Isabelle and Proof General: Preview

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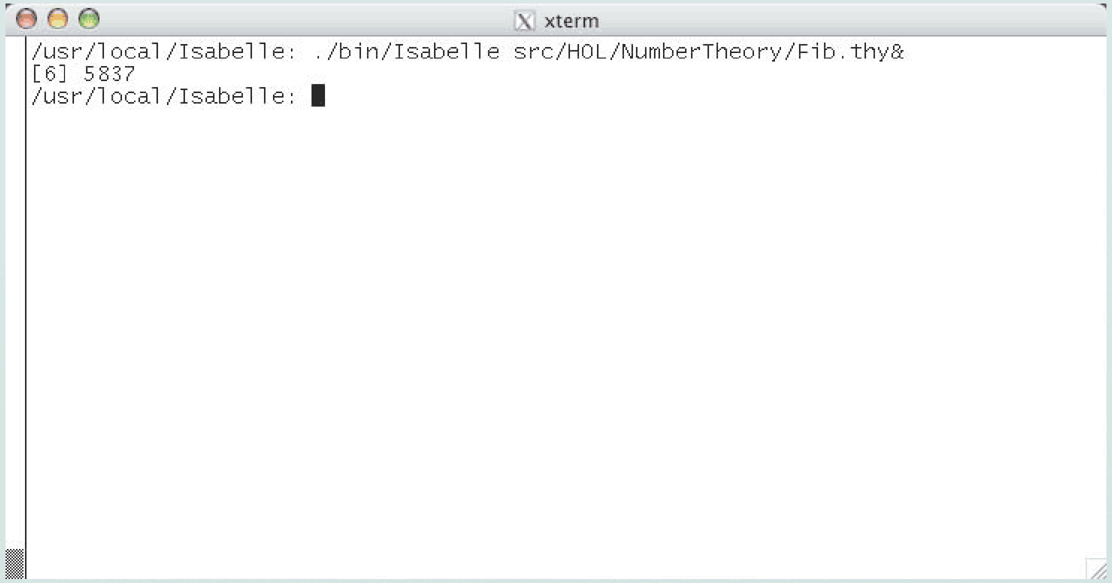
University of Cambridge

Getting Started

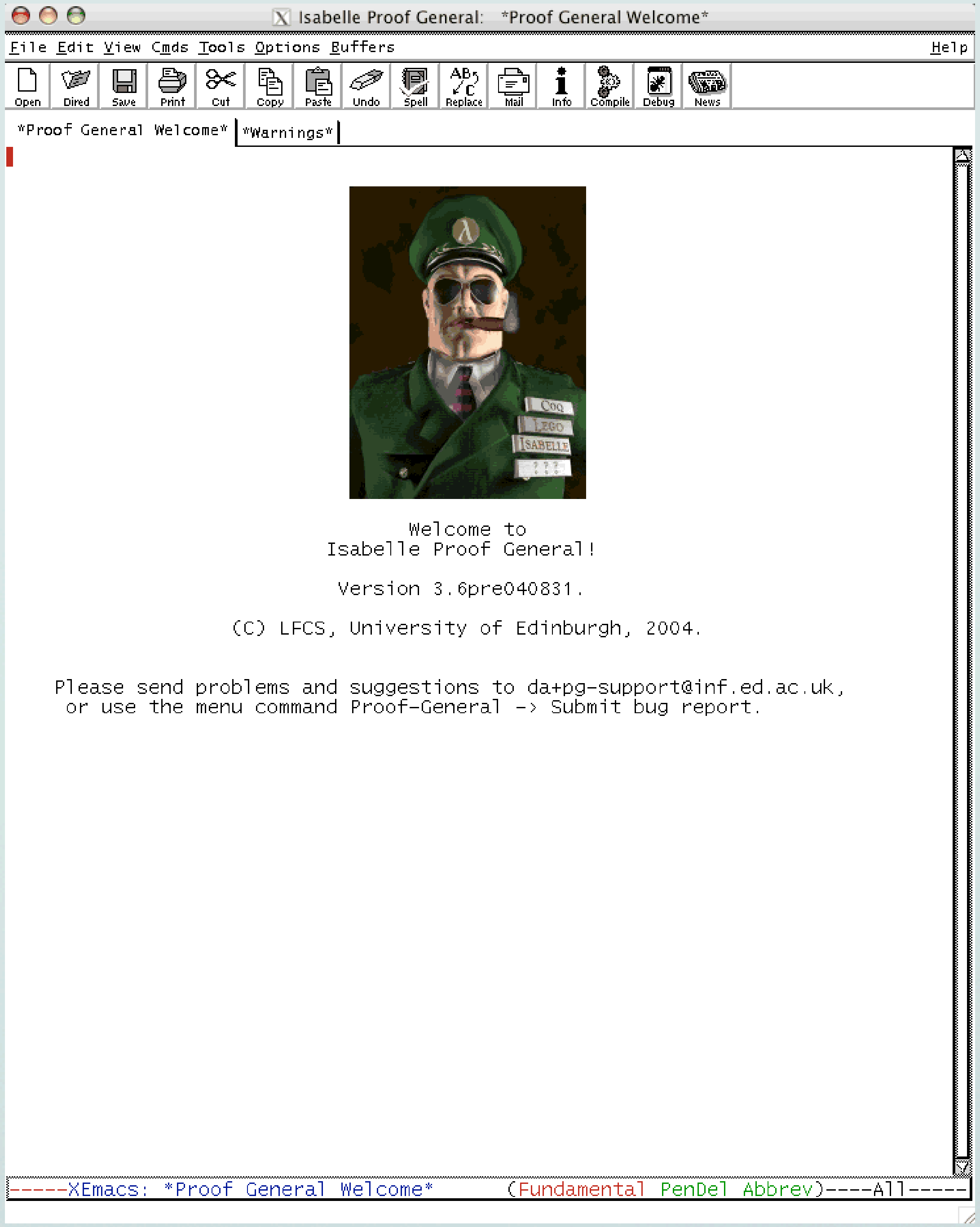
- Install Isabelle, following instructions on the download page.
- Install Proof General.
- Proof General requires the editor XEmacs to be installed.
- If you have not used XEmacs before, practice on plain text files before attempting proofs!
- Launch Isabelle from the command line.
- Here, Isabelle has been installed at `/usr/local` and is used to open one of the standard theories.

Launching Isabelle at the UNIX command line



```
/usr/local/Isabelle: ./bin/Isabelle src/HOL/NumberTheory/Fib.thy&
[6] 5837
/usr/local/Isabelle: 
```

Proof General


Isabelle Proof General: *Proof General Welcome*
File Edit View Cmds Tools Options Buffers
Help
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Compile
Debug
News
Proof General Welcome
Warnings
Coq
LEGO
ISABELLE
???
-----XEmacs: *Proof General Welcome*   (Fundamental PenDel Abbrev)----All-----

```
Welcome to
Isabelle Proof General!

Version 3.6pre040831.

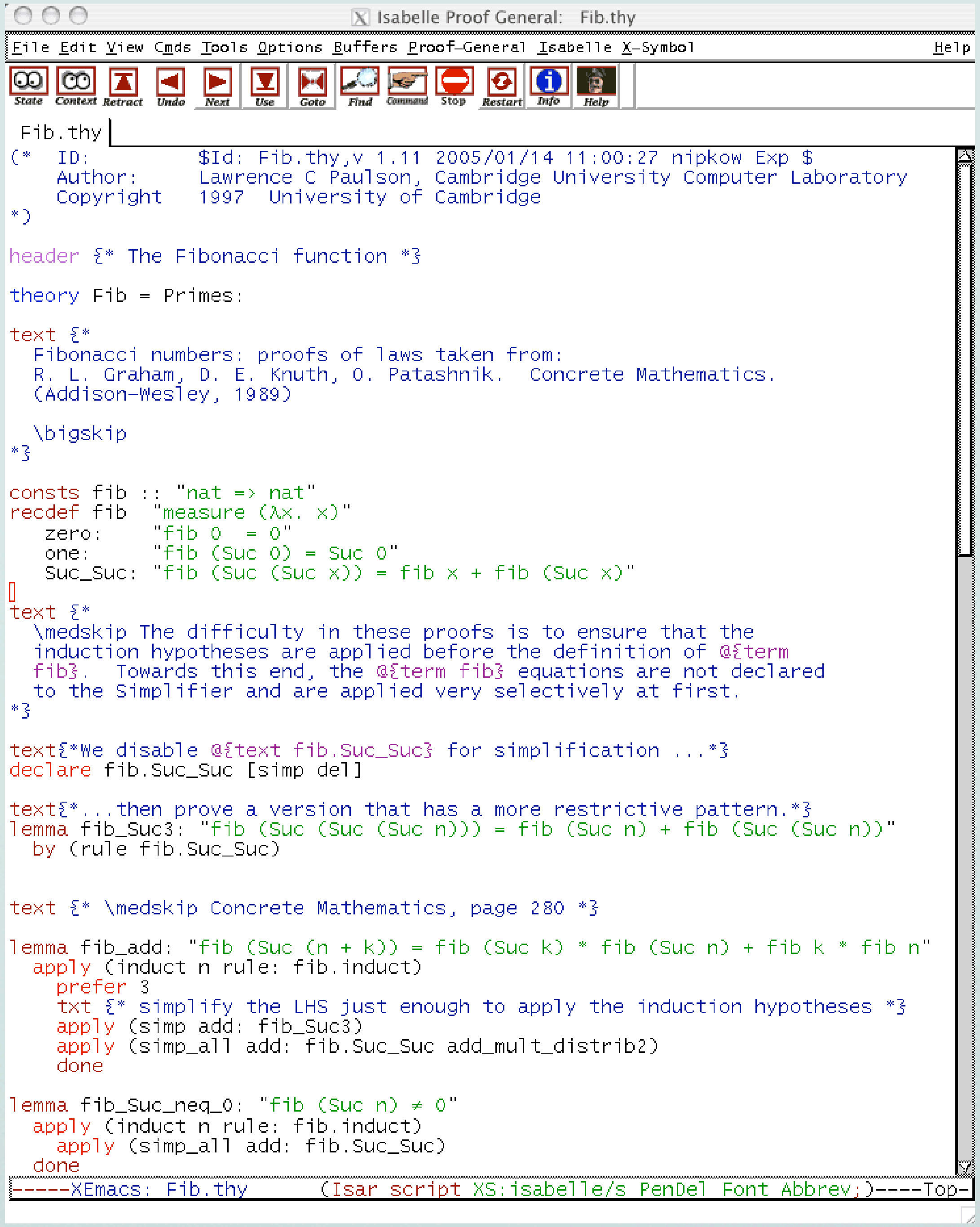
(C) LFCS, University of Edinburgh, 2004.

Please send problems and suggestions to da+pg-support@inf.ed.ac.uk,
 or use the menu command Proof-General -> Submit bug report.
```

- Proof General launches within XEmacs.
- If you don't see this splash screen, Proof General is not correctly installed.

The Theory File

- The theory opens in Proof General.
- Theory files visited from XEmacs also open in Proof General.
- Syntax colouring distinguishes constants, types, keywords, etc.
- The toolbar gives quick access to basic proof operations.
- This theory defines the Fibonacci function and proves theorems about it.


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General Isabelle X-Symbol Help
State Context Retract Undo Next Use Goto Find Command Stop Restart Info Help
Fib.thy

```
(*  ID:         $Id: Fib.thy,v 1.11 2005/01/14 11:00:27 nipkow Exp $
    Author:     Lawrence C Paulson, Cambridge University Computer Laboratory
    Copyright   1997  University of Cambridge
*)

header {* The Fibonacci function *}

theory Fib = Primes:

text {*
  Fibonacci numbers: proofs of laws taken from:
  R. L. Graham, D. E. Knuth, O. Patashnik.  Concrete Mathematics.
  (Addison-Wesley, 1989)

  \bigskip
*}

consts fib :: "nat => nat"
recdef fib  "measure (λx. x)"
   zero:    "fib 0  = 0"
   one:     "fib (Suc 0) = Suc 0"
   Suc_Suc: "fib (Suc (Suc x)) = fib x + fib (Suc x)"

text {*
  \medskip The difficulty in these proofs is to ensure that the
  induction hypotheses are applied before the definition of @{term
  fib}.  Towards this end, the @{term fib} equations are not declared
  to the Simplifier and are applied very selectively at first.
*}

text{*We disable @{text fib.Suc_Suc} for simplification ...*}
declare fib.Suc_Suc [simp del]

text{*...then prove a version that has a more restrictive pattern.*}
lemma fib_Suc3: "fib (Suc (Suc (Suc n))) = fib (Suc n) + fib (Suc (Suc n))"
  by (rule fib.Suc_Suc)

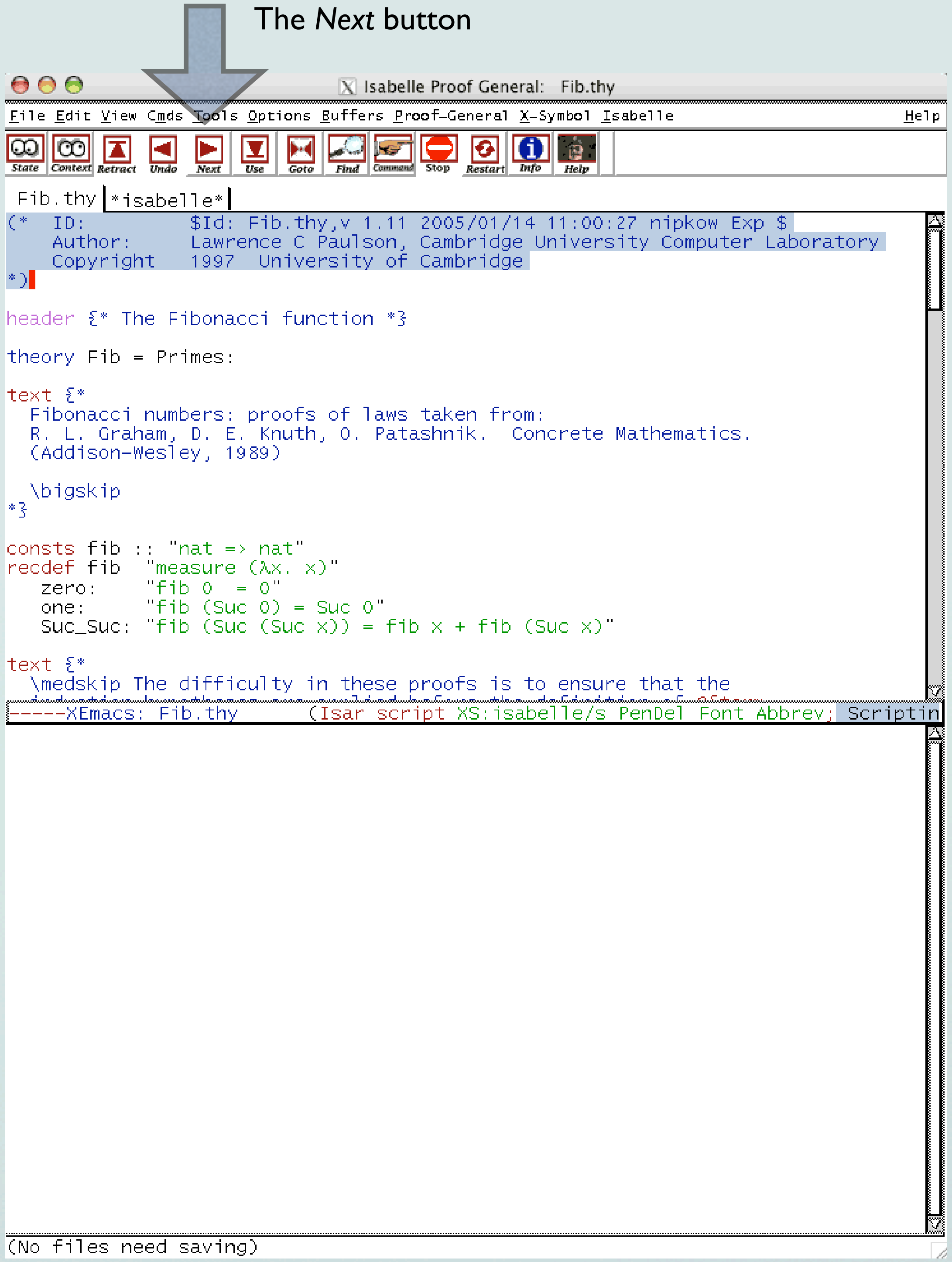

text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) ≠ 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done
-----XEmacs: Fib.thy      (Isar script XS:isabelle/s PenDel Font Abbrev;)----Top-
```

Basic Navigation

- A theory file contains definitions, proofs, LaTeX markup, and general commands.
- Clicking on *Next* starts Isabelle and processes the first item: a comment.
- Repeated clicks on *Next* step through the definitions.
- Proof General highlights material that has been processed in blue.


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle Help
State
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Use
Goto
Find
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Info
Help
Fib.thy
isabelle

The *Next* button

```
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  R. L. Graham, D. E. Knuth, O. Patashnik.  Concrete Mathematics.
  (Addison-Wesley, 1989)

  \bigskip
*}

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   Suc_Suc: "fib (Suc (Suc x)) = fib x + fib (Suc x)"

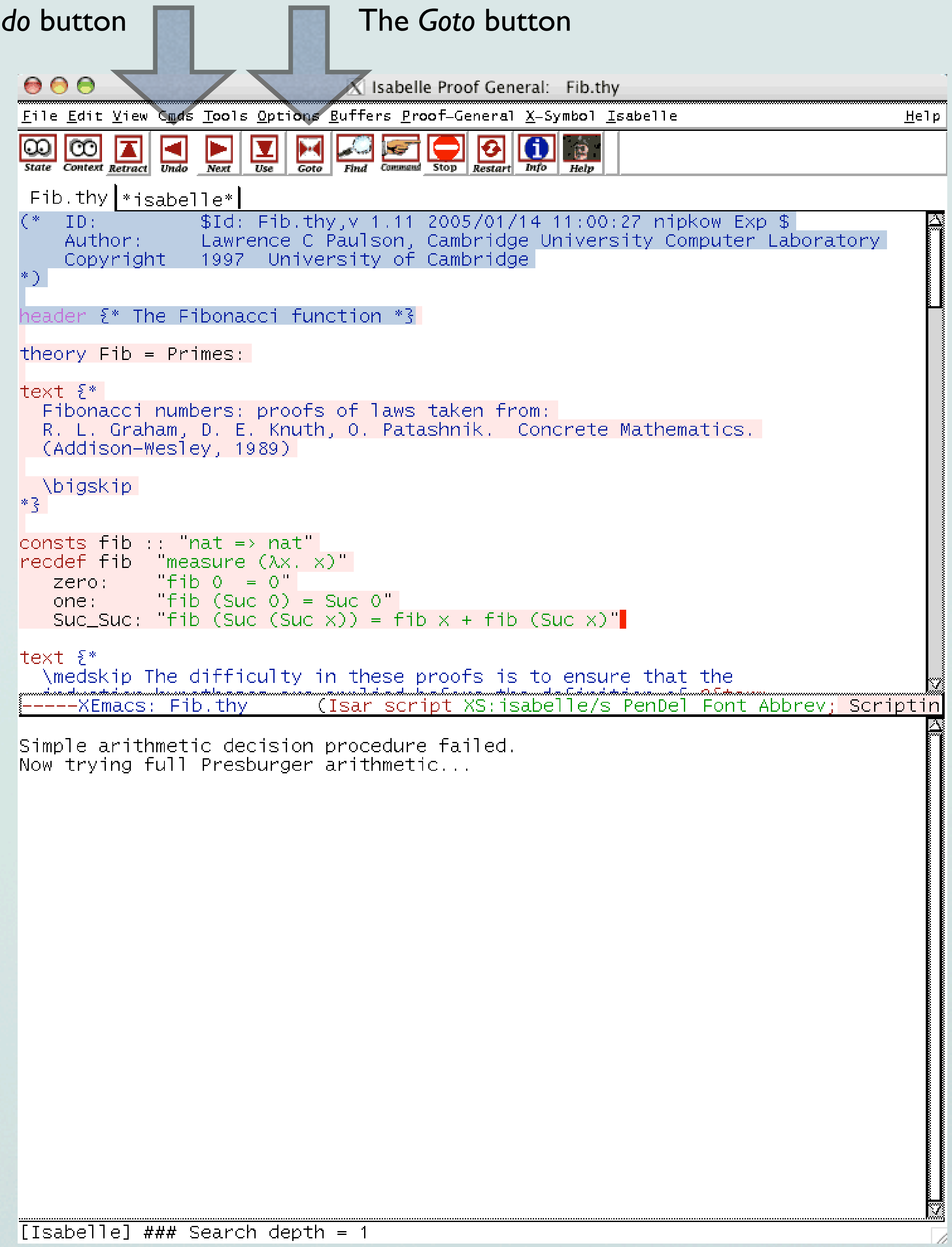
text {*
  \medskip The difficulty in these proofs is to ensure that the
```

```
-----XEmacs: Fib.thy      (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

```
(No files need saving)
```

Jumping Ahead

- You can click anywhere in the theory and then click on *Goto*.
- *Goto* can even go backward, undoing declarations and commands. (To undo a single command, use the *Undo* button.)
- The `header` command is processed quickly, but the `theory` command refers to another theory.
- While Isabelle is working, Proof General highlights the corresponding text in pink.



The *Undo* button

The *Goto* button

Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle Help
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isabelle

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  \bigskip
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text {*
  \medskip The difficulty in these proofs is to ensure that the
```

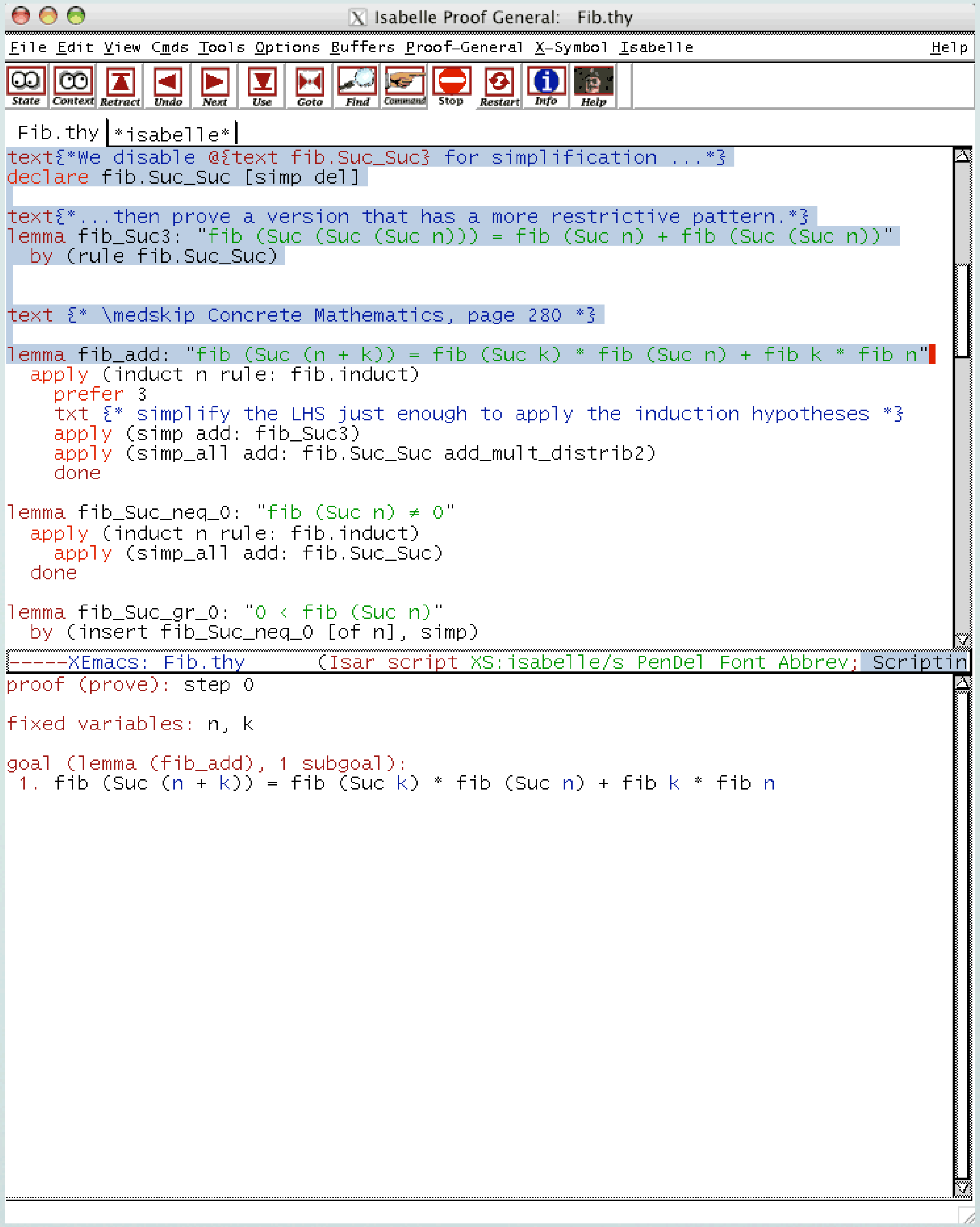
```
-----XEmacs: Fib.thy      (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

```
Simple arithmetic decision procedure failed.
Now trying full Presburger arithmetic...
```

```
[Isabelle] ### Search depth = 1
```

Running a Proof

- We are about to replay a small proof relating the Fibonacci function, addition and multiplication.
- Processing the `lemma` command displays one subgoal in the proof window.
- The commands `lemma`, `theorem` and `corollary` are essentially equivalent.


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle Help
State Context Retract Undo Next Use Goto Find Command Stop Restart Info Help
Fib.thy *isabelle*

```
text{*We disable @{text fib.Suc_Suc} for simplification ...*}
declare fib.Suc_Suc [simp del]

text{*...then prove a version that has a more restrictive pattern.*}
lemma fib_Suc3: "fib (Suc (Suc (Suc n))) = fib (Suc n) + fib (Suc (Suc n))"
  by (rule fib.Suc_Suc)


text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) ≠ 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done

lemma fib_Suc_gr_0: "0 < fib (Suc n)"
  by (insert fib_Suc_neq_0 [of n], simp)
```

-----XEmacs: Fib.thy   (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin

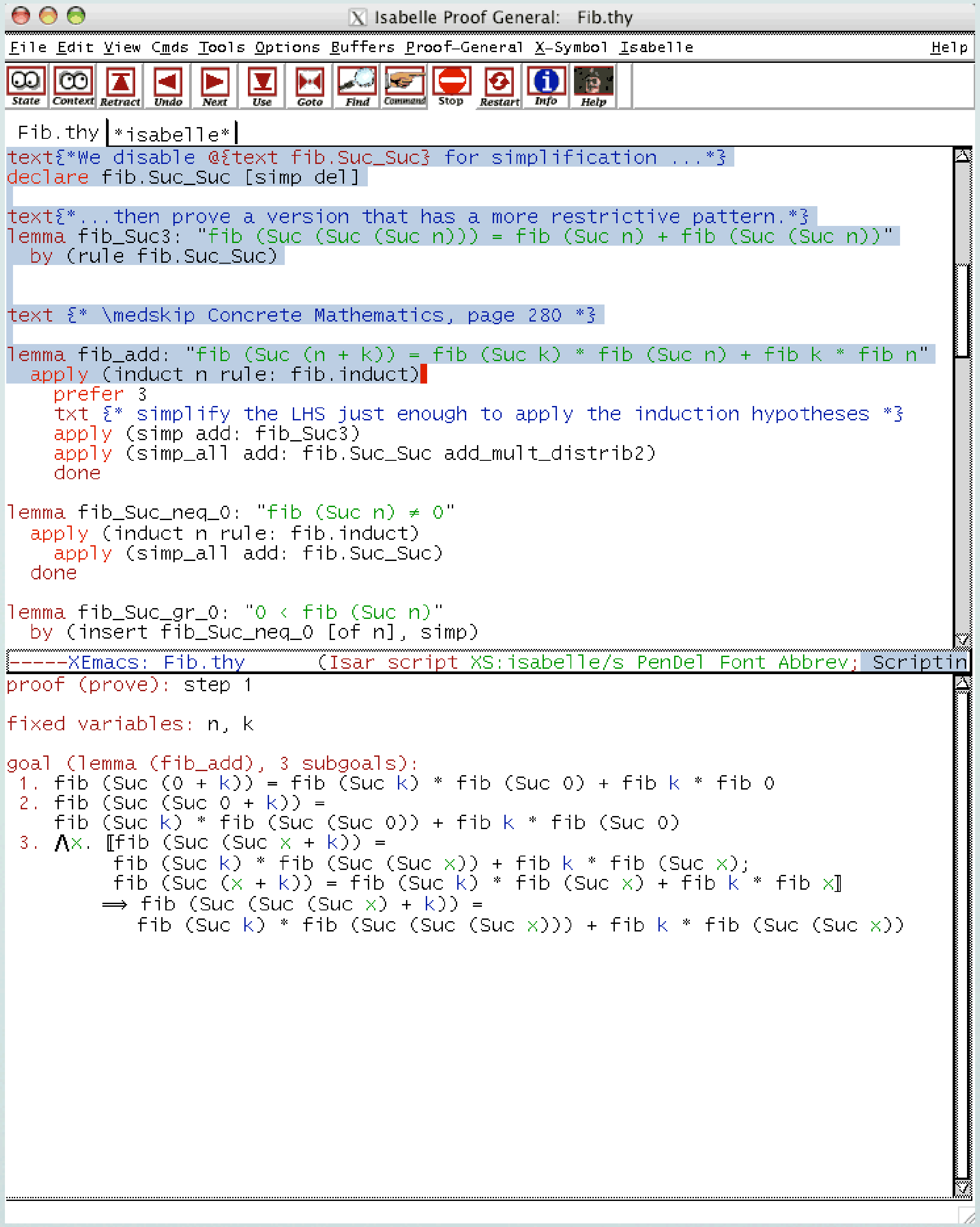
```
proof (prove): step 0

fixed variables: n, k

goal (lemma (fib_add), 1 subgoal):
 1. fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n
```

Performing Induction

- The first command performs induction on *n* using the rule `fib.induct`.
- Isabelle produced this rule while processing the recursive definition of the Fibonacci function.
- The proof window now displays three subgoals.


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle
Help
State
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isabelle

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lemma fib_Suc3: "fib (Suc (Suc (Suc n))) = fib (Suc n) + fib (Suc (Suc n))"
  by (rule fib.Suc_Suc)


text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) ≠ 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done

lemma fib_Suc_gr_0: "0 < fib (Suc n)"
  by (insert fib_Suc_neq_0 [of n], simp)
-----XEmacs: Fib.thy       (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

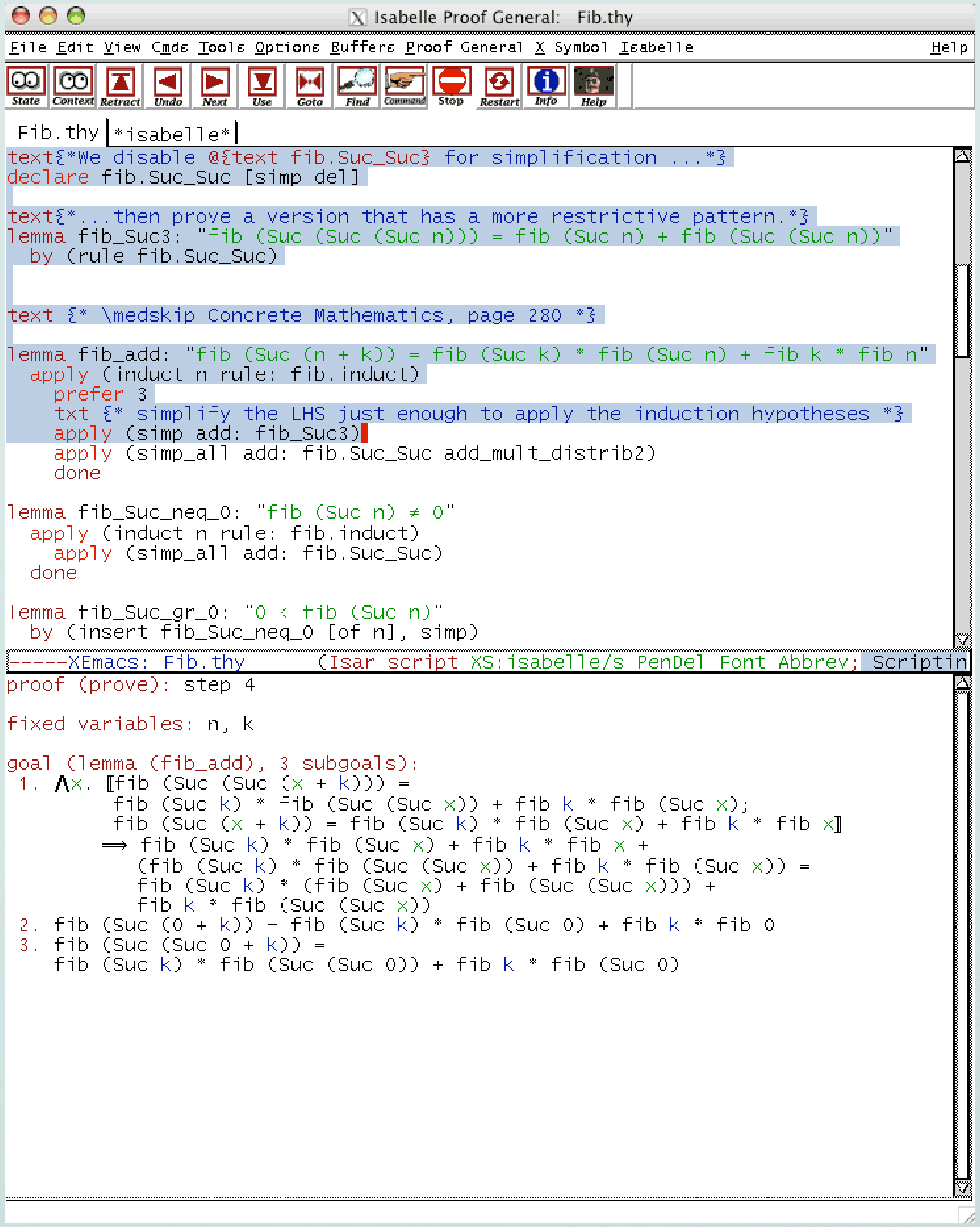
```
proof (prove): step 1

fixed variables: n, k

goal (lemma (fib_add), 3 subgoals):
 1. fib (Suc (0 + k)) = fib (Suc k) * fib (Suc 0) + fib k * fib 0
 2. fib (Suc (Suc 0 + k)) =
    fib (Suc k) * fib (Suc (Suc 0)) + fib k * fib (Suc 0)
 3. ⋀x. ⟦fib (Suc (Suc x + k)) =
         fib (Suc k) * fib (Suc (Suc x)) + fib k * fib (Suc x);
         fib (Suc (x + k)) = fib (Suc k) * fib (Suc x) + fib k * fib x⟧
        ⟹ fib (Suc (Suc (Suc x) + k)) =
           fib (Suc k) * fib (Suc (Suc (Suc x))) + fib k * fib (Suc (Suc x))
```

A Rewriting Step

- The third subgoal is selected: `prefer 3`.
- Then, it is simplified with the help of a rewrite rule called `fib_Suc3`.
- This subgoal is still rather complicated!


Isabelle Proof General: Fib.thy
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State Context Retract Undo Next Use Goto Find Command Stop Restart Info Help
Fib.thy *isabelle*

```
text{*We disable @{text fib.Suc_Suc} for simplification ...*}
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lemma fib_Suc3: "fib (Suc (Suc (Suc n))) = fib (Suc n) + fib (Suc (Suc n))"
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text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) ≠ 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done

lemma fib_Suc_gr_0: "0 < fib (Suc n)"
  by (insert fib_Suc_neq_0 [of n], simp)
```

-----XEmacs: Fib.thy   (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin

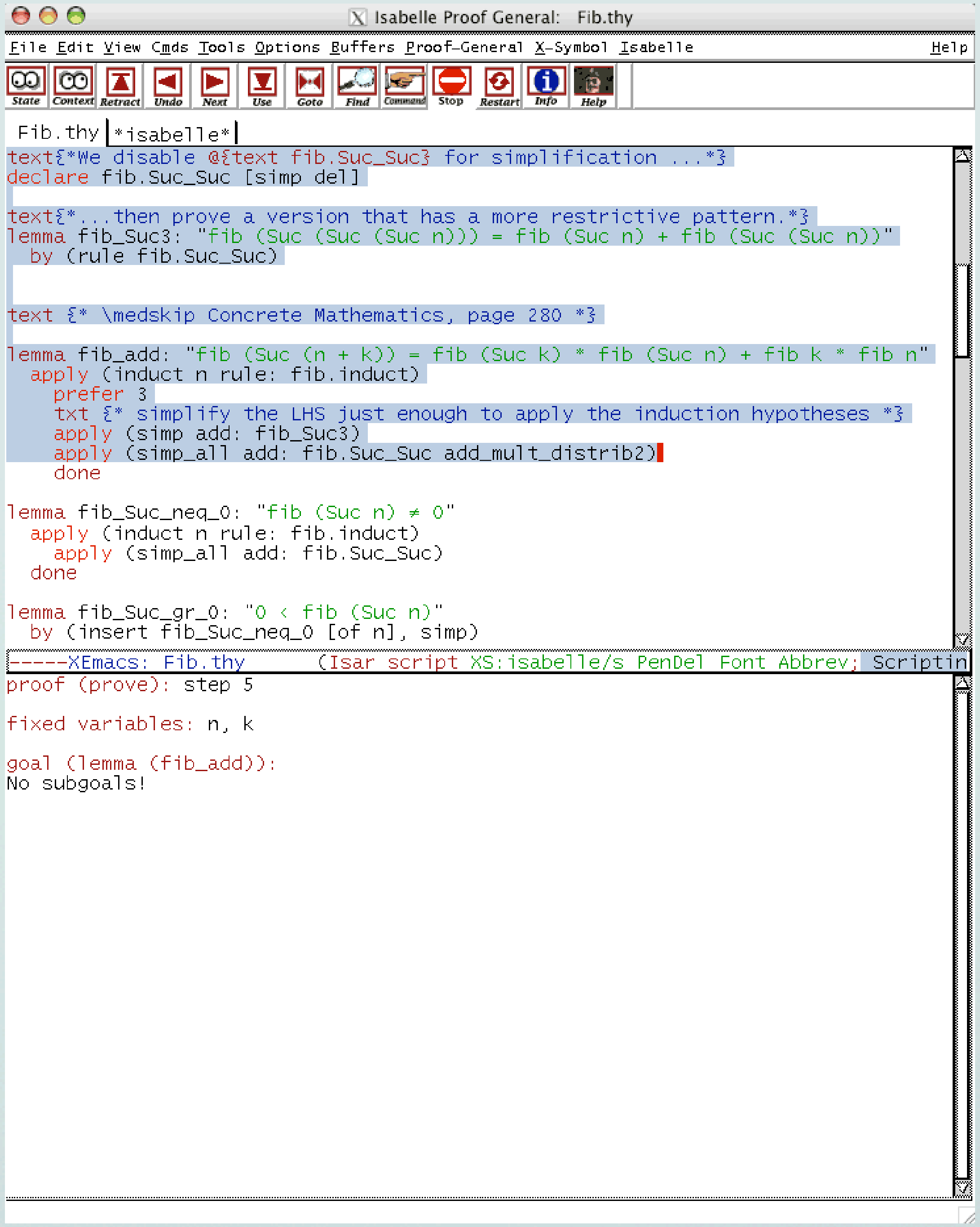
```
proof (prove): step 4

fixed variables: n, k

goal (lemma (fib_add), 3 subgoals):
 1. ⋀x. ⟦fib (Suc (Suc (x + k))) =
         fib (Suc k) * fib (Suc (Suc x)) + fib k * fib (Suc x);
         fib (Suc (x + k)) = fib (Suc k) * fib (Suc x) + fib k * fib x⟧
        ⟹ fib (Suc k) * fib (Suc x) + fib k * fib x +
           (fib (Suc k) * fib (Suc (Suc x)) + fib k * fib (Suc x)) =
           fib (Suc k) * (fib (Suc x) + fib (Suc (Suc x))) +
           fib k * fib (Suc (Suc x))
 2. fib (Suc (0 + k)) = fib (Suc k) * fib (Suc 0) + fib k * fib 0
 3. fib (Suc (Suc 0 + k)) =
    fib (Suc k) * fib (Suc (Suc 0)) + fib k * fib (Suc 0)
```

Finishing the Proof

- Next, all three subgoals are simplified, with the help of the rewrite rules shown.
- The simplifier automatically includes hundreds of other rewrite rules, as well as various decision procedures.
- This time, no subgoals remain.


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle Help
State Context Retract Undo Next Use Goto Find Command Stop Restart Info Help
Fib.thy *isabelle*

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text{*We disable @{text fib.Suc_Suc} for simplification ...*}
declare fib.Suc_Suc [simp del]

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lemma fib_Suc3: "fib (Suc (Suc (Suc n))) = fib (Suc n) + fib (Suc (Suc n))"
  by (rule fib.Suc_Suc)


text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) ≠ 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done

lemma fib_Suc_gr_0: "0 < fib (Suc n)"
  by (insert fib_Suc_neq_0 [of n], simp)
```

-----XEmacs: Fib.thy   (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin

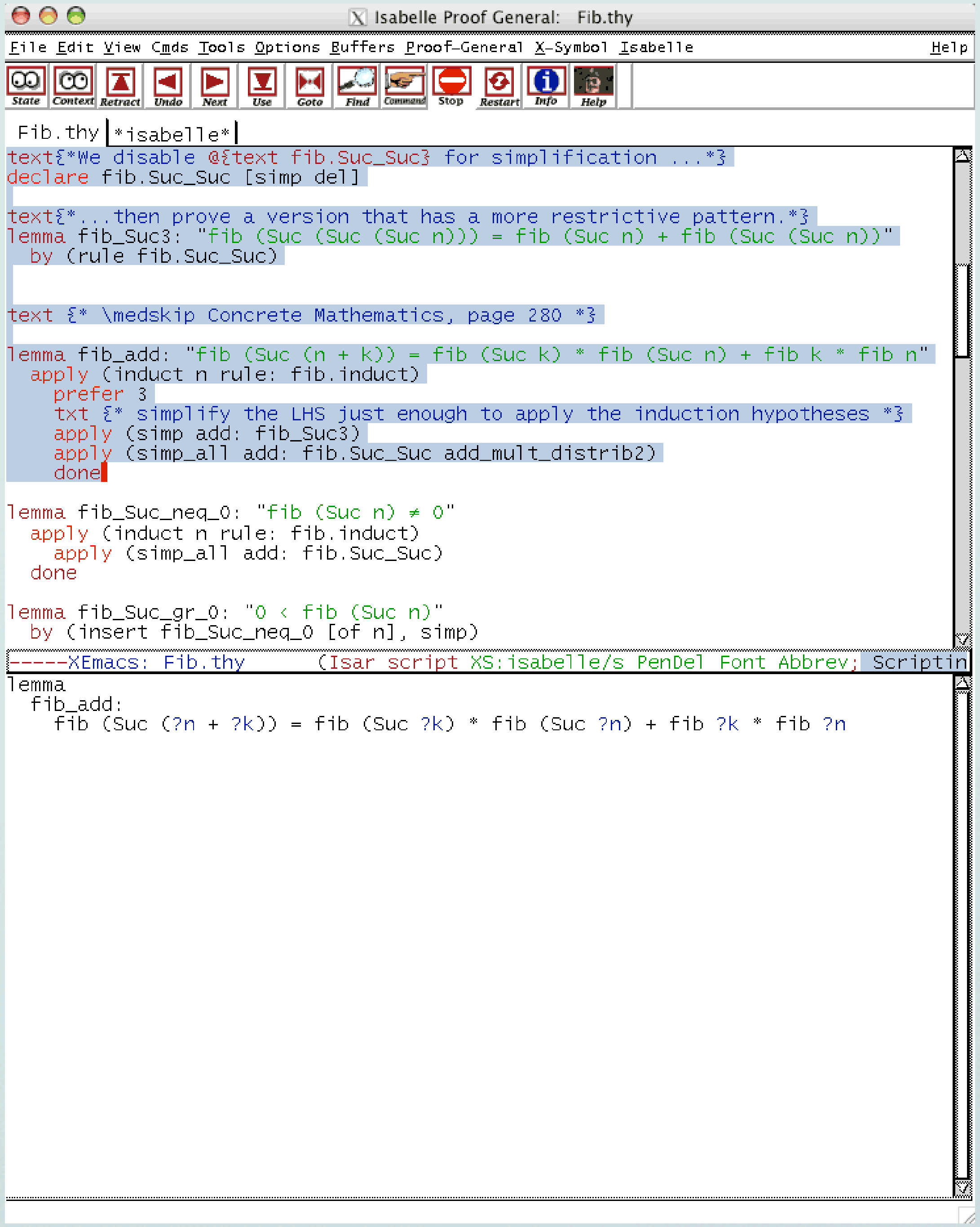
```
proof (prove): step 5

fixed variables: n, k

goal (lemma (fib_add)):
No subgoals!
```

Storing the Theorem

- The done command causes Isabelle to accept the proof, storing the theorem.
- If you were proving this theorem for the first time, you would try various commands right in the editor buffer. You would use *Undo* frequently!
- Once you have succeeded, the file will hold the final version of your proof.
- Using *Undo* on a done command moves the cursor above its proof. Isabelle "forgets" the theorem.


Isabelle Proof General: Fib.thy
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text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
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    done

lemma fib_Suc_neq_0: "fib (Suc n) ≠ 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done

lemma fib_Suc_gr_0: "0 < fib (Suc n)"
  by (insert fib_Suc_neq_0 [of n], simp)
```

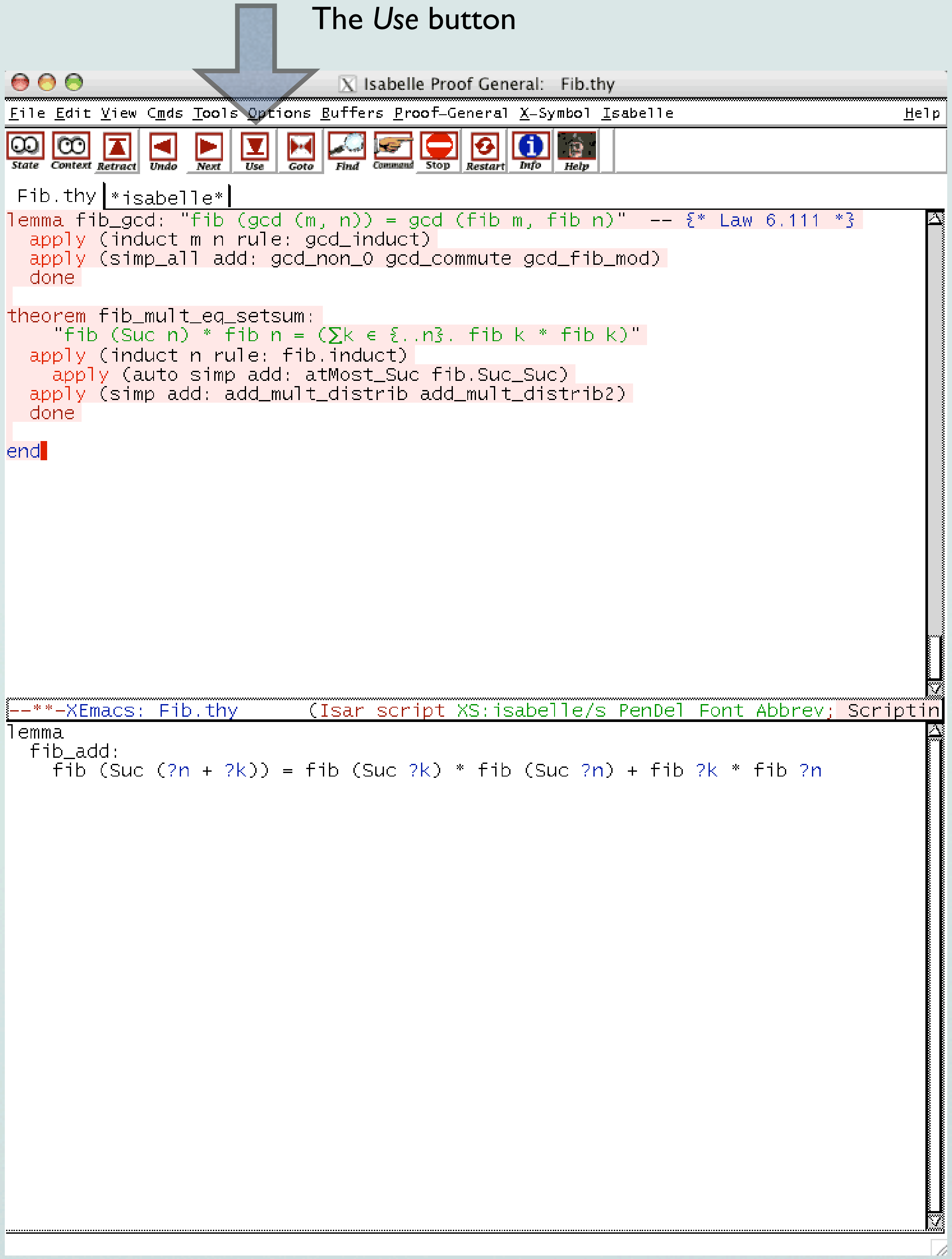
-----XEmacs: Fib.thy   (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin

```
lemma
  fib_add:
    fib (Suc (?n + ?k)) = fib (Suc ?k) * fib (Suc ?n) + fib ?k * fib ?n
```

Processing a Theory

- To run a theory right to the end, click on the *Use* button.
- Now the rest of the theory appears in pink until Isabelle can process it.

The *Use* button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Fib.thy
isabelle
--**-XEmacs: Fib.thy (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin

```
lemma fib_gcd: "fib (gcd (m, n)) = gcd (fib m, fib n)"  -- {* Law 6.111 *}
  apply (induct m n rule: gcd_induct)
  apply (simp_all add: gcd_non_0 gcd_commute gcd_fib_mod)
  done

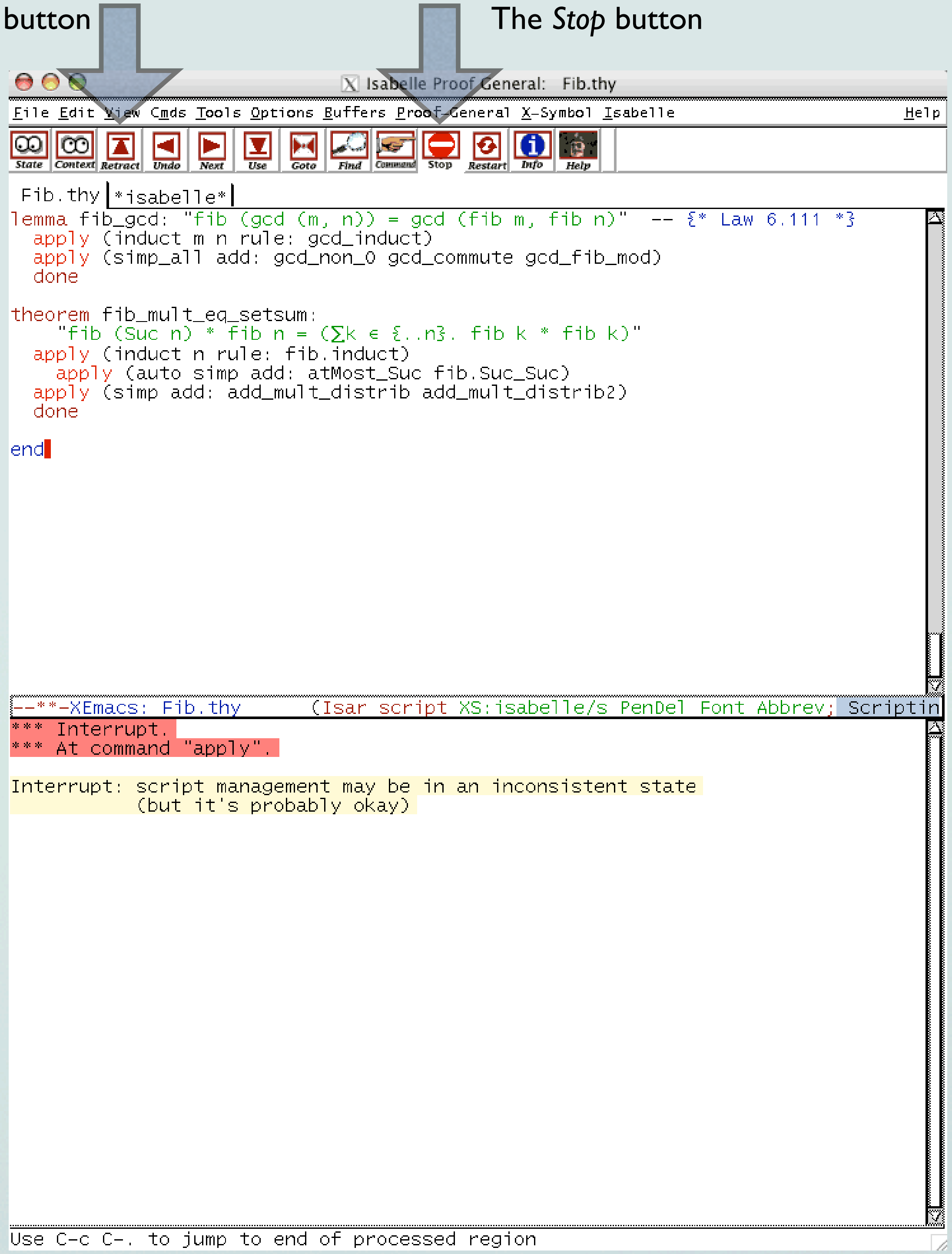
theorem fib_mult_eq_setsum:
    "fib (Suc n) * fib n = (∑k ∈ {..n}. fib k * fib k)"
  apply (induct n rule: fib.induct)
    apply (auto simp add: atMost_Suc fib.Suc_Suc)
  apply (simp add: add_mult_distrib add_mult_distrib2)
  done

end
```

```
lemma
  fib_add:
    fib (Suc (?n + ?k)) = fib (Suc ?k) * fib (Suc ?n) + fib ?k * fib ?n
```

Stop!

- Proof taking too long? Simplifier's looping? Clicked the wrong button? Just click on *Stop*.
- If things behave weirdly after this, perhaps Proof General has got out of sync with Isabelle.
- To get back into sync, use *Goto* to go back to the start of the current proof.
- You can use *Revert* to go back to the top of the theory file.



The *Revert* button

The *Stop* button

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  apply (induct n rule: fib.induct)
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  apply (simp add: add_mult_distrib add_mult_distrib2)
  done

end
```

```
--**-XEmacs: Fib.thy      (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

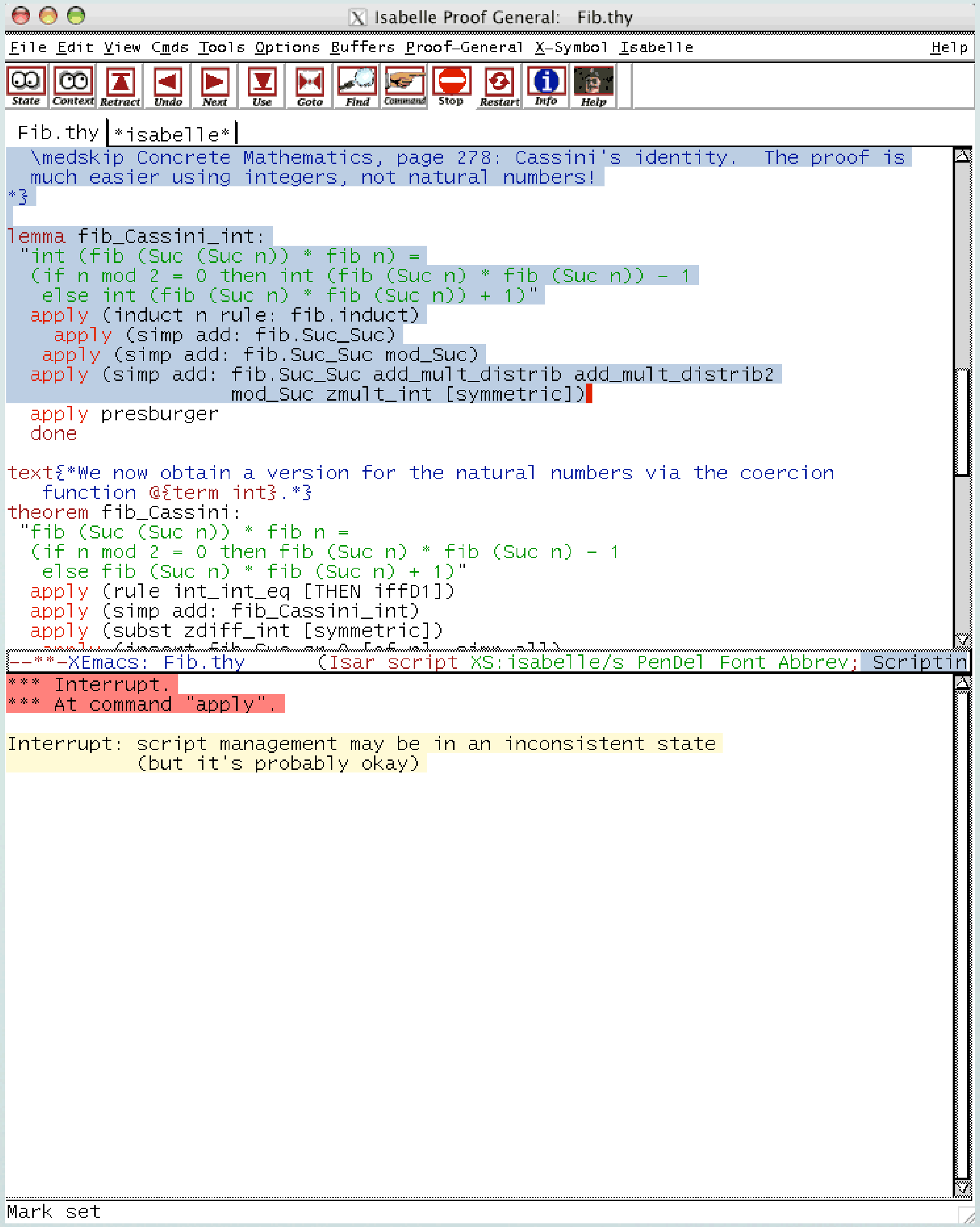
```
*** Interrupt.
*** At command "apply".

Interrupt: script management may be in an inconsistent state
           (but it's probably okay)
```

```
Use C-c C-. to jump to end of processed region
```

Where Am I?

- If a proof fails—or is interrupted—in a long theory file, how do we locate the critical spot?
- You could simply scroll through the file until you find the end of the blue region.
- To jump right there, use the menu item `Proof General > Goto Locked End`. The key combination `CTRL/C-.` does the same thing.
- The proof was interrupted during a call to `presburger`, an arithmetic decision procedure.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isabelle

```
  \medskip Concrete Mathematics, page 278: Cassini's identity.  The proof is
  much easier using integers, not natural numbers!
*}

lemma fib_Cassini_int:
 "int (fib (Suc (Suc n)) * fib n) =
  (if n mod 2 = 0 then int (fib (Suc n) * fib (Suc n)) - 1
   else int (fib (Suc n) * fib (Suc n)) + 1)"
  apply (induct n rule: fib.induct)
    apply (simp add: fib.Suc_Suc)
   apply (simp add: fib.Suc_Suc mod_Suc)
  apply (simp add: fib.Suc_Suc add_mult_distrib add_mult_distrib2
                   mod_Suc zmult_int [symmetric])
  apply presburger
  done

text{*We now obtain a version for the natural numbers via the coercion
   function @{term int}.*}
theorem fib_Cassini:
 "fib (Suc (Suc n)) * fib n =
  (if n mod 2 = 0 then fib (Suc n) * fib (Suc n) - 1
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  apply (rule int_int_eq [THEN iffD1])
  apply (simp add: fib_Cassini_int)
  apply (subst zdiff_int [symmetric])
```

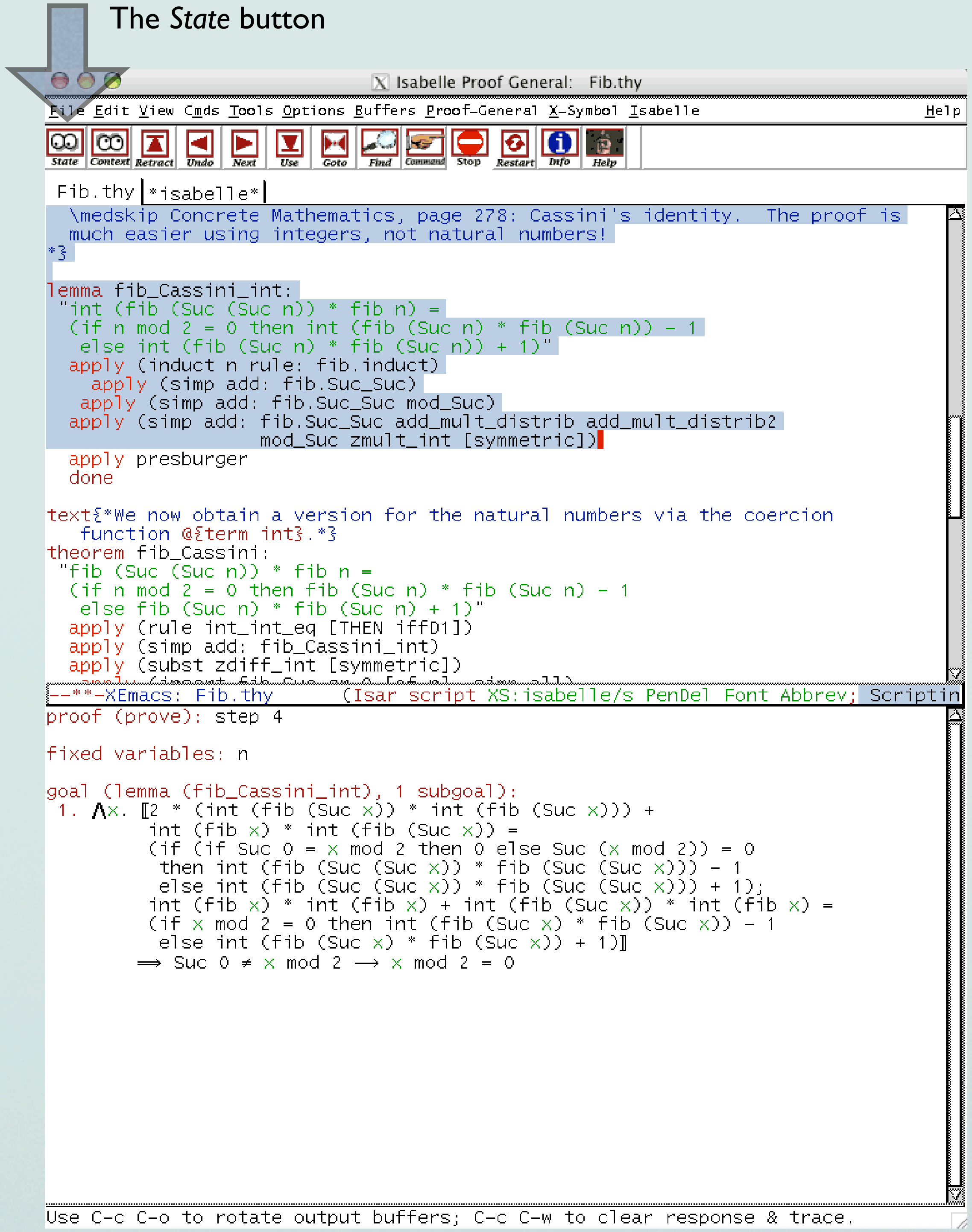
--**-XEmacs: Fib.thy   (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin

```
*** Interrupt.
*** At command "apply".

Interrupt: script management may be in an inconsistent state
           (but it's probably okay)
```

The Proof State

The *State* button


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle Help
State Context Retract Undo Next Use Goto Find Command Stop Restart Info Help
Fib.thy
isabelle

```
  \medskip Concrete Mathematics, page 278: Cassini's identity.  The proof is
  much easier using integers, not natural numbers!
*}

lemma fib_Cassini_int:
 "int (fib (Suc (Suc n)) * fib n) =
  (if n mod 2 = 0 then int (fib (Suc n) * fib (Suc n)) - 1
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  done

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  apply (rule int_int_eq [THEN iffD1])
  apply (simp add: fib_Cassini_int)
  apply (subst zdiff_int [symmetric])
```

```
--**-XEmacs: Fib.thy        (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

```
proof (prove): step 4

fixed variables: n

goal (lemma (fib_Cassini_int), 1 subgoal):
 1. ⋀x. ⟦2 * (int (fib (Suc x)) * int (fib (Suc x))) +
         int (fib x) * int (fib (Suc x)) =
         (if (if Suc 0 = x mod 2 then 0 else Suc (x mod 2)) = 0
          then int (fib (Suc (Suc x)) * fib (Suc (Suc x))) - 1
          else int (fib (Suc (Suc x)) * fib (Suc (Suc x))) + 1);
         int (fib x) * int (fib x) + int (fib (Suc x)) * int (fib x) =
         (if x mod 2 = 0 then int (fib (Suc x) * fib (Suc x)) - 1
          else int (fib (Suc x) * fib (Suc x)) + 1)⟧
        ⟹ Suc 0 ≠ x mod 2 ⟶ x mod 2 = 0
```

```
Use C-c C-o to rotate output buffers; C-c C-w to clear response & trace.
```

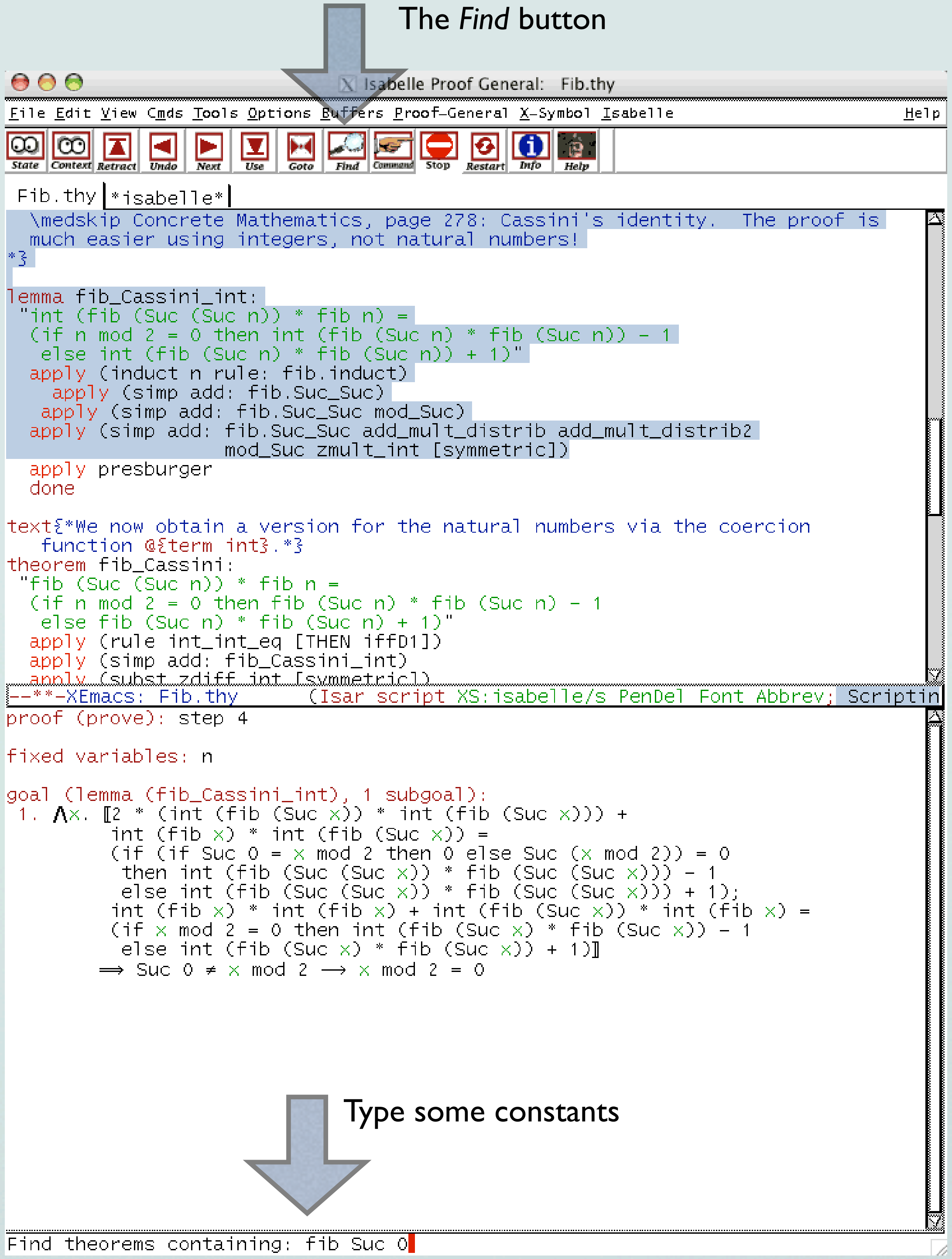
- Clicking on the *State* button reveals the proof state at the given point.

- Here, there was one subgoal left when the proof was interrupted.

Finding Theorems

- Isabelle provides thousands of lemmas. How do you find the ones you need? One way is to click the *Find* button.
- Then, type some constants—or entire terms—into the XEmacs minibuffer.
- Type the term "(_+_)*_ = _", including the quotation marks, to see all theorems containing an instance of this pattern.
- The pattern "_+_" matches all terms containing the infix operator + because _ matches any term.

The *Find* button



Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle
Help
State
Context
Retract
Undo
Next
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Goto
Find
Command
Stop
Restart
Info
Help
Fib.thy
isabelle

```
  \medskip Concrete Mathematics, page 278: Cassini's identity.  The proof is
  much easier using integers, not natural numbers!
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lemma fib_Cassini_int:
 "int (fib (Suc (Suc n)) * fib n) =
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    apply (simp add: fib.Suc_Suc)
   apply (simp add: fib.Suc_Suc mod_Suc)
  apply (simp add: fib.Suc_Suc add_mult_distrib add_mult_distrib2
                   mod_Suc zmult_int [symmetric])
  apply presburger
  done

text{*We now obtain a version for the natural numbers via the coercion
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 "fib (Suc (Suc n)) * fib n =
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  apply (simp add: fib_Cassini_int)
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--**-XEmacs: Fib.thy       (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
proof (prove): step 4

fixed variables: n

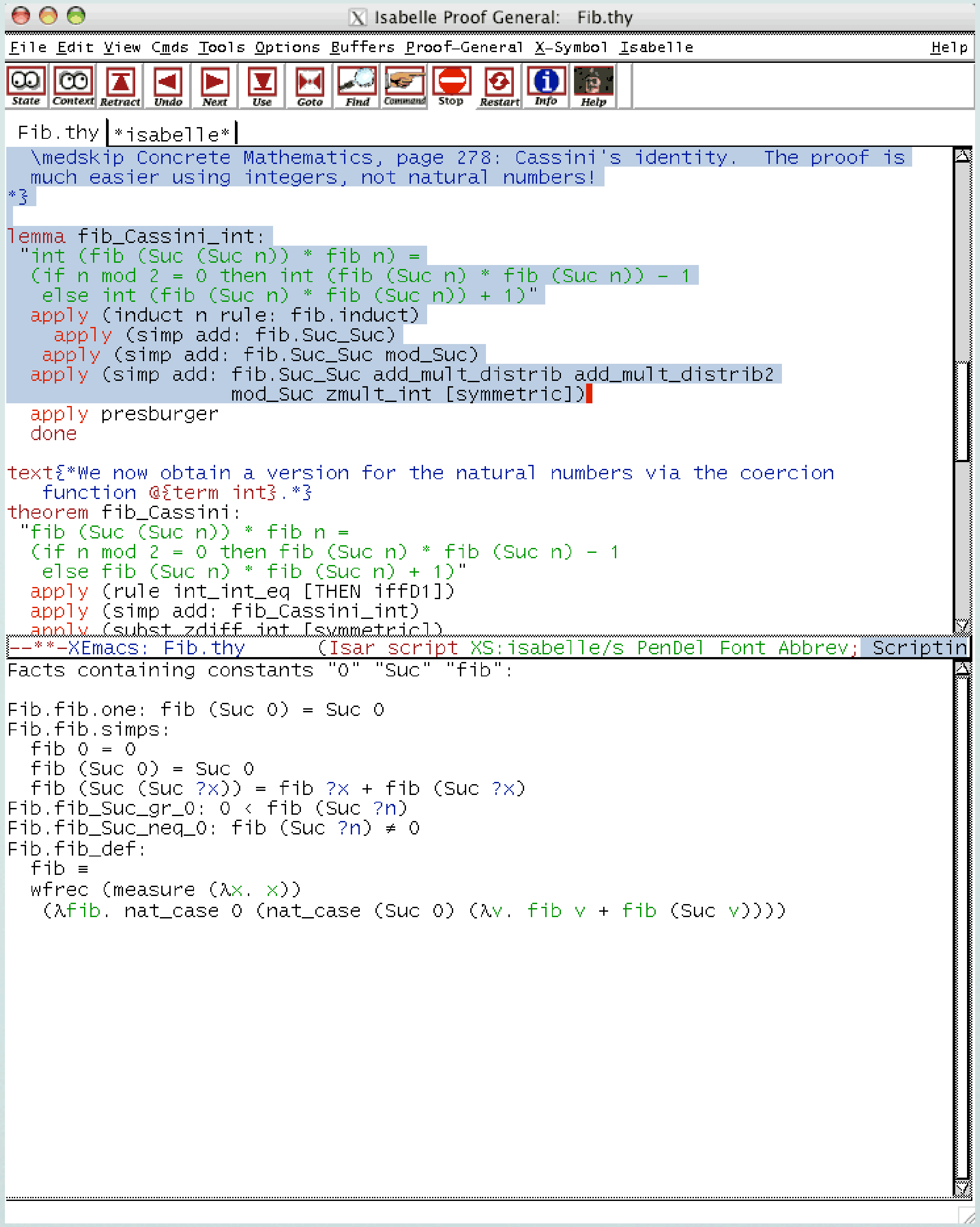
goal (lemma (fib_Cassini_int), 1 subgoal):
 1. ⋀x. ⟦2 * (int (fib (Suc x)) * int (fib (Suc x))) +
        int (fib x) * int (fib (Suc x)) =
        (if (if Suc 0 = x mod 2 then 0 else Suc (x mod 2)) = 0
         then int (fib (Suc (Suc x)) * fib (Suc (Suc x))) - 1
         else int (fib (Suc (Suc x)) * fib (Suc (Suc x))) + 1);
        int (fib x) * int (fib x) + int (fib (Suc x)) * int (fib x) =
        (if x mod 2 = 0 then int (fib (Suc x) * fib (Suc x)) - 1
         else int (fib (Suc x) * fib (Suc x)) + 1)⟧
       ⟹ Suc 0 ≠ x mod 2 ⟶ x mod 2 = 0
```

Type some constants

```
Find theorems containing: fib Suc 0
```

Theorems Found

- The response buffer lists the theorems containing *all* of the listed constants.
- If you are lucky, there will be just a few rather than hundreds!
- The more patterns you type, the fewer theorems will be displayed.
- During the search, variables mentioned in the current goal are viewed as constants.


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle
Help
State
Context
Retract
Undo
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Use
Goto
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Command
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Help
Fib.thy
isabelle

```
  \medskip Concrete Mathematics, page 278: Cassini's identity.  The proof is
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    apply (simp add: fib.Suc_Suc)
   apply (simp add: fib.Suc_Suc mod_Suc)
  apply (simp add: fib.Suc_Suc add_mult_distrib add_mult_distrib2
                   mod_Suc zmult_int [symmetric])
  apply presburger
  done

text{*We now obtain a version for the natural numbers via the coercion
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  apply (rule int_int_eq [THEN iffD1])
  apply (simp add: fib_Cassini_int)
  apply (subst zdiff_int [symmetric])
```

```
--**-XEmacs: Fib.thy       (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

```
Facts containing constants "0" "Suc" "fib":

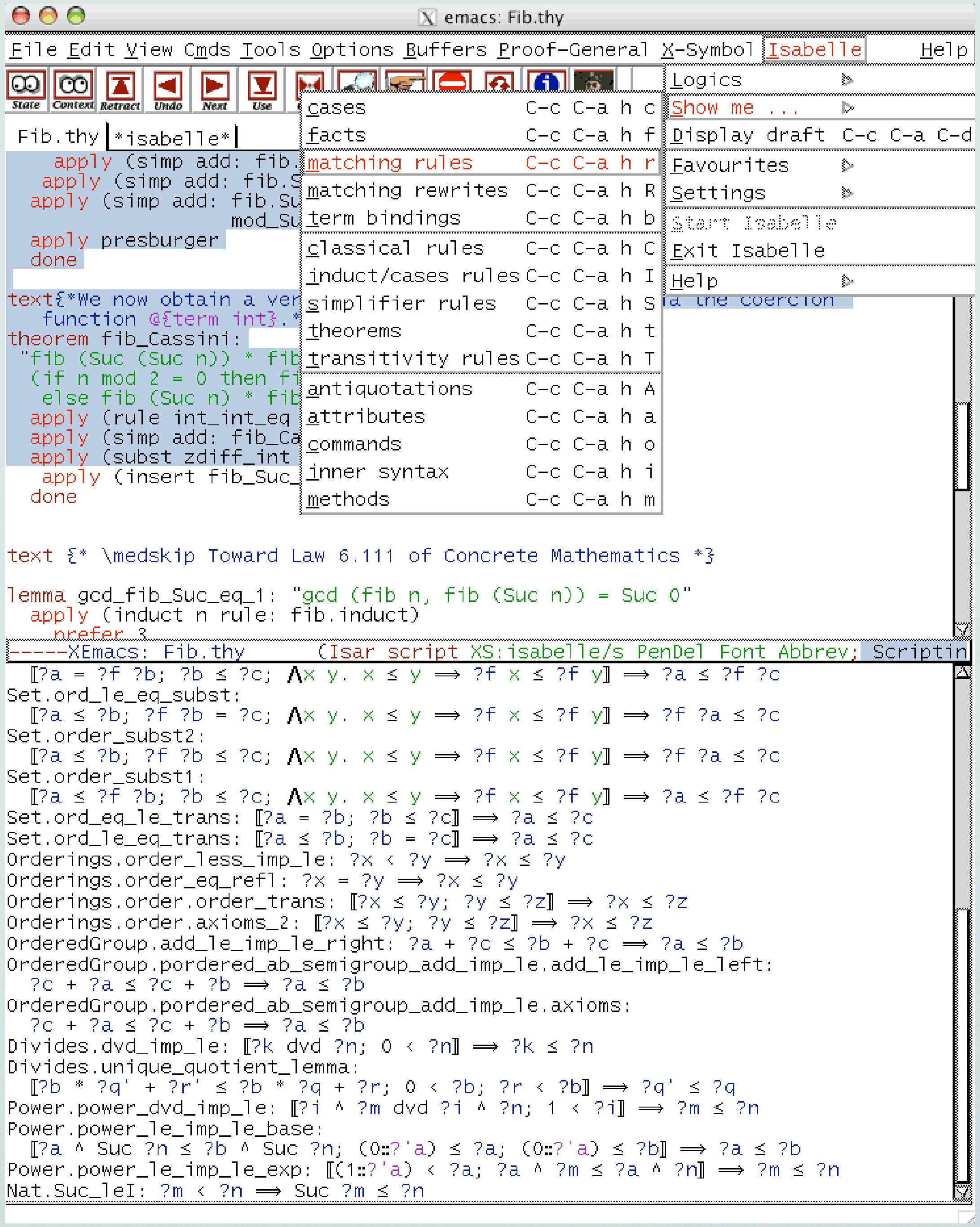
Fib.fib.one: fib (Suc 0) = Suc 0
Fib.fib.simps:
  fib 0 = 0
  fib (Suc 0) = Suc 0
  fib (Suc (Suc ?x)) = fib ?x + fib (Suc ?x)
Fib.fib_Suc_gr_0: 0 < fib (Suc ?n)
Fib.fib_Suc_neq_0: fib (Suc ?n) ≠ 0
Fib.fib_def:
  fib ≡
  wfrec (measure (λx. x))
   (λfib. nat_case 0 (nat_case (Suc 0) (λv. fib v + fib (Suc v))))
```

The Isabelle Menu

- The `Isabelle` menu gives access to Isabelle commands and information.

- `Isabelle > Show me...` provides other ways of finding theorems: `matching rules` and `matching rewrites`.

- In the example, the current subgoal has the form *x* ≤ *y*, and `matching rules` displays all known theorems that can prove a conclusion of that form.


emacs: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle Help
State
Context
Retract
Undo
Next
Use
Fib.thy
isabelle
Logics ▷
Show me ... ▷
Display draft C-c C-a C-d
Favourites ▷
Settings ▷
Start Isabelle
Exit Isabelle
Help ▷
cases C-c C-a h c
facts C-c C-a h f
matching rules C-c C-a h r
matching rewrites C-c C-a h R
term bindings C-c C-a h b
classical rules C-c C-a h C
induct/cases rules C-c C-a h I
simplifier rules C-c C-a h S
theorems C-c C-a h t
transitivity rules C-c C-a h T
antiquotations C-c C-a h A
attributes C-c C-a h a
commands C-c C-a h o
inner syntax C-c C-a h i
methods C-c C-a h m

```
    apply (simp add: fib.
   apply (simp add: fib.S
  apply (simp add: fib.Su
                   mod_Su
  apply presburger
  done

text{*We now obtain a ver                        a the coercion
   function @{term int}.*
theorem fib_Cassini:
 "fib (Suc (Suc n)) * fib
  (if n mod 2 = 0 then fi
   else fib (Suc n) * fib
  apply (rule int_int_eq
  apply (simp add: fib_Ca
  apply (subst zdiff_int
   apply (insert fib_Suc_
  done


text {* \medskip Toward Law 6.111 of Concrete Mathematics *}

lemma gcd_fib_Suc_eq_1: "gcd (fib n, fib (Suc n)) = Suc 0"
  apply (induct n rule: fib.induct)
   prefer 3
-----XEmacs: Fib.thy      (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

```
  ⟦?a = ?f ?b; ?b ≤ ?c; ⋀x y. x ≤ y ⟹ ?f x ≤ ?f y⟧ ⟹ ?a ≤ ?f ?c
Set.ord_le_eq_subst:
  ⟦?a ≤ ?b; ?f ?b = ?c; ⋀x y. x ≤ y ⟹ ?f x ≤ ?f y⟧ ⟹ ?f ?a ≤ ?c
Set.order_subst2:
  ⟦?a ≤ ?b; ?f ?b ≤ ?c; ⋀x y. x ≤ y ⟹ ?f x ≤ ?f y⟧ ⟹ ?f ?a ≤ ?c
Set.order_subst1:
  ⟦?a ≤ ?f ?b; ?b ≤ ?c; ⋀x y. x ≤ y ⟹ ?f x ≤ ?f y⟧ ⟹ ?a ≤ ?f ?c
Set.ord_eq_le_trans: ⟦?a = ?b; ?b ≤ ?c⟧ ⟹ ?a ≤ ?c
Set.ord_le_eq_trans: ⟦?a ≤ ?b; ?b = ?c⟧ ⟹ ?a ≤ ?c
Orderings.order_less_imp_le: ?x < ?y ⟹ ?x ≤ ?y
Orderings.order_eq_refl: ?x = ?y ⟹ ?x ≤ ?y
Orderings.order.order_trans: ⟦?x ≤ ?y; ?y ≤ ?z⟧ ⟹ ?x ≤ ?z
Orderings.order.axioms_2: ⟦?x ≤ ?y; ?y ≤ ?z⟧ ⟹ ?x ≤ ?z
OrderedGroup.add_le_imp_le_right: ?a + ?c ≤ ?b + ?c ⟹ ?a ≤ ?b
OrderedGroup.pordered_ab_semigroup_add_imp_le.add_le_imp_le_left:
  ?c + ?a ≤ ?c + ?b ⟹ ?a ≤ ?b
OrderedGroup.pordered_ab_semigroup_add_imp_le.axioms:
  ?c + ?a ≤ ?c + ?b ⟹ ?a ≤ ?b
Divides.dvd_imp_le: ⟦?k dvd ?n; 0 < ?n⟧ ⟹ ?k ≤ ?n
Divides.unique_quotient_lemma:
  ⟦?b * ?q' + ?r' ≤ ?b * ?q + ?r; 0 < ?b; ?r < ?b⟧ ⟹ ?q' ≤ ?q
Power.power_dvd_imp_le: ⟦?i ^ ?m dvd ?i ^ ?n; 1 < ?i⟧ ⟹ ?m ≤ ?n
Power.power_le_imp_le_base:
  ⟦?a ^ Suc ?n ≤ ?b ^ Suc ?n; (0::?'a) ≤ ?a; (0::?'a) ≤ ?b⟧ ⟹ ?a ≤ ?b
Power.power_le_imp_le_exp: ⟦(1::?'a) < ?a; ?a ^ ?m ≤ ?a ^ ?n⟧ ⟹ ?m ≤ ?n
Nat.Suc_leI: ?m < ?n ⟹ Suc ?m ≤ ?n
```

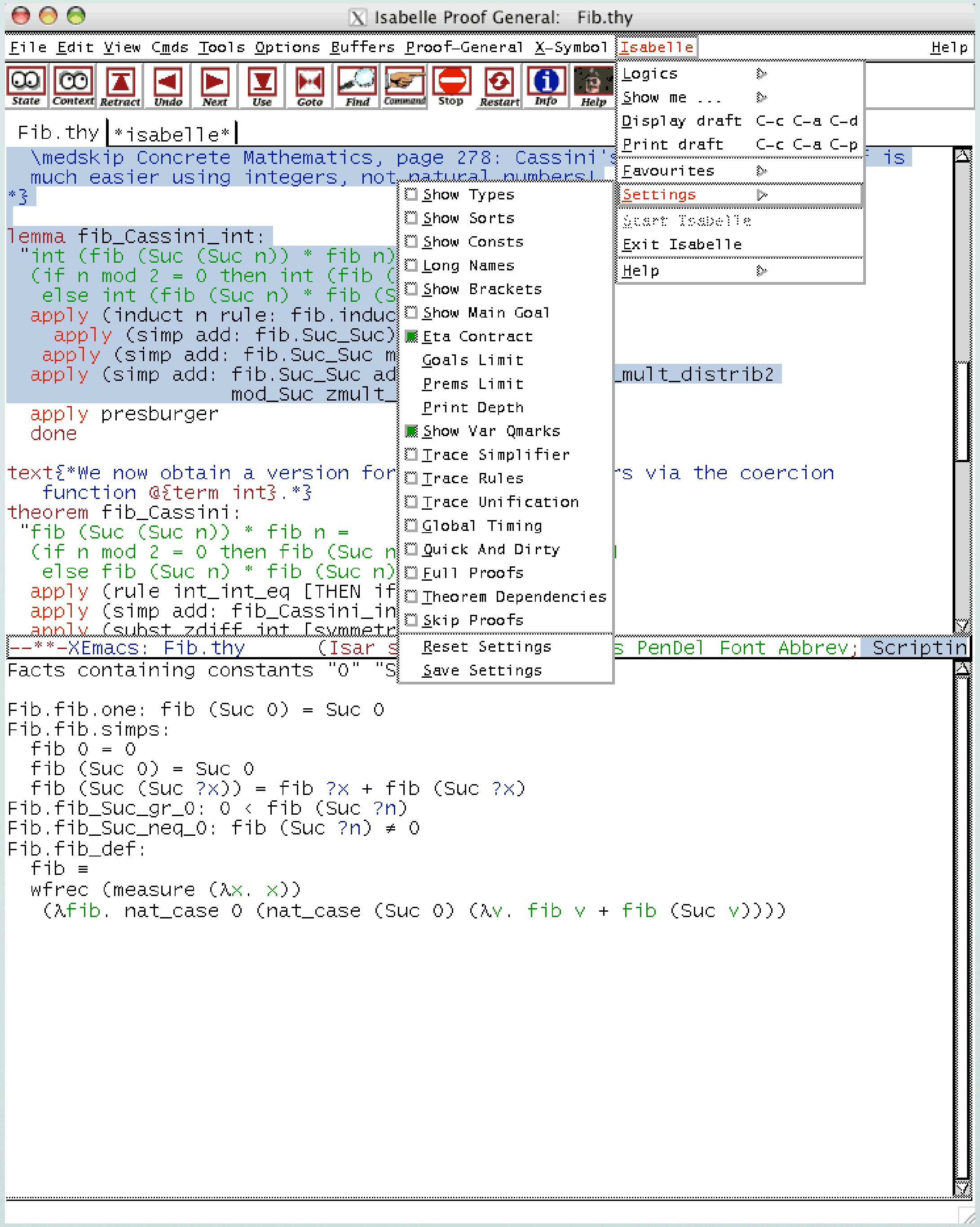
Settings

- The menu `Isabelle > Settings` can request the display of types, execution times, and various traces.

- There are printing options to suit special situations, such as enormous subgoals.

- Use `Show Types` and `Show Sorts` to cause more type information to be displayed.

- The various `Show` options make the output more verbose, but more explicit, and are helpful for diagnosing problems.



```
  \medskip Concrete Mathematics, page 278: Cassini's                       f is
  much easier using integers, not natural numbers!
*}

lemma fib_Cassini_int:
 "int (fib (Suc (Suc n)) * fib n)
  (if n mod 2 = 0 then int (fib (
   else int (fib (Suc n) * fib (S
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    apply (simp add: fib.Suc_Suc)
   apply (simp add: fib.Suc_Suc m
  apply (simp add: fib.Suc_Suc ad                             _mult_distrib2
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  apply presburger
  done

text{*We now obtain a version for                          rs via the coercion
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  (if n mod 2 = 0 then fib (Suc n
   else fib (Suc n) * fib (Suc n)
  apply (rule int_int_eq [THEN if
  apply (simp add: fib_Cassini_in
  apply (subst zdiff_int [symmetr
--**-XEmacs: Fib.thy      (Isar s                        s PenDel Font Abbrev; Scriptin
```

```
Facts containing constants "0" "S

Fib.fib.one: fib (Suc 0) = Suc 0
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  fib 0 = 0
  fib (Suc 0) = Suc 0
  fib (Suc (Suc ?x)) = fib ?x + fib (Suc ?x)
Fib.fib_Suc_gr_0: 0 < fib (Suc ?n)
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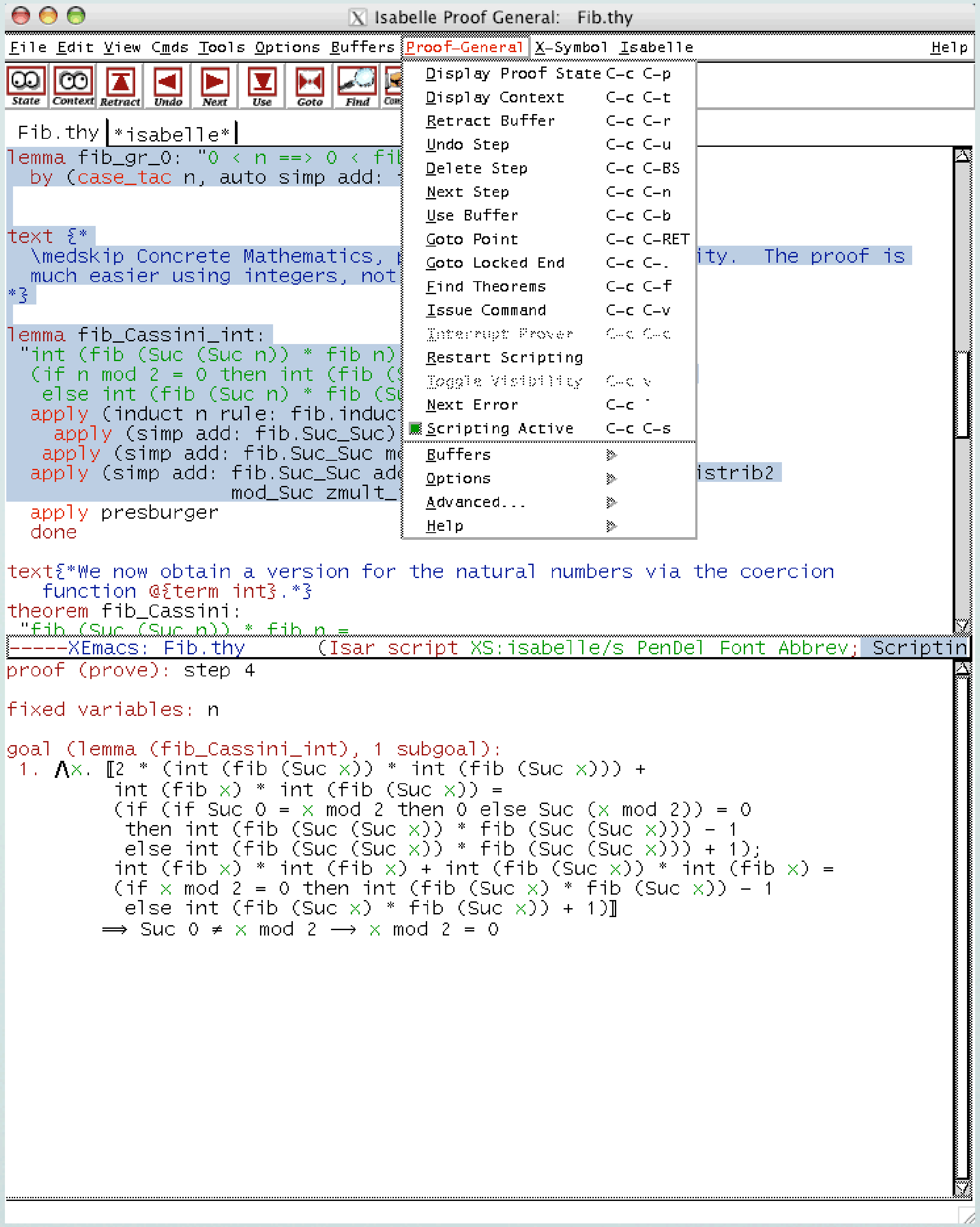
  fib ≡
  wfrec (measure (λx. x))
   (λfib. nat_case 0 (nat_case (Suc 0) (λv. fib v + fib (Suc v))))
```

The PG Menu

- The `Proof General` menu gives access to many commands.

- The main commands are available from the toolbar. A notable exception is `Goto Locked End`.

- Choose `Proof General > Buffers > Trace` to see tracing output.



```
lemma fib_gr_0: "0 < n ==> 0 < fi
  by (case_tac n, auto simp add: 


text {*
  \medskip Concrete Mathematics, p                                  ity.  The proof is
  much easier using integers, not
*}

lemma fib_Cassini_int:
 "int (fib (Suc (Suc n)) * fib n)
  (if n mod 2 = 0 then int (fib (S
   else int (fib (Suc n) * fib (Su
  apply (induct n rule: fib.induc
    apply (simp add: fib.Suc_Suc)
   apply (simp add: fib.Suc_Suc mo
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  apply presburger
  done

text{*We now obtain a version for the natural numbers via the coercion
   function @{term int}.*}
theorem fib_Cassini:
 "fib (Suc (Suc n)) * fib n =
-----XEmacs: Fib.thy       (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```

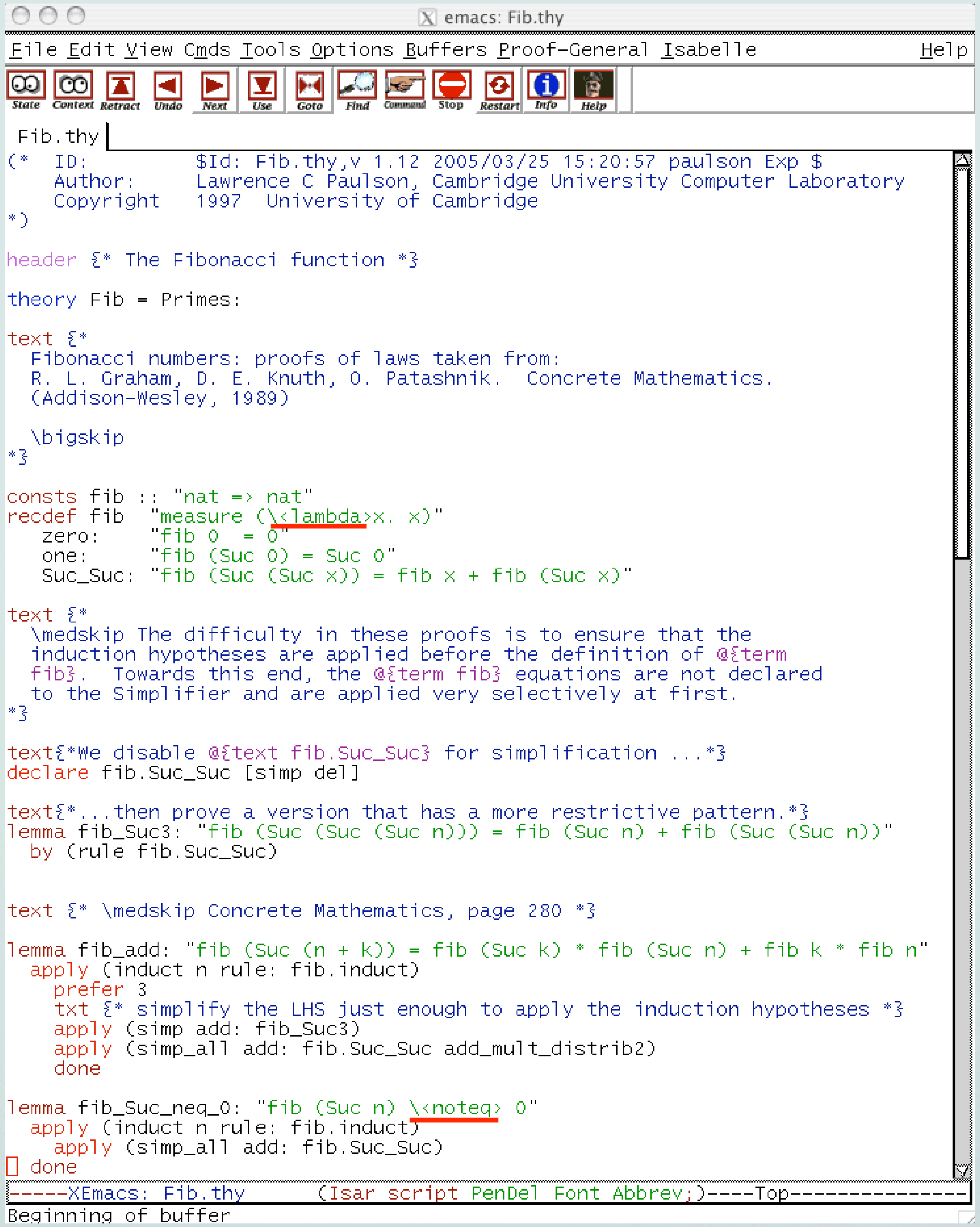
```
proof (prove): step 4

fixed variables: n

goal (lemma (fib_Cassini_int), 1 subgoal):
 1. ⋀x. ⟦2 * (int (fib (Suc x)) * int (fib (Suc x))) +
         int (fib x) * int (fib (Suc x)) =
         (if (if Suc 0 = x mod 2 then 0 else Suc (x mod 2)) = 0
          then int (fib (Suc (Suc x)) * fib (Suc (Suc x))) - 1
          else int (fib (Suc (Suc x)) * fib (Suc (Suc x))) + 1);
         int (fib x) * int (fib x) + int (fib (Suc x)) * int (fib x) =
         (if x mod 2 = 0 then int (fib (Suc x) * fib (Suc x)) - 1
          else int (fib (Suc x) * fib (Suc x)) + 1)⟧
        ⟹ Suc 0 ≠ x mod 2 ⟶ x mod 2 = 0
```

Mathematical Symbols

- Proof General uses the X-Symbol package to display mathematical symbols such as λ ≤ ≠ ∈ ∉ ∪ and ∩
- The package is included with Proof General, but may need to be switched on.
- If X-Symbol mode is off, Proof General will display ASCII escape sequences, as shown on the right.



```
(*   ID:         $Id: Fib.thy,v 1.12 2005/03/25 15:20:57 paulson Exp $
     Author:     Lawrence C Paulson, Cambridge University Computer Laboratory
     Copyright   1997  University of Cambridge
*)

header {* The Fibonacci function *}

theory Fib = Primes:

text {*
  Fibonacci numbers: proofs of laws taken from:
  R. L. Graham, D. E. Knuth, O. Patashnik.  Concrete Mathematics.
  (Addison-Wesley, 1989)

  \bigskip
*}

consts fib :: "nat => nat"
recdef fib  "measure (\<lambda>x. x)"
   zero:    "fib 0  = 0"
   one:     "fib (Suc 0) = Suc 0"
   Suc_Suc: "fib (Suc (Suc x)) = fib x + fib (Suc x)"

text {*
  \medskip The difficulty in these proofs is to ensure that the
  induction hypotheses are applied before the definition of @{term
  fib}.  Towards this end, the @{term fib} equations are not declared
  to the Simplifier and are applied very selectively at first.
*}

text{*We disable @{text fib.Suc_Suc} for simplification ...*}
declare fib.Suc_Suc [simp del]

text{*...then prove a version that has a more restrictive pattern.*}
lemma fib_Suc3: "fib (Suc (Suc (Suc n))) = fib (Suc n) + fib (Suc (Suc n))"
  by (rule fib.Suc_Suc)

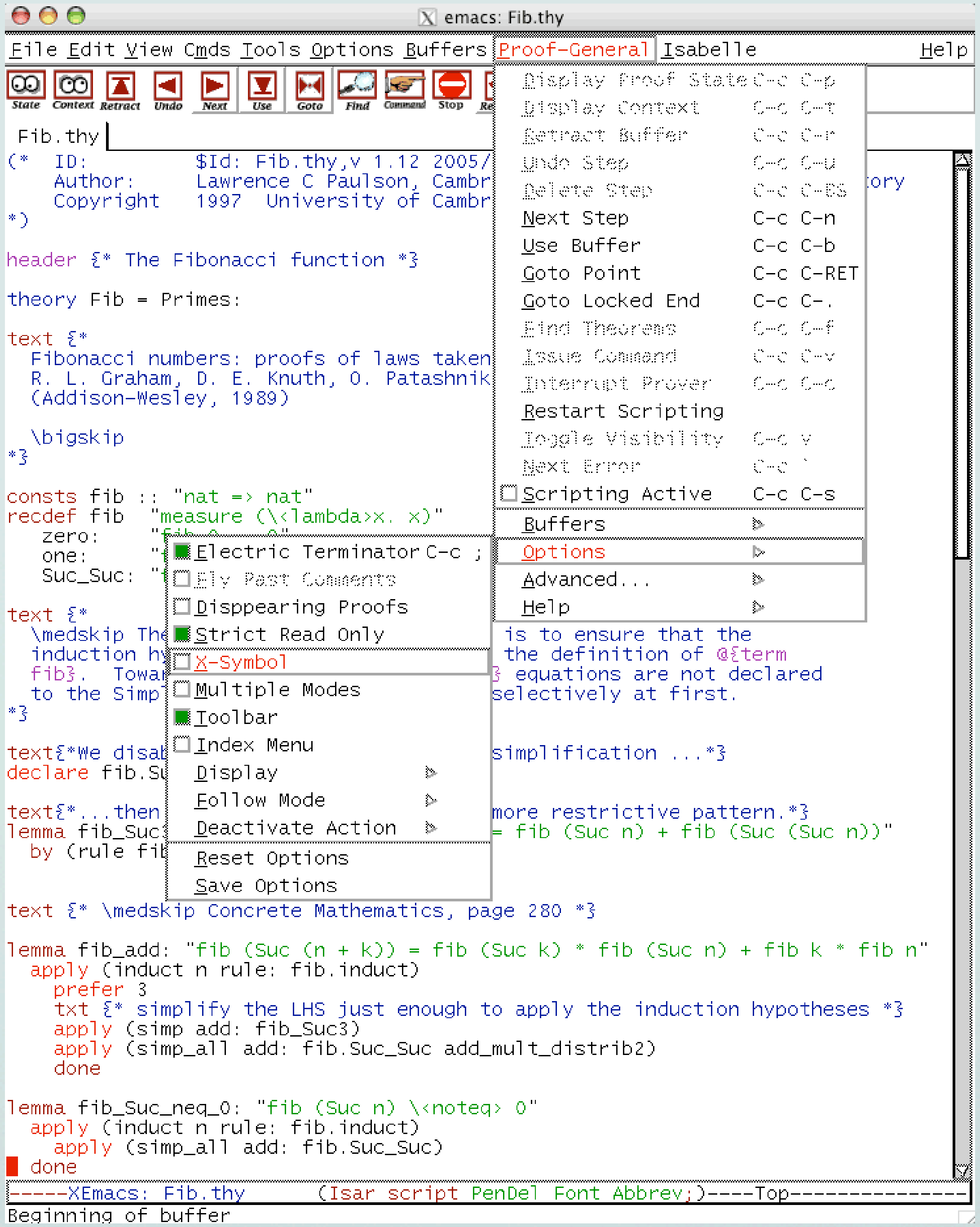

text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) \<noteq> 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done
-----XEmacs: Fib.thy       (Isar script PenDel Font Abbrev;)----Top----------------
Beginning of buffer
```

Enabling Symbols

- To enable X-Symbol mode, select the menu item `Proof General > Options > X-Symbol`.
- Then, make this setting permanent using `Proof General > Options > Save Options`.
- Take the time to explore the many other options and settings on offer.



```
Fib.thy
(*  ID:         $Id: Fib.thy,v 1.12 2005/
    Author:     Lawrence C Paulson, Cambr                                     ory
    Copyright   1997  University of Cambr
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text {*
  Fibonacci numbers: proofs of laws taken
  R. L. Graham, D. E. Knuth, O. Patashnik
  (Addison-Wesley, 1989)

  \bigskip
*}

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    zero:    "f
    one:     "f
    Suc_Suc: "f

text {*
  \medskip The                                is to ensure that the
  induction hy                                the definition of @{term
  fib}.  Towar                               } equations are not declared
  to the Simp                                selectively at first.
*}

text{*We disab                               simplification ...*}
declare fib.Su

text{*...then                                more restrictive pattern.*}
lemma fib_Suc                                = fib (Suc n) + fib (Suc (Suc n))"
  by (rule fib

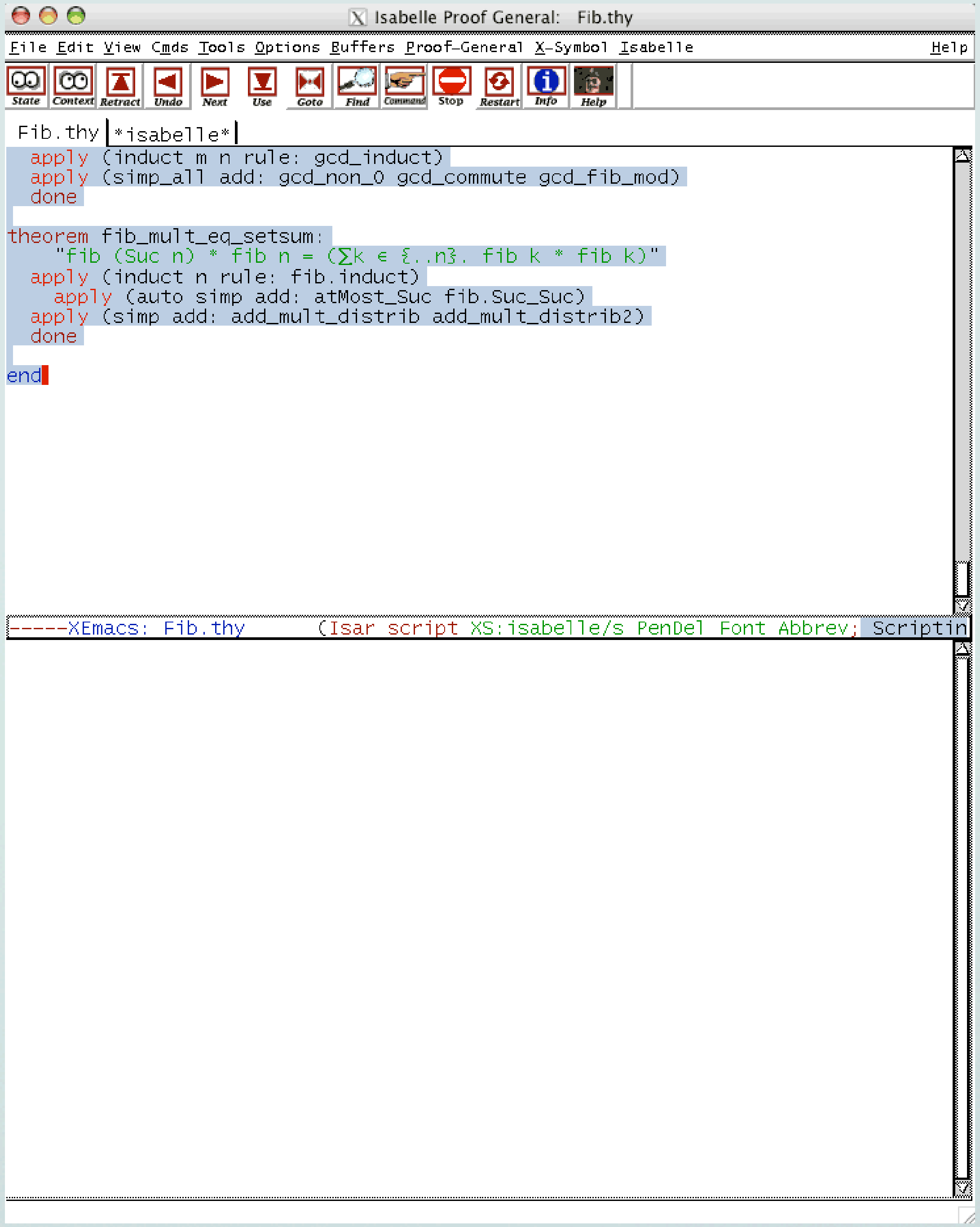
text {* \medskip Concrete Mathematics, page 280 *}

lemma fib_add: "fib (Suc (n + k)) = fib (Suc k) * fib (Suc n) + fib k * fib n"
  apply (induct n rule: fib.induct)
    prefer 3
    txt {* simplify the LHS just enough to apply the induction hypotheses *}
    apply (simp add: fib_Suc3)
    apply (simp_all add: fib.Suc_Suc add_mult_distrib2)
    done

lemma fib_Suc_neq_0: "fib (Suc n) \<noteq> 0"
  apply (induct n rule: fib.induct)
    apply (simp_all add: fib.Suc_Suc)
  done
-----XEmacs: Fib.thy      (Isar script PenDel Font Abbrev;)----Top---------------
Beginning of buffer
```

Go Forth and Prove!

- Try out this theory yourself: you will find it in `src/HOL/NumberTheory/Fib.thy`.
- For more information on Isabelle, read the documentation.
- For more information on Proof General, see its user manual.
- Have fun!


Isabelle Proof General: Fib.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle
Help
State
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Find
Command
Stop
Restart
Info
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Fib.thy
isabelle

```
  apply (induct m n rule: gcd_induct)
  apply (simp_all add: gcd_non_0 gcd_commute gcd_fib_mod)
  done

theorem fib_mult_eq_setsum:
    "fib (Suc n) * fib n = (∑k ∈ {..n}. fib k * fib k)"
  apply (induct n rule: fib.induct)
    apply (auto simp add: atMost_Suc fib.Suc_Suc)
  apply (simp add: add_mult_distrib add_mult_distrib2)
  done

end
```

```
-----XEmacs: Fib.thy       (Isar script XS:isabelle/s PenDel Font Abbrev; Scriptin
```