Isabelle/HOL Exercises
Arithmetic

Magical Methods (Computing with Natural Numbers)

A book about Vedic mathematics describes three methods to make the calculation of squares of natural numbers easier:

- *MM1*: Numbers whose predecessors have squares that are known or can easily be calculated. For example:
  Needed: 61^2
  Given: $60^2 = 3600$
  Observe: $61^2 = 3600 + 60 + 61 = 3721$
- *MM2*: Numbers greater than, but near 100. For example:
  Needed: 102^2
  Let $h = 102 - 100 = 2$, $h^2 = 4$
  Observe: $102^2 = (102 + h)$ shifted two places to the left $+h^2 = 10404$
- *MM3*: Numbers ending in 5. For example:
  Needed: 85^2
  Observe: $85^2 = (8 * 9)$ appended to $25 = 7225$
  Needed: 995^2
  Observe: $995^2 = (99 * 100)$ appended to $25 = 990025$

In this exercise we will show that these methods are not so magical after all!

- Based on *MM1* define a function `sq` that calculates the square of a natural number.
- Prove the correctness of `sq` (i.e. `sq n = n * n`).
- Formulate and prove the correctness of *MM2*.
  Hints:
  - Generalise *MM2* for an arbitrary constant (instead of 100).
  - Universally quantify all variables other than the induction variable.
- Formulate and prove the correctness of *MM3*.
  Hints:

– Try to formulate the property ‘numbers ending in 5’ such that it is easy to get to the rest of the number.
– Proving the binomial formula for $(a + b)^2$ can be of some help.