

# The Hahn-Banach Theorem for Real Vector Spaces

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## Abstract

The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

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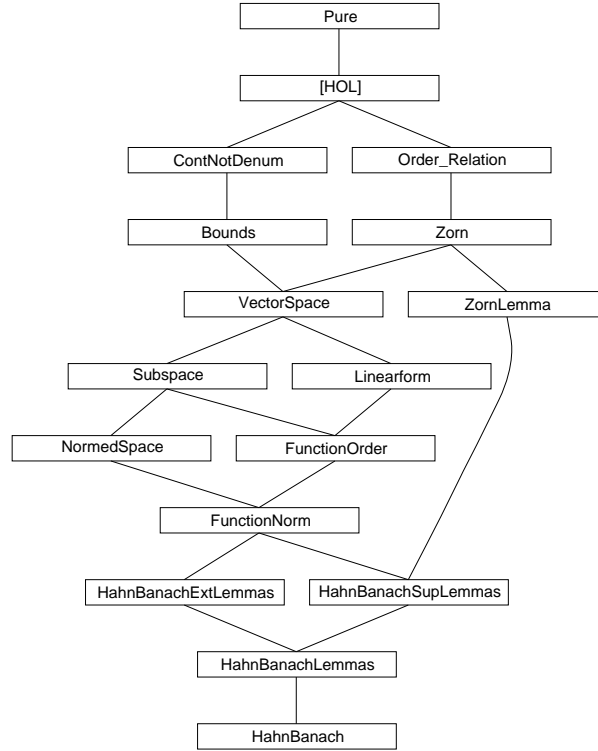
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# 1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser's textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.



## Part I

# Basic Notions

## 2 Bounds

```

theory Bounds
imports Main ContNotDenum
begin

locale lub =
  fixes A and x
  assumes least [intro?]:  $(\bigwedge a. a \in A \implies a \leq b) \implies x \leq b$ 
  and upper [intro?]:  $a \in A \implies a \leq x$ 

lemmas [elim?] = lub.least lub.upper

definition
  the-lub :: 'a::order set  $\Rightarrow$  'a where
  the-lub A = The (lub A)

notation (xsymbols)
  the-lub ( $\bigsqcup$  - [90] 90)

lemma the-lub-equality [elim?]:
  assumes lub A x
  shows  $\bigsqcup A = (x::'a::order)$ 
   $\langle proof \rangle$ 

lemma the-lubI-ex:
  assumes ex:  $\exists x. \text{lub } A \ x$ 
  shows  $\text{lub } A \ (\bigsqcup A)$ 
   $\langle proof \rangle$ 

lemma lub-compat:  $\text{lub } A \ x = \text{isLub } UNIV \ A \ x$ 
   $\langle proof \rangle$ 

lemma real-complete:
  fixes A :: real set
  assumes nonempty:  $\exists a. a \in A$ 
  and ex-upper:  $\exists y. \forall a \in A. a \leq y$ 
  shows  $\exists x. \text{lub } A \ x$ 
   $\langle proof \rangle$ 

end

```

## 3 Vector spaces

```

theory VectorSpace
imports Real Bounds Zorn
begin

```

### 3.1 Signature

For the definition of real vector spaces a type  $'a$  of the sort  $\{plus, minus, zero\}$  is considered, on which a real scalar multiplication  $\cdot$  is declared.

**consts**

$prod :: real \Rightarrow 'a::\{plus, minus, zero\} \Rightarrow 'a \quad (\text{infixr } '(\cdot)' 70)$

**notation** (*xsymbols*)

$prod \ (\text{infixr } \cdot \ 70)$

**notation** (*HTML output*)

$prod \ (\text{infixr } \cdot \ 70)$

### 3.2 Vector space laws

A *vector space* is a non-empty set  $V$  of elements from  $'a$  with the following vector space laws: The set  $V$  is closed under addition and scalar multiplication, addition is associative and commutative;  $-x$  is the inverse of  $x$  w. r. t. addition and  $0$  is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number  $1$  is the neutral element of scalar multiplication.

**locale**  $var\text{-}V = \text{fixes } V$

**locale**  $vectorspace = var\text{-}V +$

**assumes**  $non\text{-}empty$  [*iff*, *intro?*]:  $V \neq \{\}$

**and**  $add\text{-}closed$  [*iff*]:  $x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V$

**and**  $mult\text{-}closed$  [*iff*]:  $x \in V \Longrightarrow a \cdot x \in V$

**and**  $add\text{-}assoc$ :  $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z)$

**and**  $add\text{-}commute$ :  $x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x$

**and**  $diff\text{-}self$  [*simp*]:  $x \in V \Longrightarrow x - x = 0$

**and**  $add\text{-}zero\text{-}left$  [*simp*]:  $x \in V \Longrightarrow 0 + x = x$

**and**  $add\text{-}mult\text{-}distrib1$ :  $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$

**and**  $add\text{-}mult\text{-}distrib2$ :  $x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$

**and**  $mult\text{-}assoc$ :  $x \in V \Longrightarrow (a * b) \cdot x = a \cdot (b \cdot x)$

**and**  $mult\text{-}1$  [*simp*]:  $x \in V \Longrightarrow 1 \cdot x = x$

**and**  $negate\text{-}eq1$ :  $x \in V \Longrightarrow -x = (-1) \cdot x$

**and**  $diff\text{-}eq1$ :  $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + -y$

**lemma** (**in**  $vectorspace$ )  $negate\text{-}eq2$ :  $x \in V \Longrightarrow (-1) \cdot x = -x$

*<proof>*

**lemma** (**in**  $vectorspace$ )  $negate\text{-}eq2a$ :  $x \in V \Longrightarrow -1 \cdot x = -x$

*<proof>*

**lemma** (**in**  $vectorspace$ )  $diff\text{-}eq2$ :  $x \in V \Longrightarrow y \in V \Longrightarrow x + -y = x - y$

*<proof>*

**lemma** (**in**  $vectorspace$ )  $diff\text{-}closed$  [*iff*]:  $x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V$

*<proof>*

**lemma** (**in**  $vectorspace$ )  $neg\text{-}closed$  [*iff*]:  $x \in V \Longrightarrow -x \in V$

*<proof>*

**lemma** (**in**  $vectorspace$ )  $add\text{-}left\text{-}commute$ :

$x \in V \implies y \in V \implies z \in V \implies x + (y + z) = y + (x + z)$   
 $\langle \text{proof} \rangle$

**theorems** (in *vectorspace*) *add-ac* =  
*add-assoc add-commute add-left-commute*

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

**lemma** (in *vectorspace*) *zero [iff]*:  $0 \in V$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-zero-right [simp]*:  
 $x \in V \implies x + 0 = x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *mult-assoc2*:  
 $x \in V \implies a \cdot b \cdot x = (a * b) \cdot x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *diff-mult-distrib1*:  
 $x \in V \implies y \in V \implies a \cdot (x - y) = a \cdot x - a \cdot y$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *diff-mult-distrib2*:  
 $x \in V \implies (a - b) \cdot x = a \cdot x - (b \cdot x)$   
 $\langle \text{proof} \rangle$

**lemmas** (in *vectorspace*) *distrib* =  
*add-mult-distrib1 add-mult-distrib2*  
*diff-mult-distrib1 diff-mult-distrib2*

Further derived laws:

**lemma** (in *vectorspace*) *mult-zero-left [simp]*:  
 $x \in V \implies 0 \cdot x = 0$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *mult-zero-right [simp]*:  
 $a \cdot 0 = (0 :: 'a)$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *minus-mult-cancel [simp]*:  
 $x \in V \implies (- a) \cdot - x = a \cdot x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-minus-left-eq-diff*:  
 $x \in V \implies y \in V \implies - x + y = y - x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-minus [simp]*:  
 $x \in V \implies x + - x = 0$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-minus-left [simp]*:

$x \in V \implies -x + x = 0$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *minus-minus* [simp]:  
 $x \in V \implies -(-x) = x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *minus-zero* [simp]:  
 $-(0::'a) = 0$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *minus-zero-iff* [simp]:  
 $x \in V \implies (-x = 0) = (x = 0)$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-minus-cancel* [simp]:  
 $x \in V \implies y \in V \implies x + (-x + y) = y$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *minus-add-cancel* [simp]:  
 $x \in V \implies y \in V \implies -x + (x + y) = y$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *minus-add-distrib* [simp]:  
 $x \in V \implies y \in V \implies -(x + y) = -x + -y$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *diff-zero* [simp]:  
 $x \in V \implies x - 0 = x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *diff-zero-right* [simp]:  
 $x \in V \implies 0 - x = -x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-left-cancel*:  
 $x \in V \implies y \in V \implies z \in V \implies (x + y = x + z) = (y = z)$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-right-cancel*:  
 $x \in V \implies y \in V \implies z \in V \implies (y + x = z + x) = (y = z)$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *add-assoc-cong*:  
 $x \in V \implies y \in V \implies x' \in V \implies y' \in V \implies z \in V$   
 $\implies x + y = x' + y' \implies x + (y + z) = x' + (y' + z)$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *mult-left-commute*:  
 $x \in V \implies a \cdot b \cdot x = b \cdot a \cdot x$   
 $\langle \text{proof} \rangle$

**lemma** (in *vectorspace*) *mult-zero-uniq*:  
 $x \in V \implies x \neq 0 \implies a \cdot x = 0 \implies a = 0$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *mult-left-cancel*:

$$x \in V \implies y \in V \implies a \neq 0 \implies (a \cdot x = a \cdot y) = (x = y)$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *mult-right-cancel*:

$$x \in V \implies x \neq 0 \implies (a \cdot x = b \cdot x) = (a = b)$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *eq-diff-eq*:

$$x \in V \implies y \in V \implies z \in V \implies (x = z - y) = (x + y = z)$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *add-minus-eq-minus*:

$$x \in V \implies y \in V \implies x + y = 0 \implies x = -y$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *add-minus-eq*:

$$x \in V \implies y \in V \implies x - y = 0 \implies x = y$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *add-diff-swap*:

$$a \in V \implies b \in V \implies c \in V \implies d \in V \implies a + b = c + d \\ \implies a - c = d - b$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *vs-add-cancel-21*:

$$x \in V \implies y \in V \implies z \in V \implies u \in V \\ \implies (x + (y + z) = y + u) = (x + z = u)$$

$\langle proof \rangle$

**lemma** (in *vectorspace*) *add-cancel-end*:

$$x \in V \implies y \in V \implies z \in V \implies (x + (y + z) = y) = (x = -z)$$

$\langle proof \rangle$

**end**

## 4 Subspaces

**theory** *Subspace*

**imports** *VectorSpace*

**begin**

### 4.1 Definition

A non-empty subset  $U$  of a vector space  $V$  is a *subspace* of  $V$ , iff  $U$  is closed under addition and scalar multiplication.

**locale** *subspace* =

**fixes**  $U :: 'a::\{minus, plus, zero, uminus\}$  set **and**  $V$

**assumes** *non-empty* [*iff*, *intro*]:  $U \neq \{\}$

**and** *subset* [*iff*]:  $U \subseteq V$



**and** *add-closed* [iff]:  $x \in U \implies y \in U \implies x + y \in U$   
**and** *mult-closed* [iff]:  $x \in U \implies a \cdot x \in U$

**notation** (*symbols*)  
*subspace* (**infix**  $\trianglelefteq$  50)

**declare** *vectorspace.intro* [intro?] *subspace.intro* [intro?]

**lemma** *subspace-subset* [elim]:  $U \trianglelefteq V \implies U \subseteq V$   
 ⟨proof⟩

**lemma** (**in** *subspace*) *subsetD* [iff]:  $x \in U \implies x \in V$   
 ⟨proof⟩

**lemma** *subspaceD* [elim]:  $U \trianglelefteq V \implies x \in U \implies x \in V$   
 ⟨proof⟩

**lemma** *rev-subspaceD* [elim?]:  $x \in U \implies U \trianglelefteq V \implies x \in V$   
 ⟨proof⟩

**lemma** (**in** *subspace*) *diff-closed* [iff]:  
**assumes** *vectorspace*  $V$   
**assumes**  $x: x \in U$  **and**  $y: y \in U$   
**shows**  $x - y \in U$   
 ⟨proof⟩

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

**lemma** (**in** *subspace*) *zero* [intro]:  
**assumes** *vectorspace*  $V$   
**shows**  $0 \in U$   
 ⟨proof⟩

**lemma** (**in** *subspace*) *neg-closed* [iff]:  
**assumes** *vectorspace*  $V$   
**assumes**  $x: x \in U$   
**shows**  $-x \in U$   
 ⟨proof⟩

Further derived laws: every subspace is a vector space.

**lemma** (**in** *subspace*) *vectorspace* [iff]:  
**assumes** *vectorspace*  $V$   
**shows** *vectorspace*  $U$   
 ⟨proof⟩

The subspace relation is reflexive.

**lemma** (**in** *vectorspace*) *subspace-refl* [intro]:  $V \trianglelefteq V$   
 ⟨proof⟩

The subspace relation is transitive.

**lemma** (**in** *vectorspace*) *subspace-trans* [trans]:  
 $U \trianglelefteq V \implies V \trianglelefteq W \implies U \trianglelefteq W$   
 ⟨proof⟩

## 4.2 Linear closure

The *linear closure* of a vector  $x$  is the set of all scalar multiples of  $x$ .

**definition**

$lin :: ('a :: \{minus, plus, zero\}) \Rightarrow 'a \text{ set}$  **where**  
 $lin\ x = \{a \cdot x \mid a. \text{True}\}$

**lemma**  $linI$  [intro]:  $y = a \cdot x \implies y \in lin\ x$   
 $\langle proof \rangle$

**lemma**  $linI'$  [iff]:  $a \cdot x \in lin\ x$   
 $\langle proof \rangle$

**lemma**  $linE$  [elim]:  $x \in lin\ v \implies (\bigwedge a :: real. x = a \cdot v \implies C) \implies C$   
 $\langle proof \rangle$

Every vector is contained in its linear closure.

**lemma** (in *vectorspace*)  $x\text{-}lin\text{-}x$  [iff]:  $x \in V \implies x \in lin\ x$   
 $\langle proof \rangle$

**lemma** (in *vectorspace*)  $0\text{-}lin\text{-}x$  [iff]:  $x \in V \implies 0 \in lin\ x$   
 $\langle proof \rangle$

Any linear closure is a subspace.

**lemma** (in *vectorspace*)  $lin\text{-}subspace$  [intro]:  
 $x \in V \implies lin\ x \trianglelefteq V$   
 $\langle proof \rangle$

Any linear closure is a vector space.

**lemma** (in *vectorspace*)  $lin\text{-}vectorspace$  [intro]:  
**assumes**  $x \in V$   
**shows** *vectorspace* ( $lin\ x$ )  
 $\langle proof \rangle$

## 4.3 Sum of two vectorspaces

The *sum* of two vectorspaces  $U$  and  $V$  is the set of all sums of elements from  $U$  and  $V$ .

**instantiation**  $fun :: (type, type) \text{ plus}$   
**begin**

**definition**

$sum\text{-}def: plus\text{-}fun\ U\ V = \{u + v \mid u\ v. u \in U \wedge v \in V\}$

**instance**  $\langle proof \rangle$

**end**

**lemma**  $sumE$  [elim]:  
 $x \in U + V \implies (\bigwedge u\ v. x = u + v \implies u \in U \implies v \in V \implies C) \implies C$   
 $\langle proof \rangle$

**lemma** *sumI* [intro]:  
 $u \in U \implies v \in V \implies x = u + v \implies x \in U + V$   
 ⟨proof⟩

**lemma** *sumI'* [intro]:  
 $u \in U \implies v \in V \implies u + v \in U + V$   
 ⟨proof⟩

$U$  is a subspace of  $U + V$ .

**lemma** *subspace-sum1* [iff]:  
**assumes** *vectorspace*  $U$  *vectorspace*  $V$   
**shows**  $U \leq U + V$   
 ⟨proof⟩

The sum of two subspaces is again a subspace.

**lemma** *sum-subspace* [intro?]:  
**assumes** *subspace*  $U$   $E$  *vectorspace*  $E$  *subspace*  $V$   $E$   
**shows**  $U + V \leq E$   
 ⟨proof⟩

The sum of two subspaces is a vectorspace.

**lemma** *sum-vs* [intro?]:  
 $U \leq E \implies V \leq E \implies \text{vectorspace } E \implies \text{vectorspace } (U + V)$   
 ⟨proof⟩

## 4.4 Direct sums

The sum of  $U$  and  $V$  is called *direct*, iff the zero element is the only common element of  $U$  and  $V$ . For every element  $x$  of the direct sum of  $U$  and  $V$  the decomposition in  $x = u + v$  with  $u \in U$  and  $v \in V$  is unique.

**lemma** *decomp*:  
**assumes** *vectorspace*  $E$  *subspace*  $U$   $E$  *subspace*  $V$   $E$   
**assumes** *direct*:  $U \cap V = \{0\}$   
**and**  $u1: u1 \in U$  **and**  $u2: u2 \in U$   
**and**  $v1: v1 \in V$  **and**  $v2: v2 \in V$   
**and** *sum*:  $u1 + v1 = u2 + v2$   
**shows**  $u1 = u2 \wedge v1 = v2$   
 ⟨proof⟩

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page ??): for any element  $y + a \cdot x_0$  of the direct sum of a vectorspace  $H$  and the linear closure of  $x_0$  the components  $y \in H$  and  $a$  are uniquely determined.

**lemma** *decomp-H'*:  
**assumes** *vectorspace*  $E$  *subspace*  $H$   $E$   
**assumes**  $y1: y1 \in H$  **and**  $y2: y2 \in H$   
**and**  $x': x' \notin H$   $x' \in E$   $x' \neq 0$   
**and** *eq*:  $y1 + a1 \cdot x' = y2 + a2 \cdot x'$   
**shows**  $y1 = y2 \wedge a1 = a2$   
 ⟨proof⟩

Since for any element  $y + a \cdot x'$  of the direct sum of a vectorspace  $H$  and the linear closure of  $x'$  the components  $y \in H$  and  $a$  are unique, it follows from  $y \in H$  that  $a = 0$ .

**lemma** *decomp- $H'$ - $H$ :*  
**assumes** *vectorspace  $E$  subspace  $H$   $E$*   
**assumes**  *$t: t \in H$*   
**and**  *$x': x' \notin H \ x' \in E \ x' \neq 0$*   
**shows**  *$(SOME\ (y, a). t = y + a \cdot x' \wedge y \in H) = (t, 0)$*   
*<proof>*

The components  $y \in H$  and  $a$  in  $y + a \cdot x'$  are unique, so the function  $h'$  defined by  $h' (y + a \cdot x') = h\ y + a \cdot \xi$  is definite.

**lemma**  *$h'$ -definite:*  
**fixes**  *$H$*   
**assumes**  *$h'$ -def:*  
 *$h' \equiv (\lambda x. let\ (y, a) = SOME\ (y, a). (x = y + a \cdot x' \wedge y \in H)$*   
 *$in\ (h\ y) + a * xi)$*   
**and**  *$x: x = y + a \cdot x'$*   
**assumes** *vectorspace  $E$  subspace  $H$   $E$*   
**assumes**  *$y: y \in H$*   
**and**  *$x': x' \notin H \ x' \in E \ x' \neq 0$*   
**shows**  *$h' x = h\ y + a * xi$*   
*<proof>*

**end**

## 5 Normed vector spaces

**theory** *NormedSpace*  
**imports** *Subspace*  
**begin**

### 5.1 Quasinorms

A *seminorm*  $\|\cdot\|$  is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogenous and subadditive.

**locale** *norm-syntax* =  
**fixes**  *$norm :: 'a \Rightarrow real \quad (\|-\|)$*   
  
**locale** *seminorm* = *var- $V$  + norm-syntax +*  
**constrains**  *$V :: 'a::\{minus, plus, zero, uminus\}$  set*  
**assumes**  *$ge-zero$  [iff?]:  $x \in V \implies 0 \leq \|x\|$*   
**and**  *$abs-homogenous$  [iff?]:  $x \in V \implies \|a \cdot x\| = |a| * \|x\|$*   
**and**  *$subadditive$  [iff?]:  $x \in V \implies y \in V \implies \|x + y\| \leq \|x\| + \|y\|$*   
  
**declare**  *$seminorm.intro$  [intro?]*  
  
**lemma** *(in seminorm) diff-subadditive:*  
**assumes** *vectorspace  $V$*

**shows**  $x \in V \implies y \in V \implies \|x - y\| \leq \|x\| + \|y\|$   
 $\langle proof \rangle$

**lemma** (in *seminorm*) *minus*:

**assumes** *vectorspace*  $V$

**shows**  $x \in V \implies \|- x\| = \|x\|$

$\langle proof \rangle$

## 5.2 Norms

A *norm*  $\|\cdot\|$  is a seminorm that maps only the  $0$  vector to  $0$ .

**locale** *norm* = *seminorm* +

**assumes** *zero-iff* [*iff*]:  $x \in V \implies (\|x\| = 0) = (x = 0)$

## 5.3 Normed vector spaces

A vector space together with a norm is called a *normed space*.

**locale** *normed-vectorspace* = *vectorspace* + *norm*

**declare** *normed-vectorspace.intro* [*intro?*]

**lemma** (in *normed-vectorspace*) *gt-zero* [*intro?*]:

$x \in V \implies x \neq 0 \implies 0 < \|x\|$

$\langle proof \rangle$

Any subspace of a normed vector space is again a normed vectorspace.

**lemma** *subspace-normed-vs* [*intro?*]:

**fixes**  $F E$  *norm*

**assumes** *subspace*  $F E$  *normed-vectorspace*  $E$  *norm*

**shows** *normed-vectorspace*  $F$  *norm*

$\langle proof \rangle$

**end**

## 6 Linearforms

**theory** *Linearform*

**imports** *VectorSpace*

**begin**

A *linear form* is a function on a vector space into the reals that is additive and multiplicative.

**locale** *linearform* =

**fixes**  $V :: 'a :: \{minus, plus, zero, uminus\}$  *set* **and**  $f$

**assumes** *add* [*iff*]:  $x \in V \implies y \in V \implies f (x + y) = f x + f y$

**and** *mult* [*iff*]:  $x \in V \implies f (a \cdot x) = a * f x$

**declare** *linearform.intro* [*intro?*]

**lemma** (in *linearform*) *neg* [*iff*]:

```

assumes vectorspace V
shows  $x \in V \implies f (- x) = - f x$ 
 $\langle proof \rangle$ 

```

```

lemma (in linearform) diff [iff]:
  assumes vectorspace V
  shows  $x \in V \implies y \in V \implies f (x - y) = f x - f y$ 
 $\langle proof \rangle$ 

```

Every linear form yields 0 for the 0 vector.

```

lemma (in linearform) zero [iff]:
  assumes vectorspace V
  shows  $f 0 = 0$ 
 $\langle proof \rangle$ 

```

**end**

## 7 An order on functions

```

theory FunctionOrder
imports Subspace Linearform
begin

```

### 7.1 The graph of a function

We define the *graph* of a (real) function  $f$  with domain  $F$  as the set

$$\{(x, f x). x \in F\}$$

So we are modeling partial functions by specifying the domain and the mapping function. We use the term “function” also for its graph.

```

types 'a graph = ('a  $\times$  real) set

```

**definition**

```

 $graph :: 'a \ set \Rightarrow ('a \Rightarrow \ real) \Rightarrow 'a \ graph$  where
 $graph\ F\ f = \{(x, f\ x) \mid x. x \in F\}$ 

```

```

lemma graphI [intro]:  $x \in F \implies (x, f x) \in graph\ F\ f$ 
 $\langle proof \rangle$ 

```

```

lemma graphI2 [intro?]:  $x \in F \implies \exists t \in graph\ F\ f. t = (x, f x)$ 
 $\langle proof \rangle$ 

```

```

lemma graphE [elim?]:
   $(x, y) \in graph\ F\ f \implies (x \in F \implies y = f x \implies C) \implies C$ 
 $\langle proof \rangle$ 

```

### 7.2 Functions ordered by domain extension

A function  $h'$  is an extension of  $h$ , iff the graph of  $h$  is a subset of the graph of  $h'$ .

**lemma** *graph-extI*:

$(\bigwedge x. x \in H \implies h\ x = h'\ x) \implies H \subseteq H'$   
 $\implies \text{graph } H\ h \subseteq \text{graph } H'\ h'$   
 $\langle \text{proof} \rangle$

**lemma** *graph-extD1* [*dest?*]:

$\text{graph } H\ h \subseteq \text{graph } H'\ h' \implies x \in H \implies h\ x = h'\ x$   
 $\langle \text{proof} \rangle$

**lemma** *graph-extD2* [*dest?*]:

$\text{graph } H\ h \subseteq \text{graph } H'\ h' \implies H \subseteq H'$   
 $\langle \text{proof} \rangle$

### 7.3 Domain and function of a graph

The inverse functions to *graph* are *domain* and *funct*.

**definition**

*domain* :: 'a graph  $\Rightarrow$  'a set **where**  
*domain* *g* = {*x*.  $\exists y. (x, y) \in g$ }

**definition**

*funct* :: 'a graph  $\Rightarrow$  ('a  $\Rightarrow$  real) **where**  
*funct* *g* = ( $\lambda x. (\text{SOME } y. (x, y) \in g)$ )

The following lemma states that *g* is the graph of a function if the relation induced by *g* is unique.

**lemma** *graph-domain-funct*:

**assumes** *uniq*:  $\bigwedge x\ y\ z. (x, y) \in g \implies (x, z) \in g \implies z = y$   
**shows** *graph* (*domain* *g*) (*funct* *g*) = *g*  
 $\langle \text{proof} \rangle$

### 7.4 Norm-preserving extensions of a function

Given a linear form *f* on the space *F* and a seminorm *p* on *E*. The set of all linear extensions of *f*, to superspaces *H* of *F*, which are bounded by *p*, is defined as follows.

**definition**

*norm-pres-extensions* ::  
 'a::{plus, minus, uminus, zero} set  $\Rightarrow$  ('a  $\Rightarrow$  real)  $\Rightarrow$  'a set  $\Rightarrow$  ('a  $\Rightarrow$  real)  
 $\Rightarrow$  'a graph set **where**  
*norm-pres-extensions* *E* *p* *F* *f*  
 = {*g*.  $\exists H\ h. g = \text{graph } H\ h$   
 $\wedge \text{linearform } H\ h$   
 $\wedge H \trianglelefteq E$   
 $\wedge F \trianglelefteq H$   
 $\wedge \text{graph } F\ f \subseteq \text{graph } H\ h$   
 $\wedge (\forall x \in H. h\ x \leq p\ x)$ }

**lemma** *norm-pres-extensionE* [*elim*]:

*g*  $\in$  *norm-pres-extensions* *E* *p* *F* *f*  
 $\implies (\bigwedge H\ h. g = \text{graph } H\ h \implies \text{linearform } H\ h$   
 $\implies H \trianglelefteq E \implies F \trianglelefteq H \implies \text{graph } F\ f \subseteq \text{graph } H\ h$

$\implies \forall x \in H. h\ x \leq p\ x \implies C) \implies C$   
 $\langle \text{proof} \rangle$

**lemma** *norm-pres-extensionI2* [intro]:  
 $\text{linearform } H\ h \implies H \trianglelefteq E \implies F \trianglelefteq H$   
 $\implies \text{graph } F\ f \subseteq \text{graph } H\ h \implies \forall x \in H. h\ x \leq p\ x$   
 $\implies \text{graph } H\ h \in \text{norm-pres-extensions } E\ p\ F\ f$   
 $\langle \text{proof} \rangle$

**lemma** *norm-pres-extensionI*:  
 $\exists H\ h. g = \text{graph } H\ h$   
 $\wedge \text{linearform } H\ h$   
 $\wedge H \trianglelefteq E$   
 $\wedge F \trianglelefteq H$   
 $\wedge \text{graph } F\ f \subseteq \text{graph } H\ h$   
 $\wedge (\forall x \in H. h\ x \leq p\ x) \implies g \in \text{norm-pres-extensions } E\ p\ F\ f$   
 $\langle \text{proof} \rangle$

**end**

## 8 The norm of a function

**theory** *FunctionNorm*  
**imports** *NormedSpace FunctionOrder*  
**begin**

### 8.1 Continuous linear forms

A linear form  $f$  on a normed vector space  $(V, \|\cdot\|)$  is *continuous*, iff it is bounded, i.e.

$$\exists c \in \mathbb{R}. \forall x \in V. |f\ x| \leq c \cdot \|x\|$$

In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

**locale** *continuous* = *var-V* + *norm-syntax* + *linearform* +  
**assumes** *bounded*:  $\exists c. \forall x \in V. |f\ x| \leq c * \|x\|$

**declare** *continuous.intro* [intro?] *continuous-axioms.intro* [intro?]

**lemma** *continuousI* [intro]:  
**fixes** *norm* ::  $- \Rightarrow \text{real}$  ( $\|\cdot\|$ )  
**assumes** *linearform*  $V\ f$   
**assumes**  $r$ :  $\bigwedge x. x \in V \implies |f\ x| \leq c * \|x\|$   
**shows** *continuous*  $V\ \text{norm}\ f$   
 $\langle \text{proof} \rangle$

### 8.2 The norm of a linear form

The least real number  $c$  for which holds

$$\forall x \in V. |f\ x| \leq c \cdot \|x\|$$



is called the *norm* of  $f$ .

For non-trivial vector spaces  $V \neq \{0\}$  the norm can be defined as

$$\|f\| = \sup_{x \neq 0} |f x| / \|x\|$$

For the case  $V = \{0\}$  the supremum would be taken from an empty set. Since  $\mathbb{R}$  is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be  $\{ \} \geq 0$  so that *fn-norm* has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be  $0$ , as all other elements are  $\{ \} \geq 0$ .

Thus we define the set  $B$  where the supremum is taken from as follows:

$$\{0\} \cup \{|f x| / \|x\| \mid x \neq 0 \wedge x \in V\}$$

*fn-norm* is equal to the supremum of  $B$ , if the supremum exists (otherwise it is undefined).

```

locale fn-norm = norm-syntax +
  fixes  $B$  defines  $B \ V f \equiv \{0\} \cup \{|f x| / \|x\| \mid x. x \neq 0 \wedge x \in V\}$ 
  fixes fn-norm ( $\|-\|$ )  $\equiv [0, 1000] \ 999$ 
  defines  $\|f\| \equiv \bigsqcup (B \ V f)$ 

```

```

locale normed-vectorspace-with-fn-norm = normed-vectorspace + fn-norm

```

```

lemma (in fn-norm) B-not-empty [intro]:  $0 \in B \ V f$ 
  <proof>

```

The following lemma states that every continuous linear form on a normed space  $(V, \|\cdot\|)$  has a function norm.

```

lemma (in normed-vectorspace-with-fn-norm) fn-norm-works:
  assumes continuous  $V \ norm \ f$ 
  shows  $\text{lub } (B \ V f) (\|f\| \cdot V)$ 
  <proof>

```

```

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ub [iff?]:
  assumes continuous  $V \ norm \ f$ 
  assumes  $b: b \in B \ V f$ 
  shows  $b \leq \|f\| \cdot V$ 
  <proof>

```

```

lemma (in normed-vectorspace-with-fn-norm) fn-norm-leastB:
  assumes continuous  $V \ norm \ f$ 
  assumes  $b: \bigwedge b. b \in B \ V f \implies b \leq y$ 
  shows  $\|f\| \cdot V \leq y$ 
  <proof>

```

The norm of a continuous function is always  $\geq 0$ .

```

lemma (in normed-vectorspace-with-fn-norm) fn-norm-ge-zero [iff]:
  assumes continuous  $V \ norm \ f$ 
  shows  $0 \leq \|f\| \cdot V$ 
  <proof>

```

The fundamental property of function norms is:

$$|f\ x| \leq \|f\| \cdot \|x\|$$

**lemma** (in *normed-vectorspace-with-fn-norm*) *fn-norm-le-cong*:  
**assumes** *continuous V norm f linearform V f*  
**assumes** *x: x ∈ V*  
**shows**  $|f\ x| \leq \|f\| \cdot \|x\|$   
 ⟨*proof*⟩

The function norm is the least positive real number for which the following inequation holds:

$$|f\ x| \leq c \cdot \|x\|$$

**lemma** (in *normed-vectorspace-with-fn-norm*) *fn-norm-least [intro?]*:  
**assumes** *continuous V norm f*  
**assumes** *ineq: ∀ x ∈ V. |f x| ≤ c \* ||x|| and ge: 0 ≤ c*  
**shows**  $\|f\| \cdot V \leq c$   
 ⟨*proof*⟩

**end**

## 9 Zorn's Lemma

**theory** *ZornLemma*  
**imports** *Zorn*  
**begin**

Zorn's Lemmas states: if every linear ordered subset of an ordered set  $S$  has an upper bound in  $S$ , then there exists a maximal element in  $S$ . In our application,  $S$  is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn's lemma can be modified: if  $S$  is non-empty, it suffices to show that for every non-empty chain  $c$  in  $S$  the union of  $c$  also lies in  $S$ .

**theorem** *Zorn's-Lemma*:  
**assumes**  $r: \bigwedge c. c \in \text{chain } S \implies \exists x. x \in c \implies \bigcup c \in S$   
**and**  $aS: a \in S$   
**shows**  $\exists y \in S. \forall z \in S. y \subseteq z \longrightarrow y = z$   
 ⟨*proof*⟩

**end**

## Part II

# Lemmas for the Proof

## 10 The supremum w.r.t. the function order

**theory** *HahnBanachSupLemmas*  
**imports** *FunctionNorm ZornLemma*  
**begin**

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let  $E$  be a real vector space with a seminorm  $p$  on  $E$ .  $F$  is a subspace of  $E$  and  $f$  a linear form on  $F$ . We consider a chain  $c$  of norm-preserving extensions of  $f$ , such that  $\bigcup c = \text{graph } H h$ . We will show some properties about the limit function  $h$ , i.e. the supremum of the chain  $c$ .

Let  $c$  be a chain of norm-preserving extensions of the function  $f$  and let  $\text{graph } H h$  be the supremum of  $c$ . Every element in  $H$  is member of one of the elements of the chain.

**lemmas**  $[\text{dest?}] = \text{chainD}$   
**lemmas**  $\text{chainE2 } [\text{elim?}] = \text{chainD2 } [\text{elim-format, standard}]$

**lemma** *some- $H'h'$ :*  
**assumes**  $M: M = \text{norm-pres-extensions } E p F f$   
**and**  $cM: c \in \text{chain } M$   
**and**  $u: \text{graph } H h = \bigcup c$   
**and**  $x: x \in H$   
**shows**  $\exists H' h'. \text{graph } H' h' \in c$   
 $\wedge (x, h x) \in \text{graph } H' h'$   
 $\wedge \text{linearform } H' h' \wedge H' \trianglelefteq E$   
 $\wedge F \trianglelefteq H' \wedge \text{graph } F f \subseteq \text{graph } H' h'$   
 $\wedge (\forall x \in H'. h' x \leq p x)$   
 $\langle \text{proof} \rangle$

Let  $c$  be a chain of norm-preserving extensions of the function  $f$  and let  $\text{graph } H h$  be the supremum of  $c$ . Every element in the domain  $H$  of the supremum function is member of the domain  $H'$  of some function  $h'$ , such that  $h$  extends  $h'$ .

**lemma** *some- $H'h'$ :*  
**assumes**  $M: M = \text{norm-pres-extensions } E p F f$   
**and**  $cM: c \in \text{chain } M$   
**and**  $u: \text{graph } H h = \bigcup c$   
**and**  $x: x \in H$   
**shows**  $\exists H' h'. x \in H' \wedge \text{graph } H' h' \subseteq \text{graph } H h$   
 $\wedge \text{linearform } H' h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$   
 $\wedge \text{graph } F f \subseteq \text{graph } H' h' \wedge (\forall x \in H'. h' x \leq p x)$   
 $\langle \text{proof} \rangle$

Any two elements  $x$  and  $y$  in the domain  $H$  of the supremum function  $h$  are both in the domain  $H'$  of some function  $h'$ , such that  $h$  extends  $h'$ .

**lemma** *some- $H'h'2$* :  
**assumes**  $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$   
**and**  $cM: c \in \text{chain } M$   
**and**  $u: \text{graph } H \text{ } h = \bigcup c$   
**and**  $x: x \in H$   
**and**  $y: y \in H$   
**shows**  $\exists H' h'. x \in H' \wedge y \in H'$   
 $\wedge \text{graph } H' h' \subseteq \text{graph } H h$   
 $\wedge \text{linearform } H' h' \wedge H' \leq E \wedge F \leq H'$   
 $\wedge \text{graph } F f \subseteq \text{graph } H' h' \wedge (\forall x \in H'. h' x \leq p x)$   
 $\langle \text{proof} \rangle$

The relation induced by the graph of the supremum of a chain  $c$  is definite, i. e.  $t$  is the graph of a function.

**lemma** *sup-definite*:  
**assumes**  $M\text{-def}: M \equiv \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$   
**and**  $cM: c \in \text{chain } M$   
**and**  $xy: (x, y) \in \bigcup c$   
**and**  $xz: (x, z) \in \bigcup c$   
**shows**  $z = y$   
 $\langle \text{proof} \rangle$

The limit function  $h$  is linear. Every element  $x$  in the domain of  $h$  is in the domain of a function  $h'$  in the chain of norm preserving extensions. Furthermore,  $h$  is an extension of  $h'$  so the function values of  $x$  are identical for  $h'$  and  $h$ . Finally, the function  $h'$  is linear by construction of  $M$ .

**lemma** *sup-lf*:  
**assumes**  $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$   
**and**  $cM: c \in \text{chain } M$   
**and**  $u: \text{graph } H \text{ } h = \bigcup c$   
**shows**  $\text{linearform } H \text{ } h$   
 $\langle \text{proof} \rangle$

The limit of a non-empty chain of norm preserving extensions of  $f$  is an extension of  $f$ , since every element of the chain is an extension of  $f$  and the supremum is an extension for every element of the chain.

**lemma** *sup-ext*:  
**assumes**  $\text{graph}: \text{graph } H \text{ } h = \bigcup c$   
**and**  $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$   
**and**  $cM: c \in \text{chain } M$   
**and**  $ex: \exists x. x \in c$   
**shows**  $\text{graph } F \text{ } f \subseteq \text{graph } H \text{ } h$   
 $\langle \text{proof} \rangle$

The domain  $H$  of the limit function is a superspace of  $F$ , since  $F$  is a subset of  $H$ . The existence of the  $0$  element in  $F$  and the closure properties follow from the fact that  $F$  is a vector space.

**lemma** *sup-supF*:  
**assumes**  $\text{graph}: \text{graph } H \text{ } h = \bigcup c$   
**and**  $M: M = \text{norm-pres-extensions } E \text{ } p \text{ } F \text{ } f$

```

and  $cM: c \in \text{chain } M$ 
and  $ex: \exists x. x \in c$ 
and  $FE: F \leq E$ 
shows  $F \leq H$ 
<proof>

```

The domain  $H$  of the limit function is a subspace of  $E$ .

```

lemma sup-subE:
assumes  $\text{graph}: \text{graph } H \ h = \bigcup c$ 
and  $M: M = \text{norm-pres-extensions } E \ p \ F \ f$ 
and  $cM: c \in \text{chain } M$ 
and  $ex: \exists x. x \in c$ 
and  $FE: F \leq E$ 
and  $E: \text{vectorspace } E$ 
shows  $H \leq E$ 
<proof>

```

The limit function is bounded by the norm  $p$  as well, since all elements in the chain are bounded by  $p$ .

```

lemma sup-norm-pres:
assumes  $\text{graph}: \text{graph } H \ h = \bigcup c$ 
and  $M: M = \text{norm-pres-extensions } E \ p \ F \ f$ 
and  $cM: c \in \text{chain } M$ 
shows  $\forall x \in H. h \ x \leq p \ x$ 
<proof>

```

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma *abs-HahnBanach* (see page 23). For real vector spaces the following inequations are equivalent:

$$\forall x \in H. |h \ x| \leq p \ x \quad \text{and} \quad \forall x \in H. h \ x \leq p \ x$$

```

lemma abs-ineq-iff:
assumes  $\text{subspace } H \ E$  and  $\text{vectorspace } E$  and  $\text{seminorm } E \ p$ 
and  $\text{linearform } H \ h$ 
shows  $(\forall x \in H. |h \ x| \leq p \ x) = (\forall x \in H. h \ x \leq p \ x)$  (is ?L = ?R)
<proof>

```

**end**

## 11 Extending non-maximal functions

```

theory HahnBanachExtLemmas
imports FunctionNorm
begin

```

In this section the following context is presumed. Let  $E$  be a real vector space with a seminorm  $q$  on  $E$ .  $F$  is a subspace of  $E$  and  $f$  a linear function on  $F$ . We consider a subspace  $H$  of  $E$  that is a superspace of  $F$  and a linear form  $h$  on  $H$ .  $H$  is not equal to  $E$  and  $x_0$  is an element in  $E - H$ .  $H$  is extended to the

direct sum  $H' = H + \text{lin } x_0$ , so for any  $x \in H'$  the decomposition of  $x = y + a \cdot x$  with  $y \in H$  is unique.  $h'$  is defined on  $H'$  by  $h' x = h y + a \cdot \xi$  for a certain  $\xi$ .

Subsequently we show some properties of this extension  $h'$  of  $h$ .

This lemma will be used to show the existence of a linear extension of  $f$  (see page ??). It is a consequence of the completeness of  $\mathbb{R}$ . To show

$$\exists \xi. \forall y \in F. a y \leq \xi \wedge \xi \leq b y$$

it suffices to show that

$$\forall u \in F. \forall v \in F. a u \leq b v$$

**lemma** *ex-xi*:

**assumes** *vectorspace*  $F$   
**assumes**  $r: \bigwedge u v. u \in F \implies v \in F \implies a u \leq b v$   
**shows**  $\exists xi::real. \forall y \in F. a y \leq xi \wedge xi \leq b y$   
*<proof>*

The function  $h'$  is defined as a  $h' x = h y + a \cdot \xi$  where  $x = y + a \cdot \xi$  is a linear extension of  $h$  to  $H'$ .

**lemma** *h'-lf*:

**assumes** *h'-def*:  $h' \equiv \lambda x. \text{let } (y, a) =$   
     *SOME*  $(y, a). x = y + a \cdot x0 \wedge y \in H \text{ in } h y + a * xi$   
**and** *H'-def*:  $H' \equiv H + \text{lin } x0$   
**and** *HE*:  $H \sqsubseteq E$   
**assumes** *linearform*  $H h$   
**assumes**  $x0: x0 \notin H \ x0 \in E \ x0 \neq 0$   
**assumes** *E*: *vectorspace*  $E$   
**shows** *linearform*  $H' h'$   
*<proof>*

The linear extension  $h'$  of  $h$  is bounded by the seminorm  $p$ .

**lemma** *h'-norm-pres*:

**assumes** *h'-def*:  $h' \equiv \lambda x. \text{let } (y, a) =$   
     *SOME*  $(y, a). x = y + a \cdot x0 \wedge y \in H \text{ in } h y + a * xi$   
**and** *H'-def*:  $H' \equiv H + \text{lin } x0$   
**and**  $x0: x0 \notin H \ x0 \in E \ x0 \neq 0$   
**assumes** *E*: *vectorspace*  $E$  **and** *HE*: *subspace*  $H E$   
**and** *seminorm*  $E p$  **and** *linearform*  $H h$   
**assumes**  $a: \forall y \in H. h y \leq p y$   
**and**  $a': \forall y \in H. -p (y + x0) - h y \leq xi \wedge xi \leq p (y + x0) - h y$   
**shows**  $\forall x \in H'. h' x \leq p x$   
*<proof>*

**end**

## Part III

# The Main Proof

## 12 The Hahn-Banach Theorem

**theory** *HahnBanach*  
**imports** *HahnBanachLemmas*  
**begin**

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

### 12.1 The Hahn-Banach Theorem for vector spaces

**Hahn-Banach Theorem.** Let  $F$  be a subspace of a real vector space  $E$ , let  $p$  be a semi-norm on  $E$ , and  $f$  be a linear form defined on  $F$  such that  $f$  is bounded by  $p$ , i.e.  $\forall x \in F. f x \leq p x$ . Then  $f$  can be extended to a linear form  $h$  on  $E$  such that  $h$  is norm-preserving, i.e.  $h$  is also bounded by  $p$ .

#### Proof Sketch.

1. Define  $M$  as the set of norm-preserving extensions of  $f$  to subspaces of  $E$ . The linear forms in  $M$  are ordered by domain extension.
2. We show that every non-empty chain in  $M$  has an upper bound in  $M$ .
3. With Zorn's Lemma we conclude that there is a maximal function  $g$  in  $M$ .
4. The domain  $H$  of  $g$  is the whole space  $E$ , as shown by classical contradiction:
  - Assuming  $g$  is not defined on whole  $E$ , it can still be extended in a norm-preserving way to a super-space  $H'$  of  $H$ .
  - Thus  $g$  can not be maximal. Contradiction!

**theorem** *HahnBanach*:

**assumes**  $E$ : *vectorspace*  $E$  **and** *subspace*  $F$   $E$

**and** *seminorm*  $E$   $p$  **and** *linearform*  $F$   $f$

**assumes**  $fp$ :  $\forall x \in F. f x \leq p x$

**shows**  $\exists h. \text{linearform } E h \wedge (\forall x \in F. h x = f x) \wedge (\forall x \in E. h x \leq p x)$

— Let  $E$  be a vector space,  $F$  a subspace of  $E$ ,  $p$  a seminorm on  $E$ ,

— and  $f$  a linear form on  $F$  such that  $f$  is bounded by  $p$ ,

— then  $f$  can be extended to a linear form  $h$  on  $E$  in a norm-preserving way.

*<proof>*

### 12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form  $f$  and a seminorm  $p$  the following inequations are equivalent:<sup>1</sup>

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<sup>1</sup>This was shown in lemma *abs-ineq-iff* (see page 21).

$$\forall x \in H. |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H. h\ x \leq p\ x$$

**theorem** *abs-HahnBanach*:

**assumes** *E*: *vectorspace* *E* **and** *FE*: *subspace* *F* *E*

**and** *lf*: *linearform* *F* *f* **and** *sn*: *seminorm* *E* *p*

**assumes** *fp*:  $\forall x \in F. |f\ x| \leq p\ x$

**shows**  $\exists g. \text{linearform } E\ g$

$\wedge (\forall x \in F. g\ x = f\ x)$

$\wedge (\forall x \in E. |g\ x| \leq p\ x)$

*<proof>*

### 12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form  $f$  on a subspace  $F$  of a norm space  $E$ , can be extended to a continuous linear form  $g$  on  $E$  such that  $\|f\| = \|g\|$ .

**theorem** *norm-HahnBanach*:

**fixes** *V* **and** *norm* ( $\|\cdot\|$ )

**fixes** *B* **defines**  $\bigwedge V\ f. B\ V\ f \equiv \{0\} \cup \{|f\ x| / \|x\| \mid x. x \neq 0 \wedge x \in V\}$

**fixes** *fn-norm* ( $\|\cdot\|$ --  $[0, 1000]$  999)

**defines**  $\bigwedge V\ f. \|f\|_V \equiv \bigsqcup (B\ V\ f)$

**assumes** *E-norm*: *normed-vectorspace* *E* *norm* **and** *FE*: *subspace* *F* *E*

**and** *linearform*: *linearform* *F* *f* **and** *continuous* *F* *norm* *f*

**shows**  $\exists g. \text{linearform } E\ g$

$\wedge \text{continuous } E\ \text{norm } g$

$\wedge (\forall x \in F. g\ x = f\ x)$

$\wedge \|g\|_E = \|f\|_F$

*<proof>*

**end**

## References

- [1] H. Heuser. *Funktionalanalysis: Theorie und Anwendung*. Teubner, 1986.
- [2] L. Narici and E. Beckenstein. The Hahn-Banach Theorem: The life and times. In *Topology Atlas*. York University, Toronto, Ontario, Canada, 1996. <http://at.yorku.ca/topology/preprint.htm> and <http://at.yorku.ca/p/a/a/a/16.htm>.
- [3] B. Nowak and A. Trybulec. Hahn-Banach theorem. *Journal of Formalized Mathematics*, 5, 1993. <http://mizar.uwb.edu.pl/JFM/Vol5/hahnban.html>.