\ n\ 1 = 1$
by (*simp add*: *real-root-pos-unique*)

Root function is strictly monotonic, hence injective.

lemma *real-root-le-mono*: $0 < n \Longrightarrow x \leq y \Longrightarrow root\ n\ x \leq root\ n\ y$
by (*auto simp add*: *order-le-less real-root-less-mono*)

lemma *real-root-less-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x < root\ n\ y \longleftrightarrow x < y$
by (*cases* $x < y$) (*simp-all add*: *real-root-less-mono linorder-not-less real-root-le-mono*)

lemma *real-root-le-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x \leq root\ n\ y \longleftrightarrow x \leq y$
by (*cases* $x \leq y$) (*simp-all add*: *real-root-le-mono linorder-not-le real-root-less-mono*)

lemma *real-root-eq-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x = root\ n\ y \longleftrightarrow x = y$
by (*simp add*: *order-eq-iff*)

lemmas *real-root-gt-0-iff* [*simp*] = *real-root-less-iff* [**where** $x=0$, *simplified*]
lemmas *real-root-lt-0-iff* [*simp*] = *real-root-less-iff* [**where** $y=0$, *simplified*]
lemmas *real-root-ge-0-iff* [*simp*] = *real-root-le-iff* [**where** $x=0$, *simplified*]
lemmas *real-root-le-0-iff* [*simp*] = *real-root-le-iff* [**where** $y=0$, *simplified*]
lemmas *real-root-eq-0-iff* [*simp*] = *real-root-eq-iff* [**where** $y=0$, *simplified*]

lemma *real-root-gt-1-iff* [*simp*]: $0 < n \Longrightarrow 1 < root\ n\ y \longleftrightarrow 1 < y$
using *real-root-less-iff* [**where** $x=1$] **by** *simp*

lemma *real-root-lt-1-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x < 1 \longleftrightarrow x < 1$
using *real-root-less-iff* [**where** $y=1$] **by** *simp*

lemma *real-root-ge-1-iff* [*simp*]: $0 < n \Longrightarrow 1 \leq root\ n\ y \longleftrightarrow 1 \leq y$
using *real-root-le-iff* [**where** $x=1$] **by** *simp*

lemma *real-root-le-1-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x \leq 1 \longleftrightarrow x \leq 1$
using *real-root-le-iff* [**where** $y=1$] **by** *simp*

lemma *real-root-eq-1-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x = 1 \longleftrightarrow x = 1$
using *real-root-eq-iff* [**where** $y=1$] **by** *simp*

Roots of multiplication and division.

lemma *real-root-mult*: $root\ n\ (x * y) = root\ n\ x * root\ n\ y$
by (*auto split*: *split-root elim*!: *sgn-power-injE*
simp: *sgn-mult abs-mult power-mult-distrib*)

lemma *real-root-inverse*: $root\ n\ (inverse\ x) = inverse\ (root\ n\ x)$
by (*auto split*: *split-root elim*!: *sgn-power-injE*
simp: *power-inverse*)

lemma *real-root-divide*: $root\ n\ (x\ /\ y) = root\ n\ x\ /\ root\ n\ y$
by (*simp add*: *divide-inverse real-root-mult real-root-inverse*)

lemma *real-root-abs*: $0 < n \Longrightarrow root\ n\ |x| = |root\ n\ x|$
by (*simp add*: *abs-if real-root-minus*)

lemma *root-abs-power*: $n > 0 \Longrightarrow abs\ (root\ n\ (y\ \hat{}\ n)) = abs\ y$
using *root-sgn-power* [*of n*]
by (*metis abs-ge-zero power-abs real-root-abs real-root-power-cancel*)

lemma *real-root-power*: $0 < n \Longrightarrow root\ n\ (x\ \hat{}\ k) = root\ n\ x\ \hat{}\ k$
by (*induct k*) (*simp-all add*: *real-root-mult*)

Roots of roots.

lemma *real-root-Suc-0* [*simp*]: $root\ (Suc\ 0)\ x = x$

by (*simp add*: *odd-real-root-unique*)

lemma *real-root-mult-exp*: *root* (*m* ∗ *n*) *x* = *root m* (*root n x*)
by (*auto split*: *split-root elim*!: *sgn-power-injE*
simp: *sgn-zero-iff sgn-mult power-mult*[*symmetric*]
abs-mult power-mult-distrib abs-sgn-eq)

lemma *real-root-commute*: *root m* (*root n x*) = *root n* (*root m x*)
by (*simp add*: *real-root-mult-exp* [*symmetric*] *mult.commute*)

Monotonicity in first argument.

lemma *real-root-strict-decreasing*:
assumes *0* < *n n* < *N 1* < *x*
shows *root N x* < *root n x*
proof −
from *assms* **have** *root n* (*root N x*) ^ *n* < *root N* (*root n x*) ^ *N*
by (*simp add*: *real-root-commute power-strict-increasing del*: *real-root-pow-pos2*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *real-root-strict-increasing*:
assumes *0* < *n n* < *N 0* < *x x* < *1*
shows *root n x* < *root N x*
proof −
from *assms* **have** *root N* (*root n x*) ^ *N* < *root n* (*root N x*) ^ *n*
by (*simp add*: *real-root-commute power-strict-decreasing del*: *real-root-pow-pos2*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *real-root-decreasing*: *0* < *n* ⟹ *n* ≤ *N* ⟹ *1* ≤ *x* ⟹ *root N x* ≤ *root n x*
by (*auto simp add*: *order-le-less real-root-strict-decreasing*)

lemma *real-root-increasing*: *0* < *n* ⟹ *n* ≤ *N* ⟹ *0* ≤ *x* ⟹ *x* ≤ *1* ⟹ *root n x* ≤ *root N x*
by (*auto simp add*: *order-le-less real-root-strict-increasing*)

Continuity and derivatives.

lemma *isCont-real-root*: *isCont* (*root n*) *x*
proof (*cases n* > *0*)
case *True*
let *?f* = *λy*::*real*. *sgn y* ∗ |*y*|^*n*
have *continuous-on* ({*0*..} ∪ {.. *0*}) (*λx*. *if 0* < *x then x* ^ *n else* − ((−*x*) ^ *n*) :: *real*)
using *True* **by** (*intro continuous-on-If continuous-intros*) *auto*
then have *continuous-on UNIV ?f*
by (*rule continuous-on-cong*[*THEN iffD1*, *rotated 2*]) (*auto simp*: *not-less le-less True*)
then have [*simp*]: *isCont ?f x* **for** *x*
by (*simp add*: *continuous-on-eq-continuous-at*)

have *isCont* (*root n*) (*?f* (*root n x*))
by (*rule isCont-inverse-function* [**where** *f*=*?f* **and** *d*=*1*]) (*auto simp*: *root-sgn-power True*)
then show *?thesis*
by (*simp add*: *sgn-power-root True*)
next
case *False*
then show *?thesis*
by (*simp add*: *root-def*[*abs-def*])
qed

lemma *tendsto-real-root* [*tendsto-intros*]:
$(f \longrightarrow x)\ F \Longrightarrow ((\lambda x.\ root\ n\ (f\ x)) \longrightarrow root\ n\ x)\ F$
using *isCont-tendsto-compose*[*OF isCont-real-root*, *of f x F*] **.**

lemma *continuous-real-root* [*continuous-intros*]:
$continuous\ F\ f \Longrightarrow continuous\ F\ (\lambda x.\ root\ n\ (f\ x))$
unfolding *continuous-def* **by** (*rule tendsto-real-root*)

lemma *continuous-on-real-root* [*continuous-intros*]:
$continuous\text{-}on\ s\ f \Longrightarrow continuous\text{-}on\ s\ (\lambda x.\ root\ n\ (f\ x))$
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-real-root*)

lemma *DERIV-real-root*:
assumes *n*: $0 < n$
and *x*: $0 < x$
shows $DERIV\ (root\ n)\ x :> inverse\ (real\ n * root\ n\ x\ \hat{}\ (n - Suc\ 0))$
proof (*rule DERIV-inverse-function*)
show $0 < x$
using *x* **.**
show $x < x + 1$
by *simp*
show $DERIV\ (\lambda x.\ x\ \hat{}\ n)\ (root\ n\ x) :> real\ n * root\ n\ x\ \hat{}\ (n - Suc\ 0)$
by (*rule DERIV-pow*)
show $real\ n * root\ n\ x\ \hat{}\ (n - Suc\ 0) \neq 0$
using *n x* **by** *simp*
show *isCont* (*root n*) *x*
by (*rule isCont-real-root*)
qed (*use n* **in** *auto*)

lemma *DERIV-odd-real-root*:
assumes *n*: *odd n*
and *x*: $x \neq 0$
shows $DERIV\ (root\ n)\ x :> inverse\ (real\ n * root\ n\ x\ \hat{}\ (n - Suc\ 0))$
proof (*rule DERIV-inverse-function*)
show $x - 1 < x\ x < x + 1$
by *auto*
show $DERIV\ (\lambda x.\ x\ \hat{}\ n)\ (root\ n\ x) :> real\ n * root\ n\ x\ \hat{}\ (n - Suc\ 0)$
by (*rule DERIV-pow*)

show *real n ∗ root n x ^ (n − Suc 0) ≠ 0*
using *odd-pos* [*OF n*] *x* **by** *simp*
show *isCont* (*root n*) *x*
by (*rule isCont-real-root*)
qed (*use n odd-real-root-pow* **in** *auto*)

lemma *DERIV-even-real-root*:
assumes *n*: *0 < n*
and *even n*
and *x*: *x < 0*
shows *DERIV* (*root n*) *x* :> *inverse* (− *real n ∗ root n x ^ (n − Suc 0)*)
proof (*rule DERIV-inverse-function*)
show *x − 1 < x*
by *simp*
show *x < 0*
using *x* **.**
show − (*root n y ^ n*) = *y* **if** *x − 1 < y* **and** *y < 0* **for** *y*
proof −
have *root n* (−*y*) ^ *n* = −*y*
using *that* ‹*0 < n*› **by** *simp*
with *real-root-minus* **and** ‹*even n*›
show − (*root n y ^ n*) = *y* **by** *simp*
qed
show *DERIV* (λ*x*. − (*x ^ n*)) (*root n x*) :> − *real n ∗ root n x ^ (n − Suc 0)*
by (*auto intro*!: *derivative-eq-intros*)
show − *real n ∗ root n x ^ (n − Suc 0) ≠ 0*
using *n x* **by** *simp*
show *isCont* (*root n*) *x*
by (*rule isCont-real-root*)
qed

lemma *DERIV-real-root-generic*:
assumes *0 < n*
and *x ≠ 0*
and *even n* ⟹ *0 < x* ⟹ *D* = *inverse* (*real n ∗ root n x ^ (n − Suc 0)*)
and *even n* ⟹ *x < 0* ⟹ *D* = − *inverse* (*real n ∗ root n x ^ (n − Suc 0)*)
and *odd n* ⟹ *D* = *inverse* (*real n ∗ root n x ^ (n − Suc 0)*)
shows *DERIV* (*root n*) *x* :> *D*
using *assms*
by (*cases even n*, *cases 0 < x*)
(*auto intro*: *DERIV-real-root*[*THEN DERIV-cong*]
DERIV-odd-real-root[*THEN DERIV-cong*]
DERIV-even-real-root[*THEN DERIV-cong*])

lemma *power-tendsto-0-iff* [*simp*]:
fixes *f* :: *'a* ⇒ *real*
assumes *n > 0*
shows ((λ*x*. *f x ^ n*) ⟶ *0*) *F* ⟷ (*f* ⟶ *0*) *F*
proof −

have $((\lambda x.\ |root\ n\ (f\ x\ \widehat{}\ n)|) \longrightarrow 0)\ F \Longrightarrow (f \longrightarrow 0)\ F$
  by (*auto simp*: *assms root-abs-power tendsto-rabs-zero-iff*)
then have $((\lambda x.\ f\ x\ \widehat{}\ n) \longrightarrow 0)\ F \Longrightarrow (f \longrightarrow 0)\ F$
  by (*metis tendsto-real-root abs-0 real-root-zero tendsto-rabs*)
with *assms* **show** *?thesis*
  by (*auto simp*: *tendsto-null-power*)
qed

111.3 Square Root

definition *sqrt* :: *real* ⇒ *real*
  where *sqrt* = *root 2*

lemma *pos2*: $0 < (2::nat)$
  by *simp*

lemma *real-sqrt-unique*: $y^2 = x \Longrightarrow 0 \le y \Longrightarrow sqrt\ x = y$
  unfolding *sqrt-def* **by** (*rule real-root-pos-unique* [*OF pos2*])

lemma *real-sqrt-abs* [*simp*]: $sqrt\ (x^2) = |x|$
  by (*metis power2-abs abs-ge-zero real-sqrt-unique*)

lemma *real-sqrt-pow2* [*simp*]: $0 \le x \Longrightarrow (sqrt\ x)^2 = x$
  unfolding *sqrt-def* **by** (*rule real-root-pow-pos2* [*OF pos2*])

lemma *real-sqrt-pow2-iff* [*simp*]: $(sqrt\ x)^2 = x \longleftrightarrow 0 \le x$
  by (*metis real-sqrt-pow2 zero-le-power2*)

lemma *real-sqrt-zero* [*simp*]: $sqrt\ 0 = 0$
  unfolding *sqrt-def* **by** (*rule real-root-zero*)

lemma *real-sqrt-one* [*simp*]: $sqrt\ 1 = 1$
  unfolding *sqrt-def* **by** (*rule real-root-one* [*OF pos2*])

lemma *real-sqrt-four* [*simp*]: $sqrt\ 4 = 2$
  using *real-sqrt-abs*[*of 2*] **by** *simp*

lemma *real-sqrt-minus*: $sqrt\ (-\ x) = -\ sqrt\ x$
  unfolding *sqrt-def* **by** (*rule real-root-minus*)

lemma *real-sqrt-mult*: $sqrt\ (x * y) = sqrt\ x * sqrt\ y$
  unfolding *sqrt-def* **by** (*rule real-root-mult*)

lemma *real-sqrt-mult-self*[*simp*]: $sqrt\ a * sqrt\ a = |a|$
  using *real-sqrt-abs*[*of a*] **unfolding** *power2-eq-square real-sqrt-mult* **.**

lemma *real-sqrt-inverse*: $sqrt\ (inverse\ x) = inverse\ (sqrt\ x)$
  unfolding *sqrt-def* **by** (*rule real-root-inverse*)

lemma *real-sqrt-divide*: *sqrt* (*x* / *y*) = *sqrt x* / *sqrt y*
unfolding *sqrt-def* **by** (*rule real-root-divide*)

lemma *real-sqrt-power*: *sqrt* (*x* ^ *k*) = *sqrt x* ^ *k*
unfolding *sqrt-def* **by** (*rule real-root-power* [*OF pos2*])

lemma *real-sqrt-gt-zero*: *0* < *x* ⟹ *0* < *sqrt x*
unfolding *sqrt-def* **by** (*rule real-root-gt-zero* [*OF pos2*])

lemma *real-sqrt-ge-zero*: *0* ≤ *x* ⟹ *0* ≤ *sqrt x*
unfolding *sqrt-def* **by** (*rule real-root-ge-zero*)

lemma *real-sqrt-less-mono*: *x* < *y* ⟹ *sqrt x* < *sqrt y*
unfolding *sqrt-def* **by** (*rule real-root-less-mono* [*OF pos2*])

lemma *real-sqrt-le-mono*: *x* ≤ *y* ⟹ *sqrt x* ≤ *sqrt y*
unfolding *sqrt-def* **by** (*rule real-root-le-mono* [*OF pos2*])

lemma *real-sqrt-less-iff* [*simp*]: *sqrt x* < *sqrt y* ⟷ *x* < *y*
unfolding *sqrt-def* **by** (*rule real-root-less-iff* [*OF pos2*])

lemma *real-sqrt-le-iff* [*simp*]: *sqrt x* ≤ *sqrt y* ⟷ *x* ≤ *y*
unfolding *sqrt-def* **by** (*rule real-root-le-iff* [*OF pos2*])

lemma *real-sqrt-eq-iff* [*simp*]: *sqrt x* = *sqrt y* ⟷ *x* = *y*
unfolding *sqrt-def* **by** (*rule real-root-eq-iff* [*OF pos2*])

lemma *real-less-lsqrt*: *0* ≤ *y* ⟹ *x* < y^2 ⟹ *sqrt x* < *y*
using *real-sqrt-less-iff*[*of x* y^2] **by** *simp*

lemma *real-le-lsqrt*: *0* ≤ *y* ⟹ *x* ≤ y^2 ⟹ *sqrt x* ≤ *y*
using *real-sqrt-le-iff*[*of x* y^2] **by** *simp*

lemma *real-le-rsqrt*: x^2 ≤ *y* ⟹ *x* ≤ *sqrt y*
using *real-sqrt-le-mono*[*of* x^2 *y*] **by** *simp*

lemma *real-less-rsqrt*: x^2 < *y* ⟹ *x* < *sqrt y*
using *real-sqrt-less-mono*[*of* x^2 *y*] **by** *simp*

lemma *real-sqrt-power-even*:
assumes *even n x* ≥ *0*
shows *sqrt x* ^ *n* = *x* ^ (*n div 2*)
proof −
from *assms* **obtain** *k* **where** *n* = *2*∗*k* **by** (*auto elim*!: *evenE*)
with *assms* **show** *?thesis* **by** (*simp add*: *power-mult*)
qed

lemma *sqrt-le-D*: *sqrt x* ≤ *y* ⟹ *x* ≤ y^2
by (*meson not-le real-less-rsqrt*)

lemma *sqrt-ge-absD*: $|x| \leq sqrt\ y \Longrightarrow x^2 \leq y$
  using *real-sqrt-le-iff*[*of* x^2] **by** *simp*

lemma *sqrt-even-pow2*:
  assumes *n*: *even n*
  shows $sqrt\ (2\ \hat{}\ n) = 2\ \hat{}\ (n\ div\ 2)$
proof −
  from *n* **obtain** *m* **where** *m*: $n = 2 * m$ **..**
  from *m* **have** $sqrt\ (2\ \hat{}\ n) = sqrt\ ((2\ \hat{}\ m)^2)$
    by (*simp only*: *power-mult*[*symmetric*] *mult.commute*)
  then show *?thesis*
    using *m* **by** *simp*
qed

lemmas *real-sqrt-gt-0-iff* [*simp*] = *real-sqrt-less-iff* [**where** *x=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-lt-0-iff* [*simp*] = *real-sqrt-less-iff* [**where** *y=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-ge-0-iff* [*simp*] = *real-sqrt-le-iff* [**where** *x=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-le-0-iff* [*simp*] = *real-sqrt-le-iff* [**where** *y=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-eq-0-iff* [*simp*] = *real-sqrt-eq-iff* [**where** *y=0*, *unfolded real-sqrt-zero*]

lemmas *real-sqrt-gt-1-iff* [*simp*] = *real-sqrt-less-iff* [**where** *x=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-lt-1-iff* [*simp*] = *real-sqrt-less-iff* [**where** *y=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-ge-1-iff* [*simp*] = *real-sqrt-le-iff* [**where** *x=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-le-1-iff* [*simp*] = *real-sqrt-le-iff* [**where** *y=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-eq-1-iff* [*simp*] = *real-sqrt-eq-iff* [**where** *y=1*, *unfolded real-sqrt-one*]

lemma *sqrt-add-le-add-sqrt*:
  assumes $0 \leq x$ $0 \leq y$
  shows $sqrt\ (x + y) \leq sqrt\ x + sqrt\ y$
  by (*rule power2-le-imp-le*) (*simp-all add*: *power2-sum assms*)

lemma *isCont-real-sqrt*: *isCont sqrt x*
  unfolding *sqrt-def* **by** (*rule isCont-real-root*)

lemma *tendsto-real-sqrt* [*tendsto-intros*]:
  $(f \longrightarrow x)\ F \Longrightarrow ((\lambda x.\ sqrt\ (f\ x)) \longrightarrow sqrt\ x)\ F$
  unfolding *sqrt-def* **by** (*rule tendsto-real-root*)

lemma *continuous-real-sqrt* [*continuous-intros*]:
  $continuous\ F\ f \Longrightarrow continuous\ F\ (\lambda x.\ sqrt\ (f\ x))$
  unfolding *sqrt-def* **by** (*rule continuous-real-root*)

lemma *continuous-on-real-sqrt* [*continuous-intros*]:
  $continuous\text{-}on\ s\ f \Longrightarrow continuous\text{-}on\ s\ (\lambda x.\ sqrt\ (f\ x))$
  unfolding *sqrt-def* **by** (*rule continuous-on-real-root*)

lemma *DERIV-real-sqrt-generic*:
  assumes $x \neq 0$

and $x > 0 \Longrightarrow D = inverse\ (sqrt\ x)\ /\ 2$
and $x < 0 \Longrightarrow D = -\ inverse\ (sqrt\ x)\ /\ 2$
shows *DERIV sqrt x :> D*
using *assms* **unfolding** *sqrt-def*
by (*auto intro*!: *DERIV-real-root-generic*)

lemma *DERIV-real-sqrt*: $0 < x \Longrightarrow$ *DERIV sqrt x :> inverse* (*sqrt x*) / *2*
using *DERIV-real-sqrt-generic* **by** *simp*

declare
DERIV-real-sqrt-generic[*THEN DERIV-chain2*, *derivative-intros*]
DERIV-real-root-generic[*THEN DERIV-chain2*, *derivative-intros*]

lemmas *has-derivative-real-sqrt*[*derivative-intros*] = *DERIV-real-sqrt*[*THEN DE-RIV-compose-FDERIV*]

lemma *not-real-square-gt-zero* [*simp*]: $\neg\ 0 < x * x \longleftrightarrow x = 0$
for *x* :: *real*
apply *auto*
using *linorder-less-linear* [**where** $x = x$ **and** $y = 0$]
apply (*simp add*: *zero-less-mult-iff*)
done

lemma *real-sqrt-abs2* [*simp*]: $sqrt\ (x * x) = |x|$
apply (*subst power2-eq-square* [*symmetric*])
apply (*rule real-sqrt-abs*)
done

lemma *real-inv-sqrt-pow2*: $0 < x \Longrightarrow (inverse\ (sqrt\ x))^2 = inverse\ x$
by (*simp add*: *power-inverse*)

lemma *real-sqrt-eq-zero-cancel*: $0 \le x \Longrightarrow sqrt\ x = 0 \Longrightarrow x = 0$
by *simp*

lemma *real-sqrt-ge-one*: $1 \le x \Longrightarrow 1 \le sqrt\ x$
by *simp*

lemma *sqrt-divide-self-eq*:
assumes *nneg*: $0 \le x$
shows $sqrt\ x\ /\ x = inverse\ (sqrt\ x)$
proof (*cases* $x = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have *pos*: $0 < x$
using *nneg* **by** *arith*
show *?thesis*
proof (*rule right-inverse-eq* [*THEN iffD1*, *symmetric*])

show *sqrt x* / *x* ≠ *0*
by (*simp add*: *divide-inverse nneg False*)
show *inverse* (*sqrt x*) / (*sqrt x* / *x*) = *1*
by (*simp add*: *divide-inverse mult.assoc* [*symmetric*]
power2-eq-square [*symmetric*] *real-inv-sqrt-pow2 pos False*)
qed
qed

lemma *real-div-sqrt*: $0 \le x \Longrightarrow x$ / *sqrt x* = *sqrt x*
by (*cases x* = *0*) (*simp-all add*: *sqrt-divide-self-eq* [*of x*] *field-simps*)

lemma *real-divide-square-eq* [*simp*]: ($r * a$) / ($r * r$) = a / r
for *a r* :: *real*
by (*cases r* = *0*) (*simp-all add*: *divide-inverse ac-simps*)

lemma *lemma-real-divide-sqrt-less*: $0 < u \Longrightarrow u$ / *sqrt 2* < *u*
by (*simp add*: *divide-less-eq*)

lemma *four-x-squared*: $4 * x^2 = (2 * x)^2$
for *x* :: *real*
by (*simp add*: *power2-eq-square*)

lemma *sqrt-at-top*: *LIM x at-top*. *sqrt x* :: *real* :> *at-top*
by (*rule filterlim-at-top-at-top*[**where** *Q*=λ*x*. *True* **and** *P*=λ*x*. *0* < *x* **and** *g*=*power2*])
(*auto intro*: *eventually-gt-at-top*)

111.4 Square Root of Sum of Squares

lemma *sum-squares-bound*: $2 * x * y \le x^2 + y^2$
for *x y* :: *'a*::*linordered-field*
proof −
have $(x - y)^2 = x * x - 2 * x * y + y * y$
by *algebra*
then have $0 \le x^2 - 2 * x * y + y^2$
by (*metis sum-power2-ge-zero zero-le-double-add-iff-zero-le-single-add power2-eq-square*)
then show *?thesis*
by *arith*
qed

lemma *arith-geo-mean*:
fixes *u* :: *'a*::*linordered-field*
assumes $u^2 = x * y$ $x \ge 0$ $y \ge 0$
shows $u \le (x + y)/2$
apply (*rule power2-le-imp-le*)
using *sum-squares-bound assms*
apply (*auto simp*: *zero-le-mult-iff*)
apply (*auto simp*: *algebra-simps power2-eq-square*)
done

lemma *arith-geo-mean-sqrt*:
fixes x :: *real*
assumes $x \geq 0$ $y \geq 0$
shows *sqrt* $(x * y) \leq (x + y)/2$
apply (*rule arith-geo-mean*)
using *assms*
apply (*auto simp*: *zero-le-mult-iff*)
done

lemma *real-sqrt-sum-squares-mult-ge-zero* [*simp*]: $0 \leq$ *sqrt* $((x^2 + y^2) * (xa^2 + ya^2))$
by (*metis real-sqrt-ge-0-iff split-mult-pos-le sum-power2-ge-zero*)

lemma *real-sqrt-sum-squares-mult-squared-eq* [*simp*]:
(*sqrt* $((x^2 + y^2) * (xa^2 + ya^2)))^2 = (x^2 + y^2) * (xa^2 + ya^2)$
by (*simp add*: *zero-le-mult-iff*)

lemma *real-sqrt-sum-squares-eq-cancel*: *sqrt* $(x^2 + y^2) = x \Longrightarrow y = 0$
by (*drule arg-cong* [**where** $f = \lambda x.\ x^2$]) *simp*

lemma *real-sqrt-sum-squares-eq-cancel2*: *sqrt* $(x^2 + y^2) = y \Longrightarrow x = 0$
by (*drule arg-cong* [**where** $f = \lambda x.\ x^2$]) *simp*

lemma *real-sqrt-sum-squares-ge1* [*simp*]: $x \leq$ *sqrt* $(x^2 + y^2)$
by (*rule power2-le-imp-le*) *simp-all*

lemma *real-sqrt-sum-squares-ge2* [*simp*]: $y \leq$ *sqrt* $(x^2 + y^2)$
by (*rule power2-le-imp-le*) *simp-all*

lemma *real-sqrt-ge-abs1* [*simp*]: $|x| \leq$ *sqrt* $(x^2 + y^2)$
by (*rule power2-le-imp-le*) *simp-all*

lemma *real-sqrt-ge-abs2* [*simp*]: $|y| \leq$ *sqrt* $(x^2 + y^2)$
by (*rule power2-le-imp-le*) *simp-all*

lemma *le-real-sqrt-sumsq* [*simp*]: $x \leq$ *sqrt* $(x * x + y * y)$
by (*simp add*: *power2-eq-square* [*symmetric*])

lemma *sqrt-sum-squares-le-sum*:
$[\![0 \leq x;\ 0 \leq y]\!] \Longrightarrow$ *sqrt* $(x^2 + y^2) \leq x + y$
by (*rule power2-le-imp-le*) (*simp-all add*: *power2-sum*)

lemma *L2-set-mult-ineq-lemma*:
fixes $a\ b\ c\ d$ :: *real*
shows $2 * (a * c) * (b * d) \leq a^2 * d^2 + b^2 * c^2$
proof −
have $0 \leq (a * d - b * c)^2$ **by** *simp*
also have $\ldots = a^2 * d^2 + b^2 * c^2 - 2 * (a * d) * (b * c)$

by (*simp only*: *power2-diff power-mult-distrib*)
also have ... = $a^2 * d^2 + b^2 * c^2 - 2 * (a * c) * (b * d)$
by *simp*
finally show $2 * (a * c) * (b * d) \leq a^2 * d^2 + b^2 * c^2$
by *simp*
qed

lemma *sqrt-sum-squares-le-sum-abs*: $sqrt\ (x^2 + y^2) \leq |x| + |y|$
by (*rule power2-le-imp-le*) (*simp-all add*: *power2-sum*)

lemma *real-sqrt-sum-squares-triangle-ineq*:
$sqrt\ ((a + c)^2 + (b + d)^2) \leq sqrt\ (a^2 + b^2) + sqrt\ (c^2 + d^2)$
proof −
have $(a * c + b * d) \leq (sqrt\ (a^2 + b^2) * sqrt\ (c^2 + d^2))$
by (*rule power2-le-imp-le*) (*simp-all add*: *power2-sum power-mult-distrib ring-distribs L2-set-mult-ineq-lemma add.commute*)
then have $(a + c)^2 + (b + d)^2 \leq (sqrt\ (a^2 + b^2) + sqrt\ (c^2 + d^2))^2$
by (*simp add*: *power2-sum*)
then show *?thesis*
by (*auto intro*: *power2-le-imp-le*)
qed

lemma *real-sqrt-sum-squares-less*: $|x| < u\ /\ sqrt\ 2 \Longrightarrow |y| < u\ /\ sqrt\ 2 \Longrightarrow sqrt\ (x^2 + y^2) < u$
apply (*rule power2-less-imp-less*)
apply *simp*
apply (*drule power-strict-mono* [*OF - abs-ge-zero pos2*])
apply (*drule power-strict-mono* [*OF - abs-ge-zero pos2*])
apply (*simp add*: *power-divide*)
apply (*drule order-le-less-trans* [*OF abs-ge-zero*])
apply (*simp add*: *zero-less-divide-iff*)
done

lemma *sqrt2-less-2*: $sqrt\ 2 < (2::real)$
by (*metis not-less not-less-iff-gr-or-eq numeral-less-iff real-sqrt-four real-sqrt-le-iff semiring-norm*(*75*) *semiring-norm*(*78*) *semiring-norm*(*85*))

lemma *sqrt-sum-squares-half-less*:
$x < u/2 \Longrightarrow y < u/2 \Longrightarrow 0 \leq x \Longrightarrow 0 \leq y \Longrightarrow sqrt\ (x^2 + y^2) < u$
apply (*rule real-sqrt-sum-squares-less*)
apply (*auto simp add*: *abs-if field-simps*)
apply (*rule le-less-trans* [**where** $y = x*2$])
using *less-eq-real-def sqrt2-less-2* **apply** *force*
apply *assumption*
apply (*rule le-less-trans* [**where** $y = y*2$])
using *less-eq-real-def sqrt2-less-2 mult-le-cancel-left*
apply *auto*
done

lemma *LIMSEQ-root*: (λ*n*. *root n n*) ⟶ *1*
proof −
  define *x* **where** *x n* = *root n n* − *1* **for** *n*
  have *x* ⟶ *sqrt 0*
  proof (*rule tendsto-sandwich*[*OF - - tendsto-const*])
    show (λ*x*. *sqrt* (*2* / *x*)) ⟶ *sqrt 0*
      by (*intro tendsto-intros tendsto-divide-0*[*OF tendsto-const*] *filterlim-mono*[*OF filterlim-real-sequentially*])
        (*simp-all add*: *at-infinity-eq-at-top-bot*)
    have *x n* ≤ *sqrt* (*2* / *real n*) **if** *2* < *n* **for** *n* :: *nat*
    proof −
      have *1* + (*real* (*n* − *1*) ∗ *n*) / *2* ∗ (*x n*)^2 = *1* + *of-nat* (*n choose 2*) ∗ (*x n*)^2
        by (*auto simp add*: *choose-two field-char-0-class.of-nat-div mod-eq-0-iff-dvd*)
      also have ... ≤ (∑*k*∈{*0*, *2*}. *of-nat* (*n choose k*) ∗ *x n*^*k*)
        by (*simp add*: *x-def*)
      also have ... ≤ (∑*k*≤*n*. *of-nat* (*n choose k*) ∗ *x n*^*k*)
        using ‹*2* < *n*›
        by (*intro sum-mono2*) (*auto intro*!: *mult-nonneg-nonneg zero-le-power simp*: *x-def le-diff-eq*)
      also have ... = (*x n* + *1*) ^ *n*
        by (*simp add*: *binomial-ring*)
      also have ... = *n*
        using ‹*2* < *n*› **by** (*simp add*: *x-def*)
      finally have *real* (*n* − *1*) ∗ (*real n* / *2* ∗ (*x n*)^2) ≤ *real* (*n* − *1*) ∗ *1*
        using *that* **by** *auto*
      then have (*x n*)^2 ≤ *2* / *real n*
        using ‹*2* < *n*› **unfolding** *mult-le-cancel-left* **by** (*simp add*: *field-simps*)
      from *real-sqrt-le-mono*[*OF this*] **show** *?thesis*
        by *simp*
    qed
    then show *eventually* (λ*n*. *x n* ≤ *sqrt* (*2* / *real n*)) *sequentially*
      by (*auto intro*!: *exI*[*of - 3*] *simp*: *eventually-sequentially*)
    show *eventually* (λ*n*. *sqrt 0* ≤ *x n*) *sequentially*
      by (*auto intro*!: *exI*[*of - 1*] *simp*: *eventually-sequentially le-diff-eq x-def*)
  qed
  from *tendsto-add*[*OF this tendsto-const*[*of 1*]] **show** *?thesis*
    by (*simp add*: *x-def*)
qed

lemma *LIMSEQ-root-const*:
  assumes *0* < *c*
  shows (λ*n*. *root n c*) ⟶ *1*
proof −
  have *ge-1*: (λ*n*. *root n c*) ⟶ *1* **if** *1* ≤ *c* **for** *c* :: *real*
  proof −
    define *x* **where** *x n* = *root n c* − *1* **for** *n*
    have *x* ⟶ *0*
    proof (*rule tendsto-sandwich*[*OF - - tendsto-const*])
      show (λ*n*. *c* / *n*) ⟶ *0*

by (*intro tendsto-divide-0*[*OF tendsto-const*] *filterlim-mono*[*OF filterlim-real-sequentially*])
(*simp-all add*: *at-infinity-eq-at-top-bot*)
have *x n* ≤ *c* / *n* **if** *1* < *n* **for** *n* :: *nat*
proof −
have *1* + *x n* ∗ *n* = *1* + *of-nat* (*n choose 1*) ∗ *x n*^*1*
by (*simp add*: *choose-one*)
also have . . . ≤ (∑*k*∈{*0*, *1*}. *of-nat* (*n choose k*) ∗ *x n*^*k*)
by (*simp add*: *x-def*)
also have . . . ≤ (∑*k*≤*n*. *of-nat* (*n choose k*) ∗ *x n*^*k*)
using ‹*1* < *n*› ‹*1* ≤ *c*›
by (*intro sum-mono2*)
(*auto intro*!: *mult-nonneg-nonneg zero-le-power simp*: *x-def le-diff-eq*)
also have . . . = (*x n* + *1*) ^ *n*
by (*simp add*: *binomial-ring*)
also have . . . = *c*
using ‹*1* < *n*› ‹*1* ≤ *c*› **by** (*simp add*: *x-def*)
finally show *?thesis*
using ‹*1* ≤ *c*› ‹*1* < *n*› **by** (*simp add*: *field-simps*)
qed
then show *eventually* (λ*n*. *x n* ≤ *c* / *n*) *sequentially*
by (*auto intro*!: *exI*[*of - 3*] *simp*: *eventually-sequentially*)
show *eventually* (λ*n*. *0* ≤ *x n*) *sequentially*
using ‹*1* ≤ *c*›
by (*auto intro*!: *exI*[*of - 1*] *simp*: *eventually-sequentially le-diff-eq x-def*)
qed
from *tendsto-add*[*OF this tendsto-const*[*of 1*]] **show** *?thesis*
by (*simp add*: *x-def*)
qed
show *?thesis*
proof (*cases 1* ≤ *c*)
case *True*
with *ge-1* **show** *?thesis* **by** *blast*
next
case *False*
with ‹*0* < *c*› **have** *1* ≤ *1* / *c*
by *simp*
then have (λ*n*. *1* / *root n* (*1* / *c*)) ⟶ *1* / *1*
by (*intro tendsto-divide tendsto-const ge-1* ‹*1* ≤ *1* / *c*› *one-neq-zero*)
then show *?thesis*
by (*rule filterlim-cong*[*THEN iffD1*, *rotated 3*])
(*auto intro*!: *exI*[*of - 1*] *simp*: *eventually-sequentially real-root-divide*)
qed
qed

Legacy theorem names:

lemmas *real-root-pos2* = *real-root-power-cancel*
lemmas *real-root-pos-pos* = *real-root-gt-zero* [*THEN order-less-imp-le*]
lemmas *real-root-pos-pos-le* = *real-root-ge-zero*
lemmas *real-sqrt-eq-zero-cancel-iff* = *real-sqrt-eq-0-iff*

end

112 Power Series, Transcendental Functions etc.

theory *Transcendental*
imports *Series Deriv NthRoot*
begin

A theorem about the factcorial function on the reals.

lemma *square-fact-le-2-fact*: *fact n* ∗ *fact n* ≤ (*fact* (*2* ∗ *n*) :: *real*)
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  have (*fact* (*Suc n*)) ∗ (*fact* (*Suc n*)) = *of-nat* (*Suc n*) ∗ *of-nat* (*Suc n*) ∗ (*fact n* ∗ *fact n* :: *real*)
    by (*simp add*: *field-simps*)
  also have ... ≤ *of-nat* (*Suc n*) ∗ *of-nat* (*Suc n*) ∗ *fact* (*2* ∗ *n*)
    by (*rule mult-left-mono* [*OF Suc*]) *simp*
  also have ... ≤ *of-nat* (*Suc* (*Suc* (*2* ∗ *n*))) ∗ *of-nat* (*Suc* (*2* ∗ *n*)) ∗ *fact* (*2* ∗ *n*)
    by (*rule mult-right-mono*)+ (*auto simp*: *field-simps*)
  also have ... = *fact* (*2* ∗ *Suc n*) **by** (*simp add*: *field-simps*)
  finally show *?case* **.**
qed

lemma *fact-in-Reals*: *fact n* ∈ ℝ
  by (*induction n*) *auto*

lemma *of-real-fact* [*simp*]: *of-real* (*fact n*) = *fact n*
  by (*metis of-nat-fact of-real-of-nat-eq*)

lemma *pochhammer-of-real*: *pochhammer* (*of-real x*) *n* = *of-real* (*pochhammer x n*)
  by (*simp add*: *pochhammer-prod*)

lemma *norm-fact* [*simp*]: *norm* (*fact n* :: *'a*::*real-normed-algebra-1*) = *fact n*
proof −
  have (*fact n* :: *'a*) = *of-real* (*fact n*)
    by *simp*
  also have *norm* ... = *fact n*
    by (*subst norm-of-real*) *simp*
  finally show *?thesis* **.**
qed

lemma *root-test-convergence*:
  fixes *f* :: *nat* ⇒ *'a*::*banach*

assumes *f*: ($\lambda n.$ *root n* (*norm* (*f n*))) $\longrightarrow$ *x* — could be weakened to lim sup
and $x < 1$
shows *summable f*
proof –
have $0 \leq x$
by (*rule LIMSEQ-le*[*OF tendsto-const f*]) (*auto intro*!: *exI*[*of - 1*])
from ‹$x < 1$› **obtain** *z* **where** *z*: $x < z$ $z < 1$
by (*metis dense*)
from *f* ‹$x < z$› **have** *eventually* ($\lambda n.$ *root n* (*norm* (*f n*)) $< z$) *sequentially*
by (*rule order-tendstoD*)
then have *eventually* ($\lambda n.$ *norm* (*f n*) $\leq z$^*n*) *sequentially*
using *eventually-ge-at-top*
proof *eventually-elim*
fix *n*
assume *less*: *root n* (*norm* (*f n*)) $< z$ **and** *n*: $1 \leq n$
from *power-strict-mono*[*OF less*, *of n*] *n* **show** *norm* (*f n*) $\leq z$ ^ *n*
by *simp*
qed
then show *summable f*
unfolding *eventually-sequentially*
using *z* ‹$0 \leq x$› **by** (*auto intro*!: *summable-comparison-test*[*OF - summable-geometric*])
qed

112.1 Properties of Power Series

lemma *powser-zero* [*simp*]: ($\sum n.$ *f n* $*$ *0* ^ *n*) = *f 0*
for *f* :: *nat* $\Rightarrow$ *'a::real-normed-algebra-1*
proof –
have ($\sum n<1.$ *f n* $*$ *0* ^ *n*) = ($\sum n.$ *f n* $*$ *0* ^ *n*)
by (*subst suminf-finite*[**where** *N*={*0*}]) (*auto simp*: *power-0-left*)
then show *?thesis* **by** *simp*
qed

lemma *powser-sums-zero*: ($\lambda n.$ *a n* $*$ *0*^*n*) *sums a 0*
for *a* :: *nat* $\Rightarrow$ *'a::real-normed-div-algebra*
using *sums-finite* [*of* {*0*} $\lambda n.$ *a n* $*$ *0* ^ *n*]
by *simp*

lemma *powser-sums-zero-iff* [*simp*]: ($\lambda n.$ *a n* $*$ *0*^*n*) *sums x* $\longleftrightarrow$ *a 0* = *x*
for *a* :: *nat* $\Rightarrow$ *'a::real-normed-div-algebra*
using *powser-sums-zero sums-unique2* **by** *blast*

Power series has a circle or radius of convergence: if it sums for x, then it sums absolutely for z with $|z| < |x|$.

lemma *powser-insidea*:
fixes *x z* :: *'a::real-normed-div-algebra*
assumes *1*: *summable* ($\lambda n.$ *f n* $*$ *x*^*n*)
and *2*: *norm z* $<$ *norm x*
shows *summable* ($\lambda n.$ *norm* (*f n* $*$ *z* ^ *n*))

proof −
  from *2* **have** *x-neq-0*: $x \neq 0$ **by** *clarsimp*
  from *1* **have** $(\lambda n.\ f\ n * x\hat{\ }n) \longrightarrow 0$
    by (*rule summable-LIMSEQ-zero*)
  then have *convergent* $(\lambda n.\ f\ n * x\hat{\ }n)$
    by (*rule convergentI*)
  then have *Cauchy* $(\lambda n.\ f\ n * x\hat{\ }n)$
    by (*rule convergent-Cauchy*)
  then have *Bseq* $(\lambda n.\ f\ n * x\hat{\ }n)$
    by (*rule Cauchy-Bseq*)
  then obtain K **where** *3*: $0 < K$ **and** *4*: $\forall n.\ norm\ (f\ n * x\hat{\ }n) \leq K$
    by (*auto simp*: *Bseq-def*)
  have $\exists N.\ \forall n \geq N.\ norm\ (norm\ (f\ n * z\ \hat{\ }\ n)) \leq K * norm\ (z\ \hat{\ }\ n) * inverse\ (norm\ (x\hat{\ }n))$
  proof (*intro exI allI impI*)
    fix n :: *nat*
    assume $0 \leq n$
    have $norm\ (norm\ (f\ n * z\ \hat{\ }\ n)) * norm\ (x\hat{\ }n) =$
        $norm\ (f\ n * x\hat{\ }n) * norm\ (z\ \hat{\ }\ n)$
      by (*simp add*: *norm-mult abs-mult*)
    also have $\ldots \leq K * norm\ (z\ \hat{\ }\ n)$
      by (*simp only*: *mult-right-mono 4 norm-ge-zero*)
    also have $\ldots = K * norm\ (z\ \hat{\ }\ n) * (inverse\ (norm\ (x\hat{\ }n)) * norm\ (x\hat{\ }n))$
      by (*simp add*: *x-neq-0*)
    also have $\ldots = K * norm\ (z\ \hat{\ }\ n) * inverse\ (norm\ (x\hat{\ }n)) * norm\ (x\hat{\ }n)$
      by (*simp only*: *mult.assoc*)
    finally show $norm\ (norm\ (f\ n * z\ \hat{\ }\ n)) \leq K * norm\ (z\ \hat{\ }\ n) * inverse\ (norm\ (x\hat{\ }n))$
      by (*simp add*: *mult-le-cancel-right x-neq-0*)
  qed
  moreover have *summable* $(\lambda n.\ K * norm\ (z\ \hat{\ }\ n) * inverse\ (norm\ (x\hat{\ }n)))$
  proof −
    from *2* **have** $norm\ (norm\ (z * inverse\ x)) < 1$
      using *x-neq-0*
      by (*simp add*: *norm-mult nonzero-norm-inverse divide-inverse* [**where** *'a=real*, *symmetric*])
    then have *summable* $(\lambda n.\ norm\ (z * inverse\ x)\ \hat{\ }\ n)$
      by (*rule summable-geometric*)
    then have *summable* $(\lambda n.\ K * norm\ (z * inverse\ x)\ \hat{\ }\ n)$
      by (*rule summable-mult*)
    then show *summable* $(\lambda n.\ K * norm\ (z\ \hat{\ }\ n) * inverse\ (norm\ (x\hat{\ }n)))$
      using *x-neq-0*
      by (*simp add*: *norm-mult nonzero-norm-inverse power-mult-distrib*
        power-inverse norm-power mult.assoc)
  qed
  ultimately show *summable* $(\lambda n.\ norm\ (f\ n * z\ \hat{\ }\ n))$
    by (*rule summable-comparison-test*)
qed

lemma *powser-inside*:
fixes f :: *nat* ⇒ *'a*::{*real-normed-div-algebra*,*banach*}
shows
summable ($\lambda n.\ f\ n * (x\hat{}n)$) $\Longrightarrow$ *norm* $z <$ *norm* $x \Longrightarrow$
summable ($\lambda n.\ f\ n * (z\ \hat{}\ n)$)
by (*rule powser-insidea* [*THEN summable-norm-cancel*])

lemma *powser-times-n-limit-0*:
fixes x :: *'a*::{*real-normed-div-algebra*,*banach*}
assumes *norm* $x < 1$
shows ($\lambda n.$ *of-nat* $n * x\ \hat{}\ n$) $\longrightarrow 0$
proof −
have *norm* x / (1 − *norm* x) ≥ 0
using *assms* **by** (*auto simp*: *field-split-simps*)
moreover obtain N **where** N: *norm* x / (1 − *norm* x) < *of-int* N
using *ex-le-of-int* **by** (*meson ex-less-of-int*)
ultimately have *N0*: $N>0$
by *auto*
then have ∗: *real-of-int* ($N + 1$) ∗ *norm* x / *real-of-int* $N < 1$
using N *assms* **by** (*auto simp*: *field-simps*)
have ∗∗: *real-of-int* N ∗ (*norm* x ∗ (*real-of-nat* (*Suc* n) ∗ *norm* ($x\ \hat{}\ n$))) ≤
real-of-nat n ∗ (*norm* x ∗ (($1 + N$) ∗ *norm* ($x\ \hat{}\ n$))) **if** $N \leq$ *int* n **for** n :: *nat*
proof −
from *that* **have** *real-of-int* N ∗ *real-of-nat* (*Suc* n) ≤ *real-of-nat* n ∗ *real-of-int* ($1 + N$)
by (*simp add*: *algebra-simps*)
then have (*real-of-int* N ∗ *real-of-nat* (*Suc* n)) ∗ (*norm* x ∗ *norm* ($x\ \hat{}\ n$)) ≤
(*real-of-nat* n ∗ ($1 + N$)) ∗ (*norm* x ∗ *norm* ($x\ \hat{}\ n$))
using *N0 mult-mono* **by** *fastforce*
then show *?thesis*
by (*simp add*: *algebra-simps*)
qed
show *?thesis* **using** ∗
by (*rule summable-LIMSEQ-zero* [*OF summable-ratio-test*, **where** *N1*=*nat* N])
(*simp add*: *N0 norm-mult field-simps* ∗∗ *del*: *of-nat-Suc of-int-add*)
qed

corollary *lim-n-over-pown*:
fixes x :: *'a*::{*real-normed-field*,*banach*}
shows $1 <$ *norm* $x \Longrightarrow$ (($\lambda n.$ *of-nat* n / $x\hat{}n$) $\longrightarrow 0$) *sequentially*
using *powser-times-n-limit-0* [*of inverse* x]
by (*simp add*: *norm-divide field-split-simps*)

lemma *sum-split-even-odd*:
fixes f :: *nat* ⇒ *real*
shows ($\sum i<2 * n.$ *if even* i *then* $f\ i$ *else* $g\ i$) = ($\sum i<n.\ f\ (2 * i)$) + ($\sum i<n.$ $g\ (2 * i + 1)$)
proof (*induct* n)
case *0*

```
  then show ?case by simp
next
  case (Suc n)
  have (∑ i<2 ∗ Suc n. if even i then f i else g i) =
    (∑ i<n. f (2 ∗ i)) + (∑ i<n. g (2 ∗ i + 1)) + (f (2 ∗ n) + g (2 ∗ n + 1))
    using Suc.hyps unfolding One-nat-def by auto
  also have ... = (∑ i<Suc n. f (2 ∗ i)) + (∑ i<Suc n. g (2 ∗ i + 1))
    by auto
  finally show ?case .
qed

lemma sums-if':
  fixes g :: nat ⇒ real
  assumes g sums x
  shows (λ n. if even n then 0 else g ((n − 1) div 2)) sums x
  unfolding sums-def
proof (rule LIMSEQ-I)
  fix r :: real
  assume 0 < r
  from ‹g sums x›[unfolded sums-def, THEN LIMSEQ-D, OF this]
  obtain no where no-eq: ⋀n. n ≥ no ⟹ (norm (sum g {..<n} − x) < r)
    by blast

  let ?SUM = λ m. ∑ i<m. if even i then 0 else g ((i − 1) div 2)
  have (norm (?SUM m − x) < r) if m ≥ 2 ∗ no for m
  proof −
    from that have m div 2 ≥ no by auto
    have sum-eq: ?SUM (2 ∗ (m div 2)) = sum g {..< m div 2}
      using sum-split-even-odd by auto
    then have (norm (?SUM (2 ∗ (m div 2)) − x) < r)
      using no-eq unfolding sum-eq using ‹m div 2 ≥ no› by auto
    moreover
    have ?SUM (2 ∗ (m div 2)) = ?SUM m
    proof (cases even m)
      case True
      then show ?thesis
        by (auto simp: even-two-times-div-two)
    next
      case False
      then have eq: Suc (2 ∗ (m div 2)) = m by simp
      then have even (2 ∗ (m div 2)) using ‹odd m› by auto
      have ?SUM m = ?SUM (Suc (2 ∗ (m div 2))) unfolding eq ..
       also have ... = ?SUM (2 ∗ (m div 2)) using ‹even (2 ∗ (m div 2))› by
auto
      finally show ?thesis by auto
    qed
    ultimately show ?thesis by auto
  qed
  then show ∃ no. ∀ m ≥ no. norm (?SUM m − x) < r
```

by *blast*
qed

lemma *sums-if*:
fixes g :: *nat* ⇒ *real*
assumes g *sums* x **and** f *sums* y
shows (λ n. *if even* n *then* f (n *div* 2) *else* g ((n − 1) *div* 2)) *sums* (x + y)
proof −
let *?s* = λ n. *if even* n *then* 0 *else* f ((n − 1) *div* 2)
have *if-sum*: (*if* B *then* (0 :: *real*) *else* E) + (*if* B *then* T *else* 0) = (*if* B *then* T *else* E)
for B T E
by (*cases* B) *auto*
have *g-sums*: (λ n. *if even* n *then* 0 *else* g ((n − 1) *div* 2)) *sums* x
using *sums-if'*[*OF* ‹g *sums* x›] **.**
have *if-eq*: ⋀B T E. (*if* ¬ B *then* T *else* E) = (*if* B *then* E *else* T)
by *auto*
have *?s sums* y **using** *sums-if'*[*OF* ‹f *sums* y›] **.**
from *this*[*unfolded sums-def*, *THEN LIMSEQ-Suc*]
have (λn. *if even* n *then* f (n *div* 2) *else* 0) *sums* y
by (*simp add*: *lessThan-Suc-eq-insert-0 sum.atLeast1-atMost-eq image-Suc-lessThan if-eq sums-def cong del*: *if-weak-cong*)
from *sums-add*[*OF g-sums this*] **show** *?thesis*
by (*simp only*: *if-sum*)
qed

112.2 Alternating series test / Leibniz formula

lemma *sums-alternating-upper-lower*:
fixes a :: *nat* ⇒ *real*
assumes *mono*: ⋀n. a (*Suc* n) ≤ a n
and *a-pos*: ⋀n. 0 ≤ a n
and a ⟶ 0
shows ∃ l. ((∀ n. (∑ i<2∗n. (− 1)^i∗a i) ≤ l) ∧ (λ n. ∑ i<2∗n. (− 1)^i∗a i) ⟶ l) ∧
((∀ n. l ≤ (∑ i<2∗n + 1. (− 1)^i∗a i)) ∧ (λ n. ∑ i<2∗n + 1. (− 1)^i∗a i) ⟶ l)
(**is** ∃ l. ((∀ n. *?f* n ≤ l) ∧ -) ∧ ((∀ n. l ≤ *?g* n) ∧ -))
proof (*rule nested-sequence-unique*)
have *fg-diff*: ⋀n. *?f* n − *?g* n = − a (2 ∗ n) **by** *auto*

show ∀ n. *?f* n ≤ *?f* (*Suc* n)
proof
show *?f* n ≤ *?f* (*Suc* n) **for** n
using *mono*[*of* 2∗n] **by** *auto*
qed
show ∀ n. *?g* (*Suc* n) ≤ *?g* n
proof
show *?g* (*Suc* n) ≤ *?g* n **for** n

```
      using mono[of Suc (2*n)] by auto
  qed
  show ∀ n. ?f n ≤ ?g n
  proof
    show ?f n ≤ ?g n for n
      using fg-diff a-pos by auto
  qed
  show (λn. ?f n − ?g n) ⟶ 0
    unfolding fg-diff
  proof (rule LIMSEQ-I)
    fix r :: real
    assume 0 < r
     with ‹a ⟶ 0›[THEN LIMSEQ-D] obtain N where ⋀ n. n ≥ N ⟹
norm (a n − 0) < r
      by auto
    then have ∀ n ≥ N. norm (− a (2 ∗ n) − 0) < r
      by auto
    then show ∃ N. ∀ n ≥ N. norm (− a (2 ∗ n) − 0) < r
      by auto
  qed
qed

lemma summable-Leibniz':
  fixes a :: nat ⇒ real
  assumes a-zero: a ⟶ 0
    and a-pos: ⋀n. 0 ≤ a n
    and a-monotone: ⋀n. a (Suc n) ≤ a n
  shows summable: summable (λ n. (−1)^n ∗ a n)
    and ⋀n. (∑ i<2∗n. (−1)^i∗a i) ≤ (∑ i. (−1)^i∗a i)
    and (λn. ∑ i<2∗n. (−1)^i∗a i) ⟶ (∑ i. (−1)^i∗a i)
    and ⋀n. (∑ i. (−1)^i∗a i) ≤ (∑ i<2∗n+1. (−1)^i∗a i)
    and (λn. ∑ i<2∗n+1. (−1)^i∗a i) ⟶ (∑ i. (−1)^i∗a i)
proof −
  let ?S = λn. (−1)^n ∗ a n
  let ?P = λn. ∑ i<n. ?S i
  let ?f = λn. ?P (2 ∗ n)
  let ?g = λn. ?P (2 ∗ n + 1)
  obtain l :: real
    where below-l: ∀ n. ?f n ≤ l
      and ?f ⟶ l
      and above-l: ∀ n. l ≤ ?g n
      and ?g ⟶ l
    using sums-alternating-upper-lower[OF a-monotone a-pos a-zero] by blast

  let ?Sa = λm. ∑ n<m. ?S n
  have ?Sa ⟶ l
  proof (rule LIMSEQ-I)
    fix r :: real
    assume 0 < r
```

```
    with ‹?f ⟶ l›[THEN LIMSEQ-D]
    obtain f-no where f: ⋀n. n ≥ f-no ⟹ norm (?f n − l) < r
      by auto
    from ‹0 < r› ‹?g ⟶ l›[THEN LIMSEQ-D]
    obtain g-no where g: ⋀n. n ≥ g-no ⟹ norm (?g n − l) < r
      by auto
    have norm (?Sa n − l) < r if n ≥ (max (2 ∗ f-no) (2 ∗ g-no)) for n
    proof −
      from that have n ≥ 2 ∗ f-no and n ≥ 2 ∗ g-no by auto
      show ?thesis
      proof (cases even n)
        case True
        then have n-eq: 2 ∗ (n div 2) = n
          by (simp add: even-two-times-div-two)
        with ‹n ≥ 2 ∗ f-no› have n div 2 ≥ f-no
          by auto
        from f[OF this] show ?thesis
          unfolding n-eq atLeastLessThanSuc-atLeastAtMost .
      next
        case False
        then have even (n − 1) by simp
        then have n-eq: 2 ∗ ((n − 1) div 2) = n − 1
          by (simp add: even-two-times-div-two)
        then have range-eq: n − 1 + 1 = n
          using odd-pos[OF False] by auto
        from n-eq ‹n ≥ 2 ∗ g-no› have (n − 1) div 2 ≥ g-no
          by auto
        from g[OF this] show ?thesis
          by (simp only: n-eq range-eq)
      qed
    qed
    then show ∃ no. ∀ n ≥ no. norm (?Sa n − l) < r by blast
  qed
  then have sums-l: (λi. (−1)^i ∗ a i) sums l
    by (simp only: sums-def)
  then show summable ?S
    by (auto simp: summable-def)

  have l = suminf ?S by (rule sums-unique[OF sums-l])

  fix n
  show suminf ?S ≤ ?g n
    unfolding sums-unique[OF sums-l, symmetric] using above-l by auto
  show ?f n ≤ suminf ?S
    unfolding sums-unique[OF sums-l, symmetric] using below-l by auto
  show ?g ⟶ suminf ?S
    using ‹?g ⟶ l› ‹l = suminf ?S› by auto
  show ?f ⟶ suminf ?S
    using ‹?f ⟶ l› ‹l = suminf ?S› by auto
```

qed

theorem *summable-Leibniz*:
fixes $a :: nat \Rightarrow real$
assumes *a-zero*: $a \longrightarrow 0$
and *monoseq a*
shows $summable\ (\lambda\ n.\ (-1)\hat{}n * a\ n)$ (**is** *?summable*)
and $0 < a\ 0 \longrightarrow$
$(\forall n.\ (\sum i.\ (-\ 1)\hat{}i*a\ i) \in \{\ \sum i<2*n.\ (-\ 1)\hat{}i * a\ i\ ..\ \sum i<2*n+1.\ (-\ 1)\hat{}i * a\ i\})$ (**is** *?pos*)
and $a\ 0 < 0 \longrightarrow$
$(\forall n.\ (\sum i.\ (-\ 1)\hat{}i*a\ i) \in \{\ \sum i<2*n+1.\ (-\ 1)\hat{}i * a\ i\ ..\ \sum i<2*n.\ (-\ 1)\hat{}i * a\ i\})$ (**is** *?neg*)
and $(\lambda n.\ \sum i<2*n.\ (-\ 1)\hat{}i*a\ i) \longrightarrow (\sum i.\ (-\ 1)\hat{}i*a\ i)$ (**is** *?f*)
and $(\lambda n.\ \sum i<2*n+1.\ (-\ 1)\hat{}i*a\ i) \longrightarrow (\sum i.\ (-\ 1)\hat{}i*a\ i)$ (**is** *?g*)
proof −
have $?summable \wedge ?pos \wedge ?neg \wedge ?f \wedge ?g$
proof $(cases\ (\forall n.\ 0 \leq a\ n) \wedge (\forall m.\ \forall n \geq m.\ a\ n \leq a\ m))$
case *True*
then have *ord*: $\bigwedge n\ m.\ m \leq n \Longrightarrow a\ n \leq a\ m$
and *ge0*: $\bigwedge n.\ 0 \leq a\ n$
by *auto*
have *mono*: $a\ (Suc\ n) \leq a\ n$ **for** *n*
using *ord*[**where** $n=Suc\ n$ **and** $m=n$] **by** *auto*
note *leibniz* = *summable-Leibniz'*[*OF* ‹$a \longrightarrow 0$› *ge0*]
from *leibniz*[*OF mono*]
show *?thesis* **using** ‹$0 \leq a\ 0$› **by** *auto*
next
let $?a = \lambda n.\ -\ a\ n$
case *False*
with *monoseq-le*[*OF* ‹*monoseq a*› ‹$a \longrightarrow 0$›]
have $(\forall\ n.\ a\ n \leq 0) \wedge (\forall m.\ \forall n \geq m.\ a\ m \leq a\ n)$ **by** *auto*
then have *ord*: $\bigwedge n\ m.\ m \leq n \Longrightarrow ?a\ n \leq ?a\ m$ **and** *ge0*: $\bigwedge\ n.\ 0 \leq ?a\ n$
by *auto*
have *monotone*: $?a\ (Suc\ n) \leq ?a\ n$ **for** *n*
using *ord*[**where** $n=Suc\ n$ **and** $m=n$] **by** *auto*
note *leibniz* =
summable-Leibniz'[*OF* - *ge0*, *of* $\lambda x.\ x$,
OF tendsto-minus[*OF* ‹$a \longrightarrow 0$›, *unfolded minus-zero*] *monotone*]
have $summable\ (\lambda\ n.\ (-1)\hat{}n * ?a\ n)$
using *leibniz*(*1*) **by** *auto*
then obtain *l* **where** $(\lambda\ n.\ (-1)\hat{}n * ?a\ n)\ sums\ l$
unfolding *summable-def* **by** *auto*
from *this*[*THEN sums-minus*] **have** $(\lambda\ n.\ (-1)\hat{}n * a\ n)\ sums\ -l$
by *auto*
then have *?summable* **by** (*auto simp*: *summable-def*)
moreover
have $|-\ a - -\ b| = |a - b|$ **for** $a\ b :: real$
unfolding *minus-diff-minus* **by** *auto*

from *suminf-minus*[*OF leibniz*(*1*), *unfolded mult-minus-right minus-minus*]
have *move-minus*: ($\sum$ *n*. − ((− *1*) ^ *n* ∗ *a n*)) = − ($\sum$ *n*. (− *1*) ^ *n* ∗ *a n*)
by *auto*

have *?pos* **using** ‹*0* ≤ *?a 0*› **by** *auto*
moreover have *?neg*
using *leibniz*(*2*,*4*)
unfolding *mult-minus-right sum-negf move-minus neg-le-iff-le*
by *auto*
moreover have *?f* **and** *?g*
using *leibniz*(*3*,*5*)[*unfolded mult-minus-right sum-negf move-minus*, *THEN tendsto-minus-cancel*]
by *auto*
ultimately show *?thesis* **by** *auto*
qed
then show *?summable* **and** *?pos* **and** *?neg* **and** *?f* **and** *?g*
by *safe*
qed

112.3 Term-by-Term Differentiability of Power Series

definition *diffs* :: (*nat* ⇒ *'a*::*ring-1*) ⇒ *nat* ⇒ *'a*
where *diffs c* = (λ*n*. *of-nat* (*Suc n*) ∗ *c* (*Suc n*))

Lemma about distributing negation over it.

lemma *diffs-minus*: *diffs* (λ*n*. − *c n*) = (λ*n*. − *diffs c n*)
by (*simp add*: *diffs-def*)

lemma *diffs-equiv*:
fixes *x* :: *'a*::{*real-normed-vector*,*ring-1*}
shows *summable* (λ*n*. *diffs c n* ∗ *x*^*n*) ⟹
(λ*n*. *of-nat n* ∗ *c n* ∗ *x*^(*n* − *Suc 0*)) *sums* ($\sum$ *n*. *diffs c n* ∗ *x*^*n*)
unfolding *diffs-def*
by (*simp add*: *summable-sums sums-Suc-imp*)

lemma *lemma-termdiff1*:
fixes *z* :: *'a* :: {*monoid-mult*,*comm-ring*}
shows ($\sum$ *p*<*m*. (((*z* + *h*) ^ (*m* − *p*)) ∗ (*z* ^ *p*)) − (*z* ^ *m*)) =
($\sum$ *p*<*m*. (*z* ^ *p*) ∗ (((*z* + *h*) ^ (*m* − *p*)) − (*z* ^ (*m* − *p*))))
by (*auto simp*: *algebra-simps power-add* [*symmetric*])

lemma *sumr-diff-mult-const2*: *sum f* {..<*n*} − *of-nat n* ∗ *r* = ($\sum$ *i*<*n*. *f i* − *r*)
for *r* :: *'a*::*ring-1*
by (*simp add*: *sum-subtractf*)

lemma *lemma-termdiff2*:
fixes *h* :: *'a*::*field*
assumes *h*: *h* ≠ *0*

shows $((z + h) \hat{} n - z \hat{} n) / h - of\text{-}nat\ n * z \hat{} (n - Suc\ 0) =$
$h * (\sum p< n - Suc\ 0. \sum q< n - Suc\ 0 - p. (z + h) \hat{} q * z \hat{} (n - 2 - q))$
(**is** *?lhs* = *?rhs*)
proof (*cases n*)
case (*Suc m*)
have *0*: $\bigwedge x\ k. (\sum n<Suc\ k.\ h * (z \hat{} x * (z \hat{} (k - n) * (h + z) \hat{} n))) =$
$(\sum j<Suc\ k.\ h * ((h + z) \hat{} j * z \hat{} (x + k - j)))$
by (*auto simp add*: *power-add* [*symmetric*] *mult.commute intro*: *sum.cong*)
have ∗: $(\sum i<m.\ z \hat{} i * ((z + h) \hat{} (m - i) - z \hat{} (m - i))) =$
$(\sum i<m. \sum j<m - i.\ h * ((z + h) \hat{} j * z \hat{} (m - Suc\ j)))$
by (*force simp add*: *less-iff-Suc-add sum-distrib-left diff-power-eq-sum ac-simps 0*
simp del: *sum.lessThan-Suc power-Suc intro*: *sum.cong*)
have $h * ?lhs = (z + h) \hat{} n - z \hat{} n - h * of\text{-}nat\ n * z \hat{} (n - Suc\ 0)$
by (*simp add*: *right-diff-distrib diff-divide-distrib h mult.assoc* [*symmetric*])
also have ... $= h * ((\sum p<Suc\ m. (z + h) \hat{} p * z \hat{} (m - p)) - of\text{-}nat\ (Suc\ m) * z \hat{} m)$
by (*simp add*: *Suc diff-power-eq-sum h right-diff-distrib* [*symmetric*] *mult.assoc*
del: *power-Suc sum.lessThan-Suc of-nat-Suc*)
also have ... $= h * ((\sum p<Suc\ m. (z + h) \hat{} (m - p) * z \hat{} p) - of\text{-}nat\ (Suc\ m) * z \hat{} m)$
by (*subst sum.nat-diff-reindex*[*symmetric*]) *simp*
also have ... $= h * (\sum i<Suc\ m. (z + h) \hat{} (m - i) * z \hat{} i - z \hat{} m)$
by (*simp add*: *sum-subtractf*)
also have ... = *h* ∗ *?rhs*
by (*simp add*: *lemma-termdiff1 sum-distrib-left Suc* ∗)
finally have *h* ∗ *?lhs* = *h* ∗ *?rhs* **.**
then show *?thesis*
by (*simp add*: *h*)
qed *auto*

lemma *real-sum-nat-ivl-bounded2*:
fixes *K* :: *'a*::*linordered-semidom*
assumes *f*: $\bigwedge p::nat.\ p < n \Longrightarrow f\ p \le K$ **and** *K*: $0 \le K$
shows *sum f* $\{..<n-k\} \le of\text{-}nat\ n * K$
proof −
have *sum f* $\{..<n-k\} \le (\sum i<n - k.\ K)$
by (*rule sum-mono* [*OF f*]) *auto*
also have ... $\le of\text{-}nat\ n * K$
by (*auto simp*: *mult-right-mono K*)
finally show *?thesis* **.**
qed

lemma *lemma-termdiff3*:
fixes *h z* :: *'a*::*real-normed-field*
assumes *1*: $h \neq 0$
and *2*: $norm\ z \le K$

and *3*: *norm* (*z* + *h*) ≤ *K*
shows *norm* (((*z* + *h*) ^ *n* − *z* ^ *n*) / *h* − *of-nat n* ∗ *z* ^ (*n* − *Suc 0*)) ≤
of-nat n ∗ *of-nat* (*n* − *Suc 0*) ∗ *K* ^ (*n* − *2*) ∗ *norm h*
proof −
have *norm* (((*z* + *h*) ^ *n* − *z* ^ *n*) / *h* − *of-nat n* ∗ *z* ^ (*n* − *Suc 0*)) =
norm (∑ *p*<*n* − *Suc 0*. ∑ *q*<*n* − *Suc 0* − *p*. (*z* + *h*) ^ *q* ∗ *z* ^ (*n* − *2* − *q*))
∗ *norm h*
by (*metis* (*lifting*, *no-types*) *lemma-termdiff2* [*OF 1*] *mult.commute norm-mult*)
also have . . . ≤ *of-nat n* ∗ (*of-nat* (*n* − *Suc 0*) ∗ *K* ^ (*n* − *2*)) ∗ *norm h*
proof (*rule mult-right-mono* [*OF - norm-ge-zero*])
from *norm-ge-zero 2* **have** *K*: *0* ≤ *K*
by (*rule order-trans*)
have *le-Kn*: *norm* ((*z* + *h*) ^ *i* ∗ *z* ^ *j*) ≤ *K* ^ *n* **if** *i* + *j* = *n* **for** *i j n*
proof −
have *norm* (*z* + *h*) ^ *i* ∗ *norm z* ^ *j* ≤ *K* ^ *i* ∗ *K* ^ *j*
by (*intro mult-mono power-mono 2 3 norm-ge-zero zero-le-power K*)
also have ... = *K*^*n*
by (*metis power-add that*)
finally show *?thesis*
by (*simp add*: *norm-mult norm-power*)
qed
then have ⋀*p q*.
⟦*p* < *n*; *q* < *n* − *Suc 0*⟧ ⟹ *norm* ((*z* + *h*) ^ *q* ∗ *z* ^ (*n* − *2* − *q*)) ≤ *K* ^
(*n* − *2*)
by (*simp del*: *subst-all*)
then
show *norm* (∑ *p*<*n* − *Suc 0*. ∑ *q*<*n* − *Suc 0* − *p*. (*z* + *h*) ^ *q* ∗ *z* ^ (*n* − *2*
− *q*)) ≤
of-nat n ∗ (*of-nat* (*n* − *Suc 0*) ∗ *K* ^ (*n* − *2*))
by (*intro order-trans* [*OF norm-sum*]
real-sum-nat-ivl-bounded2 mult-nonneg-nonneg of-nat-0-le-iff zero-le-power
K)
qed
also have . . . = *of-nat n* ∗ *of-nat* (*n* − *Suc 0*) ∗ *K* ^ (*n* − *2*) ∗ *norm h*
by (*simp only*: *mult.assoc*)
finally show *?thesis* **.**
qed

lemma *lemma-termdiff4*:
fixes *f* :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*
and *k* :: *real*
assumes *k*: *0* < *k*
and *le*: ⋀*h*. *h* ≠ *0* ⟹ *norm h* < *k* ⟹ *norm* (*f h*) ≤ *K* ∗ *norm h*
shows *f* −*0*→ *0*
proof (*rule tendsto-norm-zero-cancel*)
show (λ*h*. *norm* (*f h*)) −*0*→ *0*
proof (*rule real-tendsto-sandwich*)
show *eventually* (λ*h*. *0* ≤ *norm* (*f h*)) (*at 0*)
by *simp*

show *eventually* ($\lambda h.$ *norm* (*f h*) $\leq$ *K* $*$ *norm h*) (*at 0*)
using *k* **by** (*auto simp*: *eventually-at dist-norm le*)
show ($\lambda h.$ *0*) $-(0::'a)\rightarrow$ ($0::real$)
by (*rule tendsto-const*)
have ($\lambda h.$ *K* $*$ *norm h*) $-(0::'a)\rightarrow$ *K* $*$ *norm* ($0::'a$)
by (*intro tendsto-intros*)
then show ($\lambda h.$ *K* $*$ *norm h*) $-(0::'a)\rightarrow$ *0*
by *simp*
qed
qed

lemma *lemma-termdiff5*:
fixes *g* :: *'a::real-normed-vector* $\Rightarrow$ *nat* $\Rightarrow$ *'b::banach*
and *k* :: *real*
assumes *k*: *0* $<$ *k*
and *f*: *summable f*
and *le*: $\bigwedge h\ n.\ h \neq 0 \Longrightarrow$ *norm h* $<$ *k* $\Longrightarrow$ *norm* (*g h n*) $\leq$ *f n* $*$ *norm h*
shows ($\lambda h.$ *suminf* (*g h*)) $-0\rightarrow$ *0*
proof (*rule lemma-termdiff4* [*OF k*])
fix *h* :: *'a*
assume $h \neq 0$ **and** *norm h* $<$ *k*
then have *1*: $\forall n.$ *norm* (*g h n*) $\leq$ *f n* $*$ *norm h*
by (*simp add*: *le*)
then have $\exists N.\ \forall n \geq N.$ *norm* (*norm* (*g h n*)) $\leq$ *f n* $*$ *norm h*
by *simp*
moreover from *f* **have** *2*: *summable* ($\lambda n.$ *f n* $*$ *norm h*)
by (*rule summable-mult2*)
ultimately have *3*: *summable* ($\lambda n.$ *norm* (*g h n*))
by (*rule summable-comparison-test*)
then have *norm* (*suminf* (*g h*)) $\leq$ ($\sum n.$ *norm* (*g h n*))
by (*rule summable-norm*)
also from *1 3 2* **have** ($\sum n.$ *norm* (*g h n*)) $\leq$ ($\sum n.$ *f n* $*$ *norm h*)
by (*simp add*: *suminf-le*)
also from *f* **have** ($\sum n.$ *f n* $*$ *norm h*) $=$ *suminf f* $*$ *norm h*
by (*rule suminf-mult2* [*symmetric*])
finally show *norm* (*suminf* (*g h*)) $\leq$ *suminf f* $*$ *norm h* **.**
qed

lemma *termdiffs-aux*:
fixes *x* :: *'a::{real-normed-field,banach}*
assumes *1*: *summable* ($\lambda n.$ *diffs* (*diffs c*) *n* $*$ *K* ^ *n*)
and *2*: *norm x* $<$ *norm K*
shows ($\lambda h.$ $\sum n.$ *c n* $*$ ((($x + h$) ^ *n* $-$ *x*^*n*) / *h* $-$ *of-nat n* $*$ *x* ^(*n* $-$ *Suc 0*)))
$-0\rightarrow$ *0*
proof $-$
from *dense* [*OF 2*] **obtain** *r* **where** *r1*: *norm x* $<$ *r* **and** *r2*: *r* $<$ *norm K*

by *fast*
from *norm-ge-zero r1* **have** *r*: $0 < r$
by (*rule order-le-less-trans*)
then have *r-neq-0*: $r \neq 0$ **by** *simp*
show *?thesis*
proof (*rule lemma-termdiff5*)
show $0 < r - norm\ x$
using *r1* **by** *simp*
from *r r2* **have** *norm* (*of-real r*::*'a*) < *norm K*
by *simp*
with *1* **have** *summable* (λ*n*. *norm* (*diffs* (*diffs c*) *n* ∗ (*of-real r* ^ *n*)))
by (*rule powser-insidea*)
then have *summable* (λ*n*. *diffs* (*diffs* (λ*n*. *norm* (*c n*))) *n* ∗ *r* ^ *n*)
using *r* **by** (*simp add*: *diffs-def norm-mult norm-power del*: *of-nat-Suc*)
then have *summable* (λ*n*. *of-nat n* ∗ *diffs* (λ*n*. *norm* (*c n*)) *n* ∗ *r* ^ (*n* − *Suc 0*))
by (*rule diffs-equiv* [*THEN sums-summable*])
also have (λ*n*. *of-nat n* ∗ *diffs* (λ*n*. *norm* (*c n*)) *n* ∗ *r* ^ (*n* − *Suc 0*)) =
(λ*n*. *diffs* (λ*m*. *of-nat* (*m* − *Suc 0*) ∗ *norm* (*c m*) ∗ *inverse r*) *n* ∗ (*r* ^ *n*))
by (*simp add*: *diffs-def r-neq-0 fun-eq-iff split*: *nat-diff-split*)
finally have *summable*
(λ*n*. *of-nat n* ∗ (*of-nat* (*n* − *Suc 0*) ∗ *norm* (*c n*) ∗ *inverse r*) ∗ *r* ^ (*n* − *Suc 0*))
by (*rule diffs-equiv* [*THEN sums-summable*])
also have
(λ*n*. *of-nat n* ∗ (*of-nat* (*n* − *Suc 0*) ∗ *norm* (*c n*) ∗ *inverse r*) ∗ *r* ^ (*n* − *Suc 0*)) =
(λ*n*. *norm* (*c n*) ∗ *of-nat n* ∗ *of-nat* (*n* − *Suc 0*) ∗ *r* ^ (*n* − *2*))
by (*rule ext*) (*simp add*: *r-neq-0 split*: *nat-diff-split*)
finally show *summable* (λ*n*. *norm* (*c n*) ∗ *of-nat n* ∗ *of-nat* (*n* − *Suc 0*) ∗ *r* ^ (*n* − *2*)) **.**
next
fix *h* :: *'a* **and** *n*
assume *h*: $h \neq 0$
assume *norm h* < *r* − *norm x*
then have *norm x* + *norm h* < *r* **by** *simp*
with *norm-triangle-ineq*
have *xh*: *norm* (*x* + *h*) < *r*
by (*rule order-le-less-trans*)
have *norm* (((*x* + *h*) ^ *n* − *x* ^ *n*) / *h* − *of-nat n* ∗ *x* ^ (*n* − *Suc 0*))
≤ *real n* ∗ (*real* (*n* − *Suc 0*) ∗ (*r* ^ (*n* − *2*) ∗ *norm h*))
by (*metis* (*mono-tags*, *lifting*) *h mult.assoc lemma-termdiff3 less-eq-real-def r1 xh*)
then show *norm* (*c n* ∗ (((*x* + *h*) ^ *n* − *x*^*n*) / *h* − *of-nat n* ∗ *x* ^ (*n* − *Suc 0*))) ≤
norm (*c n*) ∗ *of-nat n* ∗ *of-nat* (*n* − *Suc 0*) ∗ *r* ^ (*n* − *2*) ∗ *norm h*
by (*simp only*: *norm-mult mult.assoc mult-left-mono* [*OF* - *norm-ge-zero*])
qed

qed

lemma *termdiffs*:
  fixes $K\ x$ :: $'a$::{*real-normed-field*,*banach*}
  assumes *1*: *summable* $(\lambda n.\ c\ n * K\ \hat{}\ n)$
    and *2*: *summable* $(\lambda n.\ (diffs\ c)\ n * K\ \hat{}\ n)$
    and *3*: *summable* $(\lambda n.\ (diffs\ (diffs\ c))\ n * K\ \hat{}\ n)$
    and *4*: $norm\ x < norm\ K$
  shows $DERIV\ (\lambda x.\ \sum n.\ c\ n * x\hat{}n)\ x :> (\sum n.\ (diffs\ c)\ n * x\hat{}n)$
  unfolding *DERIV-def*
proof (*rule LIM-zero-cancel*)
  show $(\lambda h.\ (suminf\ (\lambda n.\ c\ n * (x + h)\ \hat{}\ n) - suminf\ (\lambda n.\ c\ n * x\hat{}n))\ /\ h$
    $- suminf\ (\lambda n.\ diffs\ c\ n * x\hat{}n))\ -0\rightarrow\ 0$
  proof (*rule LIM-equal2*)
    show $0 < norm\ K - norm\ x$
      using *4* **by** (*simp add*: *less-diff-eq*)
  next
    fix h :: $'a$
    assume $norm\ (h - 0) < norm\ K - norm\ x$
    then have $norm\ x + norm\ h < norm\ K$ **by** *simp*
    then have *5*: $norm\ (x + h) < norm\ K$
      by (*rule norm-triangle-ineq* [*THEN order-le-less-trans*])
    have *summable* $(\lambda n.\ c\ n * x\hat{}n)$
      and *summable* $(\lambda n.\ c\ n * (x + h)\ \hat{}\ n)$
      and *summable* $(\lambda n.\ diffs\ c\ n * x\hat{}n)$
      using *1 2 4 5* **by** (*auto elim*: *powser-inside*)
    then have $((\sum n.\ c\ n * (x + h)\ \hat{}\ n) - (\sum n.\ c\ n * x\hat{}n))\ /\ h - (\sum n.\ diffs\ c\ n * x\hat{}n) =$
      $(\sum n.\ (c\ n * (x + h)\ \hat{}\ n - c\ n * x\hat{}n)\ /\ h - of\text{-}nat\ n * c\ n * x\ \hat{}\ (n - Suc\ 0))$
      by (*intro sums-unique sums-diff sums-divide diffs-equiv summable-sums*)
    then show $((\sum n.\ c\ n * (x + h)\ \hat{}\ n) - (\sum n.\ c\ n * x\hat{}n))\ /\ h - (\sum n.\ diffs\ c\ n * x\hat{}n) =$
      $(\sum n.\ c\ n * (((x + h)\ \hat{}\ n - x\hat{}n)\ /\ h - of\text{-}nat\ n * x\ \hat{}\ (n - Suc\ 0)))$
      by (*simp add*: *algebra-simps*)
  next
    show $(\lambda h.\ \sum n.\ c\ n * (((x + h)\ \hat{}\ n - x\hat{}n)\ /\ h - of\text{-}nat\ n * x\ \hat{}\ (n - Suc\ 0)))\ -0\rightarrow\ 0$
      by (*rule termdiffs-aux* [*OF 3 4*])
  qed
qed

112.4 The Derivative of a Power Series Has the Same Radius of Convergence

lemma *termdiff-converges*:
  fixes x :: $'a$::{*real-normed-field*,*banach*}
  assumes *K*: $norm\ x < K$
    and *sm*: $\bigwedge x.\ norm\ x < K \Longrightarrow summable(\lambda n.\ c\ n * x\ \hat{}\ n)$

shows *summable* (λ*n*. *diffs c n* ∗ *x* ^ *n*)
proof (*cases x* = *0*)
case *True*
then show *?thesis*
using *powser-sums-zero sums-summable* **by** *auto*
next
case *False*
then have *K* > *0*
using *K less-trans zero-less-norm-iff* **by** *blast*
then obtain *r* :: *real* **where** *r*: *norm x* < *norm r norm r* < *K r* > *0*
using *K False*
by (*auto simp*: *field-simps abs-less-iff add-pos-pos intro*: *that* [*of* (*norm x* + *K*) / *2*])
have *to0*: (λ*n*. *of-nat n* ∗ (*x* / *of-real r*) ^ *n*) ⟶ *0*
using *r* **by** (*simp add*: *norm-divide powser-times-n-limit-0* [*of x* / *of-real r*])
obtain *N* **where** *N*: ⋀*n*. *n*≥*N* ⟹ *real-of-nat n* ∗ *norm x* ^ *n* < *r* ^ *n*
using *r LIMSEQ-D* [*OF to0*, *of 1*]
by (*auto simp*: *norm-divide norm-mult norm-power field-simps*)
have *summable* (λ*n*. (*of-nat n* ∗ *c n*) ∗ *x* ^ *n*)
proof (*rule summable-comparison-test'*)
show *summable* (λ*n*. *norm* (*c n* ∗ *of-real r* ^ *n*))
apply (*rule powser-insidea* [*OF sm* [*of of-real* ((*r*+*K*)/*2*)]])
using *N r norm-of-real* [*of r* + *K*, **where** *'a* = *'a*] **by** *auto*
show ⋀*n*. *N* ≤ *n* ⟹ *norm* (*of-nat n* ∗ *c n* ∗ *x* ^ *n*) ≤ *norm* (*c n* ∗ *of-real r* ^ *n*)
using *N r* **by** (*fastforce simp add*: *norm-mult norm-power less-eq-real-def*)
qed
then have *summable* (λ*n*. (*of-nat* (*Suc n*) ∗ *c*(*Suc n*)) ∗ *x* ^ *Suc n*)
using *summable-iff-shift* [*of* λ*n*. *of-nat n* ∗ *c n* ∗ *x* ^ *n 1*]
by *simp*
then have *summable* (λ*n*. (*of-nat* (*Suc n*) ∗ *c*(*Suc n*)) ∗ *x* ^ *n*)
using *False summable-mult2* [*of* λ*n*. (*of-nat* (*Suc n*) ∗ *c*(*Suc n*) ∗ *x* ^ *n*) ∗ *x inverse x*]
by (*simp add*: *mult.assoc*) (*auto simp*: *ac-simps*)
then show *?thesis*
by (*simp add*: *diffs-def*)
qed

lemma *termdiff-converges-all*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
assumes ⋀*x*. *summable* (λ*n*. *c n* ∗ *x*^*n*)
shows *summable* (λ*n*. *diffs c n* ∗ *x*^*n*)
by (*rule termdiff-converges* [**where** *K* = *1* + *norm x*]) (*use assms* **in** *auto*)

lemma *termdiffs-strong*:
fixes *K x* :: *'a*::{*real-normed-field*,*banach*}
assumes *sm*: *summable* (λ*n*. *c n* ∗ *K* ^ *n*)
and *K*: *norm x* < *norm K*
shows *DERIV* (λ*x*. ∑*n*. *c n* ∗ *x*^*n*) *x* :> (∑*n*. *diffs c n* ∗ *x*^*n*)

proof –
  have *norm K* + *norm x* < *norm K* + *norm K*
    using *K* **by** *force*
  then have *K2*: *norm* ((*of-real* (*norm K*) + *of-real* (*norm x*)) / *2* :: *'a*) < *norm K*
    by (*auto simp*: *norm-triangle-lt norm-divide field-simps*)
  then have [*simp*]: *norm* ((*of-real* (*norm K*) + *of-real* (*norm x*)) :: *'a*) < *norm K* ∗ *2*
    by *simp*
  have *summable* (λ*n*. *c n* ∗ (*of-real* (*norm x* + *norm K*) / *2*) ^ *n*)
   by (*metis K2 summable-norm-cancel* [*OF powser-insidea* [*OF sm*]] *add.commute of-real-add*)
  moreover have ⋀*x*. *norm x* < *norm K* ⟹ *summable* (λ*n*. *diffs c n* ∗ *x* ^ *n*)
    by (*blast intro*: *sm termdiff-converges powser-inside*)
  moreover have ⋀*x*. *norm x* < *norm K* ⟹ *summable* (λ*n*. *diffs*(*diffs c*) *n* ∗ *x* ^ *n*)
    by (*blast intro*: *sm termdiff-converges powser-inside*)
  ultimately show *?thesis*
    by (*rule termdiffs* [**where** *K* = *of-real* (*norm x* + *norm K*) / *2*])
      (*use K* **in** ‹*auto simp*: *field-simps simp flip*: *of-real-add*›)
qed

lemma *termdiffs-strong-converges-everywhere*:
  fixes *K x* :: *'a*::{*real-normed-field*,*banach*}
  assumes ⋀*y*. *summable* (λ*n*. *c n* ∗ *y* ^ *n*)
  shows ((λ*x*. ∑*n*. *c n* ∗ *x*^*n*) *has-field-derivative* (∑*n*. *diffs c n* ∗ *x*^*n*)) (*at x*)
  using *termdiffs-strong*[*OF assms*[*of of-real* (*norm x* + *1*)], *of x*]
  by (*force simp del*: *of-real-add*)

lemma *termdiffs-strong'*:
  fixes *z* :: *'a* :: {*real-normed-field*,*banach*}
  assumes ⋀*z*. *norm z* < *K* ⟹ *summable* (λ*n*. *c n* ∗ *z* ^ *n*)
  assumes *norm z* < *K*
  shows ((λ*z*. ∑*n*. *c n* ∗ *z*^*n*) *has-field-derivative* (∑*n*. *diffs c n* ∗ *z*^*n*)) (*at z*)
proof (*rule termdiffs-strong*)
  define *L* :: *real* **where** *L* = (*norm z* + *K*) / *2*
  have *0* ≤ *norm z* **by** *simp*
  also note ‹*norm z* < *K*›
  finally have *K*: *K* ≥ *0* **by** *simp*
  from *assms K* **have** *L*: *L* ≥ *0* *norm z* < *L* *L* < *K* **by** (*simp-all add*: *L-def*)
  from *L* **show** *norm z* < *norm* (*of-real L* :: *'a*) **by** *simp*
  from *L* **show** *summable* (λ*n*. *c n* ∗ *of-real L* ^ *n*) **by** (*intro assms*(*1*)) *simp-all*
qed

lemma *termdiffs-sums-strong*:
  fixes *z* :: *'a* :: {*banach*,*real-normed-field*}
  assumes *sums*: ⋀*z*. *norm z* < *K* ⟹ (λ*n*. *c n* ∗ *z* ^ *n*) *sums f z*
  assumes *deriv*: (*f has-field-derivative f'*) (*at z*)
  assumes *norm*: *norm z* < *K*

shows $(\lambda n.\ diffs\ c\ n * z \widehat{\ } n)$ *sums* f'
proof –
have *summable*: *summable* $(\lambda n.\ diffs\ c\ n * z \widehat{\ } n)$
by (*intro termdiff-converges*[*OF norm*] *sums-summable*[*OF sums*])
from *norm* **have** *eventually* $(\lambda z.\ z \in norm -`\ \{..<K\})$ (*nhds z*)
by (*intro eventually-nhds-in-open open-vimage*)
(*simp-all add*: *continuous-on-norm*)
hence *eq*: *eventually* $(\lambda z.\ (\sum n.\ c\ n * z \widehat{\ } n) = f\ z)$ (*nhds z*)
by *eventually-elim* (*insert sums*, *simp add*: *sums-iff*)

have $((\lambda z.\ \sum n.\ c\ n * z \widehat{\ } n)$ *has-field-derivative* $(\sum n.\ diffs\ c\ n * z \widehat{\ } n))$ (*at z*)
by (*intro termdiffs-strong*'[*OF - norm*] *sums-summable*[*OF sums*])
hence (*f has-field-derivative* $(\sum n.\ diffs\ c\ n * z \widehat{\ } n))$ (*at z*)
by (*subst* (*asm*) *DERIV-cong-ev*[*OF refl eq refl*])
from *this* **and** *deriv* **have** $(\sum n.\ diffs\ c\ n * z \widehat{\ } n) = f'$ **by** (*rule DERIV-unique*)
with *summable* **show** *?thesis* **by** (*simp add*: *sums-iff*)
qed

lemma *isCont-powser*:
fixes *K x* :: '*a*::{*real-normed-field*,*banach*}
assumes *summable* $(\lambda n.\ c\ n * K \widehat{\ } n)$
assumes *norm* $x <$ *norm* K
shows *isCont* $(\lambda x.\ \sum n.\ c\ n * x \widehat{\ } n)\ x$
using *termdiffs-strong*[*OF assms*] **by** (*blast intro*!: *DERIV-isCont*)

lemmas *isCont-powser'* = *isCont-o2*[*OF - isCont-powser*]

lemma *isCont-powser-converges-everywhere*:
fixes *K x* :: '*a*::{*real-normed-field*,*banach*}
assumes $\bigwedge y.$ *summable* $(\lambda n.\ c\ n * y \widehat{\ } n)$
shows *isCont* $(\lambda x.\ \sum n.\ c\ n * x \widehat{\ } n)\ x$
using *termdiffs-strong*[*OF assms*[*of of-real* (*norm* x + *1*)], *of x*]
by (*force intro*!: *DERIV-isCont simp del*: *of-real-add*)

lemma *powser-limit-0*:
fixes *a* :: *nat* ⇒ '*a*::{*real-normed-field*,*banach*}
assumes *s*: $0 < s$
and *sm*: $\bigwedge x.$ *norm* $x < s \Longrightarrow (\lambda n.\ a\ n * x \widehat{\ } n)$ *sums* $(f\ x)$
shows $(f \longrightarrow a\ 0)$ (*at 0*)
proof –
have *norm* (*of-real* s / *2* :: '*a*) $< s$
using *s* **by** (*auto simp*: *norm-divide*)
then have *summable* $(\lambda n.\ a\ n * ($*of-real* $s / 2) \widehat{\ } n)$
by (*rule sums-summable* [*OF sm*])
then have $((\lambda x.\ \sum n.\ a\ n * x \widehat{\ } n)$ *has-field-derivative* $(\sum n.\ diffs\ a\ n * 0 \widehat{\ } n))$
(*at 0*)
by (*rule termdiffs-strong*) (*use s* **in** ‹*auto simp*: *norm-divide*›)
then have *isCont* $(\lambda x.\ \sum n.\ a\ n * x \widehat{\ } n)\ 0$
by (*blast intro*: *DERIV-continuous*)

then have $((\lambda x. \sum n.\ a\ n * x \widehat{\ } n) \longrightarrow a\ 0)\ (at\ 0)$
by (*simp add*: *continuous-within*)
moreover have $(\lambda x. f\ x - (\sum n.\ a\ n * x \widehat{\ } n))\ -0\rightarrow\ 0$
apply (*clarsimp simp*: *LIM-eq*)
apply (*rule-tac x=s* **in** *exI*)
using *s sm sums-unique* **by** *fastforce*
ultimately show *?thesis*
by (*rule Lim-transform*)
qed

lemma *powser-limit-0-strong*:
fixes $a :: nat \Rightarrow 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes *s*: $0 < s$
and *sm*: $\bigwedge x.\ x \neq 0 \Longrightarrow norm\ x < s \Longrightarrow (\lambda n.\ a\ n * x \widehat{\ } n)\ sums\ (f\ x)$
shows $(f \longrightarrow a\ 0)\ (at\ 0)$
proof −
have ∗: $((\lambda x.\ if\ x = 0\ then\ a\ 0\ else\ f\ x) \longrightarrow a\ 0)\ (at\ 0)$
by (*rule powser-limit-0* [*OF s*]) (*auto simp*: *powser-sums-zero sm*)
show *?thesis*
using ∗ **by** (*auto cong*: *Lim-cong-within*)
qed

112.5 Derivability of power series

lemma *DERIV-series'*:
fixes $f :: real \Rightarrow nat \Rightarrow real$
assumes *DERIV-f*: $\bigwedge n.\ DERIV\ (\lambda\ x.\ f\ x\ n)\ x0 :> (f'\ x0\ n)$
and *allf-summable*: $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow summable\ (f\ x)$
and *x0-in-I*: $x0 \in \{a <..< b\}$
and *summable* $(f'\ x0)$
and *summable L*
and *L-def*: $\bigwedge n\ x\ y.\ x \in \{a <..< b\} \Longrightarrow y \in \{a <..< b\} \Longrightarrow |f\ x\ n - f\ y\ n| \leq L\ n * |x - y|$
shows $DERIV\ (\lambda\ x.\ suminf\ (f\ x))\ x0 :> (suminf\ (f'\ x0))$
unfolding *DERIV-def*
proof (*rule LIM-I*)
fix $r :: real$
assume $0 < r$ **then have** $0 < r/3$ **by** *auto*

obtain *N-L* **where** *N-L*: $\bigwedge n.\ N\text{-}L \leq n \Longrightarrow |\ \sum\ i.\ L\ (i + n)\ | < r/3$
using *suminf-exist-split*[*OF* ‹$0 < r/3$› ‹*summable L*›] **by** *auto*

obtain *N-f'* **where** *N-f'*: $\bigwedge n.\ N\text{-}f' \leq n \Longrightarrow |\ \sum\ i.\ f'\ x0\ (i + n)\ | < r/3$
using *suminf-exist-split*[*OF* ‹$0 < r/3$› ‹*summable* $(f'\ x0)$›] **by** *auto*

let $?N = Suc\ (max\ N\text{-}L\ N\text{-}f')$
have $|\ \sum\ i.\ f'\ x0\ (i + ?N)\ | < r/3$ (**is** $?f'\text{-}part < r/3$)
and *L-estimate*: $|\ \sum\ i.\ L\ (i + ?N)\ | < r/3$
using *N-L*[*of ?N*] **and** *N-f'* [*of ?N*] **by** *auto*

let *?diff* = *λi x*. (*f* (*x0* + *x*) *i* − *f x0 i*) / *x*

let *?r* = *r* / (*3* ∗ *real ?N*)
from ‹*0* < *r*› **have** *0* < *?r* **by** *simp*

let *?s* = *λn*. *SOME s*. *0* < *s* ∧ (∀ *x*. *x* ≠ *0* ∧ | *x* | < *s* ⟶ | *?diff n x* − *f′ x0 n* | < *?r*)
define *S′* **where** *S′* = *Min* (*?s* ‘ {..< *?N* })

have *0* < *S′*
unfolding *S′-def*
proof (*rule iffD2*[*OF Min-gr-iff*])
show ∀ *x* ∈ (*?s* ‘ {..< *?N* }). *0* < *x*
proof
fix *x*
assume *x* ∈ *?s* ‘ {..<*?N*}
then obtain *n* **where** *x* = *?s n* **and** *n* ∈ {..<*?N*}
using *image-iff*[*THEN iffD1*] **by** *blast*
from *DERIV-D*[*OF DERIV-f*[**where** *n*=*n*], *THEN LIM-D*, *OF* ‹*0* < *?r*›, *unfolded real-norm-def*]
obtain *s* **where** *s-bound*: *0* < *s* ∧ (∀ *x*. *x* ≠ *0* ∧ |*x*| < *s* ⟶ |*?diff n x* − *f′ x0 n*| < *?r*)
by *auto*
have *0* < *?s n*
by (*rule someI2*[**where** *a*=*s*]) (*auto simp*: *s-bound simp del*: *of-nat-Suc*)
then show *0* < *x* **by** (*simp only*: ‹*x* = *?s n*›)
qed
qed *auto*

define *S* **where** *S* = *min* (*min* (*x0* − *a*) (*b* − *x0*)) *S′*
then have *0* < *S* **and** *S-a*: *S* ≤ *x0* − *a* **and** *S-b*: *S* ≤ *b* − *x0*
and *S* ≤ *S′* **using** *x0-in-I* **and** ‹*0* < *S′*›
by *auto*

have |(*suminf* (*f* (*x0* + *x*)) − *suminf* (*f x0*)) / *x* − *suminf* (*f′ x0*)| < *r*
if *x* ≠ *0* **and** |*x*| < *S* **for** *x*
proof −
from *that* **have** *x-in-I*: *x0* + *x* ∈ {*a* <..< *b*}
using *S-a S-b* **by** *auto*

note *diff-smbl* = *summable-diff*[*OF allf-summable*[*OF x-in-I*] *allf-summable*[*OF x0-in-I*]]
note *div-smbl* = *summable-divide*[*OF diff-smbl*]
note *all-smbl* = *summable-diff*[*OF div-smbl* ‹*summable* (*f′ x0*)›]
note *ign* = *summable-ignore-initial-segment*[**where** *k*=*?N*]
note *diff-shft-smbl* = *summable-diff*[*OF ign*[*OF allf-summable*[*OF x-in-I*]] *ign*[*OF allf-summable*[*OF x0-in-I*]]]
note *div-shft-smbl* = *summable-divide*[*OF diff-shft-smbl*]

note *all-shft-smbl* = *summable-diff*[*OF div-smbl ign*[*OF* ‹*summable* (f' *x0*)›]]

have *1*: |(|*?diff* (n + *?N*) x|)| ≤ L (n + *?N*) **for** n
proof −
have |*?diff* (n + *?N*) x| ≤ L (n + *?N*) ∗ |(*x0* + x) − *x0*| / |x|
using *divide-right-mono*[*OF L-def*[*OF x-in-I x0-in-I*] *abs-ge-zero*]
by (*simp only*: *abs-divide*)
with ‹$x \neq 0$› **show** *?thesis* **by** *auto*
qed
note *2* = *summable-rabs-comparison-test*[*OF* - *ign*[*OF* ‹*summable L*›]]
from *1* **have** | $\sum$ *i*. *?diff* (i + *?N*) x | ≤ ($\sum$ *i*. L (i + *?N*))
by (*metis* (*lifting*) *abs-idempotent*
order-trans[*OF summable-rabs*[*OF 2*] *suminf-le*[*OF* - *2 ign*[*OF* ‹*summable* *L*›]]]])
then have |$\sum$ *i*. *?diff* (i + *?N*) x| ≤ r / *3* (**is** *?L-part* ≤ r/*3*)
using *L-estimate* **by** *auto*

have |$\sum$ *n*<*?N*. *?diff n x* − f' *x0 n*| ≤ ($\sum$ *n*<*?N*. |*?diff n x* − f' *x0 n*|) **..**
also have ... < ($\sum$ *n*<*?N*. *?r*)
proof (*rule sum-strict-mono*)
fix n
assume n ∈ {..< *?N*}
have |x| < S **using** ‹|x| < S› **.**
also have $S \leq S'$ **using** ‹$S \leq S'$› **.**
also have S' ≤ *?s n*
unfolding *S'-def*
proof (*rule Min-le-iff*[*THEN iffD2*])
have *?s n* ∈ (*?s* ‘ {..<*?N*}) ∧ *?s n* ≤ *?s n*
using ‹n ∈ {..< *?N*}› **by** *auto*
then show ∃ a ∈ (*?s* ‘ {..<*?N*}). a ≤ *?s n*
by *blast*
qed *auto*
finally have |x| < *?s n* **.**

from *DERIV-D*[*OF DERIV-f*[**where** *n*=*n*], *THEN LIM-D*, *OF* ‹*0* < *?r*›,
unfolded real-norm-def diff-0-right, *unfolded some-eq-ex*[*symmetric*], *THEN* *conjunct2*]
have ∀ *x*. $x \neq 0$ ∧ |x| < *?s n* ⟶ |*?diff n x* − f' *x0 n*| < *?r* **.**
with ‹$x \neq 0$› **and** ‹|x| < *?s n*› **show** |*?diff n x* − f' *x0 n*| < *?r*
by *blast*
qed *auto*
also have ... = *of-nat* (*card* {..<*?N*}) ∗ *?r*
by (*rule sum-constant*)
also have ... = *real ?N* ∗ *?r*
by *simp*
also have ... = r/*3*
by (*auto simp del*: *of-nat-Suc*)
finally have |$\sum$ *n*<*?N*. *?diff n x* − f' *x0 n* | < r / *3* (**is** *?diff-part* < r / *3*) **.**

from *suminf-diff*[*OF allf-summable*[*OF x-in-I*] *allf-summable*[*OF x0-in-I*]]
have |(*suminf* (*f* (*x0* + *x*)) − (*suminf* (*f x0*))) / *x* − *suminf* (*f′ x0*)| =
|∑ *n*. *?diff n x* − *f′ x0 n*|
unfolding *suminf-diff*[*OF div-smbl* ‹*summable* (*f′ x0*)›, *symmetric*]
using *suminf-divide*[*OF diff-smbl*, *symmetric*] **by** *auto*
also have ... ≤ *?diff-part* + |(∑ *n*. *?diff* (*n* + *?N*) *x*) − (∑ *n*. *f′ x0* (*n* + *?N*))|
unfolding *suminf-split-initial-segment*[*OF all-smbl*, **where** *k*=*?N*]
unfolding *suminf-diff*[*OF div-shft-smbl ign*[*OF* ‹*summable* (*f′ x0*)›]]
apply (*simp only*: *add.commute*)
using *abs-triangle-ineq* **by** *blast*
also have ... ≤ *?diff-part* + *?L-part* + *?f′-part*
using *abs-triangle-ineq4* **by** *auto*
also have ... < *r* /*3* + *r*/*3* + *r*/*3*
using ‹*?diff-part* < *r*/*3*› ‹*?L-part* ≤ *r*/*3*› **and** ‹*?f′-part* < *r*/*3*›
by (*rule add-strict-mono* [*OF add-less-le-mono*])
finally show *?thesis*
by *auto*
qed
then show ∃ *s* > *0*. ∀ *x*. *x* ≠ *0* ∧ *norm* (*x* − *0*) < *s* ⟶
norm (((∑ *n*. *f* (*x0* + *x*) *n*) − (∑ *n*. *f x0 n*)) / *x* − (∑ *n*. *f′ x0 n*)) < *r*
using ‹*0* < *S*› **by** *auto*
qed

lemma *DERIV-power-series′*:
fixes *f* :: *nat* ⇒ *real*
assumes *converges*: ⋀*x*. *x* ∈ {−*R* <..< *R*} ⟹ *summable* (λ*n*. *f n* ∗ *real* (*Suc n*) ∗ *x*^*n*)
and *x0-in-I*: *x0* ∈ {−*R* <..< *R*}
and *0* < *R*
shows *DERIV* (λ*x*. (∑ *n*. *f n* ∗ *x*^(*Suc n*))) *x0* :> (∑ *n*. *f n* ∗ *real* (*Suc n*) ∗ *x0*^*n*)
(**is** *DERIV* (λ*x*. *suminf* (*?f x*)) *x0* :> *suminf* (*?f′ x0*))
proof −
have *for-subinterval*: *DERIV* (λ*x*. *suminf* (*?f x*)) *x0* :> *suminf* (*?f′ x0*)
if *0* < *R′* **and** *R′* < *R* **and** −*R′* < *x0* **and** *x0* < *R′* **for** *R′*
proof −
from *that* **have** *x0* ∈ {−*R′* <..< *R′*} **and** *R′* ∈ {−*R* <..< *R*} **and** *x0* ∈ {−*R* <..< *R*}
by *auto*
show *?thesis*
proof (*rule DERIV-series′*)
show *summable* (λ *n*. |*f n* ∗ *real* (*Suc n*) ∗ *R′*^*n*|)
proof −
have (*R′* + *R*) / *2* < *R* **and** *0* < (*R′* + *R*) / *2*
using ‹*0* < *R′*› ‹*0* < *R*› ‹*R′* < *R*› **by** (*auto simp*: *field-simps*)
then have *in-Rball*: (*R′* + *R*) / *2* ∈ {−*R* <..< *R*}
using ‹*R′* < *R*› **by** *auto*
have *norm R′* < *norm* ((*R′* + *R*) / *2*)

using ‹$0 < R'$› ‹$0 < R$› ‹$R' < R$› **by** (*auto simp*: *field-simps*)
from *powser-insidea*[*OF converges*[*OF in-Rball*] *this*] **show** *?thesis*
by *auto*
qed
next
fix *n x y*
assume $x \in \{-R' <..< R'\}$ **and** $y \in \{-R' <..< R'\}$
show $|?f\ x\ n - ?f\ y\ n| \leq |f\ n * real\ (Suc\ n) * R'^{\frown}n| * |x-y|$
proof −
have $|f\ n * x\ ^{\frown}(Suc\ n) - f\ n * y\ ^{\frown}(Suc\ n)| =$
$(|f\ n| * |x-y|) * |\sum p{<}Suc\ n.\ x\ ^{\frown}\ p * y\ ^{\frown}(n - p)|$
unfolding *right-diff-distrib*[*symmetric*] *diff-power-eq-sum abs-mult*
by *auto*
also have ... $\leq (|f\ n| * |x-y|) * (|real\ (Suc\ n)| * |R'\ ^{\frown}\ n|)$
proof (*rule mult-left-mono*)
have $|\sum p{<}Suc\ n.\ x\ ^{\frown}\ p * y\ ^{\frown}(n - p)| \leq (\sum p{<}Suc\ n.\ |x\ ^{\frown}\ p * y\ ^{\frown}(n - p)|)$
by (*rule sum-abs*)
also have ... $\leq (\sum p{<}Suc\ n.\ R'\ ^{\frown}\ n)$
proof (*rule sum-mono*)
fix *p*
assume $p \in \{..{<}Suc\ n\}$
then have $p \leq n$ **by** *auto*
have $|x^{\frown}n| \leq R'^{\frown}n$ **if** $x \in \{-R'{<}..{<}R'\}$ **for** *n* **and** *x* :: *real*
proof −
from *that* **have** $|x| \leq R'$ **by** *auto*
then show *?thesis*
unfolding *power-abs* **by** (*rule power-mono*) *auto*
qed
from *mult-mono*[*OF this*[*OF* ‹$x \in \{-R'{<}..{<}R'\}$›, *of p*] *this*[*OF* ‹$y \in \{-R'{<}..{<}R'\}$›, *of n−p*]]
and ‹$0 < R'$›
have $|x^{\frown}p * y^{\frown}(n - p)| \leq R'^{\frown}p * R'^{\frown}(n - p)$
unfolding *abs-mult* **by** *auto*
then show $|x^{\frown}p * y^{\frown}(n - p)| \leq R'^{\frown}n$
unfolding *power-add*[*symmetric*] **using** ‹$p \leq n$› **by** *auto*
qed
also have ... $= real\ (Suc\ n) * R'\ ^{\frown}\ n$
unfolding *sum-constant card-atLeastLessThan* **by** *auto*
finally show $|\sum p{<}Suc\ n.\ x\ ^{\frown}\ p * y\ ^{\frown}(n - p)| \leq |real\ (Suc\ n)| * |R'\ ^{\frown}\ n|$
unfolding *abs-of-nonneg*[*OF zero-le-power*[*OF less-imp-le*[*OF* ‹$0 < R'$›]]]
by *linarith*
show $0 \leq |f\ n| * |x - y|$
unfolding *abs-mult*[*symmetric*] **by** *auto*
qed
also have ... $= |f\ n * real\ (Suc\ n) * R'\ ^{\frown}\ n| * |x - y|$
unfolding *abs-mult mult.assoc*[*symmetric*] **by** *algebra*
finally show *?thesis* **.**
qed

```
    next
      show DERIV (λx. ?f x n) x0 :> ?f' x0 n for n
        by (auto intro!: derivative-eq-intros simp del: power-Suc)
    next
      fix x
      assume x ∈ {−R' <..< R'}
      then have R' ∈ {−R <..< R} and norm x < norm R'
        using assms ‹R' < R› by auto
      have summable (λn. f n ∗ x^n)
      proof (rule summable-comparison-test, intro exI allI impI)
        fix n
        have le: |f n| ∗ 1 ≤ |f n| ∗ real (Suc n)
          by (rule mult-left-mono) auto
        show norm (f n ∗ x^n) ≤ norm (f n ∗ real (Suc n) ∗ x^n)
          unfolding real-norm-def abs-mult
          using le mult-right-mono by fastforce
      qed (rule powser-insidea[OF converges[OF ‹R' ∈ {−R <..< R}›] ‹norm x < norm R'›])
      from this[THEN summable-mult2[where c=x], simplified mult.assoc, simplified mult.commute]
      show summable (?f x) by auto
    next
      show summable (?f' x0)
        using converges[OF ‹x0 ∈ {−R <..< R}›] .
      show x0 ∈ {−R' <..< R'}
        using ‹x0 ∈ {−R' <..< R'}› .
    qed
  qed
  let ?R = (R + |x0|) / 2
  have |x0| < ?R
    using assms by (auto simp: field-simps)
  then have − ?R < x0
  proof (cases x0 < 0)
    case True
    then have − x0 < ?R
      using ‹|x0| < ?R› by auto
    then show ?thesis
      unfolding neg-less-iff-less[symmetric, of − x0] by auto
  next
    case False
    have − ?R < 0 using assms by auto
    also have . . . ≤ x0 using False by auto
    finally show ?thesis .
  qed
  then have 0 < ?R ?R < R − ?R < x0 and x0 < ?R
    using assms by (auto simp: field-simps)
  from for-subinterval[OF this] show ?thesis .
qed
```

lemma *geometric-deriv-sums*:
  fixes *z* :: *'a* :: {*real-normed-field*,*banach*}
  assumes *norm z* < *1*
  shows (λ*n*. *of-nat* (*Suc n*) * *z* ^ *n*) *sums* (*1* / (*1* − *z*)^*2*)
proof −
  have (λ*n*. *diffs* (λ*n*. *1*) *n* * *z*^*n*) *sums* (*1* / (*1* − *z*)^*2*)
  proof (*rule termdiffs-sums-strong*)
    fix *z* :: *'a* **assume** *norm z* < *1*
    thus (λ*n*. *1* * *z*^*n*) *sums* (*1* / (*1* − *z*)) **by** (*simp add*: *geometric-sums*)
  qed (*insert assms*, *auto intro*!: *derivative-eq-intros simp*: *power2-eq-square*)
  thus *?thesis* **unfolding** *diffs-def* **by** *simp*
qed

lemma *isCont-pochhammer* [*continuous-intros*]: *isCont* (λ*z*. *pochhammer z n*) *z*
  for *z* :: *'a*::*real-normed-field*
  by (*induct n*) (*auto simp*: *pochhammer-rec'*)

lemma *continuous-on-pochhammer* [*continuous-intros*]: *continuous-on A* (λ*z*. *pochhammer z n*)
  for *A* :: *'a*::*real-normed-field set*
  by (*intro continuous-at-imp-continuous-on ballI isCont-pochhammer*)

lemmas *continuous-on-pochhammer'* [*continuous-intros*] =
  continuous-on-compose2[*OF continuous-on-pochhammer - subset-UNIV*]

112.6 Exponential Function

definition *exp* :: *'a* ⇒ *'a*::{*real-normed-algebra-1*,*banach*}
  where *exp* = (λ*x*. ∑*n*. *x*^*n* /$_R$ *fact n*)

lemma *summable-exp-generic*:
  fixes *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
  defines *S-def*: *S* ≡ λ*n*. *x*^*n* /$_R$ *fact n*
  shows *summable S*
proof −
  have *S-Suc*: ⋀*n*. *S* (*Suc n*) = (*x* * *S n*) /$_R$ (*Suc n*)
    unfolding *S-def* **by** (*simp del*: *mult-Suc*)
  obtain *r* :: *real* **where** *r0*: *0* < *r* **and** *r1*: *r* < *1*
    using *dense* [*OF zero-less-one*] **by** *fast*
  obtain *N* :: *nat* **where** *N*: *norm x* < *real N* * *r*
    using *ex-less-of-nat-mult r0* **by** *auto*
  from *r1* **show** *?thesis*
  proof (*rule summable-ratio-test* [*rule-format*])
    fix *n* :: *nat*
    assume *n*: *N* ≤ *n*
    have *norm x* ≤ *real N* * *r*
      using *N* **by** (*rule order-less-imp-le*)
    also have *real N* * *r* ≤ *real* (*Suc n*) * *r*
      using *r0 n* **by** (*simp add*: *mult-right-mono*)

finally have *norm x ∗ norm (S n) ≤ real (Suc n) ∗ r ∗ norm (S n)*
using *norm-ge-zero* **by** (*rule mult-right-mono*)
then have *norm (x ∗ S n) ≤ real (Suc n) ∗ r ∗ norm (S n)*
by (*rule order-trans* [*OF norm-mult-ineq*])
then have *norm (x ∗ S n) / real (Suc n) ≤ r ∗ norm (S n)*
by (*simp add*: *pos-divide-le-eq ac-simps*)
then show *norm (S (Suc n)) ≤ r ∗ norm (S n)*
by (*simp add*: *S-Suc inverse-eq-divide*)
qed
qed

lemma *summable-norm-exp*: *summable (λn. norm (x^n /_R fact n))*
for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
proof (*rule summable-norm-comparison-test* [*OF exI*, *rule-format*])
show *summable (λn. norm x^n /$_R$ fact n)*
by (*rule summable-exp-generic*)
show *norm (x^n /$_R$ fact n) ≤ norm x^n /$_R$ fact n* **for** *n*
by (*simp add*: *norm-power-ineq*)
qed

lemma *summable-exp*: *summable (λn. inverse (fact n) ∗ x^n)*
for *x* :: *'a*::{*real-normed-field*,*banach*}
using *summable-exp-generic* [**where** *x=x*]
by (*simp add*: *scaleR-conv-of-real nonzero-of-real-inverse*)

lemma *exp-converges*: (*λn. x^n /$_R$ fact n*) *sums exp x*
unfolding *exp-def* **by** (*rule summable-exp-generic* [*THEN summable-sums*])

lemma *exp-fdiffs*:
diffs (λn. inverse (fact n)) = (λn. inverse (fact n :: *'a*::{*real-normed-field*,*banach*}))
by (*simp add*: *diffs-def mult-ac nonzero-inverse-mult-distrib nonzero-of-real-inverse*
del: *mult-Suc of-nat-Suc*)

lemma *diffs-of-real*: *diffs (λn. of-real (f n)) = (λn. of-real (diffs f n))*
by (*simp add*: *diffs-def*)

lemma *DERIV-exp* [*simp*]: *DERIV exp x :> exp x*
unfolding *exp-def scaleR-conv-of-real*
proof (*rule DERIV-cong*)
have *sinv*: *summable (λn. of-real (inverse (fact n)) ∗ x ^ n)* **for** *x*::*'a*
by (*rule exp-converges* [*THEN sums-summable*, *unfolded scaleR-conv-of-real*])
note *xx = exp-converges* [*THEN sums-summable*, *unfolded scaleR-conv-of-real*]
show ((*λx. ∑n. of-real (inverse (fact n)) ∗ x ^ n*) *has-field-derivative*
(*∑n. diffs (λn. of-real (inverse (fact n))) n ∗ x ^ n*)) (*at x*)
by (*rule termdiffs* [**where** *K=of-real (1 + norm x)*]) (*simp-all only*: *diffs-of-real exp-fdiffs sinv norm-of-real*)
show (*∑n. diffs (λn. of-real (inverse (fact n))) n ∗ x ^ n*) = (*∑n. of-real (inverse (fact n)) ∗ x ^ n*)
by (*simp add*: *diffs-of-real exp-fdiffs*)

qed

declare *DERIV-exp*[*THEN DERIV-chain2*, *derivative-intros*]
 and *DERIV-exp*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-exp*[*derivative-intros*] = *DERIV-exp*[*THEN DERIV-compose-FDERIV*]

lemma *norm-exp*: *norm* (*exp x*) ≤ *exp* (*norm x*)
proof −
 from *summable-norm*[*OF summable-norm-exp*, *of x*]
 have *norm* (*exp x*) ≤ (∑ *n*. *inverse* (*fact n*) ∗ *norm* (*x*^*n*))
  by (*simp add*: *exp-def*)
 also have . . . ≤ *exp* (*norm x*)
  using *summable-exp-generic*[*of norm x*] *summable-norm-exp*[*of x*]
  by (*auto simp*: *exp-def intro*!: *suminf-le norm-power-ineq*)
 finally show *?thesis* **.**
qed

lemma *isCont-exp*: *isCont exp x*
 for *x* :: *'a*::{*real-normed-field*,*banach*}
 by (*rule DERIV-exp* [*THEN DERIV-isCont*])

lemma *isCont-exp'* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *exp* (*f x*)) *a*
 for *f* :: - ⇒*'a*::{*real-normed-field*,*banach*}
 by (*rule isCont-o2* [*OF - isCont-exp*])

lemma *tendsto-exp* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *exp* (*f x*)) ⟶ *exp a*) *F*
 for *f*:: - ⇒*'a*::{*real-normed-field*,*banach*}
 by (*rule isCont-tendsto-compose* [*OF isCont-exp*])

lemma *continuous-exp* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *exp* (*f x*))
 for *f* :: - ⇒*'a*::{*real-normed-field*,*banach*}
 unfolding *continuous-def* **by** (*rule tendsto-exp*)

lemma *continuous-on-exp* [*continuous-intros*]: *continuous-on s f* ⟹ *continuous-on s* (λ*x*. *exp* (*f x*))
 for *f* :: - ⇒*'a*::{*real-normed-field*,*banach*}
 unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-exp*)

112.6.1 Properties of the Exponential Function

lemma *exp-zero* [*simp*]: *exp 0* = *1*
 unfolding *exp-def* **by** (*simp add*: *scaleR-conv-of-real*)

lemma *exp-series-add-commuting*:
 fixes *x y* :: *'a*::{*real-normed-algebra-1*,*banach*}

defines *S-def*: $S \equiv \lambda x\ n.\ x\hat{}n\ /_R\ fact\ n$
assumes *comm*: $x * y = y * x$
shows $S\ (x + y)\ n = (\sum i{\le}n.\ S\ x\ i * S\ y\ (n - i))$
proof (*induct n*)
case *0*
show *?case*
unfolding *S-def* **by** *simp*
next
case (*Suc n*)
have *S-Suc*: $\bigwedge x\ n.\ S\ x\ (Suc\ n) = (x * S\ x\ n)\ /_R\ real\ (Suc\ n)$
unfolding *S-def* **by** (*simp del*: *mult-Suc*)
then have *times-S*: $\bigwedge x\ n.\ x * S\ x\ n = real\ (Suc\ n) *_R S\ x\ (Suc\ n)$
by *simp*
have *S-comm*: $\bigwedge n.\ S\ x\ n * y = y * S\ x\ n$
by (*simp add*: *power-commuting-commutes comm S-def*)

have $real\ (Suc\ n) *_R S\ (x + y)\ (Suc\ n) = (x + y) * (\sum i{\le}n.\ S\ x\ i * S\ y\ (n - i))$
by (*metis Suc.hyps times-S*)
also have $\ldots = x * (\sum i{\le}n.\ S\ x\ i * S\ y\ (n - i)) + y * (\sum i{\le}n.\ S\ x\ i * S\ y\ (n - i))$
by (*rule distrib-right*)
also have $\ldots = (\sum i{\le}n.\ x * S\ x\ i * S\ y\ (n - i)) + (\sum i{\le}n.\ S\ x\ i * y * S\ y\ (n - i))$
by (*simp add*: *sum-distrib-left ac-simps S-comm*)
also have $\ldots = (\sum i{\le}n.\ x * S\ x\ i * S\ y\ (n - i)) + (\sum i{\le}n.\ S\ x\ i * (y * S\ y\ (n - i)))$
by (*simp add*: *ac-simps*)
also have $\ldots = (\sum i{\le}n.\ real\ (Suc\ i) *_R (S\ x\ (Suc\ i) * S\ y\ (n - i)))$
$+ (\sum i{\le}n.\ real\ (Suc\ n - i) *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
by (*simp add*: *times-S Suc-diff-le*)
also have $(\sum i{\le}n.\ real\ (Suc\ i) *_R (S\ x\ (Suc\ i) * S\ y\ (n - i)))$
$= (\sum i{\le}Suc\ n.\ real\ i *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
by (*subst sum.atMost-Suc-shift*) *simp*
also have $(\sum i{\le}n.\ real\ (Suc\ n - i) *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
$= (\sum i{\le}Suc\ n.\ real\ (Suc\ n - i) *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
by *simp*
also have $(\sum i{\le}Suc\ n.\ real\ i *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
$+ (\sum i{\le}Suc\ n.\ real\ (Suc\ n - i) *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
$= (\sum i{\le}Suc\ n.\ real\ (Suc\ n) *_R (S\ x\ i * S\ y\ (Suc\ n - i)))$
by (*simp flip*: *sum.distrib scaleR-add-left of-nat-add*)
also have $\ldots = real\ (Suc\ n) *_R (\sum i{\le}Suc\ n.\ S\ x\ i * S\ y\ (Suc\ n - i))$
by (*simp only*: *scaleR-right.sum*)
finally show $S\ (x + y)\ (Suc\ n) = (\sum i{\le}Suc\ n.\ S\ x\ i * S\ y\ (Suc\ n - i))$
by (*simp del*: *sum.cl-ivl-Suc*)
qed

lemma *exp-add-commuting*: $x * y = y * x \Longrightarrow exp\ (x + y) = exp\ x * exp\ y$
by (*simp only*: *exp-def Cauchy-product summable-norm-exp exp-series-add-commuting*)

lemma *exp-times-arg-commute*: *exp A* ∗ *A* = *A* ∗ *exp A*
by (*simp add*: *exp-def suminf-mult*[*symmetric*] *summable-exp-generic power-commutes suminf-mult2*)

lemma *exp-add*: *exp* (*x* + *y*) = *exp x* ∗ *exp y*
for *x y* :: *'a*::{*real-normed-field*,*banach*}
by (*rule exp-add-commuting*) (*simp add*: *ac-simps*)

lemma *exp-double*: *exp*(*2* ∗ *z*) = *exp z* ^ *2*
by (*simp add*: *exp-add-commuting mult-2 power2-eq-square*)

lemmas *mult-exp-exp* = *exp-add* [*symmetric*]

lemma *exp-of-real*: *exp* (*of-real x*) = *of-real* (*exp x*)
unfolding *exp-def*
apply (*subst suminf-of-real* [*OF summable-exp-generic*])
apply (*simp add*: *scaleR-conv-of-real*)
done

lemmas *of-real-exp* = *exp-of-real*[*symmetric*]

corollary *exp-in-Reals* [*simp*]: *z* ∈ ℝ ⟹ *exp z* ∈ ℝ
by (*metis Reals-cases Reals-of-real exp-of-real*)

lemma *exp-not-eq-zero* [*simp*]: *exp x* ≠ *0*
proof
have *exp x* ∗ *exp* (− *x*) = *1*
by (*simp add*: *exp-add-commuting*[*symmetric*])
also assume *exp x* = *0*
finally show *False* **by** *simp*
qed

lemma *exp-minus-inverse*: *exp x* ∗ *exp* (− *x*) = *1*
by (*simp add*: *exp-add-commuting*[*symmetric*])

lemma *exp-minus*: *exp* (− *x*) = *inverse* (*exp x*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*intro inverse-unique* [*symmetric*] *exp-minus-inverse*)

lemma *exp-diff*: *exp* (*x* − *y*) = *exp x* / *exp y*
for *x* :: *'a*::{*real-normed-field*,*banach*}
using *exp-add* [*of x* − *y*] **by** (*simp add*: *exp-minus divide-inverse*)

lemma *exp-of-nat-mult*: *exp* (*of-nat n* ∗ *x*) = *exp x* ^ *n*
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*induct n*) (*auto simp*: *distrib-left exp-add mult.commute*)

corollary *exp-of-nat2-mult*: *exp* (*x* ∗ *of-nat n*) = *exp x* ^ *n*

for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*metis exp-of-nat-mult mult-of-nat-commute*)

lemma *exp-sum*: *finite I* $\Longrightarrow$ *exp* (*sum f I*) = *prod* (λx. *exp* (*f x*)) *I*
by (*induct I rule*: *finite-induct*) (*auto simp*: *exp-add-commuting mult.commute*)

lemma *exp-divide-power-eq*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
assumes $n > 0$
shows *exp* (*x* / *of-nat n*) ^ *n* = *exp x*
using *assms*
proof (*induction n arbitrary*: *x*)
case (*Suc n*)
show *?case*
proof (*cases n* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have [*simp*]: *1* + (*of-nat n* ∗ *of-nat n* + *of-nat n* ∗ *2*) ≠ (*0*::*'a*)
using *of-nat-eq-iff* [*of 1* + *n* ∗ *n* + *n* ∗ *2 0*]
by *simp*
from *False* **have** [*simp*]: *x* ∗ *of-nat n* / (*1* + *of-nat n*) / *of-nat n* = *x* / (*1* + *of-nat n*)
by *simp*
have [*simp*]: *x* / (*1* + *of-nat n*) + *x* ∗ *of-nat n* / (*1* + *of-nat n*) = *x*
using *of-nat-neq-0*
by (*auto simp add*: *field-split-simps*)
show *?thesis*
using *Suc.IH* [*of x* ∗ *of-nat n* / (*1* + *of-nat n*)] *False*
by (*simp add*: *exp-add* [*symmetric*])
qed
qed *simp*

lemma *exp-power-int*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows *exp x powi n* = *exp* (*of-int n* ∗ *x*)
proof (*cases* $n \geq 0$)
case *True*
have *exp x powi n* = *exp x* ^ *nat n*
using *True* **by** (*simp add*: *power-int-def*)
thus *?thesis*
using *True* **by** (*subst* (*asm*) *exp-of-nat-mult* [*symmetric*]) *auto*
next
case *False*
have *exp x powi n* = *inverse* (*exp x* ^ *nat* (−*n*))
using *False* **by** (*simp add*: *power-int-def field-simps*)
also have *exp x* ^ *nat* (−*n*) = *exp* (*of-nat* (*nat* (−*n*)) ∗ *x*)
using *False* **by** (*subst exp-of-nat-mult*) *auto*

also have *inverse* ... = *exp* (−(*of-nat* (*nat* (−*n*)) ∗ *x*))
  by (*subst exp-minus*) (*auto simp*: *field-simps*)
also have −(*of-nat* (*nat* (−*n*)) ∗ *x*) = *of-int n* ∗ *x*
  using *False* **by** *simp*
finally show *?thesis* **.**
qed

112.6.2 Properties of the Exponential Function on Reals

Comparisons of *exp x* with zero.

Proof: because every exponential can be seen as a square.

lemma *exp-ge-zero* [*simp*]: *0* ≤ *exp x*
  for *x* :: *real*
proof −
  have *0* ≤ *exp* (*x*/*2*) ∗ *exp* (*x*/*2*)
    by *simp*
  then show *?thesis*
    by (*simp add*: *exp-add* [*symmetric*])
qed

lemma *exp-gt-zero* [*simp*]: *0* < *exp x*
  for *x* :: *real*
  by (*simp add*: *order-less-le*)

lemma *not-exp-less-zero* [*simp*]: ¬ *exp x* < *0*
  for *x* :: *real*
  by (*simp add*: *not-less*)

lemma *not-exp-le-zero* [*simp*]: ¬ *exp x* ≤ *0*
  for *x* :: *real*
  by (*simp add*: *not-le*)

lemma *abs-exp-cancel* [*simp*]: |*exp x*| = *exp x*
  for *x* :: *real*
  by *simp*

Strict monotonicity of exponential.

lemma *exp-ge-add-one-self-aux*:
  fixes *x* :: *real*
  assumes *0* ≤ *x*
  shows *1* + *x* ≤ *exp x*
  using *order-le-imp-less-or-eq* [*OF assms*]
proof
  assume *0* < *x*
  have *1* + *x* ≤ (∑ *n*<*2*. *inverse* (*fact n*) ∗ *x*^*n*)
    by (*auto simp*: *numeral-2-eq-2*)
  also have ... ≤ (∑ *n*. *inverse* (*fact n*) ∗ *x*^*n*)

using ‹*0* < *x*› **by** (*auto simp add*: *zero-le-mult-iff intro*: *sum-le-suminf* [*OF summable-exp*])
finally show *1* + *x* ≤ *exp x*
by (*simp add*: *exp-def*)
qed *auto*

lemma *exp-gt-one*: *0* < *x* ⟹ *1* < *exp x*
for *x* :: *real*
proof −
assume *x*: *0* < *x*
then have *1* < *1* + *x* **by** *simp*
also from *x* **have** *1* + *x* ≤ *exp x*
by (*simp add*: *exp-ge-add-one-self-aux*)
finally show *?thesis* **.**
qed

lemma *exp-less-mono*:
fixes *x y* :: *real*
assumes *x* < *y*
shows *exp x* < *exp y*
proof −
from ‹*x* < *y*› **have** *0* < *y* − *x* **by** *simp*
then have *1* < *exp* (*y* − *x*) **by** (*rule exp-gt-one*)
then have *1* < *exp y* / *exp x* **by** (*simp only*: *exp-diff*)
then show *exp x* < *exp y* **by** *simp*
qed

lemma *exp-less-cancel*: *exp x* < *exp y* ⟹ *x* < *y*
for *x y* :: *real*
unfolding *linorder-not-le* [*symmetric*]
by (*auto simp*: *order-le-less exp-less-mono*)

lemma *exp-less-cancel-iff* [*iff*]: *exp x* < *exp y* ⟷ *x* < *y*
for *x y* :: *real*
by (*auto intro*: *exp-less-mono exp-less-cancel*)

lemma *exp-le-cancel-iff* [*iff*]: *exp x* ≤ *exp y* ⟷ *x* ≤ *y*
for *x y* :: *real*
by (*auto simp*: *linorder-not-less* [*symmetric*])

lemma *exp-mono*:
fixes *x y* :: *real*
assumes *x* ≤ *y*
shows *exp x* ≤ *exp y*
using *assms exp-le-cancel-iff* **by** *fastforce*

lemma *exp-minus′*: *exp* (−*x*) = *1* / (*exp x*)
for *x* :: *′a*::{*real-normed-field*,*banach*}
by (*simp add*: *exp-minus inverse-eq-divide*)

lemma *exp-inj-iff* [*iff*]: *exp x* = *exp y* ⟷ *x* = *y*
  for *x y* :: *real*
  by (*simp add*: *order-eq-iff*)

Comparisons of *exp x* with one.

lemma *one-less-exp-iff* [*simp*]: *1* < *exp x* ⟷ *0* < *x*
  for *x* :: *real*
  using *exp-less-cancel-iff* [**where** *x* = *0* **and** *y* = *x*] **by** *simp*

lemma *exp-less-one-iff* [*simp*]: *exp x* < *1* ⟷ *x* < *0*
  for *x* :: *real*
  using *exp-less-cancel-iff* [**where** *x* = *x* **and** *y* = *0*] **by** *simp*

lemma *one-le-exp-iff* [*simp*]: *1* ≤ *exp x* ⟷ *0* ≤ *x*
  for *x* :: *real*
  using *exp-le-cancel-iff* [**where** *x* = *0* **and** *y* = *x*] **by** *simp*

lemma *exp-le-one-iff* [*simp*]: *exp x* ≤ *1* ⟷ *x* ≤ *0*
  for *x* :: *real*
  using *exp-le-cancel-iff* [**where** *x* = *x* **and** *y* = *0*] **by** *simp*

lemma *exp-eq-one-iff* [*simp*]: *exp x* = *1* ⟷ *x* = *0*
  for *x* :: *real*
  using *exp-inj-iff* [**where** *x* = *x* **and** *y* = *0*] **by** *simp*

lemma *lemma-exp-total*: *1* ≤ *y* ⟹ ∃ *x*. *0* ≤ *x* ∧ *x* ≤ *y* − *1* ∧ *exp x* = *y*
  for *y* :: *real*
proof (*rule IVT*)
  assume *1* ≤ *y*
  then have *0* ≤ *y* − *1* **by** *simp*
  then have *1* + (*y* − *1*) ≤ *exp* (*y* − *1*)
    by (*rule exp-ge-add-one-self-aux*)
  then show *y* ≤ *exp* (*y* − *1*) **by** *simp*
qed (*simp-all add*: *le-diff-eq*)

lemma *exp-total*: *0* < *y* ⟹ ∃ *x*. *exp x* = *y*
  for *y* :: *real*
proof (*rule linorder-le-cases* [*of 1 y*])
  assume *1* ≤ *y*
  then show ∃ *x*. *exp x* = *y*
    by (*fast dest*: *lemma-exp-total*)
next
  assume *0* < *y* **and** *y* ≤ *1*
  then have *1* ≤ *inverse y*
    by (*simp add*: *one-le-inverse-iff*)
  then obtain *x* **where** *exp x* = *inverse y*
    by (*fast dest*: *lemma-exp-total*)
  then have *exp* (− *x*) = *y*

by (*simp add*: *exp-minus*)
then show $\exists x.$ *exp* $x = y$ **..**
qed

112.7 Natural Logarithm

class *ln* = *real-normed-algebra-1* + *banach* +
fixes *ln* :: $'a \Rightarrow 'a$
assumes *ln-one* [*simp*]: *ln 1* = *0*

definition *powr* :: $'a \Rightarrow 'a \Rightarrow 'a$::*ln* (**infixr** ‹*powr*› *80*)
— exponentation via ln and exp
where *x powr a* ≡ *if x* = *0 then 0 else exp* (*a* ∗ *ln x*)

lemma *powr-0* [*simp*]: *0 powr z* = *0*
by (*simp add*: *powr-def*)

We totalise *ln* over all reals exactly as done in Mathlib

instantiation *real* :: *ln*
begin

definition *raw-ln-real* :: *real* ⇒ *real*
where *raw-ln-real x* ≡ (*THE u. exp u* = *x*)

definition *ln-real* :: *real* ⇒ *real*
where *ln-real* ≡ λ*x. if x*=*0 then 0 else raw-ln-real* |*x*|

instance
by *intro-classes* (*simp add*: *ln-real-def raw-ln-real-def*)

end

lemma *powr-eq-0-iff* [*simp*]: *w powr z* = *0* ⟷ *w* = *0*
by (*simp add*: *powr-def*)

lemma *raw-ln-exp* [*simp*]: *raw-ln-real* (*exp x*) = *x*
by (*simp add*: *raw-ln-real-def*)

lemma *exp-raw-ln* [*simp*]: *0* < *x* ⟹ *exp* (*raw-ln-real x*) = *x*
by (*auto dest*: *exp-total*)

lemma *raw-ln-unique*: *exp y* = *x* ⟹ *raw-ln-real x* = *y*
by *auto*

lemma *abs-raw-ln*: *x* ≠ *0* ⟹ *raw-ln-real*|*x*| = *ln x*
by (*simp add*: *ln-real-def*)

lemma *ln-0* [*simp*]: *ln* (*0*::*real*) = *0*
by (*simp add*: *ln-real-def*)

lemma *ln-minus*: *ln* (−*x*) = *ln x*
for *x* :: *real*
by (*simp add*: *ln-real-def*)

lemma *ln-exp* [*simp*]: *ln* (*exp x*) = *x*
for *x* :: *real*
by (*simp add*: *ln-real-def*)

lemma *exp-ln-abs*:
fixes *x*::*real*
shows $x \neq 0 \Longrightarrow exp\ (ln\ x) = |x|$
by (*simp add*: *ln-real-def*)

lemma *exp-ln* [*simp*]: $0 < x \Longrightarrow exp\ (ln\ x) = x$
for *x* :: *real*
using *exp-ln-abs* **by** *fastforce*

lemma *exp-ln-iff* [*simp*]: $exp\ (ln\ x) = x \longleftrightarrow 0 < x$
for *x* :: *real*
by (*metis exp-gt-zero exp-ln*)

lemma *ln-unique*: $exp\ y = x \Longrightarrow ln\ x = y$
for *x* :: *real*
by *auto*

lemma *ln-unique'*: $exp\ y = |x| \Longrightarrow ln\ x = y$
for *x* :: *real*
by (*metis abs-raw-ln abs-zero exp-not-eq-zero raw-ln-exp*)

lemma *raw-ln-mult*: $x>0 \Longrightarrow y>0 \Longrightarrow raw\text{-}ln\text{-}real\ (x * y) = raw\text{-}ln\text{-}real\ x + raw\text{-}ln\text{-}real\ y$
by (*metis exp-add exp-ln raw-ln-exp*)

lemma *ln-mult*: $ln\ (x * y) = (if\ x\neq 0 \wedge y\neq 0\ then\ ln\ x + ln\ y\ else\ 0)$
for *x* :: *real*
by (*simp add*: *ln-real-def abs-mult raw-ln-mult*)

lemma *ln-mult-pos*: $x>0 \Longrightarrow y>0 \Longrightarrow ln\ (x * y) = ln\ x + ln\ y$
for *x* :: *real*
by (*simp add*: *ln-mult*)

lemma *ln-prod*: $finite\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i \neq 0) \Longrightarrow ln\ (prod\ f\ I) = sum\ (\lambda x.\ ln(f\ x))\ I$
for f :: $'a \Rightarrow real$
by (*induct I rule*: *finite-induct*) (*auto simp*: *ln-mult prod-pos*)

lemma *ln-inverse*: *ln* (*inverse x*) = − *ln x*
for *x* :: *real*

by (*smt* (*verit*) *inverse-nonzero-iff-nonzero ln-mult ln-one ln-real-def right-inverse*)

lemma *ln-div*: $ln\ (x/y) = (if\ x{\neq}0 \wedge y{\neq}0\ then\ ln\ x - ln\ y\ else\ 0)$
for *x* :: *real*
by (*simp add*: *divide-inverse ln-inverse ln-mult*)

lemma *ln-divide-pos*: $x{>}0 \Longrightarrow y{>}0 \Longrightarrow ln\ (x/y) = ln\ x - ln\ y$
for *x* :: *real*
by (*simp add*: *divide-inverse ln-inverse ln-mult*)

lemma *ln-realpow*: $ln\ (x\hat{}n) = real\ n * ln\ x$
proof (*cases x=0*)
case *True*
then show *?thesis* **by** (*auto simp*: *power-0-left*)
next
case *False*
then show *?thesis*
by (*induction n*) (*auto simp*: *ln-mult distrib-right*)
qed

lemma *ln-less-cancel-iff* [*simp*]: $0 < x \Longrightarrow 0 < y \Longrightarrow ln\ x < ln\ y \longleftrightarrow x < y$
for *x* :: *real*
by (*subst exp-less-cancel-iff* [*symmetric*]) *simp*

lemma *ln-le-cancel-iff* [*simp*]: $0 < x \Longrightarrow 0 < y \Longrightarrow ln\ x \le ln\ y \longleftrightarrow x \le y$
for *x* :: *real*
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *ln-mono*: $\bigwedge x{::}real.\ [\![x \le y;\ 0 < x]\!] \Longrightarrow ln\ x \le ln\ y$
by *simp*

lemma *ln-strict-mono*: $\bigwedge x{::}real.\ [\![x < y;\ 0 < x]\!] \Longrightarrow ln\ x < ln\ y$
by *simp*

lemma *ln-inj-iff* [*simp*]: $0 < x \Longrightarrow 0 < y \Longrightarrow ln\ x = ln\ y \longleftrightarrow x = y$
for *x* :: *real*
by (*simp add*: *order-eq-iff*)

lemma *ln-add-one-self-le-self*: $0 \le x \Longrightarrow ln\ (1 + x) \le x$
for *x* :: *real*
by (*rule exp-le-cancel-iff* [*THEN iffD1*]) (*simp add*: *exp-ge-add-one-self-aux*)

lemma *ln-less-self* [*simp*]: $0 < x \Longrightarrow ln\ x < x$
for *x* :: *real*
by (*rule order-less-le-trans* [**where** $y = ln\ (1 + x)$]) (*simp-all add*: *ln-add-one-self-le-self*)

lemma *ln-ge-iff*: $\bigwedge x{::}real.\ 0 < x \Longrightarrow y \le ln\ x \longleftrightarrow exp\ y \le x$
using *exp-le-cancel-iff exp-total* **by** *force*

lemma *ln-ge-zero* [*simp*]: $1 \leq x \Longrightarrow 0 \leq ln\ x$
for x :: *real*
using *ln-le-cancel-iff* [*of 1 x*] **by** *simp*

lemma *ln-ge-zero-imp-ge-one*: $0 \leq ln\ x \Longrightarrow 0 < x \Longrightarrow 1 \leq x$
for x :: *real*
using *ln-le-cancel-iff* [*of 1 x*] **by** *simp*

lemma *ln-ge-zero-iff* [*simp*]: $0 < x \Longrightarrow 0 \leq ln\ x \longleftrightarrow 1 \leq x$
for x :: *real*
using *ln-le-cancel-iff* [*of 1 x*] **by** *simp*

lemma *ln-less-zero-iff* [*simp*]: $0 < x \Longrightarrow ln\ x < 0 \longleftrightarrow x < 1$
for x :: *real*
using *ln-less-cancel-iff* [*of x 1*] **by** *simp*

lemma *ln-le-zero-iff* [*simp*]: $0 < x \Longrightarrow ln\ x \leq 0 \longleftrightarrow x \leq 1$
for x :: *real*
by (*metis less-numeral-extra*(*1*) *ln-le-cancel-iff ln-one*)

lemma *ln-gt-zero*: $1 < x \Longrightarrow 0 < ln\ x$
for x :: *real*
using *ln-less-cancel-iff* [*of 1 x*] **by** *simp*

lemma *ln-gt-zero-imp-gt-one*: $0 < ln\ x \Longrightarrow 0 < x \Longrightarrow 1 < x$
for x :: *real*
using *ln-less-cancel-iff* [*of 1 x*] **by** *simp*

lemma *ln-gt-zero-iff* [*simp*]: $0 < x \Longrightarrow 0 < ln\ x \longleftrightarrow 1 < x$
for x :: *real*
using *ln-less-cancel-iff* [*of 1 x*] **by** *simp*

lemma *ln-eq-zero-iff* [*simp*]: $0 < x \Longrightarrow ln\ x = 0 \longleftrightarrow x = 1$
for x :: *real*
using *ln-inj-iff* [*of x 1*] **by** *simp*

lemma *ln-less-zero*: $0 < x \Longrightarrow x < 1 \Longrightarrow ln\ x < 0$
for x :: *real*
by *simp*

lemma *powr-eq-one-iff* [*simp*]:
$a\ powr\ x = 1 \longleftrightarrow x = 0$ **if** $a > 1$ **for** $a\ x$ :: *real*
using *that* **by** (*auto simp*: *powr-def split*: *if-splits*)

A consequence of our "totalising" of ln

lemma *uminus-powr-eq*: $(-a)\ powr\ x = a\ powr\ x$ **for** x::*real*
by (*simp add*: *powr-def ln-minus*)

lemma *isCont-ln-pos*:

fixes *x* :: *real*
assumes *x* > *0*
shows *isCont ln x*
by (*metis assms exp-ln isCont-exp isCont-inverse-function ln-exp*)

lemma *isCont-ln*:
fixes *x* :: *real*
assumes *x* ≠ *0*
shows *isCont ln x*
proof (*cases 0* < *x*)
case *False*
then have *isCont* (*ln o uminus*) *x*
using *isCont-minus* [*OF continuous-ident*] *assms continuous-at-compose is-Cont-ln-pos*
by *force*
then show *?thesis*
by (*simp add*: *comp-def ln-minus*)
qed (*simp add*: *isCont-ln-pos*)

lemma *tendsto-ln* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ *a* ≠ *0* ⟹ ((λ*x*. *ln* (*f x*)) ⟶ *ln a*) *F*
for *a* :: *real*
by (*rule isCont-tendsto-compose* [*OF isCont-ln*])

lemma *continuous-ln*:
continuous F f ⟹ *f* (*Lim F* (λ*x*. *x*)) ≠ *0* ⟹ *continuous F* (λ*x*. *ln* (*f x* :: *real*))
unfolding *continuous-def* **by** (*rule tendsto-ln*)

lemma *isCont-ln′* [*continuous-intros*]:
continuous (*at x*) *f* ⟹ *f x* ≠ *0* ⟹ *continuous* (*at x*) (λ*x*. *ln* (*f x* :: *real*))
unfolding *continuous-at* **by** (*rule tendsto-ln*)

lemma *continuous-within-ln* [*continuous-intros*]:
continuous (*at x within s*) *f* ⟹ *f x* ≠ *0* ⟹ *continuous* (*at x within s*) (λ*x*. *ln* (*f x* :: *real*))
unfolding *continuous-within* **by** (*rule tendsto-ln*)

lemma *continuous-on-ln* [*continuous-intros*]:
continuous-on s f ⟹ (∀ *x*∈*s*. *f x* ≠ *0*) ⟹ *continuous-on s* (λ*x*. *ln* (*f x* :: *real*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-ln*)

lemma *DERIV-ln*: *0* < *x* ⟹ *DERIV ln x* :> *inverse x*
for *x* :: *real*
by (*rule DERIV-inverse-function* [**where** *f*=*exp* **and** *a*=*0* **and** *b*=*x*+*1*])
(*auto intro*: *DERIV-cong* [*OF DERIV-exp exp-ln*] *isCont-ln*)

lemma *DERIV-ln-divide*: *0* < *x* ⟹ *DERIV ln x* :> *1*/*x*
for *x* :: *real*
by (*rule DERIV-ln*[*THEN DERIV-cong*]) (*simp-all add*: *divide-inverse*)

declare *DERIV-ln-divide*[*THEN DERIV-chain2*, *derivative-intros*]
  and *DERIV-ln-divide*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-ln*[*derivative-intros*] = *DERIV-ln*[*THEN DERIV-compose-FDERIV*]

lemma *ln-series*:
  assumes $0 < x$ **and** $x < 2$
  shows $ln\ x = (\sum\ n.\ (-1)\hat{}n * (1\ /\ real\ (n + 1)) * (x - 1)\hat{}(Suc\ n))$
    (**is** $ln\ x = suminf\ (?f\ (x - 1))$)
proof −
  let $?f' = \lambda x\ n.\ (-1)\hat{}n * (x - 1)\hat{}n$

  have $ln\ x - suminf\ (?f\ (x - 1)) = ln\ 1 - suminf\ (?f\ (1 - 1))$
  proof (*rule DERIV-isconst3* [**where** $x = x$])
    fix $x :: real$
    assume $x \in \{0 <..< 2\}$
    then have $0 < x$ **and** $x < 2$ **by** *auto*
    have $norm\ (1 - x) < 1$
      using ‹$0 < x$› **and** ‹$x < 2$› **by** *auto*
    have $1/x = 1\ /\ (1 - (1 - x))$ **by** *auto*
    also have $\ldots = (\sum\ n.\ (1 - x)\hat{}n)$
      using *geometric-sums*[*OF* ‹$norm\ (1 - x) < 1$›] **by** (*rule sums-unique*)
    also have $\ldots = suminf\ (?f'\ x)$
      unfolding *power-mult-distrib*[*symmetric*]
      by (*rule arg-cong*[**where** *f=suminf*], *rule arg-cong*[**where** *f=(^)*], *auto*)
    finally have $DERIV\ ln\ x :> suminf\ (?f'\ x)$
      using *DERIV-ln*[*OF* ‹$0 < x$›] **unfolding** *divide-inverse* **by** *auto*
    moreover
    have *repos*: $\bigwedge\ h\ x :: real.\ h - 1 + x = h + x - 1$ **by** *auto*
    have $DERIV\ (\lambda x.\ suminf\ (?f\ x))\ (x - 1) :>$
      $(\sum n.\ (-1)\hat{}n * (1\ /\ real\ (n + 1)) * real\ (Suc\ n) * (x - 1)\ \hat{}\ n)$
    proof (*rule DERIV-power-series'*)
      show $x - 1 \in \{- 1<..<1\}$ **and** $(0 :: real) < 1$
        using ‹$0 < x$› ‹$x < 2$› **by** *auto*
    next
      fix $x :: real$
      assume $x \in \{- 1<..<1\}$
      then show $summable\ (\lambda n.\ (- 1)\ \hat{}\ n * (1\ /\ real\ (n + 1)) * real\ (Suc\ n) * x\hat{}n)$
        by (*simp add*: *abs-if flip*: *power-mult-distrib*)
    qed
    then have $DERIV\ (\lambda x.\ suminf\ (?f\ x))\ (x - 1) :> suminf\ (?f'\ x)$
      unfolding *One-nat-def* **by** *auto*
    then have $DERIV\ (\lambda x.\ suminf\ (?f\ (x - 1)))\ x :> suminf\ (?f'\ x)$
      unfolding *DERIV-def repos* **.**
    ultimately have $DERIV\ (\lambda x.\ ln\ x - suminf\ (?f\ (x - 1)))\ x :> suminf\ (?f'\ x) - suminf\ (?f'\ x)$

by (*rule DERIV-diff*)
then show *DERIV* ($\lambda x.$ *ln x* − *suminf* (*?f* (*x* − *1*))) *x* :> *0* **by** *auto*
qed (*auto simp*: *assms*)
then show *?thesis* **by** *auto*
qed

lemma *exp-first-terms*:
fixes *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
shows *exp x* = ($\sum$ *n*<*k*. *inverse*(*fact n*) $*_R$ (*x* ^ *n*)) + ($\sum$ *n*. *inverse*(*fact* (*n* + *k*)) $*_R$ (*x* ^(*n* + *k*)))
proof −
have *exp x* = *suminf* ($\lambda n.$ *inverse*(*fact n*) $*_R$ (*x*^*n*))
by (*simp add*: *exp-def*)
also from *summable-exp-generic* **have** ... = ($\sum$ *n*. *inverse*(*fact*(*n*+*k*)) $*_R$ (*x* ^ (*n* + *k*))) +
($\sum$ *n*::*nat*<*k*. *inverse*(*fact n*) $*_R$ (*x*^*n*)) (**is** - = - + *?a*)
by (*rule suminf-split-initial-segment*)
finally show *?thesis* **by** *simp*
qed

lemma *exp-first-term*: *exp x* = *1* + ($\sum$ *n*. *inverse* (*fact* (*Suc n*)) $*_R$ (*x* ^ *Suc n*))
for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
using *exp-first-terms*[*of x 1*] **by** *simp*

lemma *exp-first-two-terms*: *exp x* = *1* + *x* + ($\sum$ *n*. *inverse* (*fact* (*n* + *2*)) $*_R$ (*x* ^(*n* + *2*)))
for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
using *exp-first-terms*[*of x 2*] **by** (*simp add*: *eval-nat-numeral*)

lemma *exp-bound*:
fixes *x* :: *real*
assumes *a*: $0 \le x$
and *b*: $x \le 1$
shows *exp x* $\le 1 + x + x^2$
proof −
have *suminf* ($\lambda n.$ *inverse*(*fact* (*n*+*2*)) ∗ (*x* ^(*n* + *2*))) $\le x^2$
proof −
have ($\lambda n.$ x^2 / *2* ∗ (*1*/*2*) ^ *n*) *sums* (x^2 / *2* ∗ (*1* / (*1* − *1*/*2*)))
by (*intro sums-mult geometric-sums*) *simp*
then have *sumsx*: ($\lambda n.$ x^2 / *2* ∗ (*1*/*2*) ^ *n*) *sums* x^2
by *simp*
have *suminf* ($\lambda n.$ *inverse*(*fact* (*n*+*2*)) ∗ (*x* ^(*n* + *2*))) ≤ *suminf* ($\lambda n.$ (x^2/*2*) ∗ ((*1*/*2*)^*n*))
proof (*intro suminf-le allI*)
show *inverse* (*fact* (*n* + *2*)) ∗ *x* ^(*n* + *2*) ≤ (x^2/*2*) ∗ ((*1*/*2*)^*n*) **for** *n* :: *nat*
proof −
have (*2*::*nat*) ∗ *2* ^ *n* ≤ *fact* (*n* + *2*)
by (*induct n*) *simp-all*
then have *real* ((*2*::*nat*) ∗ *2* ^ *n*) ≤ *real-of-nat* (*fact* (*n* + *2*))

by (*simp only*: *of-nat-le-iff*)
then have $((2::real) * 2 \hat{} n) \leq fact\ (n + 2)$
unfolding *of-nat-fact* **by** *simp*
then have $inverse\ (fact\ (n + 2)) \leq inverse\ ((2::real) * 2 \hat{} n)$
by (*rule le-imp-inverse-le*) *simp*
then have $inverse\ (fact\ (n + 2)) \leq 1/(2::real) * (1/2)\hat{}n$
by (*simp add*: *power-inverse* [*symmetric*])
then have $inverse\ (fact\ (n + 2)) * (x\hat{}n * x^2) \leq 1/2 * (1/2)\hat{}n * (1 * x^2)$
by (*rule mult-mono*) (*rule mult-mono*, *simp-all add*: *power-le-one a b*)
then show *?thesis*
unfolding *power-add* **by** (*simp add*: *ac-simps del*: *fact-Suc*)
qed
show $summable\ (\lambda n.\ inverse\ (fact\ (n + 2)) * x \hat{} (n + 2))$
by (*rule summable-exp* [*THEN summable-ignore-initial-segment*])
show $summable\ (\lambda n.\ x^2\ /\ 2 * (1/2) \hat{} n)$
by (*rule sums-summable* [*OF sumsx*])
qed
also have $\ldots = x^2$
by (*rule sums-unique* [*THEN sym*]) (*rule sumsx*)
finally show *?thesis* **.**
qed
then show *?thesis*
unfolding *exp-first-two-terms* **by** *auto*
qed

corollary *exp-half-le2*: $exp(1/2) \leq (2::real)$
using *exp-bound* [*of 1/2*]
by (*simp add*: *field-simps*)

corollary *exp-le*: $exp\ 1 \leq (3::real)$
using *exp-bound* [*of 1*]
by (*simp add*: *field-simps*)

lemma *exp-bound-half*: $norm\ z \leq 1/2 \Longrightarrow norm\ (exp\ z) \leq 2$
by (*blast intro*: *order-trans intro*!: *exp-half-le2 norm-exp*)

lemma *exp-bound-lemma*:
assumes $norm\ z \leq 1/2$
shows $norm\ (exp\ z) \leq 1 + 2 * norm\ z$
proof −
have ∗: $(norm\ z)^2 \leq norm\ z * 1$
unfolding *power2-eq-square*
by (*rule mult-left-mono*) (*use assms* **in** *auto*)
have $norm\ (exp\ z) \leq exp\ (norm\ z)$
by (*rule norm-exp*)
also have $\ldots \leq 1 + (norm\ z) + (norm\ z)^2$
using *assms exp-bound* **by** *auto*
also have $\ldots \leq 1 + 2 * norm\ z$
using ∗ **by** *auto*

finally show *?thesis* **.**
qed

lemma *real-exp-bound-lemma*: $0 \le x \Longrightarrow x \le 1/2 \Longrightarrow exp\ x \le 1 + 2 * x$
for *x* :: *real*
using *exp-bound-lemma* [*of x*] **by** *simp*

lemma *ln-one-minus-pos-upper-bound*:
fixes *x* :: *real*
assumes *a*: $0 \le x$ **and** *b*: $x < 1$
shows $ln\ (1 - x) \le - x$
proof −
have $(1 - x) * (1 + x + x^2) = 1 - x\hat{}3$
by (*simp add*: *algebra-simps power2-eq-square power3-eq-cube*)
also have ... ≤ 1
by (*auto simp*: *a*)
finally have $(1 - x) * (1 + x + x^2) \le 1$ **.**
moreover have *c*: $0 < 1 + x + x^2$
by (*simp add*: *add-pos-nonneg a*)
ultimately have $1 - x \le 1 / (1 + x + x^2)$
by (*elim mult-imp-le-div-pos*)
also have ... $\le 1 / exp\ x$
by (*metis a abs-one b exp-bound exp-gt-zero frac-le less-eq-real-def real-sqrt-abs real-sqrt-pow2-iff real-sqrt-power*)
also have ... $= exp\ (- x)$
by (*auto simp*: *exp-minus divide-inverse*)
finally have $1 - x \le exp\ (- x)$ **.**
also have $1 - x = exp\ (ln\ (1 - x))$
by (*metis b diff-0 exp-ln-iff less-iff-diff-less-0 minus-diff-eq*)
finally have $exp\ (ln\ (1 - x)) \le exp\ (- x)$ **.**
then show *?thesis*
by (*auto simp only*: *exp-le-cancel-iff*)
qed

lemma *exp-ge-add-one-self* [*simp*]: $1 + x \le exp\ x$
for *x* :: *real*
proof (*cases* $0 \le x \vee x \le -1$)
case *True*
then show *?thesis*
by (*meson exp-ge-add-one-self-aux exp-ge-zero order.trans real-add-le-0-iff*)
next
case *False*
then have *ln1*: $ln\ (1 + x) \le x$
using *ln-one-minus-pos-upper-bound* [*of* $-x$] **by** *simp*
have $1 + x = exp\ (ln\ (1 + x))$
using *False* **by** *auto*
also have ... $\le exp\ x$
by (*simp add*: *ln1*)
finally show *?thesis* **.**

qed

lemma *exp-gt-self*: $x < exp\ (x::real)$
  using *exp-gt-zero ln-less-self* **by** *fastforce*

lemma *ln-one-plus-pos-lower-bound*:
  fixes x :: *real*
  assumes *a*: $0 \leq x$ **and** *b*: $x \leq 1$
  shows $x - x^2 \leq ln\ (1 + x)$
proof −
  have $exp\ (x - x^2) = exp\ x\ /\ exp\ (x^2)$
    by (*rule exp-diff*)
  also have $\ldots \leq (1 + x + x^2)\ /\ exp\ (x\ ^2)$
    by (*metis a b divide-right-mono exp-bound exp-ge-zero*)
  also have $\ldots \leq (1 + x + x^2)\ /\ (1 + x^2)$
    by (*simp add*: *a divide-left-mono add-pos-nonneg*)
  also from *a* **have** $\ldots \leq 1 + x$
    by (*simp add*: *field-simps add-strict-increasing zero-le-mult-iff*)
  finally have $exp\ (x - x^2) \leq 1 + x$ **.**
  also have $\ldots = exp\ (ln\ (1 + x))$
  proof −
    from *a* **have** $0 < 1 + x$ **by** *auto*
    then show *?thesis*
      by (*auto simp only*: *exp-ln-iff* [*THEN sym*])
  qed
  finally have $exp\ (x - x^2) \leq exp\ (ln\ (1 + x))$ **.**
  then show *?thesis*
    by (*metis exp-le-cancel-iff*)
qed

lemma *ln-one-minus-pos-lower-bound*:
  fixes x :: *real*
  assumes *a*: $0 \leq x$ **and** *b*: $x \leq 1/2$
  shows $- x - 2 * x^2 \leq ln\ (1 - x)$
proof −
  from *b* **have** *c*: $x < 1$ **by** *auto*
  then have $ln\ (1 - x) = - ln\ (1 + x\ /\ (1 - x))$
   by (*auto simp*: *ln-inverse* [*symmetric*] *field-simps intro*: *arg-cong* [**where** *f=ln*])
  also have $- (x\ /\ (1 - x)) \leq \ldots$
  proof −
    have $ln\ (1 + x\ /\ (1 - x)) \leq x\ /\ (1 - x)$
      using *a c* **by** (*intro ln-add-one-self-le-self*) *auto*
    then show *?thesis*
      by *auto*
  qed
  also have $- (x\ /\ (1 - x)) = - x\ /\ (1 - x)$
    by *auto*
  finally have *d*: $- x\ /\ (1 - x) \leq ln\ (1 - x)$ **.**
  have $0 < 1 - x$ **using** *a b* **by** *simp*

then have *e*: $-\ x\ -\ 2 * x^2 \leq -\ x\ /\ (1\ -\ x)$
  using *mult-right-le-one-le*[*of* $x * x$ $2 * x$] *a b*
  by (*simp add*: *field-simps power2-eq-square*)
from *e d* **show** $-\ x\ -\ 2 * x^2 \leq ln\ (1\ -\ x)$
  by (*rule order-trans*)
qed

lemma *ln-add-one-self-le-self2*:
  fixes *x* :: *real*
  shows $-1 < x \Longrightarrow ln\ (1 + x) \leq x$
  by (*metis diff-gt-0-iff-gt diff-minus-eq-add exp-ge-add-one-self exp-le-cancel-iff exp-ln minus-less-iff*)

lemma *abs-ln-one-plus-x-minus-x-bound-nonneg*:
  fixes *x* :: *real*
  assumes *x*: $0 \leq x$ **and** *x1*: $x \leq 1$
  shows $|ln\ (1 + x) - x| \leq x^2$
proof −
  from *x* **have** $ln\ (1 + x) \leq x$
    by (*rule ln-add-one-self-le-self*)
  then have $ln\ (1 + x) - x \leq 0$
    by *simp*
  then have $|ln(1 + x) - x| = -\ (ln(1 + x) - x)$
    by (*rule abs-of-nonpos*)
  also have $\ldots = x - ln\ (1 + x)$
    by *simp*
  also have $\ldots \leq x^2$
  proof −
    from *x x1* **have** $x - x^2 \leq ln\ (1 + x)$
      by (*intro ln-one-plus-pos-lower-bound*)
    then show *?thesis*
      by *simp*
  qed
  finally show *?thesis* **.**
qed

lemma *abs-ln-one-plus-x-minus-x-bound-nonpos*:
  fixes *x* :: *real*
  assumes *a*: $-(1/2) \leq x$ **and** *b*: $x \leq 0$
  shows $|ln\ (1 + x) - x| \leq 2 * x^2$
proof −
  have $*$: $-\ (-x) - 2 * (-x)^2 \leq ln\ (1 - (-\ x))$
    by (*metis a b diff-zero ln-one-minus-pos-lower-bound minus-diff-eq neg-le-iff-le*)

  have $|ln\ (1 + x) - x| = x - ln\ (1 - (-\ x))$
    using *a ln-add-one-self-le-self2* [*of x*] **by** (*simp add*: *abs-if*)
  also have $\ldots \leq 2 * x^2$
    using $*$ **by** (*simp add*: *algebra-simps*)
  finally show *?thesis* **.**

qed

lemma *abs-ln-one-plus-x-minus-x-bound*:
  fixes x :: *real*
  assumes $|x| \leq 1/2$
  shows $|ln\ (1 + x) - x| \leq 2 * x^2$
proof (*cases* $0 \leq x$)
  case *True*
  then show *?thesis*
    using *abs-ln-one-plus-x-minus-x-bound-nonneg assms* **by** *fastforce*
next
  case *False*
  then show *?thesis*
    using *abs-ln-one-plus-x-minus-x-bound-nonpos assms* **by** *auto*
qed

lemma *ln-x-over-x-mono*:
  fixes x :: *real*
  assumes x: $exp\ 1 \leq x\ x \leq y$
  shows $ln\ y\ /\ y \leq ln\ x\ /\ x$
proof −
  note x
  moreover have $0 < exp\ (1::real)$ **by** *simp*
  ultimately have a: $0 < x$ **and** b: $0 < y$
    by (*fast intro*: *less-le-trans order-trans*)+
  have $x * ln\ y - x * ln\ x = x * (ln\ y - ln\ x)$
    by (*simp add*: *algebra-simps*)
  also have $\ldots = x * ln\ (y\ /\ x)$
    using *a b ln-div* **by** *force*
  also have $y\ /\ x = (x + (y - x))\ /\ x$
    by *simp*
  also have $\ldots = 1 + (y - x)\ /\ x$
    using *x a* **by** (*simp add*: *field-simps*)
  also have $x * ln\ (1 + (y - x)\ /\ x) \leq x * ((y - x)\ /\ x)$
    using *x a*
    by (*intro mult-left-mono ln-add-one-self-le-self*) *simp-all*
  also have $\ldots = y - x$
    using *a* **by** *simp*
  also have $\ldots = (y - x) * ln\ (exp\ 1)$ **by** *simp*
  also have $\ldots \leq (y - x) * ln\ x$
    using *a x exp-total of-nat-1 x(1)* **by** (*fastforce intro*: *mult-left-mono*)
  also have $\ldots = y * ln\ x - x * ln\ x$
    by (*rule left-diff-distrib*)
  finally have $x * ln\ y \leq y * ln\ x$
    by *arith*
  then have $ln\ y \leq (y * ln\ x)\ /\ x$
    using *a* **by** (*simp add*: *field-simps*)
  also have $\ldots = y * (ln\ x\ /\ x)$ **by** *simp*
  finally show *?thesis*

using *b* **by** (*simp add*: *field-simps*)
qed

lemma *ln-le-minus-one*: $0 < x \Longrightarrow ln\ x \le x - 1$
for *x* :: *real*
using *exp-ge-add-one-self*[*of ln x*] **by** *simp*

corollary *ln-diff-le*: $0 < x \Longrightarrow 0 < y \Longrightarrow ln\ x - ln\ y \le (x - y) / y$
for *x* :: *real*
by (*metis diff-divide-distrib divide-pos-pos divide-self ln-divide-pos ln-le-minus-one order-less-irrefl*)

lemma *ln-eq-minus-one*:
fixes *x* :: *real*
assumes $0 < x$ $ln\ x = x - 1$
shows $x = 1$
proof −
let $?l = \lambda y.\ ln\ y - y + 1$
have *D*: $\bigwedge x{::}real.\ 0 < x \Longrightarrow DERIV\ ?l\ x :> (1/x - 1)$
by (*auto intro*!: *derivative-eq-intros*)
show *?thesis*
proof (*cases rule*: *linorder-cases*)
assume $x < 1$
from *dense*[*OF* ‹$x < 1$›] **obtain** *a* **where** $x < a$ $a < 1$ **by** *blast*
from ‹$x < a$› **have** $?l\ x < ?l\ a$
proof (*rule DERIV-pos-imp-increasing*)
fix *y*
assume $x \le y$ $y \le a$
with ‹$0 < x$› ‹$a < 1$› **have** $0 < 1 / y - 1$ $0 < y$
by (*auto simp*: *field-simps*)
with *D* **show** $\exists z.\ DERIV\ ?l\ y :> z \wedge 0 < z$ **by** *blast*
qed
also have … ≤ 0
using *ln-le-minus-one* ‹$0 < x$› ‹$x < a$› **by** (*auto simp*: *field-simps*)
finally show $x = 1$ **using** *assms* **by** *auto*
next
assume $1 < x$
from *dense*[*OF this*] **obtain** *a* **where** $1 < a$ $a < x$ **by** *blast*
from ‹$a < x$› **have** $?l\ x < ?l\ a$
proof (*rule DERIV-neg-imp-decreasing*)
fix *y*
assume $a \le y$ $y \le x$
with ‹$1 < a$› **have** $1 / y - 1 < 0$ $0 < y$
by (*auto simp*: *field-simps*)
with *D* **show** $\exists z.\ DERIV\ ?l\ y :> z \wedge z < 0$
by *blast*
qed
also have … ≤ 0
using *ln-le-minus-one* ‹$1 < a$› **by** (*auto simp*: *field-simps*)

finally show $x = 1$ **using** *assms* **by** *auto*
next
assume $x = 1$
then show *?thesis* **by** *simp*
qed
qed

lemma *ln-add-one-self-less-self*:
fixes x :: *real*
assumes $x > 0$
shows *ln* $(1 + x) < x$
by (*smt* (*verit*, *best*) *assms ln-eq-minus-one ln-le-minus-one*)

lemma *ln-x-over-x-tendsto-0*: $((\lambda x::real.\ ln\ x\ /\ x) \longrightarrow 0)$ *at-top*
proof (*rule lhospital-at-top-at-top*[**where** $f' = inverse$ **and** $g' = \lambda$-. *1*])
from *eventually-gt-at-top*[*of 0*::*real*]
show $\forall_F$ *x in at-top*. (*ln has-real-derivative inverse x*) (*at x*)
by *eventually-elim* (*auto intro*!: *derivative-eq-intros simp*: *field-simps*)
qed (*use tendsto-inverse-0* **in**
‹*auto simp*: *filterlim-ident dest*!: *tendsto-mono*[*OF at-top-le-at-infinity*]›)

corollary *exp-1-gt-powr*:
assumes $x > (0::real)$
shows *exp 1* $> (1 + 1/x)$ *powr x*
proof −
have *ln* $(1 + 1/x) < 1/x$
using *ln-add-one-self-less-self assms* **by** *simp*
thus *exp 1* $> (1 + 1/x)$ *powr x* **using** *assms*
by (*simp add*: *field-simps powr-def*)
qed

lemma *exp-ge-one-plus-x-over-n-power-n*:
assumes $x \geq -$ *real n* $n > 0$
shows $(1 + x$ / *of-nat n*$)$ ^ $n \leq$ *exp x*
proof (*cases* $x = -$ *of-nat n*)
case *False*
from *assms False* **have** $(1 + x$ / *of-nat n*$)$ ^ $n =$ *exp* (*of-nat n* $*$ *ln* $(1 + x$ / *of-nat n*$)$)
by (*subst exp-of-nat-mult*, *subst exp-ln*) (*simp-all add*: *field-simps*)
also from *assms False* **have** *ln* $(1 + x$ / *real n*$) \leq x$ / *real n*
by (*intro ln-add-one-self-le-self2*) (*simp-all add*: *field-simps*)
with *assms* **have** *exp* (*of-nat n* $*$ *ln* $(1 + x$ / *of-nat n*$)) \leq$ *exp x*
by (*simp add*: *field-simps*)
finally show *?thesis* **.**
next
case *True*
then show *?thesis* **by** (*simp add*: *zero-power*)
qed

lemma *exp-ge-one-minus-x-over-n-power-n*:
assumes $x \le$ *real n n* > 0
shows $(1 - x$ / *of-nat n*) ^ $n \le$ *exp* $(-x)$
using *exp-ge-one-plus-x-over-n-power-n*[*of n* $-x$] *assms* **by** *simp*

lemma *exp-at-bot*: (*exp* $\longrightarrow$ (*0*::*real*)) *at-bot*
unfolding *tendsto-Zfun-iff*
proof (*rule ZfunI*, *simp add*: *eventually-at-bot-dense*)
fix *r* :: *real*
assume $0 < r$
have *exp x* $<$ *r* **if** $x <$ *ln r* **for** *x*
by (*metis* ‹$0 < r$› *exp-less-mono exp-ln that*)
then show $\exists k. \forall n<k.$ *exp n* $<$ *r* **by** *auto*
qed

lemma *exp-at-top*: *LIM x at-top. exp x* :: *real* :> *at-top*
by (*rule filterlim-at-top-at-top*[**where** *Q*=λx. *True* **and** *P*=λx. $0 < x$ **and** *g*=*ln*])
(*auto intro*: *eventually-gt-at-top*)

lemma *lim-exp-minus-1*: ((λz::'*a*. (*exp*(*z*) $-$ *1*) / *z*) $\longrightarrow$ *1*) (*at 0*)
for *x* :: '*a*::{*real-normed-field*,*banach*}
proof $-$
have ((λz::'*a*. *exp*(*z*) $-$ *1*) *has-field-derivative 1*) (*at 0*)
by (*intro derivative-eq-intros* | *simp*)+
then show *?thesis*
by (*simp add*: *Deriv.has-field-derivative-iff*)
qed

lemma *ln-at-0*: *LIM x at-right 0. ln* (*x*::*real*) :> *at-bot*
by (*rule filterlim-at-bot-at-right*[**where** *Q*=λx. $0 < x$ **and** *P*=λx. *True* **and** *g*=*exp*])
(*auto simp*: *eventually-at-filter*)

lemma *ln-at-top*: *LIM x at-top. ln* (*x*::*real*) :> *at-top*
by (*rule filterlim-at-top-at-top*[**where** *Q*=λx. $0 < x$ **and** *P*=λx. *True* **and** *g*=*exp*])
(*auto intro*: *eventually-gt-at-top*)

lemma *filtermap-ln-at-top*: *filtermap* (*ln*::*real* $\Rightarrow$ *real*) *at-top* = *at-top*
by (*intro filtermap-fun-inverse*[*of exp*] *exp-at-top ln-at-top*) *auto*

lemma *filtermap-exp-at-top*: *filtermap* (*exp*::*real* $\Rightarrow$ *real*) *at-top* = *at-top*
by (*intro filtermap-fun-inverse*[*of ln*] *exp-at-top ln-at-top*)
(*auto simp*: *eventually-at-top-dense*)

lemma *filtermap-ln-at-right*: *filtermap ln* (*at-right* (*0*::*real*)) = *at-bot*
by (*auto intro*!: *filtermap-fun-inverse*[**where** *g*=λx. *exp x*] *ln-at-0*
simp: *filterlim-at exp-at-bot*)

lemma *tendsto-power-div-exp-0*: $((\lambda x.\ x \ \hat{}\ k\ /\ exp\ x) \longrightarrow (0::real))\ at\text{-}top$
proof (*induct k*)
  case *0*
  show $((\lambda x.\ x \ \hat{}\ 0\ /\ exp\ x) \longrightarrow (0::real))\ at\text{-}top$
    by (*simp add*: *inverse-eq-divide*[*symmetric*])
      (*metis filterlim-compose*[*OF tendsto-inverse-0*] *exp-at-top filterlim-mono at-top-le-at-infinity order-refl*)
next
  case (*Suc k*)
  show *?case*
  proof (*rule lhospital-at-top-at-top*)
    show *eventually* $(\lambda x.\ DERIV\ (\lambda x.\ x \ \hat{}\ Suc\ k)\ x :> (real\ (Suc\ k) * x\hat{}k))\ at\text{-}top$
      by *eventually-elim* (*intro derivative-eq-intros*, *auto*)
    show *eventually* $(\lambda x.\ DERIV\ exp\ x :> exp\ x)\ at\text{-}top$
      by *eventually-elim auto*
    show *eventually* $(\lambda x.\ exp\ x \neq 0)\ at\text{-}top$
      by *auto*
    from *tendsto-mult*[*OF tendsto-const Suc*, *of real* (*Suc k*)]
    show $((\lambda x.\ real\ (Suc\ k) * x \ \hat{}\ k\ /\ exp\ x) \longrightarrow 0)\ at\text{-}top$
      by *simp*
  qed (*rule exp-at-top*)
qed

112.7.1 A couple of simple bounds

lemma *exp-plus-inverse-exp*:
  fixes *x*::*real*
  shows $2 \leq exp\ x + inverse\ (exp\ x)$
proof −
  have $2 \leq exp\ x + exp\ (-x)$
    using *exp-ge-add-one-self* [*of x*] *exp-ge-add-one-self* [*of* −*x*]
    by *linarith*
  then show *?thesis*
    by (*simp add*: *exp-minus*)
qed

lemma *real-le-x-sinh*:
  fixes *x*::*real*
  assumes $0 \leq x$
  shows $x \leq (exp\ x - inverse(exp\ x))\ /\ 2$
proof −
  have ∗: $exp\ a - inverse(exp\ a) - 2{*}a \leq exp\ b - inverse(exp\ b) - 2{*}b$ **if** $a \leq b$
for *a b*::*real*
    using *exp-plus-inverse-exp*
    by (*fastforce intro*: *derivative-eq-intros DERIV-nonneg-imp-nondecreasing* [*OF that*])
  show *?thesis*
    using∗[*OF assms*] **by** *simp*
qed

lemma *real-le-abs-sinh*:
  fixes *x*::*real*
  shows *abs x* ≤ *abs*((*exp x* − *inverse*(*exp x*)) / *2*)
proof (*cases 0* ≤ *x*)
  case *True*
  show *?thesis*
    using *real-le-x-sinh* [*OF True*] *True* **by** (*simp add*: *abs-if*)
next
  case *False*
  have −*x* ≤ (*exp*(−*x*) − *inverse*(*exp*(−*x*))) / *2*
    by (*meson False linear neg-le-0-iff-le real-le-x-sinh*)
  also have ... ≤ |(*exp x* − *inverse* (*exp x*)) / *2*|
    by (*metis* (*no-types*, *opaque-lifting*) *abs-divide abs-le-iff abs-minus-cancel*
      add.inverse-inverse exp-minus minus-diff-eq order-refl)
  finally show *?thesis*
    using *False* **by** *linarith*
qed

112.8 The general logarithm

definition *log* :: *real* ⇒ *real* ⇒ *real*
  — logarithm of *x* to base *a*
  where *log a x* = *ln x* / *ln a*

lemma *log-exp* [*simp*]: *log b* (*exp x*) = *x* / *ln b*
  by (*simp add*: *log-def*)

lemma *tendsto-log* [*tendsto-intros*]:
  (*f* ⟶ *a*) *F* ⟹ (*g* ⟶ *b*) *F* ⟹ *0* < *a* ⟹ *a* ≠ *1* ⟹ *b*≠*0* ⟹
    ((λ*x*. *log* (*f x*) (*g x*)) ⟶ *log a b*) *F*
  unfolding *log-def* **by** (*intro tendsto-intros*) *auto*

lemma *continuous-log*:
  assumes *continuous F f*
    and *continuous F g*
    and *f* (*Lim F* (λ*x*. *x*)) > *0*
    and *f* (*Lim F* (λ*x*. *x*)) ≠ *1*
    and *g* (*Lim F* (λ*x*. *x*)) ≠ *0*
  shows *continuous F* (λ*x*. *log* (*f x*) (*g x*))
  using *assms* **by** (*simp add*: *continuous-def tendsto-log*)

lemma *continuous-at-within-log*[*continuous-intros*]:
  assumes *continuous* (*at a within s*) *f*
    and *continuous* (*at a within s*) *g*
    and *0* < *f a*
    and *f a* ≠ *1*
    and *g a* ≠ *0*
  shows *continuous* (*at a within s*) (λ*x*. *log* (*f x*) (*g x*))

using *assms* **unfolding** *continuous-within* **by** (*rule tendsto-log*)

lemma *continuous-on-log*[*continuous-intros*]:
assumes *continuous-on S f continuous-on S g*
and $\forall x\in S.\ 0 < f\ x$ $\forall x\in S.\ f\ x \neq 1$ $\forall x\in S.\ g\ x \neq 0$
shows *continuous-on S* ($\lambda x.$ *log* (*f x*) (*g x*))
using *assms* **unfolding** *continuous-on-def* **by** (*fast intro*: *tendsto-log*)

lemma *exp-powr-real*:
fixes *x*::*real* **shows** *exp x powr y* = *exp* (*x*∗*y*)
by (*simp add*: *powr-def*)

lemma *powr-one-eq-one* [*simp*]: *1 powr a* = *1*
by (*simp add*: *powr-def*)

lemma *powr-zero-eq-one* [*simp*]: *x powr 0* = (*if x* = *0 then 0 else 1*)
by (*simp add*: *powr-def*)

lemma *powr-one-gt-zero-iff* [*simp*]: *x powr 1* = *x* $\longleftrightarrow$ *0* ≤ *x*
for *x* :: *real*
by (*auto simp*: *powr-def*)
declare *powr-one-gt-zero-iff* [*THEN iffD2*, *simp*]

lemma *powr-diff*:
fixes *w*:: *'a*::{*ln*,*real-normed-field*}
shows *w powr* (*z1* − *z2*) = *w powr z1* / *w powr z2*
by (*simp add*: *powr-def algebra-simps exp-diff*)

lemma *powr-mult*: (*x* ∗ *y*) *powr a* = (*x powr a*) ∗ (*y powr a*)
for *a x y* :: *real*
by (*simp add*: *powr-def exp-add* [*symmetric*] *ln-mult distrib-left*)

lemma *prod-powr-distrib*:
fixes *x* :: *'a* ⇒ *real*
shows (*prod x I*) *powr r* = ($\prod$ *i*∈*I*. *x i powr r*)
by (*induction I rule*: *infinite-finite-induct*) (*auto simp add*: *powr-mult prod-nonneg*)

lemma *powr-ge-zero* [*simp*]: *0* ≤ *x powr y*
for *x y* :: *real*
by (*simp add*: *powr-def*)

lemma *powr-non-neg*[*simp*]: ¬ *a powr x* < *0* **for** *a x*::*real*
using *powr-ge-zero*[*of a x*] **by** *arith*

lemma *inverse-powr*: $\bigwedge$*y*::*real*. *inverse y powr a* = *inverse* (*y powr a*)
by (*simp add*: *exp-minus ln-inverse powr-def*)

lemma *powr-divide*: (*x* / *y*) *powr a* = (*x powr a*) / (*y powr a*)
for *a b x* :: *real*

by (*simp add*: *divide-inverse powr-mult inverse-powr*)

lemma *powr-add*: *x powr* (*a* + *b*) = (*x powr a*) ∗ (*x powr b*)
for *a b x* :: *'a*::{*ln*,*real-normed-field*}
by (*simp add*: *powr-def exp-add* [*symmetric*] *distrib-right*)

lemma *powr-mult-base*: *0* ≤ *x* ⟹*x* ∗ *x powr y* = *x powr* (*1* + *y*)
for *x* :: *real*
by (*auto simp*: *powr-add*)

lemma *powr-mult-base'*: *abs x* ∗ *x powr y* = *x powr* (*1* + *y*)
for *x* :: *real*
by (*smt* (*verit*) *powr-mult-base uminus-powr-eq*)

lemma *powr-powr*: (*x powr a*) *powr b* = *x powr* (*a* ∗ *b*)
for *a b x* :: *real*
by (*simp add*: *powr-def*)

lemma *powr-power*:
fixes *z*:: *'a*::{*real-normed-field*,*ln*}
shows *z* ≠ *0* ⟹ (*z powr u*) ^ *n* = *z powr* (*of-nat n* ∗ *u*)
by (*induction n*) (*auto simp*: *algebra-simps powr-add*)

lemma *powr-powr-swap*: (*x powr a*) *powr b* = (*x powr b*) *powr a*
for *a b x* :: *real*
by (*simp add*: *powr-powr mult.commute*)

lemma *powr-minus*: *x powr* (− *a*) = *inverse* (*x powr a*)
for *a x* :: *'a*::{*ln*,*real-normed-field*}
by (*simp add*: *powr-def exp-minus* [*symmetric*])

lemma *powr-minus-divide*: *x powr* (− *a*) = *1*/(*x powr a*)
for *a x* :: *'a*::{*ln*,*real-normed-field*}
by (*simp add*: *divide-inverse powr-minus*)

lemma *powr-sum*:
assumes *x* ≠ *0*
shows *x powr sum f A* = (∏ *y*∈*A*. *x powr f y*)
proof (*cases finite A*)
case *True*
with *assms* **show** *?thesis*
by (*simp add*: *powr-def exp-sum sum-distrib-right*)
next
case *False*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma *divide-powr-uminus*: *a* / *b powr c* = *a* ∗ *b powr* (− *c*)
for *a b c* :: *real*

by (*simp add*: *powr-minus-divide*)

lemma *powr-less-mono*: $a < b \Longrightarrow 1 < x \Longrightarrow x\ powr\ a < x\ powr\ b$
for $a\ b\ x :: real$
by (*simp add*: *powr-def*)

lemma *powr-less-cancel*: $x\ powr\ a < x\ powr\ b \Longrightarrow 1 < x \Longrightarrow a < b$
for $a\ b\ x :: real$
by (*simp add*: *powr-def*)

lemma *powr-less-cancel-iff* [*simp*]: $1 < x \Longrightarrow x\ powr\ a < x\ powr\ b \longleftrightarrow a < b$
for $a\ b\ x :: real$
by (*blast intro*: *powr-less-cancel powr-less-mono*)

lemma *powr-le-cancel-iff* [*simp*]: $1 < x \Longrightarrow x\ powr\ a \leq x\ powr\ b \longleftrightarrow a \leq b$
for $a\ b\ x :: real$
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *powr-realpow*: $0 < x \Longrightarrow x\ powr\ (real\ n) = x\hat{}n$
by (*induction n*) (*simp-all add*: *ac-simps powr-add*)

lemma *powr-realpow'*: $(z :: real) \geq 0 \Longrightarrow n \neq 0 \Longrightarrow z\ powr\ of\text{-}nat\ n = z\ \hat{}\ n$
by (*cases* $z = 0$) (*auto simp*: *powr-realpow*)

lemma *powr-real-of-int'*:
assumes $x \geq 0\ x \neq 0 \lor n > 0$
shows $x\ powr\ real\text{-}of\text{-}int\ n = power\text{-}int\ x\ n$
by (*metis assms exp-ln-iff exp-power-int nless-le power-int-eq-0-iff powr-def*)

lemma *exp-minus-ge*:
fixes *x*::*real* **shows** $1 - x \leq exp\ (-x)$
by (*smt* (*verit*) *exp-ge-add-one-self*)

lemma *exp-minus-greater*:
fixes *x*::*real* **shows** $1 - x < exp\ (-x) \longleftrightarrow x \neq 0$
by (*smt* (*verit*) *exp-minus-ge exp-eq-one-iff exp-gt-zero ln-eq-minus-one ln-exp*)

lemma *log-ln*: $ln\ x = log\ (exp\ 1)\ x$
by (*simp add*: *log-def*)

lemma *DERIV-log*:
assumes $x > 0$
shows $DERIV\ (\lambda y.\ log\ b\ y)\ x :> 1\ /\ (ln\ b * x)$
proof −
define *lb* **where** $lb = 1\ /\ ln\ b$
moreover have $DERIV\ (\lambda y.\ lb * ln\ y)\ x :> lb\ /\ x$
using ‹$x > 0$› **by** (*auto intro*!: *derivative-eq-intros*)
ultimately show *?thesis*
by (*simp add*: *log-def*)

qed

lemmas *DERIV-log*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-log*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemma *powr-log-cancel* [*simp*]: $0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow a\ powr\ (log\ a\ x) = x$
by (*simp add*: *powr-def log-def*)

lemma *log-powr-cancel* [*simp*]: $0 < a \Longrightarrow a \neq 1 \Longrightarrow log\ a\ (a\ powr\ x) = x$
by (*simp add*: *log-def powr-def*)

lemma *powr-eq-iff*: $[\![y>0;\ a>1]\!] \Longrightarrow a\ powr\ x = y \longleftrightarrow log\ a\ y = x$
by *auto*

lemma *log-mult*:
$log\ a\ (x * y) = (if\ x\neq 0 \wedge y\neq 0\ then\ log\ a\ x + log\ a\ y\ else\ 0)$
by (*simp add*: *log-def ln-mult divide-inverse distrib-right*)

lemma *log-mult-pos*:
$x>0 \Longrightarrow y>0 \Longrightarrow log\ a\ (x * y) = log\ a\ x + log\ a\ y$
by (*simp add*: *log-def ln-mult divide-inverse distrib-right*)

lemma *log-eq-div-ln-mult-log*:
$0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow log\ a\ x = (ln\ b/ln\ a) * log\ b\ x$
by (*simp add*: *log-def divide-inverse*)

Base 10 logarithms

lemma *log-base-10-eq1*: $0 < x \Longrightarrow log\ 10\ x = (ln\ (exp\ 1)\ /\ ln\ 10) * ln\ x$
by (*simp add*: *log-def*)

lemma *log-base-10-eq2*: $0 < x \Longrightarrow log\ 10\ x = (log\ 10\ (exp\ 1)) * ln\ x$
by (*simp add*: *log-def*)

lemma *log-one* [*simp*]: $log\ a\ 1 = 0$
by (*simp add*: *log-def*)

lemma *log-eq-one* [*simp*]: $0 < a \Longrightarrow a \neq 1 \Longrightarrow log\ a\ a = 1$
by (*simp add*: *log-def*)

lemma *log-inverse*: $log\ a\ (inverse\ x) = -\ log\ a\ x$
by (*simp add*: *ln-inverse log-def*)

lemma *log-recip*: $log\ a\ (1/x) = -\ log\ a\ x$
by (*simp add*: *divide-inverse log-inverse*)

lemma *log-divide*:
$log\ a\ (x\ /\ y) = (if\ x\neq 0 \wedge y\neq 0\ then\ log\ a\ x - log\ a\ y\ else\ 0)$

by (*simp add*: *diff-divide-distrib ln-div log-def*)

lemma *log-divide-pos*:
x>0 $\Longrightarrow$ *y>0* $\Longrightarrow$ *log a* (*x* / *y*) = *log a x* − *log a y*
using *log-divide* **by** *auto*

lemma *powr-gt-zero* [*simp*]: *0* < *x powr a* $\longleftrightarrow$ *x* ≠ *0*
for *a x* :: *real*
by (*simp add*: *powr-def*)

lemma *powr-nonneg-iff*[*simp*]: *a powr x* ≤ *0* $\longleftrightarrow$ *a* = *0*
for *a x*::*real*
by (*meson not-less powr-gt-zero*)

lemma *log-add-eq-powr*: *0* < *b* $\Longrightarrow$ *b* ≠ *1* $\Longrightarrow$ *x*≠*0* $\Longrightarrow$ *log b x* + *y* = *log b* (*x* ∗ *b powr y*)
and *add-log-eq-powr*: *0* < *b* $\Longrightarrow$ *b* ≠ *1* $\Longrightarrow$ *x*≠*0* $\Longrightarrow$ *y* + *log b x* = *log b* (*b powr y* ∗ *x*)
and *log-minus-eq-powr*: *0* < *b* $\Longrightarrow$ *b* ≠ *1* $\Longrightarrow$ *x*≠*0* $\Longrightarrow$ *log b x* − *y* = *log b* (*x* ∗ *b powr* −*y*)
by (*simp-all add*: *log-mult log-divide*)

lemma *minus-log-eq-powr*: *0* < *b* $\Longrightarrow$ *b* ≠ *1* $\Longrightarrow$ *x*≠*0* $\Longrightarrow$ *y* − *log b x* = *log b* (*b powr y* / *x*)
by (*simp add*: *diff-divide-eq-iff ln-div log-def powr-def*)

lemma *log-less-cancel-iff* [*simp*]: *1* < *a* $\Longrightarrow$ *0* < *x* $\Longrightarrow$ *0* < *y* $\Longrightarrow$ *log a x* < *log a y* $\longleftrightarrow$ *x* < *y*
using *powr-less-cancel-iff* [*of a*] *powr-log-cancel* [*of a x*] *powr-log-cancel* [*of a y*]
by (*metis less-eq-real-def less-trans not-le zero-less-one*)

lemma *log-inj*:
assumes *1* < *b*
shows *inj-on* (*log b*) {*0* <..}
proof (*rule inj-onI*, *simp*)
fix *x y*
assume *pos*: *0* < *x 0* < *y* **and** ∗: *log b x* = *log b y*
show *x* = *y*
proof (*cases rule*: *linorder-cases*)
assume *x* = *y*
then show *?thesis* **by** *simp*
next
assume *x* < *y*
then have *log b x* < *log b y*
using *log-less-cancel-iff*[*OF* ‹*1* < *b*›] *pos* **by** *simp*
then show *?thesis* **using** ∗ **by** *simp*
next
assume *y* < *x*
then have *log b y* < *log b x*

using *log-less-cancel-iff* [*OF* ‹*1* < *b*›] *pos* **by** *simp*
then show *?thesis* **using** ∗ **by** *simp*
qed
qed

lemma *log-le-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *0* < *y* ⟹ *log a x* ≤ *log a y* ⟷ *x* ≤ *y*
by (*simp flip*: *linorder-not-less*)

lemma *log-mono*: *1* < *a* ⟹ *0* < *x* ⟹ *x* ≤ *y* ⟹ *log a x* ≤ *log a y*
by *simp*

lemma *log-less*: *1* < *a* ⟹ *0* < *x* ⟹ *x* < *y* ⟹ *log a x* < *log a y*
by *simp*

lemma *zero-less-log-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *0* < *log a x* ⟷ *1* < *x*
using *log-less-cancel-iff* [*of a 1 x*] **by** *simp*

lemma *zero-le-log-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *0* ≤ *log a x* ⟷ *1* ≤ *x*
using *log-le-cancel-iff* [*of a 1 x*] **by** *simp*

lemma *log-less-zero-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *log a x* < *0* ⟷ *x* < *1*
using *log-less-cancel-iff* [*of a x 1*] **by** *simp*

lemma *log-le-zero-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *log a x* ≤ *0* ⟷ *x* ≤ *1*
using *log-le-cancel-iff* [*of a x 1*] **by** *simp*

lemma *one-less-log-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *1* < *log a x* ⟷ *a* < *x*
using *log-less-cancel-iff* [*of a a x*] **by** *simp*

lemma *one-le-log-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *1* ≤ *log a x* ⟷ *a* ≤ *x*
using *log-le-cancel-iff* [*of a a x*] **by** *simp*

lemma *log-less-one-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *log a x* < *1* ⟷ *x* < *a*
using *log-less-cancel-iff* [*of a x a*] **by** *simp*

lemma *log-le-one-cancel-iff* [*simp*]: *1* < *a* ⟹ *0* < *x* ⟹ *log a x* ≤ *1* ⟷ *x* ≤ *a*
using *log-le-cancel-iff* [*of a x a*] **by** *simp*

lemma *le-log-iff*:
fixes *b x y* :: *real*
assumes *1* < *b x* > *0*
shows *y* ≤ *log b x* ⟷ *b powr y* ≤ *x*
using *assms*
by (*metis less-irrefl less-trans powr-le-cancel-iff powr-log-cancel zero-less-one*)

lemma *less-log-iff*:

assumes *1 < b x > 0*
shows *y < log b x ⟷ b powr y < x*
by (*metis assms dual-order.strict-trans less-irrefl powr-less-cancel-iff powr-log-cancel zero-less-one*)

lemma
assumes *1 < b x > 0*
shows *log-less-iff*: *log b x < y ⟷ x < b powr y*
and *log-le-iff*: *log b x ≤ y ⟷ x ≤ b powr y*
using *le-log-iff*[*OF assms*, *of y*] *less-log-iff*[*OF assms*, *of y*]
by *auto*

lemmas *powr-le-iff = le-log-iff*[*symmetric*]
and *powr-less-iff = less-log-iff*[*symmetric*]
and *less-powr-iff = log-less-iff*[*symmetric*]
and *le-powr-iff = log-le-iff*[*symmetric*]

lemma *le-log-of-power*:
assumes *b ^ n ≤ m 1 < b*
shows *n ≤ log b m*
proof −
from *assms* **have** *0 < m* **by** (*metis less-trans zero-less-power less-le-trans zero-less-one*)
thus *?thesis* **using** *assms* **by** (*simp add*: *le-log-iff powr-realpow*)
qed

lemma *le-log2-of-power*: *2 ^ n ≤ m ⟹ n ≤ log 2 m* **for** *m n* :: *nat*
using *le-log-of-power*[*of 2*] **by** *simp*

lemma *log-of-power-le*: ⟦ *m ≤ b ^ n*; *b > 1*; *m > 0* ⟧ ⟹ *log b* (*real m*) ≤ *n*
by (*simp add*: *log-le-iff powr-realpow*)

lemma *log2-of-power-le*: ⟦ *m ≤ 2 ^ n*; *m > 0* ⟧ ⟹ *log 2 m ≤ n* **for** *m n* :: *nat*
using *log-of-power-le*[*of - 2*] **by** *simp*

lemma *log-of-power-less*: ⟦ *m < b ^ n*; *b > 1*; *m > 0* ⟧ ⟹ *log b* (*real m*) < *n*
by (*simp add*: *log-less-iff powr-realpow*)

lemma *log2-of-power-less*: ⟦ *m < 2 ^ n*; *m > 0* ⟧ ⟹ *log 2 m < n* **for** *m n* :: *nat*
using *log-of-power-less*[*of - 2*] **by** *simp*

lemma *less-log-of-power*:
assumes *b ^ n < m 1 < b*
shows *n < log b m*
proof −
have *0 < m* **by** (*metis assms less-trans zero-less-power zero-less-one*)
thus *?thesis* **using** *assms* **by** (*simp add*: *less-log-iff powr-realpow*)
qed

lemma *less-log2-of-power*: *2 ^ n < m ⟹ n < log 2 m* **for** *m n* :: *nat*

using *less-log-of-power*[*of 2*] **by** *simp*

lemma *gr-one-powr*[*simp*]:
fixes *x y* :: *real* **shows** $[\![x > 1; y > 0]\!] \Longrightarrow 1 < x$ *powr* y
by(*simp add*: *less-powr-iff*)

lemma *log-pow-cancel* [*simp*]:
$a > 0 \Longrightarrow a \neq 1 \Longrightarrow log\ a\ (a\ \hat{}\ b) = b$
by (*simp add*: *ln-realpow log-def*)

lemma *floor-log-eq-powr-iff*: $x > 0 \Longrightarrow b > 1 \Longrightarrow \lfloor log\ b\ x \rfloor = k \longleftrightarrow b$ *powr* $k \leq x \wedge x < b$ *powr* $(k + 1)$
by (*auto simp*: *floor-eq-iff powr-le-iff less-powr-iff*)

lemma *floor-log-nat-eq-powr-iff*:
fixes *b n k* :: *nat*
shows $[\![b \geq 2; k > 0]\!] \Longrightarrow floor\ (log\ b\ (real\ k)) = n \longleftrightarrow b\hat{}n \leq k \wedge k < b\hat{}(n+1)$
by (*auto simp*: *floor-log-eq-powr-iff powr-add powr-realpow*
of-nat-power[*symmetric*] *of-nat-mult*[*symmetric*] *ac-simps*
simp del: *of-nat-power of-nat-mult*)

lemma *floor-log-nat-eq-if*:
fixes *b n k* :: *nat*
assumes $b\hat{}n \leq k$ $k < b\hat{}(n+1)$ $b \geq 2$
shows *floor* (*log b* (*real k*)) = *n*
proof −
have $k \geq 1$
using *assms linorder-le-less-linear* **by** *force*
with *assms* **show** *?thesis*
by(*simp add*: *floor-log-nat-eq-powr-iff*)
qed

lemma *ceiling-log-eq-powr-iff*:
$[\![x > 0; b > 1]\!] \Longrightarrow \lceil log\ b\ x \rceil = int\ k + 1 \longleftrightarrow b$ *powr* $k < x \wedge x \leq b$ *powr* $(k + 1)$
by (*auto simp*: *ceiling-eq-iff powr-less-iff le-powr-iff*)

lemma *ceiling-log-nat-eq-powr-iff*:
fixes *b n k* :: *nat*
shows $[\![b \geq 2; k > 0]\!] \Longrightarrow \lceil log\ b\ (real\ k) \rceil = int\ n + 1 \longleftrightarrow (b\hat{}n < k \wedge k \leq b\hat{}(n+1))$
using *ceiling-log-eq-powr-iff*
by (*auto simp*: *powr-add powr-realpow of-nat-power*[*symmetric*] *of-nat-mult*[*symmetric*] *ac-simps*
simp del: *of-nat-power of-nat-mult*)

lemma *ceiling-log-nat-eq-if*:
fixes *b n k* :: *nat*

assumes $b\hat{\ }n < k$ $k \leq b\hat{\ }(n+1)$ $b \geq 2$
shows $\lceil log\ (real\ b)\ (real\ k)\rceil = int\ n + 1$
using *assms ceiling-log-nat-eq-powr-iff* **by** *force*

lemma *floor-log2-div2*:
fixes $n :: nat$
assumes $n \geq 2$
shows $\lfloor log\ 2\ (real\ n)\rfloor = \lfloor log\ 2\ (n\ div\ 2)\rfloor + 1$
proof *cases*
assume $n=2$ **thus** *?thesis* **by** *simp*
next
let $?m = n\ div\ 2$
assume $n\neq 2$
hence $1 \leq ?m$ **using** *assms* **by** *arith*
then obtain i **where** i: $2\ \hat{\ }\ i \leq ?m$ $?m < 2\ \hat{\ }(i + 1)$
using *ex-power-ivl1*[*of* 2 ?m] **by** *auto*
have $2\hat{\ }(i+1) \leq 2*?m$ **using** $i(1)$ **by** *simp*
also have $2*?m \leq n$ **by** *arith*
finally have $*$: $2\hat{\ }(i+1) \leq \ldots$.
have $n < 2\hat{\ }(i+1+1)$ **using** $i(2)$ **by** *simp*
from *floor-log-nat-eq-if*[*OF* ∗ *this*] *floor-log-nat-eq-if*[*OF i*]
show *?thesis* **by** *simp*
qed

lemma *ceiling-log2-div2*:
assumes $n \geq 2$
shows $\lceil log\ 2\ (real\ n)\rceil = \lceil log\ 2\ ((n-1)\ div\ 2 + 1)\rceil + 1$
proof *cases*
assume $n=2$ **thus** *?thesis* **by** *simp*
next
let $?m = (n-1)\ div\ 2 + 1$
assume $n\neq 2$
hence $2 \leq ?m$ **using** *assms* **by** *arith*
then obtain i **where** i: $2\ \hat{\ }\ i < ?m$ $?m \leq 2\ \hat{\ }(i + 1)$
using *ex-power-ivl2*[*of* 2 ?m] **by** *auto*
have $n \leq 2*?m$ **by** *arith*
also have $2*?m \leq 2\ \hat{\ }((i+1)+1)$ **using** $i(2)$ **by** *simp*
finally have $*$: $n \leq \ldots$.
have $2\hat{\ }(i+1) < n$ **using** $i(1)$ **by** (*auto simp*: *less-Suc-eq-0-disj*)
from *ceiling-log-nat-eq-if*[*OF this* ∗] *ceiling-log-nat-eq-if*[*OF i*]
show *?thesis* **by** *simp*
qed

lemma *powr-real-of-int*:
$x > 0 \Longrightarrow x\ powr\ real\text{-}of\text{-}int\ n = (if\ n \geq 0\ then\ x\ \hat{\ }\ nat\ n\ else\ inverse\ (x\ \hat{\ }\ nat\ (-\ n)))$
using *powr-realpow*[*of x nat n*] *powr-realpow*[*of x nat* (−*n*)]
by (*auto simp*: *field-simps powr-minus*)

lemma *powr-numeral* [*simp*]: *0* ≤ *x* ⟹ *x powr* (*numeral n* :: *real*) = *x* ^(*numeral n*)
  by (*metis less-le power-zero-numeral powr-0 of-nat-numeral powr-realpow*)

lemma *powr-int*:
  assumes *x* > *0*
  shows *x powr i* = (*if i* ≥ *0 then x* ^ *nat i else 1*/*x* ^ *nat* (−*i*))
  by (*simp add*: *assms inverse-eq-divide powr-real-of-int*)

lemma *power-of-nat-log-ge*: *b* > *1* ⟹ *b* ^ *nat* ⌈*log b x*⌉ ≥ *x*
  by (*smt* (*verit*) *less-log-of-power of-nat-ceiling*)

lemma *power-of-nat-log-le*:
  assumes *b* > *1 x*≥*1*
  shows *b* ^ *nat* ⌊*log b x*⌋ ≤ *x*
proof −
  have ⌊*log b x*⌋ ≥ *0*
    using *assms* **by** *auto*
  then show *?thesis*
    by (*smt* (*verit*) *assms le-log-iff of-int-floor-le powr-int*)
qed

definition *powr-real* :: *real* ⇒ *real* ⇒ *real*
  where [*code-abbrev*, *simp*]: *powr-real* = *Transcendental.powr*

lemma *compute-powr-real* [*code*]:
  powr-real b i =
   (*if b* ≤ *0 then Code.abort* (*STR ''powr-real with nonpositive base''*) (λ-. *powr-real b i*)
     else if ⌊*i*⌋ = *i then* (*if 0* ≤ *i then b* ^ *nat* ⌊*i*⌋ *else 1* / *b* ^ *nat* ⌊− *i*⌋)
     else Code.abort (*STR ''powr-real with non−integer exponent''*) (λ-. *powr-real b i*))
    for *b i* :: *real*
  by (*auto simp*: *powr-int*)

lemma *powr-one*: *0* ≤ *x* ⟹ *x powr 1* = *x*
  for *x* :: *real*
  using *powr-realpow* [*of x 1*] **by** *simp*

lemma *powr-one'* [*simp*]: *x powr 1* = |*x*|
  for *x* :: *real*
  by (*simp add*: *ln-real-def powr-def*)

lemma *powr-neg-one*: *0* < *x* ⟹ *x powr* −*1* = *1*/*x*
  for *x* :: *real*
  using *powr-int* [*of x* − *1*] **by** *simp*

lemma *powr-neg-one'* [*simp*]: *x powr* −*1* = *1*/|*x*|
  for *x* :: *real*

by (*simp add*: *powr-minus-divide*)

lemma *powr-neg-numeral*: $0 < x \Longrightarrow x$ *powr* $-$ *numeral n* $= 1/x$ ^ *numeral n*
for *x* :: *real*
using *powr-int* [*of x* $-$ *numeral n*] **by** *simp*

lemma *root-powr-inverse*: $0 < n \Longrightarrow 0 \leq x \Longrightarrow$ *root n x* $= x$ *powr* $(1/n)$
by (*simp add*: *exp-divide-power-eq powr-def real-root-pos-unique*)

lemma *powr-inverse-root*: $0 < n \Longrightarrow x$ *powr* $(1/n) = |$*root n x*$|$
by (*metis abs-ge-zero mult-1 powr-one′ powr-powr real-root-abs root-powr-inverse*)

lemma *ln-powr* [*simp*]: *ln* (*x powr y*) $= y *$ *ln x*
for *x* :: *real*
by (*simp add*: *powr-def*)

lemma *ln-root*: $n > 0 \Longrightarrow$ *ln* (*root n b*) $=$ *ln b* / *n*
by (*metis ln-powr mult-1 powr-inverse-root powr-one′ times-divide-eq-left*)

lemma *ln-sqrt*: $0 \leq x \Longrightarrow$ *ln* (*sqrt x*) $=$ *ln x* / *2*
by (*metis* (*full-types*) *divide-inverse inverse-eq-divide ln-powr mult.commute of-nat-numeral pos2 root-powr-inverse sqrt-def*)

lemma *log-root*: $n > 0 \Longrightarrow a \geq 0 \Longrightarrow$ *log b* (*root n a*) $=$ *log b a* / *n*
by (*simp add*: *log-def ln-root*)

lemma *log-powr*: *log b* (*x powr y*) $= y *$ *log b x*
by (*simp add*: *log-def*)

lemma *log-nat-power*: $0 \leq x \Longrightarrow$ *log b* (*x*^*n*) $=$ *real n* $*$ *log b x*
by (*simp add*: *ln-realpow log-def*)

lemma *log-of-power-eq*:
assumes $m = b$ ^ n $b > 1$
shows $n =$ *log b* (*real m*)
proof $-$
have $n =$ *log b* (*b* ^ *n*) **using** *assms*(*2*) **by** (*simp add*: *log-nat-power*)
also have ... $=$ *log b m* **using** *assms* **by** *simp*
finally show *?thesis* **.**
qed

lemma *log2-of-power-eq*: $m = 2$ ^ $n \Longrightarrow n =$ *log 2 m* **for** *m n* :: *nat*
using *log-of-power-eq*[*of - 2*] **by** *simp*

lemma *log-base-change*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow$ *log b x* $=$ *log a x* / *log a b*
by (*simp add*: *log-def*)

lemma *log-base-pow*: $0 < a \Longrightarrow$ *log* (*a* ^ *n*) *x* $=$ *log a x* / *n*

by (*simp add*: *log-def ln-realpow*)

lemma *log-base-powr*: $a \neq 0 \Longrightarrow log\ (a\ powr\ b)\ x = log\ a\ x\ /\ b$
by (*simp add*: *log-def ln-powr*)

lemma *log-base-root*: $n > 0 \Longrightarrow log\ (root\ n\ b)\ x = n * (log\ b\ x)$
by (*simp add*: *log-def ln-root*)

lemma *ln-bound*: $0 < x \Longrightarrow ln\ x \leq x$ **for** *x* :: *real*
using *ln-le-minus-one* **by** *force*

lemma *powr-less-one*:
fixes *x*::*real*
assumes $1 < x$ $y < 0$
shows $x\ powr\ y < 1$
using *assms less-log-iff* **by** *force*

lemma *powr-le-one-le*: $\bigwedge x\ y$::*real*. $0 < x \Longrightarrow x \leq 1 \Longrightarrow 1 \leq y \Longrightarrow x\ powr\ y \leq x$
by (*smt* (*verit*) *ln-gt-zero-imp-gt-one ln-le-cancel-iff ln-powr mult-le-cancel-right2*)

lemma *powr-mono*:
fixes *x* :: *real*
assumes $a \leq b$ **and** $1 \leq x$ **shows** $x\ powr\ a \leq x\ powr\ b$
using *assms less-eq-real-def* **by** *auto*

lemma *ge-one-powr-ge-zero*: $1 \leq x \Longrightarrow 0 \leq a \Longrightarrow 1 \leq x\ powr\ a$
for *x* :: *real*
using *powr-mono* **by** *fastforce*

lemma *powr-less-mono2*: $0 < a \Longrightarrow 0 \leq x \Longrightarrow x < y \Longrightarrow x\ powr\ a < y\ powr\ a$
for *x* :: *real*
by (*simp add*: *powr-def*)

lemma *powr-less-mono2-neg*: $a < 0 \Longrightarrow 0 < x \Longrightarrow x < y \Longrightarrow y\ powr\ a < x\ powr\ a$
for *x* :: *real*
by (*simp add*: *powr-def*)

lemma *powr-mono2*: $x\ powr\ a \leq y\ powr\ a$ **if** $0 \leq a$ $0 \leq x$ $x \leq y$
for *x* :: *real*
using *less-eq-real-def powr-less-mono2 that* **by** *auto*

lemma *powr01-less-one*:
fixes *x*::*real*
assumes $0 < x$ $x < 1$
shows $x\ powr\ a < 1 \longleftrightarrow a>0$
proof
show $x\ powr\ a < 1 \Longrightarrow a>0$
using *assms not-less-iff-gr-or-eq powr-less-mono2-neg* **by** *fastforce*

show *a>0* ⟹ *x powr a* < *1*
by (*metis assms less-eq-real-def powr-less-mono2 powr-one-eq-one*)
qed

lemma *powr-le1*: *0* ≤ *a* ⟹ |*x*| ≤ *1* ⟹ *x powr a* ≤ *1*
for *x* :: *real*
by (*smt* (*verit*, *best*) *powr-mono2 powr-one-eq-one uminus-powr-eq*)

lemma *powr-mono2'*:
fixes *a x y* :: *real*
assumes *a* ≤ *0 x* > *0 x* ≤ *y*
shows *x powr a* ≥ *y powr a*
proof −
from *assms* **have** *x powr* − *a* ≤ *y powr* − *a*
by (*intro powr-mono2*) *simp-all*
with *assms* **show** *?thesis*
by (*auto simp*: *powr-minus field-simps*)
qed

lemma *powr-mono'*: *a* ≤ (*b*::*real*) ⟹ *x* ≥ *0* ⟹ *x* ≤ *1* ⟹ *x powr b* ≤ *x powr a*
using *powr-mono*[*of* −*b* −*a inverse x*] **by** (*auto simp*: *powr-def ln-inverse ln-div field-split-simps*)

lemma *powr-mono-both*:
fixes *x* :: *real*
assumes *0* ≤ *a a* ≤ *b 1* ≤ *x x* ≤ *y*
shows *x powr a* ≤ *y powr b*
by (*meson assms order.trans powr-mono powr-mono2 zero-le-one*)

lemma *powr-mono-both'*:
fixes *x* :: *real*
assumes *a* ≥ *b b*≥*0 0* < *x x* ≤ *y y* ≤ *1*
shows *x powr a* ≤ *y powr b*
by (*meson assms nless-le order.trans powr-mono' powr-mono2*)

lemma *powr-less-mono'*:
assumes (*x*::*real*) > *0 x* < *1 a* < *b*
shows *x powr b* < *x powr a*
by (*metis assms log-powr-cancel order.strict-iff-order powr-mono'*)

lemma *powr-inj*: *0* < *a* ⟹ *a* ≠ *1* ⟹ *a powr x* = *a powr y* ⟷ *x* = *y*
for *x* :: *real*
by (*metis log-powr-cancel*)

lemma *powr-half-sqrt*: *0* ≤ *x* ⟹ *x powr* (*1*/*2*) = *sqrt x*
by (*simp add*: *powr-def root-powr-inverse sqrt-def*)

lemma *powr-half-sqrt-powr*: *0* ≤ *x* ⟹ *x powr* (*a*/*2*) = *sqrt*(*x powr a*)
by (*metis divide-inverse mult.left-neutral powr-ge-zero powr-half-sqrt powr-powr*)

lemma *square-powr-half* [*simp*]:
  fixes *x*::*real* **shows** x^2 *powr* (*1/2*) = |*x*|
  by (*simp add*: *powr-half-sqrt*)

lemma *ln-powr-bound*: *1* ≤ *x* ⟹ *0* < *a* ⟹ *ln x* ≤ (*x powr a*) / *a*
  for *x* :: *real*
  by (*metis exp-gt-zero linear ln-eq-zero-iff ln-exp ln-less-self ln-powr mult.commute mult-imp-le-div-pos not-less powr-gt-zero*)

lemma *ln-powr-bound2*:
  fixes *x* :: *real*
  assumes *1* < *x* **and** *0* < *a*
  shows (*ln x*) *powr a* ≤ (*a powr a*) ∗ *x*
proof −
  from *assms* **have** *ln x* ≤ (*x powr* (*1* / *a*)) / (*1* / *a*)
    by (*metis less-eq-real-def ln-powr-bound zero-less-divide-1-iff*)
  also have . . . = *a* ∗ (*x powr* (*1* / *a*))
    by *simp*
  finally have (*ln x*) *powr a* ≤ (*a* ∗ (*x powr* (*1* / *a*))) *powr a*
    by (*metis assms less-imp-le ln-gt-zero powr-mono2*)
  also have . . . = (*a powr a*) ∗ ((*x powr* (*1* / *a*)) *powr a*)
    using *assms powr-mult* **by** *auto*
  also have (*x powr* (*1* / *a*)) *powr a* = *x powr* ((*1* / *a*) ∗ *a*)
    by (*rule powr-powr*)
  also have . . . = *x* **using** *assms*
    by *auto*
  finally show *?thesis* **.**
qed

lemma *tendsto-powr*:
  fixes *a b* :: *real*
  assumes *f*: (*f* ⟶ *a*) *F*
    and *g*: (*g* ⟶ *b*) *F*
    and *a*: *a* ≠ *0*
  shows ((λ*x*. *f x powr g x*) ⟶ *a powr b*) *F*
  unfolding *powr-def*
proof (*rule filterlim-If*)
  show ((λ*x*. *0*) ⟶ (*if a* = *0 then 0 else exp* (*b* ∗ *ln a*))) (*inf F* (*principal* {*x*. *f x* = *0*}))
    using *tendsto-imp-eventually-ne* [*OF f*] *a*
    by (*simp add*: *filterlim-iff eventually-inf-principal frequently-def*)
  from *f g a* **show** ((λ*x*. *exp* (*g x* ∗ *ln* (*f x*))) ⟶ (*if a* = *0 then 0 else exp* (*b* ∗ *ln a*)))
      (*inf F* (*principal* {*x*. *f x* ≠ *0*}))
    by (*auto intro*!: *tendsto-intros intro*: *tendsto-mono inf-le1*)
qed

lemma *tendsto-powr'*[*tendsto-intros*]:

fixes *a* :: *real*
assumes *f*: (*f* ⟶ *a*) *F*
and *g*: (*g* ⟶ *b*) *F*
and *a*: *a* ≠ *0* ∨ (*b* > *0* ∧ *eventually* (λ*x*. *f x* ≥ *0*) *F*)
shows ((λ*x*. *f x powr g x*) ⟶ *a powr b*) *F*
proof −
from *a* **consider** *a* ≠ *0* | *a* = *0 b* > *0 eventually* (λ*x*. *f x* ≥ *0*) *F*
by *auto*
then show *?thesis*
proof *cases*
case *1*
with *f g* **show** *?thesis* **by** (*rule tendsto-powr*)
next
case *2*
have ((λ*x*. *if f x* = *0 then 0 else exp* (*g x* ∗ *ln* (*f x*))) ⟶ *0*) *F*
proof (*intro filterlim-If*)
have *filterlim f* (*principal* {*0*<..}) (*inf F* (*principal* {*z*. *f z* ≠ *0*}))
using ‹*eventually* (λ*x*. *f x* ≥ *0*) *F*›
by (*auto simp*: *filterlim-iff eventually-inf-principal*
eventually-principal elim: *eventually-mono*)
moreover have *filterlim f* (*nhds a*) (*inf F* (*principal* {*z*. *f z* ≠ *0*}))
by (*rule tendsto-mono*[*OF* - *f*]) *simp-all*
ultimately have *f*: *filterlim f* (*at-right 0*) (*inf F* (*principal* {*x*. *f x* ≠ *0*}))
by (*simp add*: *at-within-def filterlim-inf* ‹*a* = *0*›)
have *g*: (*g* ⟶ *b*) (*inf F* (*principal* {*z*. *f z* ≠ *0*}))
by (*rule tendsto-mono*[*OF* - *g*]) *simp-all*
show ((λ*x*. *exp* (*g x* ∗ *ln* (*f x*))) ⟶ *0*) (*inf F* (*principal* {*x*. *f x* ≠ *0*}))
by (*rule filterlim-compose*[*OF exp-at-bot*] *filterlim-tendsto-pos-mult-at-bot*
filterlim-compose[*OF ln-at-0*] *f g* ‹*b* > *0*›)+
qed *simp-all*
with ‹*a* = *0*› **show** *?thesis*
by (*simp add*: *powr-def*)
qed
qed

lemma *continuous-powr*:
assumes *continuous F f*
and *continuous F g*
and *f* (*Lim F* (λ*x*. *x*)) ≠ *0*
shows *continuous F* (λ*x*. (*f x*) *powr* (*g x* :: *real*))
using *assms* **unfolding** *continuous-def* **by** (*rule tendsto-powr*)

lemma *continuous-at-within-powr*[*continuous-intros*]:
fixes *f g* :: - ⇒ *real*
assumes *continuous* (*at a within s*) *f*
and *continuous* (*at a within s*) *g*
and *f a* ≠ *0*
shows *continuous* (*at a within s*) (λ*x*. (*f x*) *powr* (*g x*))
using *assms* **unfolding** *continuous-within* **by** (*rule tendsto-powr*)

lemma *continuous-on-powr*[*continuous-intros*]:
fixes *f g* :: - ⇒ *real*
assumes *continuous-on s f continuous-on s g* **and** ∀ *x*∈*s*. *f x* ≠ *0*
shows *continuous-on s* (λ*x*. (*f x*) *powr* (*g x*))
using *assms* **unfolding** *continuous-on-def* **by** (*fast intro*: *tendsto-powr*)

lemma *tendsto-powr2*:
fixes *a* :: *real*
assumes *f*: (*f* ⟶ *a*) *F*
and *g*: (*g* ⟶ *b*) *F*
and ∀$_F$ *x in F*. *0* ≤ *f x*
and *b*: *0* < *b*
shows ((λ*x*. *f x powr g x*) ⟶ *a powr b*) *F*
using *tendsto-powr′*[*of f a F g b*] *assms* **by** *auto*

lemma *has-derivative-powr*[*derivative-intros*]:
assumes *g*[*derivative-intros*]: (*g has-derivative g′*) (*at x within X*)
and *f*[*derivative-intros*]:(*f has-derivative f′*) (*at x within X*)
assumes *pos*: *0* < *g x* **and** *x* ∈ *X*
shows ((λ*x*. *g x powr f x*::*real*) *has-derivative* (λ*h*. (*g x powr f x*) ∗ (*f′ h* ∗ *ln* (*g x*) + *g′ h* ∗ *f x* / *g x*))) (*at x within X*)
proof −
have ∀$_F$ *x in at x within X*. *g x* > *0*
by (*rule order-tendstoD*[*OF* - *pos*])
(*rule has-derivative-continuous*[*OF g*, *unfolded continuous-within*])
then obtain *d* **where** *d* > *0* **and** *pos′*: ⋀*x′*. *x′* ∈ *X* ⟹ *dist x′ x* < *d* ⟹ *0* < *g x′*
using *pos* **unfolding** *eventually-at* **by** *force*
have ((λ*x*. *exp* (*f x* ∗ *ln* (*g x*))) *has-derivative*
(λ*h*. (*g x powr f x*) ∗ (*f′ h* ∗ *ln* (*g x*) + *g′ h* ∗ *f x* / *g x*))) (*at x within X*)
using *pos*
by (*auto intro*!: *derivative-eq-intros simp*: *field-split-simps powr-def*)
then show *?thesis*
by (*rule has-derivative-transform-within*[*OF* - ‹*d* > *0*› ‹*x* ∈ *X*›]) (*auto simp*: *powr-def dest*: *pos′*)
qed

lemma *has-derivative-const-powr* [*derivative-intros*]:
fixes *a*::*real*
assumes ⋀*x*. (*f has-derivative f′*) (*at x*)
shows ((λ*x*. *a powr* (*f x*)) *has-derivative* (λ*y*. *f′ y* ∗ *ln a* ∗ *a powr* (*f x*))) (*at x*)
using *assms*
apply (*simp add*: *powr-def*)
using *DERIV-compose-FDERIV DERIV-exp has-derivative-mult-left* **by** *blast*

lemma *has-real-derivative-const-powr* [*derivative-intros*]:
fixes *a*::*real*
assumes ⋀*x*. (*f has-real-derivative f′ x*) (*at x*)

shows ((*λx. a powr* (*f x*)) *has-real-derivative* (*f′ x* ∗ *ln a* ∗ *a powr* (*f x*))) (*at x*)
using *assms*
apply (*simp add*: *powr-def*)
apply (*rule assms impI derivative-eq-intros refl* | *simp*)+
done

lemma *DERIV-powr*:
fixes *r* :: *real*
assumes *g*: *DERIV g x* :> *m*
and *pos*: *g x* > *0*
and *f*: *DERIV f x* :> *r*
shows *DERIV* (*λx. g x powr f x*) *x* :> (*g x powr f x*) ∗ (*r* ∗ *ln* (*g x*) + *m* ∗ *f x* / *g x*)
using *assms*
by (*auto intro*!: *derivative-eq-intros ext simp*: *has-field-derivative-def algebra-simps*)

lemma *DERIV-fun-powr*:
fixes *r* :: *real*
assumes *g*: *DERIV g x* :> *m*
and *pos*: *g x* > *0*
shows *DERIV* (*λx.* (*g x*) *powr r*) *x* :> *r* ∗ (*g x*) *powr* (*r* − *of-nat 1*) ∗ *m*
using *DERIV-powr*[*OF g pos DERIV-const*, *of r*] *pos*
by (*simp add*: *powr-diff field-simps*)

lemma *has-real-derivative-powr*:
assumes *z* > *0*
shows ((*λz. z powr r*) *has-real-derivative r* ∗ *z powr* (*r* − *1*)) (*at z*)
proof (*subst DERIV-cong-ev*[*OF refl - refl*])
from *assms* **have** *eventually* (*λz. z* ≠ *0*) (*nhds z*)
by (*intro t1-space-nhds*) *auto*
then show *eventually* (*λz. z powr r* = *exp* (*r* ∗ *ln z*)) (*nhds z*)
unfolding *powr-def* **by** *eventually-elim simp*
from *assms* **show** ((*λz. exp* (*r* ∗ *ln z*)) *has-real-derivative r* ∗ *z powr* (*r* − *1*)) (*at z*)
by (*auto intro*!: *derivative-eq-intros simp*: *powr-def field-simps exp-diff*)
qed

declare *has-real-derivative-powr*[*THEN DERIV-chain2*, *derivative-intros*]

A more general version, by Johannes Hölzl

lemma *has-real-derivative-powr′*:
fixes *f g* :: *real* ⇒ *real*
assumes (*f has-real-derivative f′*) (*at x*)
assumes (*g has-real-derivative g′*) (*at x*)
assumes *f x* > *0*
defines *h* ≡ *λx. f x powr g x* ∗ (*g′* ∗ *ln* (*f x*) + *f′* ∗ *g x* / *f x*)
shows ((*λx. f x powr g x*) *has-real-derivative h x*) (*at x*)
proof (*subst DERIV-cong-ev*[*OF refl - refl*])
from *assms* **have** *isCont f x*

by (*simp add*: *DERIV-continuous*)
hence *f* −*x*→ *f x* **by** (*simp add*: *continuous-at*)
with ‹*f x* > *0*› **have** *eventually* (λ*x*. *f x* > *0*) (*nhds x*)
by (*auto simp*: *tendsto-at-iff-tendsto-nhds dest*: *order-tendstoD*)
thus *eventually* (λ*x*. *f x powr g x* = *exp* (*g x* ∗ *ln* (*f x*))) (*nhds x*)
by *eventually-elim* (*simp add*: *powr-def*)
next
from *assms* **show** ((λ*x*. *exp* (*g x* ∗ *ln* (*f x*))) *has-real-derivative h x*) (*at x*)
by (*auto intro*!: *derivative-eq-intros simp*: *h-def powr-def*)
qed

lemma *tendsto-zero-powrI*:
assumes (*f* ⟶ (*0*::*real*)) *F* (*g* ⟶ *b*) *F* ∀$_F$ *x in F*. *0* ≤ *f x 0* < *b*
shows ((λ*x*. *f x powr g x*) ⟶ *0*) *F*
using *tendsto-powr2*[*OF assms*] **by** *simp*

lemma *continuous-on-powr′*:
fixes *f g* :: - ⇒ *real*
assumes *continuous-on s f continuous-on s g*
and ∀ *x*∈*s*. *f x* ≥ *0* ∧ (*f x* = *0* ⟶ *g x* > *0*)
shows *continuous-on s* (λ*x*. (*f x*) *powr* (*g x*))
unfolding *continuous-on-def*
proof
fix *x*
assume *x*: *x* ∈ *s*
from *assms x* **show** ((λ*x*. *f x powr g x*) ⟶ *f x powr g x*) (*at x within s*)
proof (*cases f x* = *0*)
case *True*
from *assms*(*3*) **have** *eventually* (λ*x*. *f x* ≥ *0*) (*at x within s*)
by (*auto simp*: *at-within-def eventually-inf-principal*)
with *True x assms* **show** *?thesis*
by (*auto intro*!: *tendsto-zero-powrI*[*of f - g g x*] *simp*: *continuous-on-def*)
next
case *False*
with *assms x* **show** *?thesis*
by (*auto intro*!: *tendsto-powr′ simp*: *continuous-on-def*)
qed
qed

lemma *tendsto-neg-powr*:
assumes *s* < *0*
and *f*: *LIM x F*. *f x* :> *at-top*
shows ((λ*x*. *f x powr s*) ⟶ (*0*::*real*)) *F*
proof −
have ((λ*x*. *exp* (*s* ∗ *ln* (*f x*))) ⟶ (*0*::*real*)) *F* (**is** *?X*)
by (*auto intro*!: *filterlim-compose*[*OF exp-at-bot*] *filterlim-compose*[*OF ln-at-top*]
filterlim-tendsto-neg-mult-at-bot assms)
also have *?X* ⟷ ((λ*x*. *f x powr s*) ⟶ (*0*::*real*)) *F*
using *f filterlim-at-top-dense*[*of f F*]

by (*intro filterlim-cong*[*OF refl refl*]) (*auto simp*: *neq-iff powr-def elim*: *eventually-mono*)
finally show *?thesis* **.**
qed

lemma *tendsto-exp-limit-at-right*: ((λ*y*. (*1* + *x* ∗ *y*) *powr* (*1* / *y*)) ⟶ *exp x*) (*at-right 0*)
for *x* :: *real*
proof (*cases x* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have ((λ*y*. *ln* (*1* + *x* ∗ *y*)::*real*) *has-real-derivative 1* ∗ *x*) (*at 0*)
by (*auto intro*!: *derivative-eq-intros*)
then have ((λ*y*. *ln* (*1* + *x* ∗ *y*) / *y*) ⟶ *x*) (*at 0*)
by (*auto simp*: *has-field-derivative-def field-has-derivative-at*)
then have ∗: ((λ*y*. *exp* (*ln* (*1* + *x* ∗ *y*) / *y*)) ⟶ *exp x*) (*at 0*)
by (*rule tendsto-intros*)
then show *?thesis*
proof (*rule filterlim-mono-eventually*)
show *eventually* (λ*xa*. *exp* (*ln* (*1* + *x* ∗ *xa*) / *xa*) = (*1* + *x* ∗ *xa*) *powr* (*1* / *xa*)) (*at-right 0*)
unfolding *eventually-at-right*[*OF zero-less-one*]
using *False*
by (*intro exI*[*of* - *1* / |*x*|]) (*auto simp*: *field-simps powr-def abs-if add-nonneg-eq-0-iff*)
qed (*simp-all add*: *at-eq-sup-left-right*)
qed

lemma *tendsto-exp-limit-at-top*: ((λ*y*. (*1* + *x* / *y*) *powr y*) ⟶ *exp x*) *at-top*
for *x* :: *real*
by (*simp add*: *filterlim-at-top-to-right inverse-eq-divide tendsto-exp-limit-at-right*)

lemma *tendsto-exp-limit-sequentially*: (λ*n*. (*1* + *x* / *n*) ^ *n*) ⟶ *exp x*
for *x* :: *real*
proof (*rule filterlim-mono-eventually*)
from *reals-Archimedean2* [*of* |*x*|] **obtain** *n* :: *nat* **where** ∗: *real n* > |*x*| **..**
then have *eventually* (λ*n* :: *nat*. *0* < *1* + *x* / *real n*) *at-top*
by (*intro eventually-sequentiallyI* [*of n*]) (*auto simp*: *field-split-simps*)
then show *eventually* (λ*n*. (*1* + *x* / *n*) *powr n* = (*1* + *x* / *n*) ^ *n*) *at-top*
by (*rule eventually-mono*) (*erule powr-realpow*)
show (λ*n*. (*1* + *x* / *real n*) *powr real n*) ⟶ *exp x*
by (*rule filterlim-compose* [*OF tendsto-exp-limit-at-top filterlim-real-sequentially*])
qed *auto*

112.9 Sine and Cosine

definition *sin-coeff* :: *nat* ⇒ *real*
where *sin-coeff* = (λ*n*. *if even n then 0 else* (− *1*) ^((*n* − *Suc 0*) *div 2*) / (*fact*

n))

definition *cos-coeff* :: *nat* ⇒ *real*
  where *cos-coeff* = (*λn. if even n then* ((− *1*) ^ (*n div 2*)) / (*fact n*) *else 0*)

definition *sin* :: *'a* ⇒ *'a*::{*real-normed-algebra-1*,*banach*}
  where *sin* = (*λx.* ∑ *n. sin-coeff n* $*_R$ *x*^*n*)

definition *cos* :: *'a* ⇒ *'a*::{*real-normed-algebra-1*,*banach*}
  where *cos* = (*λx.* ∑ *n. cos-coeff n* $*_R$ *x*^*n*)

lemma *sin-coeff-0* [*simp*]: *sin-coeff 0* = *0*
  unfolding *sin-coeff-def* **by** *simp*

lemma *cos-coeff-0* [*simp*]: *cos-coeff 0* = *1*
  unfolding *cos-coeff-def* **by** *simp*

lemma *sin-coeff-Suc*: *sin-coeff* (*Suc n*) = *cos-coeff n* / *real* (*Suc n*)
  unfolding *cos-coeff-def sin-coeff-def*
  by (*simp del*: *mult-Suc*)

lemma *cos-coeff-Suc*: *cos-coeff* (*Suc n*) = − *sin-coeff n* / *real* (*Suc n*)
  unfolding *cos-coeff-def sin-coeff-def*
  by (*simp del*: *mult-Suc*) (*auto elim*: *oddE*)

lemma *summable-norm-sin*: *summable* (*λn. norm* (*sin-coeff n* $*_R$ *x*^*n*))
  for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
proof (*rule summable-comparison-test* [*OF - summable-norm-exp*])
  show ∃ *N*. ∀ *n*≥*N*. *norm* (*norm* (*sin-coeff n* $*_R$ *x* ^ *n*)) ≤ *norm* (*x* ^ *n* $/_R$ *fact n*)
    unfolding *sin-coeff-def*
    by (*auto simp*: *divide-inverse abs-mult power-abs* [*symmetric*] *zero-le-mult-iff*)
qed

lemma *summable-norm-cos*: *summable* (*λn. norm* (*cos-coeff n* $*_R$ *x*^*n*))
  for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
proof (*rule summable-comparison-test* [*OF - summable-norm-exp*])
  show ∃ *N*. ∀ *n*≥*N*. *norm* (*norm* (*cos-coeff n* $*_R$ *x* ^ *n*)) ≤ *norm* (*x* ^ *n* $/_R$ *fact n*)
    unfolding *cos-coeff-def*
    by (*auto simp*: *divide-inverse abs-mult power-abs* [*symmetric*] *zero-le-mult-iff*)
qed

lemma *sin-converges*: (*λn. sin-coeff n* $*_R$ *x*^*n*) *sums sin x*
  unfolding *sin-def*
  by (*metis* (*full-types*) *summable-norm-cancel summable-norm-sin summable-sums*)

lemma *cos-converges*: (*λn. cos-coeff n* $*_R$ *x*^*n*) *sums cos x*

unfolding *cos-def*
by (*metis* (*full-types*) *summable-norm-cancel summable-norm-cos summable-sums*)

lemma *sin-of-real*: *sin* (*of-real x*) = *of-real* (*sin x*)
for *x* :: *real*
proof −
have ($\lambda n.$ *of-real* (*sin-coeff n* $*_R$ *x*^*n*)) = ($\lambda n.$ *sin-coeff n* $*_R$ (*of-real x*)^*n*)
proof
show *of-real* (*sin-coeff n* $*_R$ *x*^*n*) = *sin-coeff n* $*_R$ *of-real x*^*n* **for** *n*
by (*simp add*: *scaleR-conv-of-real*)
qed
also have ... *sums* (*sin* (*of-real x*))
by (*rule sin-converges*)
finally have ($\lambda n.$ *of-real* (*sin-coeff n* $*_R$ *x*^*n*)) *sums* (*sin* (*of-real x*)) **.**
then show *?thesis*
using *sums-unique2 sums-of-real* [*OF sin-converges*] **by** *blast*
qed

corollary *sin-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow$ *sin* $z \in \mathbb{R}$
by (*metis Reals-cases Reals-of-real sin-of-real*)

lemma *cos-of-real*: *cos* (*of-real x*) = *of-real* (*cos x*)
for *x* :: *real*
proof −
have ($\lambda n.$ *of-real* (*cos-coeff n* $*_R$ *x*^*n*)) = ($\lambda n.$ *cos-coeff n* $*_R$ (*of-real x*)^*n*)
proof
show *of-real* (*cos-coeff n* $*_R$ *x*^*n*) = *cos-coeff n* $*_R$ *of-real x*^*n* **for** *n*
by (*simp add*: *scaleR-conv-of-real*)
qed
also have ... *sums* (*cos* (*of-real x*))
by (*rule cos-converges*)
finally have ($\lambda n.$ *of-real* (*cos-coeff n* $*_R$ *x*^*n*)) *sums* (*cos* (*of-real x*)) **.**
then show *?thesis*
using *sums-unique2 sums-of-real* [*OF cos-converges*]
by *blast*
qed

corollary *cos-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow$ *cos* $z \in \mathbb{R}$
by (*metis Reals-cases Reals-of-real cos-of-real*)

lemma *diffs-sin-coeff*: *diffs sin-coeff* = *cos-coeff*
by (*simp add*: *diffs-def sin-coeff-Suc del*: *of-nat-Suc*)

lemma *diffs-cos-coeff*: *diffs cos-coeff* = ($\lambda n.$ − *sin-coeff n*)
by (*simp add*: *diffs-def cos-coeff-Suc del*: *of-nat-Suc*)

lemma *sin-int-times-real*: *sin* (*of-int m* ∗ *of-real x*) = *of-real* (*sin* (*of-int m* ∗ *x*))
by (*metis sin-of-real of-real-mult of-real-of-int-eq*)

lemma *cos-int-times-real*: *cos* (*of-int m* ∗ *of-real x*) = *of-real* (*cos* (*of-int m* ∗ *x*))
by (*metis cos-of-real of-real-mult of-real-of-int-eq*)

Now at last we can get the derivatives of exp, sin and cos.

lemma *DERIV-sin* [*simp*]: *DERIV sin x* :> *cos x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
unfolding *sin-def cos-def scaleR-conv-of-real*
apply (*rule DERIV-cong*)
apply (*rule termdiffs* [**where** *K*=*of-real* (*norm x*) + *1* :: *'a*])
apply (*simp-all add*: *norm-less-p1 diffs-of-real diffs-sin-coeff diffs-cos-coeff summable-minus-iff scaleR-conv-of-real* [*symmetric*] *summable-norm-sin* [*THEN summable-norm-cancel*] *summable-norm-cos* [*THEN summable-norm-cancel*])
done

declare *DERIV-sin*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-sin*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-sin*[*derivative-intros*] = *DERIV-sin*[*THEN DERIV-compose-FDERIV*]

lemma *DERIV-cos* [*simp*]: *DERIV cos x* :> − *sin x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
unfolding *sin-def cos-def scaleR-conv-of-real*
apply (*rule DERIV-cong*)
apply (*rule termdiffs* [**where** *K*=*of-real* (*norm x*) + *1* :: *'a*])
apply (*simp-all add*: *norm-less-p1 diffs-of-real diffs-minus suminf-minus diffs-sin-coeff diffs-cos-coeff summable-minus-iff scaleR-conv-of-real* [*symmetric*] *summable-norm-sin* [*THEN summable-norm-cancel*] *summable-norm-cos* [*THEN summable-norm-cancel*])
done

declare *DERIV-cos*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-cos*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-cos*[*derivative-intros*] = *DERIV-cos*[*THEN DERIV-compose-FDERIV*]

lemma *isCont-sin*: *isCont sin x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*rule DERIV-sin* [*THEN DERIV-isCont*])

lemma *continuous-on-sin-real*: *continuous-on* {*a*..*b*} *sin* **for** *a*::*real*
using *continuous-at-imp-continuous-on isCont-sin* **by** *blast*

lemma *isCont-cos*: *isCont cos x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*rule DERIV-cos* [*THEN DERIV-isCont*])

lemma *continuous-on-cos-real*: *continuous-on* {*a..b*} *cos* **for** *a::real*
  using *continuous-at-imp-continuous-on isCont-cos* **by** *blast*

context
  fixes f :: $'a$::*t2-space* ⇒ $'b$::{*real-normed-field*,*banach*}
begin

lemma *isCont-sin'* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *sin* (*f x*)) *a*
  by (*rule isCont-o2* [*OF - isCont-sin*])

lemma *isCont-cos'* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *cos* (*f x*)) *a*
  by (*rule isCont-o2* [*OF - isCont-cos*])

lemma *tendsto-sin* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *sin* (*f x*)) ⟶ *sin* *a*) *F*
  by (*rule isCont-tendsto-compose* [*OF isCont-sin*])

lemma *tendsto-cos* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *cos* (*f x*)) ⟶ *cos* *a*) *F*
  by (*rule isCont-tendsto-compose* [*OF isCont-cos*])

lemma *continuous-sin* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *sin* (*f x*))
  unfolding *continuous-def* **by** (*rule tendsto-sin*)

lemma *continuous-on-sin* [*continuous-intros*]: *continuous-on s f* ⟹ *continuous-on* *s* (λ*x*. *sin* (*f x*))
  unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-sin*)

lemma *continuous-cos* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *cos* (*f x*))
  unfolding *continuous-def* **by** (*rule tendsto-cos*)

lemma *continuous-on-cos* [*continuous-intros*]: *continuous-on s f* ⟹ *continuous-on* *s* (λ*x*. *cos* (*f x*))
  unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-cos*)

end

lemma *continuous-within-sin*: *continuous* (*at z within s*) *sin*
  for *z* :: $'a$::{*real-normed-field*,*banach*}
  by (*simp add*: *continuous-within tendsto-sin*)

lemma *continuous-within-cos*: *continuous* (*at z within s*) *cos*
  for *z* :: $'a$::{*real-normed-field*,*banach*}
  by (*simp add*: *continuous-within tendsto-cos*)

112.10 Properties of Sine and Cosine

lemma *sin-zero* [*simp*]: *sin 0 = 0*
by (*simp add*: *sin-def sin-coeff-def scaleR-conv-of-real*)

lemma *cos-zero* [*simp*]: *cos 0 = 1*
by (*simp add*: *cos-def cos-coeff-def scaleR-conv-of-real*)

lemma *DERIV-fun-sin*: *DERIV g x :> m* $\Longrightarrow$ *DERIV* (λx. *sin* (*g x*)) *x :> cos* (*g x*) $*$ *m*
by (*fact derivative-intros*)

lemma *DERIV-fun-cos*: *DERIV g x :> m* $\Longrightarrow$ *DERIV* (λx. *cos*(*g x*)) *x :>* $-$ *sin* (*g x*) $*$ *m*
by (*fact derivative-intros*)

112.11 Deriving the Addition Formulas

The product of two cosine series.

lemma *cos-x-cos-y*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows
(λp. $\sum n \le p$.
if even p $\wedge$ *even n*
then (($-$*1*) ^ (*p div 2*) $*$ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) $*$ *y*^(*p*$-$*n*) *else 0*)
sums (*cos x* $*$ *cos y*)
proof $-$
have (*cos-coeff n* $*$ *cos-coeff* (*p* $-$ *n*)) $*_R$ (*x*^*n* $*$ *y*^(*p* $-$ *n*)) =
(*if even p* $\wedge$ *even n then* (($-$*1*) ^ (*p div 2*) $*$ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) $*$ *y*^(*p* $-$ *n*)
else 0)
if *n* $\le$ *p* **for** *n p* :: *nat*
proof $-$
from *that* **have** $*$: *even n* $\Longrightarrow$ *even p* $\Longrightarrow$
($-$*1*) ^ (*n div 2*) $*$ ($-$*1*) ^ ((*p* $-$ *n*) *div 2*) = ($-$*1* :: *real*) ^ (*p div 2*)
by (*metis div-add power-add le-add-diff-inverse odd-add*)
with *that* **show** *?thesis*
by (*auto simp*: *algebra-simps cos-coeff-def binomial-fact*)
qed
then have (λp. $\sum n \le p$. *if even p* $\wedge$ *even n*
then (($-$*1*) ^ (*p div 2*) $*$ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) $*$ *y*^(*p*$-$*n*) *else 0*) =
(λp. $\sum n \le p$. (*cos-coeff n* $*$ *cos-coeff* (*p* $-$ *n*)) $*_R$ (*x*^*n* $*$ *y*^(*p*$-$*n*)))
by *simp*
also have ... = (λp. $\sum n \le p$. (*cos-coeff n* $*_R$ *x*^*n*) $*$ (*cos-coeff* (*p* $-$ *n*) $*_R$ *y*^(*p*$-$*n*)))
by (*simp add*: *algebra-simps*)
also have ... *sums* (*cos x* $*$ *cos y*)

using *summable-norm-cos*
by (*auto simp*: *cos-def scaleR-conv-of-real intro*!: *Cauchy-product-sums*)
finally show *?thesis* **.**
qed

The product of two sine series.

lemma *sin-x-sin-y*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows
($\lambda p.\ \sum n{\le}p.$
if even p $\wedge$ *odd n*
then $-$ (($-$*1*) ^(*p div 2*) $*$ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) $*$ *y*^(*p*$-$*n*)
else 0)
sums (*sin x* $*$ *sin y*)
proof $-$
have (*sin-coeff n* $*$ *sin-coeff* (*p* $-$ *n*)) $*_R$ (*x*^*n* $*$ *y*^(*p*$-$*n*)) =
(*if even p* $\wedge$ *odd n*
then $-$(($-$*1*) ^(*p div 2*) $*$ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) $*$ *y*^(*p*$-$*n*)
else 0)
if *n* $\le$ *p* **for** *n p* :: *nat*
proof $-$
have ($-$*1*) ^((*n* $-$ *Suc 0*) *div 2*) $*$ ($-$*1*) ^((*p* $-$ *Suc n*) *div 2*) = $-$ (($-$*1* :: *real*) ^(*p div 2*))
if *np*: *odd n even p*
proof $-$
have *p* > *0*
using ‹*n* $\le$ *p*› *neq0-conv that*(*1*) **by** *blast*
then have §: ($-$ *1*::*real*) ^(*p div 2* $-$ *Suc 0*) = $-$ (($-$ *1*) ^(*p div 2*))
using ‹*even p*› **by** (*auto simp add*: *dvd-def power-eq-if*)
from ‹*n* $\le$ *p*› *np* **have** $*$: *n* $-$ *Suc 0* + (*p* $-$ *Suc n*) = *p* $-$ *Suc* (*Suc 0*) *Suc* (*Suc 0*) $\le$ *p*
by *arith*+
have (*p* $-$ *Suc* (*Suc 0*)) *div 2* = *p div 2* $-$ *Suc 0*
by *simp*
with ‹*n* $\le$ *p*› *np* § $*$ **show** *?thesis*
by (*simp add*: *flip*: *div-add power-add*)
qed
then show *?thesis*
using ‹*n*$\le$*p*› **by** (*auto simp*: *algebra-simps sin-coeff-def binomial-fact*)
qed
then have ($\lambda p.\ \sum n{\le}p.$ *if even p* $\wedge$ *odd n*
then $-$ (($-$*1*) ^(*p div 2*) $*$ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) $*$ *y*^(*p*$-$*n*) *else 0*) =
($\lambda p.\ \sum n{\le}p.$ (*sin-coeff n* $*$ *sin-coeff* (*p* $-$ *n*)) $*_R$ (*x*^*n* $*$ *y*^(*p*$-$*n*)))
by *simp*
also have ... = ($\lambda p.\ \sum n{\le}p.$ (*sin-coeff n* $*_R$ *x*^*n*) $*$ (*sin-coeff* (*p* $-$ *n*) $*_R$ *y*^(*p*$-$*n*)))
by (*simp add*: *algebra-simps*)
also have ... *sums* (*sin x* $*$ *sin y*)

using *summable-norm-sin*
by (*auto simp*: *sin-def scaleR-conv-of-real intro*!: *Cauchy-product-sums*)
finally show *?thesis* **.**
qed

lemma *sums-cos-x-plus-y*:
fixes $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
shows
$(\lambda p. \sum n \le p.$
if even p
then $((-1)\ \hat{}\ (p\ div\ 2) * (p\ choose\ n)\ /\ (fact\ p)) *_R (x\hat{}n) * y\hat{}(p-n)$
else $0)$
sums cos $(x + y)$
proof −
have
$(\sum n \le p.$
if even p then $((-1)\ \hat{}\ (p\ div\ 2) * (p\ choose\ n)\ /\ (fact\ p)) *_R (x\hat{}n) * y\hat{}(p-n)$
else $0) = cos\text{-}coeff\ p *_R ((x + y)\ \hat{}\ p)$
for $p :: nat$
proof −
have
$(\sum n \le p.$ *if even p then* $((-1)\ \hat{}\ (p\ div\ 2) * (p\ choose\ n)\ /\ (fact\ p)) *_R (x\hat{}n)$
$* y\hat{}(p-n)$ *else* $0) =$
(*if even p then* $\sum n \le p. ((-1)\ \hat{}\ (p\ div\ 2) * (p\ choose\ n)\ /\ (fact\ p)) *_R (x\hat{}n)$
$* y\hat{}(p-n)$ *else* $0)$
by *simp*
also have ... =
(*if even p*
then of-real $((-1)\ \hat{}\ (p\ div\ 2)\ /\ (fact\ p)) * (\sum n \le p. (p\ choose\ n) *_R (x\hat{}n)$
$* y\hat{}(p-n))$
else $0)$
by (*auto simp*: *sum-distrib-left field-simps scaleR-conv-of-real nonzero-of-real-divide*)
also have ... $= cos\text{-}coeff\ p *_R ((x + y)\ \hat{}\ p)$
by (*simp add*: *cos-coeff-def binomial-ring* [*of x y*] *scaleR-conv-of-real atLeast0At-Most*)
finally show *?thesis* **.**
qed
then have
$(\lambda p. \sum n \le p.$
if even p
then $((-1)\ \hat{}\ (p\ div\ 2) * (p\ choose\ n)\ /\ (fact\ p)) *_R (x\hat{}n) * y\hat{}(p-n)$
else $0) = (\lambda p.\ cos\text{-}coeff\ p *_R ((x+y)\hat{}p))$
by *simp*
also have ... *sums cos* $(x + y)$
by (*rule cos-converges*)
finally show *?thesis* **.**
qed

theorem *cos-add*:

fixes x :: $'a$::{*real-normed-field*,*banach*}
shows *cos* $(x + y) =$ *cos* $x *$ *cos* $y -$ *sin* $x *$ *sin* y
proof −
have
(*if even p* ∧ *even n*
then ((− *1*) ^ (*p div 2*) ∗ *int* (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) ∗ *y*^(*p*−*n*)
else 0) −
(*if even p* ∧ *odd n*
then − ((− *1*) ^ (*p div 2*) ∗ *int* (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) ∗ *y*^(*p*−*n*)
else 0) =
(*if even p then* ((−*1*) ^ (*p div 2*) ∗ (*p choose n*) / (*fact p*)) $*_R$ (*x*^*n*) ∗ *y*^(*p*−*n*)
else 0)
if $n \leq p$ **for** *n p* :: *nat*
by *simp*
then have
(λ*p*. $\sum$ *n*≤*p*. (*if even p then* ((−*1*) ^ (*p div 2*) ∗ (*p choose n*) / (*fact p*)) $*_R$
(*x*^*n*) ∗ *y*^(*p*−*n*) *else 0*))
sums (*cos x* ∗ *cos y* − *sin x* ∗ *sin y*)
using *sums-diff* [*OF cos-x-cos-y* [*of x y*] *sin-x-sin-y* [*of x y*]]
by (*simp add*: *sum-subtractf* [*symmetric*])
then show *?thesis*
by (*blast intro*: *sums-cos-x-plus-y sums-unique2*)
qed

lemma *sin-minus-converges*: (λ*n*. − (*sin-coeff n* $*_R$ (−*x*)^*n*)) *sums sin x*
proof −
have [*simp*]: ⋀*n*. − (*sin-coeff n* $*_R$ (−*x*)^*n*) = (*sin-coeff n* $*_R$ *x*^*n*)
by (*auto simp*: *sin-coeff-def elim*!: *oddE*)
show *?thesis*
by (*simp add*: *sin-def summable-norm-sin* [*THEN summable-norm-cancel*,
THEN summable-sums])
qed

lemma *sin-minus* [*simp*]: *sin* (− *x*) = − *sin x*
for x :: $'a$::{*real-normed-algebra-1*,*banach*}
using *sin-minus-converges* [*of x*]
by (*auto simp*: *sin-def summable-norm-sin* [*THEN summable-norm-cancel*]
suminf-minus sums-iff equation-minus-iff)

lemma *cos-minus-converges*: (λ*n*. (*cos-coeff n* $*_R$ (−*x*)^*n*)) *sums cos x*
proof −
have [*simp*]: ⋀*n*. (*cos-coeff n* $*_R$ (−*x*)^*n*) = (*cos-coeff n* $*_R$ *x*^*n*)
by (*auto simp*: *Transcendental.cos-coeff-def elim*!: *evenE*)
show *?thesis*
by (*simp add*: *cos-def summable-norm-cos* [*THEN summable-norm-cancel*,
THEN summable-sums])
qed

lemma *cos-minus* [*simp*]: *cos* (−*x*) = *cos x*

for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
using *cos-minus-converges* [*of x*] **by** (*metis cos-def sums-unique*)

lemma *cos-abs-real* [*simp*]: *cos* |*x* :: *real*| = *cos x*
by (*simp add*: *abs-if*)

lemma *sin-cos-squared-add* [*simp*]: $(sin\ x)^2 + (cos\ x)^2 = 1$
for *x* :: *'a*::{*real-normed-field*,*banach*}
using *cos-add* [*of x −x*]
by (*simp add*: *power2-eq-square algebra-simps*)

lemma *sin-cos-squared-add2* [*simp*]: $(cos\ x)^2 + (sin\ x)^2 = 1$
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*subst add.commute*, *rule sin-cos-squared-add*)

lemma *sin-cos-squared-add3* [*simp*]: *cos x* ∗ *cos x* + *sin x* ∗ *sin x* = *1*
for *x* :: *'a*::{*real-normed-field*,*banach*}
using *sin-cos-squared-add2* [*unfolded power2-eq-square*] **.**

lemma *sin-squared-eq*: $(sin\ x)^2 = 1 - (cos\ x)^2$
for *x* :: *'a*::{*real-normed-field*,*banach*}
unfolding *eq-diff-eq* **by** (*rule sin-cos-squared-add*)

lemma *cos-squared-eq*: $(cos\ x)^2 = 1 - (sin\ x)^2$
for *x* :: *'a*::{*real-normed-field*,*banach*}
unfolding *eq-diff-eq* **by** (*rule sin-cos-squared-add2*)

lemma *abs-sin-le-one* [*simp*]: |*sin x*| ≤ *1*
for *x* :: *real*
by (*rule power2-le-imp-le*) (*simp-all add*: *sin-squared-eq*)

lemma *sin-ge-minus-one* [*simp*]: − *1* ≤ *sin x*
for *x* :: *real*
using *abs-sin-le-one* [*of x*] **by** (*simp add*: *abs-le-iff*)

lemma *sin-le-one* [*simp*]: *sin x* ≤ *1*
for *x* :: *real*
using *abs-sin-le-one* [*of x*] **by** (*simp add*: *abs-le-iff*)

lemma *abs-cos-le-one* [*simp*]: |*cos x*| ≤ *1*
for *x* :: *real*
by (*rule power2-le-imp-le*) (*simp-all add*: *cos-squared-eq*)

lemma *cos-ge-minus-one* [*simp*]: − *1* ≤ *cos x*
for *x* :: *real*
using *abs-cos-le-one* [*of x*] **by** (*simp add*: *abs-le-iff*)

lemma *cos-le-one* [*simp*]: *cos x* ≤ *1*
for *x* :: *real*

using *abs-cos-le-one* [*of x*] **by** (*simp add*: *abs-le-iff*)

lemma *cos-diff*: $cos\ (x - y) = cos\ x * cos\ y + sin\ x * sin\ y$
for x :: $'a$::{*real-normed-field*,*banach*}
using *cos-add* [*of x* $-$ *y*] **by** *simp*

lemma *cos-double*: $cos(2*x) = (cos\ x)^2 - (sin\ x)^2$
for x :: $'a$::{*real-normed-field*,*banach*}
using *cos-add* [**where** $x{=}x$ **and** $y{=}x$] **by** (*simp add*: *power2-eq-square*)

lemma *sin-cos-le1*: $|sin\ x * sin\ y + cos\ x * cos\ y| \le 1$
for x :: *real*
using *cos-diff* [*of x y*] **by** (*metis abs-cos-le-one add.commute*)

lemma *DERIV-fun-pow*: $DERIV\ g\ x :> m \Longrightarrow DERIV\ (\lambda x.\ (g\ x)\ \hat{}\ n)\ x :> real\ n * (g\ x)\ \hat{}\,(n - 1) * m$
by (*auto intro*!: *derivative-eq-intros simp*:)

lemma *DERIV-fun-exp*: $DERIV\ g\ x :> m \Longrightarrow DERIV\ (\lambda x.\ exp\ (g\ x))\ x :> exp\ (g\ x) * m$
by (*auto intro*!: *derivative-intros*)

112.12 The Constant Pi

definition *pi* :: *real*
where $pi = 2 * (THE\ x.\ 0 \le x \wedge x \le 2 \wedge cos\ x = 0)$

Show that there's a least positive x with $cos\ x = 0$; hence define pi.

lemma *sin-paired*: $(\lambda n.\ (-\ 1)\ \hat{}\ n\ /\ (fact\ (2 * n + 1)) * x\ \hat{}\,(2 * n + 1))$ *sums* *sin x*
for x :: *real*
proof $-$
have $(\lambda n.\ \sum k = n{*}2..{<}n * 2 + 2.\ sin\text{-}coeff\ k * x\ \hat{}\ k)$ *sums sin x*
by (*rule sums-group*) (*use sin-converges* [*of x*, *unfolded scaleR-conv-of-real*] **in** *auto*)
then show *?thesis*
by (*simp add*: *sin-coeff-def ac-simps*)
qed

lemma *sin-gt-zero-02*:
fixes x :: *real*
assumes $0 < x$ **and** $x < 2$
shows $0 < sin\ x$
proof $-$
let $?f = \lambda n{::}nat.\ \sum k = n{*}2..{<}n{*}2{+}2.\ (-\ 1)\ \hat{}\,k\ /\ (fact\ (2{*}k{+}1)) * x\hat{}(2{*}k{+}1)$
have *pos*: $\forall n.\ 0 < ?f\ n$
proof
fix n :: *nat*
let $?k2 = real\ (Suc\ (Suc\ (4 * n)))$

let *?k3* = *real* (*Suc* (*Suc* (*Suc* (*4* ∗ *n*))))
have *x* ∗ *x* < *?k2* ∗ *?k3*
using *assms* **by** (*intro mult-strict-mono′*, *simp-all*)
then have *x* ∗ *x* ∗ *x* ∗ *x* ^ (*n* ∗ *4*) < *?k2* ∗ *?k3* ∗ *x* ∗ *x* ^ (*n* ∗ *4*)
by (*intro mult-strict-right-mono zero-less-power* ‹*0* < *x*›)
then show *0* < *?f n*
by (*simp add*: *ac-simps divide-less-eq*)
qed
have *sums*: *?f sums sin x*
by (*rule sin-paired* [*THEN sums-group*]) *simp*
show *0* < *sin x*
unfolding *sums-unique* [*OF sums*] **using** *sums-summable* [*OF sums*] *pos* **by** (*simp add*: *suminf-pos*)
qed

lemma *cos-double-less-one*: *0* < *x* ⟹ *x* < *2* ⟹ *cos* (*2* ∗ *x*) < *1*
for *x* :: *real*
using *sin-gt-zero-02* [**where** *x* = *x*] **by** (*auto simp*: *cos-squared-eq cos-double*)

lemma *cos-paired*: (λ*n*. (− *1*) ^ *n* / (*fact* (*2* ∗ *n*)) ∗ *x* ^ (*2* ∗ *n*)) *sums cos x*
for *x* :: *real*
proof −
have (λ*n*. ∑ *k* = *n* ∗ *2*..<*n* ∗ *2* + *2*. *cos-coeff k* ∗ *x* ^ *k*) *sums cos x*
by (*rule sums-group*) (*use cos-converges* [*of x*, *unfolded scaleR-conv-of-real*] **in** *auto*)
then show *?thesis*
by (*simp add*: *cos-coeff-def ac-simps*)
qed

lemma *sum-pos-lt-pair*:
fixes *f* :: *nat* ⇒ *real*
assumes *f*: *summable f* **and** *fplus*: ⋀*d*. *0* < *f* (*k* + (*Suc*(*Suc 0*) ∗ *d*)) + *f* (*k* + ((*Suc* (*Suc 0*) ∗ *d*) + *1*))
shows *sum f* {..<*k*} < *suminf f*
proof −
have (λ*n*. ∑ *n* = *n* ∗ *Suc* (*Suc 0*)..<*n* ∗ *Suc* (*Suc 0*) + *Suc* (*Suc 0*). *f* (*n* + *k*))
sums (∑ *n*. *f* (*n* + *k*))
proof (*rule sums-group*)
show (λ*n*. *f* (*n* + *k*)) *sums* (∑ *n*. *f* (*n* + *k*))
by (*simp add*: *f summable-iff-shift summable-sums*)
qed *auto*
with *fplus* **have** *0* < (∑ *n*. *f* (*n* + *k*))
apply (*simp add*: *add.commute*)
apply (*metis* (*no-types*, *lifting*) *suminf-pos summable-def sums-unique*)
done
then show *?thesis*
by (*simp add*: *f suminf-minus-initial-segment*)
qed

lemma *cos-two-less-zero* [*simp*]: *cos 2* < (*0*::*real*)
proof −
  note *fact-Suc* [*simp del*]
  from *sums-minus* [*OF cos-paired*]
  have ∗: (λ*n*. − ((− *1*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / *fact* (*2* ∗ *n*))) *sums* − *cos* (*2*::*real*)
    by *simp*
  then have *sm*: *summable* (λ*n*. − ((− *1*::*real*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / (*fact* (*2* ∗ *n*))))
    by (*rule sums-summable*)
  have *0* < (∑ *n*<*Suc* (*Suc* (*Suc 0*)). − ((− *1*::*real*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / (*fact* (*2* ∗ *n*))))
    by (*simp add*: *fact-num-eq-if power-eq-if*)
  moreover have (∑ *n*<*Suc* (*Suc* (*Suc 0*)). − ((− *1*::*real*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / (*fact* (*2* ∗ *n*)))) <
    (∑ *n*. − ((− *1*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / (*fact* (*2* ∗ *n*))))
  proof −
    {
      fix *d*
      let *?six4d* = *Suc* (*Suc* (*Suc* (*Suc* (*Suc* (*Suc* (*4* ∗ *d*))))))
      have (*4*::*real*) ∗ (*fact* (*?six4d*)) < (*Suc* (*Suc* (*?six4d*)) ∗ *fact* (*Suc* (*?six4d*)))
      unfolding *of-nat-mult* **by** (*rule mult-strict-mono*) (*simp-all add*: *fact-less-mono*)
      then have (*4*::*real*) ∗ (*fact* (*?six4d*)) < (*fact* (*Suc* (*Suc* (*?six4d*))))
        by (*simp only*: *fact-Suc* [*of Suc* (*?six4d*)] *of-nat-mult of-nat-fact*)
      then have (*4*::*real*) ∗ *inverse* (*fact* (*Suc* (*Suc* (*?six4d*)))) < *inverse* (*fact* (*?six4d*))
        by (*simp add*: *inverse-eq-divide less-divide-eq*)
    }
    then show *?thesis*
      by (*force intro*!: *sum-pos-lt-pair* [*OF sm*] *simp add*: *divide-inverse algebra-simps*)
  qed
  ultimately have *0* < (∑ *n*. − ((− *1*::*real*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / (*fact* (*2* ∗ *n*))))
    by (*rule order-less-trans*)
  moreover from ∗ **have** − *cos 2* = (∑ *n*. − ((− *1*::*real*) ^ *n* ∗ *2* ^ (*2* ∗ *n*) / (*fact* (*2* ∗ *n*))))
    by (*rule sums-unique*)
  ultimately have (*0*::*real*) < − *cos 2* **by** *simp*
  then show *?thesis* **by** *simp*
qed

lemmas *cos-two-neq-zero* [*simp*] = *cos-two-less-zero* [*THEN less-imp-neq*]
lemmas *cos-two-le-zero* [*simp*] = *cos-two-less-zero* [*THEN order-less-imp-le*]

lemma *cos-is-zero*: ∃!*x*::*real*. *0* ≤ *x* ∧ *x* ≤ *2* ∧ *cos x* = *0*
proof (*rule ex-ex1I*)
  show ∃ *x*::*real*. *0* ≤ *x* ∧ *x* ≤ *2* ∧ *cos x* = *0*
    by (*rule IVT2*) *simp-all*
next

fix *a b* :: *real*
assume *ab*: $0 \le a \wedge a \le 2 \wedge cos\ a = 0$ $0 \le b \wedge b \le 2 \wedge cos\ b = 0$
have *cosd*: $\bigwedge x$::*real*. *cos differentiable* (*at x*)
unfolding *real-differentiable-def* **by** (*auto intro*: *DERIV-cos*)
show $a = b$
proof (*cases a b rule*: *linorder-cases*)
case *less*
then obtain *z* **where** $a < z$ $z < b$ (*cos has-real-derivative 0*) (*at z*)
using *Rolle* **by** (*metis cosd continuous-on-cos-real ab*)
then have $sin\ z = 0$
using *DERIV-cos DERIV-unique neg-equal-0-iff-equal* **by** *blast*
then show *?thesis*
by (*metis* ‹$a < z$› ‹$z < b$› *ab order-less-le-trans less-le sin-gt-zero-02*)
next
case *greater*
then obtain *z* **where** $b < z$ $z < a$ (*cos has-real-derivative 0*) (*at z*)
using *Rolle* **by** (*metis cosd continuous-on-cos-real ab*)
then have $sin\ z = 0$
using *DERIV-cos DERIV-unique neg-equal-0-iff-equal* **by** *blast*
then show *?thesis*
by (*metis* ‹$b < z$› ‹$z < a$› *ab order-less-le-trans less-le sin-gt-zero-02*)
qed *auto*
qed

lemma *pi-half*: $pi/2 = (THE\ x.\ 0 \le x \wedge x \le 2 \wedge cos\ x = 0)$
by (*simp add*: *pi-def*)

lemma *cos-pi-half* [*simp*]: $cos\ (pi/2) = 0$
by (*simp add*: *pi-half cos-is-zero* [*THEN theI'*])

lemma *cos-of-real-pi-half* [*simp*]: $cos\ ((of\text{-}real\ pi/2) :: 'a) = 0$
if *SORT-CONSTRAINT*('*a*::{*real-field*,*banach*,*real-normed-algebra-1*})
by (*metis cos-pi-half cos-of-real eq-numeral-simps*(*4*)
nonzero-of-real-divide of-real-0 of-real-numeral)

lemma *pi-half-gt-zero* [*simp*]: $0 < pi/2$
proof −
have $0 \le pi/2$
by (*simp add*: *pi-half cos-is-zero* [*THEN theI'*])
then show *?thesis*
by (*metis cos-pi-half cos-zero less-eq-real-def one-neq-zero*)
qed

lemmas *pi-half-neq-zero* [*simp*] = *pi-half-gt-zero* [*THEN less-imp-neq*, *symmetric*]
lemmas *pi-half-ge-zero* [*simp*] = *pi-half-gt-zero* [*THEN order-less-imp-le*]

lemma *pi-half-less-two* [*simp*]: $pi/2 < 2$
proof −
have $pi/2 \le 2$

by (*simp add*: *pi-half cos-is-zero* [*THEN theI'*])
then show *?thesis*
by (*metis cos-pi-half cos-two-neq-zero le-less*)
qed

lemmas *pi-half-neq-two* [*simp*] = *pi-half-less-two* [*THEN less-imp-neq*]
lemmas *pi-half-le-two* [*simp*] = *pi-half-less-two* [*THEN order-less-imp-le*]

lemma *pi-gt-zero* [*simp*]: $0 < pi$
using *pi-half-gt-zero* **by** *simp*

lemma *pi-ge-zero* [*simp*]: $0 \leq pi$
by (*rule pi-gt-zero* [*THEN order-less-imp-le*])

lemma *pi-neq-zero* [*simp*]: $pi \neq 0$
by (*rule pi-gt-zero* [*THEN less-imp-neq*, *symmetric*])

lemma *pi-not-less-zero* [*simp*]: $\neg\ pi < 0$
by (*simp add*: *linorder-not-less*)

lemma *minus-pi-half-less-zero*: $-(pi/2) < 0$
by *simp*

lemma *m2pi-less-pi*: $-\ (2*pi) < pi$
by *simp*

lemma *sin-pi-half* [*simp*]: $sin(pi/2) = 1$
using *sin-cos-squared-add2* [**where** $x = pi/2$]
using *sin-gt-zero-02* [*OF pi-half-gt-zero pi-half-less-two*]
by (*simp add*: *power2-eq-1-iff*)

lemma *sin-of-real-pi-half* [*simp*]: *sin* ((*of-real pi/2*) :: *'a*) = *1*
if *SORT-CONSTRAINT*(*'a*::{*real-field*,*banach*,*real-normed-algebra-1*})
using *sin-pi-half*
by (*metis sin-pi-half eq-numeral-simps*(*4*) *nonzero-of-real-divide of-real-1 of-real-numeral sin-of-real*)

lemma *sin-cos-eq*: *sin x* = *cos* (*of-real pi/2* − *x*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*simp add*: *cos-diff*)

lemma *minus-sin-cos-eq*: − *sin x* = *cos* (*x* + *of-real pi/2*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*simp add*: *cos-add nonzero-of-real-divide*)

lemma *cos-sin-eq*: *cos x* = *sin* (*of-real pi/2* − *x*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
using *sin-cos-eq* [*of of-real pi/2* − *x*] **by** *simp*

lemma *sin-add*: *sin* (*x* + *y*) = *sin x* ∗ *cos y* + *cos x* ∗ *sin y*
for *x* :: ′*a*::{*real-normed-field*,*banach*}
using *cos-add* [*of of-real pi*/*2* − *x* −*y*]
by (*simp add*: *cos-sin-eq*) (*simp add*: *sin-cos-eq*)

lemma *sin-diff*: *sin* (*x* − *y*) = *sin x* ∗ *cos y* − *cos x* ∗ *sin y*
for *x* :: ′*a*::{*real-normed-field*,*banach*}
using *sin-add* [*of x* − *y*] **by** *simp*

lemma *sin-double*: *sin*(*2* ∗ *x*) = *2* ∗ *sin x* ∗ *cos x*
for *x* :: ′*a*::{*real-normed-field*,*banach*}
using *sin-add* [**where** *x*=*x* **and** *y*=*x*] **by** *simp*

lemma *cos-of-real-pi* [*simp*]: *cos* (*of-real pi*) = −*1*
using *cos-add* [**where** *x* = *pi*/*2* **and** *y* = *pi*/*2*]
by (*simp add*: *cos-of-real*)

lemma *sin-of-real-pi* [*simp*]: *sin* (*of-real pi*) = *0*
using *sin-add* [**where** *x* = *pi*/*2* **and** *y* = *pi*/*2*]
by (*simp add*: *sin-of-real*)

lemma *cos-pi* [*simp*]: *cos pi* = −*1*
using *cos-add* [**where** *x* = *pi*/*2* **and** *y* = *pi*/*2*] **by** *simp*

lemma *sin-pi* [*simp*]: *sin pi* = *0*
using *sin-add* [**where** *x* = *pi*/*2* **and** *y* = *pi*/*2*] **by** *simp*

lemma *sin-periodic-pi* [*simp*]: *sin* (*x* + *pi*) = − *sin x*
by (*simp add*: *sin-add*)

lemma *sin-periodic-pi2* [*simp*]: *sin* (*pi* + *x*) = − *sin x*
by (*simp add*: *sin-add*)

lemma *cos-periodic-pi* [*simp*]: *cos* (*x* + *pi*) = − *cos x*
by (*simp add*: *cos-add*)

lemma *cos-periodic-pi2* [*simp*]: *cos* (*pi* + *x*) = − *cos x*
by (*simp add*: *cos-add*)

lemma *sin-periodic* [*simp*]: *sin* (*x* + *2* ∗ *pi*) = *sin x*
by (*simp add*: *sin-add sin-double cos-double*)

lemma *cos-periodic* [*simp*]: *cos* (*x* + *2* ∗ *pi*) = *cos x*
by (*simp add*: *cos-add sin-double cos-double*)

lemma *cos-npi* [*simp*]: *cos* (*real n* ∗ *pi*) = (− *1*) ^ *n*
by (*induct n*) (*auto simp*: *distrib-right*)

lemma *cos-npi2* [*simp*]: *cos* (*pi* ∗ *real n*) = (− *1*) ^ *n*

by (*metis cos-npi mult.commute*)

lemma *sin-npi* [*simp*]: *sin* (*real n* ∗ *pi*) = *0*
for *n* :: *nat*
by (*induct n*) (*auto simp*: *distrib-right*)

lemma *sin-npi2* [*simp*]: *sin* (*pi* ∗ *real n*) = *0*
for *n* :: *nat*
by (*simp add*: *mult.commute* [*of pi*])

lemma *sin-npi-numeral* [*simp*]: *sin*(*Num.numeral n* ∗ *pi*) = *0*
by (*metis of-nat-numeral sin-npi*)

lemma *sin-npi2-numeral* [*simp*]: *sin* (*pi* ∗ *Num.numeral n*) = *0*
by (*metis of-nat-numeral sin-npi2*)

lemma *cos-npi-numeral* [*simp*]: *cos* (*Num.numeral n* ∗ *pi*) = (− *1*) ^ *Num.numeral n*
by (*metis cos-npi of-nat-numeral*)

lemma *cos-npi2-numeral* [*simp*]: *cos* (*pi* ∗ *Num.numeral n*) = (− *1*) ^ *Num.numeral n*
by (*metis cos-npi2 of-nat-numeral*)

lemma *cos-two-pi* [*simp*]: *cos* (*2* ∗ *pi*) = *1*
by (*simp add*: *cos-double*)

lemma *sin-two-pi* [*simp*]: *sin* (*2* ∗ *pi*) = *0*
by (*simp add*: *sin-double*)

context
fixes *w* :: *'a*::{*real-normed-field*,*banach*}

begin

lemma *sin-times-sin*: *sin w* ∗ *sin z* = (*cos* (*w* − *z*) − *cos* (*w* + *z*)) / *2*
by (*simp add*: *cos-diff cos-add*)

lemma *sin-times-cos*: *sin w* ∗ *cos z* = (*sin* (*w* + *z*) + *sin* (*w* − *z*)) / *2*
by (*simp add*: *sin-diff sin-add*)

lemma *cos-times-sin*: *cos w* ∗ *sin z* = (*sin* (*w* + *z*) − *sin* (*w* − *z*)) / *2*
by (*simp add*: *sin-diff sin-add*)

lemma *cos-times-cos*: *cos w* ∗ *cos z* = (*cos* (*w* − *z*) + *cos* (*w* + *z*)) / *2*
by (*simp add*: *cos-diff cos-add*)

lemma *cos-double-cos*: *cos* (*2* ∗ *w*) = *2* ∗ *cos w* ^ *2* − *1*
by (*simp add*: *cos-double sin-squared-eq*)

lemma *cos-double-sin*: $cos\ (2 * w) = 1 - 2 * sin\ w\ \hat{}\ 2$
  by (*simp add*: *cos-double sin-squared-eq*)

end

lemma *sin-plus-sin*: $sin\ w + sin\ z = 2 * sin\ ((w + z) / 2) * cos\ ((w - z) / 2)$
  for w :: $'a$::{*real-normed-field*,*banach*}
  apply (*simp add*: *mult.assoc sin-times-cos*)
  apply (*simp add*: *field-simps*)
  done

lemma *sin-diff-sin*: $sin\ w - sin\ z = 2 * sin\ ((w - z) / 2) * cos\ ((w + z) / 2)$
  for w :: $'a$::{*real-normed-field*,*banach*}
  apply (*simp add*: *mult.assoc sin-times-cos*)
  apply (*simp add*: *field-simps*)
  done

lemma *cos-plus-cos*: $cos\ w + cos\ z = 2 * cos\ ((w + z) / 2) * cos\ ((w - z) / 2)$
  for w :: $'a$::{*real-normed-field*,*banach*,*field*}
  apply (*simp add*: *mult.assoc cos-times-cos*)
  apply (*simp add*: *field-simps*)
  done

lemma *cos-diff-cos*: $cos\ w - cos\ z = 2 * sin\ ((w + z) / 2) * sin\ ((z - w) / 2)$
  for w :: $'a$::{*real-normed-field*,*banach*,*field*}
  apply (*simp add*: *mult.assoc sin-times-sin*)
  apply (*simp add*: *field-simps*)
  done

lemma *sin-pi-minus* [*simp*]: $sin\ (pi - x) = sin\ x$
  by (*metis sin-minus sin-periodic-pi minus-minus uminus-add-conv-diff*)

lemma *cos-pi-minus* [*simp*]: $cos\ (pi - x) = - (cos\ x)$
  by (*metis cos-minus cos-periodic-pi uminus-add-conv-diff*)

lemma *sin-minus-pi* [*simp*]: $sin\ (x - pi) = - (sin\ x)$
  by (*simp add*: *sin-diff*)

lemma *cos-minus-pi* [*simp*]: $cos\ (x - pi) = - (cos\ x)$
  by (*simp add*: *cos-diff*)

lemma *sin-2pi-minus* [*simp*]: $sin\ (2 * pi - x) = - (sin\ x)$
  by (*metis sin-periodic-pi2 add-diff-eq mult-2 sin-pi-minus*)

lemma *cos-2pi-minus* [*simp*]: $cos\ (2 * pi - x) = cos\ x$
  by (*metis* (*no-types*, *opaque-lifting*) *cos-add cos-minus cos-two-pi sin-minus sin-two-pi diff-0-right minus-diff-eq mult-1 mult-zero-left uminus-add-conv-diff*)

lemma *sin-gt-zero2*: $0 < x \Longrightarrow x < pi/2 \Longrightarrow 0 < sin\ x$
by (*metis sin-gt-zero-02 order-less-trans pi-half-less-two*)

lemma *sin-less-zero*:
assumes $-\ pi/2 < x$ **and** $x < 0$
shows $sin\ x < 0$
proof −
have $0 < sin\ (-\ x)$
using *assms* **by** (*simp only*: *sin-gt-zero2*)
then show *?thesis* **by** *simp*
qed

lemma *pi-less-4*: $pi < 4$
using *pi-half-less-two* **by** *auto*

lemma *cos-gt-zero*: $0 < x \Longrightarrow x < pi/2 \Longrightarrow 0 < cos\ x$
by (*simp add*: *cos-sin-eq sin-gt-zero2*)

lemma *cos-gt-zero-pi*: $-(pi/2) < x \Longrightarrow x < pi/2 \Longrightarrow 0 < cos\ x$
using *cos-gt-zero* [*of x*] *cos-gt-zero* [*of* $-x$]
by (*cases rule*: *linorder-cases* [*of x 0*]) *auto*

lemma *cos-ge-zero*: $-(pi/2) \leq x \Longrightarrow x \leq pi/2 \Longrightarrow 0 \leq cos\ x$
by (*auto simp*: *order-le-less cos-gt-zero-pi*)
(*metis cos-pi-half eq-divide-eq eq-numeral-simps*(*4*))

lemma *sin-gt-zero*: $0 < x \Longrightarrow x < pi \Longrightarrow 0 < sin\ x$
by (*simp add*: *sin-cos-eq cos-gt-zero-pi*)

lemma *sin-lt-zero*: $pi < x \Longrightarrow x < 2 * pi \Longrightarrow sin\ x < 0$
using *sin-gt-zero* [*of* $x - pi$]
by (*simp add*: *sin-diff*)

lemma *pi-ge-two*: $2 \leq pi$
proof (*rule ccontr*)
assume ¬ *?thesis*
then have $pi < 2$ **by** *auto*
have $\exists y > pi.\ y < 2 \wedge y < 2 * pi$
proof (*cases* $2 < 2 * pi$)
case *True*
with *dense*[*OF* ‹$pi < 2$›] **show** *?thesis* **by** *auto*
next
case *False*
have $pi < 2 * pi$ **by** *auto*
from *dense*[*OF this*] **and** *False* **show** *?thesis* **by** *auto*
qed
then obtain y **where** $pi < y$ **and** $y < 2$ **and** $y < 2 * pi$
by *blast*
then have $0 < sin\ y$

using *sin-gt-zero-02* **by** *auto*
moreover have *sin y < 0*
using *sin-gt-zero*[*of y − pi*] ‹*pi < y*› **and** ‹*y < 2 ∗ pi*› *sin-periodic-pi*[*of y − pi*]
by *auto*
ultimately show *False* **by** *auto*
qed

lemma *sin-ge-zero*: *0 ≤ x =⇒ x ≤ pi =⇒ 0 ≤ sin x*
by (*auto simp*: *order-le-less sin-gt-zero*)

lemma *sin-le-zero*: *pi ≤ x =⇒ x < 2 ∗ pi =⇒ sin x ≤ 0*
using *sin-ge-zero* [*of x − pi*] **by** (*simp add*: *sin-diff*)

lemma *sin-pi-divide-n-ge-0* [*simp*]:
assumes *n ≠ 0*
shows *0 ≤ sin* (*pi/real n*)
by (*rule sin-ge-zero*) (*use assms* **in** ‹*simp-all add*: *field-split-simps*›)

lemma *sin-pi-divide-n-gt-0*:
assumes *2 ≤ n*
shows *0 < sin* (*pi/real n*)
by (*rule sin-gt-zero*) (*use assms* **in** ‹*simp-all add*: *field-split-simps*›)

Proof resembles that of *cos-is-zero* but with *pi* for the upper bound

lemma *cos-total*:
assumes *y*: *−1 ≤ y y ≤ 1*
shows ∃!*x*. *0 ≤ x ∧ x ≤ pi ∧ cos x = y*
proof (*rule ex-ex1I*)
show ∃ *x*::*real*. *0 ≤ x ∧ x ≤ pi ∧ cos x = y*
by (*rule IVT2*) (*simp-all add*: *y*)
next
fix *a b* :: *real*
assume *ab*: *0 ≤ a ∧ a ≤ pi ∧ cos a = y 0 ≤ b ∧ b ≤ pi ∧ cos b = y*
have *cosd*: ⋀*x*::*real*. *cos differentiable* (*at x*)
unfolding *real-differentiable-def* **by** (*auto intro*: *DERIV-cos*)
show *a = b*
proof (*cases a b rule*: *linorder-cases*)
case *less*
then obtain *z* **where** *a < z z < b* (*cos has-real-derivative 0*) (*at z*)
using *Rolle* **by** (*metis cosd continuous-on-cos-real ab*)
then have *sin z = 0*
using *DERIV-cos DERIV-unique neg-equal-0-iff-equal* **by** *blast*
then show *?thesis*
by (*metis* ‹*a < z*› ‹*z < b*› *ab order-less-le-trans less-le sin-gt-zero*)
next
case *greater*
then obtain *z* **where** *b < z z < a* (*cos has-real-derivative 0*) (*at z*)
using *Rolle* **by** (*metis cosd continuous-on-cos-real ab*)

then have *sin z = 0*
using *DERIV-cos DERIV-unique neg-equal-0-iff-equal* **by** *blast*
then show *?thesis*
by (*metis* ‹*b* < *z*› ‹*z* < *a*› *ab order-less-le-trans less-le sin-gt-zero*)
qed *auto*
qed

lemma *sin-total*:
assumes *y*: $-1 \leq y$ $y \leq 1$
shows $\exists!x.\ -(pi/2) \leq x \wedge x \leq pi/2 \wedge sin\ x = y$
proof −
from *cos-total* [*OF y*]
obtain *x* **where** *x*: $0 \leq x$ $x \leq pi$ $cos\ x = y$
and *uniq*: $\bigwedge x'.\ 0 \leq x' \Longrightarrow x' \leq pi \Longrightarrow cos\ x' = y \Longrightarrow x' = x$
by *blast*
show *?thesis*
unfolding *sin-cos-eq*
proof (*rule ex1I* [**where** *a*=*pi*/*2* − *x*])
show $-(pi/2) \leq z \wedge z \leq pi/2 \wedge cos\ (of\text{-}real\ pi/2 - z) = y \Longrightarrow$
$z = pi/2 - x$ **for** *z*
using *uniq* [*of pi*/*2* −*z*] **by** *auto*
qed (*use x* **in** *auto*)
qed

lemma *cos-zero-lemma*:
assumes $0 \leq x$ $cos\ x = 0$
shows $\exists n.\ odd\ n \wedge x = of\text{-}nat\ n * (pi/2)$
proof −
have *xle*: $x < (1 + real\text{-}of\text{-}int\ \lfloor x/pi \rfloor) * pi$
using *floor-correct* [*of x*/*pi*]
by (*simp add*: *add.commute divide-less-eq*)
obtain *n* **where** $real\ n * pi \leq x$ $x < real\ (Suc\ n) * pi$
proof
show $real\ (nat\ \lfloor x\ /\ pi \rfloor) * pi \leq x$
using *assms floor-divide-lower* [*of pi x*] **by** *auto*
show $x < real\ (Suc\ (nat\ \lfloor x\ /\ pi \rfloor)) * pi$
using *assms floor-divide-upper* [*of pi x*] **by** (*simp add*: *xle*)
qed
then have *x*: $0 \leq x - n * pi$ $(x - n * pi) \leq pi$ $cos\ (x - n * pi) = 0$
by (*auto simp*: *algebra-simps cos-diff assms*)
then have $\exists!x.\ 0 \leq x \wedge x \leq pi \wedge cos\ x = 0$
by (*auto simp*: *intro*!: *cos-total*)
then obtain ϑ **where** ϑ: $0 \leq \vartheta$ $\vartheta \leq pi$ $cos\ \vartheta = 0$
and *uniq*: $\bigwedge\varphi.\ 0 \leq \varphi \Longrightarrow \varphi \leq pi \Longrightarrow cos\ \varphi = 0 \Longrightarrow \varphi = \vartheta$
by *blast*
then have $x - real\ n * pi = \vartheta$
using *x* **by** *blast*
moreover have $pi/2 = \vartheta$
using *pi-half-ge-zero uniq* **by** *fastforce*

ultimately show *?thesis*
by (*rule-tac x* = *Suc* (*2* ∗ *n*) **in** *exI*) (*simp add*: *algebra-simps*)
qed

lemma *sin-zero-lemma*:
assumes *0* ≤ *x sin x* = *0*
shows ∃ *n*::*nat*. *even n* ∧ *x* = *real n* ∗ (*pi*/*2*)
proof −
obtain *n* **where** *odd n* **and** *n*: *x* + *pi*/*2* = *of-nat n* ∗ (*pi*/*2*) *n* > *0*
using *cos-zero-lemma* [*of x* + *pi*/*2*] *assms* **by** (*auto simp add*: *cos-add*)
then have *x* = *real* (*n* − *1*) ∗ (*pi*/*2*)
by (*simp add*: *algebra-simps of-nat-diff*)
then show *?thesis*
by (*simp add*: ‹*odd n*›)
qed

lemma *cos-zero-iff*:
cos x = *0* ⟷ ((∃ *n*. *odd n* ∧ *x* = *real n* ∗ (*pi*/*2*)) ∨ (∃ *n*. *odd n* ∧ *x* = − (*real n* ∗ (*pi*/*2*))))
(**is** *?lhs* = *?rhs*)
proof −
have ∗: *cos* (*real n* ∗ *pi*/*2*) = *0* **if** *odd n* **for** *n* :: *nat*
proof −
from *that* **obtain** *m* **where** *n* = *2* ∗ *m* + *1* **..**
then show *?thesis*
by (*simp add*: *field-simps*) (*simp add*: *cos-add add-divide-distrib*)
qed
show *?thesis*
proof
show *?rhs* **if** *?lhs*
using *that cos-zero-lemma* [*of x*] *cos-zero-lemma* [*of* −*x*] **by** *force*
show *?lhs* **if** *?rhs*
using *that* **by** (*auto dest*: ∗ *simp del*: *eq-divide-eq-numeral1*)
qed
qed

lemma *sin-zero-iff*:
sin x = *0* ⟷ ((∃ *n*. *even n* ∧ *x* = *real n* ∗ (*pi*/*2*)) ∨ (∃ *n*. *even n* ∧ *x* = − (*real n* ∗ (*pi*/*2*))))
(**is** *?lhs* = *?rhs*)
proof
show *?rhs* **if** *?lhs*
using *that sin-zero-lemma* [*of x*] *sin-zero-lemma* [*of* −*x*] **by** *force*
show *?lhs* **if** *?rhs*
using *that* **by** (*auto elim*: *evenE*)
qed

lemma *sin-zero-pi-iff*:
fixes *x*::*real*

assumes $|x| < pi$
shows *sin* $x = 0 \longleftrightarrow x = 0$
proof
show $x = 0$ **if** *sin* $x = 0$
using *that assms* **by** (*auto simp*: *sin-zero-iff*)
qed *auto*

lemma *cos-zero-iff-int*: *cos* $x = 0 \longleftrightarrow (\exists i.$ *odd* $i \wedge x =$ *of-int* $i * (pi/2))$
proof −
have *1*: $\bigwedge n.$ *odd* $n \Longrightarrow \exists i.$ *odd* $i \wedge$ *real* $n =$ *real-of-int* i
by (*metis even-of-nat-iff of-int-of-nat-eq*)
have *2*: $\bigwedge n.$ *odd* $n \Longrightarrow \exists i.$ *odd* $i \wedge -$ (*real* $n * pi$) $=$ *real-of-int* $i * pi$
by (*metis even-minus even-of-nat-iff mult.commute mult-minus-right of-int-minus of-int-of-nat-eq*)
have *3*: ⟦*odd* i; $\forall n.$ *even* $n \vee$ *real-of-int* $i \neq -$ (*real* n)⟧
$\Longrightarrow \exists n.$ *odd* $n \wedge$ *real-of-int* $i =$ *real* n **for** i
by (*cases i rule*: *int-cases2*) *auto*
show *?thesis*
by (*force simp*: *cos-zero-iff intro*!: *1 2 3*)
qed

lemma *sin-zero-iff-int*: *sin* $x = 0 \longleftrightarrow (\exists i.$ *even* $i \wedge x =$ *of-int* $i * (pi/2))$ (**is** *?lhs* = *?rhs*)
proof *safe*
assume *?lhs*
then consider (*plus*) n **where** *even* n $x =$ *real* $n * (pi/2)$ | (*minus*) n **where** *even* n $x = -$ (*real* $n * (pi/2)$)
using *sin-zero-iff* **by** *auto*
then show $\exists n.$ *even* $n \wedge x =$ *of-int* $n * (pi/2)$
proof *cases*
case *plus*
then show *?rhs*
by (*metis even-of-nat-iff of-int-of-nat-eq*)
next
case *minus*
then show *?thesis*
by (*rule-tac x=*− (*int n*) **in** *exI*) *simp*
qed
next
fix i :: *int*
assume *even* i
then show *sin* (*of-int* $i * (pi/2)$) $= 0$
by (*cases i rule*: *int-cases2*, *simp-all add*: *sin-zero-iff*)
qed

lemma *sin-zero-iff-int2*: *sin* $x = 0 \longleftrightarrow (\exists i$::*int*. $x =$ *of-int* $i * pi)$
proof −
have *sin* $x = 0 \longleftrightarrow (\exists i.$ *even* $i \wedge x =$ *real-of-int* $i * (pi/2))$
by (*auto simp*: *sin-zero-iff-int*)

also have ... = (∃*j*. *x* = *real-of-int* (*2*∗*j*) ∗ (*pi*/*2*))
  using *dvd-triv-left* **by** *blast*
also have ... = (∃*i*::*int*. *x* = *of-int i* ∗ *pi*)
  by *auto*
finally show *?thesis* **.**
qed

lemma *cos-zero-iff-int2*:
fixes *x*::*real*
shows *cos x* = *0* ⟷ (∃*n*::*int*. *x* = *n* ∗ *pi* + *pi*/*2*)
using *sin-zero-iff-int2*[*of x*−*pi*/*2*] **unfolding** *sin-cos-eq*
by (*auto simp add*: *algebra-simps*)

lemma *sin-npi-int* [*simp*]: *sin* (*pi* ∗ *of-int n*) = *0*
by (*simp add*: *sin-zero-iff-int2*)

lemma *cos-monotone-0-pi*:
assumes *0* ≤ *y* **and** *y* < *x* **and** *x* ≤ *pi*
shows *cos x* < *cos y*
proof −
have − (*x* − *y*) < *0* **using** *assms* **by** *auto*
from *MVT2*[*OF* ‹*y* < *x*› *DERIV-cos*]
obtain *z* **where** *y* < *z* **and** *z* < *x* **and** *cos-diff*: *cos x* − *cos y* = (*x* − *y*) ∗ − *sin z*
  by *auto*
then have *0* < *z* **and** *z* < *pi*
  using *assms* **by** *auto*
then have *0* < *sin z*
  using *sin-gt-zero* **by** *auto*
then have *cos x* − *cos y* < *0*
  unfolding *cos-diff minus-mult-commute*[*symmetric*]
  using ‹− (*x* − *y*) < *0*› **by** (*rule mult-pos-neg2*)
then show *?thesis* **by** *auto*
qed

lemma *cos-monotone-0-pi-le*:
assumes *0* ≤ *y* **and** *y* ≤ *x* **and** *x* ≤ *pi*
shows *cos x* ≤ *cos y*
proof (*cases y* < *x*)
case *True*
show *?thesis*
  using *cos-monotone-0-pi*[*OF* ‹*0* ≤ *y*› *True* ‹*x* ≤ *pi*›] **by** *auto*
next
case *False*
then have *y* = *x* **using** ‹*y* ≤ *x*› **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *cos-monotone-minus-pi-0*:

assumes $- pi \le y$ **and** $y < x$ **and** $x \le 0$
shows *cos y* $<$ *cos x*
proof −
have $0 \le - x$ **and** $- x < - y$ **and** $- y \le pi$
using *assms* **by** *auto*
from *cos-monotone-0-pi*[*OF this*] **show** *?thesis*
unfolding *cos-minus* **.**
qed

lemma *cos-monotone-minus-pi-0'*:
assumes $- pi \le y$ **and** $y \le x$ **and** $x \le 0$
shows *cos y* $\le$ *cos x*
proof (*cases y* $<$ *x*)
case *True*
show *?thesis* **using** *cos-monotone-minus-pi-0*[*OF* ‹$-pi \le y$› *True* ‹$x \le 0$›]
by *auto*
next
case *False*
then have $y = x$ **using** ‹$y \le x$› **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *sin-monotone-2pi*:
assumes $- (pi/2) \le y$ **and** $y < x$ **and** $x \le pi/2$
shows *sin y* $<$ *sin x*
unfolding *sin-cos-eq*
using *assms* **by** (*auto intro*: *cos-monotone-0-pi*)

lemma *sin-monotone-2pi-le*:
assumes $- (pi/2) \le y$ **and** $y \le x$ **and** $x \le pi/2$
shows *sin y* $\le$ *sin x*
by (*metis assms le-less sin-monotone-2pi*)

lemma *sin-x-le-x*:
fixes *x* :: *real*
assumes $x \ge 0$
shows *sin x* $\le$ *x*
proof −
let *?f* = $\lambda x.\ x -$ *sin x*
have $\bigwedge u.\ [\![0 \le u;\ u \le x]\!] \Longrightarrow \exists\, y.$ (*?f has-real-derivative 1* $-$ *cos u*) (*at u*)
by (*auto intro*!: *derivative-eq-intros simp*: *field-simps*)
then have *?f x* $\ge$ *?f 0*
by (*metis cos-le-one diff-ge-0-iff-ge DERIV-nonneg-imp-nondecreasing* [*OF assms*])
then show *sin x* $\le$ *x* **by** *simp*
qed

lemma *sin-x-ge-neg-x*:
fixes *x* :: *real*

assumes *x*: $x \geq 0$
shows *sin* $x \geq -x$
proof –
let *?f* = $\lambda x.\ x$ + *sin* x
have §: $\bigwedge u.$ ⟦$0 \leq u$; $u \leq x$⟧ $\Longrightarrow \exists y.$ (*?f has-real-derivative 1* + *cos u*) (*at u*)
by (*auto intro*!: *derivative-eq-intros simp*: *field-simps*)
have *?f x* ≥ *?f 0*
by (*rule DERIV-nonneg-imp-nondecreasing* [*OF assms*]) (*use* § *real-0-le-add-iff* **in** *force*)
then show *sin* $x \geq -x$ **by** *simp*
qed

lemma *abs-sin-x-le-abs-x*: |*sin x*| ≤ |*x*|
for *x* :: *real*
using *sin-x-ge-neg-x* [*of x*] *sin-x-le-x* [*of x*] *sin-x-ge-neg-x* [*of* −*x*] *sin-x-le-x* [*of* −*x*]
by (*auto simp*: *abs-real-def*)

112.13 More Corollaries about Sine and Cosine

lemma *sin-cos-npi* [*simp*]: *sin* (*real* (*Suc* (*2* ∗ *n*)) ∗ *pi*/*2*) = (−*1*) ^ *n*
proof –
have *sin* ((*real n* + *1*/*2*) ∗ *pi*) = *cos* (*real n* ∗ *pi*)
by (*auto simp*: *algebra-simps sin-add*)
then show *?thesis*
by (*simp add*: *distrib-right add-divide-distrib add.commute mult.commute* [*of pi*])
qed

lemma *cos-2npi* [*simp*]: *cos* (*2* ∗ *real n* ∗ *pi*) = *1*
for *n* :: *nat*
by (*cases even n*) (*simp-all add*: *cos-double mult.assoc*)

lemma *cos-3over2-pi* [*simp*]: *cos* (*3*/*2*∗*pi*) = *0*
proof –
have *cos* (*3*/*2*∗*pi*) = *cos* (*pi* + *pi*/*2*)
by *simp*
also have ... = *0*
by (*subst cos-add*, *simp*)
finally show *?thesis* **.**
qed

lemma *sin-2npi* [*simp*]: *sin* (*2* ∗ *real n* ∗ *pi*) = *0*
for *n* :: *nat*
by (*auto simp*: *mult.assoc sin-double*)

lemma *sin-3over2-pi* [*simp*]: *sin* (*3*/*2*∗*pi*) = − *1*
proof –
have *sin* (*3*/*2*∗*pi*) = *sin* (*pi* + *pi*/*2*)

by *simp*
also have ... = -1
by (*subst sin-add*, *simp*)
finally show *?thesis* **.**
qed

lemma *cos-pi-eq-zero* [*simp*]: *cos* (*pi* ∗ *real* (*Suc* (*2* ∗ *m*)) / *2*) = *0*
by (*simp only*: *cos-add sin-add of-nat-Suc distrib-right distrib-left add-divide-distrib*, *auto*)

lemma *DERIV-cos-add* [*simp*]: *DERIV* (λ*x*. *cos* (*x* + *k*)) *xa* :> − *sin* (*xa* + *k*)
by (*auto intro*!: *derivative-eq-intros*)

lemma *sin-zero-norm-cos-one*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
assumes *sin x* = *0*
shows *norm* (*cos x*) = *1*
using *sin-cos-squared-add* [*of x*, *unfolded assms*]
by (*simp add*: *square-norm-one*)

lemma *sin-zero-abs-cos-one*: *sin x* = *0* ⟹ |*cos x*| = (*1*::*real*)
using *sin-zero-norm-cos-one* **by** *fastforce*

lemma *cos-one-sin-zero*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
assumes *cos x* = *1*
shows *sin x* = *0*
using *sin-cos-squared-add* [*of x*, *unfolded assms*]
by *simp*

lemma *sin-times-pi-eq-0*: *sin* (*x* ∗ *pi*) = *0* ⟷ *x* ∈ ℤ
by (*simp add*: *sin-zero-iff-int2*) (*metis Ints-cases Ints-of-int*)

lemma *cos-one-2pi*: *cos x* = *1* ⟷ (∃ *n*::*nat*. *x* = *n* ∗ *2* ∗ *pi*) ∨ (∃ *n*::*nat*. *x* = − (*n* ∗ *2* ∗ *pi*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have *sin x* = *0*
by (*simp add*: *cos-one-sin-zero*)
then show *?rhs*
proof (*simp only*: *sin-zero-iff*, *elim exE disjE conjE*)
fix *n* :: *nat*
assume *n*: *even n x* = *real n* ∗ (*pi*/*2*)
then obtain *m* **where** *m*: *n* = *2* ∗ *m*
using *dvdE* **by** *blast*
then have *me*: *even m* **using** ‹*?lhs*› *n*
by (*auto simp*: *field-simps*) (*metis one-neq-neg-one power-minus-odd power-one*)
show *?rhs*

```
      using m me n
      by (auto simp: field-simps elim!: evenE)
  next
    fix n :: nat
    assume n: even n x = − (real n ∗ (pi/2))
    then obtain m where m: n = 2 ∗ m
      using dvdE by blast
    then have me: even m using ‹?lhs› n
     by (auto simp: field-simps) (metis one-neq-neg-one power-minus-odd power-one)
    show ?rhs
      using m me n
      by (auto simp: field-simps elim!: evenE)
  qed
next
  assume ?rhs
  then show cos x = 1
    by (metis cos-2npi cos-minus mult.assoc mult.left-commute)
qed

lemma cos-one-2pi-int: cos x = 1 ⟷ (∃ n::int. x = n ∗ 2 ∗ pi) (is ?lhs = ?rhs)
proof
  assume cos x = 1
  then show ?rhs
   by (metis cos-one-2pi mult.commute mult-minus-right of-int-minus of-int-of-nat-eq)
next
  assume ?rhs
  then show cos x = 1
    by (clarsimp simp add: cos-one-2pi) (metis mult-minus-right of-int-of-nat)
qed

lemma cos-npi-int [simp]:
  fixes n::int shows cos (pi ∗ of-int n) = (if even n then 1 else −1)
    by (auto simp: algebra-simps cos-one-2pi-int elim!: oddE evenE)

lemma sin-cos-sqrt: 0 ≤ sin x =⇒ sin x = sqrt (1 − (cos(x) ^ 2))
  using sin-squared-eq real-sqrt-unique by fastforce

lemma sin-eq-0-pi: − pi < x =⇒ x < pi =⇒ sin x = 0 =⇒ x = 0
 by (metis sin-gt-zero sin-minus minus-less-iff neg-0-less-iff-less not-less-iff-gr-or-eq)

lemma cos-treble-cos: cos (3 ∗ x) = 4 ∗ cos x ^ 3 − 3 ∗ cos x
  for x :: 'a::{real-normed-field,banach}
proof −
  have ∗: (sin x ∗ (sin x ∗ 3)) = 3 − (cos x ∗ (cos x ∗ 3))
   by (simp add: mult.assoc [symmetric] sin-squared-eq [unfolded power2-eq-square])
  have cos(3 ∗ x) = cos(2∗x + x)
    by simp
  also have . . . = 4 ∗ cos x ^ 3 − 3 ∗ cos x
    unfolding cos-add cos-double sin-double
```

by (*simp add*: ∗ *field-simps power2-eq-square power3-eq-cube*)
finally show *?thesis* **.**
qed

lemma *cos-45*: *cos* (*pi*/*4*) = *sqrt 2* / *2*
proof −
let *?c* = *cos* (*pi*/*4*)
let *?s* = *sin* (*pi*/*4*)
have *nonneg*: *0* ≤ *?c*
by (*simp add*: *cos-ge-zero*)
have *0* = *cos* (*pi*/*4* + *pi*/*4*)
by *simp*
also have *cos* (*pi*/*4* + *pi*/*4*) = $?c^2$ − $?s^2$
by (*simp only*: *cos-add power2-eq-square*)
also have ... = *2* ∗ $?c^2$ − *1*
by (*simp add*: *sin-squared-eq*)
finally have $?c^2$ = $(sqrt\ 2\ /\ 2)^2$
by (*simp add*: *power-divide*)
then show *?thesis*
using *nonneg* **by** (*rule power2-eq-imp-eq*) *simp*
qed

lemma *cos-30*: *cos* (*pi*/*6*) = *sqrt 3*/*2*
proof −
let *?c* = *cos* (*pi*/*6*)
let *?s* = *sin* (*pi*/*6*)
have *pos-c*: *0* < *?c*
by (*rule cos-gt-zero*) *simp-all*
have *0* = *cos* (*pi*/*6* + *pi*/*6* + *pi*/*6*)
by *simp*
also have ... = (*?c* ∗ *?c* − *?s* ∗ *?s*) ∗ *?c* − (*?s* ∗ *?c* + *?c* ∗ *?s*) ∗ *?s*
by (*simp only*: *cos-add sin-add*)
also have ... = *?c* ∗ ($?c^2$ − *3* ∗ $?s^2$)
by (*simp add*: *algebra-simps power2-eq-square*)
finally have $?c^2$ = $(sqrt\ 3/2)^2$
using *pos-c* **by** (*simp add*: *sin-squared-eq power-divide*)
then show *?thesis*
using *pos-c* [*THEN order-less-imp-le*]
by (*rule power2-eq-imp-eq*) *simp*
qed

lemma *sin-45*: *sin* (*pi*/*4*) = *sqrt 2* / *2*
by (*simp add*: *sin-cos-eq cos-45*)

lemma *sin-60*: *sin* (*pi*/*3*) = *sqrt 3*/*2*
by (*simp add*: *sin-cos-eq cos-30*)

lemma *cos-60*: *cos* (*pi*/*3*) = *1*/*2*
proof −

have *0 ≤ cos (pi/3)*
by (*rule cos-ge-zero*) (*use pi-half-ge-zero* **in** ‹*linarith+*›)
then show *?thesis*
by (*simp add*: *cos-squared-eq sin-60 power-divide power2-eq-imp-eq*)
qed

lemma *sin-30*: *sin (pi/6) = 1/2*
by (*simp add*: *sin-cos-eq cos-60*)

lemma *cos-120*: *cos (2 ∗ pi/3) = −1/2*
and *sin-120*: *sin (2 ∗ pi/3) = sqrt 3 / 2*
using *sin-double*[*of pi/3*] *cos-double*[*of pi/3*]
by (*simp-all add*: *power2-eq-square sin-60 cos-60*)

lemma *cos-120'*: *cos (pi ∗ 2 / 3) = −1/2*
using *cos-120* **by** (*subst mult.commute*)

lemma *sin-120'*: *sin (pi ∗ 2 / 3) = sqrt 3 / 2*
using *sin-120* **by** (*subst mult.commute*)

lemma *cos-integer-2pi*: *n ∈ ℤ ⟹ cos(2 ∗ pi ∗ n) = 1*
by (*metis Ints-cases cos-one-2pi-int mult.assoc mult.commute*)

lemma *sin-integer-2pi*: *n ∈ ℤ ⟹ sin(2 ∗ pi ∗ n) = 0*
by (*metis sin-two-pi Ints-mult mult.assoc mult.commute sin-times-pi-eq-0*)

lemma *cos-int-2pin* [*simp*]: *cos ((2 ∗ pi) ∗ of-int n) = 1*
by (*simp add*: *cos-one-2pi-int*)

lemma *sin-int-2pin* [*simp*]: *sin ((2 ∗ pi) ∗ of-int n) = 0*
by (*metis Ints-of-int sin-integer-2pi*)

lemma *sin-cos-eq-iff*: *sin y = sin x ∧ cos y = cos x ⟷ (∃ n::int. y = x + 2 ∗ pi ∗ n)* (**is** *?L=?R*)
proof
assume *?L*
then have *cos (y−x) = 1*
using *cos-add* [*of y −x*] **by** *simp*
then show *?R*
by (*metis cos-one-2pi-int add.commute diff-add-cancel mult.assoc mult.commute*)

next
assume *?R*
then show *?L*
by (*auto simp*: *sin-add cos-add*)
qed

lemma *sincos-principal-value*: *∃ y. (− pi < y ∧ y ≤ pi) ∧ (sin y = sin x ∧ cos y = cos x)*

proof −
 define *y* **where** $y \equiv pi - (2 * pi) * frac\ ((pi - x) / (2 * pi))$
 have $-pi < y$ $y \leq pi$
  by (*auto simp*: *field-simps frac-lt-1 y-def*)
 moreover
 have $sin\ y = sin\ x$ $cos\ y = cos\ x$
  by (*simp-all add*: *y-def frac-def divide-simps sin-add cos-add mult-of-int-commute*)
 ultimately
 show *?thesis* **by** *metis*
qed

112.14 Tangent

definition *tan* :: $'a \Rightarrow 'a::\{real\text{-}normed\text{-}field,banach\}$
 where $tan = (\lambda x.\ sin\ x / cos\ x)$

lemma *tan-of-real*: *of-real* (*tan x*) = (*tan* (*of-real x*) :: $'a::\{real\text{-}normed\text{-}field,banach\}$)
 by (*simp add*: *tan-def sin-of-real cos-of-real*)

lemma *tan-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow tan\ z \in \mathbb{R}$
 for *z* :: $'a::\{real\text{-}normed\text{-}field,banach\}$
 by (*simp add*: *tan-def*)

lemma *tan-zero* [*simp*]: $tan\ 0 = 0$
 by (*simp add*: *tan-def*)

lemma *tan-pi* [*simp*]: $tan\ pi = 0$
 by (*simp add*: *tan-def*)

lemma *tan-npi* [*simp*]: $tan\ (real\ n * pi) = 0$
 for *n* :: *nat*
 by (*simp add*: *tan-def*)

lemma *tan-pi-half* [*simp*]: $tan\ (pi / 2) = 0$
 by (*simp add*: *tan-def*)

lemma *tan-minus* [*simp*]: $tan\ (-\ x) = -\ tan\ x$
 by (*simp add*: *tan-def*)

lemma *tan-periodic* [*simp*]: $tan\ (x + 2 * pi) = tan\ x$
 by (*simp add*: *tan-def*)

lemma *lemma-tan-add1*: $cos\ x \neq 0 \Longrightarrow cos\ y \neq 0 \Longrightarrow 1 - tan\ x * tan\ y = cos\ (x + y)/(cos\ x * cos\ y)$
 by (*simp add*: *tan-def cos-add field-simps*)

lemma *add-tan-eq*: $cos\ x \neq 0 \Longrightarrow cos\ y \neq 0 \Longrightarrow tan\ x + tan\ y = sin(x + y)/(cos\ x * cos\ y)$
 for *x* :: $'a::\{real\text{-}normed\text{-}field,banach\}$

by (*simp add*: *tan-def sin-add field-simps*)

lemma *tan-eq-0-cos-sin*: *tan x = 0* ⟷ *cos x = 0* ∨ *sin x = 0*
by (*auto simp*: *tan-def*)

Note: half of these zeros would normally be regarded as undefined cases.

lemma *tan-eq-0-Ex*:
assumes *tan x = 0*
obtains *k*::*int* **where** *x* = (*k*/*2*) ∗ *pi*
using *assms*
by (*metis cos-zero-iff-int mult.commute sin-zero-iff-int tan-eq-0-cos-sin times-divide-eq-left*)

lemma *tan-add*:
cos x ≠ *0* ⟹ *cos y* ≠ *0* ⟹ *cos* (*x* + *y*) ≠ *0* ⟹ *tan* (*x* + *y*) = (*tan x* + *tan y*)/(*1* − *tan x* ∗ *tan y*)
for *x* :: ′*a*::{*real-normed-field*,*banach*}
by (*simp add*: *add-tan-eq lemma-tan-add1 field-simps*) (*simp add*: *tan-def*)

lemma *tan-double*: *cos x* ≠ *0* ⟹ *cos* (*2* ∗ *x*) ≠ *0* ⟹ *tan* (*2* ∗ *x*) = (*2* ∗ *tan x*) / (*1* − (*tan x*)^2)
for *x* :: ′*a*::{*real-normed-field*,*banach*}
using *tan-add* [*of x x*] **by** (*simp add*: *power2-eq-square*)

lemma *tan-gt-zero*: *0* < *x* ⟹ *x* < *pi*/*2* ⟹ *0* < *tan x*
by (*simp add*: *tan-def zero-less-divide-iff sin-gt-zero2 cos-gt-zero-pi*)

lemma *tan-less-zero*:
assumes − *pi*/*2* < *x* **and** *x* < *0*
shows *tan x* < *0*
proof −
have *0* < *tan* (− *x*)
using *assms* **by** (*simp only*: *tan-gt-zero*)
then show *?thesis* **by** *simp*
qed

lemma *tan-half*: *tan x* = *sin* (*2* ∗ *x*) / (*cos* (*2* ∗ *x*) + *1*)
for *x* :: ′*a*::{*real-normed-field*,*banach*,*field*}
unfolding *tan-def sin-double cos-double sin-squared-eq*
by (*simp add*: *power2-eq-square*)

lemma *tan-30*: *tan* (*pi*/*6*) = *1* / *sqrt 3*
unfolding *tan-def* **by** (*simp add*: *sin-30 cos-30*)

lemma *tan-45*: *tan* (*pi*/*4*) = *1*
unfolding *tan-def* **by** (*simp add*: *sin-45 cos-45*)

lemma *tan-60*: *tan* (*pi*/*3*) = *sqrt 3*
unfolding *tan-def* **by** (*simp add*: *sin-60 cos-60*)

lemma *DERIV-tan* [*simp*]: *cos x* ≠ *0* ⟹ *DERIV tan x* :> *inverse* ((*cos x*)^2)
for *x* :: *'a*::{*real-normed-field*,*banach*}
unfolding *tan-def*
by (*auto intro*!: *derivative-eq-intros*, *simp add*: *divide-inverse power2-eq-square*)

declare *DERIV-tan*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-tan*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-tan*[*derivative-intros*] = *DERIV-tan*[*THEN DERIV-compose-FDERIV*]

lemma *isCont-tan*: *cos x* ≠ *0* ⟹ *isCont tan x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
by (*rule DERIV-tan* [*THEN DERIV-isCont*])

lemma *isCont-tan'* [*simp*,*continuous-intros*]:
fixes *a* :: *'a*::{*real-normed-field*,*banach*} **and** *f* :: *'a* ⇒ *'a*
shows *isCont f a* ⟹ *cos* (*f a*) ≠ *0* ⟹ *isCont* (λ*x*. *tan* (*f x*)) *a*
by (*rule isCont-o2* [*OF - isCont-tan*])

lemma *tendsto-tan* [*tendsto-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows (*f* ⟶ *a*) *F* ⟹ *cos a* ≠ *0* ⟹ ((λ*x*. *tan* (*f x*)) ⟶ *tan a*) *F*
by (*rule isCont-tendsto-compose* [*OF isCont-tan*])

lemma *continuous-tan*:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous F f* ⟹ *cos* (*f* (*Lim F* (λ*x*. *x*))) ≠ *0* ⟹ *continuous F* (λ*x*. *tan* (*f x*))
unfolding *continuous-def* **by** (*rule tendsto-tan*)

lemma *continuous-on-tan* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous-on s f* ⟹ (∀ *x*∈*s*. *cos* (*f x*) ≠ *0*) ⟹ *continuous-on s* (λ*x*. *tan* (*f x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-tan*)

lemma *continuous-within-tan* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous* (*at x within s*) *f* ⟹
cos (*f x*) ≠ *0* ⟹ *continuous* (*at x within s*) (λ*x*. *tan* (*f x*))
unfolding *continuous-within* **by** (*rule tendsto-tan*)

lemma *LIM-cos-div-sin*: (λ*x*. *cos*(*x*)/*sin*(*x*)) −*pi*/*2*→ *0*
by (*rule tendsto-cong-limit*, (*rule tendsto-intros*)+, *simp-all*)

lemma *lemma-tan-total*:
assumes *0* < *y* **shows** ∃ *x*. *0* < *x* ∧ *x* < *pi*/*2* ∧ *y* < *tan x*

proof −
  obtain *s* **where** *0 < s*
    and *s*: ⋀*x*. ⟦*x* ≠ *pi/2*; *norm* (*x* − *pi/2*) < *s*⟧ ⟹ *norm* (*cos x* / *sin x* − *0*) < *inverse y*
    using *LIM-D* [*OF LIM-cos-div-sin*, *of inverse y*] *that assms* **by** *force*
  obtain *e* **where** *e*: *0 < e e < s e < pi/2*
    using ‹*0 < s*› *field-lbound-gt-zero pi-half-gt-zero* **by** *blast*
  show *?thesis*
  proof (*intro exI conjI*)
    have *0 < sin e 0 < cos e*
      using *e* **by** (*auto intro*: *cos-gt-zero sin-gt-zero2 simp*: *mult.commute*)
    then
    show *y < tan* (*pi/2* − *e*)
      using *s* [*of pi/2* − *e*] *e assms*
     by (*simp add*: *tan-def sin-diff cos-diff*) (*simp add*: *field-simps split*: *if-split-asm*)
  qed (*use e* **in** *auto*)
qed

lemma *tan-total-pos*:
  assumes *0* ≤ *y* **shows** ∃*x*. *0* ≤ *x* ∧ *x* < *pi/2* ∧ *tan x* = *y*
proof (*cases y* = *0*)
  case *True*
  then show *?thesis*
    using *pi-half-gt-zero tan-zero* **by** *blast*
next
  case *False*
  with *assms* **have** *y > 0*
    by *linarith*
  obtain *x* **where** *x*: *0 < x x < pi/2 y < tan x*
    using *lemma-tan-total* ‹*0 < y*› **by** *blast*
  have ∃*u*≥*0*. *u* ≤ *x* ∧ *tan u* = *y*
  proof (*intro IVT allI impI*)
    show *isCont tan u* **if** *0* ≤ *u* ∧ *u* ≤ *x* **for** *u*
    proof −
      have *cos u* ≠ *0*
        using *antisym-conv2 cos-gt-zero that x*(*2*) **by** *fastforce*
      with *assms* **show** *?thesis*
        by (*auto intro*!: *DERIV-tan* [*THEN DERIV-isCont*])
    qed
  qed (*use assms x* **in** *auto*)
  then show *?thesis*
    using *x*(*2*) **by** *auto*
qed

lemma *lemma-tan-total1*: ∃*x*. −(*pi/2*) < *x* ∧ *x* < (*pi/2*) ∧ *tan x* = *y*
proof (*cases 0*::*real y rule*: *le-cases*)
  case *le*
  then show *?thesis*
    by (*meson less-le-trans minus-pi-half-less-zero tan-total-pos*)

next
case *ge*
with *tan-total-pos* [*of* −*y*] **obtain** *x* **where** *0* ≤ *x* *x* < *pi*/*2* *tan x* = − *y*
by *force*
then show *?thesis*
by (*rule-tac x*=−*x* **in** *exI*) *auto*
qed

proposition *tan-total*: ∃! *x*. −(*pi*/*2*) < *x* ∧ *x* < (*pi*/*2*) ∧ *tan x* = *y*
proof −
have *u* = *v* **if** *u*: − (*pi*/*2*) < *u* *u* < *pi*/*2* **and** *v*: − (*pi*/*2*) < *v* *v* < *pi*/*2*
and *eq*: *tan u* = *tan v* **for** *u v*
proof (*cases u v rule*: *linorder-cases*)
case *less*
have ⋀*x*. *u* ≤ *x* ∧ *x* ≤ *v* ⟶ *isCont tan x*
by (*metis cos-gt-zero-pi isCont-tan le-less-trans less-irrefl less-le-trans u*(*1*) *v*(*2*))
then have *continuous-on* {*u*..*v*} *tan*
by (*simp add*: *continuous-at-imp-continuous-on*)
moreover have ⋀*x*. *u* < *x* ∧ *x* < *v* ⟹ *tan differentiable* (*at x*)
by (*metis DERIV-tan cos-gt-zero-pi real-differentiable-def less-numeral-extra*(*3*) *order*.*strict-trans u*(*1*) *v*(*2*))
ultimately obtain *z* **where** *u* < *z* *z* < *v* *DERIV tan z* :> *0*
by (*metis less Rolle eq*)
moreover have *cos z* ≠ *0*
by (*metis* (*no-types*) ‹*u* < *z*› ‹*z* < *v*› *cos-gt-zero-pi less-le-trans linorder-not-less not-less-iff-gr-or-eq u*(*1*) *v*(*2*))
ultimately show *?thesis*
using *DERIV-unique* [*OF* - *DERIV-tan*] **by** *fastforce*
next
case *greater*
have ⋀*x*. *v* ≤ *x* ∧ *x* ≤ *u* ⟹ *isCont tan x*
by (*metis cos-gt-zero-pi isCont-tan le-less-trans less-irrefl less-le-trans u*(*2*) *v*(*1*))
then have *continuous-on* {*v*..*u*} *tan*
by (*simp add*: *continuous-at-imp-continuous-on*)
moreover have ⋀*x*. *v* < *x* ∧ *x* < *u* ⟹ *tan differentiable* (*at x*)
by (*metis DERIV-tan cos-gt-zero-pi real-differentiable-def less-numeral-extra*(*3*) *order*.*strict-trans u*(*2*) *v*(*1*))
ultimately obtain *z* **where** *v* < *z* *z* < *u* *DERIV tan z* :> *0*
by (*metis greater Rolle eq*)
moreover have *cos z* ≠ *0*
by (*metis* ‹*v* < *z*› ‹*z* < *u*› *cos-gt-zero-pi less-eq-real-def less-le-trans order-less-irrefl u*(*2*) *v*(*1*))
ultimately show *?thesis*
using *DERIV-unique* [*OF* - *DERIV-tan*] **by** *fastforce*
qed *auto*
then have ∃!*x*. − (*pi*/*2*) < *x* ∧ *x* < *pi*/*2* ∧ *tan x* = *y*
if *x*: − (*pi*/*2*) < *x* *x* < *pi*/*2* *tan x* = *y* **for** *x*

```
    using that by auto
  then show ?thesis
    using lemma-tan-total1 [where y = y]
    by auto
qed

lemma tan-monotone:
  assumes − (pi/2) < y and y < x and x < pi/2
  shows tan y < tan x
proof −
  have DERIV tan x' :> inverse ((cos x')^2) if y ≤ x' x' ≤ x for x'
  proof −
    have −(pi/2) < x' and x' < pi/2
      using that assms by auto
    with cos-gt-zero-pi have cos x' ≠ 0 by force
    then show DERIV tan x' :> inverse ((cos x')^2)
      by (rule DERIV-tan)
  qed
  from MVT2[OF ‹y < x› this]
  obtain z where y < z and z < x
    and tan-diff: tan x − tan y = (x − y) ∗ inverse ((cos z)^2) by auto
  then have − (pi/2) < z and z < pi/2
    using assms by auto
  then have 0 < cos z
    using cos-gt-zero-pi by auto
  then have inv-pos: 0 < inverse ((cos z)^2)
    by auto
  have 0 < x − y using ‹y < x› by auto
  with inv-pos have 0 < tan x − tan y
    unfolding tan-diff by auto
  then show ?thesis by auto
qed

lemma tan-monotone':
  assumes − (pi/2) < y
    and y < pi/2
    and − (pi/2) < x
    and x < pi/2
  shows y < x ⟷ tan y < tan x
proof
  assume y < x
  then show tan y < tan x
    using tan-monotone and ‹− (pi/2) < y› and ‹x < pi/2› by auto
next
  assume tan y < tan x
  show y < x
  proof (rule ccontr)
    assume ¬ ?thesis
    then have x ≤ y by auto
```

then have *tan x* ≤ *tan y*
proof (*cases x* = *y*)
case *True*
then show *?thesis* **by** *auto*
next
case *False*
then have *x* < *y* **using** ‹*x* ≤ *y*› **by** *auto*
from *tan-monotone*[*OF* ‹− (*pi*/*2*) < *x*› *this* ‹*y* < *pi*/*2*›] **show** *?thesis*
by *auto*
qed
then show *False*
using ‹*tan y* < *tan x*› **by** *auto*
qed
qed

lemma *tan-inverse*: *1* / (*tan y*) = *tan* (*pi*/*2* − *y*)
unfolding *tan-def sin-cos-eq*[*of y*] *cos-sin-eq*[*of y*] **by** *auto*

lemma *tan-periodic-pi*[*simp*]: *tan* (*x* + *pi*) = *tan x*
by (*simp add*: *tan-def*)

lemma *tan-periodic-nat*[*simp*]: *tan* (*x* + *real n* ∗ *pi*) = *tan x*
proof (*induct n arbitrary*: *x*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
have *split-pi-off*: *x* + *real* (*Suc n*) ∗ *pi* = (*x* + *real n* ∗ *pi*) + *pi*
unfolding *Suc-eq-plus1 of-nat-add distrib-right* **by** *auto*
show *?case*
unfolding *split-pi-off* **using** *Suc* **by** *auto*
qed

lemma *tan-periodic-int*[*simp*]: *tan* (*x* + *of-int i* ∗ *pi*) = *tan x*
proof (*cases 0* ≤ *i*)
case *False*
then have *i-nat*: *of-int i* = − *of-int* (*nat* (− *i*)) **by** *auto*
then show *?thesis*
by (*smt* (*verit*, *best*) *mult-minus-left of-int-of-nat-eq tan-periodic-nat*)
qed (*use zero-le-imp-eq-int* **in** *fastforce*)

lemma *tan-periodic-n*[*simp*]: *tan* (*x* + *numeral n* ∗ *pi*) = *tan x*
using *tan-periodic-int*[*of* - *numeral n*] **by** *simp*

lemma *tan-minus-45* [*simp*]: *tan* (−(*pi*/*4*)) = −*1*
unfolding *tan-def* **by** (*simp add*: *sin-45 cos-45*)

lemma *tan-diff*:
cos x ≠ *0* ⟹ *cos y* ≠ *0* ⟹ *cos* (*x* − *y*) ≠ *0* ⟹ *tan* (*x* − *y*) = (*tan x* − *tan*

y)/(*1* + *tan x* ∗ *tan y*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
using *tan-add* [*of x* −*y*] **by** *simp*

lemma *tan-pos-pi2-le*: *0* ≤ *x* ⟹ *x* < *pi*/*2* ⟹ *0* ≤ *tan x*
using *less-eq-real-def tan-gt-zero* **by** *auto*

lemma *cos-tan*: |*x*| < *pi*/*2* ⟹ *cos x* = *1* / *sqrt* (*1* + *tan x* ^ *2*)
using *cos-gt-zero-pi* [*of x*]
by (*simp add*: *field-split-simps tan-def real-sqrt-divide abs-if split*: *if-split-asm*)

lemma *cos-tan-half*: *cos x* ≠*0* ⟹ *cos* (*2*∗*x*) = (*1* − (*tan x*)^*2*) / (*1* + (*tan x*)^*2*)
unfolding *cos-double tan-def* **by** (*auto simp add*:*field-simps*)

lemma *sin-tan*: |*x*| < *pi*/*2* ⟹ *sin x* = *tan x* / *sqrt* (*1* + *tan x* ^ *2*)
using *cos-gt-zero* [*of x*] *cos-gt-zero* [*of* −*x*]
by (*force simp*: *field-split-simps tan-def real-sqrt-divide abs-if split*: *if-split-asm*)

lemma *sin-tan-half*: *sin* (*2*∗*x*) = *2* ∗ *tan x* / (*1* + (*tan x*)^*2*)
unfolding *sin-double tan-def*
by (*cases cos x*=*0*) (*auto simp add*:*field-simps power2-eq-square*)

lemma *tan-mono-le*: −(*pi*/*2*) < *x* ⟹ *x* ≤ *y* ⟹ *y* < *pi*/*2* ⟹ *tan x* ≤ *tan y*
using *less-eq-real-def tan-monotone* **by** *auto*

lemma *tan-mono-lt-eq*:
−(*pi*/*2*) < *x* ⟹ *x* < *pi*/*2* ⟹ −(*pi*/*2*) < *y* ⟹ *y* < *pi*/*2* ⟹ *tan x* < *tan y* ⟷ *x* < *y*
using *tan-monotone′* **by** *blast*

lemma *tan-mono-le-eq*:
−(*pi*/*2*) < *x* ⟹ *x* < *pi*/*2* ⟹ −(*pi*/*2*) < *y* ⟹ *y* < *pi*/*2* ⟹ *tan x* ≤ *tan y* ⟷ *x* ≤ *y*
by (*meson tan-mono-le not-le tan-monotone*)

lemma *tan-bound-pi2*: |*x*| < *pi*/*4* ⟹ |*tan x*| < *1*
using *tan-45 tan-monotone* [*of x pi*/*4*] *tan-monotone* [*of* −*x pi*/*4*]
by (*auto simp*: *abs-if split*: *if-split-asm*)

lemma *tan-cot*: *tan*(*pi*/*2* − *x*) = *inverse*(*tan x*)
by (*simp add*: *tan-def sin-diff cos-diff*)

112.15 Cotangent

definition *cot* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
where *cot* = (*λx*. *cos x* / *sin x*)

lemma *cot-of-real*: *of-real* (*cot x*) = (*cot* (*of-real x*) :: *'a*::{*real-normed-field*,*banach*})

by (*simp add*: *cot-def sin-of-real cos-of-real*)

lemma *cot-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow cot\ z \in \mathbb{R}$
for z :: $'a$::{*real-normed-field*,*banach*}
by (*simp add*: *cot-def*)

lemma *cot-zero* [*simp*]: $cot\ 0 = 0$
by (*simp add*: *cot-def*)

lemma *cot-pi* [*simp*]: $cot\ pi = 0$
by (*simp add*: *cot-def*)

lemma *cot-npi* [*simp*]: $cot\ (real\ n * pi) = 0$
for n :: *nat*
by (*simp add*: *cot-def*)

lemma *cot-minus* [*simp*]: $cot\ (-\ x) = -\ cot\ x$
by (*simp add*: *cot-def*)

lemma *cot-periodic* [*simp*]: $cot\ (x + 2 * pi) = cot\ x$
by (*simp add*: *cot-def*)

lemma *cot-altdef*: $cot\ x = inverse\ (tan\ x)$
by (*simp add*: *cot-def tan-def*)

lemma *tan-altdef*: $tan\ x = inverse\ (cot\ x)$
by (*simp add*: *cot-def tan-def*)

lemma *tan-cot'*: $tan\ (pi/2 - x) = cot\ x$
by (*simp add*: *tan-cot cot-altdef*)

lemma *cot-gt-zero*: $0 < x \Longrightarrow x < pi/2 \Longrightarrow 0 < cot\ x$
by (*simp add*: *cot-def zero-less-divide-iff sin-gt-zero2 cos-gt-zero-pi*)

lemma *cot-less-zero*:
assumes *lb*: $-\ pi/2 < x$ **and** $x < 0$
shows $cot\ x < 0$
by (*smt* (*verit*) *assms cot-gt-zero cot-minus divide-minus-left*)

lemma *DERIV-cot* [*simp*]: $sin\ x \neq 0 \Longrightarrow DERIV\ cot\ x :> -inverse\ ((sin\ x)^2)$
for x :: $'a$::{*real-normed-field*,*banach*}
unfolding *cot-def* **using** *cos-squared-eq*[*of x*]
by (*auto intro*!: *derivative-eq-intros*) (*simp add*: *divide-inverse power2-eq-square*)

lemma *isCont-cot*: $sin\ x \neq 0 \Longrightarrow isCont\ cot\ x$
for x :: $'a$::{*real-normed-field*,*banach*}
by (*rule DERIV-cot* [*THEN DERIV-isCont*])

lemma *isCont-cot'* [*simp*,*continuous-intros*]:

isCont f a ⟹ *sin* (*f a*) ≠ *0* ⟹ *isCont* (λ*x. cot* (*f x*)) *a*
for *a* :: *'a*::{*real-normed-field*,*banach*} **and** *f* :: *'a* ⇒ *'a*
by (*rule isCont-o2* [*OF - isCont-cot*])

lemma *tendsto-cot* [*tendsto-intros*]: (*f* ⟶ *a*) *F* ⟹ *sin a* ≠ *0* ⟹ ((λ*x. cot* (*f x*)) ⟶ *cot a*) *F*
for *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
by (*rule isCont-tendsto-compose* [*OF isCont-cot*])

lemma *continuous-cot*:
continuous F f ⟹ *sin* (*f* (*Lim F* (λ*x. x*))) ≠ *0* ⟹ *continuous F* (λ*x. cot* (*f x*))
for *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
unfolding *continuous-def* **by** (*rule tendsto-cot*)

lemma *continuous-on-cot* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous-on s f* ⟹ (∀ *x*∈*s. sin* (*f x*) ≠ *0*) ⟹ *continuous-on s* (λ*x. cot* (*f x*))
unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-cot*)

lemma *continuous-within-cot* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous* (*at x within s*) *f* ⟹ *sin* (*f x*) ≠ *0* ⟹ *continuous* (*at x within s*) (λ*x. cot* (*f x*))
unfolding *continuous-within* **by** (*rule tendsto-cot*)

112.16 Inverse Trigonometric Functions

definition *arcsin* :: *real* ⇒ *real*
where *arcsin y* = (*THE x.* −(*pi*/*2*) ≤ *x* ∧ *x* ≤ *pi*/*2* ∧ *sin x* = *y*)

definition *arccos* :: *real* ⇒ *real*
where *arccos y* = (*THE x. 0* ≤ *x* ∧ *x* ≤ *pi* ∧ *cos x* = *y*)

definition *arctan* :: *real* ⇒ *real*
where *arctan y* = (*THE x.* −(*pi*/*2*) < *x* ∧ *x* < *pi*/*2* ∧ *tan x* = *y*)

lemma *arcsin*: − *1* ≤ *y* ⟹ *y* ≤ *1* ⟹ − (*pi*/*2*) ≤ *arcsin y* ∧ *arcsin y* ≤ *pi*/*2* ∧ *sin* (*arcsin y*) = *y*
unfolding *arcsin-def* **by** (*rule theI'* [*OF sin-total*])

lemma *arcsin-pi*: − *1* ≤ *y* ⟹ *y* ≤ *1* ⟹ − (*pi*/*2*) ≤ *arcsin y* ∧ *arcsin y* ≤ *pi* ∧ *sin* (*arcsin y*) = *y*
by (*drule* (*1*) *arcsin*) (*force intro*: *order-trans*)

lemma *sin-arcsin* [*simp*]: − *1* ≤ *y* ⟹ *y* ≤ *1* ⟹ *sin* (*arcsin y*) = *y*
by (*blast dest*: *arcsin*)

lemma *arcsin-bounded*: − *1* ≤ *y* ⟹ *y* ≤ *1* ⟹ − (*pi*/*2*) ≤ *arcsin y* ∧ *arcsin y*

$\leq$ *pi/2*
by (*blast dest*: *arcsin*)

lemma *arcsin-lbound*: $-$ *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ $-$ (*pi/2*) $\leq$ *arcsin y*
by (*blast dest*: *arcsin*)

lemma *arcsin-ubound*: $-$ *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ *arcsin y* $\leq$ *pi/2*
by (*blast dest*: *arcsin*)

lemma *arcsin-lt-bounded*:
assumes $-$ *1* $<$ *y y* $<$ *1*
shows $-$ (*pi/2*) $<$ *arcsin y* $\wedge$ *arcsin y* $<$ *pi/2*
proof $-$
have *arcsin y* $\neq$ *pi/2*
by (*metis arcsin assms not-less not-less-iff-gr-or-eq sin-pi-half*)
moreover have *arcsin y* $\neq$ $-$ *pi/2*
by (*metis arcsin assms minus-divide-left not-less not-less-iff-gr-or-eq sin-minus sin-pi-half*)
ultimately show *?thesis*
using *arcsin-bounded* [*of y*] *assms* **by** *auto*
qed

lemma *arcsin-sin*: $-$ (*pi/2*) $\leq$ *x* $\Longrightarrow$ *x* $\leq$ *pi/2* $\Longrightarrow$ *arcsin* (*sin x*) = *x*
unfolding *arcsin-def*
using *the1-equality* [*OF sin-total*] **by** *simp*

lemma *arcsin-unique*:
assumes $-$*pi/2* $\leq$ *x* **and** *x* $\leq$ *pi/2* **and** *sin x* = *y* **shows** *arcsin y* = *x*
using *arcsin-sin*[*of x*] *assms* **by** *force*

lemma *arcsin-0* [*simp*]: *arcsin 0* = *0*
using *arcsin-sin* [*of 0*] **by** *simp*

lemma *arcsin-1* [*simp*]: *arcsin 1* = *pi/2*
using *arcsin-sin* [*of pi/2*] **by** *simp*

lemma *arcsin-minus-1* [*simp*]: *arcsin* ($-$ *1*) = $-$ (*pi/2*)
using *arcsin-sin* [*of* $-$ *pi/2*] **by** *simp*

lemma *arcsin-minus*: $-$ *1* $\leq$ *x* $\Longrightarrow$ *x* $\leq$ *1* $\Longrightarrow$ *arcsin* ($-$ *x*) = $-$ *arcsin x*
by (*metis* (*no-types*, *opaque-lifting*) *arcsin arcsin-sin minus-minus neg-le-iff-le sin-minus*)

lemma *arcsin-one-half* [*simp*]: *arcsin* (*1/2*) = *pi* / *6*
and *arcsin-minus-one-half* [*simp*]: *arcsin* ($-$(*1/2*)) = $-$*pi* / *6*
by (*intro arcsin-unique*; *simp add*: *sin-30 field-simps*)+

lemma *arcsin-one-over-sqrt-2*: *arcsin* (*1* / *sqrt 2*) = *pi* / *4*
by (*rule arcsin-unique*) (*auto simp*: *sin-45 field-simps*)

lemma *arcsin-eq-iff*: $|x| \le 1 \Longrightarrow |y| \le 1 \Longrightarrow arcsin\ x = arcsin\ y \longleftrightarrow x = y$
  by (*metis abs-le-iff arcsin minus-le-iff*)

lemma *cos-arcsin-nonzero*: $-\ 1 < x \Longrightarrow x < 1 \Longrightarrow cos\ (arcsin\ x) \neq 0$
  using *arcsin-lt-bounded cos-gt-zero-pi* **by** *force*

lemma *arccos*: $-\ 1 \le y \Longrightarrow y \le 1 \Longrightarrow 0 \le arccos\ y \wedge arccos\ y \le pi \wedge cos\ (arccos\ y) = y$
  unfolding *arccos-def* **by** (*rule theI'* [*OF cos-total*])

lemma *cos-arccos* [*simp*]: $-\ 1 \le y \Longrightarrow y \le 1 \Longrightarrow cos\ (arccos\ y) = y$
  by (*blast dest*: *arccos*)

lemma *arccos-bounded*: $-\ 1 \le y \Longrightarrow y \le 1 \Longrightarrow 0 \le arccos\ y \wedge arccos\ y \le pi$
  by (*blast dest*: *arccos*)

lemma *arccos-lbound*: $-\ 1 \le y \Longrightarrow y \le 1 \Longrightarrow 0 \le arccos\ y$
  by (*blast dest*: *arccos*)

lemma *arccos-ubound*: $-\ 1 \le y \Longrightarrow y \le 1 \Longrightarrow arccos\ y \le pi$
  by (*blast dest*: *arccos*)

lemma *arccos-lt-bounded*:
  assumes $-\ 1 < y\ y < 1$
  shows $0 < arccos\ y \wedge arccos\ y < pi$
proof −
  have $arccos\ y \neq 0$
   by (*metis* (*no-types*) *arccos assms*(*1*) *assms*(*2*) *cos-zero less-eq-real-def less-irrefl*)
  moreover have $arccos\ y \neq -pi$
   by (*metis arccos assms*(*1*) *assms*(*2*) *cos-minus cos-pi not-less not-less-iff-gr-or-eq*)
  ultimately show *?thesis*
    using *arccos-bounded* [*of y*] *assms*
    by (*metis arccos cos-pi not-less not-less-iff-gr-or-eq*)
qed

lemma *arccos-cos*: $0 \le x \Longrightarrow x \le pi \Longrightarrow arccos\ (cos\ x) = x$
  by (*auto simp*: *arccos-def intro*!: *the1-equality cos-total*)

lemma *arccos-cos2*: $x \le 0 \Longrightarrow -\ pi \le x \Longrightarrow arccos\ (cos\ x) = -x$
  by (*auto simp*: *arccos-def intro*!: *the1-equality cos-total*)

lemma *arccos-unique*:
  assumes $0 \le x$ **and** $x \le pi$ **and** $cos\ x = y$ **shows** $arccos\ y = x$
  using *arccos-cos assms* **by** *blast*

lemma *cos-arcsin*:
  assumes $-\ 1 \le x\ x \le 1$
  shows $cos\ (arcsin\ x) = sqrt\ (1 - x^2)$

proof (*rule power2-eq-imp-eq*)
  show (*cos* (*arcsin x*))^2 = (*sqrt* (*1* − x^2))^2
    by (*simp add*: *square-le-1 assms cos-squared-eq*)
  show *0* ≤ *cos* (*arcsin x*)
    using *arcsin assms cos-ge-zero* **by** *blast*
  show *0* ≤ *sqrt* (*1* − x^2)
    by (*simp add*: *square-le-1 assms*)
qed

lemma *sin-arccos*:
  assumes − *1* ≤ *x x* ≤ *1*
  shows *sin* (*arccos x*) = *sqrt* (*1* − x^2)
proof (*rule power2-eq-imp-eq*)
  show (*sin* (*arccos x*))^2 = (*sqrt* (*1* − x^2))^2
    by (*simp add*: *square-le-1 assms sin-squared-eq*)
  show *0* ≤ *sin* (*arccos x*)
    by (*simp add*: *arccos-bounded assms sin-ge-zero*)
  show *0* ≤ *sqrt* (*1* − x^2)
    by (*simp add*: *square-le-1 assms*)
qed

lemma *arccos-0* [*simp*]: *arccos 0* = *pi*/*2*
  using *arccos-cos pi-half-ge-zero* **by** *fastforce*

lemma *arccos-1* [*simp*]: *arccos 1* = *0*
  using *arccos-cos* **by** *force*

lemma *arccos-minus-1* [*simp*]: *arccos* (− *1*) = *pi*
  by (*metis arccos-cos cos-pi order-refl pi-ge-zero*)

lemma *arccos-minus*: −*1* ≤ *x* ⟹ *x* ≤ *1* ⟹ *arccos* (− *x*) = *pi* − *arccos x*
  by (*smt* (*verit*, *ccfv-threshold*) *arccos arccos-cos cos-minus cos-minus-pi*)

lemma *arccos-one-half* [*simp*]: *arccos* (*1*/*2*) = *pi* / *3*
  and *arccos-minus-one-half* [*simp*]: *arccos* (−(*1*/*2*)) = *2* ∗ *pi* / *3*
  by (*intro arccos-unique*; *simp add*: *cos-60 cos-120*)+

lemma *arccos-one-over-sqrt-2*: *arccos* (*1* / *sqrt 2*) = *pi* / *4*
  by (*rule arccos-unique*) (*auto simp*: *cos-45 field-simps*)

corollary *arccos-minus-abs*:
  assumes |*x*| ≤ *1*
  shows *arccos* (− *x*) = *pi* − *arccos x*
using *assms* **by** (*simp add*: *arccos-minus*)

lemma *sin-arccos-nonzero*: − *1* < *x* ⟹ *x* < *1* ⟹ *sin* (*arccos x*) ≠ *0*
  using *arccos-lt-bounded sin-gt-zero* **by** *force*

lemma *arctan*: − (*pi*/*2*) < *arctan y* ∧ *arctan y* < *pi*/*2* ∧ *tan* (*arctan y*) = *y*

unfolding *arctan-def* **by** (*rule theI'* [*OF tan-total*])

lemma *tan-arctan*: *tan* (*arctan y*) = *y*
by (*simp add*: *arctan*)

lemma *arctan-bounded*: − (*pi*/*2*) < *arctan y* ∧ *arctan y* < *pi*/*2*
by (*auto simp only*: *arctan*)

lemma *arctan-lbound*: − (*pi*/*2*) < *arctan y*
by (*simp add*: *arctan*)

lemma *arctan-ubound*: *arctan y* < *pi*/*2*
by (*auto simp only*: *arctan*)

lemma *arctan-unique*:
assumes −(*pi*/*2*) < *x*
and *x* < *pi*/*2*
and *tan x* = *y*
shows *arctan y* = *x*
using *assms arctan* [*of y*] *tan-total* [*of y*] **by** (*fast elim*: *ex1E*)

lemma *arctan-tan*: −(*pi*/*2*) < *x* ⟹ *x* < *pi*/*2* ⟹ *arctan* (*tan x*) = *x*
by (*rule arctan-unique*) *simp-all*

lemma *arctan-zero-zero* [*simp*]: *arctan 0* = *0*
by (*rule arctan-unique*) *simp-all*

lemma *arctan-minus*: *arctan* (− *x*) = − *arctan x*
using *arctan* [*of x*] **by** (*auto simp*: *arctan-unique*)

lemma *cos-arctan-not-zero* [*simp*]: *cos* (*arctan x*) ≠ *0*
by (*intro less-imp-neq* [*symmetric*] *cos-gt-zero-pi arctan-lbound arctan-ubound*)

lemma *tan-eq-arctan-Ex*:
shows *tan x* = *y* ⟷ (∃ *k*::*int*. *x* = *arctan y* + *k*∗*pi* ∨ (*x* = *pi*/*2* + *k*∗*pi* ∧ *y*=*0*))
proof
assume *lhs*: *tan x* = *y*
obtain *k*::*int* **where** *k*:−*pi*/*2* < *x*−*k*∗*pi* *x*−*k*∗*pi* ≤ *pi*/*2*
proof
define *k* **where** *k* ≡ *ceiling* (*x*/*pi* − *1*/*2*)
show − *pi* / *2* < *x* − *real-of-int k* ∗ *pi*
using *ceiling-divide-lower* [*of pi*∗*2* (*x* ∗ *2* − *pi*)] **by** (*auto simp*: *k-def field-simps*)
show *x*−*k*∗*pi* ≤ *pi*/*2*
using *ceiling-divide-upper* [*of pi*∗*2* (*x* ∗ *2* − *pi*)] **by** (*auto simp*: *k-def field-simps*)
qed
have *x* = *arctan y* + *of-int k* ∗ *pi* **when** *x* ≠ *pi*/*2* + *k*∗*pi*

proof −
have *tan* (*x* − *k* ∗ *pi*) = *y* **using** *lhs tan-periodic-int*[*of* - −*k*] **by** *auto*
then have *arctan y* = *x* − *real-of-int k* ∗ *pi*
by (*smt* (*verit*) *arctan-tan lhs divide-minus-left k mult-minus-left of-int-minus tan-periodic-int that*)
then show *?thesis* **by** *auto*
qed
then show ∃ *k*. *x* = *arctan y* + *of-int k* ∗ *pi* ∨ (*x* = *pi*/*2* + *k*∗*pi* ∧ *y*=*0*)
using *lhs k* **by** *force*
qed (*auto simp*: *arctan*)

lemma *arctan-tan-eq-abs-pi*:
assumes *cos* ϑ ≠ *0*
obtains *k* **where** *arctan* (*tan* ϑ) = ϑ − *of-int k* ∗ *pi*
by (*metis add.commute assms cos-zero-iff-int2 eq-diff-eq tan-eq-arctan-Ex*)

lemma *tan-eq*:
assumes *tan x* = *tan y tan x* ≠ *0*
obtains *k*::*int* **where** *x* = *y* + *k* ∗ *pi*
proof −
obtain *k0* **where** *k0*: *x* = *arctan* (*tan y*) + *real-of-int k0* ∗ *pi*
using *assms tan-eq-arctan-Ex*[*of x tan y*] **by** *auto*
obtain *k1* **where** *k1*: *arctan* (*tan y*) = *y* − *of-int k1* ∗ *pi*
using *arctan-tan-eq-abs-pi assms tan-eq-0-cos-sin* **by** *auto*
have *x* = *y* + (*k0*−*k1*)∗*pi*
using *k0 k1* **by** (*auto simp*: *algebra-simps*)
with *that* **show** *?thesis*
by *blast*
qed

lemma *cos-arctan*: *cos* (*arctan x*) = *1* / *sqrt* (*1* + x^2)
proof (*rule power2-eq-imp-eq*)
have *0* < *1* + x^2 **by** (*simp add*: *add-pos-nonneg*)
show *0* ≤ *1* / *sqrt* (*1* + x^2) **by** *simp*
show *0* ≤ *cos* (*arctan x*)
by (*intro less-imp-le cos-gt-zero-pi arctan-lbound arctan-ubound*)
have (*cos* (*arctan x*))^2 ∗ (*1* + (*tan* (*arctan x*))^2) = *1*
unfolding *tan-def* **by** (*simp add*: *distrib-left power-divide*)
then show (*cos* (*arctan x*))^2 = (*1* / *sqrt* (*1* + x^2))^2
using ‹*0* < *1* + x^2› **by** (*simp add*: *arctan power-divide eq-divide-eq*)
qed

lemma *sin-arctan*: *sin* (*arctan x*) = *x* / *sqrt* (*1* + x^2)
using *add-pos-nonneg* [*OF zero-less-one zero-le-power2* [*of x*]]
using *tan-arctan* [*of x*] **unfolding** *tan-def cos-arctan*
by (*simp add*: *eq-divide-eq*)

lemma *tan-sec*: *cos x* ≠ *0* ⟹ *1* + (*tan x*)^2 = (*inverse* (*cos x*))^2
for *x* :: ′*a*::{*real-normed-field*,*banach*,*field*}

by (*simp add*: *add-divide-eq-iff inverse-eq-divide power2-eq-square tan-def*)

lemma *arctan-less-iff*: *arctan x* < *arctan y* ⟷ *x* < *y*
by (*metis tan-monotone′ arctan-lbound arctan-ubound tan-arctan*)

lemma *arctan-le-iff*: *arctan x* ≤ *arctan y* ⟷ *x* ≤ *y*
by (*simp only*: *not-less* [*symmetric*] *arctan-less-iff*)

lemma *arctan-eq-iff*: *arctan x* = *arctan y* ⟷ *x* = *y*
by (*simp only*: *eq-iff* [**where** ′*a*=*real*] *arctan-le-iff*)

lemma *zero-less-arctan-iff* [*simp*]: *0* < *arctan x* ⟷ *0* < *x*
using *arctan-less-iff* [*of 0 x*] **by** *simp*

lemma *arctan-less-zero-iff* [*simp*]: *arctan x* < *0* ⟷ *x* < *0*
using *arctan-less-iff* [*of x 0*] **by** *simp*

lemma *zero-le-arctan-iff* [*simp*]: *0* ≤ *arctan x* ⟷ *0* ≤ *x*
using *arctan-le-iff* [*of 0 x*] **by** *simp*

lemma *arctan-le-zero-iff* [*simp*]: *arctan x* ≤ *0* ⟷ *x* ≤ *0*
using *arctan-le-iff* [*of x 0*] **by** *simp*

lemma *arctan-eq-zero-iff* [*simp*]: *arctan x* = *0* ⟷ *x* = *0*
using *arctan-eq-iff* [*of x 0*] **by** *simp*

lemma *continuous-on-arcsin′*: *continuous-on* {−*1* .. *1*} *arcsin*
proof −
have *continuous-on* (*sin* ‘ {− *pi*/*2* .. *pi*/*2*}) *arcsin*
by (*rule continuous-on-inv*) (*auto intro*: *continuous-intros simp*: *arcsin-sin*)
also have *sin* ‘ {− *pi*/*2* .. *pi*/*2*} = {−*1* .. *1*}
proof *safe*
fix *x* :: *real*
assume *x* ∈ {−*1*..*1*}
then show *x* ∈ *sin* ‘ {− *pi*/*2*..*pi*/*2*}
using *arcsin-lbound arcsin-ubound*
by (*intro image-eqI*[**where** *x*=*arcsin x*]) *auto*
qed *simp*
finally show *?thesis* **.**
qed

lemma *continuous-on-arcsin* [*continuous-intros*]:
continuous-on s f ⟹ (∀*x*∈*s*. −*1* ≤ *f x* ∧ *f x* ≤ *1*) ⟹ *continuous-on s* (λ*x*. *arcsin* (*f x*))
using *continuous-on-compose*[*of s f*, *OF* - *continuous-on-subset*[*OF continuous-on-arcsin′*]]
by (*auto simp*: *comp-def subset-eq*)

lemma *isCont-arcsin*: −*1* < *x* ⟹ *x* < *1* ⟹ *isCont arcsin x*

using *continuous-on-arcsin'*[*THEN continuous-on-subset*, *of* { −*1* <..< *1* }]
by (*auto simp*: *continuous-on-eq-continuous-at subset-eq*)

lemma *continuous-on-arccos'*: *continuous-on* {−*1* .. *1*} *arccos*
proof −
have *continuous-on* (*cos* ' {*0* .. *pi*}) *arccos*
by (*rule continuous-on-inv*) (*auto intro*: *continuous-intros simp*: *arccos-cos*)
also have *cos* ' {*0* .. *pi*} = {−*1* .. *1*}
proof *safe*
fix *x* :: *real*
assume $x \in \{-1..1\}$
then show $x \in$ *cos* ' {*0*..*pi*}
using *arccos-lbound arccos-ubound*
by (*intro image-eqI*[**where** *x*=*arccos x*]) *auto*
qed *simp*
finally show *?thesis* **.**
qed

lemma *continuous-on-arccos* [*continuous-intros*]:
continuous-on s f ⟹ (∀ *x*∈*s*. −*1* ≤ *f x* ∧ *f x* ≤ *1*) ⟹ *continuous-on s* (λ*x*. *arccos* (*f x*))
using *continuous-on-compose*[*of s f*, *OF* - *continuous-on-subset*[*OF continuous-on-arccos'*]]
by (*auto simp*: *comp-def subset-eq*)

lemma *isCont-arccos*: −*1* < *x* ⟹ *x* < *1* ⟹ *isCont arccos x*
using *continuous-on-arccos'*[*THEN continuous-on-subset*, *of* { −*1* <..< *1* }]
by (*auto simp*: *continuous-on-eq-continuous-at subset-eq*)

lemma *isCont-arctan*: *isCont arctan x*
proof −
obtain *u* **where** *u*: − (*pi*/*2*) < *u u* < *arctan x*
by (*meson arctan arctan-less-iff linordered-field-no-lb*)
obtain *v* **where** *v*: *arctan x* < *v v* < *pi*/*2*
by (*meson arctan-less-iff arctan-ubound linordered-field-no-ub*)
have *isCont arctan* (*tan* (*arctan x*))
proof (*rule isCont-inverse-function2* [*of u arctan x v*])
show ⋀*z*. ⟦*u* ≤ *z*; *z* ≤ *v*⟧ ⟹ *arctan* (*tan z*) = *z*
using *arctan-unique u*(*1*) *v*(*2*) **by** *auto*
then show ⋀*z*. ⟦*u* ≤ *z*; *z* ≤ *v*⟧ ⟹ *isCont tan z*
by (*metis arctan cos-gt-zero-pi isCont-tan less-irrefl*)
qed (*use u v* **in** *auto*)
then show *?thesis*
by (*simp add*: *arctan*)
qed

lemma *tendsto-arctan* [*tendsto-intros*]: (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *arctan* (*f x*)) ⟶ *arctan x*) *F*
by (*rule isCont-tendsto-compose* [*OF isCont-arctan*])

lemma *continuous-arctan* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.$ *arctan* (*f x*))
  unfolding *continuous-def* **by** (*rule tendsto-arctan*)

lemma *continuous-on-arctan* [*continuous-intros*]:
  continuous-on s f $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *arctan* (*f x*))
  unfolding *continuous-on-def* **by** (*auto intro*: *tendsto-arctan*)

lemma *DERIV-arcsin*:
  assumes $-$ *1* $<$ *x x* $<$ *1*
  shows *DERIV arcsin x* :> *inverse* (*sqrt* (*1* $-$ x^2))
proof (*rule DERIV-inverse-function*)
  show (*sin has-real-derivative sqrt* (*1* $-$ x^2)) (*at* (*arcsin x*))
    by (*rule derivative-eq-intros* | *use assms cos-arcsin* **in** *force*)+
  show *sqrt* (*1* $-$ x^2) $\neq$ *0*
    using *abs-square-eq-1 assms* **by** *force*
qed (*use assms isCont-arcsin* **in** *auto*)

lemma *DERIV-arccos*:
  assumes $-$ *1* $<$ *x x* $<$ *1*
  shows *DERIV arccos x* :> *inverse* ($-$ *sqrt* (*1* $-$ x^2))
proof (*rule DERIV-inverse-function*)
  show (*cos has-real-derivative* $-$ *sqrt* (*1* $-$ x^2)) (*at* (*arccos x*))
    by (*rule derivative-eq-intros* | *use assms sin-arccos* **in** *force*)+
  show $-$ *sqrt* (*1* $-$ x^2) $\neq$ *0*
    using *abs-square-eq-1 assms* **by** *force*
qed (*use assms isCont-arccos* **in** *auto*)

lemma *DERIV-arctan*: *DERIV arctan x* :> *inverse* (*1* + x^2)
proof (*rule DERIV-inverse-function*)
  have *inverse* ((*cos* (*arctan x*))^2) = *1* + x^2
    by (*metis arctan cos-arctan-not-zero power-inverse tan-sec*)
  then show (*tan has-real-derivative 1* + x^2) (*at* (*arctan x*))
    by (*auto intro*!: *derivative-eq-intros*)
  show $\bigwedge y.$ $[\![x - 1 < y;\ y < x + 1]\!] \Longrightarrow$ *tan* (*arctan y*) = *y*
    using *tan-arctan* **by** *blast*
  show *1* + x^2 $\neq$ *0*
    by (*metis power-one sum-power2-eq-zero-iff zero-neq-one*)
qed (*use isCont-arctan* **in** *auto*)

declare
  DERIV-arcsin[*THEN DERIV-chain2*, *derivative-intros*]
  DERIV-arcsin[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]
  DERIV-arccos[*THEN DERIV-chain2*, *derivative-intros*]
  DERIV-arccos[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]
  DERIV-arctan[*THEN DERIV-chain2*, *derivative-intros*]

DERIV-arctan[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-arctan*[*derivative-intros*] = *DERIV-arctan*[*THEN DERIV-compose-FDERIV*]
and *has-derivative-arccos*[*derivative-intros*] = *DERIV-arccos*[*THEN DERIV-compose-FDERIV*]
and *has-derivative-arcsin*[*derivative-intros*] = *DERIV-arcsin*[*THEN DERIV-compose-FDERIV*]

lemma *filterlim-tan-at-right*: *filterlim tan at-bot* (*at-right* (− (*pi*/*2*)))
by (*rule filterlim-at-bot-at-right*[**where** *Q*=λ*x*. − *pi*/*2* < *x* ∧ *x* < *pi*/*2* **and** *P*=λ*x*. *True* **and** *g*=*arctan*])
(*auto simp*: *arctan le-less eventually-at dist-real-def simp del*: *less-divide-eq-numeral1*
intro!: *tan-monotone exI*[*of* - *pi*/*2*])

lemma *filterlim-tan-at-left*: *filterlim tan at-top* (*at-left* (*pi*/*2*))
by (*rule filterlim-at-top-at-left*[**where** *Q*=λ*x*. − *pi*/*2* < *x* ∧ *x* < *pi*/*2* **and** *P*=λ*x*. *True* **and** *g*=*arctan*])
(*auto simp*: *arctan le-less eventually-at dist-real-def simp del*: *less-divide-eq-numeral1*
intro!: *tan-monotone exI*[*of* - *pi*/*2*])

lemma *tendsto-arctan-at-top*: (*arctan* ⟶ (*pi*/*2*)) *at-top*
proof (*rule tendstoI*)
fix *e* :: *real*
assume *0* < *e*
define *y* **where** *y* = *pi*/*2* − *min* (*pi*/*2*) *e*
then have *y*: *0* ≤ *y y* < *pi*/*2 pi*/*2* ≤ *e* + *y*
using ‹*0* < *e*› **by** *auto*
show *eventually* (λ*x*. *dist* (*arctan x*) (*pi*/*2*) < *e*) *at-top*
proof (*intro eventually-at-top-dense*[*THEN iffD2*] *exI allI impI*)
fix *x*
assume *tan y* < *x*
then have *arctan* (*tan y*) < *arctan x*
by (*simp add*: *arctan-less-iff*)
with *y* **have** *y* < *arctan x*
by (*subst* (*asm*) *arctan-tan*) *simp-all*
with *arctan-ubound*[*of x*, *arith*] *y* ‹*0* < *e*›
show *dist* (*arctan x*) (*pi*/*2*) < *e*
by (*simp add*: *dist-real-def*)
qed
qed

lemma *tendsto-arctan-at-bot*: (*arctan* ⟶ − (*pi*/*2*)) *at-bot*
unfolding *filterlim-at-bot-mirror arctan-minus*
by (*intro tendsto-minus tendsto-arctan-at-top*)

lemma *sin-multiple-reduce*:
sin (*x* ∗ *numeral n* :: ′*a* :: {*real-normed-field*, *banach*}) =
sin x ∗ *cos* (*x* ∗ *of-nat* (*pred-numeral n*)) + *cos x* ∗ *sin* (*x* ∗ *of-nat* (*pred-numeral n*))
proof −

have *numeral n* = *of-nat* (*pred-numeral n*) + (*1* :: *'a*)
by (*metis add.commute numeral-eq-Suc of-nat-Suc of-nat-numeral*)
also have *sin* (*x* ∗ . . .) = *sin* (*x* ∗ *of-nat* (*pred-numeral n*) + *x*)
unfolding *of-nat-Suc* **by** (*simp add*: *ring-distribs*)
finally show *?thesis*
by (*simp add*: *sin-add*)
qed

lemma *cos-multiple-reduce*:
cos (*x* ∗ *numeral n* :: *'a* :: {*real-normed-field*, *banach*}) =
cos (*x* ∗ *of-nat* (*pred-numeral n*)) ∗ *cos x* − *sin* (*x* ∗ *of-nat* (*pred-numeral n*)) ∗ *sin x*
proof −
have *numeral n* = *of-nat* (*pred-numeral n*) + (*1* :: *'a*)
by (*metis add.commute numeral-eq-Suc of-nat-Suc of-nat-numeral*)
also have *cos* (*x* ∗ . . .) = *cos* (*x* ∗ *of-nat* (*pred-numeral n*) + *x*)
unfolding *of-nat-Suc* **by** (*simp add*: *ring-distribs*)
finally show *?thesis*
by (*simp add*: *cos-add*)
qed

lemma *arccos-eq-pi-iff*: $x \in \{-1..1\} \Longrightarrow arccos\ x = pi \longleftrightarrow x = -1$
by (*metis arccos arccos-minus-1 atLeastAtMost-iff cos-pi*)

lemma *arccos-eq-0-iff*: $x \in \{-1..1\} \Longrightarrow arccos\ x = 0 \longleftrightarrow x = 1$
by (*metis arccos arccos-1 atLeastAtMost-iff cos-zero*)

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lemma *cos-arccos-abs*: $|y| \le 1 \Longrightarrow cos\ (arccos\ y) = y$
by (*simp add*: *abs-le-iff*)

lemma *sin-arccos-abs*: $|y| \le 1 \Longrightarrow sin\ (arccos\ y) = sqrt\ (1 - y^2)$
by (*simp add*: *sin-arccos abs-le-iff*)

lemma *sin-mono-less-eq*:
$-(pi/2) \le x \Longrightarrow x \le pi/2 \Longrightarrow -(pi/2) \le y \Longrightarrow y \le pi/2 \Longrightarrow sin\ x < sin\ y \longleftrightarrow x < y$
by (*metis not-less-iff-gr-or-eq sin-monotone-2pi*)

lemma *sin-mono-le-eq*:
$-(pi/2) \le x \Longrightarrow x \le pi/2 \Longrightarrow -(pi/2) \le y \Longrightarrow y \le pi/2 \Longrightarrow sin\ x \le sin\ y \longleftrightarrow x \le y$
by (*meson leD le-less-linear sin-monotone-2pi sin-monotone-2pi-le*)

lemma *sin-inj-pi*:
$-(pi/2) \le x \Longrightarrow x \le pi/2 \Longrightarrow -(pi/2) \le y \Longrightarrow y \le pi/2 \Longrightarrow sin\ x = sin\ y \Longrightarrow x = y$
by (*metis arcsin-sin*)

lemma *arcsin-le-iff*:
assumes $x \geq -1$ $x \leq 1$ $y \geq -pi/2$ $y \leq pi/2$
shows *arcsin* $x \leq y \longleftrightarrow x \leq$ *sin* y
proof −
have *arcsin* $x \leq y \longleftrightarrow$ *sin* (*arcsin* x) $\leq$ *sin* y
using *arcsin-bounded*[*of* x] *assms* **by** (*subst sin-mono-le-eq*) *auto*
also from *assms* **have** *sin* (*arcsin* x) = x **by** *simp*
finally show *?thesis* **.**
qed

lemma *le-arcsin-iff*:
assumes $x \geq -1$ $x \leq 1$ $y \geq -pi/2$ $y \leq pi/2$
shows *arcsin* $x \geq y \longleftrightarrow x \geq$ *sin* y
proof −
have *arcsin* $x \geq y \longleftrightarrow$ *sin* (*arcsin* x) $\geq$ *sin* y
using *arcsin-bounded*[*of* x] *assms* **by** (*subst sin-mono-le-eq*) *auto*
also from *assms* **have** *sin* (*arcsin* x) = x **by** *simp*
finally show *?thesis* **.**
qed

lemma *cos-mono-less-eq*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow 0 \leq y \Longrightarrow y \leq pi \Longrightarrow cos\ x < cos\ y \longleftrightarrow y < x$
by (*meson cos-monotone-0-pi cos-monotone-0-pi-le leD le-less-linear*)

lemma *cos-mono-le-eq*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow 0 \leq y \Longrightarrow y \leq pi \Longrightarrow cos\ x \leq cos\ y \longleftrightarrow y \leq x$
by (*metis arccos-cos cos-monotone-0-pi-le eq-iff linear*)

lemma *cos-inj-pi*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow 0 \leq y \Longrightarrow y \leq pi \Longrightarrow cos\ x = cos\ y \Longrightarrow x = y$
by (*metis arccos-cos*)

lemma *arccos-le-pi2*: $[\![0 \leq y;\ y \leq 1]\!] \Longrightarrow arccos\ y \leq pi/2$
by (*metis* (*mono-tags*) *arccos-0 arccos cos-le-one cos-monotone-0-pi-le cos-pi cos-pi-half pi-half-ge-zero antisym-conv less-eq-neg-nonpos linear minus-minus order.trans order-refl*)

lemma *sincos-total-pi-half*:
assumes $0 \leq x$ $0 \leq y$ $x^2 + y^2 = 1$
shows $\exists t.\ 0 \leq t \wedge t \leq pi/2 \wedge x = cos\ t \wedge y = sin\ t$
proof −
have *x1*: $x \leq 1$
using *assms* **by** (*metis le-add-same-cancel1 power2-le-imp-le power-one zero-le-power2*)
with *assms* **have** $*$: $0 \leq arccos\ x$ $cos\ (arccos\ x) = x$
by (*auto simp*: *arccos*)
from *assms* **have** $y = sqrt\ (1 - x^2)$
by (*metis abs-of-nonneg add.commute add-diff-cancel real-sqrt-abs*)
with *x1* $*$ *assms arccos-le-pi2* [*of* x] **show** *?thesis*

by (*rule-tac x=arccos x* **in** *exI*) (*auto simp*: *sin-arccos*)
qed

lemma *sincos-total-pi*:
assumes $0 \le y$ $x^2 + y^2 = 1$
shows $\exists t.\ 0 \le t \wedge t \le pi \wedge x = cos\ t \wedge y = sin\ t$
proof (*cases rule*: *le-cases* [*of 0 x*])
case *le*
from *sincos-total-pi-half* [*OF le*] **show** *?thesis*
by (*metis pi-ge-two pi-half-le-two add.commute add-le-cancel-left add-mono assms*)
next
case *ge*
then have $0 \le -x$
by *simp*
then obtain *t* **where** *t*: $t \ge 0$ $t \le pi/2$ $-x = cos\ t$ $y = sin\ t$
using *sincos-total-pi-half assms*
by *auto* (*metis* ‹$0 \le -\ x$› *power2-minus*)
show *?thesis*
by (*rule exI* [**where** $x = pi - t$]) (*use t* **in** *auto*)
qed

lemma *sincos-total-2pi-le*:
assumes $x^2 + y^2 = 1$
shows $\exists t.\ 0 \le t \wedge t \le 2 * pi \wedge x = cos\ t \wedge y = sin\ t$
proof (*cases rule*: *le-cases* [*of 0 y*])
case *le*
from *sincos-total-pi* [*OF le*] **show** *?thesis*
by (*metis assms le-add-same-cancel1 mult.commute mult-2-right order.trans*)
next
case *ge*
then have $0 \le -y$
by *simp*
then obtain *t* **where** *t*: $t \ge 0$ $t \le pi$ $x = cos\ t$ $-y = sin\ t$
using *sincos-total-pi assms*
by *auto* (*metis* ‹$0 \le -\ y$› *power2-minus*)
show *?thesis*
by (*rule exI* [**where** $x = 2 * pi - t$]) (*use t* **in** *auto*)
qed

lemma *sincos-total-2pi*:
assumes $x^2 + y^2 = 1$
obtains *t* **where** $0 \le t$ $t < 2*pi$ $x = cos\ t$ $y = sin\ t$
proof −
from *sincos-total-2pi-le* [*OF assms*]
obtain *t* **where** *t*: $0 \le t$ $t \le 2*pi$ $x = cos\ t$ $y = sin\ t$
by *blast*
show *?thesis*
by (*cases* $t = 2 * pi$) (*use t that* **in** ‹*force+*›)

qed

lemma *arcsin-less-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arcsin\ x < arcsin\ y \longleftrightarrow x < y$
by (*rule trans* [*OF sin-mono-less-eq* [*symmetric*]]) (*use arcsin-ubound arcsin-lbound* **in** *auto*)

lemma *arcsin-le-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arcsin\ x \leq arcsin\ y \longleftrightarrow x \leq y$
using *arcsin-less-mono not-le* **by** *blast*

lemma *arcsin-less-arcsin*: $-\ 1 \leq x \Longrightarrow x < y \Longrightarrow y \leq 1 \Longrightarrow arcsin\ x < arcsin\ y$
using *arcsin-less-mono* **by** *auto*

lemma *arcsin-le-arcsin*: $-\ 1 \leq x \Longrightarrow x \leq y \Longrightarrow y \leq 1 \Longrightarrow arcsin\ x \leq arcsin\ y$
using *arcsin-le-mono* **by** *auto*

lemma *arcsin-nonneg*: $x \in \{0..1\} \Longrightarrow arcsin\ x \geq 0$
using *arcsin-le-arcsin*[*of 0 x*] **by** *simp*

lemma *arccos-less-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arccos\ x < arccos\ y \longleftrightarrow y < x$
by (*rule trans* [*OF cos-mono-less-eq* [*symmetric*]]) (*use arccos-ubound arccos-lbound* **in** *auto*)

lemma *arccos-le-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arccos\ x \leq arccos\ y \longleftrightarrow y \leq x$
using *arccos-less-mono* [*of y x*] **by** (*simp add*: *not-le* [*symmetric*])

lemma *arccos-less-arccos*: $-\ 1 \leq x \Longrightarrow x < y \Longrightarrow y \leq 1 \Longrightarrow arccos\ y < arccos\ x$
using *arccos-less-mono* **by** *auto*

lemma *arccos-le-arccos*: $-\ 1 \leq x \Longrightarrow x \leq y \Longrightarrow y \leq 1 \Longrightarrow arccos\ y \leq arccos\ x$
using *arccos-le-mono* **by** *auto*

lemma *arccos-eq-iff*: $|x| \leq 1 \land |y| \leq 1 \Longrightarrow arccos\ x = arccos\ y \longleftrightarrow x = y$
using *cos-arccos-abs* **by** *fastforce*

lemma *arccos-cos-eq-abs*:
assumes $|\vartheta| \leq pi$
shows $arccos\ (cos\ \vartheta) = |\vartheta|$
unfolding *arccos-def*
proof (*intro the-equality conjI*; *clarify?*)
show $cos\ |\vartheta| = cos\ \vartheta$
by (*simp add*: *abs-real-def*)
show $x = |\vartheta|$ **if** $cos\ x = cos\ \vartheta\ \ 0 \leq x\ \ x \leq pi$ **for** x
by (*simp add*: ‹$cos\ |\vartheta| = cos\ \vartheta$› *assms cos-inj-pi that*)
qed (*use assms* **in** *auto*)

lemma *arccos-cos-eq-abs-2pi*:
obtains k **where** $arccos\ (cos\ \vartheta) = |\vartheta - of\text{-}int\ k * (2 * pi)|$
proof −

```
  define k where k ≡ ⌊(ϑ + pi) / (2 * pi)⌋
  have lepi: |ϑ − of-int k * (2 * pi)| ≤ pi
    using floor-divide-lower [of 2*pi ϑ + pi] floor-divide-upper [of 2*pi ϑ + pi]
    by (auto simp: k-def abs-if algebra-simps)
  have arccos (cos ϑ) = arccos (cos (ϑ − of-int k * (2 * pi)))
    using cos-int-2pin sin-int-2pin by (simp add: cos-diff mult.commute)
  also have ... = |ϑ − of-int k * (2 * pi)|
    using arccos-cos-eq-abs lepi by blast
  finally show ?thesis
    using that by metis
qed

lemma arccos-arctan:
  assumes −1 < x x < 1
  shows arccos x = pi/2 − arctan(x / sqrt(1 − x^2))
proof −
  have arctan(x / sqrt(1 − x^2)) − (pi/2 − arccos x) = 0
  proof (rule sin-eq-0-pi)
    show − pi < arctan (x / sqrt (1 − x^2)) − (pi/2 − arccos x)
      using arctan-lbound [of x / sqrt(1 − x^2)] arccos-bounded [of x] assms
      by (simp add: algebra-simps)
  next
    show arctan (x / sqrt (1 − x^2)) − (pi/2 − arccos x) < pi
      using arctan-ubound [of x / sqrt(1 − x^2)] arccos-bounded [of x] assms
      by (simp add: algebra-simps)
  next
    show sin (arctan (x / sqrt (1 − x^2)) − (pi/2 − arccos x)) = 0
      using assms
      by (simp add: algebra-simps sin-diff cos-add sin-arccos sin-arctan cos-arctan
                power2-eq-square square-eq-1-iff)
  qed
  then show ?thesis
    by simp
qed

lemma arcsin-plus-arccos:
  assumes −1 ≤ x x ≤ 1
    shows arcsin x + arccos x = pi/2
proof −
  have arcsin x = pi/2 − arccos x
    apply (rule sin-inj-pi)
    using assms arcsin [OF assms] arccos [OF assms]
    by (auto simp: algebra-simps sin-diff)
  then show ?thesis
    by (simp add: algebra-simps)
qed

lemma arcsin-arccos-eq: −1 ≤ x ⟹ x ≤ 1 ⟹ arcsin x = pi/2 − arccos x
  using arcsin-plus-arccos by force
```

lemma *arccos-arcsin-eq*: $-1 \leq x \Longrightarrow x \leq 1 \Longrightarrow arccos\ x = pi/2 - arcsin\ x$
using *arcsin-plus-arccos* **by** *force*

lemma *arcsin-arctan*: $-1 < x \Longrightarrow x < 1 \Longrightarrow arcsin\ x = arctan(x\ /\ sqrt(1 - x^2))$
by (*simp add*: *arccos-arctan arcsin-arccos-eq*)

lemma *arcsin-arccos-sqrt-pos*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow arcsin\ x = arccos(sqrt(1 - x^2))$
by (*smt* (*verit*, *del-insts*) *arccos-cos arcsin-0 arcsin-le-arcsin arcsin-pi cos-arcsin*)

lemma *arcsin-arccos-sqrt-neg*: $-1 \leq x \Longrightarrow x \leq 0 \Longrightarrow arcsin\ x = -arccos(sqrt(1 - x^2))$
using *arcsin-arccos-sqrt-pos* [*of* $-x$]
by (*simp add*: *arcsin-minus*)

lemma *arccos-arcsin-sqrt-pos*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow arccos\ x = arcsin(sqrt(1 - x^2))$
by (*smt* (*verit*, *del-insts*) *arccos-lbound arccos-le-pi2 arcsin-sin sin-arccos*)

lemma *arccos-arcsin-sqrt-neg*: $-1 \leq x \Longrightarrow x \leq 0 \Longrightarrow arccos\ x = pi - arcsin(sqrt(1 - x^2))$
using *arccos-arcsin-sqrt-pos* [*of* $-x$]
by (*simp add*: *arccos-minus*)

lemma *cos-limit-1*:
assumes $(\lambda j.\ cos\ (\vartheta\ j)) \longrightarrow 1$
shows $\exists k.\ (\lambda j.\ \vartheta\ j - of\text{-}int\ (k\ j) * (2 * pi)) \longrightarrow 0$
proof −
have $\forall_F\ j\ in\ sequentially.\ cos\ (\vartheta\ j) \in \{-\ 1..1\}$
by *auto*
then have $(\lambda j.\ arccos\ (cos\ (\vartheta\ j))) \longrightarrow arccos\ 1$
using *continuous-on-tendsto-compose* [*OF continuous-on-arccos′ assms*] **by** *auto*
moreover have $\bigwedge j.\ \exists k.\ arccos\ (cos\ (\vartheta\ j)) = |\vartheta\ j - of\text{-}int\ k * (2 * pi)|$
using *arccos-cos-eq-abs-2pi* **by** *metis*
then have $\exists k.\ \forall j.\ arccos\ (cos\ (\vartheta\ j)) = |\vartheta\ j - of\text{-}int\ (k\ j) * (2 * pi)|$
by *metis*
ultimately have $\exists k.\ (\lambda j.\ |\vartheta\ j - of\text{-}int\ (k\ j) * (2 * pi)|) \longrightarrow 0$
by *auto*
then show *?thesis*
by (*simp add*: *tendsto-rabs-zero-iff*)
qed

lemma *cos-diff-limit-1*:
assumes $(\lambda j.\ cos\ (\vartheta\ j - \Theta)) \longrightarrow 1$
obtains k **where** $(\lambda j.\ \vartheta\ j - of\text{-}int\ (k\ j) * (2 * pi)) \longrightarrow \Theta$
proof −
obtain k **where** $(\lambda j.\ (\vartheta\ j - \Theta) - of\text{-}int\ (k\ j) * (2 * pi)) \longrightarrow 0$
using *cos-limit-1* [*OF assms*] **by** *auto*

then have $(\lambda j.\ \Theta + ((\vartheta\ j - \Theta) - \textit{of-int}\ (k\ j) * (2 * pi))) \longrightarrow \Theta + 0$
  by (*rule tendsto-add* [*OF tendsto-const*])
  with *that* **show** *?thesis*
  by *auto*
qed

112.18 Machin's formula

lemma *arctan-one*: $arctan\ 1 = pi/4$
  by (*rule arctan-unique*) (*simp-all add*: *tan-45 m2pi-less-pi*)

lemma *tan-total-pi4*:
  assumes $|x| < 1$
  shows $\exists z. - (pi/4) < z \wedge z < pi/4 \wedge tan\ z = x$
proof
  show $- (pi/4) < arctan\ x \wedge arctan\ x < pi/4 \wedge tan\ (arctan\ x) = x$
    unfolding *arctan-one* [*symmetric*] *arctan-minus* [*symmetric*]
    unfolding *arctan-less-iff*
    using *assms* **by** (*auto simp*: *arctan*)
qed

lemma *arctan-add*:
  assumes $|x| \le 1$ $|y| < 1$
  shows $arctan\ x + arctan\ y = arctan\ ((x + y) / (1 - x * y))$
proof (*rule arctan-unique* [*symmetric*])
  have $- (pi/4) \le arctan\ x$ $- (pi/4) < arctan\ y$
    unfolding *arctan-one* [*symmetric*] *arctan-minus* [*symmetric*]
    unfolding *arctan-le-iff arctan-less-iff*
    using *assms* **by** *auto*
  from *add-le-less-mono* [*OF this*] **show** *1*: $- (pi/2) < arctan\ x + arctan\ y$
    by *simp*
  have $arctan\ x \le pi/4$ $arctan\ y < pi/4$
    unfolding *arctan-one* [*symmetric*]
    unfolding *arctan-le-iff arctan-less-iff*
    using *assms* **by** *auto*
  from *add-le-less-mono* [*OF this*] **show** *2*: $arctan\ x + arctan\ y < pi/2$
    by *simp*
  show $tan\ (arctan\ x + arctan\ y) = (x + y) / (1 - x * y)$
    using *cos-gt-zero-pi* [*OF 1 2*] **by** (*simp add*: *arctan tan-add*)
qed

lemma *arctan-double*: $|x| < 1 \Longrightarrow 2 * arctan\ x = arctan\ ((2 * x) / (1 - x^2))$
  by (*metis arctan-add linear mult-2 not-less power2-eq-square*)

theorem *machin*: $pi/4 = 4 * arctan\ (1 / 5) - arctan\ (1/239)$
proof −
  have $|1 / 5| < (1 :: real)$
    by *auto*
  from *arctan-add*[*OF less-imp-le*[*OF this*] *this*] **have** $2 * arctan\ (1 / 5) = arctan$

(*5* / *12*)
by *auto*
moreover
have |*5* / *12*| < (*1* :: *real*)
by *auto*
from *arctan-add*[*OF less-imp-le*[*OF this*] *this*] **have** *2* ∗ *arctan* (*5* / *12*) = *arctan* (*120* / *119*)
by *auto*
moreover
have |*1*| ≤ (*1*::*real*) **and** |*1*/*239*| < (*1*::*real*)
by *auto*
from *arctan-add*[*OF this*] **have** *arctan 1* + *arctan* (*1*/*239*) = *arctan* (*120* / *119*)
by *auto*
ultimately have *arctan 1* + *arctan* (*1*/*239*) = *4* ∗ *arctan* (*1* / *5*)
by *auto*
then show *?thesis*
unfolding *arctan-one* **by** *algebra*
qed

lemma *machin-Euler*: *5* ∗ *arctan* (*1* / *7*) + *2* ∗ *arctan* (*3* / *79*) = *pi*/*4*
proof −
have *17*: |*1* / *7*| < (*1* :: *real*) **by** *auto*
with *arctan-double* **have** *2* ∗ *arctan* (*1* / *7*) = *arctan* (*7* / *24*)
by *simp* (*simp add*: *field-simps*)
moreover
have |*7* / *24*| < (*1* :: *real*) **by** *auto*
with *arctan-double* **have** *2* ∗ *arctan* (*7* / *24*) = *arctan* (*336* / *527*)
by *simp* (*simp add*: *field-simps*)
moreover
have |*336* / *527*| < (*1* :: *real*) **by** *auto*
from *arctan-add*[*OF less-imp-le*[*OF 17*] *this*]
have *arctan*(*1*/*7*) + *arctan* (*336* / *527*) = *arctan* (*2879* / *3353*)
by *auto*
ultimately have *I*: *5* ∗ *arctan* (*1* / *7*) = *arctan* (*2879* / *3353*) **by** *auto*
have *379*: |*3* / *79*| < (*1* :: *real*) **by** *auto*
with *arctan-double* **have** *II*: *2* ∗ *arctan* (*3* / *79*) = *arctan* (*237* / *3116*)
by *simp* (*simp add*: *field-simps*)
have ∗: |*2879* / *3353*| < (*1* :: *real*) **by** *auto*
have |*237* / *3116*| < (*1* :: *real*) **by** *auto*
from *arctan-add*[*OF less-imp-le*[*OF* ∗] *this*] **have** *arctan* (*2879*/*3353*) + *arctan* (*237*/*3116*) = *pi*/*4*
by (*simp add*: *arctan-one*)
with *I II* **show** *?thesis* **by** *auto*
qed

112.19 Introducing the inverse tangent power series

lemma *monoseq-arctan-series*:

fixes *x* :: *real*
assumes $|x| \leq 1$
shows *monoseq* (λn. *1* / *real* ($n * 2 + 1$) $*$ $x\hat{}(n * 2 + 1)$)
(**is** *monoseq ?a*)
proof (*cases* $x = 0$)
case *True*
then show *?thesis* **by** (*auto simp*: *monoseq-def*)
next
case *False*
have *norm* $x \leq 1$ **and** $x \leq 1$ **and** $-1 \leq x$
using *assms* **by** *auto*
show *monoseq ?a*
proof −
have *mono*: *1* / *real* (*Suc* (*Suc n* $*$ *2*)) $*$ *x* ^ *Suc* (*Suc n* $*$ *2*) $\leq$
1 / *real* (*Suc* (*n* $*$ *2*)) $*$ *x* ^ *Suc* (*n* $*$ *2*)
if $0 \leq x$ **and** $x \leq 1$ **for** *n* **and** *x* :: *real*
proof (*rule mult-mono*)
show *1* / *real* (*Suc* (*Suc n* $*$ *2*)) $\leq$ *1* / *real* (*Suc* (*n* $*$ *2*))
by (*rule frac-le*) *simp-all*
show $0 \leq$ *1* / *real* (*Suc* (*n* $*$ *2*))
by *auto*
show *x* ^ *Suc* (*Suc n* $*$ *2*) $\leq$ *x* ^ *Suc* (*n* $*$ *2*)
by (*rule power-decreasing*) (*simp-all add*: ‹$0 \leq x$› ‹$x \leq 1$›)
show $0 \leq$ *x* ^ *Suc* (*Suc n* $*$ *2*)
by (*rule zero-le-power*) (*simp add*: ‹$0 \leq x$›)
qed
show *?thesis*
proof (*cases* $0 \leq x$)
case *True*
from *mono*[*OF this* ‹$x \leq 1$›, *THEN allI*]
show *?thesis*
unfolding *Suc-eq-plus1*[*symmetric*] **by** (*rule mono-SucI2*)
next
case *False*
then have $0 \leq - x$ **and** $- x \leq 1$
using ‹$-1 \leq x$› **by** *auto*
from *mono*[*OF this*]
have *1* / *real* (*Suc* (*Suc n* $*$ *2*)) $*$ *x* ^ *Suc* (*Suc n* $*$ *2*) $\geq$
1 / *real* (*Suc* (*n* $*$ *2*)) $*$ *x* ^ *Suc* (*n* $*$ *2*) **for** *n*
using ‹$0 \leq -x$› **by** *auto*
then show *?thesis*
unfolding *Suc-eq-plus1*[*symmetric*] **by** (*rule mono-SucI1*[*OF allI*])
qed
qed
qed

lemma *zeroseq-arctan-series*:
fixes *x* :: *real*
assumes $|x| \leq 1$

shows $(\lambda n.\ 1 \ /\ real\ (n * 2 + 1) * x\hat{\ }(n * 2 + 1)) \longrightarrow 0$
(**is** $?a \longrightarrow 0$)
proof (*cases* $x = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have $norm\ x \le 1$ **and** $x \le 1$ **and** $-1 \le x$
using *assms* **by** *auto*
show $?a \longrightarrow 0$
proof (*cases* $|x| < 1$)
case *True*
then have $norm\ x < 1$ **by** *auto*
from *tendsto-mult*[*OF LIMSEQ-inverse-real-of-nat LIMSEQ-power-zero*[*OF* ‹$norm\ x < 1$›, *THEN LIMSEQ-Suc*]]
have $(\lambda n.\ 1 \ /\ real\ (n + 1) * x\ \hat{\ }(n + 1)) \longrightarrow 0$
unfolding *inverse-eq-divide Suc-eq-plus1* **by** *simp*
then show *?thesis*
using *pos2* **by** (*rule LIMSEQ-linear*)
next
case *False*
then have $x = -1 \lor x = 1$
using ‹$|x| \le 1$› **by** *auto*
then have *n-eq*: $\bigwedge n.\ x\ \hat{\ }(n * 2 + 1) = x$
unfolding *One-nat-def* **by** *auto*
from *tendsto-mult*[*OF LIMSEQ-inverse-real-of-nat*[*THEN LIMSEQ-linear*, *OF pos2*, *unfolded inverse-eq-divide*] *tendsto-const*[*of x*]]
show *?thesis*
unfolding *n-eq Suc-eq-plus1* **by** *auto*
qed
qed

lemma *summable-arctan-series*:
fixes $n :: nat$
assumes $|x| \le 1$
shows *summable* $(\lambda\ k.\ (-1)\hat{\ }k * (1 \ /\ real\ (k*2+1) * x\ \hat{\ }(k*2+1)))$
(**is** *summable* $(?c\ x)$)
by (*rule summable-Leibniz*(*1*),
rule zeroseq-arctan-series[*OF assms*],
rule monoseq-arctan-series[*OF assms*])

lemma *DERIV-arctan-series*:
assumes $|x| < 1$
shows $DERIV\ (\lambda x'.\ \sum k.\ (-1)\hat{\ }k * (1 \ /\ real\ (k * 2 + 1) * x'\ \hat{\ }(k * 2 + 1)))$ x :>
$(\sum k.\ (-1)\hat{\ }k * x\hat{\ }(k * 2))$
(**is** *DERIV ?arctan - :> ?Int*)
proof −
let $?f = \lambda n.\ if\ even\ n\ then\ (-1)\hat{\ }(n\ div\ 2) * 1 \ /\ real\ (Suc\ n)\ else\ 0$

have *n-even*: *even n* $\Longrightarrow$ *2* $*$ (*n div 2*) = *n* **for** *n* :: *nat*
by *presburger*
then have *if-eq*: *?f n* $*$ *real* (*Suc n*) $*$ *x'* ^*n* =
(*if even n then* (−*1*) ^(*n div 2*) $*$ *x'* ^(*2* $*$ (*n div 2*)) *else 0*)
for *n x'*
by *auto*

have *summable-Integral*: *summable* (λ *n*. (− *1*) ^ *n* $*$ *x* ^(*2* $*$ *n*)) **if** |*x*| < *1* **for** *x* :: *real*
proof −
from *that* **have** $x^2 < 1$
by (*simp add*: *abs-square-less-1*)
have *summable* (λ *n*. (− *1*) ^ *n* $*$ (x^2) ^*n*)
by (*rule summable-Leibniz*(*1*))
(*auto intro*!: *LIMSEQ-realpow-zero monoseq-realpow* ‹$x^2 < 1$› *order-less-imp-le*[*OF* ‹$x^2 < 1$›])
then show *?thesis*
by (*simp only*: *power-mult*)
qed

have *sums-even*: (*sums*) *f* = (*sums*) (λ *n*. *if even n then f* (*n div 2*) *else 0*)
for *f* :: *nat* ⇒ *real*
proof −
have *f sums x* = (λ *n*. *if even n then f* (*n div 2*) *else 0*) *sums x* **for** *x* :: *real*
proof
assume *f sums x*
from *sums-if*[*OF sums-zero this*] **show** (λ*n*. *if even n then f* (*n div 2*) *else 0*) *sums x*
by *auto*
next
assume (λ *n*. *if even n then f* (*n div 2*) *else 0*) *sums x*
from *LIMSEQ-linear*[*OF this*[*simplified sums-def*] *pos2*, *simplified sum-split-even-odd*[*simplified mult.commute*]]
show *f sums x*
unfolding *sums-def* **by** *auto*
qed
then show *?thesis* **..**
qed

have *Int-eq*: (∑ *n*. *?f n* $*$ *real* (*Suc n*) $*$ *x* ^*n*) = *?Int*
unfolding *if-eq mult.commute*[*of* - *2*]
suminf-def sums-even[*of* λ *n*. (− *1*) ^ *n* $*$ *x* ^ (*2* $*$ *n*), *symmetric*]
by *auto*

have *arctan-eq*: (∑ *n*. *?f n* $*$ *x* ^(*Suc n*)) = *?arctan x* **for** *x*
proof −
have *if-eq'*: ⋀*n*. (*if even n then* (− *1*) ^(*n div 2*) $*$ *1* / *real* (*Suc n*) *else 0*) $*$ *x* ^ *Suc n* =

```
    (if even n then (− 1) ^ (n div 2) ∗ (1 / real (Suc (2 ∗ (n div 2))) ∗ x ^ Suc
(2 ∗ (n div 2))) else 0)
      using n-even by auto
    have idx-eq: ⋀n. n ∗ 2 + 1 = Suc (2 ∗ n)
      by auto
    then show ?thesis
      unfolding if-eq' idx-eq suminf-def
        sums-even[of λ n. (− 1) ^ n ∗ (1 / real (Suc (2 ∗ n)) ∗ x ^ Suc (2 ∗ n)),
symmetric]
      by auto
  qed

  have DERIV (λ x. ∑ n. ?f n ∗ x^(Suc n)) x :> (∑ n. ?f n ∗ real (Suc n) ∗
x^n)
  proof (rule DERIV-power-series')
    show x ∈ {− 1 <..< 1}
      using ‹| x | < 1› by auto
    show summable (λ n. ?f n ∗ real (Suc n) ∗ x'^n)
      if x'-bounds: x' ∈ {− 1 <..< 1} for x' :: real
    proof −
      from that have |x'| < 1 by auto
      then show ?thesis
        using that sums-summable sums-if [OF sums-0 [of λx. 0] summable-sums
[OF summable-Integral]]
        by (auto simp add: if-distrib [of λx. x ∗ y for y] cong: if-cong)
    qed
  qed auto
  then show ?thesis
    by (simp only: Int-eq arctan-eq)
qed

lemma arctan-series:
  assumes |x| ≤ 1
  shows arctan x = (∑ k. (−1)^k ∗ (1 / real (k ∗ 2 + 1) ∗ x ^ (k ∗ 2 + 1)))
    (is - = suminf (λ n. ?c x n))
proof −
  let ?c' = λx n. (−1)^n ∗ x^(n∗2)

  have DERIV-arctan-suminf: DERIV (λ x. suminf (?c x)) x :> (suminf (?c' x))
    if 0 < r and r < 1 and |x| < r for r x :: real
  proof (rule DERIV-arctan-series)
    from that show |x| < 1
      using ‹r < 1› and ‹|x| < r› by auto
  qed

  {
    fix x :: real
    assume |x| ≤ 1
    note summable-Leibniz[OF zeroseq-arctan-series[OF this] monoseq-arctan-series[OF
```

this]]
} **note** *arctan-series-borders* = *this*

have *when-less-one*: *arctan* $x = (\sum k.\ ?c\ x\ k)$ **if** $|x| < 1$ **for** x :: *real*
proof −
obtain r **where** $|x| < r$ **and** $r < 1$
using *dense*[*OF* ‹$|x| < 1$›] **by** *blast*
then have $0 < r$ **and** $- r < x$ **and** $x < r$ **by** *auto*

have *suminf-eq-arctan-bounded*: *suminf* (*?c x*) − *arctan x* = *suminf* (*?c a*) − *arctan a*
if $-r < a$ **and** $b < r$ **and** $a < b$ **and** $a \leq x$ **and** $x \leq b$ **for** *x a b*
proof −
from *that* **have** $|x| < r$ **by** *auto*
show *suminf* (*?c x*) − *arctan x* = *suminf* (*?c a*) − *arctan a*
proof (*rule DERIV-isconst2*[*of a b*])
show $a < b$ **and** $a \leq x$ **and** $x \leq b$
using ‹$a < b$› ‹$a \leq x$› ‹$x \leq b$› **by** *auto*
have $\forall x. - r < x \wedge x < r \longrightarrow$ *DERIV* (λ *x*. *suminf* (*?c x*) − *arctan x*) *x* :> *0*
proof (*rule allI*, *rule impI*)
fix *x*
assume $-r < x \wedge x < r$
then have $|x| < r$ **by** *auto*
with ‹$r < 1$› **have** $|x| < 1$ **by** *auto*
have $|- (x^2)| < 1$ **using** *abs-square-less-1* ‹$|x| < 1$› **by** *auto*
then have (λn. $(- (x^2))$ ^ *n*) *sums* $(1 / (1 - (- (x^2))))$
unfolding *real-norm-def*[*symmetric*] **by** (*rule geometric-sums*)
then have (*?c' x*) *sums* $(1 / (1 - (- (x^2))))$
unfolding *power-mult-distrib*[*symmetric*] *power-mult mult.commute*[*of* -*2*] **by** *auto*
then have *suminf-c'-eq-geom*: *inverse* $(1 + x^2)$ = *suminf* (*?c' x*)
using *sums-unique* **unfolding** *inverse-eq-divide* **by** *auto*
have *DERIV* (λ *x*. *suminf* (*?c x*)) *x* :> (*inverse* $(1 + x^2)$)
unfolding *suminf-c'-eq-geom*
by (*rule DERIV-arctan-suminf*[*OF* ‹$0 < r$› ‹$r < 1$› ‹$|x| < r$›])
from *DERIV-diff* [*OF this DERIV-arctan*] **show** *DERIV* (λx. *suminf* (*?c x*) − *arctan x*) *x* :> *0*
by *auto*
qed
then have *DERIV-in-rball*: $\forall y.\ a \leq y \wedge y \leq b \longrightarrow$ *DERIV* (λx. *suminf* (*?c x*) − *arctan x*) *y* :> *0*
using ‹$-r < a$› ‹$b < r$› **by** *auto*
then show $\bigwedge y.\ [\![a < y;\ y < b]\!] \Longrightarrow$ *DERIV* (λx. *suminf* (*?c x*) − *arctan x*) *y* :> *0*
using ‹$|x| < r$› **by** *auto*
show *continuous-on* {*a*..*b*} (λx. *suminf* (*?c x*) − *arctan x*)
using *DERIV-in-rball DERIV-atLeastAtMost-imp-continuous-on* **by** *blast*
qed

qed

have *suminf-arctan-zero*: *suminf* (*?c 0*) − *arctan 0* = *0*
unfolding *Suc-eq-plus1*[*symmetric*] *power-Suc2 mult-zero-right arctan-zero-zero suminf-zero*
by *auto*

have *suminf* (*?c x*) − *arctan x* = *0*
proof (*cases x* = *0*)
case *True*
then show *?thesis*
using *suminf-arctan-zero* **by** *auto*
next
case *False*
then have *0* < |*x*| **and** − |*x*| < |*x*|
by *auto*
have *suminf* (*?c* (− |*x*|)) − *arctan* (− |*x*|) = *suminf* (*?c 0*) − *arctan 0*
by (*rule suminf-eq-arctan-bounded*[**where** *x1*=*0* **and** *a1*=−|*x*| **and** *b1*=|*x*|, *symmetric*])
(*simp-all only*: ‹|*x*| < *r*› ‹−|*x*| < |*x*|› *neg-less-iff-less*)
moreover
have *suminf* (*?c x*) − *arctan x* = *suminf* (*?c* (− |*x*|)) − *arctan* (− |*x*|)
by (*rule suminf-eq-arctan-bounded*[**where** *x1*=*x* **and** *a1*=− |*x*| **and** *b1*=|*x*|])
(*simp-all only*: ‹|*x*| < *r*› ‹− |*x*| < |*x*|› *neg-less-iff-less*)
ultimately show *?thesis*
using *suminf-arctan-zero* **by** *auto*
qed
then show *?thesis* **by** *auto*
qed

show *arctan x* = *suminf* (λ*n*. *?c x n*)
proof (*cases* |*x*| < *1*)
case *True*
then show *?thesis* **by** (*rule when-less-one*)
next
case *False*
then have |*x*| = *1* **using** ‹|*x*| ≤ *1*› **by** *auto*
let *?a* = λ*x n*. |*1* / *real* (*n* ∗ *2* + *1*) ∗ *x*^(*n* ∗ *2* + *1*)|
let *?diff* = λ*x n*. |*arctan x* − (∑ *i*<*n*. *?c x i*)|
have *?diff 1 n* ≤ *?a 1 n* **for** *n* :: *nat*
proof −
have *0* < (*1* :: *real*) **by** *auto*
moreover
have *?diff x n* ≤ *?a x n* **if** *0* < *x* **and** *x* < *1* **for** *x* :: *real*
proof −
from *that* **have** |*x*| ≤ *1* **and** |*x*| < *1*
by *auto*
from ‹*0* < *x*› **have** *0* < *1* / *real* (*0* ∗ *2* + (*1*::*nat*)) ∗ *x* ^ (*0* ∗ *2* + *1*)
by *auto*

note *bounds* = *mp*[*OF arctan-series-borders*(*2*)[*OF* ‹|*x*| ≤ *1*›] *this*, *unfolded when-less-one*[*OF* ‹|*x*| < *1*›, *symmetric*], *THEN spec*]
have *0* < *1* / *real* (*n*∗*2*+*1*) ∗ *x*^(*n*∗*2*+*1*)
by (*rule mult-pos-pos*) (*simp-all only*: *zero-less-power*[*OF* ‹*0* < *x*›], *auto*)
then have *a-pos*: *?a x n* = *1* / *real* (*n*∗*2*+*1*) ∗ *x*^(*n*∗*2*+*1*)
by (*rule abs-of-pos*)
show *?thesis*
proof (*cases even n*)
case *True*
then have *sgn-pos*: (−*1*)^*n* = (*1*::*real*) **by** *auto*
from ‹*even n*› **obtain** *m* **where** *n* = *2* ∗ *m* **..**
then have *2* ∗ *m* = *n* **..**
from *bounds*[*of m*, *unfolded this atLeastAtMost-iff*]
have |*arctan x* − (∑ *i*<*n*. (*?c x i*))| ≤ (∑ *i*<*n* + *1*. (*?c x i*)) − (∑ *i*<*n*. (*?c x i*))
by *auto*
also have . . . = *?c x n* **by** *auto*
also have . . . = *?a x n* **unfolding** *sgn-pos a-pos* **by** *auto*
finally show *?thesis* **.**
next
case *False*
then have *sgn-neg*: (−*1*)^*n* = (−*1*::*real*) **by** *auto*
from ‹*odd n*› **obtain** *m* **where** *n* = *2* ∗ *m* + *1* **..**
then have *m-def*: *2* ∗ *m* + *1* = *n* **..**
then have *m-plus*: *2* ∗ (*m* + *1*) = *n* + *1* **by** *auto*
from *bounds*[*of m* + *1*, *unfolded this atLeastAtMost-iff*, *THEN conjunct1*] *bounds*[*of m*, *unfolded m-def atLeastAtMost-iff*, *THEN conjunct2*]
have |*arctan x* − (∑ *i*<*n*. (*?c x i*))| ≤ (∑ *i*<*n*. (*?c x i*)) − (∑ *i*<*n*+*1*. (*?c x i*)) **by** *auto*
also have . . . = − *?c x n* **by** *auto*
also have . . . = *?a x n* **unfolding** *sgn-neg a-pos* **by** *auto*
finally show *?thesis* **.**
qed
qed
hence ∀ *x* ∈ { *0* <..< *1* }. *0* ≤ *?a x n* − *?diff x n* **by** *auto*
moreover have *isCont* (λ *x*. *?a x n* − *?diff x n*) *x* **for** *x*
unfolding *diff-conv-add-uminus divide-inverse*
by (*auto intro*!: *isCont-add isCont-rabs continuous-ident isCont-minus isCont-arctan*
continuous-at-within-inverse isCont-mult isCont-power continuous-const isCont-sum
simp del: *add-uminus-conv-diff*)
ultimately have *0* ≤ *?a 1 n* − *?diff 1 n*
by (*rule LIM-less-bound*)
then show *?thesis* **by** *auto*
qed
have *?a 1* ⟶ *0*
unfolding *tendsto-rabs-zero-iff power-one divide-inverse One-nat-def*
by (*auto intro*!: *tendsto-mult LIMSEQ-linear LIMSEQ-inverse-real-of-nat simp*

```
del: of-nat-Suc)
  have ?diff 1 ⟶ 0
  proof (rule LIMSEQ-I)
   fix r :: real
   assume 0 < r
   obtain N :: nat where N-I: N ≤ n ⟹ ?a 1 n < r for n
    using LIMSEQ-D[OF ‹?a 1 ⟶ 0› ‹0 < r›] by auto
   have norm (?diff 1 n − 0) < r if N ≤ n for n
    using ‹?diff 1 n ≤ ?a 1 n› N-I[OF that] by auto
   then show ∃N. ∀ n ≥ N. norm (?diff 1 n − 0) < r by blast
  qed
 from this [unfolded tendsto-rabs-zero-iff, THEN tendsto-add [OF - tendsto-const],
of − arctan 1, THEN tendsto-minus]
  have (?c 1) sums (arctan 1) unfolding sums-def by auto
  then have arctan 1 = (∑ i. ?c 1 i) by (rule sums-unique)

  show ?thesis
  proof (cases x = 1)
   case True
   then show ?thesis by (simp add: ‹arctan 1 = (∑ i. ?c 1 i)›)
  next
   case False
   then have x = −1 using ‹|x| = 1› by auto

   have − (pi/2) < 0 using pi-gt-zero by auto
   have − (2 ∗ pi) < 0 using pi-gt-zero by auto

   have c-minus-minus: ?c (− 1) i = − ?c 1 i for i by auto

   have arctan (− 1) = arctan (tan (−(pi/4)))
    unfolding tan-45 tan-minus ..
   also have ... = − (pi/4)
      by (rule arctan-tan) (auto simp: order-less-trans[OF ‹− (pi/2) < 0›
pi-gt-zero])
   also have ... = − (arctan (tan (pi/4)))
    unfolding neg-equal-iff-equal
    by (rule arctan-tan[symmetric]) (auto simp: order-less-trans[OF ‹− (2 ∗ pi)
< 0› pi-gt-zero])
   also have ... = − (arctan 1)
    unfolding tan-45 ..
   also have ... = − (∑ i. ?c 1 i)
    using ‹arctan 1 = (∑ i. ?c 1 i)› by auto
   also have ... = (∑ i. ?c (− 1) i)
    using suminf-minus[OF sums-summable[OF ‹(?c 1) sums (arctan 1)›]]
    unfolding c-minus-minus by auto
   finally show ?thesis using ‹x = −1› by auto
  qed
 qed
qed
```

lemma *arctan-half*: $arctan\ x = 2 * arctan\ (x\ /\ (1 + sqrt(1 + x^2)))$
for x :: *real*
proof –
obtain y **where** *low*: $-\ (pi/2) < y$ **and** *high*: $y < pi/2$ **and** *y-eq*: $tan\ y = x$
using *tan-total* **by** *blast*
then have *low2*: $-\ (pi/2) < y\ /\ 2$ **and** *high2*: $y\ /\ 2 < pi/2$
by *auto*

have $0 < cos\ y$ **by** (*rule cos-gt-zero-pi*[*OF low high*])
then have $cos\ y \neq 0$ **and** *cos-sqrt*: $sqrt\ ((cos\ y)^2) = cos\ y$
by *auto*

have $1 + (tan\ y)^2 = 1 + (sin\ y)^2\ /\ (cos\ y)^2$
unfolding *tan-def power-divide* **..**
also have $\ldots = (cos\ y)^2\ /\ (cos\ y)^2 + (sin\ y)^2\ /\ (cos\ y)^2$
using ‹$cos\ y \neq 0$› **by** *auto*
also have $\ldots = 1\ /\ (cos\ y)^2$
unfolding *add-divide-distrib*[*symmetric*] *sin-cos-squared-add2* **..**
finally have $1 + (tan\ y)^2 = 1\ /\ (cos\ y)^2$ **.**

have $sin\ y\ /\ (cos\ y + 1) = tan\ y\ /\ ((cos\ y + 1)\ /\ cos\ y)$
unfolding *tan-def* **using** ‹$cos\ y \neq 0$› **by** (*simp add*: *field-simps*)
also have $\ldots = tan\ y\ /\ (1 + 1\ /\ cos\ y)$
using ‹$cos\ y \neq 0$› **unfolding** *add-divide-distrib* **by** *auto*
also have $\ldots = tan\ y\ /\ (1 + 1\ /\ sqrt\ ((cos\ y)^2))$
unfolding *cos-sqrt* **..**
also have $\ldots = tan\ y\ /\ (1 + sqrt\ (1\ /\ (cos\ y)^2))$
unfolding *real-sqrt-divide* **by** *auto*
finally have *eq*: $sin\ y\ /\ (cos\ y + 1) = tan\ y\ /\ (1 + sqrt(1 + (tan\ y)^2))$
unfolding ‹$1 + (tan\ y)^2 = 1\ /\ (cos\ y)^2$› **.**

have $arctan\ x = y$
using *arctan-tan low high y-eq* **by** *auto*
also have $\ldots = 2 * (arctan\ (tan\ (y/2)))$
using *arctan-tan*[*OF low2 high2*] **by** *auto*
also have $\ldots = 2 * (arctan\ (sin\ y\ /\ (cos\ y + 1)))$
unfolding *tan-half* **by** *auto*
finally show *?thesis*
unfolding *eq* ‹$tan\ y = x$› **.**
qed

lemma *arctan-monotone*: $x < y \Longrightarrow arctan\ x < arctan\ y$
by (*simp only*: *arctan-less-iff*)

lemma *arctan-monotone'*: $x \leq y \Longrightarrow arctan\ x \leq arctan\ y$
by (*simp only*: *arctan-le-iff*)

lemma *arctan-inverse*:

assumes $x \neq 0$
shows *arctan* $(1/x) =$ *sgn* $x *$ *pi*$/2 -$ *arctan* x
proof (*rule arctan-unique*)
have §: $x > 0 \Longrightarrow$ *arctan* $x <$ *pi*
using *arctan-bounded* [*of* x] **by** *linarith*
show $-$ (*pi*$/2$) $<$ *sgn* $x *$ *pi*$/2 -$ *arctan* x
using *assms* **by** (*auto simp*: *sgn-real-def arctan algebra-simps* §)
show *sgn* $x *$ *pi*$/2 -$ *arctan* $x <$ *pi*$/2$
using *arctan-bounded* [*of* $- x$] *assms*
by (*auto simp*: *algebra-simps sgn-real-def arctan-minus*)
show *tan* (*sgn* $x *$ *pi*$/2 -$ *arctan* x) $= 1/x$
unfolding *tan-inverse* [*of arctan* x, *unfolded tan-arctan*] *sgn-real-def*
by (*simp add*: *tan-def cos-arctan sin-arctan sin-diff cos-diff*)
qed

theorem *pi-series*: *pi*$/4 = (\sum k. (-1)$^$k * 1 /$ *real* $(k * 2 + 1))$
(**is** - $=$ *?SUM*)
proof $-$
have *pi*$/4 =$ *arctan* 1
using *arctan-one* **by** *auto*
also have ... $=$ *?SUM*
using *arctan-series*[*of* 1] **by** *auto*
finally show *?thesis* **by** *auto*
qed

112.20 Existence of Polar Coordinates

lemma *cos-x-y-le-one*: $|x /$ *sqrt* $(x^2 + y^2)| \leq 1$
by (*rule power2-le-imp-le* [*OF - zero-le-one*])
(*simp add*: *power-divide divide-le-eq not-sum-power2-lt-zero*)

lemma *polar-Ex*: $\exists$ *r*::*real*. $\exists$ *a*. $x = r *$ *cos* $a \wedge y = r *$ *sin* a
proof $-$
have *polar-ex1*: $\exists$ *r a*. $x = r *$ *cos* $a \wedge y = r *$ *sin* a **if** $0 < y$ **for** y
proof $-$
have $x =$ *sqrt* $(x^2 + y^2) *$ *cos* (*arccos* ($x /$ *sqrt* $(x^2 + y^2)$))
by (*simp add*: *cos-arccos-abs* [*OF cos-x-y-le-one*])
moreover have $y =$ *sqrt* $(x^2 + y^2) *$ *sin* (*arccos* ($x /$ *sqrt* $(x^2 + y^2)$))
using *that*
by (*simp add*: *sin-arccos-abs* [*OF cos-x-y-le-one*] *power-divide right-diff-distrib flip*: *real-sqrt-mult*)
ultimately show *?thesis*
by *blast*
qed
show *?thesis*
proof (*cases* 0::*real* y *rule*: *linorder-cases*)
case *less*
then show *?thesis*
by (*rule polar-ex1*)

next
case *equal*
then show *?thesis*
by (*force simp*: *intro*!: *cos-zero sin-zero*)
next
case *greater*
with *polar-ex1* [**where** $y=-y$] **show** *?thesis*
by *auto* (*metis cos-minus minus-minus minus-mult-right sin-minus*)
qed
qed

112.21 Basics about polynomial functions: products, extremal behaviour and root counts

lemma *polynomial-product-nat*:
fixes x :: *nat*
assumes m: $\bigwedge i.\ i > m \Longrightarrow int\ (a\ i) = 0$
and n: $\bigwedge j.\ j > n \Longrightarrow int\ (b\ j) = 0$
shows $(\sum i\le m.\ (a\ i) * x\ \hat{}\ i) * (\sum j\le n.\ (b\ j) * x\ \hat{}\ j) =$
$(\sum r\le m + n.\ (\sum k\le r.\ (a\ k) * (b\ (r - k))) * x\ \hat{}\ r)$
using *polynomial-product* [*of m a n b x*] *assms*
by (*simp only*: *of-nat-mult* [*symmetric*] *of-nat-power* [*symmetric*]
of-nat-eq-iff Int.int-sum [*symmetric*])

lemma *polyfun-diff*:
fixes x :: *'a::idom*
assumes $1 \le n$
shows $(\sum i\le n.\ a\ i * x\hat{}i) - (\sum i\le n.\ a\ i * y\hat{}i) =$
$(x - y) * (\sum j<n.\ (\sum i=Suc\ j..n.\ a\ i * y\hat{}(i - j - 1)) * x\hat{}j)$
proof −
have h: *bij-betw* $(\lambda(i,j).\ (j,i))$ $((SIGMA\ i : atMost\ n.\ lessThan\ i))$ $(SIGMA\ j : lessThan\ n.\ \{Suc\ j..n\})$
by (*auto simp*: *bij-betw-def inj-on-def*)
have $(\sum i\le n.\ a\ i * x\hat{}i) - (\sum i\le n.\ a\ i * y\hat{}i) = (\sum i\le n.\ a\ i * (x\hat{}i - y\hat{}i))$
by (*simp add*: *right-diff-distrib sum-subtractf*)
also have $\ldots = (\sum i\le n.\ a\ i * (x - y) * (\sum j<i.\ y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *power-diff-sumr2 mult.assoc*)
also have $\ldots = (\sum i\le n.\ \sum j<i.\ a\ i * (x - y) * (y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum-distrib-left*)
also have $\ldots = (\sum (i,j) \in (SIGMA\ i : atMost\ n.\ lessThan\ i).\ a\ i * (x - y) * (y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum.Sigma*)
also have $\ldots = (\sum (j,i) \in (SIGMA\ j : lessThan\ n.\ \{Suc\ j..n\}).\ a\ i * (x - y) * (y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*auto simp*: *sum.reindex-bij-betw* [*OF h*, *symmetric*] *intro*: *sum.cong-simp*)
also have $\ldots = (\sum j<n.\ \sum i=Suc\ j..n.\ a\ i * (x - y) * (y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum.Sigma*)
also have $\ldots = (x - y) * (\sum j<n.\ (\sum i=Suc\ j..n.\ a\ i * y\hat{}(i - j - 1)) * x\hat{}j)$
by (*simp add*: *sum-distrib-left mult-ac*)

finally show *?thesis* **.**
qed

lemma *polyfun-diff-alt*:
fixes x :: $'a$::*idom*
assumes $1 \le n$
shows $(\sum i \le n.\ a\ i * x\widehat{\ }i) - (\sum i \le n.\ a\ i * y\widehat{\ }i) =$
$(x - y) * ((\sum j<n.\ \sum k<n-j.\ a(j + k + 1) * y\widehat{\ }k * x\widehat{\ }j))$
proof −
have $(\sum i=Suc\ j..n.\ a\ i * y\widehat{\ }(i - j - 1)) = (\sum k<n-j.\ a(j+k+1) * y\widehat{\ }k)$
if $j < n$ **for** j :: *nat*
proof −
have $\bigwedge k.\ k < n - j \Longrightarrow k \in (\lambda i.\ i - Suc\ j)$ ` $\{Suc\ j..n\}$
by (*rule-tac* $x=k + Suc\ j$ **in** *image-eqI*, *auto*)
then have h: *bij-betw* $(\lambda i.\ i - (j + 1))$ $\{Suc\ j..n\}$ (*lessThan* $(n-j)$)
by (*auto simp*: *bij-betw-def inj-on-def*)
then show *?thesis*
by (*auto simp*: *sum.reindex-bij-betw* [*OF h*, *symmetric*] *intro*: *sum.cong-simp*)
qed
then show *?thesis*
by (*simp add*: *polyfun-diff* [*OF assms*] *sum-distrib-right*)
qed

lemma *polyfun-linear-factor*:
fixes a :: $'a$::*idom*
shows $\exists\, b.\ \forall\, z.\ (\sum i \le n.\ c(i) * z\widehat{\ }i) = (z - a) * (\sum i<n.\ b(i) * z\widehat{\ }i) + (\sum i \le n.\ c(i) * a\widehat{\ }i)$
proof (*cases* $n = 0$)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
have $(\exists\, b.\ \forall\, z.\ (\sum i \le n.\ c\ i * z\widehat{\ }i) = (z - a) * (\sum i<n.\ b\ i * z\widehat{\ }i) + (\sum i \le n.\ c\ i * a\widehat{\ }i)) \longleftrightarrow$
$(\exists\, b.\ \forall\, z.\ (\sum i \le n.\ c\ i * z\widehat{\ }i) - (\sum i \le n.\ c\ i * a\widehat{\ }i) = (z - a) * (\sum i<n.\ b\ i * z\widehat{\ }i))$
by (*simp add*: *algebra-simps*)
also have $\ldots \longleftrightarrow$
$(\exists\, b.\ \forall\, z.\ (z - a) * (\sum j<n.\ (\sum i = Suc\ j..n.\ c\ i * a\widehat{\ }(i - Suc\ j)) * z\widehat{\ }j) =$
$(z - a) * (\sum i<n.\ b\ i * z\widehat{\ }i))$
using *False* **by** (*simp add*: *polyfun-diff*)
also have $\ldots = True$ **by** *auto*
finally show *?thesis*
by *simp*
qed

lemma *polyfun-linear-factor-root*:
fixes a :: $'a$::*idom*
assumes $(\sum i \le n.\ c(i) * a\widehat{\ }i) = 0$

obtains b **where** $\bigwedge z.\ (\sum i{\le}n.\ c\ i * z\hat{\ }i) = (z - a) * (\sum i{<}n.\ b\ i * z\hat{\ }i)$
using *polyfun-linear-factor* [*of c n a*] *assms* **by** *auto*

lemma *isCont-polynom*: *isCont* ($\lambda w.\ \sum i{\le}n.\ c\ i * w\hat{\ }i$) *a*
for $c :: nat \Rightarrow 'a::real\text{-}normed\text{-}div\text{-}algebra$
by *simp*

lemma *zero-polynom-imp-zero-coeffs*:
fixes $c :: nat \Rightarrow 'a::\{ab\text{-}semigroup\text{-}mult,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $\bigwedge w.\ (\sum i{\le}n.\ c\ i * w\hat{\ }i) = 0 \quad k \le n$
shows $c\ k = 0$
using *assms*
proof (*induction n arbitrary*: *c k*)
case *0*
then show *?case*
by *simp*
next
case (*Suc n c k*)
have [*simp*]: $c\ 0 = 0$ **using** *Suc.prems*(*1*) [*of 0*]
by *simp*
have $(\sum i{\le}Suc\ n.\ c\ i * w\hat{\ }i) = w * (\sum i{\le}n.\ c\ (Suc\ i) * w\hat{\ }i)$ **for** w
proof −
have $(\sum i{\le}Suc\ n.\ c\ i * w\hat{\ }i) = (\sum i{\le}n.\ c\ (Suc\ i) * w\ \hat{\ }\ Suc\ i)$
unfolding *Set-Interval.sum.atMost-Suc-shift*
by *simp*
also have $\ldots = w * (\sum i{\le}n.\ c\ (Suc\ i) * w\hat{\ }i)$
by (*simp add*: *sum-distrib-left ac-simps*)
finally show *?thesis* **.**
qed
then have w: $\bigwedge w.\ w \ne 0 \Longrightarrow (\sum i{\le}n.\ c\ (Suc\ i) * w\hat{\ }i) = 0$
using *Suc* **by** *auto*
then have $(\lambda h.\ \sum i{\le}n.\ c\ (Suc\ i) * h\hat{\ }i)\ {-0\rightarrow}\ 0$
by (*simp cong*: *LIM-cong*) — the case $w = 0$ by continuity
then have $(\sum i{\le}n.\ c\ (Suc\ i) * 0\hat{\ }i) = 0$
using *isCont-polynom* [*of 0* $\lambda i.\ c\ (Suc\ i)$ *n*] *LIM-unique*
by (*force simp*: *Limits.isCont-iff*)
then have $\bigwedge w.\ (\sum i{\le}n.\ c\ (Suc\ i) * w\hat{\ }i) = 0$
using w **by** *metis*
then have $\bigwedge i.\ i \le n \Longrightarrow c\ (Suc\ i) = 0$
using *Suc.IH* [*of* $\lambda i.\ c\ (Suc\ i)$] **by** *blast*
then show *?case* **using** ‹$k \le Suc\ n$›
by (*cases k*) *auto*
qed

lemma *polyfun-rootbound*:
fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $c\ k \ne 0\ k{\le}n$

shows *finite* $\{z.\ (\sum i{\le}n.\ c(i) * z\widehat{\ }i) = 0\} \wedge$ *card* $\{z.\ (\sum i{\le}n.\ c(i) * z\widehat{\ }i) = 0\} \le n$
using *assms*
proof (*induction n arbitrary*: *c k*)
case *0*
then show *?case*
by *simp*
next
case (*Suc m c k*)
let *?succase* = *?case*
show *?case*
proof (*cases* $\{z.\ (\sum i{\le}Suc\ m.\ c(i) * z\widehat{\ }i) = 0\} = \{\}$)
case *True*
then show *?succase*
by *simp*
next
case *False*
then obtain *z0* **where** *z0*: $(\sum i{\le}Suc\ m.\ c(i) * z0\widehat{\ }i) = 0$
by *blast*
then obtain *b* **where** *b*: $\bigwedge w.\ (\sum i{\le}Suc\ m.\ c\ i * w\widehat{\ }i) = (w - z0) * (\sum i{\le}m.\ b\ i * w\widehat{\ }i)$
using *polyfun-linear-factor-root* [*OF z0*, *unfolded lessThan-Suc-atMost*]
by *blast*
then have *eq*: $\{z.\ (\sum i{\le}Suc\ m.\ c\ i * z\widehat{\ }i) = 0\} =$ *insert z0* $\{z.\ (\sum i{\le}m.\ b\ i * z\widehat{\ }i) = 0\}$
by *auto*
have $\neg\ (\forall k{\le}m.\ b\ k = 0)$
proof
assume [*simp*]: $\forall k{\le}m.\ b\ k = 0$
then have $\bigwedge w.\ (\sum i{\le}m.\ b\ i * w\widehat{\ }i) = 0$
by *simp*
then have $\bigwedge w.\ (\sum i{\le}Suc\ m.\ c\ i * w\widehat{\ }i) = 0$
using *b* **by** *simp*
then have $\bigwedge k.\ k \le Suc\ m \Longrightarrow c\ k = 0$
using *zero-polynom-imp-zero-coeffs* **by** *blast*
then show *False* **using** *Suc.prems* **by** *blast*
qed
then obtain k' **where** bk': $b\ k' \ne 0\ k' \le m$
by *blast*
show *?succase*
using *Suc.IH* [*of b k'*] bk'
by (*simp add*: *eq card-insert-if del*: *sum.atMost-Suc*)
qed
qed

lemma
fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $c\ k \ne 0\ k{\le}n$
shows *polyfun-roots-finite*: *finite* $\{z.\ (\sum i{\le}n.\ c(i) * z\widehat{\ }i) = 0\}$

and *polyfun-roots-card*: *card* $\{z.\ (\sum i{\leq}n.\ c(i) * z\widehat{\ }i) = 0\} \leq n$
using *polyfun-rootbound assms* **by** *auto*

lemma *polyfun-finite-roots*:
fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
shows *finite* $\{x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = 0\} \longleftrightarrow (\exists\, i{\leq}n.\ c\ i \neq 0)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
moreover have $\neg$ *finite* $\{x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = 0\}$ **if** $\forall\, i{\leq}n.\ c\ i = 0$
proof −
from *that* **have** $\bigwedge x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = 0$
by *simp*
then show *?thesis*
using *ex-new-if-finite* [*OF infinite-UNIV-char-0* [**where** *'a='a*]]
by *auto*
qed
ultimately show *?rhs* **by** *metis*
next
assume *?rhs*
with *polyfun-rootbound* **show** *?lhs* **by** *blast*
qed

lemma *polyfun-eq-0*: $(\forall\, x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = 0) \longleftrightarrow (\forall\, i{\leq}n.\ c\ i = 0)$
for $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$

using *zero-polynom-imp-zero-coeffs* **by** *auto*

lemma *polyfun-eq-coeffs*: $(\forall\, x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = (\sum i{\leq}n.\ d\ i * x\widehat{\ }i)) \longleftrightarrow (\forall\, i{\leq}n.\ c\ i = d\ i)$
for $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
proof −
have $(\forall\, x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = (\sum i{\leq}n.\ d\ i * x\widehat{\ }i)) \longleftrightarrow (\forall\, x.\ (\sum i{\leq}n.\ (c\ i - d\ i) * x\widehat{\ }i) = 0)$
by (*simp add*: *left-diff-distrib Groups-Big.sum-subtractf*)
also have $\ldots \longleftrightarrow (\forall\, i{\leq}n.\ c\ i - d\ i = 0)$
by (*rule polyfun-eq-0*)
finally show *?thesis*
by *simp*
qed

lemma *polyfun-eq-const*:
fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
shows $(\forall\, x.\ (\sum i{\leq}n.\ c\ i * x\widehat{\ }i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall\, i \in \{1..n\}.\ c\ i = 0)$
(**is** *?lhs* = *?rhs*)
proof −
have $*$: $\forall\, x.\ (\sum i{\leq}n.\ (\textit{if } i{=}0 \textit{ then } k \textit{ else } 0) * x\widehat{\ }i) = k$
by (*induct n*) *auto*
show *?thesis*

proof
assume *?lhs*
with $*$ **have** $(\forall\, i \le n.\ c\ i = (\mathit{if}\ i{=}0\ \mathit{then}\ k\ \mathit{else}\ 0))$
by (*simp add*: *polyfun-eq-coeffs* [*symmetric*])
then show *?rhs* **by** *simp*
next
assume *?rhs*
then show *?lhs* **by** (*induct n*) *auto*
qed
qed

lemma *root-polyfun*:
fixes $z :: {'a}::\mathit{idom}$
assumes $1 \le n$
shows $z\hat{\ }n = a \longleftrightarrow (\sum i{\le}n.\ (\mathit{if}\ i = 0\ \mathit{then}\ -a\ \mathit{else}\ \mathit{if}\ i{=}n\ \mathit{then}\ 1\ \mathit{else}\ 0) * z\hat{\ }i) = 0$
using *assms* **by** (*cases n*) (*simp-all add*: *sum.atLeast-Suc-atMost atLeast0AtMost* [*symmetric*])

lemma
assumes *SORT-CONSTRAINT*('*a*::{*idom*,*real-normed-div-algebra*})
and $1 \le n$
shows *finite-roots-unity*: *finite* $\{z::'a.\ z\hat{\ }n = 1\}$
and *card-roots-unity*: *card* $\{z::'a.\ z\hat{\ }n = 1\} \le n$
using *polyfun-rootbound* [*of* $\lambda i.\ \mathit{if}\ i = 0\ \mathit{then}\ -1\ \mathit{else}\ \mathit{if}\ i{=}n\ \mathit{then}\ 1\ \mathit{else}\ 0\ n\ n$] *assms*(*2*)
by (*auto simp*: *root-polyfun* [*OF assms*(*2*)])

112.22 Hyperbolic functions

definition *sinh* :: '*a* :: {*banach*, *real-normed-algebra-1*} ⇒ '*a* **where**
$\mathit{sinh}\ x = (\mathit{exp}\ x - \mathit{exp}\ (-x))\ /_R\ 2$

definition *cosh* :: '*a* :: {*banach*, *real-normed-algebra-1*} ⇒ '*a* **where**
$\mathit{cosh}\ x = (\mathit{exp}\ x + \mathit{exp}\ (-x))\ /_R\ 2$

definition *tanh* :: '*a* :: {*banach*, *real-normed-field*} ⇒ '*a* **where**
$\mathit{tanh}\ x = \mathit{sinh}\ x\ /\ \mathit{cosh}\ x$

definition *arsinh* :: '*a* :: {*banach*, *real-normed-algebra-1*, *ln*} ⇒ '*a* **where**
$\mathit{arsinh}\ x = \mathit{ln}\ (x + (x\hat{\ }2 + 1)\ \mathit{powr}\ \mathit{of\text{-}real}\ (1/2))$

definition *arcosh* :: '*a* :: {*banach*, *real-normed-algebra-1*, *ln*} ⇒ '*a* **where**
$\mathit{arcosh}\ x = \mathit{ln}\ (x + (x\hat{\ }2 - 1)\ \mathit{powr}\ \mathit{of\text{-}real}\ (1/2))$

definition *artanh* :: '*a* :: {*banach*, *real-normed-field*, *ln*} ⇒ '*a* **where**
$\mathit{artanh}\ x = \mathit{ln}\ ((1 + x)\ /\ (1 - x))\ /\ 2$

lemma *arsinh-0* [*simp*]: $\mathit{arsinh}\ 0 = 0$

by (*simp add*: *arsinh-def*)

lemma *arcosh-1* [*simp*]: *arcosh 1 = 0*
by (*simp add*: *arcosh-def*)

lemma *artanh-0* [*simp*]: *artanh 0 = 0*
by (*simp add*: *artanh-def*)

lemma *tanh-altdef*:
tanh x = (*exp x* − *exp* (−*x*)) / (*exp x* + *exp* (−*x*))
proof −
have *tanh x* = (*2* $*_R$ *sinh x*) / (*2* $*_R$ *cosh x*)
by (*simp add*: *tanh-def scaleR-conv-of-real*)
also have *2* $*_R$ *sinh x* = *exp x* − *exp* (−*x*)
by (*simp add*: *sinh-def*)
also have *2* $*_R$ *cosh x* = *exp x* + *exp* (−*x*)
by (*simp add*: *cosh-def*)
finally show *?thesis* **.**
qed

lemma *tanh-real-altdef*: *tanh* (*x*::*real*) = (*1* − *exp* (− *2* ∗ *x*)) / (*1* + *exp* (− *2* ∗ *x*))
proof −
have [*simp*]: *exp* (*2* ∗ *x*) = *exp x* ∗ *exp x exp* (*x* ∗ *2*) = *exp x* ∗ *exp x*
by (*subst exp-add* [*symmetric*]; *simp*)+
have *tanh x* = (*2* ∗ *exp* (−*x*) ∗ *sinh x*) / (*2* ∗ *exp* (−*x*) ∗ *cosh x*)
by (*simp add*: *tanh-def*)
also have *2* ∗ *exp* (−*x*) ∗ *sinh x* = *1* − *exp* (−*2*∗*x*)
by (*simp add*: *exp-minus field-simps sinh-def*)
also have *2* ∗ *exp* (−*x*) ∗ *cosh x* = *1* + *exp* (−*2*∗*x*)
by (*simp add*: *exp-minus field-simps cosh-def*)
finally show *?thesis* **.**
qed

lemma *sinh-converges*: (λ*n*. *if even n then 0 else x* ^ *n* $/_R$ *fact n*) *sums sinh x*
proof −
have (λ*n*. (*x* ^ *n* $/_R$ *fact n* − (−*x*) ^ *n* $/_R$ *fact n*) $/_R$ *2*) *sums sinh x*
unfolding *sinh-def* **by** (*intro sums-scaleR-right sums-diff exp-converges*)
also have (λ*n*. (*x* ^ *n* $/_R$ *fact n* − (−*x*) ^ *n* $/_R$ *fact n*) $/_R$ *2*) =
(λ*n*. *if even n then 0 else x* ^ *n* $/_R$ *fact n*) **by** *auto*
finally show *?thesis* **.**
qed

lemma *cosh-converges*: (λ*n*. *if even n then x* ^ *n* $/_R$ *fact n else 0*) *sums cosh x*
proof −
have (λ*n*. (*x* ^ *n* $/_R$ *fact n* + (−*x*) ^ *n* $/_R$ *fact n*) $/_R$ *2*) *sums cosh x*
unfolding *cosh-def* **by** (*intro sums-scaleR-right sums-add exp-converges*)
also have (λ*n*. (*x* ^ *n* $/_R$ *fact n* + (−*x*) ^ *n* $/_R$ *fact n*) $/_R$ *2*) =

$(\lambda n.$ *if even n then* $x \mathbin{\hat{}} n /_R$ *fact n else 0*) **by** *auto*
finally show *?thesis* **.**
qed

lemma *sinh-0* [*simp*]: *sinh 0 = 0*
by (*simp add*: *sinh-def*)

lemma *cosh-0* [*simp*]: *cosh 0 = 1*
proof −
have *cosh 0* = (*1/2*) $*_R$ (*1* + *1*) **by** (*simp add*: *cosh-def*)
also have ... = *1* **by** (*rule scaleR-half-double*)
finally show *?thesis* **.**
qed

lemma *tanh-0* [*simp*]: *tanh 0 = 0*
by (*simp add*: *tanh-def*)

lemma *sinh-minus* [*simp*]: *sinh* (− *x*) = −*sinh x*
by (*simp add*: *sinh-def algebra-simps*)

lemma *cosh-minus* [*simp*]: *cosh* (− *x*) = *cosh x*
by (*simp add*: *cosh-def algebra-simps*)

lemma *tanh-minus* [*simp*]: *tanh* (−*x*) = −*tanh x*
by (*simp add*: *tanh-def*)

lemma *sinh-ln-real*: *x* > *0* ⟹ *sinh* (*ln x* :: *real*) = (*x* − *inverse x*) / *2*
by (*simp add*: *sinh-def exp-minus*)

lemma *cosh-ln-real*: *x* > *0* ⟹ *cosh* (*ln x* :: *real*) = (*x* + *inverse x*) / *2*
by (*simp add*: *cosh-def exp-minus*)

lemma *tanh-ln-real*:
tanh (*ln x* :: *real*) = (*x* ^ *2* − *1*) / (*x* ^ *2* + *1*) **if** *x* > *0*
proof −
from *that* **have** (*x* ∗ *2* − *inverse x* ∗ *2*) ∗ (x^2 + *1*) =
(x^2 − *1*) ∗ (*2* ∗ *x* + *2* ∗ *inverse x*)
by (*simp add*: *field-simps power2-eq-square*)
moreover have x^2 + *1* > *0*
using *that* **by** (*simp add*: *ac-simps add-pos-nonneg*)
moreover have *2* ∗ *x* + *2* ∗ *inverse x* > *0*
using *that* **by** (*simp add*: *add-pos-pos*)
ultimately have (*x* ∗ *2* − *inverse x* ∗ *2*) /
(*2* ∗ *x* + *2* ∗ *inverse x*) =
(x^2 − *1*) / (x^2 + *1*)
by (*simp add*: *frac-eq-eq*)
with *that* **show** *?thesis*
by (*simp add*: *tanh-def sinh-ln-real cosh-ln-real*)
qed

lemma *has-field-derivative-scaleR-right* [*derivative-intros*]:
(*f has-field-derivative D*) *F* $\Longrightarrow$ (($\lambda x.$ *c* $*_R$ *f x*) *has-field-derivative* (*c* $*_R$ *D*)) *F*
unfolding *has-field-derivative-def*
using *has-derivative-scaleR-right*[*of f* $\lambda x.$ *D* $*$ *x F c*]
by (*simp add*: *mult-scaleR-left* [*symmetric*] *del*: *mult-scaleR-left*)

lemma *has-field-derivative-sinh* [*THEN DERIV-chain2*, *derivative-intros*]:
(*sinh has-field-derivative cosh x*) (*at* (*x* :: *'a* :: {*banach*, *real-normed-field*}))
unfolding *sinh-def cosh-def* **by** (*auto intro*!: *derivative-eq-intros*)

lemma *has-field-derivative-cosh* [*THEN DERIV-chain2*, *derivative-intros*]:
(*cosh has-field-derivative sinh x*) (*at* (*x* :: *'a* :: {*banach*, *real-normed-field*}))
unfolding *sinh-def cosh-def* **by** (*auto intro*!: *derivative-eq-intros*)

lemma *has-field-derivative-tanh* [*THEN DERIV-chain2*, *derivative-intros*]:
cosh x $\neq$ *0* $\Longrightarrow$ (*tanh has-field-derivative 1* $-$ *tanh x* ^ *2*)
(*at* (*x* :: *'a* :: {*banach*, *real-normed-field*}))
unfolding *tanh-def* **by** (*auto intro*!: *derivative-eq-intros simp*: *power2-eq-square field-split-simps*)

lemma *has-derivative-sinh* [*derivative-intros*]:
fixes *g* :: *'a* $\Rightarrow$ (*'a* :: {*banach*, *real-normed-field*})
assumes (*g has-derivative* ($\lambda x.$ *Db* $*$ *x*)) (*at x within s*)
shows (($\lambda x.$ *sinh* (*g x*)) *has-derivative* ($\lambda y.$ (*cosh* (*g x*) $*$ *Db*) $*$ *y*)) (*at x within s*)
proof $-$
have (($\lambda x.$ $-$ *g x*) *has-derivative* ($\lambda y.$ $-$(*Db* $*$ *y*))) (*at x within s*)
using *assms* **by** (*intro derivative-intros*)
also have ($\lambda y.$ $-$(*Db* $*$ *y*)) = ($\lambda x.$ ($-$*Db*) $*$ *x*) **by** (*simp add*: *fun-eq-iff*)
finally have (($\lambda x.$ *sinh* (*g x*)) *has-derivative*
($\lambda y.$ (*exp* (*g x*) $*$ *Db* $*$ *y* $-$ *exp* ($-$*g x*) $*$ ($-$*Db*) $*$ *y*) $/_R$ *2*)) (*at x within s*)
unfolding *sinh-def* **by** (*intro derivative-intros assms*)
also have ($\lambda y.$ (*exp* (*g x*) $*$ *Db* $*$ *y* $-$ *exp* ($-$*g x*) $*$ ($-$*Db*) $*$ *y*) $/_R$ *2*) = ($\lambda y.$ (*cosh* (*g x*) $*$ *Db*) $*$ *y*)
by (*simp add*: *fun-eq-iff cosh-def algebra-simps*)
finally show *?thesis* **.**
qed

lemma *has-derivative-cosh* [*derivative-intros*]:
fixes *g* :: *'a* $\Rightarrow$ (*'a* :: {*banach*, *real-normed-field*})
assumes (*g has-derivative* ($\lambda y.$ *Db* $*$ *y*)) (*at x within s*)
shows (($\lambda x.$ *cosh* (*g x*)) *has-derivative* ($\lambda y.$ (*sinh* (*g x*) $*$ *Db*) $*$ *y*)) (*at x within s*)
proof $-$
have (($\lambda x.$ $-$ *g x*) *has-derivative* ($\lambda y.$ $-$(*Db* $*$ *y*))) (*at x within s*)
using *assms* **by** (*intro derivative-intros*)
also have ($\lambda y.$ $-$(*Db* $*$ *y*)) = ($\lambda y.$ ($-$*Db*) $*$ *y*) **by** (*simp add*: *fun-eq-iff*)
finally have (($\lambda x.$ *cosh* (*g x*)) *has-derivative*

(λy. (exp (g x) ∗ Db ∗ y + exp (−g x) ∗ (−Db) ∗ y) /$_R$ 2)) (at x within s)
unfolding cosh-def **by** (intro derivative-intros assms)
also have (λy. (exp (g x) ∗ Db ∗ y + exp (−g x) ∗ (−Db) ∗ y) /$_R$ 2) = (λy. (sinh (g x) ∗ Db) ∗ y)
by (simp add: fun-eq-iff sinh-def algebra-simps)
finally show ?thesis **.**
qed

lemma sinh-plus-cosh: sinh x + cosh x = exp x
proof −
have sinh x + cosh x = (1/2) ∗$_R$ (exp x + exp x)
by (simp add: sinh-def cosh-def algebra-simps)
also have . . . = exp x **by** (rule scaleR-half-double)
finally show ?thesis **.**
qed

lemma cosh-plus-sinh: cosh x + sinh x = exp x
by (subst add.commute) (rule sinh-plus-cosh)

lemma cosh-minus-sinh: cosh x − sinh x = exp (−x)
proof −
have cosh x − sinh x = (1/2) ∗$_R$ (exp (−x) + exp (−x))
by (simp add: sinh-def cosh-def algebra-simps)
also have . . . = exp (−x) **by** (rule scaleR-half-double)
finally show ?thesis **.**
qed

lemma sinh-minus-cosh: sinh x − cosh x = −exp (−x)
using cosh-minus-sinh[of x] **by** (simp add: algebra-simps)

context
fixes x :: 'a :: {real-normed-field, banach}
begin

lemma sinh-zero-iff: sinh x = 0 ⟷ exp x ∈ {1, −1}
by (auto simp: sinh-def field-simps exp-minus power2-eq-square square-eq-1-iff)

lemma cosh-zero-iff: cosh x = 0 ⟷ exp x ^ 2 = −1
by (auto simp: cosh-def exp-minus field-simps power2-eq-square eq-neg-iff-add-eq-0)

lemma cosh-square-eq: cosh x ^ 2 = sinh x ^ 2 + 1
by (simp add: cosh-def sinh-def algebra-simps power2-eq-square exp-add [symmetric] scaleR-conv-of-real)

lemma sinh-square-eq: sinh x ^ 2 = cosh x ^ 2 − 1
by (simp add: cosh-square-eq)

lemma hyperbolic-pythagoras: cosh x ^ 2 − sinh x ^ 2 = 1

by (*simp add*: *cosh-square-eq*)

lemma *sinh-add*: *sinh* (*x* + *y*) = *sinh x* ∗ *cosh y* + *cosh x* ∗ *sinh y*
by (*simp add*: *sinh-def cosh-def algebra-simps scaleR-conv-of-real exp-add* [*symmetric*])

lemma *sinh-diff*: *sinh* (*x* − *y*) = *sinh x* ∗ *cosh y* − *cosh x* ∗ *sinh y*
by (*simp add*: *sinh-def cosh-def algebra-simps scaleR-conv-of-real exp-add* [*symmetric*])

lemma *cosh-add*: *cosh* (*x* + *y*) = *cosh x* ∗ *cosh y* + *sinh x* ∗ *sinh y*
by (*simp add*: *sinh-def cosh-def algebra-simps scaleR-conv-of-real exp-add* [*symmetric*])

lemma *cosh-diff*: *cosh* (*x* − *y*) = *cosh x* ∗ *cosh y* − *sinh x* ∗ *sinh y*
by (*simp add*: *sinh-def cosh-def algebra-simps scaleR-conv-of-real exp-add* [*symmetric*])

lemma *tanh-add*:
tanh (*x* + *y*) = (*tanh x* + *tanh y*) / (*1* + *tanh x* ∗ *tanh y*)
if *cosh x* ≠ *0 cosh y* ≠ *0*
proof −
have (*sinh x* ∗ *cosh y* + *cosh x* ∗ *sinh y*) ∗ (*1* + *sinh x* ∗ *sinh y* / (*cosh x* ∗ *cosh y*)) =
(*cosh x* ∗ *cosh y* + *sinh x* ∗ *sinh y*) ∗ ((*sinh x* ∗ *cosh y* + *sinh y* ∗ *cosh x*) / (*cosh y* ∗ *cosh x*))
using *that* **by** (*simp add*: *field-split-simps*)
also have (*sinh x* ∗ *cosh y* + *sinh y* ∗ *cosh x*) / (*cosh y* ∗ *cosh x*) = *sinh x* / *cosh x* + *sinh y* / *cosh y*
using *that* **by** (*simp add*: *field-split-simps*)
finally have (*sinh x* ∗ *cosh y* + *cosh x* ∗ *sinh y*) ∗ (*1* + *sinh x* ∗ *sinh y* / (*cosh x* ∗ *cosh y*)) =
(*sinh x* / *cosh x* + *sinh y* / *cosh y*) ∗ (*cosh x* ∗ *cosh y* + *sinh x* ∗ *sinh y*)
by *simp*
then show *?thesis*
using *that* **by** (*auto simp add*: *tanh-def sinh-add cosh-add eq-divide-eq*)
(*simp-all add*: *field-split-simps*)
qed

lemma *sinh-double*: *sinh* (*2* ∗ *x*) = *2* ∗ *sinh x* ∗ *cosh x*
using *sinh-add*[*of x*] **by** *simp*

lemma *cosh-double*: *cosh* (*2* ∗ *x*) = *cosh x* ^ *2* + *sinh x* ^ *2*
using *cosh-add*[*of x*] **by** (*simp add*: *power2-eq-square*)

end

lemma *sinh-field-def*: *sinh z* = (*exp z* − *exp* (−*z*)) / (*2* :: ′*a* :: {*banach*, *real-normed-field*})
by (*simp add*: *sinh-def scaleR-conv-of-real*)

lemma *cosh-field-def*: *cosh z* = (*exp z* + *exp* (−*z*)) / (*2* :: ′*a* :: {*banach*, *real-normed-field*})
by (*simp add*: *cosh-def scaleR-conv-of-real*)

112.22.1 More specific properties of the real functions

lemma *plus-inverse-ge-2*:
  fixes x :: *real*
  assumes $x > 0$
  shows $x + inverse\ x \geq 2$
proof −
  have $0 \leq (x - 1)\ \hat{}\ 2$ **by** *simp*
  also have ... $= x\hat{}2 - 2{*}x + 1$ **by** (*simp add*: *power2-eq-square algebra-simps*)
  finally show *?thesis* **using** *assms* **by** (*simp add*: *field-simps power2-eq-square*)
qed

lemma *sinh-real-nonneg-iff* [*simp*]: $sinh\ (x :: real) \geq 0 \longleftrightarrow x \geq 0$
  by (*simp add*: *sinh-def*)

lemma *sinh-real-pos-iff* [*simp*]: $sinh\ (x :: real) > 0 \longleftrightarrow x > 0$
  by (*simp add*: *sinh-def*)

lemma *sinh-real-nonpos-iff* [*simp*]: $sinh\ (x :: real) \leq 0 \longleftrightarrow x \leq 0$
  by (*simp add*: *sinh-def*)

lemma *sinh-real-neg-iff* [*simp*]: $sinh\ (x :: real) < 0 \longleftrightarrow x < 0$
  by (*simp add*: *sinh-def*)

lemma *cosh-real-ge-1*: $cosh\ (x :: real) \geq 1$
  using *plus-inverse-ge-2*[*of exp x*] **by** (*simp add*: *cosh-def exp-minus*)

lemma *cosh-real-pos* [*simp*]: $cosh\ (x :: real) > 0$
  using *cosh-real-ge-1*[*of x*] **by** *simp*

lemma *cosh-real-nonneg*[*simp*]: $cosh\ (x :: real) \geq 0$
  using *cosh-real-ge-1*[*of x*] **by** *simp*

lemma *cosh-real-nonzero* [*simp*]: $cosh\ (x :: real) \neq 0$
  using *cosh-real-ge-1*[*of x*] **by** *simp*

lemma *arsinh-real-def*: $arsinh\ (x{::}real) = ln\ (x + sqrt\ (x\hat{}2 + 1))$
  by (*simp add*: *arsinh-def powr-half-sqrt*)

lemma *arcosh-real-def*: $x \geq 1 \Longrightarrow arcosh\ (x{::}real) = ln\ (x + sqrt\ (x\hat{}2 - 1))$
  by (*simp add*: *arcosh-def powr-half-sqrt*)

lemma *arsinh-real-aux*: $0 < x + sqrt\ (x\ \hat{}\ 2 + 1 :: real)$
proof (*cases* $x < 0$)
  case *True*
  have $(-x)\ \hat{}\ 2 = x\ \hat{}\ 2$ **by** *simp*
  also have $x\ \hat{}\ 2 < x\ \hat{}\ 2 + 1$ **by** *simp*
  finally have $sqrt\ ((-x)\ \hat{}\ 2) < sqrt\ (x\ \hat{}\ 2 + 1)$
    by (*rule real-sqrt-less-mono*)
  thus *?thesis* **using** *True* **by** *simp*

qed (*auto simp*: *add-nonneg-pos*)

lemma *arsinh-minus-real* [*simp*]: *arsinh* (−*x*::*real*) = −*arsinh x*
proof −
  have *arsinh* (−*x*) = *ln* (*sqrt* (x^2 + *1*) − *x*)
    by (*simp add*: *arsinh-real-def*)
  also have *sqrt* (*x*^*2* + *1*) − *x* = *inverse* (*sqrt* (*x*^*2* + *1*) + *x*)
   using *arsinh-real-aux*[*of x*] **by** (*simp add*: *field-split-simps algebra-simps power2-eq-square*)
  also have *ln* ... = −*arsinh x*
    using *arsinh-real-aux*[*of x*] **by** (*simp add*: *arsinh-real-def ln-inverse*)
  finally show *?thesis* **.**
qed

lemma *artanh-minus-real* [*simp*]:
  assumes *abs x* < *1*
  shows *artanh* (−*x*::*real*) = −*artanh x*
  by (*smt* (*verit*) *artanh-def assms field-sum-of-halves ln-div*)

lemma *sinh-less-cosh-real*: *sinh* (*x* :: *real*) < *cosh x*
  by (*simp add*: *sinh-def cosh-def*)

lemma *sinh-le-cosh-real*: *sinh* (*x* :: *real*) ≤ *cosh x*
  by (*simp add*: *sinh-def cosh-def*)

lemma *tanh-real-lt-1*: *tanh* (*x* :: *real*) < *1*
  by (*simp add*: *tanh-def sinh-less-cosh-real*)

lemma *tanh-real-gt-neg1*: *tanh* (*x* :: *real*) > −*1*
proof −
  have − *cosh x* < *sinh x* **by** (*simp add*: *sinh-def cosh-def field-split-simps*)
  thus *?thesis* **by** (*simp add*: *tanh-def field-simps*)
qed

lemma *tanh-real-bounds*: *tanh* (*x* :: *real*) ∈ {−*1*<..<*1*}
  using *tanh-real-lt-1 tanh-real-gt-neg1* **by** *simp*

context
  fixes *x* :: *real*
begin

lemma *arsinh-sinh-real*: *arsinh* (*sinh x*) = *x*
  by (*simp add*: *arsinh-real-def powr-def sinh-square-eq sinh-plus-cosh*)

lemma *arcosh-cosh-real*: *x* ≥ *0* ⟹ *arcosh* (*cosh x*) = *x*
 by (*simp add*: *arcosh-real-def powr-def cosh-square-eq cosh-real-ge-1 cosh-plus-sinh*)

lemma *artanh-tanh-real*: *artanh* (*tanh x*) = *x*
proof −
  have *artanh* (*tanh x*) = *ln* (*cosh x* ∗ (*cosh x* + *sinh x*) / (*cosh x* ∗ (*cosh x* −

```
sinh x))) / 2
    by (simp add: artanh-def tanh-def field-split-simps)
  also have cosh x * (cosh x + sinh x) / (cosh x * (cosh x − sinh x)) =
             (cosh x + sinh x) / (cosh x − sinh x) by simp
  also have ... = (exp x)^2
   by (simp add: cosh-plus-sinh cosh-minus-sinh exp-minus field-simps power2-eq-square)
  also have ln ((exp x)^2) / 2 = x by (simp add: ln-realpow)
  finally show ?thesis .
qed

lemma sinh-real-zero-iff [simp]: sinh x = 0 ⟷ x = 0
  by (metis arsinh-0 arsinh-sinh-real sinh-0)

lemma cosh-real-one-iff [simp]: cosh x = 1 ⟷ x = 0
  by (smt (verit, best) Transcendental.arcosh-cosh-real cosh-0 cosh-minus)

lemma tanh-real-nonneg-iff [simp]: tanh x ≥ 0 ⟷ x ≥ 0
  by (simp add: tanh-def field-simps)

lemma tanh-real-pos-iff [simp]: tanh x > 0 ⟷ x > 0
  by (simp add: tanh-def field-simps)

lemma tanh-real-nonpos-iff [simp]: tanh x ≤ 0 ⟷ x ≤ 0
  by (simp add: tanh-def field-simps)

lemma tanh-real-neg-iff [simp]: tanh x < 0 ⟷ x < 0
  by (simp add: tanh-def field-simps)

lemma tanh-real-zero-iff [simp]: tanh x = 0 ⟷ x = 0
  by (simp add: tanh-def field-simps)

end

lemma sinh-real-strict-mono: strict-mono (sinh :: real ⇒ real)
  by (force intro: strict-monoI DERIV-pos-imp-increasing [where f=sinh] deriva-
tive-intros)

lemma cosh-real-strict-mono:
  assumes 0 ≤ x and x < (y::real)
  shows   cosh x < cosh y
proof −
  from assms have ∃z>x. z < y ∧ cosh y − cosh x = (y − x) * sinh z
    by (intro MVT2) (auto dest: connectedD-interval intro!: derivative-eq-intros)
   then obtain z where z: z > x z < y cosh y − cosh x = (y − x) * sinh z by
blast
  note ‹cosh y − cosh x = (y − x) * sinh z›
  also from ‹z > x› and assms have (y − x) * sinh z > 0 by (intro mult-pos-pos)
auto
  finally show cosh x < cosh y by simp
```

qed

lemma *tanh-real-strict-mono*: *strict-mono* (*tanh* :: *real* ⇒ *real*)
proof −
  have *tanh x* ^ *2* < *1* **for** *x* :: *real*
    using *tanh-real-bounds*[*of x*] **by** (*simp add*: *abs-square-less-1 abs-if*)
  then show *?thesis*
   by (*force intro*!: *strict-monoI DERIV-pos-imp-increasing* [**where** *f*=*tanh*] *derivative-intros*)
qed

lemma *sinh-real-abs* [*simp*]: *sinh* (*abs x* :: *real*) = *abs* (*sinh x*)
  by (*simp add*: *abs-if*)

lemma *cosh-real-abs* [*simp*]: *cosh* (*abs x* :: *real*) = *cosh x*
  by (*simp add*: *abs-if*)

lemma *tanh-real-abs* [*simp*]: *tanh* (*abs x* :: *real*) = *abs* (*tanh x*)
  by (*auto simp*: *abs-if*)

lemma *sinh-real-eq-iff* [*simp*]: *sinh x* = *sinh y* ⟷ *x* = (*y* :: *real*)
  using *sinh-real-strict-mono* **by** (*simp add*: *strict-mono-eq*)

lemma *tanh-real-eq-iff* [*simp*]: *tanh x* = *tanh y* ⟷ *x* = (*y* :: *real*)
  using *tanh-real-strict-mono* **by** (*simp add*: *strict-mono-eq*)

lemma *cosh-real-eq-iff* [*simp*]: *cosh x* = *cosh y* ⟷ *abs x* = *abs* (*y* :: *real*)
proof −
  have *cosh x* = *cosh y* ⟷ *x* = *y* **if** *x* ≥ *0* *y* ≥ *0* **for** *x y* :: *real*
    using *cosh-real-strict-mono*[*of x y*] *cosh-real-strict-mono*[*of y x*] *that*
    by (*cases x y rule*: *linorder-cases*) *auto*
  from *this*[*of abs x abs y*] **show** *?thesis* **by** *simp*
qed

lemma *sinh-real-le-iff* [*simp*]: *sinh x* ≤ *sinh y* ⟷ *x* ≤ (*y*::*real*)
  using *sinh-real-strict-mono* **by** (*simp add*: *strict-mono-less-eq*)

lemma *cosh-real-nonneg-le-iff*: *x* ≥ *0* ⟹ *y* ≥ *0* ⟹ *cosh x* ≤ *cosh y* ⟷ *x* ≤ (*y*::*real*)
  using *cosh-real-strict-mono*[*of x y*] *cosh-real-strict-mono*[*of y x*]
  by (*cases x y rule*: *linorder-cases*) *auto*

lemma *cosh-real-nonpos-le-iff*: *x* ≤ *0* ⟹ *y* ≤ *0* ⟹ *cosh x* ≤ *cosh y* ⟷ *x* ≥ (*y*::*real*)
  using *cosh-real-nonneg-le-iff*[*of* −*x* −*y*] **by** *simp*

lemma *tanh-real-le-iff* [*simp*]: *tanh x* ≤ *tanh y* ⟷ *x* ≤ (*y*::*real*)
  using *tanh-real-strict-mono* **by** (*simp add*: *strict-mono-less-eq*)

lemma *sinh-real-less-iff* [*simp*]: *sinh x* < *sinh y* ⟷ *x* < (*y*::*real*)
using *sinh-real-strict-mono* **by** (*simp add*: *strict-mono-less*)

lemma *cosh-real-nonneg-less-iff*: *x* ≥ *0* ⟹ *y* ≥ *0* ⟹ *cosh x* < *cosh y* ⟷ *x* < (*y*::*real*)
using *cosh-real-strict-mono*[*of x y*] *cosh-real-strict-mono*[*of y x*]
by (*cases x y rule*: *linorder-cases*) *auto*

lemma *cosh-real-nonpos-less-iff*: *x* ≤ *0* ⟹ *y* ≤ *0* ⟹ *cosh x* < *cosh y* ⟷ *x* > (*y*::*real*)
using *cosh-real-nonneg-less-iff*[*of* −*x* −*y*] **by** *simp*

lemma *tanh-real-less-iff* [*simp*]: *tanh x* < *tanh y* ⟷ *x* < (*y*::*real*)
using *tanh-real-strict-mono* **by** (*simp add*: *strict-mono-less*)

112.22.2 Limits

lemma *sinh-real-at-top*: *filterlim* (*sinh* :: *real* ⇒ *real*) *at-top at-top*
proof −
have ∗: ((λ*x*. − *exp* (− *x*)) ⟶ (−*0*::*real*)) *at-top*
by (*intro tendsto-minus filterlim-compose*[*OF exp-at-bot*] *filterlim-uminus-at-bot-at-top*)
have *filterlim* (λ*x*. (*1*/*2*) ∗ (−*exp* (−*x*) + *exp x*) :: *real*) *at-top at-top*
by (*rule filterlim-tendsto-pos-mult-at-top*[*OF* - - *filterlim-tendsto-add-at-top*[*OF* ∗]] *tendsto-const*)+ (*auto simp*: *exp-at-top*)
also have (λ*x*. (*1*/*2*) ∗ (−*exp* (−*x*) + *exp x*) :: *real*) = *sinh*
by (*simp add*: *fun-eq-iff sinh-def*)
finally show *?thesis* **.**
qed

lemma *sinh-real-at-bot*: *filterlim* (*sinh* :: *real* ⇒ *real*) *at-bot at-bot*
proof −
have *filterlim* (λ*x*. −*sinh x* :: *real*) *at-bot at-top*
by (*simp add*: *filterlim-uminus-at-top* [*symmetric*] *sinh-real-at-top*)
also have (λ*x*. −*sinh x* :: *real*) = (λ*x*. *sinh* (−*x*)) **by** *simp*
finally show *?thesis* **by** (*subst filterlim-at-bot-mirror*)
qed

lemma *cosh-real-at-top*: *filterlim* (*cosh* :: *real* ⇒ *real*) *at-top at-top*
proof −
have ∗: ((λ*x*. *exp* (− *x*)) ⟶ (*0*::*real*)) *at-top*
by (*intro filterlim-compose*[*OF exp-at-bot*] *filterlim-uminus-at-bot-at-top*)
have *filterlim* (λ*x*. (*1*/*2*) ∗ (*exp* (−*x*) + *exp x*) :: *real*) *at-top at-top*
by (*rule filterlim-tendsto-pos-mult-at-top*[*OF* - - *filterlim-tendsto-add-at-top*[*OF* ∗]] *tendsto-const*)+ (*auto simp*: *exp-at-top*)
also have (λ*x*. (*1*/*2*) ∗ (*exp* (−*x*) + *exp x*) :: *real*) = *cosh*
by (*simp add*: *fun-eq-iff cosh-def*)

finally show *?thesis* **.**
qed

lemma *cosh-real-at-bot*: *filterlim* (*cosh* :: *real* ⇒ *real*) *at-top at-bot*
proof −
  have *filterlim* (λ*x*. *cosh* (−*x*) :: *real*) *at-top at-top*
    by (*simp add*: *cosh-real-at-top*)
  thus *?thesis* **by** (*subst filterlim-at-bot-mirror*)
qed

lemma *tanh-real-at-top*: (*tanh* ⟶ (*1*::*real*)) *at-top*
proof −
  have ((λ*x*::*real*. (*1* − *exp* (− *2* ∗ *x*)) / (*1* + *exp* (− *2* ∗ *x*))) ⟶ (*1* − *0*) / (*1* + *0*)) *at-top*
    by (*intro tendsto-intros filterlim-compose*[*OF exp-at-bot*]
          filterlim-tendsto-neg-mult-at-bot[*OF tendsto-const*] *filterlim-ident*) *auto*
  also have (λ*x*::*real*. (*1* − *exp* (− *2* ∗ *x*)) / (*1* + *exp* (− *2* ∗ *x*))) = *tanh*
    by (*rule ext*) (*simp add*: *tanh-real-altdef*)
  finally show *?thesis* **by** *simp*
qed

lemma *tanh-real-at-bot*: (*tanh* ⟶ (−*1*::*real*)) *at-bot*
proof −
  have ((λ*x*::*real*. −*tanh x*) ⟶ −*1*) *at-top*
    by (*intro tendsto-minus tanh-real-at-top*)
  also have (λ*x*. −*tanh x* :: *real*) = (λ*x*. *tanh* (−*x*)) **by** *simp*
  finally show *?thesis* **by** (*subst filterlim-at-bot-mirror*)
qed

112.22.3 Properties of the inverse hyperbolic functions

lemma *isCont-sinh*: *isCont sinh* (*x* :: *'a* :: {*real-normed-field*, *banach*})
  unfolding *sinh-def* [*abs-def*] **by** (*auto intro*!: *continuous-intros*)

lemma *isCont-cosh*: *isCont cosh* (*x* :: *'a* :: {*real-normed-field*, *banach*})
  unfolding *cosh-def* [*abs-def*] **by** (*auto intro*!: *continuous-intros*)

lemma *isCont-tanh*: *cosh x* ≠ *0* ⟹ *isCont tanh* (*x* :: *'a* :: {*real-normed-field*, *banach*})
  unfolding *tanh-def* [*abs-def*]
  by (*auto intro*!: *continuous-intros isCont-divide isCont-sinh isCont-cosh*)

lemma *continuous-on-sinh* [*continuous-intros*]:
  fixes *f* :: - ⇒*'a*::{*real-normed-field*,*banach*}
  assumes *continuous-on A f*
  shows *continuous-on A* (λ*x*. *sinh* (*f x*))
  unfolding *sinh-def* **using** *assms* **by** (*intro continuous-intros*)

lemma *continuous-on-cosh* [*continuous-intros*]:

fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
assumes *continuous-on A f*
shows *continuous-on A* (λ*x*. *cosh* (*f x*))
unfolding *cosh-def* **using** *assms* **by** (*intro continuous-intros*)

lemma *continuous-sinh* [*continuous-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
assumes *continuous F f*
shows *continuous F* (λ*x*. *sinh* (*f x*))
unfolding *sinh-def* **using** *assms* **by** (*intro continuous-intros*)

lemma *continuous-cosh* [*continuous-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
assumes *continuous F f*
shows *continuous F* (λ*x*. *cosh* (*f x*))
unfolding *cosh-def* **using** *assms* **by** (*intro continuous-intros*)

lemma *continuous-on-tanh* [*continuous-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
assumes *continuous-on A f* ⋀*x*. *x* ∈ *A* ⟹ *cosh* (*f x*) ≠ *0*
shows *continuous-on A* (λ*x*. *tanh* (*f x*))
unfolding *tanh-def* **using** *assms* **by** (*intro continuous-intros*) *auto*

lemma *continuous-at-within-tanh* [*continuous-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
assumes *continuous* (*at x within A*) *f cosh* (*f x*) ≠ *0*
shows *continuous* (*at x within A*) (λ*x*. *tanh* (*f x*))
unfolding *tanh-def* **using** *assms* **by** (*intro continuous-intros continuous-divide*) *auto*

lemma *continuous-tanh* [*continuous-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
assumes *continuous F f cosh* (*f* (*Lim F* (λ*x*. *x*))) ≠ *0*
shows *continuous F* (λ*x*. *tanh* (*f x*))
unfolding *tanh-def* **using** *assms* **by** (*intro continuous-intros continuous-divide*) *auto*

lemma *tendsto-sinh* [*tendsto-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
shows (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *sinh* (*f x*)) ⟶ *sinh a*) *F*
by (*rule isCont-tendsto-compose* [*OF isCont-sinh*])

lemma *tendsto-cosh* [*tendsto-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}
shows (*f* ⟶ *a*) *F* ⟹ ((λ*x*. *cosh* (*f x*)) ⟶ *cosh a*) *F*
by (*rule isCont-tendsto-compose* [*OF isCont-cosh*])

lemma *tendsto-tanh* [*tendsto-intros*]:
fixes *f* :: - ⇒'*a*::{*real-normed-field*,*banach*}

shows $(f \longrightarrow a)\ F \Longrightarrow cosh\ a \neq 0 \Longrightarrow ((\lambda x.\ tanh\ (f\ x)) \longrightarrow tanh\ a)\ F$
by (*rule isCont-tendsto-compose* [*OF isCont-tanh*])

lemma *arsinh-real-has-field-derivative* [*derivative-intros*]:
fixes *x* :: *real*
shows (*arsinh has-field-derivative* (*1* / (*sqrt* (*x* ^ *2* + *1*)))) (*at x within A*)
proof −
have *pos*: *1* + *x* ^ *2* > *0* **by** (*intro add-pos-nonneg*) *auto*
from *pos arsinh-real-aux*[*of x*] **show** *?thesis* **unfolding** *arsinh-def* [*abs-def*]
by (*auto intro*!: *derivative-eq-intros simp*: *powr-minus powr-half-sqrt field-split-simps*)
qed

lemma *arcosh-real-has-field-derivative* [*derivative-intros*]:
fixes *x* :: *real*
assumes *x* > *1*
shows (*arcosh has-field-derivative* (*1* / (*sqrt* (*x* ^ *2* − *1*)))) (*at x within A*)
proof −
from *assms* **have** $x + sqrt\ (x^2 - 1) > 0$ **by** (*simp add*: *add-pos-pos*)
thus *?thesis* **using** *assms* **unfolding** *arcosh-def* [*abs-def*]
by (*auto intro*!: *derivative-eq-intros*
simp: *powr-minus powr-half-sqrt field-split-simps power2-eq-1-iff*)
qed

lemma *artanh-real-has-field-derivative* [*derivative-intros*]:
(*artanh has-field-derivative* (*1* / (*1* − *x* ^ *2*))) (*at x within A*) **if**
$|x| < 1$ **for** *x* :: *real*
proof −
from *that* **have** − *1* < *x x* < *1* **by** *linarith*+
hence (*artanh has-field-derivative* (*4* − *4* ∗ *x*) / ((*1* + *x*) ∗ (*1* − *x*) ∗ (*1* − *x*) ∗ *4*))
(*at x within A*) **unfolding** *artanh-def* [*abs-def*]
by (*auto intro*!: *derivative-eq-intros simp*: *powr-minus powr-half-sqrt*)
also have (*4* − *4* ∗ *x*) / ((*1* + *x*) ∗ (*1* − *x*) ∗ (*1* − *x*) ∗ *4*) = *1* / ((*1* + *x*) ∗ (*1* − *x*))
using ‹−*1* < *x*› ‹*x* < *1*› **by** (*simp add*: *frac-eq-eq*)
also have (*1* + *x*) ∗ (*1* − *x*) = *1* − *x* ^ *2*
by (*simp add*: *algebra-simps power2-eq-square*)
finally show *?thesis* **.**
qed

lemma *cosh-double-cosh*: *cosh* (*2* ∗ *x* :: *'a* :: {*banach*, *real-normed-field*}) = *2* ∗ $(cosh\ x)^2 - 1$
using *cosh-double*[*of x*] **by** (*simp add*: *sinh-square-eq*)

lemma *sinh-multiple-reduce*:
sinh (*x* ∗ *numeral n* :: *'a* :: {*real-normed-field*, *banach*}) =
sinh x ∗ *cosh* (*x* ∗ *of-nat* (*pred-numeral n*)) + *cosh x* ∗ *sinh* (*x* ∗ *of-nat* (*pred-numeral n*))

proof −
  have *numeral n* = *of-nat* (*pred-numeral n*) + (*1* :: *'a*)
    by (*metis add.commute numeral-eq-Suc of-nat-Suc of-nat-numeral*)
  also have *sinh* (*x* ∗ . . .) = *sinh* (*x* ∗ *of-nat* (*pred-numeral n*) + *x*)
    unfolding *of-nat-Suc* **by** (*simp add*: *ring-distribs*)
  finally show *?thesis*
    by (*simp add*: *sinh-add*)
qed

lemma *cosh-multiple-reduce*:
  cosh (*x* ∗ *numeral n* :: *'a* :: {*real-normed-field*, *banach*}) =
    cosh (*x* ∗ *of-nat* (*pred-numeral n*)) ∗ *cosh x* + *sinh* (*x* ∗ *of-nat* (*pred-numeral n*)) ∗ *sinh x*
proof −
  have *numeral n* = *of-nat* (*pred-numeral n*) + (*1* :: *'a*)
    by (*metis add.commute numeral-eq-Suc of-nat-Suc of-nat-numeral*)
  also have *cosh* (*x* ∗ . . .) = *cosh* (*x* ∗ *of-nat* (*pred-numeral n*) + *x*)
    unfolding *of-nat-Suc* **by** (*simp add*: *ring-distribs*)
  finally show *?thesis*
    by (*simp add*: *cosh-add*)
qed

lemma *cosh-arcosh-real* [*simp*]:
  assumes $x \geq (1 :: real)$
  shows *cosh* (*arcosh x*) = *x*
proof −
  have *eventually* (λ*t*::*real*. *cosh t* ≥ *x*) *at-top*
    using *cosh-real-at-top* **by** (*simp add*: *filterlim-at-top*)
  then obtain *t* **where** *t* ≥ *1* *cosh t* ≥ *x*
    by (*metis eventually-at-top-linorder linorder-not-le order-le-less*)
  moreover have *isCont cosh* (*y* :: *real*) **for** *y*
    by (*intro continuous-intros*)
  ultimately obtain *y* **where** *y* ≥ *0* *x* = *cosh y*
    using *IVT*[*of cosh 0 x t*] *assms* **by** *auto*
  thus *?thesis*
    by (*simp add*: *arcosh-cosh-real*)
qed

lemma *arcosh-eq-0-iff-real* [*simp*]: *x* ≥ *1* ⟹ *arcosh x* = *0* ⟷ *x* = (*1* :: *real*)
  using *cosh-arcosh-real* **by** *fastforce*

lemma *arcosh-nonneg-real* [*simp*]:
  assumes *x* ≥ *1*
  shows *arcosh* (*x* :: *real*) ≥ *0*
proof −
  have $1 + 0 \leq x + (x^2 - 1)$ *powr* (*1* / *2*)
    using *assms* **by** (*intro add-mono*) *auto*
  thus *?thesis* **unfolding** *arcosh-def* **by** *simp*
qed

lemma *arcosh-real-strict-mono*:
  fixes $x\ y$ :: *real*
  assumes $1 \leq x$ $x < y$
  shows *arcosh* $x <$ *arcosh* y
proof −
  have *cosh* (*arcosh* x) < *cosh* (*arcosh* y)
    by (*subst* (*1 2*) *cosh-arcosh-real*) (*use assms* **in** *auto*)
  thus *?thesis*
    using *assms* **by** (*subst* (*asm*) *cosh-real-nonneg-less-iff*) *auto*
qed

lemma *arcosh-less-iff-real* [*simp*]:
  fixes $x\ y$ :: *real*
  assumes $1 \leq x$ $1 \leq y$
  shows *arcosh* $x <$ *arcosh* $y \longleftrightarrow x < y$
  using *arcosh-real-strict-mono*[*of x y*] *arcosh-real-strict-mono*[*of y x*] *assms*
  by (*cases x y rule*: *linorder-cases*) *auto*

lemma *arcosh-real-gt-1-iff* [*simp*]: $x \geq 1 \Longrightarrow$ *arcosh* $x > 0 \longleftrightarrow x \neq (1$ :: *real*)
  using *arcosh-less-iff-real*[*of 1 x*] **by** (*auto simp del*: *arcosh-less-iff-real*)

lemma *sinh-arcosh-real*: $x \geq 1 \Longrightarrow$ *sinh* (*arcosh* x) = *sqrt* ($x^2 - 1$)
  by (*rule sym*, *rule real-sqrt-unique*) (*auto simp*: *sinh-square-eq*)

lemma *sinh-arsinh-real* [*simp*]: *sinh* (*arsinh* x :: *real*) = x
proof −
  have *eventually* (λt::*real*. *sinh* $t \geq x$) *at-top*
    using *sinh-real-at-top* **by** (*simp add*: *filterlim-at-top*)
  then obtain t **where** *sinh* $t \geq x$
    by (*metis eventually-at-top-linorder linorder-not-le order-le-less*)
  moreover have *eventually* (λt::*real*. *sinh* $t \leq x$) *at-bot*
    using *sinh-real-at-bot* **by** (*simp add*: *filterlim-at-bot*)
  then obtain t' **where** $t' \leq t$ *sinh* $t' \leq x$
    by (*metis eventually-at-bot-linorder nle-le*)
  moreover have *isCont sinh* (y :: *real*) **for** y
    by (*intro continuous-intros*)
  ultimately obtain y **where** $x =$ *sinh* y
    using *IVT*[*of sinh t' x t*] **by** *auto*
  thus *?thesis*
    by (*simp add*: *arsinh-sinh-real*)
qed

lemma *arsinh-real-strict-mono*:
  fixes $x\ y$ :: *real*
  assumes $x < y$
  shows *arsinh* $x <$ *arsinh* y
proof −

have *sinh* (*arsinh x*) < *sinh* (*arsinh y*)
  by (*subst* (*1 2*) *sinh-arsinh-real*) (*use assms* **in** *auto*)
thus *?thesis*
  using *assms* **by** (*subst* (*asm*) *sinh-real-less-iff*) *auto*
qed

lemma *arsinh-less-iff-real* [*simp*]:
fixes *x y* :: *real*
shows *arsinh x* < *arsinh y* ⟷ *x* < *y*
using *arsinh-real-strict-mono*[*of x y*] *arsinh-real-strict-mono*[*of y x*]
by (*cases x y rule*: *linorder-cases*) *auto*

lemma *arsinh-real-eq-0-iff* [*simp*]: *arsinh x* = *0* ⟷ *x* = (*0* :: *real*)
by (*metis arsinh-0 sinh-arsinh-real*)

lemma *arsinh-real-pos-iff* [*simp*]: *arsinh x* > *0* ⟷ *x* > (*0* :: *real*)
using *arsinh-less-iff-real*[*of 0 x*] **by** (*simp del*: *arsinh-less-iff-real*)

lemma *arsinh-real-neg-iff* [*simp*]: *arsinh x* < *0* ⟷ *x* < (*0* :: *real*)
using *arsinh-less-iff-real*[*of x 0*] **by** (*simp del*: *arsinh-less-iff-real*)

lemma *cosh-arsinh-real*: *cosh* (*arsinh x*) = *sqrt* (x^2 + *1*)
by (*rule sym*, *rule real-sqrt-unique*) (*auto simp*: *cosh-square-eq*)

lemma *continuous-on-arsinh* [*continuous-intros*]: *continuous-on A* (*arsinh* :: *real* ⇒ *real*)
by (*rule DERIV-continuous-on derivative-intros*)+

lemma *continuous-on-arcosh* [*continuous-intros*]:
assumes *A* ⊆ {*1*..}
shows *continuous-on A* (*arcosh* :: *real* ⇒ *real*)
proof −
have *pos*: *x* + *sqrt* (*x* ^ *2* − *1*) > *0* **if** *x* ≥ *1* **for** *x*
  using *that* **by** (*intro add-pos-nonneg*) *auto*
show *?thesis*
unfolding *arcosh-def* [*abs-def*]
by (*intro continuous-on-subset* [*OF - assms*] *continuous-on-ln continuous-on-add*
      continuous-on-id continuous-on-powr′)
    (*auto dest*: *pos simp*: *powr-half-sqrt intro*!: *continuous-intros*)
qed

lemma *continuous-on-artanh* [*continuous-intros*]:
assumes *A* ⊆ {−*1*<..<*1*}
shows *continuous-on A* (*artanh* :: *real* ⇒ *real*)
unfolding *artanh-def* [*abs-def*]
by (*intro continuous-on-subset* [*OF - assms*]) (*auto intro*!: *continuous-intros*)

lemma *continuous-on-arsinh′* [*continuous-intros*]:
fixes *f* :: *real* ⇒ *real*

assumes *continuous-on A f*
shows *continuous-on A* ($\lambda x.$ *arsinh* ($f\,x$))
by (*rule continuous-on-compose2*[*OF continuous-on-arsinh assms*]) *auto*

lemma *continuous-on-arcosh'* [*continuous-intros*]:
fixes f :: *real* $\Rightarrow$ *real*
assumes *continuous-on A f* $\bigwedge x.\ x \in A \Longrightarrow f\,x \geq 1$
shows *continuous-on A* ($\lambda x.$ *arcosh* ($f\,x$))
by (*rule continuous-on-compose2*[*OF continuous-on-arcosh assms*(*1*) *order.refl*])
(*use assms*(*2*) **in** *auto*)

lemma *continuous-on-artanh'* [*continuous-intros*]:
fixes f :: *real* $\Rightarrow$ *real*
assumes *continuous-on A f* $\bigwedge x.\ x \in A \Longrightarrow f\,x \in \{-1<..<1\}$
shows *continuous-on A* ($\lambda x.$ *artanh* ($f\,x$))
by (*rule continuous-on-compose2*[*OF continuous-on-artanh assms*(*1*) *order.refl*])
(*use assms*(*2*) **in** *auto*)

lemma *isCont-arsinh* [*continuous-intros*]: *isCont arsinh* (x :: *real*)
using *continuous-on-arsinh*[*of UNIV*] **by** (*auto simp*: *continuous-on-eq-continuous-at*)

lemma *isCont-arcosh* [*continuous-intros*]:
assumes $x > 1$
shows *isCont arcosh* (x :: *real*)
proof −
have *continuous-on* {*1*::*real*<..} *arcosh*
by (*rule continuous-on-arcosh*) *auto*
with *assms* **show** *?thesis* **by** (*auto simp*: *continuous-on-eq-continuous-at*)
qed

lemma *isCont-artanh* [*continuous-intros*]:
assumes $x > -1$ $x < 1$
shows *isCont artanh* (x :: *real*)
proof −
have *continuous-on* {−*1*<..<(*1*::*real*)} *artanh*
by (*rule continuous-on-artanh*) *auto*
with *assms* **show** *?thesis* **by** (*auto simp*: *continuous-on-eq-continuous-at*)
qed

lemma *tendsto-arsinh* [*tendsto-intros*]: ($f \longrightarrow a$) $F \Longrightarrow$ (($\lambda x.$ *arsinh* ($f\,x$)) $\longrightarrow$ *arsinh a*) F
for f :: - $\Rightarrow$ *real*
by (*rule isCont-tendsto-compose* [*OF isCont-arsinh*])

lemma *tendsto-arcosh-strong* [*tendsto-intros*]:
fixes f :: - $\Rightarrow$ *real*
assumes ($f \longrightarrow a$) F $a \geq 1$ *eventually* ($\lambda x.\ f\,x \geq 1$) F
shows (($\lambda x.$ *arcosh* ($f\,x$)) $\longrightarrow$ *arcosh a*) F
by (*rule continuous-on-tendsto-compose*[*OF continuous-on-arcosh*[*OF order.refl*]])

(*use assms* **in** *auto*)

lemma *tendsto-arcosh*:
fixes *f* :: - ⇒ *real*
assumes (*f* ⟶ *a*) *F a* > *1*
shows ((λ*x*. *arcosh* (*f x*)) ⟶ *arcosh a*) *F*
by (*rule isCont-tendsto-compose* [*OF isCont-arcosh*]) (*use assms* **in** *auto*)

lemma *tendsto-arcosh-at-left-1*: (*arcosh* ⟶ *0*) (*at-right* (*1*::*real*))
proof −
have (*arcosh* ⟶ *arcosh 1*) (*at-right* (*1*::*real*))
by (*rule tendsto-arcosh-strong*) (*auto simp*: *eventually-at intro*!: *exI*[*of* - *1*])
thus *?thesis* **by** *simp*
qed

lemma *tendsto-artanh* [*tendsto-intros*]:
fixes *f* :: ′*a* ⇒ *real*
assumes (*f* ⟶ *a*) *F a* > −*1 a* < *1*
shows ((λ*x*. *artanh* (*f x*)) ⟶ *artanh a*) *F*
by (*rule isCont-tendsto-compose* [*OF isCont-artanh*]) (*use assms* **in** *auto*)

lemma *continuous-arsinh* [*continuous-intros*]:
continuous F f ⟹ *continuous F* (λ*x*. *arsinh* (*f x* :: *real*))
unfolding *continuous-def* **by** (*rule tendsto-arsinh*)

lemma *continuous-arcosh-strong* [*continuous-intros*]:
assumes *continuous F f eventually* (λ*x*. *f x* ≥ *1*) *F*
shows *continuous F* (λ*x*. *arcosh* (*f x* :: *real*))
proof (*cases F* = *bot*)
case *False*
show *?thesis*
unfolding *continuous-def*
proof (*intro tendsto-arcosh-strong*)
show *1* ≤ *f* (*Lim F* (λ*x*. *x*))
using *assms False* **unfolding** *continuous-def* **by** (*rule tendsto-lowerbound*)
qed (*insert assms*, *auto simp*: *continuous-def*)
qed *auto*

lemma *continuous-arcosh*:
continuous F f ⟹ *f* (*Lim F* (λ*x*. *x*)) > *1* ⟹ *continuous F* (λ*x*. *arcosh* (*f x* :: *real*))
unfolding *continuous-def* **by** (*rule tendsto-arcosh*) *auto*

lemma *continuous-artanh* [*continuous-intros*]:
continuous F f ⟹ *f* (*Lim F* (λ*x*. *x*)) ∈ {−*1*<..<*1*} ⟹ *continuous F* (λ*x*. *artanh* (*f x* :: *real*))
unfolding *continuous-def* **by** (*rule tendsto-artanh*) *auto*

lemma *arsinh-real-at-top*:
filterlim (*arsinh* :: *real* ⇒ *real*) *at-top at-top*
proof (*subst filterlim-cong*[*OF refl refl*])
show *filterlim* (λ*x*. *ln* (*x* + *sqrt* (*1* + x^2))) *at-top at-top*
by (*intro filterlim-compose*[*OF ln-at-top filterlim-at-top-add-at-top*] *filterlim-ident*
filterlim-compose[*OF sqrt-at-top*] *filterlim-tendsto-add-at-top*[*OF tend-sto-const*]
filterlim-pow-at-top) *auto*
qed (*auto intro*!: *eventually-mono*[*OF eventually-ge-at-top*[*of 1*]] *simp*: *arsinh-real-def add-ac*)

lemma *arsinh-real-at-bot*:
filterlim (*arsinh* :: *real* ⇒ *real*) *at-bot at-bot*
proof −
have *filterlim* (λ*x*::*real*. −*arsinh x*) *at-bot at-top*
by (*subst filterlim-uminus-at-top* [*symmetric*]) (*rule arsinh-real-at-top*)
also have (λ*x*::*real*. −*arsinh x*) = (λ*x*. *arsinh* (−*x*)) **by** *simp*
finally show *?thesis*
by (*subst filterlim-at-bot-mirror*)
qed

lemma *arcosh-real-at-top*:
filterlim (*arcosh* :: *real* ⇒ *real*) *at-top at-top*
proof (*subst filterlim-cong*[*OF refl refl*])
show *filterlim* (λ*x*. *ln* (*x* + *sqrt* (−*1* + x^2))) *at-top at-top*
by (*intro filterlim-compose*[*OF ln-at-top filterlim-at-top-add-at-top*] *filterlim-ident*
filterlim-compose[*OF sqrt-at-top*] *filterlim-tendsto-add-at-top*[*OF tend-sto-const*]
filterlim-pow-at-top) *auto*
qed (*auto intro*!: *eventually-mono*[*OF eventually-ge-at-top*[*of 1*]] *simp*: *arcosh-real-def*)

lemma *artanh-real-at-left-1*:
filterlim (*artanh* :: *real* ⇒ *real*) *at-top* (*at-left 1*)
proof −
have ∗: *filterlim* (λ*x*::*real*. (*1* + *x*) / (*1* − *x*)) *at-top* (*at-left 1*)
by (*rule LIM-at-top-divide*)
(*auto intro*!: *tendsto-eq-intros eventually-mono*[*OF eventually-at-left-real*[*of 0*]])
have *filterlim* (λ*x*::*real*. (*1*/*2*) ∗ *ln* ((*1* + *x*) / (*1* − *x*))) *at-top* (*at-left 1*)
by (*intro filterlim-tendsto-pos-mult-at-top*[*OF tendsto-const*] ∗
filterlim-compose[*OF ln-at-top*]) *auto*
also have (λ*x*::*real*. (*1*/*2*) ∗ *ln* ((*1* + *x*) / (*1* − *x*))) = *artanh*
by (*simp add*: *artanh-def* [*abs-def*])
finally show *?thesis* **.**
qed

lemma *artanh-real-at-right-1*:
filterlim (*artanh* :: *real* ⇒ *real*) *at-bot* (*at-right* (−*1*))
proof −

have *?thesis* ⟷ *filterlim* (λ*x*::*real*. −*artanh x*) *at-top* (*at-right* (−*1*))
by (*simp add*: *filterlim-uminus-at-bot*)
also have ... ⟷ *filterlim* (λ*x*::*real*. *artanh* (−*x*)) *at-top* (*at-right* (−*1*))
by (*intro filterlim-cong refl eventually-mono*[*OF eventually-at-right-real*[*of* −*1* *1*]]) *auto*
also have ... ⟷ *filterlim* (*artanh* :: *real* ⇒ *real*) *at-top* (*at-left 1*)
by (*simp add*: *filterlim-at-left-to-right*)
also have ... **by** (*rule artanh-real-at-left-1*)
finally show *?thesis* **.**
qed

112.23 Simprocs for root and power literals

lemma *numeral-powr-numeral-real* [*simp*]:
numeral m powr numeral n = (*numeral m* ^ *numeral n* :: *real*)
by (*simp add*: *powr-numeral*)

context
begin

private lemma *sqrt-numeral-simproc-aux*:
assumes *m* ∗ *m* ≡ *n*
shows *sqrt* (*numeral n* :: *real*) ≡ *numeral m*
proof −
have *numeral n* ≡ *numeral m* ∗ (*numeral m* :: *real*) **by** (*simp add*: *assms* [*symmetric*])
moreover have *sqrt* ... ≡ *numeral m* **by** (*subst real-sqrt-abs2*) *simp*
ultimately show *sqrt* (*numeral n* :: *real*) ≡ *numeral m* **by** *simp*
qed

private lemma *root-numeral-simproc-aux*:
assumes *Num.pow m n* ≡ *x*
shows *root* (*numeral n*) (*numeral x* :: *real*) ≡ *numeral m*
by (*subst assms* [*symmetric*], *subst numeral-pow*, *subst real-root-pos2*) *simp-all*

private lemma *powr-numeral-simproc-aux*:
assumes *Num.pow y n* = *x*
shows *numeral x powr* (*m* / *numeral n* :: *real*) ≡ *numeral y powr m*
by (*subst assms* [*symmetric*], *subst numeral-pow*, *subst powr-numeral* [*symmetric*])
(*simp*, *subst powr-powr*, *simp-all*)

private lemma *numeral-powr-inverse-eq*:
numeral x powr (*inverse* (*numeral n*)) = *numeral x powr* (*1* / *numeral n* :: *real*)
by *simp*

ML ‹

signature ROOT-NUMERAL-SIMPROC = *sig*

```
val sqrt : int option −> int −> int option
val sqrt' : int option −> int −> int option
val nth-root : int option −> int −> int −> int option
val nth-root' : int option −> int −> int −> int option
val sqrt-proc : Simplifier.proc
val root-proc : int ∗ int −> Simplifier.proc
val powr-proc : int ∗ int −> Simplifier.proc

end

structure Root-Numeral-Simproc : ROOT-NUMERAL-SIMPROC = struct

fun iterate NONE p f x =
      let
        fun go x = if p x then x else go (f x)
      in
        SOME (go x)
      end
  | iterate (SOME threshold) p f x =
      let
        fun go (threshold, x) =
          if p x then SOME x else if threshold = 0 then NONE else go (threshold −
1, f x)
      in
        go (threshold, x)
      end


fun nth-root - 1 x = SOME x
  | nth-root - - 0 = SOME 0
  | nth-root - - 1 = SOME 1
  | nth-root threshold n x =
  let
    fun newton-step y = ((n − 1) ∗ y + x div Integer.pow (n − 1) y) div n
    fun is-root y = Integer.pow n y <= x andalso x < Integer.pow n (y + 1)
  in
    if x < n then
      SOME 1
    else if x < Integer.pow n 2 then
      SOME 1
    else
      let
        val y = Real.floor (Math.pow (Real.fromInt x, Real.fromInt 1 / Real.fromInt
n))
      in
        if is-root y then
          SOME y
        else
```

```
        iterate threshold is-root newton-step ((x + n − 1) div n)
      end
  end

fun nth-root′ - 1 x = SOME x
  | nth-root′ - - 0 = SOME 0
  | nth-root′ - - 1 = SOME 1
  | nth-root′ threshold n x = if x < n then NONE else if x < Integer.pow n 2 then
NONE else
      case nth-root threshold n x of
        NONE => NONE
      | SOME y => if Integer.pow n y = x then SOME y else NONE

fun sqrt - 0 = SOME 0
  | sqrt - 1 = SOME 1
  | sqrt threshold n =
    let
      fun aux (a, b) = if n >= b ∗ b then aux (b, b ∗ b) else (a, b)
      val (lower-root, lower-n) = aux (1, 2)
      fun newton-step x = (x + n div x) div 2
      fun is-sqrt r = r∗r <= n andalso n < (r+1)∗(r+1)
      val y = Real.floor (Math.sqrt (Real.fromInt n))
    in
      if is-sqrt y then
        SOME y
      else
        Option.mapPartial (iterate threshold is-sqrt newton-step o (fn x => x ∗
lower-root))
        (sqrt threshold (n div lower-n))
    end

fun sqrt′ threshold x =
  case sqrt threshold x of
    NONE => NONE
  | SOME y => if y ∗ y = x then SOME y else NONE

fun sqrt-proc ctxt ct =
  let
     val n = ct |> Thm.term-of |> dest-comb |> snd |> dest-comb |> snd |>
HOLogic.dest-numeral
  in
    case sqrt′ (SOME 10000) n of
      NONE => NONE
    | SOME m =>
          SOME (Thm.instantiate′ [] (map (SOME o Thm.cterm-of ctxt o HO-
Logic.mk-numeral) [m, n])
                @{thm sqrt-numeral-simproc-aux})
  end
    handle TERM - => NONE
```

```
fun root-proc (threshold1, threshold2) ctxt ct =
  let
    val [n, x] =
      ct |> Thm.term-of |> strip-comb |> snd |> map (dest-comb #> snd #>
HOLogic.dest-numeral)
  in
    if n > threshold1 orelse x > threshold2 then NONE else
      case nth-root' (SOME 100) n x of
        NONE => NONE
      | SOME m =>
          SOME (Thm.instantiate' [] (map (SOME o Thm.cterm-of ctxt o HO-
Logic.mk-numeral) [m, n, x])
          @{thm root-numeral-simproc-aux})
  end
    handle TERM - => NONE
      | Match => NONE

fun powr-proc (threshold1, threshold2) ctxt ct =
  let
    val eq-thm = Conv.try-conv (Conv.rewr-conv @{thm numeral-powr-inverse-eq})
ct
    val ct = Thm.dest-equals-rhs (Thm.cprop-of eq-thm)
    val (-, [x, t]) = strip-comb (Thm.term-of ct)
    val (-, [m, n]) = strip-comb t
    val [x, n] = map (dest-comb #> snd #> HOLogic.dest-numeral) [x, n]
  in
    if n > threshold1 orelse x > threshold2 then NONE else
      case nth-root' (SOME 100) n x of
        NONE => NONE
      | SOME y =>
          let
            val [y, n, x] = map HOLogic.mk-numeral [y, n, x]
            val thm = Thm.instantiate' [] (map (SOME o Thm.cterm-of ctxt) [y, n,
x, m])
            @{thm powr-numeral-simproc-aux}
          in
            SOME (@{thm transitive} OF [eq-thm, thm])
          end
  end
    handle TERM - => NONE
      | Match => NONE

end
›
```

end

simproc-setup *sqrt-numeral* (*sqrt* (*numeral n*)) =

‹*K Root-Numeral-Simproc.sqrt-proc*›

simproc-setup *root-numeral* (*root* (*numeral n*) (*numeral x*)) =
‹*K* (*Root-Numeral-Simproc.root-proc* (*200*, *Integer.pow 200 2*))›

simproc-setup *powr-divide-numeral*
(*numeral x powr* (*m* / *numeral n* :: *real*) | *numeral x powr* (*inverse* (*numeral n*) :: *real*)) =
‹*K* (*Root-Numeral-Simproc.powr-proc* (*200*, *Integer.pow 200 2*))›

lemma *root 100 1267650600228229401496703205376 = 2*
by *simp*

lemma *sqrt 196 = 14*
by *simp*

lemma *256 powr* (*7* / *4* :: *real*) = *16384*
by *simp*

lemma *27 powr* (*inverse 3*) = (*3*::*real*)
by *simp*

end

113 Complex Numbers: Rectangular and Polar Representations

theory *Complex*
imports *Transcendental Real-Vector-Spaces*
begin

We use the **codatatype** command to define the type of complex numbers. This allows us to use **primcorec** to define complex functions by defining their real and imaginary result separately.

codatatype *complex* = *Complex* (*Re*: *real*) (*Im*: *real*)

lemma *complex-surj*: *Complex* (*Re z*) (*Im z*) = *z*
by (*rule complex.collapse*)

lemma *complex-eqI* [*intro?*]: *Re x* = *Re y* ⟹ *Im x* = *Im y* ⟹ *x* = *y*
by (*rule complex.expand*) *simp*

lemma *complex-eq-iff*: *x* = *y* ⟷ *Re x* = *Re y* ∧ *Im x* = *Im y*
by (*auto intro*: *complex.expand*)

113.1 Addition and Subtraction

instantiation *complex* :: *ab-group-add*
begin

primcorec *zero-complex*
  where
    $Re\ 0 = 0$
  $|\ Im\ 0 = 0$

primcorec *plus-complex*
  where
    $Re\ (x + y) = Re\ x + Re\ y$
  $|\ Im\ (x + y) = Im\ x + Im\ y$

primcorec *uminus-complex*
  where
    $Re\ (-\ x) = -\ Re\ x$
  $|\ Im\ (-\ x) = -\ Im\ x$

primcorec *minus-complex*
  where
    $Re\ (x - y) = Re\ x - Re\ y$
  $|\ Im\ (x - y) = Im\ x - Im\ y$

instance
  by *standard* (*simp-all add*: *complex-eq-iff*)

end

113.2 Multiplication and Division

instantiation *complex* :: *field*
begin

primcorec *one-complex*
  where
    $Re\ 1 = 1$
  $|\ Im\ 1 = 0$

primcorec *times-complex*
  where
    $Re\ (x * y) = Re\ x * Re\ y - Im\ x * Im\ y$
  $|\ Im\ (x * y) = Re\ x * Im\ y + Im\ x * Re\ y$

primcorec *inverse-complex*
  where
    $Re\ (inverse\ x) = Re\ x\ /\ ((Re\ x)^2 + (Im\ x)^2)$
  $|\ Im\ (inverse\ x) = -\ Im\ x\ /\ ((Re\ x)^2 + (Im\ x)^2)$

definition *x div y* = *x* ∗ *inverse y* **for** *x y* :: *complex*

instance
  by *standard*
    (*simp-all add*: *complex-eq-iff divide-complex-def*
     distrib-left distrib-right right-diff-distrib left-diff-distrib
     power2-eq-square add-divide-distrib [*symmetric*])

end

lemma *Re-divide*: *Re* (*x* / *y*) = (*Re x* ∗ *Re y* + *Im x* ∗ *Im y*) / ((*Re y*)^2 + (*Im y*)^2)
  by (*simp add*: *divide-complex-def add-divide-distrib*)

lemma *Im-divide*: *Im* (*x* / *y*) = (*Im x* ∗ *Re y* − *Re x* ∗ *Im y*) / ((*Re y*)^2 + (*Im y*)^2)
  by (*simp add*: *divide-complex-def diff-divide-distrib*)

lemma *Complex-divide*:
  (*x* / *y*) = *Complex* ((*Re x* ∗ *Re y* + *Im x* ∗ *Im y*) / ((*Re y*)^2 + (*Im y*)^2))
    ((*Im x* ∗ *Re y* − *Re x* ∗ *Im y*) / ((*Re y*)^2 + (*Im y*)^2))
  by (*metis Im-divide Re-divide complex-surj*)

lemma *Re-power2*: *Re* (*x* ^ *2*) = (*Re x*)^*2* − (*Im x*)^*2*
  by (*simp add*: *power2-eq-square*)

lemma *Im-power2*: *Im* (*x* ^ *2*) = *2* ∗ *Re x* ∗ *Im x*
  by (*simp add*: *power2-eq-square*)

lemma *Re-power-real* [*simp*]: *Im x* = *0* ⟹ *Re* (*x* ^ *n*) = *Re x* ^ *n*
  by (*induct n*) *simp-all*

lemma *Im-power-real* [*simp*]: *Im x* = *0* ⟹ *Im* (*x* ^ *n*) = *0*
  by (*induct n*) *simp-all*

113.3 Scalar Multiplication

instantiation *complex* :: *real-field*
begin

primcorec *scaleR-complex*
  where
    Re (*scaleR r x*) = *r* ∗ *Re x*
  | *Im* (*scaleR r x*) = *r* ∗ *Im x*

instance
proof
  fix *a b* :: *real* **and** *x y* :: *complex*
  show *scaleR a* (*x* + *y*) = *scaleR a x* + *scaleR a y*

  by (*simp add*: *complex-eq-iff distrib-left*)
 show *scaleR* (*a* + *b*) *x* = *scaleR a x* + *scaleR b x*
  by (*simp add*: *complex-eq-iff distrib-right*)
 show *scaleR a* (*scaleR b x*) = *scaleR* (*a* ∗ *b*) *x*
  by (*simp add*: *complex-eq-iff mult.assoc*)
 show *scaleR 1 x* = *x*
  by (*simp add*: *complex-eq-iff*)
 show *scaleR a x* ∗ *y* = *scaleR a* (*x* ∗ *y*)
  by (*simp add*: *complex-eq-iff algebra-simps*)
 show *x* ∗ *scaleR a y* = *scaleR a* (*x* ∗ *y*)
  by (*simp add*: *complex-eq-iff algebra-simps*)
qed

end

113.4 Numerals, Arithmetic, and Embedding from R

declare [[*coercion of-real* :: *real* ⇒ *complex*]]
declare [[*coercion of-rat* :: *rat* ⇒ *complex*]]
declare [[*coercion of-int* :: *int* ⇒ *complex*]]
declare [[*coercion of-nat* :: *nat* ⇒ *complex*]]

abbreviation *complex-of-nat*::*nat* ⇒ *complex*
 where *complex-of-nat* ≡ *of-nat*

abbreviation *complex-of-int*::*int* ⇒ *complex*
 where *complex-of-int* ≡ *of-int*

abbreviation *complex-of-rat*::*rat* ⇒ *complex*
 where *complex-of-rat* ≡ *of-rat*

abbreviation *complex-of-real* :: *real* ⇒ *complex*
 where *complex-of-real* ≡ *of-real*

lemma *complex-Re-of-nat* [*simp*]: *Re* (*of-nat n*) = *of-nat n*
 by (*induct n*) *simp-all*

lemma *complex-Im-of-nat* [*simp*]: *Im* (*of-nat n*) = *0*
 by (*induct n*) *simp-all*

lemma *complex-Re-of-int* [*simp*]: *Re* (*of-int z*) = *of-int z*
 by (*cases z rule*: *int-diff-cases*) *simp*

lemma *complex-Im-of-int* [*simp*]: *Im* (*of-int z*) = *0*
 by (*cases z rule*: *int-diff-cases*) *simp*

lemma *complex-Re-numeral* [*simp*]: *Re* (*numeral v*) = *numeral v*
 using *complex-Re-of-int* [*of numeral v*] **by** *simp*

lemma *complex-Im-numeral* [*simp*]: *Im* (*numeral v*) = *0*
using *complex-Im-of-int* [*of numeral v*] **by** *simp*

lemma *Re-complex-of-real* [*simp*]: *Re* (*complex-of-real z*) = *z*
by (*simp add*: *of-real-def*)

lemma *Im-complex-of-real* [*simp*]: *Im* (*complex-of-real z*) = *0*
by (*simp add*: *of-real-def*)

lemma *Re-divide-numeral* [*simp*]: *Re* (*z* / *numeral w*) = *Re z* / *numeral w*
by (*simp add*: *Re-divide sqr-conv-mult*)

lemma *Im-divide-numeral* [*simp*]: *Im* (*z* / *numeral w*) = *Im z* / *numeral w*
by (*simp add*: *Im-divide sqr-conv-mult*)

lemma *Re-divide-of-nat* [*simp*]: *Re* (*z* / *of-nat n*) = *Re z* / *of-nat n*
by (*cases n*) (*simp-all add*: *Re-divide field-split-simps power2-eq-square del*: *of-nat-Suc*)

lemma *Im-divide-of-nat* [*simp*]: *Im* (*z* / *of-nat n*) = *Im z* / *of-nat n*
by (*cases n*) (*simp-all add*: *Im-divide field-split-simps power2-eq-square del*: *of-nat-Suc*)

lemma *Re-inverse* [*simp*]: $r \in \mathbb{R} \Longrightarrow$ *Re* (*inverse r*) = *inverse* (*Re r*)
by (*metis Re-complex-of-real Reals-cases of-real-inverse*)

lemma *Im-inverse* [*simp*]: $r \in \mathbb{R} \Longrightarrow$ *Im* (*inverse r*) = *0*
by (*metis Im-complex-of-real Reals-cases of-real-inverse*)

lemma *of-real-Re* [*simp*]: $z \in \mathbb{R} \Longrightarrow$ *of-real* (*Re z*) = *z*
by (*auto simp*: *Reals-def*)

lemma *complex-Re-fact* [*simp*]: *Re* (*fact n*) = *fact n*
proof −
have (*fact n* :: *complex*) = *of-real* (*fact n*)
by *simp*
also have *Re* . . . = *fact n*
by (*subst Re-complex-of-real*) *simp-all*
finally show *?thesis* **.**
qed

lemma *surj-Re*: *surj Re*
by (*metis Re-complex-of-real surj-def*)

lemma *surj-Im*: *surj Im*
by (*metis complex.sel*(*2*) *surj-def*)

lemma *complex-Im-fact* [*simp*]: *Im* (*fact n*) = *0*
by (*metis complex-Im-of-nat of-nat-fact*)

lemma *Re-prod-Reals*: ($\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{R}$) $\Longrightarrow$ *Re* (*prod f A*) = *prod* ($\lambda x.$

Re (*f x*)) *A*
proof (*induction A rule*: *infinite-finite-induct*)
  case (*insert x A*)
  hence *Re* (*prod f* (*insert x A*)) = *Re* (*f x*) ∗ *Re* (*prod f A*) − *Im* (*f x*) ∗ *Im* (*prod f A*)
    by *simp*
  also from *insert.prems* **have** $f\ x \in \mathbb{R}$ **by** *simp*
  hence *Im* (*f x*) = *0* **by** (*auto elim*!: *Reals-cases*)
  also have *Re* (*prod f A*) = ($\prod$ *x*∈*A*. *Re* (*f x*))
    by (*intro insert.IH insert.prems*) *auto*
  finally show *?case* **using** *insert.hyps* **by** *simp*
qed *auto*

113.5 The Complex Number *i*

primcorec *imaginary-unit* :: *complex* (‹i›)
  where
    Re i = *0*
  | *Im* i = *1*

lemma *Complex-eq*: *Complex a b* = *a* + i ∗ *b*
  by (*simp add*: *complex-eq-iff*)

lemma *complex-eq*: *a* = *Re a* + i ∗ *Im a*
  by (*simp add*: *complex-eq-iff*)

lemma *fun-complex-eq*: *f* = (λ*x*. *Re* (*f x*) + i ∗ *Im* (*f x*))
  by (*simp add*: *fun-eq-iff complex-eq*)

lemma *i-squared* [*simp*]: i ∗ i = −*1*
  by (*simp add*: *complex-eq-iff*)

lemma *power2-i* [*simp*]: i^2 = −*1*
  by (*simp add*: *power2-eq-square*)

lemma *inverse-i* [*simp*]: *inverse* i = − i
  by (*rule inverse-unique*) *simp*

lemma *divide-i* [*simp*]: *x* / i = − i ∗ *x*
  by (*simp add*: *divide-complex-def*)

lemma *complex-i-mult-minus* [*simp*]: i ∗ (i ∗ *x*) = − *x*
  by (*simp add*: *mult.assoc* [*symmetric*])

lemma *complex-i-not-zero* [*simp*]: i ≠ *0*
  by (*simp add*: *complex-eq-iff*)

lemma *complex-i-not-one* [*simp*]: i ≠ *1*
  by (*simp add*: *complex-eq-iff*)

lemma *complex-i-not-numeral* [*simp*]: i ≠ *numeral w*
  by (*simp add*: *complex-eq-iff*)

lemma *complex-i-not-neg-numeral* [*simp*]: i ≠ − *numeral w*
  by (*simp add*: *complex-eq-iff*)

lemma *complex-split-polar*: ∃ *r a*. *z* = *complex-of-real r* ∗ (*cos a* + i ∗ *sin a*)
  by (*simp add*: *complex-eq-iff polar-Ex*)

lemma *i-even-power* [*simp*]: i ^ (*n* ∗ *2*) = (−*1*) ^ *n*
  by (*metis mult.commute power2-i power-mult*)

lemma *i-even-power′* [*simp*]: *even n* ⟹ i ^ *n* = (−*1*) ^ (*n div 2*)
  by (*metis dvd-mult-div-cancel power2-i power-mult*)

lemma *Re-i-times* [*simp*]: *Re* (i ∗ *z*) = − *Im z*
  by *simp*

lemma *Im-i-times* [*simp*]: *Im* (i ∗ *z*) = *Re z*
  by *simp*

lemma *i-times-eq-iff*: i ∗ *w* = *z* ⟷ *w* = − (i ∗ *z*)
  by *auto*

lemma *divide-numeral-i* [*simp*]: *z* / (*numeral n* ∗ i) = − (i ∗ *z*) / *numeral n*
  by (*metis divide-divide-eq-left divide-i mult.commute mult-minus-right*)

lemma *imaginary-eq-real-iff* [*simp*]:
  assumes *y* ∈ *Reals x* ∈ *Reals*
  shows i ∗ *y* = *x* ⟷ *x=0* ∧ *y=0*
 by (*metis Im-complex-of-real Im-i-times assms mult-zero-right of-real-0 of-real-Re*)

lemma *real-eq-imaginary-iff* [*simp*]:
  assumes *y* ∈ *Reals x* ∈ *Reals*
  shows *x* = i ∗ *y* ⟷ *x=0* ∧ *y=0*
    using *assms imaginary-eq-real-iff* **by** *fastforce*

113.6 Vector Norm

instantiation *complex* :: *real-normed-field*
begin

definition *norm z* = *sqrt* ((*Re z*)^2 + (*Im z*)^2)

abbreviation *cmod* :: *complex* ⇒ *real*
  where *cmod* ≡ *norm*

definition *complex-sgn-def*: *sgn x* = *x* $/_R$ *cmod x*

definition *dist-complex-def*: *dist x y* = *cmod* (*x* − *y*)

definition *uniformity-complex-def* [*code del*]:
(*uniformity* :: (*complex* × *complex*) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-complex-def* [*code del*]:
open (*U* :: *complex set*) ⟷ (∀ *x*∈*U*. *eventually* (λ(*x′*, *y*). *x′* = *x* ⟶ *y* ∈ *U*) *uniformity*)

instance
proof
 fix *r* :: *real* **and** *x y* :: *complex* **and** *S* :: *complex set*
 show (*norm x* = *0*) = (*x* = *0*)
  by (*simp add*: *norm-complex-def complex-eq-iff*)
 show *norm* (*x* + *y*) ≤ *norm x* + *norm y*
  by (*simp add*: *norm-complex-def complex-eq-iff real-sqrt-sum-squares-triangle-ineq*)
 show *norm* (*scaleR r x*) = |*r*| ∗ *norm x*
  by (*simp add*: *norm-complex-def complex-eq-iff power-mult-distrib distrib-left* [*symmetric*]
   real-sqrt-mult)
 show *norm* (*x* ∗ *y*) = *norm x* ∗ *norm y*
  by (*simp add*: *norm-complex-def complex-eq-iff real-sqrt-mult* [*symmetric*]
   power2-eq-square algebra-simps)
qed (*rule complex-sgn-def dist-complex-def open-complex-def uniformity-complex-def*)+

end

declare *uniformity-Abort*[**where** *′a* = *complex*, *code*]

lemma *norm-ii* [*simp*]: *norm* i = *1*
 by (*simp add*: *norm-complex-def*)

lemma *cmod-unit-one*: *cmod* (*cos a* + i ∗ *sin a*) = *1*
 by (*simp add*: *norm-complex-def*)

lemma *cmod-complex-polar*: *cmod* (*r* ∗ (*cos a* + i ∗ *sin a*)) = |*r*|
 by (*simp add*: *norm-mult cmod-unit-one*)

lemma *complex-Re-le-cmod*: *Re x* ≤ *cmod x*
 unfolding *norm-complex-def* **by** (*rule real-sqrt-sum-squares-ge1*)

lemma *complex-mod-minus-le-complex-mod*: − *cmod x* ≤ *cmod x*
 by (*rule order-trans* [*OF* - *norm-ge-zero*]) *simp*

lemma *complex-mod-triangle-ineq2*: *cmod* (*b* + *a*) − *cmod b* ≤ *cmod a*
 by (*rule ord-le-eq-trans* [*OF norm-triangle-ineq2*]) *simp*

lemma *abs-Re-le-cmod*: $|Re\ x| \le cmod\ x$
by (*simp add*: *norm-complex-def*)

lemma *abs-Im-le-cmod*: $|Im\ x| \le cmod\ x$
by (*simp add*: *norm-complex-def*)

lemma *cmod-le*: $cmod\ z \le |Re\ z| + |Im\ z|$
using *norm-complex-def sqrt-sum-squares-le-sum-abs* **by** *presburger*

lemma *cmod-eq-Re*: $Im\ z = 0 \Longrightarrow cmod\ z = |Re\ z|$
by (*simp add*: *norm-complex-def*)

lemma *cmod-eq-Im*: $Re\ z = 0 \Longrightarrow cmod\ z = |Im\ z|$
by (*simp add*: *norm-complex-def*)

lemma *cmod-power2*: $(cmod\ z)^2 = (Re\ z)^2 + (Im\ z)^2$
by (*simp add*: *norm-complex-def*)

lemma *cmod-plus-Re-le-0-iff*: $cmod\ z + Re\ z \le 0 \longleftrightarrow Re\ z = -\ cmod\ z$
using *abs-Re-le-cmod*[*of z*] **by** *auto*

lemma *cmod-Re-le-iff*: $Im\ x = Im\ y \Longrightarrow cmod\ x \le cmod\ y \longleftrightarrow |Re\ x| \le |Re\ y|$
by (*metis add.commute add-le-cancel-left norm-complex-def real-sqrt-abs real-sqrt-le-iff*)

lemma *cmod-Im-le-iff*: $Re\ x = Re\ y \Longrightarrow cmod\ x \le cmod\ y \longleftrightarrow |Im\ x| \le |Im\ y|$
by (*metis add-le-cancel-left norm-complex-def real-sqrt-abs real-sqrt-le-iff*)

lemma *Im-eq-0*: $|Re\ z| = cmod\ z \Longrightarrow Im\ z = 0$
by (*subst* (*asm*) *power-eq-iff-eq-base*[*symmetric*, **where** *n=2*]) (*auto simp add*: *norm-complex-def*)

lemma *abs-sqrt-wlog*: $(\bigwedge x.\ x \ge 0 \Longrightarrow P\ x\ (x^2)) \Longrightarrow P\ |x|\ (x^2)$
for *x*::*'a*::*linordered-idom*
by (*metis abs-ge-zero power2-abs*)

lemma *complex-abs-le-norm*: $|Re\ z| + |Im\ z| \le sqrt\ 2 * norm\ z$
unfolding *norm-complex-def*
apply (*rule abs-sqrt-wlog* [**where** *x=Re z*])
apply (*rule abs-sqrt-wlog* [**where** *x=Im z*])
apply (*rule power2-le-imp-le*)
apply (*simp-all add*: *power2-sum add.commute sum-squares-bound real-sqrt-mult* [*symmetric*])
done

lemma *complex-unit-circle*: $z \ne 0 \Longrightarrow (Re\ z\ /\ cmod\ z)^2 + (Im\ z\ /\ cmod\ z)^2 = 1$
by (*simp add*: *norm-complex-def complex-eq-iff power2-eq-square add-divide-distrib* [*symmetric*])

Properties of complex signum.

lemma *sgn-eq*: *sgn z* = *z* / *complex-of-real* (*cmod z*)
by (*simp add*: *sgn-div-norm divide-inverse scaleR-conv-of-real mult.commute*)

lemma *Re-sgn* [*simp*]: *Re*(*sgn z*) = *Re*(*z*)/*cmod z*
by (*simp add*: *complex-sgn-def divide-inverse*)

lemma *Im-sgn* [*simp*]: *Im*(*sgn z*) = *Im*(*z*)/*cmod z*
by (*simp add*: *complex-sgn-def divide-inverse*)

113.7 Absolute value

instantiation *complex* :: *field-abs-sgn*
begin

definition *abs-complex* :: *complex* ⇒ *complex*
where *abs-complex* = *of-real* ∘ *norm*

instance
proof qed (*auto simp add*: *abs-complex-def complex-sgn-def norm-divide norm-mult scaleR-conv-of-real field-simps*)
end

113.8 Completeness of the Complexes

lemma *bounded-linear-Re*: *bounded-linear Re*
by (*rule bounded-linear-intro* [**where** *K=1*]) (*simp-all add*: *norm-complex-def*)

lemma *bounded-linear-Im*: *bounded-linear Im*
by (*rule bounded-linear-intro* [**where** *K=1*]) (*simp-all add*: *norm-complex-def*)

lemmas *Cauchy-Re* = *bounded-linear.Cauchy* [*OF bounded-linear-Re*]
lemmas *Cauchy-Im* = *bounded-linear.Cauchy* [*OF bounded-linear-Im*]
lemmas *tendsto-Re* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Re*]
lemmas *tendsto-Im* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im*]
lemmas *isCont-Re* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Re*]
lemmas *isCont-Im* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im*]
lemmas *continuous-Re* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Re*]
lemmas *continuous-Im* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im*]
lemmas *continuous-on-Re* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Re*]
lemmas *continuous-on-Im* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im*]
lemmas *has-derivative-Re* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Re*]
lemmas *has-derivative-Im* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im*]
lemmas *sums-Re* = *bounded-linear.sums* [*OF bounded-linear-Re*]
lemmas *sums-Im* = *bounded-linear.sums* [*OF bounded-linear-Im*]
lemmas *Re-suminf* = *bounded-linear.suminf*[*OF bounded-linear-Re*]
lemmas *Im-suminf* = *bounded-linear.suminf*[*OF bounded-linear-Im*]

lemma *continuous-on-Complex* [*continuous-intros*]:
continuous-on A f $\Longrightarrow$ *continuous-on A g* $\Longrightarrow$ *continuous-on A* (λx. *Complex* (*f x*) (*g x*))
unfolding *Complex-eq* **by** (*intro continuous-intros*)

lemma *tendsto-Complex* [*tendsto-intros*]:
(*f* $\longrightarrow$ *a*) *F* $\Longrightarrow$ (*g* $\longrightarrow$ *b*) *F* $\Longrightarrow$ ((λx. *Complex* (*f x*) (*g x*)) $\longrightarrow$ *Complex a b*) *F*
unfolding *Complex-eq* **by** (*auto intro*!: *tendsto-intros*)

lemma *tendsto-complex-iff*:
(*f* $\longrightarrow$ *x*) *F* $\longleftrightarrow$ (((λx. *Re* (*f x*)) $\longrightarrow$ *Re x*) *F* $\wedge$ ((λx. *Im* (*f x*)) $\longrightarrow$ *Im x*) *F*)
proof *safe*
assume ((λx. *Re* (*f x*)) $\longrightarrow$ *Re x*) *F* ((λx. *Im* (*f x*)) $\longrightarrow$ *Im x*) *F*
from *tendsto-Complex*[*OF this*] **show** (*f* $\longrightarrow$ *x*) *F*
unfolding *complex.collapse* **.**
qed (*auto intro*: *tendsto-intros*)

lemma *continuous-complex-iff*:
continuous F f $\longleftrightarrow$ *continuous F* (λx. *Re* (*f x*)) $\wedge$ *continuous F* (λx. *Im* (*f x*))
by (*simp only*: *continuous-def tendsto-complex-iff*)

lemma *continuous-on-of-real-o-iff* [*simp*]:
continuous-on S (λx. *complex-of-real* (*g x*)) = *continuous-on S g*
using *continuous-on-Re continuous-on-of-real* **by** *fastforce*

lemma *continuous-on-of-real-id* [*simp*]:
continuous-on S (*of-real* :: *real* $\Rightarrow$ *'a*::*real-normed-algebra-1*)
by (*rule continuous-on-of-real* [*OF continuous-on-id*])

lemma *has-vector-derivative-complex-iff*: (*f has-vector-derivative x*) *F* $\longleftrightarrow$
((λx. *Re* (*f x*)) *has-field-derivative* (*Re x*)) *F* $\wedge$
((λx. *Im* (*f x*)) *has-field-derivative* (*Im x*)) *F*
by (*simp add*: *has-vector-derivative-def has-field-derivative-def has-derivative-def tendsto-complex-iff algebra-simps bounded-linear-scaleR-left bounded-linear-mult-right*)

lemma *has-field-derivative-Re*[*derivative-intros*]:
(*f has-vector-derivative D*) *F* $\Longrightarrow$ ((λx. *Re* (*f x*)) *has-field-derivative* (*Re D*)) *F*
unfolding *has-vector-derivative-complex-iff* **by** *safe*

lemma *has-field-derivative-Im*[*derivative-intros*]:
(*f has-vector-derivative D*) *F* $\Longrightarrow$ ((λx. *Im* (*f x*)) *has-field-derivative* (*Im D*)) *F*
unfolding *has-vector-derivative-complex-iff* **by** *safe*

instance *complex* :: *banach*
proof
fix *X* :: *nat* $\Rightarrow$ *complex*

assume *X*: *Cauchy X*
then have (λ*n*. *Complex* (*Re* (*X n*)) (*Im* (*X n*))) ⟶
Complex (*lim* (λ*n*. *Re* (*X n*))) (*lim* (λ*n*. *Im* (*X n*)))
by (*intro tendsto-Complex convergent-LIMSEQ-iff*[*THEN iffD1*]
Cauchy-convergent-iff[*THEN iffD1*] *Cauchy-Re Cauchy-Im*)
then show *convergent X*
unfolding *complex.collapse* **by** (*rule convergentI*)
qed

declare *DERIV-power*[**where** *'a*=*complex*, *unfolded of-nat-def*[*symmetric*], *derivative-intros*]

113.9 Complex Conjugation

primcorec *cnj* :: *complex* ⇒ *complex*
where
Re (*cnj z*) = *Re z*
| *Im* (*cnj z*) = − *Im z*

lemma *complex-cnj-cancel-iff* [*simp*]: *cnj x* = *cnj y* ⟷ *x* = *y*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-cnj* [*simp*]: *cnj* (*cnj z*) = *z*
by (*simp add*: *complex-eq-iff*)

lemma *in-image-cnj-iff*: *z* ∈ *cnj* ‘ *A* ⟷ *cnj z* ∈ *A*
by (*metis complex-cnj-cnj image-iff*)

lemma *image-cnj-conv-vimage-cnj*: *cnj* ‘ *A* = *cnj* −‘ *A*
using *in-image-cnj-iff* **by** *blast*

lemma *complex-cnj-zero* [*simp*]: *cnj 0* = *0*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-zero-iff* [*iff*]: *cnj z* = *0* ⟷ *z* = *0*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-one-iff* [*simp*]: *cnj z* = *1* ⟷ *z* = *1*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-add* [*simp*]: *cnj* (*x* + *y*) = *cnj x* + *cnj y*
by (*simp add*: *complex-eq-iff*)

lemma *cnj-sum* [*simp*]: *cnj* (*sum f s*) = (∑*x*∈*s*. *cnj* (*f x*))
by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *complex-cnj-diff* [*simp*]: *cnj* (*x* − *y*) = *cnj x* − *cnj y*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-minus* [*simp*]: *cnj* (− *x*) = − *cnj x*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-one* [*simp*]: *cnj 1* = *1*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-mult* [*simp*]: *cnj* (*x* ∗ *y*) = *cnj x* ∗ *cnj y*
by (*simp add*: *complex-eq-iff*)

lemma *cnj-prod* [*simp*]: *cnj* (*prod f s*) = (∏ *x*∈*s*. *cnj* (*f x*))
by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *complex-cnj-inverse* [*simp*]: *cnj* (*inverse x*) = *inverse* (*cnj x*)
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-divide* [*simp*]: *cnj* (*x* / *y*) = *cnj x* / *cnj y*
by (*simp add*: *divide-complex-def*)

lemma *complex-cnj-power* [*simp*]: *cnj* (*x* ^ *n*) = *cnj x* ^ *n*
by (*induct n*) *simp-all*

lemma *complex-cnj-of-nat* [*simp*]: *cnj* (*of-nat n*) = *of-nat n*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-of-int* [*simp*]: *cnj* (*of-int z*) = *of-int z*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-numeral* [*simp*]: *cnj* (*numeral w*) = *numeral w*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-neg-numeral* [*simp*]: *cnj* (− *numeral w*) = − *numeral w*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-scaleR* [*simp*]: *cnj* (*scaleR r x*) = *scaleR r* (*cnj x*)
by (*simp add*: *complex-eq-iff*)

lemma *complex-mod-cnj* [*simp*]: *cmod* (*cnj z*) = *cmod z*
by (*simp add*: *norm-complex-def*)

lemma *complex-cnj-complex-of-real* [*simp*]: *cnj* (*of-real x*) = *of-real x*
by (*simp add*: *complex-eq-iff*)

lemma *complex-cnj-i* [*simp*]: *cnj* i = − i
by (*simp add*: *complex-eq-iff*)

lemma *complex-add-cnj*: *z* + *cnj z* = *complex-of-real* (*2* ∗ *Re z*)
by (*simp add*: *complex-eq-iff*)

lemma *complex-diff-cnj*: *z* − *cnj z* = *complex-of-real* (*2* ∗ *Im z*) ∗ i

by (*simp add*: *complex-eq-iff*)

lemma *Ints-cnj* [*intro*]: $x \in \mathbb{Z} \Longrightarrow cnj\ x \in \mathbb{Z}$
by (*auto elim*!: *Ints-cases*)

lemma *cnj-in-Ints-iff* [*simp*]: $cnj\ x \in \mathbb{Z} \longleftrightarrow x \in \mathbb{Z}$
using *Ints-cnj*[*of x*] *Ints-cnj*[*of cnj x*] **by** *auto*

lemma *complex-mult-cnj*: $z * cnj\ z = complex\text{-}of\text{-}real\ ((Re\ z)^2 + (Im\ z)^2)$
by (*simp add*: *complex-eq-iff power2-eq-square*)

lemma *cnj-add-mult-eq-Re*: $z * cnj\ w + cnj\ z * w = 2 * Re\ (z * cnj\ w)$
by (*rule complex-eqI*) *auto*

lemma *complex-mod-mult-cnj*: $cmod\ (z * cnj\ z) = (cmod\ z)^2$
by (*simp add*: *norm-mult power2-eq-square*)

lemma *complex-mod-sqrt-Re-mult-cnj*: $cmod\ z = sqrt\ (Re\ (z * cnj\ z))$
by (*simp add*: *norm-complex-def power2-eq-square*)

lemma *complex-In-mult-cnj-zero* [*simp*]: $Im\ (z * cnj\ z) = 0$
by *simp*

lemma *complex-cnj-fact* [*simp*]: *cnj* (*fact n*) = *fact n*
by (*subst of-nat-fact* [*symmetric*], *subst complex-cnj-of-nat*) *simp*

lemma *complex-cnj-pochhammer* [*simp*]: *cnj* (*pochhammer z n*) = *pochhammer* (*cnj z*) *n*
by (*induct n arbitrary*: *z*) (*simp-all add*: *pochhammer-rec*)

lemma *bounded-linear-cnj*: *bounded-linear cnj*
using *complex-cnj-add complex-cnj-scaleR* **by** (*rule bounded-linear-intro* [**where** *K=1*]) *simp*

lemma *linear-cnj*: *linear cnj*
using *bounded-linear.linear*[*OF bounded-linear-cnj*] **.**

lemmas *tendsto-cnj* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-cnj*]
and *isCont-cnj* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-cnj*]
and *continuous-cnj* [*simp*, *continuous-intros*] = *bounded-linear.continuous* [*OF bounded-linear-cnj*]
and *continuous-on-cnj* [*simp*, *continuous-intros*] = *bounded-linear.continuous-on* [*OF bounded-linear-cnj*]
and *has-derivative-cnj* [*simp*, *derivative-intros*] = *bounded-linear.has-derivative* [*OF bounded-linear-cnj*]

lemma *lim-cnj*: $((\lambda x.\ cnj(f\ x)) \longrightarrow cnj\ l)\ F \longleftrightarrow (f \longrightarrow l)\ F$
by (*simp add*: *tendsto-iff dist-complex-def complex-cnj-diff* [*symmetric*] *del*: *complex-cnj-diff*)

lemma *sums-cnj*: ((λ*x*. *cnj*(*f x*)) *sums cnj l*) ⟷ (*f sums l*)
  by (*simp add*: *sums-def lim-cnj cnj-sum* [*symmetric*] *del*: *cnj-sum*)

lemma *differentiable-cnj-iff*:
  (λ*z*. *cnj* (*f z*)) *differentiable at x within A* ⟷ *f differentiable at x within A*
proof
  assume (λ*z*. *cnj* (*f z*)) *differentiable at x within A*
  then obtain *D* **where** ((λ*z*. *cnj* (*f z*)) *has-derivative D*) (*at x within A*)
    by (*auto simp*: *differentiable-def*)
  from *has-derivative-cnj*[*OF this*] **show** *f differentiable at x within A*
    by (*auto simp*: *differentiable-def*)
next
  assume *f differentiable at x within A*
  then obtain *D* **where** (*f has-derivative D*) (*at x within A*)
    by (*auto simp*: *differentiable-def*)
  from *has-derivative-cnj*[*OF this*] **show** (λ*z*. *cnj* (*f z*)) *differentiable at x within*
A
    by (*auto simp*: *differentiable-def*)
qed

lemma *has-vector-derivative-cnj* [*derivative-intros*]:
  assumes (*f has-vector-derivative f'*) (*at z within A*)
  shows ((λ*z*. *cnj* (*f z*)) *has-vector-derivative cnj f'*) (*at z within A*)
  using *assms* **by** (*auto simp*: *has-vector-derivative-complex-iff intro*: *derivative-intros*)

lemma *has-field-derivative-cnj-cnj*:
  assumes (*f has-field-derivative F*) (*at* (*cnj z*))
  shows ((*cnj* ∘ *f* ∘ *cnj*) *has-field-derivative cnj F*) (*at z*)
proof −
  have *cnj* −*0*→ *cnj 0*
    by (*subst lim-cnj*) *auto*
  also have *cnj 0* = *0*
    by *simp*
  finally have ∗: *filterlim cnj* (*at 0*) (*at 0*)
    by (*auto simp*: *filterlim-at eventually-at-filter*)
  have (λ*h*. (*f* (*cnj z* + *cnj h*) − *f* (*cnj z*)) / *cnj h*) −*0*→ *F*
    by (*rule filterlim-compose*[*OF* - ∗]) (*use assms* **in** ‹*auto simp*: *DERIV-def*›)
  thus *?thesis*
    by (*subst* (*asm*) *lim-cnj* [*symmetric*]) (*simp add*: *DERIV-def*)
qed

113.10 Basic Lemmas

lemma *complex-of-real-code*[*code-unfold*]: *of-real* = (λ*x*. *Complex x 0*)
  by (*intro ext*, *auto simp*: *complex-eq-iff*)

lemma *complex-eq-0*: *z*=*0* ⟷ (*Re z*)^2 + (*Im z*)^2 = *0*
  by (*metis zero-complex.sel complex-eqI sum-power2-eq-zero-iff*)

lemma *complex-neq-0*: $z \neq 0 \longleftrightarrow (Re\ z)^2 + (Im\ z)^2 > 0$
by (*metis complex-eq-0 less-numeral-extra*(*3*) *sum-power2-gt-zero-iff*)

lemma *complex-norm-square*: *of-real* $((norm\ z)^2) = z * cnj\ z$
by (*cases z*)
(*auto simp*: *complex-eq-iff norm-complex-def power2-eq-square*[*symmetric*] *of-real-power*[*symmetric*]
simp del: *of-real-power*)

lemma *complex-div-cnj*: $a\ /\ b = (a * cnj\ b)\ /\ (norm\ b)^2$
using *complex-norm-square* **by** *auto*

lemma *Re-complex-div-eq-0*: $Re\ (a\ /\ b) = 0 \longleftrightarrow Re\ (a * cnj\ b) = 0$
by (*auto simp add*: *Re-divide*)

lemma *Im-complex-div-eq-0*: $Im\ (a\ /\ b) = 0 \longleftrightarrow Im\ (a * cnj\ b) = 0$
by (*auto simp add*: *Im-divide*)

lemma *complex-div-gt-0*: $(Re\ (a\ /\ b) > 0 \longleftrightarrow Re\ (a * cnj\ b) > 0) \wedge (Im\ (a\ /\ b)$
$> 0 \longleftrightarrow Im\ (a * cnj\ b) > 0)$
proof (*cases* $b = 0$)
case *True*
then show *?thesis* **by** *auto*
next
case *False*
then have $0 < (Re\ b)^2 + (Im\ b)^2$
by (*simp add*: *complex-eq-iff sum-power2-gt-zero-iff*)
then show *?thesis*
by (*simp add*: *Re-divide Im-divide zero-less-divide-iff*)
qed

lemma *Re-complex-div-gt-0*: $Re\ (a\ /\ b) > 0 \longleftrightarrow Re\ (a * cnj\ b) > 0$
and *Im-complex-div-gt-0*: $Im\ (a\ /\ b) > 0 \longleftrightarrow Im\ (a * cnj\ b) > 0$
using *complex-div-gt-0* **by** *auto*

lemma *Re-complex-div-ge-0*: $Re\ (a\ /\ b) \geq 0 \longleftrightarrow Re\ (a * cnj\ b) \geq 0$
by (*metis le-less Re-complex-div-eq-0 Re-complex-div-gt-0*)

lemma *Im-complex-div-ge-0*: $Im\ (a\ /\ b) \geq 0 \longleftrightarrow Im\ (a * cnj\ b) \geq 0$
by (*metis Im-complex-div-eq-0 Im-complex-div-gt-0 le-less*)

lemma *Re-complex-div-lt-0*: $Re\ (a\ /\ b) < 0 \longleftrightarrow Re\ (a * cnj\ b) < 0$
by (*metis less-asym neq-iff Re-complex-div-eq-0 Re-complex-div-gt-0*)

lemma *Im-complex-div-lt-0*: $Im\ (a\ /\ b) < 0 \longleftrightarrow Im\ (a * cnj\ b) < 0$
by (*metis Im-complex-div-eq-0 Im-complex-div-gt-0 less-asym neq-iff*)

lemma *Re-complex-div-le-0*: $Re\ (a\ /\ b) \leq 0 \longleftrightarrow Re\ (a * cnj\ b) \leq 0$
by (*metis not-le Re-complex-div-gt-0*)

lemma *Im-complex-div-le-0*: *Im* (*a* / *b*) ≤ *0* ⟷ *Im* (*a* ∗ *cnj b*) ≤ *0*
by (*metis Im-complex-div-gt-0 not-le*)

lemma *Re-divide-of-real* [*simp*]: *Re* (*z* / *of-real r*) = *Re z* / *r*
by (*simp add*: *Re-divide power2-eq-square*)

lemma *Im-divide-of-real* [*simp*]: *Im* (*z* / *of-real r*) = *Im z* / *r*
by (*simp add*: *Im-divide power2-eq-square*)

lemma *Re-divide-Reals* [*simp*]: *r* ∈ ℝ ⟹ *Re* (*z* / *r*) = *Re z* / *Re r*
by (*metis Re-divide-of-real of-real-Re*)

lemma *Im-divide-Reals* [*simp*]: *r* ∈ ℝ ⟹ *Im* (*z* / *r*) = *Im z* / *Re r*
by (*metis Im-divide-of-real of-real-Re*)

lemma *Re-sum*[*simp*]: *Re* (*sum f s*) = (∑ *x*∈*s*. *Re* (*f x*))
by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *Im-sum*[*simp*]: *Im* (*sum f s*) = (∑ *x*∈*s*. *Im*(*f x*))
by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *Rats-complex-of-real-iff* [*iff*]: *complex-of-real x* ∈ ℚ ⟷ *x* ∈ ℚ
proof −
have ⋀*a b*. ⟦*0* < *b*; *x* = *complex-of-int a* / *complex-of-int b*⟧ ⟹ *x* ∈ ℚ
by (*metis Rats-divide Rats-of-int Re-complex-of-real Re-divide-of-real of-real-of-int-eq*)
then show *?thesis*
by (*auto simp*: *elim*!: *Rats-cases′*)
qed

lemma *sum-Re-le-cmod*: (∑ *i*∈*I*. *Re* (*z i*)) ≤ *cmod* (∑ *i*∈*I*. *z i*)
by (*metis Re-sum complex-Re-le-cmod*)

lemma *sum-Im-le-cmod*: (∑ *i*∈*I*. *Im* (*z i*)) ≤ *cmod* (∑ *i*∈*I*. *z i*)
by (*smt* (*verit*, *best*) *Im-sum abs-Im-le-cmod sum.cong*)

lemma *sums-complex-iff*: *f sums x* ⟷ ((λ*x*. *Re* (*f x*)) *sums Re x*) ∧ ((λ*x*. *Im* (*f x*)) *sums Im x*)
unfolding *sums-def tendsto-complex-iff Im-sum Re-sum* **..**

lemma *summable-complex-iff*: *summable f* ⟷ *summable* (λ*x*. *Re* (*f x*)) ∧ *summable* (λ*x*. *Im* (*f x*))
unfolding *summable-def sums-complex-iff*[*abs-def*] **by** (*metis complex.sel*)

lemma *summable-complex-of-real* [*simp*]: *summable* (λ*n*. *complex-of-real* (*f n*)) ⟷ *summable f*
unfolding *summable-complex-iff* **by** *simp*

lemma *summable-Re*: *summable f* ⟹ *summable* (λ*x*. *Re* (*f x*))

unfolding *summable-complex-iff* **by** *blast*

lemma *summable-Im*: *summable f* $\Longrightarrow$ *summable* ($\lambda x.$ *Im* (*f x*))
unfolding *summable-complex-iff* **by** *blast*

lemma *complex-is-Nat-iff*: $z \in \mathbb{N} \longleftrightarrow Im\ z = 0 \wedge (\exists\, i.\ Re\ z = of\text{-}nat\ i)$
by (*auto simp*: *Nats-def complex-eq-iff*)

lemma *complex-is-Int-iff*: $z \in \mathbb{Z} \longleftrightarrow Im\ z = 0 \wedge (\exists\, i.\ Re\ z = of\text{-}int\ i)$
by (*auto simp*: *Ints-def complex-eq-iff*)

lemma *complex-is-Real-iff*: $z \in \mathbb{R} \longleftrightarrow Im\ z = 0$
by (*auto simp*: *Reals-def complex-eq-iff*)

lemma *Reals-cnj-iff*: $z \in \mathbb{R} \longleftrightarrow cnj\ z = z$
by (*auto simp*: *complex-is-Real-iff complex-eq-iff*)

lemma *in-Reals-norm*: $z \in \mathbb{R} \Longrightarrow norm\ z = |Re\ z|$
by (*simp add*: *complex-is-Real-iff norm-complex-def*)

lemma *Re-Reals-divide*: $r \in \mathbb{R} \Longrightarrow Re\ (r\ /\ z) = Re\ r * Re\ z\ /\ (norm\ z)^2$
by (*simp add*: *Re-divide complex-is-Real-iff cmod-power2*)

lemma *Im-Reals-divide*: $r \in \mathbb{R} \Longrightarrow Im\ (r\ /\ z) = -Re\ r * Im\ z\ /\ (norm\ z)^2$
by (*simp add*: *Im-divide complex-is-Real-iff cmod-power2*)

lemma *series-comparison-complex*:
fixes f:: *nat* $\Rightarrow$ *'a::banach*
assumes *sg*: *summable g*
and $\bigwedge n.\ g\ n \in \mathbb{R}\ \bigwedge n.\ Re\ (g\ n) \geq 0$
and *fg*: $\bigwedge n.\ n \geq N \Longrightarrow norm(f\ n) \leq norm(g\ n)$
shows *summable f*
proof −
have *g*: $\bigwedge n.\ cmod\ (g\ n) = Re\ (g\ n)$
using *assms* **by** (*metis abs-of-nonneg in-Reals-norm*)
show *?thesis*
by (*metis fg g sg summable-comparison-test summable-complex-iff*)
qed

113.11 Polar Form for Complex Numbers

lemma *complex-unimodular-polar*:
assumes *norm z = 1*
obtains *t* **where** $0 \leq t\ t < 2 * pi\ z = Complex\ (cos\ t)\ (sin\ t)$
by (*metis cmod-power2 one-power2 complex-surj sincos-total-2pi* [*of Re z Im z*] *assms*)

113.11.1 $\cos\theta + i\sin\theta$

primcorec *cis* :: *real* $\Rightarrow$ *complex*

where
Re (*cis a*) = *cos a*
| *Im* (*cis a*) = *sin a*

lemma *cis-zero* [*simp*]: *cis 0* = *1*
by (*simp add*: *complex-eq-iff*)

lemma *norm-cis* [*simp*]: *norm* (*cis a*) = *1*
by (*simp add*: *norm-complex-def*)

lemma *sgn-cis* [*simp*]: *sgn* (*cis a*) = *cis a*
by (*simp add*: *sgn-div-norm*)

lemma *cis-2pi* [*simp*]: *cis* (*2* ∗ *pi*) = *1*
by (*simp add*: *cis.ctr complex-eq-iff*)

lemma *cis-neq-zero* [*simp*]: *cis a* ≠ *0*
by (*metis norm-cis norm-zero zero-neq-one*)

lemma *cis-cnj*: *cnj* (*cis t*) = *cis* (−*t*)
by (*simp add*: *complex-eq-iff*)

lemma *cis-mult*: *cis a* ∗ *cis b* = *cis* (*a* + *b*)
by (*simp add*: *complex-eq-iff cos-add sin-add*)

lemma *DeMoivre*: (*cis a*) ^ *n* = *cis* (*real n* ∗ *a*)
by (*induct n*) (*simp-all add*: *algebra-simps cis-mult*)

lemma *cis-inverse* [*simp*]: *inverse* (*cis a*) = *cis* (− *a*)
by (*simp add*: *complex-eq-iff*)

lemma *cis-divide*: *cis a* / *cis b* = *cis* (*a* − *b*)
by (*simp add*: *divide-complex-def cis-mult*)

lemma *divide-conv-cnj*: *norm z* = *1* ⟹ *x* / *z* = *x* ∗ *cnj z*
by (*metis complex-div-cnj div-by-1 mult-1 of-real-1 power2-eq-square*)

lemma *i-not-in-Reals* [*simp*, *intro*]: i ∉ ℝ
by (*auto simp*: *complex-is-Real-iff*)

lemma *cos-n-Re-cis-pow-n*: *cos* (*real n* ∗ *a*) = *Re* (*cis a* ^ *n*)
by (*auto simp add*: *DeMoivre*)

lemma *sin-n-Im-cis-pow-n*: *sin* (*real n* ∗ *a*) = *Im* (*cis a* ^ *n*)
by (*auto simp add*: *DeMoivre*)

lemma *cis-pi* [*simp*]: *cis pi* = −*1*
by (*simp add*: *complex-eq-iff*)

lemma *cis-pi-half*[*simp*]: *cis* (*pi* / *2*) = i
by (*simp add*: *cis.ctr complex-eq-iff*)

lemma *cis-minus-pi-half*[*simp*]: *cis* (−(*pi* / *2*)) = −i
by (*simp add*: *cis.ctr complex-eq-iff*)

lemma *cis-multiple-2pi*[*simp*]: $n \in \mathbb{Z} \Longrightarrow$ *cis* (*2* ∗ *pi* ∗ *n*) = *1*
by (*auto elim*!: *Ints-cases simp*: *cis.ctr one-complex.ctr*)

lemma *minus-cis*: −*cis x* = *cis* (*x* + *pi*)
by (*simp flip*: *cis-mult*)

lemma *minus-cis'*: −*cis x* = *cis* (*x* − *pi*)
by (*simp flip*: *cis-divide*)

113.11.2 $r(\cos\theta + i\sin\theta)$

definition *rcis* :: *real* ⇒ *real* ⇒ *complex*
where *rcis r a* = *complex-of-real r* ∗ *cis a*

lemma *Re-rcis* [*simp*]: *Re*(*rcis r a*) = *r* ∗ *cos a*
by (*simp add*: *rcis-def*)

lemma *Im-rcis* [*simp*]: *Im*(*rcis r a*) = *r* ∗ *sin a*
by (*simp add*: *rcis-def*)

lemma *rcis-Ex*: ∃ *r a*. *z* = *rcis r a*
by (*simp add*: *complex-eq-iff polar-Ex*)

lemma *complex-mod-rcis* [*simp*]: *cmod* (*rcis r a*) = |*r*|
by (*simp add*: *rcis-def norm-mult*)

lemma *cis-rcis-eq*: *cis a* = *rcis 1 a*
by (*simp add*: *rcis-def*)

lemma *rcis-mult*: *rcis r1 a* ∗ *rcis r2 b* = *rcis* (*r1* ∗ *r2*) (*a* + *b*)
by (*simp add*: *rcis-def cis-mult*)

lemma *rcis-zero-mod* [*simp*]: *rcis 0 a* = *0*
by (*simp add*: *rcis-def*)

lemma *rcis-zero-arg* [*simp*]: *rcis r 0* = *complex-of-real r*
by (*simp add*: *rcis-def*)

lemma *rcis-eq-zero-iff* [*simp*]: *rcis r a* = *0* ⟷ *r* = *0*
by (*simp add*: *rcis-def*)

lemma *DeMoivre2*: (*rcis r a*) ^ *n* = *rcis* (*r* ^ *n*) (*real n* ∗ *a*)
by (*simp add*: *rcis-def power-mult-distrib DeMoivre*)

lemma *rcis-inverse*: *inverse*(*rcis r a*) = *rcis* (*1* / *r*) (− *a*)
  by (*simp add*: *divide-inverse rcis-def*)

lemma *rcis-divide*: *rcis r1 a* / *rcis r2 b* = *rcis* (*r1* / *r2*) (*a* − *b*)
  by (*simp add*: *rcis-def cis-divide* [*symmetric*])

113.11.3 Complex exponential

lemma *exp-Reals-eq*:
  assumes $z \in \mathbb{R}$
  shows *exp z* = *of-real* (*exp* (*Re z*))
  using *assms* **by** (*auto elim*!: *Reals-cases simp*: *exp-of-real*)

lemma *cis-conv-exp*: *cis b* = *exp* (i ∗ *b*)
proof −
  have (i ∗ *complex-of-real b*) ^ *n* $/_R$ *fact n* =
      of-real (*cos-coeff n* ∗ *b*^*n*) + i ∗ *of-real* (*sin-coeff n* ∗ *b*^*n*)
    for *n* :: *nat*
  proof −
    have i ^ *n* = *fact n* $*_R$ (*cos-coeff n* + i ∗ *sin-coeff n*)
      by (*induct n*)
        (*simp-all add*: *sin-coeff-Suc cos-coeff-Suc complex-eq-iff Re-divide Im-divide field-simps*
          power2-eq-square add-nonneg-eq-0-iff)
    then show *?thesis*
      by (*simp add*: *field-simps*)
  qed
  then show *?thesis*
    using *sin-converges* [*of b*] *cos-converges* [*of b*]
    by (*auto simp add*: *Complex-eq cis.ctr exp-def simp del*: *of-real-mult*
        intro!: *sums-unique sums-add sums-mult sums-of-real*)
qed

lemma *exp-eq-polar*: *exp z* = *exp* (*Re z*) ∗ *cis* (*Im z*)
  unfolding *cis-conv-exp exp-of-real* [*symmetric*] *mult-exp-exp*
  by (*cases z*) (*simp add*: *Complex-eq*)

lemma *Re-exp*: *Re* (*exp z*) = *exp* (*Re z*) ∗ *cos* (*Im z*)
  unfolding *exp-eq-polar* **by** *simp*

lemma *Im-exp*: *Im* (*exp z*) = *exp* (*Re z*) ∗ *sin* (*Im z*)
  unfolding *exp-eq-polar* **by** *simp*

lemma *norm-cos-sin* [*simp*]: *norm* (*Complex* (*cos t*) (*sin t*)) = *1*
  by (*simp add*: *norm-complex-def*)

lemma *norm-exp-eq-Re* [*simp*]: *norm* (*exp z*) = *exp* (*Re z*)
  by (*simp add*: *cis.code cmod-complex-polar exp-eq-polar Complex-eq*)

lemma *complex-exp-exists*: $\exists\, a\ r.\ z = \textit{complex-of-real}\ r * \textit{exp}\ a$
using *cis-conv-exp rcis-Ex rcis-def* **by** *force*

lemma *exp-pi-i* [*simp*]: *exp* (*of-real pi* ∗ i) = −*1*
by (*metis cis-conv-exp cis-pi mult.commute*)

lemma *exp-pi-i′* [*simp*]: *exp* (i ∗ *of-real pi*) = −*1*
using *cis-conv-exp cis-pi* **by** *auto*

lemma *exp-two-pi-i* [*simp*]: *exp* (*2* ∗ *of-real pi* ∗ i) = *1*
by (*simp add*: *exp-eq-polar complex-eq-iff*)

lemma *exp-two-pi-i′* [*simp*]: *exp* (i ∗ (*of-real pi* ∗ *2*)) = *1*
by (*metis exp-two-pi-i mult.commute*)

lemma *continuous-on-cis* [*continuous-intros*]:
continuous-on A f ⟹ *continuous-on A* (λ*x*. *cis* (*f x*))
by (*auto simp*: *cis-conv-exp intro*!: *continuous-intros*)

lemma *tendsto-exp-0-Re-at-bot*: (*exp* ⟶ *0*) (*filtercomap Re at-bot*)
proof −
have ((λ*z*. *cmod* (*exp z*)) ⟶ *0*) (*filtercomap Re at-bot*)
by (*auto intro*!: *filterlim-filtercomapI exp-at-bot*)
thus *?thesis*
using *tendsto-norm-zero-iff* **by** *blast*
qed

lemma *filterlim-exp-at-infinity-Re-at-top*: *filterlim exp at-infinity* (*filtercomap Re at-top*)
proof −
have *filterlim* (λ*z*. *norm* (*exp z*)) *at-top* (*filtercomap Re at-top*)
by (*auto intro*!: *filterlim-filtercomapI exp-at-top*)
thus *?thesis*
using *filterlim-norm-at-top-imp-at-infinity* **by** *blast*
qed

113.11.4 Complex argument

definition *Arg* :: *complex* ⇒ *real*
where *Arg z* = (*if z* = *0 then 0 else* (*SOME a*. *sgn z* = *cis a* ∧ − *pi* < *a* ∧ *a* ≤ *pi*))

lemma *Arg-zero*: *Arg 0* = *0*
by (*simp add*: *Arg-def*)

lemma *cis-Arg-unique*:
assumes *sgn z* = *cis x* **and** −*pi* < *x* **and** *x* ≤ *pi*
shows *Arg z* = *x*

proof −
  from *assms* **have** $z \neq 0$ **by** *auto*
  have $(SOME\ a.\ sgn\ z = cis\ a \wedge -pi < a \wedge a \leq pi) = x$
  proof
    fix *a*
    define *d* **where** $d = a - x$
    assume *a*: $sgn\ z = cis\ a \wedge -\ pi < a \wedge a \leq pi$
    from *a assms* **have** $-\ (2{*}pi) < d \wedge d < 2{*}pi$
      unfolding *d-def* **by** *simp*
    moreover
    from *a assms* **have** $cos\ a = cos\ x$ **and** $sin\ a = sin\ x$
      by (*simp-all add*: *complex-eq-iff*)
    then have *cos*: $cos\ d = 1$
      by (*simp add*: *d-def cos-diff*)
    moreover from *cos* **have** $sin\ d = 0$
      by (*rule cos-one-sin-zero*)
    ultimately have $d = 0$
      by (*auto simp*: *sin-zero-iff elim*!: *evenE dest*!: *less-2-cases*)
    then show $a = x$
      by (*simp add*: *d-def*)
  qed (*simp add*: *assms del*: *Re-sgn Im-sgn*)
  with ‹$z \neq 0$› **show** $Arg\ z = x$
    by (*simp add*: *Arg-def*)
qed

lemma *Arg-correct*:
  assumes $z \neq 0$
  shows $sgn\ z = cis\ (Arg\ z) \wedge -pi < Arg\ z \wedge Arg\ z \leq pi$
proof (*simp add*: *Arg-def assms*, *rule someI-ex*)
  obtain *r a* **where** *z*: $z = rcis\ r\ a$
    using *rcis-Ex* **by** *fast*
  with *assms* **have** $r \neq 0$ **by** *auto*
  define *b* **where** $b = (if\ 0 < r\ then\ a\ else\ a + pi)$
  have *b*: $sgn\ z = cis\ b$
    using ‹$r \neq 0$› **by** (*simp add*: *z b-def rcis-def of-real-def sgn-scaleR sgn-if complex-eq-iff*)
  have *cis-2pi-nat*: $cis\ (2 * pi * real\text{-}of\text{-}nat\ n) = 1$ **for** *n*
    by (*induct n*) (*simp-all add*: *distrib-left cis-mult* [*symmetric*] *complex-eq-iff*)
  have *cis-2pi-int*: $cis\ (2 * pi * real\text{-}of\text{-}int\ x) = 1$ **for** *x*
    by (*cases x rule*: *int-diff-cases*)
      (*simp add*: *right-diff-distrib cis-divide* [*symmetric*] *cis-2pi-nat*)
  define *c* **where** $c = b - 2 * pi * of\text{-}int\ \lceil (b - pi) / (2 * pi) \rceil$
  have $sgn\ z = cis\ c$
    by (*simp add*: *b c-def cis-divide* [*symmetric*] *cis-2pi-int*)
  moreover have $-\ pi < c \wedge c \leq pi$
    using *ceiling-correct* [*of* $(b - pi) / (2{*}pi)$]
   by (*simp add*: *c-def less-divide-eq divide-le-eq algebra-simps del*: *le-of-int-ceiling*)
  ultimately show $\exists a.\ sgn\ z = cis\ a \wedge -pi < a \wedge a \leq pi$
    by *fast*

qed

lemma *Arg-bounded*: $-\ pi < Arg\ z \wedge Arg\ z \leq pi$
by (*cases* $z = 0$) (*simp-all add*: *Arg-zero Arg-correct*)

lemma *cis-Arg*: $z \neq 0 \Longrightarrow cis\ (Arg\ z) = sgn\ z$
by (*simp add*: *Arg-correct*)

lemma *rcis-cmod-Arg*: $rcis\ (cmod\ z)\ (Arg\ z) = z$
by (*cases* $z = 0$) (*simp-all add*: *rcis-def cis-Arg sgn-div-norm of-real-def*)

lemma *rcis-cnj*:
shows $cnj\ a = rcis\ (cmod\ a)\ (-\ Arg\ a)$
by (*metis cis-cnj complex-cnj-complex-of-real complex-cnj-mult rcis-cmod-Arg rcis-def*)

lemma *cos-Arg-i-mult-zero* [*simp*]: $y \neq 0 \Longrightarrow Re\ y = 0 \Longrightarrow cos\ (Arg\ y) = 0$
using *cis-Arg* [*of y*] **by** (*simp add*: *complex-eq-iff*)

lemma *Arg-ii* [*simp*]: $Arg\ \mathrm{i} = pi/2$
by (*rule cis-Arg-unique*; *simp add*: *sgn-eq*)

lemma *Arg-minus-ii* [*simp*]: $Arg\ (-\mathrm{i}) = -pi/2$
proof (*rule cis-Arg-unique*)
show $sgn\ (-\ \mathrm{i}) = cis\ (-\ pi\ /\ 2)$
by (*simp add*: *sgn-eq*)
show $-\ pi\ /\ 2 \leq pi$
using *pi-not-less-zero* **by** *linarith*
qed *auto*

lemma *cos-Arg*: $z \neq 0 \Longrightarrow cos\ (Arg\ z) = Re\ z\ /\ norm\ z$
by (*metis Re-sgn cis.sel*(*1*) *cis-Arg*)

lemma *sin-Arg*: $z \neq 0 \Longrightarrow sin\ (Arg\ z) = Im\ z\ /\ norm\ z$
by (*metis Im-sgn cis.sel*(*2*) *cis-Arg*)

113.12 Complex n-th roots

lemma *bij-betw-roots-unity*:
assumes $n > 0$
shows *bij-betw* $(\lambda k.\ cis\ (2 * pi * real\ k\ /\ real\ n))\ \{..<n\}\ \{z.\ z\ \hat{}\ n = 1\}$
(**is** *bij-betw ?f - -*)
unfolding *bij-betw-def*
proof (*intro conjI*)
show *inj*: *inj-on ?f* $\{..<n\}$ **unfolding** *inj-on-def*
proof (*safe*, *goal-cases*)
case (*1 k l*)
hence *kl*: $k < n\ l < n$ **by** *simp-all*
from *1* **have** $1 = ?f\ k\ /\ ?f\ l$ **by** *simp*

also have ... = *cis* (*2*∗*pi*∗(*real k* − *real l*)/*n*)
using *assms* **by** (*simp add*: *field-simps cis-divide*)
finally have *cos* (*2*∗*pi*∗(*real k* − *real l*) / *n*) = *1*
by (*simp add*: *complex-eq-iff*)
then obtain *m* :: *int* **where** *2* ∗ *pi* ∗ (*real k* − *real l*) / *real n* = *real-of-int m* ∗ *2* ∗ *pi*
by (*subst* (*asm*) *cos-one-2pi-int*) *blast*
hence *real-of-int* (*int k* − *int l*) = *real-of-int* (*m* ∗ *int n*)
unfolding *of-int-diff of-int-mult* **using** *assms*
by (*simp add*: *nonzero-divide-eq-eq*)
also note *of-int-eq-iff*
finally have ∗: *abs m* ∗ *n* = *abs* (*int k* − *int l*) **by** (*simp add*: *abs-mult*)
also have ... < *int n* **using** *kl* **by** *linarith*
finally have *m* = *0* **using** *assms* **by** *simp*
with ∗ **show** *k* = *l* **by** *simp*
qed

have *subset*: *?f* ' {..<*n*} ⊆ {*z*. *z* ^ *n* = *1*}
proof *safe*
fix *k* :: *nat*
have *cis* (*2* ∗ *pi* ∗ *real k* / *real n*) ^ *n* = *cis* (*2* ∗ *pi*) ^ *k*
using *assms* **by** (*simp add*: *DeMoivre mult-ac*)
also have *cis* (*2* ∗ *pi*) = *1* **by** (*simp add*: *complex-eq-iff*)
finally show *?f k* ^ *n* = *1* **by** *simp*
qed

have *n* = *card* {..<*n*} **by** *simp*
also from *assms* **and** *subset* **have** ... ≤ *card* {*z*::*complex*. *z* ^ *n* = *1*}
by (*intro card-inj-on-le*[*OF inj*]) (*auto simp*: *finite-roots-unity*)
finally have *card*: *card* {*z*::*complex*. *z* ^ *n* = *1*} = *n*
using *assms* **by** (*intro antisym card-roots-unity*) *auto*

have *card* (*?f* ' {..<*n*}) = *card* {*z*::*complex*. *z* ^ *n* = *1*}
using *card inj* **by** (*subst card-image*) *auto*
with *subset* **and** *assms* **show** *?f* ' {..<*n*} = {*z*::*complex*. *z* ^ *n* = *1*}
by (*intro card-subset-eq finite-roots-unity*) *auto*
qed

lemma *card-roots-unity-eq*:
assumes *n* > *0*
shows *card* {*z*::*complex*. *z* ^ *n* = *1*} = *n*
using *bij-betw-same-card* [*OF bij-betw-roots-unity* [*OF assms*]] **by** *simp*

lemma *bij-betw-nth-root-unity*:
fixes *c* :: *complex* **and** *n* :: *nat*
assumes *c*: *c* ≠ *0* **and** *n*: *n* > *0*
defines *c'* ≡ *root n* (*norm c*) ∗ *cis* (*Arg c* / *n*)
shows *bij-betw* (λ*z*. *c'* ∗ *z*) {*z*. *z* ^ *n* = *1*} {*z*. *z* ^ *n* = *c*}
proof −

have *c′* ^ *n* = *of-real* (*root n* (*norm c*) ^ *n*) ∗ *cis* (*Arg c*)
unfolding *of-real-power* **using** *n* **by** (*simp add*: *c′-def power-mult-distrib DeMoivre*)
also from *n* **have** *root n* (*norm c*) ^ *n* = *norm c* **by** *simp*
also from *c* **have** *of-real* . . . ∗ *cis* (*Arg c*) = *c* **by** (*simp add*: *cis-Arg Complex.sgn-eq*)
finally have [*simp*]: *c′* ^ *n* = *c* **.**

show *?thesis* **unfolding** *bij-betw-def inj-on-def*
proof *safe*
fix *z* :: *complex* **assume** *z* ^ *n* = *1*
hence (*c′* ∗ *z*) ^ *n* = *c′* ^ *n* **by** (*simp add*: *power-mult-distrib*)
also have *c′* ^ *n* = *of-real* (*root n* (*norm c*) ^ *n*) ∗ *cis* (*Arg c*)
unfolding *of-real-power* **using** *n* **by** (*simp add*: *c′-def power-mult-distrib DeMoivre*)
also from *n* **have** *root n* (*norm c*) ^ *n* = *norm c* **by** *simp*
also from *c* **have** . . . ∗ *cis* (*Arg c*) = *c* **by** (*simp add*: *cis-Arg Complex.sgn-eq*)
finally show (*c′* ∗ *z*) ^ *n* = *c* **.**
next
fix *z* **assume** *z*: *c* = *z* ^ *n*
define *z′* **where** *z′* = *z* / *c′*
from *c* **and** *n* **have** *c′* ≠ *0* **by** (*auto simp*: *c′-def*)
with *n c* **have** *z* = *c′* ∗ *z′* **and** *z′* ^ *n* = *1*
by (*auto simp*: *z′-def power-divide z*)
thus *z* ∈ (λ*z*. *c′* ∗ *z*) ‘ {*z*. *z* ^ *n* = *1*} **by** *blast*
qed (*insert c n*, *auto simp*: *c′-def*)
qed

lemma *finite-nth-roots* [*intro*]:
assumes *n* > *0*
shows *finite* {*z*::*complex*. *z* ^ *n* = *c*}
proof (*cases c* = *0*)
case *True*
with *assms* **have** {*z*::*complex*. *z* ^ *n* = *c*} = {*0*} **by** *auto*
thus *?thesis* **by** *simp*
next
case *False*
from *assms* **have** *finite* {*z*::*complex*. *z* ^ *n* = *1*} **by** (*intro finite-roots-unity*) *simp-all*
also have *?this* ⟷ *?thesis*
by (*rule bij-betw-finite*, *rule bij-betw-nth-root-unity*) *fact+*
finally show *?thesis* **.**
qed

lemma *card-nth-roots*:
assumes *c* ≠ *0* *n* > *0*
shows *card* {*z*::*complex*. *z* ^ *n* = *c*} = *n*
proof −
have *card* {*z*. *z* ^ *n* = *c*} = *card* {*z*::*complex*. *z* ^ *n* = *1*}

by (*rule sym*, *rule bij-betw-same-card*, *rule bij-betw-nth-root-unity*) *fact+*
also have ... = *n* **by** (*rule card-roots-unity-eq*) *fact+*
finally show *?thesis* **.**
qed

lemma *sum-roots-unity*:
assumes $n > 1$
shows $\sum\{z::complex.\ z \mathbin{\hat{}} n = 1\} = 0$
proof −
define ω **where** $\omega = cis\ (2 * pi\ /\ real\ n)$
have [*simp*]: $\omega \neq 1$
proof
assume $\omega = 1$
with *assms* **obtain** $k :: int$ **where** $2 * pi\ /\ real\ n = 2 * pi * of\text{-}int\ k$
by (*auto simp*: *ω-def complex-eq-iff cos-one-2pi-int*)
with *assms* **have** $real\ n * of\text{-}int\ k = 1$ **by** (*simp add*: *field-simps*)
also have $real\ n * of\text{-}int\ k = of\text{-}int\ (int\ n * k)$ **by** *simp*
also have $1 = (of\text{-}int\ 1 :: real)$ **by** *simp*
also note *of-int-eq-iff*
finally show *False* **using** *assms* **by** (*auto simp*: *zmult-eq-1-iff*)
qed

have $(\sum z \mid z \mathbin{\hat{}} n = 1.\ z :: complex) = (\sum k<n.\ cis\ (2 * pi * real\ k\ /\ real\ n))$
using *assms* **by** (*intro sum.reindex-bij-betw* [*symmetric*] *bij-betw-roots-unity*) *auto*
also have ... $= (\sum k<n.\ \omega \mathbin{\hat{}} k)$
by (*intro sum.cong refl*) (*auto simp*: *ω-def DeMoivre mult-ac*)
also have ... $= (\omega \mathbin{\hat{}} n - 1)\ /\ (\omega - 1)$
by (*subst geometric-sum*) *auto*
also have $\omega \mathbin{\hat{}} n - 1 = cis\ (2 * pi) - 1$ **using** *assms* **by** (*auto simp*: *ω-def DeMoivre*)
also have ... = *0* **by** (*simp add*: *complex-eq-iff*)
finally show *?thesis* **by** *simp*
qed

lemma *sum-nth-roots*:
assumes $n > 1$
shows $\sum\{z::complex.\ z \mathbin{\hat{}} n = c\} = 0$
proof (*cases* $c = 0$)
case *True*
with *assms* **have** $\{z::complex.\ z \mathbin{\hat{}} n = c\} = \{0\}$ **by** *auto*
also have $\sum ... = 0$ **by** *simp*
finally show *?thesis* **.**
next
case *False*
define c' **where** $c' = root\ n\ (norm\ c) * cis\ (Arg\ c\ /\ n)$
from *False* **and** *assms* **have** $(\sum\{z.\ z \mathbin{\hat{}} n = c\}) = (\sum z \mid z \mathbin{\hat{}} n = 1.\ c' * z)$
by (*subst sum.reindex-bij-betw* [*OF bij-betw-nth-root-unity*, *symmetric*])
(*auto simp*: *sum-distrib-left finite-roots-unity c′-def*)

also from *assms* **have** ... = *0*
by (*simp add*: *sum-distrib-left* [*symmetric*] *sum-roots-unity*)
finally show *?thesis* **.**
qed

113.13 Square root of complex numbers

primcorec *csqrt* :: *complex* ⇒ *complex*
where
Re (*csqrt z*) = *sqrt* ((*cmod z* + *Re z*) / *2*)
| *Im* (*csqrt z*) = (*if Im z* = *0 then 1 else sgn* (*Im z*)) ∗ *sqrt* ((*cmod z* − *Re z*) / *2*)

lemma *csqrt-of-real-nonneg* [*simp*]: *Im x* = *0* ⟹ *Re x* ≥ *0* ⟹ *csqrt x* = *sqrt* (*Re x*)
by (*simp add*: *complex-eq-iff norm-complex-def*)

lemma *csqrt-of-real-nonpos* [*simp*]: *Im x* = *0* ⟹ *Re x* ≤ *0* ⟹ *csqrt x* = i ∗ *sqrt* |*Re x*|
by (*simp add*: *complex-eq-iff norm-complex-def*)

lemma *of-real-sqrt*: *x* ≥ *0* ⟹ *of-real* (*sqrt x*) = *csqrt* (*of-real x*)
by (*simp add*: *complex-eq-iff norm-complex-def*)

lemma *csqrt-0* [*simp*]: *csqrt 0* = *0*
by *simp*

lemma *csqrt-1* [*simp*]: *csqrt 1* = *1*
by *simp*

lemma *csqrt-ii* [*simp*]: *csqrt* i = (*1* + i) / *sqrt 2*
by (*simp add*: *complex-eq-iff Re-divide Im-divide real-sqrt-divide real-div-sqrt*)

lemma *power2-csqrt*[*simp*,*algebra*]: $(csqrt\ z)^2 = z$
proof (*cases Im z* = *0*)
case *True*
then show *?thesis*
using *real-sqrt-pow2*[*of Re z*] *real-sqrt-pow2*[*of* − *Re z*]
by (*cases 0*::*real Re z rule*: *linorder-cases*)
(*simp-all add*: *complex-eq-iff Re-power2 Im-power2 power2-eq-square cmod-eq-Re*)
next
case *False*
moreover have *cmod z* ∗ *cmod z* − *Re z* ∗ *Re z* = *Im z* ∗ *Im z*
by (*simp add*: *norm-complex-def power2-eq-square*)
moreover have |*Re z*| ≤ *cmod z*
by (*simp add*: *norm-complex-def*)
ultimately show *?thesis*
by (*simp add*: *Re-power2 Im-power2 complex-eq-iff real-sgn-eq*
field-simps real-sqrt-mult[*symmetric*] *real-sqrt-divide*)

qed

lemma *csqrt-power-even*:
assumes *even n*
shows *csqrt z ^ n = z ^ (n div 2)*
by (*metis assms dvd-mult-div-cancel power2-csqrt power-mult*)

lemma *norm-csqrt* [*simp*]: *norm* (*csqrt z*) = *sqrt* (*norm z*)
by (*metis abs-of-nonneg norm-ge-zero norm-mult power2-csqrt power2-eq-square real-sqrt-abs*)

lemma *csqrt-eq-0* [*simp*]: *csqrt z* = *0* ⟷ *z* = *0*
by *auto* (*metis power2-csqrt power-eq-0-iff*)

lemma *csqrt-eq-1* [*simp*]: *csqrt z* = *1* ⟷ *z* = *1*
by *auto* (*metis power2-csqrt power2-eq-1-iff*)

lemma *csqrt-principal*: *0* < *Re* (*csqrt z*) ∨ *Re* (*csqrt z*) = *0* ∧ *0* ≤ *Im* (*csqrt z*)
by (*auto simp add*: *not-less cmod-plus-Re-le-0-iff Im-eq-0*)

lemma *Re-csqrt*: *0* ≤ *Re* (*csqrt z*)
by (*metis csqrt-principal le-less*)

lemma *csqrt-square*:
assumes *0* < *Re b* ∨ (*Re b* = *0* ∧ *0* ≤ *Im b*)
shows *csqrt* (*b^2*) = *b*
proof −
have *csqrt* (*b^2*) = *b* ∨ *csqrt* (*b^2*) = − *b*
by (*simp add*: *power2-eq-iff*[*symmetric*])
moreover have *csqrt* (*b^2*) ≠ −*b* ∨ *b* = *0*
using *csqrt-principal*[*of b ^ 2*] *assms*
by (*intro disjCI notI*) (*auto simp*: *complex-eq-iff*)
ultimately show *?thesis*
by *auto*
qed

lemma *csqrt-unique*: w^2 = *z* ⟹ *0* < *Re w* ∨ *Re w* = *0* ∧ *0* ≤ *Im w* ⟹ *csqrt z* = *w*
by (*auto simp*: *csqrt-square*)

lemma *csqrt-minus* [*simp*]:
assumes *Im x* < *0* ∨ (*Im x* = *0* ∧ *0* ≤ *Re x*)
shows *csqrt* (− *x*) = i ∗ *csqrt x*
proof −
have *csqrt* ((i ∗ *csqrt x*)^*2*) = i ∗ *csqrt x*
proof (*rule csqrt-square*)
have *Im* (*csqrt x*) ≤ *0*
using *assms* **by** (*auto simp add*: *cmod-eq-Re mult-le-0-iff field-simps complex-Re-le-cmod*)

then show $0 < Re$ (i $*$ *csqrt x*) $\vee$ *Re* (i $*$ *csqrt x*) $= 0 \wedge 0 \leq$ *Im* (i $*$ *csqrt x*)
by (*auto simp add*: *Re-csqrt simp del*: *csqrt.simps*)
qed
also have (i $*$ *csqrt x*)^*2* $= -$ *x*
by (*simp add*: *power-mult-distrib*)
finally show *?thesis* **.**
qed

Legacy theorem names

lemmas *cmod-def* = *norm-complex-def*

lemma *legacy-Complex-simps*:
shows *Complex-eq-0*: *Complex a b* = *0* $\longleftrightarrow$ *a* = *0* $\wedge$ *b* = *0*
and *complex-add*: *Complex a b* + *Complex c d* = *Complex* (*a* + *c*) (*b* + *d*)
and *complex-minus*: $-$ (*Complex a b*) = *Complex* ($-$ *a*) ($-$ *b*)
and *complex-diff*: *Complex a b* $-$ *Complex c d* = *Complex* (*a* $-$ *c*) (*b* $-$ *d*)
and *Complex-eq-1*: *Complex a b* = *1* $\longleftrightarrow$ *a* = *1* $\wedge$ *b* = *0*
and *Complex-eq-neg-1*: *Complex a b* = $-$ *1* $\longleftrightarrow$ *a* = $-$ *1* $\wedge$ *b* = *0*
and *complex-mult*: *Complex a b* $*$ *Complex c d* = *Complex* (*a* $*$ *c* $-$ *b* $*$ *d*) (*a* $*$ *d* + *b* $*$ *c*)
and *complex-inverse*: *inverse* (*Complex a b*) = *Complex* (*a* / (a^2 + b^2)) ($-$ *b* / (a^2 + b^2))
and *Complex-eq-numeral*: *Complex a b* = *numeral w* $\longleftrightarrow$ *a* = *numeral w* $\wedge$ *b* = *0*
and *Complex-eq-neg-numeral*: *Complex a b* = $-$ *numeral w* $\longleftrightarrow$ *a* = $-$ *numeral w* $\wedge$ *b* = *0*
and *complex-scaleR*: *scaleR r* (*Complex a b*) = *Complex* (*r* $*$ *a*) (*r* $*$ *b*)
and *Complex-eq-i*: *Complex x y* = i $\longleftrightarrow$ *x* = *0* $\wedge$ *y* = *1*
and *i-mult-Complex*: i $*$ *Complex a b* = *Complex* ($-$ *b*) *a*
and *Complex-mult-i*: *Complex a b* $*$ i = *Complex* ($-$ *b*) *a*
and *i-complex-of-real*: i $*$ *complex-of-real r* = *Complex 0 r*
and *complex-of-real-i*: *complex-of-real r* $*$ i = *Complex 0 r*
and *Complex-add-complex-of-real*: *Complex x y* + *complex-of-real r* = *Complex* (*x*+*r*) *y*
and *complex-of-real-add-Complex*: *complex-of-real r* + *Complex x y* = *Complex* (*r*+*x*) *y*
and *Complex-mult-complex-of-real*: *Complex x y* $*$ *complex-of-real r* = *Complex* (*x*$*$*r*) (*y*$*$*r*)
and *complex-of-real-mult-Complex*: *complex-of-real r* $*$ *Complex x y* = *Complex* (*r*$*$*x*) (*r*$*$*y*)
and *complex-eq-cancel-iff2*: (*Complex x y* = *complex-of-real xa*) = (*x* = *xa* $\wedge$ *y* = *0*)
and *complex-cnj*: *cnj* (*Complex a b*) = *Complex a* ($-$ *b*)
and *Complex-sum'*: *sum* (λ*x*. *Complex* (*f x*) *0*) *s* = *Complex* (*sum f s*) *0*
and *Complex-sum*: *Complex* (*sum f s*) *0* = *sum* (λ*x*. *Complex* (*f x*) *0*) *s*
and *complex-of-real-def*: *complex-of-real r* = *Complex r 0*
and *complex-norm*: *cmod* (*Complex x y*) = *sqrt* (x^2 + y^2)
by (*simp-all add*: *norm-complex-def field-simps complex-eq-iff Re-divide Im-divide*)

```
lemma Complex-in-Reals: Complex x 0 ∈ ℝ
  by (metis Reals-of-real complex-of-real-def)
```

Express a complex number as a linear combination of two others, not collinear with the origin

```
lemma complex-axes:
  assumes Im (y/x) ≠ 0
  obtains a b where z = of-real a ∗ x + of-real b ∗ y
proof −
  define dd where dd ≡ Re y ∗ Im x −  Im y ∗ Re x
  define a where a = (Im z ∗ Re y − Re z ∗ Im y) / dd
  define b where b = (Re z ∗ Im x − Im z ∗ Re x) / dd
  have dd ≠ 0
    using assms by (auto simp: dd-def Im-complex-div-eq-0)
  have a ∗ Re x + b ∗ Re y = Re z
    using ‹dd ≠ 0›
    apply (simp add: a-def b-def field-simps)
    by (metis dd-def diff-add-cancel distrib-right mult.assoc mult.commute)
  moreover have a ∗ Im x + b ∗ Im y = Im z
    using ‹dd ≠ 0›
    apply (simp add: a-def b-def field-simps)
   by (metis (no-types) dd-def diff-add-cancel distrib-right mult.assoc mult.commute)
  ultimately have z = of-real a ∗ x + of-real b ∗ y
    by (simp add: complex-eqI)
  then show ?thesis using that by simp
qed

end
```

114 MacLaurin and Taylor Series

```
theory MacLaurin
imports Transcendental
begin
```

114.1 Maclaurin's Theorem with Lagrange Form of Remainder

This is a very long, messy proof even now that it's been broken down into lemmas.

```
lemma Maclaurin-lemma:
  0 < h ⟹
    ∃ B::real. f h = (∑ m<n. (j m / (fact m)) ∗ (h^m)) + (B ∗ ((h^n) /(fact n)))
  by (rule exI[where x = (f h − (∑ m<n. (j m / (fact m)) ∗ h^m)) ∗ (fact n) /
(h^n)]) simp

lemma eq-diff-eq': x = y − z ⟷ y = x + z
  for x y z :: real
```

by *arith*

lemma *fact-diff-Suc*: $n < Suc\ m \Longrightarrow fact\ (Suc\ m - n) = (Suc\ m - n) * fact\ (m - n)$
by (*subst fact-reduce*) *auto*

lemma *Maclaurin-lemma2*:
fixes *B*
assumes *DERIV*: $\forall m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
and *INIT*: $n = Suc\ k$
defines $difg \equiv$
$(\lambda m\ t{::}real.\ diff\ m\ t -$
$((\sum p<n - m.\ diff\ (m + p)\ 0\ /\ fact\ p * t\ \hat{}\ p) + B * (t\ \hat{}\ (n - m)\ /\ fact\ (n - m))))$
(**is** $difg \equiv (\lambda m\ t.\ diff\ m\ t - ?difg\ m\ t)$)
shows $\forall m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow DERIV\ (difg\ m)\ t :> difg\ (Suc\ m)\ t$
proof (*rule allI impI*)+
fix *m t*
assume *INIT2*: $m < n \wedge 0 \leq t \wedge t \leq h$
have $DERIV\ (difg\ m)\ t :> diff\ (Suc\ m)\ t -$
$((\sum x<n - m.\ real\ x * t\ \hat{}\ (x - Suc\ 0) * diff\ (m + x)\ 0\ /\ fact\ x) +$
$real\ (n - m) * t\ \hat{}\ (n - Suc\ m) * B\ /\ fact\ (n - m))$
by (*auto simp*: *difg-def intro*!: *derivative-eq-intros DERIV*[*rule-format*, *OF INIT2*])
moreover
from *INIT2* **have** *intvl*: $\{..<n - m\} = insert\ 0\ (Suc\ `\ \{..<n - Suc\ m\})$ **and** $0 < n - m$
unfolding *atLeast0LessThan*[*symmetric*] **by** *auto*
have $(\sum x<n - m.\ real\ x * t\ \hat{}\ (x - Suc\ 0) * diff\ (m + x)\ 0\ /\ fact\ x) =$
$(\sum x<n - Suc\ m.\ real\ (Suc\ x) * t\ \hat{}\ x * diff\ (Suc\ m + x)\ 0\ /\ fact\ (Suc\ x))$
unfolding *intvl* **by** (*subst sum.insert*) (*auto simp*: *sum.reindex*)
moreover
have *fact-neq-0*: $\bigwedge x.\ (fact\ x) + real\ x * (fact\ x) \neq 0$
by (*metis add-pos-pos fact-gt-zero less-add-same-cancel1 less-add-same-cancel2 less-numeral-extra*(*3*) *mult-less-0-iff of-nat-less-0-iff*)
have $\bigwedge x.\ (Suc\ x) * t\ \hat{}\ x * diff\ (Suc\ m + x)\ 0\ /\ fact\ (Suc\ x) = diff\ (Suc\ m + x)\ 0 * t\hat{}x\ /\ fact\ x$
by (*rule nonzero-divide-eq-eq*[*THEN iffD2*]) *auto*
moreover
have $(n - m) * t\ \hat{}\ (n - Suc\ m) * B\ /\ fact\ (n - m) = B * (t\ \hat{}\ (n - Suc\ m)\ /\ fact\ (n - Suc\ m))$
using ‹$0 < n - m$› **by** (*simp add*: *field-split-simps fact-reduce*)
ultimately show $DERIV\ (difg\ m)\ t :> difg\ (Suc\ m)\ t$
unfolding *difg-def* **by** (*simp add*: *mult.commute*)
qed

lemma *Maclaurin*:
assumes *h*: $0 < h$

and *n*: $0 < n$
and *diff-0*: $diff\ 0 = f$
and *diff-Suc*: $\forall m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
shows
$\exists t{::}real.\ 0 < t \wedge t < h \wedge$
$f\ h = sum\ (\lambda m.\ (diff\ m\ 0\ /\ fact\ m) * h\ \hat{}\ m)\ \{..{<}n\} + (diff\ n\ t\ /\ fact\ n) * h\ \hat{}\ n$
proof −
from *n* **obtain** *m* **where** *m*: $n = Suc\ m$
by (*cases n*) (*simp add*: *n*)
from *m* **have** $m < n$ **by** *simp*

obtain *B* **where** *f-h*: $f\ h = (\sum m{<}n.\ diff\ m\ 0\ /\ fact\ m * h\ \hat{}\ m) + B * (h\ \hat{}\ n\ /\ fact\ n)$
using *Maclaurin-lemma* [*OF h*] **..**

define *g* **where** [*abs-def*]: $g\ t =$
$f\ t - (sum\ (\lambda m.\ (diff\ m\ 0\ /\ fact\ m) * t\hat{}m)\ \{..{<}n\} + B * (t\hat{}n\ /\ fact\ n))$ **for** *t*
have *g2*: $g\ 0 = 0\ g\ h = 0$
by (*simp-all add*: *m f-h g-def lessThan-Suc-eq-insert-0 image-iff diff-0 sum.reindex*)

define *difg* **where** [*abs-def*]: $difg\ m\ t =$
$diff\ m\ t - (sum\ (\lambda p.\ (diff\ (m + p)\ 0\ /\ fact\ p) * (t\ \hat{}\ p))\ \{..{<}n{-}m\} +$
$B * ((t\ \hat{}\ (n - m))\ /\ fact\ (n - m)))$ **for** *m t*
have *difg-0*: $difg\ 0 = g$
by (*simp add*: *difg-def g-def diff-0*)
have *difg-Suc*: $\forall m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow DERIV\ (difg\ m)\ t :> difg\ (Suc\ m)\ t$
using *diff-Suc m* **unfolding** *difg-def* [*abs-def*] **by** (*rule Maclaurin-lemma2*)
have *difg-eq-0*: $\forall m{<}n.\ difg\ m\ 0 = 0$
by (*auto simp*: *difg-def m Suc-diff-le lessThan-Suc-eq-insert-0 image-iff sum.reindex*)
have *isCont-difg*: $\bigwedge m\ x.\ m < n \Longrightarrow 0 \leq x \Longrightarrow x \leq h \Longrightarrow isCont\ (difg\ m)\ x$
by (*rule DERIV-isCont* [*OF difg-Suc* [*rule-format*]]) *simp*
have *differentiable-difg*: $\bigwedge m\ x.\ m < n \Longrightarrow 0 \leq x \Longrightarrow x \leq h \Longrightarrow$ *difg m differentiable* (*at x*)
using *difg-Suc real-differentiable-def* **by** *auto*
have *difg-Suc-eq-0*:
$\bigwedge m\ t.\ m < n \Longrightarrow 0 \leq t \Longrightarrow t \leq h \Longrightarrow DERIV\ (difg\ m)\ t :> 0 \Longrightarrow difg\ (Suc\ m)\ t = 0$
by (*rule DERIV-unique* [*OF difg-Suc* [*rule-format*]]) *simp*

have $\exists t.\ 0 < t \wedge t < h \wedge DERIV\ (difg\ m)\ t :> 0$
using ‹$m < n$›
proof (*induct m*)
case *0*
show *?case*
proof (*rule Rolle*)
show $0 < h$ **by** *fact*

```
      show difg 0 0 = difg 0 h
        by (simp add: difg-0 g2)
      show continuous-on {0..h} (difg 0)
        by (simp add: continuous-at-imp-continuous-on isCont-difg n)
    qed (simp add: differentiable-difg n)
  next
    case (Suc m′)
    then obtain t where t: 0 < t t < h DERIV (difg m′) t :> 0
      by force
    have ∃ t′. 0 < t′ ∧ t′ < t ∧ DERIV (difg (Suc m′)) t′ :> 0
    proof (rule Rolle)
      show 0 < t by fact
      show difg (Suc m′) 0 = difg (Suc m′) t
        using t ‹Suc m′ < n› by (simp add: difg-Suc-eq-0 difg-eq-0)
      have ⋀x. 0 ≤ x ∧ x ≤ t ⟹ isCont (difg (Suc m′)) x
        using ‹t < h› ‹Suc m′ < n› by (simp add: isCont-difg)
      then show continuous-on {0..t} (difg (Suc m′))
        by (simp add: continuous-at-imp-continuous-on)
    qed (use ‹t < h› ‹Suc m′ < n› in ‹simp add: differentiable-difg›)
    with ‹t < h› show ?case
      by auto
  qed
  then obtain t where 0 < t t < h difg (Suc m) t = 0
    using ‹m < n› difg-Suc-eq-0 by force
  show ?thesis
  proof (intro exI conjI)
    show 0 < t t < h by fact+
    show f h = (∑ m<n. diff m 0 / (fact m) ∗ h ^ m) + diff n t / (fact n) ∗ h ^ n
      using ‹difg (Suc m) t = 0› by (simp add: m f-h difg-def)
  qed
qed

lemma Maclaurin2:
  fixes n :: nat
    and h :: real
  assumes INIT1: 0 < h
    and INIT2: diff 0 = f
    and DERIV: ∀ m t. m < n ∧ 0 ≤ t ∧ t ≤ h ⟶ DERIV (diff m) t :> diff
(Suc m) t
  shows ∃ t. 0 < t ∧ t ≤ h ∧ f h = (∑ m<n. diff m 0 / (fact m) ∗ h ^ m) + diff
n t / fact n ∗ h ^ n
proof (cases n)
  case 0
  with INIT1 INIT2 show ?thesis by fastforce
next
  case Suc
  then have n > 0 by simp
  from Maclaurin [OF INIT1 this INIT2 DERIV]
  show ?thesis by fastforce
```

qed

lemma *Maclaurin-minus*:
fixes n :: *nat* **and** h :: *real*
assumes $h < 0$ $0 < n$ *diff* $0 = f$
and *DERIV*: $\forall m\ t.\ m < n \wedge h \leq t \wedge t \leq 0 \longrightarrow$ *DERIV* (*diff* m) t :> *diff* (*Suc* m) t
shows $\exists t.\ h < t \wedge t < 0 \wedge f\ h = (\sum m{<}n.\ \mathit{diff}\ m\ 0\ /\ \mathit{fact}\ m * h\ \hat{}\ m) + \mathit{diff}\ n\ t\ /\ \mathit{fact}\ n * h\ \hat{}\ n$
proof −

Transform *ABL'* into *derivative-intros* format.

note *DERIV'* = *DERIV-chain'*[*OF* - *DERIV*[*rule-format*], *THEN DERIV-cong*]
let *?sum* = $\lambda t.$
$(\sum m{<}n.\ (-\ 1)\ \hat{}\ m * \mathit{diff}\ m\ (-\ 0)\ /\ (\mathit{fact}\ m) * (-\ h)\ \hat{}\ m) +$
$(-\ 1)\ \hat{}\ n * \mathit{diff}\ n\ (-\ t)\ /\ (\mathit{fact}\ n) * (-\ h)\ \hat{}\ n$
from *assms* **have** $\exists t{>}0.\ t < -\ h \wedge f\ (-\ (-\ h)) =$ *?sum* t
by (*intro Maclaurin*) (*auto intro*!: *derivative-eq-intros DERIV'*)
then obtain t **where** $0 < t$ $t < -\ h$ $f\ (-\ (-\ h)) =$ *?sum* t
by *blast*
moreover have $(-\ 1)\ \hat{}\ n * \mathit{diff}\ n\ (-\ t) * (-\ h)\ \hat{}\ n\ /\ \mathit{fact}\ n = \mathit{diff}\ n\ (-\ t) * h\ \hat{}\ n\ /\ \mathit{fact}\ n$
by (*auto simp*: *power-mult-distrib*[*symmetric*])
moreover
have $(\sum m{<}n.\ (-\ 1)\ \hat{}\ m * \mathit{diff}\ m\ 0 * (-\ h)\ \hat{}\ m\ /\ \mathit{fact}\ m) = (\sum m{<}n.\ \mathit{diff}\ m\ 0 * h\ \hat{}\ m\ /\ \mathit{fact}\ m)$
by (*auto intro*: *sum.cong simp add*: *power-mult-distrib*[*symmetric*])
ultimately have $h < -\ t \wedge -\ t < 0 \wedge$
$f\ h = (\sum m{<}n.\ \mathit{diff}\ m\ 0\ /\ (\mathit{fact}\ m) * h\ \hat{}\ m) + \mathit{diff}\ n\ (-\ t)\ /\ (\mathit{fact}\ n) * h\ \hat{}\ n$
by *auto*
then show *?thesis* **..**
qed

114.2 More Convenient "Bidirectional" Version.

lemma *Maclaurin-bi-le*:
fixes n :: *nat* **and** x :: *real*
assumes *diff* $0 = f$
and *DERIV* : $\forall m\ t.\ m < n \wedge |t| \leq |x| \longrightarrow$ *DERIV* (*diff* m) t :> *diff* (*Suc* m) t
shows $\exists t.\ |t| \leq |x| \wedge f\ x = (\sum m{<}n.\ \mathit{diff}\ m\ 0\ /\ (\mathit{fact}\ m) * x\ \hat{}\ m) + \mathit{diff}\ n\ t\ /\ (\mathit{fact}\ n) * x\ \hat{}\ n$
(**is** $\exists t.$ - $\wedge f\ x =$ *?f* x t)
proof (*cases* $n = 0$)
case *True*
with ‹*diff* $0 = f$› **show** *?thesis* **by** *force*
next
case *False*
show *?thesis*
proof (*cases rule*: *linorder-cases*)

```
    assume x = 0
    with ‹n ≠ 0› ‹diff 0 = f› DERIV have |0| ≤ |x| ∧ f x = ?f x 0
      by auto
    then show ?thesis ..
  next
    assume x < 0
    with ‹n ≠ 0› DERIV have ∃ t>x. t < 0 ∧ diff 0 x = ?f x t
      by (intro Maclaurin-minus) auto
    then obtain t where x < t t < 0
      diff 0 x = (∑ m<n. diff m 0 / fact m * x ^ m) + diff n t / fact n * x ^ n
      by blast
    with ‹x < 0› ‹diff 0 = f› show ?thesis by force
  next
    assume x > 0
    with ‹n ≠ 0› ‹diff 0 = f› DERIV have ∃ t>0. t < x ∧ diff 0 x = ?f x t
      by (intro Maclaurin) auto
    then obtain t where 0 < t t < x
      diff 0 x = (∑ m<n. diff m 0 / fact m * x ^ m) + diff n t / fact n * x ^ n
      by blast
    with ‹x > 0› ‹diff 0 = f› have |t| ≤ |x| ∧ f x = ?f x t by simp
    then show ?thesis ..
  qed
qed

lemma Maclaurin-all-lt:
  fixes x :: real
  assumes INIT1: diff 0 = f
    and INIT2: 0 < n
    and INIT3: x ≠ 0
    and DERIV: ∀ m x. DERIV (diff m) x :> diff(Suc m) x
  shows ∃ t. 0 < |t| ∧ |t| < |x| ∧ f x =
      (∑ m<n. (diff m 0 / fact m) * x ^ m) + (diff n t / fact n) * x ^ n
    (is ∃ t. - ∧ - ∧ f x = ?f x t)
proof (cases rule: linorder-cases)
  assume x = 0
  with INIT3 show ?thesis ..
next
  assume x < 0
  with assms have ∃ t>x. t < 0 ∧ f x = ?f x t
    by (intro Maclaurin-minus) auto
  then show ?thesis by force
next
  assume x > 0
  with assms have ∃ t>0. t < x ∧ f x = ?f x t
    by (intro Maclaurin) auto
  then show ?thesis by force
qed

lemma Maclaurin-zero: x = 0 ⟹ n ≠ 0 ⟹ (∑ m<n. (diff m 0 / fact m) * x
```

$\hat{}$ *m*) = *diff 0 0*
for *x* :: *real* **and** *n* :: *nat*
by *simp*

lemma *Maclaurin-all-le*:
fixes *x* :: *real* **and** *n* :: *nat*
assumes *INIT*: *diff 0 = f*
and *DERIV*: ∀ *m x*. *DERIV* (*diff m*) *x* :> *diff* (*Suc m*) *x*
shows ∃ *t*. |*t*| ≤ |*x*| ∧ *f x* = ($\sum$ *m*<*n*. (*diff m 0* / *fact m*) ∗ *x* $\hat{}$ *m*) + (*diff n t* / *fact n*) ∗ *x* $\hat{}$ *n*
(**is** ∃ *t*. - ∧ *f x* = *?f x t*)
proof (*cases n* = *0*)
case *True*
with *INIT* **show** *?thesis* **by** *force*
next
case *False*
show *?thesis*
using *DERIV INIT Maclaurin-bi-le* **by** *auto*
qed

lemma *Maclaurin-all-le-objl*:
diff 0 = *f* ∧ (∀ *m x*. *DERIV* (*diff m*) *x* :> *diff* (*Suc m*) *x*) ⟶
(∃ *t*::*real*. |*t*| ≤ |*x*| ∧ *f x* = ($\sum$ *m*<*n*. (*diff m 0* / *fact m*) ∗ *x* $\hat{}$ *m*) + (*diff n t* / *fact n*) ∗ *x* $\hat{}$ *n*)
for *x* :: *real* **and** *n* :: *nat*
by (*blast intro*: *Maclaurin-all-le*)

114.3 Version for Exponential Function

lemma *Maclaurin-exp-lt*:
fixes *x* :: *real* **and** *n* :: *nat*
shows
x ≠ *0* ⟹ *n* > *0* ⟹
(∃ *t*. *0* < |*t*| ∧ |*t*| < |*x*| ∧ *exp x* = ($\sum$ *m*<*n*. (*x* $\hat{}$ *m*) / *fact m*) + (*exp t* / *fact n*) ∗ *x* $\hat{}$ *n*)
using *Maclaurin-all-lt* [**where** *diff* = λ*n*. *exp* **and** *f* = *exp* **and** *x* = *x* **and** *n* = *n*] **by** *auto*

lemma *Maclaurin-exp-le*:
fixes *x* :: *real* **and** *n* :: *nat*
shows ∃ *t*. |*t*| ≤ |*x*| ∧ *exp x* = ($\sum$ *m*<*n*. (*x* $\hat{}$ *m*) / *fact m*) + (*exp t* / *fact n*) ∗ *x* $\hat{}$ *n*
using *Maclaurin-all-le-objl* [**where** *diff* = λ*n*. *exp* **and** *f* = *exp* **and** *x* = *x* **and** *n* = *n*] **by** *auto*

corollary *exp-lower-Taylor-quadratic*: *0* ≤ *x* ⟹ *1* + *x* + x^2 / *2* ≤ *exp x*
for *x* :: *real*
using *Maclaurin-exp-le* [*of x 3*] **by** (*auto simp*: *numeral-3-eq-3 power2-eq-square*)

corollary *ln-2-less-1*: *ln 2* < (*1*::*real*)
by (*smt* (*verit*) *ln-eq-minus-one ln-le-minus-one*)

114.4 Version for Sine Function

lemma *mod-exhaust-less-4*: *m mod 4* = *0* ∨ *m mod 4* = *1* ∨ *m mod 4* = *2* ∨ *m mod 4* = *3*
for *m* :: *nat*
by *auto*

It is unclear why so many variant results are needed.

lemma *sin-expansion-lemma*: *sin* (*x* + *real* (*Suc m*) ∗ *pi* / *2*) = *cos* (*x* + *real m* ∗ *pi* / *2*)
by (*auto simp*: *cos-add sin-add add-divide-distrib distrib-right*)

lemma *Maclaurin-sin-expansion2*:
∃ *t*. |*t*| ≤ |*x*| ∧
sin x = (∑ *m*<*n*. *sin-coeff m* ∗ *x* ^ *m*) + (*sin* (*t* + *1*/*2* ∗ *real n* ∗ *pi*) / *fact n*) ∗ *x* ^ *n*
proof (*cases n* = *0* ∨ *x* = *0*)
case *False*
let *?diff* = λ*n x*. *sin* (*x* + *1*/*2* ∗ *real n* ∗ *pi*)
have ∃ *t*. *0* < |*t*| ∧ |*t*| < |*x*| ∧ *sin x* =
(∑ *m*<*n*. (*?diff m 0* / *fact m*) ∗ *x* ^ *m*) + (*?diff n t* / *fact n*) ∗ *x* ^ *n*
proof (*rule Maclaurin-all-lt*)
show ∀ *m x*. ((λ*t*. *sin* (*t* + *1*/*2* ∗ *real m* ∗ *pi*)) *has-real-derivative*
sin (*x* + *1*/*2* ∗ *real* (*Suc m*) ∗ *pi*)) (*at x*)
by (*rule allI derivative-eq-intros* | *use sin-expansion-lemma* **in** *force*)+
qed (*use False* **in** *auto*)
then show *?thesis*
apply (*rule ex-forward*, *simp*)
apply (*rule sum.cong*[*OF refl*])
apply (*auto simp*: *sin-coeff-def sin-zero-iff elim*: *oddE simp del*: *of-nat-Suc*)
done
qed *auto*

lemma *Maclaurin-sin-expansion*:
∃ *t*. *sin x* = (∑ *m*<*n*. *sin-coeff m* ∗ *x* ^ *m*) + (*sin* (*t* + *1*/*2* ∗ *real n* ∗ *pi*) / *fact n*) ∗ *x* ^ *n*
using *Maclaurin-sin-expansion2* [*of x n*] **by** *blast*

lemma *Maclaurin-sin-expansion3*:
assumes *n* > *0 x* > *0*
shows ∃ *t*. *0* < *t* ∧ *t* < *x* ∧
sin x = (∑ *m*<*n*. *sin-coeff m* ∗ *x* ^ *m*) + (*sin* (*t* + *1*/*2* ∗ *real n* ∗ *pi*) / *fact n*) ∗ *x* ^ *n*
proof −
let *?diff* = λ*n x*. *sin* (*x* + *1*/*2* ∗ *real n* ∗ *pi*)

have ∃ *t*. *0* < *t* ∧ *t* < *x* ∧ *sin x* = (∑ *m*<*n*. *?diff m 0* / (*fact m*) ∗ *x* ^ *m*) + *?diff n t* / *fact n* ∗ *x* ^ *n*
proof (*rule Maclaurin*)
show ∀ *m t*. *m* < *n* ∧ *0* ≤ *t* ∧ *t* ≤ *x* ⟶
((λ*u*. *sin* (*u* + *1/2* ∗ *real m* ∗ *pi*)) *has-real-derivative*
sin (*t* + *1/2* ∗ *real* (*Suc m*) ∗ *pi*)) (*at t*)
using *DERIV-shift sin-expansion-lemma* **by** *fastforce*
qed (*use assms* **in** *auto*)
then show *?thesis*
apply (*rule ex-forward*, *simp*)
apply (*rule sum.cong*[*OF refl*])
apply (*auto simp*: *sin-coeff-def sin-zero-iff elim*: *oddE simp del*: *of-nat-Suc*)
done
qed

lemma *Maclaurin-sin-expansion4*:
assumes *0* < *x*
shows ∃ *t*. *0* < *t* ∧ *t* ≤ *x* ∧ *sin x* = (∑ *m*<*n*. *sin-coeff m* ∗ *x* ^ *m*) + (*sin* (*t* + *1/2* ∗ *real n* ∗ *pi*) / *fact n*) ∗ *x* ^ *n*
proof −
let *?diff* = λ*n x*. *sin* (*x* + *1/2* ∗ *real n* ∗ *pi*)
have ∃ *t*. *0* < *t* ∧ *t* ≤ *x* ∧ *sin x* = (∑ *m*<*n*. *?diff m 0* / (*fact m*) ∗ *x* ^ *m*) + *?diff n t* / *fact n* ∗ *x* ^ *n*
proof (*rule Maclaurin2*)
show ∀ *m t*. *m* < *n* ∧ *0* ≤ *t* ∧ *t* ≤ *x* ⟶
((λ*u*. *sin* (*u* + *1/2* ∗ *real m* ∗ *pi*)) *has-real-derivative*
sin (*t* + *1/2* ∗ *real* (*Suc m*) ∗ *pi*)) (*at t*)
using *DERIV-shift sin-expansion-lemma* **by** *fastforce*
qed (*use assms* **in** *auto*)
then show *?thesis*
apply (*rule ex-forward*, *simp*)
apply (*rule sum.cong*[*OF refl*])
apply (*auto simp*: *sin-coeff-def sin-zero-iff elim*: *oddE simp del*: *of-nat-Suc*)
done
qed

114.5 Maclaurin Expansion for Cosine Function

lemma *sumr-cos-zero-one* [*simp*]: (∑ *m*<*Suc n*. *cos-coeff m* ∗ *0* ^ *m*) = *1*
by (*induct n*) *auto*

lemma *cos-expansion-lemma*: *cos* (*x* + *real* (*Suc m*) ∗ *pi* / *2*) = − *sin* (*x* + *real m* ∗ *pi* / *2*)
by (*auto simp*: *cos-add sin-add distrib-right add-divide-distrib*)

lemma *Maclaurin-cos-expansion*:
∃ *t*::*real*. |*t*| ≤ |*x*| ∧
cos x = (∑ *m*<*n*. *cos-coeff m* ∗ *x* ^ *m*) + (*cos*(*t* + *1/2* ∗ *real n* ∗ *pi*) / *fact n*) ∗ *x* ^ *n*

proof (*cases n = 0 ∨ x = 0*)
  case *False*
  let *?diff = λn x. cos (x + 1/2 ∗ real n ∗ pi)*
  have ∃ *t. 0 < |t| ∧ |t| < |x| ∧ cos x =*
    (∑ *m<n. (?diff m 0 / fact m) ∗ x ^ m) + (?diff n t / fact n) ∗ x ^ n*
  proof (*rule Maclaurin-all-lt*)
    show ∀ *m x. ((λt. cos (t + 1/2 ∗ real m ∗ pi)) has-real-derivative*
      cos (x + 1/2 ∗ real (Suc m) ∗ pi)) (at x)
      using *cos-expansion-lemma*
      by (*intro allI derivative-eq-intros | simp*)+
  qed (*use False* **in** *auto*)
  then show *?thesis*
    apply (*rule ex-forward*, *simp*)
    apply (*rule sum.cong*[*OF refl*])
    apply (*auto simp*: *cos-coeff-def cos-zero-iff elim*: *evenE simp del*: *of-nat-Suc*)
    done
qed *auto*

lemma *Maclaurin-cos-expansion2*:
  assumes *x > 0 n > 0*
  shows ∃ *t. 0 < t ∧ t < x ∧*
    cos x = (∑ *m<n. cos-coeff m ∗ x ^ m) + (cos (t + 1/2 ∗ real n ∗ pi) / fact n) ∗ x ^ n*
proof −
  let *?diff = λn x. cos (x + 1/2 ∗ real n ∗ pi)*
  have ∃ *t. 0 < t ∧ t < x ∧ cos x =* (∑ *m<n. ?diff m 0 / (fact m) ∗ x ^ m) + ?diff n t / fact n ∗ x ^ n*
  proof (*rule Maclaurin*)
    show ∀ *m t. m < n ∧ 0 ≤ t ∧ t ≤ x ⟶*
      ((λu. cos (u + 1 / 2 ∗ real m ∗ pi)) has-real-derivative
      cos (t + 1 / 2 ∗ real (Suc m) ∗ pi)) (at t)
      by (*simp add*: *cos-expansion-lemma del*: *of-nat-Suc*)
  qed (*use assms* **in** *auto*)
  then show *?thesis*
    apply (*rule ex-forward*, *simp*)
    apply (*rule sum.cong*[*OF refl*])
    apply (*auto simp*: *cos-coeff-def cos-zero-iff elim*: *evenE*)
    done
qed

lemma *Maclaurin-minus-cos-expansion*:
  assumes *n > 0 x < 0*
  shows ∃ *t. x < t ∧ t < 0 ∧*
    cos x = (∑ *m<n. cos-coeff m ∗ x ^ m) + ((cos (t + 1/2 ∗ real n ∗ pi) / fact n) ∗ x ^ n)*
proof −
  let *?diff = λn x. cos (x + 1/2 ∗ real n ∗ pi)*
  have ∃ *t. x < t ∧ t < 0 ∧ cos x =* (∑ *m<n. ?diff m 0 / (fact m) ∗ x ^ m) + ?diff n t / fact n ∗ x ^ n*

proof (*rule Maclaurin-minus*)
show ∀ *m t*. *m* < *n* ∧ *x* ≤ *t* ∧ *t* ≤ *0* ⟶
((λ*u*. *cos* (*u* + *1* / *2* ∗ *real m* ∗ *pi*)) *has-real-derivative*
cos (*t* + *1* / *2* ∗ *real* (*Suc m*) ∗ *pi*)) (*at t*)
by (*simp add*: *cos-expansion-lemma del*: *of-nat-Suc*)
qed (*use assms* **in** *auto*)
then show *?thesis*
apply (*rule ex-forward*, *simp*)
apply (*rule sum.cong*[*OF refl*])
apply (*auto simp*: *cos-coeff-def cos-zero-iff elim*: *evenE*)
done
qed

lemma *sin-bound-lemma*: *x* = *y* ⟹ |*u*| ≤ *v* ⟹ |(*x* + *u*) − *y*| ≤ *v*
for *x y u v* :: *real*
by *auto*

lemma *Maclaurin-sin-bound*: |*sin x* − (∑ *m*<*n*. *sin-coeff m* ∗ *x* ^ *m*)| ≤ *inverse* (*fact n*) ∗ |*x*| ^ *n*
proof −
have *est*: *x* ≤ *1* ⟹ *0* ≤ *y* ⟹ *x* ∗ *y* ≤ *1* ∗ *y* **for** *x y* :: *real*
by (*rule mult-right-mono*) *simp-all*
let *?diff* = λ(*n*::*nat*) (*x*::*real*).
if n mod 4 = 0 then sin x
else if n mod 4 = 1 then cos x
else if n mod 4 = 2 then − sin x
else − cos x
have *diff-0*: *?diff 0* = *sin* **by** *simp*
have *DERIV* (*?diff m*) *x* :> *?diff* (*Suc m*) *x* **for** *m* **and** *x*
using *mod-exhaust-less-4* [*of m*]
by (*auto simp*: *mod-Suc intro*!: *derivative-eq-intros*)
then have *DERIV-diff*: ∀ *m x*. *DERIV* (*?diff m*) *x* :> *?diff* (*Suc m*) *x*
by *blast*
from *Maclaurin-all-le* [*OF diff-0 DERIV-diff*]
obtain *t* **where** *t1*: |*t*| ≤ |*x*|
and *t2*: *sin x* = (∑ *m*<*n*. *?diff m 0* / (*fact m*) ∗ *x* ^ *m*) + *?diff n t* / (*fact n*) ∗ *x* ^ *n*
by *fast*
have *diff-m-0*: *?diff m 0* = (*if even m then 0 else* (− *1*) ^((*m* − *Suc 0*) *div 2*)) **for** *m*
using *mod-exhaust-less-4* [*of m*]
by (*auto simp*: *minus-one-power-iff even-even-mod-4-iff* [*of m*] *dest*: *even-mod-4-div-2 odd-mod-4-div-2*)
show *?thesis*
apply (*subst t2*)

apply (*rule sin-bound-lemma*)
 apply (*rule sum.cong*[*OF refl*])
apply (*simp add*: *diff-m-0 sin-coeff-def*)
using *est*
apply (*auto intro*: *mult-right-mono* [**where** *b*=*1*, *simplified*] *mult-right-mono* *simp*: *ac-simps divide-inverse power-abs* [*symmetric*] *abs-mult*)
done
qed

115 Taylor series

We use MacLaurin and the translation of the expansion point *c* to *0* to prove Taylor's theorem.

lemma *Taylor-up*:
assumes *INIT*: $n > 0$ *diff* $0 = f$
and *DERIV*: $\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq b \longrightarrow$ *DERIV* (*diff m*) *t* :> (*diff* (*Suc m*) *t*)
and *INTERV*: $a \leq c$ $c < b$
shows $\exists$ *t*::*real*. $c < t \wedge t < b \wedge$
$f\ b = (\sum m{<}n.\ (diff\ m\ c\ /\ fact\ m) * (b - c)\hat{}m) + (diff\ n\ t\ /\ fact\ n) * (b - c)\hat{}n$
proof −
from *INTERV* **have** $0 < b - c$ **by** *arith*
moreover from *INIT* **have** $n > 0$ $(\lambda m\ x.\ diff\ m\ (x + c))\ 0 = (\lambda x.\ f\ (x + c))$
by *auto*
moreover
have $\forall m\ t.\ m < n \wedge 0 \leq t \wedge t \leq b - c \longrightarrow$ *DERIV* $(\lambda x.\ diff\ m\ (x + c))$ *t* :> *diff* (*Suc m*) (*t* + *c*)
proof (*intro strip*)
fix *m t*
assume $m < n \wedge 0 \leq t \wedge t \leq b - c$
with *DERIV* **and** *INTERV* **have** *DERIV* (*diff m*) (*t* + *c*) :> *diff* (*Suc m*) (*t* + *c*)
by *auto*
moreover from *DERIV-ident* **and** *DERIV-const* **have** *DERIV* $(\lambda x.\ x + c)$ *t* :> *1* + *0*
by (*rule DERIV-add*)
ultimately have *DERIV* $(\lambda x.\ diff\ m\ (x + c))$ *t* :> *diff* (*Suc m*) (*t* + *c*) ∗ (*1* + *0*)
by (*rule DERIV-chain2*)
then show *DERIV* $(\lambda x.\ diff\ m\ (x + c))$ *t* :> *diff* (*Suc m*) (*t* + *c*)
by *simp*
qed
ultimately obtain *x* **where**
$0 < x \wedge x < b - c \wedge$
$f\ (b - c + c) =$
$(\sum m{<}n.\ diff\ m\ (0 + c)\ /\ fact\ m * (b - c)\ \hat{}\ m) + diff\ n\ (x + c)\ /\ fact\ n * (b - c)\ \hat{}\ n$

by (*rule Maclaurin* [*THEN exE*])
then show *?thesis*
by (*smt* (*verit*) *sum.cong*)
qed

lemma *Taylor-down*:
fixes *a* :: *real* **and** *n* :: *nat*
assumes *INIT*: $n > 0$ *diff* $0 = f$
and *DERIV*: ($\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq b \longrightarrow$ *DERIV* (*diff m*) *t* :> *diff* (*Suc m*) *t*)
and *INTERV*: $a < c$ $c \leq b$
shows $\exists t.\ a < t \wedge t < c \wedge$
f a = ($\sum$ *m*<*n*. (*diff m c* / *fact m*) ∗ (*a* − *c*)^*m*) + (*diff n t* / *fact n*) ∗ (*a* − *c*)^*n*
proof −
from *INTERV* **have** $a-c < 0$ **by** *arith*
moreover from *INIT* **have** $n > 0$ (λ*m x*. *diff m* (*x* + *c*)) *0* = (λ*x*. *f* (*x* + *c*))
by *auto*
moreover
have $\forall m\ t.\ m < n \wedge a - c \leq t \wedge t \leq 0 \longrightarrow$ *DERIV* (λ*x*. *diff m* (*x* + *c*)) *t* :> *diff* (*Suc m*) (*t* + *c*)
proof (*rule allI impI*)+
fix *m t*
assume $m < n \wedge a - c \leq t \wedge t \leq 0$
with *DERIV* **and** *INTERV* **have** *DERIV* (*diff m*) (*t* + *c*) :> *diff* (*Suc m*) (*t* + *c*)
by *auto*
moreover from *DERIV-ident* **and** *DERIV-const* **have** *DERIV* (λ*x*. *x* + *c*) *t* :> *1* + *0*
by (*rule DERIV-add*)
ultimately show *DERIV* (λ*x*. *diff m* (*x* + *c*)) *t* :> *diff* (*Suc m*) (*t* + *c*)
using *DERIV-chain2 DERIV-shift* **by** *blast*
qed
ultimately obtain *x* **where**
$a - c < x \wedge x < 0 \wedge$
f (*a* − *c* + *c*) =
($\sum$ *m*<*n*. *diff m* (*0* + *c*) / *fact m* ∗ (*a* − *c*) ^ *m*) + *diff n* (*x* + *c*) / *fact n* ∗ (*a* − *c*) ^ *n*
by (*rule Maclaurin-minus* [*THEN exE*])
then have $a < x + c \wedge x + c < c \wedge$
f a = ($\sum$ *m*<*n*. *diff m c* / *fact m* ∗ (*a* − *c*) ^ *m*) + *diff n* (*x* + *c*) / *fact n* ∗ (*a* − *c*) ^ *n*
by *fastforce*
then show *?thesis* **by** *fastforce*
qed

theorem *Taylor*:
fixes *a* :: *real* **and** *n* :: *nat*
assumes *INIT*: $n > 0$ *diff* $0 = f$

and *DERIV*: $\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
and *INTERV*: $a \leq c\ \ c \leq b\ a \leq x\ x \leq b\ x \neq c$
shows $\exists\, t.$
$(if\ x < c\ then\ x < t \wedge t < c\ else\ c < t \wedge t < x) \wedge$
$f\ x = (\sum m{<}n.\ (diff\ m\ c\ /\ fact\ m) * (x - c)\hat{}m) + (diff\ n\ t\ /\ fact\ n) * (x - c)\hat{}n$
proof (*cases* $x < c$)
case *True*
note *INIT*
moreover have $\forall m\ t.\ m < n \wedge x \leq t \wedge t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
using *DERIV* **and** *INTERV* **by** *fastforce*
ultimately show *?thesis*
using *True INTERV Taylor-down* **by** *simp*
next
case *False*
note *INIT*
moreover have $\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq x \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
using *DERIV* **and** *INTERV* **by** *fastforce*
ultimately show *?thesis*
using *Taylor-up INTERV False* **by** *simp*
qed

end

116 More facts about binomial coefficients

These facts could have been proven before, but having real numbers makes the proofs a lot easier. Thanks to Alexander Maletzky among others.

theory *Binomial-Plus*
imports *Real*
begin

116.1 More facts about binomial coefficients

These facts could have been proven before, but having real numbers makes the proofs a lot easier.

lemma *central-binomial-odd*:
$odd\ n \Longrightarrow n\ choose\ (Suc\ (n\ div\ 2)) = n\ choose\ (n\ div\ 2)$
proof –
assume *odd n*
hence $Suc\ (n\ div\ 2) \leq n$ **by** *presburger*
hence $n\ choose\ (Suc\ (n\ div\ 2)) = n\ choose\ (n - Suc\ (n\ div\ 2))$
by (*rule binomial-symmetric*)
also from ‹*odd n*› **have** $n - Suc\ (n\ div\ 2) = n\ div\ 2$ **by** *presburger*

finally show *?thesis* **.**
qed

lemma *binomial-less-binomial-Suc*:
assumes *k*: $k < n$ *div 2*
shows *n choose k* $<$ *n choose* (*Suc k*)
proof –
from *k* **have** *k'*: $k \leq n$ *Suc k* $\leq n$ **by** *simp-all*
from *k'* **have** *real* (*n choose k*) = *fact n* / (*fact k* ∗ *fact* (*n* − *k*))
by (*simp add*: *binomial-fact*)
also from *k'* **have** *n* − *k* = *Suc* (*n* − *Suc k*) **by** *simp*
also from *k'* **have** *fact* . . . = (*real n* − *real k*) ∗ *fact* (*n* − *Suc k*)
by (*subst fact-Suc*) (*simp-all add*: *of-nat-diff*)
also from *k* **have** *fact k* = *fact* (*Suc k*) / (*real k* + *1*) **by** (*simp add*: *field-simps*)
also have *fact n* / (*fact* (*Suc k*) / (*real k* + *1*) ∗ ((*real n* − *real k*) ∗ *fact* (*n* − *Suc k*))) =
(*n choose* (*Suc k*)) ∗ ((*real k* + *1*) / (*real n* − *real k*))
using *k* **by** (*simp add*: *field-split-simps binomial-fact*)
also from *assms* **have** (*real k* + *1*) / (*real n* − *real k*) < *1* **by** *simp*
finally show *?thesis* **using** *k* **by** (*simp add*: *mult-less-cancel-left*)
qed

lemma *binomial-strict-mono*:
assumes $k < k'$ $2{*}k' \leq n$
shows *n choose k* $<$ *n choose k'*
proof –
from *assms* **have** $k \leq k' - 1$ **by** *simp*
thus *?thesis*
proof (*induction rule*: *inc-induct*)
case *base*
with *assms binomial-less-binomial-Suc*[*of k'* − *1 n*]
show *?case* **by** *simp*
next
case (*step k*)
from *step.prems step.hyps assms* **have** *n choose k* $<$ *n choose* (*Suc k*)
by (*intro binomial-less-binomial-Suc*) *simp-all*
also have . . . $<$ *n choose k'* **by** (*rule step.IH*)
finally show *?case* **.**
qed
qed

lemma *binomial-mono*:
assumes $k \leq k'$ $2{*}k' \leq n$
shows *n choose k* $\leq$ *n choose k'*
using *assms binomial-strict-mono*[*of k k' n*] **by** (*cases k* = *k'*) *simp-all*

lemma *binomial-strict-antimono*:
assumes $k < k'$ $2 * k \geq n$ $k' \leq n$
shows *n choose k* $>$ *n choose k'*

```
proof -
  from assms have n choose (n - k) > n choose (n - k')
    by (intro binomial-strict-mono) (simp-all add: algebra-simps)
  with assms show ?thesis by (simp add: binomial-symmetric [symmetric])
qed

lemma binomial-antimono:
  assumes k ≤ k' k ≥ n div 2 k' ≤ n
  shows   n choose k ≥ n choose k'
proof (cases k = k')
  case False
  note not-eq = False
  show ?thesis
  proof (cases k = n div 2 ∧ odd n)
    case False
    with assms(2) have 2*k ≥ n by presburger
    with not-eq assms binomial-strict-antimono[of k k' n]
      show ?thesis by simp
  next
    case True
    have n choose k' ≤ n choose (Suc (n div 2))
    proof (cases k' = Suc (n div 2))
      case False
      with assms True not-eq have Suc (n div 2) < k' by simp
      with assms binomial-strict-antimono[of Suc (n div 2) k' n] True
        show ?thesis by auto
    qed simp-all
    also from True have ... = n choose k by (simp add: central-binomial-odd)
    finally show ?thesis .
  qed
qed simp-all

lemma binomial-maximum: n choose k ≤ n choose (n div 2)
proof -
  have k ≤ n div 2 ⟷ 2*k ≤ n by linarith
  consider 2*k ≤ n | 2*k ≥ n k ≤ n | k > n by linarith
  thus ?thesis
  proof cases
    case 1
    thus ?thesis by (intro binomial-mono) linarith+
  next
    case 2
    thus ?thesis by (intro binomial-antimono) simp-all
  qed (simp-all add: binomial-eq-0)
qed

lemma binomial-maximum': (2*n) choose k ≤ (2*n) choose n
  using binomial-maximum[of 2*n] by simp
```

lemma *central-binomial-lower-bound*:
  assumes $n > 0$
  shows $4^n / (2*real\ n) \leq real\ ((2*n)\ choose\ n)$
proof −
  from *binomial*[*of 1 1 2*∗*n*]
    have $4\ \hat{}\ n = (\sum k \leq 2*n.\ (2*n)\ choose\ k)$
    by (*simp add*: *power-mult power2-eq-square One-nat-def* [*symmetric*] *del*: *One-nat-def*)
  also have $\{..2*n\} = \{0<..<2*n\} \cup \{0,2*n\}$ **by** *auto*
  also have $(\sum k\in\ldots\ (2*n)\ choose\ k) =$
    $(\sum k\in\{0<..<2*n\}.\ (2*n)\ choose\ k) + (\sum k\in\{0,2*n\}.\ (2*n)\ choose\ k)$
    by (*subst sum.union-disjoint*) *auto*
  also have $(\sum k\in\{0,2*n\}.\ (2*n)\ choose\ k) \leq (\sum k\leq 1.\ (n\ choose\ k)^2)$
    by (*cases n*) *simp-all*
  also from *assms* **have** $\ldots \leq (\sum k\leq n.\ (n\ choose\ k)^2)$
    by (*intro sum-mono2*) *auto*
  also have $\ldots = (2*n)\ choose\ n$ **by** (*rule choose-square-sum*)
  also have $(\sum k\in\{0<..<2*n\}.\ (2*n)\ choose\ k) \leq (\sum k\in\{0<..<2*n\}.\ (2*n)\ choose\ n)$
    by (*intro sum-mono binomial-maximum′*)
  also have $\ldots = card\ \{0<..<2*n\} * ((2*n)\ choose\ n)$ **by** *simp*
  also have $card\ \{0<..<2*n\} \leq 2*n - 1$ **by** (*cases n*) *simp-all*
  also have $(2 * n - 1) * (2 * n\ choose\ n) + (2 * n\ choose\ n) = ((2*n)\ choose\ n) * (2*n)$
    using *assms* **by** (*simp add*: *algebra-simps*)
  finally have $4\ \hat{}\ n \leq (2 * n\ choose\ n) * (2 * n)$ **by** *simp-all*
  hence $real\ (4\ \hat{}\ n) \leq real\ ((2 * n\ choose\ n) * (2 * n))$
    by (*subst of-nat-le-iff*)
  with *assms* **show** *?thesis* **by** (*simp add*: *field-simps*)
qed

lemma *upper-le-binomial*:
  assumes $0 < k$ **and** $k < n$
  shows $n \leq n\ choose\ k$
proof −
  from *assms* **have** $1 \leq n$ **by** *simp*
  define k' **where** $k' = (if\ n\ div\ 2 \leq k\ then\ k\ else\ n - k)$
  from *assms* **have** *1*: $k' \leq n - 1$ **and** *2*: $n\ div\ 2 \leq k'$ **by** (*auto simp*: *k′-def*)
  from *assms*(*2*) **have** $k \leq n$ **by** *simp*
  have $n\ choose\ k = n\ choose\ k'$ **by** (*simp add*: *k′-def binomial-symmetric*[*OF* ‹$k \leq n$›])
  have $n = n\ choose\ 1$ **by** (*simp only*: *choose-one*)
  also from ‹$1 \leq n$› **have** $\ldots = n\ choose\ (n - 1)$ **by** (*rule binomial-symmetric*)
  also from *1 2* **have** $\ldots \leq n\ choose\ k'$ **by** (*rule binomial-antimono*) *simp*
  also have $\ldots = n\ choose\ k$ **by** (*simp add*: *k′-def binomial-symmetric*[*OF* ‹$k \leq n$›])
  finally show *?thesis* **.**
qed

116.2 Results about binomials and integers, thanks to Alexander Maletzky

Restore original sort constraints: semidom rather than field of char 0

setup ‹*Sign.add-const-constraint* (@{*const-name gbinomial*}, *SOME* @{*typ* 'a::{*semidom-divide,semiring-char*
⇒ *nat* ⇒ 'a})›

lemma *gbinomial-eq-0-int*:
  assumes *n* < *k*
  shows (*int n*) *gchoose k* = *0*
  by (*simp add*: *assms gbinomial-prod-rev prod-zero*)

corollary *gbinomial-eq-0*: *0* ≤ *a* ⟹ *a* < *int k* ⟹ *a gchoose k* = *0*
  by (*metis nat-eq-iff2 nat-less-iff gbinomial-eq-0-int*)

lemma *gbinomial-mono*:
  fixes *k*::*nat* **and** *a*::*real*
  assumes *of-nat k* ≤ *a a* ≤ *b* **shows** *a gchoose k* ≤ *b gchoose k*
  using *assms*
  by (*force simp*: *gbinomial-prod-rev intro*!: *divide-right-mono prod-mono*)

lemma *int-binomial*: *int* (*n choose k*) = (*int n*) *gchoose k*
proof (*cases k* ≤ *n*)
  case *True*
  from *refl* **have** *eq*: (∏ *i* = *0*..<*k*. *int* (*n* − *i*)) = (∏ *i* = *0*..<*k*. *int n* − *int i*)
  proof (*rule prod.cong*)
    fix *i*
    assume *i* ∈ {*0*..<*k*}
    with *True* **show** *int* (*n* − *i*) = *int n* − *int i* **by** *simp*
  qed
  show *?thesis*
    by (*simp add*: *gbinomial-binomial*[*symmetric*] *gbinomial-prod-rev zdiv-int eq*)
next
  case *False*
  thus *?thesis* **by** (*simp add*: *gbinomial-eq-0-int*)
qed

lemma *falling-fact-pochhammer*: *prod* (λ*i*. *a* − *int i*) {*0*..<*k*} = (− *1*) ^ *k* ∗ *pochhammer* (− *a*) *k*
proof −
  have *eq*: *z* ^ *Suc n* ∗ *prod f* {*0*..*n*} = *prod* (λ*x*. *z* ∗ *f x*) {*0*..*n*} **for** *z*::*int* **and** *n f*
    by (*induct n*) (*simp-all add*: *ac-simps*)
  show *?thesis*
  proof (*cases k*)
    case *0*
    thus *?thesis* **by** (*simp add*: *pochhammer-minus*)
  next
    case (*Suc n*)
    thus *?thesis*

by (*simp only*: *pochhammer-prod atLeastLessThanSuc-atLeastAtMost prod.atLeast-Suc-atMost-Suc-shift eq flip*: *power-mult-distrib*) (*simp add*: *of-nat-diff*)
qed
qed

lemma *falling-fact-pochhammer′*: *prod* (*λi. a − int i*) {*0..<k*} = *pochhammer* (*a − int k + 1*) *k*
by (*simp add*: *falling-fact-pochhammer pochhammer-minus′*)

lemma *gbinomial-int-pochhammer*: (*a*::*int*) *gchoose k* = (− *1*) ^ *k* ∗ *pochhammer* (− *a*) *k div fact k*
by (*simp only*: *gbinomial-prod-rev falling-fact-pochhammer*)

lemma *gbinomial-int-pochhammer′*: *a gchoose k* = *pochhammer* (*a − int k + 1*) *k div fact k*
by (*simp only*: *gbinomial-prod-rev falling-fact-pochhammer′*)

lemma *fact-dvd-pochhammer*: *fact k dvd pochhammer* (*a*::*int*) *k*
proof −
have *dvd*: *y ≠ 0* ⟹ ((*of-int* (*x div y*))::′*a*::*field-char-0*) = *of-int x* / *of-int y* ⟹ *y dvd x*
for *x y* :: *int*
by (*metis dvd-triv-right nonzero-eq-divide-eq of-int-0-eq-iff of-int-eq-iff of-int-mult*)
show *?thesis*
proof (*cases 0 < a*)
case *True*
moreover define *n* **where** *n* = *nat* (*a − 1*) + *k*
ultimately have *a*: *a* = *int n − int k + 1* **by** *simp*
from *fact-nonzero* **show** *?thesis* **unfolding** *a*
proof (*rule dvd*)
have *of-int* (*pochhammer* (*int n − int k + 1*) *k div fact k*) = (*of-int* (*int n gchoose k*)::*rat*)
by (*simp only*: *gbinomial-int-pochhammer′*)
also have ... = *of-nat* (*n choose k*)
by (*metis int-binomial of-int-of-nat-eq*)
also have ... = (*of-nat n*) *gchoose k* **by** (*fact binomial-gbinomial*)
also have ... = *pochhammer* (*of-nat n − of-nat k + 1*) *k* / *fact k*
by (*fact gbinomial-pochhammer′*)
also have ... = *pochhammer* (*of-int* (*int n − int k + 1*)) *k* / *fact k* **by** *simp*
also have ... = (*of-int* (*pochhammer* (*int n − int k + 1*) *k*)) / (*of-int* (*fact k*))
by (*simp only*: *of-int-fact pochhammer-of-int*)
finally show *of-int* (*pochhammer* (*int n − int k + 1*) *k div fact k*) = *of-int* (*pochhammer* (*int n − int k + 1*) *k*) / *rat-of-int* (*fact k*) **.**
qed
next
case *False*
moreover define *n* **where** *n* = *nat* (− *a*)

ultimately have *a*: *a* = − *int n* **by** *simp*
from *fact-nonzero* **have** *fact k dvd* (*−1*)^*k* ∗ *pochhammer* (− *int n*) *k*
proof (*rule dvd*)
have *of-int* ((*−1*)^*k* ∗ *pochhammer* (− *int n*) *k div fact k*) = (*of-int* (*int n gchoose k*)::*rat*)
by (*metis falling-fact-pochhammer gbinomial-prod-rev*)
also have ... = *of-int* (*int* (*n choose k*)) **by** (*simp only*: *int-binomial*)
also have ... = *of-nat* (*n choose k*) **by** *simp*
also have ... = (*of-nat n*) *gchoose k* **by** (*fact binomial-gbinomial*)
also have ... = (*−1*)^*k* ∗ *pochhammer* (− *of-nat n*) *k* / *fact k*
by (*fact gbinomial-pochhammer*)
also have ... = (*−1*)^*k* ∗ *pochhammer* (*of-int* (− *int n*)) *k* / *fact k* **by** *simp*
also have ... = (*−1*)^*k* ∗ (*of-int* (*pochhammer* (− *int n*) *k*)) / (*of-int* (*fact k*))
by (*simp only*: *of-int-fact pochhammer-of-int*)
also have ... = (*of-int* ((*−1*)^*k* ∗ *pochhammer* (− *int n*) *k*)) / (*of-int* (*fact k*)) **by** *simp*
finally show *of-int* ((− *1*) ^ *k* ∗ *pochhammer* (− *int n*) *k div fact k*) =
of-int ((− *1*) ^ *k* ∗ *pochhammer* (− *int n*) *k*) / *rat-of-int* (*fact k*) **.**
qed
thus *?thesis* **unfolding** *a* **by** (*metis dvdI dvd-mult-unit-iff′ minus-one-mult-self*)
qed
qed

lemma *gbinomial-int-negated-upper*: (*a gchoose k*) = (*−1*) ^ *k* ∗ ((*int k* − *a* − *1*) *gchoose k*)
by (*simp add*: *gbinomial-int-pochhammer pochhammer-minus algebra-simps fact-dvd-pochhammer div-mult-swap*)

lemma *gbinomial-int-mult-fact*: *fact k* ∗ (*a gchoose k*) = (∏ *i* = *0*..<*k*. *a* − *int i*)
by (*simp only*: *gbinomial-int-pochhammer′ fact-dvd-pochhammer dvd-mult-div-cancel falling-fact-pochhammer′*)

corollary *gbinomial-int-mult-fact′*: (*a gchoose k*) ∗ *fact k* = (∏ *i* = *0*..<*k*. *a* − *int i*)
using *gbinomial-int-mult-fact*[*of k a*] **by** (*simp add*: *ac-simps*)

lemma *gbinomial-int-binomial*:
a gchoose k = (*if 0* ≤ *a then int* ((*nat a*) *choose k*) *else* (*−1*::*int*)^*k* ∗ *int* ((*k* + (*nat* (− *a*)) − *1*) *choose k*))
by (*auto simp*: *int-binomial gbinomial-int-negated-upper*[*of a*] *int-ops*(*6*))

corollary *gbinomial-nneg*: *0* ≤ *a* ⟹ *a gchoose k* = *int* ((*nat a*) *choose k*)
by (*simp add*: *gbinomial-int-binomial*)

corollary *gbinomial-neg*: *a* < *0* ⟹ *a gchoose k* = (*−1*::*int*)^*k* ∗ *int* ((*k* + (*nat* (− *a*)) − *1*) *choose k*)
by (*simp add*: *gbinomial-int-binomial*)

lemma *of-int-gbinomial*: *of-int* (*a gchoose k*) = (*of-int a* :: *'a*::*field-char-0*) *gchoose k*
proof −
  have *of-int-div*: *y dvd x* ⟹ *of-int* (*x div y*) = *of-int x* / (*of-int y* :: *'a*) **for** *x y* :: *int* **by** *auto*
  show *?thesis*
    by (*simp add*: *gbinomial-int-pochhammer'* *gbinomial-pochhammer'* *of-int-div* *fact-dvd-pochhammer*
      pochhammer-of-int[*symmetric*])
qed

lemma *uminus-one-gbinomial* [*simp*]: (− *1*::*int*) *gchoose k* = (− *1*) ^ *k*
  by (*simp add*: *gbinomial-int-binomial*)

lemma *gbinomial-int-Suc-Suc*: (*x* + *1*::*int*) *gchoose* (*Suc k*) = (*x gchoose k*) + (*x gchoose* (*Suc k*))
proof (*rule linorder-cases*)
  assume *1*: *x* + *1* < *0*
  hence *2*: *x* < *0* **by** *simp*
  then obtain *n* **where** *3*: *nat* (− *x*) = *Suc n* **using** *not0-implies-Suc* **by** *fastforce*
  hence *4*: *nat* (− *x* − *1*) = *n* **by** *simp*
  show *?thesis*
  proof (*cases k*)
    case *0*
    show *?thesis* **by** (*simp add*: ‹*k* = *0*›)
  next
    case (*Suc k'*)
    from *1 2 3 4* **show** *?thesis* **by** (*simp add*: ‹*k* = *Suc k'*› *gbinomial-int-binomial* *int-distrib*(*2*))
  qed
next
  assume *x* + *1* = *0*
  hence *x* = − *1* **by** *simp*
  thus *?thesis* **by** *simp*
next
  assume *0* < *x* + *1*
  hence *0* ≤ *x* + *1* **and** *0* ≤ *x* **and** *nat* (*x* + *1*) = *Suc* (*nat x*) **by** *simp-all*
  thus *?thesis* **by** (*simp add*: *gbinomial-int-binomial*)
qed

corollary *plus-Suc-gbinomial*:
  (*x* + (*1* + *int k*)) *gchoose* (*Suc k*) = ((*x* + *int k*) *gchoose k*) + ((*x* + *int k*) *gchoose* (*Suc k*))
    (**is** *?l* = *?r*)
proof −
  have *?l* = (*x* + *int k* + *1*) *gchoose* (*Suc k*) **by** (*simp only*: *ac-simps*)
  also have … = *?r* **by** (*fact gbinomial-int-Suc-Suc*)
  finally show *?thesis* **.**
qed

lemma *gbinomial-int-n-n* [*simp*]: (*int n*) *gchoose n* = *1*
proof (*induct n*)
  case *0*
  show *?case* **by** *simp*
next
  case (*Suc n*)
  have *int* (*Suc n*) *gchoose Suc n* = (*int n* + *1*) *gchoose Suc n* **by** (*simp add*: *add.commute*)
  also have ... = (*int n gchoose n*) + (*int n gchoose* (*Suc n*)) **by** (*fact gbinomial-int-Suc-Suc*)
  finally show *?case* **by** (*simp add*: *Suc gbinomial-eq-0*)
qed

lemma *gbinomial-int-Suc-n* [*simp*]: (*1* + *int n*) *gchoose n* = *1* + *int n*
proof (*induct n*)
  case *0*
  show *?case* **by** *simp*
next
  case (*Suc n*)
  have *1* + *int* (*Suc n*) *gchoose Suc n* = (*1* + *int n*) + *1 gchoose Suc n* **by** *simp*
  also have ... = (*1* + *int n gchoose n*) + (*1* + *int n gchoose* (*Suc n*)) **by** (*fact gbinomial-int-Suc-Suc*)
  also have ... = *1* + *int n* + (*int* (*Suc n*) *gchoose* (*Suc n*)) **by** (*simp add*: *Suc*)
  also have ... = *1* + *int* (*Suc n*) **by** (*simp only*: *gbinomial-int-n-n*)
  finally show *?case* **.**
qed

lemma *zbinomial-eq-0-iff* [*simp*]: *a gchoose k* = *0* ⟷ (*0* ≤ *a* ∧ *a* < *int k*)
proof
  assume *a*: *a gchoose k* = *0*
  have *1*: *b* < *int k* **if** *b gchoose k* = *0* **for** *b*
  proof (*rule ccontr*)
    assume ¬ *b* < *int k*
    hence *0* ≤ *b* **and** *k* ≤ *nat b* **by** *simp-all*
    from *this*(*1*) **have** *int* ((*nat b*) *choose k*) = *b gchoose k* **by** (*simp add*: *gbinomial-int-binomial*)
    also have ... = *0* **by** (*fact that*)
    finally show *False* **using** ‹*k* ≤ *nat b*› **by** *simp*
  qed
  show *0* ≤ *a* ∧ *a* < *int k*
  proof
    show *0* ≤ *a*
    proof (*rule ccontr*)
      assume ¬ *0* ≤ *a*
      hence (−*1*) ^ *k* ∗ ((*int k* − *a* − *1*) *gchoose k*) = *a gchoose k*
        by (*simp add*: *gbinomial-int-negated-upper*[*of a*])
      also have ... = *0* **by** (*fact a*)
      finally have (*int k* − *a* − *1*) *gchoose k* = *0* **by** *simp*

```
    hence int k − a − 1 < int k by (rule 1)
    with ‹¬ 0 ≤ a› show False by simp
   qed
  next
   from a show a < int k by (rule 1)
  qed
qed (auto intro: gbinomial-eq-0)
```

116.3 Sums

```
lemma gchoose-rising-sum-nat: (∑j≤n. int j + int k gchoose k) = (int n + int k
+ 1) gchoose (Suc k)
proof −
  have (∑j≤n. int j + int k gchoose k) = int (∑j≤n. k + j choose k)
    by (simp add: int-binomial add.commute)
  also have (∑j≤n. k + j choose k) = (k + n + 1) choose (k + 1) by (fact
choose-rising-sum(1))
  also have int ... = (int n + int k + 1) gchoose (Suc k)
    by (simp add: int-binomial ac-simps del: binomial-Suc-Suc)
  finally show ?thesis .
qed

lemma gchoose-rising-sum:
  assumes 0 ≤ n  — Necessary condition.
  shows (∑j=0..n. j + int k gchoose k) = (n + int k + 1) gchoose (Suc k)
proof −
  from - refl have (∑j=0..n. j + int k gchoose k) = (∑j∈int ‘ {0..nat n}. j +
int k gchoose k)
  proof (rule sum.cong)
   from assms show {0..n} = int ‘ {0..nat n} by (simp add: image-int-atLeastAtMost)
  qed
  also have ... = (∑j≤nat n. int j + int k gchoose k) by (simp add: sum.reindex
atMost-atLeast0)
  also have ... = (int (nat n) + int k + 1) gchoose (Suc k) by (fact gchoose-rising-sum-nat)
  also from assms have ... = (n + int k + 1) gchoose (Suc k) by (simp add:
add.assoc add.commute)
  finally show ?thesis .
qed

end
```

117 Comprehensive Complex Theory

```
theory Complex-Main
imports
  Complex
  MacLaurin
  Binomial-Plus
begin
```

end

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