Welcome!

- Files and Programme at:

  `http://isabelle.in.tum.de/nominal/ijcar-09.html`

- Have you already installed Nominal Isabelle?

- Can you step through Minimal.thy without getting an error message?

If yes, then very good.
If not, then please ask us **now!**

Nominal Isabelle

Stefan Berghofer and Christian Urban
TU Munich

Quick overview: a formalisation of a CK machine:

$$\begin{tikzcd}
_ \Downarrow _ \arrow[rr] & & CK \arrow[dl] \\
& _ \longrightarrow_{cbv} _ \arrow[ul] &
\end{tikzcd}$$

A Quick and Dirty Overview of Nominal Isabelle

- Nominal Isabelle is a definitional extension of Isabelle/HOL (i.e. no additional axioms, only HOL),

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- provides an infrastructure for reasoning about named binders,
- for example lets you define

```
nominal_datatype lam =
    Var "name"
  | App "lam" "lam"
  | Lam "«name»lam" ("Lam [_]._")
```

- which give you named α-equivalence classes:

Lam [x].(Var x) = Lam [y].(Var y)

A Quick and Dirty Overview of Nominal Isabelle

- Nomi
  Isab
  HOL)
- provi
  name
- for e

  nor

  |
  | Lam

That means Nominal Isabelle is aimed at helping you with formalising results from:

- programming language theory
- term-rewriting
- logic
- ...

- which give you named α-equivalence classes:

  Lam [x].(Var x) = Lam [y].(Var y)

A Quick and Dirty Overview of Nominal Isabelle

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  | Lam "«name»lam" ("Lam [_]._")

That means Nominal Isabelle is aimed at helping you with formalising results from:

- programming language theory
- term-rewriting
- logic
- ...

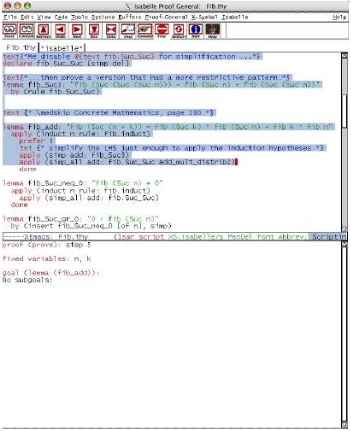
...not just the lambda-calculus!

- which give you named α-equivalence classes:

Lam [x].(Var x) = Lam [y].(Var y)

A Six-Slides Crash-Course on How to Use Isabelle

Proof General



Important buttons:

- $\boxed{\textbf{Next}}$ and $\boxed{\textbf{Undo}}$ advance / retract the processed part
- $\boxed{\textbf{Goto}}$ jumps to the current cursor position, same as **ctrl-c/ctrl-return**

Feedback:

- warning messages are given in yellow
- error messages in red

X-Symbols

- ... provide a nice way to input non-ascii characters; for example:

$$\forall\,,\ \exists\,,\ \Downarrow,\ \#,\ \bigwedge,\ \boldsymbol{\Gamma},\ \times\,,\ \neq,\ \in,\ \ldots$$

- they need to be input via the combination
  \<name-of-x-symbol>

X-Symbols

- …provide a nice way to input non-ascii characters; for example:

$$\forall\,,\ \exists\,,\Downarrow, \#, \bigwedge, \boldsymbol{\Gamma}, \times\,, \neq, \in, \ldots$$

- they need to be input via the combination
  \<name-of-x-symbol>

- short-cuts for often used symbols

$$\begin{array}{ccccccccccc} [| & \ldots & [\![& & \texttt{==>} & \ldots & \Longrightarrow & & /\backslash & \ldots & \wedge \\ |] & \ldots &]\!] & & \texttt{=>} & \ldots & \Rightarrow & & \backslash/ & \ldots & \vee \end{array}$$

Isabelle Proof-Scripts

- Every proof-script (theory) is of the form

```
theory Name
  imports T1...Tn
begin
...
end
```

Isabelle Proof-Scripts

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```
theory Name
  imports T1...Tn
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- For Nominal Isabelle proof-scripts, T_1 will normally be the theory **Nominal**.
- We use here the theory Lambda.thy, which contains the definition for lambda-terms and for capture-avoiding substitution.

Types

- Isabelle is typed, has polymorphism and overloading.
    - Base types: nat, bool, string, lam, ...
    - Type-formers: 'a list, 'a $\times$ 'b, 'c set, ...
    - Type-variables: 'a, 'b, 'c, ...

Types

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  - Type-variables: 'a, 'b, 'c, ...

- Types can be queried in Isabelle using:

  typ nat
  typ bool
  typ lam
  typ "('a $\times$ 'b)"
  typ "'c set"
  typ "nat $\Rightarrow$ bool"

Terms

- The well-formedness of terms can be queried using:

```
term c
term "1::nat"
term 1
term "{1, 2, 3::nat}"
term "[1, 2, 3]"
term "Lam [x].(Var x)"
term "App t₁ t₂"
```

Terms

- The well-formedness of terms can be queried using:

  term c
  term "1::nat"
  term 1
  term "{1, 2, 3::nat}"
  term "[1, 2, 3]"
  term "Lam [x].(Var x)"
  term "App t_1 t_2"

- Isabelle provides some useful colour feedback

term "True"	gives	"True" :: "bool"
term "true"	gives	"true" :: "'a"
term "∀ x. P x"	gives	"∀ x. P x" :: "bool"

Formulae

- Every formula in Isabelle needs to be of type bool

```
term "True"
term "True ∧ False"
term "{1,2,3} = {3,2,1}"
term "∀ x. P x"
term "A ⟶ B"
```

Formulae

- Every formula in Isabelle needs to be of type bool

  term "True"
  term "True $\wedge$ False"
  term "{1,2,3} = {3,2,1}"
  term "$\forall$ x. P x"
  term "$A \longrightarrow B$"

- When working with Isabelle, you are confronted with an objet logic (HOL) and a meta-logic (Pure)

  term "$A \longrightarrow B$" '=' **term** "$A \Longrightarrow B$"
  term "$\forall$ x. P x" '=' **term** "$\bigwedge$x. P x"

Formulae

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term "True"
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- When working with Isabelle, you are confronted with an objet logic (HOL) and a meta-logic (Pure)

term "$A \longrightarrow B$" '=' term "$A \Longrightarrow B$"
term "$\forall$ x. P x" '=' term "$\bigwedge$x. P x"

term "$A \Longrightarrow B \Longrightarrow C$" = term "$[\![A; B]\!] \Longrightarrow C$"

Definition for the Evaluation Relation, Contexts and the CK Machine on Six Slides

Evaluation Relation

```
inductive
  eval :: "lam ⇒ lam ⇒ bool"   ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
| e_App: "⟦t₁⇓ Lam [x].t; t₂⇓ v'; t[x::=v']⇓ v⟧ ⟹ App t₁ t₂ ⇓ v"
```

Evaluation Relation

a name

inductive

  eval :: "lam ⇒ lam ⇒ bool"   ("_ ⇓ _")

where

  e_Lam: "Lam [x].t ⇓ Lam [x].t"

| e_App: "$\llbracket t_1 \Downarrow$ Lam [x].t; $t_2 \Downarrow v'$; t[x::=v'] ⇓ v$\rrbracket \Longrightarrow$ App t_1 t_2 ⇓ v"

Evaluation Relation

a type

inductive
  eval :: "lam ⇒ lam ⇒ bool"   ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
| e_App: "$\llbracket t_1 \Downarrow$ Lam [x].t; $t_2 \Downarrow v'$; t[x::=v'] $\Downarrow v \rrbracket \Longrightarrow$ App t_1 $t_2 \Downarrow v$"

Evaluation Relation

optionally
pretty syntax

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eval :: "lam ⇒ lam ⇒ bool" ("_ ⇓ _")
where
e_Lam: "Lam [x].t ⇓ Lam [x].t"
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```

a clause

another clause

Evaluation Relation

```
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  eval :: "lam ⇒ lam ⇒ bool"   ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
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```

$$\frac{}{\text{Lam }[x].t \Downarrow \text{Lam }[x].t}$$

$$\frac{t_1 \Downarrow \text{Lam }[x].t \qquad t_2 \Downarrow v' \qquad t[x{::=}v'] \Downarrow v}{\text{App } t_1\ t_2 \Downarrow v}$$

Evaluation Relation

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where
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```

```
inductive
  val :: "lam ⇒ bool"
where
  v_Lam[intro]:   "val (Lam [x].t)"
```

Evaluation Relation

```
inductive
  eval :: "lam ⇒ lam ⇒ bool"   ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
| e_App: "⟦t₁⇓ Lam [x].t; t₂⇓ v'; t[x::=v']⇓ v⟧ ⟹ App t₁ t₂ ⇓ v"
```

```
inductive
  val :: "lam ⇒ bool"
where
  v_Lam[intro]:   "val (Lam [x].t)"
```

- The attribute [intro] adds the corresponding clause to the hint theorem base (later more).

Evaluation Relation

```
inductive
  eval :: "lam ⇒ lam ⇒ bool"   ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
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declare eval.intros[intro]

inductive
  val :: "lam ⇒ bool"
where
  v_Lam[intro]:   "val (Lam [x].t)"
```

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Theorems

- Isabelle's theorem database can be querried using

  thm e_Lam
  thm e_App
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e_Lam: $\mathsf{Lam}\ [?x].?t \Downarrow \mathsf{Lam}\ [?x].?t$

e_App: $[\![?t_1 \Downarrow \mathsf{Lam}\ [?x].?t;\ ?t_2 \Downarrow ?v';\ ?t[?x{::=}?v'] \Downarrow ?v]\!] \Longrightarrow \mathsf{App}\ ?t_1\ ?t_2 \Downarrow ?v$

conjI: $[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$

conjunct1: $?P \wedge ?Q \Longrightarrow ?P$

Theorems

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  thm e_Lam
  thm e_App
  thm conjI
  thm conjunct1

schematic variables

e_Lam:	$\mathsf{Lam}\ [?x].?t \Downarrow \mathsf{Lam}\ [?x].?t$
e_App:	$[\![?t_1 \Downarrow \mathsf{Lam}\ [?x].?t;\ ?t_2 \Downarrow ?v';\ ?t[?x::=?v'] \Downarrow ?v]\!]$
	$\Longrightarrow \mathsf{App}\ ?t_1\ ?t_2 \Downarrow ?v$
conjI:	$[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$
conjunct1:	$?P \wedge ?Q \Longrightarrow ?P$

Theorems

- Isabelle's theorem database can be querried using

  thm e_Lam[no_vars]
  thm e_App[no_vars]
  thm conjI[no_vars]
  thm conjunct1[no_vars] ← attributes

e_Lam:	$\mathrm{Lam}\ [x].t \Downarrow \mathrm{Lam}\ [x].t$
e_App:	$[\![t_1 \Downarrow \mathrm{Lam}\ [x].t;\ t_2 \Downarrow v';\ t[x::=v'] \Downarrow v]\!] \Longrightarrow$
	$\mathrm{App}\ t_1\ t_2 \Downarrow v$
conjI:	$[\![P;\ Q]\!] \Longrightarrow P \wedge Q$
conjunct1:	$P \wedge Q \Longrightarrow P$

Generated Theorems

- Most definitions result in automatically generated theorems; for example

```
thm eval.intros[no_vars]
thm eval.induct[no_vars]
```

Generated Theorems

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intr's: $\mathrm{Lam}\ [x].t \Downarrow \mathrm{Lam}\ [x].t$

$[\![t_1 \Downarrow \mathrm{Lam}\ [x].t;\ t_2 \Downarrow v';\ t[x::=v'] \Downarrow v]\!] \Longrightarrow \mathrm{App}\ t_1\ t_2 \Downarrow v$

ind'ct: $[\![x_1 \Downarrow x_2;$

$\bigwedge x\ t.\ P\ \mathrm{Lam}\ [x].t\ \mathrm{Lam}\ [x].t;$

$\bigwedge t_1\ x\ t\ t_2\ v'\ v.\ [\![t_1 \Downarrow \mathrm{Lam}\ [x].t;\ P\ t_1\ \mathrm{Lam}\ [x].t;\ t_2 \Downarrow v';\ P\ t_2\ v';\ t[x::=v'] \Downarrow v;\ P\ t[x::=v']\ v]\!] \Longrightarrow P\ (\mathrm{App}\ t_1\ t_2)\ v;]\!]$

$\Longrightarrow P\ x_1\ x_2$

Theorem / Lemma / Corollary

- ... they are of the form:

```
theorem theorem_name:
 fixes      x::"type"
 ...
 assumes    "assm1"
 and        "assm2"
 ...
 shows    "statement"
 ...
```

- Grey parts are optional.
- Assumptions and the (goal)statement must be of type bool. Assumptions can have labels.

Theorem / Lemma / Corollary

- ...they are of the form:

```
lemma alpha_equ:
  shows "Lam [x].Var x = Lam [y].Var y"
...

lemma Lam_freshness:
  assumes a: "x ≠ y"
  shows "y # Lam [x].t ⟹ y # t"
...

lemma neutral_element:
  fixes x::"nat"
  shows "x + 0 = x"
...
```

- Grey parts
- Assumptio ... of type bool.

Datatypes

- We define contexts with a single hole as the datatype:

```
datatype ctx =
    Hole    ("□")
  | CAppL "ctx" "lam"
  | CAppR "lam" "ctx"
```

Datatypes

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a name

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constr's

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arg type   arg type

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```

pretty syntax

Datatypes

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datatype ctx =
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```

- Isabelle now knows about:

```
typ ctx
term "□"
term "CAppL"
term "CAppL □ (Var x)"
```

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- Isabelle now knows about:

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term "CAppL"
term "CAppL □ (Var x)"

types ctxs = "ctx list"            (a type abbreviation)
```

CK Machine

- A CK machine works on configurations $\langle \_,\_ \rangle$ consisting of a lambda-term and a framestack.

inductive
machine :: "lam⇒ctxs⇒lam⇒ctxs⇒bool" ("$\langle \_,\_ \rangle \mapsto \langle \_,\_ \rangle$")
where
m_1: "$\langle$App e_1 e_2,Es$\rangle \mapsto \langle e_1$,(CAppL $\square$ e_2)#Es$\rangle$"
| m_2: "val v $\Longrightarrow$ $\langle$v,(CAppL $\square$ e_2)#Es$\rangle \mapsto \langle e_2$,(CAppR v $\square$)#Es$\rangle$"
| m_3: "val v $\Longrightarrow$ $\langle$v,(CAppR (Lam [x].e) $\square$)#Es$\rangle \mapsto \langle$e[x::=v],Es$\rangle$"

CK Machine

- A CK machine works on configurations $\langle\_,\_\rangle$ consisting of a lambda-term and a framestack.

inductive
machine :: "lam⇒ctxs⇒lam⇒ctxs⇒bool" ("⟨_,_⟩ ↦ ⟨_,_⟩")
where
m_1: "⟨App e_1 e_2,Es⟩ ↦ ⟨e_1,(CAppL □ e_2)#Es⟩"
| m_2: "val v ⟹ ⟨v,(CAppL □ e_2)#Es⟩ ↦ ⟨e_2,(CAppR v □)#Es⟩"
| m_3: "val v ⟹ ⟨v,(CAppR (Lam [x].e) □)#Es⟩ ↦ ⟨e[x::=v],Es⟩"

Initial state of the CK machine:
⟨t,[]⟩

CK Machine

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  machine :: "lam⇒ctxs⇒lam⇒ctxs⇒bool"   ("⟨_,_⟩ ↦ ⟨_,_⟩")
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| m_3: "val v ⟹ ⟨v,(CAppR (Lam [x].e) □)#Es⟩ ↦ ⟨e[x::=v],Es⟩"

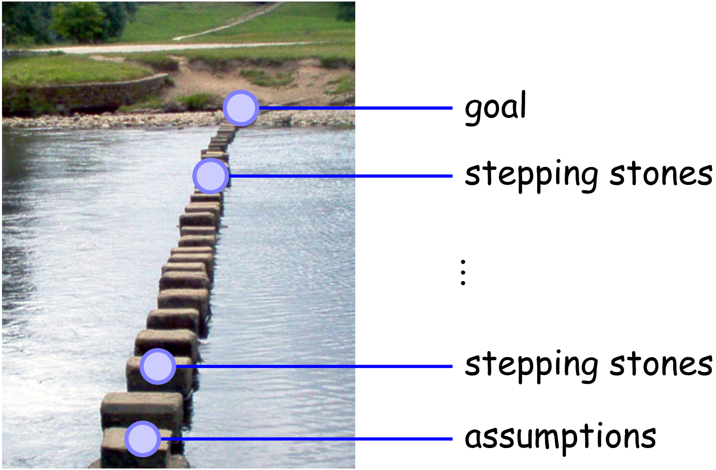
inductive
  machines :: "lam⇒ctxs⇒lam⇒ctxs⇒bool" ("⟨_,_⟩ ↦* ⟨_,_⟩")
where
  ms_1: "⟨e,Es⟩ ↦* ⟨e,Es⟩"
| ms_2: "⟦⟨e_1,Es_1⟩ ↦ ⟨e_2,Es_2⟩; ⟨e_2,Es_2⟩ ↦* ⟨e_3,Es_3⟩⟧
        ⟹ ⟨e_1,Es_1⟩ ↦* ⟨e_3,Es_3⟩"

An Isar Proof for Evaluation implying the CK Machine

An Isar Proof . . .

- The Isar proof language has been conceived by Markus Wenzel, the main developer behind Isabelle.

An Isar Proof . . .



- The Isar proof language has been conceived by Markus Wenzel, the main developer behind Isabelle.

An Isar Proof . . .

- A Rough Schema of an Isar Proof:

```
have     "assumption"
have     "assumption"
...
have     "statement"
have     "statement"
...
show "statement"
qed
```

An Isar Proof . . .

- A Rough Schema of an Isar Proof:

```
have n1: "assumption"
have n2: "assumption"
...
have  n:  "statement"
have  m:  "statement"
...
show "statement"
qed
```

- each have-statement can be given a label

An Isar Proof . . .

- A Rough Schema of an Isar Proof:

```
have n1: "assumption" by justification
have n2: "assumption" by justification
...
have  n:  "statement" by justification
have  m:  "statement" by justification
...
show "statement" by justification
qed
```

- each have-statement can be given a label
- obviously, everything needs to have a justifiation

Justifications

- Omitting proofs

  sorry

- Assumptions

  by fact

- Automated proofs

by simp	simplification (equations, definitions)
by auto	simplification & proof search (many goals)
by force	simplification & proof search (first goal)
by blast	proof search

  ...

Justifications

- Omitting proofs

  sorry

- Assumptions

  by fact

- Automated proofs

  by simp
  by auto

  by force

  by blast

  ...

Automatic justifications can also be:

using ... **by** ...

using ih **by** ...
using n1 n2 n3 **by** ...
using lemma_name... **by** ...

First Exercise

- Lets try to prove a simple lemma. Remember we defined

Transitive Closure of the CK Machine:

$$\frac{}{\langle e,Es\rangle \mapsto^* \langle e,Es\rangle}\,ms_1$$

$$\frac{\langle e_1,Es_1\rangle \mapsto \langle e_2,Es_2\rangle \qquad \langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle}{\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle}\,ms_2$$

lemma
  assumes a: "$\langle e_1,Es_1\rangle \mapsto^* \langle e_2,Es_2\rangle$"
  and b: "$\langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle$"
  shows "$\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle$"

First Exercise

- Lets try to prove a simple lemma. Remember we defined

Transitive Closure of the CK Machine:

$$\frac{}{\langle e,Es\rangle \mapsto^* \langle e,Es\rangle}\,ms_1$$

$$\frac{\langle e_1,Es_1\rangle \mapsto \langle e_2,Es_2\rangle \quad \langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle}{\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle}\,ms_2$$

lemma
  assumes a: "$\langle e_1,Es_1\rangle \mapsto^* \langle e_2,Es_2\rangle$"
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  shows "$\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle$"
using a b
proof (induct)

Proofs by Induction

- Proofs by induction involve cases, which are of the form:

```
proof (induct)
 case (Case-Name x...)
  have "assumption" by justification
  ...
  have "statment" by justification
  ...
  show "statment" by justification
next
 case (Another-Case-Name y...)
  ...
```

Your Turn

$$\frac{}{\langle e, Es\rangle \mapsto^* \langle e, Es\rangle}\,ms_1$$

$$\frac{\langle e_1, Es_1\rangle \mapsto \langle e_2, Es_2\rangle \qquad \langle e_2, Es_2\rangle \mapsto^* \langle e_3, Es_3\rangle}{\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle}\,ms_2$$

```
lemma
  assumes a: "⟨e1,Es1⟩ ↦* ⟨e2,Es2⟩"
  and     b: "⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩"
  shows "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩"
using a b
proof (induct)
  case (ms1 e1 Es1)
  have c: "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" by fact
  show "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" sorry
next
  case (ms2 e1 Es1 e2 Es2 e2' Es2')
  have ih: "⟨e2',Es2'⟩ ↦* ⟨e3,Es3⟩ ⟹ ⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩" by fact
  have d1: "⟨e2',Es2'⟩ ↦* ⟨e3,Es3⟩" by fact
  have d2: "⟨e1,Es1⟩ ↦ ⟨e2,Es2⟩" by fact

  show "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" sorry
qed
```

Your Turn

$$\frac{}{\langle e, Es\rangle \mapsto^* \langle e, Es\rangle}\ ms_1$$

$$\frac{\langle e_1, Es_1\rangle \mapsto \langle e_2, Es_2\rangle \quad \langle e_2, Es_2\rangle \mapsto^* \langle e_3, Es_3\rangle}{\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle}\ ms_2$$

lemma
  assumes a: "$\langle e_1, Es_1\rangle \mapsto^* \langle e_2, Es_2\rangle$"
  and b: "$\langle e_2, Es_2\rangle \mapsto^* \langle e_3, Es_3\rangle$"
  shows "$\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle$"
using a b
proof (induct)
  case (ms_1 e_1 Es_1)
  have c: "$\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle$" **by** fact
  show "$\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle$" **sorry**
next
  case (ms_2 e_1 Es_1 e_2 Es_2 e_2' Es_2')
  have ih: "$\langle e_2', Es_2'\rangle \mapsto^* \langle e_3, Es_3\rangle \Longrightarrow \langle e_2, Es_2\rangle \mapsto^* \langle e_3, Es_3\rangle$" **by** fact
  have d1: "$\langle e_2', Es_2'\rangle \mapsto^* \langle e_3, Es_3\rangle$" **by** fact
  have d2: "$\langle e_1, Es_1\rangle \mapsto \langle e_2, Es_2\rangle$" **by** fact

  show "$\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle$" **sorry**
qed

Your Turn

$$\frac{}{\langle e,Es\rangle \mapsto^* \langle e,Es\rangle}\,ms_1$$

$$\frac{\langle e_1,Es_1\rangle \mapsto \langle e_2,Es_2\rangle \quad \langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle}{\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle}\,ms_2$$

```
lemma
  assumes a: "⟨e1,Es1⟩ ↦* ⟨e2,Es2⟩"
  and     b: "⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩"
  shows "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩"
using a b
proof (induct)
  case (ms1 e1 Es1)
  have c: "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" by fact
  show "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" using c by simp
next
  case (ms2 e1 Es1 e2 Es2 e2' Es2')
  have ih: "⟨e2',Es2'⟩ ↦* ⟨e3,Es3⟩ ⟹ ⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩" by fact
  have d1: "⟨e2',Es2'⟩ ↦* ⟨e3,Es3⟩" by fact
  have d2: "⟨e1,Es1⟩ ↦ ⟨e2,Es2⟩" by fact

  show "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" sorry
qed
```

Your Turn

```
lemma
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩"
  and     b: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a b
proof (induct)
  case (ms₁ e₁ Es₁)
  have c: "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by fact
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" using c by simp
next
  case (ms₂ e₁ Es₁ e₂ Es₂ e₂' Es₂')
  have ih: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩ ⟹ ⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d1: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d2: "⟨e₁,Es₁⟩ ↦ ⟨e₂,Es₂⟩" by fact
  have d3: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" using ih d1 by auto
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" sorry
qed
```

$$\frac{}{\langle e,Es\rangle \mapsto^* \langle e,Es\rangle}\,ms_1$$

$$\frac{\langle e_1,Es_1\rangle \mapsto \langle e_2,Es_2\rangle \quad \langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle}{\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle}\,ms_2$$

Your Turn

```
lemma
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩"
  and     b: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a b
proof (induct)
  case (ms₁ e₁ Es₁)
  have c: "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by fact
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" using c by simp
next
  case (ms₂ e₁ Es₁ e₂ Es₂ e₂' Es₂')
  have ih: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩ ⟹ ⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d1: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d2: "⟨e₁,Es₁⟩ ↦ ⟨e₂,Es₂⟩" by fact
  have d3: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" using ih d1 by auto
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"  using d2 d3 by auto
qed
```

$$\frac{}{\langle e,Es\rangle \mapsto^* \langle e,Es\rangle}\,ms_1$$

$$\frac{\langle e_1,Es_1\rangle \mapsto \langle e_2,Es_2\rangle \quad \langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle}{\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle}\,ms_2$$

Your Turn

```
lemma
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩"
  and       b: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a b
proof (induct)
  case (ms₁ e₁ Es₁)
  have c: "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by fact
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" using c by simp
next
  case (ms₂ e₁ Es₁ e₂ Es₂ e₂' Es₂')
  have ih: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩ ⟹ ⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d1: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d2: "⟨e₁,Es₁⟩ ↦ ⟨e₂,Es₂⟩" by fact
  have d3: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" using ih d1 by auto
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"  using d2 d3 by auto
qed
```

Your Turn

```
lemma
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩"
  and     b: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a b
proof (induct)
  case (ms₁ e₁ Es₁)
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by fact
next
  case (ms₂ e₁ Es₁ e₂ Es₂ e₂' Es₂')
  have ih: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩ ⟹ ⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d1: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d2: "⟨e₁,Es₁⟩ ↦ ⟨e₂,Es₂⟩" by fact
  have d3: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" using ih d1 by auto
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" using d2 d3 by auto
qed
```

Your Turn

```
lemma
  assumes a: "⟨e1,Es1⟩ ↦* ⟨e2,Es2⟩"
  and     b: "⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩"
  shows "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩"
using a b
proof (induct)
  case (ms1 e1 Es1)
  show "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" by fact
next
  case (ms2 e1 Es1 e2 Es2 e2' Es2')
  have ih: "⟨e2',Es2'⟩ ↦* ⟨e3,Es3⟩ ⟹ ⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩" by fact
  have d2: "⟨e1,Es1⟩ ↦ ⟨e2,Es2⟩" by fact
  have d1: "⟨e2',Es2'⟩ ↦* ⟨e3,Es3⟩" by fact
  have d3: "⟨e2,Es2⟩ ↦* ⟨e3,Es3⟩" using ih d1 by auto
  show "⟨e1,Es1⟩ ↦* ⟨e3,Es3⟩" using d2 d3 by auto
qed
```

Your Turn

```
lemma
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩"
  and      b: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a b
proof (induct)
  case (ms₁ e₁ Es₁)
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by fact
next
  case (ms₂ e₁ Es₁ e₂ Es₂ e₂' Es₂')
  have ih: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩ ⟹ ⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have d2: "⟨e₁,Es₁⟩ ↦ ⟨e₂,Es₂⟩" by fact
  have "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩" by fact
  then have d3: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" using ih by auto
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" using d2 d3 by auto
qed
```

A Chain of Facts

- Isar allows you to build a chain of facts as follows:

have n1: "..."
have n2: "..."

...

have ni: "..."
have "..." **using** n1 n2 ... ni

have "..."
moreover have "..."

...

moreover have "..."
ultimately have "..."

- also works for **show**

Your Turn

```
lemma
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩"
  and     b: "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a b
proof (induct)
  case (ms₁ e₁ Es₁)
  show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by fact
next
  case (ms₂ e₁ Es₁ e₂ Es₂ e₂' Es₂')
  have ih: "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩ ⟹ ⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" by fact
  have "⟨e₁,Es₁⟩ ↦ ⟨e₂,Es₂⟩" by fact
  moreover
  have "⟨e₂',Es₂'⟩ ↦* ⟨e₃,Es₃⟩" by fact
  then have "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩" using ih by auto
  ultimately show "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩" by auto
qed
```

Automatic Proofs

- Do not expect Isabelle to be able to solve automatically **show** "P=NP", but...

lemma
  assumes a: "$\langle e_1, Es_1\rangle \mapsto^* \langle e_2, Es_2\rangle$"
  and b: "$\langle e_2, Es_2\rangle \mapsto^* \langle e_3, Es_3\rangle$"
  shows "$\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle$"
using a b
by (induct) (auto)

Eval Implies CK

```
theorem
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a
proof (induct)
  case (e_Lam x t)                                    (no assumption avail.)
  show "⟨Lam [x].t,[]⟩ ↦* ⟨Lam [x].t,[]⟩" sorry
next
  case (e_App t₁ x t t₂ v' v)
  have a1: "t₁ ⇓ Lam [x].t" by fact                  (all assumptions)
  have ih1: "⟨t₁,[]⟩ ↦* ⟨Lam [x].t,[]⟩" by fact
  have a2: "t₂ ⇓ v'" by fact
  have ih2: "⟨t₂,[]⟩ ↦* ⟨v',[]⟩" by fact
  have a3: "t[x::=v'] ⇓ v" by fact
  have ih3: "⟨t[x::=v'],[]⟩ ↦* ⟨v,[]⟩" by fact

  show "⟨App t₁ t₂,[]⟩ ↦* ⟨v,[]⟩" sorry
qed
```

Eval Implies CK

```
theorem
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a
proof (induct)
  case (e_Lam x t)                                   (no assumption avail.)
  show "⟨Lam [x].t,[]⟩ ↦* ⟨Lam [x].t,[]⟩" sorry
next
  case (e_App t1 x t t2 v' v)
  have a1: "t1 ⇓ Lam [x].t" by fact                   (all assumptions)
  have ih1: "⟨t1,[]⟩ ↦* ⟨Lam [x].t,[]⟩" by fact
  have a2: "t2 ⇓ v'" by fact
  have ih2: "⟨t2,[]⟩ ↦* ⟨v',[]⟩" by fact
  have a3: "t[x::=v'] ⇓ v" by fact
  have ih3: "⟨t[x::=v'],[]⟩ ↦* ⟨v,[]⟩" by fact

  show "⟨App t1 t2,[]⟩ ↦* ⟨v,[]⟩" sorry
qed
```

Eval Implies CK

```
theorem
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a
proof (induct)
  case (e_Lam x t)                                   (no assumption avail.)
  show "⟨Lam [x].t,[]⟩ ↦* ⟨Lam [x].t,[]⟩" sorry
next
  case (e_App t₁ x t t₂ v' v)
  have a1: "t₁ ⇓ Lam [x].t" by fact                  (all assumptions)
  have ih1: "⟨t₁,[]⟩ ↦* ⟨Lam [x].t,[]⟩" by fact
  have a2: "t₂ ⇓ v'" by fact
  have ih2: "⟨t₂,[]⟩ ↦* ⟨v',[]⟩" by fact
  have a3: "t[x::=v'] ⇓ v" by fact
  have ih3: "⟨t[x::=v'],[]⟩ ↦* ⟨v,[]⟩" by fact

  show "⟨App t₁ t₂,[]⟩ ↦* ⟨v,[]⟩" sorry
qed
```

```
thm machine.intros
thm machines.intros
thm eval_to_val
```

Eval Implies CK

Proof Idea:

$\langle \text{App } t_1\ t_2, [] \rangle$
$\mapsto^* \langle t_1, [\text{CAppL } \Box\ t_2] \rangle$
$\mapsto^* \langle \text{Lam } [x].t, [\text{CAppL } \Box\ t_2] \rangle$
$\mapsto^* \langle t_2, [\text{CAppR } (\text{Lam } [x].t)\ \Box] \rangle$
$\mapsto^* \langle v', [\text{CAppR } (\text{Lam } [x].t)\ \Box] \rangle$
$\mapsto^* \langle t[x::=v'], [] \rangle$
$\mapsto^* \langle v, [] \rangle$

thm machine.intros
thm machines.intros
thm eval_to_val

(no assumption avail.)

m $[x].t, []\rangle$" **sorry**

next
  case (e_App t_1 x t t_2 v' v) (all assumptions)
  have a1: "$t_1 \Downarrow \text{Lam } [x].t$" **by** fact
  have ih1: "$\langle t_1, [] \rangle \mapsto^* \langle \text{Lam } [x].t, [] \rangle$" **by** fact
  have a2: "$t_2 \Downarrow v'$" **by** fact
  have ih2: "$\langle t_2, [] \rangle \mapsto^* \langle v', [] \rangle$" **by** fact
  have a3: "$t[x::=v'] \Downarrow v$" **by** fact
  have ih3: "$\langle t[x::=v'], [] \rangle \mapsto^* \langle v, [] \rangle$" **by** fact

  show "$\langle \text{App } t_1\ t_2, [] \rangle \mapsto^* \langle v, [] \rangle$" **sorry**
qed

Eval Implies CK

```
theorem
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a
proof (induct)
  case (e_Lam x t)                                  (no assumption avail.)
  show "⟨Lam [x].t,[]⟩ ↦* ⟨Lam [x].t,[]⟩" sorry
next
  case (e_App t₁ x t t₂ v' v)
  have a1: "t₁ ⇓ Lam [x].t" by fact                 (all assumptions)
  have ih1: "⟨t₁,[]⟩ ↦* ⟨Lam [x].t,[]⟩" by fact
  have a2: "t₂ ⇓ v'" by fact
  have ih2: "⟨t₂,[]⟩ ↦* ⟨v',[]⟩" by fact
  have a3: "t[x::=v'] ⇓ v" by fact
  have ih3: "⟨t[x::=v'],[]⟩ ↦* ⟨v,[]⟩" by fact

  show "⟨App t₁ t₂,[]⟩ ↦* ⟨v,[]⟩" sorry
qed
```

```
thm machine.intros
thm machines.intros
thm eval_to_val
```

Eval Implies CK

```
theorem
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a
proof (induct)
  case (e_Lam x t)                                   (no assumption avail.)
  show "⟨Lam [x].t,[]⟩ ↦* ⟨Lam [x].t,[]⟩" sorry
next
  case (e_App t₁ x t t₂ v' v)
  have a1: "t₁ ⇓ Lam [x].t" by fact                  (all assumptions)
  have ih1: "⟨t₁,[]⟩ ↦* ⟨Lam [x].t,[]⟩" by fact
  have a2: "t₂ ⇓ v'" by fact
  have ih2: "⟨t₂,[]⟩ ↦* ⟨v',[]⟩" by fact
  have a3: "t[x::=v'] ⇓ v" by fact
  have ih3: "⟨t[x::=v'],[]⟩ ↦* ⟨v,[]⟩" by fact

  show "⟨App t₁ t₂,[]⟩ ↦* ⟨v,[]⟩" sorry
qed
```

```
thm machine.intros
thm machines.intros
thm eval_to_val
```

Eval Implies CK

```
theorem
  assumes a: "t ⇓ t'"
  shows "⟨t,Es⟩ ↦* ⟨t',Es⟩"
using a
proof (induct arbitrary: Es)
  case (e_Lam x t)                                  (no assumption avail.)
  show "⟨Lam [x].t,Es⟩ ↦* ⟨Lam [x].t,Es⟩" sorry
next
  case (e_App t₁ x t t₂ v' v)
  have a1: "t₁ ⇓ Lam [x].t" by fact                 (all assumptions)
  have ih1: "⋀Es. ⟨t₁,Es⟩ ↦* ⟨Lam [x].t,Es⟩" by fact
  have a2: "t₂ ⇓ v'" by fact
  have ih2: "⋀Es. ⟨t₂,Es⟩ ↦* ⟨v',Es⟩" by fact
  have a3: "t[x::=v'] ⇓ v" by fact
  have ih3: "⋀Es. ⟨t[x::=v'],Es⟩ ↦* ⟨v,Es⟩" by fact

  show "⟨App t₁ t₂,Es⟩ ↦* ⟨v,Es⟩" sorry
qed
```

```
thm machine.intros
thm machines.intros
thm eval_to_val
```

Finally: Eval Implies CK

```
theorem eval_implies_machines_ctx:
  assumes a: "t ⇓ t'"
  shows "⟨t,Es⟩ ↦* ⟨t',Es⟩"
using a
proof (induct arbitrary: Es)
...

corollary eval_implies_machines:
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a eval_implies_machines_ctx by auto
```

Finally: Eval Implies CK

```
theorem eval_implies_machines_ctx:
  assumes a: "t ⇓ t'"
  shows "⟨t,Es⟩ ↦* ⟨t',Es⟩"
using a
proof (induct arbitrary: Es)
...

corollary eval_implies_machines:
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a eval_implies_machines_ctx by auto
```

```
thm eval_implies_machines_ctx
```

gives

$?t \Downarrow ?t' \Longrightarrow \langle ?t, ?Es \rangle \mapsto^* \langle ?t', ?Es \rangle$

Weakening Lemma (trivial / routine)

Definition of Types

```
nominal_datatype ty =
  tVar "string"
| tArr "ty" "ty" ("_ → _")
```

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```

$$\frac{(x\!:\!T) \in \Gamma \quad \mathsf{valid}\ \Gamma}{\Gamma \vdash x : T}
\qquad
\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

$$\frac{x \# \Gamma \quad (x\!:\!T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}$$

$$\frac{}{\mathsf{valid}\ []}
\qquad
\frac{x \# \Gamma \quad \mathsf{valid}\ \Gamma}{\mathsf{valid}\ (x\!:\!T)::\Gamma}$$

Typing Judgements

```
types ty_ctx = "(name×ty) list"

inductive
  valid :: "ty_ctx ⇒ bool"
where
  v₁: "valid []"
| v₂: "⟦valid Γ; x#Γ⟧⟹ valid ((x,T)#Γ)"

inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t₁ : T₁→T₂; Γ ⊢ t₂ : T₁⟧ ⟹ Γ ⊢ App t₁ t₂ : T₂"
| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"
```

Typing Judgements

#: list cons
#: freshness (\<sharp>)

types ty_ctx = "(name $\times$ ty) list"

inductive
  valid :: "ty_ctx $\Rightarrow$ bool"
where
  v_1: "valid []"
| v_2: "$[\![$valid Γ; $\boxed{x\#\Gamma}$$]\!] \Longrightarrow$ valid $\boxed{((x,T)\#\Gamma)}$"

inductive
  typing :: "ty_ctx $\Rightarrow$ lam $\Rightarrow$ ty $\Rightarrow$ bool" ("_ $\vdash$ _ : _")
where
  t_Var: "$[\![$valid Γ; $(x,T) \in$ set $\Gamma]\!] \Longrightarrow \Gamma \vdash$ Var x : T"
| t_App: "$[\![\Gamma \vdash t_1 : T_1 \rightarrow T_2; \Gamma \vdash t_2 : T_1]\!] \Longrightarrow \Gamma \vdash$ App t_1 t_2 : T_2"
| t_Lam: "$[\![\boxed{x\#\Gamma}; \boxed{(x,T_1)\#\Gamma} \vdash t : T_2]\!] \Longrightarrow \Gamma \vdash$ Lam [x].t : $T_1 \rightarrow T_2$"

Freshness

- Freshness is a concept automatically defined in Nominal Isabelle; it corresponds roughly to the notion of "not-free-in".

```
lemma
  fixes x::"name"
  shows "x#Lam [x].t"
  and   "x#t₁ ∧ x#t₂ ⟹ x#App t₁ t₂"
  and   "x#(Var y) ⟹ x#y"
  and   "⟦x#t₁; x#t₂⟧ ⟹ x#(t₁,t₂)"
  and   "⟦x#l₁; x#l₂⟧ ⟹ x#(l₁@l₂)"
  and   "x#y ⟹ x≠y"
by (simp_all add: abs_fresh fresh_list_append fresh_atm)
```

Freshness

- Freshness is a concept automatically defined in Nominal Isabelle; it corresponds roughly to the notion of "not-free-in".

```
lemma ty_fresh:
  fixes x::"name"
  and   T::"ty"
  shows "x#T"
by (induct T rule: ty.induct)
   (simp_all add: fresh_string)
```

Freshness

- Freshness is a concept automatically defined in Nominal Isabelle; it corresponds roughly to the notion of “not-free-in”.

```
lemma ty_fresh:
  fixes x::"name"
  and   T::"ty"
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by (induct T rule: ty.induct)
   (simp_all add: fresh_string)
```

```
nominal_datatype ty =
  tVar "string"
| tArr "ty" "ty" ("_ → _")
```

The Weakening Lemma

- We can overload ⊆ for typing contexts, but this means we have to give explicit type-annotations.

```
abbreviation
  "sub_ty_ctx" :: "ty_ctx ⇒ ty_ctx ⇒ bool" ("_ ⊆ _")
where
  "Γ₁ ⊆ Γ₂ ≡ ∀x. x ∈ set Γ₁ ⟶ x ∈ set Γ₂"

lemma weakening:
  fixes Γ₁ Γ₂::"(name×ty) list"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof (induct arbitrary: Γ₂)
```

Your Turn: Variable Case

```
lemma
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof (induct arbitrary: Γ₂)
  case (t_Var Γ₁ x T)
  have a1: "valid Γ₁" by fact
  have a2: "(x,T) ∈ set Γ₁" by fact
  have a3: "valid Γ₂" by fact
  have a4: "Γ₁ ⊆ Γ₂" by fact
    . . .
  show "Γ₂ ⊢ Var x : T" sorry
```

My Proof for the Variable Case

```
lemma
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof (induct arbitrary: Γ₂)
  case (t_Var Γ₁ x T)
  have "Γ₁ ⊆ Γ₂" by fact
  moreover
  have "valid Γ₂" by fact
  moreover
  have "(x,T)∈ set Γ₁" by fact
  ultimately show "Γ₂ ⊢ Var x : T" by auto
```

Induction Principle for Typing

- The induction principle that comes with the typing definition is as follows:

$$\forall \Gamma\, x\, T.\ (x\!:\!T) \in \Gamma \wedge \mathsf{valid}\,\Gamma \Rightarrow P\,\Gamma\,(x)\,T$$

$$\begin{array}{l}\forall \Gamma\, t_1\, t_2\, T_1\, T_2. \\ P\,\Gamma\,t_1\,(T_1 \to T_2) \wedge P\,\Gamma\,t_2\,T_1 \Rightarrow P\,\Gamma\,(t_1\,t_2)\,T_2\end{array}$$

$$\begin{array}{l}\forall \Gamma\, x\, t\, T_1\, T_2. \\ x \# \Gamma \wedge P\,((x\!:\!T_1)::\Gamma)\,t\,T_2 \Rightarrow P\,\Gamma(\lambda x.t)\,(T_1 \to T_2)\end{array}$$

$$\overline{\quad \Gamma \vdash t : T \Rightarrow P\,\Gamma\,t\,T \quad}$$

Note the quantifiers!

Proof Idea for the Lambda Cs.

$$\frac{x \# \Gamma \quad (x:T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \rightarrow T_2}$$

- If $\Gamma_1 \vdash t : T_1$ then $\forall \Gamma_2.$ valid $\Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t : T_2$

Proof Idea for the Lambda Cs.

$$\frac{x \# \Gamma \quad (x{:}T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}$$

- If $\Gamma_1 \vdash t : T_1$ then $\forall \Gamma_2.\ \text{valid}\ \Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t : T_2$

<u>For all Γ_1, x, t, T_1 and T_2:</u>

- We know:
  $\forall \Gamma_3.\ \text{valid}\ \Gamma_3 \wedge (x{:}T_1)::\Gamma_1 \subseteq \Gamma_3 \Rightarrow \Gamma_3 \vdash t : T_1$
  $x \# \Gamma_1$
  $\text{valid}\ \Gamma_2$
  $\Gamma_1 \subseteq \Gamma_2$

- We have to show:
  $\Gamma_2 \vdash \lambda x.t : T_1 \to T_2$

Proof Idea for the Lambda Cs.

$$\frac{x \# \Gamma \quad (x{:}T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \rightarrow T_2}$$

- If $\Gamma_1 \vdash t : T_1$ then $\forall \Gamma_2.\,\text{valid}\,\Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t : T_2$

For all Γ_1, x, t, T_1 and T_2:

- We know:
  $\forall \Gamma_3.\,\text{valid}\,\Gamma_3 \wedge (x{:}T_1)::\Gamma_1 \subseteq \Gamma_3 \Rightarrow \Gamma_3 \vdash t : T_1$
  $x \# \Gamma_1$
  $\text{valid}\,\Gamma_2$
  $\Gamma_1 \subseteq \Gamma_2$

- We have to show:
  $\Gamma_2 \vdash \lambda x.t : T_1 \rightarrow T_2$

Proof Idea for the Lambda Cs.

$$\frac{x \# \Gamma \quad (x:T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \rightarrow T_2}$$

- If $\Gamma_1 \vdash t : T_1$ then $\forall \Gamma_2.\ \text{valid}\ \Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t : T_2$

For all Γ_1, x, t, T_1 and T_2:

$\boxed{\Gamma_3 \mapsto (x:T_1)::\Gamma_2}$

- We know:
  $\forall \Gamma_3.\ \text{valid}\ \Gamma_3 \wedge (x:T_1)::\Gamma_1 \subseteq \Gamma_3 \Rightarrow \Gamma_3 \vdash t : T_1$
  $x \# \Gamma_1$
  valid Γ_2
  $\Gamma_1 \subseteq \Gamma_2$

- We have to show:
  $\Gamma_2 \vdash \lambda x.t : T_1 \rightarrow T_2$

Your Turn: Lambda Case

```
lemma
  fixes Γ1 Γ2::"ty_ctx"
  assumes a: "Γ1 ⊢ t : T"
  and     b: "valid Γ2"
  and     c: "Γ1 ⊆ Γ2"
  shows "Γ2 ⊢ t : T"
using a b c
proof (induct arbitrary: Γ2)
  case (t_Lam x Γ1 T1 t T2)
  have ih: "⋀Γ3. ⟦valid Γ3; (x,T1)#Γ1 ⊆ Γ3⟧ ⟹ Γ3 ⊢ t : T2" by fact
  have a0: "x#Γ1" by fact
  have a1: "valid Γ2" by fact
  have a2: "Γ1 ⊆ Γ2" by fact
    ...
  show "Γ2 ⊢ Lam [x].t : T1 → T2" sorry
```

Strong Induction Principle

$$\forall \Gamma\, x\, T\,.\ (x\!:\!T) \in \Gamma \wedge \mathsf{valid}\,\Gamma \Rightarrow P\ \Gamma\,(x)\,T$$

$$\begin{array}{l} \forall \Gamma\, t_1\, t_2\, T_1\, T_2\,. \\ P\,\Gamma\, t_1\,(T_1 \to T_2) \wedge P\,\Gamma\, t_2\, T_1 \\ \qquad\qquad\qquad\qquad \Rightarrow P\,\Gamma\,(t_1\, t_2)\, T_2 \end{array}$$

$$\begin{array}{l} \forall \Gamma\, x\, t\, T_1\, T_2\,. \\ x \,\#\, \Gamma\ \ \wedge \\ P\,((x\!:\!T_1)\!::\!\Gamma)\, t\, T_2 \Rightarrow P\,\Gamma(\lambda x.t)\,(T_1 \to T_2) \end{array}$$

$$\overline{\qquad\qquad \Gamma \vdash t : T \Rightarrow P\,\Gamma\, t\, T \qquad\qquad}$$

- Instead we are going to use the strong induction principle and set up the induction so that the binder "avoids" Γ_2.

2nd Attempt

```
lemma
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof (induct arbitrary: Γ₂)
  case (t_Lam x Γ₁ T₁ t T₂)
  have ih: "⋀Γ₃. ⟦valid Γ₃; (x,T₁)#Γ₁ ⊆ Γ₃⟧ ⟹ Γ₃ ⊢ t : T₂" by fact
  have a0: "x#Γ₁" by fact
  have a1: "valid Γ₂" by fact
  have a2: "Γ₁ ⊆ Γ₂" by fact
    ...
  show "Γ₂ ⊢ Lam [x].t : T₁ → T₂" sorry
```

2nd Attempt

```
lemma
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof (nominal_induct avoiding: Γ₂ rule: typing.strong_induct)
  case (t_Lam x Γ₁ T₁ t T₂)
  have vc: "x#Γ₂" by fact
  have ih: "⋀Γ₃. ⟦valid Γ₃; (x,T₁)#Γ₁ ⊆ Γ₃⟧ ⟹ Γ₃ ⊢ t : T₂" by fact
  have a0: "x#Γ₁" by fact
  have a1: "valid Γ₂" by fact
  have a2: "Γ₁ ⊆ Γ₂" by fact
   . . .
  show "Γ₂ ⊢ Lam [x].t : T₁ → T₂" sorry
```

```
lemma weakening:
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T" and b: "valid Γ₂" and c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof (nominal_induct avoiding: Γ₂ rule: typing.strong_induct)
  case (t_Lam x Γ₁ T₁ t T₂)
  have vc: "x#Γ₂" by fact
  have ih: "⟦valid ((x,T₁)#Γ₂); (x,T₁)#Γ₁⊆(x,T₁)#Γ₂⟧
                                    ⟹ (x,T₁)#Γ₂ ⊢ t:T₂" by fact
  have "Γ₁ ⊆ Γ₂" by fact
  then have "(x,T₁)#Γ₁ ⊆ (x,T₁)#Γ₂" by simp
  moreover
  have "valid Γ₂" by fact
  then have "valid ((x,T₁)#Γ₂)" using vc by auto
  ultimately have "(x,T₁)#Γ₂ ⊢ t : T₂" using ih by simp
  then show "Γ₂ ⊢ Lam [x].t : T₁→T₂" using vc by auto
qed (auto)
```

```
lemma weakening:
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T" and b: "valid Γ₂" and c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
by (nominal_induct avoiding: Γ₂ rule: typing.strong_induct)
   (auto)
```

```
lemma weakening:
  fixes Γ1 Γ2::"ty_ctx"
  assumes a: "Γ1 ⊢ t : T" and b: "valid Γ2" and c: "Γ1 ⊆ Γ2"
  shows "Γ2 ⊢ t : T"
using a b c
by (nominal_induct avoiding: Γ2 rule: typing.strong_induct)
   (auto)
```

- Perhaps the weakening lemma is after all trivial / routine / obvious ;o)
- We shall late see that the work we put into the stronger induction principle needs a bit of thinking. For you, of course, it is provided automatially.

Function Definitions and the Simplifier

Function Definitions

- Later on we will need a few functions about contexts:

```
fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

Function Definitions

- Later on we will need a few functions about contexts:

a name

```
fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

Function Definitions

- Later on we will need a few functions about contexts:

a type

```
fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

Function Definitions

- Later on we will need a few functions about contexts:

pretty syntax

```
fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

Function Definitions

- Later on we will need a few functions about contexts:

```
fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

char. eqs

Function Definitions

- Later on we will need a few functions about contexts:

```
fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

- Once a function is defined, the simplifier will be able to solve equations like

```
lemma
  shows "(CAppL □ (Var x))⟦Var y⟧ = App (Var y) (Var x)"
  by simp
```

Context Composition

```
fun
 ctx_compose :: "ctx ⇒ ctx ⇒ ctx" ("_ ∘ _" [101,100] 100)
where
  "□ ∘ E' = E'"
| "(CAppL E t') ∘ E' = CAppL (E ∘ E') t'"
| "(CAppR t' E) ∘ E' = CAppR t' (E ∘ E')"

fun
  ctx_composes :: "ctxs ⇒ ctx" ("_↓" [110] 110)
where
    "[]↓ = □"
  | "(E#Es)↓ = (Es↓) ∘ E"
```

Context Composition

```
fun
 ctx_compose :: "ctx ⇒ ctx ⇒ ctx" ("_ ∘ _" [101,100] 100)
where
  "□ ∘ E' = E'"
| "(CAppL E t') ∘ E' = CAppL (E ∘ E') t'"
| "(CAppR t' E) ∘ E' = CAppR t' (E ∘ E')"
```

precedence

```
fun
  ctx_composes :: "ctxs ⇒ ctx" ("_↓" [110] 110)
where
   "[]↓ = □"
 | "(E#Es)↓ = (Es↓) ∘ E"
```

precedence

- Explicit preedences are given in order to enforce the notation:

$$(E_1 \circ E_2) \circ E_3 \qquad (E_1 \circ E_2)\downarrow$$

Context Composition

```
fun
 ctx_compose :: "ctx ⇒ ctx ⇒ ctx" ("_ ∘ _" [101,100] 100)
where
  "□ ∘ E' = E'"
| "(CAppL E t') ∘ E' = CAppL (E ∘ E') t'"
| "(CAppR t' E) ∘ E' = CAppR t' (E ∘ E')"
```

precedence

```
fun
  ctx_composes :: "ctxs ⇒ ctx" ("_↓" [110] 110)
where
   "[]↓ = □"
 | "(E#Es)↓ = (Es↓) ∘ E"
```

precedence

- Explicit preedences are given in order to enforce the notation:

$$(E_1 \circ E_2) \circ E_3 \qquad (E_1 \circ E_2)\!\downarrow$$

Your Turn

```
datatype ctx =
  Hole
| CAppL "ctx" "lam"
| CAppR "lam" "ctx"
```

```
lemma ctx_compose:
  shows "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧"
proof (induct E₁)
  case Hole
  show "□ ∘ E₂⟦t⟧ = □⟦E₂⟦t⟧⟧" sorry
next
  case (CAppL E₁ t')
  have ih: "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧" by fact
  show "((CAppL E₁ t') ∘ E₂)⟦t⟧ = (CAppL E₁ t')⟦E₂⟦t⟧⟧" sorry
next
  case (CAppR t' E₁)
  have ih: "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧" by fact
  show "((CAppR t' E₁) ∘ E₂)⟦t⟧ = (CAppR t' E₁)⟦E₂⟦t⟧⟧" sorry
qed
```

Your Turn

```
datatype ctx =
  Hole
| CAppL "ctx" "lam"
| CAppR "lam" "ctx"
```

```
lemma ctx_compose:
  shows "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧"
proof (induct E₁)
  case Hole
  show "□ ∘ E₂⟦t⟧ = □⟦E₂⟦t⟧⟧" sorry
next
  case (CAppL E₁ t')
  have ih: "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧" by fact
  show "((CAppL E₁ t') ∘ E₂)⟦t⟧ = (CAppL E₁ t')⟦E₂⟦t⟧⟧" sorry
next
  case (CAppR t' E₁)
  have ih: "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧" by fact
  show "((CAppR t' E₁) ∘ E₂)⟦t⟧ = (CAppR t' E₁)⟦E₂⟦t⟧⟧" sorry
qed
```

```
thm filling.simps[no_vars]
thm ctx_compose.simps[no_vars]
```

Your Turn Again

- Assuming:

```
lemma neut_hole: shows "E ∘ □ = E"
lemma circ_assoc:  shows "(E₁ ∘ E₂) ∘ E₃ = E₁ ∘ (E₂ ∘ E₃)"
```

- Prove

```
lemma shows "(Es₁ @ Es₂)↓ = (Es₂↓) ∘ (Es₁↓)"
proof (induct Es₁)
  case Nil
  show "([] @ Es₂)↓ = Es₂↓ ∘ []↓" sorry
next
  case (Cons E Es₁)
  have ih: "(Es₁ @ Es₂)↓ = Es₂↓ ∘ Es₁↓" by fact

  show "((E#Es₁) @ Es₂)↓ = Es₂↓ ∘ (E#Es₁)↓" sorry
qed
```

Your Turn Again

- Assuming:

```
lemma neut_hole: shows "E ∘ □ = E"
lemma circ_assoc:  shows "(E₁ ∘ E₂) ∘ E₃ = E₁ ∘ (E₂ ∘ E₃)"
```

- Prove

```
lemma shows "(Es₁ @ Es₂)↓ = (Es₂↓) ∘ (Es₁↓)"
proof (induct Es₁)
  case Nil
  show "([] @ Es₂)↓ = Es₂↓ ∘ []↓" sorry
next
  case (Cons E Es₁)
  have ih: "(Es₁ @ Es₂)↓ = Es₂↓ ∘ Es₁↓" by fact

  show "((E#Es₁) @ Es₂)↓ = Es₂↓ ∘ (E#Es₁)↓" sorry
qed
```

My Solution

```
lemma
  shows "(Es₁ @ Es₂)↓ = (Es₂↓) ∘ (Es₁↓)"
proof (induct Es₁)
  case Nil
  show "([]@Es₂)↓ = Es₂↓ ∘ []↓" using neut_hole by simp
next
  case (Cons E Es₁)
  have ih: "(Es₁ @ Es₂)↓ = Es₂↓ ∘ Es₁↓" by fact
  have lhs: "((E#Es₁) @ Es₂)↓ = (Es₁ @ Es₂)↓ ∘ E" by simp
  have lhs': "(Es₁ @ Es₂)↓ ∘ E = (Es₂↓ ∘ Es₁↓) ∘ E" using ih by simp
  have rhs: "Es₂↓ ∘ (E#Es₁)↓ = Es₂↓ ∘ (Es₁↓ ∘ E)" by simp
  show "((E#Es₁) @ Es₂)↓ = Es₂↓ ∘ (E#Es₁)↓"
    using lhs lhs' rhs circ_assoc by simp
qed
```

Equational Reasoning in Isar

- One frequently wants to prove an equation $t_1 = t_n$ by means of a chain of equations, like

$$t_1 = t_2 = t_3 = t_4 = \ldots = t_n$$

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$$t_1 = t_2 = t_3 = t_4 = \ldots = t_n$$

- This kind of reasoning is supported in Isar as:

```
have "t1 = t2" by just.
also have "... = t3" by just.
also have "... = t4" by just.
...
also have "... = tn" by just.
finally have "t1 = tn" by simp
```

A Readable Solution

```
lemma
  shows "(Es₁ @ Es₂)↓ = (Es₂↓) ∘ (Es₁↓)"
proof (induct Es₁)
  case Nil
  show "([]@Es₂)↓ = Es₂↓ ∘ []↓" using neut_hole by simp
next
  case (Cons E Es₁)
  have ih: "(Es₁ @ Es₂)↓ = Es₂↓ ∘ Es₁↓" by fact
  have "((E#Es₁) @ Es₂)↓ = (Es₁ @ Es₂)↓ ∘ E" by simp
  also have "... = (Es₂↓ ∘ Es₁↓) ∘ E" using ih by simp
  also have "... = Es₂↓ ∘ (Es₁↓ ∘ E)" using circ_assoc by simp
  also have "... = Es₂↓ ∘ (E#Es₁)↓" by simp
  finally show "((E#Es₁) @ Es₂)↓ = Es₂↓ ∘ (E#Es₁)↓" by simp
qed
```

Capture-Avoiding Substitution and the Substitution Lemma

Capture-Avoiding Subst.

- Lambda.thy contains a definition of capture-avoiding substitution with the characteristic equations:

"(Var x)[y::=s] = (if x=y then s else (Var x))"

"(App t_1 t_2)[y::=s] = App (t_1[y::=s]) (t_2[y::=s])"

"x#(y,s) $\Longrightarrow$ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"

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"x#(y,s) $\Longrightarrow$ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"

- Despite its looks, this is a total function!

Substitution Lemma: If $x \not\equiv y$ and $x \notin \mathsf{fv}(L)$, then
$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]]$$

Proof: By induction on the structure of M.

- **Case 1:** M is a variable.

  Case 1.1. $M \equiv x$. Then both sides equal $N[y := L]$ since $x \not\equiv y$.

  Case 1.2. $M \equiv y$. Then both sides equal L, for $x \notin \mathsf{fv}(L)$ implies $L[x := \ldots] \equiv L$.

  Case 1.3. $M \equiv z \not\equiv x, y$. Then both sides equal z.

- **Case 2:** $M \equiv \lambda z.M_1$. By the variable convention we may assume that $z \not\equiv x, y$ and z is not free in N, L.

$$\begin{aligned}(\lambda z.M_1)[x := N][y := L] &\equiv \lambda z.(M_1[x := N][y := L]) \\ &\equiv \lambda z.(M_1[y := L][x := N[y := L]]) \\ &\equiv (\lambda z.M_1)[y := L][x := N[y := L]].\end{aligned}$$

- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis. $\square$

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- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis. □

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Proof: By induction on the structure of M.

- **Case 1:** M
  Case 1.1. M
  x
  Case 1.2. M
  in
  Case 1.3. M
- **Case 2:** M
  assume th
  $(\lambda z.M_1)[x$

Remember only if $y \neq x$ and $x \notin \mathrm{fv}(N)$ then
$$(\lambda y.M)[x := N] = \lambda y.(M[x := N])$$

$$\begin{array}{lll} & (\lambda z.M_1)[x := N][y := L] & \\ \equiv & (\lambda z.(M_1[x := N]))[y := L] & \xleftarrow{1} \\ \equiv & \lambda z.(M_1[x := N][y := L]) & \xleftarrow{2} \\ \equiv & \lambda z.(M_1[y := L][x := N[y := L]]) & \text{IH} \\ \equiv & (\lambda z.(M_1[y := L]))[x := N[y := L]] & \xrightarrow{2} \ ! \\ \equiv & (\lambda z.M_1)[y := L][x := N[y := L]]. & \xrightarrow{1} \end{array}$$

- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis. $\square$

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Case Distintions

- Assuming $P_1 \lor P_2 \lor P_3$ is true then:

```
{ assume "P₁"
  ...
  have "something" ...}
moreover
{ assume "P₂"
  ...
  have "something" ...}
moreover
{ assume "P₃"
  ...
  have "something" ...}
ultimately have "something" by blast
```

Case Distintions

- Assuming $P_1 \vee P_2 \vee P_3$ is true then:

$P_1 \mapsto (z{=}x)$
$P_2 \mapsto (z{=}y) \wedge (z{\neq}x)$
$P_3 \mapsto (z{\neq}y) \wedge (z{\neq}x)$

```
{ assume "P1"
  ...
  have "something" ...}
moreover
{ assume "P2"
  ...
  have "something" ...}
moreover
{ assume "P3"
  ...
  have "something" ...}
ultimately have "something" by blast
```

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- Assuming $P_1 \vee P_2 \vee P_3$ is true then:

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{ assume "P1"
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moreover
{ assume "P2"
  ...
  have "something" ...}
moreover
{ assume "P3"
  ...
  have "something" ...}
ultimately have "something" by blast
```

$$\frac{P_1 \Longrightarrow smth \quad P_2 \Longrightarrow smth \quad P_3 \Longrightarrow smth}{smth}$$

```
lemma substitution_lemma:
  assumes a: "x≠y" "x # L"
  shows "M[x::=N][y::=L] = M[y::=L][x::=N[y::=L]]"
using a proof (nominal_induct M avoiding: x y N L rule: lam.strong_induct)
case (Var z)
  have a1: "x≠y" by fact
  have a2: "x#L" by fact
  show "Var z[x::=N][y::=L] = Var z[y::=L][x::=N[y::=L]]" (is "?LHS = ?RHS")
  proof -
   { assume c1: "z=x"
     have "(1)": "?LHS = N[y::=L]" using c1 by simp
     have "(2)": "?RHS = N[y::=L]" using c1 a1 by simp
      have "?LHS = ?RHS" using "(1)" "(2)" by simp }
   moreover
   { assume c2: "z=y" "z≠x"

     have "?LHS = ?RHS" sorry }
   moreover
   { assume c3: "z≠x" "z≠y"

     have "?LHS = ?RHS" sorry }
   ultimately show "?LHS = ?RHS" by blast
  qed
```

```
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   { assume c2: "z=y" "z≠x"

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   moreover
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   moreover
   { assume c3: "z≠x" "z≠y"

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     have "?LHS = ?RHS" sorry }
   ultimately show "?LHS = ?RHS" by blast
  qed
```

```
thm forget:
x # L ⟹ L[x::=P] = L
```

```
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  assumes a: "x≠y" "x # L"
  shows "M[x::=N][y::=L] = M[y::=L][x::=N[y::=L]]"
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using a proof (nominal_induct M avoiding: x y N L rule: lam.strong_induct)
 case (Lam z M₁)
 have ih: "⟦x≠y; x#L⟧ ⟹ M₁[x::=N][y::=L] = M₁[y::=L][x::=N[y::=L]]" by fact
 have "x≠y" by fact
 have "x#L" by fact
 have vc: "z#x" "z#y" "z#N" "z#L" by fact+
 then have "z#N[y::=L]" by (simp add: fresh_fact)
 show "(Lam [z].M₁)[x::=N][y::=L]=(Lam [z].M₁)[y::=L][x::=N[y::=L]]" (is "?LHS=?RHS")
 proof -
   have "?LHS = ..." sorry

   also have "... = ?RHS" sorry
   finally show "?LHS = ?RHS" by simp
 qed
next
```

```
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 have "x≠y" by fact
 have "x#L" by fact
 have vc: "z#x" "z#y" "z#N" "z#L" by fact+
 then have "z#N[y::=L]" by (simp add: fresh_fact)
 show "(Lam [z].M₁)[x::=N][y::=L]=(Lam [z].M₁)[y::=L][x::=N[y::=L]]" (is "?LHS=?RHS")
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```

```
lemma substitution_lemma:
  assumes a: "x≠y" "x # L"
  shows "M[x::=N][y::=L] = M[y::=L][x::=N[y::=L]]"
using a proof (nominal_induct M avoiding: x y N L rule: lam.strong_induct)
 case (Lam z M1)
 have ih: "⟦x≠y; x#L⟧ ⟹ M1[x::=N][y::=L] = M1[y::=L][x::=N[y::=L]]" by fact
 have "x≠y" by fact
 have "x#L" by fact
 have vc: "z#x" "z#y" "z#N" "z#L" by fact+
 then have "z#N[y::=L]" by (simp add: fresh_fact)
 show "(Lam [z].M1)[x::=N][y::=L]=(Lam [z].M1)[y::=L][x::=N[y::=L]]" (is "?LHS=?RHS")
 proof -
   have "?LHS = ..." sorry

   also have "... = ?RHS" sorry
   finally show "?LHS = ?RHS" by simp
 qed
next
```

```
lemma substitution_lemma:
  assumes a: "x≠y" "x # L"
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 have ih: "⟦x≠y; x#L⟧ ⟹ M1[x::=N][y::=L] = M1[y::=L][x::=N[y::=L]]" by fact
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 have "x#L" by fact
 have vc: "z#x" "z#y" "z#N" "z#L" by fact+
 then have "z#N[y::=L]" by (simp add: fresh_fact)
 show "(Lam [z].M1)[x::=N][y::=L]=(Lam [z].M1)[y::=L][x::=N[y::=L]]" (is "?LHS=?RHS")
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  have "?LHS = ..." sorry

  also have "... = ?RHS" sorry
  finally show "?LHS = ?RHS" by simp
 qed
next
```

```
lemma substitution_lemma:
  assumes a: "x≠y" "x # L"
  shows "M[x::=N][y::=L] = M[y::=L][x::=N[y::=L]]"
using a proof (nominal_induct M avoiding: x y N L rule: lam.strong_induct)
 case (Lam z M1)
 have ih: "⟦x≠y; x#L⟧ ⟹ M1[x::=N][y::=L] = M1[y::=L][x::=N[y::=L]]" by fact
 have "x≠y" by fact
 have "x#L" by fact
 have vc: "z#x" "z#y" "z#N" "z#L" by fact+
 then have "z#N[y::=L]" by (simp add: fresh_fact)
 show "(Lam [z].M1)[x::=N][y::=L]=(Lam [z].M1)[y::=L][x::=N[y::=L]]" (is "?LHS=?RHS")
 proof -
   have "?LHS = ..." sorry

   also have "... = ?RHS" sorry
   finally show "?LHS = ?RHS" by simp
 qed
next
```

Substitution Lemma: If $x \not\equiv y$ and $x \notin \mathsf{fv}(L)$, then
$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]]$$

Proof: By induction on the structure of M.

- **Case 1:** M is a variable.

  Case 1.1. $M \equiv x$. Then both sides equal $N[y := L]$ since $x \not\equiv y$.

  Case 1.2. $M \equiv y$. Then both sides equal L, for $x \notin \mathsf{fv}(L)$ implies $L[x := \ldots] \equiv L$.

  Case 1.3. $M \equiv z \not\equiv x, y$. Then both sides equal z.

- **Case 2:** $M \equiv \lambda z.M_1$. By the variable convention we may assume that $z \not\equiv x, y$ and z is not free in N, L.

$$\begin{aligned}(\lambda z.M_1)[x := N][y := L] &\equiv \lambda z.(M_1[x := N][y := L]) \\ &\equiv \lambda z.(M_1[y := L][x := N[y := L]]) \\ &\equiv (\lambda z.M_1)[y := L][x := N[y := L]].\end{aligned}$$

- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis. $\square$

Substitution Lemma

- The strong structural induction principle for lambda-terms allowed us to follow Barendregt's proof quite closely. It also enables Isabelle to find this proof automatically:

```
lemma substitution_lemma:
  assumes asm: "x≠y" "x#L"
  shows "M[x::=N][y::=L] = M[y::=L][x::=N[y::=L]]"
  using asm
by (nominal_induct M avoiding: x y N L rule: lam.strong_induct)
   (auto simp add: fresh_fact forget)
```

How To Prove False Using the Variable Convention (on Paper)

So Far So Good

- A Faulty Lemma with the Variable Convention?

> **Variable Convention:**
>
> If $M_1, \ldots, M_n$ occur in a certain mathematical context (e.g. definition, proof), then in these terms all bound variables are chosen to be different from the free variables.
>
> Barendregt in "The Lambda-Calculus: Its Syntax and Semantics"

Inductive Definitions:

$$\frac{\mathsf{prem}_1 \ldots \mathsf{prem}_n \;\; \mathsf{scs}}{\mathsf{concl}}$$

Rule Inductions:

1.) Assume the property for the premises. Assume the side-conditions.

2.) Show the property for the conclusion.

Faulty Reasoning

- Consider the two-place relation foo:

$$\frac{}{x \mapsto x} \qquad \frac{}{t_1\ t_2 \mapsto t_1\ t_2} \qquad \frac{t \mapsto t'}{\lambda x.t \mapsto t'}$$

Faulty Reasoning

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- The lemma we going to prove:

  Let $t \mapsto t'$. If $y \# t$ then $y \# t'$.

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- The lemma we going to prove:

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- Cases 1 and 2 are trivial:
  - If $y \# x$ then $y \# x$.
  - If $y \# t_1\, t_2$ then $y \# t_1\, t_2$.

Faulty Reasoning

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  Let $t \mapsto t'$. If $y \;\#\; t$ then $y \;\#\; t'$.

- Case 3:
  - We know $y \;\#\; \lambda x.t$. We have to show $y \;\#\; t'$.
  - The IH says: if $y \;\#\; t$ then $y \;\#\; t'$.

Variable Convention:

If $M_1, \ldots, M_n$ occur in a certain mathematical context (e.g. definition, proof), then in these terms all bound variables are chosen to be different from the free variables.

In our case:

The free variables are y and t'; the bound one is x.

By the variable convention we conclude that $x \neq y$.

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Let $t \mapsto t'$. If $y \# t$ then $y \# t'$

$$y \not\in \mathsf{fv}(\lambda x.t) \iff y \not\in \mathsf{fv}(t) - \{x\} \overset{x \neq y}{\iff} y \not\in \mathsf{fv}(t)$$

- Case 3:
  - We know $y \# \lambda x.t$. We have to show $y \# t'$.
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If $M_1, \ldots, M_n$ occur in a certain mathematical context (e.g. definition, proof), then in these terms all bound variables are chosen to be different from the free variables.

In our case:

The free variables are y and t'; the bound one is x.

By the variable convention we conclude that $x \neq y$.

Let $t \mapsto t'$. If $y \# t$ then $y \# t'$.

$$y \notin \mathsf{fv}(\lambda x.t) \iff y \notin \mathsf{fv}(t) - \{x\} \overset{x \neq y}{\iff} y \notin \mathsf{fv}(t)$$

- Case 3:
  - We know $y \# \lambda x.t$. We have to show $y \# t'$.
  - The IH says: if $y \# t$ then $y \# t'$.
  - So we have $y \# t$. Hence $y \# t'$ by IH. Done!

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VC-Compatibility

- We introduced two conditions that make the VC safe to use in rule inductions:
  - the relation needs to be **equivariant**, and
  - the binder is not allowed to occur in the **support** of the conclusion (not free in the conclusion)

VC-Compatibility

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  - the relation needs to be **equivariant**, and
  - the binder is not allowed to occur in the

A relation R is **equivariant** iff

$$\forall \pi\, t_1 \ldots t_n$$
$$R\, t_1 \ldots t_n \Rightarrow R(\pi \bullet t_1) \ldots (\pi \bullet t_n)$$

This means the relation has to be invariant under permutative renaming of variables.

VC-Compatibility

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  - the relation needs to be **equivariant**, and
  - the binder is not allowed to occur in the **support** of the conclusion (not free in the conclusion)

Typing Judgements (2)

```
inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t₁ : T₁→T₂; Γ ⊢ t₂ : T₁⟧ ⟹ Γ ⊢ App t₁ t₂ : T₂"
| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"

equivariance typing
nominal_inductive typing
```

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equivariance typing
nominal_inductive typing
```

Subgoals

1. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1){::}\Gamma \vdash t : T_2]\!] \Longrightarrow x \# \Gamma$
2. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1){::}\Gamma \vdash t : T_2]\!] \Longrightarrow x \# \mathrm{Lam}\ [x].t$
3. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1){::}\Gamma \vdash t : T_2]\!] \Longrightarrow x \# T_1 \to T_2$

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| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"

equivariance typing
nominal_inductive typing
  by (simp_all add: abs_fresh ty_fresh)
```

Subgoals

1. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1)::\Gamma \vdash t : T_2]\!] \Longrightarrow x \# \Gamma$
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3. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1)::\Gamma \vdash t : T_2]\!] \Longrightarrow x \# T_1 \to T_2$

CK Machine Implies the Evaluation Relation (Via A Small-Step Reduction)

A Direct Attempt

- The statement for the other direction is as follows:

```
lemma machines_implies_eval:
  assumes a: "⟨t,[]⟩ ↦* ⟨v,[]⟩"
  and       b: "val v"
  shows "t ⇓ v"
```

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  oops
```

- We can prove this direction by introducing a small-step reduction relation.

CBV-Reduction

inductive
cbv :: "lam⇒lam⇒bool" ("_ ⟶cbv _")
where
cbv_1: "val v ⟹ App (Lam [x].t) v ⟶cbv t[x::=v]"
| cbv_2: "t ⟶cbv t' ⟹ App t t_2 ⟶cbv App t' t_2"
| cbv_3: "t ⟶cbv t' ⟹ App t_2 t ⟶cbv App t_2 t'"

- Later on we like to use the strong induction principle for this relation.

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| cbv$_3$: "t ⟶cbv t' ⟹ App t_2 t ⟶cbv App t_2 t'"

- Later on we like to use the strong induction principle for this relation.

Conditions:

1. ⋀v x t. val v ⟹ x # App Lam [x].t v
2. ⋀v x t. val v ⟹ x # t[x::=v]

CBV-Reduction

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  cbv :: "lam⇒lam⇒bool" ("_ ⟶cbv _")
where
  cbv$_1$: "⟦val v; x#v⟧ ⟹ App (Lam [x].t) v ⟶cbv t[x::=v]"
| cbv$_2$[intro]: "t ⟶cbv t' ⟹ App t t$_2$ ⟶cbv App t' t$_2$"
| cbv$_3$[intro]: "t ⟶cbv t' ⟹ App t$_2$ t ⟶cbv App t$_2$ t'"

- The conditions that give us automatically the strong induction principle require us to add the assumption x # v. This makes this rule less useful.

Better Introduction Rule

```
lemma better_cbv₁[intro]:
  assumes a: "val v"
  shows "App (Lam [x].t) v ⟶cbv t[x::=v]"
proof -
  obtain y::"name" where fs: "y#(x,t,v)"
     by (rule exists_fresh) (auto simp add: fs_name1)
  have "App (Lam [x].t) v = App (Lam [y].([(y,x)]•t)) v" using fs
     by (auto simp add: lam.inject alpha' fresh_prod fresh_atm)
  also have "... ⟶cbv  ([(y,x)]•t)[y::=v]" using fs a
     by (auto simp add: cbv₁ fresh_prod)
  also have "... = t[x::=v]" using fs
     by (simp add: subst_rename[symmetric] fresh_prod)
  finally show "App (Lam [x].t) v ⟶cbv t[x::=v]" by simp
qed
```

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  finally show "App (Lam [x].t) v ⟶cbv t[x::=v]" by simp
qed
```

CBV-Reduction*

inductive
"cbvs" :: "lam ⇒ lam ⇒ bool" ("_ ⟶cbv* _")
where
cbvs$_1$[intro]: "e ⟶cbv* e"
| cbvs$_2$[intro]: "⟦e_1 ⟶cbv e_2; e_2 ⟶cbv* e_3⟧ ⟹ e_1 ⟶cbv* e_3"

lemma cbvs$_3$[intro]:
assumes a: "e_1 ⟶cbv* e_2" "e_2 ⟶cbv* e_3"
shows "e_1 ⟶cbv* e_3"
using a **by** (induct) (auto)

CBV-Reduction*

```
inductive
  "cbvs" :: "lam ⇒ lam ⇒ bool" (" _ ⟶cbv* _")
where
  cbvs₁[intro]: "e ⟶cbv* e"
| cbvs₂[intro]: "⟦e₁ ⟶cbv e₂; e₂ ⟶cbv* e₃⟧ ⟹ e₁ ⟶cbv* e₃"

lemma cbvs₃[intro]:
  assumes a: "e₁ ⟶cbv* e₂" "e₂ ⟶cbv* e₃"
  shows "e₁ ⟶cbv* e₃"
using a by (induct) (auto)

lemma cbv_in_ctx:
  assumes a: "t ⟶cbv t'"
  shows "E⟦t⟧ ⟶cbv E⟦t'⟧"
using a by (induct E) (auto)
```

Is another such exercise needed?

CK Machine Implies CBV*

```
lemma machines_implies_cbvs:
  assumes a: "⟨e,[]⟩ ↦* ⟨e',[]⟩"
  shows "e ⟶cbv* e'"
using a by (auto dest: machines_implies_cbvs_ctx)
```

CK Machine Implies CBV*

```
lemma machine_implies_cbvs_ctx:
  assumes a: "⟨e,Es⟩ ↦ ⟨e',Es'⟩"
  shows "(Es↓)⟦e⟧ ⟶cbv* (Es'↓)⟦e'⟧"
using a by (induct) (auto simp add: ctx_compose intro: cbv_in_ctx)
```

```
lemma machines_implies_cbvs:
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using a by (induct) (auto simp add: ctx_compose intro: cbv_in_ctx)
```

If we had not derived the better cbv-rule, then we would have to do an explicit renaming here.

```
lemma machines_implies_cbvs:
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using a by (induct) (auto simp add: ctx_compose intro: cbv_in_ctx)

lemma machines_implies_cbvs_ctx:
  assumes a: "⟨e,Es⟩ ↦* ⟨e',Es'⟩"
  shows "(Es↓)⟦e⟧ ⟶cbv* (Es'↓)⟦e'⟧"
using a by (induct) (auto dest: machine_implies_cbvs_ctx)

lemma machines_implies_cbvs:
  assumes a: "⟨e,[]⟩ ↦* ⟨e',[]⟩"
  shows "e ⟶cbv* e'"
using a by (auto dest: machines_implies_cbvs_ctx)
```

Your Turn

```
lemma machine_implies_cbvs_ctx:
  assumes a: "⟨e,Es⟩ ↦ ⟨e',Es'⟩"
  shows "(Es↓)⟦e⟧ ⟶cbv* (Es'↓)⟦e'⟧"
using a proof (induct)
  case (m1 t1 t2 Es)

  show "Es↓⟦App t1 t2⟧ ⟶cbv* (CAppL □ t2#Es)↓⟦t1⟧"  sorry
next
  case (m2 v t2 Es)
  have "val v" by fact

  show "(CAppL □ t2#Es)↓⟦v⟧ ⟶cbv* (CAppR v □#Es)↓⟦t2⟧" sorry
next
  case (m3 v x t Es)
  have "val v" by fact

  show "(CAppR Lam [x].t □#Es)↓⟦v⟧ ⟶cbv* (Es↓)⟦t[x::=v]⟧" sorry
qed
```

CBV* Implies Evaluation

- We need the following auxiliary lemmas in order to show that cbv-reduction implies evaluation.

```
lemma eval_val:
  assumes a: "val t"
  shows "t ⇓ t"
using a by (induct) (auto)

lemma e_App_elim:
  assumes a: "App t1 t2 ⇓ v"
  shows "∃ x t v'. t1 ⇓ Lam [x].t ∧ t2 ⇓ v' ∧ t[x::=v'] ⇓ v"
using a by (cases) (auto simp add: lam.inject)
```

```
lemma cbv_eval:
  assumes a: "t1 ⟶cbv t2" "t2 ⇓ t3"
  shows "t1 ⇓ t3"
using a proof(induct arbitrary: t3)
  case (cbv1 v x t t3)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t3" by fact
  show "App Lam [x].t v ⇓ t3" sorry
next
  case (cbv2 t t' t2 t3)
  have ih: "⋀t3. t' ⇓ t3 ⟹ t ⇓ t3" by fact
  have "App t' t2 ⇓ t3" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
      and a2: "t2 ⇓ v'"
      and a3: "t''[x::=v'] ⇓ t3" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t2 ⇓ t3" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma cbv_eval:
  assumes a: "t₁ ⟶cbv t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a proof(induct arbitrary: t₃)
  case (cbv₁ v x t t₃)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t₃" by fact
  show "App Lam [x].t v ⇓ t₃" using eval_val a1 a2 by auto
next
  case (cbv₂ t t' t₂ t₃)
  have ih: "⋀t₃. t' ⇓ t₃ ⟹ t ⇓ t₃" by fact
  have "App t' t₂ ⇓ t₃" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
      and a2: "t₂ ⇓ v'"
      and a3: "t''[x::=v'] ⇓ t₃" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t₂ ⇓ t₃" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma cbv_eval:
  assumes a: "t₁ ⟶cbv t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a proof(induct arbitrary: t₃)
  case (cbv₁ v x t t₃)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t₃" by fact
  show "App Lam [x].t v ⇓ t₃" using eval_val a1 a2 by auto
next
  case (cbv₂ t t' t₂ t₃)
  have ih: "⋀t₃. t' ⇓ t₃ ⟹ t ⇓ t₃" by fact
  have "App t' t₂ ⇓ t₃" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
      and a2: "t₂ ⇓ v'"
      and a3: "t''[x::=v'] ⇓ t₃" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t₂ ⇓ t₃" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma cbv_eval:
  assumes a: "t1 ⟶cbv t2" "t2 ⇓ t3"
  shows "t1 ⇓ t3"
using a proof(induct
  case (cbv1 v x t t3)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t3" by fact
  show "App Lam [x].t v ⇓ t3" using eval_val a1 a2 by auto
next
  case (cbv2 t t' t2 t3)
  have ih: "⋀t3. t' ⇓ t3 ⟹ t ⇓ t3" by fact
  have "App t' t2 ⇓ t3" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
      and a2: "t2 ⇓ v'"
      and a3: "t''[x::=v'] ⇓ t3" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t2 ⇓ t3" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma e_App_elim:
assumes a: "App t1 t2 ⇓ v"
shows "∃ x t v'. t1 ⇓ Lam [x].t ∧ t2 ⇓ v' ∧ t[x::=v'] ⇓ v"
```

```
lemma cbv_eval:
  assumes a: "t1 ⟶cbv t2" "t2 ⇓ t3"
  shows "t1 ⇓ t3"
using a proof(induct
  case (cbv1 v x t t3)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t3" by fact
  show "App Lam [x].t v ⇓ t3" using eval_val a1 a2 by auto
next
  case (cbv2 t t' t2 t3)
  have ih: "⋀t3. t' ⇓ t3 ⟹ t ⇓ t3" by fact
  have "App t' t2 ⇓ t3" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
      and a2: "t2 ⇓ v'"
      and a3: "t''[x::=v'] ⇓ t3" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t2 ⇓ t3" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma e_App_elim:
assumes a: "App t1 t2 ⇓ v"
shows "∃ x t v'. t1 ⇓ Lam [x].t ∧ t2 ⇓ v' ∧ t[x::=v'] ⇓ v"
```

```
lemma cbv_eval:
  assumes a: "t₁ ⟶cbv t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a proof(induct arbitrary: t₃)
  case (cbv₁ v x t t₃)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t₃" by fact
  show "App Lam [x].t v ⇓ t₃" using eval_val a1 a2 by auto
next
  case (cbv₂ t t' t₂ t₃)
  have ih: "⋀t₃. t' ⇓ t₃ ⟹ t ⇓ t₃" by fact
  have "App t' t₂ ⇓ t₃" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
        and a2: "t₂ ⇓ v'"
        and a3: "t''[x::=v'] ⇓ t₃" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t₂ ⇓ t₃" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma cbv_eval:
  assumes a: "t₁ ⟶cbv t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a proof(induct arbitrary: t₃)
  case (cbv₁ v x t t₃)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t₃" by fact
  show "App Lam [x].t v ⇓ t₃" using eval_val a1 a2 by auto
next
  case (cbv₂ t t' t₂ t₃)
  have ih: "⋀t₃. t' ⇓ t₃ ⟹ t ⇓ t₃" by fact
  have "App t' t₂ ⇓ t₃" by fact
  then obtain x t'' v'
    where a1: "t' ⇓ Lam [x].t''"
      and a2: "t₂ ⇓ v'"
      and a3: "t''[x::=v'] ⇓ t₃" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t₂ ⇓ t₃" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

```
lemma cbv_eval:
  assumes a: "t₁ ⟶cbv t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a proof(induct arbitrary: t₃)
  case (cbv₁ v x t t₃)
  have a1: "val v" by fact
  have a2: "t[x::=v] ⇓ t₃" by fact
  show "App Lam [x].t v ⇓ t₃" using eval_val a1 a2 by auto
next
  case (cbv₂ t t' t₂ t₃)
  have ih: "⋀t₃. t' ⇓ t₃ ⟹ t ⇓ t₃" by fact
  have "App t' t₂ ⇓ t₃" by fact
  then obtain x t'' v'
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      and a3: "t''[x::=v'] ⇓ t₃" using e_App_elim by blast
  have "t ⇓  Lam [x].t''" using ih a1 by auto
  then show "App t t₂ ⇓ t₃" using a2 a3 by auto
qed (auto dest!: e_App_elim)
```

Nothing Interesting

```
lemma cbvs_eval:
  assumes a: "t₁ ⟶cbv* t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a by (induct) (auto intro: cbv_eval)

lemma cbvs_implies_eval:
  assumes a: "t ⟶cbv* v" "val v"
  shows "t ⇓ v"
using a by (induct) (auto intro: eval_val cbvs_eval)

theorem machines_implies_eval:
  assumes a: "⟨t₁,[]⟩ ↦* ⟨t₂,[]⟩" and b: "val t₂"
  shows "t₁ ⇓ t₂"
proof -
  have "t₁ ⟶cbv* t₂" using a by (simp add: machines_implies_cbvs)
  then show "t₁ ⇓ t₂" using b by (simp add: cbvs_implies_eval)
qed
```

Extensions

- With only minimal modifications the proofs can be extended to the language given by:

```
nominal_datatype lam =
   Var "name"
 | App "lam" "lam"
 | Lam "«name»lam" ("Lam [_]._")
 | Num "nat"
 | Minus "lam" "lam" ("_ -- _")
 | Plus "lam" "lam" ("_ ++ _")
 | TRUE
 | FALSE
 | IF "lam" "lam" "lam"
 | Fix "«name»lam" ("Fix [_]._")
 | Zet "lam"
 | Eqi "lam" "lam"
```

Honest Toil, No Theft!

- The sacred principle of HOL:

> "The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil."
>
> B. Russell, Introduction of Mathematical Philosophy

- I will show next that the weak structural induction principle implies the strong structural induction principle.

  (I am only going to show the lambda-case.)

Permutations

A permutation **acts** on variable names as follows:

$$
\begin{aligned}
[]\bullet a &\stackrel{\text{def}}{=} a \\
((a_1\,a_2)::\pi)\bullet a &\stackrel{\text{def}}{=} \begin{cases} a_1 & \text{if } \pi\bullet a = a_2 \\ a_2 & \text{if } \pi\bullet a = a_1 \\ \pi\bullet a & \text{otherwise} \end{cases}
\end{aligned}
$$

- $[]$ stands for the empty list (the identity permutation), and
- $(a_1\,a_2)::\pi$ stands for the permutation π followed by the swapping $(a_1\,a_2)$.

Permutations on Lambda-Terms

- Permutations act on lambda-terms as follows:

$$
\begin{array}{rcl}
\pi \bullet x & \stackrel{\text{def}}{=} & \text{“action on variables”} \\
\pi \bullet (t_1\ t_2) & \stackrel{\text{def}}{=} & (\pi \bullet t_1)\ (\pi \bullet t_2) \\
\pi \bullet (\lambda x.t) & \stackrel{\text{def}}{=} & \lambda(\pi \bullet x).(\pi \bullet t)
\end{array}
$$

- Alpha-equivalence can be defined as:

$$\frac{t_1 = t_2}{\lambda x.t_1 = \lambda x.t_2}$$

$$\frac{x \neq y \quad t_1 = (x\,y) \bullet t_2 \quad x \# t_2}{\lambda x.t_1 = \lambda y.t_2}$$

Permutations on Lambda-Terms

- Permutations act on lambda-terms as follows:

$$\begin{aligned}\pi \bullet x &\stackrel{\text{def}}{=} \text{"action on variables"}\\ \pi \bullet (t_1\ t_2) &\stackrel{\text{def}}{=} (\pi \bullet t_1)\ (\pi \bullet t_2)\\ \pi \bullet (\lambda x.t) &\stackrel{\text{def}}{=} \lambda(\pi \bullet x).(\pi \bullet t)\end{aligned}$$

- Alpha-equivalence can be defined as:

$$\frac{t_1 = t_2}{\lambda x.t_1 = \lambda x.t_2}$$

$$\frac{x \neq y \quad t_1 = (x\ y) \bullet t_2 \quad x \,\#\, t_2}{\lambda x.t_1 = \lambda y.t_2}$$

Notice, I wrote equality here!

My Claim

$$\frac{\forall x.\, P\ x \qquad \forall t_1\, t_2.\, P\ t_1 \wedge P\ t_2 \Rightarrow P\ (t_1\ t_2) \qquad \forall x\, t.\, P\ t \Rightarrow P\ (\lambda x.t)}{P\ t}$$

implies

$$\frac{\forall x\, c.\, P\, c\, x \qquad \forall t_1\, t_2\, c.\, (\forall d.\, P\, d\, t_1) \wedge (\forall d.\, P\, d\, t_2) \Rightarrow P\, c\, (t_1\, t_2) \qquad \forall x\, t\, c.\, x \# c \wedge (\forall d.\, P\, d\, t) \Rightarrow P\, c\, (\lambda x.t)}{P\, c\, t}$$

Proof for the Strong Induction Principle

- We prove $P\,c\,t$ by induction on t.

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction on t.

Proof for the Strong Induction Principle

- We prove $\forall \pi\ c.\ P c\,(\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\,(\pi \bullet (\lambda x.t))$.

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\, \lambda(\pi \bullet x).(\pi \bullet t)$.

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction on t.
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Proof for the Strong Induction Principle

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- We have $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction.
- Our weaker precondition says that:

$$\forall x\, t\, c.\, x \,\#\, c \wedge (\forall c.\, P c\, t) \Rightarrow P c\, (\lambda x.t)$$

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\, \lambda(\pi \bullet x).(\pi \bullet t)$.
- We have $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction.
- Our weaker precondition says that:

$$\forall x\, t\, c.\, x \;\#\; c \wedge (\forall c.\, P c\, t) \Rightarrow P c\, (\lambda x.t)$$

- We choose a fresh y such that $y \;\#\; (\pi \bullet x, \pi \bullet t, c)$.

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P\, c\, (\pi \bullet t)$ by induction on t.
- I.e., we have to show $P\, c\, \lambda(\pi \bullet x).(\pi \bullet t)$.
- We have $\forall \pi\, c.\ P\, c\, (\pi \bullet t)$ by induction.
- Our weaker precondition says that:

$$\forall x\, t\, c.\, x \;\#\; c \wedge (\forall c.\, P\, c\, t) \Rightarrow P\, c\, (\lambda x.t)$$

- We choose a fresh y such that $y \;\#\; (\pi \bullet x, \pi \bullet t, c)$.
- Now we can use $\forall c.\ P\, c\, (((y\ \ \pi \bullet x) :: \pi) \bullet t)$

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\, \lambda(\pi \bullet x).(\pi \bullet t)$.
- We have $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction.
- Our weaker precondition says that:

$$\forall x\, t\, c.\, x \# c \wedge (\forall c.\, P c\, t) \Rightarrow P c\, (\lambda x.t)$$

- We choose a fresh y such that $y \# (\pi \bullet x, \pi \bullet t, c)$.
- Now we can use $\forall c.\ P c\, ((y\ \ \pi \bullet x) \bullet \pi \bullet t)$

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\, \lambda(\pi \bullet x).(\pi \bullet t)$.
- We have $\forall \pi\, c.\ P c\, (\pi \bullet t)$ by induction.
- Our weaker precondition says that:

$$\forall x\, t\, c.\, x \ \# \ c \wedge (\forall c.\, P c\, t) \Rightarrow P c\, (\lambda x.t)$$

- We choose a fresh y such that $y \ \# \ (\pi \bullet x, \pi \bullet t, c)$.
- Now we can use $\forall c.\ P c\, ((y\ \ \pi \bullet x) \bullet \pi \bullet t)$ to infer

$$P\, c\, \lambda y.((y\ \ \pi \bullet x) \bullet \pi \bullet t)$$

Proof for the Strong Induction Principle

- We prove $\forall \pi\ c.\ P c\,(\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\, \lambda(\pi \bullet x).(\pi \bullet t)$.
- We have $\forall \pi\ c.\ P c\,(\pi \bullet t)$ by induction.
- Our weak

$$\frac{x \neq y \quad t_1 = (x\ y)\bullet t_2 \quad y \,\#\, t_2}{\lambda y.t_1 = \lambda x.t_2}$$

$\forall x\ t$ ... $t)$

- We choose a fresh y such that $y \,\#\, (\pi \bullet x, \pi \bullet t, c)$.
- Now we can use $\forall c.\ P c\,((y\ \ \pi \bullet x)\bullet \pi \bullet t)$ to infer

$$P\,c\,\lambda y.((y\ \ \pi \bullet x)\bullet \pi \bullet t)$$

- However

$$\lambda y.((y\ \ \pi \bullet x)\bullet \pi \bullet t) = \lambda(\pi \bullet x).(\pi \bullet t)$$

Proof for the Strong Induction Principle

- We prove $\forall \pi\, c.\; P c\, (\pi \bullet t)$ by induction on t.
- I.e., we have to show $P c\, \lambda(\pi \bullet x).(\pi \bullet t)$.
- We have $\forall \pi\, c.\; P c\, (\pi \bullet t)$ by induction.
- Our weaker precondition says that:

$$\forall x\, t\, c.\, x \;\#\; c \wedge (\forall c.\, P c\, t) \Rightarrow P c\, (\lambda x.t)$$

- We choose a fresh y such that $y \;\#\; (\pi \bullet x, \pi \bullet t, c)$.
- Now we can use $\forall c.\; P c\, ((y\;\; \pi \bullet x) \bullet \pi \bullet t)$ to infer

$$P\, c\, \lambda y.((y\;\; \pi \bullet x) \bullet \pi \bullet t)$$

- However

$$\lambda y.((y\;\; \pi \bullet x) \bullet \pi \bullet t) = \lambda(\pi \bullet x).(\pi \bullet t)$$

- Therefore $P\, c\, \lambda(\pi \bullet x).(\pi \bullet t)$ and we are done.

This Proof in Isabelle

```
lemma lam_strong_induct:
  fixes c::"'a::fs_name"
  assumes h₁: "⋀x c. P c (Var x)"
  and     h₂: "⋀t₁ t₂ c. ⟦∀d. P d t₁; ∀d. P d t₂⟧ ⟹ P c (App t₁ t₂)"
  and     h₃: "⋀x t c. ⟦x#c; ∀d. P d t⟧ ⟹ P c (Lam [x].t)"
  shows "P c t"
proof -
  have "∀(π::name prm) c. P c (π•t)"   . . .
  then have "P c (([]::name prm)•t)" by blast
  then show "P c t" by simp
qed
```

interesting bit

Interesting Bit

```
...
have "∀ (π::name prm) c. P c (π • t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀ (π::name prm) c. P c (π • t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π • x,π • t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀ c. P c (([(y,π • x)]@π) • t)" by simp
    then have "∀ c. P c ([(y,π • x)] • (π • t))" by (auto simp only: pt_name2)
    with h3 have "P c (Lam [y].[(y,π • x)] • (π • t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π • x)] • (π • t) = Lam [(π • x)].(π • t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
    ultimately have "P c (Lam [(π • x)].(π • t))" by simp
  }
  then have "∀ (π::name prm) c. P c (Lam [(π • x)].(π • t))" by simp
  then show "∀ (π::name prm) c. P c (π • (Lam [x].t))" by simp
qed (auto intro: h1 h2)
...
```

Interesting Bit

```
...
have "∀ (π::name prm) c. P c (π • t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀ (π::name prm) c. P c (π • t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π • x,π • t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀ c. P c (([(y,π • x)]@π) • t)" by simp
    then have "∀ c. P c ([(y,π • x)] • (π • t))" by (auto simp only: pt_name2)
    with h3 have "P c (Lam [y].[(y,π • x)] • (π • t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π • x)] • (π • t) = Lam [(π • x)].(π • t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
    ultimately have "P c (Lam [(π • x)].(π • t))" by simp
  }
  then have "∀ (π::name prm) c. P c (Lam [(π • x)].(π • t))" by simp
  then show "∀ (π::name prm) c. P c (π • (Lam [x].t))" by simp
qed (auto intro: h1 h2)
...
```

Interesting Bit

```
...
have "∀ (π::name prm) c. P c (π•t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀ (π::name prm) c. P c (π•t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π•x,π•t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀ c. P c (([(y,π•x)]@π)•t)" by simp
    then have "∀ c. P c ([(y,π•x)]•(π•t))" by (auto simp only: pt_name2)
    with h3 have "P c (Lam [y].[(y,π•x)]•(π•t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π•x)]•(π•t) = Lam [(π•x)].(π•t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
    ultimately have "P c (Lam [(π•x)].(π•t))" by simp
  }
  then have "∀ (π::name prm) c. P c (Lam [(π•x)].(π•t))" by simp
  then show "∀ (π::name prm) c. P c (π•(Lam [x].t))" by simp
qed (auto intro: h1 h2)
...
```

Interesting Bit

```
...
have "∀(π::name prm) c. P c (π•t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀(π::name prm) c. P c (π•t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π•x,π•t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀c. P c (([(y,π•x)]@π)•t)" by simp
    then have "∀c. P c ([(y,π•x)]•(π•t))" by (auto simp only: pt_name2)
    with h₃ have "P c (Lam [y].[(y,π•x)]•(π•t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π•x)]•(π•t) = Lam [(π•x)].(π•t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
    ultimately have "P c (Lam [(π•x)].(π•t))" by simp
  }
  then have "∀(π::name prm) c. P c (Lam [(π•x)].(π•t))" by simp
  then show "∀(π::name prm) c. P c (π•(Lam [x].t))" by simp
qed (auto intro: h₁ h₂)
...
```

Interesting Bit

```
...
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proof (induct t rule: lam.induct)
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    from ih have "∀c. P c (([(y,π•x)]@π)•t)" by simp
    then have "∀c. P c ([(y,π•x)]•(π•t))" by (auto simp only: pt_name2)
    with h3 have "P c (Lam [y].[(y,π•x)]•(π•t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π•x)]•(π•t) = Lam [(π•x)].(π•t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
    ultimately have "P c (Lam [(π•x)].(π•t))" by simp
  }
  then have "∀(π::name prm) c. P c (Lam [(π•x)].(π•t))" by simp
  then show "∀(π::name prm) c. P c (π•(Lam [x].t))" by simp
qed (auto intro: h1 h2)
...
```

Interesting Bit

```
...
have "∀(π::name prm) c. P c (π•t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀(π::name prm) c. P c (π•t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π•x,π•t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀c. P c (([(y,π•x)]@π)•t)" by simp
    then have "∀c. P c ([(y,π•x)]•(π•t))" by (auto simp only: pt_name2)
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      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
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  then have "∀(π::name prm) c. P c (Lam [(π•x)].(π•t))" by simp
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    ultimately have "P c (Lam [(π•x)].(π•t))" by simp
  }
  then have "∀(π::name prm) c. P c (Lam [(π•x)].(π•t))" by simp
  then show "∀(π::name prm) c. P c (π•(Lam [x].t))" by simp
qed (auto intro: h₁ h₂)
...
```

h_3: "$\bigwedge$x t c. ⟦x # c; ∀d. P d t⟧ $\Longrightarrow$ P c Lam [x].t"

Interesting Bit

```
...
have "∀(π::name prm) c. P c (π•t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀(π::name prm) c. P c (π•t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π•x,π•t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀c. P c (([(y,π•x)]@π)•t)" by simp
    then have "∀c. P c ([(y,π•x)]•(π•t))" by (auto simp only: pt_name2)
    with h₃ have "P c (Lam [y].[(y,π•x)]•(π•t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π•x)]•(π•t) = Lam [(π•x)].(π•t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
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  then show "∀(π::name prm) c. P c (π•(Lam [x].t))" by simp
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```

Interesting Bit

```
...
have "∀(π::name prm) c. P c (π•t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀(π::name prm) c. P c (π•t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
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    ultimately have "P c (Lam [(π•x)].(π•t))" by simp
  }
  then have "∀(π::name prm) c. P c (Lam [(π•x)].(π•t))" by simp
  then show "∀(π::name prm) c. P c (π•(Lam [x].t))" by simp
qed (auto intro: h₁ h₂)
...
```

Interesting Bit

```
...
have "∀ (π::name prm) c. P c (π • t)"
proof (induct t rule: lam.induct)
  case (Lam x t)
  have ih: "∀ (π::name prm) c. P c (π • t)" by fact
  { fix π::"name prm" and c::"'a::fs_name"
    obtain y::"name" where fc: "y#(π • x,π • t,c)"
      by (rule exists_fresh) (auto simp add: fs_name1)
    from ih have "∀ c. P c (([(y,π • x)]@π) • t)" by simp
    then have "∀ c. P c ([(y,π • x)] • (π • t))" by (auto simp only: pt_name2)
    with h₃ have "P c (Lam [y].[(y,π • x)] • (π • t))" using fc by (simp add: fresh_prod)
    moreover
    have "Lam [y].[(y,π • x)] • (π • t) = Lam [(π • x)].(π • t)"
      using fc by (simp add: lam.inject alpha fresh_atm fresh_prod)
    ultimately have "P c (Lam [(π • x)].(π • t))" by simp
  }
  then have "∀ (π::name prm) c. P c (Lam [(π • x)].(π • t))" by simp
  then show "∀ (π::name prm) c. P c (π • (Lam [x].t))" by simp
qed (auto intro: h₁ h₂)
...
```

Some Examples

$$\frac{x \# \Gamma \quad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathsf{Lam}\,[x].t : T_1 \rightarrow T_2}$$

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$$\frac{x \# \Gamma \quad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathrm{Lam}\,[x].t : T_1 \rightarrow T_2}$$

$$\frac{t \mapsto t'}{\mathrm{Lam}\,[x].t \mapsto t'}$$

Some Examples

$$\frac{x \# \Gamma \quad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathrm{Lam}\ [x].t : T_1 \to T_2}$$

$$\frac{t \mapsto t'}{\mathrm{Lam}\ [x].t \mapsto t'}$$

Some Examples

$$\frac{x \# \Gamma \quad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathsf{Lam}\,[x].t : T_1 \rightarrow T_2}$$

$$\frac{t \mapsto t'}{\mathsf{Lam}\,[x].t \mapsto t'}$$

$$\frac{\Gamma \vdash_\Sigma A_1 : \mathsf{Type} \quad (x, A_1)::\Gamma \vdash_\Sigma M_2 : A_2 \quad x \# (\Gamma, A_1)}{\Gamma \vdash_\Sigma \mathsf{Lam}\,[x{:}A_1].M_2 : \Pi[x{:}A_1].A_2}$$

Some Examples

$$\frac{x \# \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathsf{Lam}\ [x].t : T_1 \to T_2}$$

$$\frac{t \mapsto t'}{\ldots[x].t \mapsto t'}$$

free

$$\frac{\Gamma \vdash_{\Sigma} A_1 : \mathsf{Type} \qquad (x, A_1)::\Gamma \vdash_{\Sigma} M_2 : A_2 \qquad x \# (\Gamma, A_1)}{\Gamma \vdash_{\Sigma} \mathsf{Lam}\ [x{:}A_1].M_2 : \Pi[x{:}A_1].A_2}$$

bound

bound

Some Examples

$$\frac{x \;\#\; \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathsf{Lam}\,[x].t : T_1 \to T_2}$$

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$$\frac{\Gamma \vdash_\Sigma A_1 : \mathsf{Type} \qquad (x, A_1)::\Gamma \vdash_\Sigma M_2 : A_2 \qquad x \;\#\; (\Gamma, A_1)}{\Gamma \vdash_\Sigma \mathsf{Lam}\,[x \ldots \Pi[x{:}A_1].\ldots}$$

free free free

$$\frac{(x, \tau_1)::\Delta \vdash_\Sigma \mathsf{App}\,M\,(\mathsf{Var}\,x) \Leftrightarrow \mathsf{App}\,N\,(\mathsf{Var}\,x) : \tau_2 \qquad x \;\#\; (\Delta, M, N)}{\Delta \vdash_\Sigma M \Leftrightarrow N : \tau_1 \to \tau_2}$$

Some Examples

$$\frac{x \# \Gamma \quad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathsf{Lam}\,[x].t : T_1 \to T_2}$$

$$\frac{t \mapsto t'}{\mathsf{Lam}\,[x].t \mapsto t'}$$

$$\frac{\Gamma \vdash_\Sigma A_1 : \mathsf{Type} \quad (x, A_1)::\Gamma \vdash_\Sigma M_2 : A_2 \quad x \# (\Gamma, A_1)}{\Gamma \vdash_\Sigma \mathsf{Lam}\,[x{:}A_1].M_2 : \Pi[x{:}A_1].A_2}$$

$$\frac{\begin{array}{c}(x, \tau_1)::\Delta \vdash_\Sigma \mathsf{App}\,M\,(\mathsf{Var}\,x) \Leftrightarrow \mathsf{App}\,N\,(\mathsf{Var}\,x) : \tau_2 \\ x \# (\Delta, M, N)\end{array}}{\Delta \vdash_\Sigma M \Leftrightarrow N : \tau_1 \to \tau_2}$$

Formalisation of LF

```
nominal_datatype
    kind =   Type
           | KPi "ty" "«name»kind"
and ty =   TConst "id"
           | TApp "ty" "trm"
           | TPi "ty" "«name»ty"
and trm = Const "id"
           | Var "name"
           | App "trm" "trm"
           | Lam "ty" "«name»trm"

abbreviation KPi_syn :: "name ⇒ ty ⇒ kind ⇒ kind" ("Π[_:_]._")
where "Π[x:A].K ≡ KPi A x K"

abbreviation TPi_syn :: "name ⇒ ty ⇒ ty ⇒ ty" ("Π[_:_]._")
where "Π[x:A1].A2 ≡ TPi A1 x A2"

abbreviation Lam_syn :: "name ⇒ ty ⇒ trm ⇒ trm" ("Lam [_:_]._")
where  "Lam [x:A].M ≡ Lam A x M"
```

Formalisation of LF

(joint work with Cheney and Berghofer)

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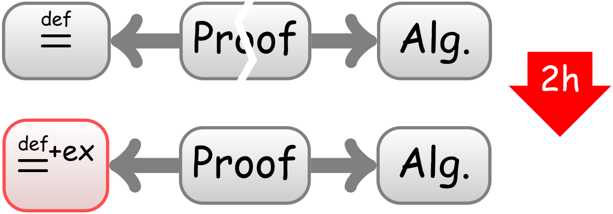
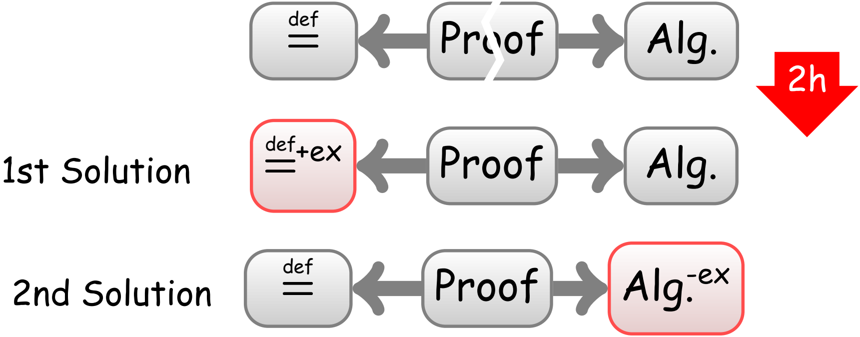
1st Solution



(each time one needs to check ∼31pp of informal paper proofs)

Formalisation of LF

(joint work with Cheney and Berghofer)



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Formalisation of LF

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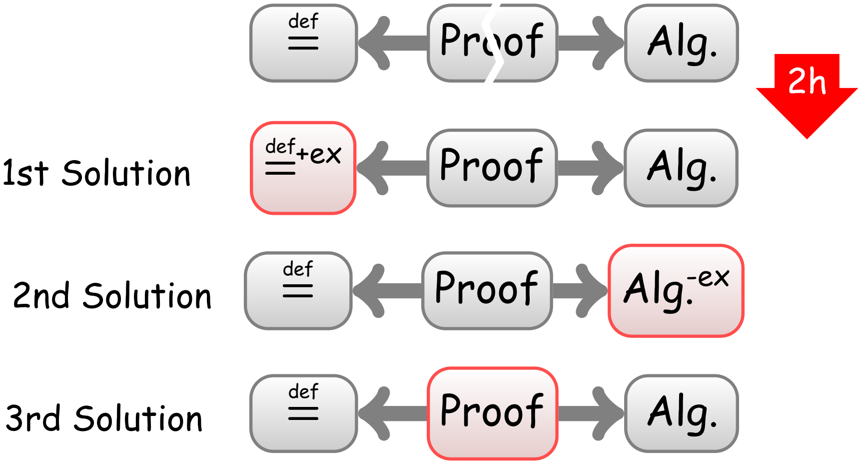

(each time one needs to check ∼31pp of informal paper proofs)

In My PhD

```
nominal_datatype trm =
    Ax   "name" "coname"
  | Cut  "«coname»trm" "«name»trm"                  ("Cut ⟨_⟩._ (_)._")
  | NotR "«name»trm" "coname"                       ("NotR (_)._ _")
  | NotL "«coname»trm" "name"                       ("NotL ⟨_⟩._ _")
  | AndR "«coname»trm" "«coname»trm" "coname"       ("AndR ⟨_⟩._ ⟨_⟩._ _")
  | AndL1 "«name»trm" "name"                        ("AndL1 (_)._ _")
  | AndL2 "«name»trm" "name"                        ("AndL2 (_)._ _")
  | OrR1 "«coname»trm" "coname"                     ("OrR1 ⟨_⟩._ _")
  | OrR2 "«coname»trm" "coname"                     ("OrR2 ⟨_⟩._ _")
  | OrL "«name»trm" "«name»trm" "name"              ("OrL (_)._ (_)._ _")
  | ImpR "«name»(«coname»trm)" "coname"             ("ImpR (_).⟨_⟩._ _")
  | ImpL "«coname»trm" "«name»trm" "name"           ("ImpL ⟨_⟩._ (_)._ _")
```

- A SN-result for cut-elimination in CL: reviewed by Henk Barendregt and Andy Pitts, and reviewers of conference and journal paper. Still, I found errors in central lemmas; fortunately the main claim was correct :o)

Two Health Warnings ;o)

Theorem provers should come with two health warnings:

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- Theorem provers are addictive!

  (Xavier Leroy: "Building [proof] scripts is surprisingly addictive, in a videogame kind of way...")

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Theorem provers should come with two health warnings:

- Theorem provers are addictive!

  (Xavier Leroy: “Building [proof] scripts is surprisingly addictive, in a videogame kind of way...”)

- Theorem provers cause you to lose faith in your proofs done by hand!

  (Michael Norrish, Mike Gordon, me, very possibly others)

Conclusions

- The Nominal Isabelle automatically derives the strong structural induction principle for **all** nominal datatypes (not just the lambda-calculus);
- also for rule inductions (though they have to satisfy a vc-condition).
- They are easy to use: you just have to think carefully what the variable convention should be.
- We can explore the dark corners of the variable convention: when and where it can actually be used.

Conclusions

- The Nominal Isabelle automatically derives the strong structural induction principle for <u>**all**</u> nominal datatypes (not just the lambda-calculus);
- also for rule inductions (though they have to satisfy a vc-condition).
- They are easy to use: you just have to think carefully what the variable convention should be.
- We can explore the dark corners of the variable convention: when and where it can actually be used.
- **Main Point:** Actually these proofs using the variable convention are all trivial / obvious / routine... **provided** you use Nominal Isabelle. ;o)

Thank you very much!