The Hahn-Banach Theorem for Real Vector Spaces

Gertrud Bauer

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Abstract

The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

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1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser’s textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.
Part I

Basic Notions

2 Bounds

theory Bounds
imports Main HOL–Analysis. Continuum-Not-Denumerable
begin

locale lub =
  fixes A and x
  assumes least [intro?] : (∀ a. a ∈ A → a ≤ b) → x ≤ b
  and upper [intro?]: a ∈ A → a ≤ x

lemmas [elim?] = lub.least lub.upper

definition the-lub :: 'a::order set ⇒ 'a (⨆ - [90] 90)
  where the-lub A = The (lub A)

lemma the-lub-equality [elim?]:
  assumes lub A x
  shows ⨆ A = (x::'a::order)
  ⟨proof⟩

lemma the-lubI-ex:
  assumes ex: ∃ x. lub A x
  shows lub A (⨆ A)
  ⟨proof⟩

lemma real-complete: ∃ a::real. a ∈ A → ∃ y. ∀ a ∈ A. a ≤ y → ∃ x. lub A x
  ⟨proof⟩

end

3 Vector spaces

theory Vector-Space
imports Complex-Main Bounds
begin

3.1 Signature

For the definition of real vector spaces a type 'a of the sort {plus, minus, zero} is considered, on which a real scalar multiplication · is declared.

consts
  prod :: real ⇒ 'a::{plus, minus, zero} ⇒ 'a (infixr · 70)
3.2 Vector space laws

A vector space is a non-empty set $V$ of elements from 'a with the following vector space laws: The set $V$ is closed under addition and scalar multiplication, addition is associative and commutative; $-x$ is the inverse of $x$ wrt. addition and $0$ is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number $1$ is the neutral element of scalar multiplication.

locale vectorspace =

  fixes V
  assumes non-empty [iff, intro?]: V ≠ {} and add-closed [iff]: x ∈ V ⇒ y ∈ V ⇒ x + y ∈ V
  and mult-closed [iff]: x ∈ V ⇒ a · x ∈ V
  and add-assoc: x ∈ V ⇒ y ∈ V ⇒ z ∈ V ⇒ (x + y) + z = x + (y + z)
  and add-commute: x ∈ V ⇒ y ∈ V ⇒ x + y = y + x
  and diff-self [simp]: x ∈ V ⇒ x − x = 0
  and add-zero-left [simp]: x ∈ V ⇒ 0 + x = x
  and add-mult-distrib1: x ∈ V ⇒ y ∈ V ⇒ a · (x + y) = a · x + a · y
  and add-mult-distrib2: x ∈ V ⇒ (a + b) · x = a · x + b · x
  and mult-assoc: x ∈ V ⇒ (a * b) · x = a · (b · x)
  and mult-1 [simp]: x ∈ V ⇒ 1 · x = x
  and negate-eq1: x ∈ V ⇒ −x = (−1) · x
  and diff-eq1: x ∈ V ⇒ y ∈ V ⇒ x − y = x + (−y)

begin

lemma negate-eq2: x ∈ V ⇒ (−1) · x = −x
⟨proof⟩

lemma negate-eq2a: x ∈ V ⇒ −1 · x = −x
⟨proof⟩

lemma diff-eq2: x ∈ V ⇒ y ∈ V ⇒ x − y = x − y
⟨proof⟩

lemma diff-closed [iff]: x ∈ V ⇒ y ∈ V ⇒ x − y ∈ V
⟨proof⟩

lemma neg-closed [iff]: x ∈ V ⇒ −x ∈ V
⟨proof⟩

lemma add-left-commute:
  x ∈ V ⇒ y ∈ V ⇒ z ∈ V ⇒ x + (y + z) = y + (x + z)
⟨proof⟩

lemmas add-ac = add-associative add-commutative add-left-commute

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

lemma zero [iff]: 0 ∈ V
⟨proof⟩

lemma add-zero-right [simp]: x ∈ V ⇒ x + 0 = x
⟨proof⟩
3.2 Vector space laws

**lemma** mult-assoc2: \( x \in V \implies a \cdot b \cdot x = (a * b) \cdot x \)

(proof)

**lemma** diff-mult-distrib1: \( x \in V \implies y \in V \implies a \cdot (x - y) = a \cdot x - a \cdot y \)

(proof)

**lemma** diff-mult-distrib2: \( x \in V \implies (a - b) \cdot x = a \cdot x - (b \cdot x) \)

(proof)

**lemmas** distrib =

  add-mult-distrib1 add-mult-distrib2
diff-mult-distrib1 diff-mult-distrib2

Further derived laws:

**lemma** mult-zero-left [simp]: \( x \in V \implies 0 \cdot x = 0 \)

(proof)

**lemma** mult-zero-right [simp]: \( a \cdot 0 = (0 :: 'a) \)

(proof)

**lemma** minus-mult-cancel [simp]: \( x \in V \implies (- a) \cdot - x = a \cdot x \)

(proof)

**lemma** add-minus-left-eq-diff: \( x \in V \implies y \in V \implies - x + y = y - x \)

(proof)

**lemma** add-minus [simp]: \( x \in V \implies x + - x = 0 \)

(proof)

**lemma** add-minus-left [simp]: \( x \in V \implies - x + x = 0 \)

(proof)

**lemma** minus-minus [simp]: \( x \in V \implies - (- x) = x \)

(proof)

**lemma** minus-zero [simp]: \( - (0 :: 'a) = 0 \)

(proof)

**lemma** minus-zero-iff [simp]:

assumes \( x: x \in V \)

shows \((- x = 0) = (x = 0)\)

(proof)

**lemma** add-minus-cancel [simp]: \( x \in V \implies y \in V \implies x + (- x + y) = y \)

(proof)

**lemma** minus-add-cancel [simp]: \( x \in V \implies y \in V \implies - x + (x + y) = y \)

(proof)

**lemma** minus-add-distrib [simp]: \( x \in V \implies y \in V \implies - (x + y) = - x + - y \)

(proof)

**lemma** diff-zero [simp]: \( x \in V \implies x - 0 = x \)

(proof)
lemma diff-zero-right [simp]: \( x \in V \Rightarrow 0 - x = -x \)

(\textit{proof})

lemma add-left-cancel:
\begin{itemize}
  \item assumes \( x: x \in V \) and \( y: y \in V \) and \( z: z \in V \)
  \item shows \( (x + y = x + z) = (y = z) \)
\end{itemize}

(\textit{proof})

lemma add-right-cancel:
\begin{itemize}
  \item \( x \in V \Rightarrow y \in V \Rightarrow z \in V \Rightarrow (y + x = z + x) = (y = z) \)
\end{itemize}

(\textit{proof})

lemma add-assoc-cong:
\begin{itemize}
  \item \( x \in V \Rightarrow y \in V \Rightarrow x' \in V \Rightarrow y' \in V \Rightarrow z \in V \Rightarrow (x + (y + z)) = (x' + (y' + z)) \)
\end{itemize}

(\textit{proof})

lemma mult-left-commute:
\begin{itemize}
  \item \( x \in V \Rightarrow a \cdot b \cdot x = b \cdot a \cdot x \)
\end{itemize}

(\textit{proof})

lemma mult-zero-uniq:
\begin{itemize}
  \item assumes \( x: x \in V \) \( x \neq 0 \) and \( ax: a \cdot x = 0 \)
  \item shows \( a = 0 \)
\end{itemize}

(\textit{proof})

lemma add-left-cancel:
\begin{itemize}
  \item assumes \( x: x \in V \) and \( y: y \in V \) and \( a: a \neq 0 \)
  \item shows \( (a \cdot x = a \cdot y) = (x = y) \)
\end{itemize}

(\textit{proof})

lemma mult-right-cancel:
\begin{itemize}
  \item assumes \( x: x \in V \) and \( neq: x \neq 0 \)
  \item shows \( (a \cdot x = b \cdot x) = (a = b) \)
\end{itemize}

(\textit{proof})

lemma eq-diff-eq:
\begin{itemize}
  \item assumes \( x: x \in V \) and \( y: y \in V \) and \( z: z \in V \)
  \item shows \( (x = z - y) = (x + y = z) \)
\end{itemize}

(\textit{proof})

lemma add-minus-eq-minus:
\begin{itemize}
  \item assumes \( x: x \in V \) and \( y: y \in V \) and \( xy: x + y = 0 \)
  \item shows \( x = -y \)
\end{itemize}

(\textit{proof})

lemma add-minus-eq:
\begin{itemize}
  \item assumes \( x: x \in V \) and \( y: y \in V \) and \( xy: x - y = 0 \)
  \item shows \( x = y \)
\end{itemize}

(\textit{proof})

lemma add-diff-swap:
\begin{itemize}
  \item assumes \( vs: a \in V \) \( b \in V \) \( c \in V \) \( d \in V \)
\end{itemize}
and eq: \( a + b = c + d \)
shows \( a - c = d - b \)

\( \langle \text{proof} \rangle \)

lemma vs-add-cancel-21:
assumes vs: \( x \in V \ y \in V \ z \in V \ u \in V \)
shows \( (x + (y + z) = y + u) = (x + z = u) \)

\( \langle \text{proof} \rangle \)

lemma add-cancel-end:
assumes vs: \( x \in V \ y \in V \ z \in V \)
shows \( (x + (y + z) = y) = (x = -z) \)

\( \langle \text{proof} \rangle \)

end

end

4 Subspaces

theory Subspace
imports Vector-Space HOL-Library.Set-Algebras
begin

4.1 Definition

A non-empty subset \( U \) of a vector space \( V \) is a \emph{subspace} of \( V \), iff \( U \) is closed under addition and scalar multiplication.

locale subspace =
fixes \( U :\{\text{minus, plus, zero, uminus}\} \text{ set and} \ V \)
assumes non-empty [iff, intro]: \( U \neq \{\} \)
and subset [iff]: \( U \subseteq V \)
and add-closed [iff]: \( x \in U \implies y \in U \implies x + y \in U \)
and mult-closed [iff]: \( x \in U \implies a \cdot x \in U \)

notation (symbols)
\text{subspace} (infix \( \subseteq \) 50)

declare vectorspace.intro [intro?] subspace.intro [intro?]

lemma subspace-subset [elim]: \( U \subseteq V \implies U \subseteq V \)
\( \langle \text{proof} \rangle \)

lemma (in subspace) subsetD [iff]: \( x \in U \implies x \in V \)
\( \langle \text{proof} \rangle \)

lemma subspaceD [elim]: \( U \subseteq V \implies x \in U \implies x \in V \)
\( \langle \text{proof} \rangle \)

lemma rev-subspaceD [elim?]: \( x \in U \implies U \subseteq V \implies x \in V \)
\( \langle \text{proof} \rangle \)

lemma (in subspace) diff-closed [iff]:

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

**Lemma (in subspace) zero [intro]:**
- Assumes vectorspace $V$
- Shows $0 \in U$

**Lemma (in subspace) neg-closed [iff]:**
- Assumes vectorspace $V$
- Assumes $x \in U$
- Shows $-x \in U$

Further derived laws: every subspace is a vector space.

**Lemma (in subspace) vectorspace [iff]:**
- Assumes vectorspace $V$
- Shows vectorspace $U$

The subspace relation is reflexive.

**Lemma (in vectorspace) subspace-refl [intro]:** $V \subseteq V$

The subspace relation is transitive.

**Lemma (in vectorspace) subspace-trans [trans]:**
- $U \subseteq V \Rightarrow V \subseteq W \Rightarrow U \subseteq W$

### 4.2 Linear closure

The **linear closure** of a vector $x$ is the set of all scalar multiples of $x$.

**Definition lin :: (‘a::{minus,plus,zero}) ⇒ ‘a set**
- Where $lin x = \{a \cdot x \mid a.\ True\}$

**Lemma linI [intro]:** $y = a \cdot x \Rightarrow y \in lin x$

**Lemma linI’ [iff]:** $a \cdot x \in lin x$

**Lemma linE [elim]:**
- Assumes $x \in lin v$
- Obtains $a :: real$ where $x = a \cdot v$

Every vector is contained in its linear closure.
4.3 Sum of two vectorspaces

**Lemma** (in vectorspace) \( x \text{-lin-} x \) [iff]: \( x \in V \implies x \in \text{lin } x \)

(\text{proof})

**Lemma** (in vectorspace) \( 0 \text{-lin-} x \) [iff]: \( x \in V \implies 0 \in \text{lin } x \)

(\text{proof})

Any linear closure is a subspace.

**Lemma** (in vectorspace) \( \text{lin-subspace} \) [intro]:
\[\text{assumes } x : x \in V \]
\[\text{shows } \text{lin } x \subseteq V \]

(\text{proof})

Any linear closure is a vector space.

**Lemma** (in vectorspace) \( \text{lin-vectorspace} \) [intro]:
\[\text{assumes } x \in V \]
\[\text{shows } \text{vectorspace } (\text{lin } x) \]

(\text{proof})

### 4.3 Sum of two vectorspaces

The *sum* of two vectorspaces \( U \) and \( V \) is the set of all sums of elements from \( U \) and \( V \).

**Lemma** \( \text{sum-def} \): \( U + V = \{ u + v \mid u \in U \land v \in V \} \)

(\text{proof})

**Lemma** \( \text{sumE} \) [elim]:
\[ x \in U + V \implies (\bigwedge u v. x = u + v \implies u \in U \implies v \in V \implies C) \implies C \]

(\text{proof})

**Lemma** \( \text{sumI} \) [intro]:
\[ u \in U \implies v \in V \implies x = u + v \implies x \in U + V \]

(\text{proof})

**Lemma** \( \text{sumI}' \) [intro]:
\[ u \in U \implies v \in V \implies u + v \in U + V \]

(\text{proof})

\( U \) is a subspace of \( U + V \).

**Lemma** \( \text{subspace-sum1} \) [iff]:
\[\text{assumes } \text{vectorspace } U \text{ vectorspace } V \]
\[\text{shows } U \subseteq U + V \]

(\text{proof})

The sum of two subspaces is again a subspace.

**Lemma** \( \text{sum-subspace} \) [intro]:
\[\text{assumes } \text{subspace } U E \text{ vectorspace } V E \]
\[\text{shows } U + V \subseteq E \]

(\text{proof})

The sum of two subspaces is a vectorspace.

**Lemma** \( \text{sum-vs} \) [intro]:
\[ U \subseteq E \implies V \subseteq E \implies \text{vectorspace } E \implies \text{vectorspace } (U + V) \]

(\text{proof})
4.4 Direct sums

The sum of $U$ and $V$ is called direct, iff the zero element is the only common element of $U$ and $V$. For every element $x$ of the direct sum of $U$ and $V$ the decomposition in $x = u + v$ with $u \in U$ and $v \in V$ is unique.

**Lemma decomp:**

- Assumes vectorspace $E$ subspace $U$ $E$ subspace $V$
- Assumes direct: $U \cap V = \{0\}$
  - and $u1$: $u1 \in U$
  - and $u2$: $u2 \in U$
  - and $v1$: $v1 \in V$ and $v2$: $v2 \in V$
  - and sum: $u1 + v1 = u2 + v2$
- Shows $u1 = u2 \land v1 = v2$

(proof)

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page ??): for any element $y + a \cdot x_0$ of the direct sum of a vectorspace $H$ and the linear closure of $x_0$ the components $y \in H$ and $a$ are uniquely determined.

**Lemma decomp-$H$:**

- Assumes vectorspace $E$ subspace $H$
- Assumes $y1$: $y1 \in H$ and $y2$: $y2 \in H$
  - and $x'$: $x' \notin H$ $x' \in E$ $x' \neq 0$
  - and eq: $y1 + a1 \cdot x' = y2 + a2 \cdot x'$
- Shows $y1 = y2 \land a1 = a2$

(proof)

Since for any element $y + a \cdot x'$ of the direct sum of a vectorspace $H$ and the linear closure of $x'$ the components $y \in H$ and $a$ are unique, it follows from $y \in H$ that $a = 0$.

**Lemma decomp-$H$'-$H$:**

- Assumes vectorspace $E$ subspace $H$
- Assumes $t$: $t \in H$
  - and $x'$: $x' \notin H$ $x' \in E$ $x' \neq 0$
- Shows (SOME $y, a). t = y + a \cdot x' \land y \in H = (t, 0)$

(proof)

The components $y \in H$ and $a$ in $y + a \cdot x'$ are unique, so the function $h'$ defined by $h'(y + a \cdot x') = h y + a \cdot \xi$ is definite.

**Lemma $h'$-definite:**

- Fixes $H$
- Assumes $h'$-def:
  - $\forall x. h' x =$
    - (let $(y, a) = SOME (y, a). (x = y + a \cdot x' \land y \in H)$
      - in $(h y) + a \ast xi$)
  - and $x$: $x = y + a \cdot x'$
- Assumes vectorspace $E$ subspace $H$
- Assumes $y$: $y \in H$
  - and $x'$: $x' \notin H$ $x' \in E$ $x' \neq 0$
- Shows $h' x = h y + a \ast xi$

(proof)

end
5 Normed vector spaces

theory Normed-Space
imports Subspace
begin

5.1 Quasinorms

A \textit{seminorm} $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogeneous and subadditive.

locale seminorm =
  fixes V :: 'a::{minus, plus, zero, uminus} set
  fixes norm :: 'a \Rightarrow real (\|\cdot\|)
assumes ge-zero [iff?]: \(x \in V \Rightarrow 0 \leq \|x\|\)
  and abs-homogenous [iff?]: \(x \in V \Rightarrow \|a \cdot x\| = |a| \cdot \|x\|\)
  and subadditive [iff?]: \(x \in V \Rightarrow y \in V \Rightarrow \|x + y\| \leq \|x\| + \|y\|\)
declare seminorm.intro [intro?]

lemma (in seminorm) diff-subadditive:
  assumes vectorspace V
  shows \(x \in V \Rightarrow y \in V \Rightarrow \|x - y\| \leq \|x\| + \|y\|\)
(\textit{proof})

lemma (in seminorm) minus:
  assumes vectorspace V
  shows \(x \in V \Rightarrow \|-x\| = \|x\|\)
(\textit{proof})

5.2 Norms

A \textit{norm} $\|\cdot\|$ is a seminorm that maps only the 0 vector to 0.

locale norm = seminorm +
  assumes zero-iff [iff?]: \(x \in V \Rightarrow (\|x\| = 0) = (x = 0)\)

5.3 Normed vector spaces

A vector space together with a norm is called a \textit{normed space}.

locale normed-vectorspace = vectorspace + norm
declare normed-vectorspace.intro [intro?]

lemma (in normed-vectorspace) gt-zero [intro?]:
  assumes x: \(x \in V\) and neq: \(x \neq 0\)
  shows \(0 < \|x\|\)
(\textit{proof})

Any subspace of a normed vector space is again a normed vectorspace.

lemma subspace-normed-vs [intro?):
  fixes F E norm
  assumes subspace F E normed-vectorspace E norm
shows normed-vectorspace F norm
⟨proof⟩
end

6 Linearforms

theory Linearform
imports Vector-Space
begin

A linear form is a function on a vector space into the reals that is additive and multiplicative.

locale linearform =
fixes V :: 'a::{minus, plus, zero, uminus} set and f
assumes add [iff]: \( x \in V \implies y \in V \implies f(x + y) = f(x) + f(y) \)
and mult [iff]: \( x \in V \implies f(a \cdot x) = a \cdot f(x) \)

declare linearform.intro [intro?]

lemma (in linearform) neg [iff]:
assumes vectorspace V
shows \( x \in V \implies f(-x) = -f(x) \)
⟨proof⟩
end

phrase

7 An order on functions

theory Function-Order
imports Subspace Linearform
begin

7.1 The graph of a function

We define the graph of a (real) function \( f \) with domain \( F \) as the set
\[
\{(x, f(x)). x \in F\}
\]
So we are modeling partial functions by specifying the domain and the mapping function. We use the term “function” also for its graph.
7.2 Functions ordered by domain extension

A function \( h' \) is an extension of \( h \), iff the graph of \( h \) is a subset of the graph of \( h' \).

\[ \text{lemma graph-extI: } \forall x. x \in H \implies h x = h' x \implies H \subseteq H' \]
\[ \implies \text{graph } H \ h \subseteq \text{graph } H' \ h' \]
\[ \langle \text{proof} \rangle \]

\[ \text{lemma graph-extD1 [dest?]: } \text{graph } H \ h \subseteq \text{graph } H' \ h' \implies x \in H \implies h x = h' x \]
\[ \langle \text{proof} \rangle \]

\[ \text{lemma graph-extD2 [dest?]: } \text{graph } H \ h \subseteq \text{graph } H' \ h' \implies H \subseteq H' \]
\[ \langle \text{proof} \rangle \]

7.3 Domain and function of a graph

The inverse functions to \( \text{graph} \) are \( \text{domain} \) and \( \text{funct} \).

\[ \text{definition domain : 'a graph } \Rightarrow \text{'a set } \]
\[ \text{where domain } g = \{ x. \exists y. (x, y) \in g \} \]

\[ \text{definition funct : 'a graph } \Rightarrow \text{('a } \Rightarrow \text{real) } \]
\[ \text{where funct } g = (\lambda x. \text{SOME } y. (x, y) \in g)) \]

The following lemma states that \( g \) is the graph of a function if the relation induced by \( g \) is unique.

\[ \text{lemma graph-domain-funct: } \]
\[ \text{assumes uniq: } \forall x \ y \ z. (x, y) \in g \implies (x, z) \in g \implies z = y \]
\[ \text{shows } \text{graph } (\text{domain } g) \ (\text{funct } g) = g \]
\[ \langle \text{proof} \rangle \]

7.4 Norm-preserving extensions of a function

Given a linear form \( f \) on the space \( F \) and a seminorm \( p \) on \( E \). The set of all linear extensions of \( f \), to superspaces \( H \) of \( F \), which are bounded by \( p \), is defined as follows.
8.1 Continuous linear forms

A linear form \( f \) on a normed vector space \((V, \|\cdot\|)\) is continuous, iff it is bounded, i.e.

\[
\exists c \in \mathbb{R}. \forall x \in V. |f x| \leq c \cdot \|x\|
\]
In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

```plaintext
textlocale continuous = linearform +
  fixes norm :: - ⇒ real (∥∥)
  assumes bounded: ∃c. ∀x ∈ V. |f x| ≤ c * ∥x∥

debute locale
```

8.2 The norm of a linear form

The least real number \( c \) for which holds

\[ \forall x ∈ V. |f x| ≤ c ∗ ∥x∥ \]

is called the norm of \( f \).

For non-trivial vector spaces \( V \neq \{0\} \) the norm can be defined as

\[ ∥f∥ = \sup x \neq 0. |f x| / ∥x∥ \]

For the case \( V = \{0\} \) the supremum would be taken from an empty set. Since \( \mathbb{R} \) is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be \( \{\} \geq 0 \) so that \( fn-norm \) has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be \( 0 \), as all other elements are \( \{\} \geq 0 \).

Thus we define the set \( B \) where the supremum is taken from as follows:

\[ \{0\} \cup \{ |f x| / ∥x∥. x \neq 0 ∧ x ∈ F\} \]

\( fn-norm \) is equal to the supremum of \( B \), if the supremum exists (otherwise it is undefined).

```plaintext
textlocale fn-norm =
  fixes norm :: - ⇒ real (∥∥)
  fixes B defines B V f ≡ \{0\} ∪ \{ |f x| / ∥x∥. x \neq 0 ∧ x ∈ V\}
  fixes fn-norm (∥∥- -) \{0, 1000\} 999
  defines ||f||-V ≡ ∪(B V f)

debute locale
```

8.2 The norm of a linear form

The following lemma states that every continuous linear form on a normed space \((V, ∥∥)\) has a function norm.

```plaintext
textlocale normed-vectorspace-with-fn-norm = normed-vectorspace + fn-norm

debute lemma (in fn-norm) B-not-empty [intro]: 0 ∈ B V f
  ⟨proof⟩
```

The following lemma states that every continuous linear form on a normed space \((V, ∥∥)\) has a function norm.
assumes continuous \( V f \) norm
shows lub \((B V f) (\|f\| - V)\)
(proof)

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ub [iff]:
assumes continuous \( V f \) norm
assumes \( b: b \in B V f \)
shows \( b \leq \|f\| - V \)
(proof)

lemma (in normed-vectorspace-with-fn-norm) fn-norm-leastB:
assumes continuous \( V f \) norm
assumes \( b: \bigwedge b. b \in B V f \implies b \leq y \)
shows \( \|f\| - V \leq y \)
(proof)

The norm of a continuous function is always \( \geq 0 \).

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ge-zero [iff]:
assumes continuous \( V f \) norm
shows \( 0 \leq \|f\| - V \)
(proof)

The fundamental property of function norms is:
\[
|f x| \leq \|f\| \cdot \|x\|
\]

lemma (in normed-vectorspace-with-fn-norm) fn-norm-le-cong:
assumes continuous \( V f \) norm linearform \( V f \)
assumes \( x: x \in V \)
shows \( |f x| \leq \|f\| - V \ast \|x\| \)
(proof)

The function norm is the least positive real number for which the following inequality holds:
\[
|f x| \leq c \cdot \|x\|
\]

lemma (in normed-vectorspace-with-fn-norm) fn-norm-least [intro]:
assumes continuous \( V f \) norm
assumes ineq: \( \bigwedge x. x \in V \implies |f x| \leq c \ast \|x\| \) and ge: \( 0 \leq c \)
shows \( \|f\| - V \leq c \)
(proof)

end

9 Zorn’s Lemma

theory Zorn-Lemma
imports Main
begin

Zorn’s Lemmas states: if every linear ordered subset of an ordered set \( S \) has an upper bound in \( S \), then there exists a maximal element in \( S \). In our application,
$S$ is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn’s lemma can be modified: if $S$ is non-empty, it suffices to show that for every non-empty chain $c$ in $S$ the union of $c$ also lies in $S$.

**Theorem Zorn’s-Lemma:**
- **Assumes $r$:** $\forall c. \ c \in \text{chains } S \implies \exists x. \ x \in c \implies \bigcup c \in S$
  - **Assumes $a$:** $a \in S$
  - **Shows $\exists y \in S. \ \forall z \in S. \ y \subseteq z \implies z = y$**

**Proof**

**End**
Part II

Lemmas for the Proof

10 The supremum wrt. the function order

theory Hahn-Banach-Sup-Lemmas
imports Function-Norm Zorn-Lemma
begin

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let \( E \) be a real vector space with a seminorm \( p \) on \( E \). \( F \) is a subspace of \( E \) and \( f \) a linear form on \( F \). We consider a chain \( c \) of norm-preserving extensions of \( f \), such that \( \bigcup c = \text{graph } H \ h \). We will show some properties about the limit function \( h \), i.e. the supremum of the chain \( c \).

Let \( c \) be a chain of norm-preserving extensions of the function \( f \) and let \( \text{graph } H \ h \) be the supremum of \( c \). Every element in \( H \) is member of one of the elements of the chain.

lemma [dest?] = chainsD
lemma chainsE2 [elim?] = chainsD2 [elim-format]

lemma some-H h':
  assumes M: M = norm-pres-extensions E p F f
  and cM: c \in chains M
  and u: graph H h = \bigcup c
  and x: x \in H
  shows \( \exists H' \ h', \text{graph } H' \ h' \in c \)
  \( \land (x, h x) \in \text{graph } H' \ h' \)
  \( \land \text{linearform } H' \ h' \land H' \leq E \)
  \( \land F \leq H' \land \text{graph } F \ f \subseteq \text{graph } H' \ h' \)
  \( \land (\forall x \in H'. \ h' x \leq p x) \)
⟨proof⟩

Any two elements \( x \) and \( y \) in the domain \( H \) of the supremum function \( h \) are both in the domain \( H' \) of some function \( h' \), such that \( h \) extends \( h' \).
lemma some-H’h’:2:
assumes M: M = norm-pres-extensions E p F f 
and cM: c ∈ chains M 
and u: graph H h = ⋃ c 
and x: x ∈ H 
and y: y ∈ H 
shows ∃ H’ h’. x ∈ H’ ∧ y ∈ H’ 
∧ graph H’ h’ ⊆ graph H h 
∧ linearform H’ h’ ∧ H’ ⊆ E ∧ F ⊆ H’ 
∧ graph F f ⊆ graph H’ h’ ∧ (∀ x ∈ H’, h’ x ≤ p x) 
(proof)

The relation induced by the graph of the supremum of a chain c is definite, i.e. it is the graph of a function.

lemma sup-definite:
assumes M-def: M = norm-pres-extensions E p F f 
and cM: c ∈ chains M 
and xy: (x, y) ∈ ⋃ c 
and z: (x, z) ∈ ⋃ c 
shows z = y 
(proof)

The limit function h is linear. Every element x in the domain of h is in the domain of a function h’ in the chain of norm preserving extensions. Furthermore, h is an extension of h’ so the function values of x are identical for h’ and h. Finally, the function h’ is linear by construction of M.

lemma sup-lf:
assumes M: M = norm-pres-extensions E p F f 
and cM: c ∈ chains M 
and u: graph H h = ⋃ c 
shows linearform H h 
(proof)

The limit of a non-empty chain of norm preserving extensions of f is an extension of f, since every element of the chain is an extension of f and the supremum is an extension for every element of the chain.

lemma sup-ext:
assumes graph: graph H h = ⋃ c 
and M: M = norm-pres-extensions E p F f 
and cM: c ∈ chains M 
and ex: ∃ x. x ∈ c 
shows graph F f ⊆ graph H h 
(proof)

The domain H of the limit function is a superspace of F, since F is a subset of H. The existence of the 0 element in F and the closure properties follow from the fact that F is a vector space.

lemma sup-supF:
assumes graph: graph H h = ⋃ c 
and M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and ex: ∃x. x ∈ c
and FE: F ≤ E
shows F ≤ H
(proof)

The domain H of the limit function is a subspace of E.

lemma sup-subE:
assumes graph: graph H h = ∪c
and M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
and ex: ∃x. x ∈ c
and FE: F ≤ E
and E: vectorspace E
shows H ≤ E
(proof)

The limit function is bounded by the norm p as well, since all elements in the chain are bounded by p.

lemma sup-norm-pres:
assumes graph: graph H h = ∪c
and M: M = norm-pres-extensions E p F f
and cM: c ∈ chains M
shows ∀x ∈ H. h x ≤ p x
(proof)

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma abs-Hahn-Banach (see page 24). For real vector spaces the following inequality are equivalent:

∀x ∈ H. |h x| ≤ p x and ∀x ∈ H. h x ≤ p x

lemma abs-ineq-iff:
assumes subspace H E and vectorspace E and seminorm E p
and linearform H h
shows (∀x ∈ H. |h x| ≤ p x) = (∀x ∈ H. h x ≤ p x) (is ?L = ?R)
(proof)

end

11 Extending non-maximal functions

theory Hahn-Banach-Ext-Lemmas
imports Function-Norm
begin

In this section the following context is presumed. Let E be a real vector space with a seminorm q on E. F is a subspace of E and f a linear function on F. We consider a subspace H of E that is a superspace of F and a linear form h on H. H is a not equal to E and x₀ is an element in E − H. H is extended to the direct sum H' = H + lin x₀, so for any x ∈ H' the decomposition of x = y +
$a \cdot x$ with $y \in H$ is unique. $h'$ is defined on $H'$ by $h' \ x = h \ y + a \cdot \xi$ for a certain $\xi$.
Subsequently we show some properties of this extension $h'$ of $h$.

This lemma will be used to show the existence of a linear extension of $f$ (see page ??). It is a consequence of the completeness of $\mathbb{R}$. To show

$$\exists \xi. \forall y \in F. \ a \ y \leq \xi \wedge \xi \leq b \ y$$

it suffices to show that

$$\forall u \in F. \forall v \in F. \ a \ u \leq b \ v$$

**lemma ex-xi:**

*assumes* vectorspace $F$
*assumes* $v : \bigwedge u \ u \in F \implies v \in F \implies a \ u \leq b \ v$
*shows* $\exists \xi : \text{real}. \forall y \in F. \ a \ y \leq \xi \wedge \xi \leq b \ y$

(proof)

The function $h'$ is defined as a $h' \ x = h \ y + a \cdot \xi$ where $x = y + a \cdot \xi$ is a linear extension of $h$ to $H'$.

**lemma h'-lf:**

*assumes* $h'\text{-def} : \bigwedge x. \ h' \ x = \ (\text{let} \ (y, a) = \ \text{SOME} \ (y, a). \ x = y + a \cdot x0 \wedge y \in H \ in \ h \ y + a \cdot xi) \wedge H'\text{-def}: H' = H + \text{lin} x0$
*assumes* $HE: H \subseteq E$
*assumes* $x0: x0 \notin H \ x0 \in E \ x0 \neq 0$
*assumes* $E: \text{vectorspace} E$
*shows* $\text{linearform} H' h'$

(proof)

The linear extension $h'$ of $h$ is bounded by the seminorm $p$.

**lemma h'-norm-pres:**

*assumes* $h'\text{-def} : \bigwedge x. \ h' \ x = \ (\text{let} \ (y, a) = \ \text{SOME} \ (y, a). \ x = y + a \cdot x0 \wedge y \in H \ in \ h \ y + a \cdot xi) \wedge H'\text{-def}: H' = H + \text{lin} x0$
*assumes* $x0: x0 \notin H \ x0 \in E \ x0 \neq 0$
*assumes* $E: \text{vectorspace} E \text{ and } HE: \text{subspace} H E$
*assumes* $p: \forall y \in H. \ h \ y \leq p \ y$
*shows* $\forall x \in H'. \ h' \ x \leq p \ x$

(proof)

end
Part III
The Main Proof

12 The Hahn-Banach Theorem

theory Hahn-Banach
imports Hahn-Banach-Lemmas
begin

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

12.1 The Hahn-Banach Theorem for vector spaces

Hahn-Banach Theorem. Let $F$ be a subspace of a real vector space $E$, let $p$ be a semi-norm on $E$, and $f$ be a linear form defined on $F$ such that $f$ is bounded by $p$, i.e. $\forall x \in F. \ f x \leq p x$. Then $f$ can be extended to a linear form $h$ on $E$ such that $h$ is norm-preserving, i.e. $h$ is also bounded by $p$.

Proof Sketch.

1. Define $M$ as the set of norm-preserving extensions of $f$ to subspaces of $E$. The linear forms in $M$ are ordered by domain extension.

2. We show that every non-empty chain in $M$ has an upper bound in $M$.

3. With Zorn’s Lemma we conclude that there is a maximal function $g$ in $M$.

4. The domain $H$ of $g$ is the whole space $E$, as shown by classical contradiction:
   - Assuming $g$ is not defined on whole $E$, it can still be extended in a norm-preserving way to a super-space $H'$ of $H$.
   - Thus $g$ can not be maximal. Contradiction!

12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form $f$ and a seminorm $p$ the following inequality are equivalent.\(^{1}\)

\(^{1}\)This was shown in lemma abs-ineq-iff (see page 22).
12.3 The Hahn-Banach Theorem for normed spaces

\[ \forall x \in H. \, |h x| \leq p x \quad \text{and} \quad \forall x \in H. \, h x \leq p x \]

**Theorem abs-Hahn-Banach:**

**Assumes**:
- \( E \): vectorspace \( E \)
- \( FE \): subspace \( F \in E \)
- \( lf \): linearform \( f \in F \)
- \( sn \): seminorm \( p \) in \( E \)

**Assumes** \( fp \):
- \( \forall x \in F. \, |f x| \leq p x \)

**Shows**:
- \( \exists g. \) linearform \( g \) in \( E \)
- \( \forall x \in F. \, g x = f x \)
- \( \forall x \in E. \, |g x| \leq p x \)

(proof)

12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form \( f \) on a subspace \( F \) of a norm space \( E \), can be extended to a continuous linear form \( g \) on \( E \) such that \( \|f\| = \|g\| \).

**Theorem norm-Hahn-Banach:**

**Fixes**:
- \( V \)
- \( \|\cdot\| \)

**Fixes** \( B \)

**Defines** \( \bigvee_{V} f \)

**Assumes** \( E \)-norm:
- \( E \): normed-vectorspace \( E \)
- \( FE \): subspace \( F \in E \)
- \( lf \): linearform \( f \) in \( F \)
- \( sn \): continuous \( f \) in \( F \)

**Shows**:
- \( \exists g. \) linearform \( g \) in \( E \)
- \( \forall x \in F. \, g x = f x \)
- \( \forall x \in E. \, \|g x\| = \|f x\| \)

(proof)

References

