The Hahn-Banach Theorem for Real Vector Spaces

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Abstract

The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

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1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser’s textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.
Part I

Basic Notions

2 Bounds

theory Bounds
imports Main HOL-Analysis. Continuum-Not-Denumerable
begin

locale lub =
fixes A and x
assumes least [intro?]: (∀a. a ∈ A ⇒ a ≤ b) ⇒ x ≤ b
and upper [intro?]: a ∈ A ⇒ a ≤ x

lemmas [elim?] = lub.least lub.upper

definition the-lub :: 'a::order set ⇒ 'a
where the-lub A = The (lub A)

lemma the-lub-equality [elim?):
assumes lub A x
shows lub A (⨆ A)
⟨proof⟩

lemma the-lubI-ex:
assumes ex: ∃x. lub A x
shows lub A (⨆ A)
⟨proof⟩

lemma real-complete: ∃a::real. a ∈ A ⇒ ∃y. ∀a ∈ A. a ≤ y ⇒ ∃x. lub A x
⟨proof⟩
end

3 Vector spaces

theory Vector-Space
imports Complex-Main Bounds
begin

3.1 Signature

For the definition of real vector spaces a type 'a of the sort {plus, minus, zero}
is considered, on which a real scalar multiplication · is declared.

consts
prod :: real ⇒ 'a::{plus,minus,zero} ⇒ 'a (infixr · 70)
3.2 Vector space laws

A vector space is a non-empty set \( V \) of elements from 'a with the following vector space laws: The set \( V \) is closed under addition and scalar multiplication, addition is associative and commutative; \(- x\) is the inverse of \( x \) wrt. addition and \( 0 \) is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number \( 1 \) is the neutral element of scalar multiplication.

locale vectorspace =
  fixes \( V \)
  assumes non-empty [iff, intro?): \( V \neq \{\} \)
  and add-closed [iff]: \( x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V \)
  and mult-closed [iff]: \( x \in V \Longrightarrow a \cdot x \in V \)
  and add-assoc: \( x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z) \)
  and add-commute: \( x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x \)
  and diff-self [simp]: \( x \in V \Longrightarrow x - x = 0 \)
  and add-assoc [simp]: \( x \in V \Longrightarrow 0 + x = x \)
  and add-mult-distrib1: \( x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y \)
  and add-mult-distrib2: \( x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x \)
  and mult-assoc: \( x \in V \Longrightarrow (a \cdot b) \cdot x = a \cdot (b \cdot x) \)
  and mult-1 [simp]: \( x \in V \Longrightarrow 1 \cdot x = x \)
  and negate-eq1: \( x \in V \Longrightarrow -x = (\neg 1) \cdot x \)
  and diff-eq1: \( x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + -y \)

begin

lemma negate-eq2: \( x \in V \Longrightarrow (-1) \cdot x = -x \)
  ⟨proof⟩

lemma negate-eq2a: \( x \in V \Longrightarrow -1 \cdot x = -x \)
  ⟨proof⟩

lemma diff-eq2: \( x \in V \Longrightarrow y \in V \Longrightarrow x + -y = x - y \)
  ⟨proof⟩

lemma diff-closed [iff]: \( x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V \)
  ⟨proof⟩

lemma neg-closed [iff]: \( x \in V \Longrightarrow -x \in V \)
  ⟨proof⟩

lemma add-left-commute:
  \( x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow x + (y + z) = y + (x + z) \)
  ⟨proof⟩

lemmas add-ac = add-assoc add-commute add-left-commute

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

lemma zero [iff]: \( 0 \in V \)
  ⟨proof⟩

lemma add-zero-right [simp]: \( x \in V \Longrightarrow x + 0 = x \)
  ⟨proof⟩
3.2 Vector space laws

\textbf{lemma} mult-assoc2: \( x \in V \implies a \cdot b \cdot x = (a \ast b) \cdot x \)
\textbf{(proof)}

\textbf{lemma} diff-mult-distrib1: \( x \in V \implies y \in V \implies a \cdot (x - y) = a \cdot x - a \cdot y \)
\textbf{(proof)}

\textbf{lemma} diff-mult-distrib2: \( x \in V \implies (a - b) \cdot x = a \cdot x - (b \cdot x) \)
\textbf{(proof)}

\textbf{lemmas} distrib =
\begin{itemize}
  \item add-mult-distrib1
  \item add-mult-distrib2
  \item diff-mult-distrib1
  \item diff-mult-distrib2
\end{itemize}

Further derived laws:

\textbf{lemma} mult-zero-left [simp]: \( x \in V \implies 0 \cdot x = 0 \)
\textbf{(proof)}

\textbf{lemma} mult-zero-right [simp]: \( a \cdot 0 = (0::'a) \)
\textbf{(proof)}

\textbf{lemma} minus-mult-cancel [simp]: \( x \in V \implies (-a) \cdot -x = a \cdot x \)
\textbf{(proof)}

\textbf{lemma} add-minus-left-eq-diff: \( x \in V \implies y \in V \implies -x + y = y - x \)
\textbf{(proof)}

\textbf{lemma} add-minus [simp]: \( x \in V \implies x + -x = 0 \)
\textbf{(proof)}

\textbf{lemma} add-minus-left [simp]: \( x \in V \implies -x + x = 0 \)
\textbf{(proof)}

\textbf{lemma} minus-minus [simp]: \( x \in V \implies -(-x) = x \)
\textbf{(proof)}

\textbf{lemma} minus-zero [simp]: \( -(0::'a) = 0 \)
\textbf{(proof)}

\textbf{lemma} minus-zero-iff [simp]:
\begin{itemize}
  \item assumes \( x: x \in V \)
  \item shows \( (-x = 0) = (x = 0) \)
\end{itemize}
\textbf{(proof)}

\textbf{lemma} add-minus-cancel [simp]: \( x \in V \implies y \in V \implies x + (-x + y) = y \)
\textbf{(proof)}

\textbf{lemma} minus-add-cancel [simp]: \( x \in V \implies y \in V \implies -x + (x + y) = y \)
\textbf{(proof)}

\textbf{lemma} minus-add-distrib [simp]: \( x \in V \implies y \in V \implies -(x + y) = -x + -y \)
\textbf{(proof)}

\textbf{lemma} diff-zero [simp]: \( x \in V \implies x - 0 = x \)
lemma diff-zero-right [simp]: \( x \in V \implies 0 - x = -x \)

lemma add-left-cancel:
assumes \( x: x \in V \) and \( y: y \in V \) and \( z: z \in V \)
shows \((x + y = x + z) = (y = z)\)

lemma add-right-cancel:
\( x \in V \implies y \in V \implies z \in V \implies (y + x = z + x) = (y = z) \)

lemma add-assoc-cong:
\( x \in V \implies y \in V \implies x' \in V \implies y' \in V \implies z \in V \)
\( \implies x + y = x' + y' \implies x + (y + z) = x' + (y' + z) \)

lemma mult-left-commute:
\( x \in V \implies a \cdot b \cdot x = b \cdot a \cdot x \)

lemma mult-zero-uniq:
assumes \( x: x \in V \) and \( x \neq 0 \) and \( ax: a \cdot x = 0 \)
shows \( a = 0 \)

lemma mult-left-cancel:
assumes \( x: x \in V \) and \( y: y \in V \) and \( a: a \neq 0 \)
shows \( (a \cdot x = a \cdot y) = (x = y) \)

lemma mult-right-cancel:
assumes \( x: x \in V \) and \( x \neq 0 \)
shows \( (a \cdot x = b \cdot x) = (a = b) \)

lemma eq-diff-eq:
assumes \( x: x \in V \) and \( y: y \in V \) and \( z: z \in V \)
shows \( (x = z - y) = (x + y = z) \)

lemma add-minus-eq-minus:
assumes \( x: x \in V \) and \( y: y \in V \) and \( xy: x + y = 0 \)
shows \( x = -y \)

lemma add-minus-eq:
assumes \( x: x \in V \) and \( y: y \in V \) and \( xy: x - y = 0 \)
shows \( x = y \)

lemma add-diff-swap:
assumes vs: \( a \in V \) \( b \in V \) \( c \in V \) \( d \in V \)
and eq: \( a + b = c + d \)
shows \( a - c = d - b \)
\( \langle \text{proof} \rangle \)

lemma \( \text{vs-add-cancel-21} \):
assumes \( \text{vs}: x \in V \ y \in V \ z \in V \ u \in V \)
shows \( (x + (y + z)) = (y + u) = (x + z = u) \)
\( \langle \text{proof} \rangle \)

lemma \( \text{add-cancel-end} \):
assumes \( \text{vs}: x \in V \ y \in V \ z \in V \)
shows \( (x + (y + z) = y) = (x = - z) \)
\( \langle \text{proof} \rangle \)

end

4 Subspaces

theory Subspace
imports Vector-Space HOL-Library.Set-Algebras
begin

4.1 Definition

A non-empty subset \( U \) of a vector space \( V \) is a \textit{subspace} of \( V \), if \( U \) is closed under addition and scalar multiplication.

locale subspace =
fixes \( U \) :: \( 'a::{\text{minus, plus, zero, uminus}} \text{ set and} \ V \)
assumes non-empty \( \langle \text{iff, intro}\rangle \): \( U \neq \{\} \)
and subset \( \langle \text{iff}\rangle \): \( U \subseteq V \)
and add-closed \( \langle \text{iff}\rangle \): \( x \in U \implies y \in U \implies x + y \in U \)
and mult-closed \( \langle \text{iff}\rangle \): \( x \in U \implies a \cdot x \in U \)

notation \( \text{(symbols)} \)
subspace \( \langle \text{infix} \leq 50 \rangle \)
declare vectorspace.intro [intro?] subspace.intro [intro?]

lemma subspace-subset \( \langle \text{elim}\rangle \): \( U \leq V \implies U \subseteq V \)
\( \langle \text{proof} \rangle \)

lemma (in subspace) subsetD \( \langle \text{iff}\rangle \): \( x \in U \implies x \in V \)
\( \langle \text{proof} \rangle \)

lemma subspaceD \( \langle \text{elim}\rangle \): \( U \leq V \implies x \in U \implies x \in V \)
\( \langle \text{proof} \rangle \)

lemma rev-subspaceD \( \langle \text{elim}\rangle \): \( x \in U \implies U \leq V \implies x \in V \)
\( \langle \text{proof} \rangle \)

lemma (in subspace) diff-closed \( \langle \text{iff}\rangle \):

assumes vectorspace $V$
assumes $x$: $x \in U$ and $y$: $y \in U$
shows $x - y \in U$
(proof)

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

lemma (in subspace) zero [intro]:
assumes vectorspace $V$
shows $0 \in U$
(proof)

lemma (in subspace) neg-closed [iff]:
assumes vectorspace $V$
assumes $x$: $x \in U$
shows $-x \in U$
(proof)

Further derived laws: every subspace is a vector space.

lemma (in subspace) vectorspace [iff]:
assumes vectorspace $V$
shows vectorspace $U$
(proof)

The subspace relation is reflexive.

lemma (in vectorspace) subspace-refl [intro]: $V \subseteq V$
(proof)

The subspace relation is transitive.

lemma (in vectorspace) subspace-trans [trans]:
$U \subseteq V \Rightarrow V \subseteq W \Rightarrow U \subseteq W$
(proof)

4.2 Linear closure

The **linear closure** of a vector $x$ is the set of all scalar multiples of $x$.

definition lin :: ('a::{minus, plus, zero}) \Rightarrow 'a set
  where lin $x = \{ a \cdot x \mid a. \mbox{True} \}$

lemma linI [intro]: $y = a \cdot x \Rightarrow y \in \mbox{lin} x$
(proof)

lemma linI' [iff]: $a \cdot x \in \mbox{lin} x$
(proof)

lemma linE [elim]:
  assumes $x \in \mbox{lin} v$
  obtains $a :: \mbox{real}$ where $x = a \cdot v$
(proof)

Every vector is contained in its linear closure.
4.3 Sum of two vectorspaces

Lemma (in vectorspace) \( x \)-lin-x [iff]: \( x \in V \Rightarrow x \in \text{lin} x \)

(\text{proof})

Lemma (in vectorspace) 0-lin-x [iff]: \( x \in V \Rightarrow 0 \in \text{lin} x \)

(\text{proof})

Any linear closure is a subspace.

Lemma (in vectorspace) lin-subspace [intro]:
\begin{itemize}
    \item assumes \( x \in V \)
    \item shows lin \( x \) \( \subseteq V \)
\end{itemize}

(\text{proof})

Any linear closure is a vector space.

Lemma (in vectorspace) lin-vectorspace [intro]:
\begin{itemize}
    \item assumes \( x \in V \)
    \item shows vectorspace (lin \( x \))
\end{itemize}

(\text{proof})

4.3 Sum of two vectorspaces

The sum of two vectorspaces \( U \) and \( V \) is the set of all sums of elements from \( U \) and \( V \).

Lemma sum-def: \( U + V = \{ u + v \mid u \in U \land v \in V \} \)

(\text{proof})

Lemma sumE [elim]:
\[
x \in U + V \implies (\forall u, v. x = u + v \implies u \in U \implies v \in V \implies C) \implies C
\]

(\text{proof})

Lemma sumI [intro]:
\[
u \in U \implies v \in V \implies x = u + v \implies x \in U + V
\]

(\text{proof})

Lemma sumI’ [intro]:
\[
u \in U \implies v \in V \implies u + v \in U + V
\]

(\text{proof})

\( U \) is a subspace of \( U + V \).

Lemma subspace-sum1 [iff]:
\begin{itemize}
    \item assumes vectorspace \( U \) vectorspace \( V \)
    \item shows \( U \subseteq U + V \)
\end{itemize}

(\text{proof})

The sum of two subspaces is again a subspace.

Lemma sum-subspace [intro]:
\begin{itemize}
    \item assumes subspace \( U \subseteq E \) vectorspace \( V \subseteq E \)
    \item shows \( U + V \subseteq E \)
\end{itemize}

(\text{proof})

The sum of two subspaces is a vectorspace.

Lemma sum-vs [intro]:
\begin{itemize}
    \item \( U \subseteq E \Rightarrow \) \( V \subseteq E \Rightarrow \text{vectorspace} \) \( E \Rightarrow \text{vectorspace} \) \( U + V \)
\end{itemize}

(\text{proof})
4.4 Direct sums

The sum of $U$ and $V$ is called direct, iff the zero element is the only common element of $U$ and $V$. For every element $x$ of the direct sum of $U$ and $V$ the decomposition in $x = u + v$ with $u \in U$ and $v \in V$ is unique.

**Lemma decomp:**

- Assumes vectorspace $E$ subspace $U$ $E$ subspace $V$
- Assumes direct: $U \cap V = \{0\}$
- And $u1: u1 \in U$ and $u2: u2 \in U$
- And $v1: v1 \in V$ and $v2: v2 \in V$
- And sum: $u1 + v1 = u2 + v2$
- Shows $u1 = u2 \land v1 = v2$

**Proof**

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page ??): for any element $y + a \cdot x_0$ of the direct sum of a vectorspace $H$ and the linear closure of $x_0$ the components $y \in H$ and $a$ are uniquely determined.

**Lemma decomp-$H'$:**

- Assumes vectorspace $E$ subspace $H$ $E$
- Assumes $y1: y1 \in H$ and $y2: y2 \in H$
- And $x1: x1 \notin H$ $x1 \in E$ $x1 \neq 0$
- And eq: $y1 + a1 \cdot x1 = y2 + a2 \cdot x'$
- Shows $y1 = y2 \land a1 = a2$

**Proof**

Since for any element $y + a \cdot x'$ of the direct sum of a vectorspace $H$ and the linear closure of $x'$ the components $y \in H$ and $a$ are unique, it follows from $y \in H$ that $a = 0$.

**Lemma decomp-$H'$-H:**

- Assumes vectorspace $E$ subspace $H$ $E$
- Assumes $t: t \in H$
- And $x1: x1 \notin H$ $x1 \in E$ $x1 \neq 0$
- Shows $(\text{SOME} (y, a). t = y + a \cdot x' \land y \in H) = (t, 0)$

**Proof**

The components $y \in H$ and $a$ in $y + a \cdot x'$ are unique, so the function $h'$ defined by $h'(y + a \cdot x') = h y + a \cdot \xi$ is definite.

**Lemma $h'$-definite:**

- Fixes $H$
- Assumes $h'$-def:
  \[
  \land x. h' x = \\
  (\text{let} (y, a) = \text{SOME} (y, a). (x = y + a \cdot x' \land y \in H) \\
  \text{in} (h y) + a \ast x1)
  \]
- And $x: x = y + a \cdot x'$
- Assumes vectorspace $E$ subspace $H$ $E$
- Assumes $y: y \in H$
- And $x1: x1 \notin H$ $x1 \in E$ $x1 \neq 0$
- Shows $h' x = h y + a \ast x1$

**Proof**

end
5 Normed vector spaces

theory Normed-Space
imports Subspace
begin

5.1 Quasinorms

A seminorm $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogeneous and subadditive.

locale seminorm =
  fixes $V :: 'a::{\text{minus}, \text{plus}, \text{zero}, \text{uminus}}}$ set
  fixes $\text{norm} :: 'a \Rightarrow \text{real} \ (\|\cdot\|)$
assumes ge-zero [iff?]: $x \in V \Longrightarrow 0 \leq \|x\|
  and abs-homogenous [iff?]: $x \in V \Longrightarrow |a \cdot x| = |a| \cdot \|x\|
  and subadditive [iff?]: $x \in V \Longrightarrow y \in V \Longrightarrow \|x + y\| \leq \|x\| + \|y\|
declare seminorm.intro [intro?]

lemma (in seminorm) diff-subadditive:
  assumes vectorspace $V$
  shows $x \in V \Longrightarrow y \in V \Longrightarrow \|x - y\| \leq \|x\| + \|y\|
⟨proof⟩
lemma (in seminorm) minus:
  assumes vectorspace $V$
  shows $x \in V \Longrightarrow \|-x\| = \|x\|
⟨proof⟩

5.2 Norms

A norm $\|\cdot\|$ is a seminorm that maps only the 0 vector to 0.

locale norm = seminorm +
  assumes zero-iff [iff?]: $x \in V \Longrightarrow (\|x\| = 0) = (x = 0)$

5.3 Normed vector spaces

A vector space together with a norm is called a normed space.

locale normed-vectorspace = vectorspace + norm
declare normed-vectorspace.intro [intro?]

lemma (in normed-vectorspace) gt-zero [intro?]:
  assumes $x : x \in V \text{ and neg: } x \neq 0$
  shows $0 < \|x\|
⟨proof⟩

Any subspace of a normed vector space is again a normed vectorspace.

lemma subspace-normed-vs [intro?):
  fixes $F E \text{ norm}$
  assumes subspace $F E \text{ normed-vectorspace} E \text{ norm}$
6 Linearforms

theory Linearform
imports Vector-Space
begin

A linear form is a function on a vector space into the reals that is additive and multiplicative.

locale linearform =
  fixes V :: 'a::{minus, plus, zero, uminus} set and f
  assumes add [iff]: \( x \in V \implies y \in V \implies f(x + y) = f(x) + f(y) \)
  and mult [iff]: \( x \in V \implies f(a \cdot x) = a \cdot f(x) \)

declare linearform.intro [intro?]

lemma (in linearform) neg [iff]:
  assumes vectorspace V
  shows \( x \in V \implies f(-x) = -f(x) \)
(proof)

lemma (in linearform) diff [iff]:
  assumes vectorspace V
  shows \( x \in V \implies y \in V \implies f(x - y) = f(x) - f(y) \)
(proof)

Every linear form yields 0 for the 0 vector.

lemma (in linearform) zero [iff]:
  assumes vectorspace V
  shows \( f(0) = 0 \)
(proof)

end

7 An order on functions

theory Function-Order
imports Subspace Linearform
begin

7.1 The graph of a function

We define the graph of a (real) function \( f \) with domain \( F \) as the set

\[ \{(x, f(x)) : x \in F\} \]

So we are modeling partial functions by specifying the domain and the mapping function. We use the term “function” also for its graph.
type-synonym 'a graph = ('a × real) set

definition graph :: 'a set ⇒ ('a ⇒ real) ⇒ 'a graph
where graph F f = { (x, f x) | x ∈ F }

lemma graphI [intro]: x ∈ F ⇒ (x, f x) ∈ graph F f
⟨proof⟩

lemma graphI2 [intro]: x ∈ F ⇒ ∃ t ∈ graph F f. t = (x, f x)
⟨proof⟩

lemma graphE [elim]:
assumes (x, y) ∈ graph F f
obtains x ∈ F and y = f x
⟨proof⟩

7.2 Functions ordered by domain extension

A function h' is an extension of h, iff the graph of h is a subset of the graph of h'.

lemma graph-extI:
(∀ x. x ∈ H ⇒ h x = h' x) ⇒ H ⊆ H'
⇒ graph H h ⊆ graph H' h'
⟨proof⟩

lemma graph-extD1 [dest]: graph H h ⊆ graph H' h' ⇒ x ∈ H ⇒ h x = h' x
⟨proof⟩

lemma graph-extD2 [dest]: graph H h ⊆ graph H' h' ⇒ H ⊆ H'
⟨proof⟩

7.3 Domain and function of a graph

The inverse functions to graph are domain and funct.

definition domain :: 'a graph ⇒ 'a set
where domain g = { x. ∃ y. (x, y) ∈ g }

definition funct :: 'a graph ⇒ ('a ⇒ real)
where funct g = (λ x. (SOME y. (x, y) ∈ g))

The following lemma states that g is the graph of a function if the relation induced by g is unique.

lemma graph-domain-funct:
assumes uniq: ∀ x y z. (x, y) ∈ g ⇒ (x, z) ∈ g ⇒ z = y
shows graph (domain g) (funct g) = g
⟨proof⟩

7.4 Norm-preserving extensions of a function

Given a linear form f on the space F and a seminorm p on E. The set of all linear extensions of f, to superspaces H of F, which are bounded by p, is defined as follows.
8 The norm of a function

theory Function-Norm
imports Normed-Space Function-Order
begin

8.1 Continuous linear forms

A linear form \( f \) on a normed vector space \((V, \|\cdot\|)\) is continuous, iff it is bounded, i.e.

\[ \exists c \in \mathbb{R}. \forall x \in V. |f x| \leq c \cdot \|x\| \]
In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

```plaintext
locale continuous = linearform +
fixes norm :: - ⇒ real (∥-∥)
assumes bounded: ∃c. ∀x ∈ V. |f x| ≤ c * ∥x∥
```

```plaintext
declare continuous.intro [intro?] continuous-axioms.intro [intro?]
```

```plaintext
lemma continuousI [intro]:
fixes norm :: - ⇒ real (∥-∥)
assumes linearform V f
assumes r: ∀x. x ∈ V =⇒ |f x| ≤ c * ∥x∥
shows continuous V f norm
```

### 8.2 The norm of a linear form

The least real number $c$ for which holds

$$\forall x ∈ V. |f x| ≤ c · ∥x∥$$

is called the norm of $f$.

For non-trivial vector spaces $V ≠ \{0\}$ the norm can be defined as

$$∥f∥ = \text{sup } x ≠ 0. |f x| / ∥x∥$$

For the case $V = \{0\}$ the supremum would be taken from an empty set. Since $\mathbb{R}$ is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be \(|\} ≥ 0\) so that fn-norm has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be $0$, as all other elements are \(\{\} ≥ 0\).

Thus we define the set $B$ where the supremum is taken from as follows:

$$\{0\} \cup \{|f x| / ∥x∥. x ≠ 0 ∧ x ∈ F\}$$

fn-norm is equal to the supremum of $B$, if the supremum exists (otherwise it is undefined).

```plaintext
locale fn-norm =
fixes norm :: - ⇒ real (∥-∥)
fixes B defines B V f ≡ \{0\} ∪ \{|f x| / ∥x∥ | x ≠ 0 ∧ x ∈ V\}
fixes fn-norm (∥-∥)-\[0, 1000\] 999)
defines ∥f∥-V ≡ ∪(B V f)
```

```plaintext
locale normed-vectorspace-with-fn-norm = normed-vectorspace + fn-norm
```

```plaintext
lemma (in fn-norm) B-not-empty [intro]: 0 ∈ B V f
(proof)
```

The following lemma states that every continuous linear form on a normed space $(V, ∥-∥)$ has a function norm.

```plaintext
lemma (in normed-vectorspace-with-fn-norm) fn-norm-works:
```
assumes continuous $Vf$ norm
shows lub $(B \cup V f) (\|f\| \cdot V)$
(proof)

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ub [iff]:
  assumes continuous $Vf$ norm
  assumes $b: b \in B \cup V f$
  shows $b \leq \|f\| \cdot V$
(proof)

lemma (in normed-vectorspace-with-fn-norm) fn-norm-leastB:
  assumes continuous $Vf$ norm
  assumes $b: \forall b. b \in B \cup V f \Rightarrow b \leq y$
  shows $\|f\| \cdot V \leq y$
(proof)

The norm of a continuous function is always $\geq 0$.

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ge-zero [iff]:
  assumes continuous $Vf$ norm
  shows $0 \leq \|f\| \cdot V$
(proof)

The fundamental property of function norms is:

$$|f x| \leq \|f\| \cdot \|x\|$$

lemma (in normed-vectorspace-with-fn-norm) fn-norm-le-cong:
  assumes continuous $Vf$ norm linearform $Vf$
  assumes $x: x \in V$
  shows $|f x| \leq \|f\| \cdot V * \|x\|$
(proof)

The function norm is the least positive real number for which the following inequality holds:

$$|f x| \leq c \cdot \|x\|$$

lemma (in normed-vectorspace-with-fn-norm) fn-norm-least [intro]:
  assumes continuous $Vf$ norm
  assumes ineq: $\forall x. x \in V \Rightarrow |f x| \leq c \cdot \|x\|$ and ge: $0 \leq c$
  shows $\|f\| \cdot V \leq c$
(proof)

end

9 Zorn’s Lemma

theory Zorn-Lemma
imports Main
begin

Zorn’s Lemmas states: if every linear ordered subset of an ordered set $S$ has an upper bound in $S$, then there exists a maximal element in $S$. In our application,
$S$ is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn’s lemma can be modified: if $S$ is non-empty, it suffices to show that for every non-empty chain $c$ in $S$ the union of $c$ also lies in $S$.

**Theorem Zorn’s Lemma:**

**Assumes** $\forall c. c \in \text{chains } S \implies \exists x. x \in c \implies \bigcup c \in S$

**And** $a \in S$

**Shows** $\exists y \in S. \forall z \in S. y \subseteq z \implies z = y$

(proof)

end
Part II
Lemmas for the Proof

10 The supremum wrt. the function order

theory Hahn-Banach-Sup-Lemmas
imports Function-Norm Zorn-Lemma
begin

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let $E$ be a real vector space with a seminorm $p$ on $E$. $F$ is a subspace of $E$ and $f$ a linear form on $F$. We consider a chain $c$ of norm-preserving extensions of $f$, such that $\bigcup c = \text{graph } H h$. We will show some properties about the limit function $h$, i.e. the supremum of the chain $c$.

Let $c$ be a chain of norm-preserving extensions of the function $f$ and let $\text{graph } H h$ be the supremum of $c$. Every element in $H$ is member of one of the elements of the chain.

lemma [dest?] = chainsD
lemma chainsE2 [elim?] = chainsD2 [elim-format]

lemma some-H h':
  assumes $M: M = \text{norm-pres-extensions } E p F f$
  and $cM: c \in \text{chains } M$
  and $u: \text{graph } H h = \bigcup c$
  and $x: x \in H$
  shows $\exists H' h', \text{graph } H' h' \subseteq c$
  $\land (x, h x) \in \text{graph } H' h'$
  $\land \text{linearform } H' h' \land H' \leq E$
  $\land F \leq H' \land \text{graph } F f \subseteq \text{graph } H' h'$
  $\land (\forall x \in H', h' x \leq p x)$
⟨proof⟩

Let $c$ be a chain of norm-preserving extensions of the function $f$ and let $\text{graph } H h$ be the supremum of $c$. Every element in the domain $H$ of the supremum function is member of the domain $H'$ of some function $h'$, such that $h$ extends $h'$.

lemma some-H h':
  assumes $M: M = \text{norm-pres-extensions } E p F f$
  and $cM: c \in \text{chains } M$
  and $u: \text{graph } H h = \bigcup c$
  and $x: x \in H$
  shows $\exists H' h', x \in H' \land \text{graph } H' h' \subseteq \text{graph } H h$
  $\land \text{linearform } H' h' \land H' \leq E \land F \leq H'$
  $\land \text{graph } F f \subseteq \text{graph } H' h' \land (\forall x \in H', h' x \leq p x)$
⟨proof⟩

Any two elements $x$ and $y$ in the domain $H$ of the supremum function $h$ are both in the domain $H'$ of some function $h'$, such that $h$ extends $h'$. 
lemma some-H' h':
assumes M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and u: graph H h = \bigcup c
and x: x ∈ H
and y: y ∈ H
shows ∃ H' h'. x ∈ H' ∧ y ∈ H'
∧ graph H' h' ⊆ graph H h
∧ linearform H' h' ∧ H' ≤ E ∧ F ≤ H'
∧ graph F f ⊆ graph H' h' ∧ (\forall x ∈ H', h' x ≤ p x)
(proof)

The relation induced by the graph of the supremum of a chain c is definite, i.e. it is the graph of a function.

lemma sup-definite:
assumes M-def: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and xy: (x, y) ∈ \bigcup c
and xz: (x, z) ∈ \bigcup c
shows z = y
(proof)

The limit function h is linear. Every element x in the domain of h is in the domain of a function h' in the chain of norm preserving extensions. Furthermore, h is an extension of h' so the function values of x are identical for h' and h. Finally, the function h' is linear by construction of M.

lemma sup-lf:
assumes M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and u: graph H h = \bigcup c
shows linearform H h
(proof)

The limit of a non-empty chain of norm preserving extensions of f is an extension of f, since every element of the chain is an extension of f and the supremum is an extension for every element of the chain.

lemma sup-ext:
assumes graph: graph H h = \bigcup c
and M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and ex: ∃ x. x ∈ c
shows graph F f ⊆ graph H h
(proof)

The domain H of the limit function is a superspace of F, since F is a subset of H. The existence of the 0 element in F and the closure properties follow from the fact that F is a vector space.

lemma sup-supF:
assumes graph: graph H h = \bigcup c
and M: M = norm-pres-extensions E p F f
and \( cM: c \in \text{chains } M \)
and \( ex: \exists x. x \in c \)
and \( FE: F \leq E \)
shows \( F \leq H \)
(proof)

The domain \( H \) of the limit function is a subspace of \( E \).

**lemma sup-subE:**
assumes \( \text{graph: } \text{graph } H h = \bigcup c \)
and \( M: \text{M = norm-pres-extensions } E p F f \)
and \( cM: c \in \text{chains } M \)
and \( ex: \exists x. x \in c \)
and \( FE: F \leq E \)
and \( E: \text{vectorspace } E \)
shows \( H \leq E \)
(proof)

The limit function is bounded by the norm \( p \) as well, since all elements in the chain are bounded by \( p \).

**lemma sup-norm-pres:**
assumes \( \text{graph: } \text{graph } H h = \bigcup c \)
and \( M: \text{M = norm-pres-extensions } E p F f \)
and \( cM: c \in \text{chains } M \)
shows \( \forall x \in H. h x \leq p x \)
(proof)

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma \( \text{abs-Hahn-Banach} \) (see page 24). For real vector spaces the following inequality are equivalent:

\[
\forall x \in H. \ |h x| \leq p x \quad \text{and} \quad \forall x \in H. h x \leq p x
\]

**lemma abs-ineq-iff:**
assumes \( \text{subspace } H E \text{ and vectorspace } E \text{ and seminorm } E p \)
and \( \text{linearform } H h \)
shows \( (\forall x \in H. \ |h x| \leq p x) = (\forall x \in H. h x \leq p x) \) (is \(?L = ?R\) )
(proof)

end

11 Extending non-maximal functions

**theory Hahn-Banach-Ext-Lemmas**
**imports** Function-Norm
**begin**

In this section the following context is presumed. Let \( E \) be a real vector space with a seminorm \( q \) on \( E \). \( F \) is a subspace of \( E \) and \( f \) a linear function on \( F \). We consider a subspace \( H \) of \( E \) that is a superspace of \( F \) and a linear form \( h \) on \( H \). \( H \) is not equal to \( E \) and \( x_0 \) is an element in \( E - H \). \( H \) is extended to the direct sum \( H' = H + \text{lin } x_0 \), so for any \( x \in H' \) the decomposition of \( x = y + \)
\[ a \cdot x \text{ with } y \in H \text{ is unique. } h' \text{ is defined on } H' \text{ by } h' x = h y + a \cdot \xi \text{ for a certain } \xi. \]

Subsequently we show some properties of this extension \( h' \) of \( h \).

This lemma will be used to show the existence of a linear extension of \( f \) (see page ??). It is a consequence of the completeness of \( \mathbb{R} \). To show

\[ \exists \xi. \forall y \in F. \ a \ y \leq \xi \wedge \xi \leq b \ y \]

it suffices to show that

\[ \forall u \in F. \forall v \in F. \ a \ u \leq b \ v \]

lemma \texttt{ex-xi}:

assumes \texttt{vectorspace } \( F \)

assumes \( r: \forall u \in F. \forall v \in F. \ a \ u \leq b \ v \)

shows \( \exists xi::\text{real}. \forall y \in F. \ a \ y \leq xi \wedge xi \leq b \ y \)

(proof)

The function \( h' \) is defined as a \( h' x = h y + a \cdot \xi \) where \( x = y + a \cdot \xi \) is a linear extension of \( h \) to \( H' \).

lemma \texttt{h'-lf}:

assumes \texttt{h'-def}: \( \forall x. \ h' x = (\text{let } (y, a) = \text{SOME} (y, a). x = y + a \cdot x0 \wedge y \in H \in h y + a \ast xi) \)

and \( H'\text{-def}: H' = H + \text{lin } x0 \)

and \( HE: H \subseteq E \)

assumes \texttt{linearform } \( H \)

assumes \( x0: x0 \notin H \ x0 \in E \ x0 \neq 0 \)

assumes \texttt{E: vectorspace } \( E \)

shows \texttt{linearform } \( H' \)

(proof)

The linear extension \( h' \) of \( h \) is bounded by the seminorm \( p \).

lemma \texttt{h'-norm-pres}:

assumes \texttt{h'-def}: \( \forall x. \ h' x = (\text{let } (y, a) = \text{SOME} (y, a). x = y + a \cdot x0 \wedge y \in H \in h y + a \ast xi) \)

and \( H'\text{-def}: H' = H + \text{lin } x0 \)

and \( x0: x0 \notin H \ x0 \in E \ x0 \neq 0 \)

assumes \texttt{E: vectorspace } \( E \) and \( HE: \text{subspace } H \)

and \( \text{seminorm } E \)

and \( \texttt{linearform } H \)

assumes \( a: \forall y \in H. \ h y \leq p \ y \)

and \( a': \forall y \in H. \ -p \ (y + x0) - h y \leq xi \wedge xi \leq p \ (y + x0) - h y \)

shows \( \forall x \in H'. \ h' x \leq p \ x \)

(proof)

end
Part III
The Main Proof

12 The Hahn-Banach Theorem

theory Hahn-Banach
imports Hahn-Banach-Lemmas
begin

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

12.1 The Hahn-Banach Theorem for vector spaces

Hahn-Banach Theorem. Let \( F \) be a subspace of a real vector space \( E \), let \( p \) be a semi-norm on \( E \), and \( f \) be a linear form defined on \( F \) such that \( f \) is bounded by \( p \), i.e. \( \forall x \in F. \ f x \leq p x \). Then \( f \) can be extended to a linear form \( h \) on \( E \) such that \( h \) is norm-preserving, i.e. \( h \) is also bounded by \( p \).

Proof Sketch.
1. Define \( M \) as the set of norm-preserving extensions of \( f \) to subspaces of \( E \). The linear forms in \( M \) are ordered by domain extension.
2. We show that every non-empty chain in \( M \) has an upper bound in \( M \).
3. With Zorn's Lemma we conclude that there is a maximal function \( g \) in \( M \).
4. The domain \( H \) of \( g \) is the whole space \( E \), as shown by classical contradiction:
   - Assuming \( g \) is not defined on whole \( E \), it can still be extended in a norm-preserving way to a super-space \( H' \) of \( H \).
   - Thus \( g \) can not be maximal. Contradiction!

theorem Hahn-Banach:
assumes E: vectorspace E and subspace F E and seminorm E p and linearform F f
assumes fp: \( \forall x \in F. \ f x \leq p x \)
shows \( \exists h. \ linearform E h \land (\forall x \in F. \ h x = f x) \land (\forall x \in E. \ h x \leq p x) \)
| --- 
| | --- 
| | Let \( E \) be a vector space, \( F \) a subspace of \( E \), \( p \) a seminorm on \( E \),
| | and \( f \) a linear form on \( F \) such that \( f \) is bounded by \( p \),
| | then \( f \) can be extended to a linear form \( h \) on \( E \) in a norm-preserving way.

⟨proof⟩

12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form \( f \) and a seminorm \( p \) the following inequality are equivalent:\(^1\)

\(^1\)This was shown in lemma abs-inq-iff (see page 22).
∀ x ∈ H. |h x| ≤ p x and ∀ x ∈ H. h x ≤ p x

theorem abs-Hahn-Banach:
assumes E: vectorspace E and FE: subspace F E
and lf: linearform F f and sn: seminorm E p
assumes fp: ∀ x ∈ F. |f x| ≤ p x
shows ∃ g. linearform E g ∧ (∀ x ∈ F. g x = f x)
∧ (∀ x ∈ E. |g x| ≤ p x)
(proof)

12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form f on a subspace F of a norm space E, can be extended to a continuous linear form g on E such that ∥f∥ = ∥g∥.

theorem norm-Hahn-Banach:
fixes V and norm (∥-∥)
fixes B defines \( \bigvee V \ f \equiv \{0\} \cup \{|f x| / \|x\| | x \neq 0 \land x \in V\}\)
defines fn-norm (∥-∥-∥-∥) (0, 1000) 999
assumes E-norm: normed-vectorspace E norm and FE: subspace F E
and linearform: linearform F f and continuous F f norm
shows ∃ g. linearform E g ∧ continuous E g norm
∧ (∀ x ∈ F. g x = f x)
∧ ∥g∥-E = ∥f∥-F
(proof)

end

References