# Isabelle/HOL — Higher-Order Logic

April 19, 2009

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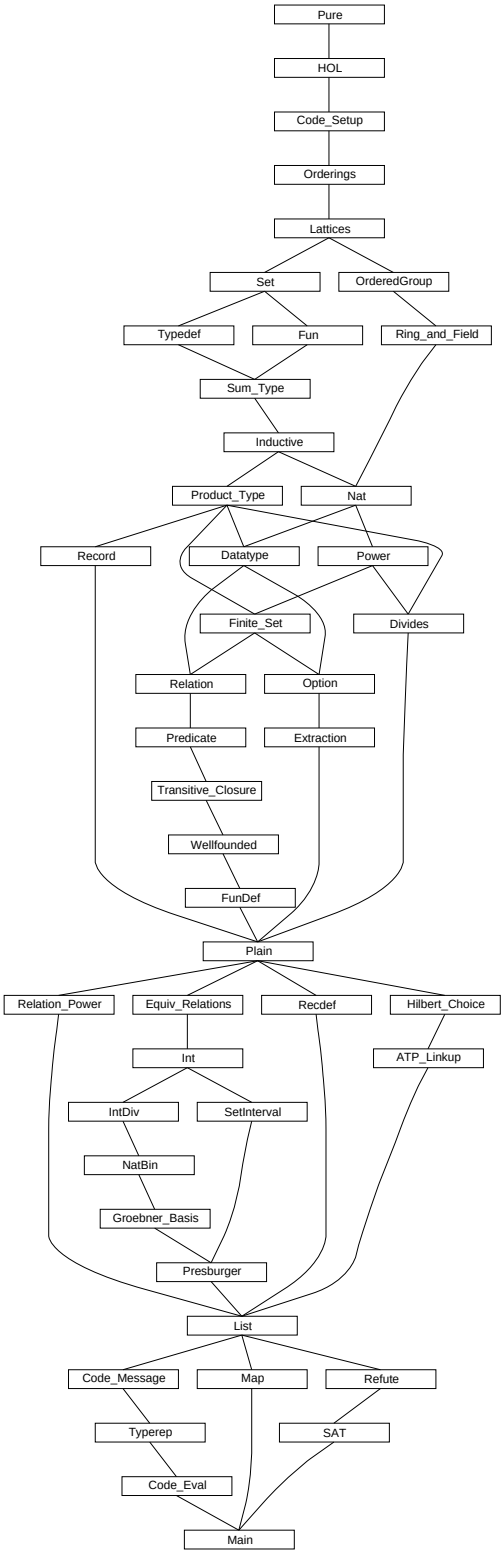
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# 1 HOL: The basis of Higher-Order Logic

**theory** *HOL*
**imports** *Pure*
**uses**
(*Tools/hologic.ML*)
~~/*src/Tools/IsaPlanner/zipper.ML*
~~/*src/Tools/IsaPlanner/isand.ML*
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~~/*src/Provers/clasimp.ML*
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~~/*src/Tools/code/code-name.ML*
~~/*src/Tools/code/code-funcgr.ML*
~~/*src/Tools/code/code-wellsorted.ML*
~~/*src/Tools/code/code-thingol.ML*
~~/*src/Tools/code/code-printer.ML*
~~/*src/Tools/code/code-target.ML*
~~/*src/Tools/code/code-ml.ML*
~~/*src/Tools/code/code-haskell.ML*
~~/*src/Tools/nbe.ML*
(*Tools/recfun-codegen.ML*)
**begin**

⟨*ML*⟩

## 1.1 Primitive logic

### 1.1.1 Core syntax

**classes** *type*
**defaultsort** *type*
⟨*ML*⟩

**arities**
*fun* :: (*type*, *type*) *type*

*itself* :: (*type*) *type*

**global**

**typedecl** *bool*

**judgment**
*Trueprop* :: *bool* => *prop* ((-) *5*)

**consts**
*Not* :: *bool* => *bool* (~ - [*40*] *40*)
*True* :: *bool*
*False* :: *bool*

*The* :: ('*a* => *bool*) => '*a*
*All* :: ('*a* => *bool*) => *bool* (**binder** *ALL* *10*)
*Ex* :: ('*a* => *bool*) => *bool* (**binder** *EX* *10*)
*Ex1* :: ('*a* => *bool*) => *bool* (**binder** *EX*! *10*)
*Let* :: ['*a*, '*a* => '*b*] => '*b*

*op* = :: ['*a*, '*a*] => *bool* (**infixl** = *50*)
*op* & :: [*bool*, *bool*] => *bool* (**infixr** & *35*)
*op* | :: [*bool*, *bool*] => *bool* (**infixr** | *30*)
*op* --> :: [*bool*, *bool*] => *bool* (**infixr** --> *25*)

**local**

**consts**
*If* :: [*bool*, '*a*, '*a*] => '*a* ((*if* (-)/ *then* (-)/ *else* (-)) *10*)

### 1.1.2 Additional concrete syntax

**notation** (**output**)
*op* = (**infix** = *50*)

**abbreviation**
*not-equal* :: ['*a*, '*a*] => *bool* (**infixl** ~= *50*) **where**
*x* ~= *y* == ~ (*x* = *y*)

**notation** (**output**)
*not-equal* (**infix** ~= *50*)

**notation** (*xsymbols*)
*Not* (¬ - [*40*] *40*) **and**
*op* & (**infixr** ∧ *35*) **and**
*op* | (**infixr** ∨ *30*) **and**
*op* --> (**infixr** ⟶ *25*) **and**
*not-equal* (**infix** ≠ *50*)

**notation** (*HTML* **output**)
*Not* (¬ - [*40*] *40*) **and**
*op &* (**infixr** ∧ *35*) **and**
*op |* (**infixr** ∨ *30*) **and**
*not-equal* (**infix** ≠ *50*)

**abbreviation** (*iff*)
*iff* :: [*bool*, *bool*] => *bool* (**infixr** <–> *25*) **where**
*A* <–> *B* == *A* = *B*

**notation** (*xsymbols*)
*iff* (**infixr** ⟷ *25*)

**nonterminals**
*letbinds* *letbind*
*case-syn* *cases-syn*

**syntax**
*-The* :: [*pttrn*, *bool*] => *'a* ((*3THE -./ -*) [*0*, *10*] *10*)

*-bind* :: [*pttrn*, *'a*] => *letbind* ((*2- =/ -*) *10*)
:: *letbind* => *letbinds* (-)
*-binds* :: [*letbind*, *letbinds*] => *letbinds* (-;/ -)
*-Let* :: [*letbinds*, *'a*] => *'a* ((*let* (-)/ *in* (-)) *10*)

*-case-syntax*:: [*'a*, *cases-syn*] => *'b* ((*case - of/ -*) *10*)
*-case1* :: [*'a*, *'b*] => *case-syn* ((*2- =>/ -*) *10*)
:: *case-syn* => *cases-syn* (-)
*-case2* :: [*case-syn*, *cases-syn*] => *cases-syn* (-/ | -)

**translations**
*THE x. P* == *The* (%*x. P*)
*-Let* (*-binds b bs*) *e* == *-Let b* (*-Let bs e*)
*let x = a in e* == *Let a* (%*x. e*)

⟨*ML*⟩

**syntax** (*xsymbols*)
*-case1* :: [*'a*, *'b*] => *case-syn* ((*2- ⇒/ -*) *10*)

**notation** (*xsymbols*)
*All* (**binder** ∀ *10*) **and**
*Ex* (**binder** ∃ *10*) **and**
*Ex1* (**binder** ∃! *10*)

**notation** (*HTML* **output**)
*All* (**binder** ∀ *10*) **and**
*Ex* (**binder** ∃ *10*) **and**

*Ex1* (**binder** $\exists!$ *10*)

**notation** (*HOL*)
*All* (**binder** ! *10*) **and**
*Ex* (**binder** *?* *10*) **and**
*Ex1* (**binder** *?!* *10*)

### 1.1.3 Axioms and basic definitions

**axioms**
*refl*: $t = (t::'a)$
*subst*: $s = t \Longrightarrow P\ s \Longrightarrow P\ t$
*ext*: $(!!x::'a.\ (f\ x ::'b) = g\ x) ==> (\%x.\ f\ x) = (\%x.\ g\ x)$
— Extensionality is built into the meta-logic, and this rule expresses a related property. It is an eta-expanded version of the traditional rule, and similar to the ABS rule of HOL

*the-eq-trivial*: $(THE\ x.\ x = a) = (a::'a)$

*impI*: $(P ==> Q) ==> P-->Q$
*mp*: $[|\ P-->Q;\ P\ |] ==> Q$

**defs**
*True-def*: $True == ((\%x::bool.\ x) = (\%x.\ x))$
*All-def*: $All(P) == (P = (\%x.\ True))$
*Ex-def*: $Ex(P) == !Q.\ (!x.\ P\ x --> Q) --> Q$
*False-def*: $False == (!P.\ P)$
*not-def*: $\sim P == P-->False$
*and-def*: $P\ \&\ Q == !R.\ (P-->Q-->R) --> R$
*or-def*: $P \mid Q == !R.\ (P-->R) --> (Q-->R) --> R$
*Ex1-def*: $Ex1(P) == ?\ x.\ P(x)\ \&\ (!\ y.\ P(y) --> y=x)$

**axioms**
*iff*: $(P-->Q) --> (Q-->P) --> (P=Q)$
*True-or-False*: $(P=True) \mid (P=False)$

**defs**
*Let-def*: $Let\ s\ f == f(s)$
*if-def*: $If\ P\ x\ y == THE\ z::'a.\ (P=True --> z=x)\ \&\ (P=False --> z=y)$

**finalconsts**
*op* =
*op* -->
*The*

**axiomatization**
*undefined* :: *'a*

**abbreviation** (*input*)
*arbitrary* ≡ *undefined*

### 1.1.4 Generic classes and algebraic operations

**class** *default* =
**fixes** *default* :: *'a*

**class** *zero* =
**fixes** *zero* :: *'a* (*0*)

**class** *one* =
**fixes** *one* :: *'a* (*1*)

**hide** (**open**) *const zero one*

**class** *plus* =
**fixes** *plus* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** + *65*)

**class** *minus* =
**fixes** *minus* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** − *65*)

**class** *uminus* =
**fixes** *uminus* :: *'a* ⇒ *'a* (− - [*81*] *80*)

**class** *times* =
**fixes** *times* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ∗ *70*)

**class** *inverse* =
**fixes** *inverse* :: *'a* ⇒ *'a*
**and** *divide* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *'/ 70*)

**class** *abs* =
**fixes** *abs* :: *'a* ⇒ *'a*
**begin**

**notation** (*xsymbols*)
*abs* (|-|)

**notation** (*HTML* **output**)
*abs* (|-|)

**end**

**class** *sgn* =
**fixes** *sgn* :: *'a* ⇒ *'a*

**class** *ord* =

**fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$
**and** *less* :: $'a \Rightarrow 'a \Rightarrow bool$
**begin**

**notation**
*less-eq* (*op* <=) **and**
*less-eq* ((-/ <= -) [*51*, *51*] *50*) **and**
*less* (*op* <) **and**
*less* ((-/ < -) [*51*, *51*] *50*)

**notation** (*xsymbols*)
*less-eq* (*op* $\leq$) **and**
*less-eq* ((-/ $\leq$ -) [*51*, *51*] *50*)

**notation** (*HTML* **output**)
*less-eq* (*op* $\leq$) **and**
*less-eq* ((-/ $\leq$ -) [*51*, *51*] *50*)

**abbreviation** (*input*)
*greater-eq* (**infix** >= *50*) **where**
$x >= y \equiv y <= x$

**notation** (*input*)
*greater-eq* (**infix** $\geq$ *50*)

**abbreviation** (*input*)
*greater* (**infix** > *50*) **where**
$x > y \equiv y < x$

**end**

**syntax**
*-index1* :: *index* ($_1$)
**translations**
(*index*) $_1$ => (*index*) $\diamond$

⟨*ML*⟩

## 1.2 Fundamental rules

### 1.2.1 Equality

**lemma** *sym*: $s = t \Longrightarrow t = s$
⟨*proof*⟩

**lemma** *ssubst*: $t = s \Longrightarrow P\ s \Longrightarrow P\ t$
⟨*proof*⟩

**lemma** *trans*: $[|\ r=s;\ s=t\ |] \Longrightarrow r=t$
⟨*proof*⟩

**lemma** *meta-eq-to-obj-eq*:
**assumes** *meq*: $A == B$
**shows** $A = B$
⟨*proof*⟩

Useful with *erule* for proving equalities from known equalities.

**lemma** *box-equals*: [| $a=b$; $a=c$; $b=d$ |] ==> $c=d$
⟨*proof*⟩

For calculational reasoning:

**lemma** *forw-subst*: $a = b$ ==> $P\ b$ ==> $P\ a$
⟨*proof*⟩

**lemma** *back-subst*: $P\ a$ ==> $a = b$ ==> $P\ b$
⟨*proof*⟩

### 1.2.2 Congruence rules for application

**lemma** *fun-cong*: $(f::'a=>'b) = g$ ==> $f(x)=g(x)$
⟨*proof*⟩

**lemma** *arg-cong*: $x=y$ ==> $f(x)=f(y)$
⟨*proof*⟩

**lemma** *arg-cong2*: $[\![ a = b; c = d ]\!] \Longrightarrow f\ a\ c = f\ b\ d$
⟨*proof*⟩

**lemma** *cong*: [| $f = g$; $(x::'a) = y$ |] ==> $f(x) = g(y)$
⟨*proof*⟩

### 1.2.3 Equality of booleans – iff

**lemma** *iffI*: **assumes** $P$ ==> $Q$ **and** $Q$ ==> $P$ **shows** $P=Q$
⟨*proof*⟩

**lemma** *iffD2*: [| $P=Q$; $Q$ |] ==> $P$
⟨*proof*⟩

**lemma** *rev-iffD2*: [| $Q$; $P=Q$ |] ==> $P$
⟨*proof*⟩

**lemma** *iffD1*: $Q = P \Longrightarrow Q \Longrightarrow P$
⟨*proof*⟩

**lemma** *rev-iffD1*: $Q \Longrightarrow Q = P \Longrightarrow P$
⟨*proof*⟩

**lemma** *iffE*:
**assumes** *major*: $P=Q$
**and** *minor*: $[|\ P \dashrightarrow Q;\ Q \dashrightarrow P\ |] \Longrightarrow R$
**shows** $R$
⟨*proof*⟩

### 1.2.4 True

**lemma** *TrueI*: *True*
⟨*proof*⟩

**lemma** *eqTrueI*: $P \Longrightarrow P = True$
⟨*proof*⟩

**lemma** *eqTrueE*: $P = True \Longrightarrow P$
⟨*proof*⟩

### 1.2.5 Universal quantifier

**lemma** *allI*: **assumes** $!!x::'a.\ P(x)$ **shows** $ALL\ x.\ P(x)$
⟨*proof*⟩

**lemma** *spec*: $ALL\ x::'a.\ P(x) \Longrightarrow P(x)$
⟨*proof*⟩

**lemma** *allE*:
**assumes** *major*: $ALL\ x.\ P(x)$
**and** *minor*: $P(x) \Longrightarrow R$
**shows** $R$
⟨*proof*⟩

**lemma** *all-dupE*:
**assumes** *major*: $ALL\ x.\ P(x)$
**and** *minor*: $[|\ P(x);\ ALL\ x.\ P(x)\ |] \Longrightarrow R$
**shows** $R$
⟨*proof*⟩

### 1.2.6 False

Depends upon *spec*; it is impossible to do propositional logic before quantifiers!

**lemma** *FalseE*: $False \Longrightarrow P$
⟨*proof*⟩

**lemma** *False-neq-True*: $False = True \Longrightarrow P$
⟨*proof*⟩

### 1.2.7 Negation

**lemma** *notI*:
**assumes** $P$ ==> *False*
**shows** ~$P$
⟨*proof*⟩

**lemma** *False-not-True*: *False* ~= *True*
⟨*proof*⟩

**lemma** *True-not-False*: *True* ~= *False*
⟨*proof*⟩

**lemma** *notE*: [| ~$P$; $P$ |] ==> $R$
⟨*proof*⟩

**lemma** *notI2*: $(P \Longrightarrow \neg Pa) \Longrightarrow (P \Longrightarrow Pa) \Longrightarrow \neg P$
⟨*proof*⟩

### 1.2.8 Implication

**lemma** *impE*:
**assumes** $P$-->$Q$ $P$ $Q$ ==> $R$
**shows** $R$
⟨*proof*⟩

**lemma** *rev-mp*: [| $P$; $P$ --> $Q$ |] ==> $Q$
⟨*proof*⟩

**lemma** *contrapos-nn*:
**assumes** *major*: ~$Q$
**and** *minor*: $P$==>$Q$
**shows** ~$P$
⟨*proof*⟩

**lemma** *contrapos-pn*:
**assumes** *major*: $Q$
**and** *minor*: $P$ ==> ~$Q$
**shows** ~$P$
⟨*proof*⟩

**lemma** *not-sym*: $t$ ~= $s$ ==> $s$ ~= $t$
⟨*proof*⟩

**lemma** *eq-neq-eq-imp-neq*: [| $x$ = $a$ ; $a$ ~= $b$; $b$ = $y$ |] ==> $x$ ~= $y$
⟨*proof*⟩

**lemma** *rev-contrapos*:
**assumes** *pq*: $P \Longrightarrow Q$
**and** *nq*: $\sim Q$
**shows** $\sim P$
⟨*proof*⟩

### 1.2.9 Existential quantifier

**lemma** *exI*: $P\ x \Longrightarrow EX\ x::'a.\ P\ x$
⟨*proof*⟩

**lemma** *exE*:
**assumes** *major*: $EX\ x::'a.\ P(x)$
**and** *minor*: $!!x.\ P(x) \Longrightarrow Q$
**shows** $Q$
⟨*proof*⟩

### 1.2.10 Conjunction

**lemma** *conjI*: $[|\ P;\ Q\ |] \Longrightarrow P\&Q$
⟨*proof*⟩

**lemma** *conjunct1*: $[|\ P\ \&\ Q\ |] \Longrightarrow P$
⟨*proof*⟩

**lemma** *conjunct2*: $[|\ P\ \&\ Q\ |] \Longrightarrow Q$
⟨*proof*⟩

**lemma** *conjE*:
**assumes** *major*: $P\&Q$
**and** *minor*: $[|\ P;\ Q\ |] \Longrightarrow R$
**shows** $R$
⟨*proof*⟩

**lemma** *context-conjI*:
**assumes** $P\ P \Longrightarrow Q$ **shows** $P\ \&\ Q$
⟨*proof*⟩

### 1.2.11 Disjunction

**lemma** *disjI1*: $P \Longrightarrow P|Q$
⟨*proof*⟩

**lemma** *disjI2*: $Q \Longrightarrow P|Q$
⟨*proof*⟩

**lemma** *disjE*:
**assumes** *major*: $P|Q$
**and** *minorP*: $P \Longrightarrow R$
**and** *minorQ*: $Q \Longrightarrow R$

**shows** $R$
⟨*proof*⟩

### 1.2.12 Classical logic

**lemma** *classical*:
**assumes** *prem*: $\sim P \Longrightarrow P$
**shows** $P$
⟨*proof*⟩

**lemmas** *ccontr* = *FalseE* [*THEN classical*, *standard*]

**lemma** *rev-notE*:
**assumes** *premp*: $P$
**and** *premnot*: $\sim R \Longrightarrow \sim P$
**shows** $R$
⟨*proof*⟩

**lemma** *notnotD*: $\sim\sim P \Longrightarrow P$
⟨*proof*⟩

**lemma** *contrapos-pp*:
**assumes** *p1*: $Q$
**and** *p2*: $\sim P \Longrightarrow \sim Q$
**shows** $P$
⟨*proof*⟩

### 1.2.13 Unique existence

**lemma** *ex1I*:
**assumes** $P\ a$ $!!x.\ P(x) \Longrightarrow x=a$
**shows** $EX!\ x.\ P(x)$
⟨*proof*⟩

Sometimes easier to use: the premises have no shared variables. Safe!

**lemma** *ex-ex1I*:
**assumes** *ex-prem*: $EX\ x.\ P(x)$
**and** *eq*: $!!x\ y.\ [|\ P(x);\ P(y)\ |] \Longrightarrow x=y$
**shows** $EX!\ x.\ P(x)$
⟨*proof*⟩

**lemma** *ex1E*:
**assumes** *major*: $EX!\ x.\ P(x)$
**and** *minor*: $!!x.\ [|\ P(x);\ \ ALL\ y.\ P(y) \longrightarrow y=x\ |] \Longrightarrow R$
**shows** $R$
⟨*proof*⟩

**lemma** *ex1-implies-ex*: $EX!\ x.\ P\ x \Longrightarrow EX\ x.\ P\ x$

⟨*proof*⟩

### 1.2.14 THE: definite description operator

**lemma** *the-equality*:
**assumes** *prema*: $P\ a$
**and** *premx*: $!!x.\ P\ x\ ==>\ x=a$
**shows** $(THE\ x.\ P\ x) = a$
⟨*proof*⟩

**lemma** *theI*:
**assumes** $P\ a$ **and** $!!x.\ P\ x\ ==>\ x=a$
**shows** $P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *theI'*: $EX!\ x.\ P\ x\ ==>\ P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *theI2*:
**assumes** $P\ a\ !!x.\ P\ x\ ==>\ x=a\ !!x.\ P\ x\ ==>\ Q\ x$
**shows** $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *the1I2*: **assumes** $EX!\ x.\ P\ x\ \bigwedge x.\ P\ x \Longrightarrow Q\ x$ **shows** $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *the1-equality* [*elim?*]: $[|\ EX!x.\ P\ x;\ P\ a\ |] ==>\ (THE\ x.\ P\ x) = a$
⟨*proof*⟩

**lemma** *the-sym-eq-trivial*: $(THE\ y.\ x=y) = x$
⟨*proof*⟩

### 1.2.15 Classical intro rules for disjunction and existential quantifiers

**lemma** *disjCI*:
**assumes** $^{\sim}Q\ ==>\ P$ **shows** $P|Q$
⟨*proof*⟩

**lemma** *excluded-middle*: $^{\sim}P\ |\ P$
⟨*proof*⟩

case distinction as a natural deduction rule. Note that $\neg\ P$ is the second case, not the first

**lemma** *case-split* [*case-names True False*]:
**assumes** *prem1*: $P\ ==>\ Q$
**and** *prem2*: $^{\sim}P\ ==>\ Q$
**shows** $Q$

⟨*proof*⟩

**lemma** *impCE*:
**assumes** *major*: $P-->Q$
**and** *minor*: $\sim P ==> R\ Q ==> R$
**shows** $R$
⟨*proof*⟩

**lemma** *impCE'*:
**assumes** *major*: $P-->Q$
**and** *minor*: $Q ==> R\ \sim P ==> R$
**shows** $R$
⟨*proof*⟩

**lemma** *iffCE*:
**assumes** *major*: $P=Q$
**and** *minor*: $[|\ P;\ Q\ |] ==> R\ \ [|\ \sim P;\ \sim Q\ |] ==> R$
**shows** $R$
⟨*proof*⟩

**lemma** *exCI*:
**assumes** $ALL\ x.\ \sim P(x) ==> P(a)$
**shows** $EX\ x.\ P(x)$
⟨*proof*⟩

### 1.2.16 Intuitionistic Reasoning

**lemma** *impE'*:
**assumes** *1*: $P\ -->\ Q$
**and** *2*: $Q ==> R$
**and** *3*: $P\ -->\ Q ==> P$
**shows** $R$
⟨*proof*⟩

**lemma** *allE'*:
**assumes** *1*: $ALL\ x.\ P\ x$
**and** *2*: $P\ x ==> ALL\ x.\ P\ x ==> Q$
**shows** $Q$
⟨*proof*⟩

**lemma** *notE'*:
**assumes** *1*: $\sim\ P$
**and** *2*: $\sim\ P ==> P$
**shows** $R$
⟨*proof*⟩

**lemma** *TrueE*: *True* ==> *P* ==> *P* ⟨*proof*⟩
**lemma** *notFalseE*: ~ *False* ==> *P* ==> *P* ⟨*proof*⟩

**lemmas** [*Pure.elim*!] = *disjE iffE FalseE conjE exE TrueE notFalseE*
**and** [*Pure.intro*!] = *iffI conjI impI TrueI notI allI refl*
**and** [*Pure.elim 2*] = *allE notE' impE'*
**and** [*Pure.intro*] = *exI disjI2 disjI1*

**lemmas** [*trans*] = *trans*
**and** [*sym*] = *sym not-sym*
**and** [*Pure.elim?*] = *iffD1 iffD2 impE*

⟨*ML*⟩

### 1.2.17 Atomizing meta-level connectives

**axiomatization where**
*eq-reflection*: *x* = *y* ⟹ *x* ≡ *y*

**lemma** *atomize-all* [*atomize*]: (!!*x*. *P x*) == *Trueprop* (*ALL x*. *P x*)
⟨*proof*⟩

**lemma** *atomize-imp* [*atomize*]: (*A* ==> *B*) == *Trueprop* (*A* --> *B*)
⟨*proof*⟩

**lemma** *atomize-not*: (*A* ==> *False*) == *Trueprop* (~*A*)
⟨*proof*⟩

**lemma** *atomize-eq* [*atomize*]: (*x* == *y*) == *Trueprop* (*x* = *y*)
⟨*proof*⟩

**lemma** *atomize-conj* [*atomize*]: (*A* &&& *B*) == *Trueprop* (*A* & *B*)
⟨*proof*⟩

**lemmas** [*symmetric*, *rulify*] = *atomize-all atomize-imp*
**and** [*symmetric*, *defn*] = *atomize-all atomize-imp atomize-eq*

### 1.2.18 Atomizing elimination rules

⟨*ML*⟩

**lemma** *atomize-exL*[*atomize-elim*]: (!!*x*. *P x* ==> *Q*) == ((*EX x*. *P x*) ==> *Q*)
⟨*proof*⟩

**lemma** *atomize-conjL*[*atomize-elim*]: (*A* ==> *B* ==> *C*) == (*A* & *B* ==> *C*)
⟨*proof*⟩

**lemma** *atomize-disjL*[*atomize-elim*]: ((*A* ==> *C*) ==> (*B* ==> *C*) ==> *C*)
== ((*A* | *B* ==> *C*) ==> *C*)
⟨*proof*⟩

**lemma** *atomize-elimL*[*atomize-elim*]: (!!*B*. (*A* ==> *B*) ==> *B*) == *Trueprop A*
⟨*proof*⟩

## 1.3 Package setup

### 1.3.1 Classical Reasoner setup

**lemma** *imp-elim*: *P* --> *Q* ==> (~ *R* ==> *P*) ==> (*Q* ==> *R*) ==> *R*
⟨*proof*⟩

**lemma** *swap*: ~ *P* ==> (~ *R* ==> *P*) ==> *R*
⟨*proof*⟩

**lemma** *thin-refl*:
⋀*X*. ⟦ *x*=*x*; *PROP W* ⟧ ⟹ *PROP W* ⟨*proof*⟩

⟨*ML*⟩

ResBlacklist holds theorems blacklisted to sledgehammer. These theorems typically produce clauses that are prolific (match too many equality or membership literals) and relate to seldom-used facts. Some duplicate other rules.

⟨*ML*⟩

**declare** *iffI* [*intro*!]
**and** *notI* [*intro*!]
**and** *impI* [*intro*!]
**and** *disjCI* [*intro*!]
**and** *conjI* [*intro*!]
**and** *TrueI* [*intro*!]
**and** *refl* [*intro*!]

**declare** *iffCE* [*elim*!]
**and** *FalseE* [*elim*!]
**and** *impCE* [*elim*!]
**and** *disjE* [*elim*!]
**and** *conjE* [*elim*!]
**and** *conjE* [*elim*!]

**declare** *ex-ex1I* [*intro*!]
**and** *allI* [*intro*!]
**and** *the-equality* [*intro*]
**and** *exI* [*intro*]

**declare** *exE* [*elim*!]
*allE* [*elim*]

⟨*ML*⟩

**lemma** *contrapos-np*: $\sim Q \Longrightarrow (\sim P \Longrightarrow Q) \Longrightarrow P$
⟨*proof*⟩

**declare** *ex-ex1I* [*rule del*, *intro*! *2*]
**and** *ex1I* [*intro*]

**lemmas** [*intro?*] = *ext*
**and** [*elim?*] = *ex1-implies-ex*

**lemma** *alt-ex1E* [*elim*!]:
**assumes** *major*: $\exists!x.\ P\ x$
**and** *prem*: $\bigwedge x.\ [\![\ P\ x;\ \forall y\ y'.\ P\ y \wedge P\ y' \longrightarrow y = y'\ ]\!] \Longrightarrow R$
**shows** $R$
⟨*proof*⟩

⟨*ML*⟩

### 1.3.2 Simplifier

**lemma** *eta-contract-eq*: $(\%s.\ f\ s) = f$ ⟨*proof*⟩

**lemma** *simp-thms*:
**shows** *not-not*: $(\sim \sim P) = P$
**and** *Not-eq-iff*: $((\sim P) = (\sim Q)) = (P = Q)$
**and**
$(P \sim= Q) = (P = (\sim Q))$
$(P \mid \sim P) = True \quad (\sim P \mid P) = True$
$(x = x) = True$
**and** *not-True-eq-False*: $(\neg\ True) = False$
**and** *not-False-eq-True*: $(\neg\ False) = True$
**and**
$(\sim P) \sim= P \quad P \sim= (\sim P)$
$(True=P) = P$
**and** *eq-True*: $(P = True) = P$
**and** $(False=P) = (\sim P)$
**and** *eq-False*: $(P = False) = (\neg\ P)$
**and**
$(True \dashrightarrow P) = P \quad (False \dashrightarrow P) = True$
$(P \dashrightarrow True) = True \quad (P \dashrightarrow P) = True$
$(P \dashrightarrow False) = (\sim P) \quad (P \dashrightarrow \sim P) = (\sim P)$
$(P \ \&\ True) = P \quad (True \ \&\ P) = P$
$(P \ \&\ False) = False \quad (False \ \&\ P) = False$
$(P \ \&\ P) = P \quad (P \ \&\ (P \ \&\ Q)) = (P \ \&\ Q)$
$(P \ \&\ \sim P) = False \quad (\sim P \ \&\ P) = False$
$(P \mid True) = True \quad (True \mid P) = True$
$(P \mid False) = P \quad (False \mid P) = P$
$(P \mid P) = P \quad (P \mid (P \mid Q)) = (P \mid Q)$ **and**
$(ALL\ x.\ P) = P \quad (EX\ x.\ P) = P \quad EX\ x.\ x=t \quad EX\ x.\ t=x$

— needed for the one-point-rule quantifier simplification procs
— essential for termination!! **and**
!!$P$. $(EX\ x.\ x{=}t\ \&\ P(x)) = P(t)$
!!$P$. $(EX\ x.\ t{=}x\ \&\ P(x)) = P(t)$
!!$P$. $(ALL\ x.\ x{=}t\ {-}{-}{>}\ P(x)) = P(t)$
!!$P$. $(ALL\ x.\ t{=}x\ {-}{-}{>}\ P(x)) = P(t)$
⟨*proof*⟩

**lemma** *disj-absorb*: $(A \mid A) = A$
⟨*proof*⟩

**lemma** *disj-left-absorb*: $(A \mid (A \mid B)) = (A \mid B)$
⟨*proof*⟩

**lemma** *conj-absorb*: $(A\ \&\ A) = A$
⟨*proof*⟩

**lemma** *conj-left-absorb*: $(A\ \&\ (A\ \&\ B)) = (A\ \&\ B)$
⟨*proof*⟩

**lemma** *eq-ac*:
**shows** *eq-commute*: $(a{=}b) = (b{=}a)$
**and** *eq-left-commute*: $(P{=}(Q{=}R)) = (Q{=}(P{=}R))$
**and** *eq-assoc*: $((P{=}Q){=}R) = (P{=}(Q{=}R))$ ⟨*proof*⟩
**lemma** *neq-commute*: $(a^{\sim}{=}b) = (b^{\sim}{=}a)$ ⟨*proof*⟩

**lemma** *conj-comms*:
**shows** *conj-commute*: $(P\&Q) = (Q\&P)$
**and** *conj-left-commute*: $(P\&(Q\&R)) = (Q\&(P\&R))$ ⟨*proof*⟩
**lemma** *conj-assoc*: $((P\&Q)\&R) = (P\&(Q\&R))$ ⟨*proof*⟩

**lemmas** *conj-ac* = *conj-commute conj-left-commute conj-assoc*

**lemma** *disj-comms*:
**shows** *disj-commute*: $(P|Q) = (Q|P)$
**and** *disj-left-commute*: $(P|(Q|R)) = (Q|(P|R))$ ⟨*proof*⟩
**lemma** *disj-assoc*: $((P|Q)|R) = (P|(Q|R))$ ⟨*proof*⟩

**lemmas** *disj-ac* = *disj-commute disj-left-commute disj-assoc*

**lemma** *conj-disj-distribL*: $(P\&(Q|R)) = (P\&Q \mid P\&R)$ ⟨*proof*⟩
**lemma** *conj-disj-distribR*: $((P|Q)\&R) = (P\&R \mid Q\&R)$ ⟨*proof*⟩

**lemma** *disj-conj-distribL*: $(P|(Q\&R)) = ((P|Q)\ \&\ (P|R))$ ⟨*proof*⟩
**lemma** *disj-conj-distribR*: $((P\&Q)|R) = ((P|R)\ \&\ (Q|R))$ ⟨*proof*⟩

**lemma** *imp-conjR*: $(P\ {-}{-}{>}\ (Q\&R)) = ((P{-}{-}{>}Q)\ \&\ (P{-}{-}{>}R))$ ⟨*proof*⟩
**lemma** *imp-conjL*: $((P\&Q)\ {-}{-}{>}R)\ = (P\ {-}{-}{>}\ (Q\ {-}{-}{>}\ R))$ ⟨*proof*⟩
**lemma** *imp-disjL*: $((P|Q)\ {-}{-}{>}\ R) = ((P{-}{-}{>}R)\&(Q{-}{-}{>}R))$ ⟨*proof*⟩

These two are specialized, but *imp-disj-not1* is useful in *Auth/Yahalom*.

**lemma** *imp-disj-not1*: $(P \dashrightarrow Q \mid R) = (\sim Q \dashrightarrow P \dashrightarrow R)$ ⟨*proof*⟩
**lemma** *imp-disj-not2*: $(P \dashrightarrow Q \mid R) = (\sim R \dashrightarrow P \dashrightarrow Q)$ ⟨*proof*⟩

**lemma** *imp-disj1*: $((P\dashrightarrow Q)|R) = (P\dashrightarrow Q|R)$ ⟨*proof*⟩
**lemma** *imp-disj2*: $(Q|(P\dashrightarrow R)) = (P\dashrightarrow Q|R)$ ⟨*proof*⟩

**lemma** *imp-cong*: $(P = P') ==> (P' ==> (Q = Q')) ==> ((P \dashrightarrow Q) = (P' \dashrightarrow Q'))$
⟨*proof*⟩

**lemma** *de-Morgan-disj*: $(\sim(P \mid Q)) = (\sim P \,\&\, \sim Q)$ ⟨*proof*⟩
**lemma** *de-Morgan-conj*: $(\sim(P \,\&\, Q)) = (\sim P \mid \sim Q)$ ⟨*proof*⟩
**lemma** *not-imp*: $(\sim(P \dashrightarrow Q)) = (P \,\&\, \sim Q)$ ⟨*proof*⟩
**lemma** *not-iff*: $(P \tilde{=} Q) = (P = (\sim Q))$ ⟨*proof*⟩
**lemma** *disj-not1*: $(\sim P \mid Q) = (P \dashrightarrow Q)$ ⟨*proof*⟩
**lemma** *disj-not2*: $(P \mid \sim Q) = (Q \dashrightarrow P)$ — changes orientation :-(
⟨*proof*⟩
**lemma** *imp-conv-disj*: $(P \dashrightarrow Q) = ((\sim P) \mid Q)$ ⟨*proof*⟩

**lemma** *iff-conv-conj-imp*: $(P = Q) = ((P \dashrightarrow Q) \,\&\, (Q \dashrightarrow P))$ ⟨*proof*⟩

**lemma** *cases-simp*: $((P \dashrightarrow Q) \,\&\, (\sim P \dashrightarrow Q)) = Q$
— Avoids duplication of subgoals after *split-if*, when the true and false
— cases boil down to the same thing.
⟨*proof*⟩

**lemma** *not-all*: $(\sim (!\ x.\ P(x))) = (?\ x.\sim P(x))$ ⟨*proof*⟩
**lemma** *imp-all*: $((!\ x.\ P\ x) \dashrightarrow Q) = (?\ x.\ P\ x \dashrightarrow Q)$ ⟨*proof*⟩
**lemma** *not-ex*: $(\sim (?\ x.\ P(x))) = (!\ x.\sim P(x))$ ⟨*proof*⟩
**lemma** *imp-ex*: $((?\ x.\ P\ x) \dashrightarrow Q) = (!\ x.\ P\ x \dashrightarrow Q)$ ⟨*proof*⟩
**lemma** *all-not-ex*: $(ALL\ x.\ P\ x) = (\sim (EX\ x.\ \sim\ P\ x\ ))$ ⟨*proof*⟩

**declare** *All-def* [*noatp*]

**lemma** *ex-disj-distrib*: $(?\ x.\ P(x) \mid Q(x)) = ((?\ x.\ P(x)) \mid (?\ x.\ Q(x)))$ ⟨*proof*⟩
**lemma** *all-conj-distrib*: $(!x.\ P(x) \,\&\, Q(x)) = ((!\ x.\ P(x)) \,\&\, (!\ x.\ Q(x)))$ ⟨*proof*⟩

The & congruence rule: not included by default! May slow rewrite proofs down by as much as 50%

**lemma** *conj-cong*:
$(P = P') ==> (P' ==> (Q = Q')) ==> ((P \,\&\, Q) = (P' \,\&\, Q'))$
⟨*proof*⟩

**lemma** *rev-conj-cong*:
$(Q = Q') ==> (Q' ==> (P = P')) ==> ((P \,\&\, Q) = (P' \,\&\, Q'))$
⟨*proof*⟩

The | congruence rule: not included by default!

**lemma** *disj-cong*:
  $(P = P') ==> (\sim P' ==> (Q = Q')) ==> ((P \mid Q) = (P' \mid Q'))$
⟨*proof*⟩

if-then-else rules

**lemma** *if-True*: $(if\ True\ then\ x\ else\ y) = x$
⟨*proof*⟩

**lemma** *if-False*: $(if\ False\ then\ x\ else\ y) = y$
⟨*proof*⟩

**lemma** *if-P*: $P ==> (if\ P\ then\ x\ else\ y) = x$
⟨*proof*⟩

**lemma** *if-not-P*: $\sim P ==> (if\ P\ then\ x\ else\ y) = y$
⟨*proof*⟩

**lemma** *split-if*: $P\ (if\ Q\ then\ x\ else\ y) = ((Q --> P(x)) \&\ (\sim Q --> P(y)))$
⟨*proof*⟩

**lemma** *split-if-asm*: $P\ (if\ Q\ then\ x\ else\ y) = (\sim((Q \&\ \sim P\ x) \mid (\sim Q \&\ \sim P\ y)))$
⟨*proof*⟩

**lemmas** *if-splits* [*noatp*] = *split-if split-if-asm*

**lemma** *if-cancel*: $(if\ c\ then\ x\ else\ x) = x$
⟨*proof*⟩

**lemma** *if-eq-cancel*: $(if\ x = y\ then\ y\ else\ x) = x$
⟨*proof*⟩

**lemma** *if-bool-eq-conj*: $(if\ P\ then\ Q\ else\ R) = ((P-->Q) \&\ (\sim P-->R))$
  — This form is useful for expanding *if*s on the RIGHT of the ==> symbol.
⟨*proof*⟩

**lemma** *if-bool-eq-disj*: $(if\ P\ then\ Q\ else\ R) = ((P\&Q) \mid (\sim P\&R))$
  — And this form is useful for expanding *if*s on the LEFT.
⟨*proof*⟩

**lemma** *Eq-TrueI*: $P ==> P == True$ ⟨*proof*⟩
**lemma** *Eq-FalseI*: $\sim P ==> P == False$ ⟨*proof*⟩

let rules for simproc

**lemma** *Let-folded*: $f\ x \equiv g\ x \Longrightarrow Let\ x\ f \equiv Let\ x\ g$
⟨*proof*⟩

**lemma** *Let-unfold*: $f\ x \equiv g \Longrightarrow Let\ x\ f \equiv g$

⟨*proof*⟩

The following copy of the implication operator is useful for fine-tuning congruence rules. It instructs the simplifier to simplify its premise.

**constdefs**
*simp-implies* :: [*prop*, *prop*] => *prop* (**infixr** =*simp*=> *1*)
[*code del*]: *simp-implies* ≡ *op* ==>

**lemma** *simp-impliesI*:
**assumes** *PQ*: (*PROP P* ⟹ *PROP Q*)
**shows** *PROP P* =*simp*=> *PROP Q*
⟨*proof*⟩

**lemma** *simp-impliesE*:
**assumes** *PQ*: *PROP P* =*simp*=> *PROP Q*
**and** *P*: *PROP P*
**and** *QR*: *PROP Q* ⟹ *PROP R*
**shows** *PROP R*
⟨*proof*⟩

**lemma** *simp-implies-cong*:
**assumes** *PP'* :*PROP P* == *PROP P'*
**and** *P'QQ'*: *PROP P'* ==> (*PROP Q* == *PROP Q'*)
**shows** (*PROP P* =*simp*=> *PROP Q*) == (*PROP P'* =*simp*=> *PROP Q'*)
⟨*proof*⟩

**lemma** *uncurry*:
**assumes** *P* ⟶ *Q* ⟶ *R*
**shows** *P* ∧ *Q* ⟶ *R*
⟨*proof*⟩

**lemma** *iff-allI*:
**assumes** ⋀*x*. *P x* = *Q x*
**shows** (∀ *x*. *P x*) = (∀ *x*. *Q x*)
⟨*proof*⟩

**lemma** *iff-exI*:
**assumes** ⋀*x*. *P x* = *Q x*
**shows** (∃ *x*. *P x*) = (∃ *x*. *Q x*)
⟨*proof*⟩

**lemma** *all-comm*:
(∀ *x y*. *P x y*) = (∀ *y x*. *P x y*)
⟨*proof*⟩

**lemma** *ex-comm*:
(∃ *x y*. *P x y*) = (∃ *y x*. *P x y*)
⟨*proof*⟩

⟨*ML*⟩

Simproc for proving $(y = x) == False$ from premise $\sim(x = y)$:

⟨*ML*⟩

**lemma** *True-implies-equals*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

**lemma** *ex-simps*:
$!!P\ Q.\ (EX\ x.\ P\ x\ \&\ Q) = ((EX\ x.\ P\ x)\ \&\ Q)$
$!!P\ Q.\ (EX\ x.\ P\ \&\ Q\ x) = (P\ \&\ (EX\ x.\ Q\ x))$
$!!P\ Q.\ (EX\ x.\ P\ x \mid Q) = ((EX\ x.\ P\ x) \mid Q)$
$!!P\ Q.\ (EX\ x.\ P \mid Q\ x) = (P \mid (EX\ x.\ Q\ x))$
$!!P\ Q.\ (EX\ x.\ P\ x \longrightarrow Q) = ((ALL\ x.\ P\ x) \longrightarrow Q)$
$!!P\ Q.\ (EX\ x.\ P \longrightarrow Q\ x) = (P \longrightarrow (EX\ x.\ Q\ x))$
— Miniscoping: pushing in existential quantifiers.
⟨*proof*⟩

**lemma** *all-simps*:
$!!P\ Q.\ (ALL\ x.\ P\ x\ \&\ Q) = ((ALL\ x.\ P\ x)\ \&\ Q)$
$!!P\ Q.\ (ALL\ x.\ P\ \&\ Q\ x) = (P\ \&\ (ALL\ x.\ Q\ x))$
$!!P\ Q.\ (ALL\ x.\ P\ x \mid Q) = ((ALL\ x.\ P\ x) \mid Q)$
$!!P\ Q.\ (ALL\ x.\ P \mid Q\ x) = (P \mid (ALL\ x.\ Q\ x))$
$!!P\ Q.\ (ALL\ x.\ P\ x \longrightarrow Q) = ((EX\ x.\ P\ x) \longrightarrow Q)$
$!!P\ Q.\ (ALL\ x.\ P \longrightarrow Q\ x) = (P \longrightarrow (ALL\ x.\ Q\ x))$
— Miniscoping: pushing in universal quantifiers.
⟨*proof*⟩

**lemmas** [*simp*] =
*triv-forall-equality*
*True-implies-equals*
*if-True*
*if-False*
*if-cancel*
*if-eq-cancel*
*imp-disjL*

*conj-assoc*
*disj-assoc*
*de-Morgan-conj*
*de-Morgan-disj*
*imp-disj1*
*imp-disj2*
*not-imp*
*disj-not1*
*not-all*
*not-ex*
*cases-simp*
*the-eq-trivial*

*the-sym-eq-trivial*
*ex-simps*
*all-simps*
*simp-thms*

**lemmas** [*cong*] = *imp-cong simp-implies-cong*
**lemmas** [*split*] = *split-if*

⟨*ML*⟩

Simplifies x assuming c and y assuming c

**lemma** *if-cong*:
**assumes** $b = c$
**and** $c \Longrightarrow x = u$
**and** $\neg\ c \Longrightarrow y = v$
**shows** (*if b then x else y*) = (*if c then u else v*)
⟨*proof*⟩

Prevents simplification of x and y: faster and allows the execution of functional programs.

**lemma** *if-weak-cong* [*cong*]:
**assumes** $b = c$
**shows** (*if b then x else y*) = (*if c then x else y*)
⟨*proof*⟩

Prevents simplification of t: much faster

**lemma** *let-weak-cong*:
**assumes** $a = b$
**shows** (*let x = a in t x*) = (*let x = b in t x*)
⟨*proof*⟩

To tidy up the result of a simproc. Only the RHS will be simplified.

**lemma** *eq-cong2*:
**assumes** $u = u'$
**shows** $(t \equiv u) \equiv (t \equiv u')$
⟨*proof*⟩

**lemma** *if-distrib*:
*f* (*if c then x else y*) = (*if c then f x else f y*)
⟨*proof*⟩

This lemma restricts the effect of the rewrite rule u=v to the left-hand side of an equality. Used in {*Integ*,*Real*}/*simproc.ML*

**lemma** *restrict-to-left*:
**assumes** $x = y$
**shows** $(x = z) = (y = z)$
⟨*proof*⟩

### 1.3.3 Generic cases and induction

Rule projections:

⟨*ML*⟩

**constdefs**
*induct-forall* **where** *induct-forall P* == ∀ *x*. *P x*
*induct-implies* **where** *induct-implies A B* == *A* ⟶ *B*
*induct-equal* **where** *induct-equal x y* == *x* = *y*
*induct-conj* **where** *induct-conj A B* == *A* ∧ *B*

**lemma** *induct-forall-eq*: (!!*x*. *P x*) == *Trueprop* (*induct-forall* (λ*x*. *P x*))
⟨*proof*⟩

**lemma** *induct-implies-eq*: (*A* ==> *B*) == *Trueprop* (*induct-implies A B*)
⟨*proof*⟩

**lemma** *induct-equal-eq*: (*x* == *y*) == *Trueprop* (*induct-equal x y*)
⟨*proof*⟩

**lemma** *induct-conj-eq*: (*A* &&& *B*) == *Trueprop* (*induct-conj A B*)
⟨*proof*⟩

**lemmas** *induct-atomize* = *induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq*
**lemmas** *induct-rulify* [*symmetric*, *standard*] = *induct-atomize*
**lemmas** *induct-rulify-fallback* =
*induct-forall-def induct-implies-def induct-equal-def induct-conj-def*

**lemma** *induct-forall-conj*: *induct-forall* (λ*x*. *induct-conj* (*A x*) (*B x*)) =
*induct-conj* (*induct-forall A*) (*induct-forall B*)
⟨*proof*⟩

**lemma** *induct-implies-conj*: *induct-implies C* (*induct-conj A B*) =
*induct-conj* (*induct-implies C A*) (*induct-implies C B*)
⟨*proof*⟩

**lemma** *induct-conj-curry*: (*induct-conj A B* ==> *PROP C*) == (*A* ==> *B* ==> *PROP C*)
⟨*proof*⟩

**lemmas** *induct-conj* = *induct-forall-conj induct-implies-conj induct-conj-curry*

**hide** *const induct-forall induct-implies induct-equal induct-conj*

Method setup.

⟨*ML*⟩

### 1.3.4 Coherent logic

⟨*ML*⟩

## 1.4 Other simple lemmas and lemma duplicates

**lemma** *Let-0* [*simp*]: *Let 0 f = f 0*
⟨*proof*⟩

**lemma** *Let-1* [*simp*]: *Let 1 f = f 1*
⟨*proof*⟩

**lemma** *ex1-eq* [*iff*]: *EX! x. x = t EX! x. t = x*
⟨*proof*⟩

**lemma** *choice-eq*: *(ALL x. EX! y. P x y) = (EX! f. ALL x. P x (f x))*
⟨*proof*⟩

**lemma** *mk-left-commute*:
**fixes** *f* (**infix** ⊗ *60*)
**assumes** *a*: ⋀*x y z*. *(x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)* **and**
*c*: ⋀*x y*. *x ⊗ y = y ⊗ x*
**shows** *x ⊗ (y ⊗ z) = y ⊗ (x ⊗ z)*
⟨*proof*⟩

**lemmas** *eq-sym-conv = eq-commute*

**lemma** *nnf-simps*:
*(¬(P ∧ Q)) = (¬ P ∨ ¬ Q) (¬ (P ∨ Q)) = (¬ P ∧ ¬Q) (P ⟶ Q) = (¬P ∨ Q)*
*(P = Q) = ((P ∧ Q) ∨ (¬P ∧ ¬ Q)) (¬(P = Q)) = ((P ∧ ¬ Q) ∨ (¬P ∧ Q))*
*(¬ ¬(P)) = P*
⟨*proof*⟩

## 1.5 Basic ML bindings

⟨*ML*⟩

## 1.6 Code generator basics – see further theory *Code-Setup*

Equality

**class** *eq* =
**fixes** *eq* :: *'a ⇒ 'a ⇒ bool*
**assumes** *eq-equals*: *eq x y ⟷ x = y*
**begin**

**lemma** *eq*: *eq = (op =)*
⟨*proof*⟩

**lemma** *eq-refl*: *eq x x* $\longleftrightarrow$ *True*
⟨*proof*⟩

**end**

Module setup

⟨*ML*⟩

### 1.7 Nitpick hooks

This will be relocated once Nitpick is moved to HOL.

⟨*ML*⟩

### 1.8 Legacy tactics and ML bindings

⟨*ML*⟩

**end**

## 2 Code-Setup: Setup of code generators and related tools

**theory** *Code-Setup*
**imports** *HOL*
**begin**

### 2.1 Generic code generator foundation

Datatypes

**code-datatype** *True False*

**code-datatype** *TYPE*('*a*::{})

**code-datatype** *Trueprop prop*

Code equations

**lemma** [*code*]:
**shows** (*True* $\Longrightarrow$ *PROP P*) $\equiv$ *PROP P*
**and** (*False* $\Longrightarrow$ *Q*) $\equiv$ *Trueprop True*
**and** (*PROP P* $\Longrightarrow$ *True*) $\equiv$ *Trueprop True*
**and** (*Q* $\Longrightarrow$ *False*) $\equiv$ *Trueprop* ($\neg$ *Q*) ⟨*proof*⟩

**lemma** [*code*]:
**shows** *False* $\wedge$ *x* $\longleftrightarrow$ *False*
**and** *True* $\wedge$ *x* $\longleftrightarrow$ *x*
**and** *x* $\wedge$ *False* $\longleftrightarrow$ *False*

**and** $x \wedge \mathit{True} \longleftrightarrow x$ ⟨*proof*⟩

**lemma** [*code*]:
**shows** $\mathit{False} \vee x \longleftrightarrow x$
**and** $\mathit{True} \vee x \longleftrightarrow \mathit{True}$
**and** $x \vee \mathit{False} \longleftrightarrow x$
**and** $x \vee \mathit{True} \longleftrightarrow \mathit{True}$ ⟨*proof*⟩

**lemma** [*code*]:
**shows** $\neg\ \mathit{True} \longleftrightarrow \mathit{False}$
**and** $\neg\ \mathit{False} \longleftrightarrow \mathit{True}$ ⟨*proof*⟩

**lemmas** [*code*] = *Let-def if-True if-False*

**lemmas** [*code*, *code unfold*, *symmetric*, *code post*] = *imp-conv-disj*

Equality

**context** *eq*
**begin**

**lemma** *equals-eq* [*code inline*, *code*]: $op = \equiv eq$
⟨*proof*⟩

**declare** *eq* [*code unfold*, *code inline del*]

**declare** *equals-eq* [*symmetric*, *code post*]

**end**

**declare** *simp-thms*(*6*) [*code nbe*]

**hide** (**open**) *const eq*
**hide** *const eq*

⟨*ML*⟩

Cases

**lemma** *Let-case-cert*:
**assumes** $\mathit{CASE} \equiv (\lambda x.\ \mathit{Let}\ x\ f)$
**shows** $\mathit{CASE}\ x \equiv f\ x$
⟨*proof*⟩

**lemma** *If-case-cert*:
**assumes** $\mathit{CASE} \equiv (\lambda b.\ \mathit{If}\ b\ f\ g)$
**shows** $(\mathit{CASE}\ \mathit{True} \equiv f)$ &&& $(\mathit{CASE}\ \mathit{False} \equiv g)$
⟨*proof*⟩

⟨*ML*⟩

**code-abort** *undefined*

## 2.2 Generic code generator preprocessor

⟨*ML*⟩

## 2.3 Generic code generator target languages

type bool

**code-type** *bool*
(*SML bool*)
(*OCaml bool*)
(*Haskell Bool*)

**code-const** *True* **and** *False* **and** *Not* **and** *op &* **and** *op |* **and** *If*
(*SML true* **and** *false* **and** *not*
**and infixl** *1 andalso* **and infixl** *0 orelse*
**and** !(*if* (-)/ *then* (-)/ *else* (-)))
(*OCaml true* **and** *false* **and** *not*
**and infixl** *4* && **and infixl** *2* ||
**and** !(*if* (-)/ *then* (-)/ *else* (-)))
(*Haskell True* **and** *False* **and** *not*
**and infixl** *3* && **and infixl** *2* ||
**and** !(*if* (-)/ *then* (-)/ *else* (-)))

**code-reserved** *SML*
*bool true false not*

**code-reserved** *OCaml*
*bool not*

using built-in Haskell equality

**code-class** *eq*
(*Haskell Eq*)

**code-const** *eq-class.eq*
(*Haskell* **infixl** *4* ==)

**code-const** *op =*
(*Haskell* **infixl** *4* ==)

undefined

**code-const** *undefined*
(*SML* !(*raise*/ *Fail*/ *undefined*))
(*OCaml failwith*/ *undefined*)
(*Haskell error*/ *undefined*)

### 2.4 SML code generator setup

**types-code**
*bool* (*bool*)
**attach** (*term-of*) ⟪
*fun term-of-bool b = if b then HOLogic.true-const else HOLogic.false-const*;
⟫
**attach** (*test*) ⟪
*fun gen-bool i =*
*let val b = one-of* [*false*, *true*]
*in* (*b*, *fn* () => *term-of-bool b*) *end*;
⟫
*prop* (*bool*)
**attach** (*term-of*) ⟪
*fun term-of-prop b =*
*HOLogic.mk-Trueprop* (*if b then HOLogic.true-const else HOLogic.false-const*);
⟫

**consts-code**
*Trueprop* ((-))
*True* (*true*)
*False* (*false*)
*Not* (*Bool.not*)
*op |* ((- *orelse*/ -))
*op &* ((- *andalso*/ -))
*If* ((*if* -/ *then* -/ *else* -))

⟨*ML*⟩

### 2.5 Evaluation and normalization by evaluation

⟨*ML*⟩

### 2.6 Quickcheck

⟨*ML*⟩

**quickcheck-params** [*size* = *5*, *iterations* = *50*]

**end**

## 3 Orderings: Abstract orderings

**theory** *Orderings*
**imports** *Code-Setup*
**uses** ~~/*src*/*Provers*/*order.ML*
**begin**

## 3.1 Quasi orders

**class** *preorder* = *ord* +
  **assumes** *less-le-not-le*: $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$
  **and** *order-refl* [*iff*]: $x \leq x$
  **and** *order-trans*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
**begin**

Reflexivity.

**lemma** *eq-refl*: $x = y \Longrightarrow x \leq y$
  — This form is useful with the classical reasoner.
⟨*proof*⟩

**lemma** *less-irrefl* [*iff*]: $\neg\ x < x$
⟨*proof*⟩

**lemma** *less-imp-le*: $x < y \Longrightarrow x \leq y$
⟨*proof*⟩

Asymmetry.

**lemma** *less-not-sym*: $x < y \Longrightarrow \neg\ (y < x)$
⟨*proof*⟩

**lemma** *less-asym*: $x < y \Longrightarrow (\neg\ P \Longrightarrow y < x) \Longrightarrow P$
⟨*proof*⟩

Transitivity.

**lemma** *less-trans*: $x < y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

**lemma** *le-less-trans*: $x \leq y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

**lemma** *less-le-trans*: $x < y \Longrightarrow y \leq z \Longrightarrow x < z$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

**lemma** *less-imp-not-less*: $x < y \Longrightarrow (\neg\ y < x) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

**lemma** *less-imp-triv*: $x < y \Longrightarrow (y < x \longrightarrow P) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

**lemma** *less-asym'*: $a < b \Longrightarrow b < a \Longrightarrow P$
⟨*proof*⟩

Dual order

**lemma** *dual-preorder*:
*preorder* $(op \geq)$ $(op >)$
⟨*proof*⟩

**end**

## 3.2 Partial orders

**class** *order* = *preorder* +
**assumes** *antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
**begin**

Reflexivity.

**lemma** *less-le*: $x < y \longleftrightarrow x \leq y \wedge x \neq y$
⟨*proof*⟩

**lemma** *le-less*: $x \leq y \longleftrightarrow x < y \vee x = y$
— NOT suitable for iff, since it can cause PROOF FAILED.
⟨*proof*⟩

**lemma** *le-imp-less-or-eq*: $x \leq y \Longrightarrow x < y \vee x = y$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

**lemma** *less-imp-not-eq*: $x < y \Longrightarrow (x = y) \longleftrightarrow \mathit{False}$
⟨*proof*⟩

**lemma** *less-imp-not-eq2*: $x < y \Longrightarrow (y = x) \longleftrightarrow \mathit{False}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

**lemma** *neq-le-trans*: $a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$
⟨*proof*⟩

**lemma** *le-neq-trans*: $a \leq b \Longrightarrow a \neq b \Longrightarrow a < b$
⟨*proof*⟩

Asymmetry.

**lemma** *eq-iff*: $x = y \longleftrightarrow x \leq y \wedge y \leq x$
⟨*proof*⟩

**lemma** *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *less-imp-neq*: $x < y \Longrightarrow x \neq y$
⟨*proof*⟩

Least value operator

**definition** (**in** *ord*)
*Least* :: (*'a* ⇒ *bool*) ⇒ *'a* (**binder** *LEAST* *10*) **where**
*Least P* = (*THE x*. *P x* ∧ (∀ *y*. *P y* ⟶ *x* ≤ *y*))

**lemma** *Least-equality*:
**assumes** *P x*
**and** ⋀*y*. *P y* ⟹ *x* ≤ *y*
**shows** *Least P* = *x*
⟨*proof*⟩

**lemma** *LeastI2-order*:
**assumes** *P x*
**and** ⋀*y*. *P y* ⟹ *x* ≤ *y*
**and** ⋀*x*. *P x* ⟹ ∀ *y*. *P y* ⟶ *x* ≤ *y* ⟹ *Q x*
**shows** *Q* (*Least P*)
⟨*proof*⟩

Dual order

**lemma** *dual-order*:
*order* (*op* ≥) (*op* >)
⟨*proof*⟩

**end**

## 3.3 Linear (total) orders

**class** *linorder* = *order* +
**assumes** *linear*: *x* ≤ *y* ∨ *y* ≤ *x*
**begin**

**lemma** *less-linear*: *x* < *y* ∨ *x* = *y* ∨ *y* < *x*
⟨*proof*⟩

**lemma** *le-less-linear*: *x* ≤ *y* ∨ *y* < *x*
⟨*proof*⟩

**lemma** *le-cases* [*case-names le ge*]:
(*x* ≤ *y* ⟹ *P*) ⟹ (*y* ≤ *x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

**lemma** *linorder-cases* [*case-names less equal greater*]:
(*x* < *y* ⟹ *P*) ⟹ (*x* = *y* ⟹ *P*) ⟹ (*y* < *x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

**lemma** *not-less*: ¬ *x* < *y* ⟷ *y* ≤ *x*
⟨*proof*⟩

**lemma** *not-less-iff-gr-or-eq*:
¬(*x* < *y*) ⟷ (*x* > *y* | *x* = *y*)

⟨*proof*⟩

**lemma** *not-le*: $\neg\ x \leq y \longleftrightarrow y < x$
⟨*proof*⟩

**lemma** *neq-iff*: $x \neq y \longleftrightarrow x < y \vee y < x$
⟨*proof*⟩

**lemma** *neqE*: $x \neq y \Longrightarrow (x < y \Longrightarrow R) \Longrightarrow (y < x \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

**lemma** *antisym-conv1*: $\neg\ x < y \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *antisym-conv2*: $x \leq y \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *antisym-conv3*: $\neg\ y < x \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *leI*: $\neg\ x < y \Longrightarrow y \leq x$
⟨*proof*⟩

**lemma** *leD*: $y \leq x \Longrightarrow \neg\ x < y$
⟨*proof*⟩

**lemma** *not-leE*: $\neg\ y \leq x \Longrightarrow x < y$
⟨*proof*⟩

Dual order

**lemma** *dual-linorder*:
  *linorder* $(op \geq)$ $(op >)$
⟨*proof*⟩

min/max

**definition** (**in** *ord*) *min* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
  [*code del*]: *min a b* = (*if a* $\leq$ *b then a else b*)

**definition** (**in** *ord*) *max* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
  [*code del*]: *max a b* = (*if a* $\leq$ *b then b else a*)

**lemma** *min-le-iff-disj*:
  *min x y* $\leq z \longleftrightarrow x \leq z \vee y \leq z$
⟨*proof*⟩

**lemma** *le-max-iff-disj*:
  $z \leq$ *max x y* $\longleftrightarrow z \leq x \vee z \leq y$
⟨*proof*⟩

**lemma** *min-less-iff-disj*:
$min\ x\ y < z \longleftrightarrow x < z \vee y < z$
⟨*proof*⟩

**lemma** *less-max-iff-disj*:
$z < max\ x\ y \longleftrightarrow z < x \vee z < y$
⟨*proof*⟩

**lemma** *min-less-iff-conj* [*simp*]:
$z < min\ x\ y \longleftrightarrow z < x \wedge z < y$
⟨*proof*⟩

**lemma** *max-less-iff-conj* [*simp*]:
$max\ x\ y < z \longleftrightarrow x < z \wedge y < z$
⟨*proof*⟩

**lemma** *split-min* [*noatp*]:
$P\ (min\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ i) \wedge (\neg\ i \leq j \longrightarrow P\ j)$
⟨*proof*⟩

**lemma** *split-max* [*noatp*]:
$P\ (max\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ j) \wedge (\neg\ i \leq j \longrightarrow P\ i)$
⟨*proof*⟩

**end**

Explicit dictionaries for code generation

**lemma** *min-ord-min* [*code*, *code unfold*, *code inline del*]:
$min = ord.min\ (op \leq)$
⟨*proof*⟩

**declare** *ord.min-def* [*code*]

**lemma** *max-ord-max* [*code*, *code unfold*, *code inline del*]:
$max = ord.max\ (op \leq)$
⟨*proof*⟩

**declare** *ord.max-def* [*code*]

## 3.4 Reasoning tools setup

⟨*ML*⟩

Declarations to set up transitivity reasoner of partial and linear orders.

**context** *order*
**begin**

**declare** *less-irrefl* [*THEN notE*, *order add less-reflE*: *order op* = :: *'a* ⇒ *'a* ⇒ *bool op* <= *op* <]

**declare** *order-refl* [*order add le-refl*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-imp-le* [*order add less-imp-le*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *antisym* [*order add eqI*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *eq-refl* [*order add eqD1*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *sym* [*THEN eq-refl*, *order add eqD2*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-trans* [*order add less-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-le-trans* [*order add less-le-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *le-less-trans* [*order add le-less-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *order-trans* [*order add le-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *le-neq-trans* [*order add le-neq-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *neq-le-trans* [*order add neq-le-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-imp-neq* [*order add less-imp-neq*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *eq-neq-eq-imp-neq* [*order add eq-neq-eq-imp-neq*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *not-sym* [*order add not-sym*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

**end**

**context** *linorder*
**begin**

**declare** [[*order del*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]]

**declare** *less-irrefl* [*THEN notE*, *order add less-reflE*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *order-refl* [*order add le-refl*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-imp-le* [*order add less-imp-le*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *not-less* [*THEN iffD2*, *order add not-lessI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *not-le* [*THEN iffD2*, *order add not-leI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *not-less* [*THEN iffD1*, *order add not-lessD*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *not-le* [*THEN iffD1*, *order add not-leD*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *antisym* [*order add eqI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *eq-refl* [*order add eqD1*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *sym* [*THEN eq-refl*, *order add eqD2*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-trans* [*order add less-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-le-trans* [*order add less-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *le-less-trans* [*order add le-less-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *order-trans* [*order add le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *le-neq-trans* [*order add le-neq-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *neq-le-trans* [*order add neq-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *less-imp-neq* [*order add less-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *eq-neq-eq-imp-neq* [*order add eq-neq-eq-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**declare** *not-sym* [*order add not-sym*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

**end**

⟨*ML*⟩

## 3.5 Name duplicates

**lemmas** *order-less-le* = *less-le*
**lemmas** *order-eq-refl* = *preorder-class.eq-refl*
**lemmas** *order-less-irrefl* = *preorder-class.less-irrefl*
**lemmas** *order-le-less* = *order-class.le-less*
**lemmas** *order-le-imp-less-or-eq* = *order-class.le-imp-less-or-eq*
**lemmas** *order-less-imp-le* = *preorder-class.less-imp-le*
**lemmas** *order-less-imp-not-eq* = *order-class.less-imp-not-eq*
**lemmas** *order-less-imp-not-eq2* = *order-class.less-imp-not-eq2*
**lemmas** *order-neq-le-trans* = *order-class.neq-le-trans*
**lemmas** *order-le-neq-trans* = *order-class.le-neq-trans*

**lemmas** *order-antisym* = *antisym*
**lemmas** *order-less-not-sym* = *preorder-class.less-not-sym*
**lemmas** *order-less-asym* = *preorder-class.less-asym*
**lemmas** *order-eq-iff* = *order-class.eq-iff*
**lemmas** *order-antisym-conv* = *order-class.antisym-conv*
**lemmas** *order-less-trans* = *preorder-class.less-trans*
**lemmas** *order-le-less-trans* = *preorder-class.le-less-trans*
**lemmas** *order-less-le-trans* = *preorder-class.less-le-trans*
**lemmas** *order-less-imp-not-less* = *preorder-class.less-imp-not-less*
**lemmas** *order-less-imp-triv* = *preorder-class.less-imp-triv*
**lemmas** *order-less-asym'* = *preorder-class.less-asym'*

**lemmas** *linorder-linear* = *linear*
**lemmas** *linorder-less-linear* = *linorder-class.less-linear*
**lemmas** *linorder-le-less-linear* = *linorder-class.le-less-linear*
**lemmas** *linorder-le-cases* = *linorder-class.le-cases*
**lemmas** *linorder-not-less* = *linorder-class.not-less*
**lemmas** *linorder-not-le* = *linorder-class.not-le*
**lemmas** *linorder-neq-iff* = *linorder-class.neq-iff*
**lemmas** *linorder-neqE* = *linorder-class.neqE*
**lemmas** *linorder-antisym-conv1* = *linorder-class.antisym-conv1*
**lemmas** *linorder-antisym-conv2* = *linorder-class.antisym-conv2*
**lemmas** *linorder-antisym-conv3* = *linorder-class.antisym-conv3*

## 3.6 Bounded quantifiers

**syntax**
*-All-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
*-All-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)

*-All-greater* :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-greater* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->-./ -*) [*0*, *0*, *10*] *10*)
*-All-greater-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->=-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-greater-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->=-./ -*) [*0*, *0*, *10*] *10*)

**syntax** (*xsymbols*)
*-All-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-<-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-<-./ -*) [*0*, *0*, *10*] *10*)
*-All-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≤*-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≤*-./ -*) [*0*, *0*, *10*] *10*)

*-All-greater* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*->-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-greater* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*->-./ -*) [*0*, *0*, *10*] *10*)
*-All-greater-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≥*-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-greater-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≥*-./ -*) [*0*, *0*, *10*] *10*)

**syntax** (*HOL*)
*-All-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<-./ -*) [*0*, *0*, *10*] *10*)
*-All-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<=-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<=-./ -*) [*0*, *0*, *10*] *10*)

**syntax** (*HTML* **output**)
*-All-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-<-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-<-./ -*) [*0*, *0*, *10*] *10*)
*-All-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≤*-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-less-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≤*-./ -*) [*0*, *0*, *10*] *10*)

*-All-greater* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*->-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-greater* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*->-./ -*) [*0*, *0*, *10*] *10*)
*-All-greater-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≥*-./ -*) [*0*, *0*, *10*] *10*)
*-Ex-greater-eq* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≥*-./ -*) [*0*, *0*, *10*] *10*)

**translations**
*ALL x<y. P* => *ALL x. x < y* ⟶ *P*
*EX x<y. P* => *EX x. x < y* ∧ *P*
*ALL x<=y. P* => *ALL x. x <= y* ⟶ *P*
*EX x<=y. P* => *EX x. x <= y* ∧ *P*
*ALL x>y. P* => *ALL x. x > y* ⟶ *P*
*EX x>y. P* => *EX x. x > y* ∧ *P*
*ALL x>=y. P* => *ALL x. x >= y* ⟶ *P*
*EX x>=y. P* => *EX x. x >= y* ∧ *P*

⟨*ML*⟩

## 3.7 Transitivity reasoning

**context** *ord*
**begin**

**lemma** *ord-le-eq-trans*: $a \leq b \Longrightarrow b = c \Longrightarrow a \leq c$
⟨*proof*⟩

**lemma** *ord-eq-le-trans*: $a = b \Longrightarrow b \leq c \Longrightarrow a \leq c$
⟨*proof*⟩

**lemma** *ord-less-eq-trans*: $a < b \Longrightarrow b = c \Longrightarrow a < c$
⟨*proof*⟩

**lemma** *ord-eq-less-trans*: $a = b \Longrightarrow b < c \Longrightarrow a < c$
⟨*proof*⟩

**end**

**lemma** *order-less-subst2*: $(a::'a::order) < b ==> f\ b < (c::'c::order) ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

**lemma** *order-less-subst1*: $(a::'a::order) < f\ b ==> (b::'b::order) < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

**lemma** *order-le-less-subst2*: $(a::'a::order) <= b ==> f\ b < (c::'c::order) ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a < c$
⟨*proof*⟩

**lemma** *order-le-less-subst1*: $(a::'a::order) <= f\ b ==> (b::'b::order) < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

**lemma** *order-less-le-subst2*: $(a::'a::order) < b ==> f\ b <= (c::'c::order) ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

**lemma** *order-less-le-subst1*: $(a::'a::order) < f\ b ==> (b::'b::order) <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a < f\ c$
⟨*proof*⟩

**lemma** *order-subst1*: $(a::'a::order) <= f\ b ==> (b::'b::order) <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a <= f\ c$
⟨*proof*⟩

**lemma** *order-subst2*: $(a::'a::order) <= b ==> f\ b <= (c::'c::order) ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a <= c$
⟨*proof*⟩

**lemma** *ord-le-eq-subst*: $a <= b ==> f\ b = c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a <= c$
⟨*proof*⟩

**lemma** *ord-eq-le-subst*: $a = f\ b ==> b <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a <= f\ c$
⟨*proof*⟩

**lemma** *ord-less-eq-subst*: $a < b ==> f\ b = c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

**lemma** *ord-eq-less-subst*: $a = f\ b ==> b < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

Note that this list of rules is in reverse order of priorities.

**lemmas** [*trans*] =
*order-less-subst2*
*order-less-subst1*
*order-le-less-subst2*
*order-le-less-subst1*
*order-less-le-subst2*
*order-less-le-subst1*
*order-subst2*
*order-subst1*
*ord-le-eq-subst*
*ord-eq-le-subst*
*ord-less-eq-subst*
*ord-eq-less-subst*
*forw-subst*
*back-subst*
*rev-mp*
*mp*

**lemmas** (**in** *order*) [*trans*] =
*neq-le-trans*
*le-neq-trans*

**lemmas** (**in** *preorder*) [*trans*] =
*less-trans*
*less-asym'*
*le-less-trans*
*less-le-trans*

*order-trans*

**lemmas** (**in** *order*) [*trans*] =
*antisym*

**lemmas** (**in** *ord*) [*trans*] =
*ord-le-eq-trans*
*ord-eq-le-trans*
*ord-less-eq-trans*
*ord-eq-less-trans*

**lemmas** [*trans*] =
*trans*

**lemmas** *order-trans-rules* =
*order-less-subst2*
*order-less-subst1*
*order-le-less-subst2*
*order-le-less-subst1*
*order-less-le-subst2*
*order-less-le-subst1*
*order-subst2*
*order-subst1*
*ord-le-eq-subst*
*ord-eq-le-subst*
*ord-less-eq-subst*
*ord-eq-less-subst*
*forw-subst*
*back-subst*
*rev-mp*
*mp*
*neq-le-trans*
*le-neq-trans*
*less-trans*
*less-asym'*
*le-less-trans*
*less-le-trans*
*order-trans*
*antisym*
*ord-le-eq-trans*
*ord-eq-le-trans*
*ord-less-eq-trans*
*ord-eq-less-trans*
*trans*

These support proving chains of decreasing inequalities a ¿= b ¿= c ... in Isar proofs.

**lemma** *xt1*:
$a = b \Longrightarrow b > c \Longrightarrow a > c$

$a > b \Longrightarrow b = c \Longrightarrow a > c$
$a = b \Longrightarrow b >= c \Longrightarrow a >= c$
$a >= b \Longrightarrow b = c \Longrightarrow a >= c$
$(x::'a::order) >= y \Longrightarrow y >= x \Longrightarrow x = y$
$(x::'a::order) >= y \Longrightarrow y >= z \Longrightarrow x >= z$
$(x::'a::order) > y \Longrightarrow y >= z \Longrightarrow x > z$
$(x::'a::order) >= y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) > b \Longrightarrow b > a \Longrightarrow P$
$(x::'a::order) > y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) >= b \Longrightarrow a \sim= b \Longrightarrow a > b$
$(a::'a::order) \sim= b \Longrightarrow a >= b \Longrightarrow a > b$
$a = f\ b \Longrightarrow b > c \Longrightarrow (!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
$a > b \Longrightarrow f\ b = c \Longrightarrow (!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
$a = f\ b \Longrightarrow b >= c \Longrightarrow (!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a >= f\ c$
$a >= b \Longrightarrow f\ b = c \Longrightarrow (!!\ x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a >= c$
⟨*proof*⟩

**lemma** *xt2*:
$(a::'a::order) >= f\ b \Longrightarrow b >= c \Longrightarrow (!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a >= f\ c$
⟨*proof*⟩

**lemma** *xt3*: $(a::'a::order) >= b \Longrightarrow (f\ b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a >= c$
⟨*proof*⟩

**lemma** *xt4*: $(a::'a::order) > f\ b \Longrightarrow (b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a > f\ c$
⟨*proof*⟩

**lemma** *xt5*: $(a::'a::order) > b \Longrightarrow (f\ b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
⟨*proof*⟩

**lemma** *xt6*: $(a::'a::order) >= f\ b \Longrightarrow b > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
⟨*proof*⟩

**lemma** *xt7*: $(a::'a::order) >= b \Longrightarrow (f\ b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a > c$
⟨*proof*⟩

**lemma** *xt8*: $(a::'a::order) > f\ b \Longrightarrow (b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
⟨*proof*⟩

**lemma** *xt9*: $(a::'a::order) > b \Longrightarrow (f\ b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
⟨*proof*⟩

**lemmas** *xtrans* = *xt1 xt2 xt3 xt4 xt5 xt6 xt7 xt8 xt9*

## 3.8 Monotonicity, least value operator and min/max

**context** *order*
**begin**

**definition** *mono* :: (*'a* ⇒ *'b*::*order*) ⇒ *bool* **where**
*mono f* ⟷ (∀ *x y*. *x* ≤ *y* ⟶ *f x* ≤ *f y*)

**lemma** *monoI* [*intro?*]:
**fixes** *f* :: *'a* ⇒ *'b*::*order*
**shows** (⋀*x y*. *x* ≤ *y* ⟹ *f x* ≤ *f y*) ⟹ *mono f*
⟨*proof*⟩

**lemma** *monoD* [*dest?*]:
**fixes** *f* :: *'a* ⇒ *'b*::*order*
**shows** *mono f* ⟹ *x* ≤ *y* ⟹ *f x* ≤ *f y*
⟨*proof*⟩

**definition** *strict-mono* :: (*'a* ⇒ *'b*::*order*) ⇒ *bool* **where**
*strict-mono f* ⟷ (∀ *x y*. *x* < *y* ⟶ *f x* < *f y*)

**lemma** *strict-monoI* [*intro?*]:
**assumes** ⋀*x y*. *x* < *y* ⟹ *f x* < *f y*
**shows** *strict-mono f*
⟨*proof*⟩

**lemma** *strict-monoD* [*dest?*]:
*strict-mono f* ⟹ *x* < *y* ⟹ *f x* < *f y*
⟨*proof*⟩

**lemma** *strict-mono-mono* [*dest?*]:
**assumes** *strict-mono f*
**shows** *mono f*
⟨*proof*⟩

**end**

**context** *linorder*
**begin**

**lemma** *strict-mono-eq*:
**assumes** *strict-mono f*
**shows** *f x* = *f y* ⟷ *x* = *y*
⟨*proof*⟩

**lemma** *strict-mono-less-eq*:

**assumes** *strict-mono f*
**shows** $f\ x \leq f\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

**lemma** *strict-mono-less*:
**assumes** *strict-mono f*
**shows** $f\ x < f\ y \longleftrightarrow x < y$
⟨*proof*⟩

**lemma** *min-of-mono*:
**fixes** $f :: 'a \Rightarrow 'b$::*linorder*
**shows** *mono f* $\Longrightarrow$ *min* $(f\ m)\ (f\ n) = f\ (min\ m\ n)$
⟨*proof*⟩

**lemma** *max-of-mono*:
**fixes** $f :: 'a \Rightarrow 'b$::*linorder*
**shows** *mono f* $\Longrightarrow$ *max* $(f\ m)\ (f\ n) = f\ (max\ m\ n)$
⟨*proof*⟩

**end**

**lemma** *min-leastL*: $(!!x.\ least <= x) ==> min\ least\ x = least$
⟨*proof*⟩

**lemma** *max-leastL*: $(!!x.\ least <= x) ==> max\ least\ x = x$
⟨*proof*⟩

**lemma** *min-leastR*: $(\bigwedge x::'a::order.\ least \leq x) \Longrightarrow min\ x\ least = least$
⟨*proof*⟩

**lemma** *max-leastR*: $(\bigwedge x::'a::order.\ least \leq x) \Longrightarrow max\ x\ least = x$
⟨*proof*⟩

## 3.9 Top and bottom elements

**class** *top* = *preorder* +
**fixes** *top* :: $'a$
**assumes** *top-greatest* [*simp*]: $x \leq top$

**class** *bot* = *preorder* +
**fixes** *bot* :: $'a$
**assumes** *bot-least* [*simp*]: $bot \leq x$

## 3.10 Dense orders

**class** *dense-linear-order* = *linorder* +
**assumes** *gt-ex*: $\exists\, y.\ x < y$
**and** *lt-ex*: $\exists\, y.\ y < x$
**and** *dense*: $x < y \Longrightarrow (\exists\, z.\ x < z \wedge z < y)$

## 3.11 Wellorders

**class** *wellorder* = *linorder* +
  **assumes** *less-induct* [*case-names less*]: $(\bigwedge x.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$
**begin**

**lemma** *wellorder-Least-lemma*:
  **fixes** $k :: 'a$
  **assumes** $P\ k$
  **shows** $P\ (LEAST\ x.\ P\ x)$ **and** $(LEAST\ x.\ P\ x) \leq k$
⟨*proof*⟩

**lemmas** *LeastI* = *wellorder-Least-lemma*(*1*)
**lemmas** *Least-le* = *wellorder-Least-lemma*(*2*)

— The following 3 lemmas are due to Brian Huffman
**lemma** *LeastI-ex*: $\exists\, x.\ P\ x \Longrightarrow P\ (Least\ P)$
  ⟨*proof*⟩

**lemma** *LeastI2*:
  $P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
  ⟨*proof*⟩

**lemma** *LeastI2-ex*:
  $\exists\, a.\ P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
  ⟨*proof*⟩

**lemma** *not-less-Least*: $k < (LEAST\ x.\ P\ x) \Longrightarrow \neg\ P\ k$
⟨*proof*⟩

**end**

## 3.12 Order on bool

**instantiation** *bool* :: {*order*, *top*, *bot*}
**begin**

**definition**
  *le-bool-def* [*code del*]: $P \leq Q \longleftrightarrow P \longrightarrow Q$

**definition**
  *less-bool-def* [*code del*]: $(P::bool) < Q \longleftrightarrow \neg\ P \wedge Q$

**definition**
  *top-bool-eq*: $top = True$

**definition**
  *bot-bool-eq*: $bot = False$

**instance** ⟨*proof*⟩

**end**

**lemma** *le-boolI*: $(P \Longrightarrow Q) \Longrightarrow P \leq Q$
⟨*proof*⟩

**lemma** *le-boolI'*: $P \longrightarrow Q \Longrightarrow P \leq Q$
⟨*proof*⟩

**lemma** *le-boolE*: $P \leq Q \Longrightarrow P \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

**lemma** *le-boolD*: $P \leq Q \Longrightarrow P \longrightarrow Q$
⟨*proof*⟩

**lemma** [*code*]:
$False \leq b \longleftrightarrow True$
$True \leq b \longleftrightarrow b$
$False < b \longleftrightarrow b$
$True < b \longleftrightarrow False$
⟨*proof*⟩

## 3.13 Order on functions

**instantiation** *fun* :: (*type*, *ord*) *ord*
**begin**

**definition**
*le-fun-def* [*code del*]: $f \leq g \longleftrightarrow (\forall x.\ f\ x \leq g\ x)$

**definition**
*less-fun-def* [*code del*]: $(f::'a \Rightarrow 'b) < g \longleftrightarrow f \leq g \wedge \neg\ (g \leq f)$

**instance** ⟨*proof*⟩

**end**

**instance** *fun* :: (*type*, *preorder*) *preorder* ⟨*proof*⟩

**instance** *fun* :: (*type*, *order*) *order* ⟨*proof*⟩

**instantiation** *fun* :: (*type*, *top*) *top*
**begin**

**definition**
*top-fun-eq*: $top = (\lambda x.\ top)$

**instance** ⟨*proof*⟩

**end**

**instantiation** *fun* :: (*type*, *bot*) *bot*
**begin**

**definition**
*bot-fun-eq*: $bot = (\lambda x.\ bot)$

**instance** ⟨*proof*⟩

**end**

**lemma** *le-funI*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow f \leq g$
⟨*proof*⟩

**lemma** *le-funE*: $f \leq g \Longrightarrow (f\ x \leq g\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *le-funD*: $f \leq g \Longrightarrow f\ x \leq g\ x$
⟨*proof*⟩

Handy introduction and elimination rules for ≤ on unary and binary predicates

**lemma** *predicate1I*:
**assumes** *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
**shows** $P \leq Q$
⟨*proof*⟩

**lemma** *predicate1D* [*Pure.dest*, *dest*]: $P \leq Q \Longrightarrow P\ x \Longrightarrow Q\ x$
⟨*proof*⟩

**lemma** *predicate2I* [*Pure.intro*!, *intro*!]:
**assumes** *PQ*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$
**shows** $P \leq Q$
⟨*proof*⟩

**lemma** *predicate2D* [*Pure.dest*, *dest*]: $P \leq Q \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
⟨*proof*⟩

**lemma** *rev-predicate1D*: $P\ x$ ==> $P$ <= $Q$ ==> $Q\ x$
⟨*proof*⟩

**lemma** *rev-predicate2D*: $P\ x\ y$ ==> $P$ <= $Q$ ==> $Q\ x\ y$
⟨*proof*⟩

**end**

# 4 Lattices: Abstract lattices

**theory** *Lattices*
**imports** *Orderings*
**begin**

## 4.1 Lattices

**notation**
  *less-eq* (**infix** $\sqsubseteq$ *50*) **and**
  *less* (**infix** $\sqsubset$ *50*)

**class** *lower-semilattice* = *order* +
  **fixes** *inf* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcap$ *70*)
  **assumes** *inf-le1* [*simp*]: $x \sqcap y \sqsubseteq x$
  **and** *inf-le2* [*simp*]: $x \sqcap y \sqsubseteq y$
  **and** *inf-greatest*: $x \sqsubseteq y \Longrightarrow x \sqsubseteq z \Longrightarrow x \sqsubseteq y \sqcap z$

**class** *upper-semilattice* = *order* +
  **fixes** *sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcup$ *65*)
  **assumes** *sup-ge1* [*simp*]: $x \sqsubseteq x \sqcup y$
  **and** *sup-ge2* [*simp*]: $y \sqsubseteq x \sqcup y$
  **and** *sup-least*: $y \sqsubseteq x \Longrightarrow z \sqsubseteq x \Longrightarrow y \sqcup z \sqsubseteq x$
**begin**

Dual lattice

**lemma** *dual-lattice*:
  *lower-semilattice* (*op* $\geq$) (*op* $>$) *sup*
⟨*proof*⟩

**end**

**class** *lattice* = *lower-semilattice* + *upper-semilattice*

### 4.1.1 Intro and elim rules

**context** *lower-semilattice*
**begin**

**lemma** *le-infI1*[*intro*]:
  **assumes** $a \sqsubseteq x$
  **shows** $a \sqcap b \sqsubseteq x$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-infI1*

**lemma** *le-infI2*[*intro*]:
  **assumes** $b \sqsubseteq x$
  **shows** $a \sqcap b \sqsubseteq x$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-infI2*

**lemma** *le-infI*[*intro*!]: $x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcap b$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-infI*

**lemma** *le-infE* [*elim*!]: $x \sqsubseteq a \sqcap b \Longrightarrow (x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-infE*

**lemma** *le-inf-iff* [*simp*]:
$x \sqsubseteq y \sqcap z = (x \sqsubseteq y \wedge x \sqsubseteq z)$
⟨*proof*⟩

**lemma** *le-iff-inf*: $(x \sqsubseteq y) = (x \sqcap y = x)$
⟨*proof*⟩

**lemma** *mono-inf*:
**fixes** $f :: 'a \Rightarrow 'b$::*lower-semilattice*
**shows** *mono* $f \Longrightarrow f\ (A \sqcap B) \leq f\ A \sqcap f\ B$
⟨*proof*⟩

**end**

**context** *upper-semilattice*
**begin**

**lemma** *le-supI1*[*intro*]: $x \sqsubseteq a \Longrightarrow x \sqsubseteq a \sqcup b$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-supI1*

**lemma** *le-supI2*[*intro*]: $x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcup b$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-supI2*

**lemma** *le-supI*[*intro*!]: $a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow a \sqcup b \sqsubseteq x$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-supI*

**lemma** *le-supE*[*elim*!]: $a \sqcup b \sqsubseteq x \Longrightarrow (a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩
**lemmas** (**in** −) [*rule del*] = *le-supE*

**lemma** *ge-sup-conv*[*simp*]:
$x \sqcup y \sqsubseteq z = (x \sqsubseteq z \wedge y \sqsubseteq z)$
⟨*proof*⟩

**lemma** *le-iff-sup*: $(x \sqsubseteq y) = (x \sqcup y = y)$
⟨*proof*⟩

**lemma** *mono-sup*:
**fixes** $f :: 'a \Rightarrow 'b$::*upper-semilattice*
**shows** *mono* $f \Longrightarrow f\ A \sqcup f\ B \leq f\ (A \sqcup B)$
⟨*proof*⟩

**end**

### 4.1.2 Equational laws

**context** *lower-semilattice*
**begin**

**lemma** *inf-commute*: $(x \sqcap y) = (y \sqcap x)$
⟨*proof*⟩

**lemma** *inf-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
⟨*proof*⟩

**lemma** *inf-idem*[*simp*]: $x \sqcap x = x$
⟨*proof*⟩

**lemma** *inf-left-idem*[*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
⟨*proof*⟩

**lemma** *inf-absorb1*: $x \sqsubseteq y \Longrightarrow x \sqcap y = x$
⟨*proof*⟩

**lemma** *inf-absorb2*: $y \sqsubseteq x \Longrightarrow x \sqcap y = y$
⟨*proof*⟩

**lemma** *inf-left-commute*: $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$
⟨*proof*⟩

**lemmas** *inf-ACI* = *inf-commute inf-assoc inf-left-commute inf-left-idem*

**end**

**context** *upper-semilattice*
**begin**

**lemma** *sup-commute*: $(x \sqcup y) = (y \sqcup x)$
⟨*proof*⟩

**lemma** *sup-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
⟨*proof*⟩

**lemma** *sup-idem*[*simp*]: $x \sqcup x = x$
⟨*proof*⟩

**lemma** *sup-left-idem*[*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
⟨*proof*⟩

**lemma** *sup-absorb1*: $y \sqsubseteq x \Longrightarrow x \sqcup y = x$
⟨*proof*⟩

**lemma** *sup-absorb2*: $x \sqsubseteq y \Longrightarrow x \sqcup y = y$
⟨*proof*⟩

**lemma** *sup-left-commute*: $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
⟨*proof*⟩

**lemmas** *sup-ACI* = *sup-commute sup-assoc sup-left-commute sup-left-idem*

**end**

**context** *lattice*
**begin**

**lemma** *inf-sup-absorb*: $x \sqcap (x \sqcup y) = x$
⟨*proof*⟩

**lemma** *sup-inf-absorb*: $x \sqcup (x \sqcap y) = x$
⟨*proof*⟩

**lemmas** *ACI* = *inf-ACI sup-ACI*

**lemmas** *inf-sup-ord* = *inf-le1 inf-le2 sup-ge1 sup-ge2*

Towards distributivity

**lemma** *distrib-sup-le*: $x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

**lemma** *distrib-inf-le*: $(x \sqcap y) \sqcup (x \sqcap z) \sqsubseteq x \sqcap (y \sqcup z)$
⟨*proof*⟩

If you have one of them, you have them all.

**lemma** *distrib-imp1*:
**assumes** *D*: $!!x\ y\ z.\ x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
**shows** $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

**lemma** *distrib-imp2*:
**assumes** *D*: $!!x\ y\ z.\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
**shows** $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

**lemma** *modular-le*: $x \sqsubseteq z \Longrightarrow x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap z$
⟨*proof*⟩

**end**

## 4.2 Distributive lattices

**class** *distrib-lattice* = *lattice* +
  **assumes** *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

**context** *distrib-lattice*
**begin**

**lemma** *sup-inf-distrib2*:
 $(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
⟨*proof*⟩

**lemma** *inf-sup-distrib1*:
 $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

**lemma** *inf-sup-distrib2*:
 $(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
⟨*proof*⟩

**lemmas** *distrib* =
  *sup-inf-distrib1 sup-inf-distrib2 inf-sup-distrib1 inf-sup-distrib2*

**end**

## 4.3 Uniqueness of inf and sup

**lemma** (**in** *lower-semilattice*) *inf-unique*:
  **fixes** $f$ (**infixl** $\triangle$ *70*)
  **assumes** *le1*: $\bigwedge x\ y.\ x \triangle y \leq x$ **and** *le2*: $\bigwedge x\ y.\ x \triangle y \leq y$
  **and** *greatest*: $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \triangle z$
  **shows** $x \sqcap y = x \triangle y$
⟨*proof*⟩

**lemma** (**in** *upper-semilattice*) *sup-unique*:
  **fixes** $f$ (**infixl** $\nabla$ *70*)
  **assumes** *ge1* [*simp*]: $\bigwedge x\ y.\ x \leq x \nabla y$ **and** *ge2*: $\bigwedge x\ y.\ y \leq x \nabla y$
  **and** *least*: $\bigwedge x\ y\ z.\ y \leq x \Longrightarrow z \leq x \Longrightarrow y \nabla z \leq x$
  **shows** $x \sqcup y = x \nabla y$
⟨*proof*⟩

## 4.4 *min*/*max* on linear orders as special case of *op* ⊓/*op* ⊔

**lemma** (**in** *linorder*) *distrib-lattice-min-max*:
  *distrib-lattice* (*op* ≤) (*op* <) *min max*

⟨*proof*⟩

**interpretation** *min-max*: *distrib-lattice op* ≤ :: *'a*::*linorder* ⇒ *'a* ⇒ *bool op* < *min max*
⟨*proof*⟩

**lemma** *inf-min*: *inf* = (*min* :: *'a*::{*lower-semilattice*, *linorder*} ⇒ *'a* ⇒ *'a*)
⟨*proof*⟩

**lemma** *sup-max*: *sup* = (*max* :: *'a*::{*upper-semilattice*, *linorder*} ⇒ *'a* ⇒ *'a*)
⟨*proof*⟩

**lemmas** *le-maxI1* = *min-max.sup-ge1*
**lemmas** *le-maxI2* = *min-max.sup-ge2*

**lemmas** *max-ac* = *min-max.sup-assoc min-max.sup-commute*
*mk-left-commute* [*of max*, *OF min-max.sup-assoc min-max.sup-commute*]

**lemmas** *min-ac* = *min-max.inf-assoc min-max.inf-commute*
*mk-left-commute* [*of min*, *OF min-max.inf-assoc min-max.inf-commute*]

Now we have inherited antisymmetry as an intro-rule on all linear orders. This is a problem because it applies to bool, which is undesirable.

**lemmas** [*rule del*] = *min-max.le-infI min-max.le-supI*
*min-max.le-supE min-max.le-infE min-max.le-supI1 min-max.le-supI2*
*min-max.le-infI1 min-max.le-infI2*

## 4.5 Bool as lattice

**instantiation** *bool* :: *distrib-lattice*
**begin**

**definition**
*inf-bool-eq*: *P* ⊓ *Q* ⟷ *P* ∧ *Q*

**definition**
*sup-bool-eq*: *P* ⊔ *Q* ⟷ *P* ∨ *Q*

**instance**
⟨*proof*⟩

**end**

## 4.6 Fun as lattice

**instantiation** *fun* :: (*type*, *lattice*) *lattice*
**begin**

**definition**

*inf-fun-eq* [*code del*]: $f \sqcap g = (\lambda x.\ f\ x \sqcap g\ x)$

**definition**
*sup-fun-eq* [*code del*]: $f \sqcup g = (\lambda x.\ f\ x \sqcup g\ x)$

**instance**
⟨*proof*⟩

**end**

**instance** *fun* :: (*type*, *distrib-lattice*) *distrib-lattice*
⟨*proof*⟩

redundant bindings

**lemmas** *inf-aci* = *inf-ACI*
**lemmas** *sup-aci* = *sup-ACI*

**no-notation**
*less-eq* (**infix** $\sqsubseteq$ *50*) **and**
*less* (**infix** $\sqsubset$ *50*) **and**
*inf* (**infixl** $\sqcap$ *70*) **and**
*sup* (**infixl** $\sqcup$ *65*)

**end**

# 5 Set: Set theory for higher-order logic

**theory** *Set*
**imports** *Lattices*
**begin**

A set in HOL is simply a predicate.

## 5.1 Basic syntax

**global**

**types** *'a set* = *'a* => *bool*

**consts**
*Collect* :: (*'a* => *bool*) => *'a set* — comprehension
*op* : :: *'a* => *'a set* => *bool* — membership
*insert* :: *'a* => *'a set* => *'a set*
*Ball* :: *'a set* => (*'a* => *bool*) => *bool* — bounded universal quantifiers

*Bex* :: *'a set* => (*'a* => *bool*) => *bool* — bounded existential quantifiers

*Bex1* :: *'a set => ('a => bool) => bool* — bounded unique existential quantifiers
*Pow* :: *'a set => 'a set set* — powerset
*image* :: *('a => 'b) => 'a set => 'b set* (**infixr** *‘ 90*)

**local**

**notation**
*op :* (*op :*) **and**
*op :* (*(-/ : -) [50, 51] 50*)

**abbreviation**
*not-mem x A == ~ (x : A)* — non-membership

**notation**
*not-mem* (*op ~:*) **and**
*not-mem* (*(-/ ~: -) [50, 51] 50*)

**notation** (*xsymbols*)
*op :* (*op ∈*) **and**
*op :* (*(-/ ∈ -) [50, 51] 50*) **and**
*not-mem* (*op ∉*) **and**
*not-mem* (*(-/ ∉ -) [50, 51] 50*)

**notation** (*HTML* **output**)
*op :* (*op ∈*) **and**
*op :* (*(-/ ∈ -) [50, 51] 50*) **and**
*not-mem* (*op ∉*) **and**
*not-mem* (*(-/ ∉ -) [50, 51] 50*)

**syntax**
*@Coll* :: *pttrn => bool => 'a set* (*(1{-./ -})*)

**translations**
*{x. P}* == *Collect (%x. P)*

**definition** *empty* :: *'a set* ({}) **where**
*empty ≡ {x. False}*

**definition** *UNIV* :: *'a set* **where**
*UNIV ≡ {x. True}*

**syntax**
*@Finset* :: *args => 'a set* ({(-)})

**translations**
*{x, xs}* == *insert x {xs}*
*{x}* == *insert x {}*

**definition** *Int* :: *'a set* ⇒ *'a set* ⇒ *'a set* (**infixl** *Int 70*) **where**
*A Int B* ≡ {*x*. *x* ∈ *A* ∧ *x* ∈ *B*}

**definition** *Un* :: *'a set* ⇒ *'a set* ⇒ *'a set* (**infixl** *Un 65*) **where**
*A Un B* ≡ {*x*. *x* ∈ *A* ∨ *x* ∈ *B*}

**notation** (*xsymbols*)
*Int* (**infixl** ∩ *70*) **and**
*Un* (**infixl** ∪ *65*)

**notation** (*HTML* **output**)
*Int* (**infixl** ∩ *70*) **and**
*Un* (**infixl** ∪ *65*)

**syntax**
*-Ball* :: *pttrn => 'a set => bool => bool* (*3ALL -:-./ -) [0, 0, 10] 10*)
*-Bex* :: *pttrn => 'a set => bool => bool* (*3EX -:-./ -) [0, 0, 10] 10*)
*-Bex1* :: *pttrn => 'a set => bool => bool* (*3EX! -:-./ -) [0, 0, 10] 10*)
*-Bleast* :: *id => 'a set => bool => 'a* (*3LEAST -:-./ -) [0, 0, 10] 10*)

**syntax** (*HOL*)
*-Ball* :: *pttrn => 'a set => bool => bool* (*3! -:-./ -) [0, 0, 10] 10*)
*-Bex* :: *pttrn => 'a set => bool => bool* (*3? -:-./ -) [0, 0, 10] 10*)
*-Bex1* :: *pttrn => 'a set => bool => bool* (*3?! -:-./ -) [0, 0, 10] 10*)

**syntax** (*xsymbols*)
*-Ball* :: *pttrn => 'a set => bool => bool* (*3∀ -∈-./ -) [0, 0, 10] 10*)
*-Bex* :: *pttrn => 'a set => bool => bool* (*3∃ -∈-./ -) [0, 0, 10] 10*)
*-Bex1* :: *pttrn => 'a set => bool => bool* (*3∃!-∈-./ -) [0, 0, 10] 10*)
*-Bleast* :: *id => 'a set => bool => 'a* (*3LEAST-∈-./ -) [0, 0, 10] 10*)

**syntax** (*HTML* **output**)
*-Ball* :: *pttrn => 'a set => bool => bool* (*3∀ -∈-./ -) [0, 0, 10] 10*)
*-Bex* :: *pttrn => 'a set => bool => bool* (*3∃ -∈-./ -) [0, 0, 10] 10*)
*-Bex1* :: *pttrn => 'a set => bool => bool* (*3∃!-∈-./ -) [0, 0, 10] 10*)

**translations**
*ALL x:A. P* == *Ball A* (%*x*. *P*)
*EX x:A. P* == *Bex A* (%*x*. *P*)
*EX! x:A. P* == *Bex1 A* (%*x*. *P*)
*LEAST x:A. P* => *LEAST x. x:A & P*

**definition** *INTER* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ *'b set* **where**
*INTER A B* ≡ {*y*. ∀*x*∈*A*. *y* ∈ *B x*}

**definition** *UNION* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ *'b set* **where**
*UNION A B* ≡ {*y*. ∃*x*∈*A*. *y* ∈ *B x*}

**definition** *Inter* :: *'a set set ⇒ 'a set* **where**
*Inter S ≡ INTER S (λx. x)*

**definition** *Union* :: *'a set set ⇒ 'a set* **where**
*Union S ≡ UNION S (λx. x)*

**notation** (*xsymbols*)
*Inter* (⋂- [*90*] *90*) **and**
*Union* (⋃- [*90*] *90*)

## 5.2 Additional concrete syntax

**syntax**
*@SetCompr* :: *'a => idts => bool => 'a set* ((*1*{- |/-./ -}))
*@Collect* :: *idt => 'a set => bool => 'a set* ((*1*{- :/ -./ -}))
*@INTER1* :: *pttrns => 'b set => 'b set* ((*3INT* -./ -) [*0*, *10*] *10*)
*@UNION1* :: *pttrns => 'b set => 'b set* ((*3UN* -./ -) [*0*, *10*] *10*)
*@INTER* :: *pttrn => 'a set => 'b set => 'b set* ((*3INT* -:-./ -) [*0*, *10*] *10*)
*@UNION* :: *pttrn => 'a set => 'b set => 'b set* ((*3UN* -:-./ -) [*0*, *10*] *10*)

**syntax** (*xsymbols*)
*@Collect* :: *idt => 'a set => bool => 'a set* ((*1*{- ∈/ -./ -}))
*@INTER1* :: *pttrns => 'b set => 'b set* ((*3*⋂-./ -) [*0*, *10*] *10*)
*@UNION1* :: *pttrns => 'b set => 'b set* ((*3*⋃-./ -) [*0*, *10*] *10*)
*@INTER* :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋂-∈-./ -) [*0*, *10*] *10*)
*@UNION* :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋃-∈-./ -) [*0*, *10*] *10*)

**syntax** (*latex* **output**)
*@INTER1* :: *pttrns => 'b set => 'b set* ((*3*⋂(*00*-)/ -) [*0*, *10*] *10*)
*@UNION1* :: *pttrns => 'b set => 'b set* ((*3*⋃(*00*-)/ -) [*0*, *10*] *10*)
*@INTER* :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋂(*00*-∈-)/ -) [*0*, *10*] *10*)
*@UNION* :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋃(*00*-∈-)/ -) [*0*, *10*] *10*)

**translations**
{*x*:*A*. *P*} => {*x*. *x*:*A* & *P*}
*INT x y*. *B* == *INT x*. *INT y*. *B*
*INT x*. *B* == *CONST INTER CONST UNIV* (%*x*. *B*)
*INT x*. *B* == *INT x*:*CONST UNIV*. *B*
*INT x*:*A*. *B* == *CONST INTER A* (%*x*. *B*)
*UN x y*. *B* == *UN x*. *UN y*. *B*
*UN x*. *B* == *CONST UNION CONST UNIV* (%*x*. *B*)
*UN x*. *B* == *UN x*:*CONST UNIV*. *B*
*UN x*:*A*. *B* == *CONST UNION A* (%*x*. *B*)

Note the difference between ordinary xsymbol syntax of indexed unions and intersections (e.g. $\bigcup a_1{\in}A_1.\ B$) and their LaTeX rendition: $\bigcup_{a_1 \in A_1} B$. The former does not make the index expression a subscript of the union/intersection

symbol because this leads to problems with nested subscripts in Proof General.

**abbreviation**
*subset* :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
*subset* ≡ *less*

**abbreviation**
*subset-eq* :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
*subset-eq* ≡ *less-eq*

**notation** (**output**)
*subset* (*op* <) **and**
*subset* ((-/ < -) [*50*, *51*] *50*) **and**
*subset-eq* (*op* <=) **and**
*subset-eq* ((-/ <= -) [*50*, *51*] *50*)

**notation** (*xsymbols*)
*subset* (*op* ⊂) **and**
*subset* ((-/ ⊂ -) [*50*, *51*] *50*) **and**
*subset-eq* (*op* ⊆) **and**
*subset-eq* ((-/ ⊆ -) [*50*, *51*] *50*)

**notation** (*HTML* **output**)
*subset* (*op* ⊂) **and**
*subset* ((-/ ⊂ -) [*50*, *51*] *50*) **and**
*subset-eq* (*op* ⊆) **and**
*subset-eq* ((-/ ⊆ -) [*50*, *51*] *50*)

**abbreviation** (*input*)
*supset* :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
*supset* ≡ *greater*

**abbreviation** (*input*)
*supset-eq* :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
*supset-eq* ≡ *greater-eq*

**notation** (*xsymbols*)
*supset* (*op* ⊃) **and**
*supset* ((-/ ⊃ -) [*50*, *51*] *50*) **and**
*supset-eq* (*op* ⊇) **and**
*supset-eq* ((-/ ⊇ -) [*50*, *51*] *50*)

**abbreviation**
*range* :: (*'a* => *'b*) => *'b set* **where** — of function
*range f* == *f* ‘ *UNIV*

### 5.2.1 Bounded quantifiers

**syntax** (**output**)

*-setlessAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
*-setlessEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
*-setleAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
*-setleEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)
*-setleEx1* :: [*idt*, *'a*, *bool*] => *bool* ((*3EX!* -<=-./ -) [*0*, *0*, *10*] *10*)

**syntax** (*xsymbols*)
*-setlessAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊂-./ -) [*0*, *0*, *10*] *10*)
*-setlessEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊂-./ -) [*0*, *0*, *10*] *10*)
*-setleAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊆-./ -) [*0*, *0*, *10*] *10*)
*-setleEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊆-./ -) [*0*, *0*, *10*] *10*)
*-setleEx1* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!-⊆-./ -) [*0*, *0*, *10*] *10*)

**syntax** (*HOL* **output**)
*-setlessAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<-./ -) [*0*, *0*, *10*] *10*)
*-setlessEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3?* -<-./ -) [*0*, *0*, *10*] *10*)
*-setleAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<=-./ -) [*0*, *0*, *10*] *10*)
*-setleEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3?* -<=-./ -) [*0*, *0*, *10*] *10*)
*-setleEx1* :: [*idt*, *'a*, *bool*] => *bool* ((*3?*! -<=-./ -) [*0*, *0*, *10*] *10*)

**syntax** (*HTML* **output**)
*-setlessAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊂-./ -) [*0*, *0*, *10*] *10*)
*-setlessEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊂-./ -) [*0*, *0*, *10*] *10*)
*-setleAll* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊆-./ -) [*0*, *0*, *10*] *10*)
*-setleEx* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊆-./ -) [*0*, *0*, *10*] *10*)
*-setleEx1* :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!-⊆-./ -) [*0*, *0*, *10*] *10*)

**translations**
∀ *A*⊂*B*. *P* => *ALL A*. *A* ⊂ *B* −−> *P*
∃ *A*⊂*B*. *P* => *EX A*. *A* ⊂ *B* & *P*
∀ *A*⊆*B*. *P* => *ALL A*. *A* ⊆ *B* −−> *P*
∃ *A*⊆*B*. *P* => *EX A*. *A* ⊆ *B* & *P*
∃!*A*⊆*B*. *P* => *EX*! *A*. *A* ⊆ *B* & *P*

⟨*ML*⟩

Translate between {*e* | *x1...xn*. *P*} and {*u*. *EX x1..xn*. *u* = *e* & *P*}; {*y*. *EX x1..xn*. *y* = *e* & *P*} is only translated if [*0..n*] *subset bvs*(*e*).

⟨*ML*⟩

## 5.3 Rules and definitions

Isomorphisms between predicates and sets.

**defs**
*mem-def* [*code*]: *x* : *S* == *S x*
*Collect-def* [*code*]: *Collect P* == *P*

**defs**

*Ball-def*: $Ball\ A\ P \quad == ALL\ x.\ x{:}A\ --{>}\ P(x)$
*Bex-def*: $Bex\ A\ P \quad == EX\ x.\ x{:}A\ \&\ P(x)$
*Bex1-def*: $Bex1\ A\ P \quad == EX!\ x.\ x{:}A\ \&\ P(x)$

**instantiation** *fun* :: (*type*, *minus*) *minus*
**begin**

**definition**
*fun-diff-def*: $A - B = (\%x.\ A\ x - B\ x)$

**instance** ⟨*proof*⟩

**end**

**instantiation** *bool* :: *minus*
**begin**

**definition**
*bool-diff-def*: $A - B = (A\ \&\ \sim B)$

**instance** ⟨*proof*⟩

**end**

**instantiation** *fun* :: (*type*, *uminus*) *uminus*
**begin**

**definition**
*fun-Compl-def*: $- A = (\%x.\ - A\ x)$

**instance** ⟨*proof*⟩

**end**

**instantiation** *bool* :: *uminus*
**begin**

**definition**
*bool-Compl-def*: $- A = (\sim A)$

**instance** ⟨*proof*⟩

**end**

**defs**
*Pow-def*: $Pow\ A \quad == \{B.\ B <= A\}$
*insert-def*: $insert\ a\ B \quad == \{x.\ x{=}a\}\ Un\ B$
*image-def*: $f`A \quad == \{y.\ EX\ x{:}A.\ y = f(x)\}$

## 5.4 Lemmas and proof tool setup

### 5.4.1 Relating predicates and sets

**lemma** *mem-Collect-eq* [*iff*]: $(a : \{x.\ P(x)\}) = P(a)$
⟨*proof*⟩

**lemma** *Collect-mem-eq* [*simp*]: $\{x.\ x{:}A\} = A$
⟨*proof*⟩

**lemma** *CollectI*: $P(a) \Longrightarrow a : \{x.\ P(x)\}$
⟨*proof*⟩

**lemma** *CollectD*: $a : \{x.\ P(x)\} \Longrightarrow P(a)$
⟨*proof*⟩

**lemma** *Collect-cong*: $(!!x.\ P\ x = Q\ x) \Longrightarrow \{x.\ P(x)\} = \{x.\ Q(x)\}$
⟨*proof*⟩

**lemmas** *CollectE* = *CollectD* [*elim-format*]

### 5.4.2 Bounded quantifiers

**lemma** *ballI* [*intro*!]: $(!!x.\ x{:}A \Longrightarrow P\ x) \Longrightarrow ALL\ x{:}A.\ P\ x$
⟨*proof*⟩

**lemmas** *strip* = *impI allI ballI*

**lemma** *bspec* [*dest?*]: $ALL\ x{:}A.\ P\ x \Longrightarrow x{:}A \Longrightarrow P\ x$
⟨*proof*⟩

**lemma** *ballE* [*elim*]: $ALL\ x{:}A.\ P\ x \Longrightarrow (P\ x \Longrightarrow Q) \Longrightarrow (x\ \tilde{}{:}\ A \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

⟨*ML*⟩

This tactic takes assumptions $\forall x{\in}A.\ P\ x$ and $a \in A$; creates assumption $P\ a$.

⟨*ML*⟩
Gives better instantiation for bound:

⟨*ML*⟩

**lemma** *bexI* [*intro*]: $P\ x \Longrightarrow x{:}A \Longrightarrow EX\ x{:}A.\ P\ x$
— Normally the best argument order: $P\ x$ constrains the choice of $x \in A$.
⟨*proof*⟩

**lemma** *rev-bexI* [*intro?*]: $x{:}A \Longrightarrow P\ x \Longrightarrow EX\ x{:}A.\ P\ x$
— The best argument order when there is only one $x \in A$.

⟨*proof*⟩

**lemma** *bexCI*: (*ALL* x:A. ~P x ==> P a) ==> a:A ==> *EX* x:A. P x
⟨*proof*⟩

**lemma** *bexE* [*elim*!]: *EX* x:A. P x ==> (!!x. x:A ==> P x ==> Q) ==> Q
⟨*proof*⟩

**lemma** *ball-triv* [*simp*]: (*ALL* x:A. P) = ((*EX* x. x:A) --> P)
— Trival rewrite rule.
⟨*proof*⟩

**lemma** *bex-triv* [*simp*]: (*EX* x:A. P) = ((*EX* x. x:A) & P)
— Dual form for existentials.
⟨*proof*⟩

**lemma** *bex-triv-one-point1* [*simp*]: (*EX* x:A. x = a) = (a:A)
⟨*proof*⟩

**lemma** *bex-triv-one-point2* [*simp*]: (*EX* x:A. a = x) = (a:A)
⟨*proof*⟩

**lemma** *bex-one-point1* [*simp*]: (*EX* x:A. x = a & P x) = (a:A & P a)
⟨*proof*⟩

**lemma** *bex-one-point2* [*simp*]: (*EX* x:A. a = x & P x) = (a:A & P a)
⟨*proof*⟩

**lemma** *ball-one-point1* [*simp*]: (*ALL* x:A. x = a --> P x) = (a:A --> P a)
⟨*proof*⟩

**lemma** *ball-one-point2* [*simp*]: (*ALL* x:A. a = x --> P x) = (a:A --> P a)
⟨*proof*⟩

⟨*ML*⟩

### 5.4.3 Congruence rules

**lemma** *ball-cong*:
A = B ==> (!!x. x:B ==> P x = Q x) ==>
(*ALL* x:A. P x) = (*ALL* x:B. Q x)
⟨*proof*⟩

**lemma** *strong-ball-cong* [*cong*]:
A = B ==> (!!x. x:B =*simp*=> P x = Q x) ==>
(*ALL* x:A. P x) = (*ALL* x:B. Q x)
⟨*proof*⟩

**lemma** *bex-cong*:

$A = B \Longrightarrow (!!x.\ x{:}B \Longrightarrow P\ x = Q\ x) \Longrightarrow$
$(EX\ x{:}A.\ P\ x) = (EX\ x{:}B.\ Q\ x)$
⟨*proof*⟩

**lemma** *strong-bex-cong* [*cong*]:
$A = B \Longrightarrow (!!x.\ x{:}B\ =simp=>\ P\ x = Q\ x) \Longrightarrow$
$(EX\ x{:}A.\ P\ x) = (EX\ x{:}B.\ Q\ x)$
⟨*proof*⟩

### 5.4.4 Subsets

**lemma** *subsetI* [*atp*,*intro*!]: $(!!x.\ x{:}A \Longrightarrow x{:}B) \Longrightarrow A \subseteq B$
⟨*proof*⟩

Map the type $'a\ set \Rightarrow anything$ to just $'a$; for overloading constants whose first argument has type $'a\ set$.

**lemma** *subsetD* [*elim*, *intro?*]: $A \subseteq B \Longrightarrow c \in A \Longrightarrow c \in B$
— Rule in Modus Ponens style.
⟨*proof*⟩

**lemma** *rev-subsetD* [*intro?*]: $c \in A \Longrightarrow A \subseteq B \Longrightarrow c \in B$
— The same, with reversed premises for use with *erule* – cf *rev-mp*.
⟨*proof*⟩

Converts $A \subseteq B$ to $x \in A \Longrightarrow x \in B$.

⟨*ML*⟩

**lemma** *subsetCE* [*elim*]: $A \subseteq B \Longrightarrow (c \notin A \Longrightarrow P) \Longrightarrow (c \in B \Longrightarrow P) \Longrightarrow P$
— Classical elimination rule.
⟨*proof*⟩

**lemma** *subset-eq*: $A \leq B = (\forall x{\in}A.\ x \in B)$ ⟨*proof*⟩

Takes assumptions $A \subseteq B$; $c \in A$ and creates the assumption $c \in B$.

⟨*ML*⟩

**lemma** *contra-subsetD*: $A \subseteq B \Longrightarrow c \notin B \Longrightarrow c \notin A$
⟨*proof*⟩

**lemma** *subset-refl* [*simp*,*atp*]: $A \subseteq A$
⟨*proof*⟩

**lemma** *subset-trans*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow A \subseteq C$
⟨*proof*⟩

### 5.4.5 Equality

**lemma** *set-ext*: **assumes** *prem*: (!!$x$. ($x$:$A$) = ($x$:$B$)) **shows** $A = B$
⟨*proof*⟩

**lemma** *expand-set-eq*: $(A = B) = (ALL\ x.\ (x{:}A) = (x{:}B))$
⟨*proof*⟩

**lemma** *subset-antisym* [*intro*!]: $A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow A = B$
— Anti-symmetry of the subset relation.
⟨*proof*⟩

Equality rules from ZF set theory – are they appropriate here?

**lemma** *equalityD1*: $A = B \Longrightarrow A \subseteq B$
⟨*proof*⟩

**lemma** *equalityD2*: $A = B \Longrightarrow B \subseteq A$
⟨*proof*⟩

Be careful when adding this to the claset as *subset-empty* is in the simpset: $A = \{\}$ goes to $\{\} \subseteq A$ and $A \subseteq \{\}$ and then back to $A = \{\}$!

**lemma** *equalityE*: $A = B \Longrightarrow (A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *equalityCE* [*elim*]:
$A = B \Longrightarrow (c \in A \Longrightarrow c \in B \Longrightarrow P) \Longrightarrow (c \notin A \Longrightarrow c \notin B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *eqset-imp-iff*: $A = B \Longrightarrow (x : A) = (x : B)$
⟨*proof*⟩

**lemma** *eqelem-imp-iff*: $x = y \Longrightarrow (x : A) = (y : A)$
⟨*proof*⟩

### 5.4.6 The universal set – UNIV

**lemma** *UNIV-I* [*simp*]: $x : UNIV$
⟨*proof*⟩

**declare** *UNIV-I* [*intro*] — unsafe makes it less likely to cause problems

**lemma** *UNIV-witness* [*intro?*]: $EX\ x.\ x : UNIV$
⟨*proof*⟩

**lemma** *subset-UNIV* [*simp*]: $A \subseteq UNIV$
⟨*proof*⟩

Eta-contracting these two rules (to remove $P$) causes them to be ignored because of their interaction with congruence rules.

**lemma** *ball-UNIV* [*simp*]: $Ball\ UNIV\ P = All\ P$
⟨*proof*⟩

**lemma** *bex-UNIV* [*simp*]: $Bex\ UNIV\ P = Ex\ P$
⟨*proof*⟩

**lemma** *UNIV-eq-I*: $(\bigwedge x.\ x \in A) \Longrightarrow UNIV = A$
⟨*proof*⟩

### 5.4.7 The empty set

**lemma** *empty-iff* [*simp*]: $(c : \{\}) = False$
⟨*proof*⟩

**lemma** *emptyE* [*elim*!]: $a : \{\} \Longrightarrow P$
⟨*proof*⟩

**lemma** *empty-subsetI* [*iff*]: $\{\} \subseteq A$
— One effect is to delete the ASSUMPTION $\{\} \subseteq A$
⟨*proof*⟩

**lemma** *equals0I*: $(!!y.\ y \in A \Longrightarrow False) \Longrightarrow A = \{\}$
⟨*proof*⟩

**lemma** *equals0D*: $A = \{\} \Longrightarrow a \notin A$
— Use for reasoning about disjointness: $A \cap B = \{\}$
⟨*proof*⟩

**lemma** *ball-empty* [*simp*]: $Ball\ \{\}\ P = True$
⟨*proof*⟩

**lemma** *bex-empty* [*simp*]: $Bex\ \{\}\ P = False$
⟨*proof*⟩

**lemma** *UNIV-not-empty* [*iff*]: $UNIV \neq \{\}$
⟨*proof*⟩

### 5.4.8 The Powerset operator – Pow

**lemma** *Pow-iff* [*iff*]: $(A \in Pow\ B) = (A \subseteq B)$
⟨*proof*⟩

**lemma** *PowI*: $A \subseteq B \Longrightarrow A \in Pow\ B$
⟨*proof*⟩

**lemma** *PowD*: $A \in Pow\ B \Longrightarrow A \subseteq B$
⟨*proof*⟩

**lemma** *Pow-bottom*: $\{\} \in Pow\ B$
⟨*proof*⟩

**lemma** *Pow-top*: $A \in Pow\ A$
⟨*proof*⟩

### 5.4.9 Set complement

**lemma** *Compl-iff* [*simp*]: $(c \in -A) = (c \notin A)$
⟨*proof*⟩

**lemma** *ComplI* [*intro!*]: $(c \in A \Longrightarrow False) \Longrightarrow c \in -A$
⟨*proof*⟩

This form, with negated conclusion, works well with the Classical prover. Negated assumptions behave like formulae on the right side of the notional turnstile ...

**lemma** *ComplD* [*dest!*]: $c : -A \Longrightarrow c^{\sim}{:}A$
⟨*proof*⟩

**lemmas** *ComplE* = *ComplD* [*elim-format*]

**lemma** *Compl-eq*: $-A = \{x.\ \sim x : A\}$ ⟨*proof*⟩

### 5.4.10 Binary union – Un

**lemma** *Un-iff* [*simp*]: $(c : A\ Un\ B) = (c{:}A \mid c{:}B)$
⟨*proof*⟩

**lemma** *UnI1* [*elim?*]: $c{:}A \Longrightarrow c : A\ Un\ B$
⟨*proof*⟩

**lemma** *UnI2* [*elim?*]: $c{:}B \Longrightarrow c : A\ Un\ B$
⟨*proof*⟩

Classical introduction rule: no commitment to $A$ vs $B$.

**lemma** *UnCI* [*intro!*]: $(c^{\sim}{:}B \Longrightarrow c{:}A) \Longrightarrow c : A\ Un\ B$
⟨*proof*⟩

**lemma** *UnE* [*elim!*]: $c : A\ Un\ B \Longrightarrow (c{:}A \Longrightarrow P) \Longrightarrow (c{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

### 5.4.11 Binary intersection – Int

**lemma** *Int-iff* [*simp*]: $(c : A\ Int\ B) = (c{:}A\ \&\ c{:}B)$
⟨*proof*⟩

**lemma** *IntI* [*intro*!]: $c{:}A \Longrightarrow c{:}B \Longrightarrow c : A \mathbin{Int} B$
⟨*proof*⟩

**lemma** *IntD1*: $c : A \mathbin{Int} B \Longrightarrow c{:}A$
⟨*proof*⟩

**lemma** *IntD2*: $c : A \mathbin{Int} B \Longrightarrow c{:}B$
⟨*proof*⟩

**lemma** *IntE* [*elim*!]: $c : A \mathbin{Int} B \Longrightarrow (c{:}A \Longrightarrow c{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

### 5.4.12 Set difference

**lemma** *Diff-iff* [*simp*]: $(c : A - B) = (c{:}A \mathbin{\&} c{\sim}{:}B)$
⟨*proof*⟩

**lemma** *DiffI* [*intro*!]: $c : A \Longrightarrow c \sim : B \Longrightarrow c : A - B$
⟨*proof*⟩

**lemma** *DiffD1*: $c : A - B \Longrightarrow c : A$
⟨*proof*⟩

**lemma** *DiffD2*: $c : A - B \Longrightarrow c : B \Longrightarrow P$
⟨*proof*⟩

**lemma** *DiffE* [*elim*!]: $c : A - B \Longrightarrow (c{:}A \Longrightarrow c{\sim}{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *set-diff-eq*: $A - B = \{x.\ x : A \mathbin{\&} \sim x : B\}$ ⟨*proof*⟩

**lemma** *Compl-eq-Diff-UNIV*: $-A = (UNIV - A)$
⟨*proof*⟩

### 5.4.13 Augmenting a set – insert

**lemma** *insert-iff* [*simp*]: $(a : \mathit{insert}\ b\ A) = (a = b \mid a{:}A)$
⟨*proof*⟩

**lemma** *insertI1*: $a : \mathit{insert}\ a\ B$
⟨*proof*⟩

**lemma** *insertI2*: $a : B \Longrightarrow a : \mathit{insert}\ b\ B$
⟨*proof*⟩

**lemma** *insertE* [*elim*!]: $a : \mathit{insert}\ b\ A \Longrightarrow (a = b \Longrightarrow P) \Longrightarrow (a{:}A \Longrightarrow P)$
$\Longrightarrow P$
⟨*proof*⟩

**lemma** *insertCI* [*intro*!]: $(a^{\sim}{:}B \Longrightarrow a = b) \Longrightarrow a{:}\ \mathit{insert}\ b\ B$
— Classical introduction rule.
⟨*proof*⟩

**lemma** *subset-insert-iff*: $(A \subseteq \mathit{insert}\ x\ B) = (\mathit{if}\ x{:}A\ \mathit{then}\ A - \{x\} \subseteq B\ \mathit{else}\ A \subseteq B)$
⟨*proof*⟩

**lemma** *set-insert*:
**assumes** $x \in A$
**obtains** $B$ **where** $A = \mathit{insert}\ x\ B$ **and** $x \notin B$
⟨*proof*⟩

**lemma** *insert-ident*: $x\ ^{\sim}{:}\ A \Longrightarrow x\ ^{\sim}{:}\ B \Longrightarrow (\mathit{insert}\ x\ A = \mathit{insert}\ x\ B) = (A = B)$
⟨*proof*⟩

### 5.4.14 Singletons, using insert

**lemma** *singletonI* [*intro*!,*noatp*]: $a : \{a\}$
— Redundant? But unlike *insertCI*, it proves the subgoal immediately!
⟨*proof*⟩

**lemma** *singletonD* [*dest*!,*noatp*]: $b : \{a\} \Longrightarrow b = a$
⟨*proof*⟩

**lemmas** *singletonE* = *singletonD* [*elim-format*]

**lemma** *singleton-iff*: $(b : \{a\}) = (b = a)$
⟨*proof*⟩

**lemma** *singleton-inject* [*dest*!]: $\{a\} = \{b\} \Longrightarrow a = b$
⟨*proof*⟩

**lemma** *singleton-insert-inj-eq* [*iff*,*noatp*]:
$(\{b\} = \mathit{insert}\ a\ A) = (a = b\ \&\ A \subseteq \{b\})$
⟨*proof*⟩

**lemma** *singleton-insert-inj-eq′* [*iff*,*noatp*]:
$(\mathit{insert}\ a\ A = \{b\}) = (a = b\ \&\ A \subseteq \{b\})$
⟨*proof*⟩

**lemma** *subset-singletonD*: $A \subseteq \{x\} \Longrightarrow A = \{\} \mid A = \{x\}$
⟨*proof*⟩

**lemma** *singleton-conv* [*simp*]: $\{x.\ x = a\} = \{a\}$
⟨*proof*⟩

**lemma** *singleton-conv2* [*simp*]: $\{x.\ a = x\} = \{a\}$

⟨*proof*⟩

**lemma** *diff-single-insert*: $A - \{x\} \subseteq B$ ==> $x \in A$ ==> $A \subseteq$ *insert* $x$ $B$
⟨*proof*⟩

**lemma** *doubleton-eq-iff*: $(\{a,b\} = \{c,d\}) = (a{=}c \ \&\ b{=}d \mid a{=}d \ \&\ b{=}c)$
⟨*proof*⟩

### 5.4.15 Unions of families

*UN* $x{:}A.\ B\ x$ is $\bigcup B$ ‘ $A$.

**declare** *UNION-def* [*noatp*]

**lemma** *UN-iff* [*simp*]: ($b$: (*UN* $x{:}A.\ B\ x$)) = (*EX* $x{:}A.\ b{:}\ B\ x$)
⟨*proof*⟩

**lemma** *UN-I* [*intro*]: $a{:}A$ ==> $b{:}\ B\ a$ ==> $b{:}$ (*UN* $x{:}A.\ B\ x$)
— The order of the premises presupposes that $A$ is rigid; $b$ may be flexible.
⟨*proof*⟩

**lemma** *UN-E* [*elim*!]: $b$ : (*UN* $x{:}A.\ B\ x$) ==> (!!$x.\ x{:}A$ ==> $b{:}\ B\ x$ ==> $R$) ==> $R$
⟨*proof*⟩

**lemma** *UN-cong* [*cong*]:
$A = B$ ==> (!!$x.\ x{:}B$ ==> $C\ x = D\ x$) ==> (*UN* $x{:}A.\ C\ x$) = (*UN* $x{:}B.\ D\ x$)
⟨*proof*⟩

**lemma** *strong-UN-cong*:
$A = B$ ==> (!!$x.\ x{:}B$ =*simp*=> $C\ x = D\ x$) ==> (*UN* $x{:}A.\ C\ x$) = (*UN* $x{:}B.\ D\ x$)
⟨*proof*⟩

### 5.4.16 Intersections of families

*INT* $x{:}A.\ B\ x$ is $\bigcap B$ ‘ $A$.

**lemma** *INT-iff* [*simp*]: ($b$: (*INT* $x{:}A.\ B\ x$)) = (*ALL* $x{:}A.\ b{:}\ B\ x$)
⟨*proof*⟩

**lemma** *INT-I* [*intro*!]: (!!$x.\ x{:}A$ ==> $b{:}\ B\ x$) ==> $b$ : (*INT* $x{:}A.\ B\ x$)
⟨*proof*⟩

**lemma** *INT-D* [*elim*]: $b$ : (*INT* $x{:}A.\ B\ x$) ==> $a{:}A$ ==> $b{:}\ B\ a$
⟨*proof*⟩

**lemma** *INT-E* [*elim*]: $b$ : (*INT* $x{:}A.\ B\ x$) ==> ($b{:}\ B\ a$ ==> $R$) ==> ($a^{\sim}{:}A$ ==> $R$) ==> $R$

— "Classical" elimination – by the Excluded Middle on $a \in A$.
⟨*proof*⟩

**lemma** *INT-cong* [*cong*]:
$A = B \Longrightarrow (!!x.\ x{:}B \Longrightarrow C\ x = D\ x) \Longrightarrow (INT\ x{:}A.\ C\ x) = (INT\ x{:}B.\ D\ x)$
⟨*proof*⟩

### 5.4.17 Union

**lemma** *Union-iff* [*simp*,*noatp*]: $(A : Union\ C) = (EX\ X{:}C.\ A{:}X)$
⟨*proof*⟩

**lemma** *UnionI* [*intro*]: $X{:}C \Longrightarrow A{:}X \Longrightarrow A : Union\ C$
— The order of the premises presupposes that $C$ is rigid; $A$ may be flexible.
⟨*proof*⟩

**lemma** *UnionE* [*elim*!]: $A : Union\ C \Longrightarrow (!!X.\ A{:}X \Longrightarrow X{:}C \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

### 5.4.18 Inter

**lemma** *Inter-iff* [*simp*,*noatp*]: $(A : Inter\ C) = (ALL\ X{:}C.\ A{:}X)$
⟨*proof*⟩

**lemma** *InterI* [*intro*!]: $(!!X.\ X{:}C \Longrightarrow A{:}X) \Longrightarrow A : Inter\ C$
⟨*proof*⟩

A "destruct" rule – every $X$ in $C$ contains $A$ as an element, but $A \in X$ can hold when $X \in C$ does not! This rule is analogous to *spec*.

**lemma** *InterD* [*elim*]: $A : Inter\ C \Longrightarrow X{:}C \Longrightarrow A{:}X$
⟨*proof*⟩

**lemma** *InterE* [*elim*]: $A : Inter\ C \Longrightarrow (X^{\sim}{:}C \Longrightarrow R) \Longrightarrow (A{:}X \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination rule – does not require proving $X \in C$.
⟨*proof*⟩

Image of a set under a function. Frequently $b$ does not have the syntactic form of $f\ x$.

**declare** *image-def* [*noatp*]

**lemma** *image-eqI* [*simp*, *intro*]: $b = f\ x \Longrightarrow x{:}A \Longrightarrow b : f`A$
⟨*proof*⟩

**lemma** *imageI*: $x : A \Longrightarrow f\ x : f\ `\ A$
⟨*proof*⟩

**lemma** *rev-image-eqI*: $x{:}A \Longrightarrow b = f\,x \Longrightarrow b : f`A$
— This version's more effective when we already have the required $x$.
⟨*proof*⟩

**lemma** *imageE* [*elim*!]:
$b : (\%x.\ f\,x)`A \Longrightarrow (!!x.\ b = f\,x \Longrightarrow x{:}A \Longrightarrow P) \Longrightarrow P$
— The eta-expansion gives variable-name preservation.
⟨*proof*⟩

**lemma** *image-Un*: $f`(A\ Un\ B) = f`A\ Un\ f`B$
⟨*proof*⟩

**lemma** *image-eq-UN*: $f`A = (UN\ x{:}A.\ \{f\,x\})$
⟨*proof*⟩

**lemma** *image-iff*: $(z : f`A) = (EX\ x{:}A.\ z = f\,x)$
⟨*proof*⟩

**lemma** *image-subset-iff*: $(f`A \subseteq B) = (\forall x{\in}A.\ f\,x \in B)$
— This rewrite rule would confuse users if made default.
⟨*proof*⟩

**lemma** *subset-image-iff*: $(B \subseteq f`A) = (EX\ AA.\ AA \subseteq A\ \&\ B = f`AA)$
⟨*proof*⟩

**lemma** *image-subsetI*: $(!!x.\ x \in A \Longrightarrow f\,x \in B) \Longrightarrow f`A \subseteq B$
— Replaces the three steps *subsetI*, *imageE*, *hypsubst*, but breaks too many existing proofs.
⟨*proof*⟩

Range of a function – just a translation for image!

**lemma** *range-eqI*: $b = f\,x \Longrightarrow b \in range\ f$
⟨*proof*⟩

**lemma** *rangeI*: $f\,x \in range\ f$
⟨*proof*⟩

**lemma** *rangeE* [*elim?*]: $b \in range\ (\lambda x.\ f\,x) \Longrightarrow (!!x.\ b = f\,x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

### 5.4.19 Set reasoning tools

Rewrite rules for boolean case-splitting: faster than *split-if* [*split*].

**lemma** *split-if-eq1*: $((if\ Q\ then\ x\ else\ y) = b) = ((Q \dashrightarrow x = b)\ \&\ (\sim Q \dashrightarrow y = b))$
⟨*proof*⟩

**lemma** *split-if-eq2*: $(a = (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)) = ((Q \dashrightarrow a = x)\ \&\ (\sim Q \dashrightarrow a = y))$
⟨*proof*⟩

Split ifs on either side of the membership relation. Not for [*simp*] – can cause goals to blow up!

**lemma** *split-if-mem1*: $((\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) : b) = ((Q \dashrightarrow x : b)\ \&\ (\sim Q \dashrightarrow y : b))$
⟨*proof*⟩

**lemma** *split-if-mem2*: $(a : (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)) = ((Q \dashrightarrow a : x)\ \&\ (\sim Q \dashrightarrow a : y))$
⟨*proof*⟩

**lemmas** *split-ifs = if-bool-eq-conj split-if-eq1 split-if-eq2 split-if-mem1 split-if-mem2*

⟨*ML*⟩

### 5.4.20 The "proper subset" relation

**lemma** *psubsetI* [*intro*!,*noatp*]: $A \subseteq B \Longrightarrow A \neq B \Longrightarrow A \subset B$
⟨*proof*⟩

**lemma** *psubsetE* [*elim*!,*noatp*]:
$[|A \subset B;\ [|A \subseteq B;\ \sim (B \subseteq A)|] \Longrightarrow R|] \Longrightarrow R$
⟨*proof*⟩

**lemma** *psubset-insert-iff*:
$(A \subset \mathit{insert}\ x\ B) = (\mathit{if}\ x \in B\ \mathit{then}\ A \subset B\ \mathit{else}\ \mathit{if}\ x \in A\ \mathit{then}\ A - \{x\} \subset B\ \mathit{else}\ A \subseteq B)$
⟨*proof*⟩

**lemma** *psubset-eq*: $(A \subset B) = (A \subseteq B\ \&\ A \neq B)$
⟨*proof*⟩

**lemma** *psubset-imp-subset*: $A \subset B \Longrightarrow A \subseteq B$
⟨*proof*⟩

**lemma** *psubset-trans*: $[|\ A \subset B;\ B \subset C\ |] \Longrightarrow A \subset C$
⟨*proof*⟩

**lemma** *psubsetD*: $[|\ A \subset B;\ c \in A\ |] \Longrightarrow c \in B$
⟨*proof*⟩

**lemma** *psubset-subset-trans*: $A \subset B \Longrightarrow B \subseteq C \Longrightarrow A \subset C$
⟨*proof*⟩

**lemma** *subset-psubset-trans*: $A \subseteq B \Longrightarrow B \subset C \Longrightarrow A \subset C$
⟨*proof*⟩

**lemma** *psubset-imp-ex-mem*: $A \subset B \Longrightarrow \exists b.\ b \in (B - A)$
⟨*proof*⟩

**lemma** *atomize-ball*:
$(!!x.\ x \in A \Longrightarrow P\ x) \equiv \mathit{Trueprop}\ (\forall x{\in}A.\ P\ x)$
⟨*proof*⟩

**lemmas** [*symmetric*, *rulify*] = *atomize-ball*
**and** [*symmetric*, *defn*] = *atomize-ball*

## 5.5 Further set-theory lemmas

### 5.5.1 Derived rules involving subsets.

*insert.*

**lemma** *subset-insertI*: $B \subseteq \mathit{insert}\ a\ B$
⟨*proof*⟩

**lemma** *subset-insertI2*: $A \subseteq B \Longrightarrow A \subseteq \mathit{insert}\ b\ B$
⟨*proof*⟩

**lemma** *subset-insert*: $x \notin A \Longrightarrow (A \subseteq \mathit{insert}\ x\ B) = (A \subseteq B)$
⟨*proof*⟩

Big Union – least upper bound of a set.

**lemma** *Union-upper*: $B \in A \Longrightarrow B \subseteq \mathit{Union}\ A$
⟨*proof*⟩

**lemma** *Union-least*: $(!!X.\ X \in A \Longrightarrow X \subseteq C) \Longrightarrow \mathit{Union}\ A \subseteq C$
⟨*proof*⟩

General union.

**lemma** *UN-upper*: $a \in A \Longrightarrow B\ a \subseteq (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

**lemma** *UN-least*: $(!!x.\ x \in A \Longrightarrow B\ x \subseteq C) \Longrightarrow (\bigcup x{\in}A.\ B\ x) \subseteq C$
⟨*proof*⟩

Big Intersection – greatest lower bound of a set.

**lemma** *Inter-lower*: $B \in A \Longrightarrow \mathit{Inter}\ A \subseteq B$
⟨*proof*⟩

**lemma** *Inter-subset*:
$[|\ !!X.\ X \in A \Longrightarrow X \subseteq B;\ A \neq \{\}\ |] \Longrightarrow \bigcap A \subseteq B$

⟨*proof*⟩

**lemma** *Inter-greatest*: $(!!X.\ X \in A ==> C \subseteq X) ==> C \subseteq Inter\ A$
⟨*proof*⟩

**lemma** *INT-lower*: $a \in A ==> (\bigcap x{\in}A.\ B\ x) \subseteq B\ a$
⟨*proof*⟩

**lemma** *INT-greatest*: $(!!x.\ x \in A ==> C \subseteq B\ x) ==> C \subseteq (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

Finite Union – the least upper bound of two sets.

**lemma** *Un-upper1*: $A \subseteq A \cup B$
⟨*proof*⟩

**lemma** *Un-upper2*: $B \subseteq A \cup B$
⟨*proof*⟩

**lemma** *Un-least*: $A \subseteq C ==> B \subseteq C ==> A \cup B \subseteq C$
⟨*proof*⟩

Finite Intersection – the greatest lower bound of two sets.

**lemma** *Int-lower1*: $A \cap B \subseteq A$
⟨*proof*⟩

**lemma** *Int-lower2*: $A \cap B \subseteq B$
⟨*proof*⟩

**lemma** *Int-greatest*: $C \subseteq A ==> C \subseteq B ==> C \subseteq A \cap B$
⟨*proof*⟩

Set difference.

**lemma** *Diff-subset*: $A - B \subseteq A$
⟨*proof*⟩

**lemma** *Diff-subset-conv*: $(A - B \subseteq C) = (A \subseteq B \cup C)$
⟨*proof*⟩

### 5.5.2 Equalities involving union, intersection, inclusion, etc.

{}.

**lemma** *Collect-const* [*simp*]: $\{s.\ P\} = (\textit{if } P \textit{ then } UNIV \textit{ else } \{\})$
— supersedes *Collect-False-empty*
⟨*proof*⟩

**lemma** *subset-empty* [*simp*]: $(A \subseteq \{\}) = (A = \{\})$
⟨*proof*⟩

**lemma** *not-psubset-empty* [*iff*]: $\neg\ (A < \{\})$
⟨*proof*⟩

**lemma** *Collect-empty-eq* [*simp*]: $(Collect\ P = \{\}) = (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

**lemma** *empty-Collect-eq* [*simp*]: $(\{\} = Collect\ P) = (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

**lemma** *Collect-neg-eq*: $\{x.\ \neg\ P\ x\} = -\ \{x.\ P\ x\}$
⟨*proof*⟩

**lemma** *Collect-disj-eq*: $\{x.\ P\ x\ |\ Q\ x\} = \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

**lemma** *Collect-imp-eq*: $\{x.\ P\ x\ -->\ Q\ x\} = -\{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

**lemma** *Collect-conj-eq*: $\{x.\ P\ x\ \&\ Q\ x\} = \{x.\ P\ x\} \cap \{x.\ Q\ x\}$
⟨*proof*⟩

**lemma** *Collect-all-eq*: $\{x.\ \forall y.\ P\ x\ y\} = (\bigcap y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

**lemma** *Collect-ball-eq*: $\{x.\ \forall y{\in}A.\ P\ x\ y\} = (\bigcap y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

**lemma** *Collect-ex-eq* [*noatp*]: $\{x.\ \exists y.\ P\ x\ y\} = (\bigcup y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

**lemma** *Collect-bex-eq* [*noatp*]: $\{x.\ \exists y{\in}A.\ P\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

*insert.*

**lemma** *insert-is-Un*: *insert a A* = $\{a\}$ *Un A*
— NOT SUITABLE FOR REWRITING since $\{a\}$ == *insert a* $\{\}$
⟨*proof*⟩

**lemma** *insert-not-empty* [*simp*]: *insert a A* $\neq \{\}$
⟨*proof*⟩

**lemmas** *empty-not-insert* = *insert-not-empty* [*symmetric*, *standard*]
**declare** *empty-not-insert* [*simp*]

**lemma** *insert-absorb*: *a* $\in$ *A* ==> *insert a A* = *A*
— [*simp*] causes recursive calls when there are nested inserts
— with *quadratic* running time
⟨*proof*⟩

**lemma** *insert-absorb2* [*simp*]: *insert x* (*insert x A*) = *insert x A*
⟨*proof*⟩

**lemma** *insert-commute*: *insert x* (*insert y A*) = *insert y* (*insert x A*)
⟨*proof*⟩

**lemma** *insert-subset* [*simp*]: (*insert x A* ⊆ *B*) = (*x* ∈ *B* & *A* ⊆ *B*)
⟨*proof*⟩

**lemma** *mk-disjoint-insert*: *a* ∈ *A* ==> ∃ *B*. *A* = *insert a B* & *a* ∉ *B*
— use new *B* rather than *A* − {*a*} to avoid infinite unfolding
⟨*proof*⟩

**lemma** *insert-Collect*: *insert a* (*Collect P*) = {*u*. *u* ≠ *a* −−> *P u*}
⟨*proof*⟩

**lemma** *UN-insert-distrib*: *u* ∈ *A* ==> (⋃ *x*∈*A*. *insert a* (*B x*)) = *insert a* (⋃ *x*∈*A*. *B x*)
⟨*proof*⟩

**lemma** *insert-inter-insert*[*simp*]: *insert a A* ∩ *insert a B* = *insert a* (*A* ∩ *B*)
⟨*proof*⟩

**lemma** *insert-disjoint* [*simp*,*noatp*]:
(*insert a A* ∩ *B* = {}) = (*a* ∉ *B* ∧ *A* ∩ *B* = {})
({} = *insert a A* ∩ *B*) = (*a* ∉ *B* ∧ {} = *A* ∩ *B*)
⟨*proof*⟩

**lemma** *disjoint-insert* [*simp*,*noatp*]:
(*B* ∩ *insert a A* = {}) = (*a* ∉ *B* ∧ *B* ∩ *A* = {})
({} = *A* ∩ *insert b B*) = (*b* ∉ *A* ∧ {} = *A* ∩ *B*)
⟨*proof*⟩

*image.*

**lemma** *image-empty* [*simp*]: *f*‘{} = {}
⟨*proof*⟩

**lemma** *image-insert* [*simp*]: *f* ‘ *insert a B* = *insert* (*f a*) (*f*‘*B*)
⟨*proof*⟩

**lemma** *image-constant*: *x* ∈ *A* ==> (λ*x*. *c*) ‘ *A* = {*c*}
⟨*proof*⟩

**lemma** *image-constant-conv*: (%*x*. *c*) ‘ *A* = (*if A* = {} *then* {} *else* {*c*})
⟨*proof*⟩

**lemma** *image-image*: *f* ‘ (*g* ‘ *A*) = (λ*x*. *f* (*g x*)) ‘ *A*
⟨*proof*⟩

**lemma** *insert-image* [*simp*]: $x \in A \Longrightarrow$ *insert* $(f\ x)\ (f`A) = f`A$
⟨*proof*⟩

**lemma** *image-is-empty* [*iff*]: $(f`A = \{\}) = (A = \{\})$
⟨*proof*⟩

**lemma** *image-Collect* [*noatp*]: $f\ `\ \{x.\ P\ x\} = \{f\ x \mid x.\ P\ x\}$
— NOT suitable as a default simprule: the RHS isn't simpler than the LHS, with its implicit quantifier and conjunction. Also image enjoys better equational properties than does the RHS.
⟨*proof*⟩

**lemma** *if-image-distrib* [*simp*]:
$(\lambda x.\ \textit{if}\ P\ x\ \textit{then}\ f\ x\ \textit{else}\ g\ x)\ `\ S$
$= (f\ `\ (S \cap \{x.\ P\ x\})) \cup (g\ `\ (S \cap \{x.\ \neg\ P\ x\}))$
⟨*proof*⟩

**lemma** *image-cong*: $M = N \Longrightarrow (!!x.\ x \in N \Longrightarrow f\ x = g\ x) \Longrightarrow f`M = g`N$
⟨*proof*⟩

*range.*

**lemma** *full-SetCompr-eq* [*noatp*]: $\{u.\ \exists x.\ u = f\ x\} = \textit{range}\ f$
⟨*proof*⟩

**lemma** *range-composition*: $\textit{range}\ (\lambda x.\ f\ (g\ x)) = f`\textit{range}\ g$
⟨*proof*⟩

*Int*

**lemma** *Int-absorb* [*simp*]: $A \cap A = A$
⟨*proof*⟩

**lemma** *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
⟨*proof*⟩

**lemma** *Int-commute*: $A \cap B = B \cap A$
⟨*proof*⟩

**lemma** *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
⟨*proof*⟩

**lemma** *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$
⟨*proof*⟩

**lemmas** *Int-ac = Int-assoc Int-left-absorb Int-commute Int-left-commute*
— Intersection is an AC-operator

**lemma** *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
⟨*proof*⟩

**lemma** *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
⟨*proof*⟩

**lemma** *Int-empty-left* [*simp*]: $\{\} \cap B = \{\}$
⟨*proof*⟩

**lemma** *Int-empty-right* [*simp*]: $A \cap \{\} = \{\}$
⟨*proof*⟩

**lemma** *disjoint-eq-subset-Compl*: $(A \cap B = \{\}) = (A \subseteq -B)$
⟨*proof*⟩

**lemma** *disjoint-iff-not-equal*: $(A \cap B = \{\}) = (\forall x{\in}A.\ \forall y{\in}B.\ x \neq y)$
⟨*proof*⟩

**lemma** *Int-UNIV-left* [*simp*]: $UNIV \cap B = B$
⟨*proof*⟩

**lemma** *Int-UNIV-right* [*simp*]: $A \cap UNIV = A$
⟨*proof*⟩

**lemma** *Int-eq-Inter*: $A \cap B = \bigcap \{A, B\}$
⟨*proof*⟩

**lemma** *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
⟨*proof*⟩

**lemma** *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
⟨*proof*⟩

**lemma** *Int-UNIV* [*simp*,*noatp*]: $(A \cap B = UNIV) = (A = UNIV \ \&\ B = UNIV)$
⟨*proof*⟩

**lemma** *Int-subset-iff* [*simp*]: $(C \subseteq A \cap B) = (C \subseteq A \ \&\ C \subseteq B)$
⟨*proof*⟩

**lemma** *Int-Collect*: $(x \in A \cap \{x.\ P\ x\}) = (x \in A \ \&\ P\ x)$
⟨*proof*⟩

*Un.*

**lemma** *Un-absorb* [*simp*]: $A \cup A = A$
⟨*proof*⟩

**lemma** *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
⟨*proof*⟩

**lemma** *Un-commute*: $A \cup B = B \cup A$
⟨*proof*⟩

**lemma** *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
⟨*proof*⟩

**lemma** *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$
⟨*proof*⟩

**lemmas** *Un-ac = Un-assoc Un-left-absorb Un-commute Un-left-commute*
— Union is an AC-operator

**lemma** *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
⟨*proof*⟩

**lemma** *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
⟨*proof*⟩

**lemma** *Un-empty-left* [*simp*]: $\{\} \cup B = B$
⟨*proof*⟩

**lemma** *Un-empty-right* [*simp*]: $A \cup \{\} = A$
⟨*proof*⟩

**lemma** *Un-UNIV-left* [*simp*]: $UNIV \cup B = UNIV$
⟨*proof*⟩

**lemma** *Un-UNIV-right* [*simp*]: $A \cup UNIV = UNIV$
⟨*proof*⟩

**lemma** *Un-eq-Union*: $A \cup B = \bigcup \{A, B\}$
⟨*proof*⟩

**lemma** *Un-insert-left* [*simp*]: $(insert\ a\ B) \cup C = insert\ a\ (B \cup C)$
⟨*proof*⟩

**lemma** *Un-insert-right* [*simp*]: $A \cup (insert\ a\ B) = insert\ a\ (A \cup B)$
⟨*proof*⟩

**lemma** *Int-insert-left*:
$(insert\ a\ B)\ Int\ C = (if\ a \in C\ then\ insert\ a\ (B \cap C)\ else\ B \cap C)$
⟨*proof*⟩

**lemma** *Int-insert-right*:
$A \cap (insert\ a\ B) = (if\ a \in A\ then\ insert\ a\ (A \cap B)\ else\ A \cap B)$
⟨*proof*⟩

**lemma** *Un-Int-distrib*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
⟨*proof*⟩

**lemma** *Un-Int-distrib2*: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$
⟨*proof*⟩

**lemma** *Un-Int-crazy*:
$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
⟨*proof*⟩

**lemma** *subset-Un-eq*: $(A \subseteq B) = (A \cup B = B)$
⟨*proof*⟩

**lemma** *Un-empty* [*iff*]: $(A \cup B = \{\}) = (A = \{\} \ \& \ B = \{\})$
⟨*proof*⟩

**lemma** *Un-subset-iff* [*simp*]: $(A \cup B \subseteq C) = (A \subseteq C \ \& \ B \subseteq C)$
⟨*proof*⟩

**lemma** *Un-Diff-Int*: $(A - B) \cup (A \cap B) = A$
⟨*proof*⟩

**lemma** *Diff-Int2*: $A \cap C - B \cap C = A \cap C - B$
⟨*proof*⟩

Set complement

**lemma** *Compl-disjoint* [*simp*]: $A \cap -A = \{\}$
⟨*proof*⟩

**lemma** *Compl-disjoint2* [*simp*]: $-A \cap A = \{\}$
⟨*proof*⟩

**lemma** *Compl-partition*: $A \cup -A = UNIV$
⟨*proof*⟩

**lemma** *Compl-partition2*: $-A \cup A = UNIV$
⟨*proof*⟩

**lemma** *double-complement* [*simp*]: $-(-A) = (A::'a\ set)$
⟨*proof*⟩

**lemma** *Compl-Un* [*simp*]: $-(A \cup B) = (-A) \cap (-B)$
⟨*proof*⟩

**lemma** *Compl-Int* [*simp*]: $-(A \cap B) = (-A) \cup (-B)$
⟨*proof*⟩

**lemma** *Compl-UN* [*simp*]: $-(\bigcup x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ -B\ x)$
⟨*proof*⟩

**lemma** *Compl-INT* [*simp*]: $-(\bigcap x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ -B\ x)$

⟨*proof*⟩

**lemma** *subset-Compl-self-eq*: $(A \subseteq -A) = (A = \{\})$
⟨*proof*⟩

**lemma** *Un-Int-assoc-eq*: $((A \cap B) \cup C = A \cap (B \cup C)) = (C \subseteq A)$
— Halmos, Naive Set Theory, page 16.
⟨*proof*⟩

**lemma** *Compl-UNIV-eq* [*simp*]: $-UNIV = \{\}$
⟨*proof*⟩

**lemma** *Compl-empty-eq* [*simp*]: $-\{\} = UNIV$
⟨*proof*⟩

**lemma** *Compl-subset-Compl-iff* [*iff*]: $(-A \subseteq -B) = (B \subseteq A)$
⟨*proof*⟩

**lemma** *Compl-eq-Compl-iff* [*iff*]: $(-A = -B) = (A = (B::'a\ set))$
⟨*proof*⟩

*Union.*

**lemma** *Union-empty* [*simp*]: $Union(\{\}) = \{\}$
⟨*proof*⟩

**lemma** *Union-UNIV* [*simp*]: $Union\ UNIV = UNIV$
⟨*proof*⟩

**lemma** *Union-insert* [*simp*]: $Union\ (insert\ a\ B) = a \cup \bigcup B$
⟨*proof*⟩

**lemma** *Union-Un-distrib* [*simp*]: $\bigcup(A\ Un\ B) = \bigcup A \cup \bigcup B$
⟨*proof*⟩

**lemma** *Union-Int-subset*: $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$
⟨*proof*⟩

**lemma** *Union-empty-conv* [*simp*,*noatp*]: $(\bigcup A = \{\}) = (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

**lemma** *empty-Union-conv* [*simp*,*noatp*]: $(\{\} = \bigcup A) = (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

**lemma** *Union-disjoint*: $(\bigcup C \cap A = \{\}) = (\forall B{\in}C.\ B \cap A = \{\})$
⟨*proof*⟩

*Inter.*

**lemma** *Inter-empty* [*simp*]: $\bigcap\{\} = UNIV$

⟨*proof*⟩

**lemma** *Inter-UNIV* [*simp*]: $\bigcap UNIV = \{\}$
⟨*proof*⟩

**lemma** *Inter-insert* [*simp*]: $\bigcap (insert\ a\ B) = a \cap \bigcap B$
⟨*proof*⟩

**lemma** *Inter-Un-subset*: $\bigcap A \cup \bigcap B \subseteq \bigcap (A \cap B)$
⟨*proof*⟩

**lemma** *Inter-Un-distrib*: $\bigcap (A \cup B) = \bigcap A \cap \bigcap B$
⟨*proof*⟩

**lemma** *Inter-UNIV-conv* [*simp*,*noatp*]:
$(\bigcap A = UNIV) = (\forall x{\in}A.\ x = UNIV)$
$(UNIV = \bigcap A) = (\forall x{\in}A.\ x = UNIV)$
⟨*proof*⟩

*UN* and *INT*.

Basic identities:

**lemma** *UN-empty* [*simp*,*noatp*]: $(\bigcup x{\in}\{\}.\ B\ x) = \{\}$
⟨*proof*⟩

**lemma** *UN-empty2* [*simp*]: $(\bigcup x{\in}A.\ \{\}) = \{\}$
⟨*proof*⟩

**lemma** *UN-singleton* [*simp*]: $(\bigcup x{\in}A.\ \{x\}) = A$
⟨*proof*⟩

**lemma** *UN-absorb*: $k \in I \Longrightarrow A\ k \cup (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ A\ i)$
⟨*proof*⟩

**lemma** *INT-empty* [*simp*]: $(\bigcap x{\in}\{\}.\ B\ x) = UNIV$
⟨*proof*⟩

**lemma** *INT-absorb*: $k \in I \Longrightarrow A\ k \cap (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ A\ i)$
⟨*proof*⟩

**lemma** *UN-insert* [*simp*]: $(\bigcup x{\in}insert\ a\ A.\ B\ x) = B\ a \cup UNION\ A\ B$
⟨*proof*⟩

**lemma** *UN-Un*[*simp*]: $(\bigcup i \in A \cup B.\ M\ i) = (\bigcup i{\in}A.\ M\ i) \cup (\bigcup i{\in}B.\ M\ i)$
⟨*proof*⟩

**lemma** *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B\ y).\ C\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}B\ y.\ C\ x)$
⟨*proof*⟩

**lemma** *UN-subset-iff*: $((\bigcup i{\in}I.\ A\ i) \subseteq B) = (\forall i{\in}I.\ A\ i \subseteq B)$

⟨*proof*⟩

**lemma** *INT-subset-iff*: $(B \subseteq (\bigcap i{\in}I.\ A\ i)) = (\forall i{\in}I.\ B \subseteq A\ i)$
⟨*proof*⟩

**lemma** *INT-insert* [*simp*]: $(\bigcap x \in \mathit{insert}\ a\ A.\ B\ x) = B\ a \cap \mathit{INTER}\ A\ B$
⟨*proof*⟩

**lemma** *INT-Un*: $(\bigcap i \in A \cup B.\ M\ i) = (\bigcap i \in A.\ M\ i) \cap (\bigcap i{\in}B.\ M\ i)$
⟨*proof*⟩

**lemma** *INT-insert-distrib*:
$u \in A \Longrightarrow (\bigcap x{\in}A.\ \mathit{insert}\ a\ (B\ x)) = \mathit{insert}\ a\ (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

**lemma** *Union-image-eq* [*simp*]: $\bigcup(B`A) = (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

**lemma** *image-Union*: $f\ `\ \bigcup S = (\bigcup x{\in}S.\ f\ `\ x)$
⟨*proof*⟩

**lemma** *Inter-image-eq* [*simp*]: $\bigcap(B`A) = (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

**lemma** *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ c)$
⟨*proof*⟩

**lemma** *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ c)$
⟨*proof*⟩

**lemma** *UN-eq*: $(\bigcup x{\in}A.\ B\ x) = \bigcup(\{Y.\ \exists x{\in}A.\ Y = B\ x\})$
⟨*proof*⟩

**lemma** *INT-eq*: $(\bigcap x{\in}A.\ B\ x) = \bigcap(\{Y.\ \exists x{\in}A.\ Y = B\ x\})$
— Look: it has an *existential* quantifier
⟨*proof*⟩

**lemma** *UNION-empty-conv*[*simp*]:
$(\{\} = (\mathit{UN}\ x{:}A.\ B\ x)) = (\forall x{\in}A.\ B\ x = \{\})$
$((\mathit{UN}\ x{:}A.\ B\ x) = \{\}) = (\forall x{\in}A.\ B\ x = \{\})$
⟨*proof*⟩

**lemma** *INTER-UNIV-conv*[*simp*]:
$(\mathit{UNIV} = (\mathit{INT}\ x{:}A.\ B\ x)) = (\forall x{\in}A.\ B\ x = \mathit{UNIV})$
$((\mathit{INT}\ x{:}A.\ B\ x) = \mathit{UNIV}) = (\forall x{\in}A.\ B\ x = \mathit{UNIV})$
⟨*proof*⟩

Distributive laws:

**lemma** *Int-Union*: $A \cap \bigcup B = (\bigcup C{\in}B.\ A \cap C)$

⟨*proof*⟩

**lemma** *Int-Union2*: $\bigcup B \cap A = (\bigcup C{\in}B.\ C \cap A)$
⟨*proof*⟩

**lemma** *Un-Union-image*: $(\bigcup x{\in}C.\ A\ x \cup B\ x) = \bigcup (A`C) \cup \bigcup (B`C)$
— Devlin, Fundamentals of Contemporary Set Theory, page 12, exercise 5:
— Union of a family of unions
⟨*proof*⟩

**lemma** *UN-Un-distrib*: $(\bigcup i{\in}I.\ A\ i \cup B\ i) = (\bigcup i{\in}I.\ A\ i) \cup (\bigcup i{\in}I.\ B\ i)$
— Equivalent version
⟨*proof*⟩

**lemma** *Un-Inter*: $A \cup \bigcap B = (\bigcap C{\in}B.\ A \cup C)$
⟨*proof*⟩

**lemma** *Int-Inter-image*: $(\bigcap x{\in}C.\ A\ x \cap B\ x) = \bigcap (A`C) \cap \bigcap (B`C)$
⟨*proof*⟩

**lemma** *INT-Int-distrib*: $(\bigcap i{\in}I.\ A\ i \cap B\ i) = (\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i)$
— Equivalent version
⟨*proof*⟩

**lemma** *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ B \cap A\ i)$
— Halmos, Naive Set Theory, page 35.
⟨*proof*⟩

**lemma** *Un-INT-distrib*: $B \cup (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ B \cup A\ i)$
⟨*proof*⟩

**lemma** *Int-UN-distrib2*: $(\bigcup i{\in}I.\ A\ i) \cap (\bigcup j{\in}J.\ B\ j) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A\ i \cap B\ j)$
⟨*proof*⟩

**lemma** *Un-INT-distrib2*: $(\bigcap i{\in}I.\ A\ i) \cup (\bigcap j{\in}J.\ B\ j) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A\ i \cup B\ j)$
⟨*proof*⟩

Bounded quantifiers.

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

**lemma** *ball-Un*: $(\forall x \in A \cup B.\ P\ x) = ((\forall x{\in}A.\ P\ x)\ \&\ (\forall x{\in}B.\ P\ x))$
⟨*proof*⟩

**lemma** *bex-Un*: $(\exists x \in A \cup B.\ P\ x) = ((\exists x{\in}A.\ P\ x) \mid (\exists x{\in}B.\ P\ x))$
⟨*proof*⟩

**lemma** *ball-UN*: $(\forall z \in \mathit{UNION}\ A\ B.\ P\ z) = (\forall x{\in}A.\ \forall z \in B\ x.\ P\ z)$
⟨*proof*⟩

**lemma** *bex-UN*: $(\exists z \in \mathit{UNION}\ A\ B.\ P\ z) = (\exists x{\in}A.\ \exists z{\in}B\ x.\ P\ z)$
⟨*proof*⟩

Set difference.

**lemma** *Diff-eq*: $A - B = A \cap (-B)$
⟨*proof*⟩

**lemma** *Diff-eq-empty-iff* [*simp*,*noatp*]: $(A - B = \{\}) = (A \subseteq B)$
⟨*proof*⟩

**lemma** *Diff-cancel* [*simp*]: $A - A = \{\}$
⟨*proof*⟩

**lemma** *Diff-idemp* [*simp*]: $(A - B) - B = A - (B::'a\ set)$
⟨*proof*⟩

**lemma** *Diff-triv*: $A \cap B = \{\} \Longrightarrow A - B = A$
⟨*proof*⟩

**lemma** *empty-Diff* [*simp*]: $\{\} - A = \{\}$
⟨*proof*⟩

**lemma** *Diff-empty* [*simp*]: $A - \{\} = A$
⟨*proof*⟩

**lemma** *Diff-UNIV* [*simp*]: $A - \mathit{UNIV} = \{\}$
⟨*proof*⟩

**lemma** *Diff-insert0* [*simp*,*noatp*]: $x \notin A \Longrightarrow A - \mathit{insert}\ x\ B = A - B$
⟨*proof*⟩

**lemma** *Diff-insert*: $A - \mathit{insert}\ a\ B = A - B - \{a\}$
— NOT SUITABLE FOR REWRITING since $\{a\} == \mathit{insert}\ a\ 0$
⟨*proof*⟩

**lemma** *Diff-insert2*: $A - \mathit{insert}\ a\ B = A - \{a\} - B$
— NOT SUITABLE FOR REWRITING since $\{a\} == \mathit{insert}\ a\ 0$
⟨*proof*⟩

**lemma** *insert-Diff-if*: $\mathit{insert}\ x\ A - B = (\mathit{if}\ x \in B\ \mathit{then}\ A - B\ \mathit{else}\ \mathit{insert}\ x\ (A - B))$
⟨*proof*⟩

**lemma** *insert-Diff1* [*simp*]: $x \in B \Longrightarrow \mathit{insert}\ x\ A - B = A - B$
⟨*proof*⟩

**lemma** *insert-Diff-single*[*simp*]: *insert a* $(A - \{a\}) =$ *insert a A*
⟨*proof*⟩

**lemma** *insert-Diff*: $a \in A \Longrightarrow$ *insert a* $(A - \{a\}) = A$
⟨*proof*⟩

**lemma** *Diff-insert-absorb*: $x \notin A \Longrightarrow (\textit{insert}\ x\ A) - \{x\} = A$
⟨*proof*⟩

**lemma** *Diff-disjoint* [*simp*]: $A \cap (B - A) = \{\}$
⟨*proof*⟩

**lemma** *Diff-partition*: $A \subseteq B \Longrightarrow A \cup (B - A) = B$
⟨*proof*⟩

**lemma** *double-diff*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow B - (C - A) = A$
⟨*proof*⟩

**lemma** *Un-Diff-cancel* [*simp*]: $A \cup (B - A) = A \cup B$
⟨*proof*⟩

**lemma** *Un-Diff-cancel2* [*simp*]: $(B - A) \cup A = B \cup A$
⟨*proof*⟩

**lemma** *Diff-Un*: $A - (B \cup C) = (A - B) \cap (A - C)$
⟨*proof*⟩

**lemma** *Diff-Int*: $A - (B \cap C) = (A - B) \cup (A - C)$
⟨*proof*⟩

**lemma** *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
⟨*proof*⟩

**lemma** *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
⟨*proof*⟩

**lemma** *Diff-Int-distrib*: $C \cap (A - B) = (C \cap A) - (C \cap B)$
⟨*proof*⟩

**lemma** *Diff-Int-distrib2*: $(A - B) \cap C = (A \cap C) - (B \cap C)$
⟨*proof*⟩

**lemma** *Diff-Compl* [*simp*]: $A - (-B) = A \cap B$
⟨*proof*⟩

**lemma** *Compl-Diff-eq* [*simp*]: $-(A - B) = -A \cup B$
⟨*proof*⟩

Quantification over type *bool*.

**lemma** *bool-induct*: $P\ True \Longrightarrow P\ False \Longrightarrow P\ x$
⟨*proof*⟩

**lemma** *all-bool-eq*: $(\forall b.\ P\ b) \longleftrightarrow P\ True \wedge P\ False$
⟨*proof*⟩

**lemma** *bool-contrapos*: $P\ x \Longrightarrow \neg\ P\ False \Longrightarrow P\ True$
⟨*proof*⟩

**lemma** *ex-bool-eq*: $(\exists b.\ P\ b) \longleftrightarrow P\ True \vee P\ False$
⟨*proof*⟩

**lemma** *Un-eq-UN*: $A \cup B = (\bigcup b.\ if\ b\ then\ A\ else\ B)$
⟨*proof*⟩

**lemma** *UN-bool-eq*: $(\bigcup b{::}bool.\ A\ b) = (A\ True \cup A\ False)$
⟨*proof*⟩

**lemma** *INT-bool-eq*: $(\bigcap b{::}bool.\ A\ b) = (A\ True \cap A\ False)$
⟨*proof*⟩

*Pow*

**lemma** *Pow-empty* [*simp*]: $Pow\ \{\} = \{\{\}\}$
⟨*proof*⟩

**lemma** *Pow-insert*: $Pow\ (insert\ a\ A) = Pow\ A \cup (insert\ a\ `\ Pow\ A)$
⟨*proof*⟩

**lemma** *Pow-Compl*: $Pow\ (-\ A) = \{-B \mid B.\ A \in Pow\ B\}$
⟨*proof*⟩

**lemma** *Pow-UNIV* [*simp*]: $Pow\ UNIV = UNIV$
⟨*proof*⟩

**lemma** *Un-Pow-subset*: $Pow\ A \cup Pow\ B \subseteq Pow\ (A \cup B)$
⟨*proof*⟩

**lemma** *UN-Pow-subset*: $(\bigcup x{\in}A.\ Pow\ (B\ x)) \subseteq Pow\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

**lemma** *subset-Pow-Union*: $A \subseteq Pow\ (\bigcup A)$
⟨*proof*⟩

**lemma** *Union-Pow-eq* [*simp*]: $\bigcup(Pow\ A) = A$
⟨*proof*⟩

**lemma** *Pow-Int-eq* [*simp*]: $Pow\ (A \cap B) = Pow\ A \cap Pow\ B$
⟨*proof*⟩

**lemma** *Pow-INT-eq*: $Pow\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ Pow\ (B\ x))$
⟨*proof*⟩

Miscellany.

**lemma** *set-eq-subset*: $(A = B) = (A \subseteq B\ \&\ B \subseteq A)$
⟨*proof*⟩

**lemma** *subset-iff*: $(A \subseteq B) = (\forall t.\ t \in A \longrightarrow t \in B)$
⟨*proof*⟩

**lemma** *subset-iff-psubset-eq*: $(A \subseteq B) = ((A \subset B) \mid (A = B))$
⟨*proof*⟩

**lemma** *all-not-in-conv* [*simp*]: $(\forall x.\ x \notin A) = (A = \{\})$
⟨*proof*⟩

**lemma** *ex-in-conv*: $(\exists x.\ x \in A) = (A \neq \{\})$
⟨*proof*⟩

**lemma** *distinct-lemma*: $f\ x \neq f\ y \Longrightarrow x \neq y$
⟨*proof*⟩

Miniscoping: pushing in quantifiers and big Unions and Intersections.

**lemma** *UN-simps* [*simp*]:
$!!a\ B\ C.\ (UN\ x{:}C.\ insert\ a\ (B\ x)) = (if\ C{=}\{\}\ then\ \{\}\ else\ insert\ a\ (UN\ x{:}C.\ B\ x))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x\ Un\ B) = ((if\ C{=}\{\}\ then\ \{\}\ else\ (UN\ x{:}C.\ A\ x)\ Un\ B))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ Un\ B\ x) = ((if\ C{=}\{\}\ then\ \{\}\ else\ A\ Un\ (UN\ x{:}C.\ B\ x)))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x\ Int\ B) = ((UN\ x{:}C.\ A\ x)\ Int\ B)$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ Int\ B\ x) = (A\ Int\ (UN\ x{:}C.\ B\ x))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x - B) = ((UN\ x{:}C.\ A\ x) - B)$
$!!A\ B\ C.\ (UN\ x{:}C.\ A - B\ x) = (A - (INT\ x{:}C.\ B\ x))$
$!!A\ B.\ (UN\ x{:}\ Union\ A.\ B\ x) = (UN\ y{:}A.\ UN\ x{:}y.\ B\ x)$
$!!A\ B\ C.\ (UN\ z{:}\ UNION\ A\ B.\ C\ z) = (UN\ \ x{:}A.\ UN\ z{:}\ B(x).\ C\ z)$
$!!A\ B\ f.\ (UN\ x{:}f`A.\ B\ x) = (UN\ a{:}A.\ B\ (f\ a))$
⟨*proof*⟩

**lemma** *INT-simps* [*simp*]:
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x\ Int\ B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (INT\ x{:}C.\ A\ x)\ Int\ B)$
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ Int\ B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A\ Int\ (INT\ x{:}C.\ B\ x))$
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x - B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (INT\ x{:}C.\ A\ x) - B)$
$!!A\ B\ C.\ (INT\ x{:}C.\ A - B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A - (UN\ x{:}C.\ B\ x))$

!!$a\ B\ C.\ (INT\ x{:}C.\ insert\ a\ (B\ x)) = insert\ a\ (INT\ x{:}C.\ B\ x)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x\ Un\ B) = ((INT\ x{:}C.\ A\ x)\ Un\ B)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ Un\ B\ x) = (A\ Un\ (INT\ x{:}C.\ B\ x))$
!!$A\ B.\ (INT\ x{:}\ Union\ A.\ B\ x) = (INT\ y{:}A.\ INT\ x{:}y.\ B\ x)$
!!$A\ B\ C.\ (INT\ z{:}\ UNION\ A\ B.\ C\ z) = (INT\ x{:}A.\ INT\ z{:}\ B(x).\ C\ z)$
!!$A\ B\ f.\ (INT\ x{:}f`A.\ B\ x) = (INT\ a{:}A.\ B\ (f\ a))$
⟨*proof*⟩

**lemma** *ball-simps* [*simp*,*noatp*]:
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ x\ |\ Q) = ((ALL\ x{:}A.\ P\ x)\ |\ Q)$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ |\ Q\ x) = (P\ |\ (ALL\ x{:}A.\ Q\ x))$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ --{>}\ Q\ x) = (P\ --{>}\ (ALL\ x{:}A.\ Q\ x))$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ x\ --{>}\ Q) = ((EX\ x{:}A.\ P\ x)\ --{>}\ Q)$
!!$P.\ (ALL\ x{:}\{\}.\ P\ x) = True$
!!$P.\ (ALL\ x{:}UNIV.\ P\ x) = (ALL\ x.\ P\ x)$
!!$a\ B\ P.\ (ALL\ x{:}insert\ a\ B.\ P\ x) = (P\ a\ \&\ (ALL\ x{:}B.\ P\ x))$
!!$A\ P.\ (ALL\ x{:}Union\ A.\ P\ x) = (ALL\ y{:}A.\ ALL\ x{:}y.\ P\ x)$
!!$A\ B\ P.\ (ALL\ x{:}\ UNION\ A\ B.\ P\ x) = (ALL\ a{:}A.\ ALL\ x{:}\ B\ a.\ P\ x)$
!!$P\ Q.\ (ALL\ x{:}Collect\ Q.\ P\ x) = (ALL\ x.\ Q\ x\ --{>}\ P\ x)$
!!$A\ P\ f.\ (ALL\ x{:}f`A.\ P\ x) = (ALL\ x{:}A.\ P\ (f\ x))$
!!$A\ P.\ (\sim(ALL\ x{:}A.\ P\ x)) = (EX\ x{:}A.\ \sim P\ x)$
⟨*proof*⟩

**lemma** *bex-simps* [*simp*,*noatp*]:
!!$A\ P\ Q.\ (EX\ x{:}A.\ P\ x\ \&\ Q) = ((EX\ x{:}A.\ P\ x)\ \&\ Q)$
!!$A\ P\ Q.\ (EX\ x{:}A.\ P\ \&\ Q\ x) = (P\ \&\ (EX\ x{:}A.\ Q\ x))$
!!$P.\ (EX\ x{:}\{\}.\ P\ x) = False$
!!$P.\ (EX\ x{:}UNIV.\ P\ x) = (EX\ x.\ P\ x)$
!!$a\ B\ P.\ (EX\ x{:}insert\ a\ B.\ P\ x) = (P(a)\ |\ (EX\ x{:}B.\ P\ x))$
!!$A\ P.\ (EX\ x{:}Union\ A.\ P\ x) = (EX\ y{:}A.\ EX\ x{:}y.\ P\ x)$
!!$A\ B\ P.\ (EX\ x{:}\ UNION\ A\ B.\ P\ x) = (EX\ a{:}A.\ EX\ x{:}B\ a.\ P\ x)$
!!$P\ Q.\ (EX\ x{:}Collect\ Q.\ P\ x) = (EX\ x.\ Q\ x\ \&\ P\ x)$
!!$A\ P\ f.\ (EX\ x{:}f`A.\ P\ x) = (EX\ x{:}A.\ P\ (f\ x))$
!!$A\ P.\ (\sim(EX\ x{:}A.\ P\ x)) = (ALL\ x{:}A.\ \sim P\ x)$
⟨*proof*⟩

**lemma** *ball-conj-distrib*:
$(ALL\ x{:}A.\ P\ x\ \&\ Q\ x) = ((ALL\ x{:}A.\ P\ x)\ \&\ (ALL\ x{:}A.\ Q\ x))$
⟨*proof*⟩

**lemma** *bex-disj-distrib*:
$(EX\ x{:}A.\ P\ x\ |\ Q\ x) = ((EX\ x{:}A.\ P\ x)\ |\ (EX\ x{:}A.\ Q\ x))$
⟨*proof*⟩

Maxiscoping: pulling out big Unions and Intersections.

**lemma** *UN-extend-simps*:
!!$a\ B\ C.\ insert\ a\ (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ \{a\}\ else\ (UN\ x{:}C.\ insert\ a\ (B\ x)))$
!!$A\ B\ C.\ (UN\ x{:}C.\ A\ x)\ Un\ B = (if\ C{=}\{\}\ then\ B\ else\ (UN\ x{:}C.\ A\ x\ Un\ B))$

!!*A B C*. *A Un* (*UN x*:*C*. *B x*) = (*if C*={} *then A else* (*UN x*:*C*. *A Un B x*))
!!*A B C*. ((*UN x*:*C*. *A x*) *Int B*) = (*UN x*:*C*. *A x Int B*)
!!*A B C*. (*A Int* (*UN x*:*C*. *B x*)) = (*UN x*:*C*. *A Int B x*)
!!*A B C*. ((*UN x*:*C*. *A x*) − *B*) = (*UN x*:*C*. *A x* − *B*)
!!*A B C*. (*A* − (*INT x*:*C*. *B x*)) = (*UN x*:*C*. *A* − *B x*)
!!*A B*. (*UN y*:*A*. *UN x*:*y*. *B x*) = (*UN x*: *Union A*. *B x*)
!!*A B C*. (*UN x*:*A*. *UN z*: *B*(*x*). *C z*) = (*UN z*: *UNION A B*. *C z*)
!!*A B f*. (*UN a*:*A*. *B* (*f a*)) = (*UN x*:*f*‘*A*. *B x*)
⟨*proof*⟩

**lemma** *INT-extend-simps*:
!!*A B C*. (*INT x*:*C*. *A x*) *Int B* = (*if C*={} *then B else* (*INT x*:*C*. *A x Int B*))
!!*A B C*. *A Int* (*INT x*:*C*. *B x*) = (*if C*={} *then A else* (*INT x*:*C*. *A Int B x*))
!!*A B C*. (*INT x*:*C*. *A x*) − *B* = (*if C*={} *then UNIV*−*B else* (*INT x*:*C*. *A x* − *B*))
!!*A B C*. *A* − (*UN x*:*C*. *B x*) = (*if C*={} *then A else* (*INT x*:*C*. *A* − *B x*))
!!*a B C*. *insert a* (*INT x*:*C*. *B x*) = (*INT x*:*C*. *insert a* (*B x*))
!!*A B C*. ((*INT x*:*C*. *A x*) *Un B*) = (*INT x*:*C*. *A x Un B*)
!!*A B C*. *A Un* (*INT x*:*C*. *B x*) = (*INT x*:*C*. *A Un B x*)
!!*A B*. (*INT y*:*A*. *INT x*:*y*. *B x*) = (*INT x*: *Union A*. *B x*)
!!*A B C*. (*INT x*:*A*. *INT z*: *B*(*x*). *C z*) = (*INT z*: *UNION A B*. *C z*)
!!*A B f*. (*INT a*:*A*. *B* (*f a*)) = (*INT x*:*f*‘*A*. *B x*)
⟨*proof*⟩

### 5.5.3 Monotonicity of various operations

**lemma** *image-mono*: *A* ⊆ *B* ==> *f*‘*A* ⊆ *f*‘*B*
⟨*proof*⟩

**lemma** *Pow-mono*: *A* ⊆ *B* ==> *Pow A* ⊆ *Pow B*
⟨*proof*⟩

**lemma** *Union-mono*: *A* ⊆ *B* ==> ⋃ *A* ⊆ ⋃ *B*
⟨*proof*⟩

**lemma** *Inter-anti-mono*: *B* ⊆ *A* ==> ⋂ *A* ⊆ ⋂ *B*
⟨*proof*⟩

**lemma** *UN-mono*:
*A* ⊆ *B* ==> (!!*x*. *x* ∈ *A* ==> *f x* ⊆ *g x*) ==>
(⋃ *x*∈*A*. *f x*) ⊆ (⋃ *x*∈*B*. *g x*)
⟨*proof*⟩

**lemma** *INT-anti-mono*:
*B* ⊆ *A* ==> (!!*x*. *x* ∈ *A* ==> *f x* ⊆ *g x*) ==>
(⋂ *x*∈*A*. *f x*) ⊆ (⋂ *x*∈*A*. *g x*)
— The last inclusion is POSITIVE!
⟨*proof*⟩

**lemma** *insert-mono*: $C \subseteq D \Longrightarrow \mathit{insert}\ a\ C \subseteq \mathit{insert}\ a\ D$
⟨*proof*⟩

**lemma** *Un-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cup B \subseteq C \cup D$
⟨*proof*⟩

**lemma** *Int-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cap B \subseteq C \cap D$
⟨*proof*⟩

**lemma** *Diff-mono*: $A \subseteq C \Longrightarrow D \subseteq B \Longrightarrow A - B \subseteq C - D$
⟨*proof*⟩

**lemma** *Compl-anti-mono*: $A \subseteq B \Longrightarrow -B \subseteq -A$
⟨*proof*⟩

Monotonicity of implications.

**lemma** *in-mono*: $A \subseteq B \Longrightarrow x \in A \longrightarrow x \in B$
⟨*proof*⟩

**lemma** *conj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1\ \&\ P2) \longrightarrow (Q1\ \&\ Q2)$
⟨*proof*⟩

**lemma** *disj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \mid P2) \longrightarrow (Q1 \mid Q2)$
⟨*proof*⟩

**lemma** *imp-mono*: $Q1 \longrightarrow P1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \longrightarrow P2) \longrightarrow (Q1 \longrightarrow Q2)$
⟨*proof*⟩

**lemma** *imp-refl*: $P \longrightarrow P$ ⟨*proof*⟩

**lemma** *ex-mono*: $(!!x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (EX\ x.\ P\ x) \longrightarrow (EX\ x.\ Q\ x)$
⟨*proof*⟩

**lemma** *all-mono*: $(!!x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (ALL\ x.\ P\ x) \longrightarrow (ALL\ x.\ Q\ x)$
⟨*proof*⟩

**lemma** *Collect-mono*: $(!!x.\ P\ x \longrightarrow Q\ x) \Longrightarrow \mathit{Collect}\ P \subseteq \mathit{Collect}\ Q$
⟨*proof*⟩

**lemma** *Int-Collect-mono*:
$A \subseteq B \Longrightarrow (!!x.\ x \in A \Longrightarrow P\ x \longrightarrow Q\ x) \Longrightarrow A \cap \mathit{Collect}\ P \subseteq B \cap \mathit{Collect}\ Q$
⟨*proof*⟩

**lemmas** *basic-monos* =
*subset-refl imp-refl disj-mono conj-mono*

*ex-mono Collect-mono in-mono*

**lemma** *eq-to-mono*: $a = b \Longrightarrow c = d \Longrightarrow b \dashrightarrow d \Longrightarrow a \dashrightarrow c$
⟨*proof*⟩

**lemma** *eq-to-mono2*: $a = b \Longrightarrow c = d \Longrightarrow {}^{\sim} b \dashrightarrow {}^{\sim} d \Longrightarrow {}^{\sim} a \dashrightarrow {}^{\sim} c$
⟨*proof*⟩

## 5.6 Inverse image of a function

**constdefs**
*vimage* :: $('a \Rightarrow 'b) \Rightarrow 'b\ set \Rightarrow 'a\ set$ (**infixr** $-`$ *90*)
[*code del*]: $f\ -`\ B == \{x.\ f\ x : B\}$

### 5.6.1 Basic rules

**lemma** *vimage-eq* [*simp*]: $(a : f\ -`\ B) = (f\ a : B)$
⟨*proof*⟩

**lemma** *vimage-singleton-eq*: $(a : f\ -`\ \{b\}) = (f\ a = b)$
⟨*proof*⟩

**lemma** *vimageI* [*intro*]: $f\ a = b \Longrightarrow b{:}B \Longrightarrow a : f\ -`\ B$
⟨*proof*⟩

**lemma** *vimageI2*: $f\ a : A \Longrightarrow a : f\ -`\ A$
⟨*proof*⟩

**lemma** *vimageE* [*elim!*]: $a{:}\ f\ -`\ B \Longrightarrow (!!x.\ f\ a = x \Longrightarrow x{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *vimageD*: $a : f\ -`\ A \Longrightarrow f\ a : A$
⟨*proof*⟩

### 5.6.2 Equations

**lemma** *vimage-empty* [*simp*]: $f\ -`\ \{\} = \{\}$
⟨*proof*⟩

**lemma** *vimage-Compl*: $f\ -`\ (-A) = -(f\ -`\ A)$
⟨*proof*⟩

**lemma** *vimage-Un* [*simp*]: $f\ -`\ (A\ Un\ B) = (f\ -`\ A)\ Un\ (f\ -`\ B)$
⟨*proof*⟩

**lemma** *vimage-Int* [*simp*]: $f\ -`\ (A\ Int\ B) = (f\ -`\ A)\ Int\ (f\ -`\ B)$
⟨*proof*⟩

**lemma** *vimage-Union*: $f\ -`\ (Union\ A) = (UN\ X{:}A.\ f\ -`\ X)$

⟨*proof*⟩

**lemma** *vimage-UN*: $f-`(UN\ x{:}A.\ B\ x) = (UN\ x{:}A.\ f\ -`\ B\ x)$
⟨*proof*⟩

**lemma** *vimage-INT*: $f-`(INT\ x{:}A.\ B\ x) = (INT\ x{:}A.\ f\ -`\ B\ x)$
⟨*proof*⟩

**lemma** *vimage-Collect-eq* [*simp*]: $f\ -`\ Collect\ P = \{y.\ P\ (f\ y)\}$
⟨*proof*⟩

**lemma** *vimage-Collect*: $(!!x.\ P\ (f\ x) = Q\ x) ==> f\ -`\ (Collect\ P) = Collect\ Q$
⟨*proof*⟩

**lemma** *vimage-insert*: $f-`(insert\ a\ B) = (f-`\{a\})\ Un\ (f-`B)$
— NOT suitable for rewriting because of the recurrence of $\{a\}$.
⟨*proof*⟩

**lemma** *vimage-Diff*: $f\ -`\ (A - B) = (f\ -`\ A) - (f\ -`\ B)$
⟨*proof*⟩

**lemma** *vimage-UNIV* [*simp*]: $f\ -`\ UNIV = UNIV$
⟨*proof*⟩

**lemma** *vimage-eq-UN*: $f-`B = (UN\ y{:}\ B.\ f-`\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

**lemma** *vimage-mono*: $A \subseteq B ==> f\ -`\ A \subseteq f\ -`\ B$
— monotonicity
⟨*proof*⟩

**lemma** *vimage-image-eq* [*noatp*]: $f\ -`\ (f\ `\ A) = \{y.\ EX\ x{:}A.\ f\ x = f\ y\}$
⟨*proof*⟩

**lemma** *image-vimage-subset*: $f\ `\ (f\ -`\ A) <= A$
⟨*proof*⟩

**lemma** *image-vimage-eq* [*simp*]: $f\ `\ (f\ -`\ A) = A\ Int\ range\ f$
⟨*proof*⟩

**lemma** *image-Int-subset*: $f`(A\ Int\ B) <= f`A\ Int\ f`B$
⟨*proof*⟩

**lemma** *image-diff-subset*: $f`A - f`B <= f`(A - B)$
⟨*proof*⟩

**lemma** *image-UN*: $(f\ `\ (UNION\ A\ B)) = (UN\ x{:}A.(f\ `\ (B\ x)))$
⟨*proof*⟩

## 5.7 Getting the Contents of a Singleton Set

**definition** *contents* :: *'a set* ⇒ *'a* **where**
  [*code del*]: *contents X* = (*THE x*. *X* = {*x*})

**lemma** *contents-eq* [*simp*]: *contents* {*x*} = *x*
  ⟨*proof*⟩

## 5.8 Transitivity rules for calculational reasoning

**lemma** *set-rev-mp*: *x*:*A* ==> *A* ⊆ *B* ==> *x*:*B*
  ⟨*proof*⟩

**lemma** *set-mp*: *A* ⊆ *B* ==> *x*:*A* ==> *x*:*B*
  ⟨*proof*⟩

**lemmas** *basic-trans-rules* [*trans*] =
  *order-trans-rules set-rev-mp set-mp*

## 5.9 Least value operator

**lemma** *Least-mono*:
  *mono* (*f*::*'a*::*order* => *'b*::*order*) ==> *EX x*:*S*. *ALL y*:*S*. *x* <= *y*
    ==> (*LEAST y*. *y* : *f* ‘ *S*) = *f* (*LEAST x*. *x* : *S*)
    — Courtesy of Stephan Merz
  ⟨*proof*⟩

## 5.10 Rudimentary code generation

**lemma** *empty-code* [*code*]: {} *x* ⟷ *False*
  ⟨*proof*⟩

**lemma** *UNIV-code* [*code*]: *UNIV x* ⟷ *True*
  ⟨*proof*⟩

**lemma** *insert-code* [*code*]: *insert y A x* ⟷ *y* = *x* ∨ *A x*
  ⟨*proof*⟩

**lemma** *inter-code* [*code*]: (*A* ∩ *B*) *x* ⟷ *A x* ∧ *B x*
  ⟨*proof*⟩

**lemma** *union-code* [*code*]: (*A* ∪ *B*) *x* ⟷ *A x* ∨ *B x*
  ⟨*proof*⟩

**lemma** *vimage-code* [*code*]: (*f* −‘ *A*) *x* = *A* (*f x*)
  ⟨*proof*⟩

## 5.11 Complete lattices

**notation**

*less-eq* (**infix** $\sqsubseteq$ *50*) **and**
*less* (**infix** $\sqsubset$ *50*) **and**
*inf* (**infixl** $\sqcap$ *70*) **and**
*sup* (**infixl** $\sqcup$ *65*)

**class** *complete-lattice* = *lattice* + *bot* + *top* +
**fixes** *Inf* :: *'a set* $\Rightarrow$ *'a* ($\bigsqcap$ - [*900*] *900*)
**and** *Sup* :: *'a set* $\Rightarrow$ *'a* ($\bigsqcup$ - [*900*] *900*)
**assumes** *Inf-lower*: $x \in A \Longrightarrow \bigsqcap A \sqsubseteq x$
**and** *Inf-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x) \Longrightarrow z \sqsubseteq \bigsqcap A$
**assumes** *Sup-upper*: $x \in A \Longrightarrow x \sqsubseteq \bigsqcup A$
**and** *Sup-least*: $(\bigwedge x.\ x \in A \Longrightarrow x \sqsubseteq z) \Longrightarrow \bigsqcup A \sqsubseteq z$
**begin**

**lemma** *Inf-Sup*: $\bigsqcap A = \bigsqcup \{b.\ \forall a \in A.\ b \leq a\}$
⟨*proof*⟩

**lemma** *Sup-Inf*: $\bigsqcup A = \bigsqcap \{b.\ \forall a \in A.\ a \leq b\}$
⟨*proof*⟩

**lemma** *Inf-Univ*: $\bigsqcap UNIV = \bigsqcup \{\}$
⟨*proof*⟩

**lemma** *Sup-Univ*: $\bigsqcup UNIV = \bigsqcap \{\}$
⟨*proof*⟩

**lemma** *Inf-insert*: $\bigsqcap insert\ a\ A = a \sqcap \bigsqcap A$
⟨*proof*⟩

**lemma** *Sup-insert*: $\bigsqcup insert\ a\ A = a \sqcup \bigsqcup A$
⟨*proof*⟩

**lemma** *Inf-singleton* [*simp*]:
$\bigsqcap \{a\} = a$
⟨*proof*⟩

**lemma** *Sup-singleton* [*simp*]:
$\bigsqcup \{a\} = a$
⟨*proof*⟩

**lemma** *Inf-insert-simp*:
$\bigsqcap insert\ a\ A = (if\ A = \{\}\ then\ a\ else\ a \sqcap \bigsqcap A)$
⟨*proof*⟩

**lemma** *Sup-insert-simp*:
$\bigsqcup insert\ a\ A = (if\ A = \{\}\ then\ a\ else\ a \sqcup \bigsqcup A)$
⟨*proof*⟩

**lemma** *Inf-binary*:

$\bigsqcap\{a, b\} = a \sqcap b$
⟨*proof*⟩

**lemma** *Sup-binary*:
$\bigsqcup\{a, b\} = a \sqcup b$
⟨*proof*⟩

**lemma** *bot-def*:
$bot = \bigsqcup\{\}$
⟨*proof*⟩

**lemma** *top-def*:
$top = \bigsqcap\{\}$
⟨*proof*⟩

**lemma** *sup-bot* [*simp*]:
$x \sqcup bot = x$
⟨*proof*⟩

**lemma** *inf-top* [*simp*]:
$x \sqcap top = x$
⟨*proof*⟩

**definition** *SUPR* :: $'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$ **where**
$SUPR\ A\ f == \bigsqcup\ (f\ `\ A)$

**definition** *INFI* :: $'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$ **where**
$INFI\ A\ f == \bigsqcap\ (f\ `\ A)$

**end**

**syntax**
*-SUP1* :: *pttrns* => $'b$ => $'b$ ((*3SUP -./ -*) [*0*, *10*] *10*)
*-SUP* :: *pttrn* => $'a\ set$ => $'b$ => $'b$ ((*3SUP -:-./ -*) [*0*, *10*] *10*)
*-INF1* :: *pttrns* => $'b$ => $'b$ ((*3INF -./ -*) [*0*, *10*] *10*)
*-INF* :: *pttrn* => $'a\ set$ => $'b$ => $'b$ ((*3INF -:-./ -*) [*0*, *10*] *10*)

**translations**
*SUP x y. B* == *SUP x. SUP y. B*
*SUP x. B* == *CONST SUPR CONST UNIV* (%*x. B*)
*SUP x. B* == *SUP x*:*CONST UNIV. B*
*SUP x*:*A. B* == *CONST SUPR A* (%*x. B*)
*INF x y. B* == *INF x. INF y. B*
*INF x. B* == *CONST INFI CONST UNIV* (%*x. B*)
*INF x. B* == *INF x*:*CONST UNIV. B*
*INF x*:*A. B* == *CONST INFI A* (%*x. B*)

⟨*ML*⟩

**context** *complete-lattice*
**begin**

**lemma** *le-SUPI*: $i : A \Longrightarrow M\ i \leq (SUP\ i{:}A.\ M\ i)$
⟨*proof*⟩

**lemma** *SUP-leI*: $(\bigwedge i.\ i : A \Longrightarrow M\ i \leq u) \Longrightarrow (SUP\ i{:}A.\ M\ i) \leq u$
⟨*proof*⟩

**lemma** *INF-leI*: $i : A \Longrightarrow (INF\ i{:}A.\ M\ i) \leq M\ i$
⟨*proof*⟩

**lemma** *le-INFI*: $(\bigwedge i.\ i : A \Longrightarrow u \leq M\ i) \Longrightarrow u \leq (INF\ i{:}A.\ M\ i)$
⟨*proof*⟩

**lemma** *SUP-const*[*simp*]: $A \neq \{\} \Longrightarrow (SUP\ i{:}A.\ M) = M$
⟨*proof*⟩

**lemma** *INF-const*[*simp*]: $A \neq \{\} \Longrightarrow (INF\ i{:}A.\ M) = M$
⟨*proof*⟩

**end**

## 5.12 Bool as complete lattice

**instantiation** *bool* :: *complete-lattice*
**begin**

**definition**
*Inf-bool-def*: $\bigsqcap A \longleftrightarrow (\forall x{\in}A.\ x)$

**definition**
*Sup-bool-def*: $\bigsqcup A \longleftrightarrow (\exists x{\in}A.\ x)$

**instance**
⟨*proof*⟩

**end**

**lemma** *Inf-empty-bool* [*simp*]:
$\bigsqcap \{\}$
⟨*proof*⟩

**lemma** *not-Sup-empty-bool* [*simp*]:
$\neg \bigsqcup \{\}$
⟨*proof*⟩

## 5.13 Fun as complete lattice

**instantiation** *fun* :: (*type*, *complete-lattice*) *complete-lattice*
**begin**

**definition**
*Inf-fun-def* [*code del*]: $\bigsqcap A = (\lambda x.\ \bigsqcap \{y.\ \exists f{\in}A.\ y = f\ x\})$

**definition**
*Sup-fun-def* [*code del*]: $\bigsqcup A = (\lambda x.\ \bigsqcup \{y.\ \exists f{\in}A.\ y = f\ x\})$

**instance**
⟨*proof*⟩

**end**

**lemma** *Inf-empty-fun*:
$\bigsqcap \{\} = (\lambda\text{-}.\ \bigsqcap \{\})$
⟨*proof*⟩

**lemma** *Sup-empty-fun*:
$\bigsqcup \{\} = (\lambda\text{-}.\ \bigsqcup \{\})$
⟨*proof*⟩

## 5.14 Set as lattice

**lemma** *inf-set-eq*: $A \sqcap B = A \cap B$
⟨*proof*⟩

**lemma** *sup-set-eq*: $A \sqcup B = A \cup B$
⟨*proof*⟩

**lemma** *mono-Int*: $mono\ f \Longrightarrow f\ (A \cap B) \subseteq f\ A \cap f\ B$
⟨*proof*⟩

**lemma** *mono-Un*: $mono\ f \Longrightarrow f\ A \cup f\ B \subseteq f\ (A \cup B)$
⟨*proof*⟩

**lemma** *Inf-set-eq*: $\bigsqcap S = \bigcap S$
⟨*proof*⟩

**lemma** *Sup-set-eq*: $\bigsqcup S = \bigcup S$
⟨*proof*⟩

**lemma** *top-set-eq*: $top = UNIV$
⟨*proof*⟩

**lemma** *bot-set-eq*: $bot = \{\}$
⟨*proof*⟩

**no-notation**
*less-eq* (**infix** $\sqsubseteq$ *50*) **and**
*less* (**infix** $\sqsubset$ *50*) **and**
*inf* (**infixl** $\sqcap$ *70*) **and**
*sup* (**infixl** $\sqcup$ *65*) **and**
*Inf* ($\bigsqcap$ - [*900*] *900*) **and**
*Sup* ($\bigsqcup$ - [*900*] *900*)

### 5.15 Misc theorem and ML bindings

**lemmas** *equalityI* = *subset-antisym*
**lemmas** *mem-simps* =
*insert-iff empty-iff Un-iff Int-iff Compl-iff Diff-iff*
*mem-Collect-eq UN-iff Union-iff INT-iff Inter-iff*
— Each of these has ALREADY been added [*simp*] above.

⟨*ML*⟩

**end**

## 6 Typedef: HOL type definitions

**theory** *Typedef*
**imports** *Set*
**uses**
(*Tools/typedef-package.ML*)
(*Tools/typecopy-package.ML*)
(*Tools/typedef-codegen.ML*)
**begin**

⟨*ML*⟩

**locale** *type-definition* =
**fixes** *Rep* **and** *Abs* **and** *A*
**assumes** *Rep*: *Rep x* ∈ *A*
**and** *Rep-inverse*: *Abs* (*Rep x*) = *x*
**and** *Abs-inverse*: *y* ∈ *A* ==> *Rep* (*Abs y*) = *y*
— This will be axiomatized for each typedef!
**begin**

**lemma** *Rep-inject*:
(*Rep x* = *Rep y*) = (*x* = *y*)
⟨*proof*⟩

**lemma** *Abs-inject*:
**assumes** *x*: *x* ∈ *A* **and** *y*: *y* ∈ *A*
**shows** (*Abs x* = *Abs y*) = (*x* = *y*)
⟨*proof*⟩

**lemma** *Rep-cases* [*cases set*]:
**assumes** *y*: $y \in A$
**and** *hyp*: $!!x.\ y = Rep\ x \Longrightarrow P$
**shows** $P$
⟨*proof*⟩

**lemma** *Abs-cases* [*cases type*]:
**assumes** *r*: $!!y.\ x = Abs\ y \Longrightarrow y \in A \Longrightarrow P$
**shows** $P$
⟨*proof*⟩

**lemma** *Rep-induct* [*induct set*]:
**assumes** *y*: $y \in A$
**and** *hyp*: $!!x.\ P\ (Rep\ x)$
**shows** $P\ y$
⟨*proof*⟩

**lemma** *Abs-induct* [*induct type*]:
**assumes** *r*: $!!y.\ y \in A \Longrightarrow P\ (Abs\ y)$
**shows** $P\ x$
⟨*proof*⟩

**lemma** *Rep-range*: $range\ Rep = A$
⟨*proof*⟩

**lemma** *Abs-image*: $Abs\ `\ A = UNIV$
⟨*proof*⟩

**end**

⟨*ML*⟩

**end**

# 7 Fun: Notions about functions

**theory** *Fun*
**imports** *Set*
**begin**

As a simplification rule, it replaces all function equalities by first-order equalities.

**lemma** *expand-fun-eq*: $f = g \longleftrightarrow (\forall x.\ f\ x = g\ x)$
⟨*proof*⟩

**lemma** *apply-inverse*:
$f\ x = u \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow g\ (f\ x) = x) \Longrightarrow P\ x \Longrightarrow x = g\ u$

⟨*proof*⟩

## 7.1 The Identity Function *id*

**definition**
*id* :: *'a* ⇒ *'a*
**where**
*id* = (λ*x*. *x*)

**lemma** *id-apply* [*simp*]: *id x* = *x*
⟨*proof*⟩

**lemma** *image-ident* [*simp*]: (%*x*. *x*) ‘ *Y* = *Y*
⟨*proof*⟩

**lemma** *image-id* [*simp*]: *id* ‘ *Y* = *Y*
⟨*proof*⟩

**lemma** *vimage-ident* [*simp*]: (%*x*. *x*) −‘ *Y* = *Y*
⟨*proof*⟩

**lemma** *vimage-id* [*simp*]: *id* −‘ *A* = *A*
⟨*proof*⟩

## 7.2 The Composition Operator *f* ∘ *g*

**definition**
*comp* :: (*'b* ⇒ *'c*) ⇒ (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'c* (**infixl** *o* 55)
**where**
*f o g* = (λ*x*. *f* (*g x*))

**notation** (*xsymbols*)
*comp* (**infixl** ∘ 55)

**notation** (*HTML* **output**)
*comp* (**infixl** ∘ 55)

compatibility

**lemmas** *o-def* = *comp-def*

**lemma** *o-apply* [*simp*]: (*f o g*) *x* = *f* (*g x*)
⟨*proof*⟩

**lemma** *o-assoc*: *f o* (*g o h*) = *f o g o h*
⟨*proof*⟩

**lemma** *id-o* [*simp*]: *id o g* = *g*
⟨*proof*⟩

**lemma** *o-id* [*simp*]: $f\ o\ id = f$
⟨*proof*⟩

**lemma** *image-compose*: $(f\ o\ g)\ `\ r = f`(g`r)$
⟨*proof*⟩

**lemma** *UN-o*: $UNION\ A\ (g\ o\ f) = UNION\ (f`A)\ g$
⟨*proof*⟩

## 7.3 The Forward Composition Operator *fcomp*

**definition**
$fcomp :: ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'c$ (**infixl** $o>\ 60$)
**where**
$f\ o>\ g = (\lambda x.\ g\ (f\ x))$

**lemma** *fcomp-apply*: $(f\ o>\ g)\ x = g\ (f\ x)$
⟨*proof*⟩

**lemma** *fcomp-assoc*: $(f\ o>\ g)\ o>\ h = f\ o>\ (g\ o>\ h)$
⟨*proof*⟩

**lemma** *id-fcomp* [*simp*]: $id\ o>\ g = g$
⟨*proof*⟩

**lemma** *fcomp-id* [*simp*]: $f\ o>\ id = f$
⟨*proof*⟩

**no-notation** *fcomp* (**infixl** $o>\ 60$)

## 7.4 Injectivity and Surjectivity

**constdefs**
$inj\text{-}on :: ['a => 'b, 'a\ set] => bool$ — injective
$inj\text{-}on\ f\ A == !\ x{:}A.\ !\ y{:}A.\ f(x){=}f(y) --> x{=}y$

A common special case: functions injective over the entire domain type.

**abbreviation**
$inj\ f == inj\text{-}on\ f\ UNIV$

**definition**
$bij\text{-}betw :: ('a => 'b) => 'a\ set => 'b\ set => bool$ **where** — bijective
[*code del*]: $bij\text{-}betw\ f\ A\ B \longleftrightarrow inj\text{-}on\ f\ A\ \&\ f\ `\ A = B$

**constdefs**
$surj :: ('a => 'b) => bool$
$surj\ f == !\ y.\ ?\ x.\ y{=}f(x)$

$bij :: ('a => 'b) => bool$

*bij f == inj f & surj f*

**lemma** *injI*:
**assumes** $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
**shows** *inj f*
⟨*proof*⟩

For Proofs in *Tools/datatype-rep-proofs*

**lemma** *datatype-injI*:
(!! *x*. *ALL y*. *f*(*x*) = *f*(*y*) −−> *x*=*y*) ==> *inj*(*f*)
⟨*proof*⟩

**theorem** *range-ex1-eq*: *inj f* $\Longrightarrow$ *b* : *range f* = (*EX*! *x*. *b* = *f x*)
⟨*proof*⟩

**lemma** *injD*: [| *inj*(*f*); *f*(*x*) = *f*(*y*) |] ==> *x*=*y*
⟨*proof*⟩

**lemma** *inj-eq*: *inj*(*f*) ==> (*f*(*x*) = *f*(*y*)) = (*x*=*y*)
⟨*proof*⟩

**lemma** *inj-on-id*[*simp*]: *inj-on id A*
⟨*proof*⟩

**lemma** *inj-on-id2*[*simp*]: *inj-on* (%*x*. *x*) *A*
⟨*proof*⟩

**lemma** *surj-id*[*simp*]: *surj id*
⟨*proof*⟩

**lemma** *bij-id*[*simp*]: *bij id*
⟨*proof*⟩

**lemma** *inj-onI*:
(!! *x y*. [| *x*:*A*; *y*:*A*; *f*(*x*) = *f*(*y*) |] ==> *x*=*y*) ==> *inj-on f A*
⟨*proof*⟩

**lemma** *inj-on-inverseI*: (!!*x*. *x*:*A* ==> *g*(*f*(*x*)) = *x*) ==> *inj-on f A*
⟨*proof*⟩

**lemma** *inj-onD*: [| *inj-on f A*; *f*(*x*)=*f*(*y*); *x*:*A*; *y*:*A* |] ==> *x*=*y*
⟨*proof*⟩

**lemma** *inj-on-iff*: [| *inj-on f A*; *x*:*A*; *y*:*A* |] ==> (*f*(*x*)=*f*(*y*)) = (*x*=*y*)
⟨*proof*⟩

**lemma** *comp-inj-on*:
[| *inj-on f A*; *inj-on g* (*f*‘*A*) |] ==> *inj-on* (*g o f*) *A*

⟨*proof*⟩

**lemma** *inj-on-imageI*: *inj-on* (*g o f*) *A* ⟹ *inj-on g* (*f ' A*)
⟨*proof*⟩

**lemma** *inj-on-image-iff*: ⟦ *ALL x*:*A*. *ALL y*:*A*. (*g*(*f x*) = *g*(*f y*)) = (*g x* = *g y*);
*inj-on f A* ⟧ ⟹ *inj-on g* (*f ' A*) = *inj-on g A*
⟨*proof*⟩

**lemma** *inj-on-contraD*: [| *inj-on f A*; ~*x*=*y*; *x*:*A*; *y*:*A* |] ==> ~ *f*(*x*)=*f*(*y*)
⟨*proof*⟩

**lemma** *inj-singleton*: *inj* (%*s*. {*s*})
⟨*proof*⟩

**lemma** *inj-on-empty*[*iff*]: *inj-on f* {}
⟨*proof*⟩

**lemma** *subset-inj-on*: [| *inj-on f B*; *A* <= *B* |] ==> *inj-on f A*
⟨*proof*⟩

**lemma** *inj-on-Un*:
*inj-on f* (*A Un B*) =
(*inj-on f A* & *inj-on f B* & *f*'(*A*−*B*) *Int f*'(*B*−*A*) = {})
⟨*proof*⟩

**lemma** *inj-on-insert*[*iff*]:
*inj-on f* (*insert a A*) = (*inj-on f A* & *f a* ~: *f*'(*A*−{*a*}))
⟨*proof*⟩

**lemma** *inj-on-diff*: *inj-on f A* ==> *inj-on f* (*A*−*B*)
⟨*proof*⟩

**lemma** *surjI*: (!! *x*. *g*(*f x*) = *x*) ==> *surj g*
⟨*proof*⟩

**lemma** *surj-range*: *surj f* ==> *range f* = *UNIV*
⟨*proof*⟩

**lemma** *surjD*: *surj f* ==> *EX x*. *y* = *f x*
⟨*proof*⟩

**lemma** *surjE*: *surj f* ==> (!!*x*. *y* = *f x* ==> *C*) ==> *C*
⟨*proof*⟩

**lemma** *comp-surj*: [| *surj f*; *surj g* |] ==> *surj* (*g o f*)
⟨*proof*⟩

**lemma** *bijI*: [| *inj f*; *surj f* |] ==> *bij f*

⟨*proof*⟩

**lemma** *bij-is-inj*: *bij f ==> inj f*
⟨*proof*⟩

**lemma** *bij-is-surj*: *bij f ==> surj f*
⟨*proof*⟩

**lemma** *bij-betw-imp-inj-on*: *bij-betw f A B ⟹ inj-on f A*
⟨*proof*⟩

**lemma** *bij-betw-inv*: **assumes** *bij-betw f A B* **shows** *EX g. bij-betw g B A*
⟨*proof*⟩

**lemma** *surj-image-vimage-eq*: *surj f ==> f ‘ (f −‘ A) = A*
⟨*proof*⟩

**lemma** *inj-vimage-image-eq*: *inj f ==> f −‘ (f ‘ A) = A*
⟨*proof*⟩

**lemma** *vimage-subsetD*: *surj f ==> f −‘ B <= A ==> B <= f ‘ A*
⟨*proof*⟩

**lemma** *vimage-subsetI*: *inj f ==> B <= f ‘ A ==> f −‘ B <= A*
⟨*proof*⟩

**lemma** *vimage-subset-eq*: *bij f ==> (f −‘ B <= A) = (B <= f ‘ A)*
⟨*proof*⟩

**lemma** *inj-on-image-Int*:
[| *inj-on f C*; *A<=C*; *B<=C* |] *==> f‘(A Int B) = f‘A Int f‘B*
⟨*proof*⟩

**lemma** *inj-on-image-set-diff*:
[| *inj-on f C*; *A<=C*; *B<=C* |] *==> f‘(A−B) = f‘A − f‘B*
⟨*proof*⟩

**lemma** *image-Int*: *inj f ==> f‘(A Int B) = f‘A Int f‘B*
⟨*proof*⟩

**lemma** *image-set-diff*: *inj f ==> f‘(A−B) = f‘A − f‘B*
⟨*proof*⟩

**lemma** *inj-image-mem-iff*: *inj f ==> (f a : f‘A) = (a : A)*
⟨*proof*⟩

**lemma** *inj-image-subset-iff*: *inj f ==> (f‘A <= f‘B) = (A<=B)*
⟨*proof*⟩

**lemma** *inj-image-eq-iff*: *inj f ==> (f`A = f`B) = (A = B)*
⟨*proof*⟩

**lemma** *image-INT*:
*[| inj-on f C; ALL x:A. B x <= C; j:A |]*
*==> f ` (INTER A B) = (INT x:A. f ` B x)*
⟨*proof*⟩

**lemma** *bij-image-INT*: *bij f ==> f ` (INTER A B) = (INT x:A. f ` B x)*
⟨*proof*⟩

**lemma** *surj-Compl-image-subset*: *surj f ==> -(f`A) <= f`(-A)*
⟨*proof*⟩

**lemma** *inj-image-Compl-subset*: *inj f ==> f`(-A) <= -(f`A)*
⟨*proof*⟩

**lemma** *bij-image-Compl-eq*: *bij f ==> f`(-A) = -(f`A)*
⟨*proof*⟩

## 7.5 Function Updating

**constdefs**
*fun-upd* :: *('a => 'b) => 'a => 'b => ('a => 'b)*
*fun-upd f a b == % x. if x=a then b else f x*

**nonterminals**
*updbinds updbind*
**syntax**
*-updbind* :: *['a, 'a] => updbind* (*(2- :=/ -)*)
:: *updbind => updbinds* (-)
*-updbinds*:: *[updbind, updbinds] => updbinds* (-,/ -)
*-Update* :: *['a, updbinds] => 'a* (*-/'((-)') [1000,0] 900*)

**translations**
*-Update f (-updbinds b bs)* == *-Update (-Update f b) bs*
*f(x:=y)* == *fun-upd f x y*

**lemma** *fun-upd-idem-iff*: *(f(x:=y) = f) = (f x = y)*
⟨*proof*⟩

**lemmas** *fun-upd-idem = fun-upd-idem-iff [THEN iffD2, standard]*

**lemmas** *fun-upd-triv* = *refl* [*THEN fun-upd-idem*]
**declare** *fun-upd-triv* [*iff*]

**lemma** *fun-upd-apply* [*simp*]: $(f(x:=y))z = (\mathit{if}\ z=x\ \mathit{then}\ y\ \mathit{else}\ f\ z)$
⟨*proof*⟩

**lemma** *fun-upd-same*: $(f(x:=y))\ x = y$
⟨*proof*⟩

**lemma** *fun-upd-other*: $z^{\sim}=x \Longrightarrow (f(x:=y))\ z = f\ z$
⟨*proof*⟩

**lemma** *fun-upd-upd* [*simp*]: $f(x:=y,x:=z) = f(x:=z)$
⟨*proof*⟩

**lemma** *fun-upd-twist*: $a\ ^{\sim}=\ c \Longrightarrow (m(a:=b))(c:=d) = (m(c:=d))(a:=b)$
⟨*proof*⟩

**lemma** *inj-on-fun-updI*: $[\![\ \mathit{inj\text{-}on}\ f\ A;\ y \notin f`A\ ]\!] \Longrightarrow \mathit{inj\text{-}on}\ (f(x:=y))\ A$
⟨*proof*⟩

**lemma** *fun-upd-image*:
$f(x:=y)\ `\ A = (\mathit{if}\ x \in A\ \mathit{then\ insert}\ y\ (f\ `\ (A-\{x\}))\ \mathit{else}\ f\ `\ A)$
⟨*proof*⟩

## 7.6 *override-on*

**definition**
*override-on* :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ \mathit{set} \Rightarrow 'a \Rightarrow 'b$
**where**
$\mathit{override\text{-}on}\ f\ g\ A = (\lambda a.\ \mathit{if}\ a \in A\ \mathit{then}\ g\ a\ \mathit{else}\ f\ a)$

**lemma** *override-on-emptyset*[*simp*]: *override-on f g* $\{\} = f$
⟨*proof*⟩

**lemma** *override-on-apply-notin*[*simp*]: $a\ ^{\sim}: A \Longrightarrow (\mathit{override\text{-}on}\ f\ g\ A)\ a = f\ a$
⟨*proof*⟩

**lemma** *override-on-apply-in*[*simp*]: $a : A \Longrightarrow (\mathit{override\text{-}on}\ f\ g\ A)\ a = g\ a$
⟨*proof*⟩

## 7.7 *swap*

**definition**
*swap* :: $'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)$
**where**
$\mathit{swap}\ a\ b\ f = f\ (a := f\ b,\ b:= f\ a)$

**lemma** *swap-self*: $\mathit{swap}\ a\ a\ f = f$

⟨*proof*⟩

**lemma** *swap-commute*: *swap a b f* = *swap b a f*
⟨*proof*⟩

**lemma** *swap-nilpotent* [*simp*]: *swap a b* (*swap a b f*) = *f*
⟨*proof*⟩

**lemma** *inj-on-imp-inj-on-swap*:
[|*inj-on f A*; *a* ∈ *A*; *b* ∈ *A*|] ==> *inj-on* (*swap a b f*) *A*
⟨*proof*⟩

**lemma** *inj-on-swap-iff* [*simp*]:
**assumes** *A*: *a* ∈ *A b* ∈ *A* **shows** *inj-on* (*swap a b f*) *A* = *inj-on f A*
⟨*proof*⟩

**lemma** *surj-imp-surj-swap*: *surj f* ==> *surj* (*swap a b f*)
⟨*proof*⟩

**lemma** *surj-swap-iff* [*simp*]: *surj* (*swap a b f*) = *surj f*
⟨*proof*⟩

**lemma** *bij-swap-iff*: *bij* (*swap a b f*) = *bij f*
⟨*proof*⟩

**hide** (**open**) *const swap*

## 7.8 Proof tool setup

simplifies terms of the form f(...,x:=y,...,x:=z,...) to f(...,x:=z,...)

⟨*ML*⟩

## 7.9 Code generator setup

**types-code**
*fun* ((- ->/ -))
**attach** (*term-of*) ⟪
*fun term-of-fun-type - aT - bT - = Free* (<*function*>, *aT* --> *bT*);
⟫
**attach** (*test*) ⟪
*fun gen-fun-type aF aT bG bT i* =
*let*
*val tab* = *ref* [];
*fun mk-upd* (*x*, (-, *y*)) *t* = *Const* (*Fun.fun-upd*,
(*aT* --> *bT*) --> *aT* --> *bT* --> *aT* --> *bT*) $ *t* $ *aF x* $ *y* ()
*in*
(*fn x* =>
*case AList.lookup op* = (!*tab*) *x of*
*NONE* =>

```
        let val p as (y, -) = bG i
        in (tab := (x, p) :: !tab; y) end
    | SOME (y, -) => y,
  fn () => Basics.fold mk-upd (!tab) (Const (HOL.undefined, aT --> bT)))
 end;
⟩⟩

code-const op ∘
 (SML infixl 5 o)
 (Haskell infixr 9 .)

code-const id
 (Haskell id)

end
```

# 8 Sum-Type: The Disjoint Sum of Two Types

**theory** *Sum-Type*
**imports** *Typedef Fun*
**begin**

The representations of the two injections

**constdefs**
```
 Inl-Rep :: ['a, 'a, 'b, bool] => bool
 Inl-Rep == (%a. %x y p. x=a & p)

 Inr-Rep :: ['b, 'a, 'b, bool] => bool
 Inr-Rep == (%b. %x y p. y=b & ~p)
```

**global**

**typedef** (*Sum*)
```
 ('a, 'b) +          (infixr + 10)
   = {f. (? a. f = Inl-Rep(a::'a)) | (? b. f = Inr-Rep(b::'b))}
 ⟨proof⟩
```

**local**

abstract constants and syntax

**constdefs**
```
 Inl :: 'a => 'a + 'b
  Inl == (%a. Abs-Sum(Inl-Rep(a)))

 Inr :: 'b => 'a + 'b
  Inr == (%b. Abs-Sum(Inr-Rep(b)))
```

*Plus* :: [*'a set*, *'b set*] => (*'a* + *'b*) *set* (**infixr** <+> *65*)
*A* <+> *B* == (*Inl*‘*A*) *Un* (*Inr*‘*B*)
— disjoint sum for sets; the operator + is overloaded with wrong type!

*Part* :: [*'a set*, *'b* => *'a*] => *'a set*
*Part A h* == *A Int* {*x*. *?* *z*. *x* = *h*(*z*)}
— for selecting out the components of a mutually recursive definition

**lemma** *Inl-RepI*: *Inl-Rep*(*a*) : *Sum*
⟨*proof*⟩

**lemma** *Inr-RepI*: *Inr-Rep*(*b*) : *Sum*
⟨*proof*⟩

**lemma** *inj-on-Abs-Sum*: *inj-on Abs-Sum Sum*
⟨*proof*⟩

## 8.1 Freeness Properties for *Inl* and *Inr*

Distinctness

**lemma** *Inl-Rep-not-Inr-Rep*: *Inl-Rep*(*a*) ~= *Inr-Rep*(*b*)
⟨*proof*⟩

**lemma** *Inl-not-Inr* [*iff*]: *Inl*(*a*) ~= *Inr*(*b*)
⟨*proof*⟩

**lemmas** *Inr-not-Inl* = *Inl-not-Inr* [*THEN not-sym*, *standard*]
**declare** *Inr-not-Inl* [*iff*]

**lemmas** *Inl-neq-Inr* = *Inl-not-Inr* [*THEN notE*, *standard*]
**lemmas** *Inr-neq-Inl* = *sym* [*THEN Inl-neq-Inr*, *standard*]

Injectiveness

**lemma** *Inl-Rep-inject*: *Inl-Rep*(*a*) = *Inl-Rep*(*c*) ==> *a*=*c*
⟨*proof*⟩

**lemma** *Inr-Rep-inject*: *Inr-Rep*(*b*) = *Inr-Rep*(*d*) ==> *b*=*d*
⟨*proof*⟩

**lemma** *inj-Inl* [*simp*]: *inj-on Inl A*
⟨*proof*⟩

**lemmas** *Inl-inject* = *inj-Inl* [*THEN injD*, *standard*]

**lemma** *inj-Inr* [*simp*]: *inj-on Inr A*
⟨*proof*⟩

**lemmas** *Inr-inject* = *inj-Inr* [*THEN injD*, *standard*]

**lemma** *Inl-eq* [*iff*]: (*Inl*(*x*)=*Inl*(*y*)) = (*x*=*y*)
⟨*proof*⟩

**lemma** *Inr-eq* [*iff*]: (*Inr*(*x*)=*Inr*(*y*)) = (*x*=*y*)
⟨*proof*⟩

## 8.2 The Disjoint Sum of Sets

**lemma** *InlI* [*intro*!]: *a* : *A* ==> *Inl*(*a*) : *A* <+> *B*
⟨*proof*⟩

**lemma** *InrI* [*intro*!]: *b* : *B* ==> *Inr*(*b*) : *A* <+> *B*
⟨*proof*⟩

**lemma** *PlusE* [*elim*!]:
[| *u*: *A* <+> *B*;
!!*x*. [| *x*:*A*; *u*=*Inl*(*x*) |] ==> *P*;
!!*y*. [| *y*:*B*; *u*=*Inr*(*y*) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

Exhaustion rule for sums, a degenerate form of induction

**lemma** *sumE*:
[| !!*x*::'*a*. *s* = *Inl*(*x*) ==> *P*; !!*y*::'*b*. *s* = *Inr*(*y*) ==> *P*
|] ==> *P*
⟨*proof*⟩

**lemma** *UNIV-Plus-UNIV* [*simp*]: *UNIV* <+> *UNIV* = *UNIV*
⟨*proof*⟩

## 8.3 The *Part* Primitive

**lemma** *Part-eqI* [*intro*]: [| *a* : *A*; *a*=*h*(*b*) |] ==> *a* : *Part A h*
⟨*proof*⟩

**lemmas** *PartI* = *Part-eqI* [*OF - refl*, *standard*]

**lemma** *PartE* [*elim*!]: [| *a* : *Part A h*; !!*z*. [| *a* : *A*; *a*=*h*(*z*) |] ==> *P* |] ==> *P*
⟨*proof*⟩

**lemma** *Part-subset*: *Part A h <= A*
⟨*proof*⟩

**lemma** *Part-mono*: *A<=B ==> Part A h <= Part B h*
⟨*proof*⟩

**lemmas** *basic-monos = basic-monos Part-mono*

**lemma** *PartD1*: *a : Part A h ==> a : A*
⟨*proof*⟩

**lemma** *Part-id*: *Part A (%x. x) = A*
⟨*proof*⟩

**lemma** *Part-Int*: *Part (A Int B) h = (Part A h) Int (Part B h)*
⟨*proof*⟩

**lemma** *Part-Collect*: *Part (A Int {x. P x}) h = (Part A h) Int {x. P x}*
⟨*proof*⟩

⟨*ML*⟩

**end**

# 9 Inductive: Knaster-Tarski Fixpoint Theorem and inductive definitions

**theory** *Inductive*
**imports** *Lattices Sum-Type*
**uses**
(*Tools/inductive-package.ML*)
*Tools/dseq.ML*
(*Tools/inductive-codegen.ML*)
(*Tools/datatype-aux.ML*)
(*Tools/datatype-prop.ML*)
(*Tools/datatype-rep-proofs.ML*)
(*Tools/datatype-abs-proofs.ML*)
(*Tools/datatype-case.ML*)
(*Tools/datatype-package.ML*)
(*Tools/old-primrec-package.ML*)
(*Tools/primrec-package.ML*)
(*Tools/datatype-codegen.ML*)
**begin**

## 9.1 Least and greatest fixed points

**context** *complete-lattice*
**begin**

**definition**
*lfp* :: $('a \Rightarrow 'a) \Rightarrow 'a$ **where**
*lfp f* = *Inf* $\{u.\ f\ u \leq u\}$ — least fixed point

**definition**
*gfp* :: $('a \Rightarrow 'a) \Rightarrow 'a$ **where**
*gfp f* = *Sup* $\{u.\ u \leq f\ u\}$ — greatest fixed point

## 9.2 Proof of Knaster-Tarski Theorem using *lfp*

*lfp f* is the least upper bound of the set $\{u.\ f\ u \leq u\}$

**lemma** *lfp-lowerbound*: *f A* $\leq$ *A* ==> *lfp f* $\leq$ *A*
⟨*proof*⟩

**lemma** *lfp-greatest*: (!!*u*. *f u* $\leq$ *u* ==> *A* $\leq$ *u*) ==> *A* $\leq$ *lfp f*
⟨*proof*⟩

**end**

**lemma** *lfp-lemma2*: *mono f* ==> *f* (*lfp f*) $\leq$ *lfp f*
⟨*proof*⟩

**lemma** *lfp-lemma3*: *mono f* ==> *lfp f* $\leq$ *f* (*lfp f*)
⟨*proof*⟩

**lemma** *lfp-unfold*: *mono f* ==> *lfp f* = *f* (*lfp f*)
⟨*proof*⟩

**lemma** *lfp-const*: *lfp* $(\lambda x.\ t)$ = *t*
⟨*proof*⟩

## 9.3 General induction rules for least fixed points

**theorem** *lfp-induct*:
**assumes** *mono*: *mono f* **and** *ind*: *f* (*inf* (*lfp f*) *P*) <= *P*
**shows** *lfp f* <= *P*
⟨*proof*⟩

**lemma** *lfp-induct-set*:
**assumes** *lfp*: *a*: *lfp*(*f*)
**and** *mono*: *mono*(*f*)
**and** *indhyp*: !!*x*. [| *x*: *f*(*lfp*(*f*) *Int* {*x*. *P*(*x*)}) |] ==> *P*(*x*)
**shows** *P*(*a*)
⟨*proof*⟩

**lemma** *lfp-ordinal-induct*:
**fixes** $f :: 'a::complete\text{-}lattice \Rightarrow 'a$
**assumes** *mono*: *mono f*
**and** *P-f*: $\bigwedge S.\ P\ S \Longrightarrow P\ (f\ S)$
**and** *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P\ (Sup\ M)$
**shows** $P\ (lfp\ f)$
⟨*proof*⟩

**lemma** *lfp-ordinal-induct-set*:
**assumes** *mono*: *mono f*
**and** *P-f*: $!!S.\ P\ S ==> P(f\ S)$
**and** *P-Union*: $!!M.\ !S{:}M.\ P\ S ==> P(Union\ M)$
**shows** $P(lfp\ f)$
⟨*proof*⟩

Definition forms of *lfp-unfold* and *lfp-induct*, to control unfolding

**lemma** *def-lfp-unfold*: $[|\ h{=}{=}lfp(f);\ \ mono(f)\ |] ==> h = f(h)$
⟨*proof*⟩

**lemma** *def-lfp-induct*:
$[|\ A == lfp(f);\ mono(f);$
$f\ (inf\ A\ P) \leq P$
$|] ==> A \leq P$
⟨*proof*⟩

**lemma** *def-lfp-induct-set*:
$[|\ A == lfp(f);\ \ mono(f);\ \ a{:}A;$
$!!x.\ [|\ x{:}\ f(A\ Int\ \{x.\ P(x)\})\ |] ==> P(x)$
$|] ==> P(a)$
⟨*proof*⟩

**lemma** *lfp-mono*: $(!!Z.\ f\ Z \leq g\ Z) ==> lfp\ f \leq lfp\ g$
⟨*proof*⟩

## 9.4 Proof of Knaster-Tarski Theorem using *gfp*

*gfp f* is the greatest lower bound of the set $\{u.\ u \leq f\ u\}$

**lemma** *gfp-upperbound*: $X \leq f\ X ==> X \leq gfp\ f$
⟨*proof*⟩

**lemma** *gfp-least*: $(!!u.\ u \leq f\ u ==> u \leq X) ==> gfp\ f \leq X$
⟨*proof*⟩

**lemma** *gfp-lemma2*: $mono\ f ==> gfp\ f \leq f\ (gfp\ f)$
⟨*proof*⟩

**lemma** *gfp-lemma3*: $mono\ f ==> f\ (gfp\ f) \leq gfp\ f$

⟨*proof*⟩

**lemma** *gfp-unfold*: *mono f* ==> *gfp f* = *f* (*gfp f*)
⟨*proof*⟩

## 9.5 Coinduction rules for greatest fixed points

weak version

**lemma** *weak-coinduct*: [| *a*: *X*; *X* ⊆ *f*(*X*) |] ==> *a* : *gfp*(*f*)
⟨*proof*⟩

**lemma** *weak-coinduct-image*: !!*X*. [| *a* : *X*; *g*‘*X* ⊆ *f* (*g*‘*X*) |] ==> *g a* : *gfp f*
⟨*proof*⟩

**lemma** *coinduct-lemma*:
[| *X* ≤ *f* (*sup X* (*gfp f*)); *mono f* |] ==> *sup X* (*gfp f*) ≤ *f* (*sup X* (*gfp f*))
⟨*proof*⟩

strong version, thanks to Coen and Frost

**lemma** *coinduct-set*: [| *mono*(*f*); *a*: *X*; *X* ⊆ *f*(*X Un gfp*(*f*)) |] ==> *a* : *gfp*(*f*)
⟨*proof*⟩

**lemma** *coinduct*: [| *mono*(*f*); *X* ≤ *f* (*sup X* (*gfp f*)) |] ==> *X* ≤ *gfp*(*f*)
⟨*proof*⟩

**lemma** *gfp-fun-UnI2*: [| *mono*(*f*); *a*: *gfp*(*f*) |] ==> *a*: *f*(*X Un gfp*(*f*))
⟨*proof*⟩

## 9.6 Even Stronger Coinduction Rule, by Martin Coen

Weakens the condition *X* ⊆ *f X* to one expressed using both *lfp* and *gfp*

**lemma** *coinduct3-mono-lemma*: *mono*(*f*) ==> *mono*(%*x*. *f*(*x*) *Un X Un B*)
⟨*proof*⟩

**lemma** *coinduct3-lemma*:
[| *X* ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*))); *mono*(*f*) |]
==> *lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*)) ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*)))
⟨*proof*⟩

**lemma** *coinduct3*:
[| *mono*(*f*); *a*:*X*; *X* ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*))) |] ==> *a* : *gfp*(*f*)
⟨*proof*⟩

Definition forms of *gfp-unfold* and *coinduct*, to control unfolding

**lemma** *def-gfp-unfold*: [| *A*==*gfp*(*f*); *mono*(*f*) |] ==> *A* = *f*(*A*)
⟨*proof*⟩

**lemma** *def-coinduct*:

[| $A$==*gfp*($f$); *mono*($f$); $X \leq f(sup\ X\ A)$ |] ==> $X \leq A$
⟨*proof*⟩

**lemma** *def-coinduct-set*:
[| $A$==*gfp*($f$); *mono*($f$); $a$:$X$; $X \subseteq f(X\ Un\ A)$ |] ==> $a$: $A$
⟨*proof*⟩

**lemma** *def-Collect-coinduct*:
[| $A$ == *gfp*(%$w$. *Collect*($P(w)$)); *mono*(%$w$. *Collect*($P(w)$));
$a$: $X$; !!$z$. $z$: $X$ ==> $P\ (X\ Un\ A)\ z$ |] ==>
$a$ : $A$
⟨*proof*⟩

**lemma** *def-coinduct3*:
[| $A$==*gfp*($f$); *mono*($f$); $a$:$X$; $X \subseteq f(lfp(\%x.\ f(x)\ Un\ X\ Un\ A))$ |] ==> $a$: $A$
⟨*proof*⟩

Monotonicity of *gfp*!

**lemma** *gfp-mono*: (!!$Z$. $f\ Z \leq g\ Z$) ==> *gfp* $f \leq$ *gfp* $g$
⟨*proof*⟩

## 9.7 Inductive predicates and sets

Inversion of injective functions.

**constdefs**
*myinv* :: ('$a$ => '$b$) => ('$b$ => '$a$)
*myinv* ($f$ :: '$a$ => '$b$) == $\lambda y$. *THE* $x$. $f\ x = y$

**lemma** *myinv-f-f*: *inj* $f$ ==> *myinv* $f\ (f\ x) = x$
⟨*proof*⟩

**lemma** *f-myinv-f*: *inj* $f$ ==> $y \in$ *range* $f$ ==> $f\ (myinv\ f\ y) = y$
⟨*proof*⟩

**hide** *const myinv*

Package setup.

**theorems** *basic-monos* =
*subset-refl imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj*
*Collect-mono in-mono vimage-mono*
*imp-conv-disj not-not de-Morgan-disj de-Morgan-conj*
*not-all not-ex*
*Ball-def Bex-def*
*induct-rulify-fallback*

⟨*ML*⟩

**theorems** [*mono*] =
*imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj*
*imp-conv-disj not-not de-Morgan-disj de-Morgan-conj*
*not-all not-ex*
*Ball-def Bex-def*
*induct-rulify-fallback*

### 9.8 Inductive datatypes and primitive recursion

Package setup.

⟨*ML*⟩
Lambda-abstractions with pattern matching:

**syntax**
*-lam-pats-syntax* :: *cases-syn* => *'a* => *'b* ((%-) *10*)
**syntax** (*xsymbols*)
*-lam-pats-syntax* :: *cases-syn* => *'a* => *'b* ((λ-) *10*)

⟨*ML*⟩

**end**

## 10 OrderedGroup: Ordered Groups

**theory** *OrderedGroup*
**imports** *Lattices*
**uses** ~~/*src*/*Provers*/*Arith*/*abel-cancel.ML*
**begin**

The theory of partially ordered groups is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

⟨*ML*⟩
The rewrites accumulated in *algebra-simps* deal with the classical algebraic structures of groups, rings and family. They simplify terms by multiplying everything out (in case of a ring) and bringing sums and products into a

canonical form (by ordered rewriting). As a result it decides group and ring equalities but also helps with inequalities.

Of course it also works for fields, but it knows nothing about multiplicative inverses or division. This is catered for by *field-simps*.

## 10.1 Semigroups and Monoids

**class** *semigroup-add* = *plus* +
  **assumes** *add-assoc*[*algebra-simps*]: $(a + b) + c = a + (b + c)$

**class** *ab-semigroup-add* = *semigroup-add* +
  **assumes** *add-commute*[*algebra-simps*]: $a + b = b + a$
**begin**

**lemma** *add-left-commute*[*algebra-simps*]: $a + (b + c) = b + (a + c)$
⟨*proof*⟩

**theorems** *add-ac* = *add-assoc add-commute add-left-commute*

**end**

**theorems** *add-ac* = *add-assoc add-commute add-left-commute*

**class** *semigroup-mult* = *times* +
  **assumes** *mult-assoc*[*algebra-simps*]: $(a * b) * c = a * (b * c)$

**class** *ab-semigroup-mult* = *semigroup-mult* +
  **assumes** *mult-commute*[*algebra-simps*]: $a * b = b * a$
**begin**

**lemma** *mult-left-commute*[*algebra-simps*]: $a * (b * c) = b * (a * c)$
⟨*proof*⟩

**theorems** *mult-ac* = *mult-assoc mult-commute mult-left-commute*

**end**

**theorems** *mult-ac* = *mult-assoc mult-commute mult-left-commute*

**class** *ab-semigroup-idem-mult* = *ab-semigroup-mult* +
  **assumes** *mult-idem*[*simp*]: $x * x = x$
**begin**

**lemma** *mult-left-idem*[*simp*]: $x * (x * y) = x * y$
  ⟨*proof*⟩

**end**

**class** *monoid-add* = *zero* + *semigroup-add* +

**assumes** *add-0-left* [*simp*]: $0 + a = a$
**and** *add-0-right* [*simp*]: $a + 0 = a$

**lemma** *zero-reorient*: $0 = x \longleftrightarrow x = 0$
⟨*proof*⟩

**class** *comm-monoid-add* = *zero* + *ab-semigroup-add* +
**assumes** *add-0*: $0 + a = a$
**begin**

**subclass** *monoid-add*
⟨*proof*⟩

**end**

**class** *monoid-mult* = *one* + *semigroup-mult* +
**assumes** *mult-1-left* [*simp*]: $1 * a = a$
**assumes** *mult-1-right* [*simp*]: $a * 1 = a$

**lemma** *one-reorient*: $1 = x \longleftrightarrow x = 1$
⟨*proof*⟩

**class** *comm-monoid-mult* = *one* + *ab-semigroup-mult* +
**assumes** *mult-1*: $1 * a = a$
**begin**

**subclass** *monoid-mult*
⟨*proof*⟩

**end**

**class** *cancel-semigroup-add* = *semigroup-add* +
**assumes** *add-left-imp-eq*: $a + b = a + c \Longrightarrow b = c$
**assumes** *add-right-imp-eq*: $b + a = c + a \Longrightarrow b = c$
**begin**

**lemma** *add-left-cancel* [*simp*]:
$a + b = a + c \longleftrightarrow b = c$
⟨*proof*⟩

**lemma** *add-right-cancel* [*simp*]:
$b + a = c + a \longleftrightarrow b = c$
⟨*proof*⟩

**end**

**class** *cancel-ab-semigroup-add* = *ab-semigroup-add* +
**assumes** *add-imp-eq*: $a + b = a + c \Longrightarrow b = c$
**begin**

**subclass** *cancel-semigroup-add*
⟨*proof*⟩

**end**

**class** *cancel-comm-monoid-add* = *cancel-ab-semigroup-add* + *comm-monoid-add*

## 10.2 Groups

**class** *group-add* = *minus* + *uminus* + *monoid-add* +
  **assumes** *left-minus* [*simp*]: − *a* + *a* = *0*
  **assumes** *diff-minus*: *a* − *b* = *a* + (− *b*)
**begin**

**lemma** *minus-add-cancel*: − *a* + (*a* + *b*) = *b*
⟨*proof*⟩

**lemma** *minus-zero* [*simp*]: − *0* = *0*
⟨*proof*⟩

**lemma** *minus-minus* [*simp*]: − (− *a*) = *a*
⟨*proof*⟩

**lemma** *right-minus* [*simp*]: *a* + − *a* = *0*
⟨*proof*⟩

**lemma** *right-minus-eq*: *a* − *b* = *0* ⟷ *a* = *b*
⟨*proof*⟩

**lemma** *equals-zero-I*:
  **assumes** *a* + *b* = *0* **shows** − *a* = *b*
⟨*proof*⟩

**lemma** *diff-self* [*simp*]: *a* − *a* = *0*
⟨*proof*⟩

**lemma** *diff-0* [*simp*]: *0* − *a* = − *a*
⟨*proof*⟩

**lemma** *diff-0-right* [*simp*]: *a* − *0* = *a*
⟨*proof*⟩

**lemma** *diff-minus-eq-add* [*simp*]: *a* − − *b* = *a* + *b*
⟨*proof*⟩

**lemma** *neg-equal-iff-equal* [*simp*]:
  − *a* = − *b* ⟷ *a* = *b*
⟨*proof*⟩

**lemma** *neg-equal-0-iff-equal* [*simp*]:
$- a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

**lemma** *neg-0-equal-iff-equal* [*simp*]:
$0 = - a \longleftrightarrow 0 = a$
⟨*proof*⟩

The next two equations can make the simplifier loop!

**lemma** *equation-minus-iff*:
$a = - b \longleftrightarrow b = - a$
⟨*proof*⟩

**lemma** *minus-equation-iff*:
$- a = b \longleftrightarrow - b = a$
⟨*proof*⟩

**lemma** *diff-add-cancel*: $a - b + b = a$
⟨*proof*⟩

**lemma** *add-diff-cancel*: $a + b - b = a$
⟨*proof*⟩

**declare** *diff-minus*[*symmetric*, *algebra-simps*]

**lemma** *eq-neg-iff-add-eq-0*: $a = - b \longleftrightarrow a + b = 0$
⟨*proof*⟩

**end**

**class** *ab-group-add* = *minus* + *uminus* + *comm-monoid-add* +
**assumes** *ab-left-minus*: $- a + a = 0$
**assumes** *ab-diff-minus*: $a - b = a + (- b)$
**begin**

**subclass** *group-add*
⟨*proof*⟩

**subclass** *cancel-comm-monoid-add*
⟨*proof*⟩

**lemma** *uminus-add-conv-diff*[*algebra-simps*]:
$- a + b = b - a$
⟨*proof*⟩

**lemma** *minus-add-distrib* [*simp*]:
$- (a + b) = - a + - b$
⟨*proof*⟩

**lemma** *minus-diff-eq* [*simp*]:
$-(a - b) = b - a$
⟨*proof*⟩

**lemma** *add-diff-eq*[*algebra-simps*]: $a + (b - c) = (a + b) - c$
⟨*proof*⟩

**lemma** *diff-add-eq*[*algebra-simps*]: $(a - b) + c = (a + c) - b$
⟨*proof*⟩

**lemma** *diff-eq-eq*[*algebra-simps*]: $a - b = c \longleftrightarrow a = c + b$
⟨*proof*⟩

**lemma** *eq-diff-eq*[*algebra-simps*]: $a = c - b \longleftrightarrow a + b = c$
⟨*proof*⟩

**lemma** *diff-diff-eq*[*algebra-simps*]: $(a - b) - c = a - (b + c)$
⟨*proof*⟩

**lemma** *diff-diff-eq2*[*algebra-simps*]: $a - (b - c) = (a + c) - b$
⟨*proof*⟩

**lemma** *eq-iff-diff-eq-0*: $a = b \longleftrightarrow a - b = 0$
⟨*proof*⟩

**lemma** *diff-eq-0-iff-eq* [*simp*, *noatp*]: $a - b = 0 \longleftrightarrow a = b$
⟨*proof*⟩

**end**

## 10.3 (Partially) Ordered Groups

**class** *pordered-ab-semigroup-add* = *order* + *ab-semigroup-add* +
**assumes** *add-left-mono*: $a \leq b \Longrightarrow c + a \leq c + b$
**begin**

**lemma** *add-right-mono*:
$a \leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

non-strict, in both arguments

**lemma** *add-mono*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow a + c \leq b + d$
⟨*proof*⟩

**end**

**class** *pordered-cancel-ab-semigroup-add* =

*pordered-ab-semigroup-add* + *cancel-ab-semigroup-add*
**begin**

**lemma** *add-strict-left-mono*:
$a < b \Longrightarrow c + a < c + b$
⟨*proof*⟩

**lemma** *add-strict-right-mono*:
$a < b \Longrightarrow a + c < b + c$
⟨*proof*⟩

Strict monotonicity in both arguments

**lemma** *add-strict-mono*:
$a < b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

**lemma** *add-less-le-mono*:
$a < b \Longrightarrow c \leq d \Longrightarrow a + c < b + d$
⟨*proof*⟩

**lemma** *add-le-less-mono*:
$a \leq b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

**end**

**class** *pordered-ab-semigroup-add-imp-le* =
*pordered-cancel-ab-semigroup-add* +
**assumes** *add-le-imp-le-left*: $c + a \leq c + b \Longrightarrow a \leq b$
**begin**

**lemma** *add-less-imp-less-left*:
**assumes** *less*: $c + a < c + b$ **shows** $a < b$
⟨*proof*⟩

**lemma** *add-less-imp-less-right*:
$a + c < b + c \Longrightarrow a < b$
⟨*proof*⟩

**lemma** *add-less-cancel-left* [*simp*]:
$c + a < c + b \longleftrightarrow a < b$
⟨*proof*⟩

**lemma** *add-less-cancel-right* [*simp*]:
$a + c < b + c \longleftrightarrow a < b$
⟨*proof*⟩

**lemma** *add-le-cancel-left* [*simp*]:
$c + a \leq c + b \longleftrightarrow a \leq b$

⟨*proof*⟩

**lemma** *add-le-cancel-right* [*simp*]:
$a + c \leq b + c \longleftrightarrow a \leq b$
⟨*proof*⟩

**lemma** *add-le-imp-le-right*:
$a + c \leq b + c \Longrightarrow a \leq b$
⟨*proof*⟩

**lemma** *max-add-distrib-left*:
$max\ x\ y + z = max\ (x + z)\ (y + z)$
⟨*proof*⟩

**lemma** *min-add-distrib-left*:
$min\ x\ y + z = min\ (x + z)\ (y + z)$
⟨*proof*⟩

**end**

## 10.4 Support for reasoning about signs

**class** *pordered-comm-monoid-add* =
*pordered-cancel-ab-semigroup-add* + *comm-monoid-add*
**begin**

**lemma** *add-pos-nonneg*:
**assumes** $0 < a$ **and** $0 \leq b$ **shows** $0 < a + b$
⟨*proof*⟩

**lemma** *add-pos-pos*:
**assumes** $0 < a$ **and** $0 < b$ **shows** $0 < a + b$
⟨*proof*⟩

**lemma** *add-nonneg-pos*:
**assumes** $0 \leq a$ **and** $0 < b$ **shows** $0 < a + b$
⟨*proof*⟩

**lemma** *add-nonneg-nonneg*:
**assumes** $0 \leq a$ **and** $0 \leq b$ **shows** $0 \leq a + b$
⟨*proof*⟩

**lemma** *add-neg-nonpos*:
**assumes** $a < 0$ **and** $b \leq 0$ **shows** $a + b < 0$
⟨*proof*⟩

**lemma** *add-neg-neg*:
**assumes** $a < 0$ **and** $b < 0$ **shows** $a + b < 0$
⟨*proof*⟩

**lemma** *add-nonpos-neg*:
**assumes** $a \leq 0$ **and** $b < 0$ **shows** $a + b < 0$
⟨*proof*⟩

**lemma** *add-nonpos-nonpos*:
**assumes** $a \leq 0$ **and** $b \leq 0$ **shows** $a + b \leq 0$
⟨*proof*⟩

**lemmas** *add-sign-intros* =
*add-pos-nonneg add-pos-pos add-nonneg-pos add-nonneg-nonneg*
*add-neg-nonpos add-neg-neg add-nonpos-neg add-nonpos-nonpos*

**lemma** *add-nonneg-eq-0-iff*:
**assumes** $x$: $0 \leq x$ **and** $y$: $0 \leq y$
**shows** $x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

**end**

**class** *pordered-ab-group-add* =
*ab-group-add* + *pordered-ab-semigroup-add*
**begin**

**subclass** *pordered-cancel-ab-semigroup-add* ⟨*proof*⟩

**subclass** *pordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

**subclass** *pordered-comm-monoid-add* ⟨*proof*⟩

**lemma** *max-diff-distrib-left*:
**shows** $max\ x\ y - z = max\ (x - z)\ (y - z)$
⟨*proof*⟩

**lemma** *min-diff-distrib-left*:
**shows** $min\ x\ y - z = min\ (x - z)\ (y - z)$
⟨*proof*⟩

**lemma** *le-imp-neg-le*:
**assumes** $a \leq b$ **shows** $-b \leq -a$
⟨*proof*⟩

**lemma** *neg-le-iff-le* [*simp*]: $- b \leq - a \longleftrightarrow a \leq b$
⟨*proof*⟩

**lemma** *neg-le-0-iff-le* [*simp*]: $- a \leq 0 \longleftrightarrow 0 \leq a$
⟨*proof*⟩

**lemma** *neg-0-le-iff-le* [*simp*]: $0 \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

**lemma** *neg-less-iff-less* [*simp*]: $- b < - a \longleftrightarrow a < b$
⟨*proof*⟩

**lemma** *neg-less-0-iff-less* [*simp*]: $- a < 0 \longleftrightarrow 0 < a$
⟨*proof*⟩

**lemma** *neg-0-less-iff-less* [*simp*]: $0 < - a \longleftrightarrow a < 0$
⟨*proof*⟩

The next several equations can make the simplifier loop!

**lemma** *less-minus-iff*: $a < - b \longleftrightarrow b < - a$
⟨*proof*⟩

**lemma** *minus-less-iff*: $- a < b \longleftrightarrow - b < a$
⟨*proof*⟩

**lemma** *le-minus-iff*: $a \leq - b \longleftrightarrow b \leq - a$
⟨*proof*⟩

**lemma** *minus-le-iff*: $- a \leq b \longleftrightarrow - b \leq a$
⟨*proof*⟩

**lemma** *less-iff-diff-less-0*: $a < b \longleftrightarrow a - b < 0$
⟨*proof*⟩

**lemma** *diff-less-eq*[*algebra-simps*]: $a - b < c \longleftrightarrow a < c + b$
⟨*proof*⟩

**lemma** *less-diff-eq*[*algebra-simps*]: $a < c - b \longleftrightarrow a + b < c$
⟨*proof*⟩

**lemma** *diff-le-eq*[*algebra-simps*]: $a - b \leq c \longleftrightarrow a \leq c + b$
⟨*proof*⟩

**lemma** *le-diff-eq*[*algebra-simps*]: $a \leq c - b \longleftrightarrow a + b \leq c$
⟨*proof*⟩

**lemma** *le-iff-diff-le-0*: $a \leq b \longleftrightarrow a - b \leq 0$
⟨*proof*⟩

Legacy - use *algebra-simps*

**lemmas** *group-simps*[*noatp*] = *algebra-simps*

**end**

Legacy - use *algebra-simps*

**lemmas** *group-simps*[*noatp*] = *algebra-simps*

**class** *ordered-ab-semigroup-add* =
  *linorder* + *pordered-ab-semigroup-add*

**class** *ordered-cancel-ab-semigroup-add* =
  *linorder* + *pordered-cancel-ab-semigroup-add*
**begin**

**subclass** *ordered-ab-semigroup-add* ⟨*proof*⟩

**subclass** *pordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

**end**

**class** *ordered-ab-group-add* =
  *linorder* + *pordered-ab-group-add*
**begin**

**subclass** *ordered-cancel-ab-semigroup-add* ⟨*proof*⟩

**lemma** *neg-less-eq-nonneg*:
  $- a \leq a \longleftrightarrow 0 \leq a$
⟨*proof*⟩

**lemma** *less-eq-neg-nonpos*:
  $a \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

**lemma** *equal-neg-zero*:
  $a = - a \longleftrightarrow a = 0$
⟨*proof*⟩

**lemma** *neg-equal-zero*:
  $- a = a \longleftrightarrow a = 0$
  ⟨*proof*⟩

**end**

— FIXME localize the following

**lemma** *add-increasing*:
  **fixes** *c* :: *'a*::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}
  **shows** [|*0*≤*a*; *b*≤*c*|] ==> *b* ≤ *a* + *c*
⟨*proof*⟩

**lemma** *add-increasing2*:
  **fixes** *c* :: *'a*::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}

**shows** $[|0 \leq c;\ b \leq a|] ==> b \leq a + c$
⟨*proof*⟩

**lemma** *add-strict-increasing*:
**fixes** $c :: 'a::\{$*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*$\}$
**shows** $[|0 < a;\ b \leq c|] ==> b < a + c$
⟨*proof*⟩

**lemma** *add-strict-increasing2*:
**fixes** $c :: 'a::\{$*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*$\}$
**shows** $[|0 \leq a;\ b < c|] ==> b < a + c$
⟨*proof*⟩

**class** *pordered-ab-group-add-abs* = *pordered-ab-group-add* + *abs* +
**assumes** *abs-ge-zero* [*simp*]: $|a| \geq 0$
**and** *abs-ge-self*: $a \leq |a|$
**and** *abs-leI*: $a \leq b \Longrightarrow -a \leq b \Longrightarrow |a| \leq b$
**and** *abs-minus-cancel* [*simp*]: $|-a| = |a|$
**and** *abs-triangle-ineq*: $|a + b| \leq |a| + |b|$
**begin**

**lemma** *abs-minus-le-zero*: $-|a| \leq 0$
⟨*proof*⟩

**lemma** *abs-of-nonneg* [*simp*]:
**assumes** *nonneg*: $0 \leq a$ **shows** $|a| = a$
⟨*proof*⟩

**lemma** *abs-idempotent* [*simp*]: $||a|| = |a|$
⟨*proof*⟩

**lemma** *abs-eq-0* [*simp*]: $|a| = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

**lemma** *abs-zero* [*simp*]: $|0| = 0$
⟨*proof*⟩

**lemma** *abs-0-eq* [*simp*, *noatp*]: $0 = |a| \longleftrightarrow a = 0$
⟨*proof*⟩

**lemma** *abs-le-zero-iff* [*simp*]: $|a| \leq 0 \longleftrightarrow a = 0$
⟨*proof*⟩

**lemma** *zero-less-abs-iff* [*simp*]: $0 < |a| \longleftrightarrow a \neq 0$
⟨*proof*⟩

**lemma** *abs-not-less-zero* [*simp*]: $\neg |a| < 0$
⟨*proof*⟩

**lemma** *abs-ge-minus-self*: $-\ a \leq |a|$
⟨*proof*⟩

**lemma** *abs-minus-commute*:
$|a\ -\ b| = |b\ -\ a|$
⟨*proof*⟩

**lemma** *abs-of-pos*: $0 < a \Longrightarrow |a| = a$
⟨*proof*⟩

**lemma** *abs-of-nonpos* [*simp*]:
**assumes** $a \leq 0$ **shows** $|a| = -\ a$
⟨*proof*⟩

**lemma** *abs-of-neg*: $a < 0 \Longrightarrow |a| = -\ a$
⟨*proof*⟩

**lemma** *abs-le-D1*: $|a| \leq b \Longrightarrow a \leq b$
⟨*proof*⟩

**lemma** *abs-le-D2*: $|a| \leq b \Longrightarrow -\ a \leq b$
⟨*proof*⟩

**lemma** *abs-le-iff*: $|a| \leq b \longleftrightarrow a \leq b \wedge -\ a \leq b$
⟨*proof*⟩

**lemma** *abs-triangle-ineq2*: $|a| - |b| \leq |a\ -\ b|$
⟨*proof*⟩

**lemma** *abs-triangle-ineq3*: $||a| - |b|| \leq |a\ -\ b|$
⟨*proof*⟩

**lemma** *abs-triangle-ineq4*: $|a\ -\ b| \leq |a| + |b|$
⟨*proof*⟩

**lemma** *abs-diff-triangle-ineq*: $|a\ +\ b\ -\ (c\ +\ d)| \leq |a\ -\ c| + |b\ -\ d|$
⟨*proof*⟩

**lemma** *abs-add-abs* [*simp*]:
$||a| + |b|| = |a| + |b|$ (**is** *?L = ?R*)
⟨*proof*⟩

**end**

## 10.5 Lattice Ordered (Abelian) Groups

**class** *lordered-ab-group-add-meet* = *pordered-ab-group-add* + *lower-semilattice*
**begin**

**lemma** *add-inf-distrib-left*:
$a + inf\ b\ c = inf\ (a + b)\ (a + c)$
⟨*proof*⟩

**lemma** *add-inf-distrib-right*:
$inf\ a\ b + c = inf\ (a + c)\ (b + c)$
⟨*proof*⟩

**end**

**class** *lordered-ab-group-add-join* = *pordered-ab-group-add* + *upper-semilattice*
**begin**

**lemma** *add-sup-distrib-left*:
$a + sup\ b\ c = sup\ (a + b)\ (a + c)$
⟨*proof*⟩

**lemma** *add-sup-distrib-right*:
$sup\ a\ b + c = sup\ (a+c)\ (b+c)$
⟨*proof*⟩

**end**

**class** *lordered-ab-group-add* = *pordered-ab-group-add* + *lattice*
**begin**

**subclass** *lordered-ab-group-add-meet* ⟨*proof*⟩
**subclass** *lordered-ab-group-add-join* ⟨*proof*⟩

**lemmas** *add-sup-inf-distribs* = *add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left*

**lemma** *inf-eq-neg-sup*: $inf\ a\ b = -\ sup\ (-a)\ (-b)$
⟨*proof*⟩

**lemma** *sup-eq-neg-inf*: $sup\ a\ b = -\ inf\ (-a)\ (-b)$
⟨*proof*⟩

**lemma** *neg-inf-eq-sup*: $-\ inf\ a\ b = sup\ (-a)\ (-b)$
⟨*proof*⟩

**lemma** *neg-sup-eq-inf*: $-\ sup\ a\ b = inf\ (-a)\ (-b)$
⟨*proof*⟩

**lemma** *add-eq-inf-sup*: $a + b = sup\ a\ b + inf\ a\ b$
⟨*proof*⟩

## 10.6 Positive Part, Negative Part, Absolute Value

**definition**
*nprt* :: *'a* ⇒ *'a* **where**
*nprt x* = *inf x 0*

**definition**
*pprt* :: *'a* ⇒ *'a* **where**
*pprt x* = *sup x 0*

**lemma** *pprt-neg*: *pprt* (− *x*) = − *nprt x*
⟨*proof*⟩

**lemma** *nprt-neg*: *nprt* (− *x*) = − *pprt x*
⟨*proof*⟩

**lemma** *prts*: *a* = *pprt a* + *nprt a*
⟨*proof*⟩

**lemma** *zero-le-pprt*[*simp*]: *0* ≤ *pprt a*
⟨*proof*⟩

**lemma** *nprt-le-zero*[*simp*]: *nprt a* ≤ *0*
⟨*proof*⟩

**lemma** *le-eq-neg*: *a* ≤ − *b* ⟷ *a* + *b* ≤ *0* (**is** *?l* = *?r*)
⟨*proof*⟩

**lemma** *pprt-0*[*simp*]: *pprt 0* = *0* ⟨*proof*⟩
**lemma** *nprt-0*[*simp*]: *nprt 0* = *0* ⟨*proof*⟩

**lemma** *pprt-eq-id* [*simp*, *noatp*]: *0* ≤ *x* ⟹ *pprt x* = *x*
⟨*proof*⟩

**lemma** *nprt-eq-id* [*simp*, *noatp*]: *x* ≤ *0* ⟹ *nprt x* = *x*
⟨*proof*⟩

**lemma** *pprt-eq-0* [*simp*, *noatp*]: *x* ≤ *0* ⟹ *pprt x* = *0*
⟨*proof*⟩

**lemma** *nprt-eq-0* [*simp*, *noatp*]: *0* ≤ *x* ⟹ *nprt x* = *0*
⟨*proof*⟩

**lemma** *sup-0-imp-0*: *sup a* (− *a*) = *0* ⟹ *a* = *0*
⟨*proof*⟩

**lemma** *inf-0-imp-0*: *inf a* (−*a*) = *0* ⟹ *a* = *0*
⟨*proof*⟩

**lemma** *inf-0-eq-0* [*simp*, *noatp*]: *inf a* (− *a*) = *0* ⟷ *a* = *0*

⟨*proof*⟩

**lemma** *sup-0-eq-0* [*simp*, *noatp*]: *sup a* (− *a*) = *0* ⟷ *a* = *0*
⟨*proof*⟩

**lemma** *zero-le-double-add-iff-zero-le-single-add* [*simp*]:
*0* ≤ *a* + *a* ⟷ *0* ≤ *a*
⟨*proof*⟩

**lemma** *double-zero*: *a* + *a* = *0* ⟷ *a* = *0*
⟨*proof*⟩

**lemma** *zero-less-double-add-iff-zero-less-single-add*:
*0* < *a* + *a* ⟷ *0* < *a*
⟨*proof*⟩

**lemma** *double-add-le-zero-iff-single-add-le-zero* [*simp*]:
*a* + *a* ≤ *0* ⟷ *a* ≤ *0*
⟨*proof*⟩

**lemma** *double-add-less-zero-iff-single-less-zero* [*simp*]:
*a* + *a* < *0* ⟷ *a* < *0*
⟨*proof*⟩

**declare** *neg-inf-eq-sup* [*simp*] *neg-sup-eq-inf* [*simp*]

**lemma** *le-minus-self-iff*: *a* ≤ − *a* ⟷ *a* ≤ *0*
⟨*proof*⟩

**lemma** *minus-le-self-iff*: − *a* ≤ *a* ⟷ *0* ≤ *a*
⟨*proof*⟩

**lemma** *zero-le-iff-zero-nprt*: *0* ≤ *a* ⟷ *nprt a* = *0*
⟨*proof*⟩

**lemma** *le-zero-iff-zero-pprt*: *a* ≤ *0* ⟷ *pprt a* = *0*
⟨*proof*⟩

**lemma** *le-zero-iff-pprt-id*: *0* ≤ *a* ⟷ *pprt a* = *a*
⟨*proof*⟩

**lemma** *zero-le-iff-nprt-id*: *a* ≤ *0* ⟷ *nprt a* = *a*
⟨*proof*⟩

**lemma** *pprt-mono* [*simp*, *noatp*]: *a* ≤ *b* ⟹ *pprt a* ≤ *pprt b*
⟨*proof*⟩

**lemma** *nprt-mono* [*simp*, *noatp*]: *a* ≤ *b* ⟹ *nprt a* ≤ *nprt b*
⟨*proof*⟩

**end**

**lemmas** *add-sup-inf-distribs* = *add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left*

**class** *lordered-ab-group-add-abs* = *lordered-ab-group-add* + *abs* +
  **assumes** *abs-lattice*: |*a*| = *sup a* (− *a*)
**begin**

**lemma** *abs-prts*: |*a*| = *pprt a* − *nprt a*
⟨*proof*⟩

**subclass** *pordered-ab-group-add-abs*
⟨*proof*⟩

**end**

**lemma** *sup-eq-if*:
  **fixes** *a* :: 'a::{*lordered-ab-group-add*, *linorder*}
  **shows** *sup a* (− *a*) = (*if a* < *0 then* − *a else a*)
⟨*proof*⟩

**lemma** *abs-if-lattice*:
  **fixes** *a* :: 'a::{*lordered-ab-group-add-abs*, *linorder*}
  **shows** |*a*| = (*if a* < *0 then* − *a else a*)
⟨*proof*⟩

Needed for abelian cancellation simprocs:

**lemma** *add-cancel-21*: ((*x*::'a::*ab-group-add*) + (*y* + *z*) = *y* + *u*) = (*x* + *z* = *u*)
⟨*proof*⟩

**lemma** *add-cancel-end*: (*x* + (*y* + *z*) = *y*) = (*x* = − (*z*::'a::*ab-group-add*))
⟨*proof*⟩

**lemma** *less-eqI*: (*x*::'a::*pordered-ab-group-add*) − *y* = *x'* − *y'* ⟹ (*x* < *y*) = (*x'* < *y'*)
⟨*proof*⟩

**lemma** *le-eqI*: (*x*::'a::*pordered-ab-group-add*) − *y* = *x'* − *y'* ⟹ (*y* <= *x*) = (*y'* <= *x'*)
⟨*proof*⟩

**lemma** *eq-eqI*: (*x*::'a::*ab-group-add*) − *y* = *x'* − *y'* ⟹ (*x* = *y*) = (*x'* = *y'*)
⟨*proof*⟩

**lemma** *diff-def*: (*x*::'a::*ab-group-add*) − *y* == *x* + (−*y*)
⟨*proof*⟩

**lemma** *add-minus-cancel*: $(a::'a::ab\text{-}group\text{-}add) + (-a + b) = b$
⟨*proof*⟩

**lemma** *le-add-right-mono*:
**assumes**
$a <= b + (c::'a::pordered\text{-}ab\text{-}group\text{-}add)$
$c <= d$
**shows** $a <= b + d$
⟨*proof*⟩

**lemma** *estimate-by-abs*:
$a + b <= (c::'a::lordered\text{-}ab\text{-}group\text{-}add\text{-}abs) \Longrightarrow a <= c + abs\ b$
⟨*proof*⟩

### 10.7 Tools setup

**lemma** *add-mono-thms-ordered-semiring* [*noatp*]:
**fixes** $i\ j\ k :: 'a::pordered\text{-}ab\text{-}semigroup\text{-}add$
**shows** $i \leq j \wedge k \leq l \Longrightarrow i + k \leq j + l$
**and** $i = j \wedge k \leq l \Longrightarrow i + k \leq j + l$
**and** $i \leq j \wedge k = l \Longrightarrow i + k \leq j + l$
**and** $i = j \wedge k = l \Longrightarrow i + k = j + l$
⟨*proof*⟩

**lemma** *add-mono-thms-ordered-field* [*noatp*]:
**fixes** $i\ j\ k :: 'a::pordered\text{-}cancel\text{-}ab\text{-}semigroup\text{-}add$
**shows** $i < j \wedge k = l \Longrightarrow i + k < j + l$
**and** $i = j \wedge k < l \Longrightarrow i + k < j + l$
**and** $i < j \wedge k \leq l \Longrightarrow i + k < j + l$
**and** $i \leq j \wedge k < l \Longrightarrow i + k < j + l$
**and** $i < j \wedge k < l \Longrightarrow i + k < j + l$
⟨*proof*⟩

Simplification of $x - y < (0::'a)$, etc.

**lemmas** *diff-less-0-iff-less* [*simp*, *noatp*] = *less-iff-diff-less-0* [*symmetric*]
**lemmas** *diff-le-0-iff-le* [*simp*, *noatp*] = *le-iff-diff-le-0* [*symmetric*]

⟨*ML*⟩

**end**

## 11 Ring-and-Field: (Ordered) Rings and Fields

**theory** *Ring-and-Field*
**imports** *OrderedGroup*
**begin**

The theory of partially ordered rings is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

**class** *semiring* = *ab-semigroup-add* + *semigroup-mult* +
  **assumes** *left-distrib*[*algebra-simps*]: $(a + b) * c = a * c + b * c$
  **assumes** *right-distrib*[*algebra-simps*]: $a * (b + c) = a * b + a * c$
**begin**

For the *combine-numerals* simproc

**lemma** *combine-common-factor*:
  $a * e + (b * e + c) = (a + b) * e + c$
⟨*proof*⟩

**end**

**class** *mult-zero* = *times* + *zero* +
  **assumes** *mult-zero-left* [*simp*]: $0 * a = 0$
  **assumes** *mult-zero-right* [*simp*]: $a * 0 = 0$

**class** *semiring-0* = *semiring* + *comm-monoid-add* + *mult-zero*

**class** *semiring-0-cancel* = *semiring* + *cancel-comm-monoid-add*
**begin**

**subclass** *semiring-0*
⟨*proof*⟩

**end**

**class** *comm-semiring* = *ab-semigroup-add* + *ab-semigroup-mult* +
  **assumes** *distrib*: $(a + b) * c = a * c + b * c$
**begin**

**subclass** *semiring*
⟨*proof*⟩

**end**

**class** *comm-semiring-0* = *comm-semiring* + *comm-monoid-add* + *mult-zero*

**begin**

**subclass** *semiring-0* ⟨*proof*⟩

**end**

**class** *comm-semiring-0-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
**begin**

**subclass** *semiring-0-cancel* ⟨*proof*⟩

**subclass** *comm-semiring-0* ⟨*proof*⟩

**end**

**class** *zero-neq-one* = *zero* + *one* +
  **assumes** *zero-neq-one* [*simp*]: *0* ≠ *1*
**begin**

**lemma** *one-neq-zero* [*simp*]: *1* ≠ *0*
⟨*proof*⟩

**end**

**class** *semiring-1* = *zero-neq-one* + *semiring-0* + *monoid-mult*

Abstract divisibility

**class** *dvd* = *times*
**begin**

**definition** *dvd* :: *'a* ⇒ *'a* ⇒ *bool* (**infixl** *dvd 50*) **where**
  [*code del*]: *b dvd a* ⟷ (∃ *k*. *a* = *b* ∗ *k*)

**lemma** *dvdI* [*intro?*]: *a* = *b* ∗ *k* ⟹ *b dvd a*
  ⟨*proof*⟩

**lemma** *dvdE* [*elim?*]: *b dvd a* ⟹ (⋀*k*. *a* = *b* ∗ *k* ⟹ *P*) ⟹ *P*
  ⟨*proof*⟩

**end**

**class** *comm-semiring-1* = *zero-neq-one* + *comm-semiring-0* + *comm-monoid-mult*
+ *dvd*

**begin**

**subclass** *semiring-1* ⟨*proof*⟩

**lemma** *dvd-refl*[*simp*]: *a dvd a*

⟨*proof*⟩

**lemma** *dvd-trans*:
**assumes** *a dvd b* **and** *b dvd c*
**shows** *a dvd c*
⟨*proof*⟩

**lemma** *dvd-0-left-iff* [*noatp*, *simp*]: *0 dvd a* ⟷ *a = 0*
⟨*proof*⟩

**lemma** *dvd-0-right* [*iff*]: *a dvd 0*
⟨*proof*⟩

**lemma** *one-dvd* [*simp*]: *1 dvd a*
⟨*proof*⟩

**lemma** *dvd-mult*[*simp*]: *a dvd c* ⟹ *a dvd* (*b* ∗ *c*)
⟨*proof*⟩

**lemma** *dvd-mult2*[*simp*]: *a dvd b* ⟹ *a dvd* (*b* ∗ *c*)
⟨*proof*⟩

**lemma** *dvd-triv-right* [*simp*]: *a dvd b* ∗ *a*
⟨*proof*⟩

**lemma** *dvd-triv-left* [*simp*]: *a dvd a* ∗ *b*
⟨*proof*⟩

**lemma** *mult-dvd-mono*:
**assumes** *a dvd b*
**and** *c dvd d*
**shows** *a* ∗ *c dvd b* ∗ *d*
⟨*proof*⟩

**lemma** *dvd-mult-left*: *a* ∗ *b dvd c* ⟹ *a dvd c*
⟨*proof*⟩

**lemma** *dvd-mult-right*: *a* ∗ *b dvd c* ⟹ *b dvd c*
⟨*proof*⟩

**lemma** *dvd-0-left*: *0 dvd a* ⟹ *a = 0*
⟨*proof*⟩

**lemma** *dvd-add*[*simp*]:
**assumes** *a dvd b* **and** *a dvd c* **shows** *a dvd* (*b* + *c*)
⟨*proof*⟩

**end**

**class** *no-zero-divisors* = *zero* + *times* +
  **assumes** *no-zero-divisors*: $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$

**class** *semiring-1-cancel* = *semiring* + *cancel-comm-monoid-add*
  + *zero-neq-one* + *monoid-mult*
**begin**

**subclass** *semiring-0-cancel* ⟨*proof*⟩

**subclass** *semiring-1* ⟨*proof*⟩

**end**

**class** *comm-semiring-1-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
  + *zero-neq-one* + *comm-monoid-mult*
**begin**

**subclass** *semiring-1-cancel* ⟨*proof*⟩
**subclass** *comm-semiring-0-cancel* ⟨*proof*⟩
**subclass** *comm-semiring-1* ⟨*proof*⟩

**end**

**class** *ring* = *semiring* + *ab-group-add*
**begin**

**subclass** *semiring-0-cancel* ⟨*proof*⟩

Distribution rules

**lemma** *minus-mult-left*: $-(a * b) = - a * b$
⟨*proof*⟩

**lemma** *minus-mult-right*: $-(a * b) = a * - b$
⟨*proof*⟩

Extract signs from products

**lemmas** *mult-minus-left* [*simp*, *noatp*] = *minus-mult-left* [*symmetric*]
**lemmas** *mult-minus-right* [*simp*,*noatp*] = *minus-mult-right* [*symmetric*]

**lemma** *minus-mult-minus* [*simp*]: $- a * - b = a * b$
⟨*proof*⟩

**lemma** *minus-mult-commute*: $- a * b = a * - b$
⟨*proof*⟩

**lemma** *right-diff-distrib*[*algebra-simps*]: $a * (b - c) = a * b - a * c$
⟨*proof*⟩

**lemma** *left-diff-distrib*[*algebra-simps*]: $(a - b) * c = a * c - b * c$
⟨*proof*⟩

**lemmas** *ring-distribs*[*noatp*] =
*right-distrib left-distrib left-diff-distrib right-diff-distrib*

Legacy - use *algebra-simps*

**lemmas** *ring-simps*[*noatp*] = *algebra-simps*

**lemma** *eq-add-iff1*:
$a * e + c = b * e + d \longleftrightarrow (a - b) * e + c = d$
⟨*proof*⟩

**lemma** *eq-add-iff2*:
$a * e + c = b * e + d \longleftrightarrow c = (b - a) * e + d$
⟨*proof*⟩

**end**

**lemmas** *ring-distribs*[*noatp*] =
*right-distrib left-distrib left-diff-distrib right-diff-distrib*

**class** *comm-ring* = *comm-semiring* + *ab-group-add*
**begin**

**subclass** *ring* ⟨*proof*⟩
**subclass** *comm-semiring-0-cancel* ⟨*proof*⟩

**end**

**class** *ring-1* = *ring* + *zero-neq-one* + *monoid-mult*
**begin**

**subclass** *semiring-1-cancel* ⟨*proof*⟩

**end**

**class** *comm-ring-1* = *comm-ring* + *zero-neq-one* + *comm-monoid-mult*

**begin**

**subclass** *ring-1* ⟨*proof*⟩
**subclass** *comm-semiring-1-cancel* ⟨*proof*⟩

**lemma** *dvd-minus-iff* [*simp*]: *x dvd* − *y* ⟷ *x dvd y*
⟨*proof*⟩

**lemma** *minus-dvd-iff* [*simp*]: − *x dvd y* ⟷ *x dvd y*
⟨*proof*⟩

**lemma** *dvd-diff*[*simp*]: *x dvd y* $\Longrightarrow$ *x dvd z* $\Longrightarrow$ *x dvd* (*y* − *z*)
⟨*proof*⟩

**end**

**class** *ring-no-zero-divisors* = *ring* + *no-zero-divisors*
**begin**

**lemma** *mult-eq-0-iff* [*simp*]:
**shows** *a* ∗ *b* = *0* ⟷ (*a* = *0* ∨ *b* = *0*)
⟨*proof*⟩

Cancellation of equalities with a common factor

**lemma** *mult-cancel-right* [*simp*, *noatp*]:
*a* ∗ *c* = *b* ∗ *c* ⟷ *c* = *0* ∨ *a* = *b*
⟨*proof*⟩

**lemma** *mult-cancel-left* [*simp*, *noatp*]:
*c* ∗ *a* = *c* ∗ *b* ⟷ *c* = *0* ∨ *a* = *b*
⟨*proof*⟩

**end**

**class** *ring-1-no-zero-divisors* = *ring-1* + *ring-no-zero-divisors*
**begin**

**lemma** *mult-cancel-right1* [*simp*]:
*c* = *b* ∗ *c* ⟷ *c* = *0* ∨ *b* = *1*
⟨*proof*⟩

**lemma** *mult-cancel-right2* [*simp*]:
*a* ∗ *c* = *c* ⟷ *c* = *0* ∨ *a* = *1*
⟨*proof*⟩

**lemma** *mult-cancel-left1* [*simp*]:
*c* = *c* ∗ *b* ⟷ *c* = *0* ∨ *b* = *1*
⟨*proof*⟩

**lemma** *mult-cancel-left2* [*simp*]:
*c* ∗ *a* = *c* ⟷ *c* = *0* ∨ *a* = *1*
⟨*proof*⟩

**end**

**class** *idom* = *comm-ring-1* + *no-zero-divisors*
**begin**

**subclass** *ring-1-no-zero-divisors* ⟨*proof*⟩

**lemma** *square-eq-iff*: $a * a = b * b \longleftrightarrow (a = b \vee a = - b)$
⟨*proof*⟩

**lemma** *dvd-mult-cancel-right* [*simp*]:
$a * c$ *dvd* $b * c \longleftrightarrow c = 0 \vee a$ *dvd* $b$
⟨*proof*⟩

**lemma** *dvd-mult-cancel-left* [*simp*]:
$c * a$ *dvd* $c * b \longleftrightarrow c = 0 \vee a$ *dvd* $b$
⟨*proof*⟩

**end**

**class** *division-ring* = *ring-1* + *inverse* +
**assumes** *left-inverse* [*simp*]: $a \neq 0 \Longrightarrow$ *inverse* $a * a = 1$
**assumes** *right-inverse* [*simp*]: $a \neq 0 \Longrightarrow a *$ *inverse* $a = 1$
**begin**

**subclass** *ring-1-no-zero-divisors*
⟨*proof*⟩

**lemma** *nonzero-imp-inverse-nonzero*:
$a \neq 0 \Longrightarrow$ *inverse* $a \neq 0$
⟨*proof*⟩

**lemma** *inverse-zero-imp-zero*:
*inverse* $a = 0 \Longrightarrow a = 0$
⟨*proof*⟩

**lemma** *inverse-unique*:
**assumes** *ab*: $a * b = 1$
**shows** *inverse* $a = b$
⟨*proof*⟩

**lemma** *nonzero-inverse-minus-eq*:
$a \neq 0 \Longrightarrow$ *inverse* $(- a) = -$ *inverse* $a$
⟨*proof*⟩

**lemma** *nonzero-inverse-inverse-eq*:
$a \neq 0 \Longrightarrow$ *inverse* (*inverse* $a$) $= a$
⟨*proof*⟩

**lemma** *nonzero-inverse-eq-imp-eq*:
**assumes** *inverse* $a =$ *inverse* $b$ **and** $a \neq 0$ **and** $b \neq 0$
**shows** $a = b$
⟨*proof*⟩

**lemma** *inverse-1* [*simp*]: *inverse* $1 = 1$

⟨*proof*⟩

**lemma** *nonzero-inverse-mult-distrib*:
**assumes** $a \neq 0$ **and** $b \neq 0$
**shows** $inverse\ (a * b) = inverse\ b * inverse\ a$
⟨*proof*⟩

**lemma** *division-ring-inverse-add*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ a + inverse\ b = inverse\ a * (a + b) * inverse\ b$
⟨*proof*⟩

**lemma** *division-ring-inverse-diff*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ a - inverse\ b = inverse\ a * (b - a) * inverse\ b$
⟨*proof*⟩

**end**

**class** *field* = *comm-ring-1* + *inverse* +
**assumes** *field-inverse*: $a \neq 0 \Longrightarrow inverse\ a * a = 1$
**assumes** *divide-inverse*: $a\ /\ b = a * inverse\ b$
**begin**

**subclass** *division-ring*
⟨*proof*⟩

**subclass** *idom* ⟨*proof*⟩

**lemma** *right-inverse-eq*: $b \neq 0 \Longrightarrow a\ /\ b = 1 \longleftrightarrow a = b$
⟨*proof*⟩

**lemma** *nonzero-inverse-eq-divide*: $a \neq 0 \Longrightarrow inverse\ a = 1\ /\ a$
⟨*proof*⟩

**lemma** *divide-self* [*simp*]: $a \neq 0 \Longrightarrow a\ /\ a = 1$
⟨*proof*⟩

**lemma** *divide-zero-left* [*simp*]: $0\ /\ a = 0$
⟨*proof*⟩

**lemma** *inverse-eq-divide*: $inverse\ a = 1\ /\ a$
⟨*proof*⟩

**lemma** *add-divide-distrib*: $(a+b)\ /\ c = a/c + b/c$
⟨*proof*⟩

There is no slick version using division by zero.

**lemma** *inverse-add*:
[| $a \neq 0$; $b \neq 0$ |]
==> $inverse\ a + inverse\ b = (a + b) * inverse\ a * inverse\ b$

⟨*proof*⟩

**lemma** *nonzero-mult-divide-mult-cancel-left* [*simp*, *noatp*]:
**assumes** [*simp*]: $b \neq 0$ **and** [*simp*]: $c \neq 0$ **shows** $(c*a)/(c*b) = a/b$
⟨*proof*⟩

**lemma** *nonzero-mult-divide-mult-cancel-right* [*simp*, *noatp*]:
$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (a * c) \ / \ (b * c) = a \ / \ b$
⟨*proof*⟩

**lemma** *divide-1* [*simp*]: $a \ / \ 1 = a$
⟨*proof*⟩

**lemma** *times-divide-eq-right*: $a * (b \ / \ c) = (a * b) \ / \ c$
⟨*proof*⟩

**lemma** *times-divide-eq-left*: $(b \ / \ c) * a = (b * a) \ / \ c$
⟨*proof*⟩

These are later declared as simp rules.

**lemmas** *times-divide-eq* [*noatp*] = *times-divide-eq-right times-divide-eq-left*

**lemma** *add-frac-eq*:
**assumes** $y \neq 0$ **and** $z \neq 0$
**shows** $x \ / \ y + w \ / \ z = (x * z + w * y) \ / \ (y * z)$
⟨*proof*⟩

Special Cancellation Simprules for Division

**lemma** *nonzero-mult-divide-cancel-right* [*simp*, *noatp*]:
$b \neq 0 \Longrightarrow a * b \ / \ b = a$
⟨*proof*⟩

**lemma** *nonzero-mult-divide-cancel-left* [*simp*, *noatp*]:
$a \neq 0 \Longrightarrow a * b \ / \ a = b$
⟨*proof*⟩

**lemma** *nonzero-divide-mult-cancel-right* [*simp*, *noatp*]:
$[\![a \neq 0;\ b \neq 0]\!] \Longrightarrow b \ / \ (a * b) = 1 \ / \ a$
⟨*proof*⟩

**lemma** *nonzero-divide-mult-cancel-left* [*simp*, *noatp*]:
$[\![a \neq 0;\ b \neq 0]\!] \Longrightarrow a \ / \ (a * b) = 1 \ / \ b$
⟨*proof*⟩

**lemma** *nonzero-mult-divide-mult-cancel-left2* [*simp*, *noatp*]:
$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (c * a) \ / \ (b * c) = a \ / \ b$
⟨*proof*⟩

**lemma** *nonzero-mult-divide-mult-cancel-right2* [*simp*, *noatp*]:

$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (a * c) \ / \ (c * b) = a \ / \ b$
⟨*proof*⟩

**lemma** *minus-divide-left*: $- (a \ / \ b) = (-a) \ / \ b$
⟨*proof*⟩

**lemma** *nonzero-minus-divide-right*: $b \neq 0 \Longrightarrow - (a \ / \ b) = a \ / \ (- b)$
⟨*proof*⟩

**lemma** *nonzero-minus-divide-divide*: $b \neq 0 \Longrightarrow (-a) \ / \ (-b) = a \ / \ b$
⟨*proof*⟩

**lemma** *divide-minus-left* [*simp*, *noatp*]: $(-a) \ / \ b = - (a \ / \ b)$
⟨*proof*⟩

**lemma** *diff-divide-distrib*: $(a - b) \ / \ c = a \ / \ c - b \ / \ c$
⟨*proof*⟩

**lemma** *add-divide-eq-iff*:
$z \neq 0 \Longrightarrow x + y \ / \ z = (z * x + y) \ / \ z$
⟨*proof*⟩

**lemma** *divide-add-eq-iff*:
$z \neq 0 \Longrightarrow x \ / \ z + y = (x + z * y) \ / \ z$
⟨*proof*⟩

**lemma** *diff-divide-eq-iff*:
$z \neq 0 \Longrightarrow x - y \ / \ z = (z * x - y) \ / \ z$
⟨*proof*⟩

**lemma** *divide-diff-eq-iff*:
$z \neq 0 \Longrightarrow x \ / \ z - y = (x - z * y) \ / \ z$
⟨*proof*⟩

**lemma** *nonzero-eq-divide-eq*: $c \neq 0 \Longrightarrow a = b \ / \ c \longleftrightarrow a * c = b$
⟨*proof*⟩

**lemma** *nonzero-divide-eq-eq*: $c \neq 0 \Longrightarrow b \ / \ c = a \longleftrightarrow b = a * c$
⟨*proof*⟩

**lemma** *divide-eq-imp*: $c \neq 0 \Longrightarrow b = a * c \Longrightarrow b \ / \ c = a$
⟨*proof*⟩

**lemma** *eq-divide-imp*: $c \neq 0 \Longrightarrow a * c = b \Longrightarrow a = b \ / \ c$
⟨*proof*⟩

**lemmas** *field-eq-simps*[*noatp*] = *algebra-simps*

*add-divide-eq-iff divide-add-eq-iff*

*diff-divide-eq-iff divide-diff-eq-iff*

*nonzero-eq-divide-eq nonzero-divide-eq-eq*

An example:

**lemma** $[\![a{\neq}b;\ c{\neq}d;\ e{\neq}f]\!] \Longrightarrow ((a-b)*(c-d)*(e-f))/((c-d)*(e-f)*(a-b)) = 1$
⟨*proof*⟩

**lemma** *diff-frac-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x\ /\ y - w\ /\ z = (x * z - w * y)\ /\ (y * z)$
⟨*proof*⟩

**lemma** *frac-eq-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (x\ /\ y = w\ /\ z) = (x * z = w * y)$
⟨*proof*⟩

**end**

**class** *division-by-zero* = *zero* + *inverse* +
**assumes** *inverse-zero* [*simp*]: *inverse 0 = 0*

**lemma** *divide-zero* [*simp*]:
$a\ /\ 0 = (0::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

**lemma** *divide-self-if* [*simp*]:
$a\ /\ (a::'a::\{field, division\text{-}by\text{-}zero\}) = (if\ a{=}0\ then\ 0\ else\ 1)$
⟨*proof*⟩

**class** *mult-mono* = *times* + *zero* + *ord* +
**assumes** *mult-left-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
**assumes** *mult-right-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow a * c \leq b * c$

**class** *pordered-semiring* = *mult-mono* + *semiring-0* + *pordered-ab-semigroup-add*

**begin**

**lemma** *mult-mono*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq b \Longrightarrow 0 \leq c$
$\Longrightarrow a * c \leq b * d$
⟨*proof*⟩

**lemma** *mult-mono'*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq a \Longrightarrow 0 \leq c$
$\Longrightarrow a * c \leq b * d$
⟨*proof*⟩

**end**

**class** *pordered-cancel-semiring* = *mult-mono* + *pordered-ab-semigroup-add* + *semiring* + *cancel-comm-monoid-add*
**begin**

**subclass** *semiring-0-cancel* ⟨*proof*⟩
**subclass** *pordered-semiring* ⟨*proof*⟩

**lemma** *mult-nonneg-nonneg*: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

**lemma** *mult-nonneg-nonpos*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow a * b \leq 0$
⟨*proof*⟩

**lemma** *mult-nonpos-nonneg*: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow a * b \leq 0$
⟨*proof*⟩

Legacy - use *mult-nonpos-nonneg*

**lemma** *mult-nonneg-nonpos2*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow b * a \leq 0$
⟨*proof*⟩

**lemma** *split-mult-neg-le*: $(0 \leq a \;\&\; b \leq 0) \mid (a \leq 0 \;\&\; 0 \leq b) \Longrightarrow a * b \leq 0$
⟨*proof*⟩

**end**

**class** *ordered-semiring* = *semiring* + *comm-monoid-add* + *ordered-cancel-ab-semigroup-add* + *mult-mono*
**begin**

**subclass** *pordered-cancel-semiring* ⟨*proof*⟩

**subclass** *pordered-comm-monoid-add* ⟨*proof*⟩

**lemma** *mult-left-less-imp-less*:
$c * a < c * b \Longrightarrow 0 \leq c \Longrightarrow a < b$
⟨*proof*⟩

**lemma** *mult-right-less-imp-less*:
$a * c < b * c \Longrightarrow 0 \leq c \Longrightarrow a < b$
⟨*proof*⟩

**end**

**class** *ordered-semiring-strict* = *semiring* + *comm-monoid-add* + *ordered-cancel-ab-semigroup-add* +
**assumes** *mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
**assumes** *mult-strict-right-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow a * c < b * c$
**begin**

**subclass** *semiring-0-cancel* ⟨*proof*⟩

**subclass** *ordered-semiring*
⟨*proof*⟩

**lemma** *mult-left-le-imp-le*:
$c * a \leq c * b \Longrightarrow 0 < c \Longrightarrow a \leq b$
⟨*proof*⟩

**lemma** *mult-right-le-imp-le*:
$a * c \leq b * c \Longrightarrow 0 < c \Longrightarrow a \leq b$
⟨*proof*⟩

**lemma** *mult-pos-pos*: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a * b$
⟨*proof*⟩

**lemma** *mult-pos-neg*: $0 < a \Longrightarrow b < 0 \Longrightarrow a * b < 0$
⟨*proof*⟩

**lemma** *mult-neg-pos*: $a < 0 \Longrightarrow 0 < b \Longrightarrow a * b < 0$
⟨*proof*⟩

Legacy - use *mult-neg-pos*

**lemma** *mult-pos-neg2*: $0 < a \Longrightarrow b < 0 \Longrightarrow b * a < 0$
⟨*proof*⟩

**lemma** *zero-less-mult-pos*:
$0 < a * b \Longrightarrow 0 < a \Longrightarrow 0 < b$
⟨*proof*⟩

**lemma** *zero-less-mult-pos2*:
$0 < b * a \Longrightarrow 0 < a \Longrightarrow 0 < b$
⟨*proof*⟩

Strict monotonicity in both arguments

**lemma** *mult-strict-mono*:
**assumes** $a < b$ **and** $c < d$ **and** $0 < b$ **and** $0 \leq c$
**shows** $a * c < b * d$
⟨*proof*⟩

This weaker variant has more natural premises

**lemma** *mult-strict-mono'*:
**assumes** $a < b$ **and** $c < d$ **and** $0 \leq a$ **and** $0 \leq c$
**shows** $a * c < b * d$
⟨*proof*⟩

**lemma** *mult-less-le-imp-less*:
**assumes** $a < b$ **and** $c \leq d$ **and** $0 \leq a$ **and** $0 < c$
**shows** $a * c < b * d$

⟨*proof*⟩

**lemma** *mult-le-less-imp-less*:
**assumes** $a \leq b$ **and** $c < d$ **and** $0 < a$ **and** $0 \leq c$
**shows** $a * c < b * d$
⟨*proof*⟩

**lemma** *mult-less-imp-less-left*:
**assumes** *less*: $c * a < c * b$ **and** *nonneg*: $0 \leq c$
**shows** $a < b$
⟨*proof*⟩

**lemma** *mult-less-imp-less-right*:
**assumes** *less*: $a * c < b * c$ **and** *nonneg*: $0 \leq c$
**shows** $a < b$
⟨*proof*⟩

**end**

**class** *mult-mono1* = *times* + *zero* + *ord* +
**assumes** *mult-mono1*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$

**class** *pordered-comm-semiring* = *comm-semiring-0*
+ *pordered-ab-semigroup-add* + *mult-mono1*
**begin**

**subclass** *pordered-semiring*
⟨*proof*⟩

**end**

**class** *pordered-cancel-comm-semiring* = *comm-semiring-0-cancel*
+ *pordered-ab-semigroup-add* + *mult-mono1*
**begin**

**subclass** *pordered-comm-semiring* ⟨*proof*⟩
**subclass** *pordered-cancel-semiring* ⟨*proof*⟩

**end**

**class** *ordered-comm-semiring-strict* = *comm-semiring-0* + *ordered-cancel-ab-semigroup-add*
+
**assumes** *mult-strict-left-mono-comm*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
**begin**

**subclass** *ordered-semiring-strict*
⟨*proof*⟩

**subclass** *pordered-cancel-comm-semiring*

⟨*proof*⟩

**end**

**class** *pordered-ring* = *ring* + *pordered-cancel-semiring*
**begin**

**subclass** *pordered-ab-group-add* ⟨*proof*⟩

Legacy - use *algebra-simps*

**lemmas** *ring-simps*[*noatp*] = *algebra-simps*

**lemma** *less-add-iff1*:
$a * e + c < b * e + d \longleftrightarrow (a - b) * e + c < d$
⟨*proof*⟩

**lemma** *less-add-iff2*:
$a * e + c < b * e + d \longleftrightarrow c < (b - a) * e + d$
⟨*proof*⟩

**lemma** *le-add-iff1*:
$a * e + c \leq b * e + d \longleftrightarrow (a - b) * e + c \leq d$
⟨*proof*⟩

**lemma** *le-add-iff2*:
$a * e + c \leq b * e + d \longleftrightarrow c \leq (b - a) * e + d$
⟨*proof*⟩

**lemma** *mult-left-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow c * a \leq c * b$
⟨*proof*⟩

**lemma** *mult-right-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow a * c \leq b * c$
⟨*proof*⟩

**lemma** *mult-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

**lemma** *split-mult-pos-le*:
$(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

**end**

**class** *abs-if* = *minus* + *uminus* + *ord* + *zero* + *abs* +
**assumes** *abs-if*: $|a| = (\mathit{if}\ a < 0\ \mathit{then} - a\ \mathit{else}\ a)$

**class** *sgn-if* = *minus* + *uminus* + *zero* + *one* + *ord* + *sgn* +

**assumes** *sgn-if*: $sgn\ x = (if\ x = 0\ then\ 0\ else\ if\ 0 < x\ then\ 1\ else\ -1)$

**lemma** (**in** *sgn-if*) *sgn0*[*simp*]: $sgn\ 0 = 0$
⟨*proof*⟩

**class** *ordered-ring* = *ring* + *ordered-semiring*
+ *ordered-ab-group-add* + *abs-if*
**begin**

**subclass** *pordered-ring* ⟨*proof*⟩

**subclass** *pordered-ab-group-add-abs*
⟨*proof*⟩

**end**

**class** *ordered-ring-strict* = *ring* + *ordered-semiring-strict*
+ *ordered-ab-group-add* + *abs-if*
**begin**

**subclass** *ordered-ring* ⟨*proof*⟩

**lemma** *mult-strict-left-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow c * a < c * b$
⟨*proof*⟩

**lemma** *mult-strict-right-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow a * c < b * c$
⟨*proof*⟩

**lemma** *mult-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow 0 < a * b$
⟨*proof*⟩

**subclass** *ring-no-zero-divisors*
⟨*proof*⟩

**lemma** *zero-less-mult-iff*:
$0 < a * b \longleftrightarrow 0 < a \wedge 0 < b \vee a < 0 \wedge b < 0$
⟨*proof*⟩

**lemma** *zero-le-mult-iff*:
$0 \leq a * b \longleftrightarrow 0 \leq a \wedge 0 \leq b \vee a \leq 0 \wedge b \leq 0$
⟨*proof*⟩

**lemma** *mult-less-0-iff*:
$a * b < 0 \longleftrightarrow 0 < a \wedge b < 0 \vee a < 0 \wedge 0 < b$
⟨*proof*⟩

**lemma** *mult-le-0-iff*:
$a * b \leq 0 \longleftrightarrow 0 \leq a \wedge b \leq 0 \vee a \leq 0 \wedge 0 \leq b$

⟨*proof*⟩

**lemma** *zero-le-square* [*simp*]: $0 \leq a * a$
⟨*proof*⟩

**lemma** *not-square-less-zero* [*simp*]: $\neg (a * a < 0)$
⟨*proof*⟩

Cancellation laws for $c * a < c * b$ and $a * c < b * c$, also with the relations $\leq$ and equality.

These "disjunction" versions produce two cases when the comparison is an assumption, but effectively four when the comparison is a goal.

**lemma** *mult-less-cancel-right-disj*:
$a * c < b * c \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
⟨*proof*⟩

**lemma** *mult-less-cancel-left-disj*:
$c * a < c * b \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
⟨*proof*⟩

The "conjunction of implication" lemmas produce two cases when the comparison is a goal, but give four when the comparison is an assumption.

**lemma** *mult-less-cancel-right*:
$a * c < b * c \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
⟨*proof*⟩

**lemma** *mult-less-cancel-left*:
$c * a < c * b \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
⟨*proof*⟩

**lemma** *mult-le-cancel-right*:
$a * c \leq b * c \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

**lemma** *mult-le-cancel-left*:
$c * a \leq c * b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

**lemma** *mult-le-cancel-left-pos*:
$0 < c \Longrightarrow c * a \leq c * b \longleftrightarrow a \leq b$
⟨*proof*⟩

**lemma** *mult-le-cancel-left-neg*:
$c < 0 \Longrightarrow c * a \leq c * b \longleftrightarrow b \leq a$
⟨*proof*⟩

**lemma** *mult-less-cancel-left-pos*:
$0 < c \Longrightarrow c * a < c * b \longleftrightarrow a < b$

⟨*proof*⟩

**lemma** *mult-less-cancel-left-neg*:
  $c < 0 \Longrightarrow c * a < c * b \longleftrightarrow b < a$
⟨*proof*⟩

**end**

Legacy - use *algebra-simps*

**lemmas** *ring-simps*[*noatp*] = *algebra-simps*

**lemmas** *mult-sign-intros* =
  *mult-nonneg-nonneg mult-nonneg-nonpos*
  *mult-nonpos-nonneg mult-nonpos-nonpos*
  *mult-pos-pos mult-pos-neg*
  *mult-neg-pos mult-neg-neg*

**class** *pordered-comm-ring* = *comm-ring* + *pordered-comm-semiring*
**begin**

**subclass** *pordered-ring* ⟨*proof*⟩
**subclass** *pordered-cancel-comm-semiring* ⟨*proof*⟩

**end**

**class** *ordered-semidom* = *comm-semiring-1-cancel* + *ordered-comm-semiring-strict*
+

  **assumes** *zero-less-one* [*simp*]: *0* < *1*
**begin**

**lemma** *pos-add-strict*:
  **shows** $0 < a \Longrightarrow b < c \Longrightarrow b < a + c$
  ⟨*proof*⟩

**lemma** *zero-le-one* [*simp*]: *0* ≤ *1*
⟨*proof*⟩

**lemma** *not-one-le-zero* [*simp*]: ¬ *1* ≤ *0*
⟨*proof*⟩

**lemma** *not-one-less-zero* [*simp*]: ¬ *1* < *0*
⟨*proof*⟩

**lemma** *less-1-mult*:
  **assumes** *1* < *m* **and** *1* < *n*
  **shows** *1* < *m* * *n*
  ⟨*proof*⟩

**end**

**class** *ordered-idom* = *comm-ring-1* +
*ordered-comm-semiring-strict* + *ordered-ab-group-add* +
*abs-if* + *sgn-if*

**begin**

**subclass** *ordered-ring-strict* ⟨*proof*⟩
**subclass** *pordered-comm-ring* ⟨*proof*⟩
**subclass** *idom* ⟨*proof*⟩

**subclass** *ordered-semidom*
⟨*proof*⟩

**lemma** *linorder-neqE-ordered-idom*:
**assumes** $x \neq y$ **obtains** $x < y \mid y < x$
⟨*proof*⟩

These cancellation simprules also produce two cases when the comparison is a goal.

**lemma** *mult-le-cancel-right1*:
$c \leq b * c \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
⟨*proof*⟩

**lemma** *mult-le-cancel-right2*:
$a * c \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
⟨*proof*⟩

**lemma** *mult-le-cancel-left1*:
$c \leq c * b \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
⟨*proof*⟩

**lemma** *mult-le-cancel-left2*:
$c * a \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
⟨*proof*⟩

**lemma** *mult-less-cancel-right1*:
$c < b * c \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
⟨*proof*⟩

**lemma** *mult-less-cancel-right2*:
$a * c < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
⟨*proof*⟩

**lemma** *mult-less-cancel-left1*:
$c < c * b \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
⟨*proof*⟩

**lemma** *mult-less-cancel-left2*:
$c * a < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
⟨*proof*⟩

**lemma** *sgn-sgn* [*simp*]:
$sgn\ (sgn\ a) = sgn\ a$
⟨*proof*⟩

**lemma** *sgn-0-0*:
$sgn\ a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

**lemma** *sgn-1-pos*:
$sgn\ a = 1 \longleftrightarrow a > 0$
⟨*proof*⟩

**lemma** *sgn-1-neg*:
$sgn\ a = -\ 1 \longleftrightarrow a < 0$
⟨*proof*⟩

**lemma** *sgn-pos* [*simp*]:
$0 < a \Longrightarrow sgn\ a = 1$
⟨*proof*⟩

**lemma** *sgn-neg* [*simp*]:
$a < 0 \Longrightarrow sgn\ a = -\ 1$
⟨*proof*⟩

**lemma** *sgn-times*:
$sgn\ (a * b) = sgn\ a * sgn\ b$
⟨*proof*⟩

**lemma** *abs-sgn*: $abs\ k = k * sgn\ k$
⟨*proof*⟩

**lemma** *sgn-greater* [*simp*]:
$0 < sgn\ a \longleftrightarrow 0 < a$
⟨*proof*⟩

**lemma** *sgn-less* [*simp*]:
$sgn\ a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

**lemma** *abs-dvd-iff* [*simp*]: $(abs\ m)\ dvd\ k \longleftrightarrow m\ dvd\ k$
⟨*proof*⟩

**lemma** *dvd-abs-iff* [*simp*]: $m\ dvd\ (abs\ k) \longleftrightarrow m\ dvd\ k$
⟨*proof*⟩

**end**

**class** *ordered-field* = *field* + *ordered-idom*

Simprules for comparisons where common factors can be cancelled.

**lemmas** *mult-compare-simps*[*noatp*] =
*mult-le-cancel-right mult-le-cancel-left*
*mult-le-cancel-right1 mult-le-cancel-right2*
*mult-le-cancel-left1 mult-le-cancel-left2*
*mult-less-cancel-right mult-less-cancel-left*
*mult-less-cancel-right1 mult-less-cancel-right2*
*mult-less-cancel-left1 mult-less-cancel-left2*
*mult-cancel-right mult-cancel-left*
*mult-cancel-right1 mult-cancel-right2*
*mult-cancel-left1 mult-cancel-left2*

— FIXME continue localization here

**lemma** *inverse-nonzero-iff-nonzero* [*simp*]:
(*inverse a* = *0*) = (*a* = (*0*::'*a*::{*division-ring*,*division-by-zero*}))
⟨*proof*⟩

**lemma** *inverse-minus-eq* [*simp*]:
*inverse*(−*a*) = −*inverse*(*a*::'*a*::{*division-ring*,*division-by-zero*})
⟨*proof*⟩

**lemma** *inverse-eq-imp-eq*:
*inverse a* = *inverse b* ==> *a* = (*b*::'*a*::{*division-ring*,*division-by-zero*})
⟨*proof*⟩

**lemma** *inverse-eq-iff-eq* [*simp*]:
(*inverse a* = *inverse b*) = (*a* = (*b*::'*a*::{*division-ring*,*division-by-zero*}))
⟨*proof*⟩

**lemma** *inverse-inverse-eq* [*simp*]:
*inverse*(*inverse* (*a*::'*a*::{*division-ring*,*division-by-zero*})) = *a*
⟨*proof*⟩

This version builds in division by zero while also re-orienting the right-hand side.

**lemma** *inverse-mult-distrib* [*simp*]:
*inverse*(*a*∗*b*) = *inverse*(*a*) ∗ *inverse*(*b*::'*a*::{*field*,*division-by-zero*})
⟨*proof*⟩

**lemma** *inverse-divide* [*simp*]:
*inverse* (*a*/*b*) = *b* / (*a*::'*a*::{*field*,*division-by-zero*})
⟨*proof*⟩

## 11.1 Calculations with fractions

There is a whole bunch of simp-rules just for class *field* but none for class *field* and *nonzero-divides* because the latter are covered by a simproc.

**lemma** *mult-divide-mult-cancel-left*:
$c \neq 0 \Longrightarrow (c*a) \,/\, (c*b) = a \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

**lemma** *mult-divide-mult-cancel-right*:
$c \neq 0 \Longrightarrow (a*c) \,/\, (b*c) = a \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

**lemma** *divide-divide-eq-right* [*simp*,*noatp*]:
$a \,/\, (b/c) = (a*c) \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

**lemma** *divide-divide-eq-left* [*simp*,*noatp*]:
$(a \,/\, b) \,/\, (c::'a::\{field, division\text{-}by\text{-}zero\}) = a \,/\, (b*c)$
⟨*proof*⟩

### 11.1.1 Special Cancellation Simprules for Division

**lemma** *mult-divide-mult-cancel-left-if* [*simp*,*noatp*]:
**fixes** $c :: 'a :: \{field, division\text{-}by\text{-}zero\}$
**shows** $(c*a) \,/\, (c*b) = (\textit{if } c=0 \textit{ then } 0 \textit{ else } a/b)$
⟨*proof*⟩

## 11.2 Division and Unary Minus

**lemma** *minus-divide-right*: $-(a/b) = a \,/\, -(b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

**lemma** *divide-minus-right* [*simp*, *noatp*]:
$a \,/\, -(b::'a::\{field, division\text{-}by\text{-}zero\}) = -(a \,/\, b)$
⟨*proof*⟩

**lemma** *minus-divide-divide*:
$(-a)/(-b) = a \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

**lemma** *eq-divide-eq*:
$((a::'a::\{field, division\text{-}by\text{-}zero\}) = b/c) = (\textit{if } c \neq 0 \textit{ then } a*c = b \textit{ else } a=0)$
⟨*proof*⟩

**lemma** *divide-eq-eq*:
$(b/c = (a::'a::\{field, division\text{-}by\text{-}zero\})) = (\textit{if } c \neq 0 \textit{ then } b = a*c \textit{ else } a=0)$
⟨*proof*⟩

## 11.3 Ordered Fields

**lemma** *positive-imp-inverse-positive*:
**assumes** *a-gt-0*: $0 < a$ **shows** $0 < inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *negative-imp-inverse-negative*:
$a < 0 \Longrightarrow inverse\ a < (0::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *inverse-le-imp-le*:
**assumes** *invle*: $inverse\ a \leq inverse\ b$ **and** *apos*: $0 < a$
**shows** $b \leq (a::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *inverse-positive-imp-positive*:
**assumes** *inv-gt-0*: $0 < inverse\ a$ **and** *nz*: $a \neq 0$
**shows** $0 < (a::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *inverse-positive-iff-positive* [*simp*]:
$(0 < inverse\ a) = (0 < (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *inverse-negative-imp-negative*:
**assumes** *inv-less-0*: $inverse\ a < 0$ **and** *nz*: $a \neq 0$
**shows** $a < (0::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *inverse-negative-iff-negative* [*simp*]:
$(inverse\ a < 0) = (a < (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *inverse-nonnegative-iff-nonnegative* [*simp*]:
$(0 \leq inverse\ a) = (0 \leq (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *inverse-nonpositive-iff-nonpositive* [*simp*]:
$(inverse\ a \leq 0) = (a \leq (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *ordered-field-no-lb*: $\forall\ x.\ \exists\ y.\ y < (x::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *ordered-field-no-ub*: $\forall\ x.\ \exists\ y.\ y > (x::'a::ordered\text{-}field)$
⟨*proof*⟩

## 11.4 Anti-Monotonicity of *inverse*

**lemma** *less-imp-inverse-less*:

**assumes** *less*: $a < b$ **and** *apos*: $0 < a$
**shows** *inverse* $b <$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

**lemma** *inverse-less-imp-less*:
$[|$*inverse* $a <$ *inverse* $b;\ 0 < a|] ==> b < (a::'a::$*ordered-field*$)$
⟨*proof*⟩

Both premises are essential. Consider -1 and 1.

**lemma** *inverse-less-iff-less* [*simp*,*noatp*]:
$[|0 < a;\ 0 < b|] ==>$ (*inverse* $a <$ *inverse* $b) = (b < (a::'a::$*ordered-field*$))$
⟨*proof*⟩

**lemma** *le-imp-inverse-le*:
$[|a \leq b;\ 0 < a|] ==>$ *inverse* $b \leq$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

**lemma** *inverse-le-iff-le* [*simp*,*noatp*]:
$[|0 < a;\ 0 < b|] ==>$ (*inverse* $a \leq$ *inverse* $b) = (b \leq (a::'a::$*ordered-field*$))$
⟨*proof*⟩

These results refer to both operands being negative. The opposite-sign case is trivial, since inverse preserves signs.

**lemma** *inverse-le-imp-le-neg*:
$[|$*inverse* $a \leq$ *inverse* $b;\ b < 0|] ==> b \leq (a::'a::$*ordered-field*$)$
⟨*proof*⟩

**lemma** *less-imp-inverse-less-neg*:
$[|a < b;\ b < 0|] ==>$ *inverse* $b <$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

**lemma** *inverse-less-imp-less-neg*:
$[|$*inverse* $a <$ *inverse* $b;\ b < 0|] ==> b < (a::'a::$*ordered-field*$)$
⟨*proof*⟩

**lemma** *inverse-less-iff-less-neg* [*simp*,*noatp*]:
$[|a < 0;\ b < 0|] ==>$ (*inverse* $a <$ *inverse* $b) = (b < (a::'a::$*ordered-field*$))$
⟨*proof*⟩

**lemma** *le-imp-inverse-le-neg*:
$[|a \leq b;\ b < 0|] ==>$ *inverse* $b \leq$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

**lemma** *inverse-le-iff-le-neg* [*simp*,*noatp*]:
$[|a < 0;\ b < 0|] ==>$ (*inverse* $a \leq$ *inverse* $b) = (b \leq (a::'a::$*ordered-field*$))$
⟨*proof*⟩

## 11.5 Inverses and the Number One

**lemma** *one-less-inverse-iff*:
$(1 < inverse\ x) = (0 < x\ \&\ x < (1::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *inverse-eq-1-iff* [*simp*]:
$(inverse\ x = 1) = (x = (1::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *one-le-inverse-iff*:
$(1 \leq inverse\ x) = (0 < x\ \&\ x \leq (1::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *inverse-less-1-iff*:
$(inverse\ x < 1) = (x \leq 0 \mid 1 < (x::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *inverse-le-1-iff*:
$(inverse\ x \leq 1) = (x \leq 0 \mid 1 \leq (x::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

## 11.6 Simplification of Inequalities Involving Literal Divisors

**lemma** *pos-le-divide-eq*: $0 < (c::'a::ordered\text{-}field) ==> (a \leq b/c) = (a*c \leq b)$
⟨*proof*⟩

**lemma** *neg-le-divide-eq*: $c < (0::'a::ordered\text{-}field) ==> (a \leq b/c) = (b \leq a*c)$
⟨*proof*⟩

**lemma** *le-divide-eq*:
$(a \leq b/c) =$
$(if\ 0 < c\ then\ a*c \leq b$
$\quad else\ if\ c < 0\ then\ b \leq a*c$
$\quad else\ \ a \leq (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *pos-divide-le-eq*: $0 < (c::'a::ordered\text{-}field) ==> (b/c \leq a) = (b \leq a*c)$
⟨*proof*⟩

**lemma** *neg-divide-le-eq*: $c < (0::'a::ordered\text{-}field) ==> (b/c \leq a) = (a*c \leq b)$
⟨*proof*⟩

**lemma** *divide-le-eq*:
$(b/c \leq a) =$
$(if\ 0 < c\ then\ b \leq a*c$
$\quad else\ if\ c < 0\ then\ a*c \leq b$
$\quad else\ 0 \leq (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *pos-less-divide-eq*:
$0 < (c::'a::ordered\text{-}field) ==> (a < b/c) = (a*c < b)$
⟨*proof*⟩

**lemma** *neg-less-divide-eq*:
$c < (0::'a::ordered\text{-}field) ==> (a < b/c) = (b < a*c)$
⟨*proof*⟩

**lemma** *less-divide-eq*:
$(a < b/c) =$
$(if\ 0 < c\ then\ a*c < b$
$\quad else\ if\ c < 0\ then\ b < a*c$
$\quad else\ \ a < (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *pos-divide-less-eq*:
$0 < (c::'a::ordered\text{-}field) ==> (b/c < a) = (b < a*c)$
⟨*proof*⟩

**lemma** *neg-divide-less-eq*:
$c < (0::'a::ordered\text{-}field) ==> (b/c < a) = (a*c < b)$
⟨*proof*⟩

**lemma** *divide-less-eq*:
$(b/c < a) =$
$(if\ 0 < c\ then\ b < a*c$
$\quad else\ if\ c < 0\ then\ a*c < b$
$\quad else\ 0 < (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

## 11.7 Field simplification

Lemmas *field-simps* multiply with denominators in in(equations) if they can be proved to be non-zero (for equations) or positive/negative (for inequations). Can be too aggressive and is therefore separate from the more benign *algebra-simps*.

**lemmas** *field-simps*[*noatp*] = *field-eq-simps*

*pos-divide-less-eq neg-divide-less-eq*
*pos-less-divide-eq neg-less-divide-eq*
*pos-divide-le-eq neg-divide-le-eq*
*pos-le-divide-eq neg-le-divide-eq*

Lemmas *sign-simps* is a first attempt to automate proofs of positivity/negativity needed for *field-simps*. Have not added *sign-simps* to *field-simps* because the former can lead to case explosions.

**lemmas** *sign-simps*[*noatp*] = *group-simps*
*zero-less-mult-iff* *mult-less-0-iff*

## 11.8 Division and Signs

**lemma** *zero-less-divide-iff*:
$((0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}) < a/b) = (0 < a \ \& \ 0 < b \mid a < 0 \ \& \ b < 0)$
⟨*proof*⟩

**lemma** *divide-less-0-iff*:
$(a/b < (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) =$
$(0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b)$
⟨*proof*⟩

**lemma** *zero-le-divide-iff*:
$((0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}) \leq a/b) =$
$(0 \leq a \ \& \ 0 \leq b \mid a \leq 0 \ \& \ b \leq 0)$
⟨*proof*⟩

**lemma** *divide-le-0-iff*:
$(a/b \leq (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) =$
$(0 \leq a \ \& \ b \leq 0 \mid a \leq 0 \ \& \ 0 \leq b)$
⟨*proof*⟩

**lemma** *divide-eq-0-iff* [*simp*,*noatp*]:
$(a/b = 0) = (a=0 \mid b=(0::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

**lemma** *divide-pos-pos*:
$0 < (x::'a::ordered\text{-}field) \Longrightarrow 0 < y \Longrightarrow 0 < x \ / \ y$
⟨*proof*⟩

**lemma** *divide-nonneg-pos*:
$0 <= (x::'a::ordered\text{-}field) \Longrightarrow 0 < y \Longrightarrow 0 <= x \ / \ y$
⟨*proof*⟩

**lemma** *divide-neg-pos*:
$(x::'a::ordered\text{-}field) < 0 \Longrightarrow 0 < y \Longrightarrow x \ / \ y < 0$
⟨*proof*⟩

**lemma** *divide-nonpos-pos*:
$(x::'a::ordered\text{-}field) <= 0 \Longrightarrow 0 < y \Longrightarrow x \ / \ y <= 0$
⟨*proof*⟩

**lemma** *divide-pos-neg*:
$0 < (x::'a::ordered\text{-}field) \Longrightarrow y < 0 \Longrightarrow x \ / \ y < 0$
⟨*proof*⟩

**lemma** *divide-nonneg-neg*:
$0 <= (x::'a::ordered\text{-}field) \Longrightarrow y < 0 \Longrightarrow x \ / \ y <= 0$
⟨*proof*⟩

**lemma** *divide-neg-neg*:
(*x*::'*a*::*ordered-field*) < *0* ==> *y* < *0* ==> *0* < *x* / *y*
⟨*proof*⟩

**lemma** *divide-nonpos-neg*:
(*x*::'*a*::*ordered-field*) <= *0* ==> *y* < *0* ==> *0* <= *x* / *y*
⟨*proof*⟩

## 11.9 Cancellation Laws for Division

**lemma** *divide-cancel-right* [*simp*,*noatp*]:
(*a*/*c* = *b*/*c*) = (*c* = *0* | *a* = (*b*::'*a*::{*field*,*division-by-zero*}))
⟨*proof*⟩

**lemma** *divide-cancel-left* [*simp*,*noatp*]:
(*c*/*a* = *c*/*b*) = (*c* = *0* | *a* = (*b*::'*a*::{*field*,*division-by-zero*}))
⟨*proof*⟩

## 11.10 Division and the Number One

Simplify expressions equated with 1

**lemma** *divide-eq-1-iff* [*simp*,*noatp*]:
(*a*/*b* = *1*) = (*b* ≠ *0* & *a* = (*b*::'*a*::{*field*,*division-by-zero*}))
⟨*proof*⟩

**lemma** *one-eq-divide-iff* [*simp*,*noatp*]:
(*1* = *a*/*b*) = (*b* ≠ *0* & *a* = (*b*::'*a*::{*field*,*division-by-zero*}))
⟨*proof*⟩

**lemma** *zero-eq-1-divide-iff* [*simp*,*noatp*]:
((*0*::'*a*::{*ordered-field*,*division-by-zero*}) = *1*/*a*) = (*a* = *0*)
⟨*proof*⟩

**lemma** *one-divide-eq-0-iff* [*simp*,*noatp*]:
(*1*/*a* = (*0*::'*a*::{*ordered-field*,*division-by-zero*})) = (*a* = *0*)
⟨*proof*⟩

Simplify expressions such as *0* < *1*/*x* to *0* < *x*

**lemmas** *zero-less-divide-1-iff* = *zero-less-divide-iff* [*of 1*, *simplified*]
**lemmas** *divide-less-0-1-iff* = *divide-less-0-iff* [*of 1*, *simplified*]
**lemmas** *zero-le-divide-1-iff* = *zero-le-divide-iff* [*of 1*, *simplified*]
**lemmas** *divide-le-0-1-iff* = *divide-le-0-iff* [*of 1*, *simplified*]

**declare** *zero-less-divide-1-iff* [*simp*,*noatp*]
**declare** *divide-less-0-1-iff* [*simp*,*noatp*]
**declare** *zero-le-divide-1-iff* [*simp*,*noatp*]
**declare** *divide-le-0-1-iff* [*simp*,*noatp*]

## 11.11 Ordering Rules for Division

**lemma** *divide-strict-right-mono*:
[|*a* < *b*; *0* < *c*|] ==> *a* / *c* < *b* / (*c*::'*a*::*ordered-field*)
⟨*proof*⟩

**lemma** *divide-right-mono*:
[|*a* ≤ *b*; *0* ≤ *c*|] ==> *a*/*c* ≤ *b*/(*c*::'*a*::{*ordered-field*,*division-by-zero*})
⟨*proof*⟩

**lemma** *divide-right-mono-neg*: (*a*::'*a*::{*division-by-zero*,*ordered-field*}) <= *b*
==> *c* <= *0* ==> *b* / *c* <= *a* / *c*
⟨*proof*⟩

**lemma** *divide-strict-right-mono-neg*:
[|*b* < *a*; *c* < *0*|] ==> *a* / *c* < *b* / (*c*::'*a*::*ordered-field*)
⟨*proof*⟩

The last premise ensures that *a* and *b* have the same sign

**lemma** *divide-strict-left-mono*:
[|*b* < *a*; *0* < *c*; *0* < *a*∗*b*|] ==> *c* / *a* < *c* / (*b*::'*a*::*ordered-field*)
⟨*proof*⟩

**lemma** *divide-left-mono*:
[|*b* ≤ *a*; *0* ≤ *c*; *0* < *a*∗*b*|] ==> *c* / *a* ≤ *c* / (*b*::'*a*::*ordered-field*)
⟨*proof*⟩

**lemma** *divide-left-mono-neg*: (*a*::'*a*::{*division-by-zero*,*ordered-field*}) <= *b*
==> *c* <= *0* ==> *0* < *a* ∗ *b* ==> *c* / *a* <= *c* / *b*
⟨*proof*⟩

**lemma** *divide-strict-left-mono-neg*:
[|*a* < *b*; *c* < *0*; *0* < *a*∗*b*|] ==> *c* / *a* < *c* / (*b*::'*a*::*ordered-field*)
⟨*proof*⟩

Simplify quotients that are compared with the value 1.

**lemma** *le-divide-eq-1* [*noatp*]:
**fixes** *a* :: '*a* :: {*ordered-field*,*division-by-zero*}
**shows** (*1* ≤ *b* / *a*) = ((*0* < *a* & *a* ≤ *b*) | (*a* < *0* & *b* ≤ *a*))
⟨*proof*⟩

**lemma** *divide-le-eq-1* [*noatp*]:
**fixes** *a* :: '*a* :: {*ordered-field*,*division-by-zero*}
**shows** (*b* / *a* ≤ *1*) = ((*0* < *a* & *b* ≤ *a*) | (*a* < *0* & *a* ≤ *b*) | *a*=*0*)
⟨*proof*⟩

**lemma** *less-divide-eq-1* [*noatp*]:
**fixes** *a* :: '*a* :: {*ordered-field*,*division-by-zero*}
**shows** (*1* < *b* / *a*) = ((*0* < *a* & *a* < *b*) | (*a* < *0* & *b* < *a*))
⟨*proof*⟩

**lemma** *divide-less-eq-1* [*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $(b \ / \ a < 1) = ((0 < a \ \& \ b < a) \mid (a < 0 \ \& \ a < b) \mid a=0)$
⟨*proof*⟩

## 11.12 Conditional Simplification Rules: No Case Splits

**lemma** *le-divide-eq-1-pos* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $0 < a \Longrightarrow (1 \leq b/a) = (a \leq b)$
⟨*proof*⟩

**lemma** *le-divide-eq-1-neg* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $a < 0 \Longrightarrow (1 \leq b/a) = (b \leq a)$
⟨*proof*⟩

**lemma** *divide-le-eq-1-pos* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $0 < a \Longrightarrow (b/a \leq 1) = (b \leq a)$
⟨*proof*⟩

**lemma** *divide-le-eq-1-neg* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $a < 0 \Longrightarrow (b/a \leq 1) = (a \leq b)$
⟨*proof*⟩

**lemma** *less-divide-eq-1-pos* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $0 < a \Longrightarrow (1 < b/a) = (a < b)$
⟨*proof*⟩

**lemma** *less-divide-eq-1-neg* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $a < 0 \Longrightarrow (1 < b/a) = (b < a)$
⟨*proof*⟩

**lemma** *divide-less-eq-1-pos* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $0 < a \Longrightarrow (b/a < 1) = (b < a)$
⟨*proof*⟩

**lemma** *divide-less-eq-1-neg* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $a < 0 \Longrightarrow b/a < 1 <-> a < b$
⟨*proof*⟩

**lemma** *eq-divide-eq-1* [*simp*,*noatp*]:

**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $(1 = b/a) = ((a \neq 0 \ \& \ a = b))$
⟨*proof*⟩

**lemma** *divide-eq-eq-1* [*simp*,*noatp*]:
**fixes** $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
**shows** $(b/a = 1) = ((a \neq 0 \ \& \ a = b))$
⟨*proof*⟩

## 11.13 Reasoning about inequalities with division

**lemma** *mult-right-le-one-le*: *0* <= (*x*::'*a*::*ordered-idom*) ==> *0* <= *y* ==> *y* <= *1*
==> *x* * *y* <= *x*
⟨*proof*⟩

**lemma** *mult-left-le-one-le*: *0* <= (*x*::'*a*::*ordered-idom*) ==> *0* <= *y* ==> *y* <= *1*
==> *y* * *x* <= *x*
⟨*proof*⟩

**lemma** *mult-imp-div-pos-le*: *0* < (*y*::'*a*::*ordered-field*) ==> *x* <= *z* * *y* ==>
*x* / *y* <= *z*
⟨*proof*⟩

**lemma** *mult-imp-le-div-pos*: *0* < (*y*::'*a*::*ordered-field*) ==> *z* * *y* <= *x* ==>
*z* <= *x* / *y*
⟨*proof*⟩

**lemma** *mult-imp-div-pos-less*: *0* < (*y*::'*a*::*ordered-field*) ==> *x* < *z* * *y* ==>
*x* / *y* < *z*
⟨*proof*⟩

**lemma** *mult-imp-less-div-pos*: *0* < (*y*::'*a*::*ordered-field*) ==> *z* * *y* < *x* ==>
*z* < *x* / *y*
⟨*proof*⟩

**lemma** *frac-le*: (*0*::'*a*::*ordered-field*) <= *x* ==>
*x* <= *y* ==> *0* < *w* ==> *w* <= *z* ==> *x* / *z* <= *y* / *w*
⟨*proof*⟩

**lemma** *frac-less*: (*0*::'*a*::*ordered-field*) <= *x* ==>
*x* < *y* ==> *0* < *w* ==> *w* <= *z* ==> *x* / *z* < *y* / *w*
⟨*proof*⟩

**lemma** *frac-less2*: (*0*::'*a*::*ordered-field*) < *x* ==>
*x* <= *y* ==> *0* < *w* ==> *w* < *z* ==> *x* / *z* < *y* / *w*
⟨*proof*⟩

It's not obvious whether these should be simprules or not. Their effect is

to gather terms into one big fraction, like a*b*c / x*y*z. The rationale for that is unclear, but many proofs seem to need them.

**declare** *times-divide-eq* [*simp*]

## 11.14 Ordered Fields are Dense

**context** *ordered-semidom*
**begin**

**lemma** *less-add-one*: $a < a + 1$
⟨*proof*⟩

**lemma** *zero-less-two*: $0 < 1 + 1$
⟨*proof*⟩

**end**

**lemma** *less-half-sum*: $a < b \Longrightarrow a < (a+b) / (1+1::'a::ordered\text{-}field)$
⟨*proof*⟩

**lemma** *gt-half-sum*: $a < b \Longrightarrow (a+b)/(1+1::'a::ordered\text{-}field) < b$
⟨*proof*⟩

**instance** *ordered-field* < *dense-linear-order*
⟨*proof*⟩

## 11.15 Absolute Value

**context** *ordered-idom*
**begin**

**lemma** *mult-sgn-abs*: $sgn\ x * abs\ x = x$
⟨*proof*⟩

**end**

**lemma** *abs-one* [*simp*]: $abs\ 1 = (1::'a::ordered\text{-}idom)$
⟨*proof*⟩

**class** *pordered-ring-abs* = *pordered-ring* + *pordered-ab-group-add-abs* +
**assumes** *abs-eq-mult*:
$(0 \leq a \vee a \leq 0) \wedge (0 \leq b \vee b \leq 0) \Longrightarrow |a * b| = |a| * |b|$

**class** *lordered-ring* = *pordered-ring* + *lordered-ab-group-add-abs*
**begin**

**subclass** *lordered-ab-group-add-meet* ⟨*proof*⟩
**subclass** *lordered-ab-group-add-join* ⟨*proof*⟩

**end**

**lemma** *abs-le-mult*: $abs\ (a * b) \leq (abs\ a) * (abs\ (b::'a::lordered\text{-}ring))$
⟨*proof*⟩

**instance** *lordered-ring* ⊆ *pordered-ring-abs*
⟨*proof*⟩

**instance** *ordered-idom* ⊆ *pordered-ring-abs*
⟨*proof*⟩

**lemma** *abs-mult*: $abs\ (a * b) = abs\ a * abs\ (b::'a::ordered\text{-}idom)$
⟨*proof*⟩

**lemma** *abs-mult-self*: $abs\ a * abs\ a = a * (a::'a::ordered\text{-}idom)$
⟨*proof*⟩

**lemma** *nonzero-abs-inverse*:
$a \neq 0 \Longrightarrow abs\ (inverse\ (a::'a::ordered\text{-}field)) = inverse\ (abs\ a)$
⟨*proof*⟩

**lemma** *abs-inverse* [*simp*]:
$abs\ (inverse\ (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) =$
$inverse\ (abs\ a)$
⟨*proof*⟩

**lemma** *nonzero-abs-divide*:
$b \neq 0 \Longrightarrow abs\ (a\ /\ (b::'a::ordered\text{-}field)) = abs\ a\ /\ abs\ b$
⟨*proof*⟩

**lemma** *abs-divide* [*simp*]:
$abs\ (a\ /\ (b::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) = abs\ a\ /\ abs\ b$
⟨*proof*⟩

**lemma** *abs-mult-less*:
$[|\ abs\ a < c;\ abs\ b < d\ |] \Longrightarrow abs\ a * abs\ b < c*(d::'a::ordered\text{-}idom)$
⟨*proof*⟩

**lemmas** *eq-minus-self-iff*[*noatp*] = *equal-neg-zero*

**lemma** *less-minus-self-iff*: $(a < -a) = (a < (0::'a::ordered\text{-}idom))$
⟨*proof*⟩

**lemma** *abs-less-iff*: $(abs\ a < b) = (a < b\ \&\ -a < (b::'a::ordered\text{-}idom))$
⟨*proof*⟩

**lemma** *abs-mult-pos*: $(0::'a::ordered\text{-}idom) <= x \Longrightarrow$
$(abs\ y) * x = abs\ (y * x)$

⟨*proof*⟩

**lemma** *abs-div-pos*: (*0*::'*a*::{*division-by-zero*,*ordered-field*}) < *y* ==>
*abs x* / *y* = *abs* (*x* / *y*)
⟨*proof*⟩

### 11.16 Bounds of products via negative and positive Part

**lemma** *mult-le-prts*:
**assumes**
*a1* <= (*a*::'*a*::*lordered-ring*)
*a* <= *a2*
*b1* <= *b*
*b* <= *b2*
**shows**
*a* * *b* <= *pprt a2* * *pprt b2* + *pprt a1* * *nprt b2* + *nprt a2* * *pprt b1* + *nprt a1* * *nprt b1*
⟨*proof*⟩

**lemma** *mult-ge-prts*:
**assumes**
*a1* <= (*a*::'*a*::*lordered-ring*)
*a* <= *a2*
*b1* <= *b*
*b* <= *b2*
**shows**
*a* * *b* >= *nprt a1* * *pprt b2* + *nprt a2* * *nprt b2* + *pprt a1* * *pprt b1* + *pprt a2* * *nprt b1*
⟨*proof*⟩

**end**

## 12 Nat: Natural numbers

**theory** *Nat*
**imports** *Inductive Ring-and-Field*
**uses**
~~/*src*/*Tools*/*rat.ML*
~~/*src*/*Provers*/*Arith*/*cancel-sums.ML*
*Tools*/*arith-data.ML*
(*Tools*/*nat-arith.ML*)
~~/*src*/*Provers*/*Arith*/*fast-lin-arith.ML*
(*Tools*/*lin-arith.ML*)
**begin**

### 12.1 Type *ind*

**typedecl** *ind*

**axiomatization**
  *Zero-Rep* :: *ind* **and**
  *Suc-Rep* :: *ind* => *ind*
**where**
  — the axiom of infinity in 2 parts
  *inj-Suc-Rep*: *inj Suc-Rep* **and**
  *Suc-Rep-not-Zero-Rep*: *Suc-Rep x* ≠ *Zero-Rep*

## 12.2 Type nat

Type definition

**inductive** *Nat* :: *ind* ⇒ *bool*
**where**
    *Zero-RepI*: *Nat Zero-Rep*
  | *Suc-RepI*: *Nat i* ⟹ *Nat* (*Suc-Rep i*)

**global**

**typedef** (**open** *Nat*)
  *nat* = *Nat*
  ⟨*proof*⟩

**constdefs**
  *Suc* ::  *nat* => *nat*
  *Suc-def*: *Suc* == (%*n*. *Abs-Nat* (*Suc-Rep* (*Rep-Nat n*)))

**local**

**instantiation** *nat* :: *zero*
**begin**

**definition** *Zero-nat-def* [*code del*]:
  *0* = *Abs-Nat Zero-Rep*

**instance** ⟨*proof*⟩

**end**

**lemma** *Suc-not-Zero*: *Suc m* ≠ *0*
  ⟨*proof*⟩

**lemma** *Zero-not-Suc*: *0* ≠ *Suc m*
  ⟨*proof*⟩

**rep-datatype** *0* :: *nat Suc*
  ⟨*proof*⟩

**lemma** *nat-induct* [*case-names 0 Suc*, *induct type*: *nat*]:

— for backward compatibility – names of variables differ
**fixes** *n*
**assumes** *P 0*
**and** $\bigwedge n.\ P\ n \Longrightarrow P\ (Suc\ n)$
**shows** *P n*
⟨*proof*⟩

**declare** *nat.exhaust* [*case-names 0 Suc*, *cases type*: *nat*]

**lemmas** *nat-rec-0* = *nat.recs*(*1*)
**and** *nat-rec-Suc* = *nat.recs*(*2*)

**lemmas** *nat-case-0* = *nat.cases*(*1*)
**and** *nat-case-Suc* = *nat.cases*(*2*)

Injectiveness and distinctness lemmas

**lemma** *inj-Suc*[*simp*]: *inj-on Suc N*
⟨*proof*⟩

**lemma** *Suc-neq-Zero*: $Suc\ m = 0 \Longrightarrow R$
⟨*proof*⟩

**lemma** *Zero-neq-Suc*: $0 = Suc\ m \Longrightarrow R$
⟨*proof*⟩

**lemma** *Suc-inject*: $Suc\ x = Suc\ y \Longrightarrow x = y$
⟨*proof*⟩

**lemma** *n-not-Suc-n*: $n \neq Suc\ n$
⟨*proof*⟩

**lemma** *Suc-n-not-n*: $Suc\ n \neq n$
⟨*proof*⟩

A special form of induction for reasoning about $m < n$ and $m - n$

**lemma** *diff-induct*: (!!*x*. *P x 0*) ==> (!!*y*. *P 0* (*Suc y*)) ==>
(!!*x y*. *P x y* ==> *P* (*Suc x*) (*Suc y*)) ==> *P m n*
⟨*proof*⟩

## 12.3 Arithmetic operators

**instantiation** *nat* :: {*minus*, *comm-monoid-add*}
**begin**

**primrec** *plus-nat*
**where**
*add-0*: $0 + n = (n::nat)$
| *add-Suc*: $Suc\ m + n = Suc\ (m + n)$

**lemma** *add-0-right* [*simp*]: $m + 0 = (m::nat)$
⟨*proof*⟩

**lemma** *add-Suc-right* [*simp*]: $m + Suc\ n = Suc\ (m + n)$
⟨*proof*⟩

**declare** *add-0* [*code*]

**lemma** *add-Suc-shift* [*code*]: $Suc\ m + n = m + Suc\ n$
⟨*proof*⟩

**primrec** *minus-nat*
**where**
*diff-0*: $m - 0 = (m::nat)$
| *diff-Suc*: $m - Suc\ n = (case\ m - n\ of\ 0 => 0 \mid Suc\ k => k)$

**declare** *diff-Suc* [*simp del*]
**declare** *diff-0* [*code*]

**lemma** *diff-0-eq-0* [*simp*, *code*]: $0 - n = (0::nat)$
⟨*proof*⟩

**lemma** *diff-Suc-Suc* [*simp*, *code*]: $Suc\ m - Suc\ n = m - n$
⟨*proof*⟩

**instance** ⟨*proof*⟩

**end**

**instantiation** *nat* :: *comm-semiring-1-cancel*
**begin**

**definition**
*One-nat-def* [*simp*]: $1 = Suc\ 0$

**primrec** *times-nat*
**where**
*mult-0*: $0 * n = (0::nat)$
| *mult-Suc*: $Suc\ m * n = n + (m * n)$

**lemma** *mult-0-right* [*simp*]: $(m::nat) * 0 = 0$
⟨*proof*⟩

**lemma** *mult-Suc-right* [*simp*]: $m * Suc\ n = m + (m * n)$
⟨*proof*⟩

**lemma** *add-mult-distrib*: $(m + n) * k = (m * k) + ((n * k)::nat)$
⟨*proof*⟩

**instance** ⟨*proof*⟩

**end**

### 12.3.1 Addition

**lemma** *nat-add-assoc*: $(m + n) + k = m + ((n + k)::nat)$
⟨*proof*⟩

**lemma** *nat-add-commute*: $m + n = n + (m::nat)$
⟨*proof*⟩

**lemma** *nat-add-left-commute*: $x + (y + z) = y + ((x + z)::nat)$
⟨*proof*⟩

**lemma** *nat-add-left-cancel* [*simp*]: $(k + m = k + n) = (m = (n::nat))$
⟨*proof*⟩

**lemma** *nat-add-right-cancel* [*simp*]: $(m + k = n + k) = (m=(n::nat))$
⟨*proof*⟩

Reasoning about $m + 0 = 0$, etc.

**lemma** *add-is-0* [*iff*]:
**fixes** *m n* :: *nat*
**shows** $(m + n = 0) = (m = 0 \ \& \ n = 0)$
⟨*proof*⟩

**lemma** *add-is-1*:
$(m+n= Suc\ 0) = (m= Suc\ 0 \ \& \ n=0 \mid m=0 \ \& \ n= Suc\ 0)$
⟨*proof*⟩

**lemma** *one-is-add*:
$(Suc\ 0 = m + n) = (m = Suc\ 0 \ \& \ n = 0 \mid m = 0 \ \& \ n = Suc\ 0)$
⟨*proof*⟩

**lemma** *add-eq-self-zero*:
**fixes** *m n* :: *nat*
**shows** $m + n = m \Longrightarrow n = 0$
⟨*proof*⟩

**lemma** *inj-on-add-nat*[*simp*]: *inj-on* $(\%n::nat.\ n+k)\ N$
⟨*proof*⟩

### 12.3.2 Difference

**lemma** *diff-self-eq-0* [*simp*]: $(m::nat) - m = 0$
⟨*proof*⟩

**lemma** *diff-diff-left*: $(i::nat) - j - k = i - (j + k)$
⟨*proof*⟩

**lemma** *Suc-diff-diff* [*simp*]: $(Suc\ m - n) - Suc\ k = m - n - k$
⟨*proof*⟩

**lemma** *diff-commute*: $(i::nat) - j - k = i - k - j$
⟨*proof*⟩

**lemma** *diff-add-inverse*: $(n + m) - n = (m::nat)$
⟨*proof*⟩

**lemma** *diff-add-inverse2*: $(m + n) - n = (m::nat)$
⟨*proof*⟩

**lemma** *diff-cancel*: $(k + m) - (k + n) = m - (n::nat)$
⟨*proof*⟩

**lemma** *diff-cancel2*: $(m + k) - (n + k) = m - (n::nat)$
⟨*proof*⟩

**lemma** *diff-add-0*: $n - (n + m) = (0::nat)$
⟨*proof*⟩

**lemma** *diff-Suc-1* [*simp*]: $Suc\ n - 1 = n$
⟨*proof*⟩

Difference distributes over multiplication

**lemma** *diff-mult-distrib*: $((m::nat) - n) * k = (m * k) - (n * k)$
⟨*proof*⟩

**lemma** *diff-mult-distrib2*: $k * ((m::nat) - n) = (k * m) - (k * n)$
⟨*proof*⟩

### 12.3.3 Multiplication

**lemma** *nat-mult-assoc*: $(m * n) * k = m * ((n * k)::nat)$
⟨*proof*⟩

**lemma** *nat-mult-commute*: $m * n = n * (m::nat)$
⟨*proof*⟩

**lemma** *add-mult-distrib2*: $k * (m + n) = (k * m) + ((k * n)::nat)$
⟨*proof*⟩

**lemma** *mult-is-0* [*simp*]: $((m::nat) * n = 0) = (m=0 \mid n=0)$
⟨*proof*⟩

**lemmas** *nat-distrib* =
*add-mult-distrib add-mult-distrib2 diff-mult-distrib diff-mult-distrib2*

**lemma** *mult-eq-1-iff* [*simp*]: $(m * n = Suc\ 0) = (m = Suc\ 0\ \&\ n = Suc\ 0)$
⟨*proof*⟩

**lemma** *one-eq-mult-iff* [*simp*,*noatp*]: $(Suc\ 0 = m * n) = (m = Suc\ 0\ \&\ n = Suc\ 0)$
⟨*proof*⟩

**lemma** *nat-mult-eq-1-iff* [*simp*]: $m * n = (1::nat) \longleftrightarrow m = 1 \wedge n = 1$
⟨*proof*⟩

**lemma** *nat-1-eq-mult-iff* [*simp*]: $(1::nat) = m * n \longleftrightarrow m = 1 \wedge n = 1$
⟨*proof*⟩

**lemma** *mult-cancel1* [*simp*]: $(k * m = k * n) = (m = n \mid (k = (0::nat)))$
⟨*proof*⟩

**lemma** *mult-cancel2* [*simp*]: $(m * k = n * k) = (m = n \mid (k = (0::nat)))$
⟨*proof*⟩

**lemma** *Suc-mult-cancel1*: $(Suc\ k * m = Suc\ k * n) = (m = n)$
⟨*proof*⟩

## 12.4 Orders on *nat*

### 12.4.1 Operation definition

**instantiation** *nat* :: *linorder*
**begin**

**primrec** *less-eq-nat* **where**
$(0::nat) \leq n \longleftrightarrow True$
| $Suc\ m \leq n \longleftrightarrow (case\ n\ of\ 0 \Rightarrow False \mid Suc\ n \Rightarrow m \leq n)$

**declare** *less-eq-nat.simps* [*simp del*]
**lemma** [*code*]: $(0::nat) \leq n \longleftrightarrow True$ ⟨*proof*⟩
**lemma** *le0* [*iff*]: $0 \leq (n::nat)$ ⟨*proof*⟩

**definition** *less-nat* **where**
*less-eq-Suc-le*: $n < m \longleftrightarrow Suc\ n \leq m$

**lemma** *Suc-le-mono* [*iff*]: $Suc\ n \leq Suc\ m \longleftrightarrow n \leq m$
⟨*proof*⟩

**lemma** *Suc-le-eq* [*code*]: $Suc\ m \leq n \longleftrightarrow m < n$
⟨*proof*⟩

**lemma** *le-0-eq* [*iff*]: $(n::nat) \leq 0 \longleftrightarrow n = 0$
⟨*proof*⟩

**lemma** *not-less0* [*iff*]: $\neg\ n < (0::nat)$

⟨*proof*⟩

**lemma** *less-nat-zero-code* [*code*]: $n < (0::nat) \longleftrightarrow False$
⟨*proof*⟩

**lemma** *Suc-less-eq* [*iff*]: $Suc\ m < Suc\ n \longleftrightarrow m < n$
⟨*proof*⟩

**lemma** *less-Suc-eq-le* [*code*]: $m < Suc\ n \longleftrightarrow m \leq n$
⟨*proof*⟩

**lemma** *le-SucI*: $m \leq n \Longrightarrow m \leq Suc\ n$
⟨*proof*⟩

**lemma** *Suc-leD*: $Suc\ m \leq n \Longrightarrow m \leq n$
⟨*proof*⟩

**lemma** *less-SucI*: $m < n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

**lemma** *Suc-lessD*: $Suc\ m < n \Longrightarrow m < n$
⟨*proof*⟩

**instance**
⟨*proof*⟩

**end**

**instantiation** *nat* :: *bot*
**begin**

**definition** *bot-nat* :: *nat* **where**
*bot-nat* = *0*

**instance** ⟨*proof*⟩

**end**

### 12.4.2 Introduction properties

**lemma** *lessI* [*iff*]: $n < Suc\ n$
⟨*proof*⟩

**lemma** *zero-less-Suc* [*iff*]: $0 < Suc\ n$
⟨*proof*⟩

### 12.4.3 Elimination properties

**lemma** *less-not-refl*: $\sim\ n < (n::nat)$
⟨*proof*⟩

**lemma** *less-not-refl2*: $n < m \Longrightarrow m \neq (n::nat)$
⟨*proof*⟩

**lemma** *less-not-refl3*: $(s::nat) < t \Longrightarrow s \neq t$
⟨*proof*⟩

**lemma** *less-irrefl-nat*: $(n::nat) < n \Longrightarrow R$
⟨*proof*⟩

**lemma** *less-zeroE*: $(n::nat) < 0 \Longrightarrow R$
⟨*proof*⟩

**lemma** *less-Suc-eq*: $(m < Suc\ n) = (m < n \mid m = n)$
⟨*proof*⟩

**lemma** *less-Suc0* [*iff*]: $(n < Suc\ 0) = (n = 0)$
⟨*proof*⟩

**lemma** *less-one* [*iff*, *noatp*]: $(n < (1::nat)) = (n = 0)$
⟨*proof*⟩

**lemma** *Suc-mono*: $m < n \Longrightarrow Suc\ m < Suc\ n$
⟨*proof*⟩

"Less than" is antisymmetric, sort of

**lemma** *less-antisym*: $[\![ \neg\ n < m;\ n < Suc\ m\ ]\!] \Longrightarrow m = n$
⟨*proof*⟩

**lemma** *nat-neq-iff*: $((m::nat) \neq n) = (m < n \mid n < m)$
⟨*proof*⟩

**lemma** *nat-less-cases*: **assumes** *major*: $(m::nat) < n \Longrightarrow P\ n\ m$
**and** *eqCase*: $m = n \Longrightarrow P\ n\ m$ **and** *lessCase*: $n<m \Longrightarrow P\ n\ m$
**shows** $P\ n\ m$
⟨*proof*⟩

### 12.4.4 Inductive (?) properties

**lemma** *Suc-lessI*: $m < n \Longrightarrow Suc\ m \neq n \Longrightarrow Suc\ m < n$
⟨*proof*⟩

**lemma** *lessE*:
**assumes** *major*: $i < k$
**and** *p1*: $k = Suc\ i \Longrightarrow P$ **and** *p2*: $!!j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$
**shows** $P$
⟨*proof*⟩

**lemma** *less-SucE*: **assumes** *major*: $m < Suc\ n$

**and** *less*: $m < n \Longrightarrow P$ **and** *eq*: $m = n \Longrightarrow P$ **shows** $P$
⟨*proof*⟩

**lemma** *Suc-lessE*: **assumes** *major*: $Suc\ i < k$
**and** *minor*: $!!j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$ **shows** $P$
⟨*proof*⟩

**lemma** *Suc-less-SucD*: $Suc\ m < Suc\ n \Longrightarrow m < n$
⟨*proof*⟩

**lemma** *less-trans-Suc*:
**assumes** *le*: $i < j$ **shows** $j < k \Longrightarrow Suc\ i < k$
⟨*proof*⟩

Can be used with *less-Suc-eq* to get $n = m \vee n < m$

**lemma** *not-less-eq*: $\neg\ m < n \longleftrightarrow n < Suc\ m$
⟨*proof*⟩

**lemma** *not-less-eq-eq*: $\neg\ m \leq n \longleftrightarrow Suc\ n \leq m$
⟨*proof*⟩

Properties of "less than or equal"

**lemma** *le-imp-less-Suc*: $m \leq n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

**lemma** *Suc-n-not-le-n*: $\sim Suc\ n \leq n$
⟨*proof*⟩

**lemma** *le-Suc-eq*: $(m \leq Suc\ n) = (m \leq n \mid m = Suc\ n)$
⟨*proof*⟩

**lemma** *le-SucE*: $m \leq Suc\ n \Longrightarrow (m \leq n \Longrightarrow R) \Longrightarrow (m = Suc\ n \Longrightarrow R)$
$\Longrightarrow R$
⟨*proof*⟩

**lemma** *Suc-leI*: $m < n \Longrightarrow Suc(m) \leq n$
⟨*proof*⟩

Stronger version of *Suc-leD*

**lemma** *Suc-le-lessD*: $Suc\ m \leq n \Longrightarrow m < n$
⟨*proof*⟩

**lemma** *less-imp-le-nat*: $m < n \Longrightarrow m \leq (n::nat)$
⟨*proof*⟩

For instance, $(Suc\ m < Suc\ n) = (Suc\ m \leq n) = (m < n)$

**lemmas** *le-simps* = *less-imp-le-nat less-Suc-eq-le Suc-le-eq*

Equivalence of $m \leq n$ and $m < n \vee m = n$

**lemma** *less-or-eq-imp-le*: $m < n \mid m = n \Longrightarrow m \leq (n::nat)$
⟨*proof*⟩

**lemma** *le-eq-less-or-eq*: $(m \leq (n::nat)) = (m < n \mid m=n)$
⟨*proof*⟩

Useful with *blast*.

**lemma** *eq-imp-le*: $(m::nat) = n \Longrightarrow m \leq n$
⟨*proof*⟩

**lemma** *le-refl*: $n \leq (n::nat)$
⟨*proof*⟩

**lemma** *le-trans*: $[\!|\ i \leq j; j \leq k\ |\!] \Longrightarrow i \leq (k::nat)$
⟨*proof*⟩

**lemma** *le-anti-sym*: $[\!|\ m \leq n; n \leq m\ |\!] \Longrightarrow m = (n::nat)$
⟨*proof*⟩

**lemma** *nat-less-le*: $((m::nat) < n) = (m \leq n \ \&\ m \neq n)$
⟨*proof*⟩

**lemma** *le-neq-implies-less*: $(m::nat) \leq n \Longrightarrow m \neq n \Longrightarrow m < n$
⟨*proof*⟩

**lemma** *nat-le-linear*: $(m::nat) \leq n \mid n \leq m$
⟨*proof*⟩

**lemmas** *linorder-neqE-nat* = *linorder-neqE* [**where** $'a = nat$]

**lemma** *le-less-Suc-eq*: $m \leq n \Longrightarrow (n < Suc\ m) = (n = m)$
⟨*proof*⟩

**lemma** *not-less-less-Suc-eq*: $\sim n < m \Longrightarrow (n < Suc\ m) = (n = m)$
⟨*proof*⟩

**lemmas** *not-less-simps* = *not-less-less-Suc-eq le-less-Suc-eq*

These two rules ease the use of primitive recursion. NOTE USE OF ==

**lemma** *def-nat-rec-0*: $(!!n.\ f\ n == nat\text{-}rec\ c\ h\ n) \Longrightarrow f\ 0 = c$
⟨*proof*⟩

**lemma** *def-nat-rec-Suc*: $(!!n.\ f\ n == nat\text{-}rec\ c\ h\ n) \Longrightarrow f\ (Suc\ n) = h\ n\ (f\ n)$
⟨*proof*⟩

**lemma** *not0-implies-Suc*: $n \neq 0 \Longrightarrow \exists\, m.\ n = Suc\ m$
⟨*proof*⟩

**lemma** *gr0-implies-Suc*: $n > 0 \Longrightarrow \exists\, m.\ n = Suc\ m$

⟨*proof*⟩

**lemma** *gr-implies-not0*: **fixes** $n$ :: *nat* **shows** $m<n \Longrightarrow n \neq 0$
⟨*proof*⟩

**lemma** *neq0-conv*[*iff*]: **fixes** $n$ :: *nat* **shows** $(n \neq 0) = (0 < n)$
⟨*proof*⟩

This theorem is useful with *blast*

**lemma** *gr0I*: $((n::nat) = 0 \Longrightarrow False) \Longrightarrow 0 < n$
⟨*proof*⟩

**lemma** *gr0-conv-Suc*: $(0 < n) = (\exists m.\ n = Suc\ m)$
⟨*proof*⟩

**lemma** *not-gr0* [*iff*,*noatp*]: $!!n::nat.\ (\sim (0 < n)) = (n = 0)$
⟨*proof*⟩

**lemma** *Suc-le-D*: $(Suc\ n \leq m') \Longrightarrow (?\ m.\ m' = Suc\ m)$
⟨*proof*⟩

Useful in certain inductive arguments

**lemma** *less-Suc-eq-0-disj*: $(m < Suc\ n) = (m = 0 \mid (\exists j.\ m = Suc\ j\ \&\ j < n))$
⟨*proof*⟩

### 12.4.5 *min* and *max*

**lemma** *mono-Suc*: *mono Suc*
⟨*proof*⟩

**lemma** *min-0L* [*simp*]: $min\ 0\ n = (0::nat)$
⟨*proof*⟩

**lemma** *min-0R* [*simp*]: $min\ n\ 0 = (0::nat)$
⟨*proof*⟩

**lemma** *min-Suc-Suc* [*simp*]: $min\ (Suc\ m)\ (Suc\ n) = Suc\ (min\ m\ n)$
⟨*proof*⟩

**lemma** *min-Suc1*:
$min\ (Suc\ n)\ m = (case\ m\ of\ 0 \Rightarrow 0 \mid Suc\ m' \Rightarrow Suc(min\ n\ m'))$
⟨*proof*⟩

**lemma** *min-Suc2*:
$min\ m\ (Suc\ n) = (case\ m\ of\ 0 \Rightarrow 0 \mid Suc\ m' \Rightarrow Suc(min\ m'\ n))$
⟨*proof*⟩

**lemma** *max-0L* [*simp*]: $max\ 0\ n = (n::nat)$
⟨*proof*⟩

**lemma** *max-0R* [*simp*]: *max n 0* = (*n*::*nat*)
⟨*proof*⟩

**lemma** *max-Suc-Suc* [*simp*]: *max* (*Suc m*) (*Suc n*) = *Suc*(*max m n*)
⟨*proof*⟩

**lemma** *max-Suc1*:
*max* (*Suc n*) *m* = (*case m of 0* => *Suc n* | *Suc m'* => *Suc*(*max n m'*))
⟨*proof*⟩

**lemma** *max-Suc2*:
*max m* (*Suc n*) = (*case m of 0* => *Suc n* | *Suc m'* => *Suc*(*max m' n*))
⟨*proof*⟩

### 12.4.6 Monotonicity of Addition

**lemma** *Suc-pred* [*simp*]: *n*>*0* ==> *Suc* (*n* − *Suc 0*) = *n*
⟨*proof*⟩

**lemma** *Suc-diff-1* [*simp*]: *0* < *n* ==> *Suc* (*n* − *1*) = *n*
⟨*proof*⟩

**lemma** *nat-add-left-cancel-le* [*simp*]: (*k* + *m* ≤ *k* + *n*) = (*m*≤(*n*::*nat*))
⟨*proof*⟩

**lemma** *nat-add-left-cancel-less* [*simp*]: (*k* + *m* < *k* + *n*) = (*m*<(*n*::*nat*))
⟨*proof*⟩

**lemma** *add-gr-0* [*iff*]: !!*m*::*nat*. (*m* + *n* > *0*) = (*m*>*0* | *n*>*0*)
⟨*proof*⟩

strict, in 1st argument

**lemma** *add-less-mono1*: *i* < *j* ==> *i* + *k* < *j* + (*k*::*nat*)
⟨*proof*⟩

strict, in both arguments

**lemma** *add-less-mono*: [|*i* < *j*; *k* < *l*|] ==> *i* + *k* < *j* + (*l*::*nat*)
⟨*proof*⟩

Deleted *less-natE*; use *less-imp-Suc-add RS exE*

**lemma** *less-imp-Suc-add*: *m* < *n* ==> (∃ *k*. *n* = *Suc* (*m* + *k*))
⟨*proof*⟩

strict, in 1st argument; proof is by induction on *k* > *0*

**lemma** *mult-less-mono2*: (*i*::*nat*) < *j* ==> *0*<*k* ==> *k* ∗ *i* < *k* ∗ *j*
⟨*proof*⟩

The naturals form an ordered *comm-semiring-1-cancel*

**instance** *nat* :: *ordered-semidom*
⟨*proof*⟩

**instance** *nat* :: *no-zero-divisors*
⟨*proof*⟩

**lemma** *nat-mult-1*: $(1::nat) * n = n$
⟨*proof*⟩

**lemma** *nat-mult-1-right*: $n * (1::nat) = n$
⟨*proof*⟩

### 12.4.7 Additional theorems about *op* ≤

Complete induction, aka course-of-values induction

**instance** *nat* :: *wellorder* ⟨*proof*⟩

**lemma** *Least-Suc*:
$[|\ P\ n;\ \sim\ P\ 0\ |] ==> (LEAST\ n.\ P\ n) = Suc\ (LEAST\ m.\ P(Suc\ m))$
⟨*proof*⟩

**lemma** *Least-Suc2*:
$[|P\ n;\ Q\ m;\ \sim P\ 0;\ !k.\ P\ (Suc\ k) = Q\ k|] ==> Least\ P = Suc\ (Least\ Q)$
⟨*proof*⟩

**lemma** *ex-least-nat-le*: $\neg P(0) \Longrightarrow P(n::nat) \Longrightarrow \exists\, k{\leq}n.\ (\forall\, i{<}k.\ \neg P\ i)\ \&\ P(k)$
⟨*proof*⟩

**lemma** *ex-least-nat-less*: $\neg P(0) \Longrightarrow P(n::nat) \Longrightarrow \exists\, k{<}n.\ (\forall\, i{\leq}k.\ \neg P\ i)\ \&\ P(k{+}1)$
⟨*proof*⟩

**lemma** *nat-less-induct*:
**assumes** $!!n.\ \forall\, m::nat.\ m < n\ --> P\ m ==> P\ n$ **shows** $P\ n$
⟨*proof*⟩

**lemma** *measure-induct-rule* [*case-names less*]:
**fixes** $f :: 'a \Rightarrow nat$
**assumes** *step*: $\bigwedge x.\ (\bigwedge y.\ f\ y < f\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
**shows** $P\ a$
⟨*proof*⟩

old style induction rules:

**lemma** *measure-induct*:
**fixes** $f :: 'a \Rightarrow nat$
**shows** $(\bigwedge x.\ \forall\, y.\ f\ y < f\ x \longrightarrow P\ y \Longrightarrow P\ x) \Longrightarrow P\ a$
⟨*proof*⟩

**lemma** *full-nat-induct*:
**assumes** *step*: $(!!n.\ (ALL\ m.\ Suc\ m <= n\ --> P\ m) ==> P\ n)$

**shows** $P\ n$
⟨*proof*⟩

An induction rule for estabilishing binary relations

**lemma** *less-Suc-induct*:
**assumes** *less*: $i < j$
**and** *step*: $!!i.\ P\ i\ (Suc\ i)$
**and** *trans*: $!!i\ j\ k.\ P\ i\ j \Longrightarrow P\ j\ k \Longrightarrow P\ i\ k$
**shows** $P\ i\ j$
⟨*proof*⟩

The method of infinite descent, frequently used in number theory. Provided by Roelof Oosterhuis. $P(n)$ is true for all $n \in \mathbb{N}$ if

- case "0": given $n = 0$ prove $P(n)$,
- case "smaller": given $n > 0$ and $\neg P(n)$ prove there exists a smaller integer $m$ such that $\neg P(m)$.

A compact version without explicit base case:

**lemma** *infinite-descent*:
$[\![\ !!n::nat.\ \neg\ P\ n \Longrightarrow\ \exists\, m<n.\ \neg\ \ P\ m\ ]\!] \Longrightarrow\ \ P\ n$
⟨*proof*⟩

**lemma** *infinite-descent0*[*case-names 0 smaller*]:
$[\![\ P\ 0;\ !!n.\ n>0 \Longrightarrow \neg\ P\ n \Longrightarrow (\exists\, m::nat.\ m < n \wedge \neg P\ m)\ ]\!] \Longrightarrow P\ n$
⟨*proof*⟩

Infinite descent using a mapping to $\mathbb{N}$: $P(x)$ is true for all $x \in D$ if there exists a $V : D \to \mathbb{N}$ and

- case "0": given $V(x) = 0$ prove $P(x)$,
- case "smaller": given $V(x) > 0$ and $\neg P(x)$ prove there exists a $y \in D$ such that $V(y) < V(x)$ and $\neg P(y)$.

NB: the proof also shows how to use the previous lemma.

**corollary** *infinite-descent0-measure* [*case-names 0 smaller*]:
**assumes** *A0*: $!!x.\ V\ x = (0::nat) \Longrightarrow P\ x$
**and** *A1*: $!!x.\ V\ x > 0 \Longrightarrow \neg P\ x \Longrightarrow (\exists\, y.\ V\ y < V\ x \wedge \neg P\ y)$
**shows** $P\ x$
⟨*proof*⟩

Again, without explicit base case:

**lemma** *infinite-descent-measure*:
**assumes** $!!x.\ \neg\ P\ x \Longrightarrow \exists\, y.\ (V::'a \Rightarrow nat)\ y < V\ x \wedge \neg\ P\ y$ **shows** $P\ x$
⟨*proof*⟩

A [clumsy] way of lifting $<$ monotonicity to $\leq$ monotonicity

**lemma** *less-mono-imp-le-mono*:
$[\![ \; !!i\ j{::}nat.\ i < j \Longrightarrow f\ i < f\ j;\ i \leq j \;]\!] \Longrightarrow f\ i \leq ((f\ j){::}nat)$
⟨*proof*⟩

non-strict, in 1st argument

**lemma** *add-le-mono1*: $i \leq j$ ==> $i + k \leq j + (k{::}nat)$
⟨*proof*⟩

non-strict, in both arguments

**lemma** *add-le-mono*: [| $i \leq j$; $k \leq l$ |] ==> $i + k \leq j + (l{::}nat)$
⟨*proof*⟩

**lemma** *le-add2*: $n \leq ((m + n){::}nat)$
⟨*proof*⟩

**lemma** *le-add1*: $n \leq ((n + m){::}nat)$
⟨*proof*⟩

**lemma** *less-add-Suc1*: $i < Suc\ (i + m)$
⟨*proof*⟩

**lemma** *less-add-Suc2*: $i < Suc\ (m + i)$
⟨*proof*⟩

**lemma** *less-iff-Suc-add*: $(m < n) = (\exists\, k.\ n = Suc\ (m + k))$
⟨*proof*⟩

**lemma** *trans-le-add1*: $(i{::}nat) \leq j$ ==> $i \leq j + m$
⟨*proof*⟩

**lemma** *trans-le-add2*: $(i{::}nat) \leq j$ ==> $i \leq m + j$
⟨*proof*⟩

**lemma** *trans-less-add1*: $(i{::}nat) < j$ ==> $i < j + m$
⟨*proof*⟩

**lemma** *trans-less-add2*: $(i{::}nat) < j$ ==> $i < m + j$
⟨*proof*⟩

**lemma** *add-lessD1*: $i + j < (k{::}nat)$ ==> $i < k$
⟨*proof*⟩

**lemma** *not-add-less1* [*iff*]: ~ $(i + j < (i{::}nat))$
⟨*proof*⟩

**lemma** *not-add-less2* [*iff*]: ~ $(j + i < (i{::}nat))$
⟨*proof*⟩

**lemma** *add-leD1*: $m + k \leq n \Longrightarrow m \leq (n::nat)$
⟨*proof*⟩

**lemma** *add-leD2*: $m + k \leq n \Longrightarrow k \leq (n::nat)$
⟨*proof*⟩

**lemma** *add-leE*: $(m::nat) + k \leq n \Longrightarrow (m \leq n \Longrightarrow k \leq n \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

needs $!!k$ for *add-ac* to work

**lemma** *less-add-eq-less*: $!!k::nat.\ k < l \Longrightarrow m + l = k + n \Longrightarrow m < n$
⟨*proof*⟩

### 12.4.8 More results about difference

Addition is the inverse of subtraction: if $n \leq m$ then $n + (m - n) = m$.

**lemma** *add-diff-inverse*: $\sim\ m < n \Longrightarrow n + (m - n) = (m::nat)$
⟨*proof*⟩

**lemma** *le-add-diff-inverse* [*simp*]: $n \leq m \Longrightarrow n + (m - n) = (m::nat)$
⟨*proof*⟩

**lemma** *le-add-diff-inverse2* [*simp*]: $n \leq m \Longrightarrow (m - n) + n = (m::nat)$
⟨*proof*⟩

**lemma** *Suc-diff-le*: $n \leq m \Longrightarrow Suc\ m - n = Suc\ (m - n)$
⟨*proof*⟩

**lemma** *diff-less-Suc*: $m - n < Suc\ m$
⟨*proof*⟩

**lemma** *diff-le-self* [*simp*]: $m - n \leq (m::nat)$
⟨*proof*⟩

**lemma** *le-iff-add*: $(m::nat) \leq n = (\exists k.\ n = m + k)$
⟨*proof*⟩

**lemma** *less-imp-diff-less*: $(j::nat) < k \Longrightarrow j - n < k$
⟨*proof*⟩

**lemma** *diff-Suc-less* [*simp*]: $0<n \Longrightarrow n - Suc\ i < n$
⟨*proof*⟩

**lemma** *diff-add-assoc*: $k \leq (j::nat) \Longrightarrow (i + j) - k = i + (j - k)$
⟨*proof*⟩

**lemma** *diff-add-assoc2*: $k \leq (j::nat) \Longrightarrow (j + i) - k = (j - k) + i$
⟨*proof*⟩

**lemma** *le-imp-diff-is-add*: $i \leq (j::nat) \Longrightarrow (j - i = k) = (j = k + i)$
⟨*proof*⟩

**lemma** *diff-is-0-eq* [*simp*]: $((m::nat) - n = 0) = (m \leq n)$
⟨*proof*⟩

**lemma** *diff-is-0-eq'* [*simp*]: $m \leq n \Longrightarrow (m::nat) - n = 0$
⟨*proof*⟩

**lemma** *zero-less-diff* [*simp*]: $(0 < n - (m::nat)) = (m < n)$
⟨*proof*⟩

**lemma** *less-imp-add-positive*:
**assumes** $i < j$
**shows** $\exists k::nat.\ 0 < k\ \&\ i + k = j$
⟨*proof*⟩

a nice rewrite for bounded subtraction

**lemma** *nat-minus-add-max*:
**fixes** $n\ m :: nat$
**shows** $n - m + m = max\ n\ m$
⟨*proof*⟩

**lemma** *nat-diff-split*:
$P(a - b::nat) = ((a<b \longrightarrow P\ 0)\ \&\ (ALL\ d.\ a = b + d \longrightarrow P\ d))$
— elimination of $-$ on *nat*
⟨*proof*⟩

**lemma** *nat-diff-split-asm*:
$P(a - b::nat) = (\sim\ (a < b\ \&\ \sim\ P\ 0 \mid (EX\ d.\ a = b + d\ \&\ \sim\ P\ d)))$
— elimination of $-$ on *nat* in assumptions
⟨*proof*⟩

### 12.4.9 Monotonicity of Multiplication

**lemma** *mult-le-mono1*: $i \leq (j::nat) \Longrightarrow i * k \leq j * k$
⟨*proof*⟩

**lemma** *mult-le-mono2*: $i \leq (j::nat) \Longrightarrow k * i \leq k * j$
⟨*proof*⟩

$\leq$ monotonicity, BOTH arguments

**lemma** *mult-le-mono*: $i \leq (j::nat) \Longrightarrow k \leq l \Longrightarrow i * k \leq j * l$
⟨*proof*⟩

**lemma** *mult-less-mono1*: $(i::nat) < j \Longrightarrow 0 < k \Longrightarrow i * k < j * k$
⟨*proof*⟩

Differs from the standard *zero-less-mult-iff* in that there are no negative numbers.

**lemma** *nat-0-less-mult-iff* [*simp*]: $(0 < (m::nat) * n) = (0 < m \ \& \ 0 < n)$
⟨*proof*⟩

**lemma** *one-le-mult-iff* [*simp*]: $(Suc\ 0 \leq m * n) = (Suc\ 0 \leq m \ \& \ Suc\ 0 \leq n)$
⟨*proof*⟩

**lemma** *mult-less-cancel2* [*simp*]: $((m::nat) * k < n * k) = (0 < k \ \& \ m < n)$
⟨*proof*⟩

**lemma** *mult-less-cancel1* [*simp*]: $(k * (m::nat) < k * n) = (0 < k \ \& \ m < n)$
⟨*proof*⟩

**lemma** *mult-le-cancel1* [*simp*]: $(k * (m::nat) \leq k * n) = (0 < k \longrightarrow m \leq n)$
⟨*proof*⟩

**lemma** *mult-le-cancel2* [*simp*]: $((m::nat) * k \leq n * k) = (0 < k \longrightarrow m \leq n)$
⟨*proof*⟩

**lemma** *Suc-mult-less-cancel1*: $(Suc\ k * m < Suc\ k * n) = (m < n)$
⟨*proof*⟩

**lemma** *Suc-mult-le-cancel1*: $(Suc\ k * m \leq Suc\ k * n) = (m \leq n)$
⟨*proof*⟩

**lemma** *le-square*: $m \leq m * (m::nat)$
⟨*proof*⟩

**lemma** *le-cube*: $(m::nat) \leq m * (m * m)$
⟨*proof*⟩

Lemma for *gcd*

**lemma** *mult-eq-self-implies-10*: $(m::nat) = m * n \Longrightarrow n = 1 \mid m = 0$
⟨*proof*⟩

the lattice order on *nat*

**instantiation** *nat* :: *distrib-lattice*
**begin**

**definition**
$(inf :: nat \Rightarrow nat \Rightarrow nat) = min$

**definition**
$(sup :: nat \Rightarrow nat \Rightarrow nat) = max$

**instance** ⟨*proof*⟩

**end**

## 12.5 Embedding of the Naturals into any *semiring-1*: *of-nat*

**context** *semiring-1*
**begin**

**primrec**
*of-nat* :: *nat* ⇒ *'a*
**where**
*of-nat-0*: *of-nat 0 = 0*
| *of-nat-Suc*: *of-nat (Suc m) = 1 + of-nat m*

**lemma** *of-nat-1* [*simp*]: *of-nat 1 = 1*
⟨*proof*⟩

**lemma** *of-nat-add* [*simp*]: *of-nat (m + n) = of-nat m + of-nat n*
⟨*proof*⟩

**lemma** *of-nat-mult*: *of-nat (m * n) = of-nat m * of-nat n*
⟨*proof*⟩

**primrec** *of-nat-aux* :: *('a ⇒ 'a) ⇒ nat ⇒ 'a ⇒ 'a* **where**
*of-nat-aux inc 0 i = i*
| *of-nat-aux inc (Suc n) i = of-nat-aux inc n (inc i)* — tail recursive

**lemma** *of-nat-code* [*code*, *code unfold*, *code inline del*]:
*of-nat n = of-nat-aux (λi. i + 1) n 0*
⟨*proof*⟩

**end**

Class for unital semirings with characteristic zero. Includes non-ordered rings like the complex numbers.

**class** *semiring-char-0 = semiring-1 +*
**assumes** *of-nat-eq-iff* [*simp*]: *of-nat m = of-nat n ⟷ m = n*
**begin**

Special cases where either operand is zero

**lemma** *of-nat-0-eq-iff* [*simp*, *noatp*]: *0 = of-nat n ⟷ 0 = n*
⟨*proof*⟩

**lemma** *of-nat-eq-0-iff* [*simp*, *noatp*]: *of-nat m = 0 ⟷ m = 0*
⟨*proof*⟩

**lemma** *inj-of-nat*: *inj of-nat*
⟨*proof*⟩

**end**

**context** *ordered-semidom*

**begin**

**lemma** *zero-le-imp-of-nat*: $0 \leq$ *of-nat* $m$
⟨*proof*⟩

**lemma** *less-imp-of-nat-less*: $m < n \Longrightarrow$ *of-nat* $m <$ *of-nat* $n$
⟨*proof*⟩

**lemma** *of-nat-less-imp-less*: *of-nat* $m <$ *of-nat* $n \Longrightarrow m < n$
⟨*proof*⟩

**lemma** *of-nat-less-iff* [*simp*]: *of-nat* $m <$ *of-nat* $n \longleftrightarrow m < n$
⟨*proof*⟩

**lemma** *of-nat-le-iff* [*simp*]: *of-nat* $m \leq$ *of-nat* $n \longleftrightarrow m \leq n$
⟨*proof*⟩

Every *ordered-semidom* has characteristic zero.

**subclass** *semiring-char-0*
⟨*proof*⟩

Special cases where either operand is zero

**lemma** *of-nat-0-le-iff* [*simp*]: $0 \leq$ *of-nat* $n$
⟨*proof*⟩

**lemma** *of-nat-le-0-iff* [*simp*, *noatp*]: *of-nat* $m \leq 0 \longleftrightarrow m = 0$
⟨*proof*⟩

**lemma** *of-nat-0-less-iff* [*simp*]: $0 <$ *of-nat* $n \longleftrightarrow 0 < n$
⟨*proof*⟩

**lemma** *of-nat-less-0-iff* [*simp*]: $\neg$ *of-nat* $m < 0$
⟨*proof*⟩

**end**

**context** *ring-1*
**begin**

**lemma** *of-nat-diff*: $n \leq m \Longrightarrow$ *of-nat* $(m - n) =$ *of-nat* $m -$ *of-nat* $n$
⟨*proof*⟩

**end**

**context** *ordered-idom*
**begin**

**lemma** *abs-of-nat* [*simp*]: $|$*of-nat* $n| =$ *of-nat* $n$
⟨*proof*⟩

**end**

**lemma** *of-nat-id* [*simp*]: *of-nat n* = *n*
⟨*proof*⟩

**lemma** *of-nat-eq-id* [*simp*]: *of-nat* = *id*
⟨*proof*⟩

## 12.6 The Set of Natural Numbers

**context** *semiring-1*
**begin**

**definition**
*Nats* :: *'a set* **where**
[*code del*]: *Nats* = *range of-nat*

**notation** (*xsymbols*)
*Nats* (ℕ)

**lemma** *of-nat-in-Nats* [*simp*]: *of-nat n* ∈ ℕ
⟨*proof*⟩

**lemma** *Nats-0* [*simp*]: *0* ∈ ℕ
⟨*proof*⟩

**lemma** *Nats-1* [*simp*]: *1* ∈ ℕ
⟨*proof*⟩

**lemma** *Nats-add* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* + *b* ∈ ℕ
⟨*proof*⟩

**lemma** *Nats-mult* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* ∗ *b* ∈ ℕ
⟨*proof*⟩

**end**

## 12.7 Further Arithmetic Facts Concerning the Natural Numbers

**lemma** *subst-equals*:
**assumes** *1*: *t* = *s* **and** *2*: *u* = *t*
**shows** *u* = *s*
⟨*proof*⟩

⟨*ML*⟩

**lemmas** [*arith-split*] = *nat-diff-split split-min split-max*

**context** *order*
**begin**

**lemma** *lift-Suc-mono-le*:
  **assumes** *mono*: !!$n$. $f\ n \leq f(Suc\ n)$ **and** $n \leq n'$
  **shows** $f\ n \leq f\ n'$
⟨*proof*⟩

**lemma** *lift-Suc-mono-less*:
  **assumes** *mono*: !!$n$. $f\ n < f(Suc\ n)$ **and** $n < n'$
  **shows** $f\ n < f\ n'$
⟨*proof*⟩

**lemma** *lift-Suc-mono-less-iff*:
  (!!$n$. $f\ n < f(Suc\ n)$) $\Longrightarrow f(n) < f(m) \longleftrightarrow n<m$
⟨*proof*⟩

**end**

**lemma** *mono-iff-le-Suc*: *mono* $f = (\forall\, n.\ f\ n \leq f\ (Suc\ n))$
⟨*proof*⟩

**lemma** *mono-nat-linear-lb*:
  (!!$m\ n$::*nat*. $m<n \Longrightarrow f\ m < f\ n) \Longrightarrow f(m)+k \leq f(m+k)$
⟨*proof*⟩

Subtraction laws, mostly by Clemens Ballarin

**lemma** *diff-less-mono*: [| $a < (b$::*nat*); $c \leq a$ |] ==> $a-c < b-c$
⟨*proof*⟩

**lemma** *less-diff-conv*: $(i < j-k) = (i+k < (j$::*nat*))
⟨*proof*⟩

**lemma** *le-diff-conv*: $(j-k \leq (i$::*nat*)) $= (j \leq i+k)$
⟨*proof*⟩

**lemma** *le-diff-conv2*: $k \leq j$ ==> $(i \leq j-k) = (i+k \leq (j$::*nat*))
⟨*proof*⟩

**lemma** *diff-diff-cancel* [*simp*]: $i \leq (n$::*nat*) ==> $n - (n - i) = i$
⟨*proof*⟩

**lemma** *le-add-diff*: $k \leq (n$::*nat*) ==> $m \leq n + m - k$
⟨*proof*⟩

**lemma** *diff-less*[*simp*]: !!$m$::*nat*. [| $0<n$; $0<m$ |] ==> $m - n < m$
⟨*proof*⟩

Simplification of relational expressions involving subtraction

**lemma** *diff-diff-eq*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $((m-k) - (n-k)) = (m-n)$
⟨*proof*⟩

**lemma** *eq-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k = n-k) = (m=n)$
⟨*proof*⟩

**lemma** *less-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k < n-k) = (m<n)$
⟨*proof*⟩

**lemma** *le-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k \leq n-k) = (m \leq n)$
⟨*proof*⟩

(Anti)Monotonicity of subtraction – by Stephan Merz

**lemma** *diff-le-mono*: $m \leq (n::nat)$ ==> $(m-l) \leq (n-l)$
⟨*proof*⟩

**lemma** *diff-le-mono2*: $m \leq (n::nat)$ ==> $(l-n) \leq (l-m)$
⟨*proof*⟩

**lemma** *diff-less-mono2*: [| $m < (n::nat)$; $m<l$ |] ==> $(l-n) < (l-m)$
⟨*proof*⟩

**lemma** *diffs0-imp-equal*: !!$m::nat$. [| $m-n = 0$; $n-m = 0$ |] ==> $m=n$
⟨*proof*⟩

**lemma** *min-diff*: $min\ (m - (i::nat))\ (n - i) = min\ m\ n - i$
⟨*proof*⟩

**lemma** *inj-on-diff-nat*:
**assumes** *k-le-n*: $\forall n \in N.\ k \leq (n::nat)$
**shows** *inj-on* $(\lambda n.\ n - k)\ N$
⟨*proof*⟩

Rewriting to pull differences out

**lemma** *diff-diff-right* [*simp*]: $k \leq j$ --> $i - (j - k) = i + (k::nat) - j$
⟨*proof*⟩

**lemma** *diff-Suc-diff-eq1* [*simp*]: $k \leq j$ ==> $m - Suc\ (j - k) = m + k - Suc\ j$
⟨*proof*⟩

**lemma** *diff-Suc-diff-eq2* [*simp*]: $k \leq j$ ==> $Suc\ (j - k) - m = Suc\ j - (k + m)$
⟨*proof*⟩

Lemmas for ex/Factorization

**lemma** *one-less-mult*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] ==> $Suc\ 0 < m*n$
⟨*proof*⟩

**lemma** *n-less-m-mult-n*: [| *Suc 0* < *n*; *Suc 0* < *m* |] ==> *n*<*m*∗*n*
⟨*proof*⟩

**lemma** *n-less-n-mult-m*: [| *Suc 0* < *n*; *Suc 0* < *m* |] ==> *n*<*n*∗*m*
⟨*proof*⟩

Specialized induction principles that work "backwards":

**lemma** *inc-induct*[*consumes 1*, *case-names base step*]:
**assumes** *less*: *i* <= *j*
**assumes** *base*: *P j*
**assumes** *step*: !!*i*. [| *i* < *j*; *P* (*Suc i*) |] ==> *P i*
**shows** *P i*
⟨*proof*⟩

**lemma** *strict-inc-induct*[*consumes 1*, *case-names base step*]:
**assumes** *less*: *i* < *j*
**assumes** *base*: !!*i*. *j* = *Suc i* ==> *P i*
**assumes** *step*: !!*i*. [| *i* < *j*; *P* (*Suc i*) |] ==> *P i*
**shows** *P i*
⟨*proof*⟩

**lemma** *zero-induct-lemma*: *P k* ==> (!!*n*. *P* (*Suc n*) ==> *P n*) ==> *P* (*k* − *i*)
⟨*proof*⟩

**lemma** *zero-induct*: *P k* ==> (!!*n*. *P* (*Suc n*) ==> *P n*) ==> *P 0*
⟨*proof*⟩

**lemma** *nat-not-singleton*: (∀ *x*. *x* = (*0*::*nat*)) = *False*
⟨*proof*⟩

**lemmas** *add-diff-assoc* = *diff-add-assoc* [*symmetric*]
**lemmas** *add-diff-assoc2* = *diff-add-assoc2*[*symmetric*]
**declare** *diff-diff-left* [*simp*] *add-diff-assoc* [*simp*] *add-diff-assoc2*[*simp*]

At present we prove no analogue of *not-less-Least* or *Least-Suc*, since there appears to be no need.

### 12.8 size of a datatype value

**class** *size* =
**fixes** *size* :: *'a* ⇒ *nat* — see further theory *Wellfounded*

**end**

## 13 Product-Type: Cartesian products

**theory** *Product-Type*

**imports** *Inductive*
**uses**
(*Tools/split-rule.ML*)
(*Tools/inductive-set-package.ML*)
(*Tools/inductive-realizer.ML*)
(*Tools/datatype-realizer.ML*)
**begin**

## 13.1 *bool* is a datatype

**rep-datatype** *True False* ⟨*proof*⟩

**declare** *case-split* [*cases type*: *bool*]
— prefer plain propositional version

**lemma**
**shows** [*code*]: *eq-class.eq False P* ⟷ ¬ *P*
**and** [*code*]: *eq-class.eq True P* ⟷ *P*
**and** [*code*]: *eq-class.eq P False* ⟷ ¬ *P*
**and** [*code*]: *eq-class.eq P True* ⟷ *P*
**and** [*code nbe*]: *eq-class.eq P P* ⟷ *True*
⟨*proof*⟩

**code-const** *eq-class.eq* :: *bool* ⇒ *bool* ⇒ *bool*
(*Haskell* **infixl** 4 ==)

**code-instance** *bool* :: *eq*
(*Haskell* −)

## 13.2 Unit

**typedef** *unit* = {*True*}
⟨*proof*⟩

**definition**
*Unity* :: *unit* ('())
**where**
() = *Abs-unit True*

**lemma** *unit-eq* [*noatp*]: *u* = ()
⟨*proof*⟩

Simplification procedure for *unit-eq*. Cannot use this rule directly — it loops!

⟨*ML*⟩

**rep-datatype** () ⟨*proof*⟩

**lemma** *unit-all-eq1*: (!!*x*::*unit*. *PROP P x*) == *PROP P* ()

⟨*proof*⟩

**lemma** *unit-all-eq2*: (!!*x*::*unit*. *PROP P*) == *PROP P*
⟨*proof*⟩

This rewrite counters the effect of *unit-eq-proc* on %*u*::*unit*. *f u*, replacing it by *f* rather than by %*u*. *f* ().

**lemma** *unit-abs-eta-conv* [*simp*,*noatp*]: (%*u*::*unit*. *f* ()) = *f*
⟨*proof*⟩

code generator setup

**instance** *unit* :: *eq* ⟨*proof*⟩

**lemma** [*code*]:
*eq-class.eq* (*u*::*unit*) *v* ⟷ *True* ⟨*proof*⟩

**code-type** *unit*
(*SML unit*)
(*OCaml unit*)
(*Haskell* ())

**code-instance** *unit* :: *eq*
(*Haskell* −)

**code-const** *eq-class.eq* :: *unit* ⇒ *unit* ⇒ *bool*
(*Haskell* **infixl** 4 ==)

**code-const** *Unity*
(*SML* ())
(*OCaml* ())
(*Haskell* ())

**code-reserved** *SML*
*unit*

**code-reserved** *OCaml*
*unit*

## 13.3 Pairs

### 13.3.1 Product type, basic operations and concrete syntax

**definition**
*Pair-Rep* :: 'a ⇒ 'b ⇒ 'a ⇒ 'b ⇒ *bool*
**where**
*Pair-Rep a b* = (λ*x y*. *x* = *a* ∧ *y* = *b*)

**global**

**typedef** (*Prod*)
  $('a, 'b) *$  (**infixr** $*$ *20*)
    $= \{f.\ \exists\, a\ b.\ f = Pair\text{-}Rep\ (a{::}'a)\ (b{::}'b)\}$
⟨*proof*⟩

**syntax** (*xsymbols*)
  $*$  :: [*type*, *type*] => *type*  ((- ×/ -) [*21*, *20*] *20*)
**syntax** (*HTML* **output**)
  $*$  :: [*type*, *type*] => *type*  ((- ×/ -) [*21*, *20*] *20*)

**consts**
  *Pair*  :: $'a \Rightarrow 'b \Rightarrow 'a \times 'b$
  *fst*  :: $'a \times 'b \Rightarrow 'a$
  *snd*  :: $'a \times 'b \Rightarrow 'b$
  *split*  :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'c$
  *curry*  :: $('a \times 'b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'b \Rightarrow 'c$

**local**

**defs**
  *Pair-def*:  *Pair a b == Abs-Prod (Pair-Rep a b)*
  *fst-def*:  *fst p == THE a. EX b. p = Pair a b*
  *snd-def*:  *snd p == THE b. EX a. p = Pair a b*
  *split-def*:  *split == (%c p. c (fst p) (snd p))*
  *curry-def*:  *curry == (%c x y. c (Pair x y))*

Patterns – extends pre-defined type *pttrn* used in abstractions.

**nonterminals**
  *tuple-args patterns*

**syntax**
  *-tuple*  :: *'a => tuple-args => 'a * 'b*  (($1'(-,/ -')$))
  *-tuple-arg*  :: *'a => tuple-args*  (-)
  *-tuple-args*  :: *'a => tuple-args => tuple-args*  (-,/ -)
  *-pattern*  :: [*pttrn*, *patterns*] => *pttrn*  ('(-,/ -'))
    :: *pttrn => patterns*  (-)
  *-patterns*  :: [*pttrn*, *patterns*] => *patterns*  (-,/ -)

**translations**
  *(x, y)*  *== Pair x y*
  *-tuple x (-tuple-args y z) == -tuple x (-tuple-arg (-tuple y z))*
  *%(x,y,zs).b*  *== split(%x (y,zs).b)*
  *%(x,y).b*  *== split(%x y. b)*
  *-abs (Pair x y) t => %(x,y).t*

⟨*ML*⟩

Towards a datatype declaration

**lemma** *surj-pair* [*simp*]: *EX x y. p = (x, y)*
⟨*proof*⟩

**lemma** *PairE* [*cases type*: ∗]:
**obtains** *x y* **where** *p = (x, y)*
⟨*proof*⟩

**lemma** *ProdI*: *Pair-Rep a b ∈ Prod*
⟨*proof*⟩

**lemma** *Pair-Rep-inject*: *Pair-Rep a b = Pair-Rep a′ b′ ⟹ a = a′ ∧ b = b′*
⟨*proof*⟩

**lemma** *inj-on-Abs-Prod*: *inj-on Abs-Prod Prod*
⟨*proof*⟩

**lemma** *Pair-inject*:
**assumes** *(a, b) = (a′, b′)*
**and** *a = a′ ==> b = b′ ==> R*
**shows** *R*
⟨*proof*⟩

**rep-datatype** (*prod*) *Pair*
⟨*proof*⟩

**lemmas** *Pair-eq = prod.inject*

**lemma** *fst-conv* [*simp*, *code*]: *fst (a, b) = a*
⟨*proof*⟩

**lemma** *snd-conv* [*simp*, *code*]: *snd (a, b) = b*
⟨*proof*⟩

### 13.3.2 Basic rules and proof tools

**lemma** *fst-eqD*: *fst (x, y) = a ==> x = a*
⟨*proof*⟩

**lemma** *snd-eqD*: *snd (x, y) = a ==> y = a*
⟨*proof*⟩

**lemma** *pair-collapse* [*simp*]: *(fst p, snd p) = p*
⟨*proof*⟩

**lemmas** *surjective-pairing = pair-collapse* [*symmetric*]

**lemma** *split-paired-all*: *(!!x. PROP P x) == (!!a b. PROP P (a, b))*
⟨*proof*⟩

The rule *split-paired-all* does not work with the Simplifier because it also affects premises in congrence rules, where this can lead to premises of the form !!*a b*. ... = *?P*(*a*, *b*) which cannot be solved by reflexivity.

**lemmas** *split-tupled-all* = *split-paired-all unit-all-eq2*

⟨*ML*⟩

**lemma** *split-paired-All* [*simp*]: (*ALL x*. *P x*) = (*ALL a b*. *P* (*a*, *b*))
— [*iff*] is not a good idea because it makes *blast* loop
⟨*proof*⟩

**lemma** *split-paired-Ex* [*simp*]: (*EX x*. *P x*) = (*EX a b*. *P* (*a*, *b*))
⟨*proof*⟩

**lemma** *Pair-fst-snd-eq*: *s* = *t* ⟷ *fst s* = *fst t* ∧ *snd s* = *snd t*
⟨*proof*⟩

**lemma** *prod-eqI* [*intro?*]: *fst p* = *fst q* ⟹ *snd p* = *snd q* ⟹ *p* = *q*
⟨*proof*⟩

### 13.3.3 *split* and *curry*

**lemma** *split-conv* [*simp*, *code*]: *split f* (*a*, *b*) = *f a b*
⟨*proof*⟩

**lemma** *curry-conv* [*simp*, *code*]: *curry f a b* = *f* (*a*, *b*)
⟨*proof*⟩

**lemmas** *split* = *split-conv* — for backwards compatibility

**lemma** *splitI*: *f a b* ⟹ *split f* (*a*, *b*)
⟨*proof*⟩

**lemma** *splitD*: *split f* (*a*, *b*) ⟹ *f a b*
⟨*proof*⟩

**lemma** *curryI* [*intro!*]: *f* (*a*, *b*) ⟹ *curry f a b*
⟨*proof*⟩

**lemma** *curryD* [*dest!*]: *curry f a b* ⟹ *f* (*a*, *b*)
⟨*proof*⟩

**lemma** *curryE*: *curry f a b* ⟹ (*f* (*a*, *b*) ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

**lemma** *curry-split* [*simp*]: *curry* (*split f*) = *f*
⟨*proof*⟩

**lemma** *split-curry* [*simp*]: *split* (*curry f*) = *f*

⟨*proof*⟩

**lemma** *split-Pair* [*simp*]: $(\lambda(x,\ y).\ (x,\ y)) = id$
⟨*proof*⟩

**lemma** *split-eta*: $(\lambda(x,\ y).\ f\ (x,\ y)) = f$
— Subsumes the old *split-Pair* when $f$ is the identity function.
⟨*proof*⟩

**lemma** *split-comp*: $split\ (f \circ g)\ x = f\ (g\ (fst\ x))\ (snd\ x)$
⟨*proof*⟩

**lemma** *split-twice*: $split\ f\ (split\ g\ p) = split\ (\lambda x\ y.\ split\ f\ (g\ x\ y))\ p$
⟨*proof*⟩

**lemma** *split-paired-The*: $(THE\ x.\ P\ x) = (THE\ (a,\ b).\ P\ (a,\ b))$
— Can't be added to simpset: loops!
⟨*proof*⟩

**lemma** *The-split*: $The\ (split\ P) = (THE\ xy.\ P\ (fst\ xy)\ (snd\ xy))$
⟨*proof*⟩

**lemma** *split-weak-cong*: $p = q \Longrightarrow split\ c\ p = split\ c\ q$
— Prevents simplification of $c$: much faster
⟨*proof*⟩

**lemma** *cond-split-eta*: $(!!x\ y.\ f\ x\ y = g\ (x,\ y)) ==> (\%(x,\ y).\ f\ x\ y) = g$
⟨*proof*⟩

Simplification procedure for *cond-split-eta*. Using *split-eta* as a rewrite rule is not general enough, and using *cond-split-eta* directly would render some existing proofs very inefficient; similarly for *split-beta*.

⟨*ML*⟩

**lemma** *split-beta* [*mono*]: $(\%(x,\ y).\ P\ x\ y)\ z = P\ (fst\ z)\ (snd\ z)$
⟨*proof*⟩

**lemma** *split-split* [*noatp*]: $R(split\ c\ p) = (ALL\ x\ y.\ p = (x,\ y) --> R(c\ x\ y))$
— For use with *split* and the Simplifier.
⟨*proof*⟩

*split-split* could be declared as [*split*] done after the Splitter has been speeded up significantly; precompute the constants involved and don't do anything unless the current goal contains one of those constants.

**lemma** *split-split-asm* [*noatp*]: $R\ (split\ c\ p) = (\sim(EX\ x\ y.\ p = (x,\ y)\ \&\ (\sim R\ (c\ x\ y))))$
⟨*proof*⟩

*split* used as a logical connective or set former.

These rules are for use with *blast*; could instead call *simp* using *split* as rewrite.

**lemma** *splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *c a b* |] ==> *split c p*
⟨*proof*⟩

**lemma** *splitI2'*: !!*p*. [| !!*a b*. (*a*, *b*) = *p* ==> *c a b x* |] ==> *split c p x*
⟨*proof*⟩

**lemma** *splitE*: *split c p* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y* ==> *Q*) ==> *Q*
⟨*proof*⟩

**lemma** *splitE'*: *split c p z* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y z* ==> *Q*) ==> *Q*
⟨*proof*⟩

**lemma** *splitE2*:
[| *Q* (*split P z*); !!*x y*. [|*z* = (*x*, *y*); *Q* (*P x y*)|] ==> *R* |] ==> *R*
⟨*proof*⟩

**lemma** *splitD'*: *split R* (*a*,*b*) *c* ==> *R a b c*
⟨*proof*⟩

**lemma** *mem-splitI*: *z*: *c a b* ==> *z*: *split c* (*a*, *b*)
⟨*proof*⟩

**lemma** *mem-splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *z*: *c a b* |] ==> *z*: *split c p*
⟨*proof*⟩

**lemma** *mem-splitE*:
**assumes** *major*: *z*: *split c p*
**and** *cases*: !!*x y*. [| *p* = (*x*,*y*); *z*: *c x y* |] ==> *Q*
**shows** *Q*
⟨*proof*⟩

**declare** *mem-splitI2* [*intro*!] *mem-splitI* [*intro*!] *splitI2'* [*intro*!] *splitI2* [*intro*!] *splitI* [*intro*!]
**declare** *mem-splitE* [*elim*!] *splitE'* [*elim*!] *splitE* [*elim*!]

⟨*ML*⟩

**lemma** *split-eta-SetCompr* [*simp*,*noatp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P* (*x*, *y*)) = *P*
⟨*proof*⟩

**lemma** *split-eta-SetCompr2* [*simp*,*noatp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P x y*) = *split P*
⟨*proof*⟩

**lemma** *split-part* [*simp*]: (%(*a*,*b*). *P* & *Q a b*) = (%*ab*. *P* & *split Q ab*)
— Allows simplifications of nested splits in case of independent predicates.

⟨*proof*⟩

**lemma** *split-comp-eq*:
**fixes** $f :: 'a => 'b => 'c$ **and** $g :: 'd => 'a$
**shows** $(\%u.\ f\ (g\ (fst\ u))\ (snd\ u)) = (split\ (\%x.\ f\ (g\ x)))$
⟨*proof*⟩

**lemma** *pair-imageI* [*intro*]: $(a,\ b) : A ==> f\ a\ b : (\%(a,\ b).\ f\ a\ b)\ `\ A$
⟨*proof*⟩

**lemma** *The-split-eq* [*simp*]: $(THE\ (x',y').\ x = x'\ \&\ y = y') = (x,\ y)$
⟨*proof*⟩

Setup of internal *split-rule*.

**definition**
*internal-split* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'c$
**where**
*internal-split* == *split*

**lemma** *internal-split-conv*: *internal-split c* $(a,\ b) = c\ a\ b$
⟨*proof*⟩

**hide** *const internal-split*

⟨*ML*⟩

**lemmas** *prod-caseI* = *prod.cases* [*THEN iffD2*, *standard*]

**lemma** *prod-caseI2*: $!!p.\ [|\ !!a\ b.\ p = (a,\ b) ==> c\ a\ b\ |] ==>$ *prod-case c p*
⟨*proof*⟩

**lemma** *prod-caseI2'*: $!!p.\ [|\ !!a\ b.\ (a,\ b) = p ==> c\ a\ b\ x\ |] ==>$ *prod-case c p x*
⟨*proof*⟩

**lemma** *prod-caseE*: *prod-case c p* $==> (!!x\ y.\ p = (x,\ y) ==> c\ x\ y ==> Q) ==> Q$
⟨*proof*⟩

**lemma** *prod-caseE'*: *prod-case c p z* $==> (!!x\ y.\ p = (x,\ y) ==> c\ x\ y\ z ==> Q) ==> Q$
⟨*proof*⟩

**lemma** *prod-case-unfold*: *prod-case* $= (\%c\ p.\ c\ (fst\ p)\ (snd\ p))$
⟨*proof*⟩

**declare** *prod-caseI2'* [*intro*!] *prod-caseI2* [*intro*!] *prod-caseI* [*intro*!]
**declare** *prod-caseE'* [*elim*!] *prod-caseE* [*elim*!]

**lemma** *prod-case-split*:
*prod-case* = *split*
⟨*proof*⟩

**lemma** *prod-case-beta*:
*prod-case f p* = *f* (*fst p*) (*snd p*)
⟨*proof*⟩

## 13.4 Further cases/induct rules for tuples

**lemma** *prod-cases3* [*cases type*]:
**obtains** (*fields*) *a b c* **where** *y* = (*a*, *b*, *c*)
⟨*proof*⟩

**lemma** *prod-induct3* [*case-names fields*, *induct type*]:
(!!*a b c*. *P* (*a*, *b*, *c*)) ==> *P x*
⟨*proof*⟩

**lemma** *prod-cases4* [*cases type*]:
**obtains** (*fields*) *a b c d* **where** *y* = (*a*, *b*, *c*, *d*)
⟨*proof*⟩

**lemma** *prod-induct4* [*case-names fields*, *induct type*]:
(!!*a b c d*. *P* (*a*, *b*, *c*, *d*)) ==> *P x*
⟨*proof*⟩

**lemma** *prod-cases5* [*cases type*]:
**obtains** (*fields*) *a b c d e* **where** *y* = (*a*, *b*, *c*, *d*, *e*)
⟨*proof*⟩

**lemma** *prod-induct5* [*case-names fields*, *induct type*]:
(!!*a b c d e*. *P* (*a*, *b*, *c*, *d*, *e*)) ==> *P x*
⟨*proof*⟩

**lemma** *prod-cases6* [*cases type*]:
**obtains** (*fields*) *a b c d e f* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*)
⟨*proof*⟩

**lemma** *prod-induct6* [*case-names fields*, *induct type*]:
(!!*a b c d e f*. *P* (*a*, *b*, *c*, *d*, *e*, *f*)) ==> *P x*
⟨*proof*⟩

**lemma** *prod-cases7* [*cases type*]:
**obtains** (*fields*) *a b c d e f g* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*)
⟨*proof*⟩

**lemma** *prod-induct7* [*case-names fields*, *induct type*]:
(!!*a b c d e f g*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*)) ==> *P x*
⟨*proof*⟩

### 13.4.1 Derived operations

The composition-uncurry combinator.

**notation** *fcomp* (**infixl** *o>* *60*)

**definition**
*scomp* :: (*'a* ⇒ *'b* × *'c*) ⇒ (*'b* ⇒ *'c* ⇒ *'d*) ⇒ *'a* ⇒ *'d* (**infixl** *o→* *60*)
**where**
*f o→ g* = (λ*x*. *split g* (*f x*))

**lemma** *scomp-apply*: (*f o→ g*) *x* = *split g* (*f x*)
⟨*proof*⟩

**lemma** *Pair-scomp*: *Pair x o→ f* = *f x*
⟨*proof*⟩

**lemma** *scomp-Pair*: *x o→ Pair* = *x*
⟨*proof*⟩

**lemma** *scomp-scomp*: (*f o→ g*) *o→ h* = *f o→* (λ*x*. *g x o→ h*)
⟨*proof*⟩

**lemma** *scomp-fcomp*: (*f o→ g*) *o> h* = *f o→* (λ*x*. *g x o> h*)
⟨*proof*⟩

**lemma** *fcomp-scomp*: (*f o> g*) *o→ h* = *f o>* (*g o→ h*)
⟨*proof*⟩

**no-notation** *fcomp* (**infixl** *o>* *60*)
**no-notation** *scomp* (**infixl** *o→* *60*)

*prod-fun* — action of the product functor upon functions.

**definition** *prod-fun* :: (*'a* ⇒ *'c*) ⇒ (*'b* ⇒ *'d*) ⇒ *'a* × *'b* ⇒ *'c* × *'d* **where**
[*code del*]: *prod-fun f g* = (λ(*x*, *y*). (*f x*, *g y*))

**lemma** *prod-fun* [*simp*, *code*]: *prod-fun f g* (*a*, *b*) = (*f a*, *g b*)
⟨*proof*⟩

**lemma** *prod-fun-compose*: *prod-fun* (*f1 o f2*) (*g1 o g2*) = (*prod-fun f1 g1 o prod-fun f2 g2*)
⟨*proof*⟩

**lemma** *prod-fun-ident* [*simp*]: *prod-fun* (%*x*. *x*) (%*y*. *y*) = (%*z*. *z*)
⟨*proof*⟩

**lemma** *prod-fun-imageI* [*intro*]: (*a*, *b*) : *r* ==> (*f a*, *g b*) : *prod-fun f g* ‘ *r*
⟨*proof*⟩

**lemma** *prod-fun-imageE* [*elim*!]:

**assumes** *major*: *c*: (*prod-fun f g*)'*r*
**and** *cases*: !!*x y*. [| *c*=(*f*(*x*),*g*(*y*)); (*x*,*y*):*r* |] ==> *P*
**shows** *P*
⟨*proof*⟩

**definition**
*apfst* :: ('*a* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*c* × '*b*
**where**
[*code del*]: *apfst f* = *prod-fun f id*

**definition**
*apsnd* :: ('*b* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*a* × '*c*
**where**
[*code del*]: *apsnd f* = *prod-fun id f*

**lemma** *apfst-conv* [*simp*, *code*]:
*apfst f* (*x*, *y*) = (*f x*, *y*)
⟨*proof*⟩

**lemma** *upd-snd-conv* [*simp*, *code*]:
*apsnd f* (*x*, *y*) = (*x*, *f y*)
⟨*proof*⟩

Disjoint union of a family of sets – Sigma.

**definition** *Sigma* :: ['*a set*, '*a* => '*b set*] => ('*a* × '*b*) *set* **where**
*Sigma-def*: *Sigma A B* == *UN x*:*A*. *UN y*:*B x*. {*Pair x y*}

**abbreviation**
*Times* :: ['*a set*, '*b set*] => ('*a* * '*b*) *set*
(**infixr** <*> *80*) **where**
*A* <*> *B* == *Sigma A* (%-. *B*)

**notation** (*xsymbols*)
*Times* (**infixr** × *80*)

**notation** (*HTML* **output**)
*Times* (**infixr** × *80*)

**syntax**
@*Sigma* ::[*pttrn*, '*a set*, '*b set*] => ('*a* * '*b*) *set* ((*3SIGMA* -:-./ -) [*0*, *0*, *10*] *10*)

**translations**
*SIGMA x*:*A*. *B* == *Product-Type.Sigma A* (%*x*. *B*)

**lemma** *SigmaI* [*intro*!]: [| *a*:*A*; *b*:*B*(*a*) |] ==> (*a*,*b*) : *Sigma A B*
⟨*proof*⟩

**lemma** *SigmaE* [*elim*!]:
[| *c*: *Sigma A B*;

!!$x$ $y$.[| $x$:$A$; $y$:$B(x)$; $c$=$(x,y)$ |] ==> $P$
|] ==> $P$
— The general elimination rule.
⟨*proof*⟩

Elimination of $(a,\ b) \in A \times B$ – introduces no eigenvariables.

**lemma** *SigmaD1*: $(a,\ b)$ : *Sigma* $A$ $B$ ==> $a$ : $A$
⟨*proof*⟩

**lemma** *SigmaD2*: $(a,\ b)$ : *Sigma* $A$ $B$ ==> $b$ : $B$ $a$
⟨*proof*⟩

**lemma** *SigmaE2*:
[| $(a,\ b)$ : *Sigma* $A$ $B$;
[| $a$:$A$; $b$:$B(a)$ |] ==> $P$
|] ==> $P$
⟨*proof*⟩

**lemma** *Sigma-cong*:
⟦$A = B$; !!$x$. $x \in B \Longrightarrow C\ x = D\ x$⟧
$\Longrightarrow$ (*SIGMA* $x$: $A$. $C\ x$) = (*SIGMA* $x$: $B$. $D\ x$)
⟨*proof*⟩

**lemma** *Sigma-mono*: [| $A$ <= $C$; !!$x$. $x$:$A$ ==> $B\ x$ <= $D\ x$ |] ==> *Sigma* $A$ $B$ <= *Sigma* $C$ $D$
⟨*proof*⟩

**lemma** *Sigma-empty1* [*simp*]: *Sigma* {} $B$ = {}
⟨*proof*⟩

**lemma** *Sigma-empty2* [*simp*]: $A$ <*> {} = {}
⟨*proof*⟩

**lemma** *UNIV-Times-UNIV* [*simp*]: *UNIV* <*> *UNIV* = *UNIV*
⟨*proof*⟩

**lemma** *Compl-Times-UNIV1* [*simp*]: − (*UNIV* <*> $A$) = *UNIV* <*> (−$A$)
⟨*proof*⟩

**lemma** *Compl-Times-UNIV2* [*simp*]: − ($A$ <*> *UNIV*) = (−$A$) <*> *UNIV*
⟨*proof*⟩

**lemma** *mem-Sigma-iff* [*iff*]: (($a$,$b$): *Sigma* $A$ $B$) = ($a$:$A$ & $b$:$B(a)$)
⟨*proof*⟩

**lemma** *Times-subset-cancel2*: $x$:$C$ ==> ($A$ <*> $C$ <= $B$ <*> $C$) = ($A$ <= $B$)
⟨*proof*⟩

**lemma** *Times-eq-cancel2*: $x$:$C$ ==> ($A$ <*> $C$ = $B$ <*> $C$) = ($A$ = $B$)

⟨*proof*⟩

**lemma** *SetCompr-Sigma-eq*:
*Collect* (*split* (%*x y*. *P x* & *Q x y*)) = (*SIGMA x*:*Collect P*. *Collect* (*Q x*))
⟨*proof*⟩

**lemma** *Collect-split* [*simp*]: {(*a*,*b*). *P a* & *Q b*} = *Collect P* <*∗*> *Collect Q*
⟨*proof*⟩

**lemma** *UN-Times-distrib*:
(*UN* (*a*,*b*):(*A* <*∗*> *B*). *E a* <*∗*> *F b*) = (*UNION A E*) <*∗*> (*UNION B F*)
— Suggested by Pierre Chartier
⟨*proof*⟩

**lemma** *split-paired-Ball-Sigma* [*simp*,*noatp*]:
(*ALL z*: *Sigma A B*. *P z*) = (*ALL x*:*A*. *ALL y*: *B x*. *P*(*x*,*y*))
⟨*proof*⟩

**lemma** *split-paired-Bex-Sigma* [*simp*,*noatp*]:
(*EX z*: *Sigma A B*. *P z*) = (*EX x*:*A*. *EX y*: *B x*. *P*(*x*,*y*))
⟨*proof*⟩

**lemma** *Sigma-Un-distrib1*: (*SIGMA i*:*I Un J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) *Un* (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

**lemma** *Sigma-Un-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *Un B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) *Un* (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

**lemma** *Sigma-Int-distrib1*: (*SIGMA i*:*I Int J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) *Int* (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

**lemma** *Sigma-Int-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *Int B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) *Int* (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

**lemma** *Sigma-Diff-distrib1*: (*SIGMA i*:*I − J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) − (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

**lemma** *Sigma-Diff-distrib2*: (*SIGMA i*:*I*. *A*(*i*) − *B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) − (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

**lemma** *Sigma-Union*: *Sigma* (*Union X*) *B* = (*UN A*:*X*. *Sigma A B*)
⟨*proof*⟩

Non-dependent versions are needed to avoid the need for higher-order match-

ing, especially when the rules are re-oriented.

**lemma** *Times-Un-distrib1*: $(A\ Un\ B) <*> C = (A <*> C)\ Un\ (B <*> C)$
⟨*proof*⟩

**lemma** *Times-Int-distrib1*: $(A\ Int\ B) <*> C = (A <*> C)\ Int\ (B <*> C)$
⟨*proof*⟩

**lemma** *Times-Diff-distrib1*: $(A - B) <*> C = (A <*> C) - (B <*> C)$
⟨*proof*⟩

**lemma** *insert-times-insert*[*simp*]:
$insert\ a\ A \times insert\ b\ B =$
$insert\ (a,b)\ (A \times insert\ b\ B \cup insert\ a\ A \times B)$
⟨*proof*⟩

### 13.4.2 Code generator setup

**instance** $* :: (eq,\ eq)\ eq$ ⟨*proof*⟩

**lemma** [*code*]:
$eq\text{-}class.eq\ (x1::'a::eq,\ y1::'b::eq)\ (x2,\ y2) \longleftrightarrow x1 = x2 \wedge y1 = y2$ ⟨*proof*⟩

**lemma** *split-case-cert*:
**assumes** $CASE \equiv split\ f$
**shows** $CASE\ (a,\ b) \equiv f\ a\ b$
⟨*proof*⟩

⟨*ML*⟩

**code-type** $*$
(*SML* **infix** *2* $*$)
(*OCaml* **infix** *2* $*$)
(*Haskell* !((-),/ (-)))

**code-instance** $* :: eq$
(*Haskell* −)

**code-const** $eq\text{-}class.eq :: 'a::eq \times 'b::eq \Rightarrow 'a \times 'b \Rightarrow bool$
(*Haskell* **infixl** *4* ==)

**code-const** *Pair*
(*SML* !((-),/ (-)))
(*OCaml* !((-),/ (-)))
(*Haskell* !((-),/ (-)))

**code-const** *fst* **and** *snd*
(*Haskell fst* **and** *snd*)

**types-code**

```
  *     ((- */ -))
attach (term-of) ⟪
fun term-of-id-42 aF aT bF bT (x, y) = HOLogic.pair-const aT bT $ aF x $ bF
y;
⟫
attach (test) ⟪
fun gen-id-42 aG aT bG bT i =
  let
    val (x, t) = aG i;
    val (y, u) = bG i
  in ((x, y), fn () => HOLogic.pair-const aT bT $ t () $ u ()) end;
⟫

consts-code
  Pair    ((-,/ -))
```

⟨*ML*⟩

### 13.5 Legacy bindings

⟨*ML*⟩

### 13.6 Further inductive packages

⟨*ML*⟩

**end**

## 14 Datatype: Analogues of the Cartesian Product and Disjoint Sum for Datatypes

```
theory Datatype
imports Nat Product-Type
begin

typedef (Node)
  ('a,'b) node = {p. EX f x k. p = (f::nat=>'b+nat, x::'a+nat) & f k = Inr 0}
    — it is a subtype of (nat=>'b+nat) * ('a+nat)
  ⟨proof⟩
```

Datatypes will be represented by sets of type *node*

```
types 'a item        = ('a, unit) node set
      ('a, 'b) dtree = ('a, 'b) node set

consts
  Push      :: [('b + nat), nat => ('b + nat)] => (nat => ('b + nat))
```

*Push-Node* :: $[('b + nat), ('a, 'b)\ node] \Rightarrow ('a, 'b)\ node$
*ndepth* :: $('a, 'b)\ node \Rightarrow nat$

*Atom* :: $('a + nat) \Rightarrow ('a, 'b)\ dtree$
*Leaf* :: $'a \Rightarrow ('a, 'b)\ dtree$
*Numb* :: $nat \Rightarrow ('a, 'b)\ dtree$
*Scons* :: $[('a, 'b)\ dtree, ('a, 'b)\ dtree] \Rightarrow ('a, 'b)\ dtree$
*In0* :: $('a, 'b)\ dtree \Rightarrow ('a, 'b)\ dtree$
*In1* :: $('a, 'b)\ dtree \Rightarrow ('a, 'b)\ dtree$
*Lim* :: $('b \Rightarrow ('a, 'b)\ dtree) \Rightarrow ('a, 'b)\ dtree$

*ntrunc* :: $[nat, ('a, 'b)\ dtree] \Rightarrow ('a, 'b)\ dtree$

*uprod* :: $[('a, 'b)\ dtree\ set, ('a, 'b)\ dtree\ set] \Rightarrow ('a, 'b)\ dtree\ set$
*usum* :: $[('a, 'b)\ dtree\ set, ('a, 'b)\ dtree\ set] \Rightarrow ('a, 'b)\ dtree\ set$

*Split* :: $[[('a, 'b)\ dtree, ('a, 'b)\ dtree] \Rightarrow 'c, ('a, 'b)\ dtree] \Rightarrow 'c$
*Case* :: $[[('a, 'b)\ dtree] \Rightarrow 'c, [('a, 'b)\ dtree] \Rightarrow 'c, ('a, 'b)\ dtree] \Rightarrow 'c$

*dprod* :: $[(('a, 'b)\ dtree * ('a, 'b)\ dtree)set, (('a, 'b)\ dtree * ('a, 'b)\ dtree)set] \Rightarrow (('a, 'b)\ dtree * ('a, 'b)\ dtree)set$
*dsum* :: $[(('a, 'b)\ dtree * ('a, 'b)\ dtree)set, (('a, 'b)\ dtree * ('a, 'b)\ dtree)set] \Rightarrow (('a, 'b)\ dtree * ('a, 'b)\ dtree)set$

**defs**

*Push-Node-def*: *Push-Node* == (%*n x*. *Abs-Node* (*apfst* (*Push n*) (*Rep-Node x*)))

*Push-def*: *Push* == (%*b h*. *nat-case b h*)

*Atom-def*: *Atom* == (%*x*. {*Abs-Node*((%*k*. *Inr 0*, *x*))})
*Scons-def*: *Scons M N* == (*Push-Node* (*Inr 1*) ‘ *M*) *Un* (*Push-Node* (*Inr* (*Suc 1*)) ‘ *N*)

*Leaf-def*: *Leaf* == *Atom o Inl*
*Numb-def*: *Numb* == *Atom o Inr*

*In0-def*: *In0*(*M*) == *Scons* (*Numb 0*) *M*
*In1-def*: *In1*(*M*) == *Scons* (*Numb 1*) *M*

*Lim-def*: $Lim\ f == Union\ \{z.\ ?\ x.\ z = Push\text{-}Node\ (Inl\ x)\ `\ (f\ x)\}$

*ndepth-def*: $ndepth(n) == (\%(f,x).\ LEAST\ k.\ f\ k = Inr\ 0)\ (Rep\text{-}Node\ n)$
*ntrunc-def*: $ntrunc\ k\ N == \{n.\ n{:}N\ \&\ ndepth(n){<}k\}$

*uprod-def*: $uprod\ A\ B == UN\ x{:}A.\ UN\ y{:}B.\ \{\ Scons\ x\ y\ \}$
*usum-def*: $usum\ A\ B == In0`A\ Un\ In1`B$

*Split-def*: $Split\ c\ M == THE\ u.\ EX\ x\ y.\ M = Scons\ x\ y\ \&\ u = c\ x\ y$

*Case-def*: $Case\ c\ d\ M == THE\ u.\ (EX\ x\ .\ M = In0(x)\ \&\ u = c(x))$
$\quad | (EX\ y\ .\ M = In1(y)\ \&\ u = d(y))$

*dprod-def*: $dprod\ r\ s == UN\ (x,x'){:}r.\ UN\ (y,y'){:}s.\ \{(Scons\ x\ y,\ Scons\ x'\ y')\}$

*dsum-def*: $dsum\ r\ s == (UN\ (x,x'){:}r.\ \{(In0(x),In0(x'))\})\ Un$
$\quad (UN\ (y,y'){:}s.\ \{(In1(y),In1(y'))\})$

**lemma** *apfst-convE*:
$[|\ q = apfst\ f\ p;\ \ !!x\ y.\ [|\ p = (x,y);\ \ q = (f(x),y)\ |] ==> R\ |] ==> R$
⟨*proof*⟩

**lemma** *Push-inject1*: $Push\ i\ f = Push\ j\ g\ ==> i{=}j$
⟨*proof*⟩

**lemma** *Push-inject2*: $Push\ i\ f = Push\ j\ g\ ==> f{=}g$
⟨*proof*⟩

**lemma** *Push-inject*:
$[|\ Push\ i\ f\ {=}Push\ j\ g;\ \ [|\ i{=}j;\ \ f{=}g\ |] ==> P\ |] ==> P$
⟨*proof*⟩

**lemma** *Push-neq-K0*: $Push\ (Inr\ (Suc\ k))\ f = (\%z.\ Inr\ 0) ==> P$
⟨*proof*⟩

**lemmas** *Abs-Node-inj* = *Abs-Node-inject* [*THEN* [*2*] *rev-iffD1*, *standard*]

**lemma** *Node-K0-I*: $(\%k.\ Inr\ 0,\ a) : Node$
⟨*proof*⟩

**lemma** *Node-Push-I*: $p$: $Node \Longrightarrow apfst\ (Push\ i)\ p : Node$
⟨*proof*⟩

## 14.1 Freeness: Distinctness of Constructors

**lemma** *Scons-not-Atom* [*iff*]: $Scons\ M\ N \neq Atom(a)$
⟨*proof*⟩

**lemmas** *Atom-not-Scons* [*iff*] = *Scons-not-Atom* [*THEN not-sym*, *standard*]

**lemma** *inj-Atom*: $inj(Atom)$
⟨*proof*⟩
**lemmas** *Atom-inject* = *inj-Atom* [*THEN injD*, *standard*]

**lemma** *Atom-Atom-eq* [*iff*]: $(Atom(a)=Atom(b)) = (a=b)$
⟨*proof*⟩

**lemma** *inj-Leaf*: $inj(Leaf)$
⟨*proof*⟩

**lemmas** *Leaf-inject* [*dest*!] = *inj-Leaf* [*THEN injD*, *standard*]

**lemma** *inj-Numb*: $inj(Numb)$
⟨*proof*⟩

**lemmas** *Numb-inject* [*dest*!] = *inj-Numb* [*THEN injD*, *standard*]

**lemma** *Push-Node-inject*:
$[|\ Push\text{-}Node\ i\ m = Push\text{-}Node\ j\ n;\ \ [|\ i=j;\ \ m=n\ |] \Longrightarrow P\ |] \Longrightarrow P$
⟨*proof*⟩

**lemma** *Scons-inject-lemma1*: $Scons\ M\ N <= Scons\ M'\ N' \Longrightarrow M<=M'$

⟨*proof*⟩

**lemma** *Scons-inject-lemma2*: *Scons M N <= Scons M' N' ==> N<=N'*
⟨*proof*⟩

**lemma** *Scons-inject1*: *Scons M N = Scons M' N' ==> M=M'*
⟨*proof*⟩

**lemma** *Scons-inject2*: *Scons M N = Scons M' N' ==> N=N'*
⟨*proof*⟩

**lemma** *Scons-inject*:
[| *Scons M N = Scons M' N'*; [| *M=M'*; *N=N'* |] ==> *P* |] ==> *P*
⟨*proof*⟩

**lemma** *Scons-Scons-eq* [*iff*]: (*Scons M N = Scons M' N'*) = (*M=M'* & *N=N'*)
⟨*proof*⟩

**lemma** *Scons-not-Leaf* [*iff*]: *Scons M N* ≠ *Leaf*(*a*)
⟨*proof*⟩

**lemmas** *Leaf-not-Scons* [*iff*] = *Scons-not-Leaf* [*THEN not-sym*, *standard*]

**lemma** *Scons-not-Numb* [*iff*]: *Scons M N* ≠ *Numb*(*k*)
⟨*proof*⟩

**lemmas** *Numb-not-Scons* [*iff*] = *Scons-not-Numb* [*THEN not-sym*, *standard*]

**lemma** *Leaf-not-Numb* [*iff*]: *Leaf*(*a*) ≠ *Numb*(*k*)
⟨*proof*⟩

**lemmas** *Numb-not-Leaf* [*iff*] = *Leaf-not-Numb* [*THEN not-sym*, *standard*]

**lemma** *ndepth-K0*: *ndepth* (*Abs-Node*(%*k*. *Inr 0*, *x*)) = *0*
⟨*proof*⟩

**lemma** *ndepth-Push-Node-aux*:

*nat-case* (*Inr* (*Suc i*)) *f k* = *Inr 0* −−> *Suc*(*LEAST x*. *f x* = *Inr 0*) <= *k*
⟨*proof*⟩

**lemma** *ndepth-Push-Node*:
*ndepth* (*Push-Node* (*Inr* (*Suc i*)) *n*) = *Suc*(*ndepth*(*n*))
⟨*proof*⟩

**lemma** *ntrunc-0* [*simp*]: *ntrunc 0 M* = {}
⟨*proof*⟩

**lemma** *ntrunc-Atom* [*simp*]: *ntrunc* (*Suc k*) (*Atom a*) = *Atom*(*a*)
⟨*proof*⟩

**lemma** *ntrunc-Leaf* [*simp*]: *ntrunc* (*Suc k*) (*Leaf a*) = *Leaf*(*a*)
⟨*proof*⟩

**lemma** *ntrunc-Numb* [*simp*]: *ntrunc* (*Suc k*) (*Numb i*) = *Numb*(*i*)
⟨*proof*⟩

**lemma** *ntrunc-Scons* [*simp*]:
*ntrunc* (*Suc k*) (*Scons M N*) = *Scons* (*ntrunc k M*) (*ntrunc k N*)
⟨*proof*⟩

**lemma** *ntrunc-one-In0* [*simp*]: *ntrunc* (*Suc 0*) (*In0 M*) = {}
⟨*proof*⟩

**lemma** *ntrunc-In0* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In0 M*) = *In0* (*ntrunc* (*Suc k*) *M*)
⟨*proof*⟩

**lemma** *ntrunc-one-In1* [*simp*]: *ntrunc* (*Suc 0*) (*In1 M*) = {}
⟨*proof*⟩

**lemma** *ntrunc-In1* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In1 M*) = *In1* (*ntrunc* (*Suc k*) *M*)
⟨*proof*⟩

## 14.2 Set Constructions

**lemma** *uprodI* [*intro*!]: [| *M*:*A*; *N*:*B* |] ==> *Scons M N* : *uprod A B*
⟨*proof*⟩

**lemma** *uprodE* [*elim*!]:
[| $c$ : *uprod* $A$ $B$;
!!$x$ $y$. [| $x$:$A$; $y$:$B$; $c$ = *Scons* $x$ $y$ |] ==> $P$
|] ==> $P$
⟨*proof*⟩

**lemma** *uprodE2*: [| *Scons* $M$ $N$ : *uprod* $A$ $B$; [| $M$:$A$; $N$:$B$ |] ==> $P$ |] ==> $P$
⟨*proof*⟩

**lemma** *usum-In0I* [*intro*]: $M$:$A$ ==> *In0*($M$) : *usum* $A$ $B$
⟨*proof*⟩

**lemma** *usum-In1I* [*intro*]: $N$:$B$ ==> *In1*($N$) : *usum* $A$ $B$
⟨*proof*⟩

**lemma** *usumE* [*elim*!]:
[| $u$ : *usum* $A$ $B$;
!!$x$. [| $x$:$A$; $u$=*In0*($x$) |] ==> $P$;
!!$y$. [| $y$:$B$; $u$=*In1*($y$) |] ==> $P$
|] ==> $P$
⟨*proof*⟩

**lemma** *In0-not-In1* [*iff*]: *In0*($M$) ≠ *In1*($N$)
⟨*proof*⟩

**lemmas** *In1-not-In0* [*iff*] = *In0-not-In1* [*THEN not-sym*, *standard*]

**lemma** *In0-inject*: *In0*($M$) = *In0*($N$) ==> $M$=$N$
⟨*proof*⟩

**lemma** *In1-inject*: *In1*($M$) = *In1*($N$) ==> $M$=$N$
⟨*proof*⟩

**lemma** *In0-eq* [*iff*]: (*In0* $M$ = *In0* $N$) = ($M$=$N$)
⟨*proof*⟩

**lemma** *In1-eq* [*iff*]: (*In1* $M$ = *In1* $N$) = ($M$=$N$)
⟨*proof*⟩

**lemma** *inj-In0*: *inj In0*

⟨*proof*⟩

**lemma** *inj-In1*: *inj In1*
⟨*proof*⟩

**lemma** *Lim-inject*: *Lim f = Lim g ==> f = g*
⟨*proof*⟩

**lemma** *ntrunc-subsetI*: *ntrunc k M <= M*
⟨*proof*⟩

**lemma** *ntrunc-subsetD*: *(!!k. ntrunc k M <= N) ==> M<=N*
⟨*proof*⟩

**lemma** *ntrunc-equality*: *(!!k. ntrunc k M = ntrunc k N) ==> M=N*
⟨*proof*⟩

**lemma** *ntrunc-o-equality*:
*[| !!k. (ntrunc(k) o h1) = (ntrunc(k) o h2) |] ==> h1=h2*
⟨*proof*⟩

**lemma** *uprod-mono*: *[| A<=A';  B<=B' |] ==> uprod A B <= uprod A' B'*
⟨*proof*⟩

**lemma** *usum-mono*: *[| A<=A';  B<=B' |] ==> usum A B <= usum A' B'*
⟨*proof*⟩

**lemma** *Scons-mono*: *[| M<=M';  N<=N' |] ==> Scons M N <= Scons M' N'*
⟨*proof*⟩

**lemma** *In0-mono*: *M<=N ==> In0(M) <= In0(N)*
⟨*proof*⟩

**lemma** *In1-mono*: *M<=N ==> In1(M) <= In1(N)*
⟨*proof*⟩

**lemma** *Split* [*simp*]: *Split c* (*Scons M N*) = *c M N*
⟨*proof*⟩

**lemma** *Case-In0* [*simp*]: *Case c d* (*In0 M*) = *c*(*M*)
⟨*proof*⟩

**lemma** *Case-In1* [*simp*]: *Case c d* (*In1 N*) = *d*(*N*)
⟨*proof*⟩

**lemma** *ntrunc-UN1*: *ntrunc k* (*UN x. f*(*x*)) = (*UN x. ntrunc k* (*f x*))
⟨*proof*⟩

**lemma** *Scons-UN1-x*: *Scons* (*UN x. f x*) *M* = (*UN x. Scons* (*f x*) *M*)
⟨*proof*⟩

**lemma** *Scons-UN1-y*: *Scons M* (*UN x. f x*) = (*UN x. Scons M* (*f x*))
⟨*proof*⟩

**lemma** *In0-UN1*: *In0*(*UN x. f*(*x*)) = (*UN x. In0*(*f*(*x*)))
⟨*proof*⟩

**lemma** *In1-UN1*: *In1*(*UN x. f*(*x*)) = (*UN x. In1*(*f*(*x*)))
⟨*proof*⟩

**lemma** *dprodI* [*intro*!]:
[| (*M*,*M'*):*r*; (*N*,*N'*):*s* |] ==> (*Scons M N*, *Scons M' N'*) : *dprod r s*
⟨*proof*⟩

**lemma** *dprodE* [*elim*!]:
[| *c* : *dprod r s*;
!!*x y x' y'*. [| (*x*,*x'*) : *r*; (*y*,*y'*) : *s*;
*c* = (*Scons x y*, *Scons x' y'*) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

**lemma** *dsum-In0I* [*intro*]: (*M*,*M'*):*r* ==> (*In0*(*M*), *In0*(*M'*)) : *dsum r s*
⟨*proof*⟩

**lemma** *dsum-In1I* [*intro*]: (*N*,*N'*):*s* ==> (*In1*(*N*), *In1*(*N'*)) : *dsum r s*
⟨*proof*⟩

**lemma** *dsumE* [*elim*!]:
[| *w* : *dsum r s*;
!!*x x'*. [| (*x*,*x'*) : *r*; *w* = (*In0*(*x*), *In0*(*x'*)) |] ==> *P*;
!!*y y'*. [| (*y*,*y'*) : *s*; *w* = (*In1*(*y*), *In1*(*y'*)) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

**lemma** *dprod-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dprod r s* <= *dprod r' s'*
⟨*proof*⟩

**lemma** *dsum-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dsum r s* <= *dsum r' s'*
⟨*proof*⟩

**lemma** *dprod-Sigma*: (*dprod* (*A* <*> *B*) (*C* <*> *D*)) <= (*uprod A C*) <*> (*uprod B D*)
⟨*proof*⟩

**lemmas** *dprod-subset-Sigma* = *subset-trans* [*OF dprod-mono dprod-Sigma*, *standard*]

**lemma** *dprod-subset-Sigma2*:
(*dprod* (*Sigma A B*) (*Sigma C D*)) <=
*Sigma* (*uprod A C*) (*Split* (%*x y*. *uprod* (*B x*) (*D y*)))
⟨*proof*⟩

**lemma** *dsum-Sigma*: (*dsum* (*A* <*> *B*) (*C* <*> *D*)) <= (*usum A C*) <*> (*usum B D*)
⟨*proof*⟩

**lemmas** *dsum-subset-Sigma* = *subset-trans* [*OF dsum-mono dsum-Sigma*, *standard*]

hides popular names

**hide** (**open**) *type node item*
**hide** (**open**) *const Push Node Atom Leaf Numb Lim Split Case*

# 15 Datatypes

## 15.1 Representing sums

**rep-datatype** (*sum*) *Inl Inr*
⟨*proof*⟩

**lemma** *sum-case-KK*[*simp*]: *sum-case* (%*x*. *a*) (%*x*. *a*) = (%*x*. *a*)
⟨*proof*⟩

**lemma** *surjective-sum*: *sum-case* (%*x*::'*a*. *f* (*Inl x*)) (%*y*::'*b*. *f* (*Inr y*)) *s* = *f*(*s*)
⟨*proof*⟩

**lemma** *sum-case-weak-cong*: *s* = *t* ==> *sum-case f g s* = *sum-case f g t*
— Prevents simplification of *f* and *g*: much faster.
⟨*proof*⟩

**lemma** *sum-case-inject*:
*sum-case f1 f2* = *sum-case g1 g2* ==> (*f1* = *g1* ==> *f2* = *g2* ==> *P*) ==> *P*
⟨*proof*⟩

**constdefs**
*Suml* :: ('*a* => '*c*) => '*a* + '*b* => '*c*
*Suml* == (%*f*. *sum-case f undefined*)

*Sumr* :: ('*b* => '*c*) => '*a* + '*b* => '*c*
*Sumr* == *sum-case undefined*

**lemma** *Suml-inject*: *Suml f* = *Suml g* ==> *f* = *g*
⟨*proof*⟩

**lemma** *Sumr-inject*: *Sumr f* = *Sumr g* ==> *f* = *g*
⟨*proof*⟩

**primrec** *Projl* :: '*a* + '*b* => '*a*
**where** *Projl-Inl*: *Projl* (*Inl x*) = *x*

**primrec** *Projr* :: '*a* + '*b* => '*b*
**where** *Projr-Inr*: *Projr* (*Inr x*) = *x*

**hide** (**open**) *const Suml Sumr Projl Projr*

**end**

# 16 Power: Exponentiation

**theory** *Power*

**imports** *Nat*
**begin**

**class** *power* =
  **fixes** *power* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ^ *80*)

## 16.1 Powers for Arbitrary Monoids

**class** *recpower* = *monoid-mult* + *power* +
  **assumes** *power-0* [*simp*]: *a ^ 0 = 1*
  **assumes** *power-Suc* [*simp*]: *a ^ Suc n = a * (a ^ n)*

**lemma** *power-0-Suc* [*simp*]: *(0::'a::{recpower,semiring-0}) ^ (Suc n) = 0*
  ⟨*proof*⟩

It looks plausible as a simprule, but its effect can be strange.

**lemma** *power-0-left*: *0^n = (if n=0 then 1 else (0::'a::{recpower,semiring-0}))*
  ⟨*proof*⟩

**lemma** *power-one* [*simp*]: *1^n = (1::'a::recpower)*
  ⟨*proof*⟩

**lemma** *power-one-right* [*simp*]: *(a::'a::recpower) ^ 1 = a*
  ⟨*proof*⟩

**lemma** *power-commutes*: *(a::'a::recpower) ^ n * a = a * a ^ n*
  ⟨*proof*⟩

**lemma** *power-Suc2*: *(a::'a::recpower) ^ Suc n = a ^ n * a*
  ⟨*proof*⟩

**lemma** *power-add*: *(a::'a::recpower) ^ (m+n) = (a^m) * (a^n)*
  ⟨*proof*⟩

**lemma** *power-mult*: *(a::'a::recpower) ^ (m*n) = (a^m) ^ n*
  ⟨*proof*⟩

**lemma** *power-mult-distrib*: *((a::'a::{recpower,comm-monoid-mult}) * b) ^ n = (a^n) * (b^n)*
  ⟨*proof*⟩

**lemma** *zero-less-power*[*simp*]:
    *0 < (a::'a::{ordered-semidom,recpower}) ==> 0 < a^n*
⟨*proof*⟩

**lemma** *zero-le-power*[*simp*]:
    *0 ≤ (a::'a::{ordered-semidom,recpower}) ==> 0 ≤ a^n*
⟨*proof*⟩

**lemma** *one-le-power*[*simp*]:
*1* ≤ (*a*::*'a*::{*ordered-semidom*,*recpower*}) ==> *1* ≤ *a* ^ *n*
⟨*proof*⟩

**lemma** *gt1-imp-ge0*: *1* < *a* ==> *0* ≤ (*a*::*'a*::*ordered-semidom*)
⟨*proof*⟩

**lemma** *power-gt1-lemma*:
**assumes** *gt1*: *1* < (*a*::*'a*::{*ordered-semidom*,*recpower*})
**shows** *1* < *a* * *a* ^ *n*
⟨*proof*⟩

**lemma** *one-less-power*[*simp*]:
⟦*1* < (*a*::*'a*::{*ordered-semidom*,*recpower*}); *0* < *n*⟧ ⟹ *1* < *a* ^ *n*
⟨*proof*⟩

**lemma** *power-gt1*:
*1* < (*a*::*'a*::{*ordered-semidom*,*recpower*}) ==> *1* < *a* ^ (*Suc n*)
⟨*proof*⟩

**lemma** *power-le-imp-le-exp*:
**assumes** *gt1*: (*1*::*'a*::{*recpower*,*ordered-semidom*}) < *a*
**shows** !!*n*. *a* ^ *m* ≤ *a* ^ *n* ==> *m* ≤ *n*
⟨*proof*⟩

Surely we can strengthen this? It holds for *0*<*a*<*1* too.

**lemma** *power-inject-exp* [*simp*]:
*1* < (*a*::*'a*::{*ordered-semidom*,*recpower*}) ==> (*a* ^ *m* = *a* ^ *n*) = (*m*=*n*)
⟨*proof*⟩

Can relax the first premise to (*0*::*'a*) < *a* in the case of the natural numbers.

**lemma** *power-less-imp-less-exp*:
[| (*1*::*'a*::{*recpower*,*ordered-semidom*}) < *a*; *a* ^ *m* < *a* ^ *n* |] ==> *m* < *n*
⟨*proof*⟩

**lemma** *power-mono*:
[|*a* ≤ *b*; (*0*::*'a*::{*recpower*,*ordered-semidom*}) ≤ *a*|] ==> *a* ^ *n* ≤ *b* ^ *n*
⟨*proof*⟩

**lemma** *power-strict-mono* [*rule-format*]:
[|*a* < *b*; (*0*::*'a*::{*recpower*,*ordered-semidom*}) ≤ *a*|]
==> *0* < *n* --> *a* ^ *n* < *b* ^ *n*
⟨*proof*⟩

**lemma** *power-eq-0-iff* [*simp*]:
(*a* ^ *n* = *0*) ⟷
(*a* = (*0*::*'a*::{*mult-zero*,*zero-neq-one*,*no-zero-divisors*,*recpower*}) & *n*≠*0*)
⟨*proof*⟩

**lemma** *field-power-not-zero*:
$a \neq (0::'a::\{ring\text{-}1\text{-}no\text{-}zero\text{-}divisors,recpower\}) ==> a\hat{}n \neq 0$
⟨*proof*⟩

**lemma** *nonzero-power-inverse*:
**fixes** $a :: 'a::\{division\text{-}ring,recpower\}$
**shows** $a \neq 0 ==> inverse\ (a\ \hat{}\ n) = (inverse\ a)\ \hat{}\ n$
⟨*proof*⟩

Perhaps these should be simprules.

**lemma** *power-inverse*:
**fixes** $a :: 'a::\{division\text{-}ring,division\text{-}by\text{-}zero,recpower\}$
**shows** $inverse\ (a\ \hat{}\ n) = (inverse\ a)\ \hat{}\ n$
⟨*proof*⟩

**lemma** *power-one-over*: $1\ /\ (a::'a::\{field,division\text{-}by\text{-}zero,recpower\})\hat{}n =$
$(1\ /\ a)\hat{}n$
⟨*proof*⟩

**lemma** *nonzero-power-divide*:
$b \neq 0 ==> (a/b)\ \hat{}\ n = ((a::'a::\{field,recpower\})\ \hat{}\ n)\ /\ (b\ \hat{}\ n)$
⟨*proof*⟩

**lemma** *power-divide*:
$(a/b)\ \hat{}\ n = ((a::'a::\{field,division\text{-}by\text{-}zero,recpower\})\ \hat{}\ n\ /\ b\ \hat{}\ n)$
⟨*proof*⟩

**lemma** *power-abs*: $abs(a\ \hat{}\ n) = abs(a::'a::\{ordered\text{-}idom,recpower\})\ \hat{}\ n$
⟨*proof*⟩

**lemma** *abs-power-minus* [*simp*]:
**fixes** $a:: 'a::\{ordered\text{-}idom,recpower\}$ **shows** $abs((-a)\ \hat{}\ n) = abs(a\ \hat{}\ n)$
⟨*proof*⟩

**lemma** *zero-less-power-abs-iff* [*simp*,*noatp*]:
$(0 < (abs\ a)\hat{}n) = (a \neq (0::'a::\{ordered\text{-}idom,recpower\}) \mid n=0)$
⟨*proof*⟩

**lemma** *zero-le-power-abs* [*simp*]:
$(0::'a::\{ordered\text{-}idom,recpower\}) \leq (abs\ a)\hat{}n$
⟨*proof*⟩

**lemma** *power-minus*: $(-a)\ \hat{}\ n = (-\ 1)\hat{}n * (a::'a::\{ring\text{-}1,recpower\})\ \hat{}\ n$
⟨*proof*⟩

Lemma for *power-strict-decreasing*

**lemma** *power-Suc-less*:

$[|(0::'a::\{ordered\text{-}semidom,recpower\}) < a;\ a < 1|]$
$\Longrightarrow a * a\hat{}n < a\hat{}n$
⟨*proof*⟩

**lemma** *power-strict-decreasing*:
$[|n < N;\ 0 < a;\ a < (1::'a::\{ordered\text{-}semidom,recpower\})|]$
$\Longrightarrow a\hat{}N < a\hat{}n$
⟨*proof*⟩

Proof resembles that of *power-strict-decreasing*

**lemma** *power-decreasing*:
$[|n \leq N;\ 0 \leq a;\ a \leq (1::'a::\{ordered\text{-}semidom,recpower\})|]$
$\Longrightarrow a\hat{}N \leq a\hat{}n$
⟨*proof*⟩

**lemma** *power-Suc-less-one*:
$[|\ 0 < a;\ a < (1::'a::\{ordered\text{-}semidom,recpower\})\ |] \Longrightarrow a\ \hat{}\ Suc\ n < 1$
⟨*proof*⟩

Proof again resembles that of *power-strict-decreasing*

**lemma** *power-increasing*:
$[|n \leq N;\ (1::'a::\{ordered\text{-}semidom,recpower\}) \leq a|] \Longrightarrow a\hat{}n \leq a\hat{}N$
⟨*proof*⟩

Lemma for *power-strict-increasing*

**lemma** *power-less-power-Suc*:
$(1::'a::\{ordered\text{-}semidom,recpower\}) < a \Longrightarrow a\hat{}n < a * a\hat{}n$
⟨*proof*⟩

**lemma** *power-strict-increasing*:
$[|n < N;\ (1::'a::\{ordered\text{-}semidom,recpower\}) < a|] \Longrightarrow a\hat{}n < a\hat{}N$
⟨*proof*⟩

**lemma** *power-increasing-iff* [*simp*]:
$1 < (b::'a::\{ordered\text{-}semidom,recpower\}) \Longrightarrow (b\ \hat{}\ x \leq b\ \hat{}\ y) = (x \leq y)$
⟨*proof*⟩

**lemma** *power-strict-increasing-iff* [*simp*]:
$1 < (b::'a::\{ordered\text{-}semidom,recpower\}) \Longrightarrow (b\ \hat{}\ x < b\ \hat{}\ y) = (x < y)$
⟨*proof*⟩

**lemma** *power-le-imp-le-base*:
**assumes** *le*: $a\ \hat{}\ Suc\ n \leq b\ \hat{}\ Suc\ n$
**and** *ynonneg*: $(0::'a::\{ordered\text{-}semidom,recpower\}) \leq b$
**shows** $a \leq b$
⟨*proof*⟩

**lemma** *power-less-imp-less-base*:
**fixes** $a\ b :: 'a::\{ordered\text{-}semidom,recpower\}$

**assumes** *less*: $a \hat{} n < b \hat{} n$
**assumes** *nonneg*: $0 \leq b$
**shows** $a < b$
⟨*proof*⟩

**lemma** *power-inject-base*:
$[\!| a \hat{} Suc\ n = b \hat{} Suc\ n; 0 \leq a; 0 \leq b |\!] ==> a = (b::'a::\{ordered\text{-}semidom,recpower\})$
⟨*proof*⟩

**lemma** *power-eq-imp-eq-base*:
**fixes** $a\ b :: 'a::\{ordered\text{-}semidom,recpower\}$
**shows** $[\![a \hat{} n = b \hat{} n; 0 \leq a; 0 \leq b; 0 < n]\!] \Longrightarrow a = b$
⟨*proof*⟩

The divides relation

**lemma** *le-imp-power-dvd*:
**fixes** $a :: 'a::\{comm\text{-}semiring\text{-}1,recpower\}$
**assumes** $m \leq n$ **shows** $a \hat{} m\ dvd\ a \hat{} n$
⟨*proof*⟩

**lemma** *power-le-dvd*:
**fixes** $a\ b :: 'a::\{comm\text{-}semiring\text{-}1,recpower\}$
**shows** $a \hat{} n\ dvd\ b \Longrightarrow m \leq n \Longrightarrow a \hat{} m\ dvd\ b$
⟨*proof*⟩

**lemma** *dvd-power-same*:
$(x::'a::\{comm\text{-}semiring\text{-}1,recpower\})\ dvd\ y \Longrightarrow x \hat{} n\ dvd\ y \hat{} n$
⟨*proof*⟩

**lemma** *dvd-power-le*:
$(x::'a::\{comm\text{-}semiring\text{-}1,recpower\})\ dvd\ y \Longrightarrow m >= n \Longrightarrow x \hat{} n\ dvd\ y \hat{} m$
⟨*proof*⟩

**lemma** *dvd-power* [*simp*]:
$n > 0 \mid (x::'a::\{comm\text{-}semiring\text{-}1,recpower\}) = 1 \Longrightarrow x\ dvd\ x \hat{} n$
⟨*proof*⟩

## 16.2 Exponentiation for the Natural Numbers

**instantiation** *nat* :: *recpower*
**begin**

**primrec** *power-nat* **where**
$p \hat{} 0 = (1::nat)$
$\mid p \hat{} (Suc\ n) = (p::nat) * (p \hat{} n)$

**instance** ⟨*proof*⟩

**declare** *power-nat.simps* [*simp del*]

**end**

**lemma** *of-nat-power*:
*of-nat* (*m* ^ *n*) = (*of-nat m*::'*a*::{*semiring-1*,*recpower*}) ^ *n*
⟨*proof*⟩

**lemma** *nat-one-le-power* [*simp*]: *Suc 0* ≤ *i* ==> *Suc 0* ≤ *i*^*n*
⟨*proof*⟩

**lemma** *nat-zero-less-power-iff* [*simp*]: (*x*^*n* > *0*) = (*x* > (*0*::*nat*) | *n*=*0*)
⟨*proof*⟩

**lemma** *nat-power-eq-Suc-0-iff* [*simp*]:
((*x*::*nat*)^*m* = *Suc 0*) = (*m* = *0* | *x* = *Suc 0*)
⟨*proof*⟩

**lemma** *power-Suc-0*[*simp*]: (*Suc 0*)^*n* = *Suc 0*
⟨*proof*⟩

Valid for the naturals, but what if *0*<*i*<*1*? Premises cannot be weakened: consider the case where $i = (0::'a)$, $m = (1::'a)$ and $n = (0::'a)$.

**lemma** *nat-power-less-imp-less*:
**assumes** *nonneg*: *0* < (*i*::*nat*)
**assumes** *less*: *i*^*m* < *i*^*n*
**shows** *m* < *n*
⟨*proof*⟩

**lemma** *power-diff*:
**assumes** *nz*: *a* ~= *0*
**shows** *n* <= *m* ==> (*a*::'*a*::{*recpower*, *field*}) ^ (*m*−*n*) = (*a*^*m*) / (*a*^*n*)
⟨*proof*⟩

**end**

# 17 Finite-Set: Finite sets

**theory** *Finite-Set*
**imports** *Nat Product-Type Power*
**begin**

## 17.1 Definition and basic properties

**inductive** *finite* :: '*a set* => *bool*
**where**
*emptyI* [*simp*, *intro*!]: *finite* {}

| *insertI* [*simp*, *intro*!]: *finite A ==> finite* (*insert a A*)

**lemma** *ex-new-if-finite*: — does not depend on def of finite at all
**assumes** ¬ *finite* (*UNIV* :: *'a set*) **and** *finite A*
**shows** ∃ *a*::*'a*. *a* ∉ *A*
⟨*proof*⟩

**lemma** *finite-induct* [*case-names empty insert*, *induct set*: *finite*]:
*finite F ==>*
*P* {} ==> (!!*x F*. *finite F ==> x* ∉ *F ==> P F ==> P* (*insert x F*)) ==> *P F*
— Discharging *x* ∉ *F* entails extra work.
⟨*proof*⟩

**lemma** *finite-ne-induct*[*case-names singleton insert*, *consumes 2*]:
**assumes** *fin*: *finite F* **shows** *F* ≠ {} ⟹
⟦ ⋀*x*. *P*{*x*};
⋀*x F*. ⟦ *finite F*; *F* ≠ {}; *x* ∉ *F*; *P F* ⟧ ⟹ *P* (*insert x F*) ⟧
⟹ *P F*
⟨*proof*⟩

**lemma** *finite-subset-induct* [*consumes 2*, *case-names empty insert*]:
**assumes** *finite F* **and** *F* ⊆ *A*
**and** *empty*: *P* {}
**and** *insert*: !!*a F*. *finite F ==> a* ∈ *A ==> a* ∉ *F ==> P F ==> P* (*insert a F*)
**shows** *P F*
⟨*proof*⟩

A finite choice principle. Does not need the SOME choice operator.

**lemma** *finite-set-choice*:
*finite A* ⟹ *ALL x*:*A*. (*EX y*. *P x y*) ⟹ *EX f*. *ALL x*:*A*. *P x* (*f x*)
⟨*proof*⟩

Finite sets are the images of initial segments of natural numbers:

**lemma** *finite-imp-nat-seg-image-inj-on*:
**assumes** *fin*: *finite A*
**shows** ∃ (*n*::*nat*) *f*. *A* = *f* ‘ {*i*. *i*<*n*} & *inj-on f* {*i*. *i*<*n*}
⟨*proof*⟩

**lemma** *nat-seg-image-imp-finite*:
!!*f A*. *A* = *f* ‘ {*i*::*nat*. *i*<*n*} ⟹ *finite A*
⟨*proof*⟩

**lemma** *finite-conv-nat-seg-image*:
*finite A* = (∃ (*n*::*nat*) *f*. *A* = *f* ‘ {*i*::*nat*. *i*<*n*})
⟨*proof*⟩

**lemma** *finite-Collect-less-nat*[*iff*]: *finite*{*n*::*nat*. *n*<*k*}

⟨*proof*⟩

### 17.1.1 Finiteness and set theoretic constructions

**lemma** *finite-UnI*: *finite F* ==> *finite G* ==> *finite* (*F Un G*)
⟨*proof*⟩

**lemma** *finite-subset*: *A* ⊆ *B* ==> *finite B* ==> *finite A*
— Every subset of a finite set is finite.
⟨*proof*⟩

**lemma** *finite-Un* [*iff*]: *finite* (*F Un G*) = (*finite F* & *finite G*)
⟨*proof*⟩

**lemma** *finite-Collect-disjI*[*simp*]:
*finite*{*x*. *P x* | *Q x*} = (*finite*{*x*. *P x*} & *finite*{*x*. *Q x*})
⟨*proof*⟩

**lemma** *finite-Int* [*simp*, *intro*]: *finite F* | *finite G* ==> *finite* (*F Int G*)
— The converse obviously fails.
⟨*proof*⟩

**lemma** *finite-Collect-conjI* [*simp*, *intro*]:
*finite*{*x*. *P x*} | *finite*{*x*. *Q x*} ==> *finite*{*x*. *P x* & *Q x*}
— The converse obviously fails.
⟨*proof*⟩

**lemma** *finite-Collect-le-nat*[*iff*]: *finite*{*n*::*nat*. *n*<=*k*}
⟨*proof*⟩

**lemma** *finite-insert* [*simp*]: *finite* (*insert a A*) = *finite A*
⟨*proof*⟩

**lemma** *finite-Union*[*simp*, *intro*]:
⟦ *finite A*; !!*M*. *M* ∈ *A* ⟹ *finite M* ⟧ ⟹ *finite*(⋃ *A*)
⟨*proof*⟩

**lemma** *finite-empty-induct*:
**assumes** *finite A*
**and** *P A*
**and** !!*a A*. *finite A* ==> *a*:*A* ==> *P A* ==> *P* (*A* − {*a*})
**shows** *P* {}
⟨*proof*⟩

**lemma** *finite-Diff* [*simp*]: *finite A* ==> *finite* (*A* − *B*)
⟨*proof*⟩

**lemma** *finite-Diff2* [*simp*]:
**assumes** *finite B* **shows** *finite* (*A* − *B*) = *finite A*

⟨*proof*⟩

**lemma** *finite-compl*[*simp*]:
*finite*(*A*::'*a set*) ⟹ *finite*(−*A*) = *finite*(*UNIV*::'*a set*)
⟨*proof*⟩

**lemma** *finite-Collect-not*[*simp*]:
*finite*{*x*::'*a*. *P x*} ⟹ *finite*{*x*. ~*P x*} = *finite*(*UNIV*::'*a set*)
⟨*proof*⟩

**lemma** *finite-Diff-insert* [*iff*]: *finite* (*A* − *insert a B*) = *finite* (*A* − *B*)
⟨*proof*⟩

Image and Inverse Image over Finite Sets

**lemma** *finite-imageI*[*simp*]: *finite F* ==> *finite* (*h* ' *F*)
— The image of a finite set is finite.
⟨*proof*⟩

**lemma** *finite-surj*: *finite A* ==> *B* <= *f* ' *A* ==> *finite B*
⟨*proof*⟩

**lemma** *finite-range-imageI*:
*finite* (*range g*) ==> *finite* (*range* (%*x*. *f* (*g x*)))
⟨*proof*⟩

**lemma** *finite-imageD*: *finite* (*f*'*A*) ==> *inj-on f A* ==> *finite A*
⟨*proof*⟩

**lemma** *inj-vimage-singleton*: *inj f* ==> *f*−'{*a*} ⊆ {*THE x*. *f x* = *a*}
— The inverse image of a singleton under an injective function is included in a singleton.
⟨*proof*⟩

**lemma** *finite-vimageI*: [|*finite F*; *inj h*|] ==> *finite* (*h* −' *F*)
— The inverse image of a finite set under an injective function is finite.
⟨*proof*⟩

The finite UNION of finite sets

**lemma** *finite-UN-I*: *finite A* ==> (!!*a*. *a*:*A* ==> *finite* (*B a*)) ==> *finite* (*UN a*:*A*. *B a*)
⟨*proof*⟩

Strengthen RHS to (∀ *x*∈*A*. *finite* (*B x*)) ∧ *finite* {*x* ∈ *A*. *B x* ≠ {}}?
We'd need to prove *finite C* ⟹ ∀ *A B*. *UNION A B* ⊆ *C* ⟶ *finite* {*x* ∈ *A*. *B x* ≠ {}} by induction.

**lemma** *finite-UN* [*simp*]:
*finite A* ==> *finite* (*UNION A B*) = (*ALL x*:*A*. *finite* (*B x*))

⟨*proof*⟩

**lemma** *finite-Collect-bex*[*simp*]: *finite A* ⟹
*finite*{*x*. *EX y*:*A*. *Q x y*} = (*ALL y*:*A*. *finite*{*x*. *Q x y*})
⟨*proof*⟩

**lemma** *finite-Collect-bounded-ex*[*simp*]: *finite*{*y*. *P y*} ⟹
*finite*{*x*. *EX y*. *P y* & *Q x y*} = (*ALL y*. *P y* ⟶ *finite*{*x*. *Q x y*})
⟨*proof*⟩

**lemma** *finite-Plus*: [| *finite A*; *finite B* |] ==> *finite* (*A* <+> *B*)
⟨*proof*⟩

Sigma of finite sets

**lemma** *finite-SigmaI* [*simp*]:
*finite A* ==> (!!*a*. *a*:*A* ==> *finite* (*B a*)) ==> *finite* (*SIGMA a*:*A*. *B a*)
⟨*proof*⟩

**lemma** *finite-cartesian-product*: [| *finite A*; *finite B* |] ==>
*finite* (*A* <*> *B*)
⟨*proof*⟩

**lemma** *finite-Prod-UNIV*:
*finite* (*UNIV*::'*a set*) ==> *finite* (*UNIV*::'*b set*) ==> *finite* (*UNIV*::('*a* * '*b*) *set*)
⟨*proof*⟩

**lemma** *finite-cartesian-productD1*:
[| *finite* (*A* <*> *B*); *B* ≠ {} |] ==> *finite A*
⟨*proof*⟩

**lemma** *finite-cartesian-productD2*:
[| *finite* (*A* <*> *B*); *A* ≠ {} |] ==> *finite B*
⟨*proof*⟩

The powerset of a finite set

**lemma** *finite-Pow-iff* [*iff*]: *finite* (*Pow A*) = *finite A*
⟨*proof*⟩

**lemma** *finite-Collect-subsets*[*simp*,*intro*]: *finite A* ⟹ *finite*{*B*. *B* ⊆ *A*}
⟨*proof*⟩

**lemma** *finite-UnionD*: *finite*(⋃ *A*) ⟹ *finite A*
⟨*proof*⟩

## 17.2 Class *finite*

⟨*ML*⟩
**class** *finite* =
  **assumes** *finite-UNIV*: *finite* (*UNIV* :: *'a set*)
⟨*ML*⟩
**hide** *const finite*

**context** *finite*
**begin**

**lemma** *finite* [*simp*]: *finite* (*A* :: *'a set*)
  ⟨*proof*⟩

**end**

**lemma** *UNIV-unit* [*noatp*]:
  *UNIV* = {()} ⟨*proof*⟩

**instance** *unit* :: *finite*
  ⟨*proof*⟩

**lemma** *UNIV-bool* [*noatp*]:
  *UNIV* = {*False*, *True*} ⟨*proof*⟩

**instance** *bool* :: *finite*
  ⟨*proof*⟩

**instance** ∗ :: (*finite*, *finite*) *finite*
  ⟨*proof*⟩

**lemma** *inj-graph*: *inj* (%*f*. {(*x*, *y*). *y* = *f x*})
  ⟨*proof*⟩

**instance** *fun* :: (*finite*, *finite*) *finite*
⟨*proof*⟩

**instance** + :: (*finite*, *finite*) *finite*
  ⟨*proof*⟩

## 17.3 A fold functional for finite sets

The intended behaviour is *fold f z* {$x_1$, ..., $x_n$} = *f* $x_1$ (... (*f* $x_n$ *z*)...) if *f* is "left-commutative":

**locale** *fun-left-comm* =
  **fixes** *f* :: *'a* ⇒ *'b* ⇒ *'b*
  **assumes** *fun-left-comm*: *f x* (*f y z*) = *f y* (*f x z*)
**begin**

On a functional level it looks much nicer:

**lemma** *fun-comp-comm*: $f\ x \circ f\ y = f\ y \circ f\ x$
⟨*proof*⟩

**end**

**inductive** *fold-graph* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b \Rightarrow bool$
**for** $f :: 'a \Rightarrow 'b \Rightarrow 'b$ **and** $z :: 'b$ **where**
*emptyI* [*intro*]: *fold-graph f z* {} *z* |
*insertI* [*intro*]: $x \notin A \Longrightarrow$ *fold-graph f z A y*
$\Longrightarrow$ *fold-graph f z* (*insert x A*) (*f x y*)

**inductive-cases** *empty-fold-graphE* [*elim*!]: *fold-graph f z* {} *x*

**definition** *fold* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b$ **where**
[*code del*]: *fold f z A* = (*THE y*. *fold-graph f z A y*)

A tempting alternative for the definiens is *if finite A then THE y*. *fold-graph f z A y else e*. It allows the removal of finiteness assumptions from the theorems *fold-comm*, *fold-reindex* and *fold-distrib*. The proofs become ugly. It is not worth the effort. (???)

**lemma** *Diff1-fold-graph*:
*fold-graph f z* (*A* − {*x*}) *y* $\Longrightarrow$ $x \in A \Longrightarrow$ *fold-graph f z A* (*f x y*)
⟨*proof*⟩

**lemma** *fold-graph-imp-finite*: *fold-graph f z A x* $\Longrightarrow$ *finite A*
⟨*proof*⟩

**lemma** *finite-imp-fold-graph*: *finite A* $\Longrightarrow \exists x$. *fold-graph f z A x*
⟨*proof*⟩

### 17.3.1 From *fold-graph* to *fold*

**lemma** *image-less-Suc*: *h* ‘ {*i*. *i* < *Suc m*} = *insert* (*h m*) (*h* ‘ {*i*. *i* < *m*})
⟨*proof*⟩

**lemma** *insert-image-inj-on-eq*:
[|*insert* (*h m*) *A* = *h* ‘ {*i*. *i* < *Suc m*}; *h m* ∉ *A*;
*inj-on h* {*i*. *i* < *Suc m*}|]
==> *A* = *h* ‘ {*i*. *i* < *m*}
⟨*proof*⟩

**lemma** *insert-inj-onE*:
**assumes** *aA*: *insert a A* = *h*‘{*i*::*nat*. *i*<*n*} **and** *anot*: *a* ∉ *A*
**and** *inj-on*: *inj-on h* {*i*::*nat*. *i*<*n*}
**shows** $\exists$ *hm m*. *inj-on hm* {*i*::*nat*. *i*<*m*} & *A* = *hm* ‘ {*i*. *i*<*m*} & *m* < *n*
⟨*proof*⟩

**context** *fun-left-comm*

**begin**

**lemma** *fold-graph-determ-aux*:
$A = h`\{i::nat.\ i<n\} \Longrightarrow$ *inj-on* $h\ \{i.\ i<n\}$
$\Longrightarrow$ *fold-graph* $f\ z\ A\ x \Longrightarrow$ *fold-graph* $f\ z\ A\ x' \Longrightarrow x' = x$
⟨*proof*⟩

**lemma** *fold-graph-determ*:
*fold-graph* $f\ z\ A\ x \Longrightarrow$ *fold-graph* $f\ z\ A\ y \Longrightarrow y = x$
⟨*proof*⟩

**lemma** *fold-equality*:
*fold-graph* $f\ z\ A\ y \Longrightarrow$ *fold* $f\ z\ A = y$
⟨*proof*⟩

The base case for *fold*:

**lemma** (**in** −) *fold-empty* [*simp*]: *fold* $f\ z\ \{\} = z$
⟨*proof*⟩

The various recursion equations for *fold*:

**lemma** *fold-insert-aux*: $x \notin A$
$\Longrightarrow$ *fold-graph* $f\ z$ (*insert* $x\ A$) $v \longleftrightarrow$
$(\exists y.$ *fold-graph* $f\ z\ A\ y \wedge v = f\ x\ y)$
⟨*proof*⟩

**lemma** *fold-insert* [*simp*]:
*finite* $A \Longrightarrow x \notin A \Longrightarrow$ *fold* $f\ z$ (*insert* $x\ A$) $= f\ x$ (*fold* $f\ z\ A$)
⟨*proof*⟩

**lemma** *fold-fun-comm*:
*finite* $A \Longrightarrow f\ x$ (*fold* $f\ z\ A$) $=$ *fold* $f$ $(f\ x\ z)\ A$
⟨*proof*⟩

**lemma** *fold-insert2*:
*finite* $A \Longrightarrow x \notin A \Longrightarrow$ *fold* $f\ z$ (*insert* $x\ A$) $=$ *fold* $f$ $(f\ x\ z)\ A$
⟨*proof*⟩

**lemma** *fold-rec*:
**assumes** *finite* $A$ **and** $x \in A$
**shows** *fold* $f\ z\ A = f\ x$ (*fold* $f\ z$ $(A - \{x\})$)
⟨*proof*⟩

**lemma** *fold-insert-remove*:
**assumes** *finite* $A$
**shows** *fold* $f\ z$ (*insert* $x\ A$) $= f\ x$ (*fold* $f\ z$ $(A - \{x\})$)
⟨*proof*⟩

**end**

A simplified version for idempotent functions:

**locale** *fun-left-comm-idem* = *fun-left-comm* +
  **assumes** *fun-left-idem*: *f x* (*f x z*) = *f x z*
**begin**

The nice version:

**lemma** *fun-comp-idem* : *f x o f x* = *f x*
⟨*proof*⟩

**lemma** *fold-insert-idem*:
  **assumes** *fin*: *finite A*
  **shows** *fold f z* (*insert x A*) = *f x* (*fold f z A*)
⟨*proof*⟩

**declare** *fold-insert*[*simp del*] *fold-insert-idem*[*simp*]

**lemma** *fold-insert-idem2*:
  *finite A* ⟹ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
⟨*proof*⟩

**end**

### 17.3.2 The derived combinator *fold-image*

**definition** *fold-image* :: ('*b* ⇒ '*b* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*) ⇒ '*b* ⇒ '*a set* ⇒ '*b*
**where** *fold-image f g* = *fold* (%*x y*. *f* (*g x*) *y*)

**lemma** *fold-image-empty*[*simp*]: *fold-image f g z* {} = *z*
⟨*proof*⟩

**context** *ab-semigroup-mult*
**begin**

**lemma** *fold-image-insert*[*simp*]:
**assumes** *finite A* **and** *a* ∉ *A*
**shows** *fold-image times g z* (*insert a A*) = *g a* ∗ (*fold-image times g z A*)
⟨*proof*⟩

**lemma** *fold-image-reindex*:
**assumes** *fin*: *finite A*
**shows** *inj-on h A* ⟹ *fold-image times g z* (*h*‘*A*) = *fold-image times* (*g*∘*h*) *z A*
⟨*proof*⟩

**lemma** *fold-image-cong*:
  *finite A* ⟹

(!!$x$. $x$:$A$ ==> $g\ x = h\ x$) ==> *fold-image times* $g\ z\ A$ = *fold-image times* $h\ z\ A$
⟨*proof*⟩

**end**

**context** *comm-monoid-mult*
**begin**

**lemma** *fold-image-Un-Int*:
*finite* $A$ ==> *finite* $B$ ==>
*fold-image times* $g\ 1\ A * $ *fold-image times* $g\ 1\ B$ =
*fold-image times* $g\ 1$ ($A$ *Un* $B$) $*$ *fold-image times* $g\ 1$ ($A$ *Int* $B$)
⟨*proof*⟩

**corollary** *fold-Un-disjoint*:
*finite* $A$ ==> *finite* $B$ ==> $A$ *Int* $B = \{\}$ ==>
*fold-image times* $g\ 1$ ($A$ *Un* $B$) =
*fold-image times* $g\ 1\ A * $ *fold-image times* $g\ 1\ B$
⟨*proof*⟩

**lemma** *fold-image-UN-disjoint*:
⟦ *finite* $I$; *ALL* $i$:$I$. *finite* ($A\ i$);
*ALL* $i$:$I$. *ALL* $j$:$I$. $i \neq j$ --> $A\ i$ *Int* $A\ j = \{\}$ ⟧
⟹ *fold-image times* $g\ 1$ (*UNION* $I\ A$) =
*fold-image times* (%$i$. *fold-image times* $g\ 1$ ($A\ i$)) $1\ I$
⟨*proof*⟩

**lemma** *fold-image-Sigma*: *finite* $A$ ==> *ALL* $x$:$A$. *finite* ($B\ x$) ==>
*fold-image times* (%$x$. *fold-image times* ($g\ x$) $1$ ($B\ x$)) $1\ A$ =
*fold-image times* (*split* $g$) $1$ (*SIGMA* $x$:$A$. $B\ x$)
⟨*proof*⟩

**lemma** *fold-image-distrib*: *finite* $A$ ⟹
*fold-image times* (%$x$. $g\ x * h\ x$) $1\ A$ =
*fold-image times* $g\ 1\ A *$ *fold-image times* $h\ 1\ A$
⟨*proof*⟩

**lemma** *fold-image-related*:
**assumes** *Re*: $R\ e\ e$
**and** *Rop*: ∀ *x1 y1 x2 y2*. $R\ x1\ x2 \wedge R\ y1\ y2 \longrightarrow R\ (x1 * y1)\ (x2 * y2)$
**and** *fS*: *finite* $S$ **and** *Rfg*: ∀ $x$∈$S$. $R\ (h\ x)\ (g\ x)$
**shows** $R$ (*fold-image* (*op* $*$) $h\ e\ S$) (*fold-image* (*op* $*$) $g\ e\ S$)
⟨*proof*⟩

**lemma** *fold-image-eq-general*:
**assumes** *fS*: *finite* $S$
**and** $h$: ∀ $y$∈$S'$. ∃!$x$. $x$∈ $S \wedge h(x) = y$
**and** *f12*: ∀ $x$∈$S$. $h\ x \in S' \wedge$ *f2*$(h\ x) =$ *f1* $x$
**shows** *fold-image* (*op* $*$) *f1* $e\ S$ = *fold-image* (*op* $*$) *f2* $e\ S'$

⟨*proof*⟩

**lemma** *fold-image-eq-general-inverses*:
**assumes** *fS*: *finite S*
**and** *kh*: $\bigwedge y.\ y \in T \Longrightarrow k\ y \in S \wedge h\ (k\ y) = y$
**and** *hk*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in T \wedge k\ (h\ x) = x\ \wedge g\ (h\ x) = f\ x$
**shows** *fold-image* (*op* ∗) *f e S* = *fold-image* (*op* ∗) *g e T*

⟨*proof*⟩

**end**

## 17.4 Generalized summation over a set

**interpretation** *comm-monoid-add*: *comm-monoid-mult 0*::′*a*::*comm-monoid-add op* +
⟨*proof*⟩

**definition** *setsum* :: (′*a* => ′*b*) => ′*a set* => ′*b*::*comm-monoid-add*
**where** *setsum f A* == *if finite A then fold-image* (*op* +) *f 0 A else 0*

**abbreviation**
*Setsum* (∑ - [*1000*] *999*) **where**
∑ *A* == *setsum* (%*x*. *x*) *A*

Now: lot's of fancy syntax. First, *setsum* (λ*x*. *e*) *A* is written ∑ *x*∈*A*. *e*.

**syntax**
*-setsum* :: *pttrn* => ′*a set* => ′*b* => ′*b*::*comm-monoid-add* ((*3SUM* -:-. -) [*0*, *51*, *10*] *10*)
**syntax** (*xsymbols*)
*-setsum* :: *pttrn* => ′*a set* => ′*b* => ′*b*::*comm-monoid-add* ((*3*∑ -∈-. -) [*0*, *51*, *10*] *10*)
**syntax** (*HTML* **output**)
*-setsum* :: *pttrn* => ′*a set* => ′*b* => ′*b*::*comm-monoid-add* ((*3*∑ -∈-. -) [*0*, *51*, *10*] *10*)

**translations** — Beware of argument permutation!
*SUM i*:*A*. *b* == *CONST setsum* (%*i*. *b*) *A*
∑ *i*∈*A*. *b* == *CONST setsum* (%*i*. *b*) *A*

Instead of ∑ *x*∈{*x*. *P*}. *e* we introduce the shorter ∑ *x*|*P*. *e*.

**syntax**
*-qsetsum* :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a* ((*3SUM* - |/ -./ -) [*0*,*0*,*10*] *10*)
**syntax** (*xsymbols*)
*-qsetsum* :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a* ((*3*∑ - | (-)./ -) [*0*,*0*,*10*] *10*)
**syntax** (*HTML* **output**)
*-qsetsum* :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a* ((*3*∑ - | (-)./ -) [*0*,*0*,*10*] *10*)

**translations**

*SUM x|P. t => CONST setsum (%x. t) {x. P}*
$\sum x|P.\ t$ => *CONST setsum (%x. t) {x. P}*

⟨*ML*⟩

**lemma** *setsum-empty* [*simp*]: *setsum f {} = 0*
⟨*proof*⟩

**lemma** *setsum-insert* [*simp*]:
*finite F ==> a ∉ F ==> setsum f (insert a F) = f a + setsum f F*
⟨*proof*⟩

**lemma** *setsum-infinite* [*simp*]: ~ *finite A ==> setsum f A = 0*
⟨*proof*⟩

**lemma** *setsum-reindex*:
*inj-on f B ==> setsum h (f ' B) = setsum (h ∘ f) B*
⟨*proof*⟩

**lemma** *setsum-reindex-id*:
*inj-on f B ==> setsum f B = setsum id (f ' B)*
⟨*proof*⟩

**lemma** *setsum-reindex-nonzero*:
**assumes** *fS*: *finite S*
**and** *nz*: $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow x \neq y \Longrightarrow f\ x = f\ y \Longrightarrow h\ (f\ x) = 0$
**shows** *setsum h (f ' S) = setsum (h o f) S*
⟨*proof*⟩

**lemma** *setsum-cong*:
*A = B ==> (!!x. x:B ==> f x = g x) ==> setsum f A = setsum g B*
⟨*proof*⟩

**lemma** *strong-setsum-cong*[*cong*]:
*A = B ==> (!!x. x:B =simp=> f x = g x)*
*==> setsum (%x. f x) A = setsum (%x. g x) B*
⟨*proof*⟩

**lemma** *setsum-cong2*: $[\![\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x]\!] \Longrightarrow$ *setsum f A = setsum g A*
⟨*proof*⟩

**lemma** *setsum-reindex-cong*:
*[|inj-on f A; B = f ' A; !!a. a:A ==> g a = h (f a)|]*
*==> setsum h B = setsum g A*
⟨*proof*⟩

**lemma** *setsum-0*[*simp*]: *setsum (%i. 0) A = 0*

⟨*proof*⟩

**lemma** *setsum-0'*: *ALL a:A. f a = 0 ==> setsum f A = 0*
⟨*proof*⟩

**lemma** *setsum-Un-Int*: *finite A ==> finite B ==>*
*setsum g (A Un B) + setsum g (A Int B) = setsum g A + setsum g B*
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

**lemma** *setsum-Un-disjoint*: *finite A ==> finite B*
*==> A Int B = {} ==> setsum g (A Un B) = setsum g A + setsum g B*
⟨*proof*⟩

**lemma** *setsum-mono-zero-left*:
**assumes** *fT*: *finite T* **and** *ST*: $S \subseteq T$
**and** *z*: $\forall i \in T - S.\ f\ i = 0$
**shows** *setsum f S = setsum f T*
⟨*proof*⟩

**lemma** *setsum-mono-zero-right*:
*finite* $T \Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S.\ f\ i = 0 \Longrightarrow$ *setsum f T = setsum f S*
⟨*proof*⟩

**lemma** *setsum-mono-zero-cong-left*:
**assumes** *fT*: *finite T* **and** *ST*: $S \subseteq T$
**and** *z*: $\forall i \in T - S.\ g\ i = 0$
**and** *fg*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
**shows** *setsum f S = setsum g T*
⟨*proof*⟩

**lemma** *setsum-mono-zero-cong-right*:
**assumes** *fT*: *finite T* **and** *ST*: $S \subseteq T$
**and** *z*: $\forall i \in T - S.\ f\ i = 0$
**and** *fg*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
**shows** *setsum f T = setsum g S*
⟨*proof*⟩

**lemma** *setsum-delta*:
**assumes** *fS*: *finite S*
**shows** *setsum* $(\lambda k.$ *if k=a then b k else 0*$)$ $S =$ $($*if* $a \in S$ *then b a else 0*$)$
⟨*proof*⟩
**lemma** *setsum-delta'*:
**assumes** *fS*: *finite S* **shows**
*setsum* $(\lambda k.$ *if a = k then b k else 0*$)$ $S =$
$($*if* $a \in S$ *then b a else 0*$)$
⟨*proof*⟩

**lemma** *setsum-restrict-set*:

**assumes** *fA*: *finite A*
**shows** *setsum f* (*A* ∩ *B*) = *setsum* (λ*x*. *if x* ∈ *B then f x else 0*) *A*
⟨*proof*⟩

**lemma** *setsum-cases*:
**assumes** *fA*: *finite A*
**shows** *setsum* (λ*x*. *if x* ∈ *B then f x else g x*) *A* =
*setsum f* (*A* ∩ *B*) + *setsum g* (*A* ∩ − *B*)
⟨*proof*⟩

**lemma** *setsum-UN-disjoint*:
*finite I* ==> (*ALL i*:*I*. *finite* (*A i*)) ==>
(*ALL i*:*I*. *ALL j*:*I*. *i* ≠ *j* −−> *A i Int A j* = {}) ==>
*setsum f* (*UNION I A*) = (∑ *i*∈*I*. *setsum f* (*A i*))
⟨*proof*⟩

No need to assume that $C$ is finite. If infinite, the rhs is directly 0, and $\bigcup C$ is also infinite, hence the lhs is also 0.

**lemma** *setsum-Union-disjoint*:
[| (*ALL A*:*C*. *finite A*);
(*ALL A*:*C*. *ALL B*:*C*. *A* ≠ *B* −−> *A Int B* = {}) |]
==> *setsum f* (*Union C*) = *setsum* (*setsum f*) *C*
⟨*proof*⟩

**lemma** *setsum-Sigma*: *finite A* ==> *ALL x*:*A*. *finite* (*B x*) ==>
(∑ *x*∈*A*. (∑ *y*∈*B x*. *f x y*)) = (∑ (*x*,*y*)∈(*SIGMA x*:*A*. *B x*). *f x y*)
⟨*proof*⟩

Here we can eliminate the finiteness assumptions, by cases.

**lemma** *setsum-cartesian-product*:
(∑ *x*∈*A*. (∑ *y*∈*B*. *f x y*)) = (∑ (*x*,*y*) ∈ *A* <*> *B*. *f x y*)
⟨*proof*⟩

**lemma** *setsum-addf*: *setsum* (%*x*. *f x* + *g x*) *A* = (*setsum f A* + *setsum g A*)
⟨*proof*⟩

### 17.4.1 Properties in more restricted classes of structures

**lemma** *setsum-SucD*: *setsum f A* = *Suc n* ==> *EX a*:*A*. *0* < *f a*
⟨*proof*⟩

**lemma** *setsum-eq-0-iff* [*simp*]:
*finite F* ==> (*setsum f F* = *0*) = (*ALL a*:*F*. *f a* = (*0*::*nat*))
⟨*proof*⟩

**lemma** *setsum-eq-Suc0-iff*: *finite A* ⟹

(*setsum f A = Suc 0*) = (*EX a:A. f a = Suc 0 & (ALL b:A. a≠b ⟶ f b = 0*))
⟨*proof*⟩

**lemmas** *setsum-eq-1-iff = setsum-eq-Suc0-iff*[*simplified One-nat-def*[*symmetric*]]

**lemma** *setsum-Un-nat*: *finite A ==> finite B ==>*
(*setsum f* (*A Un B*) :: *nat*) = *setsum f A + setsum f B − setsum f* (*A Int B*)
— For the natural numbers, we have subtraction.
⟨*proof*⟩

**lemma** *setsum-Un*: *finite A ==> finite B ==>*
(*setsum f* (*A Un B*) :: *'a* :: *ab-group-add*) =
*setsum f A + setsum f B − setsum f* (*A Int B*)
⟨*proof*⟩

**lemma** (**in** *comm-monoid-mult*) *fold-image-1*: *finite S* ⟹ (∀ *x*∈*S*. *f x = 1*) ⟹ *fold-image op ∗ f 1 S = 1*
⟨*proof*⟩

**lemma** (**in** *comm-monoid-mult*) *fold-image-Un-one*:
**assumes** *fS*: *finite S* **and** *fT*: *finite T*
**and** *I0*: ∀ *x* ∈ *S*∩*T*. *f x = 1*
**shows** *fold-image* (*op ∗*) *f 1* (*S* ∪ *T*) = *fold-image* (*op ∗*) *f 1 S ∗ fold-image* (*op ∗*) *f 1 T*
⟨*proof*⟩

**lemma** *setsum-eq-general-reverses*:
**assumes** *fS*: *finite S* **and** *fT*: *finite T*
**and** *kh*: ⋀*y*. *y* ∈ *T* ⟹ *k y* ∈ *S* ∧ *h* (*k y*) = *y*
**and** *hk*: ⋀*x*. *x* ∈ *S* ⟹ *h x* ∈ *T* ∧ *k* (*h x*) = *x* ∧ *g* (*h x*) = *f x*
**shows** *setsum f S = setsum g T*
⟨*proof*⟩

**lemma** *setsum-Un-zero*:
**assumes** *fS*: *finite S* **and** *fT*: *finite T*
**and** *I0*: ∀ *x* ∈ *S*∩*T*. *f x = 0*
**shows** *setsum f* (*S* ∪ *T*) = *setsum f S* + *setsum f T*
⟨*proof*⟩

**lemma** *setsum-UNION-zero*:
**assumes** *fS*: *finite S* **and** *fSS*: ∀ *T* ∈ *S*. *finite T*
**and** *f0*: ⋀*T1 T2 x*. *T1*∈*S* ⟹ *T2*∈*S* ⟹ *T1* ≠ *T2* ⟹ *x* ∈ *T1* ⟹ *x* ∈ *T2* ⟹ *f x = 0*
**shows** *setsum f* (⋃ *S*) = *setsum* (λ*T*. *setsum f T*) *S*
⟨*proof*⟩

**lemma** *setsum-diff1-nat*: $(setsum\ f\ (A - \{a\}) :: nat) =$
$(if\ a{:}A\ then\ setsum\ f\ A - f\ a\ else\ setsum\ f\ A)$
⟨*proof*⟩

**lemma** *setsum-diff1*: *finite* $A \Longrightarrow$
$(setsum\ f\ (A - \{a\}) :: ('a{::}ab\text{-}group\text{-}add)) =$
$(if\ a{:}A\ then\ setsum\ f\ A - f\ a\ else\ setsum\ f\ A)$
⟨*proof*⟩

**lemma** *setsum-diff1'*[*rule-format*]:
*finite* $A \Longrightarrow a \in A \longrightarrow (\sum\ x \in A.\ f\ x) = f\ a + (\sum\ x \in (A - \{a\}).\ f\ x)$
⟨*proof*⟩

**lemma** *setsum-diff-nat*:
**assumes** *finite* $B$ **and** $B \subseteq A$
**shows** $(setsum\ f\ (A - B) :: nat) = (setsum\ f\ A) - (setsum\ f\ B)$
⟨*proof*⟩

**lemma** *setsum-diff*:
**assumes** *le*: *finite* $A$ $B \subseteq A$
**shows** $setsum\ f\ (A - B) = setsum\ f\ A - ((setsum\ f\ B){::}('a{::}ab\text{-}group\text{-}add))$
⟨*proof*⟩

**lemma** *setsum-mono*:
**assumes** *le*: $\bigwedge i.\ i{\in}K \Longrightarrow f\ (i{::}'a) \leq ((g\ i){::}('b{::}\{comm\text{-}monoid\text{-}add,\ pordered\text{-}ab\text{-}semigroup\text{-}add\}))$
**shows** $(\sum i{\in}K.\ f\ i) \leq (\sum i{\in}K.\ g\ i)$
⟨*proof*⟩

**lemma** *setsum-strict-mono*:
**fixes** $f :: 'a \Rightarrow 'b{::}\{pordered\text{-}cancel\text{-}ab\text{-}semigroup\text{-}add,comm\text{-}monoid\text{-}add\}$
**assumes** *finite* $A$ $A \neq \{\}$
**and** $!!x.\ x{:}A \Longrightarrow f\ x < g\ x$
**shows** $setsum\ f\ A < setsum\ g\ A$
⟨*proof*⟩

**lemma** *setsum-negf*:
$setsum\ (\%x.\ -\ (f\ x){::}'a{::}ab\text{-}group\text{-}add)\ A = -\ setsum\ f\ A$
⟨*proof*⟩

**lemma** *setsum-subtractf*:
$setsum\ (\%x.\ ((f\ x){::}'a{::}ab\text{-}group\text{-}add)\ -\ g\ x)\ A =$
$setsum\ f\ A\ -\ setsum\ g\ A$
⟨*proof*⟩

**lemma** *setsum-nonneg*:
**assumes** *nn*: $\forall\, x{\in}A.\ (0{::}'a{::}\{pordered\text{-}ab\text{-}semigroup\text{-}add,comm\text{-}monoid\text{-}add\}) \leq$

*f x*
**shows** *0 ≤ setsum f A*
⟨*proof*⟩

**lemma** *setsum-nonpos*:
**assumes** *np*: ∀ *x*∈*A*. *f x ≤* (*0*::*'a*::{*pordered-ab-semigroup-add*,*comm-monoid-add*})
**shows** *setsum f A ≤ 0*
⟨*proof*⟩

**lemma** *setsum-mono2*:
**fixes** *f* :: *'a ⇒ 'b* :: {*pordered-ab-semigroup-add-imp-le*,*comm-monoid-add*}
**assumes** *fin*: *finite B* **and** *sub*: *A ⊆ B* **and** *nn*: ⋀*b*. *b* ∈ *B*−*A ⟹ 0 ≤ f b*
**shows** *setsum f A ≤ setsum f B*
⟨*proof*⟩

**lemma** *setsum-mono3*: *finite B ==> A <= B ==>*
*ALL x*: *B − A*.
*0 <= ((f x)*::*'a*::{*comm-monoid-add*,*pordered-ab-semigroup-add*}) ==>
*setsum f A <= setsum f B*
⟨*proof*⟩

**lemma** *setsum-right-distrib*:
**fixes** *f* :: *'a => ('b*::*semiring-0)*
**shows** *r ∗ setsum f A = setsum* (%*n*. *r ∗ f n*) *A*
⟨*proof*⟩

**lemma** *setsum-left-distrib*:
*setsum f A ∗* (*r*::*'a*::*semiring-0*) = (∑ *n*∈*A*. *f n ∗ r*)
⟨*proof*⟩

**lemma** *setsum-divide-distrib*:
*setsum f A / (r*::*'a*::*field*) = (∑ *n*∈*A*. *f n / r*)
⟨*proof*⟩

**lemma** *setsum-abs*[*iff*]:
**fixes** *f* :: *'a => ('b*::*pordered-ab-group-add-abs)*
**shows** *abs* (*setsum f A*) ≤ *setsum* (%*i*. *abs*(*f i*)) *A*
⟨*proof*⟩

**lemma** *setsum-abs-ge-zero*[*iff*]:
**fixes** *f* :: *'a => ('b*::*pordered-ab-group-add-abs)*
**shows** *0 ≤ setsum* (%*i*. *abs*(*f i*)) *A*
⟨*proof*⟩

**lemma** *abs-setsum-abs*[*simp*]:
**fixes** *f* :: *'a => ('b*::*pordered-ab-group-add-abs)*
**shows** *abs* (∑ *a*∈*A*. *abs*(*f a*)) = (∑ *a*∈*A*. *abs*(*f a*))
⟨*proof*⟩

Commuting outer and inner summation

**lemma** *swap-inj-on*:
*inj-on* (%(*i*, *j*). (*j*, *i*)) (*A* × *B*)
⟨*proof*⟩

**lemma** *swap-product*:
(%(*i*, *j*). (*j*, *i*)) ' (*A* × *B*) = *B* × *A*
⟨*proof*⟩

**lemma** *setsum-commute*:
(∑ *i*∈*A*. ∑ *j*∈*B*. *f i j*) = (∑ *j*∈*B*. ∑ *i*∈*A*. *f i j*)
⟨*proof*⟩

**lemma** *setsum-product*:
**fixes** *f* :: *'a* => (*'b*::*semiring-0*)
**shows** *setsum f A* * *setsum g B* = (∑ *i*∈*A*. ∑ *j*∈*B*. *f i* * *g j*)
⟨*proof*⟩

## 17.5 Generalized product over a set

**definition** *setprod* :: (*'a* => *'b*) => *'a set* => *'b*::*comm-monoid-mult*
**where** *setprod f A* == *if finite A then fold-image* (*op* *) *f 1 A else 1*

**abbreviation**
*Setprod* (∏ - [*1000*] *999*) **where**
∏ *A* == *setprod* (%*x*. *x*) *A*

**syntax**
*-setprod* :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*3PROD* -:-. -) [*0*, *51*, *10*] *10*)
**syntax** (*xsymbols*)
*-setprod* :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*3*∏ -∈-. -) [*0*, *51*, *10*] *10*)
**syntax** (*HTML* **output**)
*-setprod* :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*3*∏ -∈-. -) [*0*, *51*, *10*] *10*)

**translations** — Beware of argument permutation!
*PROD i*:*A*. *b* == *CONST setprod* (%*i*. *b*) *A*
∏ *i*∈*A*. *b* == *CONST setprod* (%*i*. *b*) *A*

Instead of ∏ *x*∈{*x*. *P*}. *e* we introduce the shorter ∏ *x*|*P*. *e*.

**syntax**
*-qsetprod* :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3PROD* - |/ -./ -) [*0*,*0*,*10*] *10*)
**syntax** (*xsymbols*)
*-qsetprod* :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*∏ - | (-)./ -) [*0*,*0*,*10*] *10*)
**syntax** (*HTML* **output**)
*-qsetprod* :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*∏ - | (-)./ -) [*0*,*0*,*10*] *10*)

**translations**

*PROD x|P. t => CONST setprod (%x. t) {x. P}*
$\prod x|P.\ t$ => *CONST setprod (%x. t) {x. P}*

**lemma** *setprod-empty* [*simp*]: *setprod f {} = 1*
⟨*proof*⟩

**lemma** *setprod-insert* [*simp*]: [| *finite A*; *a* ∉ *A* |] ==>
*setprod f (insert a A) = f a * setprod f A*
⟨*proof*⟩

**lemma** *setprod-infinite* [*simp*]: ~ *finite A* ==> *setprod f A = 1*
⟨*proof*⟩

**lemma** *setprod-reindex*:
*inj-on f B* ==> *setprod h (f ' B) = setprod (h ∘ f) B*
⟨*proof*⟩

**lemma** *setprod-reindex-id*: *inj-on f B* ==> *setprod f B = setprod id (f ' B)*
⟨*proof*⟩

**lemma** *setprod-cong*:
*A = B* ==> (!!*x*. *x*:*B* ==> *f x = g x*) ==> *setprod f A = setprod g B*
⟨*proof*⟩

**lemma** *strong-setprod-cong*[*cong*]:
*A = B* ==> (!!*x*. *x*:*B* =*simp*=> *f x = g x*) ==> *setprod f A = setprod g B*
⟨*proof*⟩

**lemma** *setprod-reindex-cong*: *inj-on f A* ==>
*B = f ' A* ==> *g = h ∘ f* ==> *setprod h B = setprod g A*
⟨*proof*⟩

**lemma** *strong-setprod-reindex-cong*: **assumes** *i*: *inj-on f A*
**and** *B*: *B = f ' A* **and** *eq*: $\bigwedge x.\ x \in A \Longrightarrow g\ x = (h \circ f)\ x$
**shows** *setprod h B = setprod g A*
⟨*proof*⟩

**lemma** *setprod-Un-one*:
**assumes** *fS*: *finite S* **and** *fT*: *finite T*
**and** *I0*: ∀ *x* ∈ *S*∩*T*. *f x = 1*
**shows** *setprod f (S ∪ T) = setprod f S * setprod f T*
⟨*proof*⟩

**lemma** *setprod-1*: *setprod (%i. 1) A = 1*
⟨*proof*⟩

**lemma** *setprod-1'*: *ALL a*:*F*. *f a = 1* ==> *setprod f F = 1*

⟨*proof*⟩

**lemma** *setprod-Un-Int*: *finite A* ==> *finite B*
  ==> *setprod g* (*A Un B*) * *setprod g* (*A Int B*) = *setprod g A* * *setprod g B*
⟨*proof*⟩

**lemma** *setprod-Un-disjoint*: *finite A* ==> *finite B*
 ==> *A Int B* = {} ==> *setprod g* (*A Un B*) = *setprod g A* * *setprod g B*
⟨*proof*⟩

**lemma** *setprod-mono-one-left*:
 **assumes** *fT*: *finite T* **and** *ST*: *S* ⊆ *T*
 **and** *z*: ∀ *i* ∈ *T* − *S*. *f i* = *1*
 **shows** *setprod f S* = *setprod f T*
⟨*proof*⟩

**lemmas** *setprod-mono-one-right* = *setprod-mono-one-left* [*THEN sym*]

**lemma** *setprod-delta*:
 **assumes** *fS*: *finite S*
 **shows** *setprod* (λ*k*. *if k*=*a then b k else 1*) *S* = (*if a* ∈ *S then b a else 1*)
⟨*proof*⟩

**lemma** *setprod-delta'*:
 **assumes** *fS*: *finite S* **shows**
 *setprod* (λ*k*. *if a* = *k then b k else 1*) *S* =
  (*if a*∈ *S then b a else 1*)
 ⟨*proof*⟩

**lemma** *setprod-UN-disjoint*:
  *finite I* ==> (*ALL i*:*I*. *finite* (*A i*)) ==>
    (*ALL i*:*I*. *ALL j*:*I*. *i* ≠ *j* --> *A i Int A j* = {}) ==>
   *setprod f* (*UNION I A*) = *setprod* (%*i*. *setprod f* (*A i*)) *I*
⟨*proof*⟩

**lemma** *setprod-Union-disjoint*:
 [| (*ALL A*:*C*. *finite A*);
    (*ALL A*:*C*. *ALL B*:*C*. *A* ≠ *B* --> *A Int B* = {}) |]
  ==> *setprod f* (*Union C*) = *setprod* (*setprod f*) *C*
⟨*proof*⟩

**lemma** *setprod-Sigma*: *finite A* ==> *ALL x*:*A*. *finite* (*B x*) ==>
  (∏ *x*∈*A*. (∏ *y*∈ *B x*. *f x y*)) =
  (∏ (*x*,*y*)∈(*SIGMA x*:*A*. *B x*). *f x y*)
⟨*proof*⟩

Here we can eliminate the finiteness assumptions, by cases.

**lemma** *setprod-cartesian-product*:

$(\prod x{\in}A.\ (\prod y{\in}\ B.\ f\ x\ y)) = (\prod (x,y){\in}(A <*> B).\ f\ x\ y)$
⟨*proof*⟩

**lemma** *setprod-timesf*:
$setprod\ (\%x.\ f\ x * g\ x)\ A = (setprod\ f\ A * setprod\ g\ A)$
⟨*proof*⟩

### 17.5.1 Properties in more restricted classes of structures

**lemma** *setprod-eq-1-iff* [*simp*]:
$finite\ F ==> (setprod\ f\ F = 1) = (ALL\ a{:}F.\ f\ a = (1{::}nat))$
⟨*proof*⟩

**lemma** *setprod-zero*:
$finite\ A ==> EX\ x{:}\ A.\ f\ x = (0{::}'a{::}comm\text{-}semiring\text{-}1) ==> setprod\ f\ A = 0$
⟨*proof*⟩

**lemma** *setprod-nonneg* [*rule-format*]:
$(ALL\ x{:}\ A.\ (0{::}'a{::}ordered\text{-}semidom) \le f\ x) --> 0 \le setprod\ f\ A$
⟨*proof*⟩

**lemma** *setprod-pos* [*rule-format*]: $(ALL\ x{:}\ A.\ (0{::}'a{::}ordered\text{-}semidom) < f\ x)$
$--> 0 < setprod\ f\ A$
⟨*proof*⟩

**lemma** *setprod-zero-iff*[*simp*]: $finite\ A ==>$
$(setprod\ f\ A = (0{::}'a{::}\{comm\text{-}semiring\text{-}1,no\text{-}zero\text{-}divisors\})) =$
$(EX\ x{:}\ A.\ f\ x = 0)$
⟨*proof*⟩

**lemma** *setprod-pos-nat*:
$finite\ S ==> (ALL\ x : S.\ f\ x > (0{::}nat)) ==> setprod\ f\ S > 0$
⟨*proof*⟩

**lemma** *setprod-pos-nat-iff*[*simp*]:
$finite\ S ==> (setprod\ f\ S > 0) = (ALL\ x : S.\ f\ x > (0{::}nat))$
⟨*proof*⟩

**lemma** *setprod-Un*: $finite\ A ==> finite\ B ==> (ALL\ x{:}\ A\ Int\ B.\ f\ x \neq 0) ==>$
$(setprod\ f\ (A\ Un\ B) :: 'a ::\{field\})$
$= setprod\ f\ A * setprod\ f\ B\ /\ setprod\ f\ (A\ Int\ B)$
⟨*proof*⟩

**lemma** *setprod-diff1*: $finite\ A ==> f\ a \neq 0 ==>$
$(setprod\ f\ (A - \{a\}) :: 'a :: \{field\}) =$
$(if\ a{:}A\ then\ setprod\ f\ A\ /\ f\ a\ else\ setprod\ f\ A)$
⟨*proof*⟩

**lemma** *setprod-inversef*: $finite\ A ==>$

*ALL x: A. f x ≠ (0::'a::{field,division-by-zero}) ==>*
*setprod (inverse ∘ f) A = inverse (setprod f A)*
⟨*proof*⟩

**lemma** *setprod-dividef*:
[|*finite A*;
*∀x ∈ A. g x ≠ (0::'a::{field,division-by-zero})*|]
*==> setprod (%x. f x / g x) A = setprod f A / setprod g A*
⟨*proof*⟩

**lemma** *setprod-dvd-setprod* [*rule-format*]:
*(ALL x : A. f x dvd g x) ⟶ setprod f A dvd setprod g A*
⟨*proof*⟩

**lemma** *setprod-dvd-setprod-subset*:
*finite B ⟹ A <= B ⟹ setprod f A dvd setprod f B*
⟨*proof*⟩

**lemma** *setprod-dvd-setprod-subset2*:
*finite B ⟹ A <= B ⟹ ALL x : A. (f x::'a::comm-semiring-1) dvd g x ⟹*
*setprod f A dvd setprod g B*
⟨*proof*⟩

**lemma** *dvd-setprod*: *finite A ⟹ i:A ⟹*
*(f i ::'a::comm-semiring-1) dvd setprod f A*
⟨*proof*⟩

**lemma** *dvd-setsum* [*rule-format*]: *(ALL i : A. d dvd f i) ⟶*
*(d::'a::comm-semiring-1) dvd (SUM x : A. f x)*
⟨*proof*⟩

## 17.6 Finite cardinality

This definition, although traditional, is ugly to work with: *card A == LEAST n. EX f. A = {f i | i. i < n}*. But now that we have *setsum* things are easy:

**definition** *card* :: *'a set ⇒ nat*
**where** *card A = setsum (λx. 1) A*

**lemma** *card-empty* [*simp*]: *card {} = 0*
⟨*proof*⟩

**lemma** *card-infinite* [*simp*]: *~ finite A ==> card A = 0*
⟨*proof*⟩

**lemma** *card-eq-setsum*: *card A = setsum (%x. 1) A*
⟨*proof*⟩

**lemma** *card-insert-disjoint* [*simp*]:

*finite A ==> x ∉ A ==> card (insert x A) = Suc(card A)*
⟨*proof*⟩

**lemma** *card-insert-if*:
*finite A ==> card (insert x A) = (if x:A then card A else Suc(card(A)))*
⟨*proof*⟩

**lemma** *card-0-eq* [*simp*,*noatp*]: *finite A ==> (card A = 0) = (A = {})*
⟨*proof*⟩

**lemma** *card-eq-0-iff*: *(card A = 0) = (A = {} | ~ finite A)*
⟨*proof*⟩

**lemma** *card-Suc-Diff1*: *finite A ==> x: A ==> Suc (card (A − {x})) = card A*
⟨*proof*⟩

**lemma** *card-Diff-singleton*:
*finite A ==> x: A ==> card (A − {x}) = card A − 1*
⟨*proof*⟩

**lemma** *card-Diff-singleton-if*:
*finite A ==> card (A−{x}) = (if x : A then card A − 1 else card A)*
⟨*proof*⟩

**lemma** *card-Diff-insert*[*simp*]:
**assumes** *finite A* **and** *a*:*A* **and** *a* ~: *B*
**shows** *card(A − insert a B) = card(A − B) − 1*
⟨*proof*⟩

**lemma** *card-insert*: *finite A ==> card (insert x A) = Suc (card (A − {x}))*
⟨*proof*⟩

**lemma** *card-insert-le*: *finite A ==> card A <= card (insert x A)*
⟨*proof*⟩

**lemma** *card-mono*: ⟦ *finite B*; *A* ⊆ *B* ⟧ ⟹ *card A* ≤ *card B*
⟨*proof*⟩

**lemma** *card-seteq*: *finite B ==> (!!A. A <= B ==> card B <= card A ==> A = B)*
⟨*proof*⟩

**lemma** *psubset-card-mono*: *finite B ==> A < B ==> card A < card B*
⟨*proof*⟩

**lemma** *card-Un-Int*: *finite A ==> finite B*
*==> card A + card B = card (A Un B) + card (A Int B)*
⟨*proof*⟩

**lemma** *card-Un-disjoint*: *finite A* ==> *finite B*
==> *A Int B* = {} ==> *card* (*A Un B*) = *card A* + *card B*
⟨*proof*⟩

**lemma** *card-Diff-subset*:
*finite B* ==> *B* <= *A* ==> *card* (*A* − *B*) = *card A* − *card B*
⟨*proof*⟩

**lemma** *card-Diff1-less*: *finite A* ==> *x*: *A* ==> *card* (*A* − {*x*}) < *card A*
⟨*proof*⟩

**lemma** *card-Diff2-less*:
*finite A* ==> *x*: *A* ==> *y*: *A* ==> *card* (*A* − {*x*} − {*y*}) < *card A*
⟨*proof*⟩

**lemma** *card-Diff1-le*: *finite A* ==> *card* (*A* − {*x*}) <= *card A*
⟨*proof*⟩

**lemma** *card-psubset*: *finite B* ==> *A* ⊆ *B* ==> *card A* < *card B* ==> *A* < *B*
⟨*proof*⟩

**lemma** *insert-partition*:
⟦ *x* ∉ *F*; ∀ *c1* ∈ *insert x F*. ∀ *c2* ∈ *insert x F*. *c1* ≠ *c2* ⟶ *c1* ∩ *c2* = {} ⟧
⟹ *x* ∩ ⋃ *F* = {}
⟨*proof*⟩

main cardinality theorem

**lemma** *card-partition* [*rule-format*]:
*finite C* ==>
*finite* (⋃ *C*) −−>
(∀ *c*∈*C*. *card c* = *k*) −−>
(∀ *c1* ∈ *C*. ∀ *c2* ∈ *C*. *c1* ≠ *c2* −−> *c1* ∩ *c2* = {}) −−>
*k* ∗ *card*(*C*) = *card* (⋃ *C*)
⟨*proof*⟩

The form of a finite set of given cardinality

**lemma** *card-eq-SucD*:
**assumes** *card A* = *Suc k*
**shows** ∃ *b B*. *A* = *insert b B* & *b* ∉ *B* & *card B* = *k* & (*k*=*0* ⟶ *B*={})
⟨*proof*⟩

**lemma** *card-Suc-eq*:
(*card A* = *Suc k*) =
(∃ *b B*. *A* = *insert b B* & *b* ∉ *B* & *card B* = *k* & (*k*=*0* ⟶ *B*={}))
⟨*proof*⟩

**lemma** *setsum-constant* [*simp*]: (∑ *x* ∈ *A*. *y*) = *of-nat*(*card A*) ∗ *y*
⟨*proof*⟩

**lemma** *setprod-constant*: *finite A ==>* $(\prod x{\in}A.\ (y{::}'a{::}\{recpower, comm\text{-}monoid\text{-}mult\}))$
$= y\hat{}(card\ A)$
⟨*proof*⟩

**lemma** *setprod-gen-delta*:
**assumes** *fS*: *finite S*
**shows** *setprod* $(\lambda k.\ if\ k{=}a\ then\ b\ k\ else\ c)\ S = (if\ a \in S\ then\ (b\ a\ {::}'a{::}\{comm\text{-}monoid\text{-}mult, recpower\})\ *\ c\hat{}\ (card\ S - 1)\ else\ c\hat{}\ card\ S)$
⟨*proof*⟩

**lemma** *setsum-bounded*:
**assumes** *le*: $\bigwedge i.\ i{\in}A \Longrightarrow f\ i \leq (K{::}'a{::}\{semiring\text{-}1, pordered\text{-}ab\text{-}semigroup\text{-}add\})$
**shows** *setsum f A* $\leq$ *of-nat*$(card\ A) * K$
⟨*proof*⟩

### 17.6.1 Cardinality of unions

**lemma** *card-UN-disjoint*:
*finite I ==> (ALL i:I. finite (A i)) ==>*
*(ALL i:I. ALL j:I. i ≠ j --> A i Int A j = {}) ==> card (UNION I A) =* $(\sum i{\in}I.\ card(A\ i))$
⟨*proof*⟩

**lemma** *card-Union-disjoint*:
*finite C ==> (ALL A:C. finite A) ==>*
*(ALL A:C. ALL B:C. A ≠ B --> A Int B = {}) ==> card (Union C) = setsum card C*
⟨*proof*⟩

### 17.6.2 Cardinality of image

The image of a finite set can be expressed using *fold-image*.

**lemma** *image-eq-fold-image*:
*finite A ==> f ' A = fold-image (op Un) (%x. {f x}) {} A*
⟨*proof*⟩

**lemma** *card-image-le*: *finite A ==> card (f ' A) <= card A*
⟨*proof*⟩

**lemma** *card-image*: *inj-on f A ==> card (f ' A) = card A*
⟨*proof*⟩

**lemma** *endo-inj-surj*: *finite A ==> f ' A ⊆ A ==> inj-on f A ==> f ' A = A*
⟨*proof*⟩

**lemma** *eq-card-imp-inj-on*:
*[| finite A; card(f ' A) = card A |] ==> inj-on f A*

⟨*proof*⟩

**lemma** *inj-on-iff-eq-card*:
*finite A ==> inj-on f A = (card(f ' A) = card A)*
⟨*proof*⟩

**lemma** *card-inj-on-le*:
*[|inj-on f A; f ' A ⊆ B; finite B |] ==> card A ≤ card B*
⟨*proof*⟩

**lemma** *card-bij-eq*:
*[|inj-on f A; f ' A ⊆ B; inj-on g B; g ' B ⊆ A;*
*finite A; finite B |] ==> card A = card B*
⟨*proof*⟩

### 17.6.3 Cardinality of products

**lemma** *card-SigmaI* [*simp*]:
*⟦ finite A; ALL a:A. finite (B a) ⟧*
*⟹ card (SIGMA x: A. B x) = (∑ a∈A. card (B a))*
⟨*proof*⟩

**lemma** *card-cartesian-product*: *card (A <*> B) = card(A) * card(B)*
⟨*proof*⟩

**lemma** *card-cartesian-product-singleton*: *card({x} <*> A) = card(A)*
⟨*proof*⟩

### 17.6.4 Cardinality of sums

**lemma** *card-Plus*:
**assumes** *finite A* **and** *finite B*
**shows** *card (A <+> B) = card A + card B*
⟨*proof*⟩

### 17.6.5 Cardinality of the Powerset

**lemma** *card-Pow*: *finite A ==> card (Pow A) = Suc (Suc 0) ^ card A*
⟨*proof*⟩

Relates to equivalence classes. Based on a theorem of F. Kammüller.

**lemma** *dvd-partition*:
*finite (Union C) ==>*
*ALL c : C. k dvd card c ==>*
*(ALL c1: C. ALL c2: C. c1 ≠ c2 --> c1 Int c2 = {}) ==>*
*k dvd card (Union C)*
⟨*proof*⟩

### 17.6.6 Relating injectivity and surjectivity

**lemma** *finite-surj-inj*: *finite*(*A*) $\Longrightarrow$ *A* <= *f*'*A* $\Longrightarrow$ *inj-on f A*
⟨*proof*⟩

**lemma** *finite-UNIV-surj-inj*: **fixes** *f* :: *'a* $\Rightarrow$ *'a*
**shows** *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *surj f* $\Longrightarrow$ *inj f*
⟨*proof*⟩

**lemma** *finite-UNIV-inj-surj*: **fixes** *f* :: *'a* $\Rightarrow$ *'a*
**shows** *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *inj f* $\Longrightarrow$ *surj f*
⟨*proof*⟩

**corollary** *infinite-UNIV-nat*: ~*finite*(*UNIV*::*nat set*)
⟨*proof*⟩

**lemma** *infinite-UNIV-char-0*:
¬ *finite* (*UNIV*::*'a*::*semiring-char-0 set*)
⟨*proof*⟩

## 17.7 A fold functional for non-empty sets

Does not require start value.

**inductive**
*fold1Set* :: (*'a* => *'a* => *'a*) => *'a set* => *'a* => *bool*
**for** *f* :: *'a* => *'a* => *'a*
**where**
*fold1Set-insertI* [*intro*]:
⟦ *fold-graph f a A x*; *a* ∉ *A* ⟧ $\Longrightarrow$ *fold1Set f* (*insert a A*) *x*

**constdefs**
*fold1* :: (*'a* => *'a* => *'a*) => *'a set* => *'a*
*fold1 f A* == *THE x*. *fold1Set f A x*

**lemma** *fold1Set-nonempty*:
*fold1Set f A x* $\Longrightarrow$ *A* ≠ {}
⟨*proof*⟩

**inductive-cases** *empty-fold1SetE* [*elim*!]: *fold1Set f* {} *x*

**inductive-cases** *insert-fold1SetE* [*elim*!]: *fold1Set f* (*insert a X*) *x*

**lemma** *fold1Set-sing* [*iff*]: (*fold1Set f* {*a*} *b*) = (*a* = *b*)
⟨*proof*⟩

**lemma** *fold1-singleton* [*simp*]: *fold1 f* {*a*} = *a*
⟨*proof*⟩

**lemma** *finite-nonempty-imp-fold1Set*:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *EX x*. *fold1Set f A x*
⟨*proof*⟩

First, some lemmas about *fold-graph*.

**context** *ab-semigroup-mult*
**begin**

**lemma** *fun-left-comm*: *fun-left-comm*(*op* ∗)
⟨*proof*⟩

**lemma** *fold-graph-insert-swap*:
**assumes** *fold*: *fold-graph times* (*b*::′*a*) *A y* **and** *b* ∉ *A*
**shows** *fold-graph times z* (*insert b A*) (*z* ∗ *y*)
⟨*proof*⟩

**lemma** *fold-graph-permute-diff*:
**assumes** *fold*: *fold-graph times b A x*
**shows** !!*a*. ⟦*a* ∈ *A*; *b* ∉ *A*⟧ ⟹ *fold-graph times a* (*insert b* (*A*−{*a*})) *x*
⟨*proof*⟩

**lemma** *fold1-eq-fold*:
**assumes** *finite A a* ∉ *A* **shows** *fold1 times* (*insert a A*) = *fold times a A*
⟨*proof*⟩

**lemma** *nonempty-iff*: (*A* ≠ {}) = (∃ *x B*. *A* = *insert x B* & *x* ∉ *B*)
⟨*proof*⟩

**lemma** *fold1-insert*:
**assumes** *nonempty*: *A* ≠ {} **and** *A*: *finite A x* ∉ *A*
**shows** *fold1 times* (*insert x A*) = *x* ∗ *fold1 times A*
⟨*proof*⟩

**end**

**context** *ab-semigroup-idem-mult*
**begin**

**lemma** *fun-left-comm-idem*: *fun-left-comm-idem*(*op* ∗)
⟨*proof*⟩

**lemma** *fold1-insert-idem* [*simp*]:
**assumes** *nonempty*: *A* ≠ {} **and** *A*: *finite A*
**shows** *fold1 times* (*insert x A*) = *x* ∗ *fold1 times A*
⟨*proof*⟩

**lemma** *hom-fold1-commute*:
**assumes** *hom*: !!*x y*. *h* (*x* ∗ *y*) = *h x* ∗ *h y*

**and** $N$: *finite N* $N \neq \{\}$ **shows** $h$ (*fold1 times N*) = *fold1 times* ($h$ ' $N$)
⟨*proof*⟩

**end**

Now the recursion rules for definitions:

**lemma** *fold1-singleton-def*: $g$ = *fold1* $f \Longrightarrow g\ \{a\} = a$
⟨*proof*⟩

**lemma** (**in** *ab-semigroup-mult*) *fold1-insert-def*:
⟦ $g$ = *fold1 times*; *finite* $A$; $x \notin A$; $A \neq \{\}$ ⟧ $\Longrightarrow g$ (*insert x A*) = $x * g\ A$
⟨*proof*⟩

**lemma** (**in** *ab-semigroup-idem-mult*) *fold1-insert-idem-def*:
⟦ $g$ = *fold1 times*; *finite* $A$; $A \neq \{\}$ ⟧ $\Longrightarrow g$ (*insert x A*) = $x * g\ A$
⟨*proof*⟩

### 17.7.1 Determinacy for *fold1Set*

**declare**
*empty-fold-graphE* [*rule del*] *fold-graph.intros* [*rule del*]
*empty-fold1SetE* [*rule del*] *insert-fold1SetE* [*rule del*]
— No more proofs involve these relations.

### 17.7.2 Lemmas about *fold1*

**context** *ab-semigroup-mult*
**begin**

**lemma** *fold1-Un*:
**assumes** $A$: *finite A* $A \neq \{\}$
**shows** *finite* $B \Longrightarrow B \neq \{\} \Longrightarrow A\ Int\ B = \{\} \Longrightarrow$
*fold1 times* (*A Un B*) = *fold1 times A* $*$ *fold1 times B*
⟨*proof*⟩

**lemma** *fold1-in*:
**assumes** $A$: *finite* (*A*) $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x,y\}$
**shows** *fold1 times A* $\in A$
⟨*proof*⟩

**end**

**lemma** (**in** *ab-semigroup-idem-mult*) *fold1-Un2*:
**assumes** $A$: *finite A* $A \neq \{\}$
**shows** *finite* $B \Longrightarrow B \neq \{\} \Longrightarrow$
*fold1 times* (*A Un B*) = *fold1 times A* $*$ *fold1 times B*
⟨*proof*⟩

### 17.7.3 Fold1 in lattices with *inf* and *sup*

As an application of *fold1* we define infimum and supremum in (not necessarily complete!) lattices over (non-empty) sets by means of *fold1*.

**context** *lower-semilattice*
**begin**

**lemma** *ab-semigroup-idem-mult-inf*:
*ab-semigroup-idem-mult inf*
⟨*proof*⟩

**lemma** *below-fold1-iff*:
**assumes** *finite A* $A \neq \{\}$
**shows** $x \leq$ *fold1 inf A* $\longleftrightarrow (\forall a{\in}A.\ x \leq a)$
⟨*proof*⟩

**lemma** *fold1-belowI*:
**assumes** *finite A*
**and** $a \in A$
**shows** *fold1 inf A* $\leq a$
⟨*proof*⟩

**end**

**lemma** (**in** *upper-semilattice*) *ab-semigroup-idem-mult-sup*:
*ab-semigroup-idem-mult sup*
⟨*proof*⟩

**context** *lattice*
**begin**

**definition**
*Inf-fin* :: *'a set* ⇒ *'a* ($\bigsqcap_{fin}$- [*900*] *900*)
**where**
*Inf-fin* = *fold1 inf*

**definition**
*Sup-fin* :: *'a set* ⇒ *'a* ($\bigsqcup_{fin}$- [*900*] *900*)
**where**
*Sup-fin* = *fold1 sup*

**lemma** *Inf-le-Sup* [*simp*]: ⟦ *finite A*; $A \neq \{\}$ ⟧ ⟹ $\bigsqcap_{fin} A \leq \bigsqcup_{fin} A$
⟨*proof*⟩

**lemma** *sup-Inf-absorb* [*simp*]:
*finite A* ⟹ $a \in A$ ⟹ *sup a* ($\bigsqcap_{fin} A$) = $a$
⟨*proof*⟩

**lemma** *inf-Sup-absorb* [*simp*]:

*finite* $A \Longrightarrow a \in A \Longrightarrow$ *inf* $a$ $(\bigsqcup_{fin} A) = a$
⟨*proof*⟩

**end**

**context** *distrib-lattice*
**begin**

**lemma** *sup-Inf1-distrib*:
**assumes** *finite* $A$
**and** $A \neq \{\}$
**shows** *sup* $x$ $(\bigsqcap_{fin} A) = \bigsqcap_{fin} \{sup\ x\ a | a.\ a \in A\}$
⟨*proof*⟩

**lemma** *sup-Inf2-distrib*:
**assumes** $A$: *finite* $A$ $A \neq \{\}$ **and** $B$: *finite* $B$ $B \neq \{\}$
**shows** *sup* $(\bigsqcap_{fin} A)$ $(\bigsqcap_{fin} B) = \bigsqcap_{fin} \{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

**lemma** *inf-Sup1-distrib*:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** *inf* $x$ $(\bigsqcup_{fin} A) = \bigsqcup_{fin} \{inf\ x\ a | a.\ a \in A\}$
⟨*proof*⟩

**lemma** *inf-Sup2-distrib*:
**assumes** $A$: *finite* $A$ $A \neq \{\}$ **and** $B$: *finite* $B$ $B \neq \{\}$
**shows** *inf* $(\bigsqcup_{fin} A)$ $(\bigsqcup_{fin} B) = \bigsqcup_{fin} \{inf\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

**end**

**context** *complete-lattice*
**begin**

Coincidence on finite sets in complete lattices:

**lemma** *Inf-fin-Inf*:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $\bigsqcap_{fin} A = Inf\ A$
⟨*proof*⟩

**lemma** *Sup-fin-Sup*:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $\bigsqcup_{fin} A = Sup\ A$
⟨*proof*⟩

**end**

### 17.7.4 Fold1 in linear orders with *min* and *max*

As an application of *fold1* we define minimum and maximum in (not necessarily complete!) linear orders over (non-empty) sets by means of *fold1*.

**context** *linorder*
**begin**

**lemma** *ab-semigroup-idem-mult-min*:
*ab-semigroup-idem-mult min*
⟨*proof*⟩

**lemma** *ab-semigroup-idem-mult-max*:
*ab-semigroup-idem-mult max*
⟨*proof*⟩

**lemma** *min-lattice*:
*lower-semilattice* (*op* ≤) (*op* <) *min*
⟨*proof*⟩

**lemma** *max-lattice*:
*lower-semilattice* (*op* ≥) (*op* >) *max*
⟨*proof*⟩

**lemma** *dual-max*:
*ord.max* (*op* ≥) = *min*
⟨*proof*⟩

**lemma** *dual-min*:
*ord.min* (*op* ≥) = *max*
⟨*proof*⟩

**lemma** *strict-below-fold1-iff*:
**assumes** *finite A* **and** *A* ≠ {}
**shows** *x* < *fold1 min A* ⟷ (∀ *a*∈*A*. *x* < *a*)
⟨*proof*⟩

**lemma** *fold1-below-iff*:
**assumes** *finite A* **and** *A* ≠ {}
**shows** *fold1 min A* ≤ *x* ⟷ (∃ *a*∈*A*. *a* ≤ *x*)
⟨*proof*⟩

**lemma** *fold1-strict-below-iff*:
**assumes** *finite A* **and** *A* ≠ {}
**shows** *fold1 min A* < *x* ⟷ (∃ *a*∈*A*. *a* < *x*)
⟨*proof*⟩

**lemma** *fold1-antimono*:
**assumes** *A* ≠ {} **and** *A* ⊆ *B* **and** *finite B*
**shows** *fold1 min B* ≤ *fold1 min A*

⟨*proof*⟩

**definition**
  *Min* :: *'a set* ⇒ *'a*
**where**
  *Min* = *fold1 min*

**definition**
  *Max* :: *'a set* ⇒ *'a*
**where**
  *Max* = *fold1 max*

**lemmas** *Min-singleton* [*simp*] = *fold1-singleton-def* [*OF Min-def*]
**lemmas** *Max-singleton* [*simp*] = *fold1-singleton-def* [*OF Max-def*]

**lemma** *Min-insert* [*simp*]:
  **assumes** *finite A* **and** *A* ≠ {}
  **shows** *Min* (*insert x A*) = *min x* (*Min A*)
⟨*proof*⟩

**lemma** *Max-insert* [*simp*]:
  **assumes** *finite A* **and** *A* ≠ {}
  **shows** *Max* (*insert x A*) = *max x* (*Max A*)
⟨*proof*⟩

**lemma** *Min-in* [*simp*]:
  **assumes** *finite A* **and** *A* ≠ {}
  **shows** *Min A* ∈ *A*
⟨*proof*⟩

**lemma** *Max-in* [*simp*]:
  **assumes** *finite A* **and** *A* ≠ {}
  **shows** *Max A* ∈ *A*
⟨*proof*⟩

**lemma** *Min-Un*:
  **assumes** *finite A* **and** *A* ≠ {} **and** *finite B* **and** *B* ≠ {}
  **shows** *Min* (*A* ∪ *B*) = *min* (*Min A*) (*Min B*)
⟨*proof*⟩

**lemma** *Max-Un*:
  **assumes** *finite A* **and** *A* ≠ {} **and** *finite B* **and** *B* ≠ {}
  **shows** *Max* (*A* ∪ *B*) = *max* (*Max A*) (*Max B*)
⟨*proof*⟩

**lemma** *hom-Min-commute*:
  **assumes** ⋀*x y*. *h* (*min x y*) = *min* (*h x*) (*h y*)
    **and** *finite N* **and** *N* ≠ {}
  **shows** *h* (*Min N*) = *Min* (*h* ‘ *N*)

⟨*proof*⟩

**lemma** *hom-Max-commute*:
**assumes** $\bigwedge x\ y.\ h\ (max\ x\ y) = max\ (h\ x)\ (h\ y)$
**and** *finite* $N$ **and** $N \neq \{\}$
**shows** $h\ (Max\ N) = Max\ (h\ `\ N)$
⟨*proof*⟩

**lemma** *Min-le* [*simp*]:
**assumes** *finite* $A$ **and** $x \in A$
**shows** $Min\ A \leq x$
⟨*proof*⟩

**lemma** *Max-ge* [*simp*]:
**assumes** *finite* $A$ **and** $x \in A$
**shows** $x \leq Max\ A$
⟨*proof*⟩

**lemma** *Min-ge-iff* [*simp*, *noatp*]:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $x \leq Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

**lemma** *Max-le-iff* [*simp*, *noatp*]:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $Max\ A \leq x \longleftrightarrow (\forall\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

**lemma** *Min-gr-iff* [*simp*, *noatp*]:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $x < Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x < a)$
⟨*proof*⟩

**lemma** *Max-less-iff* [*simp*, *noatp*]:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $Max\ A < x \longleftrightarrow (\forall\, a{\in}A.\ a < x)$
⟨*proof*⟩

**lemma** *Min-le-iff* [*noatp*]:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $Min\ A \leq x \longleftrightarrow (\exists\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

**lemma** *Max-ge-iff* [*noatp*]:
**assumes** *finite* $A$ **and** $A \neq \{\}$
**shows** $x \leq Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

**lemma** *Min-less-iff* [*noatp*]:

**assumes** *finite A* **and** $A \neq \{\}$
**shows** $Min\ A < x \longleftrightarrow (\exists\, a{\in}A.\ a < x)$
⟨*proof*⟩

**lemma** *Max-gr-iff* [*noatp*]:
**assumes** *finite A* **and** $A \neq \{\}$
**shows** $x < Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x < a)$
⟨*proof*⟩

**lemma** *Min-eqI*:
**assumes** *finite A*
**assumes** $\bigwedge y.\ y \in A \Longrightarrow y \geq x$
**and** $x \in A$
**shows** $Min\ A = x$
⟨*proof*⟩

**lemma** *Max-eqI*:
**assumes** *finite A*
**assumes** $\bigwedge y.\ y \in A \Longrightarrow y \leq x$
**and** $x \in A$
**shows** $Max\ A = x$
⟨*proof*⟩

**lemma** *Min-antimono*:
**assumes** $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
**shows** $Min\ N \leq Min\ M$
⟨*proof*⟩

**lemma** *Max-mono*:
**assumes** $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
**shows** $Max\ M \leq Max\ N$
⟨*proof*⟩

**lemma** *finite-linorder-induct*[*consumes 1*, *case-names empty insert*]:
*finite A* $\Longrightarrow$ $P\ \{\} \Longrightarrow$
$(!!A\ b.\ finite\ A \Longrightarrow ALL\ a{:}A.\ a < b \Longrightarrow P\ A \Longrightarrow P(insert\ b\ A))$
$\Longrightarrow P\ A$
⟨*proof*⟩

**end**

**context** *ordered-ab-semigroup-add*
**begin**

**lemma** *add-Min-commute*:
**fixes** *k*
**assumes** *finite N* **and** $N \neq \{\}$
**shows** $k + Min\ N = Min\ \{k + m \mid m.\ m \in N\}$
⟨*proof*⟩

**lemma** *add-Max-commute*:
**fixes** $k$
**assumes** *finite* $N$ **and** $N \neq \{\}$
**shows** $k + Max\ N = Max\ \{k + m \mid m.\ m \in N\}$
⟨*proof*⟩

**end**

**end**

# 18 Relation: Relations

**theory** *Relation*
**imports** *Datatype Finite-Set*
**begin**

## 18.1 Definitions

**definition**
*converse* :: $('a * 'b)\ set \Rightarrow ('b * 'a)\ set$
((-^−1) [1000] 999) **where**
$r\hat{}-1 == \{(y, x).\ (x, y) : r\}$

**notation** (*xsymbols*)
*converse* ((-$^{-1}$) [1000] 999)

**definition**
*rel-comp* :: $[('b * 'c)\ set, ('a * 'b)\ set] \Rightarrow ('a * 'c)\ set$
(**infixr** *O* 75) **where**
$r\ O\ s == \{(x,z).\ EX\ y.\ (x, y) : s\ \&\ (y, z) : r\}$

**definition**
*Image* :: $[('a * 'b)\ set, 'a\ set] \Rightarrow 'b\ set$
(**infixl** `` 90) **where**
$r\ ``\ s == \{y.\ EX\ x{:}s.\ (x,y){:}r\}$

**definition**
*Id* :: $('a * 'a)\ set$ **where** — the identity relation
$Id == \{p.\ EX\ x.\ p = (x,x)\}$

**definition**
*Id-on* :: $'a\ set \Rightarrow ('a * 'a)\ set$ **where** — diagonal: identity over a set
$Id\text{-}on\ A == \bigcup x{\in}A.\ \{(x,x)\}$

**definition**
*Domain* :: $('a * 'b)\ set \Rightarrow 'a\ set$ **where**
$Domain\ r == \{x.\ EX\ y.\ (x,y){:}r\}$

**definition**
*Range* :: $('a * 'b)$ *set* => $'b$ *set* **where**
*Range* $r$ == *Domain*$(r\hat{}-1)$

**definition**
*Field* :: $('a * 'a)$ *set* => $'a$ *set* **where**
*Field* $r$ == *Domain* $r \cup$ *Range* $r$

**definition**
*refl-on* :: [$'a$ *set*, $('a * 'a)$ *set*] => *bool* **where** — reflexivity over a set
*refl-on* $A$ $r$ == $r \subseteq A \times A$ & (*ALL* $x$: $A$. $(x,x)$ : $r$)

**abbreviation**
*refl* :: $('a * 'a)$ *set* => *bool* **where** — reflexivity over a type
*refl* == *refl-on UNIV*

**definition**
*sym* :: $('a * 'a)$ *set* => *bool* **where** — symmetry predicate
*sym* $r$ == *ALL* $x$ $y$. $(x,y)$: $r$ −−> $(y,x)$: $r$

**definition**
*antisym* :: $('a * 'a)$ *set* => *bool* **where** — antisymmetry predicate
*antisym* $r$ == *ALL* $x$ $y$. $(x,y)$:$r$ −−> $(y,x)$:$r$ −−> $x$=$y$

**definition**
*trans* :: $('a * 'a)$ *set* => *bool* **where** — transitivity predicate
*trans* $r$ == (*ALL* $x$ $y$ $z$. $(x,y)$:$r$ −−> $(y,z)$:$r$ −−> $(x,z)$:$r$)

**definition**
*irrefl* :: $('a * 'a)$ *set* => *bool* **where**
*irrefl* $r \equiv \forall x$. $(x,x) \notin r$

**definition**
*total-on* :: $'a$ *set* => $('a * 'a)$ *set* => *bool* **where**
*total-on* $A$ $r \equiv \forall x{\in}A. \forall y{\in}A.\ x{\neq}y \longrightarrow (x,y){\in}r \vee (y,x){\in}r$

**abbreviation** *total* $\equiv$ *total-on UNIV*

**definition**
*single-valued* :: $('a * 'b)$ *set* => *bool* **where**
*single-valued* $r$ == *ALL* $x$ $y$. $(x,y)$:$r$ −−> (*ALL* $z$. $(x,z)$:$r$ −−> $y$=$z$)

**definition**
*inv-image* :: $('b * 'b)$ *set* => $('a$ => $'b) => ('a * 'a)$ *set* **where**
*inv-image* $r$ $f$ == $\{(x,\ y).\ (f\ x,\ f\ y) : r\}$

## 18.2 The identity relation

**lemma** *IdI* [*intro*]: $(a, a) : Id$
⟨*proof*⟩

**lemma** *IdE* [*elim*!]: $p : Id \Longrightarrow (!!x.\ p = (x, x) \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *pair-in-Id-conv* [*iff*]: $((a, b) : Id) = (a = b)$
⟨*proof*⟩

**lemma** *refl-Id*: *refl Id*
⟨*proof*⟩

**lemma** *antisym-Id*: *antisym Id*
— A strange result, since *Id* is also symmetric.
⟨*proof*⟩

**lemma** *sym-Id*: *sym Id*
⟨*proof*⟩

**lemma** *trans-Id*: *trans Id*
⟨*proof*⟩

## 18.3 Diagonal: identity over a set

**lemma** *Id-on-empty* [*simp*]: *Id-on* $\{\} = \{\}$
⟨*proof*⟩

**lemma** *Id-on-eqI*: $a = b \Longrightarrow a : A \Longrightarrow (a, b) : Id\text{-}on\ A$
⟨*proof*⟩

**lemma** *Id-onI* [*intro*!,*noatp*]: $a : A \Longrightarrow (a, a) : Id\text{-}on\ A$
⟨*proof*⟩

**lemma** *Id-onE* [*elim*!]:
$c : Id\text{-}on\ A \Longrightarrow (!!x.\ x : A \Longrightarrow c = (x, x) \Longrightarrow P) \Longrightarrow P$
— The general elimination rule.
⟨*proof*⟩

**lemma** *Id-on-iff*: $((x, y) : Id\text{-}on\ A) = (x = y\ \&\ x : A)$
⟨*proof*⟩

**lemma** *Id-on-subset-Times*: $Id\text{-}on\ A \subseteq A \times A$
⟨*proof*⟩

## 18.4 Composition of two relations

**lemma** *rel-compI* [*intro*]:
$(a, b) : s \Longrightarrow (b, c) : r \Longrightarrow (a, c) : r\ O\ s$

⟨*proof*⟩

**lemma** *rel-compE* [*elim*!]: $xz : r\ O\ s \Longrightarrow$
$(!!x\ y\ z.\ xz = (x,\ z) \Longrightarrow (x,\ y) : s \Longrightarrow (y,\ z) : r \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *rel-compEpair*:
$(a,\ c) : r\ O\ s \Longrightarrow (!!y.\ (a,\ y) : s \Longrightarrow (y,\ c) : r \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *R-O-Id* [*simp*]: $R\ O\ Id = R$
⟨*proof*⟩

**lemma** *Id-O-R* [*simp*]: $Id\ O\ R = R$
⟨*proof*⟩

**lemma** *rel-comp-empty1*[*simp*]: $\{\}\ O\ R = \{\}$
⟨*proof*⟩

**lemma** *rel-comp-empty2*[*simp*]: $R\ O\ \{\} = \{\}$
⟨*proof*⟩

**lemma** *O-assoc*: $(R\ O\ S)\ O\ T = R\ O\ (S\ O\ T)$
⟨*proof*⟩

**lemma** *trans-O-subset*: $trans\ r \Longrightarrow r\ O\ r \subseteq r$
⟨*proof*⟩

**lemma** *rel-comp-mono*: $r' \subseteq r \Longrightarrow s' \subseteq s \Longrightarrow (r'\ O\ s') \subseteq (r\ O\ s)$
⟨*proof*⟩

**lemma** *rel-comp-subset-Sigma*:
$s \subseteq A \times B \Longrightarrow r \subseteq B \times C \Longrightarrow (r\ O\ s) \subseteq A \times C$
⟨*proof*⟩

**lemma** *rel-comp-distrib*[*simp*]: $R\ O\ (S \cup T) = (R\ O\ S) \cup (R\ O\ T)$
⟨*proof*⟩

**lemma** *rel-comp-distrib2*[*simp*]: $(S \cup T)\ O\ R = (S\ O\ R) \cup (T\ O\ R)$
⟨*proof*⟩

## 18.5 Reflexivity

**lemma** *refl-onI*: $r \subseteq A \times A \Longrightarrow (!!x.\ x : A \Longrightarrow (x,\ x) : r) \Longrightarrow refl\text{-}on\ A\ r$
⟨*proof*⟩

**lemma** *refl-onD*: $refl\text{-}on\ A\ r \Longrightarrow a : A \Longrightarrow (a,\ a) : r$
⟨*proof*⟩

**lemma** *refl-onD1*: *refl-on A r* ==> $(x, y) : r$ ==> $x : A$
⟨*proof*⟩

**lemma** *refl-onD2*: *refl-on A r* ==> $(x, y) : r$ ==> $y : A$
⟨*proof*⟩

**lemma** *refl-on-Int*: *refl-on A r* ==> *refl-on B s* ==> *refl-on* $(A \cap B)$ $(r \cap s)$
⟨*proof*⟩

**lemma** *refl-on-Un*: *refl-on A r* ==> *refl-on B s* ==> *refl-on* $(A \cup B)$ $(r \cup s)$
⟨*proof*⟩

**lemma** *refl-on-INTER*:
*ALL x:S. refl-on* $(A\ x)$ $(r\ x)$ ==> *refl-on* $(INTER\ S\ A)$ $(INTER\ S\ r)$
⟨*proof*⟩

**lemma** *refl-on-UNION*:
*ALL x:S. refl-on* $(A\ x)$ $(r\ x) \Longrightarrow$ *refl-on* $(UNION\ S\ A)$ $(UNION\ S\ r)$
⟨*proof*⟩

**lemma** *refl-on-empty*[*simp*]: *refl-on* {} {}
⟨*proof*⟩

**lemma** *refl-on-Id-on*: *refl-on A* (*Id-on A*)
⟨*proof*⟩

## 18.6 Antisymmetry

**lemma** *antisymI*:
$(!!x\ y.\ (x, y) : r$ ==> $(y, x) : r$ ==> $x=y)$ ==> *antisym r*
⟨*proof*⟩

**lemma** *antisymD*: *antisym r* ==> $(a, b) : r$ ==> $(b, a) : r$ ==> $a = b$
⟨*proof*⟩

**lemma** *antisym-subset*: $r \subseteq s$ ==> *antisym s* ==> *antisym r*
⟨*proof*⟩

**lemma** *antisym-empty* [*simp*]: *antisym* {}
⟨*proof*⟩

**lemma** *antisym-Id-on* [*simp*]: *antisym* (*Id-on A*)
⟨*proof*⟩

## 18.7 Symmetry

**lemma** *symI*: $(!!a\ b.\ (a, b) : r$ ==> $(b, a) : r)$ ==> *sym r*
⟨*proof*⟩

**lemma** *symD*: *sym r* ==> $(a, b) : r$ ==> $(b, a) : r$

⟨*proof*⟩

**lemma** *sym-Int*: *sym r* ==> *sym s* ==> *sym* (*r* ∩ *s*)
⟨*proof*⟩

**lemma** *sym-Un*: *sym r* ==> *sym s* ==> *sym* (*r* ∪ *s*)
⟨*proof*⟩

**lemma** *sym-INTER*: *ALL x*:*S*. *sym* (*r x*) ==> *sym* (*INTER S r*)
⟨*proof*⟩

**lemma** *sym-UNION*: *ALL x*:*S*. *sym* (*r x*) ==> *sym* (*UNION S r*)
⟨*proof*⟩

**lemma** *sym-Id-on* [*simp*]: *sym* (*Id-on A*)
⟨*proof*⟩

## 18.8 Transitivity

**lemma** *transI*:
(!!*x y z*. (*x*, *y*) : *r* ==> (*y*, *z*) : *r* ==> (*x*, *z*) : *r*) ==> *trans r*
⟨*proof*⟩

**lemma** *transD*: *trans r* ==> (*a*, *b*) : *r* ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*
⟨*proof*⟩

**lemma** *trans-Int*: *trans r* ==> *trans s* ==> *trans* (*r* ∩ *s*)
⟨*proof*⟩

**lemma** *trans-INTER*: *ALL x*:*S*. *trans* (*r x*) ==> *trans* (*INTER S r*)
⟨*proof*⟩

**lemma** *trans-Id-on* [*simp*]: *trans* (*Id-on A*)
⟨*proof*⟩

**lemma** *trans-diff-Id*: *trans r* ⟹ *antisym r* ⟹ *trans* (*r*−*Id*)
⟨*proof*⟩

## 18.9 Irreflexivity

**lemma** *irrefl-diff-Id*[*simp*]: *irrefl*(*r*−*Id*)
⟨*proof*⟩

## 18.10 Totality

**lemma** *total-on-empty*[*simp*]: *total-on* {} *r*
⟨*proof*⟩

**lemma** *total-on-diff-Id*[*simp*]: *total-on A* (*r*−*Id*) = *total-on A r*
⟨*proof*⟩

## 18.11 Converse

**lemma** *converse-iff* [*iff*]: $((a,b): r\hat{}-1) = ((b,a) : r)$
⟨*proof*⟩

**lemma** *converseI*[*sym*]: $(a,\ b) : r \Longrightarrow (b,\ a) : r\hat{}-1$
⟨*proof*⟩

**lemma** *converseD*[*sym*]: $(a,b) : r\hat{}-1 \Longrightarrow (b,\ a) : r$
⟨*proof*⟩

**lemma** *converseE* [*elim*!]:
$yx : r\hat{}-1 \Longrightarrow (!!x\ y.\ yx = (y,\ x) \Longrightarrow (x,\ y) : r \Longrightarrow P) \Longrightarrow P$
— More general than *converseD*, as it "splits" the member of the relation.
⟨*proof*⟩

**lemma** *converse-converse* [*simp*]: $(r\hat{}-1)\hat{}-1 = r$
⟨*proof*⟩

**lemma** *converse-rel-comp*: $(r\ O\ s)\hat{}-1 = s\hat{}-1\ O\ r\hat{}-1$
⟨*proof*⟩

**lemma** *converse-Int*: $(r \cap s)\hat{}-1 = r\hat{}-1 \cap s\hat{}-1$
⟨*proof*⟩

**lemma** *converse-Un*: $(r \cup s)\hat{}-1 = r\hat{}-1 \cup s\hat{}-1$
⟨*proof*⟩

**lemma** *converse-INTER*: $(INTER\ S\ r)\hat{}-1 = (INT\ x{:}S.\ (r\ x)\hat{}-1)$
⟨*proof*⟩

**lemma** *converse-UNION*: $(UNION\ S\ r)\hat{}-1 = (UN\ x{:}S.\ (r\ x)\hat{}-1)$
⟨*proof*⟩

**lemma** *converse-Id* [*simp*]: $Id\hat{}-1 = Id$
⟨*proof*⟩

**lemma** *converse-Id-on* [*simp*]: $(Id\text{-}on\ A)\hat{}-1 = Id\text{-}on\ A$
⟨*proof*⟩

**lemma** *refl-on-converse* [*simp*]: $refl\text{-}on\ A\ (converse\ r) = refl\text{-}on\ A\ r$
⟨*proof*⟩

**lemma** *sym-converse* [*simp*]: $sym\ (converse\ r) = sym\ r$
⟨*proof*⟩

**lemma** *antisym-converse* [*simp*]: $antisym\ (converse\ r) = antisym\ r$
⟨*proof*⟩

**lemma** *trans-converse* [*simp*]: $trans\ (converse\ r) = trans\ r$

⟨*proof*⟩

**lemma** *sym-conv-converse-eq*: *sym r* = (*r*^−*1* = *r*)
⟨*proof*⟩

**lemma** *sym-Un-converse*: *sym* (*r* ∪ *r*^−*1*)
⟨*proof*⟩

**lemma** *sym-Int-converse*: *sym* (*r* ∩ *r*^−*1*)
⟨*proof*⟩

**lemma** *total-on-converse*[*simp*]: *total-on A* (*r*^−*1*) = *total-on A r*
⟨*proof*⟩

## 18.12 Domain

**declare** *Domain-def* [*noatp*]

**lemma** *Domain-iff*: (*a* : *Domain r*) = (*EX y*. (*a*, *y*) : *r*)
⟨*proof*⟩

**lemma** *DomainI* [*intro*]: (*a*, *b*) : *r* ==> *a* : *Domain r*
⟨*proof*⟩

**lemma** *DomainE* [*elim*!]:
*a* : *Domain r* ==> (!!*y*. (*a*, *y*) : *r* ==> *P*) ==> *P*
⟨*proof*⟩

**lemma** *Domain-empty* [*simp*]: *Domain* {} = {}
⟨*proof*⟩

**lemma** *Domain-insert*: *Domain* (*insert* (*a*, *b*) *r*) = *insert a* (*Domain r*)
⟨*proof*⟩

**lemma** *Domain-Id* [*simp*]: *Domain Id* = *UNIV*
⟨*proof*⟩

**lemma** *Domain-Id-on* [*simp*]: *Domain* (*Id-on A*) = *A*
⟨*proof*⟩

**lemma** *Domain-Un-eq*: *Domain*(*A* ∪ *B*) = *Domain*(*A*) ∪ *Domain*(*B*)
⟨*proof*⟩

**lemma** *Domain-Int-subset*: *Domain*(*A* ∩ *B*) ⊆ *Domain*(*A*) ∩ *Domain*(*B*)
⟨*proof*⟩

**lemma** *Domain-Diff-subset*: *Domain*(*A*) − *Domain*(*B*) ⊆ *Domain*(*A* − *B*)
⟨*proof*⟩

**lemma** *Domain-Union*: *Domain* (*Union S*) = (⋃ *A*∈*S*. *Domain A*)
⟨*proof*⟩

**lemma** *Domain-converse*[*simp*]: *Domain*(*r*^−1) = *Range r*
⟨*proof*⟩

**lemma** *Domain-mono*: *r* ⊆ *s* ==> *Domain r* ⊆ *Domain s*
⟨*proof*⟩

**lemma** *fst-eq-Domain*: *fst* ‘ *R* = *Domain R*
⟨*proof*⟩

**lemma** *Domain-dprod* [*simp*]: *Domain* (*dprod r s*) = *uprod* (*Domain r*) (*Domain s*)
⟨*proof*⟩

**lemma** *Domain-dsum* [*simp*]: *Domain* (*dsum r s*) = *usum* (*Domain r*) (*Domain s*)
⟨*proof*⟩

## 18.13 Range

**lemma** *Range-iff*: (*a* : *Range r*) = (*EX y*. (*y*, *a*) : *r*)
⟨*proof*⟩

**lemma** *RangeI* [*intro*]: (*a*, *b*) : *r* ==> *b* : *Range r*
⟨*proof*⟩

**lemma** *RangeE* [*elim*!]: *b* : *Range r* ==> (!!*x*. (*x*, *b*) : *r* ==> *P*) ==> *P*
⟨*proof*⟩

**lemma** *Range-empty* [*simp*]: *Range* {} = {}
⟨*proof*⟩

**lemma** *Range-insert*: *Range* (*insert* (*a*, *b*) *r*) = *insert b* (*Range r*)
⟨*proof*⟩

**lemma** *Range-Id* [*simp*]: *Range Id* = *UNIV*
⟨*proof*⟩

**lemma** *Range-Id-on* [*simp*]: *Range* (*Id-on A*) = *A*
⟨*proof*⟩

**lemma** *Range-Un-eq*: *Range*(*A* ∪ *B*) = *Range*(*A*) ∪ *Range*(*B*)
⟨*proof*⟩

**lemma** *Range-Int-subset*: *Range*(*A* ∩ *B*) ⊆ *Range*(*A*) ∩ *Range*(*B*)
⟨*proof*⟩

**lemma** *Range-Diff-subset*: $Range(A) - Range(B) \subseteq Range(A - B)$
⟨*proof*⟩

**lemma** *Range-Union*: $Range\ (Union\ S) = (\bigcup A{\in}S.\ Range\ A)$
⟨*proof*⟩

**lemma** *Range-converse*[*simp*]: $Range(r\hat{}{-}1) = Domain\ r$
⟨*proof*⟩

**lemma** *snd-eq-Range*: $snd\ `\ R = Range\ R$
⟨*proof*⟩

## 18.14 Field

**lemma** *mono-Field*: $r \subseteq s \Longrightarrow Field\ r \subseteq Field\ s$
⟨*proof*⟩

**lemma** *Field-empty*[*simp*]: $Field\ \{\} = \{\}$
⟨*proof*⟩

**lemma** *Field-insert*[*simp*]: $Field\ (insert\ (a,b)\ r) = \{a,b\} \cup Field\ r$
⟨*proof*⟩

**lemma** *Field-Un*[*simp*]: $Field\ (r \cup s) = Field\ r \cup Field\ s$
⟨*proof*⟩

**lemma** *Field-Union*[*simp*]: $Field\ (\bigcup R) = \bigcup (Field\ `\ R)$
⟨*proof*⟩

**lemma** *Field-converse*[*simp*]: $Field(r\hat{}{-}1) = Field\ r$
⟨*proof*⟩

## 18.15 Image of a set under a relation

**declare** *Image-def* [*noatp*]

**lemma** *Image-iff*: $(b : r``A) = (EX\ x{:}A.\ (x,\ b) : r)$
⟨*proof*⟩

**lemma** *Image-singleton*: $r``\{a\} = \{b.\ (a,\ b) : r\}$
⟨*proof*⟩

**lemma** *Image-singleton-iff* [*iff*]: $(b : r``\{a\}) = ((a,\ b) : r)$
⟨*proof*⟩

**lemma** *ImageI* [*intro*,*noatp*]: $(a,\ b) : r \Longrightarrow a : A \Longrightarrow b : r``A$
⟨*proof*⟩

**lemma** *ImageE* [*elim*!]:
$b : r\ ``\ A \Longrightarrow (!!x.\ (x,\ b) : r \Longrightarrow x : A \Longrightarrow P) \Longrightarrow P$

⟨*proof*⟩

**lemma** *rev-ImageI*: $a : A \Longrightarrow (a, b) : r \Longrightarrow b : r \mathbin{``} A$
— This version's more effective when we already have the required $a$
⟨*proof*⟩

**lemma** *Image-empty* [*simp*]: $R``\{\} = \{\}$
⟨*proof*⟩

**lemma** *Image-Id* [*simp*]: $Id \mathbin{``} A = A$
⟨*proof*⟩

**lemma** *Image-Id-on* [*simp*]: $Id\text{-}on\ A \mathbin{``} B = A \cap B$
⟨*proof*⟩

**lemma** *Image-Int-subset*: $R \mathbin{``} (A \cap B) \subseteq R \mathbin{``} A \cap R \mathbin{``} B$
⟨*proof*⟩

**lemma** *Image-Int-eq*:
$single\text{-}valued\ (converse\ R) \Longrightarrow R \mathbin{``} (A \cap B) = R \mathbin{``} A \cap R \mathbin{``} B$
⟨*proof*⟩

**lemma** *Image-Un*: $R \mathbin{``} (A \cup B) = R \mathbin{``} A \cup R \mathbin{``} B$
⟨*proof*⟩

**lemma** *Un-Image*: $(R \cup S) \mathbin{``} A = R \mathbin{``} A \cup S \mathbin{``} A$
⟨*proof*⟩

**lemma** *Image-subset*: $r \subseteq A \times B \Longrightarrow r``C \subseteq B$
⟨*proof*⟩

**lemma** *Image-eq-UN*: $r``B = (\bigcup y{\in}\ B.\ r``\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

**lemma** *Image-mono*: $r' \subseteq r \Longrightarrow A' \subseteq A \Longrightarrow (r' \mathbin{``} A') \subseteq (r \mathbin{``} A)$
⟨*proof*⟩

**lemma** *Image-UN*: $(r \mathbin{``} (UNION\ A\ B)) = (\bigcup x{\in}A.\ r \mathbin{``} (B\ x))$
⟨*proof*⟩

**lemma** *Image-INT-subset*: $(r \mathbin{``} INTER\ A\ B) \subseteq (\bigcap x{\in}A.\ r \mathbin{``} (B\ x))$
⟨*proof*⟩

Converse inclusion requires some assumptions

**lemma** *Image-INT-eq*:
$[|single\text{-}valued\ (r^{-1});\ A{\neq}\{\}|] \Longrightarrow r \mathbin{``} INTER\ A\ B = (\bigcap x{\in}A.\ r \mathbin{``} B\ x)$
⟨*proof*⟩

**lemma** *Image-subset-eq*: $(r``A \subseteq B) = (A \subseteq - ((r\hat{}-1) \mathbin{``} (-B)))$
⟨*proof*⟩

## 18.16 Single valued relations

**lemma** *single-valuedI*:
$ALL\ x\ y.\ (x,y):r \longrightarrow (ALL\ z.\ (x,z):r \longrightarrow y=z) \Longrightarrow single\text{-}valued\ r$
⟨*proof*⟩

**lemma** *single-valuedD*:
$single\text{-}valued\ r \Longrightarrow (x,\ y) : r \Longrightarrow (x,\ z) : r \Longrightarrow y = z$
⟨*proof*⟩

**lemma** *single-valued-rel-comp*:
$single\text{-}valued\ r \Longrightarrow single\text{-}valued\ s \Longrightarrow single\text{-}valued\ (r\ O\ s)$
⟨*proof*⟩

**lemma** *single-valued-subset*:
$r \subseteq s \Longrightarrow single\text{-}valued\ s \Longrightarrow single\text{-}valued\ r$
⟨*proof*⟩

**lemma** *single-valued-Id* [*simp*]: *single-valued Id*
⟨*proof*⟩

**lemma** *single-valued-Id-on* [*simp*]: *single-valued* (*Id-on A*)
⟨*proof*⟩

## 18.17 Graphs given by *Collect*

**lemma** *Domain-Collect-split* [*simp*]: $Domain\{(x,y).\ P\ x\ y\} = \{x.\ EX\ y.\ P\ x\ y\}$
⟨*proof*⟩

**lemma** *Range-Collect-split* [*simp*]: $Range\{(x,y).\ P\ x\ y\} = \{y.\ EX\ x.\ P\ x\ y\}$
⟨*proof*⟩

**lemma** *Image-Collect-split* [*simp*]: $\{(x,y).\ P\ x\ y\} \mathbin{``} A = \{y.\ EX\ x{:}A.\ P\ x\ y\}$
⟨*proof*⟩

## 18.18 Inverse image

**lemma** *sym-inv-image*: $sym\ r \Longrightarrow sym\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

**lemma** *trans-inv-image*: $trans\ r \Longrightarrow trans\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

## 18.19 Finiteness

**lemma** *finite-converse* [*iff*]: $finite\ (r\hat{}-1) = finite\ r$
⟨*proof*⟩

**Finiteness of transitive closure** (Thanks to Sidi Ehmety)

**lemma** *finite-Field*: *finite r ==> finite* (*Field r*)
— A finite relation has a finite field (= *domain* ∪ *range*.
⟨*proof*⟩

### 18.20 Version of *lfp-induct* for binary relations

**lemmas** *lfp-induct2* =
*lfp-induct-set* [*of* (*a*, *b*), *split-format* (*complete*)]

**end**

## 19 Predicate: Predicates as relations and enumerations

**theory** *Predicate*
**imports** *Inductive Relation*
**begin**

**notation**
*inf* (**infixl** ⊓ *70*) **and**
*sup* (**infixl** ⊔ *65*) **and**
*Inf* (⨅ - [*900*] *900*) **and**
*Sup* (⨆ - [*900*] *900*) **and**
*top* (⊤) **and**
*bot* (⊥)

### 19.1 Predicates as (complete) lattices

#### 19.1.1 *op* ⊔ **on** *bool*

**lemma** *sup-boolI1*:
$P \Longrightarrow P \sqcup Q$
⟨*proof*⟩

**lemma** *sup-boolI2*:
$Q \Longrightarrow P \sqcup Q$
⟨*proof*⟩

**lemma** *sup-boolE*:
$P \sqcup Q \Longrightarrow (P \Longrightarrow R) \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

#### 19.1.2 Equality and Subsets

**lemma** *pred-equals-eq*: $((\lambda x.\ x \in R) = (\lambda x.\ x \in S)) = (R = S)$
⟨*proof*⟩

**lemma** *pred-equals-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x,\ y) \in R) = (\lambda x\ y.\ (x,\ y) \in S)) = (R = S)$
⟨*proof*⟩

**lemma** *pred-subset-eq*: $((\lambda x.\ x \in R) <= (\lambda x.\ x \in S)) = (R <= S)$
⟨*proof*⟩

**lemma** *pred-subset-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x,\ y) \in R) <= (\lambda x\ y.\ (x,\ y) \in S)) = (R <= S)$
⟨*proof*⟩

### 19.1.3 Top and bottom elements

**lemma** *top1I* [*intro*!]: *top x*
⟨*proof*⟩

**lemma** *top2I* [*intro*!]: *top x y*
⟨*proof*⟩

**lemma** *bot1E* [*elim*!]: *bot x* $\Longrightarrow$ *P*
⟨*proof*⟩

**lemma** *bot2E* [*elim*!]: *bot x y* $\Longrightarrow$ *P*
⟨*proof*⟩

### 19.1.4 The empty set

**lemma** *bot-empty-eq*: *bot* $= (\lambda x.\ x \in \{\})$
⟨*proof*⟩

**lemma** *bot-empty-eq2*: *bot* $= (\lambda x\ y.\ (x,\ y) \in \{\})$
⟨*proof*⟩

### 19.1.5 Binary union

**lemma** *sup1-iff* [*simp*]: *sup A B x* $\longleftrightarrow$ *A x* | *B x*
⟨*proof*⟩

**lemma** *sup2-iff* [*simp*]: *sup A B x y* $\longleftrightarrow$ *A x y* | *B x y*
⟨*proof*⟩

**lemma** *sup-Un-eq* [*pred-set-conv*]: *sup* $(\lambda x.\ x \in R)\ (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cup S)$
⟨*proof*⟩

**lemma** *sup-Un-eq2* [*pred-set-conv*]: *sup* $(\lambda x\ y.\ (x,\ y) \in R)\ (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cup S)$
⟨*proof*⟩

**lemma** *sup1I1* [*elim?*]: *A x* $\Longrightarrow$ *sup A B x*

⟨*proof*⟩

**lemma** *sup2I1* [*elim?*]: $A\ x\ y \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

**lemma** *sup1I2* [*elim?*]: $B\ x \Longrightarrow sup\ A\ B\ x$
⟨*proof*⟩

**lemma** *sup2I2* [*elim?*]: $B\ x\ y \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

Classical introduction rule: no commitment to $A$ vs $B$.

**lemma** *sup1CI* [*intro*!]: $(\sim B\ x \Longrightarrow A\ x) \Longrightarrow sup\ A\ B\ x$
⟨*proof*⟩

**lemma** *sup2CI* [*intro*!]: $(\sim B\ x\ y \Longrightarrow A\ x\ y) \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

**lemma** *sup1E* [*elim*!]: $sup\ A\ B\ x \Longrightarrow (A\ x \Longrightarrow P) \Longrightarrow (B\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *sup2E* [*elim*!]: $sup\ A\ B\ x\ y \Longrightarrow (A\ x\ y \Longrightarrow P) \Longrightarrow (B\ x\ y \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

### 19.1.6 Binary intersection

**lemma** *inf1-iff* [*simp*]: $inf\ A\ B\ x \longleftrightarrow A\ x \wedge B\ x$
⟨*proof*⟩

**lemma** *inf2-iff* [*simp*]: $inf\ A\ B\ x\ y \longleftrightarrow A\ x\ y \wedge B\ x\ y$
⟨*proof*⟩

**lemma** *inf-Int-eq* [*pred-set-conv*]: $inf\ (\lambda x.\ x \in R)\ (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cap S)$
⟨*proof*⟩

**lemma** *inf-Int-eq2* [*pred-set-conv*]: $inf\ (\lambda x\ y.\ (x,\ y) \in R)\ (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cap S)$
⟨*proof*⟩

**lemma** *inf1I* [*intro*!]: $A\ x \Longrightarrow B\ x \Longrightarrow inf\ A\ B\ x$
⟨*proof*⟩

**lemma** *inf2I* [*intro*!]: $A\ x\ y \Longrightarrow B\ x\ y \Longrightarrow inf\ A\ B\ x\ y$
⟨*proof*⟩

**lemma** *inf1D1*: $inf\ A\ B\ x \Longrightarrow A\ x$
⟨*proof*⟩

**lemma** *inf2D1*: *inf A B x y ==> A x y*
⟨*proof*⟩

**lemma** *inf1D2*: *inf A B x ==> B x*
⟨*proof*⟩

**lemma** *inf2D2*: *inf A B x y ==> B x y*
⟨*proof*⟩

**lemma** *inf1E* [*elim*!]: *inf A B x ==> (A x ==> B x ==> P) ==> P*
⟨*proof*⟩

**lemma** *inf2E* [*elim*!]: *inf A B x y ==> (A x y ==> B x y ==> P) ==> P*
⟨*proof*⟩

### 19.1.7 Unions of families

**lemma** *SUP1-iff* [*simp*]: *(SUP x:A. B x) b = (EX x:A. B x b)*
⟨*proof*⟩

**lemma** *SUP2-iff* [*simp*]: *(SUP x:A. B x) b c = (EX x:A. B x b c)*
⟨*proof*⟩

**lemma** *SUP1-I* [*intro*]: *a : A ==> B a b ==> (SUP x:A. B x) b*
⟨*proof*⟩

**lemma** *SUP2-I* [*intro*]: *a : A ==> B a b c ==> (SUP x:A. B x) b c*
⟨*proof*⟩

**lemma** *SUP1-E* [*elim*!]: *(SUP x:A. B x) b ==> (!!x. x : A ==> B x b ==> R) ==> R*
⟨*proof*⟩

**lemma** *SUP2-E* [*elim*!]: *(SUP x:A. B x) b c ==> (!!x. x : A ==> B x b c ==> R) ==> R*
⟨*proof*⟩

**lemma** *SUP-UN-eq*: *(SUP i. (λx. x ∈ r i)) = (λx. x ∈ (UN i. r i))*
⟨*proof*⟩

**lemma** *SUP-UN-eq2*: *(SUP i. (λx y. (x, y) ∈ r i)) = (λx y. (x, y) ∈ (UN i. r i))*
⟨*proof*⟩

### 19.1.8 Intersections of families

**lemma** *INF1-iff* [*simp*]: *(INF x:A. B x) b = (ALL x:A. B x b)*
⟨*proof*⟩

**lemma** *INF2-iff* [*simp*]: *(INF x:A. B x) b c = (ALL x:A. B x b c)*

⟨*proof*⟩

**lemma** *INF1-I* [*intro*!]: (!!*x*. *x* : *A* ==> *B x b*) ==> (*INF x*:*A*. *B x*) *b*
⟨*proof*⟩

**lemma** *INF2-I* [*intro*!]: (!!*x*. *x* : *A* ==> *B x b c*) ==> (*INF x*:*A*. *B x*) *b c*
⟨*proof*⟩

**lemma** *INF1-D* [*elim*]: (*INF x*:*A*. *B x*) *b* ==> *a* : *A* ==> *B a b*
⟨*proof*⟩

**lemma** *INF2-D* [*elim*]: (*INF x*:*A*. *B x*) *b c* ==> *a* : *A* ==> *B a b c*
⟨*proof*⟩

**lemma** *INF1-E* [*elim*]: (*INF x*:*A*. *B x*) *b* ==> (*B a b* ==> *R*) ==> (*a* ~: *A* ==> *R*) ==> *R*
⟨*proof*⟩

**lemma** *INF2-E* [*elim*]: (*INF x*:*A*. *B x*) *b c* ==> (*B a b c* ==> *R*) ==> (*a* ~: *A* ==> *R*) ==> *R*
⟨*proof*⟩

**lemma** *INF-INT-eq*: (*INF i*. (λ*x*. *x* ∈ *r i*)) = (λ*x*. *x* ∈ (*INT i*. *r i*))
⟨*proof*⟩

**lemma** *INF-INT-eq2*: (*INF i*. (λ*x y*. (*x*, *y*) ∈ *r i*)) = (λ*x y*. (*x*, *y*) ∈ (*INT i*. *r i*))
⟨*proof*⟩

## 19.2 Predicates as relations

### 19.2.1 Composition

**inductive**
*pred-comp* :: ['*b* => '*c* => *bool*, '*a* => '*b* => *bool*] => '*a* => '*c* => *bool*
(**infixr** *OO* 75)
**for** *r* :: '*b* => '*c* => *bool* **and** *s* :: '*a* => '*b* => *bool*
**where**
*pred-compI* [*intro*]: *s a b* ==> *r b c* ==> (*r OO s*) *a c*

**inductive-cases** *pred-compE* [*elim*!]: (*r OO s*) *a c*

**lemma** *pred-comp-rel-comp-eq* [*pred-set-conv*]:
((λ*x y*. (*x*, *y*) ∈ *r*) *OO* (λ*x y*. (*x*, *y*) ∈ *s*)) = (λ*x y*. (*x*, *y*) ∈ *r O s*)
⟨*proof*⟩

### 19.2.2 Converse

**inductive**
*conversep* :: ('*a* => '*b* => *bool*) => '*b* => '*a* => *bool*
((-^--1) [1000] 1000)

**for** $r :: 'a \Rightarrow 'b \Rightarrow bool$
**where**
*conversepI*: $r\ a\ b \Longrightarrow r\hat{}{-}{-}1\ b\ a$

**notation** (*xsymbols*)
*conversep* $((\text{-}^{-1-1})\ [1000]\ 1000)$

**lemma** *conversepD*:
**assumes** *ab*: $r\hat{}{-}{-}1\ a\ b$
**shows** $r\ b\ a$ ⟨*proof*⟩

**lemma** *conversep-iff* [*iff*]: $r\hat{}{-}{-}1\ a\ b = r\ b\ a$
⟨*proof*⟩

**lemma** *conversep-converse-eq* [*pred-set-conv*]:
$(\lambda x\ y.\ (x,\ y) \in r)\hat{}{-}{-}1 = (\lambda x\ y.\ (x,\ y) \in r\hat{}{-}1)$
⟨*proof*⟩

**lemma** *conversep-conversep* [*simp*]: $(r\hat{}{-}{-}1)\hat{}{-}{-}1 = r$
⟨*proof*⟩

**lemma** *converse-pred-comp*: $(r\ OO\ s)\hat{}{-}{-}1 = s\hat{}{-}{-}1\ OO\ r\hat{}{-}{-}1$
⟨*proof*⟩

**lemma** *converse-meet*: $(inf\ r\ s)\hat{}{-}{-}1 = inf\ r\hat{}{-}{-}1\ s\hat{}{-}{-}1$
⟨*proof*⟩

**lemma** *converse-join*: $(sup\ r\ s)\hat{}{-}{-}1 = sup\ r\hat{}{-}{-}1\ s\hat{}{-}{-}1$
⟨*proof*⟩

**lemma** *conversep-noteq* [*simp*]: $(op\ \tilde{}{=})\hat{}{-}{-}1 = op\ \tilde{}{=}$
⟨*proof*⟩

**lemma** *conversep-eq* [*simp*]: $(op =)\hat{}{-}{-}1 = op =$
⟨*proof*⟩

### 19.2.3 Domain

**inductive**
*DomainP* $:: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$
**for** $r :: 'a \Rightarrow 'b \Rightarrow bool$
**where**
*DomainPI* [*intro*]: $r\ a\ b \Longrightarrow DomainP\ r\ a$

**inductive-cases** *DomainPE* [*elim*!]: $DomainP\ r\ a$

**lemma** *DomainP-Domain-eq* [*pred-set-conv*]: $DomainP\ (\lambda x\ y.\ (x,\ y) \in r) = (\lambda x.\ x \in Domain\ r)$
⟨*proof*⟩

### 19.2.4 Range

**inductive**
*RangeP* :: (*'a => 'b => bool*) => *'b => bool*
**for** *r* :: *'a => 'b => bool*
**where**
*RangePI* [*intro*]: *r a b ==> RangeP r b*

**inductive-cases** *RangePE* [*elim!*]: *RangeP r b*

**lemma** *RangeP-Range-eq* [*pred-set-conv*]: *RangeP* (λ*x y*. (*x*, *y*) ∈ *r*) = (λ*x*. *x* ∈ *Range r*)
⟨*proof*⟩

### 19.2.5 Inverse image

**definition**
*inv-imagep* :: (*'b => 'b => bool*) => (*'a => 'b*) => *'a => 'a => bool* **where**
*inv-imagep r f* == %*x y*. *r* (*f x*) (*f y*)

**lemma** [*pred-set-conv*]: *inv-imagep* (λ*x y*. (*x*, *y*) ∈ *r*) *f* = (λ*x y*. (*x*, *y*) ∈ *inv-image r f*)
⟨*proof*⟩

**lemma** *in-inv-imagep* [*simp*]: *inv-imagep r f x y* = *r* (*f x*) (*f y*)
⟨*proof*⟩

### 19.2.6 Powerset

**definition** *Powp* :: (*'a* ⇒ *bool*) ⇒ *'a set* ⇒ *bool* **where**
*Powp A* == λ*B*. ∀ *x* ∈ *B*. *A x*

**lemma** *Powp-Pow-eq* [*pred-set-conv*]: *Powp* (λ*x*. *x* ∈ *A*) = (λ*x*. *x* ∈ *Pow A*)
⟨*proof*⟩

**lemmas** *Powp-mono* [*mono*] = *Pow-mono* [*to-pred pred-subset-eq*]

### 19.2.7 Properties of relations

**abbreviation** *antisymP* :: (*'a => 'a => bool*) => *bool* **where**
*antisymP r* == *antisym* {(*x*, *y*). *r x y*}

**abbreviation** *transP* :: (*'a => 'a => bool*) => *bool* **where**
*transP r* == *trans* {(*x*, *y*). *r x y*}

**abbreviation** *single-valuedP* :: (*'a => 'b => bool*) => *bool* **where**
*single-valuedP r* == *single-valued* {(*x*, *y*). *r x y*}

## 19.3 Predicates as enumerations

### 19.3.1 The type of predicate enumerations (a monad)

**datatype** *'a pred* = *Pred 'a* ⇒ *bool*

**primrec** *eval* :: *'a pred* ⇒ *'a* ⇒ *bool* **where**
*eval-pred*: *eval* (*Pred f*) = *f*

**lemma** *Pred-eval* [*simp*]:
*Pred* (*eval x*) = *x*
⟨*proof*⟩

**lemma** *eval-inject*: *eval x* = *eval y* ⟷ *x* = *y*
⟨*proof*⟩

**definition** *single* :: *'a* ⇒ *'a pred* **where**
*single x* = *Pred* ((*op* =) *x*)

**definition** *bind* :: *'a pred* ⇒ (*'a* ⇒ *'b pred*) ⇒ *'b pred* (**infixl** ⤜ *70*) **where**
*P* ⤜ *f* = *Pred* (λ*x*. (∃ *y*. *eval P y* ∧ *eval* (*f y*) *x*))

**instantiation** *pred* :: (*type*) *complete-lattice*
**begin**

**definition**
*P* ≤ *Q* ⟷ *eval P* ≤ *eval Q*

**definition**
*P* < *Q* ⟷ *eval P* < *eval Q*

**definition**
⊥ = *Pred* ⊥

**definition**
⊤ = *Pred* ⊤

**definition**
*P* ⊓ *Q* = *Pred* (*eval P* ⊓ *eval Q*)

**definition**
*P* ⊔ *Q* = *Pred* (*eval P* ⊔ *eval Q*)

**definition**
⨅ *A* = *Pred* (*INFI A eval*)

**definition**
⨆ *A* = *Pred* (*SUPR A eval*)

**instance** ⟨*proof*⟩

**end**

**lemma** *bind-bind*:
$(P \gg\!= Q) \gg\!= R = P \gg\!= (\lambda x.\ Q\ x \gg\!= R)$
⟨*proof*⟩

**lemma** *bind-single*:
$P \gg\!= \mathit{single} = P$
⟨*proof*⟩

**lemma** *single-bind*:
$\mathit{single}\ x \gg\!= P = P\ x$
⟨*proof*⟩

**lemma** *bottom-bind*:
$\bot \gg\!= P = \bot$
⟨*proof*⟩

**lemma** *sup-bind*:
$(P \sqcup Q) \gg\!= R = P \gg\!= R \sqcup Q \gg\!= R$
⟨*proof*⟩

**lemma** *Sup-bind*: $(\bigsqcup A \gg\!= f) = \bigsqcup((\lambda x.\ x \gg\!= f)\ `\ A)$
⟨*proof*⟩

**lemma** *pred-iffI*:
**assumes** $\bigwedge x.\ \mathit{eval}\ A\ x \Longrightarrow \mathit{eval}\ B\ x$
**and** $\bigwedge x.\ \mathit{eval}\ B\ x \Longrightarrow \mathit{eval}\ A\ x$
**shows** $A = B$
⟨*proof*⟩

**lemma** *singleI*: $\mathit{eval}\ (\mathit{single}\ x)\ x$
⟨*proof*⟩

**lemma** *singleI-unit*: $\mathit{eval}\ (\mathit{single}\ ())\ x$
⟨*proof*⟩

**lemma** *singleE*: $\mathit{eval}\ (\mathit{single}\ x)\ y \Longrightarrow (y = x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *singleE'*: $\mathit{eval}\ (\mathit{single}\ x)\ y \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *bindI*: $\mathit{eval}\ P\ x \Longrightarrow \mathit{eval}\ (Q\ x)\ y \Longrightarrow \mathit{eval}\ (P \gg\!= Q)\ y$
⟨*proof*⟩

**lemma** *bindE*: $\mathit{eval}\ (R \gg\!= Q)\ y \Longrightarrow (\bigwedge x.\ \mathit{eval}\ R\ x \Longrightarrow \mathit{eval}\ (Q\ x)\ y \Longrightarrow P) \Longrightarrow P$

⟨*proof*⟩

**lemma** *botE*: *eval* ⊥ *x* ⟹ *P*
⟨*proof*⟩

**lemma** *supI1*: *eval A x* ⟹ *eval* (*A* ⊔ *B*) *x*
⟨*proof*⟩

**lemma** *supI2*: *eval B x* ⟹ *eval* (*A* ⊔ *B*) *x*
⟨*proof*⟩

**lemma** *supE*: *eval* (*A* ⊔ *B*) *x* ⟹ (*eval A x* ⟹ *P*) ⟹ (*eval B x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

### 19.3.2 Derived operations

**definition** *if-pred* :: *bool* ⇒ *unit pred* **where**
*if-pred-eq*: *if-pred b* = (*if b then single* () *else* ⊥)

**definition** *not-pred* :: *unit pred* ⇒ *unit pred* **where**
*not-pred-eq*: *not-pred P* = (*if eval P* () *then* ⊥ *else single* ())

**lemma** *if-predI*: *P* ⟹ *eval* (*if-pred P*) ()
⟨*proof*⟩

**lemma** *if-predE*: *eval* (*if-pred b*) *x* ⟹ (*b* ⟹ *x* = () ⟹ *P*) ⟹ *P*
⟨*proof*⟩

**lemma** *not-predI*: ¬ *P* ⟹ *eval* (*not-pred* (*Pred* (λ*u*. *P*))) ()
⟨*proof*⟩

**lemma** *not-predI'*: ¬ *eval P* () ⟹ *eval* (*not-pred P*) ()
⟨*proof*⟩

**lemma** *not-predE*: *eval* (*not-pred* (*Pred* (λ*u*. *P*))) *x* ⟹ (¬ *P* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

**lemma** *not-predE'*: *eval* (*not-pred P*) *x* ⟹ (¬ *eval P x* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

### 19.3.3 Implementation

**datatype** *'a seq* = *Empty* | *Insert 'a 'a pred* | *Join 'a pred 'a seq*

**primrec** *pred-of-seq* :: *'a seq* ⇒ *'a pred* **where**
*pred-of-seq Empty* = ⊥
| *pred-of-seq* (*Insert x P*) = *single x* ⊔ *P*
| *pred-of-seq* (*Join P xq*) = *P* ⊔ *pred-of-seq xq*

**definition** *Seq* :: (*unit* ⇒ *'a seq*) ⇒ *'a pred* **where**
*Seq f* = *pred-of-seq* (*f* ())

**code-datatype** *Seq*

**primrec** *member* :: *'a seq* ⇒ *'a* ⇒ *bool* **where**
*member Empty x* ⟷ *False*
| *member* (*Insert y P*) *x* ⟷ *x* = *y* ∨ *eval P x*
| *member* (*Join P xq*) *x* ⟷ *eval P x* ∨ *member xq x*

**lemma** *eval-member*:
*member xq* = *eval* (*pred-of-seq xq*)
⟨*proof*⟩

**lemma** *eval-code* [*code*]: *eval* (*Seq f*) = *member* (*f* ())
⟨*proof*⟩

**lemma** *single-code* [*code*]:
*single x* = *Seq* (λ*u*. *Insert x* ⊥)
⟨*proof*⟩

**primrec** *apply* :: (*'a* ⇒ *'b Predicate.pred*) ⇒ *'a seq* ⇒ *'b seq* **where**
*apply f Empty* = *Empty*
| *apply f* (*Insert x P*) = *Join* (*f x*) (*Join* (*P* ⤜ *f*) *Empty*)
| *apply f* (*Join P xq*) = *Join* (*P* ⤜ *f*) (*apply f xq*)

**lemma** *apply-bind*:
*pred-of-seq* (*apply f xq*) = *pred-of-seq xq* ⤜ *f*
⟨*proof*⟩

**lemma** *bind-code* [*code*]:
*Seq g* ⤜ *f* = *Seq* (λ*u*. *apply f* (*g* ()))
⟨*proof*⟩

**lemma** *bot-set-code* [*code*]:
⊥ = *Seq* (λ*u*. *Empty*)
⟨*proof*⟩

**primrec** *adjunct* :: *'a pred* ⇒ *'a seq* ⇒ *'a seq* **where**
*adjunct P Empty* = *Join P Empty*
| *adjunct P* (*Insert x Q*) = *Insert x* (*Q* ⊔ *P*)
| *adjunct P* (*Join Q xq*) = *Join Q* (*adjunct P xq*)

**lemma** *adjunct-sup*:
*pred-of-seq* (*adjunct P xq*) = *P* ⊔ *pred-of-seq xq*
⟨*proof*⟩

**lemma** *sup-code* [*code*]:
*Seq f* ⊔ *Seq g* = *Seq* (λ*u*. *case f* ()

```
    of Empty ⇒ g ()
     | Insert x P ⇒ Insert x (P ⊔ Seq g)
     | Join P xq ⇒ adjunct (Seq g) (Join P xq))
⟨proof⟩

primrec contained :: 'a seq ⇒ 'a pred ⇒ bool where
    contained Empty Q ⟷ True
  | contained (Insert x P) Q ⟷ eval Q x ∧ P ≤ Q
  | contained (Join P xq) Q ⟷ P ≤ Q ∧ contained xq Q

lemma single-less-eq-eval:
  single x ≤ P ⟷ eval P x
  ⟨proof⟩

lemma contained-less-eq:
  contained xq Q ⟷ pred-of-seq xq ≤ Q
  ⟨proof⟩

lemma less-eq-pred-code [code]:
  Seq f ≤ Q = (case f ()
   of Empty ⇒ True
    | Insert x P ⇒ eval Q x ∧ P ≤ Q
    | Join P xq ⇒ P ≤ Q ∧ contained xq Q)
  ⟨proof⟩

lemma eq-pred-code [code]:
  fixes P Q :: 'a::eq pred
  shows eq-class.eq P Q ⟷ P ≤ Q ∧ Q ≤ P
  ⟨proof⟩

lemma [code]:
  pred-case f P = f (eval P)
  ⟨proof⟩

lemma [code]:
  pred-rec f P = f (eval P)
  ⟨proof⟩

no-notation
  inf (infixl ⊓ 70) and
  sup (infixl ⊔ 65) and
  Inf (⨅- [900] 900) and
  Sup (⨆- [900] 900) and
  top (⊤) and
  bot (⊥) and
  bind (infixl ≫= 70)

hide (open) type pred seq
hide (open) const Pred eval single bind if-pred not-pred
```

*Empty Insert Join Seq member pred-of-seq apply adjunct*

**end**

# 20 Transitive-Closure: Reflexive and Transitive closure of a relation

**theory** *Transitive-Closure*
**imports** *Predicate*
**uses** ~~/*src/Provers/trancl.ML*
**begin**

*rtrancl* is reflexive/transitive closure, *trancl* is transitive closure, *reflcl* is reflexive closure.

These postfix operators have *maximum priority*, forcing their operands to be atomic.

**inductive-set**
  *rtrancl* :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set*  ((-^*) [*1000*] *999*)
  **for** $r$ :: $('a \times 'a)$ *set*
**where**
    *rtrancl-refl* [*intro*!, *Pure.intro*!, *simp*]: $(a, a) : r\hat{}*$
  | *rtrancl-into-rtrancl* [*Pure.intro*]: $(a, b) : r\hat{}* \Longrightarrow (b, c) : r \Longrightarrow (a, c) : r\hat{}*$

**inductive-set**
  *trancl* :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set*  ((-^+) [*1000*] *999*)
  **for** $r$ :: $('a \times 'a)$ *set*
**where**
    *r-into-trancl* [*intro*, *Pure.intro*]: $(a, b) : r \Longrightarrow (a, b) : r\hat{}+$
  | *trancl-into-trancl* [*Pure.intro*]: $(a, b) : r\hat{}+ \Longrightarrow (b, c) : r \Longrightarrow (a, c) : r\hat{}+$

**notation**
  *rtranclp*  ((-^**) [*1000*] *1000*) **and**
  *tranclp*  ((-^++) [*1000*] *1000*)

**abbreviation**
  *reflclp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$  ((-^==) [*1000*] *1000*)
**where**
  $r\hat{}== == sup\ r\ op =$

**abbreviation**
  *reflcl* :: $('a \times 'a)$ *set* => $('a \times 'a)$ *set*  ((-^=) [*1000*] *999*) **where**
  $r\hat{}= == r \cup Id$

**notation** (*xsymbols*)
  *rtranclp*  ((-**) [*1000*] *1000*) **and**
  *tranclp*  ((-++) [*1000*] *1000*) **and**
  *reflclp*  ((-==) [*1000*] *1000*) **and**

*rtrancl* ((-*) [*1000*] *999*) **and**
*trancl* ((-$^+$) [*1000*] *999*) **and**
*reflcl* ((-$^=$) [*1000*] *999*)

**notation** (*HTML* **output**)
*rtranclp* ((-**) [*1000*] *1000*) **and**
*tranclp* ((-$^{++}$) [*1000*] *1000*) **and**
*reflclp* ((-$^{==}$) [*1000*] *1000*) **and**
*rtrancl* ((-*) [*1000*] *999*) **and**
*trancl* ((-$^+$) [*1000*] *999*) **and**
*reflcl* ((-$^=$) [*1000*] *999*)

## 20.1 Reflexive closure

**lemma** *refl-reflcl*[*simp*]: *refl*($r$ ^ =)
⟨*proof*⟩

**lemma** *antisym-reflcl*[*simp*]: *antisym*($r$ ^ =) = *antisym* $r$
⟨*proof*⟩

**lemma** *trans-reflclI*[*simp*]: *trans* $r$ ⟹ *trans*($r$ ^ =)
⟨*proof*⟩

## 20.2 Reflexive-transitive closure

**lemma** *reflcl-set-eq* [*pred-set-conv*]: (*sup* (λ$x$ $y$. ($x$, $y$) ∈ $r$) *op* =) = (λ$x$ $y$. ($x$, $y$) ∈ $r$ *Un* *Id*)
⟨*proof*⟩

**lemma** *r-into-rtrancl* [*intro*]: !!$p$. $p$ ∈ $r$ ==> $p$ ∈ $r$ ^ *
— *rtrancl* of $r$ contains $r$
⟨*proof*⟩

**lemma** *r-into-rtranclp* [*intro*]: $r$ $x$ $y$ ==> $r$ ^ ** $x$ $y$
— *rtrancl* of $r$ contains $r$
⟨*proof*⟩

**lemma** *rtranclp-mono*: $r$ ≤ $s$ ==> $r$ ^ ** ≤ $s$ ^ **
— monotonicity of *rtrancl*
⟨*proof*⟩

**lemmas** *rtrancl-mono* = *rtranclp-mono* [*to-set*]

**theorem** *rtranclp-induct* [*consumes 1*, *case-names base step*, *induct set*: *rtranclp*]:
**assumes** *a*: $r$ ^ ** $a$ $b$
**and** *cases*: $P$ $a$ !!$y$ $z$. [| $r$ ^ ** $a$ $y$; $r$ $y$ $z$; $P$ $y$ |] ==> $P$ $z$
**shows** $P$ $b$
⟨*proof*⟩

**lemmas** *rtrancl-induct* [*induct set*: *rtrancl*] = *rtranclp-induct* [*to-set*]

**lemmas** *rtranclp-induct2* =
*rtranclp-induct*[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
*consumes 1*, *case-names refl step*]

**lemmas** *rtrancl-induct2* =
*rtrancl-induct*[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
*consumes 1*, *case-names refl step*]

**lemma** *refl-rtrancl*: *refl* (*r*^*)
⟨*proof*⟩

Transitivity of transitive closure.

**lemma** *trans-rtrancl*: *trans* (*r*^*)
⟨*proof*⟩

**lemmas** *rtrancl-trans* = *trans-rtrancl* [*THEN transD*, *standard*]

**lemma** *rtranclp-trans*:
**assumes** *xy*: *r*^** *x y*
**and** *yz*: *r*^** *y z*
**shows** *r*^** *x z* ⟨*proof*⟩

**lemma** *rtranclE* [*cases set*: *rtrancl*]:
**assumes** *major*: (*a*::'*a*, *b*) : *r*^*
**obtains**
(*base*) *a* = *b*
| (*step*) *y* **where** (*a*, *y*) : *r*^* **and** (*y*, *b*) : *r*
— elimination of *rtrancl* – by induction on a special formula
⟨*proof*⟩

**lemma** *rtrancl-Int-subset*: [| *Id* ⊆ *s*; *r O* (*r*^* ∩ *s*) ⊆ *s*|] ==> *r*^* ⊆ *s*
⟨*proof*⟩

**lemma** *converse-rtranclp-into-rtranclp*:
*r a b* ⟹ *r*** *b c* ⟹ *r*** *a c*
⟨*proof*⟩

**lemmas** *converse-rtrancl-into-rtrancl* = *converse-rtranclp-into-rtranclp* [*to-set*]

More $r^*$ equations and inclusions.

**lemma** *rtranclp-idemp* [*simp*]: (*r*^**)^** = *r*^**
⟨*proof*⟩

**lemmas** *rtrancl-idemp* [*simp*] = *rtranclp-idemp* [*to-set*]

**lemma** *rtrancl-idemp-self-comp* [*simp*]: *R*^* *O R*^* = *R*^*
⟨*proof*⟩

**lemma** *rtrancl-subset-rtrancl*: $r \subseteq s\hat{}* \Longrightarrow r\hat{}* \subseteq s\hat{}*$
⟨*proof*⟩

**lemma** *rtranclp-subset*: $R \leq S \Longrightarrow S \leq R\hat{}** \Longrightarrow S\hat{}** = R\hat{}**$
⟨*proof*⟩

**lemmas** *rtrancl-subset* = *rtranclp-subset* [*to-set*]

**lemma** *rtranclp-sup-rtranclp*: $(sup\ (R\hat{}**)\ (S\hat{}**))\hat{}** = (sup\ R\ S)\hat{}**$
⟨*proof*⟩

**lemmas** *rtrancl-Un-rtrancl* = *rtranclp-sup-rtranclp* [*to-set*]

**lemma** *rtranclp-reflcl* [*simp*]: $(R\hat{}==)\hat{}** = R\hat{}**$
⟨*proof*⟩

**lemmas** *rtrancl-reflcl* [*simp*] = *rtranclp-reflcl* [*to-set*]

**lemma** *rtrancl-r-diff-Id*: $(r - Id)\hat{}* = r\hat{}*$
⟨*proof*⟩

**lemma** *rtranclp-r-diff-Id*: $(inf\ r\ op\ \sim=)\hat{}** = r\hat{}**$
⟨*proof*⟩

**theorem** *rtranclp-converseD*:
**assumes** *r*: $(r\hat{}--1)\hat{}**\ x\ y$
**shows** $r\hat{}**\ y\ x$
⟨*proof*⟩

**lemmas** *rtrancl-converseD* = *rtranclp-converseD* [*to-set*]

**theorem** *rtranclp-converseI*:
**assumes** $r\hat{}**\ y\ x$
**shows** $(r\hat{}--1)\hat{}**\ x\ y$
⟨*proof*⟩

**lemmas** *rtrancl-converseI* = *rtranclp-converseI* [*to-set*]

**lemma** *rtrancl-converse*: $(r\hat{}-1)\hat{}* = (r\hat{}*)\hat{}-1$
⟨*proof*⟩

**lemma** *sym-rtrancl*: *sym r* ⟹ *sym* $(r\hat{}*)$
⟨*proof*⟩

**theorem** *converse-rtranclp-induct*[*consumes 1*]:
**assumes** *major*: $r\hat{}**\ a\ b$
**and** *cases*: $P\ b$ !!$y\ z$. [| $r\ y\ z$; $r\hat{}**\ z\ b$; $P\ z$ |] ⟹ $P\ y$
**shows** $P\ a$
⟨*proof*⟩

**lemmas** *converse-rtrancl-induct* = *converse-rtranclp-induct* [*to-set*]

**lemmas** *converse-rtranclp-induct2* =
*converse-rtranclp-induct* [*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
*consumes 1*, *case-names refl step*]

**lemmas** *converse-rtrancl-induct2* =
*converse-rtrancl-induct* [*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
*consumes 1*, *case-names refl step*]

**lemma** *converse-rtranclpE*:
**assumes** *major*: *r*^** *x z*
**and** *cases*: *x*=*z* ==> *P*
!!*y*. [| *r x y*; *r*^** *y z* |] ==> *P*
**shows** *P*
⟨*proof*⟩

**lemmas** *converse-rtranclE* = *converse-rtranclpE* [*to-set*]

**lemmas** *converse-rtranclpE2* = *converse-rtranclpE* [*of* - (*xa*,*xb*) (*za*,*zb*), *split-rule*]

**lemmas** *converse-rtranclE2* = *converse-rtranclE* [*of* (*xa*,*xb*) (*za*,*zb*), *split-rule*]

**lemma** *r-comp-rtrancl-eq*: *r O r*^* = *r*^* *O r*
⟨*proof*⟩

**lemma** *rtrancl-unfold*: *r*^* = *Id Un r O r*^*
⟨*proof*⟩

## 20.3 Transitive closure

**lemma** *trancl-mono*: !!*p*. *p* ∈ *r*^+ ==> *r* ⊆ *s* ==> *p* ∈ *s*^+
⟨*proof*⟩

**lemma** *r-into-trancl'*: !!*p*. *p* : *r* ==> *p* : *r*^+
⟨*proof*⟩

Conversions between *trancl* and *rtrancl*.

**lemma** *tranclp-into-rtranclp*: *r*^++ *a b* ==> *r*^** *a b*
⟨*proof*⟩

**lemmas** *trancl-into-rtrancl* = *tranclp-into-rtranclp* [*to-set*]

**lemma** *rtranclp-into-tranclp1*: **assumes** *r*: *r*^** *a b*
**shows** !!*c*. *r b c* ==> *r*^++ *a c* ⟨*proof*⟩

**lemmas** *rtrancl-into-trancl1* = *rtranclp-into-tranclp1* [*to-set*]

**lemma** *rtranclp-into-tranclp2*: [| *r a b*; *r* ^** *b c* |] ==> *r* ^++ *a c*
— intro rule from *r* and *rtrancl*
⟨*proof*⟩

**lemmas** *rtrancl-into-trancl2* = *rtranclp-into-tranclp2* [*to-set*]

Nice induction rule for *trancl*

**lemma** *tranclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *tranclp*]:
**assumes** *r* ^++ *a b*
**and** *cases*: !!*y*. *r a y* ==> *P y*
!!*y z*. *r* ^++ *a y* ==> *r y z* ==> *P y* ==> *P z*
**shows** *P b*
⟨*proof*⟩

**lemmas** *trancl-induct* [*induct set*: *trancl*] = *tranclp-induct* [*to-set*]

**lemmas** *tranclp-induct2* =
*tranclp-induct* [*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
*consumes 1*, *case-names base step*]

**lemmas** *trancl-induct2* =
*trancl-induct* [*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
*consumes 1*, *case-names base step*]

**lemma** *tranclp-trans-induct*:
**assumes** *major*: *r* ^++ *x y*
**and** *cases*: !!*x y*. *r x y* ==> *P x y*
!!*x y z*. [| *r* ^++ *x y*; *P x y*; *r* ^++ *y z*; *P y z* |] ==> *P x z*
**shows** *P x y*
— Another induction rule for trancl, incorporating transitivity
⟨*proof*⟩

**lemmas** *trancl-trans-induct* = *tranclp-trans-induct* [*to-set*]

**lemma** *tranclE* [*cases set*: *trancl*]:
**assumes** (*a*, *b*) : *r* ^+
**obtains**
(*base*) (*a*, *b*) : *r*
| (*step*) *c* **where** (*a*, *c*) : *r* ^+ **and** (*c*, *b*) : *r*
⟨*proof*⟩

**lemma** *trancl-Int-subset*: [| *r* ⊆ *s*; *r O* (*r* ^+ ∩ *s*) ⊆ *s*|] ==> *r* ^+ ⊆ *s*
⟨*proof*⟩

**lemma** *trancl-unfold*: *r* ^+ = *r Un r O r* ^+
⟨*proof*⟩

Transitivity of $r^+$

**lemma** *trans-trancl* [*simp*]: *trans* (*r* ^+)

⟨*proof*⟩

**lemmas** *trancl-trans* = *trans-trancl* [*THEN transD*, *standard*]

**lemma** *tranclp-trans*:
**assumes** *xy*: $r\hat{}{+}{+}\ x\ y$
**and** *yz*: $r\hat{}{+}{+}\ y\ z$
**shows** $r\hat{}{+}{+}\ x\ z$ ⟨*proof*⟩

**lemma** *trancl-id* [*simp*]: *trans* $r \Longrightarrow r\hat{}{+} = r$
⟨*proof*⟩

**lemma** *rtranclp-tranclp-tranclp*:
**assumes** $r\hat{}{*}{*}\ x\ y$
**shows** $!!z.\ r\hat{}{+}{+}\ y\ z \Longrightarrow r\hat{}{+}{+}\ x\ z$ ⟨*proof*⟩

**lemmas** *rtrancl-trancl-trancl* = *rtranclp-tranclp-tranclp* [*to-set*]

**lemma** *tranclp-into-tranclp2*: $r\ a\ b \Longrightarrow r\hat{}{+}{+}\ b\ c \Longrightarrow r\hat{}{+}{+}\ a\ c$
⟨*proof*⟩

**lemmas** *trancl-into-trancl2* = *tranclp-into-tranclp2* [*to-set*]

**lemma** *trancl-insert*:
$(\mathit{insert}\ (y,\ x)\ r)\hat{}{+} = r\hat{}{+} \cup \{(a,\ b).\ (a,\ y) \in r\hat{}{*} \wedge (x,\ b) \in r\hat{}{*}\}$
— primitive recursion for *trancl* over finite relations
⟨*proof*⟩

**lemma** *tranclp-converseI*: $(r\hat{}{+}{+})\hat{}{-}{-}1\ x\ y \Longrightarrow (r\hat{}{-}{-}1)\hat{}{+}{+}\ x\ y$
⟨*proof*⟩

**lemmas** *trancl-converseI* = *tranclp-converseI* [*to-set*]

**lemma** *tranclp-converseD*: $(r\hat{}{-}{-}1)\hat{}{+}{+}\ x\ y \Longrightarrow (r\hat{}{+}{+})\hat{}{-}{-}1\ x\ y$
⟨*proof*⟩

**lemmas** *trancl-converseD* = *tranclp-converseD* [*to-set*]

**lemma** *tranclp-converse*: $(r\hat{}{-}{-}1)\hat{}{+}{+} = (r\hat{}{+}{+})\hat{}{-}{-}1$
⟨*proof*⟩

**lemmas** *trancl-converse* = *tranclp-converse* [*to-set*]

**lemma** *sym-trancl*: *sym* $r \Longrightarrow$ *sym* $(r\hat{}{+})$
⟨*proof*⟩

**lemma** *converse-tranclp-induct*:
**assumes** *major*: $r\hat{}{+}{+}\ a\ b$
**and** *cases*: $!!y.\ r\ y\ b \Longrightarrow P(y)$

$!!y\ z.[|\ r\ y\ z;\ \ r\hat{}{+}{+}\ z\ b;\ \ P(z)\ |] ==> P(y)$
**shows** $P\ a$
⟨*proof*⟩

**lemmas** *converse-trancl-induct* = *converse-tranclp-induct* [*to-set*]

**lemma** *tranclpD*: $R\hat{}{+}{+}\ x\ y ==> EX\ z.\ R\ x\ z \wedge R\hat{}{*}{*}\ z\ y$
⟨*proof*⟩

**lemmas** *tranclD* = *tranclpD* [*to-set*]

**lemma** *tranclD2*:
$(x,\ y) \in R^+ \Longrightarrow \exists z.\ (x,\ z) \in R^* \wedge (z,\ y) \in R$
⟨*proof*⟩

**lemma** *irrefl-tranclI*: $r\hat{}{-}1 \cap r\hat{}{*} = \{\} ==> (x,\ x) \notin r\hat{}{+}$
⟨*proof*⟩

**lemma** *irrefl-trancl-rD*: $!!X.\ ALL\ x.\ (x,\ x) \notin r\hat{}{+} ==> (x,\ y) \in r ==> x \neq y$
⟨*proof*⟩

**lemma** *trancl-subset-Sigma-aux*:
$(a,\ b) \in r\hat{}{*} ==> r \subseteq A \times A ==> a = b \vee a \in A$
⟨*proof*⟩

**lemma** *trancl-subset-Sigma*: $r \subseteq A \times A ==> r\hat{}{+} \subseteq A \times A$
⟨*proof*⟩

**lemma** *reflcl-tranclp* [*simp*]: $(r\hat{}{+}{+})\hat{}{=}{=} = r\hat{}{*}{*}$
⟨*proof*⟩

**lemmas** *reflcl-trancl* [*simp*] = *reflcl-tranclp* [*to-set*]

**lemma** *trancl-reflcl* [*simp*]: $(r\hat{}{=})\hat{}{+} = r\hat{}{*}$
⟨*proof*⟩

**lemma** *trancl-empty* [*simp*]: $\{\}\hat{}{+} = \{\}$
⟨*proof*⟩

**lemma** *rtrancl-empty* [*simp*]: $\{\}\hat{}{*} = Id$
⟨*proof*⟩

**lemma** *rtranclpD*: $R\hat{}{*}{*}\ a\ b ==> a = b \vee a \neq b \wedge R\hat{}{+}{+}\ a\ b$
⟨*proof*⟩

**lemmas** *rtranclD* = *rtranclpD* [*to-set*]

**lemma** *rtrancl-eq-or-trancl*:
$(x,y) \in R^* = (x{=}y \vee x{\neq}y \wedge (x,y) \in R^+)$

⟨*proof*⟩

*Domain* and *Range*

**lemma** *Domain-rtrancl* [*simp*]: *Domain* $(R\hat{}*) = UNIV$
⟨*proof*⟩

**lemma** *Range-rtrancl* [*simp*]: *Range* $(R\hat{}*) = UNIV$
⟨*proof*⟩

**lemma** *rtrancl-Un-subset*: $(R\hat{}* \cup S\hat{}*) \subseteq (R\ Un\ S)\hat{}*$
⟨*proof*⟩

**lemma** *in-rtrancl-UnI*: $x \in R\hat{}* \vee x \in S\hat{}* ==> x \in (R \cup S)\hat{}*$
⟨*proof*⟩

**lemma** *trancl-domain* [*simp*]: *Domain* $(r\hat{}+) =$ *Domain* $r$
⟨*proof*⟩

**lemma** *trancl-range* [*simp*]: *Range* $(r\hat{}+) =$ *Range* $r$
⟨*proof*⟩

**lemma** *Not-Domain-rtrancl*:
$x \sim$: *Domain* $R ==> ((x,\ y) : R\hat{}*) = (x = y)$
⟨*proof*⟩

**lemma** *trancl-subset-Field2*: $r\hat{}+ <=$ *Field* $r \times$ *Field* $r$
⟨*proof*⟩

**lemma** *finite-trancl*: *finite* $(r\hat{}+) =$ *finite* $r$
⟨*proof*⟩

More about converse *rtrancl* and *trancl*, should be merged with main body.

**lemma** *single-valued-confluent*:
⟦ *single-valued* $r$; $(x,y) \in r\hat{}*$; $(x,z) \in r\hat{}*$ ⟧
$\Longrightarrow (y,z) \in r\hat{}* \vee (z,y) \in r\hat{}*$
⟨*proof*⟩

**lemma** *r-r-into-trancl*: $(a,\ b) \in R ==> (b,\ c) \in R ==> (a,\ c) \in R\hat{}+$
⟨*proof*⟩

**lemma** *trancl-into-trancl* [*rule-format*]:
$(a,\ b) \in r^{+} ==> (b,\ c) \in r --> (a,c) \in r^{+}$
⟨*proof*⟩

**lemma** *tranclp-rtranclp-tranclp*:
$r^{++}\ a\ b ==> r^{**}\ b\ c ==> r^{++}\ a\ c$
⟨*proof*⟩

**lemmas** *trancl-rtrancl-trancl* = *tranclp-rtranclp-tranclp* [*to-set*]

**lemmas** *transitive-closure-trans* [*trans*] =
*r-r-into-trancl trancl-trans rtrancl-trans*
*trancl.trancl-into-trancl trancl-into-trancl2*
*rtrancl.rtrancl-into-rtrancl converse-rtrancl-into-rtrancl*
*rtrancl-trancl-trancl trancl-rtrancl-trancl*

**lemmas** *transitive-closurep-trans'* [*trans*] =
*tranclp-trans rtranclp-trans*
*tranclp.trancl-into-trancl tranclp-into-tranclp2*
*rtranclp.rtrancl-into-rtrancl converse-rtranclp-into-rtranclp*
*rtranclp-tranclp-tranclp tranclp-rtranclp-tranclp*

**declare** *trancl-into-rtrancl* [*elim*]

### 20.4 Setup of transitivity reasoner

⟨*ML*⟩

**end**

## 21 Wellfounded: Well-founded Recursion

**theory** *Wellfounded*
**imports** *Finite-Set Transitive-Closure Nat*
**uses** (*Tools/function-package/size.ML*)
**begin**

### 21.1 Basic Definitions

**inductive**
*wfrec-rel* :: ('*a* ∗ '*a*) *set* => (('*a* => '*b*) => '*a* => '*b*) => '*a* => '*b* => *bool*
**for** *R* :: ('*a* ∗ '*a*) *set*
**and** *F* :: ('*a* => '*b*) => '*a* => '*b*
**where**
*wfrecI*: *ALL z*. (*z*, *x*) : *R* −−> *wfrec-rel R F z* (*g z*) ==>
*wfrec-rel R F x* (*F g x*)

**constdefs**
*wf* :: ('*a* ∗ '*a*)*set* => *bool*
*wf*(*r*) == (!*P*. (!*x*. (!*y*. (*y*,*x*):*r* −−> *P*(*y*)) −−> *P*(*x*)) −−> (!*x*. *P*(*x*)))

*wfP* :: ('*a* => '*a* => *bool*) => *bool*
*wfP r* == *wf* {(*x*, *y*). *r x y*}

*acyclic* :: ('*a*∗'*a*)*set* => *bool*
*acyclic r* == !*x*. (*x*,*x*) ~: *r*^+

*cut* :: ('a => 'b) => ('a * 'a)*set* => 'a => 'a => 'b
*cut f r x* == (%*y*. *if* (*y*,*x*):*r then f y else undefined*)

*adm-wf* :: ('a * 'a) *set* => (('a => 'b) => 'a => 'b) => *bool*
*adm-wf R F* == *ALL f g x*.
(*ALL z*. (*z*, *x*) : *R* --> *f z* = *g z*) --> *F f x* = *F g x*

*wfrec* :: ('a * 'a) *set* => (('a => 'b) => 'a => 'b) => 'a => 'b
[*code del*]: *wfrec R F* == %*x*. *THE y*. *wfrec-rel R* (%*f x*. *F* (*cut f R x*) *x*) *x y*

**abbreviation** *acyclicP* :: ('a => 'a => *bool*) => *bool* **where**
*acyclicP r* == *acyclic* {(*x*, *y*). *r x y*}

**lemma** *wfP-wf-eq* [*pred-set-conv*]: *wfP* (λ*x y*. (*x*, *y*) ∈ *r*) = *wf r*
⟨*proof*⟩

**lemma** *wfUNIVI*:
(!!*P x*. (*ALL x*. (*ALL y*. (*y*,*x*) : *r* --> *P*(*y*)) --> *P*(*x*)) ==> *P*(*x*)) ==> *wf*(*r*)
⟨*proof*⟩

**lemmas** *wfPUNIVI* = *wfUNIVI* [*to-pred*]

Restriction to domain $A$ and range $B$. If $r$ is well-founded over their intersection, then *wf r*

**lemma** *wfI*:
[| *r* ⊆ *A* <*> *B*;
!!*x P*. [|∀ *x*. (∀ *y*. (*y*,*x*) : *r* --> *P y*) --> *P x*; *x* : *A*; *x* : *B* |] ==> *P x* |]
==> *wf r*
⟨*proof*⟩

**lemma** *wf-induct*:
[| *wf*(*r*);
!!*x*.[| *ALL y*. (*y*,*x*): *r* --> *P*(*y*) |] ==> *P*(*x*)
|] ==> *P*(*a*)
⟨*proof*⟩

**lemmas** *wfP-induct* = *wf-induct* [*to-pred*]

**lemmas** *wf-induct-rule* = *wf-induct* [*rule-format*, *consumes 1*, *case-names less*, *induct set*: *wf*]

**lemmas** *wfP-induct-rule* = *wf-induct-rule* [*to-pred*, *induct set*: *wfP*]

**lemma** *wf-not-sym*: *wf r* ==> (*a*, *x*) : *r* ==> (*x*, *a*) ~: *r*
⟨*proof*⟩

**lemmas** *wf-asym* = *wf-not-sym* [*elim-format*]

**lemma** *wf-not-refl* [*simp*]: *wf r ==> (a, a) ~: r*
⟨*proof*⟩

**lemmas** *wf-irrefl = wf-not-refl* [*elim-format*]

**lemma** *wf-wellorderI*:
**assumes** *wf*: *wf {(x::'a::ord, y). x < y}*
**assumes** *lin*: *OFCLASS('a::ord, linorder-class)*
**shows** *OFCLASS('a::ord, wellorder-class)*
⟨*proof*⟩

**lemma** (**in** *wellorder*) *wf*:
*wf {(x, y). x < y}*
⟨*proof*⟩

## 21.2 Basic Results

transitive closure of a well-founded relation is well-founded!

**lemma** *wf-trancl*:
**assumes** *wf r*
**shows** *wf (r^+)*
⟨*proof*⟩

**lemmas** *wfP-trancl = wf-trancl* [*to-pred*]

**lemma** *wf-converse-trancl*: *wf (r^-1) ==> wf ((r^+)^-1)*
⟨*proof*⟩

Minimal-element characterization of well-foundedness

**lemma** *wf-eq-minimal*: *wf r = (∀Q x. x∈Q --> (∃z∈Q. ∀y. (y,z)∈r --> y∉Q))*
⟨*proof*⟩

**lemma** *wfE-min*:
**assumes** *wf R x ∈ Q*
**obtains** *z* **where** *z ∈ Q* ⋀*y. (y, z) ∈ R ⟹ y ∉ Q*
⟨*proof*⟩

**lemma** *wfI-min*:
*(⋀x Q. x ∈ Q ⟹ ∃z∈Q. ∀y. (y, z) ∈ R ⟶ y ∉ Q)*
*⟹ wf R*
⟨*proof*⟩

**lemmas** *wfP-eq-minimal = wf-eq-minimal* [*to-pred*]

Well-foundedness of subsets

**lemma** *wf-subset*: [| *wf(r); p<=r* |] ==> *wf(p)*

⟨*proof*⟩

**lemmas** *wfP-subset* = *wf-subset* [*to-pred*]

Well-foundedness of the empty relation

**lemma** *wf-empty* [*iff*]: *wf*({})
⟨*proof*⟩

**lemmas** *wfP-empty* [*iff*] =
*wf-empty* [*to-pred bot-empty-eq2*, *simplified bot-fun-eq bot-bool-eq*]

**lemma** *wf-Int1*: *wf r* ==> *wf* (*r Int r'*)
⟨*proof*⟩

**lemma** *wf-Int2*: *wf r* ==> *wf* (*r' Int r*)
⟨*proof*⟩

Well-foundedness of insert

**lemma** *wf-insert* [*iff*]: *wf*(*insert* (*y*,*x*) *r*) = (*wf*(*r*) & (*x*,*y*) ~: *r*^*)
⟨*proof*⟩

Well-foundedness of image

**lemma** *wf-prod-fun-image*: [| *wf r*; *inj f* |] ==> *wf*(*prod-fun f f* ` *r*)
⟨*proof*⟩

## 21.3 Well-Foundedness Results for Unions

**lemma** *wf-union-compatible*:
**assumes** *wf R wf S*
**assumes** *S O R* ⊆ *R*
**shows** *wf* (*R* ∪ *S*)
⟨*proof*⟩

Well-foundedness of indexed union with disjoint domains and ranges

**lemma** *wf-UN*: [| *ALL i*:*I*. *wf*(*r i*);
*ALL i*:*I*. *ALL j*:*I*. *r i* ~= *r j* --> *Domain*(*r i*) *Int Range*(*r j*) = {}
|] ==> *wf*(*UN i*:*I*. *r i*)
⟨*proof*⟩

**lemmas** *wfP-SUP* = *wf-UN* [**where** *I*=*UNIV* **and** *r*=λ*i*. {(*x*, *y*). *r i x y*},
*to-pred SUP-UN-eq2 bot-empty-eq pred-equals-eq*, *simplified*, *standard*]

**lemma** *wf-Union*:
[| *ALL r*:*R*. *wf r*;
*ALL r*:*R*. *ALL s*:*R*. *r* ~= *s* --> *Domain r Int Range s* = {}
|] ==> *wf*(*Union R*)
⟨*proof*⟩

**lemma** *wf-Un*:
[| *wf r*; *wf s*; *Domain r Int Range s* = {} |] ==> *wf*(*r Un s*)
⟨*proof*⟩

**lemma** *wf-union-merge*:
*wf* (*R* ∪ *S*) = *wf* (*R O R* ∪ *R O S* ∪ *S*) (**is** *wf ?A* = *wf ?B*)
⟨*proof*⟩

**lemma** *wf-comp-self*: *wf R* = *wf* (*R O R*) — special case
⟨*proof*⟩

### 21.3.1 acyclic

**lemma** *acyclicI*: *ALL x*. (*x*, *x*) ~: *r*^+ ==> *acyclic r*
⟨*proof*⟩

**lemma** *wf-acyclic*: *wf r* ==> *acyclic r*
⟨*proof*⟩

**lemmas** *wfP-acyclicP* = *wf-acyclic* [*to-pred*]

**lemma** *acyclic-insert* [*iff*]:
*acyclic*(*insert* (*y*,*x*) *r*) = (*acyclic r* & (*x*,*y*) ~: *r*^*)
⟨*proof*⟩

**lemma** *acyclic-converse* [*iff*]: *acyclic*(*r*^−1) = *acyclic r*
⟨*proof*⟩

**lemmas** *acyclicP-converse* [*iff*] = *acyclic-converse* [*to-pred*]

**lemma** *acyclic-impl-antisym-rtrancl*: *acyclic r* ==> *antisym*(*r*^*)
⟨*proof*⟩

**lemma** *acyclic-subset*: [| *acyclic s*; *r* <= *s* |] ==> *acyclic r*
⟨*proof*⟩

Wellfoundedness of finite acyclic relations

**lemma** *finite-acyclic-wf* [*rule-format*]: *finite r* ==> *acyclic r* −−> *wf r*
⟨*proof*⟩

**lemma** *finite-acyclic-wf-converse*: [|*finite r*; *acyclic r*|] ==> *wf* (*r*^−1)
⟨*proof*⟩

**lemma** *wf-iff-acyclic-if-finite*: *finite r* ==> *wf r* = *acyclic r*
⟨*proof*⟩

## 21.4 Well-Founded Recursion

cut

**lemma** *cuts-eq*: $(cut\ f\ r\ x = cut\ g\ r\ x) = (ALL\ y.\ (y,x):r\ --> f(y)=g(y))$
⟨*proof*⟩

**lemma** *cut-apply*: $(x,a):r ==> (cut\ f\ r\ a)(x) = f(x)$
⟨*proof*⟩

Inductive characterization of wfrec combinator; for details see: John Harrison, "Inductive definitions: automation and application"

**lemma** *wfrec-unique*: $[|\ adm\text{-}wf\ R\ F;\ wf\ R\ |] ==> EX!\ y.\ wfrec\text{-}rel\ R\ F\ x\ y$
⟨*proof*⟩

**lemma** *adm-lemma*: $adm\text{-}wf\ R\ (\%f\ x.\ F\ (cut\ f\ R\ x)\ x)$
⟨*proof*⟩

**lemma** *wfrec*: $wf(r) ==> wfrec\ r\ H\ a = H\ (cut\ (wfrec\ r\ H)\ r\ a)\ a$
⟨*proof*⟩

## 21.5 Code generator setup

**consts-code**
*wfrec* (⟨**module**⟩*wfrec?*)
**attach** ⟪
*fun wfrec f x = f (wfrec f) x;*
⟫

## 21.6 *nat* is well-founded

**lemma** *less-nat-rel*: $op < = (\lambda m\ n.\ n = Suc\ m)\hat{}++$
⟨*proof*⟩

**definition**
*pred-nat* :: $(nat * nat)\ set$ **where**
$pred\text{-}nat = \{(m,\ n).\ n = Suc\ m\}$

**definition**
*less-than* :: $(nat * nat)\ set$ **where**
$less\text{-}than = pred\text{-}nat\hat{}+$

**lemma** *less-eq*: $(m,\ n) \in pred\text{-}nat\hat{}+ \longleftrightarrow m < n$
⟨*proof*⟩

**lemma** *pred-nat-trancl-eq-le*:
$(m,\ n) \in pred\text{-}nat\hat{}* \longleftrightarrow m \leq n$
⟨*proof*⟩

**lemma** *wf-pred-nat*: *wf pred-nat*

⟨*proof*⟩

**lemma** *wf-less-than* [*iff*]: *wf less-than*
⟨*proof*⟩

**lemma** *trans-less-than* [*iff*]: *trans less-than*
⟨*proof*⟩

**lemma** *less-than-iff* [*iff*]: ((*x*,*y*): *less-than*) = (*x*<*y*)
⟨*proof*⟩

**lemma** *wf-less*: *wf* {(*x*, *y*::*nat*). *x* < *y*}
⟨*proof*⟩

## 21.7 Accessible Part

Inductive definition of the accessible part *acc r* of a relation; see also [**?**].

**inductive-set**
*acc* :: ('*a* ∗ '*a*) *set* => '*a set*
**for** *r* :: ('*a* ∗ '*a*) *set*
**where**
*accI*: (!!*y*. (*y*, *x*) : *r* ==> *y* : *acc r*) ==> *x* : *acc r*

**abbreviation**
*termip* :: ('*a* => '*a* => *bool*) => '*a* => *bool* **where**
*termip r* == *accp* ($r^{-1-1}$)

**abbreviation**
*termi* :: ('*a* ∗ '*a*) *set* => '*a set* **where**
*termi r* == *acc* ($r^{-1}$)

**lemmas** *accpI* = *accp.accI*

Induction rules

**theorem** *accp-induct*:
**assumes** *major*: *accp r a*
**assumes** *hyp*: !!*x*. *accp r x* ==> ∀ *y*. *r y x* −−> *P y* ==> *P x*
**shows** *P a*
⟨*proof*⟩

**theorems** *accp-induct-rule* = *accp-induct* [*rule-format*, *induct set*: *accp*]

**theorem** *accp-downward*: *accp r b* ==> *r a b* ==> *accp r a*
⟨*proof*⟩

**lemma** *not-accp-down*:
**assumes** *na*: ¬ *accp R x*
**obtains** *z* **where** *R z x* **and** ¬ *accp R z*
⟨*proof*⟩

**lemma** *accp-downwards-aux*: $r^{**}\ b\ a \Longrightarrow accp\ r\ a \longrightarrow accp\ r\ b$
⟨*proof*⟩

**theorem** *accp-downwards*: $accp\ r\ a \Longrightarrow r^{**}\ b\ a \Longrightarrow accp\ r\ b$
⟨*proof*⟩

**theorem** *accp-wfPI*: $\forall x.\ accp\ r\ x \Longrightarrow wfP\ r$
⟨*proof*⟩

**theorem** *accp-wfPD*: $wfP\ r \Longrightarrow accp\ r\ x$
⟨*proof*⟩

**theorem** *wfP-accp-iff*: $wfP\ r = (\forall x.\ accp\ r\ x)$
⟨*proof*⟩

Smaller relations have bigger accessible parts:

**lemma** *accp-subset*:
**assumes** *sub*: $R1 \leq R2$
**shows** $accp\ R2 \leq accp\ R1$
⟨*proof*⟩

This is a generalized induction theorem that works on subsets of the accessible part.

**lemma** *accp-subset-induct*:
**assumes** *subset*: $D \leq accp\ R$
**and** *dcl*: $\bigwedge x\ z.\ [\![D\ x;\ R\ z\ x]\!] \Longrightarrow D\ z$
**and** $D\ x$
**and** *istep*: $\bigwedge x.\ [\![D\ x;\ (\bigwedge z.\ R\ z\ x \Longrightarrow P\ z)]\!] \Longrightarrow P\ x$
**shows** $P\ x$
⟨*proof*⟩

Set versions of the above theorems

**lemmas** *acc-induct* = *accp-induct* [*to-set*]

**lemmas** *acc-induct-rule* = *acc-induct* [*rule-format*, *induct set*: *acc*]

**lemmas** *acc-downward* = *accp-downward* [*to-set*]

**lemmas** *not-acc-down* = *not-accp-down* [*to-set*]

**lemmas** *acc-downwards-aux* = *accp-downwards-aux* [*to-set*]

**lemmas** *acc-downwards* = *accp-downwards* [*to-set*]

**lemmas** *acc-wfI* = *accp-wfPI* [*to-set*]

**lemmas** *acc-wfD* = *accp-wfPD* [*to-set*]

**lemmas** *wf-acc-iff* = *wfP-accp-iff* [*to-set*]

**lemmas** *acc-subset* = *accp-subset* [*to-set pred-subset-eq*]

**lemmas** *acc-subset-induct* = *accp-subset-induct* [*to-set pred-subset-eq*]

## 21.8 Tools for building wellfounded relations

Inverse Image

**lemma** *wf-inv-image* [*simp*,*intro*!]: *wf*(*r*) ==> *wf*(*inv-image r* (*f*::'*a*=>'*b*))
⟨*proof*⟩

**lemma** *in-inv-image*[*simp*]: ((*x*,*y*) : *inv-image r f*) = ((*f x*, *f y*) : *r*)
⟨*proof*⟩

Measure functions into *nat*

**definition** *measure* :: ('*a* => *nat*) => ('*a* * '*a*)*set*
**where** *measure* == *inv-image less-than*

**lemma** *in-measure*[*simp*]: ((*x*,*y*) : *measure f*) = (*f x* < *f y*)
⟨*proof*⟩

**lemma** *wf-measure* [*iff*]: *wf* (*measure f*)
⟨*proof*⟩

Lexicographic combinations

**definition**
*lex-prod* :: [('*a**'*a*)*set*, ('*b**'*b*)*set*] => (('*a**'*b*)*('*a**'*b*))*set*
(**infixr** <**lex**> *80*)
**where**
*ra* <**lex**> *rb* == {((*a*,*b*),(*a*',*b*')). (*a*,*a*') : *ra* | *a*=*a*' & (*b*,*b*') : *rb*}

**lemma** *wf-lex-prod* [*intro*!]: [| *wf*(*ra*); *wf*(*rb*) |] ==> *wf*(*ra* <**lex**> *rb*)
⟨*proof*⟩

**lemma** *in-lex-prod*[*simp*]:
(((*a*,*b*),(*a*',*b*')): *r* <**lex**> *s*) = ((*a*,*a*'): *r* ∨ (*a* = *a*' ∧ (*b*, *b*') : *s*))
⟨*proof*⟩

*op* <**lex**> preserves transitivity

**lemma** *trans-lex-prod* [*intro*!]:
[| *trans R1*; *trans R2* |] ==> *trans* (*R1* <**lex**> *R2*)
⟨*proof*⟩

lexicographic combinations with measure functions

**definition**
*mlex-prod* :: ('*a* ⇒ *nat*) ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* (**infixr** <**mlex**> *80*)
**where**

$f <*mlex*> R = \mathit{inv\text{-}image}\ (\mathit{less\text{-}than} <*lex*> R)\ (\%x.\ (f\ x,\ x))$

**lemma** *wf-mlex*: $\mathit{wf}\ R \Longrightarrow \mathit{wf}\ (f <*mlex*> R)$
⟨*proof*⟩

**lemma** *mlex-less*: $f\ x < f\ y \Longrightarrow (x,\ y) \in f <*mlex*> R$
⟨*proof*⟩

**lemma** *mlex-leq*: $f\ x \leq f\ y \Longrightarrow (x,\ y) \in R \Longrightarrow (x,\ y) \in f <*mlex*> R$
⟨*proof*⟩

proper subset relation on finite sets

**definition** *finite-psubset* :: $('a\ \mathit{set} * 'a\ \mathit{set})\ \mathit{set}$
**where** $\mathit{finite\text{-}psubset} == \{(A,B).\ A < B\ \&\ \mathit{finite}\ B\}$

**lemma** *wf-finite-psubset*[*simp*]: *wf*(*finite-psubset*)
⟨*proof*⟩

**lemma** *trans-finite-psubset*: *trans finite-psubset*
⟨*proof*⟩

**lemma** *in-finite-psubset*[*simp*]: $(A,\ B) \in \mathit{finite\text{-}psubset} = (A < B\ \&\ \mathit{finite}\ B)$
⟨*proof*⟩

max- and min-extension of order to finite sets

**inductive-set** *max-ext* :: $('a \times 'a)\ \mathit{set} \Rightarrow ('a\ \mathit{set} \times 'a\ \mathit{set})\ \mathit{set}$
**for** $R$ :: $('a \times 'a)\ \mathit{set}$
**where**
*max-extI*[*intro*]: $\mathit{finite}\ X \Longrightarrow \mathit{finite}\ Y \Longrightarrow Y \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow \exists\, y{\in}Y.\ (x,\ y) \in R) \Longrightarrow (X,\ Y) \in \mathit{max\text{-}ext}\ R$

**lemma** *max-ext-wf*:
**assumes** *wf*: *wf r*
**shows** *wf* (*max-ext r*)
⟨*proof*⟩

**lemma** *max-ext-additive*:
$(A,\ B) \in \mathit{max\text{-}ext}\ R \Longrightarrow (C,\ D) \in \mathit{max\text{-}ext}\ R \Longrightarrow$
$(A \cup C,\ B \cup D) \in \mathit{max\text{-}ext}\ R$
⟨*proof*⟩

**definition**
*min-ext* :: $('a \times 'a)\ \mathit{set} \Rightarrow ('a\ \mathit{set} \times 'a\ \mathit{set})\ \mathit{set}$
**where**
[*code del*]: $\mathit{min\text{-}ext}\ r = \{(X,\ Y) \mid X\ Y.\ X \neq \{\} \wedge (\forall\, y \in Y.\ (\exists\, x \in X.\ (x,\ y) \in r))\}$

**lemma** *min-ext-wf*:

**assumes** *wf r*
**shows** *wf* (*min-ext r*)
⟨*proof*⟩

Wellfoundedness of *same-fst*

**definition**
*same-fst* :: (*'a* => *bool*) => (*'a* => (*'b* * *'b*)*set*) => ((*'a***'b*)*(*'a***'b*))*set*
**where**
*same-fst P R* == {((*x'*,*y'*),(*x*,*y*)) . *x'*=*x* & *P x* & (*y'*,*y*) : *R x*}
— For *rec-def* declarations where the first n parameters stay unchanged in the recursive call.

**lemma** *same-fstI* [*intro*!]:
[| *P x*; (*y'*,*y*) : *R x* |] ==> ((*x*,*y'*),(*x*,*y*)) : *same-fst P R*
⟨*proof*⟩

**lemma** *wf-same-fst*:
**assumes** *prem*: (!!*x*. *P x* ==> *wf*(*R x*))
**shows** *wf*(*same-fst P R*)
⟨*proof*⟩

## 21.9 Weakly decreasing sequences (w.r.t. some well-founded order) stabilize.

This material does not appear to be used any longer.

**lemma** *sequence-trans*: [| *ALL i*. (*f* (*Suc i*), *f i*) : *r*^* |] ==> (*f* (*i*+*k*), *f i*) : *r*^*
⟨*proof*⟩

**lemma** *wf-weak-decr-stable*:
**assumes** *as*: *ALL i*. (*f* (*Suc i*), *f i*) : *r*^* *wf* (*r*^+)
**shows** *EX i*. *ALL k*. *f* (*i*+*k*) = *f i*
⟨*proof*⟩

**lemma** *weak-decr-stable*:
*ALL i*. *f* (*Suc i*) <= ((*f i*)::*nat*) ==> *EX i*. *ALL k*. *f* (*i*+*k*) = *f i*
⟨*proof*⟩

## 21.10 size of a datatype value

⟨*ML*⟩

**lemma** *size-bool* [*code*]:
*size* (*b*::*bool*) = *0* ⟨*proof*⟩

**lemma** *nat-size* [*simp*, *code*]: *size* (*n*::*nat*) = *n*
⟨*proof*⟩

**declare** *prod.size* [*noatp*]

**lemma** [*code*]:
*size* (*P* :: *'a Predicate.pred*) = *0* ⟨*proof*⟩

**lemma** [*code*]:
*pred-size f P* = *0* ⟨*proof*⟩

**end**

# 22 FunDef: Function Definitions and Termination Proofs

**theory** *FunDef*
**imports** *Wellfounded*
**uses**
*Tools/prop-logic.ML*
*Tools/sat-solver.ML*
(*Tools/function-package/fundef-lib.ML*)
(*Tools/function-package/fundef-common.ML*)
(*Tools/function-package/inductive-wrap.ML*)
(*Tools/function-package/context-tree.ML*)
(*Tools/function-package/fundef-core.ML*)
(*Tools/function-package/sum-tree.ML*)
(*Tools/function-package/mutual.ML*)
(*Tools/function-package/pattern-split.ML*)
(*Tools/function-package/fundef-package.ML*)
(*Tools/function-package/auto-term.ML*)
(*Tools/function-package/measure-functions.ML*)
(*Tools/function-package/lexicographic-order.ML*)
(*Tools/function-package/fundef-datatype.ML*)
(*Tools/function-package/induction-scheme.ML*)
(*Tools/function-package/termination.ML*)
(*Tools/function-package/decompose.ML*)
(*Tools/function-package/descent.ML*)
(*Tools/function-package/scnp-solve.ML*)
(*Tools/function-package/scnp-reconstruct.ML*)
**begin**

## 22.1 Definitions with default value.

**definition**
*THE-default* :: *'a* ⇒ (*'a* ⇒ *bool*) ⇒ *'a* **where**
*THE-default d P* = (*if* (∃!*x*. *P x*) *then* (*THE x*. *P x*) *else d*)

**lemma** *THE-defaultI'*: ∃!*x*. *P x* ⟹ *P* (*THE-default d P*)
⟨*proof*⟩

**lemma** *THE-default1-equality*:
$[\![\exists!x.\ P\ x;\ P\ a]\!] \Longrightarrow$ *THE-default* $d\ P = a$
⟨*proof*⟩

**lemma** *THE-default-none*:
$\neg(\exists!x.\ P\ x) \Longrightarrow$ *THE-default* $d\ P = d$
⟨*proof*⟩

**lemma** *fundef-ex1-existence*:
**assumes** *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
**assumes** *ex1*: $\exists!y.\ G\ x\ y$
**shows** $G\ x\ (f\ x)$
⟨*proof*⟩

**lemma** *fundef-ex1-uniqueness*:
**assumes** *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
**assumes** *ex1*: $\exists!y.\ G\ x\ y$
**assumes** *elm*: $G\ x\ (h\ x)$
**shows** $h\ x = f\ x$
⟨*proof*⟩

**lemma** *fundef-ex1-iff*:
**assumes** *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
**assumes** *ex1*: $\exists!y.\ G\ x\ y$
**shows** $(G\ x\ y) = (f\ x = y)$
⟨*proof*⟩

**lemma** *fundef-default-value*:
**assumes** *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
**assumes** *graph*: $\bigwedge x\ y.\ G\ x\ y \Longrightarrow D\ x$
**assumes** $\neg\ D\ x$
**shows** $f\ x = d\ x$
⟨*proof*⟩

**definition** *in-rel-def*[*simp*]:
*in-rel* $R\ x\ y == (x,\ y) \in R$

**lemma** *wf-in-rel*:
*wf* $R \Longrightarrow$ *wfP* (*in-rel* $R$)
⟨*proof*⟩

⟨*ML*⟩

## 22.2 Measure Functions

**inductive** *is-measure* :: $('a \Rightarrow nat) \Rightarrow bool$
**where** *is-measure-trivial*: *is-measure* $f$

⟨*ML*⟩

**lemma** *measure-size*[*measure-function*]: *is-measure size*
⟨*proof*⟩

**lemma** *measure-fst*[*measure-function*]: *is-measure f* ⟹ *is-measure* (λ*p*. *f* (*fst p*))
⟨*proof*⟩
**lemma** *measure-snd*[*measure-function*]: *is-measure f* ⟹ *is-measure* (λ*p*. *f* (*snd p*))
⟨*proof*⟩

⟨*ML*⟩

## 22.3 Congruence Rules

**lemma** *let-cong* [*fundef-cong*]:
*M* = *N* ⟹ (⋀*x*. *x* = *N* ⟹ *f x* = *g x*) ⟹ *Let M f* = *Let N g*
⟨*proof*⟩

**lemmas** [*fundef-cong*] =
*if-cong image-cong INT-cong UN-cong*
*bex-cong ball-cong imp-cong*

**lemma** *split-cong* [*fundef-cong*]:
(⋀*x y*. (*x*, *y*) = *q* ⟹ *f x y* = *g x y*) ⟹ *p* = *q*
⟹ *split f p* = *split g q*
⟨*proof*⟩

**lemma** *comp-cong* [*fundef-cong*]:
*f* (*g x*) = *f* ′ (*g* ′ *x* ′) ⟹ (*f o g*) *x* = (*f* ′ *o g* ′) *x* ′
⟨*proof*⟩

## 22.4 Simp rules for termination proofs

**lemma** *termination-basic-simps*[*termination-simp*]:
*x* < (*y*::*nat*) ⟹ *x* < *y* + *z*
*x* < *z* ⟹ *x* < *y* + *z*
*x* ≤ *y* ⟹ *x* ≤ *y* + (*z*::*nat*)
*x* ≤ *z* ⟹ *x* ≤ *y* + (*z*::*nat*)
*x* < *y* ⟹ *x* ≤ (*y*::*nat*)
⟨*proof*⟩

**declare** *le-imp-less-Suc*[*termination-simp*]

**lemma** *prod-size-simp*[*termination-simp*]:
*prod-size f g p* = *f* (*fst p*) + *g* (*snd p*) + *Suc 0*
⟨*proof*⟩

## 22.5 Decomposition

**lemma** *less-by-empty*:
$A = \{\} \Longrightarrow A \subseteq B$
**and** *union-comp-emptyL*:
$[\![ A\ O\ C = \{\};\ B\ O\ C = \{\} ]\!] \Longrightarrow (A \cup B)\ O\ C = \{\}$
**and** *union-comp-emptyR*:
$[\![ A\ O\ B = \{\};\ A\ O\ C = \{\} ]\!] \Longrightarrow A\ O\ (B \cup C) = \{\}$
**and** *wf-no-loop*:
$R\ O\ R = \{\} \Longrightarrow wf\ R$
⟨*proof*⟩

## 22.6 Reduction Pairs

**definition**
*reduction-pair* $P = (wf\ (fst\ P) \wedge snd\ P\ O\ fst\ P \subseteq fst\ P)$

**lemma** *reduction-pairI*[*intro*]: $wf\ R \Longrightarrow S\ O\ R \subseteq R \Longrightarrow$ *reduction-pair* $(R, S)$
⟨*proof*⟩

**lemma** *reduction-pair-lemma*:
**assumes** *rp*: *reduction-pair* $P$
**assumes** $R \subseteq fst\ P$
**assumes** $S \subseteq snd\ P$
**assumes** $wf\ S$
**shows** $wf\ (R \cup S)$
⟨*proof*⟩

**definition**
*rp-inv-image* $= (\lambda(R,S)\ f.\ (inv\text{-}image\ R\ f,\ inv\text{-}image\ S\ f))$

**lemma** *rp-inv-image-rp*:
*reduction-pair* $P \Longrightarrow$ *reduction-pair* (*rp-inv-image* $P\ f$)
⟨*proof*⟩

## 22.7 Concrete orders for SCNP termination proofs

**definition** *pair-less* $=$ *less-than* $<*lex*>$ *less-than*
**definition** [*code del*]: *pair-leq* $=$ *pair-less*$^=$
**definition** *max-strict* $=$ *max-ext pair-less*
**definition** [*code del*]: *max-weak* $=$ *max-ext pair-leq* $\cup$ $\{(\{\}, \{\})\}$
**definition** [*code del*]: *min-strict* $=$ *min-ext pair-less*
**definition** [*code del*]: *min-weak* $=$ *min-ext pair-leq* $\cup$ $\{(\{\}, \{\})\}$

**lemma** *wf-pair-less*[*simp*]: *wf pair-less*
⟨*proof*⟩

Introduction rules for *pair-less*/*pair-leq*

**lemma** *pair-leqI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*
**and** *pair-leqI2*: $a \leq b \Longrightarrow s \leq t \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*

**and** *pair-lessI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
**and** *pair-lessI2*: $a \leq b \Longrightarrow s < t \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
⟨*proof*⟩

Introduction rules for max

**lemma** *smax-emptyI*:
*finite* $Y \Longrightarrow Y \neq \{\} \Longrightarrow (\{\}, Y) \in$ *max-strict*
**and** *smax-insertI*:
$[\![y \in Y; (x, y) \in$ *pair-less*; $(X, Y) \in$ *max-strict*$]\!] \Longrightarrow (insert\ x\ X, Y) \in$ *max-strict*
**and** *wmax-emptyI*:
*finite* $X \Longrightarrow (\{\}, X) \in$ *max-weak*
**and** *wmax-insertI*:
$[\![y \in YS; (x, y) \in$ *pair-leq*; $(XS, YS) \in$ *max-weak*$]\!] \Longrightarrow (insert\ x\ XS, YS) \in$ *max-weak*
⟨*proof*⟩

Introduction rules for min

**lemma** *smin-emptyI*:
$X \neq \{\} \Longrightarrow (X, \{\}) \in$ *min-strict*
**and** *smin-insertI*:
$[\![x \in XS; (x, y) \in$ *pair-less*; $(XS, YS) \in$ *min-strict*$]\!] \Longrightarrow (XS, insert\ y\ YS) \in$ *min-strict*
**and** *wmin-emptyI*:
$(X, \{\}) \in$ *min-weak*
**and** *wmin-insertI*:
$[\![x \in XS; (x, y) \in$ *pair-leq*; $(XS, YS) \in$ *min-weak*$]\!] \Longrightarrow (XS, insert\ y\ YS) \in$ *min-weak*
⟨*proof*⟩

Reduction Pairs

**lemma** *max-ext-compat*:
**assumes** $S\ O\ R \subseteq R$
**shows** $(max\text{-}ext\ S \cup \{(\{\},\{\})\})\ O\ max\text{-}ext\ R \subseteq max\text{-}ext\ R$
⟨*proof*⟩

**lemma** *max-rpair-set*: *reduction-pair* (*max-strict*, *max-weak*)
⟨*proof*⟩

**lemma** *min-ext-compat*:
**assumes** $S\ O\ R \subseteq R$
**shows** $(min\text{-}ext\ S \cup \{(\{\},\{\})\})\ O\ min\text{-}ext\ R \subseteq min\text{-}ext\ R$
⟨*proof*⟩

**lemma** *min-rpair-set*: *reduction-pair* (*min-strict*, *min-weak*)
⟨*proof*⟩

## 22.8 Tool setup

⟨*ML*⟩

**end**

# 23 Record: Extensible records with structural subtyping

**theory** *Record*
**imports** *Product-Type*
**uses** (*Tools/record-package.ML*)
**begin**

**lemma** *prop-subst*: $s = t \Longrightarrow PROP\ P\ t \Longrightarrow PROP\ P\ s$
⟨*proof*⟩

**lemma** *rec-UNIV-I*: $\bigwedge x.\ x{\in}UNIV \equiv True$
⟨*proof*⟩

**lemma** *rec-True-simp*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

**lemma** *K-record-comp*: $(\lambda x.\ c) \circ f = (\lambda x.\ c)$
⟨*proof*⟩

## 23.1 Concrete record syntax

**nonterminals**
*ident field-type field-types field fields update updates*
**syntax**
*-constify* :: *id* => *ident* (-)
*-constify* :: *longid* => *ident* (-)

*-field-type* :: [*ident*, *type*] => *field-type* ((*2*- ::/ -))
:: *field-type* => *field-types* (-)
*-field-types* :: [*field-type*, *field-types*] => *field-types* (-,/ -)
*-record-type* :: *field-types* => *type* ((*3*'(| - |')))
*-record-type-scheme* :: [*field-types*, *type*] => *type* ((*3*'(| -,/ (*2*... ::/ -) |')))

*-field* :: [*ident*, 'a] => *field* ((*2*- =/ -))
:: *field* => *fields* (-)
*-fields* :: [*field*, *fields*] => *fields* (-,/ -)
*-record* :: *fields* => 'a ((*3*'(| - |')))
*-record-scheme* :: [*fields*, 'a] => 'a ((*3*'(| -,/ (*2*... =/ -) |')))

*-update-name* :: *idt*
*-update* :: [*ident*, 'a] => *update* ((*2*- :=/ -))
:: *update* => *updates* (-)
*-updates* :: [*update*, *updates*] => *updates* (-,/ -)

*-record-update* :: [*'a*, *updates*] => *'b* (-/(*3'*(| - |')) [*900*,*0*] *900*)

**syntax** (*xsymbols*)
*-record-type* :: *field-types* => *type* ((*3*(⦇-⦈)))
*-record-type-scheme* :: [*field-types*, *type*] => *type* ((*3*(⦇-,/ (*2*... ::/ -)⦈)))
*-record* :: *fields* => *'a* ((*3*(⦇-⦈)))
*-record-scheme* :: [*fields*, *'a*] => *'a* ((*3*(⦇-,/ (*2*... =/ -)⦈)))
*-record-update* :: [*'a*, *updates*] => *'b* (-/(*3*(⦇-⦈)) [*900*,*0*] *900*)

⟨*ML*⟩

**end**

# 24 Option: Datatype option

**theory** *Option*
**imports** *Datatype Finite-Set*
**begin**

**datatype** *'a option* = *None* | *Some 'a*

**lemma** *not-None-eq* [*iff*]: (*x* ~= *None*) = (*EX y*. *x* = *Some y*)
⟨*proof*⟩

**lemma** *not-Some-eq* [*iff*]: (*ALL y*. *x* ~= *Some y*) = (*x* = *None*)
⟨*proof*⟩

Although it may appear that both of these equalities are helpful only when applied to assumptions, in practice it seems better to give them the uniform iff attribute.

**lemma** *option-caseE*:
**assumes** *c*: (*case x of None* => *P* | *Some y* => *Q y*)
**obtains**
(*None*) *x* = *None* **and** *P*
| (*Some*) *y* **where** *x* = *Some y* **and** *Q y*
⟨*proof*⟩

**lemma** *insert-None-conv-UNIV*: *insert None* (*range Some*) = *UNIV*
⟨*proof*⟩

**instance** *option* :: (*finite*) *finite* ⟨*proof*⟩

**lemma** *inj-Some* [*simp*]: *inj-on Some A*
⟨*proof*⟩

### 24.0.1 Operations

**primrec** *the* :: *'a option* => *'a* **where**

*the (Some x) = x*

**primrec** *set :: 'a option => 'a set* **where**
*set None = {} |*
*set (Some x) = {x}*

**lemma** *ospec [dest]: (ALL x:set A. P x) ==> A = Some x ==> P x*
⟨*proof*⟩

⟨*ML*⟩

**lemma** *elem-set [iff]: (x : set xo) = (xo = Some x)*
⟨*proof*⟩

**lemma** *set-empty-eq [simp]: (set xo = {}) = (xo = None)*
⟨*proof*⟩

**definition**
*map :: ('a ⇒ 'b) ⇒ 'a option ⇒ 'b option*
**where**
*[code del]: map = (%f y. case y of None => None | Some x => Some (f x))*

**lemma** *option-map-None [simp, code]: map f None = None*
⟨*proof*⟩

**lemma** *option-map-Some [simp, code]: map f (Some x) = Some (f x)*
⟨*proof*⟩

**lemma** *option-map-is-None [iff]:*
*(map f opt = None) = (opt = None)*
⟨*proof*⟩

**lemma** *option-map-eq-Some [iff]:*
*(map f xo = Some y) = (EX z. xo = Some z & f z = y)*
⟨*proof*⟩

**lemma** *option-map-comp:*
*map f (map g opt) = map (f o g) opt*
⟨*proof*⟩

**lemma** *option-map-o-sum-case [simp]:*
*map f o sum-case g h = sum-case (map f o g) (map f o h)*
⟨*proof*⟩

**hide** (**open**) *const set map*

### 24.0.2 Code generator setup

**definition**
*is-none* :: *'a option* ⇒ *bool* **where**
*is-none-none* [*code post*, *symmetric*, *code inline*]: *is-none x* ⟷ *x* = *None*

**lemma** *is-none-code* [*code*]:
**shows** *is-none None* ⟷ *True*
**and** *is-none* (*Some x*) ⟷ *False*
⟨*proof*⟩

**hide** (**open**) *const is-none*

**code-type** *option*
(*SML* - *option*)
(*OCaml* - *option*)
(*Haskell Maybe* -)

**code-const** *None* **and** *Some*
(*SML NONE* **and** *SOME*)
(*OCaml None* **and** *Some* -)
(*Haskell Nothing* **and** *Just*)

**code-instance** *option* :: *eq*
(*Haskell* −)

**code-const** *eq-class.eq* :: *'a*::*eq option* ⇒ *'a option* ⇒ *bool*
(*Haskell* **infixl** *4* ==)

**code-reserved** *SML*
*option NONE SOME*

**code-reserved** *OCaml*
*option None Some*

**end**

# 25 Extraction: Program extraction for HOL

**theory** *Extraction*
**imports** *Option*
**uses** *Tools/rewrite-hol-proof.ML*
**begin**

## 25.1 Setup

⟨*ML*⟩

**lemmas** [*extraction-expand*] =
*meta-spec atomize-eq atomize-all atomize-imp atomize-conj*
*allE rev-mp conjE Eq-TrueI Eq-FalseI eqTrueI eqTrueE eq-cong2*
*notE' impE' impE iffE imp-cong simp-thms eq-True eq-False*
*induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq*
*induct-forall-def induct-implies-def induct-equal-def induct-conj-def*
*induct-atomize induct-rulify induct-rulify-fallback*
*True-implies-equals TrueE*

**datatype** *sumbool* = *Left* | *Right*

## 25.2 Type of extracted program

**extract-type**
*typeof* (*Trueprop P*) ≡ *typeof P*

*typeof P* ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
*typeof* (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*Q*))

*typeof Q* ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof* (*P* ⟶ *Q*) ≡ *Type* (*TYPE*(*Null*))

*typeof P* ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
*typeof* (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*P* ⇒ '*Q*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*(*Null*))) ⟹
*typeof* (∀ *x*. *P x*) ≡ *Type* (*TYPE*(*Null*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*('*P*))) ⟹
*typeof* (∀ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a* ⇒ '*P*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*(*Null*))) ⟹
*typeof* (∃ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*('*P*))) ⟹
*typeof* (∃ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a* × '*P*))

*typeof P* ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*(*Null*)) ⟹
*typeof* (*P* ∨ *Q*) ≡ *Type* (*TYPE*(*sumbool*))

*typeof P* ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
*typeof* (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*Q option*))

*typeof P* ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*(*Null*)) ⟹
*typeof* (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*P option*))

*typeof P* ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
*typeof* (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*P* + '*Q*))

*typeof P* ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹

$typeof\ (P \wedge Q) \equiv Type\ (TYPE('Q))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P \times 'Q))$

$typeof\ (P = Q) \equiv typeof\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P))$

$typeof\ (x \in P) \equiv typeof\ P$

## 25.3 Realizability

**realizability**
$(realizes\ t\ (Trueprop\ P)) \equiv (Trueprop\ (realizes\ t\ P))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (realizes\ Null\ P \longrightarrow realizes\ t\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE('P))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x::'P.\ realizes\ x\ P \longrightarrow realizes\ Null\ Q)$

$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x.\ realizes\ x\ P \longrightarrow realizes\ (t\ x)\ Q)$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ Null\ (P\ x))$

$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ (t\ x)\ (P\ x))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ Null\ (P\ t))$

$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ (snd\ t)\ (P\ (fst\ t)))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Left \Rightarrow realizes\ Null\ P \mid Right \Rightarrow realizes\ Null\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ P \mid Some\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ Q \mid Some\ p \Rightarrow realizes\ p\ P)$

$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Inl\ p \Rightarrow realizes\ p\ P \mid Inr\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ Null\ P \wedge realizes\ t\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ t\ P \wedge realizes\ Null\ Q)$

$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ (fst\ t)\ P \wedge realizes\ (snd\ t)\ Q)$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (\neg\ P) \equiv \neg\ realizes\ Null\ P$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow$
$realizes\ t\ (\neg\ P) \equiv (\forall x{::}'P.\ \neg\ realizes\ x\ P)$

$typeof\ (P{::}bool) \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (P = Q) \equiv realizes\ Null\ P = realizes\ Null\ Q$

$(realizes\ t\ (P = Q)) \equiv (realizes\ t\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P)))$

## 25.4 Computational content of basic inference rules

**theorem** *disjE-realizer*:
**assumes** *r*: $case\ x\ of\ Inl\ p \Rightarrow P\ p \mid Inr\ q \Rightarrow Q\ q$
**and** *r1*: $\bigwedge p.\ P\ p \Longrightarrow R\ (f\ p)$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
**shows** $R\ (case\ x\ of\ Inl\ p \Rightarrow f\ p \mid Inr\ q \Rightarrow g\ q)$
⟨*proof*⟩

**theorem** *disjE-realizer2*:
**assumes** *r*: $case\ x\ of\ None \Rightarrow P \mid Some\ q \Rightarrow Q\ q$
**and** *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
**shows** $R\ (case\ x\ of\ None \Rightarrow f \mid Some\ q \Rightarrow g\ q)$
⟨*proof*⟩

**theorem** *disjE-realizer3*:
**assumes** *r*: $case\ x\ of\ Left \Rightarrow P \mid Right \Rightarrow Q$
**and** *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $Q \Longrightarrow R\ g$
**shows** $R\ (case\ x\ of\ Left \Rightarrow f \mid Right \Rightarrow g)$
⟨*proof*⟩

**theorem** *conjI-realizer*:
$P\ p \Longrightarrow Q\ q \Longrightarrow P\ (fst\ (p,\ q)) \wedge Q\ (snd\ (p,\ q))$
⟨*proof*⟩

**theorem** *exI-realizer*:
$P\ y\ x \Longrightarrow P\ (snd\ (x,\ y))\ (fst\ (x,\ y))$ ⟨*proof*⟩

**theorem** *exE-realizer*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q\ (f\ x\ y)) \Longrightarrow Q\ (let\ (x,\ y) = p\ in\ f\ x\ y)$
⟨*proof*⟩

**theorem** *exE-realizer'*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q) \Longrightarrow Q$ ⟨*proof*⟩

⟨*ML*⟩

**realizers**
*impI* $(P,\ Q)$: $\lambda pq.\ pq$
$\Lambda\ P\ Q\ pq\ (h\text{: }\text{-}).\ allI \cdot \text{-} \bullet (\Lambda\ x.\ impI \cdot \text{-} \cdot \text{-} \bullet (h \cdot x))$

*impI* $(P)$: *Null*
$\Lambda\ P\ Q\ (h\text{: }\text{-}).\ allI \cdot \text{-} \bullet (\Lambda\ x.\ impI \cdot \text{-} \cdot \text{-} \bullet (h \cdot x))$

*impI* $(Q)$: $\lambda q.\ q\ \Lambda\ P\ Q\ q.\ impI \cdot \text{-} \cdot \text{-}$

*impI*: *Null impI*

*mp* $(P,\ Q)$: $\lambda pq.\ pq$
$\Lambda\ P\ Q\ pq\ (h\text{: }\text{-})\ p.\ mp \cdot \text{-} \cdot \text{-} \bullet (spec \cdot \text{-} \cdot p \bullet h)$

*mp* $(P)$: *Null*
$\Lambda\ P\ Q\ (h\text{: }\text{-})\ p.\ mp \cdot \text{-} \cdot \text{-} \bullet (spec \cdot \text{-} \cdot p \bullet h)$

*mp* $(Q)$: $\lambda q.\ q\ \Lambda\ P\ Q\ q.\ mp \cdot \text{-} \cdot \text{-}$

*mp*: *Null mp*

*allI* $(P)$: $\lambda p.\ p\ \Lambda\ P\ p.\ allI \cdot \text{-}$

*allI*: *Null allI*

*spec* $(P)$: $\lambda x\ p.\ p\ x\ \Lambda\ P\ x\ p.\ spec \cdot \text{-} \cdot x$

*spec*: *Null spec*

*exI* $(P)$: $\lambda x\ p.\ (x,\ p)\ \Lambda\ P\ x\ p.\ exI\text{-}realizer \cdot P \cdot p \cdot x$

*exI*: $\lambda x.\ x\ \Lambda\ P\ x\ (h\text{: }\text{-}).\ h$

*exE* $(P,\ Q)$: $\lambda p\ pq.\ let\ (x,\ y) = p\ in\ pq\ x\ y$
$\Lambda\ P\ Q\ p\ (h\text{: }\text{-})\ pq.\ exE\text{-}realizer \cdot P \cdot p \cdot Q \cdot pq \bullet h$

*exE* $(P)$: *Null*
$\Lambda\ P\ Q\ p.\ exE\text{-}realizer' \cdot \text{-} \cdot \text{-} \cdot \text{-}$

*exE* (*Q*): $\lambda x\ pq.\ pq\ x$
  $\Lambda\ P\ Q\ x\ (h1\colon \text{-})\ pq\ (h2\colon \text{-}).\ h2 \cdot x \bullet h1$

*exE*: *Null*
  $\Lambda\ P\ Q\ x\ (h1\colon \text{-})\ (h2\colon \text{-}).\ h2 \cdot x \bullet h1$

*conjI* (*P*, *Q*): *Pair*
  $\Lambda\ P\ Q\ p\ (h\colon \text{-})\ q.\ conjI\text{-}realizer \cdot P \cdot p \cdot Q \cdot q \bullet h$

*conjI* (*P*): $\lambda p.\ p$
  $\Lambda\ P\ Q\ p.\ conjI \cdot \text{-} \cdot \text{-}$

*conjI* (*Q*): $\lambda q.\ q$
  $\Lambda\ P\ Q\ (h\colon \text{-})\ q.\ conjI \cdot \text{-} \cdot \text{-} \bullet h$

*conjI*: *Null conjI*

*conjunct1* (*P*, *Q*): *fst*
  $\Lambda\ P\ Q\ pq.\ conjunct1 \cdot \text{-} \cdot \text{-}$

*conjunct1* (*P*): $\lambda p.\ p$
  $\Lambda\ P\ Q\ p.\ conjunct1 \cdot \text{-} \cdot \text{-}$

*conjunct1* (*Q*): *Null*
  $\Lambda\ P\ Q\ q.\ conjunct1 \cdot \text{-} \cdot \text{-}$

*conjunct1*: *Null conjunct1*

*conjunct2* (*P*, *Q*): *snd*
  $\Lambda\ P\ Q\ pq.\ conjunct2 \cdot \text{-} \cdot \text{-}$

*conjunct2* (*P*): *Null*
  $\Lambda\ P\ Q\ p.\ conjunct2 \cdot \text{-} \cdot \text{-}$

*conjunct2* (*Q*): $\lambda p.\ p$
  $\Lambda\ P\ Q\ p.\ conjunct2 \cdot \text{-} \cdot \text{-}$

*conjunct2*: *Null conjunct2*

*disjI1* (*P*, *Q*): *Inl*
  $\Lambda\ P\ Q\ p.\ iffD2 \cdot \text{-} \cdot \text{-} \bullet (sum.cases\text{-}1 \cdot P \cdot \text{-} \cdot p)$

*disjI1* (*P*): *Some*
  $\Lambda\ P\ Q\ p.\ iffD2 \cdot \text{-} \cdot \text{-} \bullet (option.cases\text{-}2 \cdot \text{-} \cdot P \cdot p)$

*disjI1* (*Q*): *None*
  $\Lambda\ P\ Q.\ iffD2 \cdot \text{-} \cdot \text{-} \bullet (option.cases\text{-}1 \cdot \text{-} \cdot \text{-})$

*disjI1*: *Left*

```
  Λ P Q. iffD2 · - · - · (sumbool.cases-1 · - · -)

disjI2 (P, Q): Inr
  Λ Q P q. iffD2 · - · - · (sum.cases-2 · - · Q · q)

disjI2 (P): None
  Λ Q P. iffD2 · - · - · (option.cases-1 · - · -)

disjI2 (Q): Some
  Λ Q P q. iffD2 · - · - · (option.cases-2 · - · Q · q)

disjI2: Right
  Λ Q P. iffD2 · - · - · (sumbool.cases-2 · - · -)

disjE (P, Q, R): λpq pr qr.
    (case pq of Inl p ⇒ pr p | Inr q ⇒ qr q)
  Λ P Q R pq (h1: -) pr (h2: -) qr.
      disjE-realizer · - · - · pq · R · pr · qr · h1 · h2

disjE (Q, R): λpq pr qr.
    (case pq of None ⇒ pr | Some q ⇒ qr q)
  Λ P Q R pq (h1: -) pr (h2: -) qr.
      disjE-realizer2 · - · - · pq · R · pr · qr · h1 · h2

disjE (P, R): λpq pr qr.
    (case pq of None ⇒ qr | Some p ⇒ pr p)
  Λ P Q R pq (h1: -) pr (h2: -) qr (h3: -).
      disjE-realizer2 · - · - · pq · R · qr · pr · h1 · h3 · h2

disjE (R): λpq pr qr.
    (case pq of Left ⇒ pr | Right ⇒ qr)
  Λ P Q R pq (h1: -) pr (h2: -) qr.
      disjE-realizer3 · - · - · pq · R · pr · qr · h1 · h2

disjE (P, Q): Null
  Λ P Q R pq. disjE-realizer · - · - · pq · (λx. R) · - · -

disjE (Q): Null
  Λ P Q R pq. disjE-realizer2 · - · - · pq · (λx. R) · - · -

disjE (P): Null
  Λ P Q R pq (h1: -) (h2: -) (h3: -).
      disjE-realizer2 · - · - · pq · (λx. R) · - · - · h1 · h3 · h2

disjE: Null
  Λ P Q R pq. disjE-realizer3 · - · - · pq · (λx. R) · - · -

FalseE (P): default
  Λ P. FalseE · -
```

*FalseE*: *Null FalseE*

*notI* (*P*): *Null*
  Λ *P* (*h*: -). *allI* · - · (Λ *x*. *notI* · - · (*h* · *x*))

*notI*: *Null notI*

*notE* (*P*, *R*): λ*p*. *default*
  Λ *P R* (*h*: -) *p*. *notE* · - · - · (*spec* · - · *p* · *h*)

*notE* (*P*): *Null*
  Λ *P R* (*h*: -) *p*. *notE* · - · - · (*spec* · - · *p* · *h*)

*notE* (*R*): *default*
  Λ *P R*. *notE* · - · -

*notE*: *Null notE*

*subst* (*P*): λ*s t ps*. *ps*
  Λ *s t P* (*h*: -) *ps*. *subst* · *s* · *t* · *P ps* · *h*

*subst*: *Null subst*

*iffD1* (*P*, *Q*): *fst*
  Λ *Q P pq* (*h*: -) *p*.
    *mp* · - · - · (*spec* · - · *p* · (*conjunct1* · - · - · *h*))

*iffD1* (*P*): λ*p*. *p*
  Λ *Q P p* (*h*: -). *mp* · - · - · (*conjunct1* · - · - · *h*)

*iffD1* (*Q*): *Null*
  Λ *Q P q1* (*h*: -) *q2*.
    *mp* · - · - · (*spec* · - · *q2* · (*conjunct1* · - · - · *h*))

*iffD1*: *Null iffD1*

*iffD2* (*P*, *Q*): *snd*
  Λ *P Q pq* (*h*: -) *q*.
    *mp* · - · - · (*spec* · - · *q* · (*conjunct2* · - · - · *h*))

*iffD2* (*P*): λ*p*. *p*
  Λ *P Q p* (*h*: -). *mp* · - · - · (*conjunct2* · - · - · *h*)

*iffD2* (*Q*): *Null*
  Λ *P Q q1* (*h*: -) *q2*.
    *mp* · - · - · (*spec* · - · *q2* · (*conjunct2* · - · - · *h*))

*iffD2*: *Null iffD2*

*iffI* (*P*, *Q*): *Pair*
  Λ *P Q pq* (*h1* : -) *qp* (*h2* : -). *conjI-realizer* ·
    (λ*pq*. ∀ *x*. *P x* ⟶ *Q* (*pq x*)) · *pq* ·
    (λ*qp*. ∀ *x*. *Q x* ⟶ *P* (*qp x*)) · *qp* ·
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h1* · *x*))) ·
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h2* · *x*)))

*iffI* (*P*): λ*p*. *p*
  Λ *P Q* (*h1* : -) *p* (*h2* : -). *conjI* · - · - · -
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h1* · *x*))) ·
    (*impI* · - · - · - · *h2*)

*iffI* (*Q*): λ*q*. *q*
  Λ *P Q q* (*h1* : -) (*h2* : -). *conjI* · - · - · -
    (*impI* · - · - · - · *h1*) ·
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h2* · *x*)))

*iffI*: *Null iffI*

⟨*ML*⟩

**end**

# 26 Divides: The division operators div and mod

**theory** *Divides*
**imports** *Nat Power Product-Type*
**uses** ~~/*src*/*Provers*/*Arith*/*cancel-div-mod.ML*
**begin**

## 26.1 Syntactic division operations

**class** *div* = *dvd* +
  **fixes** *div* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *div 70*)
    **and** *mod* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *mod 70*)

## 26.2 Abstract division in commutative semirings.

**class** *semiring-div* = *comm-semiring-1-cancel* + *div* +
  **assumes** *mod-div-equality*: *a div b* ∗ *b* + *a mod b* = *a*
    **and** *div-by-0* [*simp*]: *a div 0* = *0*
    **and** *div-0* [*simp*]: *0 div a* = *0*
    **and** *div-mult-self1* [*simp*]: *b* ≠ *0* ⟹ (*a* + *c* ∗ *b*) *div b* = *c* + *a div b*
**begin**

*op div* and *op mod*

**lemma** *mod-div-equality2*: $b * (a\ div\ b) + a\ mod\ b = a$
⟨*proof*⟩

**lemma** *mod-div-equality'*: $a\ mod\ b + a\ div\ b * b = a$
⟨*proof*⟩

**lemma** *div-mod-equality*: $((a\ div\ b) * b + a\ mod\ b) + c = a + c$
⟨*proof*⟩

**lemma** *div-mod-equality2*: $(b * (a\ div\ b) + a\ mod\ b) + c = a + c$
⟨*proof*⟩

**lemma** *mod-by-0* [*simp*]: $a\ mod\ 0 = a$
⟨*proof*⟩

**lemma** *mod-0* [*simp*]: $0\ mod\ a = 0$
⟨*proof*⟩

**lemma** *div-mult-self2* [*simp*]:
**assumes** $b \neq 0$
**shows** $(a + b * c)\ div\ b = c + a\ div\ b$
⟨*proof*⟩

**lemma** *mod-mult-self1* [*simp*]: $(a + c * b)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

**lemma** *mod-mult-self2* [*simp*]: $(a + b * c)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

**lemma** *div-mult-self1-is-id* [*simp*]: $b \neq 0 \Longrightarrow b * a\ div\ b = a$
⟨*proof*⟩

**lemma** *div-mult-self2-is-id* [*simp*]: $b \neq 0 \Longrightarrow a * b\ div\ b = a$
⟨*proof*⟩

**lemma** *mod-mult-self1-is-0* [*simp*]: $b * a\ mod\ b = 0$
⟨*proof*⟩

**lemma** *mod-mult-self2-is-0* [*simp*]: $a * b\ mod\ b = 0$
⟨*proof*⟩

**lemma** *div-by-1* [*simp*]: $a\ div\ 1 = a$
⟨*proof*⟩

**lemma** *mod-by-1* [*simp*]: $a\ mod\ 1 = 0$
⟨*proof*⟩

**lemma** *mod-self* [*simp*]: $a\ mod\ a = 0$

⟨*proof*⟩

**lemma** *div-self* [*simp*]: $a \neq 0 \Longrightarrow a \ div \ a = 1$
⟨*proof*⟩

**lemma** *div-add-self1* [*simp*]:
**assumes** $b \neq 0$
**shows** $(b + a) \ div \ b = a \ div \ b + 1$
⟨*proof*⟩

**lemma** *div-add-self2* [*simp*]:
**assumes** $b \neq 0$
**shows** $(a + b) \ div \ b = a \ div \ b + 1$
⟨*proof*⟩

**lemma** *mod-add-self1* [*simp*]:
$(b + a) \ mod \ b = a \ mod \ b$
⟨*proof*⟩

**lemma** *mod-add-self2* [*simp*]:
$(a + b) \ mod \ b = a \ mod \ b$
⟨*proof*⟩

**lemma** *mod-div-decomp*:
**fixes** $a \ b$
**obtains** $q \ r$ **where** $q = a \ div \ b$ **and** $r = a \ mod \ b$
**and** $a = q * b + r$
⟨*proof*⟩

**lemma** *dvd-eq-mod-eq-0* [*code unfold*]: $a \ dvd \ b \longleftrightarrow b \ mod \ a = 0$
⟨*proof*⟩

**lemma** *mod-div-trivial* [*simp*]: $a \ mod \ b \ div \ b = 0$
⟨*proof*⟩

**lemma** *mod-mod-trivial* [*simp*]: $a \ mod \ b \ mod \ b = a \ mod \ b$
⟨*proof*⟩

**lemma** *dvd-imp-mod-0*: $a \ dvd \ b \Longrightarrow b \ mod \ a = 0$
⟨*proof*⟩

**lemma** *dvd-div-mult-self*: $a \ dvd \ b \Longrightarrow (b \ div \ a) * a = b$
⟨*proof*⟩

**lemma** *dvd-div-mult*: $a \ dvd \ b \Longrightarrow (b \ div \ a) * c = b * c \ div \ a$
⟨*proof*⟩

**lemma** *div-dvd-div*[*simp*]:
$a \ dvd \ b \Longrightarrow a \ dvd \ c \Longrightarrow (b \ div \ a \ dvd \ c \ div \ a) = (b \ dvd \ c)$

⟨*proof*⟩

**lemma** *dvd-mod-imp-dvd*: [| *k dvd m mod n*; *k dvd n* |] ==> *k dvd m*
⟨*proof*⟩

Addition respects modular equivalence.

**lemma** *mod-add-left-eq*: $(a + b)\ mod\ c = (a\ mod\ c + b)\ mod\ c$
⟨*proof*⟩

**lemma** *mod-add-right-eq*: $(a + b)\ mod\ c = (a + b\ mod\ c)\ mod\ c$
⟨*proof*⟩

**lemma** *mod-add-eq*: $(a + b)\ mod\ c = (a\ mod\ c + b\ mod\ c)\ mod\ c$
⟨*proof*⟩

**lemma** *mod-add-cong*:
**assumes** $a\ mod\ c = a'\ mod\ c$
**assumes** $b\ mod\ c = b'\ mod\ c$
**shows** $(a + b)\ mod\ c = (a' + b')\ mod\ c$
⟨*proof*⟩

**lemma** *div-add*[*simp*]: *z dvd x* ⟹ *z dvd y*
⟹ $(x + y)\ div\ z = x\ div\ z + y\ div\ z$
⟨*proof*⟩

Multiplication respects modular equivalence.

**lemma** *mod-mult-left-eq*: $(a * b)\ mod\ c = ((a\ mod\ c) * b)\ mod\ c$
⟨*proof*⟩

**lemma** *mod-mult-right-eq*: $(a * b)\ mod\ c = (a * (b\ mod\ c))\ mod\ c$
⟨*proof*⟩

**lemma** *mod-mult-eq*: $(a * b)\ mod\ c = ((a\ mod\ c) * (b\ mod\ c))\ mod\ c$
⟨*proof*⟩

**lemma** *mod-mult-cong*:
**assumes** $a\ mod\ c = a'\ mod\ c$
**assumes** $b\ mod\ c = b'\ mod\ c$
**shows** $(a * b)\ mod\ c = (a' * b')\ mod\ c$
⟨*proof*⟩

**lemma** *mod-mod-cancel*:
**assumes** *c dvd b*
**shows** $a\ mod\ b\ mod\ c = a\ mod\ c$
⟨*proof*⟩

**end**

**lemma** *div-mult-div-if-dvd*: $(y::'a::\{semiring\text{-}div, no\text{-}zero\text{-}divisors\})\ dvd\ x \Longrightarrow$

*z dvd w* $\Longrightarrow$ (*x div y*) $*$ (*w div z*) = (*x* $*$ *w*) *div* (*y* $*$ *z*)
⟨*proof*⟩

**lemma** *div-power*: (*y*::'*a*::{*semiring-div*,*no-zero-divisors*,*recpower*}) *dvd x* $\Longrightarrow$
(*x div y*) ^*n* = *x*^*n div y*^*n*
⟨*proof*⟩

**class** *ring-div* = *semiring-div* + *comm-ring-1*
**begin**

Negation respects modular equivalence.

**lemma** *mod-minus-eq*: (− *a*) *mod b* = (− (*a mod b*)) *mod b*
⟨*proof*⟩

**lemma** *mod-minus-cong*:
**assumes** *a mod b* = *a'* *mod b*
**shows** (− *a*) *mod b* = (− *a'*) *mod b*
⟨*proof*⟩

Subtraction respects modular equivalence.

**lemma** *mod-diff-left-eq*: (*a* − *b*) *mod c* = (*a mod c* − *b*) *mod c*
⟨*proof*⟩

**lemma** *mod-diff-right-eq*: (*a* − *b*) *mod c* = (*a* − *b mod c*) *mod c*
⟨*proof*⟩

**lemma** *mod-diff-eq*: (*a* − *b*) *mod c* = (*a mod c* − *b mod c*) *mod c*
⟨*proof*⟩

**lemma** *mod-diff-cong*:
**assumes** *a mod c* = *a'* *mod c*
**assumes** *b mod c* = *b'* *mod c*
**shows** (*a* − *b*) *mod c* = (*a'* − *b'*) *mod c*
⟨*proof*⟩

**lemma** *dvd-neg-div*: *y dvd x* $\Longrightarrow$ −*x div y* = − (*x div y*)
⟨*proof*⟩

**lemma** *dvd-div-neg*: *y dvd x* $\Longrightarrow$ *x div* −*y* = − (*x div y*)
⟨*proof*⟩

**end**

## 26.3 Division on *nat*

We define *op div* and *op mod* on *nat* by means of a characteristic relation with two input arguments *m*, *n* and two output arguments *q*(uotient) and *r*(emainder).

**definition** *divmod-rel* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
*divmod-rel m n q r* ⟷ *m* = *q* ∗ *n* + *r* ∧ (*if n* > *0 then 0* ≤ *r* ∧ *r* < *n else q* = *0*)

*divmod-rel* is total:

**lemma** *divmod-rel-ex*:
**obtains** *q r* **where** *divmod-rel m n q r*
⟨*proof*⟩

*divmod-rel* is injective:

**lemma** *divmod-rel-unique-div*:
**assumes** *divmod-rel m n q r*
**and** *divmod-rel m n q′ r′*
**shows** *q* = *q′*
⟨*proof*⟩

**lemma** *divmod-rel-unique-mod*:
**assumes** *divmod-rel m n q r*
**and** *divmod-rel m n q′ r′*
**shows** *r* = *r′*
⟨*proof*⟩

We instantiate divisibility on the natural numbers by means of *divmod-rel*:

**instantiation** *nat* :: *semiring-div*
**begin**

**definition** *divmod* :: *nat* ⇒ *nat* ⇒ *nat* × *nat* **where**
[*code del*]: *divmod m n* = (*THE* (*q*, *r*). *divmod-rel m n q r*)

**definition** *div-nat* **where**
*m div n* = *fst* (*divmod m n*)

**definition** *mod-nat* **where**
*m mod n* = *snd* (*divmod m n*)

**lemma** *divmod-div-mod*:
*divmod m n* = (*m div n*, *m mod n*)
⟨*proof*⟩

**lemma** *divmod-eq*:
**assumes** *divmod-rel m n q r*
**shows** *divmod m n* = (*q*, *r*)
⟨*proof*⟩

**lemma** *div-eq*:
**assumes** *divmod-rel m n q r*
**shows** *m div n* = *q*
⟨*proof*⟩

**lemma** *mod-eq*:
**assumes** *divmod-rel m n q r*
**shows** *m mod n = r*
⟨*proof*⟩

**lemma** *divmod-rel*: *divmod-rel m n (m div n) (m mod n)*
⟨*proof*⟩

**lemma** *divmod-zero*:
*divmod m 0 = (0, m)*
⟨*proof*⟩

**lemma** *divmod-base*:
**assumes** *m < n*
**shows** *divmod m n = (0, m)*
⟨*proof*⟩

**lemma** *divmod-step*:
**assumes** *0 < n* **and** *n ≤ m*
**shows** *divmod m n = (Suc ((m − n) div n), (m − n) mod n)*
⟨*proof*⟩

The "recursion" equations for *op div* and *op mod*

**lemma** *div-less* [*simp*]:
**fixes** *m n* :: *nat*
**assumes** *m < n*
**shows** *m div n = 0*
⟨*proof*⟩

**lemma** *le-div-geq*:
**fixes** *m n* :: *nat*
**assumes** *0 < n* **and** *n ≤ m*
**shows** *m div n = Suc ((m − n) div n)*
⟨*proof*⟩

**lemma** *mod-less* [*simp*]:
**fixes** *m n* :: *nat*
**assumes** *m < n*
**shows** *m mod n = m*
⟨*proof*⟩

**lemma** *le-mod-geq*:
**fixes** *m n* :: *nat*
**assumes** *n ≤ m*
**shows** *m mod n = (m − n) mod n*
⟨*proof*⟩

**instance** ⟨*proof*⟩

**end**

Simproc for cancelling *op div* and *op mod*

⟨*ML*⟩

code generator setup

**lemma** *divmod-if* [*code*]: *divmod m n* = (*if n* = *0* ∨ *m* < *n then* (*0*, *m*) *else*
*let* (*q*, *r*) = *divmod* (*m* − *n*) *n in* (*Suc q*, *r*))
⟨*proof*⟩

**code-modulename** *SML*
*Divides Nat*

**code-modulename** *OCaml*
*Divides Nat*

**code-modulename** *Haskell*
*Divides Nat*

### 26.3.1 Quotient

**lemma** *div-geq*: *0* < *n* ⟹ ¬ *m* < *n* ⟹ *m div n* = *Suc* ((*m* − *n*) *div n*)
⟨*proof*⟩

**lemma** *div-if*: *0* < *n* ⟹ *m div n* = (*if m* < *n then 0 else Suc* ((*m* − *n*) *div n*))
⟨*proof*⟩

**lemma** *div-mult-self-is-m* [*simp*]: *0*<*n* ==> (*m*∗*n*) *div n* = (*m*::*nat*)
⟨*proof*⟩

**lemma** *div-mult-self1-is-m* [*simp*]: *0*<*n* ==> (*n*∗*m*) *div n* = (*m*::*nat*)
⟨*proof*⟩

### 26.3.2 Remainder

**lemma** *mod-less-divisor* [*simp*]:
**fixes** *m n* :: *nat*
**assumes** *n* > *0*
**shows** *m mod n* < (*n*::*nat*)
⟨*proof*⟩

**lemma** *mod-less-eq-dividend* [*simp*]:
**fixes** *m n* :: *nat*
**shows** *m mod n* ≤ *m*
⟨*proof*⟩

**lemma** *mod-geq*: ¬ *m* < (*n*::*nat*) ⟹ *m mod n* = (*m* − *n*) *mod n*
⟨*proof*⟩

**lemma** *mod-if*: *m mod* (*n*::*nat*) = (*if m* < *n then m else* (*m* − *n*) *mod n*)

⟨*proof*⟩

**lemma** *mod-1* [*simp*]: *m mod Suc 0 = 0*
⟨*proof*⟩

**lemma** *mod-mult-distrib*: *(m mod n) * (k::nat) = (m * k) mod (n * k)*
⟨*proof*⟩

**lemma** *mod-mult-distrib2*: *(k::nat) * (m mod n) = (k*m) mod (k*n)*
⟨*proof*⟩

**lemma** *mult-div-cancel*: *(n::nat) * (m div n) = m - (m mod n)*
⟨*proof*⟩

**lemma** *mod-le-divisor*[*simp*]: *0 < n ⟹ m mod n ≤ (n::nat)*
⟨*proof*⟩

### 26.3.3 Quotient and Remainder

**lemma** *divmod-rel-mult1-eq*:
*[| divmod-rel b c q r; c > 0 |]*
*==> divmod-rel (a*b) c (a*q + a*r div c) (a*r mod c)*
⟨*proof*⟩

**lemma** *div-mult1-eq*: *(a*b) div c = a*(b div c) + a*(b mod c) div (c::nat)*
⟨*proof*⟩

**lemma** *divmod-rel-add1-eq*:
*[| divmod-rel a c aq ar; divmod-rel b c bq br; c > 0 |]*
*==> divmod-rel (a + b) c (aq + bq + (ar+br) div c) ((ar + br) mod c)*
⟨*proof*⟩

**lemma** *div-add1-eq*:
*(a+b) div (c::nat) = a div c + b div c + ((a mod c + b mod c) div c)*
⟨*proof*⟩

**lemma** *mod-lemma*: *[| (0::nat) < c; r < b |] ==> b * (q mod c) + r < b * c*
⟨*proof*⟩

**lemma** *divmod-rel-mult2-eq*: *[| divmod-rel a b q r; 0 < b; 0 < c |]*
*==> divmod-rel a (b*c) (q div c) (b*(q mod c) + r)*
⟨*proof*⟩

**lemma** *div-mult2-eq*: *a div (b*c) = (a div b) div (c::nat)*
⟨*proof*⟩

**lemma** *mod-mult2-eq*: *a mod (b*c) = b*(a div b mod c) + a mod (b::nat)*

⟨*proof*⟩

### 26.3.4 Cancellation of Common Factors in Division

**lemma** *div-mult-mult-lemma*:
[| (0::*nat*) < *b*; 0 < *c* |] ==> (*c*∗*a*) *div* (*c*∗*b*) = *a div b*
⟨*proof*⟩

**lemma** *div-mult-mult1* [*simp*]: (0::*nat*) < *c* ==> (*c*∗*a*) *div* (*c*∗*b*) = *a div b*
⟨*proof*⟩

**lemma** *div-mult-mult2* [*simp*]: (0::*nat*) < *c* ==> (*a*∗*c*) *div* (*b*∗*c*) = *a div b*
⟨*proof*⟩

### 26.3.5 Further Facts about Quotient and Remainder

**lemma** *div-1* [*simp*]: *m div Suc* 0 = *m*
⟨*proof*⟩

**lemma** *div-le-mono* [*rule-format*]:
∀ *m*::*nat*. *m* ≤ *n* −−> (*m div k*) ≤ (*n div k*)
⟨*proof*⟩

**lemma** *div-le-mono2*: !!*m*::*nat*. [| 0<*m*; *m*≤*n* |] ==> (*k div n*) ≤ (*k div m*)
⟨*proof*⟩

**lemma** *div-le-dividend* [*simp*]: *m div n* ≤ (*m*::*nat*)
⟨*proof*⟩

**lemma** *div-less-dividend* [*rule-format*]:
!!*n*::*nat*. 1<*n* ==> 0 < *m* −−> *m div n* < *m*
⟨*proof*⟩

**lemma** *nat-div-eq-0* [*simp*]: (*n*::*nat*) > 0 ==> ((*m div n*) = 0) = (*m* < *n*)
⟨*proof*⟩

**lemma** *nat-div-gt-0* [*simp*]: (*n*::*nat*) > 0 ==> ((*m div n*) > 0) = (*m* >= *n*)
⟨*proof*⟩

**declare** *div-less-dividend* [*simp*]

A fact for the mutilated chess board

**lemma** *mod-Suc*: *Suc*(*m*) *mod n* = (*if Suc*(*m mod n*) = *n then* 0 *else Suc*(*m mod n*))
⟨*proof*⟩

### 26.3.6 The Divides Relation

**lemma** *dvd-1-left* [*iff*]: *Suc 0 dvd k*
⟨*proof*⟩

**lemma** *dvd-1-iff-1* [*simp*]: (*m dvd Suc 0*) = (*m = Suc 0*)
⟨*proof*⟩

**lemma** *nat-dvd-1-iff-1* [*simp*]: *m dvd* (*1*::*nat*) ⟷ *m = 1*
⟨*proof*⟩

**lemma** *dvd-anti-sym*: [| *m dvd n*; *n dvd m* |] ==> *m* = (*n*::*nat*)
⟨*proof*⟩

*op dvd* is a partial order

**interpretation** *dvd*: *order op dvd* λ*n m* :: *nat*. *n dvd m* ∧ ¬ *m dvd n*
⟨*proof*⟩

**lemma** *nat-dvd-diff* [*simp*]: [| *k dvd m*; *k dvd n* |] ==> *k dvd* (*m*−*n* :: *nat*)
⟨*proof*⟩

**lemma** *dvd-diffD*: [| *k dvd m*−*n*; *k dvd n*; *n*≤*m* |] ==> *k dvd* (*m*::*nat*)
⟨*proof*⟩

**lemma** *dvd-diffD1*: [| *k dvd m*−*n*; *k dvd m*; *n*≤*m* |] ==> *k dvd* (*n*::*nat*)
⟨*proof*⟩

**lemma** *dvd-reduce*: (*k dvd n* + *k*) = (*k dvd* (*n*::*nat*))
⟨*proof*⟩

**lemma** *dvd-mod*: !!*n*::*nat*. [| *f dvd m*; *f dvd n* |] ==> *f dvd m mod n*
⟨*proof*⟩

**lemma** *dvd-mod-iff*: *k dvd n* ==> ((*k*::*nat*) *dvd m mod n*) = (*k dvd m*)
⟨*proof*⟩

**lemma** *dvd-mult-cancel*: !!*k*::*nat*. [| *k*∗*m dvd k*∗*n*; *0*<*k* |] ==> *m dvd n*
⟨*proof*⟩

**lemma** *dvd-mult-cancel1*: *0*<*m* ==> (*m*∗*n dvd m*) = (*n* = (*1*::*nat*))
⟨*proof*⟩

**lemma** *dvd-mult-cancel2*: *0*<*m* ==> (*n*∗*m dvd m*) = (*n* = (*1*::*nat*))
⟨*proof*⟩

**lemma** *dvd-imp-le*: [| *k dvd n*; *0* < *n* |] ==> *k* ≤ (*n*::*nat*)
⟨*proof*⟩

**lemma** *nat-dvd-not-less*: (*0*::*nat*) < *m* ⟹ *m* < *n* ⟹ ¬ *n dvd m*
⟨*proof*⟩

**lemma** *dvd-mult-div-cancel*: $n$ *dvd* $m$ ==> $n * (m$ *div* $n) = (m$::*nat*)
⟨*proof*⟩

**lemma** *nat-zero-less-power-iff* [*simp*]: $(x\hat{}n > 0) = (x > (0$::*nat*) | $n=0)$
⟨*proof*⟩

**lemma** *power-dvd-imp-le*: $[|i\hat{}m$ *dvd* $i\hat{}n;\ (1$::*nat*) $< i|]$ ==> $m \leq n$
⟨*proof*⟩

**lemma** *mod-eq-0-iff*: $(m$ *mod* $d = 0) = (\exists\, q$::*nat*. $m = d*q)$
⟨*proof*⟩

**lemmas** *mod-eq-0D* [*dest*!] = *mod-eq-0-iff* [*THEN iffD1*]

**lemma** *mod-eqD*: $(m$ *mod* $d = r)$ ==> $\exists\, q$::*nat*. $m = r + q*d$
⟨*proof*⟩

**lemma** *split-div*:
$P(n$ *div* $k$ :: *nat*) =
$((k = 0 \longrightarrow P\ 0) \wedge (k \neq 0 \longrightarrow (!i.\ !j<k.\ n = k*i + j \longrightarrow P\ i)))$
(**is** *?P* = *?Q* **is** - = (- ∧ (- ⟶ *?R*)))
⟨*proof*⟩

**lemma** *split-div-lemma*:
**assumes** $0 < n$
**shows** $n * q \leq m \wedge m < n * Suc\ q \longleftrightarrow q = ((m$::*nat*) *div* $n)$ (**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

**theorem** *split-div'*:
$P\ ((m$::*nat*) *div* $n) = ((n = 0 \wedge P\ 0) \vee$
$(\exists\, q.\ (n * q \leq m \wedge m < n * (Suc\ q)) \wedge P\ q))$
⟨*proof*⟩

**lemma** *split-mod*:
$P(n$ *mod* $k$ :: *nat*) =
$((k = 0 \longrightarrow P\ n) \wedge (k \neq 0 \longrightarrow (!i.\ !j<k.\ n = k*i + j \longrightarrow P\ j)))$
(**is** *?P* = *?Q* **is** - = (- ∧ (- ⟶ *?R*)))
⟨*proof*⟩

**theorem** *mod-div-equality'*: $(m$::*nat*$)$ *mod* $n = m - (m$ *div* $n) * n$
⟨*proof*⟩

**lemma** *div-mod-equality'*:
**fixes** $m\ n$ :: *nat*
**shows** $m$ *div* $n * n = m - m$ *mod* $n$
⟨*proof*⟩

### 26.3.7 An "induction" law for modulus arithmetic.

**lemma** *mod-induct-0*:
**assumes** *step*: $\forall i<p.\ P\ i \longrightarrow P\ ((Suc\ i)\ mod\ p)$
**and** *base*: $P\ i$ **and** *i*: $i<p$
**shows** $P\ 0$
⟨*proof*⟩

**lemma** *mod-induct*:
**assumes** *step*: $\forall i<p.\ P\ i \longrightarrow P\ ((Suc\ i)\ mod\ p)$
**and** *base*: $P\ i$ **and** *i*: $i<p$ **and** *j*: $j<p$
**shows** $P\ j$
⟨*proof*⟩

**end**

# 27 Plain: Plain HOL

**theory** *Plain*
**imports** *Datatype FunDef Record Extraction Divides*
**begin**

Plain bootstrap of fundamental HOL tools and packages; does not include *Hilbert-Choice*.

⟨*ML*⟩

**end**

# 28 Relation-Power: Powers of Relations and Functions

**theory** *Relation-Power*
**imports** *Power Transitive-Closure Plain*
**begin**

**instance**
$fun :: (type, type)\ power$ ⟨*proof*⟩

**overloading**
$relpow \equiv power :: ('a \times 'a)\ set \Rightarrow nat \Rightarrow ('a \times 'a)\ set$ (**unchecked**)
**begin**

$R\ \hat{}\ n = R\ O\ ...\ O\ R$, the n-fold composition of $R$

**primrec** *relpow* **where**
$(R :: ('a \times 'a)\ set)\ \hat{}\ 0 = Id$
| $(R :: ('a \times 'a)\ set)\ \hat{}\ Suc\ n = R\ O\ (R\ \hat{}\ n)$

**end**

**overloading**
*funpow* ≡ *power* :: $('a \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a \Rightarrow 'a$ (**unchecked**)
**begin**

$f \hat{} n = f\ o\ ...\ o\ f$, the n-fold composition of $f$

**primrec** *funpow* **where**
$(f :: 'a \Rightarrow 'a) \hat{} 0 = id$
| $(f :: 'a \Rightarrow 'a) \hat{} Suc\ n = f\ o\ (f \hat{} n)$

**end**

WARNING: due to the limits of Isabelle's type classes, exponentiation on functions and relations has too general a domain, namely $('a \times 'b)\ set$ and $'a \Rightarrow 'b$. Explicit type constraints may therefore be necessary. For example, $range\ (f \hat{} n) = A$ and $Range\ (R \hat{} n) = B$ need constraints.

Circumvent this problem for code generation:

**primrec**
*fun-pow* :: $nat \Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a$
**where**
*fun-pow 0 f* = *id*
| *fun-pow* (*Suc n*) *f* = *f o fun-pow n f*

**lemma** *funpow-fun-pow* [*code unfold*]: $f \hat{} n = fun\text{-}pow\ n\ f$
⟨*proof*⟩

**lemma** *funpow-add*: $f \hat{} (m+n) = f\hat{}m\ o\ f\hat{}n$
⟨*proof*⟩

**lemma** *funpow-swap1*: $f((f\hat{}n)\ x) = (f\hat{}n)(f\ x)$
⟨*proof*⟩

**lemma** *rel-pow-1* [*simp*]:
**fixes** $R :: ('a*'a)set$
**shows** $R\hat{}1 = R$
⟨*proof*⟩

**lemma** *rel-pow-0-I*: $(x,x) : R\hat{}0$
⟨*proof*⟩

**lemma** *rel-pow-Suc-I*: [| $(x,y) : R\hat{}n$; $(y,z):R$ |] ==> $(x,z):R\hat{}(Suc\ n)$
⟨*proof*⟩

**lemma** *rel-pow-Suc-I2*:
$(x,\ y) : R \Longrightarrow (y,\ z) : R\hat{}n \Longrightarrow (x,z) : R\hat{}(Suc\ n)$
⟨*proof*⟩

**lemma** *rel-pow-0-E*: [| (*x*,*y*) : *R* ^ *0*; *x*=*y* ==> *P* |] ==> *P*
⟨*proof*⟩

**lemma** *rel-pow-Suc-E*:
[| (*x*,*z*) : *R* ^ (*Suc n*); !!*y*. [| (*x*,*y*) : *R* ^ *n*; (*y*,*z*) : *R* |] ==> *P* |] ==> *P*
⟨*proof*⟩

**lemma** *rel-pow-E*:
[| (*x*,*z*) : *R* ^ *n*; [| *n*=*0*; *x* = *z* |] ==> *P*;
!!*y m*. [| *n* = *Suc m*; (*x*,*y*) : *R* ^ *m*; (*y*,*z*) : *R* |] ==> *P*
|] ==> *P*
⟨*proof*⟩

**lemma** *rel-pow-Suc-D2*:
(*x*, *z*) : *R* ^ (*Suc n*) ⟹ (∃ *y*. (*x*,*y*) : *R* & (*y*,*z*) : *R* ^ *n*)
⟨*proof*⟩

**lemma** *rel-pow-Suc-D2'*:
∀ *x y z*. (*x*,*y*) : *R* ^ *n* & (*y*,*z*) : *R* --> (∃ *w*. (*x*,*w*) : *R* & (*w*,*z*) : *R* ^ *n*)
⟨*proof*⟩

**lemma** *rel-pow-E2*:
[| (*x*,*z*) : *R* ^ *n*; [| *n*=*0*; *x* = *z* |] ==> *P*;
!!*y m*. [| *n* = *Suc m*; (*x*,*y*) : *R*; (*y*,*z*) : *R* ^ *m* |] ==> *P*
|] ==> *P*
⟨*proof*⟩

**lemma** *rtrancl-imp-UN-rel-pow*: !!*p*. *p*:*R* ^ * ==> *p* : (*UN n*. *R* ^ *n*)
⟨*proof*⟩

**lemma** *rel-pow-imp-rtrancl*: !!*p*. *p*:*R* ^ *n* ==> *p*:*R* ^ *
⟨*proof*⟩

**lemma** *rtrancl-is-UN-rel-pow*: *R* ^ * = (*UN n*. *R* ^ *n*)
⟨*proof*⟩

**lemma** *trancl-power*:
*x* ∈ *r* ^ + = (∃ *n* > *0*. *x* ∈ *r* ^ *n*)
⟨*proof*⟩

**lemma** *single-valued-rel-pow*:
!!*r*::('*a* * '*a*)*set*. *single-valued r* ==> *single-valued* (*r* ^ *n*)
⟨*proof*⟩

⟨*ML*⟩

**end**

# 29 Equiv-Relations: Equivalence Relations in Higher-Order Set Theory

**theory** *Equiv-Relations*
**imports** *Finite-Set Relation Plain*
**begin**

## 29.1 Equivalence relations

**locale** *equiv* =
  **fixes** $A$ **and** $r$
  **assumes** *refl-on*: *refl-on A r*
    **and** *sym*: *sym r*
    **and** *trans*: *trans r*

Suppes, Theorem 70: $r$ is an equiv relation iff $r^{-1}\ O\ r = r$.

First half: *equiv A r* ==> $r^{-1}\ O\ r = r$.

**lemma** *sym-trans-comp-subset*:
    *sym r* ==> *trans r* ==> $r^{-1}\ O\ r \subseteq r$
  ⟨*proof*⟩

**lemma** *refl-on-comp-subset*: *refl-on A r* ==> $r \subseteq r^{-1}\ O\ r$
  ⟨*proof*⟩

**lemma** *equiv-comp-eq*: *equiv A r* ==> $r^{-1}\ O\ r = r$
  ⟨*proof*⟩

Second half.

**lemma** *comp-equivI*:
    $r^{-1}\ O\ r = r$ ==> *Domain r* = *A* ==> *equiv A r*
  ⟨*proof*⟩

## 29.2 Equivalence classes

**lemma** *equiv-class-subset*:
  *equiv A r* ==> $(a, b) \in r$ ==> $r``\{a\} \subseteq r``\{b\}$
  — lemma for the next result
  ⟨*proof*⟩

**theorem** *equiv-class-eq*: *equiv A r* ==> $(a, b) \in r$ ==> $r``\{a\} = r``\{b\}$
  ⟨*proof*⟩

**lemma** *equiv-class-self*: *equiv A r* ==> $a \in A$ ==> $a \in r``\{a\}$
  ⟨*proof*⟩

**lemma** *subset-equiv-class*:
    *equiv A r* ==> $r``\{b\} \subseteq r``\{a\}$ ==> $b \in A$ ==> $(a,b) \in r$
  — lemma for the next result
  ⟨*proof*⟩

**lemma** *eq-equiv-class*:
$r``\{a\} = r``\{b\} \Longrightarrow equiv\ A\ r \Longrightarrow b \in A \Longrightarrow (a, b) \in r$
⟨*proof*⟩

**lemma** *equiv-class-nondisjoint*:
$equiv\ A\ r \Longrightarrow x \in (r``\{a\} \cap r``\{b\}) \Longrightarrow (a, b) \in r$
⟨*proof*⟩

**lemma** *equiv-type*: $equiv\ A\ r \Longrightarrow r \subseteq A \times A$
⟨*proof*⟩

**theorem** *equiv-class-eq-iff*:
$equiv\ A\ r \Longrightarrow ((x, y) \in r) = (r``\{x\} = r``\{y\} \ \&\ x \in A \ \&\ y \in A)$
⟨*proof*⟩

**theorem** *eq-equiv-class-iff*:
$equiv\ A\ r \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow (r``\{x\} = r``\{y\}) = ((x, y) \in r)$
⟨*proof*⟩

## 29.3 Quotients

**definition** *quotient* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *'a set set* (**infixl** *'//'* *90*) **where**
[*code del*]: $A//r = (\bigcup x \in A.\ \{r``\{x\}\})$ — set of equiv classes

**lemma** *quotientI*: $x \in A \Longrightarrow r``\{x\} \in A//r$
⟨*proof*⟩

**lemma** *quotientE*:
$X \in A//r \Longrightarrow (!!x.\ X = r``\{x\} \Longrightarrow x \in A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

**lemma** *Union-quotient*: $equiv\ A\ r \Longrightarrow Union\ (A//r) = A$
⟨*proof*⟩

**lemma** *quotient-disj*:
$equiv\ A\ r \Longrightarrow X \in A//r \Longrightarrow Y \in A//r \Longrightarrow X = Y \mid (X \cap Y = \{\})$
⟨*proof*⟩

**lemma** *quotient-eqI*:
$[|equiv\ A\ r;\ X \in A//r;\ Y \in A//r;\ x \in X;\ y \in Y;\ (x,y) \in r|] \Longrightarrow X = Y$
⟨*proof*⟩

**lemma** *quotient-eq-iff*:
$[|equiv\ A\ r;\ X \in A//r;\ Y \in A//r;\ x \in X;\ y \in Y|] \Longrightarrow (X = Y) = ((x,y) \in r)$
⟨*proof*⟩

**lemma** *eq-equiv-class-iff2*:

⟦ *equiv A r*; $x \in A$; $y \in A$ ⟧ $\Longrightarrow$ $(\{x\}//r = \{y\}//r) = ((x,y) : r)$
⟨*proof*⟩

**lemma** *quotient-empty* [*simp*]: $\{\}//r = \{\}$
⟨*proof*⟩

**lemma** *quotient-is-empty* [*iff*]: $(A//r = \{\}) = (A = \{\})$
⟨*proof*⟩

**lemma** *quotient-is-empty2* [*iff*]: $(\{\} = A//r) = (A = \{\})$
⟨*proof*⟩

**lemma** *singleton-quotient*: $\{x\}//r = \{r \text{ `` } \{x\}\}$
⟨*proof*⟩

**lemma** *quotient-diff1*:
⟦ *inj-on* (%*a*. $\{a\}//r$) *A*; $a \in A$ ⟧ $\Longrightarrow$ $(A - \{a\})//r = A//r - \{a\}//r$
⟨*proof*⟩

## 29.4 Defining unary operations upon equivalence classes

A congruence-preserving function

**locale** *congruent* =
**fixes** *r* **and** *f*
**assumes** *congruent*: $(y,z) \in r$ ==> $f\ y = f\ z$

**abbreviation**
*RESPECTS* :: $('a$ => $'b)$ => $('a * 'a)$ *set* => *bool*
(**infixr** *respects* *80*) **where**
*f respects r* == *congruent r f*

**lemma** *UN-constant-eq*: $a \in A$ ==> $\forall y \in A.\ f\ y = c$ ==> $(\bigcup y \in A.\ f(y))=c$
— lemma required to prove *UN-equiv-class*
⟨*proof*⟩

**lemma** *UN-equiv-class*:
*equiv A r* ==> *f respects r* ==> $a \in A$
==> $(\bigcup x \in r\text{``}\{a\}.\ f\ x) = f\ a$
— Conversion rule
⟨*proof*⟩

**lemma** *UN-equiv-class-type*:
*equiv A r* ==> *f respects r* ==> $X \in A//r$ ==>
(!!$x$. $x \in A$ ==> $f\ x \in B$) ==> $(\bigcup x \in X.\ f\ x) \in B$
⟨*proof*⟩

Sufficient conditions for injectiveness. Could weaken premises! major premise could be an inclusion; bcong could be $!!y.\ y \in A \Longrightarrow f\ y \in B$.

**lemma** *UN-equiv-class-inject*:
*equiv A r ==> f respects r ==>*
$(\bigcup x \in X.\ f\ x) = (\bigcup y \in Y.\ f\ y)$ *==> X ∈ A//r ==> Y ∈ A//r*
*==>* $(!!x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \Longrightarrow (x,\ y) \in r)$
*==> X = Y*
⟨*proof*⟩

## 29.5 Defining binary operations upon equivalence classes

A congruence-preserving function of two arguments

**locale** *congruent2* =
**fixes** *r1* **and** *r2* **and** *f*
**assumes** *congruent2*:
*(y1,z1) ∈ r1 ==> (y2,z2) ∈ r2 ==> f y1 y2 = f z1 z2*

Abbreviation for the common case where the relations are identical

**abbreviation**
*RESPECTS2*:: $['a \Rightarrow 'a \Rightarrow 'b,\ ('a * 'a)\ set] \Rightarrow bool$
(**infixr** *respects2 80*) **where**
*f respects2 r == congruent2 r r f*

**lemma** *congruent2-implies-congruent*:
*equiv A r1 ==> congruent2 r1 r2 f ==> a ∈ A ==> congruent r2 (f a)*
⟨*proof*⟩

**lemma** *congruent2-implies-congruent-UN*:
*equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f ==> a ∈ A2 ==>*
*congruent r1* $(\lambda x1.\ \bigcup x2 \in r2``\{a\}.\ f\ x1\ x2)$
⟨*proof*⟩

**lemma** *UN-equiv-class2*:
*equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f ==> a1 ∈ A1 ==> a2 ∈ A2*
$\Longrightarrow (\bigcup x1 \in r1``\{a1\}.\ \bigcup x2 \in r2``\{a2\}.\ f\ x1\ x2) = f\ a1\ a2$
⟨*proof*⟩

**lemma** *UN-equiv-class-type2*:
*equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f*
*==> X1 ∈ A1//r1 ==> X2 ∈ A2//r2*
$\Longrightarrow (!!x1\ x2.\ x1 \in A1 \Longrightarrow x2 \in A2 \Longrightarrow f\ x1\ x2 \in B)$
$\Longrightarrow (\bigcup x1 \in X1.\ \bigcup x2 \in X2.\ f\ x1\ x2) \in B$
⟨*proof*⟩

**lemma** *UN-UN-split-split-eq*:
$(\bigcup (x1,\ x2) \in X.\ \bigcup (y1,\ y2) \in Y.\ A\ x1\ x2\ y1\ y2) =$

$(\bigcup x \in X.\ \bigcup y \in Y.\ (\lambda(x1,\ x2).\ (\lambda(y1,\ y2).\ A\ x1\ x2\ y1\ y2)\ y)\ x)$
— Allows a natural expression of binary operators,
— without explicit calls to *split*
⟨*proof*⟩

**lemma** *congruent2I*:
*equiv A1 r1 ==> equiv A2 r2*
*==> (!!y z w. w ∈ A2 ==> (y,z) ∈ r1 ==> f y w = f z w)*
*==> (!!y z w. w ∈ A1 ==> (y,z) ∈ r2 ==> f w y = f w z)*
*==> congruent2 r1 r2 f*
— Suggested by John Harrison – the two subproofs may be
— *much* simpler than the direct proof.
⟨*proof*⟩

**lemma** *congruent2-commuteI*:
**assumes** *equivA*: *equiv A r*
**and** *commute*: *!!y z. y ∈ A ==> z ∈ A ==> f y z = f z y*
**and** *congt*: *!!y z w. w ∈ A ==> (y,z) ∈ r ==> f w y = f w z*
**shows** *f respects2 r*
⟨*proof*⟩

### 29.6 Quotients and finiteness

Suggested by Florian Kammüller

**lemma** *finite-quotient*: *finite A ==> r ⊆ A × A ==> finite (A//r)*
— recall *equiv ?A ?r ⟹ ?r ⊆ ?A × ?A*
⟨*proof*⟩

**lemma** *finite-equiv-class*:
*finite A ==> r ⊆ A × A ==> X ∈ A//r ==> finite X*
⟨*proof*⟩

**lemma** *equiv-imp-dvd-card*:
*finite A ==> equiv A r ==> ∀ X ∈ A//r. k dvd card X*
*==> k dvd card A*
⟨*proof*⟩

**lemma** *card-quotient-disjoint*:
⟦ *finite A*; *inj-on (λx. {x} // r) A* ⟧ ⟹ *card(A//r) = card A*
⟨*proof*⟩

**end**

# 30 Int: The Integers as Equivalence Classes over Pairs of Natural Numbers

**theory** *Int*

**imports** *Equiv-Relations Nat Wellfounded*
**uses**
(*Tools/numeral.ML*)
(*Tools/numeral-syntax.ML*)
~~/*src/Provers/Arith/assoc-fold.ML*
~~/*src/Provers/Arith/cancel-numerals.ML*
~~/*src/Provers/Arith/combine-numerals.ML*
(*Tools/int-arith.ML*)
**begin**

## 30.1 The equivalence relation underlying the integers

**definition** *intrel* :: $((nat \times nat) \times (nat \times nat))$ *set* **where**
[*code del*]: $intrel = \{((x, y), (u, v)) \mid x\ y\ u\ v.\ x + v = u + y\ \}$

**typedef** (*Integ*)
$int = UNIV//intrel$
⟨*proof*⟩

**instantiation** *int* :: {*zero*, *one*, *plus*, *minus*, *uminus*, *times*, *ord*, *abs*, *sgn*}
**begin**

**definition**
*Zero-int-def* [*code del*]: $0 = Abs\text{-}Integ\ (intrel\ ``\ \{(0, 0)\})$

**definition**
*One-int-def* [*code del*]: $1 = Abs\text{-}Integ\ (intrel\ ``\ \{(1, 0)\})$

**definition**
*add-int-def* [*code del*]: $z + w = Abs\text{-}Integ$
$(\bigcup(x, y) \in Rep\text{-}Integ\ z.\ \bigcup(u, v) \in Rep\text{-}Integ\ w.$
$intrel\ ``\ \{(x + u, y + v)\})$

**definition**
*minus-int-def* [*code del*]:
$-z = Abs\text{-}Integ\ (\bigcup(x, y) \in Rep\text{-}Integ\ z.\ intrel\ ``\ \{(y, x)\})$

**definition**
*diff-int-def* [*code del*]: $z - w = z + (-w :: int)$

**definition**
*mult-int-def* [*code del*]: $z * w = Abs\text{-}Integ$
$(\bigcup(x, y) \in Rep\text{-}Integ\ z.\ \bigcup(u,v\ ) \in Rep\text{-}Integ\ w.$
$intrel\ ``\ \{(x*u + y*v, x*v + y*u)\})$

**definition**
*le-int-def* [*code del*]:
$z \leq w \longleftrightarrow (\exists x\ y\ u\ v.\ x+v \leq u+y \wedge (x, y) \in Rep\text{-}Integ\ z \wedge (u, v) \in Rep\text{-}Integ\ w)$

**definition**
*less-int-def* [*code del*]: $(z::int) < w \longleftrightarrow z \leq w \wedge z \neq w$

**definition**
*zabs-def*: $|i::int| = (if\ i < 0\ then\ -\ i\ else\ i)$

**definition**
*zsgn-def*: $sgn\ (i::int) = (if\ i=0\ then\ 0\ else\ if\ 0<i\ then\ 1\ else\ -\ 1)$

**instance** ⟨*proof*⟩

**end**

## 30.2 Construction of the Integers

**lemma** *intrel-iff* [*simp*]: $(((x,y),(u,v)) \in intrel) = (x+v = u+y)$
⟨*proof*⟩

**lemma** *equiv-intrel*: *equiv UNIV intrel*
⟨*proof*⟩

Reduces equality of equivalence classes to the *intrel* relation: $(intrel\ ``\ \{x\} = intrel\ ``\ \{y\}) = ((x,\ y) \in intrel)$

**lemmas** *equiv-intrel-iff* [*simp*] = *eq-equiv-class-iff* [*OF equiv-intrel UNIV-I UNIV-I*]

All equivalence classes belong to set of representatives

**lemma** [*simp*]: $intrel``\{(x,y)\} \in Integ$
⟨*proof*⟩

Reduces equality on abstractions to equality on representatives: $[\![x \in Integ;\ y \in Integ]\!] \Longrightarrow (Abs\text{-}Integ\ x = Abs\text{-}Integ\ y) = (x = y)$

**declare** *Abs-Integ-inject* [*simp*,*noatp*] *Abs-Integ-inverse* [*simp*,*noatp*]

Case analysis on the representation of an integer as an equivalence class of pairs of naturals.

**lemma** *eq-Abs-Integ* [*case-names Abs-Integ*, *cases type*: *int*]:
(!!$x\ y.\ z = Abs\text{-}Integ(intrel``\{(x,y)\}) ==> P) ==> P$
⟨*proof*⟩

## 30.3 Arithmetic Operations

**lemma** *minus*: $-\ Abs\text{-}Integ(intrel``\{(x,y)\}) = Abs\text{-}Integ(intrel\ ``\ \{(y,x)\})$
⟨*proof*⟩

**lemma** *add*:
$Abs\text{-}Integ\ (intrel``\{(x,y)\}) + Abs\text{-}Integ\ (intrel``\{(u,v)\}) =$
$Abs\text{-}Integ\ (intrel``\{(x+u,\ y+v)\})$

⟨*proof*⟩

Congruence property for multiplication

**lemma** *mult-congruent2*:
(%*p1 p2*. (%(*x*,*y*). (%(*u*,*v*). *intrel*``{(*x*∗*u* + *y*∗*v*, *x*∗*v* + *y*∗*u*)}) *p2*) *p1*)
*respects2 intrel*
⟨*proof*⟩

**lemma** *mult*:
*Abs-Integ*((*intrel*``{(*x*,*y*)})) ∗ *Abs-Integ*((*intrel*``{(*u*,*v*)})) =
*Abs-Integ*(*intrel* `` {(*x*∗*u* + *y*∗*v*, *x*∗*v* + *y*∗*u*)})
⟨*proof*⟩

The integers form a *comm-ring-1*

**instance** *int* :: *comm-ring-1*
⟨*proof*⟩

**lemma** *int-def*: *of-nat m* = *Abs-Integ* (*intrel* `` {(*m*, *0*)})
⟨*proof*⟩

## 30.4 The ≤ Ordering

**lemma** *le*:
(*Abs-Integ*(*intrel*``{(*x*,*y*)}) ≤ *Abs-Integ*(*intrel*``{(*u*,*v*)})) = (*x*+*v* ≤ *u*+*y*)
⟨*proof*⟩

**lemma** *less*:
(*Abs-Integ*(*intrel*``{(*x*,*y*)}) < *Abs-Integ*(*intrel*``{(*u*,*v*)})) = (*x*+*v* < *u*+*y*)
⟨*proof*⟩

**instance** *int* :: *linorder*
⟨*proof*⟩

**instantiation** *int* :: *distrib-lattice*
**begin**

**definition**
(*inf* :: *int* ⇒ *int* ⇒ *int*) = *min*

**definition**
(*sup* :: *int* ⇒ *int* ⇒ *int*) = *max*

**instance**
⟨*proof*⟩

**end**

**instance** *int* :: *pordered-cancel-ab-semigroup-add*
⟨*proof*⟩

Strict Monotonicity of Multiplication

strict, in 1st argument; proof is by induction on k¿0

**lemma** *zmult-zless-mono2-lemma*:
$(i::int)<j \Longrightarrow 0<k \Longrightarrow of\text{-}nat\ k * i < of\text{-}nat\ k * j$
⟨*proof*⟩

**lemma** *zero-le-imp-eq-int*: $(0::int) \leq k \Longrightarrow \exists n.\ k = of\text{-}nat\ n$
⟨*proof*⟩

**lemma** *zero-less-imp-eq-int*: $(0::int) < k \Longrightarrow \exists n>0.\ k = of\text{-}nat\ n$
⟨*proof*⟩

**lemma** *zmult-zless-mono2*: $[|\ i<j;\ \ (0::int) < k\ |] \Longrightarrow k*i < k*j$
⟨*proof*⟩

The integers form an ordered integral domain

**instance** *int* :: *ordered-idom*
⟨*proof*⟩

**instance** *int* :: *lordered-ring*
⟨*proof*⟩

**lemma** *zless-imp-add1-zle*: $w < z \Longrightarrow w + (1::int) \leq z$
⟨*proof*⟩

**lemma** *zless-iff-Suc-zadd*:
$(w :: int) < z \longleftrightarrow (\exists n.\ z = w + of\text{-}nat\ (Suc\ n))$
⟨*proof*⟩

**lemmas** *int-distrib* =
*left-distrib* [*of z1*::*int z2 w*, *standard*]
*right-distrib* [*of w*::*int z1 z2*, *standard*]
*left-diff-distrib* [*of z1*::*int z2 w*, *standard*]
*right-diff-distrib* [*of w*::*int z1 z2*, *standard*]

## 30.5 Embedding of the Integers into any *ring-1*: *of-int*

**context** *ring-1*
**begin**

**definition**
*of-int* :: $int \Rightarrow 'a$
**where**
[*code del*]: $of\text{-}int\ z = contents\ (\bigcup(i, j) \in Rep\text{-}Integ\ z.\ \{\ of\text{-}nat\ i - of\text{-}nat\ j\ \})$

**lemma** *of-int*: $of\text{-}int\ (Abs\text{-}Integ\ (intrel\ ``\ \{(i,j)\})) = of\text{-}nat\ i - of\text{-}nat\ j$
⟨*proof*⟩

**lemma** *of-int-0* [*simp*]: *of-int 0 = 0*
⟨*proof*⟩

**lemma** *of-int-1* [*simp*]: *of-int 1 = 1*
⟨*proof*⟩

**lemma** *of-int-add* [*simp*]: *of-int (w+z) = of-int w + of-int z*
⟨*proof*⟩

**lemma** *of-int-minus* [*simp*]: *of-int (−z) = − (of-int z)*
⟨*proof*⟩

**lemma** *of-int-diff* [*simp*]: *of-int (w − z) = of-int w − of-int z*
⟨*proof*⟩

**lemma** *of-int-mult* [*simp*]: *of-int (w∗z) = of-int w ∗ of-int z*
⟨*proof*⟩

Collapse nested embeddings

**lemma** *of-int-of-nat-eq* [*simp*]: *of-int (of-nat n) = of-nat n*
⟨*proof*⟩

**end**

**context** *ordered-idom*
**begin**

**lemma** *of-int-le-iff* [*simp*]:
*of-int w ≤ of-int z ⟷ w ≤ z*
⟨*proof*⟩

Special cases where either operand is zero

**lemmas** *of-int-0-le-iff* [*simp*] = *of-int-le-iff* [*of 0*, *simplified*]
**lemmas** *of-int-le-0-iff* [*simp*] = *of-int-le-iff* [*of - 0*, *simplified*]

**lemma** *of-int-less-iff* [*simp*]:
*of-int w < of-int z ⟷ w < z*
⟨*proof*⟩

Special cases where either operand is zero

**lemmas** *of-int-0-less-iff* [*simp*] = *of-int-less-iff* [*of 0*, *simplified*]
**lemmas** *of-int-less-0-iff* [*simp*] = *of-int-less-iff* [*of - 0*, *simplified*]

**end**

Class for unital rings with characteristic zero. Includes non-ordered rings like the complex numbers.

**class** *ring-char-0 = ring-1 + semiring-char-0*

**begin**

**lemma** *of-int-eq-iff* [*simp*]:
  *of-int w = of-int z ⟷ w = z*
⟨*proof*⟩

Special cases where either operand is zero

**lemmas** *of-int-0-eq-iff* [*simp*] = *of-int-eq-iff* [*of 0*, *simplified*]
**lemmas** *of-int-eq-0-iff* [*simp*] = *of-int-eq-iff* [*of - 0*, *simplified*]

**end**

Every *ordered-idom* has characteristic zero.

**subclass** (**in** *ordered-idom*) *ring-char-0* ⟨*proof*⟩

**lemma** *of-int-eq-id* [*simp*]: *of-int = id*
⟨*proof*⟩

## 30.6 Magnitude of an Integer, as a Natural Number: *nat*

**definition**
  *nat* :: *int ⇒ nat*
**where**
  [*code del*]: *nat z = contents* (⋃(*x*, *y*) ∈ *Rep-Integ z*. {*x−y*})

**lemma** *nat*: *nat* (*Abs-Integ* (*intrel*``{(*x*,*y*)})) = *x−y*
⟨*proof*⟩

**lemma** *nat-int* [*simp*]: *nat* (*of-nat n*) = *n*
⟨*proof*⟩

**lemma** *nat-zero* [*simp*]: *nat 0 = 0*
⟨*proof*⟩

**lemma** *int-nat-eq* [*simp*]: *of-nat* (*nat z*) = (*if 0 ≤ z then z else 0*)
⟨*proof*⟩

**corollary** *nat-0-le*: *0 ≤ z ==> of-nat* (*nat z*) = *z*
⟨*proof*⟩

**lemma** *nat-le-0* [*simp*]: *z ≤ 0 ==> nat z = 0*
⟨*proof*⟩

**lemma** *nat-le-eq-zle*: *0 < w | 0 ≤ z ==>* (*nat w ≤ nat z*) = (*w≤z*)
⟨*proof*⟩

An alternative condition is (*0*::*'a*) ≤ *w*

**corollary** *nat-mono-iff*: *0 < z ==>* (*nat w < nat z*) = (*w < z*)
⟨*proof*⟩

**corollary** *nat-less-eq-zless*: $0 \leq w \Longrightarrow (nat\ w < nat\ z) = (w<z)$
⟨*proof*⟩

**lemma** *zless-nat-conj* [*simp*]: $(nat\ w < nat\ z) = (0 < z \ \&\ w < z)$
⟨*proof*⟩

**lemma** *nonneg-eq-int*:
**fixes** $z$ :: *int*
**assumes** $0 \leq z$ **and** $\bigwedge m.\ z = of\text{-}nat\ m \Longrightarrow P$
**shows** $P$
⟨*proof*⟩

**lemma** *nat-eq-iff*: $(nat\ w = m) = (if\ 0 \leq w\ then\ w = of\text{-}nat\ m\ else\ m=0)$
⟨*proof*⟩

**corollary** *nat-eq-iff2*: $(m = nat\ w) = (if\ 0 \leq w\ then\ w = of\text{-}nat\ m\ else\ m=0)$
⟨*proof*⟩

**lemma** *nat-less-iff*: $0 \leq w \Longrightarrow (nat\ w < m) = (w < of\text{-}nat\ m)$
⟨*proof*⟩

**lemma** *nat-0-iff* [*simp*]: $nat(i::int) = 0 \longleftrightarrow i \leq 0$
⟨*proof*⟩

**lemma** *int-eq-iff*: $(of\text{-}nat\ m = z) = (m = nat\ z \ \&\ 0 \leq z)$
⟨*proof*⟩

**lemma** *zero-less-nat-eq* [*simp*]: $(0 < nat\ z) = (0 < z)$
⟨*proof*⟩

**lemma** *nat-add-distrib*:
$[|\ (0::int) \leq z;\ \ 0 \leq z'\ |] \Longrightarrow nat\ (z+z') = nat\ z + nat\ z'$
⟨*proof*⟩

**lemma** *nat-diff-distrib*:
$[|\ (0::int) \leq z';\ \ z' \leq z\ |] \Longrightarrow nat\ (z-z') = nat\ z - nat\ z'$
⟨*proof*⟩

**lemma** *nat-zminus-int* [*simp*]: $nat\ (-\ (of\text{-}nat\ n)) = 0$
⟨*proof*⟩

**lemma** *zless-nat-eq-int-zless*: $(m < nat\ z) = (of\text{-}nat\ m < z)$
⟨*proof*⟩

**context** *ring-1*
**begin**

**lemma** *of-nat-nat*: $0 \leq z \Longrightarrow of\text{-}nat\ (nat\ z) = of\text{-}int\ z$

⟨*proof*⟩

**end**

For termination proofs:

**lemma** *measure-function-int*[*measure-function*]: *is-measure* (*nat o abs*) ⟨*proof*⟩

## 30.7 Lemmas about the Function *of-nat* and Orderings

**lemma** *negative-zless-0*: − (*of-nat* (*Suc n*)) < (*0* :: *int*)
⟨*proof*⟩

**lemma** *negative-zless* [*iff*]: − (*of-nat* (*Suc n*)) < (*of-nat m* :: *int*)
⟨*proof*⟩

**lemma** *negative-zle-0*: − *of-nat n* ≤ (*0* :: *int*)
⟨*proof*⟩

**lemma** *negative-zle* [*iff*]: − *of-nat n* ≤ (*of-nat m* :: *int*)
⟨*proof*⟩

**lemma** *not-zle-0-negative* [*simp*]: ~ (*0* ≤ − (*of-nat* (*Suc n*) :: *int*))
⟨*proof*⟩

**lemma** *int-zle-neg*: ((*of-nat n* :: *int*) ≤ − *of-nat m*) = (*n* = *0* & *m* = *0*)
⟨*proof*⟩

**lemma** *not-int-zless-negative* [*simp*]: ~ ((*of-nat n* :: *int*) < − *of-nat m*)
⟨*proof*⟩

**lemma** *negative-eq-positive* [*simp*]: ((− *of-nat n* :: *int*) = *of-nat m*) = (*n* = *0* & *m* = *0*)
⟨*proof*⟩

**lemma** *zle-iff-zadd*: (*w*::*int*) ≤ *z* ⟷ (∃ *n*. *z* = *w* + *of-nat n*)
⟨*proof*⟩

**lemma** *zadd-int-left*: *of-nat m* + (*of-nat n* + *z*) = *of-nat* (*m* + *n*) + (*z*::*int*)
⟨*proof*⟩

**lemma** *int-Suc0-eq-1*: *of-nat* (*Suc 0*) = (*1*::*int*)
⟨*proof*⟩

This version is proved for all ordered rings, not just integers! It is proved here because attribute *arith-split* is not available in theory *Ring-and-Field*. But is it really better than just rewriting with *abs-if*?

**lemma** *abs-split* [*arith-split*,*noatp*]:
*P*(*abs*(*a*::'*a*::*ordered-idom*)) = ((*0* ≤ *a* −−> *P a*) & (*a* < *0* −−> *P*(−*a*)))
⟨*proof*⟩

**lemma** *negD*: $(x :: int) < 0 \Longrightarrow \exists n.\ x = -\ (of\text{-}nat\ (Suc\ n))$
⟨*proof*⟩

## 30.8 Cases and induction

Now we replace the case analysis rule by a more conventional one: whether an integer is negative or not.

**theorem** *int-cases* [*cases type*: *int*, *case-names nonneg neg*]:
[|!! $n$. $(z :: int) = of\text{-}nat\ n$ ==> $P$; !! $n$. $z = -\ (of\text{-}nat\ (Suc\ n))$ ==> $P$ |] ==> $P$
⟨*proof*⟩

**theorem** *int-induct* [*induct type*: *int*, *case-names nonneg neg*]:
[|!! $n$. $P\ (of\text{-}nat\ n :: int)$; !!$n$. $P\ (-\ (of\text{-}nat\ (Suc\ n)))$ |] ==> $P\ z$
⟨*proof*⟩

Contributed by Brian Huffman

**theorem** *int-diff-cases*:
**obtains** (*diff*) $m\ n$ **where** $(z::int) = of\text{-}nat\ m - of\text{-}nat\ n$
⟨*proof*⟩

## 30.9 Binary representation

This formalization defines binary arithmetic in terms of the integers rather than using a datatype. This avoids multiple representations (leading zeroes, etc.) See *ZF/Tools/twos−compl.ML*, function *int-of-binary*, for the numerical interpretation.

The representation expects that $(m\ mod\ 2)$ is 0 or 1, even if m is negative; For instance, $-5\ div\ 2 = -3$ and $-5\ mod\ 2 = 1$; thus $-5 = (-3)*2 + 1$.

This two's complement binary representation derives from the paper "An Efficient Representation of Arithmetic for Term Rewriting" by Dave Cohen and Phil Watson, Rewriting Techniques and Applications, Springer LNCS 488 (240-251), 1991.

### 30.9.1 The constructors *Bit0*, *Bit1*, *Pls* and *Min*

**definition**
*Pls* :: *int* **where**
[*code del*]: $Pls = 0$

**definition**
*Min* :: *int* **where**
[*code del*]: $Min = -\ 1$

**definition**

*Bit0* :: *int* ⇒ *int* **where**
[*code del*]: *Bit0 k* = *k* + *k*

**definition**
*Bit1* :: *int* ⇒ *int* **where**
[*code del*]: *Bit1 k* = *1* + *k* + *k*

**class** *number* = — for numeric types: nat, int, real, ...
**fixes** *number-of* :: *int* ⇒ *'a*

⟨*ML*⟩

**syntax**
*-Numeral* :: *num-const* ⇒ *'a* (-)

⟨*ML*⟩

**abbreviation**
*Numeral0* ≡ *number-of Pls*

**abbreviation**
*Numeral1* ≡ *number-of* (*Bit1 Pls*)

**lemma** *Let-number-of* [*simp*]: *Let* (*number-of v*) *f* = *f* (*number-of v*)
— Unfold all *let*s involving constants
⟨*proof*⟩

**definition**
*succ* :: *int* ⇒ *int* **where**
[*code del*]: *succ k* = *k* + *1*

**definition**
*pred* :: *int* ⇒ *int* **where**
[*code del*]: *pred k* = *k* − *1*

**lemmas**
*max-number-of* [*simp*] = *max-def*
[*of number-of u number-of v*, *standard*, *simp*]
**and**
*min-number-of* [*simp*] = *min-def*
[*of number-of u number-of v*, *standard*, *simp*]
— unfolding *minx* and *max* on numerals

**lemmas** *numeral-simps* =
*succ-def pred-def Pls-def Min-def Bit0-def Bit1-def*

Removal of leading zeroes

**lemma** *Bit0-Pls* [*simp*, *code post*]:
*Bit0 Pls* = *Pls*

⟨*proof*⟩

**lemma** *Bit1-Min* [*simp*, *code post*]:
*Bit1 Min* = *Min*
⟨*proof*⟩

**lemmas** *normalize-bin-simps* =
*Bit0-Pls Bit1-Min*

### 30.9.2 Successor and predecessor functions

Successor

**lemma** *succ-Pls*:
*succ Pls* = *Bit1 Pls*
⟨*proof*⟩

**lemma** *succ-Min*:
*succ Min* = *Pls*
⟨*proof*⟩

**lemma** *succ-Bit0*:
*succ* (*Bit0 k*) = *Bit1 k*
⟨*proof*⟩

**lemma** *succ-Bit1*:
*succ* (*Bit1 k*) = *Bit0* (*succ k*)
⟨*proof*⟩

**lemmas** *succ-bin-simps* [*simp*] =
*succ-Pls succ-Min succ-Bit0 succ-Bit1*

Predecessor

**lemma** *pred-Pls*:
*pred Pls* = *Min*
⟨*proof*⟩

**lemma** *pred-Min*:
*pred Min* = *Bit0 Min*
⟨*proof*⟩

**lemma** *pred-Bit0*:
*pred* (*Bit0 k*) = *Bit1* (*pred k*)
⟨*proof*⟩

**lemma** *pred-Bit1*:
*pred* (*Bit1 k*) = *Bit0 k*
⟨*proof*⟩

**lemmas** *pred-bin-simps* [*simp*] =

*pred-Pls pred-Min pred-Bit0 pred-Bit1*

### 30.9.3 Binary arithmetic

Addition

**lemma** *add-Pls*:
*Pls + k = k*
⟨*proof*⟩

**lemma** *add-Min*:
*Min + k = pred k*
⟨*proof*⟩

**lemma** *add-Bit0-Bit0*:
*(Bit0 k) + (Bit0 l) = Bit0 (k + l)*
⟨*proof*⟩

**lemma** *add-Bit0-Bit1*:
*(Bit0 k) + (Bit1 l) = Bit1 (k + l)*
⟨*proof*⟩

**lemma** *add-Bit1-Bit0*:
*(Bit1 k) + (Bit0 l) = Bit1 (k + l)*
⟨*proof*⟩

**lemma** *add-Bit1-Bit1*:
*(Bit1 k) + (Bit1 l) = Bit0 (k + succ l)*
⟨*proof*⟩

**lemma** *add-Pls-right*:
*k + Pls = k*
⟨*proof*⟩

**lemma** *add-Min-right*:
*k + Min = pred k*
⟨*proof*⟩

**lemmas** *add-bin-simps* [*simp*] =
*add-Pls add-Min add-Pls-right add-Min-right*
*add-Bit0-Bit0 add-Bit0-Bit1 add-Bit1-Bit0 add-Bit1-Bit1*

Negation

**lemma** *minus-Pls*:
*− Pls = Pls*
⟨*proof*⟩

**lemma** *minus-Min*:
*− Min = Bit1 Pls*
⟨*proof*⟩

**lemma** *minus-Bit0*:
$-\ (Bit0\ k) = Bit0\ (-\ k)$
⟨*proof*⟩

**lemma** *minus-Bit1*:
$-\ (Bit1\ k) = Bit1\ (pred\ (-\ k))$
⟨*proof*⟩

**lemmas** *minus-bin-simps* [*simp*] =
*minus-Pls minus-Min minus-Bit0 minus-Bit1*

Subtraction

**lemma** *diff-bin-simps* [*simp*]:
$k - Pls = k$
$k - Min = succ\ k$
$Pls - (Bit0\ l) = Bit0\ (Pls - l)$
$Pls - (Bit1\ l) = Bit1\ (Min - l)$
$Min - (Bit0\ l) = Bit1\ (Min - l)$
$Min - (Bit1\ l) = Bit0\ (Min - l)$
$(Bit0\ k) - (Bit0\ l) = Bit0\ (k - l)$
$(Bit0\ k) - (Bit1\ l) = Bit1\ (pred\ k - l)$
$(Bit1\ k) - (Bit0\ l) = Bit1\ (k - l)$
$(Bit1\ k) - (Bit1\ l) = Bit0\ (k - l)$
⟨*proof*⟩

Multiplication

**lemma** *mult-Pls*:
$Pls * w = Pls$
⟨*proof*⟩

**lemma** *mult-Min*:
$Min * k = -\ k$
⟨*proof*⟩

**lemma** *mult-Bit0*:
$(Bit0\ k) * l = Bit0\ (k * l)$
⟨*proof*⟩

**lemma** *mult-Bit1*:
$(Bit1\ k) * l = (Bit0\ (k * l)) + l$
⟨*proof*⟩

**lemmas** *mult-bin-simps* [*simp*] =
*mult-Pls mult-Min mult-Bit0 mult-Bit1*

### 30.9.4 Binary comparisons

Preliminaries

**lemma** *even-less-0-iff*:
$a + a < 0 \longleftrightarrow a < (0::'a::ordered\text{-}idom)$
⟨*proof*⟩

**lemma** *le-imp-0-less*:
**assumes** *le*: $0 \leq z$
**shows** $(0::int) < 1 + z$
⟨*proof*⟩

**lemma** *odd-less-0-iff*:
$(1 + z + z < 0) = (z < (0::int))$
⟨*proof*⟩

**lemma** *bin-less-0-simps*:
$Pls < 0 \longleftrightarrow False$
$Min < 0 \longleftrightarrow True$
$Bit0\ w < 0 \longleftrightarrow w < 0$
$Bit1\ w < 0 \longleftrightarrow w < 0$
⟨*proof*⟩

**lemma** *less-bin-lemma*: $k < l \longleftrightarrow k - l < (0::int)$
⟨*proof*⟩

**lemma** *le-iff-pred-less*: $k \leq l \longleftrightarrow pred\ k < l$
⟨*proof*⟩

**lemma** *succ-pred*: $succ\ (pred\ x) = x$
⟨*proof*⟩

Less-than

**lemma** *less-bin-simps* [*simp*]:
$Pls < Pls \longleftrightarrow False$
$Pls < Min \longleftrightarrow False$
$Pls < Bit0\ k \longleftrightarrow Pls < k$
$Pls < Bit1\ k \longleftrightarrow Pls \leq k$
$Min < Pls \longleftrightarrow True$
$Min < Min \longleftrightarrow False$
$Min < Bit0\ k \longleftrightarrow Min < k$
$Min < Bit1\ k \longleftrightarrow Min < k$
$Bit0\ k < Pls \longleftrightarrow k < Pls$
$Bit0\ k < Min \longleftrightarrow k \leq Min$
$Bit1\ k < Pls \longleftrightarrow k < Pls$
$Bit1\ k < Min \longleftrightarrow k < Min$
$Bit0\ k < Bit0\ l \longleftrightarrow k < l$
$Bit0\ k < Bit1\ l \longleftrightarrow k \leq l$
$Bit1\ k < Bit0\ l \longleftrightarrow k < l$
$Bit1\ k < Bit1\ l \longleftrightarrow k < l$
⟨*proof*⟩

Less-than-or-equal

**lemma** *le-bin-simps* [*simp*]:
*Pls* ≤ *Pls* ⟷ *True*
*Pls* ≤ *Min* ⟷ *False*
*Pls* ≤ *Bit0 k* ⟷ *Pls* ≤ *k*
*Pls* ≤ *Bit1 k* ⟷ *Pls* ≤ *k*
*Min* ≤ *Pls* ⟷ *True*
*Min* ≤ *Min* ⟷ *True*
*Min* ≤ *Bit0 k* ⟷ *Min* < *k*
*Min* ≤ *Bit1 k* ⟷ *Min* ≤ *k*
*Bit0 k* ≤ *Pls* ⟷ *k* ≤ *Pls*
*Bit0 k* ≤ *Min* ⟷ *k* ≤ *Min*
*Bit1 k* ≤ *Pls* ⟷ *k* < *Pls*
*Bit1 k* ≤ *Min* ⟷ *k* ≤ *Min*
*Bit0 k* ≤ *Bit0 l* ⟷ *k* ≤ *l*
*Bit0 k* ≤ *Bit1 l* ⟷ *k* ≤ *l*
*Bit1 k* ≤ *Bit0 l* ⟷ *k* < *l*
*Bit1 k* ≤ *Bit1 l* ⟷ *k* ≤ *l*
⟨*proof*⟩

Equality

**lemma** *eq-bin-simps* [*simp*]:
*Pls* = *Pls* ⟷ *True*
*Pls* = *Min* ⟷ *False*
*Pls* = *Bit0 l* ⟷ *Pls* = *l*
*Pls* = *Bit1 l* ⟷ *False*
*Min* = *Pls* ⟷ *False*
*Min* = *Min* ⟷ *True*
*Min* = *Bit0 l* ⟷ *False*
*Min* = *Bit1 l* ⟷ *Min* = *l*
*Bit0 k* = *Pls* ⟷ *k* = *Pls*
*Bit0 k* = *Min* ⟷ *False*
*Bit1 k* = *Pls* ⟷ *False*
*Bit1 k* = *Min* ⟷ *k* = *Min*
*Bit0 k* = *Bit0 l* ⟷ *k* = *l*
*Bit0 k* = *Bit1 l* ⟷ *False*
*Bit1 k* = *Bit0 l* ⟷ *False*
*Bit1 k* = *Bit1 l* ⟷ *k* = *l*
⟨*proof*⟩

## 30.10 Converting Numerals to Rings: *number-of*

**class** *number-ring* = *number* + *comm-ring-1* +
**assumes** *number-of-eq*: *number-of k* = *of-int k*

self-embedding of the integers

**instantiation** *int* :: *number-ring*
**begin**

**definition** *int-number-of-def* [*code del*]:

*number-of w* = (*of-int w* :: *int*)

**instance** ⟨*proof*⟩

**end**

**lemma** *number-of-is-id*:
*number-of* (*k*::*int*) = *k*
⟨*proof*⟩

**lemma** *number-of-succ*:
*number-of* (*succ k*) = (*1* + *number-of k* ::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *number-of-pred*:
*number-of* (*pred w*) = (− *1* + *number-of w* ::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *number-of-minus*:
*number-of* (*uminus w*) = (− (*number-of w*)::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *number-of-add*:
*number-of* (*v* + *w*) = (*number-of v* + *number-of w*::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *number-of-diff*:
*number-of* (*v* − *w*) = (*number-of v* − *number-of w*::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *number-of-mult*:
*number-of* (*v* ∗ *w*) = (*number-of v* ∗ *number-of w*::'*a*::*number-ring*)
⟨*proof*⟩

The correctness of shifting. But it doesn't seem to give a measurable speed-up.

**lemma** *double-number-of-Bit0*:
(*1* + *1*) ∗ *number-of w* = (*number-of* (*Bit0 w*) ::'*a*::*number-ring*)
⟨*proof*⟩

Converting numerals 0 and 1 to their abstract versions.

**lemma** *numeral-0-eq-0* [*simp*]:
*Numeral0* = (*0*::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *numeral-1-eq-1* [*simp*]:
*Numeral1* = (*1*::'*a*::*number-ring*)
⟨*proof*⟩

Special-case simplification for small constants.

Unary minus for the abstract constant 1. Cannot be inserted as a simprule until later: it is *number-of-Min* re-oriented!

**lemma** *numeral-m1-eq-minus-1*:
$(-1::'a::number\text{-}ring) = -\ 1$
⟨*proof*⟩

**lemma** *mult-minus1* [*simp*]:
$-1 * z = -(z::'a::number\text{-}ring)$
⟨*proof*⟩

**lemma** *mult-minus1-right* [*simp*]:
$z * -1 = -(z::'a::number\text{-}ring)$
⟨*proof*⟩

**lemma** *minus-number-of-mult* [*simp*]:
$-\ (number\text{-}of\ w) * z = number\text{-}of\ (uminus\ w) * (z::'a::number\text{-}ring)$
⟨*proof*⟩

Subtraction

**lemma** *diff-number-of-eq*:
$number\text{-}of\ v - number\text{-}of\ w =$
$(number\text{-}of\ (v + uminus\ w)::'a::number\text{-}ring)$
⟨*proof*⟩

**lemma** *number-of-Pls*:
$number\text{-}of\ Pls = (0::'a::number\text{-}ring)$
⟨*proof*⟩

**lemma** *number-of-Min*:
$number\text{-}of\ Min = (-\ 1::'a::number\text{-}ring)$
⟨*proof*⟩

**lemma** *number-of-Bit0*:
$number\text{-}of\ (Bit0\ w) = (0::'a::number\text{-}ring) + (number\text{-}of\ w) + (number\text{-}of\ w)$
⟨*proof*⟩

**lemma** *number-of-Bit1*:
$number\text{-}of\ (Bit1\ w) = (1::'a::number\text{-}ring) + (number\text{-}of\ w) + (number\text{-}of\ w)$
⟨*proof*⟩

### 30.10.1 Equality of Binary Numbers

First version by Norbert Voelker

**definition**
$iszero :: 'a::semiring\text{-}1 \Rightarrow bool$

**where**
*iszero z* $\longleftrightarrow$ *z = 0*

**lemma** *iszero-0*: *iszero 0*
⟨*proof*⟩

**lemma** *not-iszero-1*: ~ *iszero 1*
⟨*proof*⟩

**lemma** *eq-number-of-eq*:
((*number-of x*::'*a*::*number-ring*) = *number-of y*) =
*iszero* (*number-of* (*x* + *uminus y*) :: '*a*)
⟨*proof*⟩

**lemma** *iszero-number-of-Pls*:
*iszero* ((*number-of Pls*)::'*a*::*number-ring*)
⟨*proof*⟩

**lemma** *nonzero-number-of-Min*:
~ *iszero* ((*number-of Min*)::'*a*::*number-ring*)
⟨*proof*⟩

### 30.10.2 Comparisons, for Ordered Rings

**lemmas** *double-eq-0-iff* = *double-zero*

**lemma** *odd-nonzero*:
*1* + *z* + *z* ≠ (*0*::*int*)
⟨*proof*⟩

**lemma** *iszero-number-of-Bit0*:
*iszero* (*number-of* (*Bit0 w*)::'*a*) =
*iszero* (*number-of w*::'*a*::{*ring-char-0*,*number-ring*})
⟨*proof*⟩

**lemma** *iszero-number-of-Bit1*:
~ *iszero* (*number-of* (*Bit1 w*)::'*a*::{*ring-char-0*,*number-ring*})
⟨*proof*⟩

**lemmas** *iszero-simps* =
*iszero-0 not-iszero-1*
*iszero-number-of-Pls nonzero-number-of-Min*
*iszero-number-of-Bit0 iszero-number-of-Bit1*

### 30.10.3 The Less-Than Relation

**lemma** *double-less-0-iff*:
(*a* + *a* < *0*) = (*a* < (*0*::'*a*::*ordered-idom*))
⟨*proof*⟩

**lemma** *odd-less-0*:
$(1 + z + z < 0) = (z < (0::int))$
⟨*proof*⟩

Less-Than or Equals

Reduces $a \leq b$ to $\neg\ b < a$ for ALL numerals.

**lemmas** *le-number-of-eq-not-less* =
*linorder-not-less* [*of number-of w number-of v*, *symmetric*, *standard*]

Absolute value (*abs*)

**lemma** *abs-number-of*:
$abs(number\text{-}of\ x::'a::\{ordered\text{-}idom,number\text{-}ring\}) =$
$(if\ number\text{-}of\ x < (0::'a)\ then\ -number\text{-}of\ x\ else\ number\text{-}of\ x)$
⟨*proof*⟩

Re-orientation of the equation nnn=x

**lemma** *number-of-reorient*:
$(number\text{-}of\ w = x) = (x = number\text{-}of\ w)$
⟨*proof*⟩

### 30.10.4 Simplification of arithmetic operations on integer constants.

**lemmas** *arith-extra-simps* [*standard*, *simp*] =
*number-of-add* [*symmetric*]
*number-of-minus* [*symmetric*]
*numeral-m1-eq-minus-1* [*symmetric*]
*number-of-mult* [*symmetric*]
*diff-number-of-eq abs-number-of*

For making a minimal simpset, one must include these default simprules. Also include *simp-thms*.

**lemmas** *arith-simps* =
*normalize-bin-simps pred-bin-simps succ-bin-simps*
*add-bin-simps minus-bin-simps mult-bin-simps*
*abs-zero abs-one arith-extra-simps*

Simplification of relational operations

**lemma** *less-number-of* [*simp*]:
$(number\text{-}of\ x::'a::\{ordered\text{-}idom,number\text{-}ring\}) < number\text{-}of\ y \longleftrightarrow x < y$
⟨*proof*⟩

**lemma** *le-number-of* [*simp*]:
$(number\text{-}of\ x::'a::\{ordered\text{-}idom,number\text{-}ring\}) \leq number\text{-}of\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

**lemma** *eq-number-of* [*simp*]:
(*number-of* $x$::$'a$::{*ring-char-0*,*number-ring*}) = *number-of* $y \longleftrightarrow x = y$
⟨*proof*⟩

**lemmas** *rel-simps* [*simp*] =
*less-number-of less-bin-simps*
*le-number-of le-bin-simps*
*eq-number-of-eq eq-bin-simps*
*iszero-simps*

### 30.10.5 Simplification of arithmetic when nested to the right.

**lemma** *add-number-of-left* [*simp*]:
*number-of* $v$ + (*number-of* $w$ + $z$) =
(*number-of*($v$ + $w$) + $z$::$'a$::*number-ring*)
⟨*proof*⟩

**lemma** *mult-number-of-left* [*simp*]:
*number-of* $v$ * (*number-of* $w$ * $z$) =
(*number-of*($v$ * $w$) * $z$::$'a$::*number-ring*)
⟨*proof*⟩

**lemma** *add-number-of-diff1*:
*number-of* $v$ + (*number-of* $w$ − $c$) =
*number-of*($v$ + $w$) − ($c$::$'a$::*number-ring*)
⟨*proof*⟩

**lemma** *add-number-of-diff2* [*simp*]:
*number-of* $v$ + ($c$ − *number-of* $w$) =
*number-of* ($v$ + *uminus* $w$) + ($c$::$'a$::*number-ring*)
⟨*proof*⟩

## 30.11 The Set of Integers

**context** *ring-1*
**begin**

**definition** *Ints* :: $'a$ *set* **where**
[*code del*]: *Ints* = *range of-int*

**end**

**notation** (*xsymbols*)
*Ints* (ℤ)

**context** *ring-1*
**begin**

**lemma** *Ints-0* [*simp*]: 0 ∈ ℤ
⟨*proof*⟩

**lemma** *Ints-1* [*simp*]: $1 \in \mathbb{Z}$
⟨*proof*⟩

**lemma** *Ints-add* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a + b \in \mathbb{Z}$
⟨*proof*⟩

**lemma** *Ints-minus* [*simp*]: $a \in \mathbb{Z} \Longrightarrow -a \in \mathbb{Z}$
⟨*proof*⟩

**lemma** *Ints-mult* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a * b \in \mathbb{Z}$
⟨*proof*⟩

**lemma** *Ints-cases* [*cases set*: *Ints*]:
**assumes** $q \in \mathbb{Z}$
**obtains** (*of-int*) $z$ **where** $q = of\text{-}int\ z$
⟨*proof*⟩

**lemma** *Ints-induct* [*case-names of-int*, *induct set*: *Ints*]:
$q \in \mathbb{Z} \Longrightarrow (\bigwedge z.\ P\ (of\text{-}int\ z)) \Longrightarrow P\ q$
⟨*proof*⟩

**end**

**lemma** *Ints-diff* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a-b \in \mathbb{Z}$
⟨*proof*⟩

The premise involving $\mathbb{Z}$ prevents $a = (1::'a)\ /\ (2::'a)$.

**lemma** *Ints-double-eq-0-iff*:
**assumes** *in-Ints*: $a \in Ints$
**shows** $(a + a = 0) = (a = (0::'a::ring\text{-}char\text{-}0))$
⟨*proof*⟩

**lemma** *Ints-odd-nonzero*:
**assumes** *in-Ints*: $a \in Ints$
**shows** $1 + a + a \neq (0::'a::ring\text{-}char\text{-}0)$
⟨*proof*⟩

**lemma** *Ints-number-of*:
$(number\text{-}of\ w :: 'a::number\text{-}ring) \in Ints$
⟨*proof*⟩

**lemma** *Ints-odd-less-0*:
**assumes** *in-Ints*: $a \in Ints$
**shows** $(1 + a + a < 0) = (a < (0::'a::ordered\text{-}idom))$
⟨*proof*⟩

## 30.12 *setsum* and *setprod*

**lemma** *of-nat-setsum*: *of-nat* (*setsum f A*) = ($\sum x{\in}A$. *of-nat*(*f x*))
⟨*proof*⟩

**lemma** *of-int-setsum*: *of-int* (*setsum f A*) = ($\sum x{\in}A$. *of-int*(*f x*))
⟨*proof*⟩

**lemma** *of-nat-setprod*: *of-nat* (*setprod f A*) = ($\prod x{\in}A$. *of-nat*(*f x*))
⟨*proof*⟩

**lemma** *of-int-setprod*: *of-int* (*setprod f A*) = ($\prod x{\in}A$. *of-int*(*f x*))
⟨*proof*⟩

**lemmas** *int-setsum* = *of-nat-setsum* [**where** *'a*=*int*]
**lemmas** *int-setprod* = *of-nat-setprod* [**where** *'a*=*int*]

## 30.13 Inequality Reasoning for the Arithmetic Simproc

**lemma** *add-numeral-0*: *Numeral0* + *a* = (*a*::*'a*::*number-ring*)
⟨*proof*⟩

**lemma** *add-numeral-0-right*: *a* + *Numeral0* = (*a*::*'a*::*number-ring*)
⟨*proof*⟩

**lemma** *mult-numeral-1*: *Numeral1* ∗ *a* = (*a*::*'a*::*number-ring*)
⟨*proof*⟩

**lemma** *mult-numeral-1-right*: *a* ∗ *Numeral1* = (*a*::*'a*::*number-ring*)
⟨*proof*⟩

**lemma** *divide-numeral-1*: *a* / *Numeral1* = (*a*::*'a*::{*number-ring*,*field*})
⟨*proof*⟩

**lemma** *inverse-numeral-1*:
*inverse Numeral1* = (*Numeral1*::*'a*::{*number-ring*,*field*})
⟨*proof*⟩

Theorem lists for the cancellation simprocs. The use of binary numerals for 0 and 1 reduces the number of special cases.

**lemmas** *add-0s* = *add-numeral-0 add-numeral-0-right*
**lemmas** *mult-1s* = *mult-numeral-1 mult-numeral-1-right*
*mult-minus1 mult-minus1-right*

## 30.14 Special Arithmetic Rules for Abstract 0 and 1

Arithmetic computations are defined for binary literals, which leaves 0 and 1 as special cases. Addition already has rules for 0, but not 1. Multiplication and unary minus already have rules for both 0 and 1.

**lemma** *binop-eq*: $[|f\ x\ y = g\ x\ y;\ x = x';\ y = y'|] ==> f\ x'\ y' = g\ x'\ y'$
⟨*proof*⟩

**lemmas** *add-number-of-eq* = *number-of-add* [*symmetric*]

Allow 1 on either or both sides

**lemma** *one-add-one-is-two*: *1 + 1* = (*2*::'*a*::*number-ring*)
⟨*proof*⟩

**lemmas** *add-special* =
*one-add-one-is-two*
*binop-eq* [*of op +*, *OF add-number-of-eq numeral-1-eq-1 refl*, *standard*]
*binop-eq* [*of op +*, *OF add-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 1 on either or both sides (1-1 already simplifies to 0)

**lemmas** *diff-special* =
*binop-eq* [*of op −*, *OF diff-number-of-eq numeral-1-eq-1 refl*, *standard*]
*binop-eq* [*of op −*, *OF diff-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

**lemmas** *eq-special* =
*binop-eq* [*of op =*, *OF eq-number-of-eq numeral-0-eq-0 refl*, *standard*]
*binop-eq* [*of op =*, *OF eq-number-of-eq numeral-1-eq-1 refl*, *standard*]
*binop-eq* [*of op =*, *OF eq-number-of-eq refl numeral-0-eq-0*, *standard*]
*binop-eq* [*of op =*, *OF eq-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

**lemmas** *less-special* =
*binop-eq* [*of op <*, *OF less-number-of numeral-0-eq-0 refl*, *standard*]
*binop-eq* [*of op <*, *OF less-number-of numeral-1-eq-1 refl*, *standard*]
*binop-eq* [*of op <*, *OF less-number-of refl numeral-0-eq-0*, *standard*]
*binop-eq* [*of op <*, *OF less-number-of refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

**lemmas** *le-special* =
*binop-eq* [*of op ≤*, *OF le-number-of numeral-0-eq-0 refl*, *standard*]
*binop-eq* [*of op ≤*, *OF le-number-of numeral-1-eq-1 refl*, *standard*]
*binop-eq* [*of op ≤*, *OF le-number-of refl numeral-0-eq-0*, *standard*]
*binop-eq* [*of op ≤*, *OF le-number-of refl numeral-1-eq-1*, *standard*]

**lemmas** *arith-special*[*simp*] =
*add-special diff-special eq-special less-special le-special*

**lemma** *min-max-01*: *min* (*0*::*int*) *1* = *0* & *min* (*1*::*int*) *0* = *0* &
*max* (*0*::*int*) *1* = *1* & *max* (*1*::*int*) *0* = *1*
⟨*proof*⟩

**lemmas** *min-max-special*[*simp*] =
*min-max-01*
*max-def*[*of 0*::*int number-of v*, *standard*, *simp*]
*min-def*[*of 0*::*int number-of v*, *standard*, *simp*]
*max-def*[*of number-of u 0*::*int*, *standard*, *simp*]
*min-def*[*of number-of u 0*::*int*, *standard*, *simp*]
*max-def*[*of 1*::*int number-of v*, *standard*, *simp*]
*min-def*[*of 1*::*int number-of v*, *standard*, *simp*]
*max-def*[*of number-of u 1*::*int*, *standard*, *simp*]
*min-def*[*of number-of u 1*::*int*, *standard*, *simp*]

Legacy theorems

**lemmas** *zle-int* = *of-nat-le-iff* [**where** *'a*=*int*]
**lemmas** *int-int-eq* = *of-nat-eq-iff* [**where** *'a*=*int*]

## 30.15 Setting up simplification procedures

**lemmas** *int-arith-rules* =
*neg-le-iff-le numeral-0-eq-0 numeral-1-eq-1*
*minus-zero diff-minus left-minus right-minus*
*mult-zero-left mult-zero-right mult-Bit1 mult-1-right*
*mult-minus-left mult-minus-right*
*minus-add-distrib minus-minus mult-assoc*
*of-nat-0 of-nat-1 of-nat-Suc of-nat-add of-nat-mult*
*of-int-0 of-int-1 of-int-add of-int-mult*

⟨*ML*⟩

## 30.16 Lemmas About Small Numerals

**lemma** *of-int-m1* [*simp*]: *of-int* −*1* = (−*1* :: *'a* :: *number-ring*)
⟨*proof*⟩

**lemma** *abs-minus-one* [*simp*]: *abs* (−*1*) = (*1*::*'a*::{*ordered-idom*,*number-ring*})
⟨*proof*⟩

**lemma** *abs-power-minus-one* [*simp*]:
*abs*(−*1* ^ *n*) = (*1*::*'a*::{*ordered-idom*,*number-ring*,*recpower*})
⟨*proof*⟩

**lemma** *of-int-number-of-eq* [*simp*]:
*of-int* (*number-of v*) = (*number-of v* :: *'a* :: *number-ring*)
⟨*proof*⟩

Lemmas for specialist use, NOT as default simprules

**lemma** *mult-2*: *2* ∗ *z* = (*z*+*z*::*'a*::*number-ring*)
⟨*proof*⟩

**lemma** *mult-2-right*: $z * 2 = (z+z::'a::number\text{-}ring)$
⟨*proof*⟩

## 30.17 More Inequality Reasoning

**lemma** *zless-add1-eq*: $(w < z + (1::int)) = (w<z \mid w=z)$
⟨*proof*⟩

**lemma** *add1-zle-eq*: $(w + (1::int) \leq z) = (w<z)$
⟨*proof*⟩

**lemma** *zle-diff1-eq* [*simp*]: $(w \leq z - (1::int)) = (w<z)$
⟨*proof*⟩

**lemma** *zle-add1-eq-le* [*simp*]: $(w < z + (1::int)) = (w\leq z)$
⟨*proof*⟩

**lemma** *int-one-le-iff-zero-less*: $((1::int) \leq z) = (0 < z)$
⟨*proof*⟩

## 30.18 The functions *nat* and *int*

Simplify the terms *int* $(0::'a)$, *int* $(Suc\ 0)$ and $w + - z$

**declare** *Zero-int-def* [*symmetric*, *simp*]
**declare** *One-int-def* [*symmetric*, *simp*]

**lemmas** *diff-int-def-symmetric* = *diff-int-def* [*symmetric*, *simp*]

**lemma** *nat-0*: *nat 0* = *0*
⟨*proof*⟩

**lemma** *nat-1*: *nat 1* = *Suc 0*
⟨*proof*⟩

**lemma** *nat-2*: *nat 2* = *Suc* (*Suc 0*)
⟨*proof*⟩

**lemma** *one-less-nat-eq* [*simp*]: $(Suc\ 0 < nat\ z) = (1 < z)$
⟨*proof*⟩

This simplifies expressions of the form *int n* = *z* where z is an integer literal.

**lemmas** *int-eq-iff-number-of* [*simp*] = *int-eq-iff* [*of* - *number-of v*, *standard*]

**lemma** *split-nat* [*arith-split*]:
$P(nat(i::int)) = ((\forall n.\ i = of\text{-}nat\ n \longrightarrow P\ n)\ \&\ (i < 0 \longrightarrow P\ 0))$
(**is** *?P* = (*?L* & *?R*))
⟨*proof*⟩

**context** *ring-1*

**begin**

**lemma** *of-int-of-nat* [*nitpick-const-simp*]:
*of-int k = (if k < 0 then − of-nat (nat (− k)) else of-nat (nat k))*
⟨*proof*⟩

**end**

**lemma** *nat-mult-distrib*:
**fixes** *z z′* :: *int*
**assumes** *0 ≤ z*
**shows** *nat (z ∗ z′) = nat z ∗ nat z′*
⟨*proof*⟩

**lemma** *nat-mult-distrib-neg*: *z ≤ (0*::*int) ==> nat(z∗z′) = nat(−z) ∗ nat(−z′)*
⟨*proof*⟩

**lemma** *nat-abs-mult-distrib*: *nat (abs (w ∗ z)) = nat (abs w) ∗ nat (abs z)*
⟨*proof*⟩

## 30.19 Induction principles for int

Well-founded segments of the integers

**definition**
*int-ge-less-than* :: *int => (int ∗ int) set*
**where**
*int-ge-less-than d = {(z′,z). d ≤ z′ & z′ < z}*

**theorem** *wf-int-ge-less-than*: *wf (int-ge-less-than d)*
⟨*proof*⟩

This variant looks odd, but is typical of the relations suggested by RankFinder.

**definition**
*int-ge-less-than2* :: *int => (int ∗ int) set*
**where**
*int-ge-less-than2 d = {(z′,z). d ≤ z & z′ < z}*

**theorem** *wf-int-ge-less-than2*: *wf (int-ge-less-than2 d)*
⟨*proof*⟩

**abbreviation**
*int* :: *nat ⇒ int*
**where**
*int ≡ of-nat*

**theorem** *int-ge-induct* [*case-names base step*, *induct set*: *int*]:

**fixes** $i :: int$
**assumes** *ge*: $k \leq i$ **and**
*base*: $P\ k$ **and**
*step*: $\bigwedge i.\ k \leq i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
**shows** $P\ i$
⟨*proof*⟩

**theorem** *int-gr-induct* [*case-names base step*, *induct set*: *int*]:
**assumes** *gr*: $k < (i::int)$ **and**
*base*: $P(k+1)$ **and**
*step*: $\bigwedge i.\ [\![k < i;\ P\ i]\!] \Longrightarrow P(i+1)$
**shows** $P\ i$
⟨*proof*⟩

**theorem** *int-le-induct*[*consumes 1*,*case-names base step*]:
**assumes** *le*: $i \leq (k::int)$ **and**
*base*: $P(k)$ **and**
*step*: $\bigwedge i.\ [\![i \leq k;\ P\ i]\!] \Longrightarrow P(i - 1)$
**shows** $P\ i$
⟨*proof*⟩

**theorem** *int-less-induct* [*consumes 1*,*case-names base step*]:
**assumes** *less*: $(i::int) < k$ **and**
*base*: $P(k - 1)$ **and**
*step*: $\bigwedge i.\ [\![i < k;\ P\ i]\!] \Longrightarrow P(i - 1)$
**shows** $P\ i$
⟨*proof*⟩

## 30.20 Intermediate value theorems

**lemma** *int-val-lemma*:
$(\forall\, i{<}n::nat.\ abs(f(i+1) - f\ i) \leq 1)$ −−>
$f\ 0 \leq k$ −−> $k \leq f\ n$ −−> $(\exists\, i \leq n.\ f\ i = (k::int))$
⟨*proof*⟩

**lemmas** *nat0-intermed-int-val* = *int-val-lemma* [*rule-format* (*no-asm*)]

**lemma** *nat-intermed-int-val*:
[| $\forall\, i.\ m \leq i\ \&\ i < n$ −−> $abs(f(i + 1::nat) - f\ i) \leq 1;\ m < n;$
$f\ m \leq k;\ k \leq f\ n$ |] ==> $?\, i.\ m \leq i\ \&\ i \leq n\ \&\ f\ i = (k::int)$
⟨*proof*⟩

## 30.21 Products and 1, by T. M. Rasmussen

**lemma** *zabs-less-one-iff* [*simp*]: $(|z| < 1) = (z = (0::int))$
⟨*proof*⟩

**lemma** *abs-zmult-eq-1*: $(|m * n| = 1)$ ==> $|m| = (1::int)$
⟨*proof*⟩

**lemma** *pos-zmult-eq-1-iff-lemma*: $(m * n = 1) ==> m = (1::int) \mid m = -1$
⟨*proof*⟩

**lemma** *pos-zmult-eq-1-iff*: $0 < (m::int) ==> (m * n = 1) = (m = 1 \ \& \ n = 1)$
⟨*proof*⟩

**lemma** *zmult-eq-1-iff*: $(m*n = (1::int)) = ((m = 1 \ \& \ n = 1) \mid (m = -1 \ \& \ n = -1))$
⟨*proof*⟩

**lemma** *infinite-UNIV-int*: $\sim finite(UNIV::int\ set)$
⟨*proof*⟩

## 30.22 Integer Powers

**instantiation** *int* :: *recpower*
**begin**

**primrec** *power-int* **where**
$p \ \hat{} \ 0 = (1::int)$
$\mid p \ \hat{} \ (Suc\ n) = (p::int) * (p \ \hat{} \ n)$

**instance** ⟨*proof*⟩

**declare** *power-int.simps* [*simp del*]

**end**

**lemma** *zpower-zadd-distrib*: $x \ \hat{} \ (y + z) = ((x \ \hat{} \ y) * (x \ \hat{} \ z)::int)$
⟨*proof*⟩

**lemma** *zpower-zpower*: $(x \ \hat{} \ y) \ \hat{} \ z = (x \ \hat{} \ (y * z)::int)$
⟨*proof*⟩

**lemma** *zero-less-zpower-abs-iff* [*simp*]:
$(0 < abs\ x \ \hat{} \ n) \longleftrightarrow (x \neq (0::int) \mid n = 0)$
⟨*proof*⟩

**lemma** *zero-le-zpower-abs* [*simp*]: $(0::int) \leq abs\ x \ \hat{} \ n$
⟨*proof*⟩

**lemma** *of-int-power*:
$of\text{-}int\ (z \ \hat{} \ n) = (of\text{-}int\ z \ \hat{} \ n :: 'a::\{recpower, ring\text{-}1\})$
⟨*proof*⟩

**lemma** *int-power*: $int\ (m\hat{}n) = (int\ m) \ \hat{} \ n$
⟨*proof*⟩

**lemmas** *zpower-int* = *int-power* [*symmetric*]

## 30.23 Further theorems on numerals

### 30.23.1 Special Simplification for Constants

These distributive laws move literals inside sums and differences.

**lemmas** *left-distrib-number-of* [*simp*] =
*left-distrib* [*of* - - *number-of v*, *standard*]

**lemmas** *right-distrib-number-of* [*simp*] =
*right-distrib* [*of number-of v*, *standard*]

**lemmas** *left-diff-distrib-number-of* [*simp*] =
*left-diff-distrib* [*of* - - *number-of v*, *standard*]

**lemmas** *right-diff-distrib-number-of* [*simp*] =
*right-diff-distrib* [*of number-of v*, *standard*]

These are actually for fields, like real: but where else to put them?

**lemmas** *zero-less-divide-iff-number-of* [*simp*, *noatp*] =
*zero-less-divide-iff* [*of number-of w*, *standard*]

**lemmas** *divide-less-0-iff-number-of* [*simp*, *noatp*] =
*divide-less-0-iff* [*of number-of w*, *standard*]

**lemmas** *zero-le-divide-iff-number-of* [*simp*, *noatp*] =
*zero-le-divide-iff* [*of number-of w*, *standard*]

**lemmas** *divide-le-0-iff-number-of* [*simp*, *noatp*] =
*divide-le-0-iff* [*of number-of w*, *standard*]

Replaces *inverse #nn* by *1/#nn*. It looks strange, but then other simprocs simplify the quotient.

**lemmas** *inverse-eq-divide-number-of* [*simp*] =
*inverse-eq-divide* [*of number-of w*, *standard*]

These laws simplify inequalities, moving unary minus from a term into the literal.

**lemmas** *less-minus-iff-number-of* [*simp*, *noatp*] =
*less-minus-iff* [*of number-of v*, *standard*]

**lemmas** *le-minus-iff-number-of* [*simp*, *noatp*] =
*le-minus-iff* [*of number-of v*, *standard*]

**lemmas** *equation-minus-iff-number-of* [*simp*, *noatp*] =
*equation-minus-iff* [*of number-of v*, *standard*]

**lemmas** *minus-less-iff-number-of* [*simp*, *noatp*] =
*minus-less-iff* [*of* - *number-of v*, *standard*]

**lemmas** *minus-le-iff-number-of* [*simp*, *noatp*] =
*minus-le-iff* [*of* - *number-of v*, *standard*]

**lemmas** *minus-equation-iff-number-of* [*simp*, *noatp*] =
*minus-equation-iff* [*of* - *number-of v*, *standard*]

To Simplify Inequalities Where One Side is the Constant 1

**lemma** *less-minus-iff-1* [*simp*,*noatp*]:
**fixes** *b*::'*b*::{*ordered-idom*,*number-ring*}
**shows** (*1* < − *b*) = (*b* < −*1*)
⟨*proof*⟩

**lemma** *le-minus-iff-1* [*simp*,*noatp*]:
**fixes** *b*::'*b*::{*ordered-idom*,*number-ring*}
**shows** (*1* ≤ − *b*) = (*b* ≤ −*1*)
⟨*proof*⟩

**lemma** *equation-minus-iff-1* [*simp*,*noatp*]:
**fixes** *b*::'*b*::*number-ring*
**shows** (*1* = − *b*) = (*b* = −*1*)
⟨*proof*⟩

**lemma** *minus-less-iff-1* [*simp*,*noatp*]:
**fixes** *a*::'*b*::{*ordered-idom*,*number-ring*}
**shows** (− *a* < *1*) = (−*1* < *a*)
⟨*proof*⟩

**lemma** *minus-le-iff-1* [*simp*,*noatp*]:
**fixes** *a*::'*b*::{*ordered-idom*,*number-ring*}
**shows** (− *a* ≤ *1*) = (−*1* ≤ *a*)
⟨*proof*⟩

**lemma** *minus-equation-iff-1* [*simp*,*noatp*]:
**fixes** *a*::'*b*::*number-ring*
**shows** (− *a* = *1*) = (*a* = −*1*)
⟨*proof*⟩

Cancellation of constant factors in comparisons (< and ≤)

**lemmas** *mult-less-cancel-left-number-of* [*simp*, *noatp*] =
*mult-less-cancel-left* [*of number-of v*, *standard*]

**lemmas** *mult-less-cancel-right-number-of* [*simp*, *noatp*] =
*mult-less-cancel-right* [*of* - *number-of v*, *standard*]

**lemmas** *mult-le-cancel-left-number-of* [*simp*, *noatp*] =

*mult-le-cancel-left* [*of number-of v*, *standard*]

**lemmas** *mult-le-cancel-right-number-of* [*simp*, *noatp*] =
*mult-le-cancel-right* [*of - number-of v*, *standard*]

Multiplying out constant divisors in comparisons ($<$, $\leq$ and $=$)

**lemmas** *le-divide-eq-number-of1* [*simp*] = *le-divide-eq* [*of - - number-of w*, *standard*]
**lemmas** *divide-le-eq-number-of1* [*simp*] = *divide-le-eq* [*of - number-of w*, *standard*]
**lemmas** *less-divide-eq-number-of1* [*simp*] = *less-divide-eq* [*of - - number-of w*, *standard*]
**lemmas** *divide-less-eq-number-of1* [*simp*] = *divide-less-eq* [*of - number-of w*, *standard*]
**lemmas** *eq-divide-eq-number-of1* [*simp*] = *eq-divide-eq* [*of - - number-of w*, *standard*]
**lemmas** *divide-eq-eq-number-of1* [*simp*] = *divide-eq-eq* [*of - number-of w*, *standard*]

### 30.23.2 Optional Simplification Rules Involving Constants

Simplify quotients that are compared with a literal constant.

**lemmas** *le-divide-eq-number-of* = *le-divide-eq* [*of number-of w*, *standard*]
**lemmas** *divide-le-eq-number-of* = *divide-le-eq* [*of - - number-of w*, *standard*]
**lemmas** *less-divide-eq-number-of* = *less-divide-eq* [*of number-of w*, *standard*]
**lemmas** *divide-less-eq-number-of* = *divide-less-eq* [*of - - number-of w*, *standard*]
**lemmas** *eq-divide-eq-number-of* = *eq-divide-eq* [*of number-of w*, *standard*]
**lemmas** *divide-eq-eq-number-of* = *divide-eq-eq* [*of - - number-of w*, *standard*]

Not good as automatic simprules because they cause case splits.

**lemmas** *divide-const-simps* =
*le-divide-eq-number-of divide-le-eq-number-of less-divide-eq-number-of*
*divide-less-eq-number-of eq-divide-eq-number-of divide-eq-eq-number-of*
*le-divide-eq-1 divide-le-eq-1 less-divide-eq-1 divide-less-eq-1*

Division By $-1$

**lemma** *divide-minus1* [*simp*]:
$x/-1 = -(x::'a::\{field,division\text{-}by\text{-}zero,number\text{-}ring\})$
⟨*proof*⟩

**lemma** *minus1-divide* [*simp*]:
$-1 \;/\; (x::'a::\{field,division\text{-}by\text{-}zero,number\text{-}ring\}) = -\;(1/x)$
⟨*proof*⟩

**lemma** *half-gt-zero-iff*:
$(0 < r/2) = (0 < (r::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero,number\text{-}ring\}))$
⟨*proof*⟩

**lemmas** *half-gt-zero* [*simp*] = *half-gt-zero-iff* [*THEN iffD2*, *standard*]

## 30.24 Configuration of the code generator

**code-datatype** *Pls Min Bit0 Bit1 number-of* :: *int* ⇒ *int*

**lemmas** *pred-succ-numeral-code* [*code*] =
*pred-bin-simps succ-bin-simps*

**lemmas** *plus-numeral-code* [*code*] =
*add-bin-simps*
*arith-extra-simps*(*1*) [**where** *'a* = *int*]

**lemmas** *minus-numeral-code* [*code*] =
*minus-bin-simps*
*arith-extra-simps*(*2*) [**where** *'a* = *int*]
*arith-extra-simps*(*5*) [**where** *'a* = *int*]

**lemmas** *times-numeral-code* [*code*] =
*mult-bin-simps*
*arith-extra-simps*(*4*) [**where** *'a* = *int*]

**instantiation** *int* :: *eq*
**begin**

**definition** [*code del*]: *eq-class.eq k l* ⟷ *k* − *l* = (*0*::*int*)

**instance** ⟨*proof*⟩

**end**

**lemma** *eq-number-of-int-code* [*code*]:
*eq-class.eq* (*number-of k* :: *int*) (*number-of l*) ⟷ *eq-class.eq k l*
⟨*proof*⟩

**lemma** *eq-int-code* [*code*]:
*eq-class.eq Int.Pls Int.Pls* ⟷ *True*
*eq-class.eq Int.Pls Int.Min* ⟷ *False*
*eq-class.eq Int.Pls* (*Int.Bit0 k2*) ⟷ *eq-class.eq Int.Pls k2*
*eq-class.eq Int.Pls* (*Int.Bit1 k2*) ⟷ *False*
*eq-class.eq Int.Min Int.Pls* ⟷ *False*
*eq-class.eq Int.Min Int.Min* ⟷ *True*
*eq-class.eq Int.Min* (*Int.Bit0 k2*) ⟷ *False*
*eq-class.eq Int.Min* (*Int.Bit1 k2*) ⟷ *eq-class.eq Int.Min k2*
*eq-class.eq* (*Int.Bit0 k1*) *Int.Pls* ⟷ *eq-class.eq k1 Int.Pls*
*eq-class.eq* (*Int.Bit1 k1*) *Int.Pls* ⟷ *False*
*eq-class.eq* (*Int.Bit0 k1*) *Int.Min* ⟷ *False*
*eq-class.eq* (*Int.Bit1 k1*) *Int.Min* ⟷ *eq-class.eq k1 Int.Min*
*eq-class.eq* (*Int.Bit0 k1*) (*Int.Bit0 k2*) ⟷ *eq-class.eq k1 k2*
*eq-class.eq* (*Int.Bit0 k1*) (*Int.Bit1 k2*) ⟷ *False*
*eq-class.eq* (*Int.Bit1 k1*) (*Int.Bit0 k2*) ⟷ *False*
*eq-class.eq* (*Int.Bit1 k1*) (*Int.Bit1 k2*) ⟷ *eq-class.eq k1 k2*

⟨*proof*⟩

**lemma** *eq-int-refl* [*code nbe*]:
*eq-class.eq* (*k*::*int*) *k* ⟷ *True*
⟨*proof*⟩

**lemma** *less-eq-number-of-int-code* [*code*]:
(*number-of k* :: *int*) ≤ *number-of l* ⟷ *k* ≤ *l*
⟨*proof*⟩

**lemma** *less-eq-int-code* [*code*]:
*Int.Pls* ≤ *Int.Pls* ⟷ *True*
*Int.Pls* ≤ *Int.Min* ⟷ *False*
*Int.Pls* ≤ *Int.Bit0 k* ⟷ *Int.Pls* ≤ *k*
*Int.Pls* ≤ *Int.Bit1 k* ⟷ *Int.Pls* ≤ *k*
*Int.Min* ≤ *Int.Pls* ⟷ *True*
*Int.Min* ≤ *Int.Min* ⟷ *True*
*Int.Min* ≤ *Int.Bit0 k* ⟷ *Int.Min* < *k*
*Int.Min* ≤ *Int.Bit1 k* ⟷ *Int.Min* ≤ *k*
*Int.Bit0 k* ≤ *Int.Pls* ⟷ *k* ≤ *Int.Pls*
*Int.Bit1 k* ≤ *Int.Pls* ⟷ *k* < *Int.Pls*
*Int.Bit0 k* ≤ *Int.Min* ⟷ *k* ≤ *Int.Min*
*Int.Bit1 k* ≤ *Int.Min* ⟷ *k* ≤ *Int.Min*
*Int.Bit0 k1* ≤ *Int.Bit0 k2* ⟷ *k1* ≤ *k2*
*Int.Bit0 k1* ≤ *Int.Bit1 k2* ⟷ *k1* ≤ *k2*
*Int.Bit1 k1* ≤ *Int.Bit0 k2* ⟷ *k1* < *k2*
*Int.Bit1 k1* ≤ *Int.Bit1 k2* ⟷ *k1* ≤ *k2*
⟨*proof*⟩

**lemma** *less-number-of-int-code* [*code*]:
(*number-of k* :: *int*) < *number-of l* ⟷ *k* < *l*
⟨*proof*⟩

**lemma** *less-int-code* [*code*]:
*Int.Pls* < *Int.Pls* ⟷ *False*
*Int.Pls* < *Int.Min* ⟷ *False*
*Int.Pls* < *Int.Bit0 k* ⟷ *Int.Pls* < *k*
*Int.Pls* < *Int.Bit1 k* ⟷ *Int.Pls* ≤ *k*
*Int.Min* < *Int.Pls* ⟷ *True*
*Int.Min* < *Int.Min* ⟷ *False*
*Int.Min* < *Int.Bit0 k* ⟷ *Int.Min* < *k*
*Int.Min* < *Int.Bit1 k* ⟷ *Int.Min* < *k*
*Int.Bit0 k* < *Int.Pls* ⟷ *k* < *Int.Pls*
*Int.Bit1 k* < *Int.Pls* ⟷ *k* < *Int.Pls*
*Int.Bit0 k* < *Int.Min* ⟷ *k* ≤ *Int.Min*
*Int.Bit1 k* < *Int.Min* ⟷ *k* < *Int.Min*
*Int.Bit0 k1* < *Int.Bit0 k2* ⟷ *k1* < *k2*
*Int.Bit0 k1* < *Int.Bit1 k2* ⟷ *k1* ≤ *k2*
*Int.Bit1 k1* < *Int.Bit0 k2* ⟷ *k1* < *k2*

*Int.Bit1 k1 < Int.Bit1 k2 ⟷ k1 < k2*
⟨*proof*⟩

**definition**
*nat-aux* :: *int* ⇒ *nat* ⇒ *nat* **where**
*nat-aux i n = nat i + n*

**lemma** [*code*]:
*nat-aux i n = (if i ≤ 0 then n else nat-aux (i − 1) (Suc n))* — tail recursive
⟨*proof*⟩

**lemma** [*code*]: *nat i = nat-aux i 0*
⟨*proof*⟩

**hide** (**open**) *const nat-aux*

**lemma** *zero-is-num-zero* [*code*, *code inline*, *symmetric*, *code post*]:
*(0::int) = Numeral0*
⟨*proof*⟩

**lemma** *one-is-num-one* [*code*, *code inline*, *symmetric*, *code post*]:
*(1::int) = Numeral1*
⟨*proof*⟩

**code-modulename** *SML*
*Int Integer*

**code-modulename** *OCaml*
*Int Integer*

**code-modulename** *Haskell*
*Int Integer*

**types-code**
*int* (*int*)
**attach** (*term-of*) ⟪
*val term-of-int = HOLogic.mk-number HOLogic.intT*;
⟫
**attach** (*test*) ⟪
*fun gen-int i =*
*let val j = one-of* [~*1*, *1*] ∗ *random-range 0 i*
*in* (*j*, *fn* () => *term-of-int j*) *end*;
⟫

⟨*ML*⟩

**consts-code**
*number-of* :: *int* ⇒ *int* ((-))
*0* :: *int* (*0*)

*1* :: *int* (*1*)
*uminus* :: *int* => *int* (~)
*op* + :: *int* => *int* => *int* ((- +/ -))
*op* ∗ :: *int* => *int* => *int* ((- ∗/ -))
*op* ≤ :: *int* => *int* => *bool* ((- <=/ -))
*op* < :: *int* => *int* => *bool* ((- </ -))

**quickcheck-params** [*default-type* = *int*]

**hide** (**open**) *const Pls Min Bit0 Bit1 succ pred*

## 30.25 Legacy theorems

**lemmas** *zminus-zminus* = *minus-minus* [*of z*::*int*, *standard*]
**lemmas** *zminus-0* = *minus-zero* [**where** *'a*=*int*]
**lemmas** *zminus-zadd-distrib* = *minus-add-distrib* [*of z*::*int w*, *standard*]
**lemmas** *zadd-commute* = *add-commute* [*of z*::*int w*, *standard*]
**lemmas** *zadd-assoc* = *add-assoc* [*of z1*::*int z2 z3*, *standard*]
**lemmas** *zadd-left-commute* = *add-left-commute* [*of x*::*int y z*, *standard*]
**lemmas** *zadd-ac* = *zadd-assoc zadd-commute zadd-left-commute*
**lemmas** *zmult-ac* = *OrderedGroup.mult-ac*
**lemmas** *zadd-0* = *OrderedGroup.add-0-left* [*of z*::*int*, *standard*]
**lemmas** *zadd-0-right* = *OrderedGroup.add-0-left* [*of z*::*int*, *standard*]
**lemmas** *zadd-zminus-inverse2* = *left-minus* [*of z*::*int*, *standard*]
**lemmas** *zmult-zminus* = *mult-minus-left* [*of z*::*int w*, *standard*]
**lemmas** *zmult-commute* = *mult-commute* [*of z*::*int w*, *standard*]
**lemmas** *zmult-assoc* = *mult-assoc* [*of z1*::*int z2 z3*, *standard*]
**lemmas** *zadd-zmult-distrib* = *left-distrib* [*of z1*::*int z2 w*, *standard*]
**lemmas** *zadd-zmult-distrib2* = *right-distrib* [*of w*::*int z1 z2*, *standard*]
**lemmas** *zdiff-zmult-distrib* = *left-diff-distrib* [*of z1*::*int z2 w*, *standard*]
**lemmas** *zdiff-zmult-distrib2* = *right-diff-distrib* [*of w*::*int z1 z2*, *standard*]

**lemmas** *zmult-1* = *mult-1-left* [*of z*::*int*, *standard*]
**lemmas** *zmult-1-right* = *mult-1-right* [*of z*::*int*, *standard*]

**lemmas** *zle-refl* = *order-refl* [*of w*::*int*, *standard*]
**lemmas** *zle-trans* = *order-trans* [**where** *'a*=*int* **and** *x*=*i* **and** *y*=*j* **and** *z*=*k*, *standard*]
**lemmas** *zle-anti-sym* = *order-antisym* [*of z*::*int w*, *standard*]
**lemmas** *zle-linear* = *linorder-linear* [*of z*::*int w*, *standard*]
**lemmas** *zless-linear* = *linorder-less-linear* [**where** *'a* = *int*]

**lemmas** *zadd-left-mono* = *add-left-mono* [*of i*::*int j k*, *standard*]
**lemmas** *zadd-strict-right-mono* = *add-strict-right-mono* [*of i*::*int j k*, *standard*]
**lemmas** *zadd-zless-mono* = *add-less-le-mono* [*of w'*::*int w z'* *z*, *standard*]

**lemmas** *int-0-less-1* = *zero-less-one* [**where** *'a*=*int*]
**lemmas** *int-0-neq-1* = *zero-neq-one* [**where** *'a*=*int*]

**lemmas** *inj-int* = *inj-of-nat* [**where** *'a=int*]
**lemmas** *zadd-int* = *of-nat-add* [**where** *'a=int*, *symmetric*]
**lemmas** *int-mult* = *of-nat-mult* [**where** *'a=int*]
**lemmas** *zmult-int* = *of-nat-mult* [**where** *'a=int*, *symmetric*]
**lemmas** *int-eq-0-conv* = *of-nat-eq-0-iff* [**where** *'a=int* **and** *m=n*, *standard*]
**lemmas** *zless-int* = *of-nat-less-iff* [**where** *'a=int*]
**lemmas** *int-less-0-conv* = *of-nat-less-0-iff* [**where** *'a=int* **and** *m=k*, *standard*]
**lemmas** *zero-less-int-conv* = *of-nat-0-less-iff* [**where** *'a=int*]
**lemmas** *zero-zle-int* = *of-nat-0-le-iff* [**where** *'a=int*]
**lemmas** *int-le-0-conv* = *of-nat-le-0-iff* [**where** *'a=int* **and** *m=n*, *standard*]
**lemmas** *int-0* = *of-nat-0* [**where** *'a=int*]
**lemmas** *int-1* = *of-nat-1* [**where** *'a=int*]
**lemmas** *int-Suc* = *of-nat-Suc* [**where** *'a=int*]
**lemmas** *abs-int-eq* = *abs-of-nat* [**where** *'a=int* **and** *n=m*, *standard*]
**lemmas** *of-int-int-eq* = *of-int-of-nat-eq* [**where** *'a=int*]
**lemmas** *zdiff-int* = *of-nat-diff* [**where** *'a=int*, *symmetric*]
**lemmas** *zless-le* = *less-int-def*
**lemmas** *int-eq-of-nat* = *TrueI*

**end**

# 31 IntDiv: The Division Operators div and mod

**theory** *IntDiv*
**imports** *Int Divides FunDef*
**uses**
*~~/src/Provers/Arith/cancel-numeral-factor.ML*
*~~/src/Provers/Arith/extract-common-term.ML*
(*Tools/int-factor-simprocs.ML*)
**begin**

**definition** *divmod-rel* :: *int* ⇒ *int* ⇒ *int* × *int* ⇒ *bool* **where**
— definition of quotient and remainder
[*code*]: *divmod-rel a b* = (λ(*q*, *r*). *a* = *b* ∗ *q* + *r* ∧
(*if* 0 < *b then* 0 ≤ *r* ∧ *r* < *b else b* < *r* ∧ *r* ≤ 0))

**definition** *adjust* :: *int* ⇒ *int* × *int* ⇒ *int* × *int* **where**
— for the division algorithm
[*code*]: *adjust b* = (λ(*q*, *r*). *if* 0 ≤ *r* − *b then* (2 ∗ *q* + 1, *r* − *b*)
*else* (2 ∗ *q*, *r*))

algorithm for the case $a \geq 0$, $b > 0$

**function** *posDivAlg* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
*posDivAlg a b* = (*if a* < *b* ∨ *b* ≤ 0 *then* (0, *a*)
*else adjust b* (*posDivAlg a* (2 ∗ *b*)))
⟨*proof*⟩
**termination** ⟨*proof*⟩

algorithm for the case $a<0$, $b>0$

**function** *negDivAlg* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
  *negDivAlg a b* = (*if* 0 ≤*a* + *b* ∨ *b* ≤ 0 *then* (−1, *a* + *b*)
    *else adjust b* (*negDivAlg a* (2 ∗ *b*)))
⟨*proof*⟩
**termination** ⟨*proof*⟩

algorithm for the general case $b \neq (0::'a)$

**definition** *negateSnd* :: *int* × *int* ⇒ *int* × *int* **where**
  [*code inline*]: *negateSnd* = *apsnd uminus*

**definition** *divmod* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
  — The full division algorithm considers all possible signs for a, b including the special case $a=0$, $b<0$ because *negDivAlg* requires $a < (0::'a)$.
  *divmod a b* = (*if* 0 ≤ *a then if* 0 ≤ *b then posDivAlg a b*
        *else if a* = 0 *then* (0, 0)
          *else negateSnd* (*negDivAlg* (−*a*) (−*b*))
      *else*
        *if* 0 < *b then negDivAlg a b*
        *else negateSnd* (*posDivAlg* (−*a*) (−*b*)))

**instantiation** *int* :: *Divides.div*
**begin**

**definition**
  *div-def*: *a div b* = *fst* (*divmod a b*)

**definition**
  *mod-def*: *a mod b* = *snd* (*divmod a b*)

**instance** ⟨*proof*⟩

**end**

**lemma** *divmod-mod-div*:
  *divmod p q* = (*p div q*, *p mod q*)
  ⟨*proof*⟩

Here is the division algorithm in ML:

```
    fun posDivAlg (a,b) =
      if a<b then (0,a)
      else let val (q,r) = posDivAlg(a, 2*b)
       in  if 0\<le>r-b then (2*q+1, r-b) else (2*q, r)
   end

    fun negDivAlg (a,b) =
      if 0\<le>a+b then (~1,a+b)
```

```
      else let val (q,r) = negDivAlg(a, 2*b)
       in  if 0\<le>r-b then (2*q+1, r-b) else (2*q, r)
   end;

    fun negateSnd (q,r:int) = (q,~r);

    fun divmod (a,b) = if 0\<le>a then
  if b>0 then posDivAlg (a,b)
   else if a=0 then (0,0)
else negateSnd (negDivAlg (~a,~b))
       else
  if 0<b then negDivAlg (a,b)
  else        negateSnd (posDivAlg (~a,~b));
```

## 31.1 Uniqueness and Monotonicity of Quotients and Remainders

**lemma** *unique-quotient-lemma*:
[| $b*q' + r' \le b*q + r$; $0 \le r'$; $r' < b$; $r < b$ |]
==> $q' \le (q::int)$
⟨*proof*⟩

**lemma** *unique-quotient-lemma-neg*:
[| $b*q' + r' \le b*q + r$; $r \le 0$; $b < r$; $b < r'$ |]
==> $q \le (q'::int)$
⟨*proof*⟩

**lemma** *unique-quotient*:
[| *divmod-rel a b* $(q, r)$; *divmod-rel a b* $(q', r')$; $b \neq 0$ |]
==> $q = q'$
⟨*proof*⟩

**lemma** *unique-remainder*:
[| *divmod-rel a b* $(q, r)$; *divmod-rel a b* $(q', r')$; $b \neq 0$ |]
==> $r = r'$
⟨*proof*⟩

## 31.2 Correctness of *posDivAlg*, the Algorithm for Non-Negative Dividends

And positive divisors

**lemma** *adjust-eq* [*simp*]:
*adjust b* $(q,r)$ =
($let$ *diff* $= r-b$ $in$
*if* $0 \le$ *diff then* $(2*q + 1,$ *diff*$)$

*else* (*2*q*, *r*))
⟨*proof*⟩

**declare** *posDivAlg.simps* [*simp del*]

use with a simproc to avoid repeatedly proving the premise

**lemma** *posDivAlg-eqn*:
*0 < b* ==>
*posDivAlg a b* = (*if a<b then* (*0*,*a*) *else adjust b* (*posDivAlg a* (*2*b*)))
⟨*proof*⟩

Correctness of *posDivAlg*: it computes quotients correctly

**theorem** *posDivAlg-correct*:
**assumes** *0* ≤ *a* **and** *0 < b*
**shows** *divmod-rel a b* (*posDivAlg a b*)
⟨*proof*⟩

## 31.3 Correctness of *negDivAlg*, the Algorithm for Negative Dividends

And positive divisors

**declare** *negDivAlg.simps* [*simp del*]

use with a simproc to avoid repeatedly proving the premise

**lemma** *negDivAlg-eqn*:
*0 < b* ==>
*negDivAlg a b* =
(*if 0*≤*a*+*b then* (−*1*,*a*+*b*) *else adjust b* (*negDivAlg a* (*2*b*)))
⟨*proof*⟩

**lemma** *negDivAlg-correct*:
**assumes** *a < 0* **and** *b > 0*
**shows** *divmod-rel a b* (*negDivAlg a b*)
⟨*proof*⟩

## 31.4 Existence Shown by Proving the Division Algorithm to be Correct

**lemma** *divmod-rel-0*: *b* ≠ *0* ==> *divmod-rel 0 b* (*0*, *0*)
⟨*proof*⟩

**lemma** *posDivAlg-0* [*simp*]: *posDivAlg 0 b* = (*0*, *0*)
⟨*proof*⟩

**lemma** *negDivAlg-minus1* [*simp*]: *negDivAlg* −*1 b* = (−*1*, *b* − *1*)
⟨*proof*⟩

**lemma** *negateSnd-eq* [*simp*]: *negateSnd*(*q*,*r*) = (*q*,−*r*)
⟨*proof*⟩

**lemma** *divmod-rel-neg*: *divmod-rel* (−*a*) (−*b*) *qr* ==> *divmod-rel a b* (*negateSnd qr*)
⟨*proof*⟩

**lemma** *divmod-correct*: *b* ≠ *0* ==> *divmod-rel a b* (*divmod a b*)
⟨*proof*⟩

Arbitrary definitions for division by zero. Useful to simplify certain equations.

**lemma** *DIVISION-BY-ZERO* [*simp*]: *a div* (*0*::*int*) = *0* & *a mod* (*0*::*int*) = *a*
⟨*proof*⟩

Basic laws about division and remainder

**lemma** *zmod-zdiv-equality*: (*a*::*int*) = *b* * (*a div b*) + (*a mod b*)
⟨*proof*⟩

**lemma** *zdiv-zmod-equality*: (*b* * (*a div b*) + (*a mod b*)) + *k* = (*a*::*int*)+*k*
⟨*proof*⟩

**lemma** *zdiv-zmod-equality2*: ((*a div b*) * *b* + (*a mod b*)) + *k* = (*a*::*int*)+*k*
⟨*proof*⟩

Tool setup

⟨*ML*⟩

**lemma** *pos-mod-conj* : (*0*::*int*) < *b* ==> *0* ≤ *a mod b* & *a mod b* < *b*
⟨*proof*⟩

**lemmas** *pos-mod-sign* [*simp*] = *pos-mod-conj* [*THEN conjunct1*, *standard*]
**and** *pos-mod-bound* [*simp*] = *pos-mod-conj* [*THEN conjunct2*, *standard*]

**lemma** *neg-mod-conj* : *b* < (*0*::*int*) ==> *a mod b* ≤ *0* & *b* < *a mod b*
⟨*proof*⟩

**lemmas** *neg-mod-sign* [*simp*] = *neg-mod-conj* [*THEN conjunct1*, *standard*]
**and** *neg-mod-bound* [*simp*] = *neg-mod-conj* [*THEN conjunct2*, *standard*]

## 31.5 General Properties of div and mod

**lemma** *divmod-rel-div-mod*: *b* ≠ *0* ==> *divmod-rel a b* (*a div b*, *a mod b*)
⟨*proof*⟩

**lemma** *divmod-rel-div*: [| *divmod-rel a b* (*q*, *r*); *b* ≠ *0* |] ==> *a div b* = *q*
⟨*proof*⟩

**lemma** *divmod-rel-mod*: [| *divmod-rel a b* (*q*, *r*); *b* ≠ *0* |] ==> *a mod b* = *r*

⟨*proof*⟩

**lemma** *div-pos-pos-trivial*: [| (0::*int*) ≤ *a*; *a* < *b* |] ==> *a div b* = 0
⟨*proof*⟩

**lemma** *div-neg-neg-trivial*: [| *a* ≤ (0::*int*); *b* < *a* |] ==> *a div b* = 0
⟨*proof*⟩

**lemma** *div-pos-neg-trivial*: [| (0::*int*) < *a*; *a*+*b* ≤ 0 |] ==> *a div b* = −1
⟨*proof*⟩

**lemma** *mod-pos-pos-trivial*: [| (0::*int*) ≤ *a*; *a* < *b* |] ==> *a mod b* = *a*
⟨*proof*⟩

**lemma** *mod-neg-neg-trivial*: [| *a* ≤ (0::*int*); *b* < *a* |] ==> *a mod b* = *a*
⟨*proof*⟩

**lemma** *mod-pos-neg-trivial*: [| (0::*int*) < *a*; *a*+*b* ≤ 0 |] ==> *a mod b* = *a*+*b*
⟨*proof*⟩

There is no *mod-neg-pos-trivial*.

**lemma** *zdiv-zminus-zminus* [*simp*]: (−*a*) *div* (−*b*) = *a div* (*b*::*int*)
⟨*proof*⟩

**lemma** *zmod-zminus-zminus* [*simp*]: (−*a*) *mod* (−*b*) = − (*a mod* (*b*::*int*))
⟨*proof*⟩

## 31.6 Laws for div and mod with Unary Minus

**lemma** *zminus1-lemma*:
*divmod-rel a b* (*q*, *r*)
==> *divmod-rel* (−*a*) *b* (*if r*=0 *then* −*q else* −*q* − 1,
*if r*=0 *then* 0 *else b*−*r*)
⟨*proof*⟩

**lemma** *zdiv-zminus1-eq-if*:
*b* ≠ (0::*int*)
==> (−*a*) *div b* =
(*if a mod b* = 0 *then* − (*a div b*) *else* − (*a div b*) − 1)
⟨*proof*⟩

**lemma** *zmod-zminus1-eq-if*:
(−*a*::*int*) *mod b* = (*if a mod b* = 0 *then* 0 *else b* − (*a mod b*))
⟨*proof*⟩

**lemma** *zmod-zminus1-not-zero*:
**fixes** *k l* :: *int*
**shows** $-\ k\ mod\ l \neq 0 \Longrightarrow k\ mod\ l \neq 0$
⟨*proof*⟩

**lemma** *zdiv-zminus2*: $a\ div\ (-b) = (-a::int)\ div\ b$
⟨*proof*⟩

**lemma** *zmod-zminus2*: $a\ mod\ (-b) = -\ ((-a::int)\ mod\ b)$
⟨*proof*⟩

**lemma** *zdiv-zminus2-eq-if*:
$b \neq (0::int)$
$\Longrightarrow a\ div\ (-b) =$
$(if\ a\ mod\ b = 0\ then\ -\ (a\ div\ b)\ else\ -\ (a\ div\ b) - 1)$
⟨*proof*⟩

**lemma** *zmod-zminus2-eq-if*:
$a\ mod\ (-b::int) = (if\ a\ mod\ b = 0\ then\ 0\ else\ (a\ mod\ b) - b)$
⟨*proof*⟩

**lemma** *zmod-zminus2-not-zero*:
**fixes** *k l* :: *int*
**shows** $k\ mod\ -\ l \neq 0 \Longrightarrow k\ mod\ l \neq 0$
⟨*proof*⟩

## 31.7 Division of a Number by Itself

**lemma** *self-quotient-aux1*: $[|\ (0::int) < a;\ a = r + a*q;\ r < a\ |] \Longrightarrow 1 \leq q$
⟨*proof*⟩

**lemma** *self-quotient-aux2*: $[|\ (0::int) < a;\ a = r + a*q;\ 0 \leq r\ |] \Longrightarrow q \leq 1$
⟨*proof*⟩

**lemma** *self-quotient*: $[|\ divmod\text{-}rel\ a\ a\ (q, r);\ \ a \neq (0::int)\ |] \Longrightarrow q = 1$
⟨*proof*⟩

**lemma** *self-remainder*: $[|\ divmod\text{-}rel\ a\ a\ (q, r);\ \ a \neq (0::int)\ |] \Longrightarrow r = 0$
⟨*proof*⟩

**lemma** *zdiv-self* [*simp*]: $a \neq 0 \Longrightarrow a\ div\ a = (1::int)$
⟨*proof*⟩

**lemma** *zmod-self* [*simp*]: $a\ mod\ a = (0::int)$
⟨*proof*⟩

## 31.8 Computation of Division and Remainder

**lemma** *zdiv-zero* [*simp*]: $(0::int)\ div\ b = 0$

⟨*proof*⟩

**lemma** *div-eq-minus1*: $(0::int) < b \Longrightarrow -1 \ div \ b = -1$
⟨*proof*⟩

**lemma** *zmod-zero* [*simp*]: $(0::int) \ mod \ b = 0$
⟨*proof*⟩

**lemma** *zmod-minus1*: $(0::int) < b \Longrightarrow -1 \ mod \ b = b - 1$
⟨*proof*⟩

a positive, b positive

**lemma** *div-pos-pos*: $[\!| \ 0 < a; \ \ 0 \leq b \ |\!] \Longrightarrow a \ div \ b = fst \ (posDivAlg \ a \ b)$
⟨*proof*⟩

**lemma** *mod-pos-pos*: $[\!| \ 0 < a; \ \ 0 \leq b \ |\!] \Longrightarrow a \ mod \ b = snd \ (posDivAlg \ a \ b)$
⟨*proof*⟩

a negative, b positive

**lemma** *div-neg-pos*: $[\!| \ a < 0; \ \ 0 < b \ |\!] \Longrightarrow a \ div \ b = fst \ (negDivAlg \ a \ b)$
⟨*proof*⟩

**lemma** *mod-neg-pos*: $[\!| \ a < 0; \ \ 0 < b \ |\!] \Longrightarrow a \ mod \ b = snd \ (negDivAlg \ a \ b)$
⟨*proof*⟩

a positive, b negative

**lemma** *div-pos-neg*:
$[\!| \ 0 < a; \ \ b < 0 \ |\!] \Longrightarrow a \ div \ b = fst \ (negateSnd \ (negDivAlg \ (-a) \ (-b)))$
⟨*proof*⟩

**lemma** *mod-pos-neg*:
$[\!| \ 0 < a; \ \ b < 0 \ |\!] \Longrightarrow a \ mod \ b = snd \ (negateSnd \ (negDivAlg \ (-a) \ (-b)))$
⟨*proof*⟩

a negative, b negative

**lemma** *div-neg-neg*:
$[\!| \ a < 0; \ \ b \leq 0 \ |\!] \Longrightarrow a \ div \ b = fst \ (negateSnd \ (posDivAlg \ (-a) \ (-b)))$
⟨*proof*⟩

**lemma** *mod-neg-neg*:
$[\!| \ a < 0; \ \ b \leq 0 \ |\!] \Longrightarrow a \ mod \ b = snd \ (negateSnd \ (posDivAlg \ (-a) \ (-b)))$
⟨*proof*⟩

Simplify expresions in which div and mod combine numerical constants

**lemma** *divmod-relI*:
$[\![a == b * q + r; \ if \ 0 < b \ then \ 0 \leq r \wedge r < b \ else \ b < r \wedge r \leq 0]\!]$
$\Longrightarrow divmod\text{-}rel \ a \ b \ (q, \ r)$
⟨*proof*⟩

**lemmas** *divmod-rel-div-eq* = *divmod-relI* [*THEN divmod-rel-div*, *THEN eq-reflection*]
**lemmas** *divmod-rel-mod-eq* = *divmod-relI* [*THEN divmod-rel-mod*, *THEN eq-reflection*]
**lemmas** *arithmetic-simps* =
*arith-simps*
*add-special*
*OrderedGroup.add-0-left*
*OrderedGroup.add-0-right*
*mult-zero-left*
*mult-zero-right*
*mult-1-left*
*mult-1-right*

⟨*ML*⟩

**lemmas** *posDivAlg-eqn-number-of* [*simp*] =
*posDivAlg-eqn* [*of number-of v number-of w*, *standard*]

**lemmas** *negDivAlg-eqn-number-of* [*simp*] =
*negDivAlg-eqn* [*of number-of v number-of w*, *standard*]

Special-case simplification

**lemma** *zmod-minus1-right* [*simp*]: *a mod* (−*1*::*int*) = *0*
⟨*proof*⟩

**lemma** *zdiv-minus1-right* [*simp*]: *a div* (−*1*::*int*) = −*a*
⟨*proof*⟩

**lemmas** *div-pos-pos-1-number-of* [*simp*] =
*div-pos-pos* [*OF int-0-less-1*, *of number-of w*, *standard*]

**lemmas** *div-pos-neg-1-number-of* [*simp*] =
*div-pos-neg* [*OF int-0-less-1*, *of number-of w*, *standard*]

**lemmas** *mod-pos-pos-1-number-of* [*simp*] =
*mod-pos-pos* [*OF int-0-less-1*, *of number-of w*, *standard*]

**lemmas** *mod-pos-neg-1-number-of* [*simp*] =
*mod-pos-neg* [*OF int-0-less-1*, *of number-of w*, *standard*]

**lemmas** *posDivAlg-eqn-1-number-of* [*simp*] =
*posDivAlg-eqn* [*of concl*: *1 number-of w*, *standard*]

**lemmas** *negDivAlg-eqn-1-number-of* [*simp*] =
*negDivAlg-eqn* [*of concl*: *1 number-of w*, *standard*]

## 31.9 Monotonicity in the First Argument (Dividend)

**lemma** *zdiv-mono1*: $[|\ a \leq a';\ \ 0 < (b::int)\ |] ==> a\ div\ b \leq a'\ div\ b$
⟨*proof*⟩

**lemma** *zdiv-mono1-neg*: $[|\ a \leq a';\ \ (b::int) < 0\ |] ==> a'\ div\ b \leq a\ div\ b$
⟨*proof*⟩

## 31.10 Monotonicity in the Second Argument (Divisor)

**lemma** *q-pos-lemma*:
$[|\ 0 \leq b'*q' + r';\ r' < b';\ \ 0 < b'\ |] ==> 0 \leq (q'::int)$
⟨*proof*⟩

**lemma** *zdiv-mono2-lemma*:
$[|\ b*q + r = b'*q' + r';\ \ 0 \leq b'*q' + r';$
$r' < b';\ \ 0 \leq r;\ \ 0 < b';\ \ b' \leq b\ |]$
$==> q \leq (q'::int)$
⟨*proof*⟩

**lemma** *zdiv-mono2*:
$[|\ (0::int) \leq a;\ \ 0 < b';\ \ b' \leq b\ |] ==> a\ div\ b \leq a\ div\ b'$
⟨*proof*⟩

**lemma** *q-neg-lemma*:
$[|\ b'*q' + r' < 0;\ \ 0 \leq r';\ \ 0 < b'\ |] ==> q' \leq (0::int)$
⟨*proof*⟩

**lemma** *zdiv-mono2-neg-lemma*:
$[|\ b*q + r = b'*q' + r';\ \ b'*q' + r' < 0;$
$r < b;\ \ 0 \leq r';\ \ 0 < b';\ \ b' \leq b\ |]$
$==> q' \leq (q::int)$
⟨*proof*⟩

**lemma** *zdiv-mono2-neg*:
$[|\ a < (0::int);\ \ 0 < b';\ \ b' \leq b\ |] ==> a\ div\ b' \leq a\ div\ b$
⟨*proof*⟩

## 31.11 More Algebraic Laws for div and mod

proving (a*b) div c = a * (b div c) + a * (b mod c)

**lemma** *zmult1-lemma*:
$[|\ divmod\text{-}rel\ b\ c\ (q,\ r);\ \ c \neq 0\ |]$
$==> divmod\text{-}rel\ (a * b)\ c\ (a*q + a*r\ div\ c,\ a*r\ mod\ c)$
⟨*proof*⟩

**lemma** *zdiv-zmult1-eq*: $(a*b)\ div\ c = a*(b\ div\ c) + a*(b\ mod\ c)\ div\ (c::int)$
⟨*proof*⟩

**lemma** *zmod-zmult1-eq*: $(a*b)\ mod\ c = a*(b\ mod\ c)\ mod\ (c::int)$
⟨*proof*⟩

**lemma** *zmod-zdiv-trivial*: $(a\ mod\ b)\ div\ b = (0::int)$
⟨*proof*⟩

proving (a+b) div c = a div c + b div c + ((a mod c + b mod c) div c)

**lemma** *zadd1-lemma*:
$[|\ divmod\text{-}rel\ a\ c\ (aq,\ ar);\ \ divmod\text{-}rel\ b\ c\ (bq,\ br);\ \ c \neq 0\ |]$
$\Longrightarrow divmod\text{-}rel\ (a+b)\ c\ (aq + bq + (ar+br)\ div\ c,\ (ar+br)\ mod\ c)$
⟨*proof*⟩

**lemma** *zdiv-zadd1-eq*:
$(a+b)\ div\ (c::int) = a\ div\ c + b\ div\ c + ((a\ mod\ c + b\ mod\ c)\ div\ c)$
⟨*proof*⟩

**instance** *int* :: *ring-div*
⟨*proof*⟩

**lemma** *posDivAlg-div-mod*:
**assumes** $k \geq 0$
**and** $l \geq 0$
**shows** $posDivAlg\ k\ l = (k\ div\ l,\ k\ mod\ l)$
⟨*proof*⟩

**lemma** *negDivAlg-div-mod*:
**assumes** $k < 0$
**and** $l > 0$
**shows** $negDivAlg\ k\ l = (k\ div\ l,\ k\ mod\ l)$
⟨*proof*⟩

**lemma** *zmod-eq-0-iff*: $(m\ mod\ d = 0) = (EX\ q::int.\ m = d*q)$
⟨*proof*⟩

**lemmas** *zmod-eq-0D* [*dest*!] = *zmod-eq-0-iff* [*THEN iffD1*]

## 31.12 Proving $a\ div\ (b * c) = a\ div\ b\ div\ c$

first, four lemmas to bound the remainder for the cases b¡0 and b¿0

**lemma** *zmult2-lemma-aux1*: $[|\ (0::int) < c;\ \ b < r;\ \ r \leq 0\ |] \Longrightarrow b*c < b*(q\ mod\ c) + r$
⟨*proof*⟩

**lemma** *zmult2-lemma-aux2*:
$[|\ (0::int) < c;\ \ b < r;\ \ r \leq 0\ |] \Longrightarrow b * (q\ mod\ c) + r \leq 0$
⟨*proof*⟩

**lemma** *zmult2-lemma-aux3*: [| (0::*int*) < *c*; 0 ≤ *r*; *r* < *b* |] ==> 0 ≤ *b* * (*q* *mod* *c*) + *r*
⟨*proof*⟩

**lemma** *zmult2-lemma-aux4*: [| (0::*int*) < *c*; 0 ≤ *r*; *r* < *b* |] ==> *b* * (*q* *mod* *c*) + *r* < *b* * *c*
⟨*proof*⟩

**lemma** *zmult2-lemma*: [| *divmod-rel* *a* *b* (*q*, *r*); *b* ≠ 0; 0 < *c* |]
==> *divmod-rel* *a* (*b* * *c*) (*q* *div* *c*, *b**(*q* *mod* *c*) + *r*)
⟨*proof*⟩

**lemma** *zdiv-zmult2-eq*: (0::*int*) < *c* ==> *a* *div* (*b***c*) = (*a* *div* *b*) *div* *c*
⟨*proof*⟩

**lemma** *zmod-zmult2-eq*:
(0::*int*) < *c* ==> *a* *mod* (*b***c*) = *b**(*a* *div* *b* *mod* *c*) + *a* *mod* *b*
⟨*proof*⟩

## 31.13 Cancellation of Common Factors in div

**lemma** *zdiv-zmult-zmult1-aux1*:
[| (0::*int*) < *b*; *c* ≠ 0 |] ==> (*c***a*) *div* (*c***b*) = *a* *div* *b*
⟨*proof*⟩

**lemma** *zdiv-zmult-zmult1-aux2*:
[| *b* < (0::*int*); *c* ≠ 0 |] ==> (*c***a*) *div* (*c***b*) = *a* *div* *b*
⟨*proof*⟩

**lemma** *zdiv-zmult-zmult1*: *c* ≠ (0::*int*) ==> (*c***a*) *div* (*c***b*) = *a* *div* *b*
⟨*proof*⟩

**lemma** *zdiv-zmult-zmult1-if*[*simp*]:
(*k***m*) *div* (*k***n*) = (*if* *k* = (0::*int*) *then* 0 *else* *m* *div* *n*)
⟨*proof*⟩

## 31.14 Distribution of Factors over mod

**lemma** *zmod-zmult-zmult1-aux1*:
[| (0::*int*) < *b*; *c* ≠ 0 |] ==> (*c***a*) *mod* (*c***b*) = *c* * (*a* *mod* *b*)
⟨*proof*⟩

**lemma** *zmod-zmult-zmult1-aux2*:
[| *b* < (0::*int*); *c* ≠ 0 |] ==> (*c***a*) *mod* (*c***b*) = *c* * (*a* *mod* *b*)
⟨*proof*⟩

**lemma** *zmod-zmult-zmult1*: (*c***a*) *mod* (*c***b*) = (*c*::*int*) * (*a* *mod* *b*)
⟨*proof*⟩

**lemma** *zmod-zmult-zmult2*: (*a***c*) *mod* (*b***c*) = (*a* *mod* *b*) * (*c*::*int*)

⟨*proof*⟩

## 31.15 Splitting Rules for div and mod

The proofs of the two lemmas below are essentially identical

**lemma** *split-pos-lemma*:
$0<k \Longrightarrow$
$P(n\ div\ k :: int)(n\ mod\ k) = (\forall i\ j.\ 0 \leq j\ \&\ j<k\ \&\ n = k*i + j \longrightarrow P\ i\ j)$
⟨*proof*⟩

**lemma** *split-neg-lemma*:
$k<0 \Longrightarrow$
$P(n\ div\ k :: int)(n\ mod\ k) = (\forall i\ j.\ k<j\ \&\ j \leq 0\ \&\ n = k*i + j \longrightarrow P\ i\ j)$
⟨*proof*⟩

**lemma** *split-zdiv*:
$P(n\ div\ k :: int) =$
$((k = 0 \longrightarrow P\ 0)\ \&$
$(0<k \longrightarrow (\forall i\ j.\ 0 \leq j\ \&\ j<k\ \&\ n = k*i + j \longrightarrow P\ i))\ \&$
$(k<0 \longrightarrow (\forall i\ j.\ k<j\ \&\ j \leq 0\ \&\ n = k*i + j \longrightarrow P\ i)))$
⟨*proof*⟩

**lemma** *split-zmod*:
$P(n\ mod\ k :: int) =$
$((k = 0 \longrightarrow P\ n)\ \&$
$(0<k \longrightarrow (\forall i\ j.\ 0 \leq j\ \&\ j<k\ \&\ n = k*i + j \longrightarrow P\ j))\ \&$
$(k<0 \longrightarrow (\forall i\ j.\ k<j\ \&\ j \leq 0\ \&\ n = k*i + j \longrightarrow P\ j)))$
⟨*proof*⟩

**declare** *split-zdiv* [*of - - number-of k*, *simplified*, *standard*, *arith-split*]
**declare** *split-zmod* [*of - - number-of k*, *simplified*, *standard*, *arith-split*]

## 31.16 Speeding up the Division Algorithm with Shifting

computing div by shifting

**lemma** *pos-zdiv-mult-2*: $(0::int) \leq a \Longrightarrow (1 + 2*b)\ div\ (2*a) = b\ div\ a$
⟨*proof*⟩

**lemma** *neg-zdiv-mult-2*: $a \leq (0::int) \Longrightarrow (1 + 2*b)\ div\ (2*a) = (b+1)\ div\ a$
⟨*proof*⟩

**lemma** *zdiv-number-of-Bit0* [*simp*]:
$number\text{-}of\ (Int.Bit0\ v)\ div\ number\text{-}of\ (Int.Bit0\ w) =$
$number\text{-}of\ v\ div\ (number\text{-}of\ w :: int)$
⟨*proof*⟩

**lemma** *zdiv-number-of-Bit1* [*simp*]:

*number-of (Int.Bit1 v) div number-of (Int.Bit0 w) =*
*(if (0::int) ≤ number-of w*
*then number-of v div (number-of w)*
*else (number-of v + (1::int)) div (number-of w))*
⟨*proof*⟩

## 31.17 Computing mod by Shifting (proofs resemble those for div)

**lemma** *pos-zmod-mult-2*:
*(0::int) ≤ a ==> (1 + 2*b) mod (2*a) = 1 + 2 * (b mod a)*
⟨*proof*⟩

**lemma** *neg-zmod-mult-2*:
*a ≤ (0::int) ==> (1 + 2*b) mod (2*a) = 2 * ((b+1) mod a) − 1*
⟨*proof*⟩

**lemma** *zmod-number-of-Bit0* [*simp*]:
*number-of (Int.Bit0 v) mod number-of (Int.Bit0 w) =*
*(2::int) * (number-of v mod number-of w)*
⟨*proof*⟩

**lemma** *zmod-number-of-Bit1* [*simp*]:
*number-of (Int.Bit1 v) mod number-of (Int.Bit0 w) =*
*(if (0::int) ≤ number-of w*
*then 2 * (number-of v mod number-of w) + 1*
*else 2 * ((number-of v + (1::int)) mod number-of w) − 1)*
⟨*proof*⟩

## 31.18 Quotients of Signs

**lemma** *div-neg-pos-less0*: [| *a < (0::int); 0 < b* |] ==> *a div b < 0*
⟨*proof*⟩

**lemma** *div-nonneg-neg-le0*: [| *(0::int) ≤ a; b < 0* |] ==> *a div b ≤ 0*
⟨*proof*⟩

**lemma** *div-nonpos-pos-le0*: [| *(a::int) ≤ 0; b > 0* |] ==> *a div b ≤ 0*
⟨*proof*⟩

**lemma** *pos-imp-zdiv-nonneg-iff*: *(0::int) < b ==> (0 ≤ a div b) = (0 ≤ a)*
⟨*proof*⟩

**lemma** *neg-imp-zdiv-nonneg-iff*:
*b < (0::int) ==> (0 ≤ a div b) = (a ≤ (0::int))*
⟨*proof*⟩

**lemma** *pos-imp-zdiv-neg-iff*: *(0::int) < b ==> (a div b < 0) = (a < 0)*

⟨*proof*⟩

**lemma** *neg-imp-zdiv-neg-iff*: $b < (0::int) \Longrightarrow (a\ div\ b < 0) = (0 < a)$
⟨*proof*⟩

## 31.19 The Divides Relation

**lemmas** *zdvd-iff-zmod-eq-0-number-of* [*simp*] =
*dvd-eq-mod-eq-0* [*of number-of x::int number-of y::int*, *standard*]

**lemma** *zdvd-anti-sym*:
$0 < m \Longrightarrow 0 < n \Longrightarrow m\ dvd\ n \Longrightarrow n\ dvd\ m \Longrightarrow m = (n::int)$
⟨*proof*⟩

**lemma** *zdvd-dvd-eq*: **assumes** $a \neq 0$ **and** $(a::int)\ dvd\ b$ **and** $b\ dvd\ a$
**shows** $|a| = |b|$
⟨*proof*⟩

**lemma** *zdvd-zdiffD*: $k\ dvd\ m - n \Longrightarrow k\ dvd\ n \Longrightarrow k\ dvd\ (m::int)$
⟨*proof*⟩

**lemma** *zdvd-reduce*: $(k\ dvd\ n + k * m) = (k\ dvd\ (n::int))$
⟨*proof*⟩

**lemma** *zdvd-zmod*: $f\ dvd\ m \Longrightarrow f\ dvd\ (n::int) \Longrightarrow f\ dvd\ m\ mod\ n$
⟨*proof*⟩

**lemma** *zdvd-zmod-imp-zdvd*: $k\ dvd\ m\ mod\ n \Longrightarrow k\ dvd\ n \Longrightarrow k\ dvd\ (m::int)$
⟨*proof*⟩

**lemma** *zdvd-not-zless*: $0 < m \Longrightarrow m < n \Longrightarrow \neg\ n\ dvd\ (m::int)$
⟨*proof*⟩

**lemma** *zmult-div-cancel*: $(n::int) * (m\ div\ n) = m - (m\ mod\ n)$
⟨*proof*⟩

**lemma** *zdvd-mult-div-cancel*:$(n::int)\ dvd\ m \Longrightarrow n * (m\ div\ n) = m$
⟨*proof*⟩

**lemma** *zdvd-mult-cancel*: **assumes** $d{:}k * m\ dvd\ k * n$ **and** $kz{:}k \neq (0::int)$
**shows** $m\ dvd\ n$
⟨*proof*⟩

**theorem** *ex-nat*: $(\exists\, x::nat.\ P\ x) = (\exists\, x::int.\ 0 <= x \wedge P\ (nat\ x))$
⟨*proof*⟩

**theorem** *zdvd-int*: $(x\ dvd\ y) = (int\ x\ dvd\ int\ y)$

⟨*proof*⟩

**lemma** *zdvd1-eq*[*simp*]: (*x*::*int*) *dvd* *1* = ( |*x*| = *1*)
⟨*proof*⟩
**lemma** *zdvd-mult-cancel1*:
**assumes** *mp*:*m* ≠(*0*::*int*) **shows** (*m* * *n* *dvd* *m*) = (|*n*| = *1*)
⟨*proof*⟩

**lemma** *int-dvd-iff*: (*int* *m* *dvd* *z*) = (*m* *dvd* *nat* (*abs* *z*))
⟨*proof*⟩

**lemma** *dvd-int-iff*: (*z* *dvd* *int* *m*) = (*nat* (*abs* *z*) *dvd* *m*)
⟨*proof*⟩

**lemma** *nat-dvd-iff*: (*nat* *z* *dvd* *m*) = (*if* *0* ≤ *z* *then* (*z* *dvd* *int* *m*) *else* *m* = *0*)
⟨*proof*⟩

**lemma** *zdvd-imp-le*: [| *z* *dvd* *n*; *0* < *n* |] ==> *z* ≤ (*n*::*int*)
⟨*proof*⟩

**lemma** *zpower-zmod*: ((*x*::*int*) *mod* *m*) ^*y* *mod* *m* = *x* ^*y* *mod* *m*
⟨*proof*⟩

**lemma** *zdiv-int*: *int* (*a* *div* *b*) = (*int* *a*) *div* (*int* *b*)
⟨*proof*⟩

**lemma** *zmod-int*: *int* (*a* *mod* *b*) = (*int* *a*) *mod* (*int* *b*)
⟨*proof*⟩

**lemma** *abs-div*: (*y*::*int*) *dvd* *x* ⟹ *abs* (*x* *div* *y*) = *abs* *x* *div* *abs* *y*
⟨*proof*⟩

Suggested by Matthias Daum

**lemma** *int-power-div-base*:
⟦*0* < *m*; *0* < *k*⟧ ⟹ *k* ^ *m* *div* *k* = (*k*::*int*) ^ (*m* − *Suc* *0*)
⟨*proof*⟩

by Brian Huffman

**lemma** *zminus-zmod*: − ((*x*::*int*) *mod* *m*) *mod* *m* = − *x* *mod* *m*
⟨*proof*⟩

**lemma** *zdiff-zmod-left*: (*x* *mod* *m* − *y*) *mod* *m* = (*x* − *y*) *mod* (*m*::*int*)
⟨*proof*⟩

**lemma** *zdiff-zmod-right*: (*x* − *y* *mod* *m*) *mod* *m* = (*x* − *y*) *mod* (*m*::*int*)
⟨*proof*⟩

**lemmas** *zmod-simps* =
*mod-add-left-eq* [*symmetric*]

*mod-add-right-eq* [*symmetric*]
*IntDiv.zmod-zmult1-eq* [*symmetric*]
*mod-mult-left-eq* [*symmetric*]
*IntDiv.zpower-zmod*
*zminus-zmod zdiff-zmod-left zdiff-zmod-right*

Distributive laws for function *nat*.

**lemma** *nat-div-distrib*: $0 \leq x \Longrightarrow nat\ (x\ div\ y) = nat\ x\ div\ nat\ y$
⟨*proof*⟩

**lemma** *nat-mod-distrib*:
$[\![0 \leq x;\ 0 \leq y]\!] \Longrightarrow nat\ (x\ mod\ y) = nat\ x\ mod\ nat\ y$
⟨*proof*⟩

Suggested by Matthias Daum

**lemma** *int-div-less-self*: $[\![0 < x;\ 1 < k]\!] \Longrightarrow x\ div\ k < (x::int)$
⟨*proof*⟩

code generator setup

**context** *ring-1*
**begin**

**lemma** *of-int-num* [*code*]:
$of\text{-}int\ k = (if\ k = 0\ then\ 0\ else\ if\ k < 0\ then$
$-\ of\text{-}int\ (-\ k)\ else\ let$
$(l,\ m) = divmod\ k\ 2;$
$l' = of\text{-}int\ l$
$in\ if\ m = 0\ then\ l' + l'\ else\ l' + l' + 1)$
⟨*proof*⟩

**end**

**lemma** *zmod-eq-dvd-iff*: $(x::int)\ mod\ n = y\ mod\ n \longleftrightarrow n\ dvd\ x - y$
⟨*proof*⟩

**lemma** *nat-mod-eq-lemma*: **assumes** *xyn*: $(x::nat)\ mod\ n = y\ \ mod\ n$ **and** *xy*:$y \leq x$
**shows** $\exists q.\ x = y + n * q$
⟨*proof*⟩

**lemma** *nat-mod-eq-iff*: $(x::nat)\ mod\ n = y\ mod\ n \longleftrightarrow (\exists q1\ q2.\ x + n * q1 = y + n * q2)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

## 31.20 Simproc setup

⟨*ML*⟩

### 31.21 Code generation

**definition** *pdivmod* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
*pdivmod k l* = (|*k*| *div* |*l*|, |*k*| *mod* |*l*|)

**lemma** *pdivmod-posDivAlg* [*code*]:
*pdivmod k l* = (*if l* = *0 then* (*0*, |*k*|) *else posDivAlg* |*k*| |*l*|)
⟨*proof*⟩

**lemma** *divmod-pdivmod*: *divmod k l* = (*if k* = *0 then* (*0*, *0*) *else if l* = *0 then* (*0*, *k*) *else*
*apsnd* ((*op* ∗) (*sgn l*)) (*if* 0 < *l* ∧ 0 ≤ *k* ∨ *l* < 0 ∧ *k* < 0
*then pdivmod k l*
*else* (*let* (*r*, *s*) = *pdivmod k l in*
*if s* = *0 then* (− *r*, *0*) *else* (− *r* − *1*, |*l*| − *s*))))
⟨*proof*⟩

**lemma** *divmod-code* [*code*]: *divmod k l* = (*if k* = *0 then* (*0*, *0*) *else if l* = *0 then* (*0*, *k*) *else*
*apsnd* ((*op* ∗) (*sgn l*)) (*if sgn k* = *sgn l*
*then pdivmod k l*
*else* (*let* (*r*, *s*) = *pdivmod k l in*
*if s* = *0 then* (− *r*, *0*) *else* (− *r* − *1*, |*l*| − *s*))))
⟨*proof*⟩

**code-modulename** *SML*
*IntDiv Integer*

**code-modulename** *OCaml*
*IntDiv Integer*

**code-modulename** *Haskell*
*IntDiv Integer*

**end**

# 32 NatBin: Binary arithmetic for the natural numbers

**theory** *NatBin*
**imports** *IntDiv*
**uses** (*Tools/nat-simprocs.ML*)
**begin**

Arithmetic for naturals is reduced to that for the non-negative integers.

**instantiation** *nat* :: *number*
**begin**

**definition**
*nat-number-of-def* [*code inline*, *code del*]: *number-of v* = *nat* (*number-of v*)

**instance** ⟨*proof*⟩

**end**

**lemma** [*code post*]:
*nat* (*number-of v*) = *number-of v*
⟨*proof*⟩

**abbreviation** (*xsymbols*)
*power2* :: *'a*::*power* => *'a* ((-²) [*1000*] *999*) **where**
*x*² == *x*^*2*

**notation** (*latex* **output**)
*power2* ((-²) [*1000*] *999*)

**notation** (*HTML* **output**)
*power2* ((-²) [*1000*] *999*)

## 32.1 Predicate for negative binary numbers

**definition** *neg* :: *int* ⇒ *bool* **where**
*neg Z* ⟷ *Z* < *0*

**lemma** *not-neg-int* [*simp*]: ~ *neg* (*of-nat n*)
⟨*proof*⟩

**lemma** *neg-zminus-int* [*simp*]: *neg* (− (*of-nat* (*Suc n*)))
⟨*proof*⟩

**lemmas** *neg-eq-less-0* = *neg-def*

**lemma** *not-neg-eq-ge-0*: (~*neg x*) = (*0* ≤ *x*)
⟨*proof*⟩

To simplify inequalities when Numeral1 can get simplified to 1

**lemma** *not-neg-0*: ~ *neg 0*
⟨*proof*⟩

**lemma** *not-neg-1*: ~ *neg 1*
⟨*proof*⟩

**lemma** *neg-nat*: *neg z* ==> *nat z* = *0*
⟨*proof*⟩

**lemma** *not-neg-nat*: ~ *neg z* ==> *of-nat* (*nat z*) = *z*
⟨*proof*⟩

If *Numeral0* is rewritten to 0 then this rule can't be applied: *Numeral0* IS *number-of Pls*

**lemma** *not-neg-number-of-Pls*: ~ *neg* (*number-of Int.Pls*)
⟨*proof*⟩

**lemma** *neg-number-of-Min*: *neg* (*number-of Int.Min*)
⟨*proof*⟩

**lemma** *neg-number-of-Bit0*:
*neg* (*number-of* (*Int.Bit0 w*)) = *neg* (*number-of w*)
⟨*proof*⟩

**lemma** *neg-number-of-Bit1*:
*neg* (*number-of* (*Int.Bit1 w*)) = *neg* (*number-of w*)
⟨*proof*⟩

**lemmas** *neg-simps* [*simp*] =
*not-neg-0 not-neg-1*
*not-neg-number-of-Pls neg-number-of-Min*
*neg-number-of-Bit0 neg-number-of-Bit1*

## 32.2 Function *nat*: Coercion from Type *int* to *nat*

**declare** *nat-0* [*simp*] *nat-1* [*simp*]

**lemma** *nat-number-of* [*simp*]: *nat* (*number-of w*) = *number-of w*
⟨*proof*⟩

**lemma** *nat-numeral-0-eq-0* [*simp*]: *Numeral0* = (*0*::*nat*)
⟨*proof*⟩

**lemma** *nat-numeral-1-eq-1* [*simp*]: *Numeral1* = (*1*::*nat*)
⟨*proof*⟩

**lemma** *numeral-1-eq-Suc-0*: *Numeral1* = *Suc 0*
⟨*proof*⟩

**lemma** *numeral-2-eq-2*: *2* = *Suc* (*Suc 0*)
⟨*proof*⟩

## 32.3 Function *int*: Coercion from Type *nat* to *int*

**lemma** *int-nat-number-of* [*simp*]:
*int* (*number-of v*) =
(*if neg* (*number-of v* :: *int*) *then 0*
*else* (*number-of v* :: *int*))
⟨*proof*⟩

### 32.3.1 Successor

**lemma** *Suc-nat-eq-nat-zadd1*: $(0::int) <= z ==> Suc\ (nat\ z) = nat\ (1 + z)$
⟨*proof*⟩

**lemma** *Suc-nat-number-of-add*:
$Suc\ (number\text{-}of\ v + n) =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ 1+n\ else\ number\text{-}of\ (Int.succ\ v) + n)$
⟨*proof*⟩

**lemma** *Suc-nat-number-of* [*simp*]:
$Suc\ (number\text{-}of\ v) =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ 1\ else\ number\text{-}of\ (Int.succ\ v))$
⟨*proof*⟩

### 32.3.2 Addition

**lemma** *add-nat-number-of* [*simp*]:
$(number\text{-}of\ v :: nat) + number\text{-}of\ v' =$
$(if\ v < Int.Pls\ then\ number\text{-}of\ v'$
$else\ if\ v' < Int.Pls\ then\ number\text{-}of\ v$
$else\ number\text{-}of\ (v + v'))$
⟨*proof*⟩

**lemma** *nat-number-of-add-1* [*simp*]:
$number\text{-}of\ v + (1::nat) =$
$(if\ v < Int.Pls\ then\ 1\ else\ number\text{-}of\ (Int.succ\ v))$
⟨*proof*⟩

**lemma** *nat-1-add-number-of* [*simp*]:
$(1::nat) + number\text{-}of\ v =$
$(if\ v < Int.Pls\ then\ 1\ else\ number\text{-}of\ (Int.succ\ v))$
⟨*proof*⟩

**lemma** *nat-1-add-1* [*simp*]: $1 + 1 = (2::nat)$
⟨*proof*⟩

### 32.3.3 Subtraction

**lemma** *diff-nat-eq-if*:
$nat\ z - nat\ z' =$
$(if\ neg\ z'\ then\ nat\ z$
$else\ let\ d = z-z'\ in$
$if\ neg\ d\ then\ 0\ else\ nat\ d)$
⟨*proof*⟩

**lemma** *diff-nat-number-of* [*simp*]:
$(number\text{-}of\ v :: nat) - number\text{-}of\ v' =$
$(if\ v' < Int.Pls\ then\ number\text{-}of\ v$

*else let d = number-of (v + uminus v′) in*
*if neg d then 0 else nat d)*
⟨*proof*⟩

**lemma** *nat-number-of-diff-1* [*simp*]:
*number-of v − (1::nat) =*
*(if v ≤ Int.Pls then 0 else number-of (Int.pred v))*
⟨*proof*⟩

### 32.3.4 Multiplication

**lemma** *mult-nat-number-of* [*simp*]:
*(number-of v :: nat) ∗ number-of v′ =*
*(if v < Int.Pls then 0 else number-of (v ∗ v′))*
⟨*proof*⟩

### 32.3.5 Quotient

**lemma** *div-nat-number-of* [*simp*]:
*(number-of v :: nat) div number-of v′ =*
*(if neg (number-of v :: int) then 0*
*else nat (number-of v div number-of v′))*
⟨*proof*⟩

**lemma** *one-div-nat-number-of* [*simp*]:
*Suc 0 div number-of v′ = nat (1 div number-of v′)*
⟨*proof*⟩

### 32.3.6 Remainder

**lemma** *mod-nat-number-of* [*simp*]:
*(number-of v :: nat) mod number-of v′ =*
*(if neg (number-of v :: int) then 0*
*else if neg (number-of v′ :: int) then number-of v*
*else nat (number-of v mod number-of v′))*
⟨*proof*⟩

**lemma** *one-mod-nat-number-of* [*simp*]:
*Suc 0 mod number-of v′ =*
*(if neg (number-of v′ :: int) then Suc 0*
*else nat (1 mod number-of v′))*
⟨*proof*⟩

### 32.3.7 Divisibility

**lemmas** *dvd-eq-mod-eq-0-number-of =*
*dvd-eq-mod-eq-0* [*of number-of x number-of y, standard*]

**declare** *dvd-eq-mod-eq-0-number-of* [*simp*]

⟨*ML*⟩

## 32.4 Comparisons

### 32.4.1 Equals (=)

**lemma** *eq-nat-nat-iff*:
[| (*0*::*int*) <= *z*; *0* <= *z'* |] ==> (*nat z* = *nat z'*) = (*z*=*z'*)
⟨*proof*⟩

**lemma** *eq-nat-number-of* [*simp*]:
((*number-of v* :: *nat*) = *number-of v'*) =
(*if neg* (*number-of v* :: *int*) *then* (*number-of v'* :: *int*) ≤ *0*
*else if neg* (*number-of v'* :: *int*) *then* (*number-of v* :: *int*) = *0*
*else v* = *v'*)
⟨*proof*⟩

### 32.4.2 Less-than (¡)

**lemma** *less-nat-number-of* [*simp*]:
(*number-of v* :: *nat*) < *number-of v'* ⟷
(*if v* < *v' then Int.Pls* < *v' else False*)
⟨*proof*⟩

### 32.4.3 Less-than-or-equal

**lemma** *le-nat-number-of* [*simp*]:
(*number-of v* :: *nat*) ≤ *number-of v'* ⟷
(*if v* ≤ *v' then True else v* ≤ *Int.Pls*)
⟨*proof*⟩

**lemmas** *numerals* = *nat-numeral-0-eq-0 nat-numeral-1-eq-1 numeral-2-eq-2*

## 32.5 Powers with Numeric Exponents

We cannot refer to the number *2*::*'a* in *Ring-and-Field.thy*. We cannot prove general results about the numeral −*1*::*'a*, so we have to use − (*1*::*'a*) instead.

**lemma** *power2-eq-square*: (*a*::*'a*::*recpower*)^2 = *a* * *a*
⟨*proof*⟩

**lemma** *zero-power2* [*simp*]: (*0*::*'a*::{*semiring-1*,*recpower*})^2 = *0*
⟨*proof*⟩

**lemma** *one-power2* [*simp*]: (*1*::*'a*::{*semiring-1*,*recpower*})^2 = *1*
⟨*proof*⟩

**lemma** *power3-eq-cube*: (*x*::*'a*::*recpower*) ^ *3* = *x* * *x* * *x*

⟨*proof*⟩

Squares of literal numerals will be evaluated.

**lemmas** *power2-eq-square-number-of* =
*power2-eq-square* [*of number-of w*, *standard*]
**declare** *power2-eq-square-number-of* [*simp*]

**lemma** *zero-le-power2*[*simp*]: $0 \leq (a^2::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

**lemma** *zero-less-power2*[*simp*]:
$(0 < a^2) = (a \neq (0::'a::\{ordered\text{-}idom,recpower\}))$
⟨*proof*⟩

**lemma** *power2-less-0*[*simp*]:
**fixes** $a :: 'a::\{ordered\text{-}idom,recpower\}$
**shows** $\sim (a^2 < 0)$
⟨*proof*⟩

**lemma** *zero-eq-power2*[*simp*]:
$(a^2 = 0) = (a = (0::'a::\{ordered\text{-}idom,recpower\}))$
⟨*proof*⟩

**lemma** *abs-power2*[*simp*]:
$abs(a^2) = (a^2::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

**lemma** *power2-abs*[*simp*]:
$(abs\ a)^2 = (a^2::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

**lemma** *power2-minus*[*simp*]:
$(-\ a)^2 = (a^2::'a::\{comm\text{-}ring\text{-}1,recpower\})$
⟨*proof*⟩

**lemma** *power2-le-imp-le*:
**fixes** $x\ y :: 'a::\{ordered\text{-}semidom,recpower\}$
**shows** $[\![x^2 \leq y^2;\ 0 \leq y]\!] \Longrightarrow x \leq y$
⟨*proof*⟩

**lemma** *power2-less-imp-less*:
**fixes** $x\ y :: 'a::\{ordered\text{-}semidom,recpower\}$
**shows** $[\![x^2 < y^2;\ 0 \leq y]\!] \Longrightarrow x < y$
⟨*proof*⟩

**lemma** *power2-eq-imp-eq*:
**fixes** $x\ y :: 'a::\{ordered\text{-}semidom,recpower\}$
**shows** $[\![x^2 = y^2;\ 0 \leq x;\ 0 \leq y]\!] \Longrightarrow x = y$

⟨*proof*⟩

**lemma** *power-minus1-even*[*simp*]: $(-1) \hat{} (2*n) = (1::'a::\{comm\text{-}ring\text{-}1,recpower\})$
⟨*proof*⟩

**lemma** *power-minus1-odd*: $(-1) \hat{} Suc(2*n) = -(1::'a::\{comm\text{-}ring\text{-}1,recpower\})$
⟨*proof*⟩

**lemma** *power-even-eq*: $(a::'a::recpower) \hat{} (2*n) = (a \hat{} n) \hat{} 2$
⟨*proof*⟩

**lemma** *power-odd-eq*: $(a::int) \hat{} Suc(2*n) = a * (a \hat{} n) \hat{} 2$
⟨*proof*⟩

**lemma** *power-minus-even* [*simp*]:
$(-a) \hat{} (2*n) = (a::'a::\{comm\text{-}ring\text{-}1,recpower\}) \hat{} (2*n)$
⟨*proof*⟩

**lemma** *zero-le-even-power'*[*simp*]:
$0 \leq (a::'a::\{ordered\text{-}idom,recpower\}) \hat{} (2*n)$
⟨*proof*⟩

**lemma** *odd-power-less-zero*:
$(a::'a::\{ordered\text{-}idom,recpower\}) < 0 \Longrightarrow a \hat{} Suc(2*n) < 0$
⟨*proof*⟩

**lemma** *odd-0-le-power-imp-0-le*:
$0 \leq a \hat{} Suc(2*n) \Longrightarrow 0 \leq (a::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

Simprules for comparisons where common factors can be cancelled.

**lemmas** *zero-compare-simps* =
*add-strict-increasing add-strict-increasing2 add-increasing*
*zero-le-mult-iff zero-le-divide-iff*
*zero-less-mult-iff zero-less-divide-iff*
*mult-le-0-iff divide-le-0-iff*
*mult-less-0-iff divide-less-0-iff*
*zero-le-power2 power2-less-0*

### 32.5.1 Nat

**lemma** *Suc-pred'*: $0 < n \Longrightarrow n = Suc(n - 1)$
⟨*proof*⟩

**lemmas** *expand-Suc* = *Suc-pred'* [*of number-of v*, *standard*]

### 32.5.2 Arith

**lemma** *Suc-eq-add-numeral-1*: $Suc\ n = n + 1$

⟨*proof*⟩

**lemma** *Suc-eq-add-numeral-1-left*: *Suc n = 1 + n*
⟨*proof*⟩

**lemma** *add-eq-if*: *(m::nat) + n = (if m=0 then n else Suc ((m − 1) + n))*
⟨*proof*⟩

**lemma** *mult-eq-if*: *(m::nat) * n = (if m=0 then 0 else n + ((m − 1) * n))*
⟨*proof*⟩

**lemma** *power-eq-if*: *(p ^ m :: nat) = (if m=0 then 1 else p * (p ^ (m − 1)))*
⟨*proof*⟩

## 32.6 Comparisons involving (0::nat)

Simplification already does $n < (0::'a)$, $n \leq (0::'a)$ and $(0::'a) \leq n$.

**lemma** *eq-number-of-0* [*simp*]:
*number-of v = (0::nat) ⟷ v ≤ Int.Pls*
⟨*proof*⟩

**lemma** *eq-0-number-of* [*simp*]:
*(0::nat) = number-of v ⟷ v ≤ Int.Pls*
⟨*proof*⟩

**lemma** *less-0-number-of* [*simp*]:
*(0::nat) < number-of v ⟷ Int.Pls < v*
⟨*proof*⟩

**lemma** *neg-imp-number-of-eq-0*: *neg (number-of v :: int) ==> number-of v = (0::nat)*
⟨*proof*⟩

## 32.7 Comparisons involving *Suc*

**lemma** *eq-number-of-Suc* [*simp*]:
*(number-of v = Suc n) =*
*(let pv = number-of (Int.pred v) in*
*if neg pv then False else nat pv = n)*
⟨*proof*⟩

**lemma** *Suc-eq-number-of* [*simp*]:
*(Suc n = number-of v) =*
*(let pv = number-of (Int.pred v) in*
*if neg pv then False else nat pv = n)*
⟨*proof*⟩

**lemma** *less-number-of-Suc* [*simp*]:
(*number-of v* < *Suc n*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then True else nat pv* < *n*)
⟨*proof*⟩

**lemma** *less-Suc-number-of* [*simp*]:
(*Suc n* < *number-of v*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then False else n* < *nat pv*)
⟨*proof*⟩

**lemma** *le-number-of-Suc* [*simp*]:
(*number-of v* <= *Suc n*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then True else nat pv* <= *n*)
⟨*proof*⟩

**lemma** *le-Suc-number-of* [*simp*]:
(*Suc n* <= *number-of v*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then False else n* <= *nat pv*)
⟨*proof*⟩

**lemma** *eq-number-of-Pls-Min*: (*Numeral0* ::*int*) ~= *number-of Int.Min*
⟨*proof*⟩

## 32.8 Max and Min Combined with *Suc*

**lemma** *max-number-of-Suc* [*simp*]:
*max* (*Suc n*) (*number-of v*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then Suc n else Suc*(*max n* (*nat pv*)))
⟨*proof*⟩

**lemma** *max-Suc-number-of* [*simp*]:
*max* (*number-of v*) (*Suc n*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then Suc n else Suc*(*max* (*nat pv*) *n*))
⟨*proof*⟩

**lemma** *min-number-of-Suc* [*simp*]:
*min* (*Suc n*) (*number-of v*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then 0 else Suc*(*min n* (*nat pv*)))
⟨*proof*⟩

**lemma** *min-Suc-number-of* [*simp*]:

*min (number-of v) (Suc n) =*
*(let pv = number-of (Int.pred v) in*
*if neg pv then 0 else Suc(min (nat pv) n))*
⟨*proof*⟩

## 32.9 Literal arithmetic involving powers

**lemma** *nat-power-eq*: *(0::int) <= z ==> nat (z^n) = nat z ^ n*
⟨*proof*⟩

**lemma** *power-nat-number-of*:
*(number-of v :: nat) ^ n =*
*(if neg (number-of v :: int) then 0^n else nat ((number-of v :: int) ^ n))*
⟨*proof*⟩

**lemmas** *power-nat-number-of-number-of = power-nat-number-of [of - number-of w, standard]*
**declare** *power-nat-number-of-number-of [simp]*

For arbitrary rings

**lemma** *power-number-of-even*:
**fixes** *z :: 'a::{number-ring,recpower}*
**shows** *z ^ number-of (Int.Bit0 w) = (let w = z ^ (number-of w) in w * w)*
⟨*proof*⟩

**lemma** *power-number-of-odd*:
**fixes** *z :: 'a::{number-ring,recpower}*
**shows** *z ^ number-of (Int.Bit1 w) = (if (0::int) <= number-of w*
*then (let w = z ^ (number-of w) in z * w * w) else 1)*
⟨*proof*⟩

**lemmas** *zpower-number-of-even = power-number-of-even* [**where** *'a=int*]
**lemmas** *zpower-number-of-odd = power-number-of-odd* [**where** *'a=int*]

**lemmas** *power-number-of-even-number-of [simp] =*
*power-number-of-even [of number-of v, standard]*

**lemmas** *power-number-of-odd-number-of [simp] =*
*power-number-of-odd [of number-of v, standard]*

⟨*ML*⟩

**declare** *split-div[of - - number-of k, standard, arith-split]*
**declare** *split-mod[of - - number-of k, standard, arith-split]*

**lemma** *nat-number-of-Pls*: *Numeral0* = (*0*::*nat*)
⟨*proof*⟩

**lemma** *nat-number-of-Min*: *number-of Int.Min* = (*0*::*nat*)
⟨*proof*⟩

**lemma** *nat-number-of-Bit0*:
*number-of* (*Int.Bit0 w*) = (*let n*::*nat* = *number-of w in n* + *n*)
⟨*proof*⟩

**lemma** *nat-number-of-Bit1*:
*number-of* (*Int.Bit1 w*) =
(*if neg* (*number-of w* :: *int*) *then 0*
*else let n* = *number-of w in Suc* (*n* + *n*))
⟨*proof*⟩

**lemmas** *nat-number* =
*nat-number-of-Pls nat-number-of-Min*
*nat-number-of-Bit0 nat-number-of-Bit1*

**lemma** *Let-Suc* [*simp*]: *Let* (*Suc n*) *f* == *f* (*Suc n*)
⟨*proof*⟩

**lemma** *power-m1-even*: (−*1*) ^ (*2*∗*n*) = (*1*::*'a*::{*number-ring*,*recpower*})
⟨*proof*⟩

**lemma** *power-m1-odd*: (−*1*) ^ *Suc*(*2*∗*n*) = (−*1*::*'a*::{*number-ring*,*recpower*})
⟨*proof*⟩

## 32.10 Literal arithmetic and *of-nat*

**lemma** *of-nat-double*:
*0* ≤ *x* ==> *of-nat* (*nat* (*2* ∗ *x*)) = *of-nat* (*nat x*) + *of-nat* (*nat x*)
⟨*proof*⟩

**lemma** *nat-numeral-m1-eq-0*: −*1* = (*0*::*nat*)
⟨*proof*⟩

**lemma** *of-nat-number-of-lemma*:
*of-nat* (*number-of v* :: *nat*) =
(*if 0* ≤ (*number-of v* :: *int*)
*then* (*number-of v* :: *'a* :: *number-ring*)
*else 0*)
⟨*proof*⟩

**lemma** *of-nat-number-of-eq* [*simp*]:
*of-nat* (*number-of v* :: *nat*) =
(*if neg* (*number-of v* :: *int*) *then 0*
*else* (*number-of v* :: *'a* :: *number-ring*))

⟨*proof*⟩

## 32.11 Lemmas for the Combination and Cancellation Simprocs

**lemma** *nat-number-of-add-left*:
*number-of v + (number-of v′ + (k::nat)) =*
*(if neg (number-of v :: int) then number-of v′ + k*
*else if neg (number-of v′ :: int) then number-of v + k*
*else number-of (v + v′) + k)*
⟨*proof*⟩

**lemma** *nat-number-of-mult-left*:
*number-of v * (number-of v′ * (k::nat)) =*
*(if v < Int.Pls then 0*
*else number-of (v * v′) * k)*
⟨*proof*⟩

### 32.11.1 For *combine-numerals*

**lemma** *left-add-mult-distrib*: $i*u + (j*u + k) = (i+j)*u + (k::nat)$
⟨*proof*⟩

### 32.11.2 For *cancel-numerals*

**lemma** *nat-diff-add-eq1*:
$j <= (i::nat) ==> ((i*u + m) - (j*u + n)) = (((i-j)*u + m) - n)$
⟨*proof*⟩

**lemma** *nat-diff-add-eq2*:
$i <= (j::nat) ==> ((i*u + m) - (j*u + n)) = (m - ((j-i)*u + n))$
⟨*proof*⟩

**lemma** *nat-eq-add-iff1*:
$j <= (i::nat) ==> (i*u + m = j*u + n) = ((i-j)*u + m = n)$
⟨*proof*⟩

**lemma** *nat-eq-add-iff2*:
$i <= (j::nat) ==> (i*u + m = j*u + n) = (m = (j-i)*u + n)$
⟨*proof*⟩

**lemma** *nat-less-add-iff1*:
$j <= (i::nat) ==> (i*u + m < j*u + n) = ((i-j)*u + m < n)$
⟨*proof*⟩

**lemma** *nat-less-add-iff2*:
$i <= (j::nat) ==> (i*u + m < j*u + n) = (m < (j-i)*u + n)$
⟨*proof*⟩

**lemma** *nat-le-add-iff1*:

$j <= (i::nat) ==> (i*u + m <= j*u + n) = ((i-j)*u + m <= n)$
⟨*proof*⟩

**lemma** *nat-le-add-iff2*:
$i <= (j::nat) ==> (i*u + m <= j*u + n) = (m <= (j-i)*u + n)$
⟨*proof*⟩

### 32.11.3 For *cancel-numeral-factors*

**lemma** *nat-mult-le-cancel1*: $(0::nat) < k ==> (k*m <= k*n) = (m<=n)$
⟨*proof*⟩

**lemma** *nat-mult-less-cancel1*: $(0::nat) < k ==> (k*m < k*n) = (m<n)$
⟨*proof*⟩

**lemma** *nat-mult-eq-cancel1*: $(0::nat) < k ==> (k*m = k*n) = (m=n)$
⟨*proof*⟩

**lemma** *nat-mult-div-cancel1*: $(0::nat) < k ==> (k*m)\ div\ (k*n) = (m\ div\ n)$
⟨*proof*⟩

**lemma** *nat-mult-dvd-cancel-disj*[*simp*]:
$(k*m)\ dvd\ (k*n) = (k=0 \mid m\ dvd\ (n::nat))$
⟨*proof*⟩

**lemma** *nat-mult-dvd-cancel1*: $0 < k \Longrightarrow (k*m)\ dvd\ (k*n::nat) = (m\ dvd\ n)$
⟨*proof*⟩

### 32.11.4 For *cancel-factor*

**lemma** *nat-mult-le-cancel-disj*: $(k*m <= k*n) = ((0::nat) < k --> m<=n)$
⟨*proof*⟩

**lemma** *nat-mult-less-cancel-disj*: $(k*m < k*n) = ((0::nat) < k \ \&\ m<n)$
⟨*proof*⟩

**lemma** *nat-mult-eq-cancel-disj*: $(k*m = k*n) = (k = (0::nat) \mid m=n)$
⟨*proof*⟩

**lemma** *nat-mult-div-cancel-disj*[*simp*]:
$(k*m)\ div\ (k*n) = (if\ k = (0::nat)\ then\ 0\ else\ m\ div\ n)$
⟨*proof*⟩

## 32.12 Simprocs for the Naturals

⟨*ML*⟩

### 32.12.1 For simplifying *Suc m − K* and *K − Suc m*

Where K above is a literal

**lemma** *Suc-diff-eq-diff-pred*: *Numeral0* < *n* ==> *Suc m* − *n* = *m* − (*n* − *Numeral1*)
⟨*proof*⟩

Now just instantiating *n* to *number-of v* does the right simplification, but with some redundant inequality tests.

**lemma** *neg-number-of-pred-iff-0*:
*neg* (*number-of* (*Int.pred v*)::*int*) = (*number-of v* = (*0*::*nat*))
⟨*proof*⟩

No longer required as a simprule because of the *inverse-fold* simproc

**lemma** *Suc-diff-number-of*:
*Int.Pls* < *v* ==>
*Suc m* − (*number-of v*) = *m* − (*number-of* (*Int.pred v*))
⟨*proof*⟩

**lemma** *diff-Suc-eq-diff-pred*: *m* − *Suc n* = (*m* − *1*) − *n*
⟨*proof*⟩

### 32.12.2 For *nat-case* and *nat-rec*

**lemma** *nat-case-number-of* [*simp*]:
*nat-case a f* (*number-of v*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then a else f* (*nat pv*))
⟨*proof*⟩

**lemma** *nat-case-add-eq-if* [*simp*]:
*nat-case a f* ((*number-of v*) + *n*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then nat-case a f n else f* (*nat pv* + *n*))
⟨*proof*⟩

**lemma** *nat-rec-number-of* [*simp*]:
*nat-rec a f* (*number-of v*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then a else f* (*nat pv*) (*nat-rec a f* (*nat pv*)))
⟨*proof*⟩

**lemma** *nat-rec-add-eq-if* [*simp*]:
*nat-rec a f* (*number-of v* + *n*) =
(*let pv* = *number-of* (*Int.pred v*) *in*
*if neg pv then nat-rec a f n*
*else f* (*nat pv* + *n*) (*nat-rec a f* (*nat pv* + *n*)))
⟨*proof*⟩

### 32.12.3 Various Other Lemmas

Evens and Odds, for Mutilated Chess Board

Lemmas for specialist use, NOT as default simprules

**lemma** *nat-mult-2*: $2 * z = (z+z::nat)$
⟨*proof*⟩

**lemma** *nat-mult-2-right*: $z * 2 = (z+z::nat)$
⟨*proof*⟩

Case analysis on $n < (2::'a)$

**lemma** *less-2-cases*: $(n::nat) < 2 \Longrightarrow n = 0 \mid n = Suc\ 0$
⟨*proof*⟩

**lemma** *div2-Suc-Suc* [*simp*]: $Suc(Suc\ m)\ div\ 2 = Suc\ (m\ div\ 2)$
⟨*proof*⟩

**lemma** *add-self-div-2* [*simp*]: $(m + m)\ div\ 2 = (m::nat)$
⟨*proof*⟩

**lemma** *mod2-Suc-Suc* [*simp*]: $Suc(Suc(m))\ mod\ 2 = m\ mod\ 2$
⟨*proof*⟩

**lemma** *mod2-gr-0* [*simp*]: $!!m::nat.\ (0 < m\ mod\ 2) = (m\ mod\ 2 = 1)$
⟨*proof*⟩

Removal of Small Numerals: 0, 1 and (in additive positions) 2

**lemma** *add-2-eq-Suc* [*simp*]: $2 + n = Suc\ (Suc\ n)$
⟨*proof*⟩

**lemma** *add-2-eq-Suc'* [*simp*]: $n + 2 = Suc\ (Suc\ n)$
⟨*proof*⟩

Can be used to eliminate long strings of Sucs, but not by default

**lemma** *Suc3-eq-add-3*: $Suc\ (Suc\ (Suc\ n)) = 3 + n$
⟨*proof*⟩

These lemmas collapse some needless occurrences of Suc: at least three Sucs, since two and fewer are rewritten back to Suc again! We already have some rules to simplify operands smaller than 3.

**lemma** *div-Suc-eq-div-add3* [*simp*]: $m\ div\ (Suc\ (Suc\ (Suc\ n))) = m\ div\ (3+n)$
⟨*proof*⟩

**lemma** *mod-Suc-eq-mod-add3* [*simp*]: $m\ mod\ (Suc\ (Suc\ (Suc\ n))) = m\ mod\ (3+n)$
⟨*proof*⟩

**lemma** *Suc-div-eq-add3-div*: $(Suc\ (Suc\ (Suc\ m)))\ div\ n = (3+m)\ div\ n$
⟨*proof*⟩

**lemma** *Suc-mod-eq-add3-mod*: $(Suc\ (Suc\ (Suc\ m)))\ mod\ n = (3+m)\ mod\ n$
⟨*proof*⟩

**lemmas** *Suc-div-eq-add3-div-number-of* =
*Suc-div-eq-add3-div* [*of* - *number-of v*, *standard*]
**declare** *Suc-div-eq-add3-div-number-of* [*simp*]

**lemmas** *Suc-mod-eq-add3-mod-number-of* =
*Suc-mod-eq-add3-mod* [*of* - *number-of v*, *standard*]
**declare** *Suc-mod-eq-add3-mod-number-of* [*simp*]

**end**

# 33 Groebner-Basis: Semiring normalization and Groebner Bases

**theory** *Groebner-Basis*
**imports** *NatBin*
**uses**
*Tools/Groebner-Basis/misc.ML*
*Tools/Groebner-Basis/normalizer-data.ML*
(*Tools/Groebner-Basis/normalizer.ML*)
(*Tools/Groebner-Basis/groebner.ML*)
**begin**

## 33.1 Semiring normalization

⟨*ML*⟩

**locale** *gb-semiring* =
**fixes** *add mul pwr r0 r1*
**assumes** *add-a*:(*add x* (*add y z*) = *add* (*add x y*) *z*)
**and** *add-c*: *add x y* = *add y x* **and** *add-0*:*add r0 x* = *x*
**and** *mul-a*:*mul x* (*mul y z*) = *mul* (*mul x y*) *z* **and** *mul-c*:*mul x y* = *mul y x*
**and** *mul-1*:*mul r1 x* = *x* **and** *mul-0*:*mul r0 x* = *r0*
**and** *mul-d*:*mul x* (*add y z*) = *add* (*mul x y*) (*mul x z*)
**and** *pwr-0*:*pwr x 0* = *r1* **and** *pwr-Suc*:*pwr x* (*Suc n*) = *mul x* (*pwr x n*)
**begin**

**lemma** *mul-pwr*:*mul* (*pwr x p*) (*pwr x q*) = *pwr x* (*p* + *q*)
⟨*proof*⟩

**lemma** *pwr-mul*: *pwr* (*mul x y*) *q* = *mul* (*pwr x q*) (*pwr y q*)
⟨*proof*⟩

**lemma** *pwr-pwr*: *pwr* (*pwr x p*) *q* = *pwr x* (*p* ∗ *q*)
⟨*proof*⟩

### 33.1.1 Declaring the abstract theory

**lemma** *semiring-ops*:
**shows** *TERM* (*add x y*) **and** *TERM* (*mul x y*) **and** *TERM* (*pwr x n*)
**and** *TERM r0* **and** *TERM r1* ⟨*proof*⟩

**lemma** *semiring-rules*:
*add* (*mul a m*) (*mul b m*) = *mul* (*add a b*) *m*
*add* (*mul a m*) *m* = *mul* (*add a r1*) *m*
*add m* (*mul a m*) = *mul* (*add a r1*) *m*
*add m m* = *mul* (*add r1 r1*) *m*
*add r0 a* = *a*
*add a r0* = *a*
*mul a b* = *mul b a*
*mul* (*add a b*) *c* = *add* (*mul a c*) (*mul b c*)
*mul r0 a* = *r0*
*mul a r0* = *r0*
*mul r1 a* = *a*
*mul a r1* = *a*
*mul* (*mul lx ly*) (*mul rx ry*) = *mul* (*mul lx rx*) (*mul ly ry*)
*mul* (*mul lx ly*) (*mul rx ry*) = *mul lx* (*mul ly* (*mul rx ry*))
*mul* (*mul lx ly*) (*mul rx ry*) = *mul rx* (*mul* (*mul lx ly*) *ry*)
*mul* (*mul lx ly*) *rx* = *mul* (*mul lx rx*) *ly*
*mul* (*mul lx ly*) *rx* = *mul lx* (*mul ly rx*)
*mul lx* (*mul rx ry*) = *mul* (*mul lx rx*) *ry*
*mul lx* (*mul rx ry*) = *mul rx* (*mul lx ry*)
*add* (*add a b*) (*add c d*) = *add* (*add a c*) (*add b d*)
*add* (*add a b*) *c* = *add a* (*add b c*)
*add a* (*add c d*) = *add c* (*add a d*)
*add* (*add a b*) *c* = *add* (*add a c*) *b*
*add a c* = *add c a*
*add a* (*add c d*) = *add* (*add a c*) *d*
*mul* (*pwr x p*) (*pwr x q*) = *pwr x* (*p* + *q*)
*mul x* (*pwr x q*) = *pwr x* (*Suc q*)
*mul* (*pwr x q*) *x* = *pwr x* (*Suc q*)
*mul x x* = *pwr x 2*
*pwr* (*mul x y*) *q* = *mul* (*pwr x q*) (*pwr y q*)
*pwr* (*pwr x p*) *q* = *pwr x* (*p* ∗ *q*)
*pwr x 0* = *r1*
*pwr x 1* = *x*
*mul x* (*add y z*) = *add* (*mul x y*) (*mul x z*)
*pwr x* (*Suc q*) = *mul x* (*pwr x q*)
*pwr x* (*2*∗*n*) = *mul* (*pwr x n*) (*pwr x n*)
*pwr x* (*Suc* (*2*∗*n*)) = *mul x* (*mul* (*pwr x n*) (*pwr x n*))
⟨*proof*⟩

**lemmas** *gb-semiring-axioms'* =
*gb-semiring-axioms* [*normalizer*
*semiring ops*: *semiring-ops*

*semiring rules*: *semiring-rules*]

**end**

**interpretation** *class-semiring*: *gb-semiring*
*op + op * op ^ 0*::*'a*::{*comm-semiring-1*, *recpower*} *1*
⟨*proof*⟩

**lemmas** *nat-arith* =
*add-nat-number-of*
*diff-nat-number-of*
*mult-nat-number-of*
*eq-nat-number-of*
*less-nat-number-of*

**lemma** *not-iszero-Numeral1*: ¬ *iszero* (*Numeral1*::*'a*::*number-ring*)
⟨*proof*⟩

**lemmas** *comp-arith* =
*Let-def arith-simps nat-arith rel-simps neg-simps if-False*
*if-True add-0 add-Suc add-number-of-left mult-number-of-left*
*numeral-1-eq-1*[*symmetric*] *Suc-eq-add-numeral-1*
*numeral-0-eq-0*[*symmetric*] *numerals*[*symmetric*]
*iszero-simps not-iszero-Numeral1*

**lemmas** *semiring-norm* = *comp-arith*

⟨*ML*⟩

**locale** *gb-ring* = *gb-semiring* +
**fixes** *sub* :: *'a* ⇒ *'a* ⇒ *'a*
**and** *neg* :: *'a* ⇒ *'a*
**assumes** *neg-mul*: *neg x* = *mul* (*neg r1*) *x*
**and** *sub-add*: *sub x y* = *add x* (*neg y*)
**begin**

**lemma** *ring-ops*: **shows** *TERM* (*sub x y*) **and** *TERM* (*neg x*) ⟨*proof*⟩

**lemmas** *ring-rules* = *neg-mul sub-add*

**lemmas** *gb-ring-axioms'* =
*gb-ring-axioms* [*normalizer*
*semiring ops*: *semiring-ops*
*semiring rules*: *semiring-rules*
*ring ops*: *ring-ops*
*ring rules*: *ring-rules*]

**end**

**interpretation** *class-ring*: *gb-ring op + op * op ^*
*0::'a::{comm-semiring-1,recpower,number-ring} 1 op − uminus*
⟨*proof*⟩

⟨*ML*⟩

**locale** *gb-field* = *gb-ring* +
**fixes** *divide* :: *'a ⇒ 'a ⇒ 'a*
**and** *inverse*:: *'a ⇒ 'a*
**assumes** *divide-inverse*: *divide x y* = *mul x* (*inverse y*)
**and** *inverse-divide*: *inverse x* = *divide r1 x*
**begin**

**lemma** *field-ops*: **shows** *TERM* (*divide x y*) **and** *TERM* (*inverse x*) ⟨*proof*⟩

**lemmas** *field-rules* = *divide-inverse inverse-divide*

**lemmas** *gb-field-axioms'* =
*gb-field-axioms* [*normalizer*
*semiring ops*: *semiring-ops*
*semiring rules*: *semiring-rules*
*ring ops*: *ring-ops*
*ring rules*: *ring-rules*
*field ops*: *field-ops*
*field rules*: *field-rules*]

**end**

## 33.2 Groebner Bases

**locale** *semiringb* = *gb-semiring* +
**assumes** *add-cancel*: *add* (*x::'a*) *y* = *add x z* ⟷ *y* = *z*
**and** *add-mul-solve*: *add* (*mul w y*) (*mul x z*) =
*add* (*mul w z*) (*mul x y*) ⟷ *w* = *x* ∨ *y* = *z*
**begin**

**lemma** *noteq-reduce*: *a* ≠ *b* ∧ *c* ≠ *d* ⟷ *add* (*mul a c*) (*mul b d*) ≠ *add* (*mul a d*) (*mul b c*)
⟨*proof*⟩

**lemma** *add-scale-eq-noteq*: ⟦*r* ≠ *r0* ; (*a* = *b*) ∧ ~(*c* = *d*)⟧
⟹ *add a* (*mul r c*) ≠ *add b* (*mul r d*)
⟨*proof*⟩

**lemma** *add-r0-iff*: *x* = *add x a* ⟷ *a* = *r0*

⟨*proof*⟩

**declare** *gb-semiring-axioms'* [*normalizer del*]

**lemmas** *semiringb-axioms'* = *semiringb-axioms* [*normalizer*
*semiring ops*: *semiring-ops*
*semiring rules*: *semiring-rules*
*idom rules*: *noteq-reduce add-scale-eq-noteq*]

**end**

**locale** *ringb* = *semiringb* + *gb-ring* +
**assumes** *subr0-iff*: *sub x y* = *r0* ⟷ *x* = *y*
**begin**

**declare** *gb-ring-axioms'* [*normalizer del*]

**lemmas** *ringb-axioms'* = *ringb-axioms* [*normalizer*
*semiring ops*: *semiring-ops*
*semiring rules*: *semiring-rules*
*ring ops*: *ring-ops*
*ring rules*: *ring-rules*
*idom rules*: *noteq-reduce add-scale-eq-noteq*
*ideal rules*: *subr0-iff add-r0-iff*]

**end**

**lemma** *no-zero-divirors-neq0*:
**assumes** *az*: (*a*::'*a*::*no-zero-divisors*) ≠ *0*
**and** *ab*: *a*∗*b* = *0* **shows** *b* = *0*
⟨*proof*⟩

**interpretation** *class-ringb*: *ringb*
*op* + *op* ∗ *op* ^ *0*::'*a*::{*idom*,*recpower*,*number-ring*} *1 op* − *uminus*
⟨*proof*⟩

⟨*ML*⟩

**interpretation** *natgb*: *semiringb*
*op* + *op* ∗ *op* ^ *0*::*nat 1*
⟨*proof*⟩

⟨*ML*⟩

**locale** *fieldgb* = *ringb* + *gb-field*
**begin**

**declare** *gb-field-axioms'* [*normalizer del*]

**lemmas** *fieldgb-axioms'* = *fieldgb-axioms* [*normalizer*
*semiring ops*: *semiring-ops*
*semiring rules*: *semiring-rules*
*ring ops*: *ring-ops*
*ring rules*: *ring-rules*
*field ops*: *field-ops*
*field rules*: *field-rules*
*idom rules*: *noteq-reduce add-scale-eq-noteq*
*ideal rules*: *subr0-iff add-r0-iff*]

**end**

**lemmas** *bool-simps* = *simp-thms*(1−34)
**lemma** *dnf*:
$(P \,\&\, (Q \mid R)) = ((P\&Q) \mid (P\&R))$ $((Q \mid R) \,\&\, P) = ((Q\&P) \mid (R\&P))$
$(P \wedge Q) = (Q \wedge P)$ $(P \vee Q) = (Q \vee P)$
⟨*proof*⟩

**lemmas** *weak-dnf-simps* = *dnf bool-simps*

**lemma** *nnf-simps*:
$(\neg(P \wedge Q)) = (\neg P \vee \neg Q)$ $(\neg(P \vee Q)) = (\neg P \wedge \neg Q)$ $(P \longrightarrow Q) = (\neg P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg P \wedge \neg\, Q))$ $(\neg\, \neg(P)) = P$
⟨*proof*⟩

**lemma** *PFalse*:
$P \equiv \mathit{False} \Longrightarrow \neg\, P$
$\neg\, P \Longrightarrow (P \equiv \mathit{False})$
⟨*proof*⟩
⟨*ML*⟩
**declare** *dvd-def*[*algebra*]
**declare** *dvd-eq-mod-eq-0*[*symmetric*, *algebra*]
**declare** *mod-div-trivial*[*algebra*]
**declare** *mod-mod-trivial*[*algebra*]
**declare** *conjunct1*[*OF DIVISION-BY-ZERO*, *algebra*]
**declare** *conjunct2*[*OF DIVISION-BY-ZERO*, *algebra*]
**declare** *zmod-zdiv-equality*[*symmetric*,*algebra*]
**declare** *zdiv-zmod-equality*[*symmetric*, *algebra*]
**declare** *zdiv-zminus-zminus*[*algebra*]
**declare** *zmod-zminus-zminus*[*algebra*]
**declare** *zdiv-zminus2*[*algebra*]
**declare** *zmod-zminus2*[*algebra*]
**declare** *zdiv-zero*[*algebra*]
**declare** *zmod-zero*[*algebra*]
**declare** *mod-by-1*[*algebra*]
**declare** *div-by-1*[*algebra*]
**declare** *zmod-minus1-right*[*algebra*]

**declare** *zdiv-minus1-right*[*algebra*]
**declare** *mod-div-trivial*[*algebra*]
**declare** *mod-mod-trivial*[*algebra*]
**declare** *mod-mult-self2-is-0*[*algebra*]
**declare** *mod-mult-self1-is-0*[*algebra*]
**declare** *zmod-eq-0-iff*[*algebra*]
**declare** *dvd-0-left-iff*[*algebra*]
**declare** *zdvd1-eq*[*algebra*]
**declare** *zmod-eq-dvd-iff*[*algebra*]
**declare** *nat-mod-eq-iff*[*algebra*]

### 33.3 Groebner Bases for fields

**interpretation** *class-fieldgb*:
*fieldgb op + op * op ^ 0::'a::{field,recpower,number-ring} 1 op − uminus op / inverse* ⟨*proof*⟩

**lemma** *divide-Numeral1*: (*x*::'*a*::{*field*,*number-ring*}) / *Numeral1* = *x* ⟨*proof*⟩
**lemma** *divide-Numeral0*: (*x*::'*a*::{*field*,*number-ring*, *division-by-zero*}) / *Numeral0* = *0*
⟨*proof*⟩
**lemma** *mult-frac-frac*: ((*x*::'*a*::{*field*,*division-by-zero*}) / *y*) * (*z* / *w*) = (*x**z*) / (*y***w*)
⟨*proof*⟩
**lemma** *mult-frac-num*: ((*x*::'*a*::{*field*, *division-by-zero*}) / *y*) * *z* = (*x***z*) / *y*
⟨*proof*⟩
**lemma** *mult-num-frac*: ((*x*::'*a*::{*field*, *division-by-zero*}) / *y*) * *z* = (*x***z*) / *y*
⟨*proof*⟩

**lemma** *Numeral1-eq1-nat*: (*1*::*nat*) = *Numeral1* ⟨*proof*⟩

**lemma** *add-frac-num*: *y*≠ *0* ⟹ (*x*::'*a*::{*field*, *division-by-zero*}) / *y* + *z* = (*x* + *z***y*) / *y*
⟨*proof*⟩
**lemma** *add-num-frac*: *y*≠ *0* ⟹ *z* + (*x*::'*a*::{*field*, *division-by-zero*}) / *y* = (*x* + *z***y*) / *y*
⟨*proof*⟩
⟨*ML*⟩

**end**

## 34 SetInterval: Set intervals

**theory** *SetInterval*
**imports** *Int*
**begin**

**context** *ord*

**begin**
**definition**
*lessThan* :: *'a => 'a set* ((*1*{..<-})) **where**
{..<*u*} == {*x*. *x* < *u*}

**definition**
*atMost* :: *'a => 'a set* ((*1*{..-})) **where**
{..*u*} == {*x*. *x* ≤ *u*}

**definition**
*greaterThan* :: *'a => 'a set* ((*1*{-<..})) **where**
{*l*<..} == {*x*. *l*<*x*}

**definition**
*atLeast* :: *'a => 'a set* ((*1*{-..})) **where**
{*l*..} == {*x*. *l*≤*x*}

**definition**
*greaterThanLessThan* :: *'a => 'a => 'a set* ((*1*{-<..<-})) **where**
{*l*<..<*u*} == {*l*<..} *Int* {..<*u*}

**definition**
*atLeastLessThan* :: *'a => 'a => 'a set* ((*1*{-..<-})) **where**
{*l*..<*u*} == {*l*..} *Int* {..<*u*}

**definition**
*greaterThanAtMost* :: *'a => 'a => 'a set* ((*1*{-<..-})) **where**
{*l*<..*u*} == {*l*<..} *Int* {..*u*}

**definition**
*atLeastAtMost* :: *'a => 'a => 'a set* ((*1*{-..-})) **where**
{*l*..*u*} == {*l*..} *Int* {..*u*}

**end**

A note of warning when using {..<*n*} on type *nat*: it is equivalent to {*0*..<*n*} but some lemmas involving {*m*..<*n*} may not exist in {..<*n*}-form as well.

**syntax**
@*UNION-le* :: *'a => 'a => 'b set => 'b set* ((*3UN* -<=-./ -) *10*)
@*UNION-less* :: *'a => 'a => 'b set => 'b set* ((*3UN* -<-./ -) *10*)
@*INTER-le* :: *'a => 'a => 'b set => 'b set* ((*3INT* -<=-./ -) *10*)
@*INTER-less* :: *'a => 'a => 'b set => 'b set* ((*3INT* -<-./ -) *10*)

**syntax** (*xsymbols*)
@*UNION-le* :: *'a => 'a => 'b set => 'b set* ((*3*⋃ -≤-./ -) *10*)
@*UNION-less* :: *'a => 'a => 'b set => 'b set* ((*3*⋃ -<-./ -) *10*)
@*INTER-le* :: *'a => 'a => 'b set => 'b set* ((*3*⋂ -≤-./ -) *10*)
@*INTER-less* :: *'a => 'a => 'b set => 'b set* ((*3*⋂ -<-./ -) *10*)

**syntax** (*latex* **output**)
  @*UNION-le* :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋃(*00-* ≤ *-*)/ *-*) *10*)
  @*UNION-less* :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋃(*00-* < *-*)/ *-*) *10*)
  @*INTER-le* :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋂(*00-* ≤ *-*)/ *-*) *10*)
  @*INTER-less* :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋂(*00-* < *-*)/ *-*) *10*)

**translations**
  *UN i<=n. A* == *UN i:{..n}. A*
  *UN i<n. A* == *UN i:{..<n}. A*
  *INT i<=n. A* == *INT i:{..n}. A*
  *INT i<n. A* == *INT i:{..<n}. A*

## 34.1 Various equivalences

**lemma** (**in** *ord*) *lessThan-iff* [*iff*]: (*i*: *lessThan k*) = (*i*<*k*)
⟨*proof*⟩

**lemma** *Compl-lessThan* [*simp*]:
  !!*k*:: *'a*::*linorder*. −*lessThan k* = *atLeast k*
⟨*proof*⟩

**lemma** *single-Diff-lessThan* [*simp*]: !!*k*:: *'a*::*order*. {*k*} − *lessThan k* = {*k*}
⟨*proof*⟩

**lemma** (**in** *ord*) *greaterThan-iff* [*iff*]: (*i*: *greaterThan k*) = (*k*<*i*)
⟨*proof*⟩

**lemma** *Compl-greaterThan* [*simp*]:
  !!*k*:: *'a*::*linorder*. −*greaterThan k* = *atMost k*
⟨*proof*⟩

**lemma** *Compl-atMost* [*simp*]: !!*k*:: *'a*::*linorder*. −*atMost k* = *greaterThan k*
⟨*proof*⟩

**lemma** (**in** *ord*) *atLeast-iff* [*iff*]: (*i*: *atLeast k*) = (*k*<=*i*)
⟨*proof*⟩

**lemma** *Compl-atLeast* [*simp*]:
  !!*k*:: *'a*::*linorder*. −*atLeast k* = *lessThan k*
⟨*proof*⟩

**lemma** (**in** *ord*) *atMost-iff* [*iff*]: (*i*: *atMost k*) = (*i*<=*k*)
⟨*proof*⟩

**lemma** *atMost-Int-atLeast*: !!*n*:: *'a*::*order*. *atMost n Int atLeast n* = {*n*}
⟨*proof*⟩

## 34.2 Logical Equivalences for Set Inclusion and Equality

**lemma** *atLeast-subset-iff* [*iff*]:

(*atLeast x* ⊆ *atLeast y*) = (*y* ≤ (*x*::'*a*::*order*))
⟨*proof*⟩

**lemma** *atLeast-eq-iff* [*iff*]:
(*atLeast x* = *atLeast y*) = (*x* = (*y*::'*a*::*linorder*))
⟨*proof*⟩

**lemma** *greaterThan-subset-iff* [*iff*]:
(*greaterThan x* ⊆ *greaterThan y*) = (*y* ≤ (*x*::'*a*::*linorder*))
⟨*proof*⟩

**lemma** *greaterThan-eq-iff* [*iff*]:
(*greaterThan x* = *greaterThan y*) = (*x* = (*y*::'*a*::*linorder*))
⟨*proof*⟩

**lemma** *atMost-subset-iff* [*iff*]: (*atMost x* ⊆ *atMost y*) = (*x* ≤ (*y*::'*a*::*order*))
⟨*proof*⟩

**lemma** *atMost-eq-iff* [*iff*]: (*atMost x* = *atMost y*) = (*x* = (*y*::'*a*::*linorder*))
⟨*proof*⟩

**lemma** *lessThan-subset-iff* [*iff*]:
(*lessThan x* ⊆ *lessThan y*) = (*x* ≤ (*y*::'*a*::*linorder*))
⟨*proof*⟩

**lemma** *lessThan-eq-iff* [*iff*]:
(*lessThan x* = *lessThan y*) = (*x* = (*y*::'*a*::*linorder*))
⟨*proof*⟩

## 34.3 Two-sided intervals

**context** *ord*
**begin**

**lemma** *greaterThanLessThan-iff* [*simp*,*noatp*]:
(*i* : {*l*<..<*u*}) = (*l* < *i* & *i* < *u*)
⟨*proof*⟩

**lemma** *atLeastLessThan-iff* [*simp*,*noatp*]:
(*i* : {*l*..<*u*}) = (*l* <= *i* & *i* < *u*)
⟨*proof*⟩

**lemma** *greaterThanAtMost-iff* [*simp*,*noatp*]:
(*i* : {*l*<..*u*}) = (*l* < *i* & *i* <= *u*)
⟨*proof*⟩

**lemma** *atLeastAtMost-iff* [*simp*,*noatp*]:
(*i* : {*l*..*u*}) = (*l* <= *i* & *i* <= *u*)
⟨*proof*⟩

The above four lemmas could be declared as iffs. If we do so, a call to blast in Hyperreal/Star.ML, lemma *STAR-Int* seems to take forever (more than one hour).

**end**

### 34.3.1 Emptyness and singletons

**context** *order*
**begin**

**lemma** *atLeastAtMost-empty* [*simp*]: $n < m \Longrightarrow \{m..n\} = \{\}$
⟨*proof*⟩

**lemma** *atLeastLessThan-empty*[*simp*]: $n \leq m \Longrightarrow \{m..<n\} = \{\}$
⟨*proof*⟩

**lemma** *greaterThanAtMost-empty*[*simp*]:$l \leq k \Longrightarrow \{k<..l\} = \{\}$
⟨*proof*⟩

**lemma** *greaterThanLessThan-empty*[*simp*]:$l \leq k \Longrightarrow \{k<..<l\} = \{\}$
⟨*proof*⟩

**lemma** *atLeastAtMost-singleton* [*simp*]: $\{a..a\} = \{a\}$
⟨*proof*⟩

**end**

## 34.4 Intervals of natural numbers

### 34.4.1 The Constant *lessThan*

**lemma** *lessThan-0* [*simp*]: *lessThan* (*0*::*nat*) = {}
⟨*proof*⟩

**lemma** *lessThan-Suc*: *lessThan* (*Suc k*) = *insert k* (*lessThan k*)
⟨*proof*⟩

**lemma** *lessThan-Suc-atMost*: *lessThan* (*Suc k*) = *atMost k*
⟨*proof*⟩

**lemma** *UN-lessThan-UNIV*: (*UN m*::*nat*. *lessThan m*) = *UNIV*
⟨*proof*⟩

### 34.4.2 The Constant *greaterThan*

**lemma** *greaterThan-0* [*simp*]: *greaterThan 0* = *range Suc*
⟨*proof*⟩

**lemma** *greaterThan-Suc*: *greaterThan* (*Suc k*) = *greaterThan k* − {*Suc k*}

⟨*proof*⟩

**lemma** *INT-greaterThan-UNIV*: (*INT m*::*nat*. *greaterThan m*) = {}
⟨*proof*⟩

### 34.4.3 The Constant *atLeast*

**lemma** *atLeast-0* [*simp*]: *atLeast* (*0*::*nat*) = *UNIV*
⟨*proof*⟩

**lemma** *atLeast-Suc*: *atLeast* (*Suc k*) = *atLeast k* − {*k*}
⟨*proof*⟩

**lemma** *atLeast-Suc-greaterThan*: *atLeast* (*Suc k*) = *greaterThan k*
⟨*proof*⟩

**lemma** *UN-atLeast-UNIV*: (*UN m*::*nat*. *atLeast m*) = *UNIV*
⟨*proof*⟩

### 34.4.4 The Constant *atMost*

**lemma** *atMost-0* [*simp*]: *atMost* (*0*::*nat*) = {*0*}
⟨*proof*⟩

**lemma** *atMost-Suc*: *atMost* (*Suc k*) = *insert* (*Suc k*) (*atMost k*)
⟨*proof*⟩

**lemma** *UN-atMost-UNIV*: (*UN m*::*nat*. *atMost m*) = *UNIV*
⟨*proof*⟩

### 34.4.5 The Constant *atLeastLessThan*

The orientation of the following 2 rules is tricky. The lhs is defined in terms of the rhs. Hence the chosen orientation makes sense in this theory — the reverse orientation complicates proofs (eg nontermination). But outside, when the definition of the lhs is rarely used, the opposite orientation seems preferable because it reduces a specific concept to a more general one.

**lemma** *atLeast0LessThan*: {*0*::*nat*..<*n*} = {..<*n*}
⟨*proof*⟩

**lemma** *atLeast0AtMost*: {*0*..*n*::*nat*} = {..*n*}
⟨*proof*⟩

**declare** *atLeast0LessThan*[*symmetric*, *code unfold*]
*atLeast0AtMost*[*symmetric*, *code unfold*]

**lemma** *atLeastLessThan0*: {*m*..<*0*::*nat*} = {}
⟨*proof*⟩

### 34.4.6 Intervals of nats with *Suc*

Not a simprule because the RHS is too messy.

**lemma** *atLeastLessThanSuc*:
$\{m..<Suc\ n\} = (if\ m \leq n\ then\ insert\ n\ \{m..<n\}\ else\ \{\})$
⟨*proof*⟩

**lemma** *atLeastLessThan-singleton* [*simp*]: $\{m..<Suc\ m\} = \{m\}$
⟨*proof*⟩

**lemma** *atLeastLessThanSuc-atLeastAtMost*: $\{l..<Suc\ u\} = \{l..u\}$
⟨*proof*⟩

**lemma** *atLeastSucAtMost-greaterThanAtMost*: $\{Suc\ l..u\} = \{l<..u\}$
⟨*proof*⟩

**lemma** *atLeastSucLessThan-greaterThanLessThan*: $\{Suc\ l..<u\} = \{l<..<u\}$
⟨*proof*⟩

**lemma** *atLeastAtMostSuc-conv*: $m \leq Suc\ n \Longrightarrow \{m..Suc\ n\} = insert\ (Suc\ n)\ \{m..n\}$
⟨*proof*⟩

### 34.4.7 Image

**lemma** *image-add-atLeastAtMost*:
$(\%n{::}nat.\ n+k)\ `\ \{i..j\} = \{i+k..j+k\}$ (**is** *?A* = *?B*)
⟨*proof*⟩

**lemma** *image-add-atLeastLessThan*:
$(\%n{::}nat.\ n+k)\ `\ \{i..<j\} = \{i+k..<j+k\}$ (**is** *?A* = *?B*)
⟨*proof*⟩

**corollary** *image-Suc-atLeastAtMost*[*simp*]:
$Suc\ `\ \{i..j\} = \{Suc\ i..Suc\ j\}$
⟨*proof*⟩

**corollary** *image-Suc-atLeastLessThan*[*simp*]:
$Suc\ `\ \{i..<j\} = \{Suc\ i..<Suc\ j\}$
⟨*proof*⟩

**lemma** *image-add-int-atLeastLessThan*:
$(\%x.\ x + (l{::}int))\ `\ \{0..<u-l\} = \{l..<u\}$
⟨*proof*⟩

### 34.4.8 Finiteness

**lemma** *finite-lessThan* [*iff*]: **fixes** $k :: nat$ **shows** *finite* $\{..<k\}$
⟨*proof*⟩

**lemma** *finite-atMost* [*iff*]: **fixes** $k$ :: *nat* **shows** *finite* $\{..k\}$
⟨*proof*⟩

**lemma** *finite-greaterThanLessThan* [*iff*]:
**fixes** $l$ :: *nat* **shows** *finite* $\{l<..<u\}$
⟨*proof*⟩

**lemma** *finite-atLeastLessThan* [*iff*]:
**fixes** $l$ :: *nat* **shows** *finite* $\{l..<u\}$
⟨*proof*⟩

**lemma** *finite-greaterThanAtMost* [*iff*]:
**fixes** $l$ :: *nat* **shows** *finite* $\{l<..u\}$
⟨*proof*⟩

**lemma** *finite-atLeastAtMost* [*iff*]:
**fixes** $l$ :: *nat* **shows** *finite* $\{l..u\}$
⟨*proof*⟩

A bounded set of natural numbers is finite.

**lemma** *bounded-nat-set-is-finite*:
(*ALL i:N. i* < (*n*::*nat*)) ==> *finite N*
⟨*proof*⟩

**lemma** *finite-less-ub*:
!!*f*::*nat*=>*nat*. (!!*n*. *n* ≤ *f n*) ==> *finite* {*n*. *f n* ≤ *u*}
⟨*proof*⟩

Any subset of an interval of natural numbers the size of the subset is exactly that interval.

**lemma** *subset-card-intvl-is-intvl*:
*A* <= {*k*..<*k*+*card A*} ⟹ *A* = {*k*..<*k*+*card A*} (**is** *PROP ?P*)
⟨*proof*⟩

### 34.4.9 Cardinality

**lemma** *card-lessThan* [*simp*]: *card* $\{..<u\} = u$
⟨*proof*⟩

**lemma** *card-atMost* [*simp*]: *card* $\{..u\} = Suc\ u$
⟨*proof*⟩

**lemma** *card-atLeastLessThan* [*simp*]: *card* $\{l..<u\} = u - l$
⟨*proof*⟩

**lemma** *card-atLeastAtMost* [*simp*]: *card* $\{l..u\} = Suc\ u - l$
⟨*proof*⟩

**lemma** *card-greaterThanAtMost* [*simp*]: *card* {*l*<..*u*} = *u* − *l*
⟨*proof*⟩

**lemma** *card-greaterThanLessThan* [*simp*]: *card* {*l*<..<*u*} = *u* − *Suc l*
⟨*proof*⟩

**lemma** *ex-bij-betw-nat-finite*:
*finite M* ⟹ ∃ *h*. *bij-betw h* {*0*..<*card M*} *M*
⟨*proof*⟩

**lemma** *ex-bij-betw-finite-nat*:
*finite M* ⟹ ∃ *h*. *bij-betw h M* {*0*..<*card M*}
⟨*proof*⟩

## 34.5 Intervals of integers

**lemma** *atLeastLessThanPlusOne-atLeastAtMost-int*: {*l*..<*u*+*1*} = {*l*..(*u*::*int*)}
⟨*proof*⟩

**lemma** *atLeastPlusOneAtMost-greaterThanAtMost-int*: {*l*+*1*..*u*} = {*l*<..(*u*::*int*)}
⟨*proof*⟩

**lemma** *atLeastPlusOneLessThan-greaterThanLessThan-int*:
{*l*+*1*..<*u*} = {*l*<..<*u*::*int*}
⟨*proof*⟩

### 34.5.1 Finiteness

**lemma** *image-atLeastZeroLessThan-int*: *0* ≤ *u* ==>
{(*0*::*int*)..<*u*} = *int* ‘ {..<*nat u*}
⟨*proof*⟩

**lemma** *finite-atLeastZeroLessThan-int*: *finite* {(*0*::*int*)..<*u*}
⟨*proof*⟩

**lemma** *finite-atLeastLessThan-int* [*iff*]: *finite* {*l*..<*u*::*int*}
⟨*proof*⟩

**lemma** *finite-atLeastAtMost-int* [*iff*]: *finite* {*l*..(*u*::*int*)}
⟨*proof*⟩

**lemma** *finite-greaterThanAtMost-int* [*iff*]: *finite* {*l*<..(*u*::*int*)}
⟨*proof*⟩

**lemma** *finite-greaterThanLessThan-int* [*iff*]: *finite* {*l*<..<*u*::*int*}
⟨*proof*⟩

### 34.5.2 Cardinality

**lemma** *card-atLeastZeroLessThan-int*: *card* $\{(0::int)..<u\} = nat\ u$
⟨*proof*⟩

**lemma** *card-atLeastLessThan-int* [*simp*]: *card* $\{l..<u\} = nat\ (u - l)$
⟨*proof*⟩

**lemma** *card-atLeastAtMost-int* [*simp*]: *card* $\{l..u\} = nat\ (u - l + 1)$
⟨*proof*⟩

**lemma** *card-greaterThanAtMost-int* [*simp*]: *card* $\{l<..u\} = nat\ (u - l)$
⟨*proof*⟩

**lemma** *card-greaterThanLessThan-int* [*simp*]: *card* $\{l<..<u\} = nat\ (u - (l + 1))$
⟨*proof*⟩

**lemma** *finite-M-bounded-by-nat*: *finite* $\{k.\ P\ k \wedge k < (i::nat)\}$
⟨*proof*⟩

**lemma** *card-less*:
**assumes** *zero-in-M*: $0 \in M$
**shows** *card* $\{k \in M.\ k < Suc\ i\} \neq 0$
⟨*proof*⟩

**lemma** *card-less-Suc2*: $0 \notin M \Longrightarrow card\ \{k.\ Suc\ k \in M \wedge k < i\} = card\ \{k \in M.\ k < Suc\ i\}$
⟨*proof*⟩

**lemma** *card-less-Suc*:
**assumes** *zero-in-M*: $0 \in M$
**shows** $Suc\ (card\ \{k.\ Suc\ k \in M \wedge k < i\}) = card\ \{k \in M.\ k < Suc\ i\}$
⟨*proof*⟩

## 34.6 Lemmas useful with the summation operator setsum

For examples, see Algebra/poly/UnivPoly2.thy

### 34.6.1 Disjoint Unions

Singletons and open intervals

**lemma** *ivl-disj-un-singleton*:
$\{l::'a::linorder\}\ Un\ \{l<..\} = \{l..\}$
$\{..<u\}\ Un\ \{u::'a::linorder\} = \{..u\}$
$(l::'a::linorder) < u ==> \{l\}\ Un\ \{l<..<u\} = \{l..<u\}$
$(l::'a::linorder) < u ==> \{l<..<u\}\ Un\ \{u\} = \{l<..u\}$
$(l::'a::linorder) <= u ==> \{l\}\ Un\ \{l<..u\} = \{l..u\}$
$(l::'a::linorder) <= u ==> \{l..<u\}\ Un\ \{u\} = \{l..u\}$
⟨*proof*⟩

One- and two-sided intervals

**lemma** *ivl-disj-un-one*:
$(l::'a::linorder) < u ==> \{..l\}\ Un\ \{l<..<u\} = \{..<u\}$
$(l::'a::linorder) <= u ==> \{..<l\}\ Un\ \{l..<u\} = \{..<u\}$
$(l::'a::linorder) <= u ==> \{..l\}\ Un\ \{l<..u\} = \{..u\}$
$(l::'a::linorder) <= u ==> \{..<l\}\ Un\ \{l..u\} = \{..u\}$
$(l::'a::linorder) <= u ==> \{l<..u\}\ Un\ \{u<..\} = \{l<..\}$
$(l::'a::linorder) < u ==> \{l<..<u\}\ Un\ \{u..\} = \{l<..\}$
$(l::'a::linorder) <= u ==> \{l..u\}\ Un\ \{u<..\} = \{l..\}$
$(l::'a::linorder) <= u ==> \{l..<u\}\ Un\ \{u..\} = \{l..\}$
⟨*proof*⟩

Two- and two-sided intervals

**lemma** *ivl-disj-un-two*:
$[|\ (l::'a::linorder) < m;\ m <= u\ |] ==> \{l<..<m\}\ Un\ \{m..<u\} = \{l<..<u\}$
$[|\ (l::'a::linorder) <= m;\ m < u\ |] ==> \{l<..m\}\ Un\ \{m<..<u\} = \{l<..<u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..<m\}\ Un\ \{m..<u\} = \{l..<u\}$
$[|\ (l::'a::linorder) <= m;\ m < u\ |] ==> \{l..m\}\ Un\ \{m<..<u\} = \{l..<u\}$
$[|\ (l::'a::linorder) < m;\ m <= u\ |] ==> \{l<..<m\}\ Un\ \{m..u\} = \{l<..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l<..m\}\ Un\ \{m<..u\} = \{l<..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..<m\}\ Un\ \{m..u\} = \{l..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..m\}\ Un\ \{m<..u\} = \{l..u\}$
⟨*proof*⟩

**lemmas** *ivl-disj-un = ivl-disj-un-singleton ivl-disj-un-one ivl-disj-un-two*

### 34.6.2 Disjoint Intersections

Singletons and open intervals

**lemma** *ivl-disj-int-singleton*:
$\{l::'a::order\}\ Int\ \{l<..\} = \{\}$
$\{..<u\}\ Int\ \{u\} = \{\}$
$\{l\}\ Int\ \{l<..<u\} = \{\}$
$\{l<..<u\}\ Int\ \{u\} = \{\}$
$\{l\}\ Int\ \{l<..u\} = \{\}$
$\{l..<u\}\ Int\ \{u\} = \{\}$
⟨*proof*⟩

One- and two-sided intervals

**lemma** *ivl-disj-int-one*:
$\{..l::'a::order\}\ Int\ \{l<..<u\} = \{\}$
$\{..<l\}\ Int\ \{l..<u\} = \{\}$
$\{..l\}\ Int\ \{l<..u\} = \{\}$
$\{..<l\}\ Int\ \{l..u\} = \{\}$
$\{l<..u\}\ Int\ \{u<..\} = \{\}$
$\{l<..<u\}\ Int\ \{u..\} = \{\}$
$\{l..u\}\ Int\ \{u<..\} = \{\}$
$\{l..<u\}\ Int\ \{u..\} = \{\}$

⟨*proof*⟩

Two- and two-sided intervals

**lemma** *ivl-disj-int-two*:
{*l*::'*a*::*order*<..<*m*} *Int* {*m*..<*u*} = {}
{*l*<..*m*} *Int* {*m*<..<*u*} = {}
{*l*..<*m*} *Int* {*m*..<*u*} = {}
{*l*..*m*} *Int* {*m*<..<*u*} = {}
{*l*<..<*m*} *Int* {*m*..*u*} = {}
{*l*<..*m*} *Int* {*m*<..*u*} = {}
{*l*..<*m*} *Int* {*m*..*u*} = {}
{*l*..*m*} *Int* {*m*<..*u*} = {}
⟨*proof*⟩

**lemmas** *ivl-disj-int* = *ivl-disj-int-singleton ivl-disj-int-one ivl-disj-int-two*

### 34.6.3 Some Differences

**lemma** *ivl-diff*[*simp*]:
*i* ≤ *n* ⟹ {*i*..<*m*} − {*i*..<*n*} = {*n*..<(*m*::'*a*::*linorder*)}
⟨*proof*⟩

### 34.6.4 Some Subset Conditions

**lemma** *ivl-subset* [*simp*,*noatp*]:
({*i*..<*j*} ⊆ {*m*..<*n*}) = (*j* ≤ *i* | *m* ≤ *i* & *j* ≤ (*n*::'*a*::*linorder*))
⟨*proof*⟩

## 34.7 Summation indexed over intervals

**syntax**
*-from-to-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*SUM* - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*SUM* - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setsum* :: *idt* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*SUM* -<-./ -) [*0*,*0*,*10*] *10*)
*-upto-setsum* :: *idt* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*SUM* -<=-./ -) [*0*,*0*,*10*] *10*)
**syntax** (*xsymbols*)
*-from-to-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setsum* :: *idt* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ -<-./ -) [*0*,*0*,*10*] *10*)
*-upto-setsum* :: *idt* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ -≤-./ -) [*0*,*0*,*10*] *10*)
**syntax** (*HTML* **output**)
*-from-to-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setsum* :: *idt* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ -<-./ -) [*0*,*0*,*10*] *10*)
*-upto-setsum* :: *idt* ⇒ '*a* ⇒ '*b* ⇒ '*b* ((*3*∑ -≤-./ -) [*0*,*0*,*10*] *10*)
**syntax** (*latex-sum* **output**)
*-from-to-setsum* :: *idt* ⇒ '*a* ⇒ '*a* ⇒ '*b* ⇒ '*b*

$((3\sum\text{-} = \_\ \text{-})\ [0,0,0,10]\ 10)$
*-from-upto-setsum* :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
$((3\sum\text{-}^{<} = \_\ \text{-})\ [0,0,0,10]\ 10)$
*-upt-setsum* :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
$((3\sum\text{-} < \_\ \text{-})\ [0,0,10]\ 10)$
*-upto-setsum* :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
$((3\sum\text{-} \leq \_\ \text{-})\ [0,0,10]\ 10)$

**translations**
$\sum x{=}a..b.\ t == CONST\ setsum\ (\%x.\ t)\ \{a..b\}$
$\sum x{=}a..{<}b.\ t == CONST\ setsum\ (\%x.\ t)\ \{a..{<}b\}$
$\sum i{\leq}n.\ t == CONST\ setsum\ (\lambda i.\ t)\ \{..n\}$
$\sum i{<}n.\ t == CONST\ setsum\ (\lambda i.\ t)\ \{..{<}n\}$

The above introduces some pretty alternative syntaxes for summation over intervals:

| Old | New | LaTeX |
|---|---|---|
| $\sum x{\in}\{a..b\}.\ e$ | $\sum x = a..b.\ e$ | $\sum_{x\ =\ a}^{b}\ e$ |
| $\sum x{\in}\{a..{<}b\}.\ e$ | $\sum x = a..{<}b.\ e$ | $\sum_{x\ =\ a}^{<b}\ e$ |
| $\sum x{\in}\{..b\}.\ e$ | $\sum x{\leq}b.\ e$ | $\sum_{x\ \leq\ b}\ e$ |
| $\sum x{\in}\{..{<}b\}.\ e$ | $\sum x{<}b.\ e$ | $\sum_{x\ <\ b}\ e$ |

The left column shows the term before introduction of the new syntax, the middle column shows the new (default) syntax, and the right column shows a special syntax. The latter is only meaningful for latex output and has to be activated explicitly by setting the print mode to *latex-sum* (e.g. via *mode = latex-sum* in antiquotations). It is not the default LaTeX output because it only works well with italic-style formulae, not tt-style.

Note that for uniformity on *nat* it is better to use $\sum x = 0..{<}n.\ e$ rather than $\sum x{<}n.\ e$: *setsum* may not provide all lemmas available for $\{m..{<}n\}$ also in the special form for $\{..{<}n\}$.

This congruence rule should be used for sums over intervals as the standard theorem *setsum-cong* does not work well with the simplifier who adds the unsimplified premise $x \in B$ to the context.

**lemma** *setsum-ivl-cong*:
$[\![a = c;\ b = d;\ !!x.\ [\![\ c \leq x;\ x < d\ ]\!] \Longrightarrow f\ x = g\ x\ ]\!] \Longrightarrow$
$setsum\ f\ \{a..{<}b\} = setsum\ g\ \{c..{<}d\}$
⟨*proof*⟩

**lemma** *setsum-atMost-Suc*[*simp*]: $(\sum i \leq Suc\ n.\ f\ i) = (\sum i \leq n.\ f\ i) + f(Suc\ n)$
⟨*proof*⟩

**lemma** *setsum-lessThan-Suc*[*simp*]: $(\sum i < Suc\ n.\ f\ i) = (\sum i < n.\ f\ i) + f\ n$

⟨*proof*⟩

**lemma** *setsum-cl-ivl-Suc*[*simp*]:
*setsum f {m..Suc n} = (if Suc n < m then 0 else setsum f {m..n} + f(Suc n))*
⟨*proof*⟩

**lemma** *setsum-op-ivl-Suc*[*simp*]:
*setsum f {m..<Suc n} = (if n < m then 0 else setsum f {m..<n} + f(n))*
⟨*proof*⟩

**lemma** *setsum-head*:
**fixes** *n :: nat*
**assumes** *mn*: *m <= n*
**shows** *(∑ x∈{m..n}. P x) = P m + (∑ x∈{m<..n}. P x)* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

**lemma** *setsum-head-Suc*:
*m ≤ n ⟹ setsum f {m..n} = f m + setsum f {Suc m..n}*
⟨*proof*⟩

**lemma** *setsum-head-upt-Suc*:
*m < n ⟹ setsum f {m..<n} = f m + setsum f {Suc m..<n}*
⟨*proof*⟩

**lemma** *setsum-add-nat-ivl*: *⟦ m ≤ n; n ≤ p ⟧ ⟹*
*setsum f {m..<n} + setsum f {n..<p} = setsum f {m..<p::nat}*
⟨*proof*⟩

**lemma** *setsum-diff-nat-ivl*:
**fixes** *f :: nat ⇒ 'a::ab-group-add*
**shows** *⟦ m ≤ n; n ≤ p ⟧ ⟹*
*setsum f {m..<p} − setsum f {m..<n} = setsum f {n..<p}*
⟨*proof*⟩

## 34.8 Shifting bounds

**lemma** *setsum-shift-bounds-nat-ivl*:
*setsum f {m+k..<n+k} = setsum (%i. f(i + k)){m..<n::nat}*
⟨*proof*⟩

**lemma** *setsum-shift-bounds-cl-nat-ivl*:
*setsum f {m+k..n+k} = setsum (%i. f(i + k)){m..n::nat}*
⟨*proof*⟩

**corollary** *setsum-shift-bounds-cl-Suc-ivl*:
*setsum f {Suc m..Suc n} = setsum (%i. f(Suc i)){m..n}*
⟨*proof*⟩

**corollary** *setsum-shift-bounds-Suc-ivl*:
*setsum f {Suc m..<Suc n} = setsum (%i. f(Suc i)){m..<n}*
⟨*proof*⟩

**lemma** *setsum-shift-lb-Suc0-0*:
*f(0::nat) = (0::nat) ⟹ setsum f {Suc 0..k} = setsum f {0..k}*
⟨*proof*⟩

**lemma** *setsum-shift-lb-Suc0-0-upt*:
*f(0::nat) = 0 ⟹ setsum f {Suc 0..<k} = setsum f {0..<k}*
⟨*proof*⟩

## 34.9 The formula for geometric sums

**lemma** *geometric-sum*:
*x ~= 1 ==> (∑ i=0..<n. x ^ i) =*
*(x ^ n − 1) / (x − 1::'a::{field, recpower})*
⟨*proof*⟩

## 34.10 The formula for arithmetic sums

**lemma** *gauss-sum*:
*((1::'a::comm-semiring-1) + 1)∗(∑ i∈{1..n}. of-nat i) =*
*of-nat n∗((of-nat n)+1)*
⟨*proof*⟩

**theorem** *arith-series-general*:
*((1::'a::comm-semiring-1) + 1) ∗ (∑ i∈{..<n}. a + of-nat i ∗ d) =*
*of-nat n ∗ (a + (a + of-nat(n − 1)∗d))*
⟨*proof*⟩

**lemma** *arith-series-nat*:
*Suc (Suc 0) ∗ (∑ i∈{..<n}. a+i∗d) = n ∗ (a + (a+(n − 1)∗d))*
⟨*proof*⟩

**lemma** *arith-series-int*:
*(2::int) ∗ (∑ i∈{..<n}. a + of-nat i ∗ d) =*
*of-nat n ∗ (a + (a + of-nat(n − 1)∗d))*
⟨*proof*⟩

**lemma** *sum-diff-distrib*:
**fixes** *P::nat⇒nat*
**shows**
*∀ x. Q x ≤ P x ⟹*
*(∑ x<n. P x) − (∑ x<n. Q x) = (∑ x<n. P x − Q x)*
⟨*proof*⟩

### 34.11 Products indexed over intervals

**syntax**
*-from-to-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD* - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD* - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD* -<-./ -) [*0*,*0*,*10*] *10*)
*-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD* -<=-./ -) [*0*,*0*,*10*] *10*)
**syntax** (*xsymbols*)
*-from-to-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ -<-./ -) [*0*,*0*,*10*] *10*)
*-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ -≤-./ -) [*0*,*0*,*10*] *10*)
**syntax** (*HTML* **output**)
*-from-to-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ -<-./ -) [*0*,*0*,*10*] *10*)
*-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∏ -≤-./ -) [*0*,*0*,*10*] *10*)
**syntax** (*latex-prod* **output**)
*-from-to-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∏_ = _ -) [*0*,*0*,*0*,*10*] *10*)
*-from-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∏_ <= _ -) [*0*,*0*,*0*,*10*] *10*)
*-upt-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∏_ < _ -) [*0*,*0*,*10*] *10*)
*-upto-setprod* :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∏_ ≤ _ -) [*0*,*0*,*10*] *10*)

**translations**
∏ *x*=*a*..*b*. *t* == *CONST setprod* (%*x*. *t*) {*a*..*b*}
∏ *x*=*a*..<*b*. *t* == *CONST setprod* (%*x*. *t*) {*a*..<*b*}
∏ *i*≤*n*. *t* == *CONST setprod* (λ*i*. *t*) {..*n*}
∏ *i*<*n*. *t* == *CONST setprod* (λ*i*. *t*) {..<*n*}

**end**

# 35 Presburger: Decision Procedure for Presburger Arithmetic

**theory** *Presburger*
**imports** *Groebner-Basis SetInterval*
**uses**
*Tools/Qelim/qelim.ML*
*Tools/Qelim/cooper-data.ML*
*Tools/Qelim/generated-cooper.ML*
(*Tools/Qelim/cooper.ML*)

(*Tools/Qelim/presburger.ML*)
**begin**

⟨*ML*⟩

## 35.1 The $-\infty$ and $+\infty$ Properties

**lemma** *minf*:
⟦∃(*z* ::'*a*::*linorder*).∀ *x*<*z*. *P x* = *P'* *x*; ∃ *z*.∀ *x*<*z*. *Q x* = *Q'* *x*⟧
⟹ ∃ *z*.∀ *x*<*z*. (*P x* ∧ *Q x*) = (*P'* *x* ∧ *Q'* *x*)
⟦∃(*z* ::'*a*::*linorder*).∀ *x*<*z*. *P x* = *P'* *x*; ∃ *z*.∀ *x*<*z*. *Q x* = *Q'* *x*⟧
⟹ ∃ *z*.∀ *x*<*z*. (*P x* ∨ *Q x*) = (*P'* *x* ∨ *Q'* *x*)
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* = *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≠ *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* < *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≤ *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* > *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≥ *t*) = *False*
∃ *z*.∀(*x*::'*a*::{*linorder*,*plus*,*Ring-and-Field.dvd*})<*z*. (*d dvd x* + *s*) = (*d dvd x* + *s*)
∃ *z*.∀(*x*::'*a*::{*linorder*,*plus*,*Ring-and-Field.dvd*})<*z*. (¬ *d dvd x* + *s*) = (¬ *d dvd x* + *s*)
∃ *z*.∀ *x*<*z*. *F* = *F*
⟨*proof*⟩

**lemma** *pinf*:
⟦∃(*z* ::'*a*::*linorder*).∀ *x*>*z*. *P x* = *P'* *x*; ∃ *z*.∀ *x*>*z*. *Q x* = *Q'* *x*⟧
⟹ ∃ *z*.∀ *x*>*z*. (*P x* ∧ *Q x*) = (*P'* *x* ∧ *Q'* *x*)
⟦∃(*z* ::'*a*::*linorder*).∀ *x*>*z*. *P x* = *P'* *x*; ∃ *z*.∀ *x*>*z*. *Q x* = *Q'* *x*⟧
⟹ ∃ *z*.∀ *x*>*z*. (*P x* ∨ *Q x*) = (*P'* *x* ∨ *Q'* *x*)
∃(*z* ::'*a*::{*linorder*}).∀ *x*>*z*.(*x* = *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*>*z*.(*x* ≠ *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*>*z*.(*x* < *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*>*z*.(*x* ≤ *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*>*z*.(*x* > *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*>*z*.(*x* ≥ *t*) = *True*
∃ *z*.∀(*x*::'*a*::{*linorder*,*plus*,*Ring-and-Field.dvd*})>*z*. (*d dvd x* + *s*) = (*d dvd x* + *s*)
∃ *z*.∀(*x*::'*a*::{*linorder*,*plus*,*Ring-and-Field.dvd*})>*z*. (¬ *d dvd x* + *s*) = (¬ *d dvd x* + *s*)
∃ *z*.∀ *x*>*z*. *F* = *F*
⟨*proof*⟩

**lemma** *inf-period*:
⟦∀ *x k*. *P x* = *P* (*x* − *k*∗*D*); ∀ *x k*. *Q x* = *Q* (*x* − *k*∗*D*)⟧
⟹ ∀ *x k*. (*P x* ∧ *Q x*) = (*P* (*x* − *k*∗*D*) ∧ *Q* (*x* − *k*∗*D*))
⟦∀ *x k*. *P x* = *P* (*x* − *k*∗*D*); ∀ *x k*. *Q x* = *Q* (*x* − *k*∗*D*)⟧
⟹ ∀ *x k*. (*P x* ∨ *Q x*) = (*P* (*x* − *k*∗*D*) ∨ *Q* (*x* − *k*∗*D*))
(*d*::'*a*::{*comm-ring*,*Ring-and-Field.dvd*}) *dvd D* ⟹ ∀ *x k*. (*d dvd x* + *t*) = (*d*

$dvd\ (x - k{*}D) + t)$
$(d{::}'a{::}\{comm\text{-}ring,Ring\text{-}and\text{-}Field.dvd\})\ dvd\ D \Longrightarrow \forall\, x\ k.\ (\neg d\ dvd\ x + t) = (\neg d\ dvd\ (x - k{*}D) + t)$
$\forall\, x\ k.\ F = F$
⟨*proof*⟩

## 35.2 The A and B sets

**lemma** *bset*:
$[\![\forall\, x.(\forall\, j \in \{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x - D) \wedge Q\ (x - D))$
$[\![\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x - D) \vee Q\ (x - D))$
$[\![D{>}0;\ t - 1{\in}\ B]\!] \Longrightarrow (\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (x = t) \longrightarrow (x - D = t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (x \neq t) \longrightarrow (x - D \neq t))$
$D{>}0 \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (x < t) \longrightarrow (x - D < t))$
$D{>}0 \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (x \leq t) \longrightarrow (x - D \leq t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (x > t) \longrightarrow (x - D > t))$
$[\![D{>}0\ ;\ t - 1 \in B]\!] \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (x \geq t) \longrightarrow (x - D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x - D) + t))$
$d\ dvd\ D \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg\ d\ dvd\ (x - D) + t))$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}B.\ x \neq b + j) \longrightarrow F \longrightarrow F$
⟨*proof*⟩

**lemma** *aset*:
$[\![\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x + D) \wedge Q\ (x + D))$
$[\![\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x + D) \vee Q\ (x + D))$
$[\![D{>}0;\ t + 1{\in}\ A]\!] \Longrightarrow (\forall\, x.(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow (x = t) \longrightarrow (x + D = t))$
$[\![D{>}0\ ;\ t \in A]\!] \Longrightarrow (\forall\, (x{::}int).(\forall\, j{\in}\{1\ ..\ D\}.\ \forall\, b{\in}A.\ x \neq b - j) \longrightarrow (x \neq t) \longrightarrow (x + D \neq t))$

$[\![D{>}0;\ t{\in}\ A]\!] \Longrightarrow (\forall\,(x{::}int).\ (\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow (x < t) \longrightarrow (x + D < t))$
$[\![D{>}0;\ t + 1 \in A]\!] \Longrightarrow (\forall\,(x{::}int).(\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow (x \leq t) \longrightarrow (x + D \leq t))$
$D{>}0 \Longrightarrow (\forall\,(x{::}int).(\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow (x > t) \longrightarrow (x + D > t))$
$D{>}0 \Longrightarrow (\forall\,(x{::}int).(\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow (x \geq t) \longrightarrow (x + D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall\,(x{::}int).(\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x + D) + t))$
$d\ dvd\ D \Longrightarrow (\forall\,(x{::}int).(\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg\ d\ dvd\ (x + D) + t))$
$\forall\,x.(\forall\,j{\in}\{1\ ..\ D\}.\ \forall\,b{\in}A.\ x \neq b - j) \longrightarrow F \longrightarrow F$
⟨*proof*⟩

## 35.3 Cooper's Theorem $-\infty$ and $+\infty$ Version

### 35.3.1 First some trivial facts about periodic sets or predicates

**lemma** *periodic-finite-ex*:
**assumes** *dpos*: $(0{::}int) < d$ **and** *modd*: $ALL\ x\ k.\ P\ x = P(x - k{*}d)$
**shows** $(EX\ x.\ P\ x) = (EX\ j : \{1..d\}.\ P\ j)$
(**is** *?LHS* = *?RHS*)
⟨*proof*⟩

### 35.3.2 The $-\infty$ Version

**lemma** *decr-lemma*: $0 < (d{::}int) \Longrightarrow x - (abs(x{-}z){+}1) * d < z$
⟨*proof*⟩

**lemma** *incr-lemma*: $0 < (d{::}int) \Longrightarrow z < x + (abs(x{-}z){+}1) * d$
⟨*proof*⟩

**theorem** *int-induct*[*case-names base step1 step2*]:
**assumes**
*base*: $P(k{::}int)$ **and** *step1*: $\bigwedge i.\ [\![k \leq i;\ P\ i]\!] \Longrightarrow P(i{+}1)$ **and**
*step2*: $\bigwedge i.\ [\![k \geq i;\ P\ i]\!] \Longrightarrow P(i - 1)$
**shows** $P\ i$
⟨*proof*⟩

**lemma** *decr-mult-lemma*:
**assumes** *dpos*: $(0{::}int) < d$ **and** *minus*: $\forall\,x.\ P\ x \longrightarrow P(x - d)$ **and** *knneg*: $0 <= k$
**shows** $ALL\ x.\ P\ x \longrightarrow P(x - k{*}d)$
⟨*proof*⟩

**lemma** *minusinfinity*:
**assumes** *dpos*: $0 < d$ **and**
*P1eqP1*: $ALL\ x\ k.\ P1\ x = P1(x - k{*}d)$ **and** *ePeqP1*: $EX\ z{::}int.\ ALL\ x.\ x < z \longrightarrow (P\ x = P1\ x)$

**shows** $(EX\ x.\ P1\ x) \longrightarrow (EX\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *cpmi*:
**assumes** *dp*: $0 < D$ **and** *p1*:$\exists\, z.\ \forall\ x< z.\ P\ x = P'\ x$
**and** *nb*:$\forall\, x.(\forall\ j\in \{1..D\}.\ \forall\, (b::int) \in B.\ x \neq b+j) \text{ --> } P\ (x) \text{ --> } P\ (x - D)$
**and** *pd*: $\forall\ x\ k.\ P'\ x = P'\ (x-k*D)$
**shows** $(\exists\, x.\ P\ x) = ((\exists\ j\in \{1..D\}\ .\ P'\ j) \mid (\exists\ j \in \{1..D\}.\exists\ b\in B.\ P\ (b+j)))$
(**is** *?L* = (*?R1* ∨ *?R2*))
⟨*proof*⟩

### 35.3.3 The $+\infty$ Version

**lemma** *plusinfinity*:
**assumes** *dpos*: $(0::int) < d$ **and**
*P1eqP1*: $\forall\, x\ k.\ P'\ x = P'(x - k*d)$ **and** *ePeqP1*: $\exists\ z.\ \forall\ x>z.\ P\ x = P'\ x$
**shows** $(\exists\ x.\ P'\ x) \longrightarrow (\exists\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *incr-mult-lemma*:
**assumes** *dpos*: $(0::int) < d$ **and** *plus*: $ALL\ x::int.\ P\ x \longrightarrow P(x + d)$ **and** *knneg*: $0 <= k$
**shows** $ALL\ x.\ P\ x \longrightarrow P(x + k*d)$
⟨*proof*⟩

**lemma** *cppi*:
**assumes** *dp*: $0 < D$ **and** *p1*:$\exists\, z.\ \forall\ x> z.\ P\ x = P'\ x$
**and** *nb*:$\forall\, x.(\forall\ j\in \{1..D\}.\ \forall\, (b::int) \in A.\ x \neq b - j) \text{ --> } P\ (x) \text{ --> } P\ (x + D)$
**and** *pd*: $\forall\ x\ k.\ P'\ x= P'\ (x-k*D)$
**shows** $(\exists\, x.\ P\ x) = ((\exists\ j\in \{1..D\}\ .\ P'\ j) \mid (\exists\ j \in \{1..D\}.\exists\ b\in A.\ P\ (b - j)))$
(**is** *?L* = (*?R1* ∨ *?R2*))
⟨*proof*⟩

**lemma** *simp-from-to*: $\{i..j::int\} = (if\ j < i\ then\ \{\}\ else\ insert\ i\ \{i+1..j\})$
⟨*proof*⟩

**theorem** *unity-coeff-ex*: $(\exists\, (x::'a::\{semiring\text{-}0, Ring\text{-}and\text{-}Field.dvd\}).\ P\ (l * x)) \equiv (\exists\, x.\ l\ dvd\ (x + 0) \wedge P\ x)$
⟨*proof*⟩

**lemma** *zdvd-mono*: **assumes** *not0*: $(k::int) \neq 0$
**shows** $((m::int)\ dvd\ t) \equiv (k*m\ dvd\ k*t)$
⟨*proof*⟩

**lemma** *uminus-dvd-conv*: $(d\ dvd\ (t::int)) \equiv (-d\ dvd\ t)\ (d\ dvd\ (t::int)) \equiv (d\ dvd\ -t)$
⟨*proof*⟩

Theorems for transforming predicates on nat to predicates on *int*

**lemma** *all-nat*: $(\forall x{::}nat.\ P\ x) = (\forall x{::}int.\ 0 <= x \longrightarrow P\ (nat\ x))$
⟨*proof*⟩

**lemma** *ex-nat*: $(\exists x{::}nat.\ P\ x) = (\exists x{::}int.\ 0 <= x \wedge P\ (nat\ x))$
⟨*proof*⟩

**lemma** *zdiff-int-split*: $P\ (int\ (x - y)) =$
$((y \leq x \longrightarrow P\ (int\ x - int\ y)) \wedge (x < y \longrightarrow P\ 0))$
⟨*proof*⟩

**lemma** *number-of1*: $(0{::}int) <= number\text{-}of\ n \Longrightarrow (0{::}int) <= number\text{-}of\ (Int.Bit0\ n) \wedge (0{::}int) <= number\text{-}of\ (Int.Bit1\ n)$
⟨*proof*⟩
**lemma** *number-of2*: $(0{::}int) <= Numeral0$ ⟨*proof*⟩
**lemma** *Suc-plus1*: $Suc\ n = n + 1$ ⟨*proof*⟩

Specific instances of congruence rules, to prevent simplifier from looping.

**theorem** *imp-le-cong*: $(0 <= x \Longrightarrow P = P') \Longrightarrow (0 <= (x{::}int) \longrightarrow P) = (0 <= x \longrightarrow P')$ ⟨*proof*⟩

**theorem** *conj-le-cong*: $(0 <= x \Longrightarrow P = P') \Longrightarrow (0 <= (x{::}int) \wedge P) = (0 <= x \wedge P')$
⟨*proof*⟩
**lemma** *int-eq-number-of-eq*:
$(((number\text{-}of\ v){::}int) = (number\text{-}of\ w)) = iszero\ ((number\text{-}of\ (v + (uminus\ w))){::}int)$
⟨*proof*⟩

**declare** *dvd-eq-mod-eq-0*[*symmetric*, *presburger*]
**declare** *mod-1*[*presburger*]
**declare** *mod-0*[*presburger*]
**declare** *mod-by-1*[*presburger*]
**declare** *zmod-zero*[*presburger*]
**declare** *zmod-self*[*presburger*]
**declare** *mod-self*[*presburger*]
**declare** *mod-by-0*[*presburger*]
**declare** *mod-div-trivial*[*presburger*]
**declare** *div-mod-equality2*[*presburger*]
**declare** *div-mod-equality*[*presburger*]
**declare** *mod-div-equality2*[*presburger*]
**declare** *mod-div-equality*[*presburger*]
**declare** *mod-mult-self1*[*presburger*]
**declare** *mod-mult-self2*[*presburger*]
**declare** *zdiv-zmod-equality2*[*presburger*]
**declare** *zdiv-zmod-equality*[*presburger*]
**declare** *mod2-Suc-Suc*[*presburger*]
**lemma** [*presburger*]: $(a{::}int)\ div\ 0 = 0$ **and** [*presburger*]: $a\ mod\ 0 = a$

⟨*proof*⟩

⟨*ML*⟩

**lemma** [*presburger*, *algebra*]: *m mod 2* = (*1*::*nat*) ⟷ ¬ *2 dvd m* ⟨*proof*⟩
**lemma** [*presburger*, *algebra*]: *m mod 2* = *Suc 0* ⟷ ¬ *2 dvd m* ⟨*proof*⟩
**lemma** [*presburger*, *algebra*]: *m mod* (*Suc* (*Suc 0*)) = (*1*::*nat*) ⟷ ¬ *2 dvd m* ⟨*proof*⟩
**lemma** [*presburger*, *algebra*]: *m mod* (*Suc* (*Suc 0*)) = *Suc 0* ⟷ ¬ *2 dvd m* ⟨*proof*⟩
**lemma** [*presburger*, *algebra*]: *m mod 2* = (*1*::*int*) ⟷ ¬ *2 dvd m* ⟨*proof*⟩

**lemma** *zdvd-period*:
**fixes** *a d* :: *int*
**assumes** *advdd*: *a dvd d*
**shows** *a dvd* (*x* + *t*) ⟷ *a dvd* ((*x* + *c* ∗ *d*) + *t*)
⟨*proof*⟩

**end**

# 36 Recdef: TFL: recursive function definitions

**theory** *Recdef*
**imports** *FunDef Plain*
**uses**
(*Tools*/*TFL*/*casesplit.ML*)
(*Tools*/*TFL*/*utils.ML*)
(*Tools*/*TFL*/*usyntax.ML*)
(*Tools*/*TFL*/*dcterm.ML*)
(*Tools*/*TFL*/*thms.ML*)
(*Tools*/*TFL*/*rules.ML*)
(*Tools*/*TFL*/*thry.ML*)
(*Tools*/*TFL*/*tfl.ML*)
(*Tools*/*TFL*/*post.ML*)
(*Tools*/*recdef-package.ML*)
**begin**

* This form avoids giant explosions in proofs. NOTE USE OF ==

**lemma** *def-wfrec*: [| *f*==*wfrec r H*; *wf*(*r*) |] ==> *f*(*a*) = *H* (*cut f r a*) *a*
⟨*proof*⟩

**lemma** *tfl-wf-induct*: *ALL R*. *wf R* −−>
(*ALL P*. (*ALL x*. (*ALL y*. (*y*,*x*):*R* −−> *P y*) −−> *P x*) −−> (*ALL x*. *P x*))
⟨*proof*⟩

**lemma** *tfl-cut-apply*: $ALL\ f\ R.\ (x,a){:}R\ {-}{-}{>}\ (cut\ f\ R\ a)(x) = f(x)$
⟨*proof*⟩

**lemma** *tfl-wfrec*:
$ALL\ M\ R\ f.\ (f{=}wfrec\ R\ M)\ {-}{-}{>}\ wf\ R\ {-}{-}{>}\ (ALL\ x.\ f\ x = M\ (cut\ f\ R\ x)\ x)$
⟨*proof*⟩

**lemma** *tfl-eq-True*: $(x = True)\ {-}{-}{>}\ x$
⟨*proof*⟩

**lemma** *tfl-rev-eq-mp*: $(x = y)\ {-}{-}{>}\ y\ {-}{-}{>}\ x$
⟨*proof*⟩

**lemma** *tfl-simp-thm*: $(x\ {-}{-}{>}\ y)\ {-}{-}{>}\ (x = x')\ {-}{-}{>}\ (x'\ {-}{-}{>}\ y)$
⟨*proof*⟩

**lemma** *tfl-P-imp-P-iff-True*: $P\ {=}{=}{>}\ P = True$
⟨*proof*⟩

**lemma** *tfl-imp-trans*: $(A\ {-}{-}{>}\ B)\ {=}{=}{>}\ (B\ {-}{-}{>}\ C)\ {=}{=}{>}\ (A\ {-}{-}{>}\ C)$
⟨*proof*⟩

**lemma** *tfl-disj-assoc*: $(a \vee b) \vee c == a \vee (b \vee c)$
⟨*proof*⟩

**lemma** *tfl-disjE*: $P \vee Q\ {=}{=}{>}\ P\ {-}{-}{>}\ R\ {=}{=}{>}\ Q\ {-}{-}{>}\ R\ {=}{=}{>}\ R$
⟨*proof*⟩

**lemma** *tfl-exE*: $\exists x.\ P\ x\ {=}{=}{>}\ \forall x.\ P\ x\ {-}{-}{>}\ Q\ {=}{=}{>}\ Q$
⟨*proof*⟩

⟨*ML*⟩

**lemmas** [*recdef-simp*] =
*inv-image-def*
*measure-def*
*lex-prod-def*
*same-fst-def*
*less-Suc-eq* [*THEN iffD2*]

**lemmas** [*recdef-cong*] =
*if-cong let-cong image-cong INT-cong UN-cong bex-cong ball-cong imp-cong*

**lemmas** [*recdef-wf*] =
*wf-trancl*
*wf-less-than*
*wf-lex-prod*
*wf-inv-image*
*wf-measure*

*wf-pred-nat*
*wf-same-fst*
*wf-empty*

**end**

# 37 Hilbert-Choice: Hilbert's Epsilon-Operator and the Axiom of Choice

**theory** *Hilbert-Choice*
**imports** *Nat Wellfounded Plain*
**uses** (*Tools/meson.ML*) (*Tools/specification-package.ML*)
**begin**

## 37.1 Hilbert's epsilon

**axiomatization**
  *Eps* :: (*'a* => *bool*) => *'a*
**where**
  *someI*: *P x* ==> *P* (*Eps P*)

**syntax** (*epsilon*)
  *-Eps* :: [*pttrn*, *bool*] => *'a* ((*3ϵ -./ -*) [*0*, *10*] *10*)
**syntax** (*HOL*)
  *-Eps* :: [*pttrn*, *bool*] => *'a* ((*3@ -./ -*) [*0*, *10*] *10*)
**syntax**
  *-Eps* :: [*pttrn*, *bool*] => *'a* ((*3SOME -./ -*) [*0*, *10*] *10*)
**translations**
  *SOME x. P* == *CONST Eps* (%*x. P*)

⟨*ML*⟩

**constdefs**
  *inv* :: (*'a* => *'b*) => (*'b* => *'a*)
  *inv*(*f* :: *'a* => *'b*) == %*y. SOME x. f x* = *y*

  *Inv* :: *'a set* => (*'a* => *'b*) => (*'b* => *'a*)
  *Inv A f* == %*x. SOME y. y* ∈ *A* & *f y* = *x*

## 37.2 Hilbert's Epsilon-operator

Easier to apply than *someI* if the witness comes from an existential formula

**lemma** *someI-ex* [*elim?*]: ∃ *x. P x* ==> *P* (*SOME x. P x*)
⟨*proof*⟩

Easier to apply than *someI* because the conclusion has only one occurrence of *P*.

**lemma** *someI2*: [| *P a*; !!*x*. *P x* ==> *Q x* |] ==> *Q* (*SOME x*. *P x*)
⟨*proof*⟩

Easier to apply than *someI2* if the witness comes from an existential formula

**lemma** *someI2-ex*: [| ∃ *a*. *P a*; !!*x*. *P x* ==> *Q x* |] ==> *Q* (*SOME x*. *P x*)
⟨*proof*⟩

**lemma** *some-equality* [*intro*]:
[| *P a*; !!*x*. *P x* ==> *x*=*a* |] ==> (*SOME x*. *P x*) = *a*
⟨*proof*⟩

**lemma** *some1-equality*: [| *EX*!*x*. *P x*; *P a* |] ==> (*SOME x*. *P x*) = *a*
⟨*proof*⟩

**lemma** *some-eq-ex*: *P* (*SOME x*. *P x*) = (∃ *x*. *P x*)
⟨*proof*⟩

**lemma** *some-eq-trivial* [*simp*]: (*SOME y*. *y*=*x*) = *x*
⟨*proof*⟩

**lemma** *some-sym-eq-trivial* [*simp*]: (*SOME y*. *x*=*y*) = *x*
⟨*proof*⟩

## 37.3 Axiom of Choice, Proved Using the Description Operator

Used in *Tools/meson.ML*

**lemma** *choice*: ∀ *x*. ∃ *y*. *Q x y* ==> ∃ *f*. ∀ *x*. *Q x* (*f x*)
⟨*proof*⟩

**lemma** *bchoice*: ∀ *x*∈*S*. ∃ *y*. *Q x y* ==> ∃ *f*. ∀ *x*∈*S*. *Q x* (*f x*)
⟨*proof*⟩

## 37.4 Function Inverse

**lemma** *inv-id* [*simp*]: *inv id* = *id*
⟨*proof*⟩

A one-to-one function has an inverse.

**lemma** *inv-f-f* [*simp*]: *inj f* ==> *inv f* (*f x*) = *x*
⟨*proof*⟩

**lemma** *inv-f-eq*: [| *inj f*; *f x* = *y* |] ==> *inv f y* = *x*
⟨*proof*⟩

**lemma** *inj-imp-inv-eq*: [| *inj f*; ∀ *x*. *f*(*g x*) = *x* |] ==> *inv f* = *g*
⟨*proof*⟩

But is it useful?

**lemma** *inj-transfer*:
  **assumes** *injf*: *inj f* **and** *minor*: !!*y*. *y* ∈ *range*(*f*) ==> *P*(*inv f y*)
  **shows** *P x*
⟨*proof*⟩

**lemma** *inj-iff*: (*inj f*) = (*inv f o f* = *id*)
⟨*proof*⟩

**lemma** *inv-o-cancel*[*simp*]: *inj f* ==> *inv f o f* = *id*
⟨*proof*⟩

**lemma** *o-inv-o-cancel*[*simp*]: *inj f* ==> *g o inv f o f* = *g*
⟨*proof*⟩

**lemma** *inv-image-cancel*[*simp*]:
  *inj f* ==> *inv f* ‘ *f* ‘ *S* = *S*
⟨*proof*⟩

**lemma** *inj-imp-surj-inv*: *inj f* ==> *surj* (*inv f*)
⟨*proof*⟩

**lemma** *f-inv-f*: *y* ∈ *range*(*f*) ==> *f*(*inv f y*) = *y*
⟨*proof*⟩

**lemma** *surj-f-inv-f*: *surj f* ==> *f*(*inv f y*) = *y*
⟨*proof*⟩

**lemma** *inv-injective*:
  **assumes** *eq*: *inv f x* = *inv f y*
    **and** *x*: *x*: *range f*
    **and** *y*: *y*: *range f*
  **shows** *x*=*y*
⟨*proof*⟩

**lemma** *inj-on-inv*: *A* <= *range*(*f*) ==> *inj-on* (*inv f*) *A*
⟨*proof*⟩

**lemma** *surj-imp-inj-inv*: *surj f* ==> *inj* (*inv f*)
⟨*proof*⟩

**lemma** *surj-iff*: (*surj f*) = (*f o inv f* = *id*)
⟨*proof*⟩

**lemma** *surj-imp-inv-eq*: [| *surj f*; ∀ *x*. *g*(*f x*) = *x* |] ==> *inv f* = *g*
⟨*proof*⟩

**lemma** *bij-imp-bij-inv*: *bij f* ==> *bij* (*inv f*)
⟨*proof*⟩

**lemma** *inv-equality*: [| !!$x$. $g\ (f\ x) = x$; !!$y$. $f\ (g\ y) = y$ |] ==> $inv\ f = g$
⟨*proof*⟩

**lemma** *inv-inv-eq*: $bij\ f$ ==> $inv\ (inv\ f) = f$
⟨*proof*⟩

**lemma** *o-inv-distrib*: [| $bij\ f$; $bij\ g$ |] ==> $inv\ (f\ o\ g) = inv\ g\ o\ inv\ f$
⟨*proof*⟩

**lemma** *image-surj-f-inv-f*: $surj\ f$ ==> $f\ `\ (inv\ f\ `\ A) = A$
⟨*proof*⟩

**lemma** *image-inv-f-f*: $inj\ f$ ==> $(inv\ f)\ `\ (f\ `\ A) = A$
⟨*proof*⟩

**lemma** *inv-image-comp*: $inj\ f$ ==> $inv\ f\ `\ (f`X) = X$
⟨*proof*⟩

**lemma** *bij-image-Collect-eq*: $bij\ f$ ==> $f\ `\ Collect\ P = \{y.\ P\ (inv\ f\ y)\}$
⟨*proof*⟩

**lemma** *bij-vimage-eq-inv-image*: $bij\ f$ ==> $f\ -`\ A = inv\ f\ `\ A$
⟨*proof*⟩

## 37.5 Inverse of a PI-function (restricted domain)

**lemma** *Inv-f-f*: [| $inj\text{-}on\ f\ A$; $x \in A$ |] ==> $Inv\ A\ f\ (f\ x) = x$
⟨*proof*⟩

**lemma** *f-Inv-f*: $y \in f`A$ ==> $f\ (Inv\ A\ f\ y) = y$
⟨*proof*⟩

**lemma** *Inv-injective*:
**assumes** *eq*: $Inv\ A\ f\ x = Inv\ A\ f\ y$
**and** $x$: $x$: $f`A$
**and** $y$: $y$: $f`A$
**shows** $x=y$
⟨*proof*⟩

**lemma** *inj-on-Inv*: $B <= f`A$ ==> $inj\text{-}on\ (Inv\ A\ f)\ B$
⟨*proof*⟩

**lemma** *Inv-mem*: [| $f\ `\ A = B$; $x \in B$ |] ==> $Inv\ A\ f\ x \in A$
⟨*proof*⟩

**lemma** *Inv-f-eq*: [| *inj-on f A*; *f x = y*; *x ∈ A* |] ==> *Inv A f y = x*
⟨*proof*⟩

**lemma** *Inv-comp*:
[| *inj-on f (g ' A)*; *inj-on g A*; *x ∈ f ' g ' A* |] ==>
*Inv A (f o g) x = (Inv A g o Inv (g ' A) f) x*
⟨*proof*⟩

**lemma** *bij-betw-Inv*: *bij-betw f A B* ⟹ *bij-betw (Inv A f) B A*
⟨*proof*⟩

## 37.6 Other Consequences of Hilbert's Epsilon

Hilbert's Epsilon and the *split* Operator

Looping simprule

**lemma** *split-paired-Eps*: (*SOME x. P x*) = (*SOME (a,b). P(a,b)*)
⟨*proof*⟩

**lemma** *Eps-split*: *Eps (split P)* = (*SOME xy. P (fst xy) (snd xy)*)
⟨*proof*⟩

**lemma** *Eps-split-eq* [*simp*]: (@(*x'*,*y'*). *x = x' & y = y'*) = (*x*,*y*)
⟨*proof*⟩

A relation is wellfounded iff it has no infinite descending chain

**lemma** *wf-iff-no-infinite-down-chain*:
*wf r* = (~(∃*f*. ∀ *i*. (*f*(*Suc i*),*f i*) ∈ *r*))
⟨*proof*⟩

**lemma** *wf-no-infinite-down-chainE*:
**assumes** *wf r* **obtains** *k* **where** (*f* (*Suc k*), *f k*) ∉ *r*
⟨*proof*⟩

A dynamically-scoped fact for TFL

**lemma** *tfl-some*: ∀ *P x*. *P x* --> *P* (*Eps P*)
⟨*proof*⟩

## 37.7 Least value operator

**constdefs**
*LeastM* :: ['*a* => '*b*::*ord*, '*a* => *bool*] => '*a*
*LeastM m P* == *SOME x*. *P x* & (∀ *y*. *P y* --> *m x* <= *m y*)

**syntax**
*-LeastM* :: [*pttrn*, '*a* => '*b*::*ord*, *bool*] => '*a* (*LEAST - WRT -*. - [*0*, *4*, *10*] *10*)
**translations**
*LEAST x WRT m*. *P* == *LeastM m* (%*x*. *P*)

**lemma** *LeastMI2*:
*P x ==> (!!y. P y ==> m x <= m y)*
*==> (!!x. P x ==> ∀ y. P y --> m x ≤ m y ==> Q x)*
*==> Q (LeastM m P)*
⟨*proof*⟩

**lemma** *LeastM-equality*:
*P k ==> (!!x. P x ==> m k <= m x)*
*==> m (LEAST x WRT m. P x) = (m k::'a::order)*
⟨*proof*⟩

**lemma** *wf-linord-ex-has-least*:
*wf r ==> ∀ x y. ((x,y):r^+) = ((y,x)~:r^*) ==> P k*
*==> ∃ x. P x & (!y. P y --> (m x,m y):r^*)*
⟨*proof*⟩

**lemma** *ex-has-least-nat*:
*P k ==> ∃ x. P x & (∀ y. P y --> m x <= (m y::nat))*
⟨*proof*⟩

**lemma** *LeastM-nat-lemma*:
*P k ==> P (LeastM m P) & (∀ y. P y --> m (LeastM m P) <= (m y::nat))*
⟨*proof*⟩

**lemmas** *LeastM-natI = LeastM-nat-lemma [THEN conjunct1, standard]*

**lemma** *LeastM-nat-le*: *P x ==> m (LeastM m P) <= (m x::nat)*
⟨*proof*⟩

## 37.8 Greatest value operator

**constdefs**
*GreatestM :: ['a => 'b::ord, 'a => bool] => 'a*
*GreatestM m P == SOME x. P x & (∀ y. P y --> m y <= m x)*

*Greatest :: ('a::ord => bool) => 'a* (**binder** *GREATEST 10*)
*Greatest == GreatestM (%x. x)*

**syntax**
*-GreatestM :: [pttrn, 'a=>'b::ord, bool] => 'a*
*(GREATEST - WRT -. - [0, 4, 10] 10)*

**translations**
*GREATEST x WRT m. P == GreatestM m (%x. P)*

**lemma** *GreatestMI2*:
*P x ==> (!!y. P y ==> m y <= m x)*
*==> (!!x. P x ==> ∀ y. P y --> m y ≤ m x ==> Q x)*

$==>$ $Q$ $(GreatestM\ m\ P)$
⟨*proof*⟩

**lemma** *GreatestM-equality*:
$P\ k ==> (!!x.\ P\ x ==> m\ x <= m\ k)$
$==> m\ (GREATEST\ x\ WRT\ m.\ P\ x) = (m\ k::'a::order)$
⟨*proof*⟩

**lemma** *Greatest-equality*:
$P\ (k::'a::order) ==> (!!x.\ P\ x ==> x <= k) ==> (GREATEST\ x.\ P\ x) = k$
⟨*proof*⟩

**lemma** *ex-has-greatest-nat-lemma*:
$P\ k ==> \forall x.\ P\ x --> (\exists y.\ P\ y\ \&\ \sim ((m\ y::nat) <= m\ x))$
$==> \exists y.\ P\ y\ \&\ \sim (m\ y < m\ k + n)$
⟨*proof*⟩

**lemma** *ex-has-greatest-nat*:
$P\ k ==> \forall y.\ P\ y --> m\ y < b$
$==> \exists x.\ P\ x\ \&\ (\forall y.\ P\ y --> (m\ y::nat) <= m\ x)$
⟨*proof*⟩

**lemma** *GreatestM-nat-lemma*:
$P\ k ==> \forall y.\ P\ y --> m\ y < b$
$==> P\ (GreatestM\ m\ P)\ \&\ (\forall y.\ P\ y --> (m\ y::nat) <= m\ (GreatestM\ m\ P))$
⟨*proof*⟩

**lemmas** *GreatestM-natI* = *GreatestM-nat-lemma* [*THEN conjunct1*, *standard*]

**lemma** *GreatestM-nat-le*:
$P\ x ==> \forall y.\ P\ y --> m\ y < b$
$==> (m\ x::nat) <= m\ (GreatestM\ m\ P)$
⟨*proof*⟩

Specialization to *GREATEST*.

**lemma** *GreatestI*: $P\ (k::nat) ==> \forall y.\ P\ y --> y < b ==> P\ (GREATEST\ x.\ P\ x)$
⟨*proof*⟩

**lemma** *Greatest-le*:
$P\ x ==> \forall y.\ P\ y --> y < b ==> (x::nat) <= (GREATEST\ x.\ P\ x)$
⟨*proof*⟩

## 37.9 The Meson proof procedure

### 37.9.1 Negation Normal Form

de Morgan laws

**lemma** *meson-not-conjD*: $\sim(P\&Q) ==> \sim P \mid \sim Q$
**and** *meson-not-disjD*: $\sim(P|Q) ==> \sim P \,\&\, \sim Q$
**and** *meson-not-notD*: $\sim\sim P ==> P$
**and** *meson-not-allD*: $!!P.\ \sim(\forall x.\ P(x)) ==> \exists x.\ \sim P(x)$
**and** *meson-not-exD*: $!!P.\ \sim(\exists x.\ P(x)) ==> \forall x.\ \sim P(x)$
⟨*proof*⟩

Removal of $--> $ and $<->$ (positive and negative occurrences)

**lemma** *meson-imp-to-disjD*: $P-->Q ==> \sim P \mid Q$
**and** *meson-not-impD*: $\sim(P-->Q) ==> P \,\&\, \sim Q$
**and** *meson-iff-to-disjD*: $P=Q ==> (\sim P \mid Q) \,\&\, (\sim Q \mid P)$
**and** *meson-not-iffD*: $\sim(P=Q) ==> (P \mid Q) \,\&\, (\sim P \mid \sim Q)$
— Much more efficient than $P \wedge \neg\ Q \vee Q \wedge \neg\ P$ for computing CNF
**and** *meson-not-refl-disj-D*: $x \sim= x \mid P ==> P$
⟨*proof*⟩

### 37.9.2 Pulling out the existential quantifiers

Conjunction

**lemma** *meson-conj-exD1*: $!!P\ Q.\ (\exists x.\ P(x)) \,\&\, Q ==> \exists x.\ P(x) \,\&\, Q$
**and** *meson-conj-exD2*: $!!P\ Q.\ P \,\&\, (\exists x.\ Q(x)) ==> \exists x.\ P \,\&\, Q(x)$
⟨*proof*⟩

Disjunction

**lemma** *meson-disj-exD*: $!!P\ Q.\ (\exists x.\ P(x)) \mid (\exists x.\ Q(x)) ==> \exists x.\ P(x) \mid Q(x)$
— DO NOT USE with forall-Skolemization: makes fewer schematic variables!!
— With ex-Skolemization, makes fewer Skolem constants
**and** *meson-disj-exD1*: $!!P\ Q.\ (\exists x.\ P(x)) \mid Q ==> \exists x.\ P(x) \mid Q$
**and** *meson-disj-exD2*: $!!P\ Q.\ P \mid (\exists x.\ Q(x)) ==> \exists x.\ P \mid Q(x)$
⟨*proof*⟩

### 37.9.3 Generating clauses for the Meson Proof Procedure

Disjunctions

**lemma** *meson-disj-assoc*: $(P|Q)|R ==> P|(Q|R)$
**and** *meson-disj-comm*: $P|Q ==> Q|P$
**and** *meson-disj-FalseD1*: $False|P ==> P$
**and** *meson-disj-FalseD2*: $P|False ==> P$
⟨*proof*⟩

## 37.10 Lemmas for Meson, the Model Elimination Procedure

Generation of contrapositives

Inserts negated disjunct after removing the negation; P is a literal. Model elimination requires assuming the negation of every attempted subgoal, hence the negated disjuncts.

**lemma** *make-neg-rule*: $\sim P|Q \Longrightarrow ((\sim P \Longrightarrow P) \Longrightarrow Q)$
⟨*proof*⟩

Version for Plaisted's "Postive refinement" of the Meson procedure

**lemma** *make-refined-neg-rule*: $\sim P|Q \Longrightarrow (P \Longrightarrow Q)$
⟨*proof*⟩

$P$ should be a literal

**lemma** *make-pos-rule*: $P|Q \Longrightarrow ((P \Longrightarrow \sim P) \Longrightarrow Q)$
⟨*proof*⟩

Versions of *make-neg-rule* and *make-pos-rule* that don't insert new assumptions, for ordinary resolution.

**lemmas** *make-neg-rule'* = *make-refined-neg-rule*

**lemma** *make-pos-rule'*: $[|P|Q; \sim P|] \Longrightarrow Q$
⟨*proof*⟩

Generation of a goal clause – put away the final literal

**lemma** *make-neg-goal*: $\sim P \Longrightarrow ((\sim P \Longrightarrow P) \Longrightarrow False)$
⟨*proof*⟩

**lemma** *make-pos-goal*: $P \Longrightarrow ((P \Longrightarrow \sim P) \Longrightarrow False)$
⟨*proof*⟩

### 37.10.1 Lemmas for Forward Proof

There is a similarity to congruence rules

**lemma** *conj-forward*: $[|\ P'\&Q';\ \ P' \Longrightarrow P;\ \ Q' \Longrightarrow Q\ |] \Longrightarrow P\&Q$
⟨*proof*⟩

**lemma** *disj-forward*: $[|\ P'|Q';\ \ P' \Longrightarrow P;\ \ Q' \Longrightarrow Q\ |] \Longrightarrow P|Q$
⟨*proof*⟩

**lemma** *disj-forward2*:
$[|\ P'|Q';\ \ P' \Longrightarrow P;\ \ [|\ Q';\ P \Longrightarrow False\ |] \Longrightarrow Q\ |] \Longrightarrow P|Q$
⟨*proof*⟩

**lemma** *all-forward*: $[|\ \forall x.\ P'(x);\ \ !!x.\ P'(x) \Longrightarrow P(x)\ |] \Longrightarrow \forall x.\ P(x)$
⟨*proof*⟩

**lemma** *ex-forward*: $[|\ \exists x.\ P'(x);\ \ !!x.\ P'(x) \Longrightarrow P(x)\ |] \Longrightarrow \exists x.\ P(x)$
⟨*proof*⟩

Many of these bindings are used by the ATP linkup, and not just by legacy proof scripts.

⟨*ML*⟩

### 37.11 Meson package

⟨*ML*⟩

### 37.12 Specification package – Hilbertized version

**lemma** *exE-some*: [| *Ex P* ; *c* == *Eps P* |] ==> *P c*
⟨*proof*⟩

⟨*ML*⟩

**end**

## 38 ATP-Linkup: The Isabelle-ATP Linkup

**theory** *ATP-Linkup*
**imports** *Divides Record Hilbert-Choice Plain*
**uses**
*Tools/polyhash.ML*
*Tools/res-clause.ML*
(*Tools/res-axioms.ML*)
(*Tools/res-hol-clause.ML*)
(*Tools/res-reconstruct.ML*)
(*Tools/res-atp.ML*)
(*Tools/atp-manager.ML*)
(*Tools/atp-wrapper.ML*)
~~/*src/Tools/Metis/metis.ML*
(*Tools/metis-tools.ML*)
**begin**

**definition** *COMBI* :: *'a* => *'a*
**where** *COMBI P* == *P*

**definition** *COMBK* :: *'a* => *'b* => *'a*
**where** *COMBK P Q* == *P*

**definition** *COMBB* :: (*'b* => *'c*) => (*'a* => *'b*) => *'a* => *'c*
**where** *COMBB P Q R* == *P* (*Q R*)

**definition** *COMBC* :: (*'a* => *'b* => *'c*) => *'b* => *'a* => *'c*
**where** *COMBC P Q R* == *P R Q*

**definition** *COMBS* :: (*'a* => *'b* => *'c*) => (*'a* => *'b*) => *'a* => *'c*
**where** *COMBS P Q R* == *P R* (*Q R*)

**definition** *fequal* :: *'a* => *'a* => *bool*
**where** *fequal X Y* == (*X*=*Y*)

**lemma** *fequal-imp-equal*: *fequal X Y ==> X=Y*
⟨*proof*⟩

**lemma** *equal-imp-fequal*: *X=Y ==> fequal X Y*
⟨*proof*⟩

These two represent the equivalence between Boolean equality and iff. They can't be converted to clauses automatically, as the iff would be expanded...

**lemma** *iff-positive*: *P | Q | P=Q*
⟨*proof*⟩

**lemma** *iff-negative*: *~P | ~Q | P=Q*
⟨*proof*⟩

Theorems for translation to combinators

**lemma** *abs-S*: *(%x. (f x) (g x)) == COMBS f g*
⟨*proof*⟩

**lemma** *abs-I*: *(%x. x) == COMBI*
⟨*proof*⟩

**lemma** *abs-K*: *(%x. y) == COMBK y*
⟨*proof*⟩

**lemma** *abs-B*: *(%x. a (g x)) == COMBB a g*
⟨*proof*⟩

**lemma** *abs-C*: *(%x. (f x) b) == COMBC f b*
⟨*proof*⟩

## 38.1 Setup of external ATPs

⟨*ML*⟩

basic provers

⟨*ML*⟩

provers with stuctured output

⟨*ML*⟩

on some problems better results

⟨*ML*⟩

remote provers via SystemOnTPTP

⟨*ML*⟩

## 38.2 The Metis prover

⟨*ML*⟩

**end**

# 39 List: The datatype of finite lists

**theory** *List*
**imports** *Plain Relation-Power Presburger Recdef ATP-Linkup*
**uses** *Tools/string-syntax.ML*
**begin**

**datatype** *'a list* =
  *Nil* ([])
| *Cons 'a 'a list* (**infixr** # *65*)

## 39.1 Basic list processing functions

**consts**
  *filter*:: (*'a => bool*) => *'a list => 'a list*
  *concat*:: *'a list list => 'a list*
  *foldl* :: (*'b => 'a => 'b*) => *'b => 'a list => 'b*
  *foldr* :: (*'a => 'b => 'b*) => *'a list => 'b => 'b*
  *hd*:: *'a list => 'a*
  *tl*:: *'a list => 'a list*
  *last*:: *'a list => 'a*
  *butlast* :: *'a list => 'a list*
  *set* :: *'a list => 'a set*
  *map* :: (*'a=>'b*) => (*'a list => 'b list*)
  *listsum* :: *'a list => 'a*::*monoid-add*
  *list-update* :: *'a list => nat => 'a => 'a list*
  *take*:: *nat => 'a list => 'a list*
  *drop*:: *nat => 'a list => 'a list*
  *takeWhile* :: (*'a => bool*) => *'a list => 'a list*
  *dropWhile* :: (*'a => bool*) => *'a list => 'a list*
  *rev* :: *'a list => 'a list*
  *zip* :: *'a list => 'b list => ('a * 'b) list*
  *upt* :: *nat => nat => nat list* ((*1*[-..</-']))
  *remdups* :: *'a list => 'a list*
  *remove1* :: *'a => 'a list => 'a list*
  *removeAll* :: *'a => 'a list => 'a list*
  *distinct*:: *'a list => bool*
  *replicate* :: *nat => 'a => 'a list*
  *splice* :: *'a list ⇒ 'a list ⇒ 'a list*

**nonterminals** *lupdbinds lupdbind*

**syntax**
  — list Enumeration
  @*list* :: *args => 'a list* ([(-)])

— Special syntax for filter
@*filter* :: [*pttrn*, *'a list*, *bool*] => *'a list*  ((*1*[-<−-./ -]))

— list update
-*lupdbind*:: ['*a*, *'a*] => *lupdbind*  ((*2*- :=/ -))
:: *lupdbind* => *lupdbinds*  (-)
-*lupdbinds* :: [*lupdbind*, *lupdbinds*] => *lupdbinds*  (-,/ -)
-*LUpdate* :: ['*a*, *lupdbinds*] => *'a*  (-/[(-)] [*900*,*0*] *900*)

**translations**
[*x*, *xs*] == *x*#[*xs*]
[*x*] == *x*#[]
[*x*<−*xs* . *P*]== *filter* (%*x*. *P*) *xs*

-*LUpdate xs* (-*lupdbinds b bs*)== -*LUpdate* (-*LUpdate xs b*) *bs*
*xs*[*i*:=*x*] == *list-update xs i x*

**syntax** (*xsymbols*)
@*filter* :: [*pttrn*, *'a list*, *bool*] => *'a list*((*1*[-←- ./ -]))
**syntax** (*HTML* **output**)
@*filter* :: [*pttrn*, *'a list*, *bool*] => *'a list*((*1*[-←- ./ -]))

Function *size* is overloaded for all datatypes. Users may refer to the list version as *length*.

**abbreviation**
*length* :: *'a list* => *nat* **where**
*length* == *size*

**primrec**
*hd*(*x*#*xs*) = *x*

**primrec**
*tl*([]) = []
*tl*(*x*#*xs*) = *xs*

**primrec**
*last*(*x*#*xs*) = (*if xs*=[] *then x else last xs*)

**primrec**
*butlast* []= []
*butlast*(*x*#*xs*) = (*if xs*=[] *then* [] *else x*#*butlast xs*)

**primrec**
*set* [] = {}
*set* (*x*#*xs*) = *insert x* (*set xs*)

**primrec**

*map f [] = []*
*map f (x#xs) = f(x)#map f xs*

**primrec**
*append* :: *'a list* ⇒ *'a list* ⇒ *'a list* (**infixr** @ *65*)
**where**
*append-Nil*:[] @ *ys* = *ys*
| *append-Cons*: (*x#xs*) @ *ys* = *x* # *xs* @ *ys*

**primrec**
*rev*([]) = []
*rev*(*x#xs*) = *rev*(*xs*) @ [*x*]

**primrec**
*filter P* [] = []
*filter P* (*x#xs*) = (*if P x then x#filter P xs else filter P xs*)

**primrec**
*foldl-Nil*:*foldl f a* [] = *a*
*foldl-Cons*: *foldl f a* (*x#xs*) = *foldl f* (*f a x*) *xs*

**primrec**
*foldr f* [] *a* = *a*
*foldr f* (*x#xs*) *a* = *f x* (*foldr f xs a*)

**primrec**
*concat*([]) = []
*concat*(*x#xs*) = *x* @ *concat*(*xs*)

**primrec**
*listsum* [] = *0*
*listsum* (*x* # *xs*) = *x* + *listsum xs*

**primrec**
*drop-Nil*:*drop n* [] = []
*drop-Cons*: *drop n* (*x#xs*) = (*case n of 0* => *x#xs* | *Suc*(*m*) => *drop m xs*)
— Warning: simpset does not contain this definition, but separate theorems for *n* = *0* and *n* = *Suc k*

**primrec**
*take-Nil*:*take n* [] = []
*take-Cons*: *take n* (*x#xs*) = (*case n of 0* => [] | *Suc*(*m*) => *x* # *take m xs*)
— Warning: simpset does not contain this definition, but separate theorems for *n* = *0* and *n* = *Suc k*

**primrec** *nth* :: *'a list* => *nat* => *'a* (**infixl** ! *100*) **where**
*nth-Cons*: (*x#xs*)!*n* = (*case n of 0* => *x* | (*Suc k*) => *xs*!*k*)
— Warning: simpset does not contain this definition, but separate theorems for *n* = *0* and *n* = *Suc k*

**primrec**
[][*i*:=*v*] = []
(*x*#*xs*)[*i*:=*v*] = (*case i of 0 => v # xs | Suc j => x # xs*[*j*:=*v*])

**primrec**
*takeWhile P* [] = []
*takeWhile P* (*x*#*xs*) = (*if P x then x*#*takeWhile P xs else* [])

**primrec**
*dropWhile P* [] = []
*dropWhile P* (*x*#*xs*) = (*if P x then dropWhile P xs else x*#*xs*)

**primrec**
*zip xs* [] = []
*zip-Cons*: *zip xs* (*y*#*ys*) = (*case xs of* [] => [] | *z*#*zs* => (*z*,*y*)#*zip zs ys*)
— Warning: simpset does not contain this definition, but separate theorems for *xs* = [] and *xs* = *z* # *zs*

**primrec**
*upt-0*: [*i*..<*0*] = []
*upt-Suc*: [*i*..<(*Suc j*)] = (*if i <= j then* [*i*..<*j*] @ [*j*] *else* [])

**primrec**
*distinct* [] = *True*
*distinct* (*x*#*xs*) = (*x* ~: *set xs* ∧ *distinct xs*)

**primrec**
*remdups* [] = []
*remdups* (*x*#*xs*) = (*if x* : *set xs then remdups xs else x # remdups xs*)

**primrec**
*remove1 x* [] = []
*remove1 x* (*y*#*xs*) = (*if x*=*y then xs else y # remove1 x xs*)

**primrec**
*removeAll x* [] = []
*removeAll x* (*y*#*xs*) = (*if x*=*y then removeAll x xs else y # removeAll x xs*)

**primrec**
*replicate-0*: *replicate 0 x* = []
*replicate-Suc*: *replicate* (*Suc n*) *x* = *x* # *replicate n x*

**definition**
*rotate1* :: *'a list* ⇒ *'a list* **where**
*rotate1 xs* = (*case xs of* [] ⇒ [] | *x*#*xs* ⇒ *xs* @ [*x*])

**definition**
*rotate* :: *nat* ⇒ *'a list* ⇒ *'a list* **where**

*rotate n = rotate1 ^ n*

**definition**
*list-all2* :: *('a => 'b => bool) => 'a list => 'b list => bool* **where**
[*code del*]: *list-all2 P xs ys =*
*(length xs = length ys ∧ (∀(x, y) ∈ set (zip xs ys). P x y))*

**definition**
*sublist* :: *'a list => nat set => 'a list* **where**
*sublist xs A = map fst (filter (λp. snd p ∈ A) (zip xs [0..<size xs]))*

**primrec**
*splice [] ys = ys*
*splice (x#xs) ys = (if ys=[] then x#xs else x # hd ys # splice xs (tl ys))*
— Warning: simpset does not contain the second eqn but a derived one.

Figure 1 shows characteristic examples that should give an intuitive understanding of the above functions.

The following simple sort functions are intended for proofs, not for efficient implementations.

**context** *linorder*
**begin**

**fun** *sorted* :: *'a list ⇒ bool* **where**
*sorted [] ⟷ True |*
*sorted [x] ⟷ True |*
*sorted (x#y#zs) ⟷ x <= y ∧ sorted (y#zs)*

**primrec** *insort* :: *'a ⇒ 'a list ⇒ 'a list* **where**
*insort x [] = [x] |*
*insort x (y#ys) = (if x <= y then (x#y#ys) else y#(insort x ys))*

**primrec** *sort* :: *'a list ⇒ 'a list* **where**
*sort [] = [] |*
*sort (x#xs) = insort x (sort xs)*

**end**

### 39.1.1 List comprehension

Input syntax for Haskell-like list comprehension notation. Typical example: [(*x*,*y*). *x* ← *xs*, *y* ← *ys*, *x* ≠ *y*], the list of all pairs of distinct elements from *xs* and *ys*. The syntax is as in Haskell, except that | becomes a dot (like in Isabelle's set comprehension): [*e*. *x* ← *xs*, ...] rather than `[e| x <- xs, ...]`.
The qualifiers after the dot are

$[a, b] @ [c, d] = [a, b, c, d]$
*length* $[a, b, c] = 3$
*set* $[a, b, c] = \{a, b, c\}$
*map f* $[a, b, c] = [f\ a, f\ b, f\ c]$
*rev* $[a, b, c] = [c, b, a]$
*hd* $[a, b, c, d] = a$
*tl* $[a, b, c, d] = [b, c, d]$
*last* $[a, b, c, d] = d$
*butlast* $[a, b, c, d] = [a, b, c]$
*filter* $(\lambda n{::}nat.\ n{<}2)\ [0,2,1] = [0,1]$
*concat* $[[a, b], [c, d, e], [], [f]] = [a, b, c, d, e, f]$
*foldl f x* $[a, b, c] = f\ (f\ (f\ x\ a)\ b)\ c$
*foldr f* $[a, b, c]\ x = f\ a\ (f\ b\ (f\ c\ x))$
*zip* $[a, b, c]\ [x, y, z] = [(a, x), (b, y), (c, z)]$
*zip* $[a, b]\ [x, y, z] = [(a, x), (b, y)]$
*splice* $[a, b, c]\ [x, y, z] = [a, x, b, y, c, z]$
*splice* $[a, b, c, d]\ [x, y] = [a, x, b, y, c, d]$
*take 2* $[a, b, c, d] = [a, b]$
*take 6* $[a, b, c, d] = [a, b, c, d]$
*drop 2* $[a, b, c, d] = [c, d]$
*drop 6* $[a, b, c, d] = []$
*takeWhile* $(\lambda n.\ n < 3)\ [1, 2, 3, 0] = [1, 2]$
*dropWhile* $(\lambda n.\ n < 3)\ [1, 2, 3, 0] = [3, 0]$
*distinct* $[2, 0, 1]$
*remdups* $[2, 0, 2, 1, 2] = [0, 1, 2]$
*remove1 2* $[2, 0, 2, 1, 2] = [0, 2, 1, 2]$
*removeAll 2* $[2, 0, 2, 1, 2] = [0, 1]$
$[a, b, c, d]\ !\ 2 = c$
$[a, b, c, d][2 := x] = [a, b, x, d]$
*sublist* $[a, b, c, d, e]\ \{0, 2, 3\} = [a, c, d]$
*rotate1* $[a, b, c, d] = [b, c, d, a]$
*rotate 3* $[a, b, c, d] = [d, a, b, c]$
*replicate 4 a* $= [a, a, a, a]$
$[2..{<}5] = [2, 3, 4]$
*listsum* $[1, 2, 3] = 6$

Figure 1: Characteristic examples

**generators** $p \leftarrow xs$, where $p$ is a pattern and $xs$ an expression of list type, or

**guards** $b$, where $b$ is a boolean expression.

Just like in Haskell, list comprehension is just a shorthand. To avoid misunderstandings, the translation into desugared form is not reversed upon output. Note that the translation of $[e.\ x \leftarrow xs]$ is optmized to $map\ (\lambda x.\ e)\ xs$.

It is easy to write short list comprehensions which stand for complex expressions. During proofs, they may become unreadable (and mangled). In such cases it can be advisable to introduce separate definitions for the list comprehensions in question.

**nonterminals** *lc-qual lc-quals*

**syntax**
*-listcompr* :: $'a \Rightarrow$ *lc-qual* $\Rightarrow$ *lc-quals* $\Rightarrow$ $'a$ *list* ([- . --)
*-lc-gen* :: $'a \Rightarrow$ $'a$ *list* $\Rightarrow$ *lc-qual* (- <− -)
*-lc-test* :: *bool* $\Rightarrow$ *lc-qual* (-)

*-lc-end* :: *lc-quals* (])
*-lc-quals* :: *lc-qual* $\Rightarrow$ *lc-quals* $\Rightarrow$ *lc-quals* (, --)
*-lc-abs* :: $'a$ => $'b$ *list* => $'b$ *list*

**syntax** (*xsymbols*)
*-lc-gen* :: $'a \Rightarrow$ $'a$ *list* $\Rightarrow$ *lc-qual* (- ← -)
**syntax** (*HTML* **output**)
*-lc-gen* :: $'a \Rightarrow$ $'a$ *list* $\Rightarrow$ *lc-qual* (- ← -)

⟨*ML*⟩

### 39.1.2 [] and *op* #

**lemma** *not-Cons-self* [*simp*]:
$xs \neq x\ \#\ xs$
⟨*proof*⟩

**lemmas** *not-Cons-self2* [*simp*] = *not-Cons-self* [*symmetric*]

**lemma** *neq-Nil-conv*: $(xs \neq []) = (\exists\, y\ ys.\ xs = y\ \#\ ys)$
⟨*proof*⟩

**lemma** *length-induct*:
$(\bigwedge xs.\ \forall\, ys.\ length\ ys < length\ xs \longrightarrow P\ ys \Longrightarrow P\ xs) \Longrightarrow P\ xs$
⟨*proof*⟩

### 39.1.3 *length*

Needs to come before @ because of theorem *append-eq-append-conv*.

**lemma** *length-append* [*simp*]: *length* (*xs* @ *ys*) = *length xs* + *length ys*
⟨*proof*⟩

**lemma** *length-map* [*simp*]: *length* (*map f xs*) = *length xs*
⟨*proof*⟩

**lemma** *length-rev* [*simp*]: *length* (*rev xs*) = *length xs*
⟨*proof*⟩

**lemma** *length-tl* [*simp*]: *length* (*tl xs*) = *length xs* − *1*
⟨*proof*⟩

**lemma** *length-0-conv* [*iff*]: (*length xs* = *0*) = (*xs* = []⟩
⟨*proof*⟩

**lemma** *length-greater-0-conv* [*iff*]: (*0* < *length xs*) = (*xs* ≠ []⟩
⟨*proof*⟩

**lemma** *length-pos-if-in-set*: *x* : *set xs* ⟹ *length xs* > *0*
⟨*proof*⟩

**lemma** *length-Suc-conv*:
(*length xs* = *Suc n*) = (∃ *y ys*. *xs* = *y* # *ys* ∧ *length ys* = *n*)
⟨*proof*⟩

**lemma** *Suc-length-conv*:
(*Suc n* = *length xs*) = (∃ *y ys*. *xs* = *y* # *ys* ∧ *length ys* = *n*)
⟨*proof*⟩

**lemma** *impossible-Cons*: *length xs* <= *length ys* ==> *xs* = *x* # *ys* = *False*
⟨*proof*⟩

**lemma** *list-induct2* [*consumes 1*, *case-names Nil Cons*]:
*length xs* = *length ys* ⟹ *P* [] [] ⟹
(⋀*x xs y ys*. *length xs* = *length ys* ⟹ *P xs ys* ⟹ *P* (*x*#*xs*) (*y*#*ys*))
⟹ *P xs ys*
⟨*proof*⟩

**lemma** *list-induct3* [*consumes 2*, *case-names Nil Cons*]:
*length xs* = *length ys* ⟹ *length ys* = *length zs* ⟹ *P* [] [] [] ⟹
(⋀*x xs y ys z zs*. *length xs* = *length ys* ⟹ *length ys* = *length zs* ⟹ *P xs ys zs*
⟹ *P* (*x*#*xs*) (*y*#*ys*) (*z*#*zs*))
⟹ *P xs ys zs*
⟨*proof*⟩

**lemma** *list-induct2'*:

⟦ $P$ [] [];
⋀$x$ $xs$. $P$ ($x$#$xs$) [];
⋀$y$ $ys$. $P$ [] ($y$#$ys$);
⋀$x$ $xs$ $y$ $ys$. $P$ $xs$ $ys$ ⟹ $P$ ($x$#$xs$) ($y$#$ys$) ⟧
⟹ $P$ $xs$ $ys$
⟨*proof*⟩

**lemma** *neq-if-length-neq*: *length* $xs \neq$ *length* $ys \Longrightarrow (xs = ys) == False$
⟨*proof*⟩

⟨*ML*⟩

### 39.1.4 @ − append

**lemma** *append-assoc* [*simp*]: $(xs\ @\ ys)\ @\ zs = xs\ @\ (ys\ @\ zs)$
⟨*proof*⟩

**lemma** *append-Nil2* [*simp*]: $xs\ @\ [] = xs$
⟨*proof*⟩

**lemma** *append-is-Nil-conv* [*iff*]: $(xs\ @\ ys = []) = (xs = [] \wedge ys = [])$
⟨*proof*⟩

**lemma** *Nil-is-append-conv* [*iff*]: $([] = xs\ @\ ys) = (xs = [] \wedge ys = [])$
⟨*proof*⟩

**lemma** *append-self-conv* [*iff*]: $(xs\ @\ ys = xs) = (ys = [])$
⟨*proof*⟩

**lemma** *self-append-conv* [*iff*]: $(xs = xs\ @\ ys) = (ys = [])$
⟨*proof*⟩

**lemma** *append-eq-append-conv* [*simp*, *noatp*]:
*length* $xs$ = *length* $ys \vee$ *length* $us$ = *length* $vs$
==> $(xs@us = ys@vs) = (xs{=}ys \wedge us{=}vs)$
⟨*proof*⟩

**lemma** *append-eq-append-conv2*: $(xs\ @\ ys = zs\ @\ ts) =$
($EX$ $us$. $xs = zs\ @\ us$ & $us\ @\ ys = ts$ | $xs\ @\ us = zs$ & $ys = us@\ ts$)
⟨*proof*⟩

**lemma** *same-append-eq* [*iff*]: $(xs\ @\ ys = xs\ @\ zs) = (ys = zs)$
⟨*proof*⟩

**lemma** *append1-eq-conv* [*iff*]: $(xs\ @\ [x] = ys\ @\ [y]) = (xs = ys \wedge x = y)$
⟨*proof*⟩

**lemma** *append-same-eq* [*iff*]: $(ys\ @\ xs = zs\ @\ xs) = (ys = zs)$
⟨*proof*⟩

**lemma** *append-self-conv2* [*iff*]: $(xs \ @ \ ys = ys) = (xs = [])$
⟨*proof*⟩

**lemma** *self-append-conv2* [*iff*]: $(ys = xs \ @ \ ys) = (xs = [])$
⟨*proof*⟩

**lemma** *hd-Cons-tl* [*simp*,*noatp*]: $xs \neq [] \Longrightarrow hd\ xs \ \# \ tl\ xs = xs$
⟨*proof*⟩

**lemma** *hd-append*: $hd\ (xs \ @ \ ys) = (if\ xs = []\ then\ hd\ ys\ else\ hd\ xs)$
⟨*proof*⟩

**lemma** *hd-append2* [*simp*]: $xs \neq [] \Longrightarrow hd\ (xs \ @ \ ys) = hd\ xs$
⟨*proof*⟩

**lemma** *tl-append*: $tl\ (xs \ @ \ ys) = (case\ xs\ of\ [] \Rightarrow tl\ ys \mid z\#zs \Rightarrow zs \ @ \ ys)$
⟨*proof*⟩

**lemma** *tl-append2* [*simp*]: $xs \neq [] \Longrightarrow tl\ (xs \ @ \ ys) = tl\ xs \ @ \ ys$
⟨*proof*⟩

**lemma** *Cons-eq-append-conv*: $x\#xs = ys@zs =$
$(ys = [] \ \& \ x\#xs = zs \mid (EX\ ys'.\ x\#ys' = ys \ \& \ xs = ys'@zs))$
⟨*proof*⟩

**lemma** *append-eq-Cons-conv*: $(ys@zs = x\#xs) =$
$(ys = [] \ \& \ zs = x\#xs \mid (EX\ ys'.\ ys = x\#ys' \ \& \ ys'@zs = xs))$
⟨*proof*⟩

Trivial rules for solving @-equations automatically.

**lemma** *eq-Nil-appendI*: $xs = ys \Longrightarrow xs = [] \ @ \ ys$
⟨*proof*⟩

**lemma** *Cons-eq-appendI*:
$[|\ x \ \# \ xs1 = ys;\ xs = xs1 \ @ \ zs\ |] \Longrightarrow x \ \# \ xs = ys \ @ \ zs$
⟨*proof*⟩

**lemma** *append-eq-appendI*:
$[|\ xs \ @ \ xs1 = zs;\ ys = xs1 \ @ \ us\ |] \Longrightarrow xs \ @ \ ys = zs \ @ \ us$
⟨*proof*⟩

Simplification procedure for all list equalities. Currently only tries to rearrange @ to see if - both lists end in a singleton list, - or both lists end in the same list.

⟨*ML*⟩

### 39.1.5 *map*

**lemma** *map-ext*: $(!!x.\ x : set\ xs \longrightarrow f\ x = g\ x) \Longrightarrow map\ f\ xs = map\ g\ xs$
⟨*proof*⟩

**lemma** *map-ident* [*simp*]: $map\ (\lambda x.\ x) = (\lambda xs.\ xs)$
⟨*proof*⟩

**lemma** *map-append* [*simp*]: $map\ f\ (xs\ @\ ys) = map\ f\ xs\ @\ map\ f\ ys$
⟨*proof*⟩

**lemma** *map-compose*: $map\ (f\ o\ g)\ xs = map\ f\ (map\ g\ xs)$
⟨*proof*⟩

**lemma** *rev-map*: $rev\ (map\ f\ xs) = map\ f\ (rev\ xs)$
⟨*proof*⟩

**lemma** *map-eq-conv*[*simp*]: $(map\ f\ xs = map\ g\ xs) = (!x : set\ xs.\ f\ x = g\ x)$
⟨*proof*⟩

**lemma** *map-cong* [*fundef-cong*, *recdef-cong*]:
$xs = ys \Longrightarrow (!!x.\ x : set\ ys \Longrightarrow f\ x = g\ x) \Longrightarrow map\ f\ xs = map\ g\ ys$
— a congruence rule for *map*
⟨*proof*⟩

**lemma** *map-is-Nil-conv* [*iff*]: $(map\ f\ xs = []) = (xs = [])$
⟨*proof*⟩

**lemma** *Nil-is-map-conv* [*iff*]: $([] = map\ f\ xs) = (xs = [])$
⟨*proof*⟩

**lemma** *map-eq-Cons-conv*:
$(map\ f\ xs = y\#ys) = (\exists z\ zs.\ xs = z\#zs \wedge f\ z = y \wedge map\ f\ zs = ys)$
⟨*proof*⟩

**lemma** *Cons-eq-map-conv*:
$(x\#xs = map\ f\ ys) = (\exists z\ zs.\ ys = z\#zs \wedge x = f\ z \wedge xs = map\ f\ zs)$
⟨*proof*⟩

**lemmas** *map-eq-Cons-D* = *map-eq-Cons-conv* [*THEN iffD1*]
**lemmas** *Cons-eq-map-D* = *Cons-eq-map-conv* [*THEN iffD1*]
**declare** *map-eq-Cons-D* [*dest*!] *Cons-eq-map-D* [*dest*!]

**lemma** *ex-map-conv*:
$(EX\ xs.\ ys = map\ f\ xs) = (ALL\ y : set\ ys.\ EX\ x.\ y = f\ x)$
⟨*proof*⟩

**lemma** *map-eq-imp-length-eq*:
**assumes** $map\ f\ xs = map\ f\ ys$
**shows** $length\ xs = length\ ys$

⟨*proof*⟩

**lemma** *map-inj-on*:
[| *map f xs* = *map f ys*; *inj-on f* (*set xs Un set ys*) |]
==> *xs* = *ys*
⟨*proof*⟩

**lemma** *inj-on-map-eq-map*:
*inj-on f* (*set xs Un set ys*) ⟹ (*map f xs* = *map f ys*) = (*xs* = *ys*)
⟨*proof*⟩

**lemma** *map-injective*:
*map f xs* = *map f ys* ==> *inj f* ==> *xs* = *ys*
⟨*proof*⟩

**lemma** *inj-map-eq-map*[*simp*]: *inj f* ⟹ (*map f xs* = *map f ys*) = (*xs* = *ys*)
⟨*proof*⟩

**lemma** *inj-mapI*: *inj f* ==> *inj* (*map f*)
⟨*proof*⟩

**lemma** *inj-mapD*: *inj* (*map f*) ==> *inj f*
⟨*proof*⟩

**lemma** *inj-map*[*iff*]: *inj* (*map f*) = *inj f*
⟨*proof*⟩

**lemma** *inj-on-mapI*: *inj-on f* (⋃(*set* ` *A*)) ⟹ *inj-on* (*map f*) *A*
⟨*proof*⟩

**lemma** *map-idI*: (⋀*x*. *x* ∈ *set xs* ⟹ *f x* = *x*) ⟹ *map f xs* = *xs*
⟨*proof*⟩

**lemma** *map-fun-upd* [*simp*]: *y* ∉ *set xs* ⟹ *map* (*f*(*y*:=*v*)) *xs* = *map f xs*
⟨*proof*⟩

**lemma** *map-fst-zip*[*simp*]:
*length xs* = *length ys* ⟹ *map fst* (*zip xs ys*) = *xs*
⟨*proof*⟩

**lemma** *map-snd-zip*[*simp*]:
*length xs* = *length ys* ⟹ *map snd* (*zip xs ys*) = *ys*
⟨*proof*⟩

### 39.1.6 *rev*

**lemma** *rev-append* [*simp*]: *rev* (*xs* @ *ys*) = *rev ys* @ *rev xs*
⟨*proof*⟩

**lemma** *rev-rev-ident* [*simp*]: *rev* (*rev xs*) = *xs*
⟨*proof*⟩

**lemma** *rev-swap*: (*rev xs* = *ys*) = (*xs* = *rev ys*)
⟨*proof*⟩

**lemma** *rev-is-Nil-conv* [*iff*]: (*rev xs* = []) = (*xs* = [])
⟨*proof*⟩

**lemma** *Nil-is-rev-conv* [*iff*]: ([] = *rev xs*) = (*xs* = [])
⟨*proof*⟩

**lemma** *rev-singleton-conv* [*simp*]: (*rev xs* = [*x*]) = (*xs* = [*x*])
⟨*proof*⟩

**lemma** *singleton-rev-conv* [*simp*]: ([*x*] = *rev xs*) = (*xs* = [*x*])
⟨*proof*⟩

**lemma** *rev-is-rev-conv* [*iff*]: (*rev xs* = *rev ys*) = (*xs* = *ys*)
⟨*proof*⟩

**lemma** *inj-on-rev*[*iff*]: *inj-on rev A*
⟨*proof*⟩

**lemma** *rev-induct* [*case-names Nil snoc*]:
[| *P* []; !!*x xs*. *P xs* ==> *P* (*xs* @ [*x*]) |] ==> *P xs*
⟨*proof*⟩

**lemma** *rev-exhaust* [*case-names Nil snoc*]:
(*xs* = [] ==> *P*) ==>(!!*ys y*. *xs* = *ys* @ [*y*] ==> *P*) ==> *P*
⟨*proof*⟩

**lemmas** *rev-cases* = *rev-exhaust*

**lemma** *rev-eq-Cons-iff*[*iff*]: (*rev xs* = *y*#*ys*) = (*xs* = *rev ys* @ [*y*])
⟨*proof*⟩

### 39.1.7 *set*

**lemma** *finite-set* [*iff*]: *finite* (*set xs*)
⟨*proof*⟩

**lemma** *set-append* [*simp*]: *set* (*xs* @ *ys*) = (*set xs* ∪ *set ys*)
⟨*proof*⟩

**lemma** *hd-in-set*[*simp*]: *xs* ≠ [] ⟹ *hd xs* : *set xs*
⟨*proof*⟩

**lemma** *set-subset-Cons*: *set xs* ⊆ *set* (*x* # *xs*)

⟨*proof*⟩

**lemma** *set-ConsD*: $y \in set\ (x \# xs) \Longrightarrow y{=}x \vee y \in set\ xs$
⟨*proof*⟩

**lemma** *set-empty* [*iff*]: $(set\ xs = \{\}) = (xs = [])$
⟨*proof*⟩

**lemma** *set-empty2*[*iff*]: $(\{\} = set\ xs) = (xs = [])$
⟨*proof*⟩

**lemma** *set-rev* [*simp*]: $set\ (rev\ xs) = set\ xs$
⟨*proof*⟩

**lemma** *set-map* [*simp*]: $set\ (map\ f\ xs) = f`(set\ xs)$
⟨*proof*⟩

**lemma** *set-filter* [*simp*]: $set\ (filter\ P\ xs) = \{x.\ x : set\ xs \wedge P\ x\}$
⟨*proof*⟩

**lemma** *set-upt* [*simp*]: $set[i..{<}j] = \{k.\ i \leq k \wedge k < j\}$
⟨*proof*⟩

**lemma** *split-list*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x \# zs$
⟨*proof*⟩

**lemma** *in-set-conv-decomp*: $x \in set\ xs \longleftrightarrow (\exists\, ys\ zs.\ xs = ys\ @\ x \# zs)$
⟨*proof*⟩

**lemma** *split-list-first*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x \# zs \wedge x \notin set\ ys$
⟨*proof*⟩

**lemma** *in-set-conv-decomp-first*:
$(x : set\ xs) = (\exists\, ys\ zs.\ xs = ys\ @\ x \# zs \wedge x \notin set\ ys)$
⟨*proof*⟩

**lemma** *split-list-last*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x \# zs \wedge x \notin set\ zs$
⟨*proof*⟩

**lemma** *in-set-conv-decomp-last*:
$(x : set\ xs) = (\exists\, ys\ zs.\ xs = ys\ @\ x \# zs \wedge x \notin set\ zs)$
⟨*proof*⟩

**lemma** *split-list-prop*: $\exists\, x \in set\ xs.\ P\ x \Longrightarrow \exists\, ys\ x\ zs.\ xs = ys\ @\ x \# zs\ \&\ P\ x$
⟨*proof*⟩

**lemma** *split-list-propE*:
**assumes** $\exists\, x \in set\ xs.\ P\ x$

**obtains** *ys x zs* **where** $xs = ys \mathbin{@} x \# zs$ **and** $P\ x$
⟨*proof*⟩

**lemma** *split-list-first-prop*:
$\exists x \in set\ xs.\ P\ x \Longrightarrow$
$\exists ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall y \in set\ ys.\ \neg\ P\ y)$
⟨*proof*⟩

**lemma** *split-list-first-propE*:
**assumes** $\exists x \in set\ xs.\ P\ x$
**obtains** *ys x zs* **where** $xs = ys \mathbin{@} x \# zs$ **and** $P\ x$ **and** $\forall y \in set\ ys.\ \neg\ P\ y$
⟨*proof*⟩

**lemma** *split-list-first-prop-iff*:
$(\exists x \in set\ xs.\ P\ x) \longleftrightarrow$
$(\exists ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall y \in set\ ys.\ \neg\ P\ y))$
⟨*proof*⟩

**lemma** *split-list-last-prop*:
$\exists x \in set\ xs.\ P\ x \Longrightarrow$
$\exists ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall z \in set\ zs.\ \neg\ P\ z)$
⟨*proof*⟩

**lemma** *split-list-last-propE*:
**assumes** $\exists x \in set\ xs.\ P\ x$
**obtains** *ys x zs* **where** $xs = ys \mathbin{@} x \# zs$ **and** $P\ x$ **and** $\forall z \in set\ zs.\ \neg\ P\ z$
⟨*proof*⟩

**lemma** *split-list-last-prop-iff*:
$(\exists x \in set\ xs.\ P\ x) \longleftrightarrow$
$(\exists ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall z \in set\ zs.\ \neg\ P\ z))$
⟨*proof*⟩

**lemma** *finite-list*: *finite A ==> EX xs. set xs = A*
⟨*proof*⟩

**lemma** *card-length*: $card\ (set\ xs) \leq length\ xs$
⟨*proof*⟩

**lemma** *set-minus-filter-out*:
$set\ xs - \{y\} = set\ (filter\ (\lambda x.\ \neg\ (x = y))\ xs)$
⟨*proof*⟩

### 39.1.8 *filter*

**lemma** *filter-append* [*simp*]: $filter\ P\ (xs \mathbin{@} ys) = filter\ P\ xs \mathbin{@} filter\ P\ ys$
⟨*proof*⟩

**lemma** *rev-filter*: $rev\ (filter\ P\ xs) = filter\ P\ (rev\ xs)$

⟨*proof*⟩

**lemma** *filter-filter* [*simp*]: *filter P* (*filter Q xs*) = *filter* (λ*x*. *Q x* ∧ *P x*) *xs*
⟨*proof*⟩

**lemma** *length-filter-le* [*simp*]: *length* (*filter P xs*) ≤ *length xs*
⟨*proof*⟩

**lemma** *sum-length-filter-compl*:
*length*(*filter P xs*) + *length*(*filter* (%*x*. ~*P x*) *xs*) = *length xs*
⟨*proof*⟩

**lemma** *filter-True* [*simp*]: ∀ *x* ∈ *set xs*. *P x* ==> *filter P xs* = *xs*
⟨*proof*⟩

**lemma** *filter-False* [*simp*]: ∀ *x* ∈ *set xs*. ¬ *P x* ==> *filter P xs* = []
⟨*proof*⟩

**lemma** *filter-empty-conv*: (*filter P xs* = []) = (∀ *x*∈*set xs*. ¬ *P x*)
⟨*proof*⟩

**lemma** *filter-id-conv*: (*filter P xs* = *xs*) = (∀ *x*∈*set xs*. *P x*)
⟨*proof*⟩

**lemma** *filter-map*:
*filter P* (*map f xs*) = *map f* (*filter* (*P o f*) *xs*)
⟨*proof*⟩

**lemma** *length-filter-map*[*simp*]:
*length* (*filter P* (*map f xs*)) = *length*(*filter* (*P o f*) *xs*)
⟨*proof*⟩

**lemma** *filter-is-subset* [*simp*]: *set* (*filter P xs*) ≤ *set xs*
⟨*proof*⟩

**lemma** *length-filter-less*:
⟦ *x* : *set xs*; ~ *P x* ⟧ ⟹ *length*(*filter P xs*) < *length xs*
⟨*proof*⟩

**lemma** *length-filter-conv-card*:
*length*(*filter p xs*) = *card*{*i*. *i* < *length xs* & *p*(*xs*!*i*)}
⟨*proof*⟩

**lemma** *Cons-eq-filterD*:
*x*#*xs* = *filter P ys* ⟹
∃ *us vs*. *ys* = *us* @ *x* # *vs* ∧ (∀ *u*∈*set us*. ¬ *P u*) ∧ *P x* ∧ *xs* = *filter P vs*
(**is** - ⟹ ∃ *us vs*. *?P ys us vs*)
⟨*proof*⟩

**lemma** *filter-eq-ConsD*:
*filter P ys = x#xs ⟹*
*∃us vs. ys = us @ x # vs ∧ (∀u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs*
⟨*proof*⟩

**lemma** *filter-eq-Cons-iff*:
*(filter P ys = x#xs) =*
*(∃us vs. ys = us @ x # vs ∧ (∀u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs)*
⟨*proof*⟩

**lemma** *Cons-eq-filter-iff*:
*(x#xs = filter P ys) =*
*(∃us vs. ys = us @ x # vs ∧ (∀u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs)*
⟨*proof*⟩

**lemma** *filter-cong*[*fundef-cong*, *recdef-cong*]:
*xs = ys ⟹ (⋀x. x ∈ set ys ⟹ P x = Q x) ⟹ filter P xs = filter Q ys*
⟨*proof*⟩

### 39.1.9 List partitioning

**primrec** *partition* :: *('a ⇒ bool) ⇒'a list ⇒ 'a list × 'a list* **where**
*partition P [] = ([], [])*
*| partition P (x # xs) =*
*(let (yes, no) = partition P xs*
*in if P x then (x # yes, no) else (yes, x # no))*

**lemma** *partition-filter1*:
*fst (partition P xs) = filter P xs*
⟨*proof*⟩

**lemma** *partition-filter2*:
*snd (partition P xs) = filter (Not o P) xs*
⟨*proof*⟩

**lemma** *partition-P*:
**assumes** *partition P xs = (yes, no)*
**shows** *(∀p ∈ set yes. P p) ∧ (∀p ∈ set no. ¬ P p)*
⟨*proof*⟩

**lemma** *partition-set*:
**assumes** *partition P xs = (yes, no)*
**shows** *set yes ∪ set no = set xs*
⟨*proof*⟩

### 39.1.10 *concat*

**lemma** *concat-append* [*simp*]: *concat (xs @ ys) = concat xs @ concat ys*
⟨*proof*⟩

**lemma** *concat-eq-Nil-conv* [*simp*]: $(concat\ xss = []) = (\forall xs \in set\ xss.\ xs = [])$
⟨*proof*⟩

**lemma** *Nil-eq-concat-conv* [*simp*]: $([] = concat\ xss) = (\forall xs \in set\ xss.\ xs = [])$
⟨*proof*⟩

**lemma** *set-concat* [*simp*]: $set\ (concat\ xs) = (UN\ x{:}set\ xs.\ set\ x)$
⟨*proof*⟩

**lemma** *concat-map-singleton*[*simp*]: $concat(map\ (\%x.\ [f\ x])\ xs) = map\ f\ xs$
⟨*proof*⟩

**lemma** *map-concat*: $map\ f\ (concat\ xs) = concat\ (map\ (map\ f)\ xs)$
⟨*proof*⟩

**lemma** *filter-concat*: $filter\ p\ (concat\ xs) = concat\ (map\ (filter\ p)\ xs)$
⟨*proof*⟩

**lemma** *rev-concat*: $rev\ (concat\ xs) = concat\ (map\ rev\ (rev\ xs))$
⟨*proof*⟩

### 39.1.11 *nth*

**lemma** *nth-Cons-0* [*simp*, *code*]: $(x\ \#\ xs)!0 = x$
⟨*proof*⟩

**lemma** *nth-Cons-Suc* [*simp*, *code*]: $(x\ \#\ xs)!(Suc\ n) = xs!n$
⟨*proof*⟩

**declare** *nth.simps* [*simp del*]

**lemma** *nth-append*:
$(xs\ @\ ys)!n = (if\ n < length\ xs\ then\ xs!n\ else\ ys!(n - length\ xs))$
⟨*proof*⟩

**lemma** *nth-append-length* [*simp*]: $(xs\ @\ x\ \#\ ys)\ !\ length\ xs = x$
⟨*proof*⟩

**lemma** *nth-append-length-plus*[*simp*]: $(xs\ @\ ys)\ !\ (length\ xs + n) = ys\ !\ n$
⟨*proof*⟩

**lemma** *nth-map* [*simp*]: $n < length\ xs ==> (map\ f\ xs)!n = f(xs!n)$
⟨*proof*⟩

**lemma** *hd-conv-nth*: $xs \neq [] \Longrightarrow hd\ xs = xs!0$
⟨*proof*⟩

**lemma** *list-eq-iff-nth-eq*:

$(xs = ys) = (length\ xs = length\ ys \wedge (ALL\ i{<}length\ xs.\ xs!i = ys!i))$
⟨*proof*⟩

**lemma** *set-conv-nth*: $set\ xs = \{xs!i \mid i.\ i < length\ xs\}$
⟨*proof*⟩

**lemma** *in-set-conv-nth*: $(x \in set\ xs) = (\exists\, i < length\ xs.\ xs!i = x)$
⟨*proof*⟩

**lemma** *list-ball-nth*: $[|\ n < length\ xs;\ !x : set\ xs.\ P\ x|] ==> P(xs!n)$
⟨*proof*⟩

**lemma** *nth-mem* [*simp*]: $n < length\ xs ==> xs!n : set\ xs$
⟨*proof*⟩

**lemma** *all-nth-imp-all-set*:
$[|\ !i < length\ xs.\ P(xs!i);\ x : set\ xs|] ==> P\ x$
⟨*proof*⟩

**lemma** *all-set-conv-all-nth*:
$(\forall\, x \in set\ xs.\ P\ x) = (\forall\, i.\ i < length\ xs\ --> P\ (xs\ !\ i))$
⟨*proof*⟩

**lemma** *rev-nth*:
$n < size\ xs \Longrightarrow rev\ xs\ !\ n = xs\ !\ (length\ xs - Suc\ n)$
⟨*proof*⟩

### 39.1.12 *list-update*

**lemma** *length-list-update* [*simp*]: $length(xs[i:=x]) = length\ xs$
⟨*proof*⟩

**lemma** *nth-list-update*:
$i < length\ xs ==> (xs[i:=x])!j = (if\ i = j\ then\ x\ else\ xs!j)$
⟨*proof*⟩

**lemma** *nth-list-update-eq* [*simp*]: $i < length\ xs ==> (xs[i:=x])!i = x$
⟨*proof*⟩

**lemma** *nth-list-update-neq* [*simp*]: $i \neq j ==> xs[i:=x]!j = xs!j$
⟨*proof*⟩

**lemma** *list-update-id*[*simp*]: $xs[i := xs!i] = xs$
⟨*proof*⟩

**lemma** *list-update-beyond*[*simp*]: $length\ xs \leq i \Longrightarrow xs[i:=x] = xs$
⟨*proof*⟩

**lemma** *list-update-same-conv*:

$i < length\ xs \Longrightarrow (xs[i := x] = xs) = (xs!i = x)$
⟨*proof*⟩

**lemma** *list-update-append1*:
$i < size\ xs \Longrightarrow (xs\ @\ ys)[i:=x] = xs[i:=x]\ @\ ys$
⟨*proof*⟩

**lemma** *list-update-append*:
$(xs\ @\ ys)\ [n:=\ x] =$
$(if\ n < length\ xs\ then\ xs[n:=\ x]\ @\ ys\ else\ xs\ @\ (ys\ [n-length\ xs:=\ x]))$
⟨*proof*⟩

**lemma** *list-update-length* [*simp*]:
$(xs\ @\ x\ \#\ ys)[length\ xs := y] = (xs\ @\ y\ \#\ ys)$
⟨*proof*⟩

**lemma** *update-zip*:
$length\ xs = length\ ys \Longrightarrow$
$(zip\ xs\ ys)[i:=xy] = zip\ (xs[i:=fst\ xy])\ (ys[i:=snd\ xy])$
⟨*proof*⟩

**lemma** *set-update-subset-insert*: $set(xs[i:=x]) <= insert\ x\ (set\ xs)$
⟨*proof*⟩

**lemma** *set-update-subsetI*: $[|\ set\ xs <= A;\ x:A\ |] \Longrightarrow set(xs[i := x]) <= A$
⟨*proof*⟩

**lemma** *set-update-memI*: $n < length\ xs \Longrightarrow x \in set\ (xs[n := x])$
⟨*proof*⟩

**lemma** *list-update-overwrite*:
$xs\ [i := x,\ i := y] = xs\ [i := y]$
⟨*proof*⟩

**lemma** *list-update-swap*:
$i \neq i' \Longrightarrow xs\ [i := x,\ i' := x'] = xs\ [i' := x',\ i := x]$
⟨*proof*⟩

**lemma** *list-update-code* [*code*]:
$[][i := y] = []$
$(x\ \#\ xs)[0 := y] = y\ \#\ xs$
$(x\ \#\ xs)[Suc\ i := y] = x\ \#\ xs[i := y]$
⟨*proof*⟩

### 39.1.13 *last* and *butlast*

**lemma** *last-snoc* [*simp*]: $last\ (xs\ @\ [x]) = x$
⟨*proof*⟩

**lemma** *butlast-snoc* [*simp*]: *butlast* (*xs* @ [*x*]) = *xs*
⟨*proof*⟩

**lemma** *last-ConsL*: *xs* = [] ⟹ *last*(*x*#*xs*) = *x*
⟨*proof*⟩

**lemma** *last-ConsR*: *xs* ≠ [] ⟹ *last*(*x*#*xs*) = *last xs*
⟨*proof*⟩

**lemma** *last-append*: *last*(*xs* @ *ys*) = (*if ys* = [] *then last xs else last ys*)
⟨*proof*⟩

**lemma** *last-appendL*[*simp*]: *ys* = [] ⟹ *last*(*xs* @ *ys*) = *last xs*
⟨*proof*⟩

**lemma** *last-appendR*[*simp*]: *ys* ≠ [] ⟹ *last*(*xs* @ *ys*) = *last ys*
⟨*proof*⟩

**lemma** *hd-rev*: *xs* ≠ [] ⟹ *hd*(*rev xs*) = *last xs*
⟨*proof*⟩

**lemma** *last-rev*: *xs* ≠ [] ⟹ *last*(*rev xs*) = *hd xs*
⟨*proof*⟩

**lemma** *last-in-set*[*simp*]: *as* ≠ [] ⟹ *last as* ∈ *set as*
⟨*proof*⟩

**lemma** *length-butlast* [*simp*]: *length* (*butlast xs*) = *length xs* − *1*
⟨*proof*⟩

**lemma** *butlast-append*:
*butlast* (*xs* @ *ys*) = (*if ys* = [] *then butlast xs else xs* @ *butlast ys*)
⟨*proof*⟩

**lemma** *append-butlast-last-id* [*simp*]:
*xs* ≠ [] ==> *butlast xs* @ [*last xs*] = *xs*
⟨*proof*⟩

**lemma** *in-set-butlastD*: *x* : *set* (*butlast xs*) ==> *x* : *set xs*
⟨*proof*⟩

**lemma** *in-set-butlast-appendI*:
*x* : *set* (*butlast xs*) | *x* : *set* (*butlast ys*) ==> *x* : *set* (*butlast* (*xs* @ *ys*))
⟨*proof*⟩

**lemma** *last-drop*[*simp*]: *n* < *length xs* ⟹ *last* (*drop n xs*) = *last xs*
⟨*proof*⟩

**lemma** *last-conv-nth*: *xs*≠[] ⟹ *last xs* = *xs*!(*length xs* − *1*)

⟨*proof*⟩

**lemma** *butlast-conv-take*: *butlast xs* = *take* (*length xs* − *1*) *xs*
⟨*proof*⟩

### 39.1.14 *take* and *drop*

**lemma** *take-0* [*simp*]: *take 0 xs* = []
⟨*proof*⟩

**lemma** *drop-0* [*simp*]: *drop 0 xs* = *xs*
⟨*proof*⟩

**lemma** *take-Suc-Cons* [*simp*]: *take* (*Suc n*) (*x* # *xs*) = *x* # *take n xs*
⟨*proof*⟩

**lemma** *drop-Suc-Cons* [*simp*]: *drop* (*Suc n*) (*x* # *xs*) = *drop n xs*
⟨*proof*⟩

**declare** *take-Cons* [*simp del*] **and** *drop-Cons* [*simp del*]

**lemma** *take-1-Cons* [*simp*]: *take 1* (*x* # *xs*) = [*x*]
⟨*proof*⟩

**lemma** *drop-1-Cons* [*simp*]: *drop 1* (*x* # *xs*) = *xs*
⟨*proof*⟩

**lemma** *take-Suc*: *xs* ~= [] ==> *take* (*Suc n*) *xs* = *hd xs* # *take n* (*tl xs*)
⟨*proof*⟩

**lemma** *drop-Suc*: *drop* (*Suc n*) *xs* = *drop n* (*tl xs*)
⟨*proof*⟩

**lemma** *take-tl*: *take n* (*tl xs*) = *tl* (*take* (*Suc n*) *xs*)
⟨*proof*⟩

**lemma** *drop-tl*: *drop n* (*tl xs*) = *tl*(*drop n xs*)
⟨*proof*⟩

**lemma** *tl-take*: *tl* (*take n xs*) = *take* (*n* − *1*) (*tl xs*)
⟨*proof*⟩

**lemma** *tl-drop*: *tl* (*drop n xs*) = *drop n* (*tl xs*)
⟨*proof*⟩

**lemma** *nth-via-drop*: *drop n xs* = *y*#*ys* ⟹ *xs*!*n* = *y*
⟨*proof*⟩

**lemma** *take-Suc-conv-app-nth*:

$i < length\ xs \Longrightarrow take\ (Suc\ i)\ xs = take\ i\ xs\ @\ [xs!i]$
⟨*proof*⟩

**lemma** *drop-Suc-conv-tl*:
$i < length\ xs \Longrightarrow (xs!i)\ \#\ (drop\ (Suc\ i)\ xs) = drop\ i\ xs$
⟨*proof*⟩

**lemma** *length-take* [*simp*]: $length\ (take\ n\ xs) = min\ (length\ xs)\ n$
⟨*proof*⟩

**lemma** *length-drop* [*simp*]: $length\ (drop\ n\ xs) = (length\ xs - n)$
⟨*proof*⟩

**lemma** *take-all* [*simp*]: $length\ xs <= n ==> take\ n\ xs = xs$
⟨*proof*⟩

**lemma** *drop-all* [*simp*]: $length\ xs <= n ==> drop\ n\ xs = []$
⟨*proof*⟩

**lemma** *take-append* [*simp*]:
$take\ n\ (xs\ @\ ys) = (take\ n\ xs\ @\ take\ (n - length\ xs)\ ys)$
⟨*proof*⟩

**lemma** *drop-append* [*simp*]:
$drop\ n\ (xs\ @\ ys) = drop\ n\ xs\ @\ drop\ (n - length\ xs)\ ys$
⟨*proof*⟩

**lemma** *take-take* [*simp*]: $take\ n\ (take\ m\ xs) = take\ (min\ n\ m)\ xs$
⟨*proof*⟩

**lemma** *drop-drop* [*simp*]: $drop\ n\ (drop\ m\ xs) = drop\ (n + m)\ xs$
⟨*proof*⟩

**lemma** *take-drop*: $take\ n\ (drop\ m\ xs) = drop\ m\ (take\ (n + m)\ xs)$
⟨*proof*⟩

**lemma** *drop-take*: $drop\ n\ (take\ m\ xs) = take\ (m-n)\ (drop\ n\ xs)$
⟨*proof*⟩

**lemma** *append-take-drop-id* [*simp*]: $take\ n\ xs\ @\ drop\ n\ xs = xs$
⟨*proof*⟩

**lemma** *take-eq-Nil*[*simp*]: $(take\ n\ xs = []) = (n = 0 \vee xs = [])$
⟨*proof*⟩

**lemma** *drop-eq-Nil*[*simp*]: $(drop\ n\ xs = []) = (length\ xs <= n)$
⟨*proof*⟩

**lemma** *take-map*: $take\ n\ (map\ f\ xs) = map\ f\ (take\ n\ xs)$

⟨*proof*⟩

**lemma** *drop-map*: *drop n* (*map f xs*) = *map f* (*drop n xs*)
⟨*proof*⟩

**lemma** *rev-take*: *rev* (*take i xs*) = *drop* (*length xs* − *i*) (*rev xs*)
⟨*proof*⟩

**lemma** *rev-drop*: *rev* (*drop i xs*) = *take* (*length xs* − *i*) (*rev xs*)
⟨*proof*⟩

**lemma** *nth-take* [*simp*]: *i* < *n* ==> (*take n xs*)!*i* = *xs*!*i*
⟨*proof*⟩

**lemma** *nth-drop* [*simp*]:
*n* + *i* <= *length xs* ==> (*drop n xs*)!*i* = *xs*!(*n* + *i*)
⟨*proof*⟩

**lemma** *butlast-take*:
*n* <= *length xs* ==> *butlast* (*take n xs*) = *take* (*n* − *1*) *xs*
⟨*proof*⟩

**lemma** *butlast-drop*: *butlast* (*drop n xs*) = *drop n* (*butlast xs*)
⟨*proof*⟩

**lemma** *take-butlast*: *n* < *length xs* ==> *take n* (*butlast xs*) = *take n xs*
⟨*proof*⟩

**lemma** *drop-butlast*: *drop n* (*butlast xs*) = *butlast* (*drop n xs*)
⟨*proof*⟩

**lemma** *hd-drop-conv-nth*: ⟦ *xs* ≠ []; *n* < *length xs* ⟧ ⟹ *hd*(*drop n xs*) = *xs*!*n*
⟨*proof*⟩

**lemma** *set-take-subset*: *set*(*take n xs*) ⊆ *set xs*
⟨*proof*⟩

**lemma** *set-drop-subset*: *set*(*drop n xs*) ⊆ *set xs*
⟨*proof*⟩

**lemma** *in-set-takeD*: *x* : *set*(*take n xs*) ⟹ *x* : *set xs*
⟨*proof*⟩

**lemma** *in-set-dropD*: *x* : *set*(*drop n xs*) ⟹ *x* : *set xs*
⟨*proof*⟩

**lemma** *append-eq-conv-conj*:
(*xs* @ *ys* = *zs*) = (*xs* = *take* (*length xs*) *zs* ∧ *ys* = *drop* (*length xs*) *zs*)
⟨*proof*⟩

**lemma** *take-add*:
$i+j \leq length(xs) \Longrightarrow take\ (i+j)\ xs = take\ i\ xs\ @\ take\ j\ (drop\ i\ xs)$
⟨*proof*⟩

**lemma** *append-eq-append-conv-if*:
$(xs_1\ @\ xs_2 = ys_1\ @\ ys_2) =$
$(if\ size\ xs_1 \leq size\ ys_1$
$then\ xs_1 = take\ (size\ xs_1)\ ys_1 \wedge xs_2 = drop\ (size\ xs_1)\ ys_1\ @\ ys_2$
$else\ take\ (size\ ys_1)\ xs_1 = ys_1 \wedge drop\ (size\ ys_1)\ xs_1\ @\ xs_2 = ys_2)$
⟨*proof*⟩

**lemma** *take-hd-drop*:
$n < length\ xs \Longrightarrow take\ n\ xs\ @\ [hd\ (drop\ n\ xs)] = take\ (Suc\ n)\ xs$
⟨*proof*⟩

**lemma** *id-take-nth-drop*:
$i < length\ xs \Longrightarrow xs = take\ i\ xs\ @\ xs!i\ \#\ drop\ (Suc\ i)\ xs$
⟨*proof*⟩

**lemma** *upd-conv-take-nth-drop*:
$i < length\ xs \Longrightarrow xs[i:=a] = take\ i\ xs\ @\ a\ \#\ drop\ (Suc\ i)\ xs$
⟨*proof*⟩

**lemma** *nth-drop'*:
$i < length\ xs \Longrightarrow xs\ !\ i\ \#\ drop\ (Suc\ i)\ xs = drop\ i\ xs$
⟨*proof*⟩

### 39.1.15 *takeWhile* and *dropWhile*

**lemma** *takeWhile-dropWhile-id* [*simp*]: $takeWhile\ P\ xs\ @\ dropWhile\ P\ xs = xs$
⟨*proof*⟩

**lemma** *takeWhile-append1* [*simp*]:
$[|\ x{:}set\ xs;\ {\sim}P(x)|] ==> takeWhile\ P\ (xs\ @\ ys) = takeWhile\ P\ xs$
⟨*proof*⟩

**lemma** *takeWhile-append2* [*simp*]:
$(!!x.\ x : set\ xs ==> P\ x) ==> takeWhile\ P\ (xs\ @\ ys) = xs\ @\ takeWhile\ P\ ys$
⟨*proof*⟩

**lemma** *takeWhile-tail*: $\neg\ P\ x ==> takeWhile\ P\ (xs\ @\ (x\#l)) = takeWhile\ P\ xs$
⟨*proof*⟩

**lemma** *dropWhile-append1* [*simp*]:
$[|\ x : set\ xs;\ {\sim}P(x)|] ==> dropWhile\ P\ (xs\ @\ ys) = (dropWhile\ P\ xs)@ys$
⟨*proof*⟩

**lemma** *dropWhile-append2* [*simp*]:

(!!$x$. $x$:*set xs* ==> $P(x)$) ==> *dropWhile* $P$ (*xs* @ *ys*) = *dropWhile* $P$ *ys*
⟨*proof*⟩

**lemma** *set-takeWhileD*: $x$ : *set* (*takeWhile* $P$ *xs*) ==> $x$ : *set xs* ∧ $P$ $x$
⟨*proof*⟩

**lemma** *takeWhile-eq-all-conv*[*simp*]:
(*takeWhile* $P$ *xs* = *xs*) = (∀ $x$ ∈ *set xs*. $P$ $x$)
⟨*proof*⟩

**lemma** *dropWhile-eq-Nil-conv*[*simp*]:
(*dropWhile* $P$ *xs* = []) = (∀ $x$ ∈ *set xs*. $P$ $x$)
⟨*proof*⟩

**lemma** *dropWhile-eq-Cons-conv*:
(*dropWhile* $P$ *xs* = $y$#*ys*) = (*xs* = *takeWhile* $P$ *xs* @ $y$ # *ys* & ¬ $P$ $y$)
⟨*proof*⟩

The following two lemmmas could be generalized to an arbitrary property.

**lemma** *takeWhile-neq-rev*: ⟦*distinct xs*; $x$ ∈ *set xs*⟧ ⟹
*takeWhile* (λ$y$. $y$ ≠ $x$) (*rev xs*) = *rev* (*tl* (*dropWhile* (λ$y$. $y$ ≠ $x$) *xs*))
⟨*proof*⟩

**lemma** *dropWhile-neq-rev*: ⟦*distinct xs*; $x$ ∈ *set xs*⟧ ⟹
*dropWhile* (λ$y$. $y$ ≠ $x$) (*rev xs*) = $x$ # *rev* (*takeWhile* (λ$y$. $y$ ≠ $x$) *xs*)
⟨*proof*⟩

**lemma** *takeWhile-not-last*:
⟦ *xs* ≠ []; *distinct xs*⟧ ⟹ *takeWhile* (λ$y$. $y$ ≠ *last xs*) *xs* = *butlast xs*
⟨*proof*⟩

**lemma** *takeWhile-cong* [*fundef-cong*, *recdef-cong*]:
[| $l$ = $k$; !!$x$. $x$ : *set* $l$ ==> $P$ $x$ = $Q$ $x$ |]
==> *takeWhile* $P$ $l$ = *takeWhile* $Q$ $k$
⟨*proof*⟩

**lemma** *dropWhile-cong* [*fundef-cong*, *recdef-cong*]:
[| $l$ = $k$; !!$x$. $x$ : *set* $l$ ==> $P$ $x$ = $Q$ $x$ |]
==> *dropWhile* $P$ $l$ = *dropWhile* $Q$ $k$
⟨*proof*⟩

### 39.1.16 *zip*

**lemma** *zip-Nil* [*simp*]: *zip* [] *ys* = []
⟨*proof*⟩

**lemma** *zip-Cons-Cons* [*simp*]: *zip* ($x$ # *xs*) ($y$ # *ys*) = ($x$, $y$) # *zip xs ys*
⟨*proof*⟩

**declare** *zip-Cons* [*simp del*]

**lemma** *zip-Cons1*:
*zip* (*x*#*xs*) *ys* = (*case ys of* [] ⇒ [] | *y*#*ys* ⇒ (*x*,*y*)#*zip xs ys*)
⟨*proof*⟩

**lemma** *length-zip* [*simp*]:
*length* (*zip xs ys*) = *min* (*length xs*) (*length ys*)
⟨*proof*⟩

**lemma** *zip-append1*:
*zip* (*xs* @ *ys*) *zs* =
*zip xs* (*take* (*length xs*) *zs*) @ *zip ys* (*drop* (*length xs*) *zs*)
⟨*proof*⟩

**lemma** *zip-append2*:
*zip xs* (*ys* @ *zs*) =
*zip* (*take* (*length ys*) *xs*) *ys* @ *zip* (*drop* (*length ys*) *xs*) *zs*
⟨*proof*⟩

**lemma** *zip-append* [*simp*]:
[| *length xs* = *length us*; *length ys* = *length vs* |] ==>
*zip* (*xs*@*ys*) (*us*@*vs*) = *zip xs us* @ *zip ys vs*
⟨*proof*⟩

**lemma** *zip-rev*:
*length xs* = *length ys* ==> *zip* (*rev xs*) (*rev ys*) = *rev* (*zip xs ys*)
⟨*proof*⟩

**lemma** *map-zip-map*:
*map f* (*zip* (*map g xs*) *ys*) = *map* (%(*x*,*y*). *f*(*g x*, *y*)) (*zip xs ys*)
⟨*proof*⟩

**lemma** *map-zip-map2*:
*map f* (*zip xs* (*map g ys*)) = *map* (%(*x*,*y*). *f*(*x*, *g y*)) (*zip xs ys*)
⟨*proof*⟩

**lemma** *nth-zip* [*simp*]:
[| *i* < *length xs*; *i* < *length ys*|] ==> (*zip xs ys*)!*i* = (*xs*!*i*, *ys*!*i*)
⟨*proof*⟩

**lemma** *set-zip*:
*set* (*zip xs ys*) = {(*xs*!*i*, *ys*!*i*) | *i*. *i* < *min* (*length xs*) (*length ys*)}
⟨*proof*⟩

**lemma** *zip-update*:
*length xs* = *length ys* ==> *zip* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*]) = (*zip xs ys*)[*i*:=(*x*,*y*)]
⟨*proof*⟩

**lemma** *zip-replicate* [*simp*]:
*zip* (*replicate i x*) (*replicate j y*) = *replicate* (*min i j*) (*x*,*y*)
⟨*proof*⟩

**lemma** *take-zip*:
*take n* (*zip xs ys*) = *zip* (*take n xs*) (*take n ys*)
⟨*proof*⟩

**lemma** *drop-zip*:
*drop n* (*zip xs ys*) = *zip* (*drop n xs*) (*drop n ys*)
⟨*proof*⟩

**lemma** *set-zip-leftD*:
(*x*,*y*)∈ *set* (*zip xs ys*) ⟹ *x* ∈ *set xs*
⟨*proof*⟩

**lemma** *set-zip-rightD*:
(*x*,*y*)∈ *set* (*zip xs ys*) ⟹ *y* ∈ *set ys*
⟨*proof*⟩

**lemma** *in-set-zipE*:
(*x*,*y*) : *set*(*zip xs ys*) ⟹ (⟦ *x* : *set xs*; *y* : *set ys* ⟧ ⟹ *R*) ⟹ *R*
⟨*proof*⟩

**lemma** *zip-map-fst-snd*:
*zip* (*map fst zs*) (*map snd zs*) = *zs*
⟨*proof*⟩

**lemma** *zip-eq-conv*:
*length xs* = *length ys* ⟹ *zip xs ys* = *zs* ⟷ *map fst zs* = *xs* ∧ *map snd zs* = *ys*
⟨*proof*⟩

### 39.1.17 *list-all2*

**lemma** *list-all2-lengthD* [*intro?*]:
*list-all2 P xs ys* ==> *length xs* = *length ys*
⟨*proof*⟩

**lemma** *list-all2-Nil* [*iff*, *code*]: *list-all2 P* [] *ys* = (*ys* = [])
⟨*proof*⟩

**lemma** *list-all2-Nil2* [*iff*, *code*]: *list-all2 P xs* [] = (*xs* = [])
⟨*proof*⟩

**lemma** *list-all2-Cons* [*iff*, *code*]:
*list-all2 P* (*x* # *xs*) (*y* # *ys*) = (*P x y* ∧ *list-all2 P xs ys*)
⟨*proof*⟩

**lemma** *list-all2-Cons1*:

*list-all2 P (x # xs) ys = (∃ z zs. ys = z # zs ∧ P x z ∧ list-all2 P xs zs)*
⟨*proof*⟩

**lemma** *list-all2-Cons2*:
*list-all2 P xs (y # ys) = (∃ z zs. xs = z # zs ∧ P z y ∧ list-all2 P zs ys)*
⟨*proof*⟩

**lemma** *list-all2-rev* [*iff*]:
*list-all2 P (rev xs) (rev ys) = list-all2 P xs ys*
⟨*proof*⟩

**lemma** *list-all2-rev1*:
*list-all2 P (rev xs) ys = list-all2 P xs (rev ys)*
⟨*proof*⟩

**lemma** *list-all2-append1*:
*list-all2 P (xs @ ys) zs =*
*(EX us vs. zs = us @ vs ∧ length us = length xs ∧ length vs = length ys ∧*
*list-all2 P xs us ∧ list-all2 P ys vs)*
⟨*proof*⟩

**lemma** *list-all2-append2*:
*list-all2 P xs (ys @ zs) =*
*(EX us vs. xs = us @ vs ∧ length us = length ys ∧ length vs = length zs ∧*
*list-all2 P us ys ∧ list-all2 P vs zs)*
⟨*proof*⟩

**lemma** *list-all2-append*:
*length xs = length ys ⟹*
*list-all2 P (xs@us) (ys@vs) = (list-all2 P xs ys ∧ list-all2 P us vs)*
⟨*proof*⟩

**lemma** *list-all2-appendI* [*intro?*, *trans*]:
*⟦ list-all2 P a b; list-all2 P c d ⟧ ⟹ list-all2 P (a@c) (b@d)*
⟨*proof*⟩

**lemma** *list-all2-conv-all-nth*:
*list-all2 P xs ys =*
*(length xs = length ys ∧ (∀ i < length xs. P (xs!i) (ys!i)))*
⟨*proof*⟩

**lemma** *list-all2-trans*:
**assumes** *tr*: !!*a b c. P1 a b ==> P2 b c ==> P3 a c*
**shows** !!*bs cs. list-all2 P1 as bs ==> list-all2 P2 bs cs ==> list-all2 P3 as cs*
(**is** !!*bs cs. PROP ?Q as bs cs*)
⟨*proof*⟩

**lemma** *list-all2-all-nthI* [*intro?*]:
*length a = length b ⟹ (⋀n. n < length a ⟹ P (a!n) (b!n)) ⟹ list-all2 P a b*

⟨*proof*⟩

**lemma** *list-all2I*:
∀ *x* ∈ *set* (*zip a b*). *split P x* ⟹ *length a* = *length b* ⟹ *list-all2 P a b*
⟨*proof*⟩

**lemma** *list-all2-nthD*:
⟦ *list-all2 P xs ys*; *p* < *size xs* ⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
⟨*proof*⟩

**lemma** *list-all2-nthD2*:
⟦*list-all2 P xs ys*; *p* < *size ys*⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
⟨*proof*⟩

**lemma** *list-all2-map1*:
*list-all2 P* (*map f as*) *bs* = *list-all2* (λ*x y*. *P* (*f x*) *y*) *as bs*
⟨*proof*⟩

**lemma** *list-all2-map2*:
*list-all2 P as* (*map f bs*) = *list-all2* (λ*x y*. *P x* (*f y*)) *as bs*
⟨*proof*⟩

**lemma** *list-all2-refl* [*intro?*]:
(⋀*x*. *P x x*) ⟹ *list-all2 P xs xs*
⟨*proof*⟩

**lemma** *list-all2-update-cong*:
⟦ *i*<*size xs*; *list-all2 P xs ys*; *P x y* ⟧ ⟹ *list-all2 P* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*])
⟨*proof*⟩

**lemma** *list-all2-update-cong2*:
⟦*list-all2 P xs ys*; *P x y*; *i* < *length ys*⟧ ⟹ *list-all2 P* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*])
⟨*proof*⟩

**lemma** *list-all2-takeI* [*simp*,*intro?*]:
*list-all2 P xs ys* ⟹ *list-all2 P* (*take n xs*) (*take n ys*)
⟨*proof*⟩

**lemma** *list-all2-dropI* [*simp*,*intro?*]:
*list-all2 P as bs* ⟹ *list-all2 P* (*drop n as*) (*drop n bs*)
⟨*proof*⟩

**lemma** *list-all2-mono* [*intro?*]:
*list-all2 P xs ys* ⟹ (⋀*xs ys*. *P xs ys* ⟹ *Q xs ys*) ⟹ *list-all2 Q xs ys*
⟨*proof*⟩

**lemma** *list-all2-eq*:
*xs* = *ys* ⟷ *list-all2* (*op* =) *xs ys*
⟨*proof*⟩

### 39.1.18 *foldl* and *foldr*

**lemma** *foldl-append* [*simp*]:
*foldl f a (xs @ ys) = foldl f (foldl f a xs) ys*
⟨*proof*⟩

**lemma** *foldr-append*[*simp*]: *foldr f (xs @ ys) a = foldr f xs (foldr f ys a)*
⟨*proof*⟩

**lemma** *foldr-map*: *foldr g (map f xs) a = foldr (g o f) xs a*
⟨*proof*⟩

For efficient code generation: avoid intermediate list.

**lemma** *foldl-map*[*code unfold*]:
*foldl g a (map f xs) = foldl (%a x. g a (f x)) a xs*
⟨*proof*⟩

**lemma** *foldl-cong* [*fundef-cong*, *recdef-cong*]:
[| *a = b*; *l = k*; !!*a x*. *x : set l ==> f a x = g a x* |]
*==> foldl f a l = foldl g b k*
⟨*proof*⟩

**lemma** *foldr-cong* [*fundef-cong*, *recdef-cong*]:
[| *a = b*; *l = k*; !!*a x*. *x : set l ==> f x a = g x a* |]
*==> foldr f l a = foldr g k b*
⟨*proof*⟩

**lemma** (**in** *semigroup-add*) *foldl-assoc*:
**shows** *foldl op+ (x+y) zs = x + (foldl op+ y zs)*
⟨*proof*⟩

**lemma** (**in** *monoid-add*) *foldl-absorb0*:
**shows** *x + (foldl op+ 0 zs) = foldl op+ x zs*
⟨*proof*⟩

The "First Duality Theorem" in Bird & Wadler:

**lemma** *foldl-foldr1-lemma*:
*foldl op + a xs = a + foldr op + xs (0::'a::monoid-add)*
⟨*proof*⟩

**corollary** *foldl-foldr1*:
*foldl op + 0 xs = foldr op + xs (0::'a::monoid-add)*
⟨*proof*⟩

The "Third Duality Theorem" in Bird & Wadler:

**lemma** *foldr-foldl*: *foldr f xs a = foldl (%x y. f y x) a (rev xs)*
⟨*proof*⟩

**lemma** *foldl-foldr*: *foldl f a xs = foldr (%x y. f y x) (rev xs) a*

⟨*proof*⟩

**lemma** (**in** *ab-semigroup-add*) *foldr-conv-foldl*: *foldr op + xs a = foldl op + a xs*
⟨*proof*⟩

Note: $n \leq$ *foldl* (*op* +) *n ns* looks simpler, but is more difficult to use because it requires an additional transitivity step.

**lemma** *start-le-sum*: (*m*::*nat*) <= *n* ==> *m* <= *foldl* (*op* +) *n ns*
⟨*proof*⟩

**lemma** *elem-le-sum*: (*n*::*nat*) : *set ns* ==> *n* <= *foldl* (*op* +) *0 ns*
⟨*proof*⟩

**lemma** *sum-eq-0-conv* [*iff*]:
(*foldl* (*op* +) (*m*::*nat*) *ns* = *0*) = (*m* = *0* ∧ (∀ *n* ∈ *set ns*. *n* = *0*))
⟨*proof*⟩

**lemma** *foldr-invariant*:
⟦*Q x* ; ∀ *x*∈ *set xs*. *P x*; ∀ *x y*. *P x* ∧ *Q y* ⟶ *Q* (*f x y*) ⟧ ⟹ *Q* (*foldr f xs x*)
⟨*proof*⟩

**lemma** *foldl-invariant*:
⟦*Q x* ; ∀ *x*∈ *set xs*. *P x*; ∀ *x y*. *P x* ∧ *Q y* ⟶ *Q* (*f y x*) ⟧ ⟹ *Q* (*foldl f x xs*)
⟨*proof*⟩

*foldl* and *concat*

**lemma** *foldl-conv-concat*:
*foldl* (*op* @) *xs xss* = *xs* @ *concat xss*
⟨*proof*⟩

**lemma** *concat-conv-foldl*: *concat xss* = *foldl* (*op* @) [] *xss*
⟨*proof*⟩

### 39.1.19 List summation: *listsum* and $\sum$

**lemma** *listsum-append* [*simp*]: *listsum* (*xs* @ *ys*) = *listsum xs* + *listsum ys*
⟨*proof*⟩

**lemma** *listsum-rev* [*simp*]:
**fixes** *xs* :: *'a*::*comm-monoid-add list*
**shows** *listsum* (*rev xs*) = *listsum xs*
⟨*proof*⟩

**lemma** *listsum-foldr*: *listsum xs* = *foldr* (*op* +) *xs 0*
⟨*proof*⟩

**lemma** *length-concat*: *length* (*concat xss*) = *listsum* (*map length xss*)
⟨*proof*⟩

For efficient code generation — *listsum* is not tail recursive but *foldl* is.

**lemma** *listsum*[*code unfold*]: *listsum xs* = *foldl* (*op* +) *0 xs*
⟨*proof*⟩

Some syntactic sugar for summing a function over a list:

**syntax**
*-listsum* :: *pttrn* => *'a list* => *'b* => *'b* ((*3SUM -<−-. -*) [*0*, *51*, *10*] *10*)
**syntax** (*xsymbols*)
*-listsum* :: *pttrn* => *'a list* => *'b* => *'b* ((*3*∑ *-←-. -*) [*0*, *51*, *10*] *10*)
**syntax** (*HTML* **output**)
*-listsum* :: *pttrn* => *'a list* => *'b* => *'b* ((*3*∑ *-←-. -*) [*0*, *51*, *10*] *10*)

**translations** — Beware of argument permutation!
*SUM x<−xs. b* == *CONST listsum* (*map* (%*x. b*) *xs*)
∑ *x←xs. b* == *CONST listsum* (*map* (%*x. b*) *xs*)

**lemma** *listsum-triv*: (∑ *x←xs. r*) = *of-nat* (*length xs*) ∗ *r*
⟨*proof*⟩

**lemma** *listsum-0* [*simp*]: (∑ *x←xs. 0*) = *0*
⟨*proof*⟩

For non-Abelian groups *xs* needs to be reversed on one side:

**lemma** *uminus-listsum-map*:
**fixes** *f* :: *'a* ⇒ *'b*::*ab-group-add*
**shows** − *listsum* (*map f xs*) = (*listsum* (*map* (*uminus o f*) *xs*))
⟨*proof*⟩

### 39.1.20 *upt*

**lemma** *upt-rec*[*code*]: [*i..<j*] = (*if i<j then i#*[*Suc i..<j*] *else* []) — simp does not terminate!
⟨*proof*⟩

**lemma** *upt-conv-Nil* [*simp*]: *j* <= *i* ==> [*i..<j*] = []
⟨*proof*⟩

**lemma** *upt-eq-Nil-conv*[*simp*]: ([*i..<j*] = []) = (*j* = *0* ∨ *j* <= *i*)
⟨*proof*⟩

**lemma** *upt-eq-Cons-conv*:
([*i..<j*] = *x#xs*) = (*i* < *j* & *i* = *x* & [*i+1..<j*] = *xs*)
⟨*proof*⟩

**lemma** *upt-Suc-append*: *i* <= *j* ==> [*i..<*(*Suc j*)] = [*i..<j*]@[*j*] — Only needed if *upt-Suc* is deleted from the simpset.
⟨*proof*⟩

**lemma** *upt-conv-Cons*: *i* < *j* ==> [*i..<j*] = *i* # [*Suc i..<j*]
⟨*proof*⟩

**lemma** *upt-add-eq-append*: $i<=j$ ==> $[i..<j+k] = [i..<j]@[j..<j+k]$
— LOOPS as a simprule, since $j <= j$.
⟨*proof*⟩

**lemma** *length-upt* [*simp*]: *length* $[i..<j] = j - i$
⟨*proof*⟩

**lemma** *nth-upt* [*simp*]: $i + k < j$ ==> $[i..<j] ! k = i + k$
⟨*proof*⟩

**lemma** *hd-upt*[*simp*]: $i < j \Longrightarrow hd[i..<j] = i$
⟨*proof*⟩

**lemma** *last-upt*[*simp*]: $i < j \Longrightarrow last[i..<j] = j - 1$
⟨*proof*⟩

**lemma** *take-upt* [*simp*]: $i+m <= n$ ==> *take* $m$ $[i..<n] = [i..<i+m]$
⟨*proof*⟩

**lemma** *drop-upt*[*simp*]: *drop* $m$ $[i..<j] = [i+m..<j]$
⟨*proof*⟩

**lemma** *map-Suc-upt*: *map Suc* $[m..<n] = [Suc\ m..<Suc\ n]$
⟨*proof*⟩

**lemma** *nth-map-upt*: $i < n-m$ ==> (*map f* $[m..<n]$) ! $i = f(m+i)$
⟨*proof*⟩

**lemma** *nth-take-lemma*:
$k <= length\ xs$ ==> $k <= length\ ys$ ==>
(!!$i$. $i < k$ --> $xs!i = ys!i$) ==> *take k xs* = *take k ys*
⟨*proof*⟩

**lemma** *nth-equalityI*:
[| *length xs* = *length ys*; *ALL* $i <$ *length xs*. $xs!i = ys!i$ |] ==> $xs = ys$
⟨*proof*⟩

**lemma** *map-nth*:
*map* ($\lambda i$. *xs* ! $i$) $[0..<length\ xs] = xs$
⟨*proof*⟩

**lemma** *list-all2-antisym*:
⟦ ($\bigwedge x\ y$. ⟦$P\ x\ y$; $Q\ y\ x$⟧ $\Longrightarrow x = y$); *list-all2 P xs ys*; *list-all2 Q ys xs* ⟧
$\Longrightarrow xs = ys$
⟨*proof*⟩

**lemma** *take-equalityI*: $(\forall i.\ take\ i\ xs = take\ i\ ys) ==> xs = ys$
— The famous take-lemma.
⟨*proof*⟩

**lemma** *take-Cons'*:
$take\ n\ (x\ \#\ xs) = (if\ n = 0\ then\ []\ else\ x\ \#\ take\ (n - 1)\ xs)$
⟨*proof*⟩

**lemma** *drop-Cons'*:
$drop\ n\ (x\ \#\ xs) = (if\ n = 0\ then\ x\ \#\ xs\ else\ drop\ (n - 1)\ xs)$
⟨*proof*⟩

**lemma** *nth-Cons'*: $(x\ \#\ xs)!n = (if\ n = 0\ then\ x\ else\ xs!(n - 1))$
⟨*proof*⟩

**lemmas** *take-Cons-number-of* = *take-Cons'*[*of number-of v*,*standard*]
**lemmas** *drop-Cons-number-of* = *drop-Cons'*[*of number-of v*,*standard*]
**lemmas** *nth-Cons-number-of* = *nth-Cons'*[*of - - number-of v*,*standard*]

**declare** *take-Cons-number-of* [*simp*]
*drop-Cons-number-of* [*simp*]
*nth-Cons-number-of* [*simp*]

### 39.1.21 *distinct* and *remdups*

**lemma** *distinct-append* [*simp*]:
$distinct\ (xs\ @\ ys) = (distinct\ xs \wedge distinct\ ys \wedge set\ xs \cap set\ ys = \{\})$
⟨*proof*⟩

**lemma** *distinct-rev*[*simp*]: $distinct(rev\ xs) = distinct\ xs$
⟨*proof*⟩

**lemma** *set-remdups* [*simp*]: $set\ (remdups\ xs) = set\ xs$
⟨*proof*⟩

**lemma** *distinct-remdups* [*iff*]: $distinct\ (remdups\ xs)$
⟨*proof*⟩

**lemma** *distinct-remdups-id*: $distinct\ xs ==> remdups\ xs = xs$
⟨*proof*⟩

**lemma** *remdups-id-iff-distinct* [*simp*]: $remdups\ xs = xs \longleftrightarrow distinct\ xs$
⟨*proof*⟩

**lemma** *finite-distinct-list*: $finite\ A \Longrightarrow EX\ xs.\ set\ xs = A\ \&\ distinct\ xs$
⟨*proof*⟩

**lemma** *remdups-eq-nil-iff* [*simp*]: $(remdups\ x = []) = (x = [])$

⟨*proof*⟩

**lemma** *remdups-eq-nil-right-iff* [*simp*]: ([] = *remdups x*) = (*x* = [])
⟨*proof*⟩

**lemma** *length-remdups-leq*[*iff*]: *length*(*remdups xs*) <= *length xs*
⟨*proof*⟩

**lemma** *length-remdups-eq*[*iff*]:
(*length* (*remdups xs*) = *length xs*) = (*remdups xs* = *xs*)
⟨*proof*⟩

**lemma** *distinct-map*:
*distinct*(*map f xs*) = (*distinct xs* & *inj-on f* (*set xs*))
⟨*proof*⟩

**lemma** *distinct-filter* [*simp*]: *distinct xs* ==> *distinct* (*filter P xs*)
⟨*proof*⟩

**lemma** *distinct-upt*[*simp*]: *distinct*[*i*..<*j*]
⟨*proof*⟩

**lemma** *distinct-take*[*simp*]: *distinct xs* ⟹ *distinct* (*take i xs*)
⟨*proof*⟩

**lemma** *distinct-drop*[*simp*]: *distinct xs* ⟹ *distinct* (*drop i xs*)
⟨*proof*⟩

**lemma** *distinct-list-update*:
**assumes** *d*: *distinct xs* **and** *a*: *a* ∉ *set xs* − {*xs*!*i*}
**shows** *distinct* (*xs*[*i*:=*a*])
⟨*proof*⟩

It is best to avoid this indexed version of distinct, but sometimes it is useful.

**lemma** *distinct-conv-nth*:
*distinct xs* = (∀ *i* < *size xs*. ∀ *j* < *size xs*. *i* ≠ *j* −−> *xs*!*i* ≠ *xs*!*j*)
⟨*proof*⟩

**lemma** *nth-eq-iff-index-eq*:
⟦ *distinct xs*; *i* < *length xs*; *j* < *length xs* ⟧ ⟹ (*xs*!*i* = *xs*!*j*) = (*i* = *j*)
⟨*proof*⟩

**lemma** *distinct-card*: *distinct xs* ==> *card* (*set xs*) = *size xs*
⟨*proof*⟩

**lemma** *card-distinct*: *card* (*set xs*) = *size xs* ==> *distinct xs*
⟨*proof*⟩

**lemma** *not-distinct-decomp*: ~ *distinct ws* ==> *EX xs ys zs y. ws* = *xs*@[*y*]@*ys*@[*y*]@*zs*
⟨*proof*⟩

**lemma** *length-remdups-concat*:
*length*(*remdups*(*concat xss*)) = *card*(⋃ *xs* ∈ *set xss. set xs*)
⟨*proof*⟩

### 39.1.22 *remove1*

**lemma** *remove1-append*:
*remove1 x* (*xs* @ *ys*) =
(*if x* ∈ *set xs then remove1 x xs* @ *ys else xs* @ *remove1 x ys*)
⟨*proof*⟩

**lemma** *in-set-remove1*[*simp*]:
*a* ≠ *b* ⟹ *a* : *set*(*remove1 b xs*) = (*a* : *set xs*)
⟨*proof*⟩

**lemma** *set-remove1-subset*: *set*(*remove1 x xs*) <= *set xs*
⟨*proof*⟩

**lemma** *set-remove1-eq* [*simp*]: *distinct xs* ==> *set*(*remove1 x xs*) = *set xs* − {*x*}
⟨*proof*⟩

**lemma** *length-remove1*:
*length*(*remove1 x xs*) = (*if x* : *set xs then length xs* − *1 else length xs*)
⟨*proof*⟩

**lemma** *remove1-filter-not*[*simp*]:
¬ *P x* ⟹ *remove1 x* (*filter P xs*) = *filter P xs*
⟨*proof*⟩

**lemma** *notin-set-remove1*[*simp*]: *x* ~: *set xs* ==> *x* ~: *set*(*remove1 y xs*)
⟨*proof*⟩

**lemma** *distinct-remove1*[*simp*]: *distinct xs* ==> *distinct*(*remove1 x xs*)
⟨*proof*⟩

### 39.1.23 *removeAll*

**lemma** *removeAll-append*[*simp*]:
*removeAll x* (*xs* @ *ys*) = *removeAll x xs* @ *removeAll x ys*
⟨*proof*⟩

**lemma** *set-removeAll*[*simp*]: *set*(*removeAll x xs*) = *set xs* − {*x*}
⟨*proof*⟩

**lemma** *removeAll-id*[*simp*]: *x* ∉ *set xs* ⟹ *removeAll x xs* = *xs*
⟨*proof*⟩

**lemma** *removeAll-filter-not*[*simp*]:
$\neg\ P\ x \Longrightarrow$ *removeAll x* (*filter P xs*) = *filter P xs*
⟨*proof*⟩

**lemma** *distinct-remove1-removeAll*:
*distinct xs* ==> *remove1 x xs* = *removeAll x xs*
⟨*proof*⟩

**lemma** *map-removeAll-inj-on*: *inj-on f* (*insert x* (*set xs*)) ⟹
*map f* (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
⟨*proof*⟩

**lemma** *map-removeAll-inj*: *inj f* ⟹
*map f* (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
⟨*proof*⟩

### 39.1.24 *replicate*

**lemma** *length-replicate* [*simp*]: *length* (*replicate n x*) = *n*
⟨*proof*⟩

**lemma** *map-replicate* [*simp*]: *map f* (*replicate n x*) = *replicate n* (*f x*)
⟨*proof*⟩

**lemma** *replicate-app-Cons-same*:
(*replicate n x*) @ (*x* # *xs*) = *x* # *replicate n x* @ *xs*
⟨*proof*⟩

**lemma** *rev-replicate* [*simp*]: *rev* (*replicate n x*) = *replicate n x*
⟨*proof*⟩

**lemma** *replicate-add*: *replicate* (*n* + *m*) *x* = *replicate n x* @ *replicate m x*
⟨*proof*⟩

Courtesy of Matthias Daum:

**lemma** *append-replicate-commute*:
*replicate n x* @ *replicate k x* = *replicate k x* @ *replicate n x*
⟨*proof*⟩

**lemma** *hd-replicate* [*simp*]: *n* ≠ *0* ==> *hd* (*replicate n x*) = *x*
⟨*proof*⟩

**lemma** *tl-replicate* [*simp*]: *n* ≠ *0* ==> *tl* (*replicate n x*) = *replicate* (*n* − *1*) *x*
⟨*proof*⟩

**lemma** *last-replicate* [*simp*]: $n \neq 0 \Longrightarrow last\ (replicate\ n\ x) = x$
⟨*proof*⟩

**lemma** *nth-replicate*[*simp*]: $i < n \Longrightarrow (replicate\ n\ x)!i = x$
⟨*proof*⟩

Courtesy of Matthias Daum (2 lemmas):

**lemma** *take-replicate*[*simp*]: $take\ i\ (replicate\ k\ x) = replicate\ (min\ i\ k)\ x$
⟨*proof*⟩

**lemma** *drop-replicate*[*simp*]: $drop\ i\ (replicate\ k\ x) = replicate\ (k-i)\ x$
⟨*proof*⟩

**lemma** *set-replicate-Suc*: $set\ (replicate\ (Suc\ n)\ x) = \{x\}$
⟨*proof*⟩

**lemma** *set-replicate* [*simp*]: $n \neq 0 \Longrightarrow set\ (replicate\ n\ x) = \{x\}$
⟨*proof*⟩

**lemma** *set-replicate-conv-if*: $set\ (replicate\ n\ x) = (if\ n = 0\ then\ \{\}\ else\ \{x\})$
⟨*proof*⟩

**lemma** *in-set-replicateD*: $x : set\ (replicate\ n\ y) \Longrightarrow x = y$
⟨*proof*⟩

**lemma** *replicate-append-same*:
$replicate\ i\ x\ @\ [x] = x\ \#\ replicate\ i\ x$
⟨*proof*⟩

**lemma** *map-replicate-trivial*:
$map\ (\lambda i.\ x)\ [0..<i] = replicate\ i\ x$
⟨*proof*⟩

**lemma** *replicate-empty*[*simp*]: $(replicate\ n\ x = []) \longleftrightarrow n=0$
⟨*proof*⟩

**lemma** *empty-replicate*[*simp*]: $([] = replicate\ n\ x) \longleftrightarrow n=0$
⟨*proof*⟩

**lemma** *replicate-eq-replicate*[*simp*]:
$(replicate\ m\ x = replicate\ n\ y) \longleftrightarrow (m=n\ \&\ (m\neq 0 \longrightarrow x=y))$
⟨*proof*⟩

### 39.1.25 *rotate1* and *rotate*

**lemma** *rotate-simps*[*simp*]: $rotate1\ [] = [] \wedge rotate1\ (x\#xs) = xs\ @\ [x]$
⟨*proof*⟩

**lemma** *rotate0*[*simp*]: *rotate 0 = id*
⟨*proof*⟩

**lemma** *rotate-Suc*[*simp*]: *rotate (Suc n) xs = rotate1*(*rotate n xs*)
⟨*proof*⟩

**lemma** *rotate-add*:
*rotate (m+n) = rotate m o rotate n*
⟨*proof*⟩

**lemma** *rotate-rotate*: *rotate m* (*rotate n xs*) = *rotate* (*m+n*) *xs*
⟨*proof*⟩

**lemma** *rotate1-rotate-swap*: *rotate1* (*rotate n xs*) = *rotate n* (*rotate1 xs*)
⟨*proof*⟩

**lemma** *rotate1-length01*[*simp*]: *length xs <= 1* ⟹ *rotate1 xs = xs*
⟨*proof*⟩

**lemma** *rotate-length01*[*simp*]: *length xs <= 1* ⟹ *rotate n xs = xs*
⟨*proof*⟩

**lemma** *rotate1-hd-tl*: *xs* ≠ [] ⟹ *rotate1 xs = tl xs* @ [*hd xs*]
⟨*proof*⟩

**lemma** *rotate-drop-take*:
*rotate n xs = drop* (*n mod length xs*) *xs* @ *take* (*n mod length xs*) *xs*
⟨*proof*⟩

**lemma** *rotate-conv-mod*: *rotate n xs = rotate* (*n mod length xs*) *xs*
⟨*proof*⟩

**lemma** *rotate-id*[*simp*]: *n mod length xs = 0* ⟹ *rotate n xs = xs*
⟨*proof*⟩

**lemma** *length-rotate1*[*simp*]: *length*(*rotate1 xs*) = *length xs*
⟨*proof*⟩

**lemma** *length-rotate*[*simp*]: *length*(*rotate n xs*) = *length xs*
⟨*proof*⟩

**lemma** *distinct1-rotate*[*simp*]: *distinct*(*rotate1 xs*) = *distinct xs*
⟨*proof*⟩

**lemma** *distinct-rotate*[*simp*]: *distinct*(*rotate n xs*) = *distinct xs*
⟨*proof*⟩

**lemma** *rotate-map*: *rotate n* (*map f xs*) = *map f* (*rotate n xs*)

⟨*proof*⟩

**lemma** *set-rotate1*[*simp*]: *set*(*rotate1 xs*) = *set xs*
⟨*proof*⟩

**lemma** *set-rotate*[*simp*]: *set*(*rotate n xs*) = *set xs*
⟨*proof*⟩

**lemma** *rotate1-is-Nil-conv*[*simp*]: (*rotate1 xs* = []) = (*xs* = [])
⟨*proof*⟩

**lemma** *rotate-is-Nil-conv*[*simp*]: (*rotate n xs* = []) = (*xs* = [])
⟨*proof*⟩

**lemma** *rotate-rev*:
*rotate n* (*rev xs*) = *rev*(*rotate* (*length xs* − (*n mod length xs*)) *xs*)
⟨*proof*⟩

**lemma** *hd-rotate-conv-nth*: *xs* ≠ [] ⟹ *hd*(*rotate n xs*) = *xs*!(*n mod length xs*)
⟨*proof*⟩

### 39.1.26 *sublist* — a generalization of *nth* to sets

**lemma** *sublist-empty* [*simp*]: *sublist xs* {} = []
⟨*proof*⟩

**lemma** *sublist-nil* [*simp*]: *sublist* [] *A* = []
⟨*proof*⟩

**lemma** *length-sublist*:
*length*(*sublist xs I*) = *card*{*i*. *i* < *length xs* ∧ *i* : *I*}
⟨*proof*⟩

**lemma** *sublist-shift-lemma-Suc*:
*map fst* (*filter* (%*p*. *P*(*Suc*(*snd p*))) (*zip xs is*)) =
*map fst* (*filter* (%*p*. *P*(*snd p*)) (*zip xs* (*map Suc is*)))
⟨*proof*⟩

**lemma** *sublist-shift-lemma*:
*map fst* [*p*<−*zip xs* [*i*..<*i* + *length xs*] . *snd p* : *A*] =
*map fst* [*p*<−*zip xs* [0..<*length xs*] . *snd p* + *i* : *A*]
⟨*proof*⟩

**lemma** *sublist-append*:
*sublist* (*l* @ *l*′) *A* = *sublist l A* @ *sublist l*′ {*j*. *j* + *length l* : *A*}
⟨*proof*⟩

**lemma** *sublist-Cons*:
*sublist* (*x* # *l*) *A* = (*if* 0:*A then* [*x*] *else* []) @ *sublist l* {*j*. *Suc j* : *A*}

⟨*proof*⟩

**lemma** *set-sublist*: *set*(*sublist xs I*) = {*xs*!*i*|*i*. *i*<*size xs* ∧ *i* ∈ *I*}
⟨*proof*⟩

**lemma** *set-sublist-subset*: *set*(*sublist xs I*) ⊆ *set xs*
⟨*proof*⟩

**lemma** *notin-set-sublistI*[*simp*]: *x* ∉ *set xs* ⟹ *x* ∉ *set*(*sublist xs I*)
⟨*proof*⟩

**lemma** *in-set-sublistD*: *x* ∈ *set*(*sublist xs I*) ⟹ *x* ∈ *set xs*
⟨*proof*⟩

**lemma** *sublist-singleton* [*simp*]: *sublist* [*x*] *A* = (*if* 0 : *A then* [*x*] *else* [])
⟨*proof*⟩

**lemma** *distinct-sublistI*[*simp*]: *distinct xs* ⟹ *distinct*(*sublist xs I*)
⟨*proof*⟩

**lemma** *sublist-upt-eq-take* [*simp*]: *sublist l* {..<*n*} = *take n l*
⟨*proof*⟩

**lemma** *filter-in-sublist*:
*distinct xs* ⟹ *filter* (%*x*. *x* ∈ *set*(*sublist xs s*)) *xs* = *sublist xs s*
⟨*proof*⟩

### 39.1.27 *splice*

**lemma** *splice-Nil2* [*simp*, *code*]:
*splice xs* [] = *xs*
⟨*proof*⟩

**lemma** *splice-Cons-Cons* [*simp*, *code*]:
*splice* (*x*#*xs*) (*y*#*ys*) = *x* # *y* # *splice xs ys*
⟨*proof*⟩

**declare** *splice.simps*(*2*) [*simp del*, *code del*]

**lemma** *length-splice*[*simp*]: *length*(*splice xs ys*) = *length xs* + *length ys*
⟨*proof*⟩

### 39.1.28 Infiniteness

**lemma** *finite-maxlen*:
*finite* (*M*::'*a list set*) ==> *EX n*. *ALL s*:*M*. *size s* < *n*
⟨*proof*⟩

**lemma** *infinite-UNIV-listI*: ~ *finite*(*UNIV*::'*a list set*)
⟨*proof*⟩

## 39.2 Sorting

Currently it is not shown that *sort* returns a permutation of its input because the nicest proof is via multisets, which are not yet available. Alternatively one could define a function that counts the number of occurrences of an element in a list and use that instead of multisets to state the correctness property.

**context** *linorder*
**begin**

**lemma** *sorted-Cons*: *sorted* (*x*#*xs*) = (*sorted xs* & (*ALL y*:*set xs*. *x* <= *y*))
⟨*proof*⟩

**lemma** *sorted-append*:
*sorted* (*xs*@*ys*) = (*sorted xs* & *sorted ys* & (∀ *x* ∈ *set xs*. ∀ *y* ∈ *set ys*. *x*≤*y*))
⟨*proof*⟩

**lemma** *set-insort*: *set*(*insort x xs*) = *insert x* (*set xs*)
⟨*proof*⟩

**lemma** *set-sort*[*simp*]: *set*(*sort xs*) = *set xs*
⟨*proof*⟩

**lemma** *distinct-insort*: *distinct* (*insort x xs*) = (*x* ∉ *set xs* ∧ *distinct xs*)
⟨*proof*⟩

**lemma** *distinct-sort*[*simp*]: *distinct* (*sort xs*) = *distinct xs*
⟨*proof*⟩

**lemma** *sorted-insort*: *sorted* (*insort x xs*) = *sorted xs*
⟨*proof*⟩

**theorem** *sorted-sort*[*simp*]: *sorted* (*sort xs*)
⟨*proof*⟩

**lemma** *insort-is-Cons*: ∀ *x*∈*set xs*. *a* ≤ *x* ⟹ *insort a xs* = *a* # *xs*
⟨*proof*⟩

**lemma** *sorted-remove1*: *sorted xs* ⟹ *sorted* (*remove1 a xs*)
⟨*proof*⟩

**lemma** *insort-remove1*: ⟦ *a* ∈ *set xs*; *sorted xs* ⟧ ⟹ *insort a* (*remove1 a xs*) = *xs*
⟨*proof*⟩

**lemma** *sorted-remdups*[*simp*]:

*sorted l* $\Longrightarrow$ *sorted* (*remdups l*)
⟨*proof*⟩

**lemma** *sorted-distinct-set-unique*:
**assumes** *sorted xs distinct xs sorted ys distinct ys set xs* = *set ys*
**shows** *xs* = *ys*
⟨*proof*⟩

**lemma** *finite-sorted-distinct-unique*:
**shows** *finite A* $\Longrightarrow$ *EX*! *xs*. *set xs* = *A* & *sorted xs* & *distinct xs*
⟨*proof*⟩

**lemma** *sorted-take*:
*sorted xs* $\Longrightarrow$ *sorted* (*take n xs*)
⟨*proof*⟩

**lemma** *sorted-drop*:
*sorted xs* $\Longrightarrow$ *sorted* (*drop n xs*)
⟨*proof*⟩

**end**

**lemma** *sorted-upt*[*simp*]: *sorted*[*i*..<*j*]
⟨*proof*⟩

### 39.2.1 *sorted-list-of-set*

This function maps (finite) linearly ordered sets to sorted lists. Warning: in most cases it is not a good idea to convert from sets to lists but one should convert in the other direction (via *set*).

**context** *linorder*
**begin**

**definition**
*sorted-list-of-set* :: *'a set* ⇒ *'a list* **where**
[*code del*]: *sorted-list-of-set A* == *THE xs*. *set xs* = *A* & *sorted xs* & *distinct xs*

**lemma** *sorted-list-of-set*[*simp*]: *finite A* $\Longrightarrow$
*set*(*sorted-list-of-set A*) = *A* &
*sorted*(*sorted-list-of-set A*) & *distinct*(*sorted-list-of-set A*)
⟨*proof*⟩

**lemma** *sorted-list-of-empty*[*simp*]: *sorted-list-of-set* {} = []
⟨*proof*⟩

**end**

### 39.2.2 *upto*: the generic interval-list

**class** *finite-intvl-succ* = *linorder* +
**fixes** *successor* :: $'a \Rightarrow 'a$
**assumes** *finite-intvl*: *finite*$\{a..b\}$
**and** *successor-incr*: $a <$ *successor* $a$
**and** *ord-discrete*: $\neg(\exists x.\ a < x\ \&\ x <$ *successor* $a)$

**context** *finite-intvl-succ*
**begin**

**definition**
*upto* :: $'a \Rightarrow 'a \Rightarrow 'a$ *list* ((*1*[-../-])) **where**
*upto* $i\ j ==$ *sorted-list-of-set* $\{i..j\}$

**lemma** *upto*[*simp*]: *set*$[a..b] = \{a..b\}$ & *sorted*$[a..b]$ & *distinct*$[a..b]$
⟨*proof*⟩

**lemma** *insert-intvl*: $i \le j \Longrightarrow$ *insert* $i$ $\{$*successor* $i..j\} = \{i..j\}$
⟨*proof*⟩

**lemma** *sorted-list-of-set-rec*: $i \le j \Longrightarrow$
*sorted-list-of-set* $\{i..j\} = i$ # *sorted-list-of-set* $\{$*successor* $i..j\}$
⟨*proof*⟩

**lemma** *sorted-list-of-set-rec2*: $i \le j \Longrightarrow$
*sorted-list-of-set* $\{i..$*successor* $j\} =$
*sorted-list-of-set* $\{i..j\}$ @ [*successor* $j$]
⟨*proof*⟩

**lemma** *upto-rec*[*code*]: $[i..j] =$ (*if* $i \le j$ *then* $i$ # [*successor* $i..j$] *else* []$)$
⟨*proof*⟩

**lemma** *upto-empty*[*simp*]: $j < i \Longrightarrow [i..j] = []$
⟨*proof*⟩

**lemma** *upto-rec2*: $i \le j \Longrightarrow [i..$*successor* $j] = [i..j]$ @ [*successor* $j$]
⟨*proof*⟩

**end**

The integers are an instance of the above class:

**instantiation** *int*:: *finite-intvl-succ*
**begin**

**definition**
*successor-int-def*: *successor* = (%$i$::*int*. *i+1*)

**instance**
⟨*proof*⟩

**end**

Now $[i..j]$ is defined for integers.

**hide** (**open**) *const successor*

**lemma** *upto-rec2-int*: $(i::int) \leq j \Longrightarrow [i..j+1] = [i..j] \ @ \ [j+1]$
⟨*proof*⟩

### 39.2.3 *lists*: the list-forming operator over sets

**inductive-set**
  *lists* :: *'a set* => *'a list set*
  **for** *A* :: *'a set*
**where**
    *Nil* [*intro*!]: []: *lists A*
  | *Cons* [*intro*!,*noatp*]: [| *a*: *A*; *l*: *lists A*|] ==> *a#l* : *lists A*

**inductive-cases** *listsE* [*elim*!,*noatp*]: *x#l* : *lists A*
**inductive-cases** *listspE* [*elim*!,*noatp*]: *listsp A* (*x* # *l*)

**lemma** *listsp-mono* [*mono*]: *A* ≤ *B* ==> *listsp A* ≤ *listsp B*
⟨*proof*⟩

**lemmas** *lists-mono* = *listsp-mono* [*to-set pred-subset-eq*]

**lemma** *listsp-infI*:
  **assumes** *l*: *listsp A l* **shows** *listsp B l* ==> *listsp* (*inf A B*) *l* ⟨*proof*⟩

**lemmas** *lists-IntI* = *listsp-infI* [*to-set*]

**lemma** *listsp-inf-eq* [*simp*]: *listsp* (*inf A B*) = *inf* (*listsp A*) (*listsp B*)
⟨*proof*⟩

**lemmas** *listsp-conj-eq* [*simp*] = *listsp-inf-eq* [*simplified inf-fun-eq inf-bool-eq*]

**lemmas** *lists-Int-eq* [*simp*] = *listsp-inf-eq* [*to-set pred-equals-eq*]

**lemma** *append-in-listsp-conv* [*iff*]:
    (*listsp A* (*xs* @ *ys*)) = (*listsp A xs* ∧ *listsp A ys*)
⟨*proof*⟩

**lemmas** *append-in-lists-conv* [*iff*] = *append-in-listsp-conv* [*to-set*]

**lemma** *in-listsp-conv-set*: (*listsp A xs*) = (∀ *x* ∈ *set xs*. *A x*)
— eliminate *listsp* in favour of *set*
⟨*proof*⟩

**lemmas** *in-lists-conv-set* = *in-listsp-conv-set* [*to-set*]

**lemma** *in-listspD* [*dest*!,*noatp*]: *listsp A xs ==> ∀x∈set xs. A x*
⟨*proof*⟩

**lemmas** *in-listsD* [*dest*!,*noatp*] = *in-listspD* [*to-set*]

**lemma** *in-listspI* [*intro*!,*noatp*]: *∀x∈set xs. A x ==> listsp A xs*
⟨*proof*⟩

**lemmas** *in-listsI* [*intro*!,*noatp*] = *in-listspI* [*to-set*]

**lemma** *lists-UNIV* [*simp*]: *lists UNIV = UNIV*
⟨*proof*⟩

### 39.2.4 Inductive definition for membership

**inductive** *ListMem* :: *'a ⇒ 'a list ⇒ bool*
**where**
*elem*: *ListMem x (x # xs)*
| *insert*: *ListMem x xs ⟹ ListMem x (y # xs)*

**lemma** *ListMem-iff*: *(ListMem x xs) = (x ∈ set xs)*
⟨*proof*⟩

### 39.2.5 Lists as Cartesian products

*set-Cons A Xs*: the set of lists with head drawn from *A* and tail drawn from *Xs*.

**constdefs**
*set-Cons* :: *'a set ⇒ 'a list set ⇒ 'a list set*
*set-Cons A XS == {z. ∃x xs. z = x#xs & x ∈ A & xs ∈ XS}*
**declare** *set-Cons-def* [*code del*]

**lemma** *set-Cons-sing-Nil* [*simp*]: *set-Cons A {[]} = (%x. [x])`A*
⟨*proof*⟩

Yields the set of lists, all of the same length as the argument and with elements drawn from the corresponding element of the argument.

**consts** *listset* :: *'a set list ⇒ 'a list set*
**primrec**
*listset [] = {[]}*
*listset(A#As) = set-Cons A (listset As)*

## 39.3 Relations on Lists

### 39.3.1 Length Lexicographic Ordering

These orderings preserve well-foundedness: shorter lists precede longer lists. These ordering are not used in dictionaries.

**consts** *lexn* :: $('a * 'a)set => nat => ('a\ list * 'a\ list)set$
— The lexicographic ordering for lists of the specified length
**primrec**
*lexn r 0* = {}
*lexn r* (*Suc n*) =
(*prod-fun* (%(*x*,*xs*). *x*#*xs*) (%(*x*,*xs*). *x*#*xs*) ` (*r* <*lex*> *lexn r n*)) *Int*
{(*xs*,*ys*). *length xs* = *Suc n* ∧ *length ys* = *Suc n*}

**constdefs**
*lex* :: $('a \times 'a)\ set => ('a\ list \times 'a\ list)\ set$
*lex r* == $\bigcup n.$ *lexn r n*
— Holds only between lists of the same length

*lenlex* :: $('a \times 'a)\ set => ('a\ list \times 'a\ list)\ set$
*lenlex r* == *inv-image* (*less-than* <*lex*> *lex r*) (%*xs*. (*length xs*, *xs*))
— Compares lists by their length and then lexicographically

**declare** *lex-def* [*code del*]

**lemma** *wf-lexn*: *wf r* ==> *wf* (*lexn r n*)
⟨*proof*⟩

**lemma** *lexn-length*:
(*xs*, *ys*) : *lexn r n* ==> *length xs* = *n* ∧ *length ys* = *n*
⟨*proof*⟩

**lemma** *wf-lex* [*intro*!]: *wf r* ==> *wf* (*lex r*)
⟨*proof*⟩

**lemma** *lexn-conv*:
*lexn r n* =
{(*xs*,*ys*). *length xs* = *n* ∧ *length ys* = *n* ∧
(∃ *xys x y xs' ys'*. *xs*= *xys* @ *x*#*xs'* ∧ *ys*= *xys* @ *y* # *ys'* ∧ (*x*, *y*):*r*)}
⟨*proof*⟩

**lemma** *lex-conv*:
*lex r* =
{(*xs*,*ys*). *length xs* = *length ys* ∧
(∃ *xys x y xs' ys'*. *xs* = *xys* @ *x* # *xs'* ∧ *ys* = *xys* @ *y* # *ys'* ∧ (*x*, *y*):*r*)}
⟨*proof*⟩

**lemma** *wf-lenlex* [*intro*!]: *wf r* ==> *wf* (*lenlex r*)
⟨*proof*⟩

**lemma** *lenlex-conv*:
*lenlex r* = {(*xs*,*ys*). *length xs* < *length ys* |
*length xs* = *length ys* ∧ (*xs*, *ys*) : *lex r*}
⟨*proof*⟩

**lemma** *Nil-notin-lex* [*iff*]: $([], ys) \notin lex\ r$
⟨*proof*⟩

**lemma** *Nil2-notin-lex* [*iff*]: $(xs, []) \notin lex\ r$
⟨*proof*⟩

**lemma** *Cons-in-lex* [*simp*]:
$((x \# xs, y \# ys) : lex\ r) =$
$((x, y) : r \wedge length\ xs = length\ ys \mid x = y \wedge (xs, ys) : lex\ r)$
⟨*proof*⟩

### 39.3.2 Lexicographic Ordering

Classical lexicographic ordering on lists, ie. "a" ¡ "ab" ¡ "b". This ordering does *not* preserve well-foundedness. Author: N. Voelker, March 2005.

**constdefs**
*lexord* :: $('a * 'a)set \Rightarrow ('a\ list * 'a\ list)\ set$
*lexord* $r == \{(x,y).\ \exists\ a\ v.\ y = x\ @\ a \# v \vee$
$(\exists\ u\ a\ b\ v\ w.\ (a,b) \in r \wedge x = u\ @\ (a \# v) \wedge y = u\ @\ (b \# w))\}$
**declare** *lexord-def* [*code del*]

**lemma** *lexord-Nil-left*[*simp*]: $([],y) \in lexord\ r = (\exists\ a\ x.\ y = a \# x)$
⟨*proof*⟩

**lemma** *lexord-Nil-right*[*simp*]: $(x,[]) \notin lexord\ r$
⟨*proof*⟩

**lemma** *lexord-cons-cons*[*simp*]:
$((a \# x, b \# y) \in lexord\ r) = ((a,b) \in r \mid (a = b\ \&\ (x,y) \in lexord\ r))$
⟨*proof*⟩

**lemmas** *lexord-simps = lexord-Nil-left lexord-Nil-right lexord-cons-cons*

**lemma** *lexord-append-rightI*: $\exists\ b\ z.\ y = b \# z \Longrightarrow (x,\ x\ @\ y) \in lexord\ r$
⟨*proof*⟩

**lemma** *lexord-append-left-rightI*:
$(a,b) \in r \Longrightarrow (u\ @\ a \# x,\ u\ @\ b \# y) \in lexord\ r$
⟨*proof*⟩

**lemma** *lexord-append-leftI*: $(u,v) \in lexord\ r \Longrightarrow (x\ @\ u,\ x\ @\ v) \in lexord\ r$
⟨*proof*⟩

**lemma** *lexord-append-leftD*:
$[\![\ (x\ @\ u,\ x\ @\ v) \in lexord\ r;\ (!\ a.\ (a,a) \notin r)\ ]\!] \Longrightarrow (u,v) \in lexord\ r$
⟨*proof*⟩

**lemma** *lexord-take-index-conv*:

$((x,y) : lexord\ r) =$
$((length\ x < length\ y \wedge take\ (length\ x)\ y = x) \vee$
$(\exists i.\ i < min(length\ x)(length\ y)\ \&\ take\ i\ x = take\ i\ y\ \&\ (x!i,y!i) \in r))$
⟨*proof*⟩

**lemma** *lexord-lex*: $(x,y) \in lex\ r = ((x,y) \in lexord\ r \wedge length\ x = length\ y)$
⟨*proof*⟩

**lemma** *lexord-irreflexive*: $(!\ x.\ (x,x) \notin r) \Longrightarrow (y,y) \notin lexord\ r$
⟨*proof*⟩

**lemma** *lexord-trans*:
$[\![ (x,\ y) \in lexord\ r; (y,\ z) \in lexord\ r; trans\ r ]\!] \Longrightarrow (x,\ z) \in lexord\ r$
⟨*proof*⟩

**lemma** *lexord-transI*: $trans\ r \Longrightarrow trans\ (lexord\ r)$
⟨*proof*⟩

**lemma** *lexord-linear*: $(!\ a\ b.\ (a,b) \in r \mid a = b \mid (b,a) \in r) \Longrightarrow (x,y) : lexord\ r \mid x = y \mid (y,x) : lexord\ r$
⟨*proof*⟩

## 39.4 Lexicographic combination of measure functions

These are useful for termination proofs

**definition**
$measures\ fs = inv\text{-}image\ (lex\ less\text{-}than)\ (\%a.\ map\ (\%f.\ f\ a)\ fs)$

**lemma** *wf-measures*[*recdef-wf*, *simp*]: $wf\ (measures\ fs)$
⟨*proof*⟩

**lemma** *in-measures*[*simp*]:
$(x,\ y) \in measures\ [] = False$
$(x,\ y) \in measures\ (f\ \#\ fs)$
$= (f\ x < f\ y \vee (f\ x = f\ y \wedge (x,\ y) \in measures\ fs))$
⟨*proof*⟩

**lemma** *measures-less*: $f\ x < f\ y \Longrightarrow (x,\ y) \in measures\ (f\#fs)$
⟨*proof*⟩

**lemma** *measures-lesseq*: $f\ x <= f\ y \Longrightarrow (x,\ y) \in measures\ fs \Longrightarrow (x,\ y) \in measures\ (f\#fs)$
⟨*proof*⟩

### 39.4.1 Lifting a Relation on List Elements to the Lists

**inductive-set**
$listrel :: ('a * 'a)set => ('a\ list * 'a\ list)set$
**for** $r :: ('a * 'a)set$
**where**

*Nil*: ([],[]) ∈ *listrel r*
| *Cons*: [| (*x*,*y*) ∈ *r*; (*xs*,*ys*) ∈ *listrel r* |] ==> (*x*#*xs*, *y*#*ys*) ∈ *listrel r*

**inductive-cases** *listrel-Nil1* [*elim*!]: ([],*xs*) ∈ *listrel r*
**inductive-cases** *listrel-Nil2* [*elim*!]: (*xs*,[]) ∈ *listrel r*
**inductive-cases** *listrel-Cons1* [*elim*!]: (*y*#*ys*,*xs*) ∈ *listrel r*
**inductive-cases** *listrel-Cons2* [*elim*!]: (*xs*,*y*#*ys*) ∈ *listrel r*

**lemma** *listrel-mono*: *r* ⊆ *s* ⟹ *listrel r* ⊆ *listrel s*
⟨*proof*⟩

**lemma** *listrel-subset*: *r* ⊆ *A* × *A* ⟹ *listrel r* ⊆ *lists A* × *lists A*
⟨*proof*⟩

**lemma** *listrel-refl-on*: *refl-on A r* ⟹ *refl-on* (*lists A*) (*listrel r*)
⟨*proof*⟩

**lemma** *listrel-sym*: *sym r* ⟹ *sym* (*listrel r*)
⟨*proof*⟩

**lemma** *listrel-trans*: *trans r* ⟹ *trans* (*listrel r*)
⟨*proof*⟩

**theorem** *equiv-listrel*: *equiv A r* ⟹ *equiv* (*lists A*) (*listrel r*)
⟨*proof*⟩

**lemma** *listrel-Nil* [*simp*]: *listrel r* `` {[]} = {[]}
⟨*proof*⟩

**lemma** *listrel-Cons*:
*listrel r* `` {*x*#*xs*} = *set-Cons* (*r*``{*x*}) (*listrel r* `` {*xs*})
⟨*proof*⟩

## 39.5 Miscellany

### 39.5.1 Characters and strings

**datatype** *nibble* =
*Nibble0* | *Nibble1* | *Nibble2* | *Nibble3* | *Nibble4* | *Nibble5* | *Nibble6* | *Nibble7*
| *Nibble8* | *Nibble9* | *NibbleA* | *NibbleB* | *NibbleC* | *NibbleD* | *NibbleE* | *NibbleF*

**lemma** *UNIV-nibble*:
*UNIV* = {*Nibble0*, *Nibble1*, *Nibble2*, *Nibble3*, *Nibble4*, *Nibble5*, *Nibble6*, *Nibble7*,
*Nibble8*, *Nibble9*, *NibbleA*, *NibbleB*, *NibbleC*, *NibbleD*, *NibbleE*, *NibbleF*} (**is** - = *?A*)
⟨*proof*⟩

**instance** *nibble* :: *finite*
⟨*proof*⟩

**datatype** *char* = *Char nibble nibble*
— Note: canonical order of character encoding coincides with standard term ordering

**lemma** *UNIV-char*:
*UNIV* = *image* (*split Char*) (*UNIV* × *UNIV*)
⟨*proof*⟩

**instance** *char* :: *finite*
⟨*proof*⟩

**lemma** *size-char* [*code*, *simp*]:
*size* (*c*::*char*) = *0* ⟨*proof*⟩

**lemma** *char-size* [*code*, *simp*]:
*char-size* (*c*::*char*) = *0* ⟨*proof*⟩

**primrec** *nibble-pair-of-char* :: *char* ⇒ *nibble* × *nibble* **where**
*nibble-pair-of-char* (*Char n m*) = (*n*, *m*)

**declare** *nibble-pair-of-char.simps* [*code del*]

⟨*ML*⟩

**lemma** *char-case-nibble-pair* [*code*, *code inline*]:
*char-case f* = *split f o nibble-pair-of-char*
⟨*proof*⟩

**lemma** *char-rec-nibble-pair* [*code*, *code inline*]:
*char-rec f* = *split f o nibble-pair-of-char*
⟨*proof*⟩

**types** *string* = *char list*

**syntax**
-*Char* :: *xstr* => *char* (*CHR* -)
-*String* :: *xstr* => *string* (-)

⟨*ML*⟩

## 39.6 Size function

**lemma** [*measure-function*]: *is-measure f* ⟹ *is-measure* (*list-size f*)
⟨*proof*⟩

**lemma** [*measure-function*]: *is-measure f* ⟹ *is-measure* (*option-size f*)
⟨*proof*⟩

**lemma** *list-size-estimation*[*termination-simp*]:
$x \in set\ xs \Longrightarrow y < f\ x \Longrightarrow y < list\text{-}size\ f\ xs$
⟨*proof*⟩

**lemma** *list-size-estimation'*[*termination-simp*]:
$x \in set\ xs \Longrightarrow y \leq f\ x \Longrightarrow y \leq list\text{-}size\ f\ xs$
⟨*proof*⟩

**lemma** *list-size-map*[*simp*]: $list\text{-}size\ f\ (map\ g\ xs) = list\text{-}size\ (f\ o\ g)\ xs$
⟨*proof*⟩

**lemma** *list-size-pointwise*[*termination-simp*]:
$(\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x < g\ x) \Longrightarrow list\text{-}size\ f\ xs \leq list\text{-}size\ g\ xs$
⟨*proof*⟩

## 39.7 Code generator

### 39.7.1 Setup

**types-code**
*list* (- *list*)
**attach** (*term-of*) ⟪
*fun term-of-list f T = HOLogic.mk-list T o map f;*
⟫
**attach** (*test*) ⟪
*fun gen-list' aG aT i j = frequency*
[(*i*, *fn* () =>
*let*
*val* (*x*, *t*) = *aG j*;
*val* (*xs*, *ts*) = *gen-list' aG aT* (*i*−*1*) *j*
*in* (*x* :: *xs*, *fn* () => *HOLogic.cons-const aT* $ *t* () $ *ts* ()) *end*),
(*1*, *fn* () => ([], *fn* () => *HOLogic.nil-const aT*))] ()
*and gen-list aG aT i = gen-list' aG aT i i;*
⟫
*char* (*string*)
**attach** (*term-of*) ⟪
*val term-of-char = HOLogic.mk-char o ord;*
⟫
**attach** (*test*) ⟪
*fun gen-char i =*
*let val j = random-range* (*ord a*) (*Int.min* (*ord a* + *i*, *ord z*))
*in* (*chr j*, *fn* () => *HOLogic.mk-char j*) *end;*
⟫

**consts-code** *Cons* ((- ::/ -))

**code-type** *list*
(*SML* - *list*)
(*OCaml* - *list*)
(*Haskell* ![-])

**code-reserved** *SML*
*list*

**code-reserved** *OCaml*
*list*

**code-const** *Nil*
(*SML* [])
(*OCaml* [])
(*Haskell* [])

⟨*ML*⟩

**code-instance** *list* :: *eq*
(*Haskell* −)

**code-const** *eq-class.eq* :: *'a*::*eq list* ⇒ *'a list* ⇒ *bool*
(*Haskell* **infixl** *4* ==)

⟨*ML*⟩

### 39.7.2 Generation of efficient code

**primrec**
*member* :: *'a* ⇒ *'a list* ⇒ *bool* (**infixl** *mem 55*)
**where**
*x mem* [] ⟷ *False*
| *x mem* (*y*#*ys*) ⟷ *x* = *y* ∨ *x mem ys*

**primrec**
*null*:: *'a list* ⇒ *bool*
**where**
*null* [] = *True*
| *null* (*x*#*xs*) = *False*

**primrec**
*list-inter* :: *'a list* ⇒ *'a list* ⇒ *'a list*
**where**
*list-inter* [] *bs* = []
| *list-inter* (*a*#*as*) *bs* =
(*if a* ∈ *set bs then a* # *list-inter as bs else list-inter as bs*)

**primrec**
*list-all* :: (*'a* ⇒ *bool*) ⇒ (*'a list* ⇒ *bool*)
**where**
*list-all P* [] = *True*
| *list-all P* (*x*#*xs*) = (*P x* ∧ *list-all P xs*)

**primrec**
*list-ex* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ *bool*
**where**
*list-ex P* [] = *False*
| *list-ex P* (*x*#*xs*) = (*P x* ∨ *list-ex P xs*)

**primrec**
*filtermap* :: ('*a* ⇒ '*b option*) ⇒ '*a list* ⇒ '*b list*
**where**
*filtermap f* [] = []
| *filtermap f* (*x*#*xs*) =
(*case f x of None* ⇒ *filtermap f xs*
| *Some y* ⇒ *y* # *filtermap f xs*)

**primrec**
*map-filter* :: ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*b list*
**where**
*map-filter f P* [] = []
| *map-filter f P* (*x*#*xs*) =
(*if P x then f x* # *map-filter f P xs else map-filter f P xs*)

**primrec**
*length-unique* :: '*a list* ⇒ *nat*
**where**
*length-unique* [] = *0*
| *length-unique* (*x*#*xs*) =
(*if x* ∈ *set xs then length-unique xs else Suc* (*length-unique xs*))

Only use *mem* for generating executable code. Otherwise use *x* ∈ *set xs* instead — it is much easier to reason about. The same is true for *list-all* and *list-ex*: write ∀ *x*∈*set xs* and ∃ *x*∈*set xs* instead because the HOL quantifiers are aleady known to the automatic provers. In fact, the declarations in the code subsection make sure that ∈, ∀ *x*∈*set xs* and ∃ *x*∈*set xs* are implemented efficiently.

Efficient emptyness check is implemented by *null*.

The functions *filtermap* and *map-filter* are just there to generate efficient code. Do not use them for modelling and proving.

**lemma** *rev-foldl-cons* [*code*]:
*rev xs* = *foldl* (λ*xs x*. *x* # *xs*) [] *xs*
⟨*proof*⟩

**lemma** *mem-iff* [*code post*]:
*x mem xs* ⟷ *x* ∈ *set xs*
⟨*proof*⟩

**lemmas** *in-set-code* [*code unfold*] = *mem-iff* [*symmetric*]

**lemma** *empty-null* [*code inline*]:

*xs = [] ⟷ null xs*
⟨*proof*⟩

**lemmas** *null-empty* [*code post*] =
*empty-null* [*symmetric*]

**lemma** *list-inter-conv*:
*set* (*list-inter xs ys*) = *set xs* ∩ *set ys*
⟨*proof*⟩

**lemma** *list-all-iff* [*code post*]:
*list-all P xs ⟷* (∀ *x* ∈ *set xs*. *P x*)
⟨*proof*⟩

**lemmas** *list-ball-code* [*code unfold*] = *list-all-iff* [*symmetric*]

**lemma** *list-all-append* [*simp*]:
*list-all P* (*xs* @ *ys*) ⟷ (*list-all P xs* ∧ *list-all P ys*)
⟨*proof*⟩

**lemma** *list-all-rev* [*simp*]:
*list-all P* (*rev xs*) ⟷ *list-all P xs*
⟨*proof*⟩

**lemma** *list-all-length*:
*list-all P xs ⟷* (∀ *n* < *length xs*. *P* (*xs* ! *n*))
⟨*proof*⟩

**lemma** *list-ex-iff* [*code post*]:
*list-ex P xs ⟷* (∃ *x* ∈ *set xs*. *P x*)
⟨*proof*⟩

**lemmas** *list-bex-code* [*code unfold*] =
*list-ex-iff* [*symmetric*]

**lemma** *list-ex-length*:
*list-ex P xs ⟷* (∃ *n* < *length xs*. *P* (*xs* ! *n*))
⟨*proof*⟩

**lemma** *filtermap-conv*:
*filtermap f xs* = *map* (λ*x*. *the* (*f x*)) (*filter* (λ*x*. *f x* ≠ *None*) *xs*)
⟨*proof*⟩

**lemma** *map-filter-conv* [*simp*]:
*map-filter f P xs* = *map f* (*filter P xs*)
⟨*proof*⟩

**lemma** *length-remdups-length-unique* [*code inline*]:
*length* (*remdups xs*) = *length-unique xs*

⟨*proof*⟩

**hide** (**open**) *const length-unique*

Code for bounded quantification and summation over nats.

**lemma** *atMost-upto* [*code unfold*]:
  $\{..n\} = set\ [0..<Suc\ n]$
⟨*proof*⟩

**lemma** *atLeast-upt* [*code unfold*]:
  $\{..<n\} = set\ [0..<n]$
⟨*proof*⟩

**lemma** *greaterThanLessThan-upt* [*code unfold*]:
  $\{n<..<m\} = set\ [Suc\ n..<m]$
⟨*proof*⟩

**lemma** *atLeastLessThan-upt* [*code unfold*]:
  $\{n..<m\} = set\ [n..<m]$
⟨*proof*⟩

**lemma** *greaterThanAtMost-upt* [*code unfold*]:
  $\{n<..m\} = set\ [Suc\ n..<Suc\ m]$
⟨*proof*⟩

**lemma** *atLeastAtMost-upt* [*code unfold*]:
  $\{n..m\} = set\ [n..<Suc\ m]$
⟨*proof*⟩

**lemma** *all-nat-less-eq* [*code unfold*]:
  $(\forall\, m<n::nat.\ P\ m) \longleftrightarrow (\forall\, m \in \{0..<n\}.\ P\ m)$
⟨*proof*⟩

**lemma** *ex-nat-less-eq* [*code unfold*]:
  $(\exists\, m<n::nat.\ P\ m) \longleftrightarrow (\exists\, m \in \{0..<n\}.\ P\ m)$
⟨*proof*⟩

**lemma** *all-nat-less* [*code unfold*]:
  $(\forall\, m\leq n::nat.\ P\ m) \longleftrightarrow (\forall\, m \in \{0..n\}.\ P\ m)$
⟨*proof*⟩

**lemma** *ex-nat-less* [*code unfold*]:
  $(\exists\, m\leq n::nat.\ P\ m) \longleftrightarrow (\exists\, m \in \{0..n\}.\ P\ m)$
⟨*proof*⟩

**lemma** *setsum-set-distinct-conv-listsum*:
  $distinct\ xs \Longrightarrow setsum\ f\ (set\ xs) = listsum\ (map\ f\ xs)$
⟨*proof*⟩

**lemma** *setsum-set-upt-conv-listsum* [*code unfold*]:
*setsum f* (*set* [*m*..<*n*]) = *listsum* (*map f* [*m*..<*n*])
⟨*proof*⟩

Code for summation over ints.

**lemma** *greaterThanLessThan-upto* [*code unfold*]:
{*i*<..<*j*::*int*} = *set* [*i*+*1*..*j* − *1*]
⟨*proof*⟩

**lemma** *atLeastLessThan-upto* [*code unfold*]:
{*i*..<*j*::*int*} = *set* [*i*..*j* − *1*]
⟨*proof*⟩

**lemma** *greaterThanAtMost-upto* [*code unfold*]:
{*i*<..*j*::*int*} = *set* [*i*+*1*..*j*]
⟨*proof*⟩

**lemma** *atLeastAtMost-upto* [*code unfold*]:
{*i*..*j*::*int*} = *set* [*i*..*j*]
⟨*proof*⟩

**lemma** *setsum-set-upto-conv-listsum* [*code unfold*]:
*setsum f* (*set* [*i*..*j*::*int*]) = *listsum* (*map f* [*i*..*j*])
⟨*proof*⟩

**end**

# 40 Code-Message: Monolithic strings (message strings) for code generation

**theory** *Code-Message*
**imports** *Plain* ~~/*src*/*HOL*/*List*
**begin**

## 40.1 Datatype of messages

**datatype** *message-string* = *STR string*

**lemmas** [*code del*] = *message-string.recs message-string.cases*

**lemma** [*code*]: *size* (*s*::*message-string*) = *0*
⟨*proof*⟩

**lemma** [*code*]: *message-string-size* (*s*::*message-string*) = *0*
⟨*proof*⟩

### 40.2 ML interface

⟨*ML*⟩

### 40.3 Code serialization

**code-type** *message-string*
(*SML string*)
(*OCaml string*)
(*Haskell String*)

⟨*ML*⟩

**code-reserved** *SML string*
**code-reserved** *OCaml string*

**code-instance** *message-string* :: *eq*
(*Haskell* −)

**code-const** *eq-class.eq* :: *message-string* ⇒ *message-string* ⇒ *bool*
(*SML* !((- : *string*) = -))
(*OCaml* !((- : *string*) = -))
(*Haskell* **infixl** *4* ==)

**end**

## 41 Typerep: Reflecting Pure types into HOL

**theory** *Typerep*
**imports** *Plain List Code-Message*
**begin**

**datatype** *typerep* = *Typerep message-string typerep list*

**class** *typerep* =
**fixes** *typerep* :: *'a*::{} *itself* ⇒ *typerep*
**begin**

**definition** *typerep-of* :: *'a* ⇒ *typerep* **where**
[*simp*]: *typerep-of x* = *typerep TYPE*(*'a*)

**end**

⟨*ML*⟩

**lemma** [*code*]:
*eq-class.eq* (*Typerep tyco1 tys1*) (*Typerep tyco2 tys2*) ⟷ *eq-class.eq tyco1 tyco2*
∧ *list-all2 eq-class.eq tys1 tys2*

⟨*proof*⟩

**code-type** *typerep*
(*SML Term.typ*)

**code-const** *Typerep*
(*SML Term.Type*/ (-, -))

**code-reserved** *SML Term*

**hide** (**open**) *const typerep Typerep*

**end**

# 42 Code-Eval: Term evaluation using the generic code generator

**theory** *Code-Eval*
**imports** *Plain Typerep*
**begin**

## 42.1 Term representation

### 42.1.1 Terms and class *term-of*

**datatype** *term* = *dummy-term*

**definition** *Const* :: *message-string* ⇒ *typerep* ⇒ *term* **where**
*Const* - - = *dummy-term*

**definition** *App* :: *term* ⇒ *term* ⇒ *term* **where**
*App* - - = *dummy-term*

**code-datatype** *Const App*

**class** *term-of* = *typerep* +
**fixes** *term-of* :: 'a::{} ⇒ *term*

**lemma** *term-of-anything*: *term-of x* ≡ *t*
⟨*proof*⟩

⟨*ML*⟩

### 42.1.2 *term-of* instances

⟨*ML*⟩

### 42.1.3 Code generator setup

**lemmas** [*code del*] = *term.recs term.cases term.size*
**lemma** [*code*, *code del*]: *eq-class.eq* (*t1*::*term*) *t2* ⟷ *eq-class.eq t1 t2* ⟨*proof*⟩

**lemma** [*code*, *code del*]: (*term-of* :: *typerep* ⇒ *term*) = *term-of* ⟨*proof*⟩
**lemma** [*code*, *code del*]: (*term-of* :: *term* ⇒ *term*) = *term-of* ⟨*proof*⟩
**lemma** [*code*, *code del*]: (*term-of* :: *message-string* ⇒ *term*) = *term-of* ⟨*proof*⟩
**lemma** [*code*, *code del*]:
(*Code-Eval.term-of* :: 'a::{*type*, *term-of*} *Predicate.pred* ⇒ *Code-Eval.term*) = *Code-Eval.term-of* ⟨*proof*⟩
**lemma** [*code*, *code del*]:
(*Code-Eval.term-of* :: 'a::{*type*, *term-of*} *Predicate.seq* ⇒ *Code-Eval.term*) = *Code-Eval.term-of* ⟨*proof*⟩

**lemma** *term-of-char* [*unfolded typerep-fun-def typerep-char-def typerep-nibble-def*, *code*]: *Code-Eval.term-of c* =
(*let* (*n*, *m*) = *nibble-pair-of-char c*
*in Code-Eval.App* (*Code-Eval.App* (*Code-Eval.Const* (*STR* ''*Pair*'') (*TYPEREP*(*nibble* ⇒ *nibble* ⇒ *char*))) (*Code-Eval.term-of n*)) (*Code-Eval.term-of m*))
⟨*proof*⟩

**code-type** *term*
(*SML Term.term*)

**code-const** *Const* **and** *App*
(*SML Term.Const*/ (-, -) **and** *Term.*$/ (-, -))

**code-const** *term-of* :: *message-string* ⇒ *term*
(*SML Message'-String.mk*)

## 42.2 Evaluation setup

⟨*ML*⟩

### 42.2.1 Syntax

⟨*ML*⟩

**hide** *const dummy-term*
**hide** (**open**) *const Const App*
**hide** (**open**) *const term-of*

**end**

# 43 Map: Maps

**theory** *Map*

**imports** *List*
**begin**

**types** $('a,'b)$ $\sim\!\Rightarrow$ $= 'a \Rightarrow 'b\ option$ (**infixr** *0*)
**translations** (*type*) $a \sim\!\Rightarrow b$ $<=$ (*type*) $a \Rightarrow b\ option$

**syntax** (*xsymbols*)
$\sim\!\Rightarrow$ :: [*type*, *type*] $\Rightarrow$ *type* (**infixr** $\rightharpoonup$ *0*)

**abbreviation**
*empty* :: $'a \sim\!\Rightarrow 'b$ **where**
*empty* $== \%x.\ None$

**definition**
*map-comp* :: $('b \sim\!\Rightarrow 'c) \Rightarrow ('a \sim\!\Rightarrow 'b) \Rightarrow ('a \sim\!\Rightarrow 'c)$ (**infixl** *o'-m 55*)
**where**
$f\ o\text{-}m\ g = (\lambda k.\ case\ g\ k\ of\ None \Rightarrow None \mid Some\ v \Rightarrow f\ v)$

**notation** (*xsymbols*)
*map-comp* (**infixl** $\circ_m$ *55*)

**definition**
*map-add* :: $('a \sim\!\Rightarrow 'b) \Rightarrow ('a \sim\!\Rightarrow 'b) \Rightarrow ('a \sim\!\Rightarrow 'b)$ (**infixl** $++$ *100*)
**where**
$m1 ++ m2 = (\lambda x.\ case\ m2\ x\ of\ None \Rightarrow m1\ x \mid Some\ y \Rightarrow Some\ y)$

**definition**
*restrict-map* :: $('a \sim\!\Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow ('a \sim\!\Rightarrow 'b)$ (**infixl** $|`$ *110*) **where**
$m|`A = (\lambda x.\ if\ x : A\ then\ m\ x\ else\ None)$

**notation** (*latex* **output**)
*restrict-map* (-$\restriction$- [*111*,*110*] *110*)

**definition**
*dom* :: $('a \sim\!\Rightarrow 'b) \Rightarrow 'a\ set$ **where**
$dom\ m = \{a.\ m\ a \sim= None\}$

**definition**
*ran* :: $('a \sim\!\Rightarrow 'b) \Rightarrow 'b\ set$ **where**
$ran\ m = \{b.\ EX\ a.\ m\ a = Some\ b\}$

**definition**
*map-le* :: $('a \sim\!\Rightarrow 'b) \Rightarrow ('a \sim\!\Rightarrow 'b) \Rightarrow bool$ (**infix** $\subseteq_m$ *50*) **where**
$(m_1 \subseteq_m m_2) = (\forall\, a \in dom\ m_1.\ m_1\ a = m_2\ a)$

**consts**
*map-of* :: $('a * 'b)\ list \Rightarrow 'a \sim\!\Rightarrow 'b$
*map-upds* :: $('a \sim\!\Rightarrow 'b) \Rightarrow 'a\ list \Rightarrow 'b\ list \Rightarrow ('a \sim\!\Rightarrow 'b)$

**nonterminals**
*maplets maplet*

**syntax**
*-maplet* :: [*'a*, *'a*] => *maplet* (- /|->/ -)
*-maplets* :: [*'a*, *'a*] => *maplet* (- /[|->]/ -)
:: *maplet* => *maplets* (-)
*-Maplets* :: [*maplet*, *maplets*] => *maplets* (-,/ -)
*-MapUpd* :: [*'a* ~=> *'b*, *maplets*] => *'a* ~=> *'b* (-/'(-') [*900*,*0*]*900*)
*-Map* :: *maplets* => *'a* ~=> *'b* ((*1*[-]))

**syntax** (*xsymbols*)
*-maplet* :: [*'a*, *'a*] => *maplet* (- /↦/ -)
*-maplets* :: [*'a*, *'a*] => *maplet* (- /[↦]/ -)

**translations**
*-MapUpd m* (*-Maplets xy ms*) == *-MapUpd* (*-MapUpd m xy*) *ms*
*-MapUpd m* (*-maplet x y*) == *m*(*x*:=*Some y*)
*-MapUpd m* (*-maplets x y*) == *map-upds m x y*
*-Map ms* == *-MapUpd* (*CONST empty*) *ms*
*-Map* (*-Maplets ms1 ms2*) <= *-MapUpd* (*-Map ms1*) *ms2*
*-Maplets ms1* (*-Maplets ms2 ms3*) <= *-Maplets* (*-Maplets ms1 ms2*) *ms3*

**primrec**
*map-of* [] = *empty*
*map-of* (*p*#*ps*) = (*map-of ps*)(*fst p* |-> *snd p*)

**declare** *map-of.simps* [*code del*]

**lemma** *map-of-Cons-code* [*code*]:
*map-of* [] *k* = *None*
*map-of* ((*l*, *v*) # *ps*) *k* = (*if l* = *k then Some v else map-of ps k*)
⟨*proof*⟩

**defs**
*map-upds-def* [*code*]: *m*(*xs* [|->] *ys*) == *m* ++ *map-of* (*rev*(*zip xs ys*))

## 43.1 *empty*

**lemma** *empty-upd-none* [*simp*]: *empty*(*x* := *None*) = *empty*
⟨*proof*⟩

## 43.2 *map-upd*

**lemma** *map-upd-triv*: *t k* = *Some x* ==> *t*(*k*|->*x*) = *t*
⟨*proof*⟩

**lemma** *map-upd-nonempty* [*simp*]: *t*(*k*|->*x*) ~= *empty*
⟨*proof*⟩

**lemma** *map-upd-eqD1*:
**assumes** $m(a\mapsto x) = n(a\mapsto y)$
**shows** $x = y$
⟨*proof*⟩

**lemma** *map-upd-Some-unfold*:
$((m(a|->b))\ x = Some\ y) = (x = a \wedge b = y \vee x \neq a \wedge m\ x = Some\ y)$
⟨*proof*⟩

**lemma** *image-map-upd* [*simp*]: $x \notin A \Longrightarrow m(x \mapsto y)\ `\ A = m\ `\ A$
⟨*proof*⟩

**lemma** *finite-range-updI*: *finite* (*range f*) ==> *finite* (*range* ($f(a|->b)$))
⟨*proof*⟩

## 43.3 *map-of*

**lemma** *map-of-eq-None-iff*:
(*map-of xys x* = *None*) = ($x \notin$ *fst* ` (*set xys*))
⟨*proof*⟩

**lemma** *map-of-is-SomeD*: *map-of xys x* = *Some y* $\Longrightarrow$ ($x$,$y$) $\in$ *set xys*
⟨*proof*⟩

**lemma** *map-of-eq-Some-iff* [*simp*]:
*distinct*(*map fst xys*) $\Longrightarrow$ (*map-of xys x* = *Some y*) = (($x$,$y$) $\in$ *set xys*)
⟨*proof*⟩

**lemma** *Some-eq-map-of-iff* [*simp*]:
*distinct*(*map fst xys*) $\Longrightarrow$ (*Some y* = *map-of xys x*) = (($x$,$y$) $\in$ *set xys*)
⟨*proof*⟩

**lemma** *map-of-is-SomeI* [*simp*]: $[\![$ *distinct*(*map fst xys*); ($x$,$y$) $\in$ *set xys* $]\!]$
$\Longrightarrow$ *map-of xys x* = *Some y*
⟨*proof*⟩

**lemma** *map-of-zip-is-None* [*simp*]:
*length xs* = *length ys* $\Longrightarrow$ (*map-of* (*zip xs ys*) *x* = *None*) = ($x \notin$ *set xs*)
⟨*proof*⟩

**lemma** *map-of-zip-is-Some*:
**assumes** *length xs* = *length ys*
**shows** $x \in$ *set xs* $\longleftrightarrow$ ($\exists\, y.$ *map-of* (*zip xs ys*) *x* = *Some y*)
⟨*proof*⟩

**lemma** *map-of-zip-upd*:
**fixes** $x$ :: $'a$ **and** *xs* :: $'a$ *list* **and** *ys zs* :: $'b$ *list*
**assumes** *length ys* = *length xs*
**and** *length zs* = *length xs*

**and** $x \notin set\ xs$
**and** $map\text{-}of\ (zip\ xs\ ys)(x \mapsto y) = map\text{-}of\ (zip\ xs\ zs)(x \mapsto z)$
**shows** $map\text{-}of\ (zip\ xs\ ys) = map\text{-}of\ (zip\ xs\ zs)$
⟨*proof*⟩

**lemma** *map-of-zip-inject*:
**assumes** $length\ ys = length\ xs$
**and** $length\ zs = length\ xs$
**and** *dist*: $distinct\ xs$
**and** *map-of*: $map\text{-}of\ (zip\ xs\ ys) = map\text{-}of\ (zip\ xs\ zs)$
**shows** $ys = zs$
⟨*proof*⟩

**lemma** *finite-range-map-of*: $finite\ (range\ (map\text{-}of\ xys))$
⟨*proof*⟩

**lemma** *map-of-SomeD*: $map\text{-}of\ xs\ k = Some\ y \Longrightarrow (k, y) \in set\ xs$
⟨*proof*⟩

**lemma** *map-of-mapk-SomeI*:
$inj\ f$ ==> $map\text{-}of\ t\ k = Some\ x$ ==>
$map\text{-}of\ (map\ (split\ (\%k.\ Pair\ (f\ k)))\ t)\ (f\ k) = Some\ x$
⟨*proof*⟩

**lemma** *weak-map-of-SomeI*: $(k, x) : set\ l$ ==> $\exists x.\ map\text{-}of\ l\ k = Some\ x$
⟨*proof*⟩

**lemma** *map-of-filter-in*:
$map\text{-}of\ xs\ k = Some\ z \Longrightarrow P\ k\ z \Longrightarrow map\text{-}of\ (filter\ (split\ P)\ xs)\ k = Some\ z$
⟨*proof*⟩

**lemma** *map-of-map*: $map\text{-}of\ (map\ (\%(a,b).\ (a,f\ b))\ xs)\ x = Option.map\ f\ (map\text{-}of\ xs\ x)$
⟨*proof*⟩

## 43.4 *Option.map* related

**lemma** *option-map-o-empty* [*simp*]: $Option.map\ f\ o\ empty = empty$
⟨*proof*⟩

**lemma** *option-map-o-map-upd* [*simp*]:
$Option.map\ f\ o\ m(a|{->}b) = (Option.map\ f\ o\ m)(a|{->}f\ b)$
⟨*proof*⟩

## 43.5 *map-comp* related

**lemma** *map-comp-empty* [*simp*]:
$m \circ_m empty = empty$
$empty \circ_m m = empty$
⟨*proof*⟩

**lemma** *map-comp-simps* [*simp*]:
*m2 k = None ⟹ (m1 ∘ₘ m2) k = None*
*m2 k = Some k' ⟹ (m1 ∘ₘ m2) k = m1 k'*
⟨*proof*⟩

**lemma** *map-comp-Some-iff*:
*((m1 ∘ₘ m2) k = Some v) = (∃ k'. m2 k = Some k' ∧ m1 k' = Some v)*
⟨*proof*⟩

**lemma** *map-comp-None-iff*:
*((m1 ∘ₘ m2) k = None) = (m2 k = None ∨ (∃ k'. m2 k = Some k' ∧ m1 k' = None))*
⟨*proof*⟩

## 43.6 ++

**lemma** *map-add-empty*[*simp*]: *m ++ empty = m*
⟨*proof*⟩

**lemma** *empty-map-add*[*simp*]: *empty ++ m = m*
⟨*proof*⟩

**lemma** *map-add-assoc*[*simp*]: *m1 ++ (m2 ++ m3) = (m1 ++ m2) ++ m3*
⟨*proof*⟩

**lemma** *map-add-Some-iff*:
*((m ++ n) k = Some x) = (n k = Some x | n k = None & m k = Some x)*
⟨*proof*⟩

**lemma** *map-add-SomeD* [*dest*!]:
*(m ++ n) k = Some x ⟹ n k = Some x ∨ n k = None ∧ m k = Some x*
⟨*proof*⟩

**lemma** *map-add-find-right* [*simp*]: !!*xx. n k = Some xx ==> (m ++ n) k = Some xx*
⟨*proof*⟩

**lemma** *map-add-None* [*iff*]: *((m ++ n) k = None) = (n k = None & m k = None)*
⟨*proof*⟩

**lemma** *map-add-upd*[*simp*]: *f ++ g(x|->y) = (f ++ g)(x|->y)*
⟨*proof*⟩

**lemma** *map-add-upds*[*simp*]: *m1 ++ (m2(xs[↦]ys)) = (m1++m2)(xs[↦]ys)*
⟨*proof*⟩

**lemma** *map-of-append*[*simp*]: *map-of (xs @ ys) = map-of ys ++ map-of xs*

⟨*proof*⟩

**lemma** *finite-range-map-of-map-add*:
*finite* (*range f*) ==> *finite* (*range* (*f* ++ *map-of l*))
⟨*proof*⟩

**lemma** *inj-on-map-add-dom* [*iff*]:
*inj-on* (*m* ++ *m'*) (*dom m'*) = *inj-on m'* (*dom m'*)
⟨*proof*⟩

## 43.7 *restrict-map*

**lemma** *restrict-map-to-empty* [*simp*]: $m|`\{\} = empty$
⟨*proof*⟩

**lemma** *restrict-map-empty* [*simp*]: $empty|`D = empty$
⟨*proof*⟩

**lemma** *restrict-in* [*simp*]: $x \in A \Longrightarrow (m|`A)\ x = m\ x$
⟨*proof*⟩

**lemma** *restrict-out* [*simp*]: $x \notin A \Longrightarrow (m|`A)\ x = None$
⟨*proof*⟩

**lemma** *ran-restrictD*: $y \in ran\ (m|`A) \Longrightarrow \exists x{\in}A.\ m\ x = Some\ y$
⟨*proof*⟩

**lemma** *dom-restrict* [*simp*]: $dom\ (m|`A) = dom\ m \cap A$
⟨*proof*⟩

**lemma** *restrict-upd-same* [*simp*]: $m(x{\mapsto}y)|`(-\{x\}) = m|`(-\{x\})$
⟨*proof*⟩

**lemma** *restrict-restrict* [*simp*]: $m|`A|`B = m|`(A{\cap}B)$
⟨*proof*⟩

**lemma** *restrict-fun-upd* [*simp*]:
$m(x := y)|`D = (if\ x \in D\ then\ (m|`(D{-}\{x\}))(x := y)\ else\ m|`D)$
⟨*proof*⟩

**lemma** *fun-upd-None-restrict* [*simp*]:
$(m|`D)(x := None) = (if\ x{:}D\ then\ m|`(D - \{x\})\ else\ m|`D)$
⟨*proof*⟩

**lemma** *fun-upd-restrict*: $(m|`D)(x := y) = (m|`(D{-}\{x\}))(x := y)$
⟨*proof*⟩

**lemma** *fun-upd-restrict-conv* [*simp*]:
$x \in D \Longrightarrow (m|`D)(x := y) = (m|`(D{-}\{x\}))(x := y)$

⟨*proof*⟩

## 43.8 *map-upds*

**lemma** *map-upds-Nil1* [*simp*]: $m([]\ [|{->}]\ bs) = m$
⟨*proof*⟩

**lemma** *map-upds-Nil2* [*simp*]: $m(as\ [|{->}]\ []) = m$
⟨*proof*⟩

**lemma** *map-upds-Cons* [*simp*]: $m(a\#as\ [|{->}]\ b\#bs) = (m(a|{->}b))(as[|{->}]bs)$
⟨*proof*⟩

**lemma** *map-upds-append1* [*simp*]: $\bigwedge ys\ m.\ size\ xs < size\ ys \Longrightarrow$
$m(xs@[x]\ [\mapsto]\ ys) = m(xs\ [\mapsto]\ ys)(x \mapsto ys!size\ xs)$
⟨*proof*⟩

**lemma** *map-upds-list-update2-drop* [*simp*]:
$[\![size\ xs \leq i;\ i < size\ ys]\!]$
$\Longrightarrow m(xs[\mapsto]ys[i:=y]) = m(xs[\mapsto]ys)$
⟨*proof*⟩

**lemma** *map-upd-upds-conv-if*:
$(f(x|{->}y))(xs\ [|{->}]\ ys) =$
$(if\ x : set(take\ (length\ ys)\ xs)\ then\ f(xs\ [|{->}]\ ys)$
$else\ (f(xs\ [|{->}]\ ys))(x|{->}y))$
⟨*proof*⟩

**lemma** *map-upds-twist* [*simp*]:
$a\ {}^{\sim}: set\ as\ ==>\ m(a|{->}b)(as[|{->}]bs) = m(as[|{->}]bs)(a|{->}b)$
⟨*proof*⟩

**lemma** *map-upds-apply-nontin* [*simp*]:
$x\ {}^{\sim}: set\ xs\ ==>\ (f(xs[|{->}]ys))\ x = f\ x$
⟨*proof*⟩

**lemma** *fun-upds-append-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs@zs[\mapsto]ys) = m(xs[\mapsto]ys)$
⟨*proof*⟩

**lemma** *fun-upds-append2-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs[\mapsto]ys@zs) = m(xs[\mapsto]ys)$
⟨*proof*⟩

**lemma** *restrict-map-upds*[*simp*]:
$[\![\ length\ xs = length\ ys;\ set\ xs \subseteq D\ ]\!]$
$\Longrightarrow m(xs\ [\mapsto]\ ys)|`D = (m|`(D - set\ xs))(xs\ [\mapsto]\ ys)$
⟨*proof*⟩

## 43.9 *dom*

**lemma** *domI*: *m a = Some b ==> a : dom m*
⟨*proof*⟩

**lemma** *domD*: *a : dom m ==> ∃ b. m a = Some b*
⟨*proof*⟩

**lemma** *domIff* [*iff*, *simp del*]: *(a : dom m) = (m a ~= None)*
⟨*proof*⟩

**lemma** *dom-empty* [*simp*]: *dom empty = {}*
⟨*proof*⟩

**lemma** *dom-fun-upd* [*simp*]:
*dom(f(x := y)) = (if y=None then dom f - {x} else insert x (dom f))*
⟨*proof*⟩

**lemma** *dom-map-of*: *dom(map-of xys) = {x. ∃ y. (x,y) : set xys}*
⟨*proof*⟩

**lemma** *dom-map-of-conv-image-fst*:
*dom(map-of xys) = fst ' (set xys)*
⟨*proof*⟩

**lemma** *dom-map-of-zip* [*simp*]: *[| length xs = length ys; distinct xs |] ==>*
*dom(map-of(zip xs ys)) = set xs*
⟨*proof*⟩

**lemma** *finite-dom-map-of*: *finite (dom (map-of l))*
⟨*proof*⟩

**lemma** *dom-map-upds* [*simp*]:
*dom(m(xs[|->]ys)) = set(take (length ys) xs) Un dom m*
⟨*proof*⟩

**lemma** *dom-map-add* [*simp*]: *dom(m++n) = dom n Un dom m*
⟨*proof*⟩

**lemma** *dom-override-on* [*simp*]:
*dom(override-on f g A) =*
*(dom f - {a. a : A - dom g}) Un {a. a : A Int dom g}*
⟨*proof*⟩

**lemma** *map-add-comm*: *dom m1 ∩ dom m2 = {} ⟹ m1++m2 = m2++m1*
⟨*proof*⟩

**lemma** *dom-const* [*simp*]:
*dom (λx. Some y) = UNIV*

⟨*proof*⟩

**lemma** *dom-if*:
*dom* (λ*x*. *if P x then f x else g x*) = *dom f* ∩ {*x*. *P x*} ∪ *dom g* ∩ {*x*. ¬ *P x*}
⟨*proof*⟩

**lemma** *finite-map-freshness*:
*finite* (*dom* (*f* :: 'a ⇀ 'b)) ⟹ ¬ *finite* (*UNIV* :: 'a *set*) ⟹
∃ *x*. *f x* = *None*
⟨*proof*⟩

**lemma** *dom-minus*:
*f x* = *None* ⟹ *dom f* − *insert x A* = *dom f* − *A*
⟨*proof*⟩

**lemma** *insert-dom*:
*f x* = *Some y* ⟹ *insert x* (*dom f*) = *dom f*
⟨*proof*⟩

## 43.10 *ran*

**lemma** *ranI*: *m a* = *Some b* ==> *b* : *ran m*
⟨*proof*⟩

**lemma** *ran-empty* [*simp*]: *ran empty* = {}
⟨*proof*⟩

**lemma** *ran-map-upd* [*simp*]: *m a* = *None* ==> *ran*(*m*(*a*|->*b*)) = *insert b* (*ran m*)
⟨*proof*⟩

## 43.11 *map-le*

**lemma** *map-le-empty* [*simp*]: *empty* ⊆$_m$ *g*
⟨*proof*⟩

**lemma** *upd-None-map-le* [*simp*]: *f*(*x* := *None*) ⊆$_m$ *f*
⟨*proof*⟩

**lemma** *map-le-upd*[*simp*]: *f* ⊆$_m$ *g* ==> *f*(*a* := *b*) ⊆$_m$ *g*(*a* := *b*)
⟨*proof*⟩

**lemma** *map-le-imp-upd-le* [*simp*]: *m1* ⊆$_m$ *m2* ⟹ *m1*(*x* := *None*) ⊆$_m$ *m2*(*x* ↦ *y*)
⟨*proof*⟩

**lemma** *map-le-upds* [*simp*]:

$f \subseteq_m g \Longrightarrow f(as\ [|->]\ bs) \subseteq_m g(as\ [|->]\ bs)$
⟨*proof*⟩

**lemma** *map-le-implies-dom-le*: $(f \subseteq_m g) \Longrightarrow (dom\ f \subseteq dom\ g)$
⟨*proof*⟩

**lemma** *map-le-refl* [*simp*]: $f \subseteq_m f$
⟨*proof*⟩

**lemma** *map-le-trans*[*trans*]: $[\![\ m1 \subseteq_m m2;\ m2 \subseteq_m m3 ]\!] \Longrightarrow m1 \subseteq_m m3$
⟨*proof*⟩

**lemma** *map-le-antisym*: $[\![\ f \subseteq_m g;\ g \subseteq_m f\ ]\!] \Longrightarrow f = g$
⟨*proof*⟩

**lemma** *map-le-map-add* [*simp*]: $f \subseteq_m (g\ ++\ f)$
⟨*proof*⟩

**lemma** *map-le-iff-map-add-commute*: $(f \subseteq_m f\ ++\ g) = (f++g = g++f)$
⟨*proof*⟩

**lemma** *map-add-le-mapE*: $f++g \subseteq_m h \Longrightarrow g \subseteq_m h$
⟨*proof*⟩

**lemma** *map-add-le-mapI*: $[\![\ f \subseteq_m h;\ g \subseteq_m h;\ f \subseteq_m f++g\ ]\!] \Longrightarrow f++g \subseteq_m h$
⟨*proof*⟩

**end**

# 44 Refute: Refute

**theory** *Refute*
**imports** *Hilbert-Choice List Record*
**uses**
*Tools/prop-logic.ML*
*Tools/sat-solver.ML*
*Tools/refute.ML*
*Tools/refute-isar.ML*
**begin**

⟨*ML*⟩

```
(* ------------------------------------------------------------------------- *)
(* REFUTE                                                                    *)
(*                                                                           *)
(* We use a SAT solver to search for a (finite) model that refutes a given   *)
(* HOL formula.                                                              *)
```

```
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* NOTE                                                                      *)
(*                                                                           *)
(* I strongly recommend that you install a stand-alone SAT solver if you     *)
(* want to use 'refute'.  For details see 'HOL/Tools/sat_solver.ML'.  If you *)
(* have installed (a supported version of) zChaff, simply set 'ZCHAFF_HOME'  *)
(* in 'etc/settings'.                                                        *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* USAGE                                                                     *)
(*                                                                           *)
(* See the file 'HOL/ex/Refute_Examples.thy' for examples.  The supported    *)
(* parameters are explained below.                                           *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* CURRENT LIMITATIONS                                                       *)
(*                                                                           *)
(* 'refute' currently accepts formulas of higher-order predicate logic (with *)
(* equality), including free/bound/schematic variables, lambda abstractions, *)
(* sets and set membership, "arbitrary", "The", "Eps", records and           *)
(* inductively defined sets.  Constants are unfolded automatically, and sort *)
(* axioms are added as well.  Other, user-asserted axioms however are        *)
(* ignored.  Inductive datatypes and recursive functions are supported, but  *)
(* may lead to spurious countermodels.                                       *)
(*                                                                           *)
(* The (space) complexity of the algorithm is non-elementary.                *)
(*                                                                           *)
(* Schematic type variables are not supported.                               *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* PARAMETERS                                                                *)
(*                                                                           *)
(* The following global parameters are currently supported (and required):   *)
(*                                                                           *)
(* Name          Type    Description                                         *)
(*                                                                           *)
(* "minsize"     int     Only search for models with size at least           *)
(*                       'minsize'.                                          *)
(* "maxsize"     int     If >0, only search for models with size at most     *)
(*                       'maxsize'.                                          *)
(* "maxvars"     int     If >0, use at most 'maxvars' boolean variables      *)
(*                       when transforming the term into a propositional     *)
(*                       formula.                                            *)
(* "maxtime"     int     If >0, terminate after at most 'maxtime' seconds.   *)
```

```
(*                       This value is ignored under some ML compilers.      *)
(* "satsolver"   string  Name of the SAT solver to be used.                  *)
(*                                                                           *)
(* See 'HOL/SAT.thy' for default values.                                     *)
(*                                                                           *)
(* The size of particular types can be specified in the form type=size       *)
(* (where 'type' is a string, and 'size' is an int).  Examples:              *)
(* "'a"=1                                                                    *)
(* "List.list"=2                                                             *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* FILES                                                                     *)
(*                                                                           *)
(* HOL/Tools/prop_logic.ML    Propositional logic                            *)
(* HOL/Tools/sat_solver.ML    SAT solvers                                    *)
(* HOL/Tools/refute.ML        Translation HOL -> propositional logic and     *)
(*                            Boolean assignment -> HOL model                *)
(* HOL/Tools/refute_isar.ML   Adds 'refute'/'refute_params' to Isabelle's    *)
(*                            syntax                                         *)
(* HOL/Refute.thy             This file: loads the ML files, basic setup,    *)
(*                            documentation                                  *)
(* HOL/SAT.thy                Sets default parameters                        *)
(* HOL/ex/RefuteExamples.thy  Examples                                       *)
(* ------------------------------------------------------------------------- *)
```

**end**

# 45 SAT: Reconstructing external resolution proofs for propositional logic

**theory** *SAT*
**imports** *Refute*
**uses**
*Tools/cnf-funcs.ML*
*Tools/sat-funcs.ML*
**begin**

Late package setup: default values for refute, see also theory *Refute*.

**refute-params**
[*itself=1*,
*minsize=1*,
*maxsize=8*,
*maxvars=10000*,
*maxtime=60*,
*satsolver=auto*]

⟨*ML*⟩

**end**

## 46 Main: Main HOL

**theory** *Main*
**imports** *Plain Code-Eval Map Recdef SAT*
**begin**

Classical Higher-order Logic – only "Main", excluding real and complex numbers etc.

See further [1]

**end**

## References

[1] T. Nipkow, L. C. Paulson, and M. Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer-Verlag, 2002.