Isabelle/HOL — Higher-Order Logic

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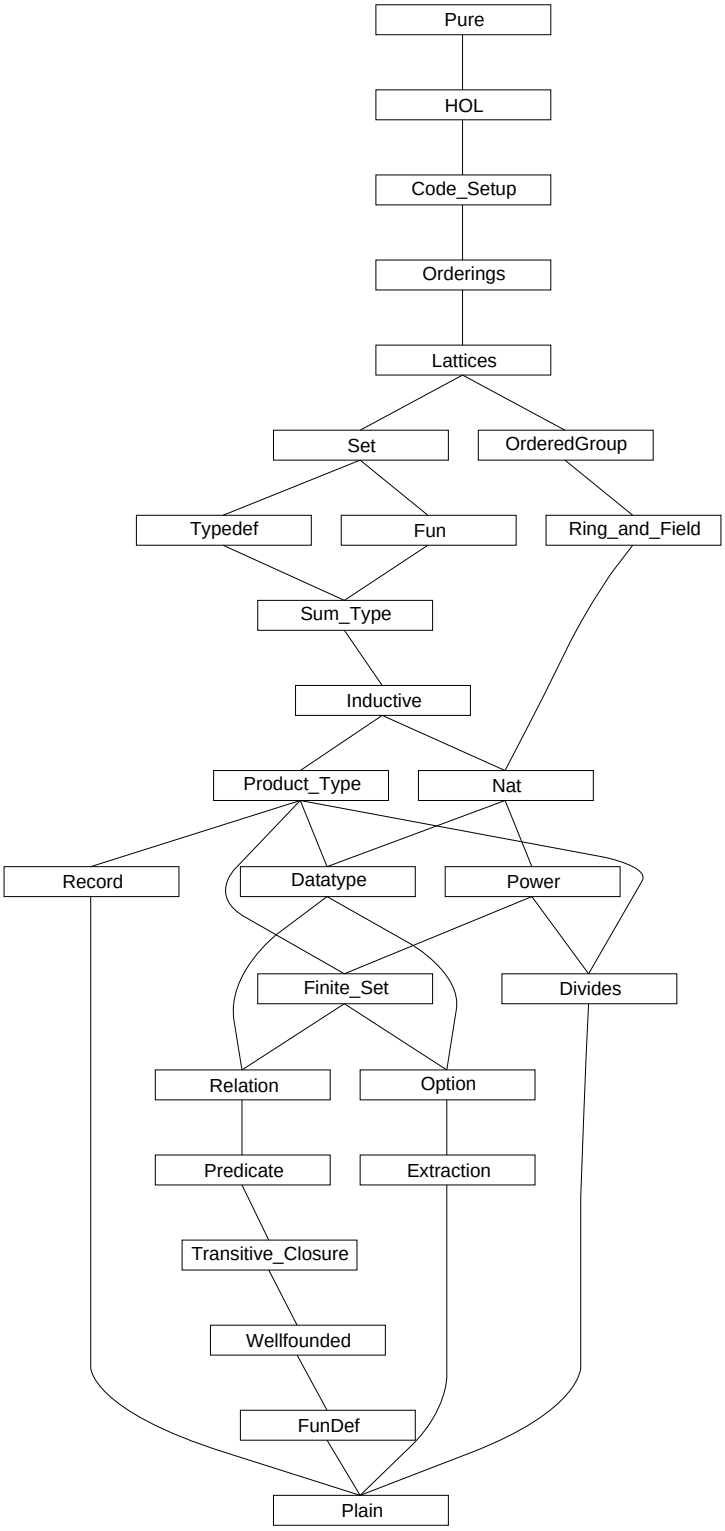
Pure
HOL
Code_Setup
Orderings
Lattices
Set
OrderedGroup
Typedef
Fun
Ring_and_Field
Sum_Type
Inductive
Product_Type
Nat
Record
Datatype
Power
Finite_Set
Divides
Relation
Option
Predicate
Extraction
Transitive_Closure
Wellfounded
FunDef
Plain

1 HOL: The basis of Higher-Order Logic

theory *HOL*
imports *Pure*
uses
(*Tools/hologic.ML*)
~~/*src/Tools/IsaPlanner/zipper.ML*
~~/*src/Tools/IsaPlanner/isand.ML*
~~/*src/Tools/IsaPlanner/rw-tools.ML*
~~/*src/Tools/IsaPlanner/rw-inst.ML*
~~/*src/Tools/intuitionistic.ML*
~~/*src/Tools/project-rule.ML*
~~/*src/Provers/hypsubst.ML*
~~/*src/Provers/splitter.ML*
~~/*src/Provers/classical.ML*
~~/*src/Provers/blast.ML*
~~/*src/Provers/clasimp.ML*
~~/*src/Tools/coherent.ML*
~~/*src/Tools/eqsubst.ML*
~~/*src/Provers/quantifier1.ML*
(*Tools/simpdata.ML*)
~~/*src/Tools/random-word.ML*
~~/*src/Tools/atomize-elim.ML*
~~/*src/Tools/induct.ML*
(~~/*src/Tools/induct-tacs.ML*)
~~/*src/Tools/value.ML*
~~/*src/Tools/code/code-name.ML*
~~/*src/Tools/code/code-funcgr.ML*
~~/*src/Tools/code/code-wellsorted.ML*
~~/*src/Tools/code/code-thingol.ML*
~~/*src/Tools/code/code-printer.ML*
~~/*src/Tools/code/code-target.ML*
~~/*src/Tools/code/code-ml.ML*
~~/*src/Tools/code/code-haskell.ML*
~~/*src/Tools/nbe.ML*
(*Tools/recfun-codegen.ML*)
begin

⟨*ML*⟩

1.1 Primitive logic

1.1.1 Core syntax

classes *type*
defaultsort *type*
⟨*ML*⟩

arities
fun :: (*type*, *type*) *type*

itself :: (*type*) *type*

global

typedecl *bool*

judgment
Trueprop :: *bool* => *prop* ((-) *5*)

consts
Not :: *bool* => *bool* (~ - [*40*] *40*)
True :: *bool*
False :: *bool*

The :: ('*a* => *bool*) => '*a*
All :: ('*a* => *bool*) => *bool* (**binder** *ALL* *10*)
Ex :: ('*a* => *bool*) => *bool* (**binder** *EX* *10*)
Ex1 :: ('*a* => *bool*) => *bool* (**binder** *EX*! *10*)
Let :: ['*a*, '*a* => '*b*] => '*b*

op = :: ['*a*, '*a*] => *bool* (**infixl** = *50*)
op & :: [*bool*, *bool*] => *bool* (**infixr** & *35*)
op | :: [*bool*, *bool*] => *bool* (**infixr** | *30*)
op --> :: [*bool*, *bool*] => *bool* (**infixr** --> *25*)

local

consts
If :: [*bool*, '*a*, '*a*] => '*a* ((*if* (-)/ *then* (-)/ *else* (-)) *10*)

1.1.2 Additional concrete syntax

notation (**output**)
op = (**infix** = *50*)

abbreviation
not-equal :: ['*a*, '*a*] => *bool* (**infixl** ~= *50*) **where**
x ~= *y* == ~ (*x* = *y*)

notation (**output**)
not-equal (**infix** ~= *50*)

notation (*xsymbols*)
Not (¬ - [*40*] *40*) **and**
op & (**infixr** ∧ *35*) **and**
op | (**infixr** ∨ *30*) **and**
op --> (**infixr** ⟶ *25*) **and**
not-equal (**infix** ≠ *50*)

notation (*HTML* **output**)
Not (¬ - [*40*] *40*) **and**
op & (**infixr** ∧ *35*) **and**
op | (**infixr** ∨ *30*) **and**
not-equal (**infix** ≠ *50*)

abbreviation (*iff*)
iff :: [*bool*, *bool*] => *bool* (**infixr** <–> *25*) **where**
A <–> *B* == *A* = *B*

notation (*xsymbols*)
iff (**infixr** ⟷ *25*)

nonterminals
letbinds letbind
case-syn cases-syn

syntax
-The :: [*pttrn*, *bool*] => *'a* ((*3THE -./ -*) [*0*, *10*] *10*)

-bind :: [*pttrn*, *'a*] => *letbind* ((*2- =/ -*) *10*)
:: *letbind* => *letbinds* (-)
-binds :: [*letbind*, *letbinds*] => *letbinds* (-;/ -)
-Let :: [*letbinds*, *'a*] => *'a* ((*let* (-)/ *in* (-)) *10*)

-case-syntax:: [*'a*, *cases-syn*] => *'b* ((*case - of/ -*) *10*)
-case1 :: [*'a*, *'b*] => *case-syn* ((*2- =>/ -*) *10*)
:: *case-syn* => *cases-syn* (-)
-case2 :: [*case-syn*, *cases-syn*] => *cases-syn* (-/ | -)

translations
THE x. P == *The* (%*x. P*)
-Let (*-binds b bs*) *e* == *-Let b* (*-Let bs e*)
let x = a in e == *Let a* (%*x. e*)

⟨*ML*⟩

syntax (*xsymbols*)
-case1 :: [*'a*, *'b*] => *case-syn* ((*2- ⇒/ -*) *10*)

notation (*xsymbols*)
All (**binder** ∀ *10*) **and**
Ex (**binder** ∃ *10*) **and**
Ex1 (**binder** ∃! *10*)

notation (*HTML* **output**)
All (**binder** ∀ *10*) **and**
Ex (**binder** ∃ *10*) **and**

Ex1 (**binder** $\exists!$ *10*)

notation (*HOL*)
All (**binder** ! *10*) **and**
Ex (**binder** *?* *10*) **and**
Ex1 (**binder** *?!* *10*)

1.1.3 Axioms and basic definitions

axioms
refl: $t = (t::'a)$
subst: $s = t \Longrightarrow P\ s \Longrightarrow P\ t$
ext: $(!!x::'a.\ (f\ x ::'b) = g\ x) ==> (\%x.\ f\ x) = (\%x.\ g\ x)$
— Extensionality is built into the meta-logic, and this rule expresses a related property. It is an eta-expanded version of the traditional rule, and similar to the ABS rule of HOL

the-eq-trivial: $(THE\ x.\ x = a) = (a::'a)$

impI: $(P ==> Q) ==> P-->Q$
mp: $[|\ P-->Q;\ P\ |] ==> Q$

defs
True-def: $True == ((\%x::bool.\ x) = (\%x.\ x))$
All-def: $All(P) == (P = (\%x.\ True))$
Ex-def: $Ex(P) == !Q.\ (!x.\ P\ x --> Q) --> Q$
False-def: $False == (!P.\ P)$
not-def: $\sim P == P-->False$
and-def: $P\ \&\ Q == !R.\ (P-->Q-->R) --> R$
or-def: $P \mid Q == !R.\ (P-->R) --> (Q-->R) --> R$
Ex1-def: $Ex1(P) == ?\ x.\ P(x)\ \&\ (!\ y.\ P(y) --> y=x)$

axioms
iff: $(P-->Q) --> (Q-->P) --> (P=Q)$
True-or-False: $(P=True) \mid (P=False)$

defs
Let-def: $Let\ s\ f == f(s)$
if-def: $If\ P\ x\ y == THE\ z::'a.\ (P=True --> z=x)\ \&\ (P=False --> z=y)$

finalconsts
op =
op -->
The

axiomatization
undefined :: *'a*

abbreviation (*input*)
arbitrary ≡ *undefined*

1.1.4 Generic classes and algebraic operations

class *default* =
fixes *default* :: *'a*

class *zero* =
fixes *zero* :: *'a* (*0*)

class *one* =
fixes *one* :: *'a* (*1*)

hide (**open**) *const zero one*

class *plus* =
fixes *plus* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** + *65*)

class *minus* =
fixes *minus* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** − *65*)

class *uminus* =
fixes *uminus* :: *'a* ⇒ *'a* (− - [*81*] *80*)

class *times* =
fixes *times* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ∗ *70*)

class *inverse* =
fixes *inverse* :: *'a* ⇒ *'a*
and *divide* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *'/ 70*)

class *abs* =
fixes *abs* :: *'a* ⇒ *'a*
begin

notation (*xsymbols*)
abs (|-|)

notation (*HTML* **output**)
abs (|-|)

end

class *sgn* =
fixes *sgn* :: *'a* ⇒ *'a*

class *ord* =

fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$
and *less* :: $'a \Rightarrow 'a \Rightarrow bool$
begin

notation
less-eq (*op* <=) **and**
less-eq ((-/ <= -) [*51*, *51*] *50*) **and**
less (*op* <) **and**
less ((-/ < -) [*51*, *51*] *50*)

notation (*xsymbols*)
less-eq (*op* $\leq$) **and**
less-eq ((-/ $\leq$ -) [*51*, *51*] *50*)

notation (*HTML* **output**)
less-eq (*op* $\leq$) **and**
less-eq ((-/ $\leq$ -) [*51*, *51*] *50*)

abbreviation (*input*)
greater-eq (**infix** >= *50*) **where**
$x >= y \equiv y <= x$

notation (*input*)
greater-eq (**infix** $\geq$ *50*)

abbreviation (*input*)
greater (**infix** > *50*) **where**
$x > y \equiv y < x$

end

syntax
-index1 :: *index* ($_1$)
translations
(*index*) $_1$ => (*index*) $_\diamond$

⟨*ML*⟩

1.2 Fundamental rules

1.2.1 Equality

lemma *sym*: $s = t \Longrightarrow t = s$
⟨*proof*⟩

lemma *ssubst*: $t = s \Longrightarrow P\ s \Longrightarrow P\ t$
⟨*proof*⟩

lemma *trans*: $[\![r=s; s=t]\!] \Longrightarrow r=t$
⟨*proof*⟩

lemma *meta-eq-to-obj-eq*:
assumes *meq*: $A == B$
shows $A = B$
⟨*proof*⟩

Useful with *erule* for proving equalities from known equalities.

lemma *box-equals*: [| $a=b$; $a=c$; $b=d$ |] ==> $c=d$
⟨*proof*⟩

For calculational reasoning:

lemma *forw-subst*: $a = b$ ==> $P\ b$ ==> $P\ a$
⟨*proof*⟩

lemma *back-subst*: $P\ a$ ==> $a = b$ ==> $P\ b$
⟨*proof*⟩

1.2.2 Congruence rules for application

lemma *fun-cong*: $(f::'a=>'b) = g$ ==> $f(x)=g(x)$
⟨*proof*⟩

lemma *arg-cong*: $x=y$ ==> $f(x)=f(y)$
⟨*proof*⟩

lemma *arg-cong2*: ⟦ $a = b$; $c = d$ ⟧ ⟹ $f\ a\ c = f\ b\ d$
⟨*proof*⟩

lemma *cong*: [| $f = g$; $(x::'a) = y$ |] ==> $f(x) = g(y)$
⟨*proof*⟩

1.2.3 Equality of booleans – iff

lemma *iffI*: **assumes** P ==> Q **and** Q ==> P **shows** $P=Q$
⟨*proof*⟩

lemma *iffD2*: [| $P=Q$; Q |] ==> P
⟨*proof*⟩

lemma *rev-iffD2*: [| Q; $P=Q$ |] ==> P
⟨*proof*⟩

lemma *iffD1*: $Q = P$ ⟹ Q ⟹ P
⟨*proof*⟩

lemma *rev-iffD1*: Q ⟹ $Q = P$ ⟹ P
⟨*proof*⟩

lemma *iffE*:
assumes *major*: $P=Q$
and *minor*: $[|\ P \dashrightarrow Q;\ Q \dashrightarrow P\ |] \Longrightarrow R$
shows R
⟨*proof*⟩

1.2.4 True

lemma *TrueI*: *True*
⟨*proof*⟩

lemma *eqTrueI*: $P \Longrightarrow P = True$
⟨*proof*⟩

lemma *eqTrueE*: $P = True \Longrightarrow P$
⟨*proof*⟩

1.2.5 Universal quantifier

lemma *allI*: **assumes** $!!x::'a.\ P(x)$ **shows** $ALL\ x.\ P(x)$
⟨*proof*⟩

lemma *spec*: $ALL\ x::'a.\ P(x) \Longrightarrow P(x)$
⟨*proof*⟩

lemma *allE*:
assumes *major*: $ALL\ x.\ P(x)$
and *minor*: $P(x) \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *all-dupE*:
assumes *major*: $ALL\ x.\ P(x)$
and *minor*: $[|\ P(x);\ ALL\ x.\ P(x)\ |] \Longrightarrow R$
shows R
⟨*proof*⟩

1.2.6 False

Depends upon *spec*; it is impossible to do propositional logic before quantifiers!

lemma *FalseE*: $False \Longrightarrow P$
⟨*proof*⟩

lemma *False-neq-True*: $False = True \Longrightarrow P$
⟨*proof*⟩

1.2.7 Negation

lemma *notI*:
assumes P ==> *False*
shows ~P
⟨*proof*⟩

lemma *False-not-True*: *False* ~= *True*
⟨*proof*⟩

lemma *True-not-False*: *True* ~= *False*
⟨*proof*⟩

lemma *notE*: [| ~P; P |] ==> R
⟨*proof*⟩

lemma *notI2*: $(P \Longrightarrow \neg Pa) \Longrightarrow (P \Longrightarrow Pa) \Longrightarrow \neg P$
⟨*proof*⟩

1.2.8 Implication

lemma *impE*:
assumes P-->Q P Q ==> R
shows R
⟨*proof*⟩

lemma *rev-mp*: [| P; P --> Q |] ==> Q
⟨*proof*⟩

lemma *contrapos-nn*:
assumes *major*: ~Q
and *minor*: P==>Q
shows ~P
⟨*proof*⟩

lemma *contrapos-pn*:
assumes *major*: Q
and *minor*: P ==> ~Q
shows ~P
⟨*proof*⟩

lemma *not-sym*: t ~= s ==> s ~= t
⟨*proof*⟩

lemma *eq-neq-eq-imp-neq*: [| x = a ; a ~= b; b = y |] ==> x ~= y
⟨*proof*⟩

lemma *rev-contrapos*:
assumes *pq*: $P \Longrightarrow Q$
and *nq*: $\sim Q$
shows $\sim P$
⟨*proof*⟩

1.2.9 Existential quantifier

lemma *exI*: $P\ x \Longrightarrow EX\ x::'a.\ P\ x$
⟨*proof*⟩

lemma *exE*:
assumes *major*: $EX\ x::'a.\ P(x)$
and *minor*: $!!x.\ P(x) \Longrightarrow Q$
shows Q
⟨*proof*⟩

1.2.10 Conjunction

lemma *conjI*: $[|\ P;\ Q\ |] \Longrightarrow P\&Q$
⟨*proof*⟩

lemma *conjunct1*: $[|\ P\ \&\ Q\ |] \Longrightarrow P$
⟨*proof*⟩

lemma *conjunct2*: $[|\ P\ \&\ Q\ |] \Longrightarrow Q$
⟨*proof*⟩

lemma *conjE*:
assumes *major*: $P\&Q$
and *minor*: $[|\ P;\ Q\ |] \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *context-conjI*:
assumes $P\ P \Longrightarrow Q$ **shows** $P\ \&\ Q$
⟨*proof*⟩

1.2.11 Disjunction

lemma *disjI1*: $P \Longrightarrow P|Q$
⟨*proof*⟩

lemma *disjI2*: $Q \Longrightarrow P|Q$
⟨*proof*⟩

lemma *disjE*:
assumes *major*: $P|Q$
and *minorP*: $P \Longrightarrow R$
and *minorQ*: $Q \Longrightarrow R$

shows R
⟨*proof*⟩

1.2.12 Classical logic

lemma *classical*:
assumes *prem*: $\sim P \Longrightarrow P$
shows P
⟨*proof*⟩

lemmas *ccontr* = *FalseE* [*THEN classical*, *standard*]

lemma *rev-notE*:
assumes *premp*: P
and *premnot*: $\sim R \Longrightarrow \sim P$
shows R
⟨*proof*⟩

lemma *notnotD*: $\sim\sim P \Longrightarrow P$
⟨*proof*⟩

lemma *contrapos-pp*:
assumes *p1*: Q
and *p2*: $\sim P \Longrightarrow \sim Q$
shows P
⟨*proof*⟩

1.2.13 Unique existence

lemma *ex1I*:
assumes $P\ a$ $!!x.\ P(x) \Longrightarrow x=a$
shows $EX!\ x.\ P(x)$
⟨*proof*⟩

Sometimes easier to use: the premises have no shared variables. Safe!

lemma *ex-ex1I*:
assumes *ex-prem*: $EX\ x.\ P(x)$
and *eq*: $!!x\ y.\ [|\ P(x);\ P(y)\ |] \Longrightarrow x=y$
shows $EX!\ x.\ P(x)$
⟨*proof*⟩

lemma *ex1E*:
assumes *major*: $EX!\ x.\ P(x)$
and *minor*: $!!x.\ [|\ P(x);\ \ ALL\ y.\ P(y) \longrightarrow y=x\ |] \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *ex1-implies-ex*: $EX!\ x.\ P\ x \Longrightarrow EX\ x.\ P\ x$

⟨*proof*⟩

1.2.14 THE: definite description operator

lemma *the-equality*:
assumes *prema*: $P\ a$
and *premx*: $!!x.\ P\ x \Longrightarrow x=a$
shows $(THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *theI*:
assumes $P\ a$ **and** $!!x.\ P\ x \Longrightarrow x=a$
shows $P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *theI'*: $EX!\ x.\ P\ x \Longrightarrow P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *theI2*:
assumes $P\ a\ !!x.\ P\ x \Longrightarrow x=a\ !!x.\ P\ x \Longrightarrow Q\ x$
shows $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *the1I2*: **assumes** $EX!\ x.\ P\ x\ \bigwedge x.\ P\ x \Longrightarrow Q\ x$ **shows** $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *the1-equality* [*elim?*]: $[|\ EX!x.\ P\ x;\ P\ a\ |] \Longrightarrow (THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *the-sym-eq-trivial*: $(THE\ y.\ x=y) = x$
⟨*proof*⟩

1.2.15 Classical intro rules for disjunction and existential quantifiers

lemma *disjCI*:
assumes $\sim Q \Longrightarrow P$ **shows** $P|Q$
⟨*proof*⟩

lemma *excluded-middle*: $\sim P \mid P$
⟨*proof*⟩

case distinction as a natural deduction rule. Note that $\neg\ P$ is the second case, not the first

lemma *case-split* [*case-names True False*]:
assumes *prem1*: $P \Longrightarrow Q$
and *prem2*: $\sim P \Longrightarrow Q$
shows Q

⟨*proof*⟩

lemma *impCE*:
assumes *major*: $P-->Q$
and *minor*: $\sim P ==> R\ Q ==> R$
shows R
⟨*proof*⟩

lemma *impCE'*:
assumes *major*: $P-->Q$
and *minor*: $Q ==> R\ \sim P ==> R$
shows R
⟨*proof*⟩

lemma *iffCE*:
assumes *major*: $P=Q$
and *minor*: $[|\ P;\ Q\ |] ==> R\ \ [|\ \sim P;\ \sim Q\ |] ==> R$
shows R
⟨*proof*⟩

lemma *exCI*:
assumes $ALL\ x.\ \sim P(x) ==> P(a)$
shows $EX\ x.\ P(x)$
⟨*proof*⟩

1.2.16 Intuitionistic Reasoning

lemma *impE'*:
assumes *1*: $P\ -->\ Q$
and *2*: $Q ==> R$
and *3*: $P\ -->\ Q ==> P$
shows R
⟨*proof*⟩

lemma *allE'*:
assumes *1*: $ALL\ x.\ P\ x$
and *2*: $P\ x ==> ALL\ x.\ P\ x ==> Q$
shows Q
⟨*proof*⟩

lemma *notE'*:
assumes *1*: $\sim\ P$
and *2*: $\sim\ P ==> P$
shows R
⟨*proof*⟩

lemma *TrueE*: *True* ==> *P* ==> *P* ⟨*proof*⟩
lemma *notFalseE*: ~ *False* ==> *P* ==> *P* ⟨*proof*⟩

lemmas [*Pure.elim*!] = *disjE iffE FalseE conjE exE TrueE notFalseE*
and [*Pure.intro*!] = *iffI conjI impI TrueI notI allI refl*
and [*Pure.elim 2*] = *allE notE' impE'*
and [*Pure.intro*] = *exI disjI2 disjI1*

lemmas [*trans*] = *trans*
and [*sym*] = *sym not-sym*
and [*Pure.elim?*] = *iffD1 iffD2 impE*

⟨*ML*⟩

1.2.17 Atomizing meta-level connectives

axiomatization where
eq-reflection: *x* = *y* ⟹ *x* ≡ *y*

lemma *atomize-all* [*atomize*]: (!!*x*. *P x*) == *Trueprop* (*ALL x*. *P x*)
⟨*proof*⟩

lemma *atomize-imp* [*atomize*]: (*A* ==> *B*) == *Trueprop* (*A* –> *B*)
⟨*proof*⟩

lemma *atomize-not*: (*A* ==> *False*) == *Trueprop* (~*A*)
⟨*proof*⟩

lemma *atomize-eq* [*atomize*]: (*x* == *y*) == *Trueprop* (*x* = *y*)
⟨*proof*⟩

lemma *atomize-conj* [*atomize*]: (*A* &&& *B*) == *Trueprop* (*A* & *B*)
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-all atomize-imp*
and [*symmetric*, *defn*] = *atomize-all atomize-imp atomize-eq*

1.2.18 Atomizing elimination rules

⟨*ML*⟩

lemma *atomize-exL*[*atomize-elim*]: (!!*x*. *P x* ==> *Q*) == ((*EX x*. *P x*) ==> *Q*)
⟨*proof*⟩

lemma *atomize-conjL*[*atomize-elim*]: (*A* ==> *B* ==> *C*) == (*A* & *B* ==> *C*)
⟨*proof*⟩

lemma *atomize-disjL*[*atomize-elim*]: ((*A* ==> *C*) ==> (*B* ==> *C*) ==> *C*)
== ((*A* | *B* ==> *C*) ==> *C*)
⟨*proof*⟩

lemma *atomize-elimL*[*atomize-elim*]: (!!*B*. (*A* ==> *B*) ==> *B*) == *Trueprop A*
⟨*proof*⟩

1.3 Package setup

1.3.1 Classical Reasoner setup

lemma *imp-elim*: *P* --> *Q* ==> (~ *R* ==> *P*) ==> (*Q* ==> *R*) ==> *R*
⟨*proof*⟩

lemma *swap*: ~ *P* ==> (~ *R* ==> *P*) ==> *R*
⟨*proof*⟩

lemma *thin-refl*:
⋀*X*. ⟦ *x*=*x*; *PROP W* ⟧ ⟹ *PROP W* ⟨*proof*⟩

⟨*ML*⟩

ResBlacklist holds theorems blacklisted to sledgehammer. These theorems typically produce clauses that are prolific (match too many equality or membership literals) and relate to seldom-used facts. Some duplicate other rules.

⟨*ML*⟩

declare *iffI* [*intro*!]
and *notI* [*intro*!]
and *impI* [*intro*!]
and *disjCI* [*intro*!]
and *conjI* [*intro*!]
and *TrueI* [*intro*!]
and *refl* [*intro*!]

declare *iffCE* [*elim*!]
and *FalseE* [*elim*!]
and *impCE* [*elim*!]
and *disjE* [*elim*!]
and *conjE* [*elim*!]
and *conjE* [*elim*!]

declare *ex-ex1I* [*intro*!]
and *allI* [*intro*!]
and *the-equality* [*intro*]
and *exI* [*intro*]

declare *exE* [*elim*!]
allE [*elim*]

⟨*ML*⟩

lemma *contrapos-np*: $\sim Q \Longrightarrow (\sim P \Longrightarrow Q) \Longrightarrow P$
⟨*proof*⟩

declare *ex-ex1I* [*rule del*, *intro*! *2*]
and *ex1I* [*intro*]

lemmas [*intro?*] = *ext*
and [*elim?*] = *ex1-implies-ex*

lemma *alt-ex1E* [*elim*!]:
assumes *major*: $\exists!x.\ P\ x$
and *prem*: $\bigwedge x.\ [\![\ P\ x;\ \forall y\ y'.\ P\ y \wedge P\ y' \longrightarrow y = y'\ ]\!] \Longrightarrow R$
shows R
⟨*proof*⟩

⟨*ML*⟩

1.3.2 Simplifier

lemma *eta-contract-eq*: $(\%s.\ f\ s) = f$ ⟨*proof*⟩

lemma *simp-thms*:
shows *not-not*: $(\sim \sim P) = P$
and *Not-eq-iff*: $((\sim P) = (\sim Q)) = (P = Q)$
and
$(P \sim= Q) = (P = (\sim Q))$
$(P \mid \sim P) = True \quad (\sim P \mid P) = True$
$(x = x) = True$
and *not-True-eq-False*: $(\neg\ True) = False$
and *not-False-eq-True*: $(\neg\ False) = True$
and
$(\sim P) \sim= P \quad P \sim= (\sim P)$
$(True=P) = P$
and *eq-True*: $(P = True) = P$
and $(False=P) = (\sim P)$
and *eq-False*: $(P = False) = (\neg\ P)$
and
$(True \dashrightarrow P) = P \quad (False \dashrightarrow P) = True$
$(P \dashrightarrow True) = True \quad (P \dashrightarrow P) = True$
$(P \dashrightarrow False) = (\sim P) \quad (P \dashrightarrow \sim P) = (\sim P)$
$(P \ \&\ True) = P \quad (True \ \&\ P) = P$
$(P \ \&\ False) = False \quad (False \ \&\ P) = False$
$(P \ \&\ P) = P \quad (P \ \&\ (P \ \&\ Q)) = (P \ \&\ Q)$
$(P \ \&\ \sim P) = False \quad (\sim P \ \&\ P) = False$
$(P \mid True) = True \quad (True \mid P) = True$
$(P \mid False) = P \quad (False \mid P) = P$
$(P \mid P) = P \quad (P \mid (P \mid Q)) = (P \mid Q)$ **and**
$(ALL\ x.\ P) = P \quad (EX\ x.\ P) = P \quad EX\ x.\ x=t \quad EX\ x.\ t=x$

— needed for the one-point-rule quantifier simplification procs
— essential for termination!! **and**
!!P. $(EX\ x.\ x{=}t\ \&\ P(x)) = P(t)$
!!P. $(EX\ x.\ t{=}x\ \&\ P(x)) = P(t)$
!!P. $(ALL\ x.\ x{=}t\ {-}{-}{>}\ P(x)) = P(t)$
!!P. $(ALL\ x.\ t{=}x\ {-}{-}{>}\ P(x)) = P(t)$
⟨*proof*⟩

lemma *disj-absorb*: $(A \mid A) = A$
⟨*proof*⟩

lemma *disj-left-absorb*: $(A \mid (A \mid B)) = (A \mid B)$
⟨*proof*⟩

lemma *conj-absorb*: $(A\ \&\ A) = A$
⟨*proof*⟩

lemma *conj-left-absorb*: $(A\ \&\ (A\ \&\ B)) = (A\ \&\ B)$
⟨*proof*⟩

lemma *eq-ac*:
shows *eq-commute*: $(a{=}b) = (b{=}a)$
and *eq-left-commute*: $(P{=}(Q{=}R)) = (Q{=}(P{=}R))$
and *eq-assoc*: $((P{=}Q){=}R) = (P{=}(Q{=}R))$ ⟨*proof*⟩
lemma *neq-commute*: $(a^{\sim}{=}b) = (b^{\sim}{=}a)$ ⟨*proof*⟩

lemma *conj-comms*:
shows *conj-commute*: $(P\&Q) = (Q\&P)$
and *conj-left-commute*: $(P\&(Q\&R)) = (Q\&(P\&R))$ ⟨*proof*⟩
lemma *conj-assoc*: $((P\&Q)\&R) = (P\&(Q\&R))$ ⟨*proof*⟩

lemmas *conj-ac* = *conj-commute conj-left-commute conj-assoc*

lemma *disj-comms*:
shows *disj-commute*: $(P|Q) = (Q|P)$
and *disj-left-commute*: $(P|(Q|R)) = (Q|(P|R))$ ⟨*proof*⟩
lemma *disj-assoc*: $((P|Q)|R) = (P|(Q|R))$ ⟨*proof*⟩

lemmas *disj-ac* = *disj-commute disj-left-commute disj-assoc*

lemma *conj-disj-distribL*: $(P\&(Q|R)) = (P\&Q \mid P\&R)$ ⟨*proof*⟩
lemma *conj-disj-distribR*: $((P|Q)\&R) = (P\&R \mid Q\&R)$ ⟨*proof*⟩

lemma *disj-conj-distribL*: $(P|(Q\&R)) = ((P|Q)\ \&\ (P|R))$ ⟨*proof*⟩
lemma *disj-conj-distribR*: $((P\&Q)|R) = ((P|R)\ \&\ (Q|R))$ ⟨*proof*⟩

lemma *imp-conjR*: $(P\ {-}{-}{>}\ (Q\&R)) = ((P{-}{-}{>}Q)\ \&\ (P{-}{-}{>}R))$ ⟨*proof*⟩
lemma *imp-conjL*: $((P\&Q)\ {-}{-}{>}R)\ = (P\ {-}{-}{>}\ (Q\ {-}{-}{>}\ R))$ ⟨*proof*⟩
lemma *imp-disjL*: $((P|Q)\ {-}{-}{>}\ R) = ((P{-}{-}{>}R)\&(Q{-}{-}{>}R))$ ⟨*proof*⟩

These two are specialized, but *imp-disj-not1* is useful in *Auth/Yahalom*.

lemma *imp-disj-not1*: $(P \dashrightarrow Q \mid R) = (\sim Q \dashrightarrow P \dashrightarrow R)$ ⟨*proof*⟩
lemma *imp-disj-not2*: $(P \dashrightarrow Q \mid R) = (\sim R \dashrightarrow P \dashrightarrow Q)$ ⟨*proof*⟩

lemma *imp-disj1*: $((P\dashrightarrow Q)|R) = (P\dashrightarrow Q|R)$ ⟨*proof*⟩
lemma *imp-disj2*: $(Q|(P\dashrightarrow R)) = (P\dashrightarrow Q|R)$ ⟨*proof*⟩

lemma *imp-cong*: $(P = P') ==> (P' ==> (Q = Q')) ==> ((P \dashrightarrow Q) = (P' \dashrightarrow Q'))$
⟨*proof*⟩

lemma *de-Morgan-disj*: $(\sim(P \mid Q)) = (\sim P \,\&\, \sim Q)$ ⟨*proof*⟩
lemma *de-Morgan-conj*: $(\sim(P \,\&\, Q)) = (\sim P \mid \sim Q)$ ⟨*proof*⟩
lemma *not-imp*: $(\sim(P \dashrightarrow Q)) = (P \,\&\, \sim Q)$ ⟨*proof*⟩
lemma *not-iff*: $(P \tilde{=} Q) = (P = (\sim Q))$ ⟨*proof*⟩
lemma *disj-not1*: $(\sim P \mid Q) = (P \dashrightarrow Q)$ ⟨*proof*⟩
lemma *disj-not2*: $(P \mid \sim Q) = (Q \dashrightarrow P)$ — changes orientation :-(
⟨*proof*⟩
lemma *imp-conv-disj*: $(P \dashrightarrow Q) = ((\sim P) \mid Q)$ ⟨*proof*⟩

lemma *iff-conv-conj-imp*: $(P = Q) = ((P \dashrightarrow Q) \,\&\, (Q \dashrightarrow P))$ ⟨*proof*⟩

lemma *cases-simp*: $((P \dashrightarrow Q) \,\&\, (\sim P \dashrightarrow Q)) = Q$
— Avoids duplication of subgoals after *split-if*, when the true and false
— cases boil down to the same thing.
⟨*proof*⟩

lemma *not-all*: $(\sim (!\ x.\ P(x))) = (?\ x.\sim P(x))$ ⟨*proof*⟩
lemma *imp-all*: $((!\ x.\ P\ x) \dashrightarrow Q) = (?\ x.\ P\ x \dashrightarrow Q)$ ⟨*proof*⟩
lemma *not-ex*: $(\sim (?\ x.\ P(x))) = (!\ x.\sim P(x))$ ⟨*proof*⟩
lemma *imp-ex*: $((?\ x.\ P\ x) \dashrightarrow Q) = (!\ x.\ P\ x \dashrightarrow Q)$ ⟨*proof*⟩
lemma *all-not-ex*: $(ALL\ x.\ P\ x) = (\sim (EX\ x.\ \sim\ P\ x\ ))$ ⟨*proof*⟩

declare *All-def* [*noatp*]

lemma *ex-disj-distrib*: $(?\ x.\ P(x) \mid Q(x)) = ((?\ x.\ P(x)) \mid (?\ x.\ Q(x)))$ ⟨*proof*⟩
lemma *all-conj-distrib*: $(!x.\ P(x) \,\&\, Q(x)) = ((!\ x.\ P(x)) \,\&\, (!\ x.\ Q(x)))$ ⟨*proof*⟩

The & congruence rule: not included by default! May slow rewrite proofs down by as much as 50%

lemma *conj-cong*:
$(P = P') ==> (P' ==> (Q = Q')) ==> ((P \,\&\, Q) = (P' \,\&\, Q'))$
⟨*proof*⟩

lemma *rev-conj-cong*:
$(Q = Q') ==> (Q' ==> (P = P')) ==> ((P \,\&\, Q) = (P' \,\&\, Q'))$
⟨*proof*⟩

The | congruence rule: not included by default!

lemma *disj-cong*:
$(P = P') ==> (\sim P' ==> (Q = Q')) ==> ((P \mid Q) = (P' \mid Q'))$
⟨*proof*⟩

if-then-else rules

lemma *if-True*: $(if\ True\ then\ x\ else\ y) = x$
⟨*proof*⟩

lemma *if-False*: $(if\ False\ then\ x\ else\ y) = y$
⟨*proof*⟩

lemma *if-P*: $P ==> (if\ P\ then\ x\ else\ y) = x$
⟨*proof*⟩

lemma *if-not-P*: $\sim P ==> (if\ P\ then\ x\ else\ y) = y$
⟨*proof*⟩

lemma *split-if*: $P\ (if\ Q\ then\ x\ else\ y) = ((Q --> P(x)) \&\ (\sim Q --> P(y)))$
⟨*proof*⟩

lemma *split-if-asm*: $P\ (if\ Q\ then\ x\ else\ y) = (\sim((Q \&\ \sim P\ x) \mid (\sim Q \&\ \sim P\ y)))$
⟨*proof*⟩

lemmas *if-splits* [*noatp*] = *split-if split-if-asm*

lemma *if-cancel*: $(if\ c\ then\ x\ else\ x) = x$
⟨*proof*⟩

lemma *if-eq-cancel*: $(if\ x = y\ then\ y\ else\ x) = x$
⟨*proof*⟩

lemma *if-bool-eq-conj*: $(if\ P\ then\ Q\ else\ R) = ((P-->Q) \&\ (\sim P-->R))$
— This form is useful for expanding *if*s on the RIGHT of the ==> symbol.
⟨*proof*⟩

lemma *if-bool-eq-disj*: $(if\ P\ then\ Q\ else\ R) = ((P\&Q) \mid (\sim P\&R))$
— And this form is useful for expanding *if*s on the LEFT.
⟨*proof*⟩

lemma *Eq-TrueI*: $P ==> P == True$ ⟨*proof*⟩
lemma *Eq-FalseI*: $\sim P ==> P == False$ ⟨*proof*⟩

let rules for simproc

lemma *Let-folded*: $f\ x \equiv g\ x \Longrightarrow Let\ x\ f \equiv Let\ x\ g$
⟨*proof*⟩

lemma *Let-unfold*: $f\ x \equiv g \Longrightarrow Let\ x\ f \equiv g$

⟨*proof*⟩

The following copy of the implication operator is useful for fine-tuning congruence rules. It instructs the simplifier to simplify its premise.

constdefs
simp-implies :: [*prop*, *prop*] => *prop* (**infixr** =*simp*=> *1*)
[*code del*]: *simp-implies* ≡ *op* ==>

lemma *simp-impliesI*:
assumes *PQ*: (*PROP P* ⟹ *PROP Q*)
shows *PROP P* =*simp*=> *PROP Q*
⟨*proof*⟩

lemma *simp-impliesE*:
assumes *PQ*: *PROP P* =*simp*=> *PROP Q*
and *P*: *PROP P*
and *QR*: *PROP Q* ⟹ *PROP R*
shows *PROP R*
⟨*proof*⟩

lemma *simp-implies-cong*:
assumes *PP'* :*PROP P* == *PROP P'*
and *P'QQ'*: *PROP P'* ==> (*PROP Q* == *PROP Q'*)
shows (*PROP P* =*simp*=> *PROP Q*) == (*PROP P'* =*simp*=> *PROP Q'*)
⟨*proof*⟩

lemma *uncurry*:
assumes *P* ⟶ *Q* ⟶ *R*
shows *P* ∧ *Q* ⟶ *R*
⟨*proof*⟩

lemma *iff-allI*:
assumes ⋀*x*. *P x* = *Q x*
shows (∀ *x*. *P x*) = (∀ *x*. *Q x*)
⟨*proof*⟩

lemma *iff-exI*:
assumes ⋀*x*. *P x* = *Q x*
shows (∃ *x*. *P x*) = (∃ *x*. *Q x*)
⟨*proof*⟩

lemma *all-comm*:
(∀ *x y*. *P x y*) = (∀ *y x*. *P x y*)
⟨*proof*⟩

lemma *ex-comm*:
(∃ *x y*. *P x y*) = (∃ *y x*. *P x y*)
⟨*proof*⟩

⟨*ML*⟩

Simproc for proving $(y = x) == False$ from premise $\sim(x = y)$:

⟨*ML*⟩

lemma *True-implies-equals*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

lemma *ex-simps*:
$!!P\ Q.\ (EX\ x.\ P\ x\ \&\ Q) = ((EX\ x.\ P\ x)\ \&\ Q)$
$!!P\ Q.\ (EX\ x.\ P\ \&\ Q\ x) = (P\ \&\ (EX\ x.\ Q\ x))$
$!!P\ Q.\ (EX\ x.\ P\ x \mid Q) = ((EX\ x.\ P\ x) \mid Q)$
$!!P\ Q.\ (EX\ x.\ P \mid Q\ x) = (P \mid (EX\ x.\ Q\ x))$
$!!P\ Q.\ (EX\ x.\ P\ x \longrightarrow Q) = ((ALL\ x.\ P\ x) \longrightarrow Q)$
$!!P\ Q.\ (EX\ x.\ P \longrightarrow Q\ x) = (P \longrightarrow (EX\ x.\ Q\ x))$
— Miniscoping: pushing in existential quantifiers.
⟨*proof*⟩

lemma *all-simps*:
$!!P\ Q.\ (ALL\ x.\ P\ x\ \&\ Q) = ((ALL\ x.\ P\ x)\ \&\ Q)$
$!!P\ Q.\ (ALL\ x.\ P\ \&\ Q\ x) = (P\ \&\ (ALL\ x.\ Q\ x))$
$!!P\ Q.\ (ALL\ x.\ P\ x \mid Q) = ((ALL\ x.\ P\ x) \mid Q)$
$!!P\ Q.\ (ALL\ x.\ P \mid Q\ x) = (P \mid (ALL\ x.\ Q\ x))$
$!!P\ Q.\ (ALL\ x.\ P\ x \longrightarrow Q) = ((EX\ x.\ P\ x) \longrightarrow Q)$
$!!P\ Q.\ (ALL\ x.\ P \longrightarrow Q\ x) = (P \longrightarrow (ALL\ x.\ Q\ x))$
— Miniscoping: pushing in universal quantifiers.
⟨*proof*⟩

lemmas [*simp*] =
triv-forall-equality
True-implies-equals
if-True
if-False
if-cancel
if-eq-cancel
imp-disjL

conj-assoc
disj-assoc
de-Morgan-conj
de-Morgan-disj
imp-disj1
imp-disj2
not-imp
disj-not1
not-all
not-ex
cases-simp
the-eq-trivial

the-sym-eq-trivial
ex-simps
all-simps
simp-thms

lemmas [*cong*] = *imp-cong simp-implies-cong*
lemmas [*split*] = *split-if*

⟨*ML*⟩

Simplifies x assuming c and y assuming c

lemma *if-cong*:
assumes $b = c$
and $c \Longrightarrow x = u$
and $\neg\ c \Longrightarrow y = v$
shows (*if b then x else y*) = (*if c then u else v*)
⟨*proof*⟩

Prevents simplification of x and y: faster and allows the execution of functional programs.

lemma *if-weak-cong* [*cong*]:
assumes $b = c$
shows (*if b then x else y*) = (*if c then x else y*)
⟨*proof*⟩

Prevents simplification of t: much faster

lemma *let-weak-cong*:
assumes $a = b$
shows (*let x = a in t x*) = (*let x = b in t x*)
⟨*proof*⟩

To tidy up the result of a simproc. Only the RHS will be simplified.

lemma *eq-cong2*:
assumes $u = u'$
shows $(t \equiv u) \equiv (t \equiv u')$
⟨*proof*⟩

lemma *if-distrib*:
f (*if c then x else y*) = (*if c then f x else f y*)
⟨*proof*⟩

This lemma restricts the effect of the rewrite rule u=v to the left-hand side of an equality. Used in {*Integ*,*Real*}/*simproc.ML*

lemma *restrict-to-left*:
assumes $x = y$
shows $(x = z) = (y = z)$
⟨*proof*⟩

1.3.3 Generic cases and induction

Rule projections:

⟨*ML*⟩

constdefs
induct-forall **where** *induct-forall P* == ∀ *x*. *P x*
induct-implies **where** *induct-implies A B* == *A* ⟶ *B*
induct-equal **where** *induct-equal x y* == *x* = *y*
induct-conj **where** *induct-conj A B* == *A* ∧ *B*

lemma *induct-forall-eq*: (!!*x*. *P x*) == *Trueprop* (*induct-forall* (λ*x*. *P x*))
⟨*proof*⟩

lemma *induct-implies-eq*: (*A* ==> *B*) == *Trueprop* (*induct-implies A B*)
⟨*proof*⟩

lemma *induct-equal-eq*: (*x* == *y*) == *Trueprop* (*induct-equal x y*)
⟨*proof*⟩

lemma *induct-conj-eq*: (*A* &&& *B*) == *Trueprop* (*induct-conj A B*)
⟨*proof*⟩

lemmas *induct-atomize* = *induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq*
lemmas *induct-rulify* [*symmetric*, *standard*] = *induct-atomize*
lemmas *induct-rulify-fallback* =
induct-forall-def induct-implies-def induct-equal-def induct-conj-def

lemma *induct-forall-conj*: *induct-forall* (λ*x*. *induct-conj* (*A x*) (*B x*)) =
induct-conj (*induct-forall A*) (*induct-forall B*)
⟨*proof*⟩

lemma *induct-implies-conj*: *induct-implies C* (*induct-conj A B*) =
induct-conj (*induct-implies C A*) (*induct-implies C B*)
⟨*proof*⟩

lemma *induct-conj-curry*: (*induct-conj A B* ==> *PROP C*) == (*A* ==> *B* ==> *PROP C*)
⟨*proof*⟩

lemmas *induct-conj* = *induct-forall-conj induct-implies-conj induct-conj-curry*

hide *const induct-forall induct-implies induct-equal induct-conj*

Method setup.

⟨*ML*⟩

1.3.4 Coherent logic

⟨*ML*⟩

1.4 Other simple lemmas and lemma duplicates

lemma *Let-0* [*simp*]: *Let 0 f = f 0*
⟨*proof*⟩

lemma *Let-1* [*simp*]: *Let 1 f = f 1*
⟨*proof*⟩

lemma *ex1-eq* [*iff*]: *EX! x. x = t EX! x. t = x*
⟨*proof*⟩

lemma *choice-eq*: *(ALL x. EX! y. P x y) = (EX! f. ALL x. P x (f x))*
⟨*proof*⟩

lemma *mk-left-commute*:
fixes *f* (**infix** ⊗ *60*)
assumes *a*: ⋀*x y z*. *(x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)* **and**
c: ⋀*x y*. *x ⊗ y = y ⊗ x*
shows *x ⊗ (y ⊗ z) = y ⊗ (x ⊗ z)*
⟨*proof*⟩

lemmas *eq-sym-conv = eq-commute*

lemma *nnf-simps*:
(¬(P ∧ Q)) = (¬ P ∨ ¬ Q) (¬ (P ∨ Q)) = (¬ P ∧ ¬Q) (P ⟶ Q) = (¬P ∨ Q)
(P = Q) = ((P ∧ Q) ∨ (¬P ∧ ¬ Q)) (¬(P = Q)) = ((P ∧ ¬ Q) ∨ (¬P ∧ Q))
(¬ ¬(P)) = P
⟨*proof*⟩

1.5 Basic ML bindings

⟨*ML*⟩

1.6 Code generator basics – see further theory *Code-Setup*

Equality

class *eq* =
fixes *eq* :: *'a ⇒ 'a ⇒ bool*
assumes *eq-equals*: *eq x y ⟷ x = y*
begin

lemma *eq*: *eq = (op =)*
⟨*proof*⟩

lemma *eq-refl*: *eq x x* $\longleftrightarrow$ *True*
⟨*proof*⟩

end

Module setup

⟨*ML*⟩

1.7 Nitpick hooks

This will be relocated once Nitpick is moved to HOL.

⟨*ML*⟩

1.8 Legacy tactics and ML bindings

⟨*ML*⟩

end

2 Code-Setup: Setup of code generators and related tools

theory *Code-Setup*
imports *HOL*
begin

2.1 Generic code generator foundation

Datatypes

code-datatype *True False*

code-datatype *TYPE*('*a*::{})

code-datatype *Trueprop prop*

Code equations

lemma [*code*]:
shows (*True* $\Longrightarrow$ *PROP P*) $\equiv$ *PROP P*
and (*False* $\Longrightarrow$ *Q*) $\equiv$ *Trueprop True*
and (*PROP P* $\Longrightarrow$ *True*) $\equiv$ *Trueprop True*
and (*Q* $\Longrightarrow$ *False*) $\equiv$ *Trueprop* ($\neg$ *Q*) ⟨*proof*⟩

lemma [*code*]:
shows *False* $\wedge$ *x* $\longleftrightarrow$ *False*
and *True* $\wedge$ *x* $\longleftrightarrow$ *x*
and *x* $\wedge$ *False* $\longleftrightarrow$ *False*

and $x \wedge \mathit{True} \longleftrightarrow x$ ⟨*proof*⟩

lemma [*code*]:
shows $\mathit{False} \vee x \longleftrightarrow x$
and $\mathit{True} \vee x \longleftrightarrow \mathit{True}$
and $x \vee \mathit{False} \longleftrightarrow x$
and $x \vee \mathit{True} \longleftrightarrow \mathit{True}$ ⟨*proof*⟩

lemma [*code*]:
shows $\neg\ \mathit{True} \longleftrightarrow \mathit{False}$
and $\neg\ \mathit{False} \longleftrightarrow \mathit{True}$ ⟨*proof*⟩

lemmas [*code*] = *Let-def if-True if-False*

lemmas [*code*, *code unfold*, *symmetric*, *code post*] = *imp-conv-disj*

Equality

context *eq*
begin

lemma *equals-eq* [*code inline*, *code*]: $op = \equiv eq$
⟨*proof*⟩

declare *eq* [*code unfold*, *code inline del*]

declare *equals-eq* [*symmetric*, *code post*]

end

declare *simp-thms*(*6*) [*code nbe*]

hide (**open**) *const eq*
hide *const eq*

⟨*ML*⟩

Cases

lemma *Let-case-cert*:
assumes $\mathit{CASE} \equiv (\lambda x.\ \mathit{Let}\ x\ f)$
shows $\mathit{CASE}\ x \equiv f\ x$
⟨*proof*⟩

lemma *If-case-cert*:
assumes $\mathit{CASE} \equiv (\lambda b.\ \mathit{If}\ b\ f\ g)$
shows $(\mathit{CASE}\ \mathit{True} \equiv f)$ &&& $(\mathit{CASE}\ \mathit{False} \equiv g)$
⟨*proof*⟩

⟨*ML*⟩

code-abort *undefined*

2.2 Generic code generator preprocessor

⟨*ML*⟩

2.3 Generic code generator target languages

type bool

code-type *bool*
(*SML bool*)
(*OCaml bool*)
(*Haskell Bool*)

code-const *True* **and** *False* **and** *Not* **and** *op &* **and** *op |* **and** *If*
(*SML true* **and** *false* **and** *not*
and infixl *1 andalso* **and infixl** *0 orelse*
and !(*if* (-)/ *then* (-)/ *else* (-)))
(*OCaml true* **and** *false* **and** *not*
and infixl *4* && **and infixl** *2* ||
and !(*if* (-)/ *then* (-)/ *else* (-)))
(*Haskell True* **and** *False* **and** *not*
and infixl *3* && **and infixl** *2* ||
and !(*if* (-)/ *then* (-)/ *else* (-)))

code-reserved *SML*
bool true false not

code-reserved *OCaml*
bool not

using built-in Haskell equality

code-class *eq*
(*Haskell Eq*)

code-const *eq-class.eq*
(*Haskell* **infixl** *4* ==)

code-const *op =*
(*Haskell* **infixl** *4* ==)

undefined

code-const *undefined*
(*SML* !(*raise*/ *Fail*/ *undefined*))
(*OCaml failwith*/ *undefined*)
(*Haskell error*/ *undefined*)

2.4 SML code generator setup

types-code
bool (*bool*)
attach (*term-of*) ⟪
fun term-of-bool b = if b then HOLogic.true-const else HOLogic.false-const;
⟫
attach (*test*) ⟪
fun gen-bool i =
let val b = one-of [*false*, *true*]
in (*b*, *fn* () => *term-of-bool b*) *end*;
⟫
prop (*bool*)
attach (*term-of*) ⟪
fun term-of-prop b =
HOLogic.mk-Trueprop (*if b then HOLogic.true-const else HOLogic.false-const*);
⟫

consts-code
Trueprop ((-))
True (*true*)
False (*false*)
Not (*Bool.not*)
op | ((- *orelse*/ -))
op & ((- *andalso*/ -))
If ((*if* -/ *then* -/ *else* -))

⟨*ML*⟩

2.5 Evaluation and normalization by evaluation

⟨*ML*⟩

2.6 Quickcheck

⟨*ML*⟩

quickcheck-params [*size* = *5*, *iterations* = *50*]

end

3 Orderings: Abstract orderings

theory *Orderings*
imports *Code-Setup*
uses ~~/*src*/*Provers*/*order.ML*
begin

3.1 Quasi orders

class *preorder* = *ord* +
  assumes *less-le-not-le*: $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$
  and *order-refl* [*iff*]: $x \leq x$
  and *order-trans*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
begin

Reflexivity.

lemma *eq-refl*: $x = y \Longrightarrow x \leq y$
  — This form is useful with the classical reasoner.
⟨*proof*⟩

lemma *less-irrefl* [*iff*]: $\neg\ x < x$
⟨*proof*⟩

lemma *less-imp-le*: $x < y \Longrightarrow x \leq y$
⟨*proof*⟩

Asymmetry.

lemma *less-not-sym*: $x < y \Longrightarrow \neg\ (y < x)$
⟨*proof*⟩

lemma *less-asym*: $x < y \Longrightarrow (\neg\ P \Longrightarrow y < x) \Longrightarrow P$
⟨*proof*⟩

Transitivity.

lemma *less-trans*: $x < y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

lemma *le-less-trans*: $x \leq y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

lemma *less-le-trans*: $x < y \Longrightarrow y \leq z \Longrightarrow x < z$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-less*: $x < y \Longrightarrow (\neg\ y < x) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

lemma *less-imp-triv*: $x < y \Longrightarrow (y < x \longrightarrow P) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

lemma *less-asym'*: $a < b \Longrightarrow b < a \Longrightarrow P$
⟨*proof*⟩

Dual order

lemma *dual-preorder*:
preorder $(op \geq)$ $(op >)$
⟨*proof*⟩

end

3.2 Partial orders

class *order* = *preorder* +
assumes *antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
begin

Reflexivity.

lemma *less-le*: $x < y \longleftrightarrow x \leq y \wedge x \neq y$
⟨*proof*⟩

lemma *le-less*: $x \leq y \longleftrightarrow x < y \vee x = y$
— NOT suitable for iff, since it can cause PROOF FAILED.
⟨*proof*⟩

lemma *le-imp-less-or-eq*: $x \leq y \Longrightarrow x < y \vee x = y$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-eq*: $x < y \Longrightarrow (x = y) \longleftrightarrow \mathit{False}$
⟨*proof*⟩

lemma *less-imp-not-eq2*: $x < y \Longrightarrow (y = x) \longleftrightarrow \mathit{False}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

lemma *neq-le-trans*: $a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$
⟨*proof*⟩

lemma *le-neq-trans*: $a \leq b \Longrightarrow a \neq b \Longrightarrow a < b$
⟨*proof*⟩

Asymmetry.

lemma *eq-iff*: $x = y \longleftrightarrow x \leq y \wedge y \leq x$
⟨*proof*⟩

lemma *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *less-imp-neq*: $x < y \Longrightarrow x \neq y$
⟨*proof*⟩

Least value operator

definition (**in** *ord*)
  Least :: (*'a* ⇒ *bool*) ⇒ *'a* (**binder** *LEAST* *10*) **where**
  Least P = (*THE x*. *P x* ∧ (∀ *y*. *P y* ⟶ *x* ≤ *y*))

lemma *Least-equality*:
  assumes *P x*
    and ⋀*y*. *P y* ⟹ *x* ≤ *y*
  shows *Least P* = *x*
⟨*proof*⟩

lemma *LeastI2-order*:
  assumes *P x*
    and ⋀*y*. *P y* ⟹ *x* ≤ *y*
    and ⋀*x*. *P x* ⟹ ∀ *y*. *P y* ⟶ *x* ≤ *y* ⟹ *Q x*
  shows *Q* (*Least P*)
⟨*proof*⟩

Dual order

lemma *dual-order*:
  order (*op* ≥) (*op* >)
⟨*proof*⟩

end

3.3 Linear (total) orders

class *linorder* = *order* +
  assumes *linear*: *x* ≤ *y* ∨ *y* ≤ *x*
begin

lemma *less-linear*: *x* < *y* ∨ *x* = *y* ∨ *y* < *x*
⟨*proof*⟩

lemma *le-less-linear*: *x* ≤ *y* ∨ *y* < *x*
⟨*proof*⟩

lemma *le-cases* [*case-names le ge*]:
  (*x* ≤ *y* ⟹ *P*) ⟹ (*y* ≤ *x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *linorder-cases* [*case-names less equal greater*]:
  (*x* < *y* ⟹ *P*) ⟹ (*x* = *y* ⟹ *P*) ⟹ (*y* < *x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *not-less*: ¬ *x* < *y* ⟷ *y* ≤ *x*
⟨*proof*⟩

lemma *not-less-iff-gr-or-eq*:
 ¬(*x* < *y*) ⟷ (*x* > *y* | *x* = *y*)

⟨*proof*⟩

lemma *not-le*: $\neg\ x \leq y \longleftrightarrow y < x$
⟨*proof*⟩

lemma *neq-iff*: $x \neq y \longleftrightarrow x < y \vee y < x$
⟨*proof*⟩

lemma *neqE*: $x \neq y \Longrightarrow (x < y \Longrightarrow R) \Longrightarrow (y < x \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *antisym-conv1*: $\neg\ x < y \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *antisym-conv2*: $x \leq y \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *antisym-conv3*: $\neg\ y < x \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *leI*: $\neg\ x < y \Longrightarrow y \leq x$
⟨*proof*⟩

lemma *leD*: $y \leq x \Longrightarrow \neg\ x < y$
⟨*proof*⟩

lemma *not-leE*: $\neg\ y \leq x \Longrightarrow x < y$
⟨*proof*⟩

Dual order

lemma *dual-linorder*:
  linorder $(op \geq)$ $(op >)$
⟨*proof*⟩

min/max

definition (**in** *ord*) *min* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
  [*code del*]: *min a b* = (*if a* $\leq$ *b then a else b*)

definition (**in** *ord*) *max* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
  [*code del*]: *max a b* = (*if a* $\leq$ *b then b else a*)

lemma *min-le-iff-disj*:
  min x y $\leq z \longleftrightarrow x \leq z \vee y \leq z$
⟨*proof*⟩

lemma *le-max-iff-disj*:
  $z \leq$ *max x y* $\longleftrightarrow z \leq x \vee z \leq y$
⟨*proof*⟩

lemma *min-less-iff-disj*:
$min\ x\ y < z \longleftrightarrow x < z \vee y < z$
⟨*proof*⟩

lemma *less-max-iff-disj*:
$z < max\ x\ y \longleftrightarrow z < x \vee z < y$
⟨*proof*⟩

lemma *min-less-iff-conj* [*simp*]:
$z < min\ x\ y \longleftrightarrow z < x \wedge z < y$
⟨*proof*⟩

lemma *max-less-iff-conj* [*simp*]:
$max\ x\ y < z \longleftrightarrow x < z \wedge y < z$
⟨*proof*⟩

lemma *split-min* [*noatp*]:
$P\ (min\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ i) \wedge (\neg\ i \leq j \longrightarrow P\ j)$
⟨*proof*⟩

lemma *split-max* [*noatp*]:
$P\ (max\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ j) \wedge (\neg\ i \leq j \longrightarrow P\ i)$
⟨*proof*⟩

end

Explicit dictionaries for code generation

lemma *min-ord-min* [*code*, *code unfold*, *code inline del*]:
$min = ord.min\ (op \leq)$
⟨*proof*⟩

declare *ord.min-def* [*code*]

lemma *max-ord-max* [*code*, *code unfold*, *code inline del*]:
$max = ord.max\ (op \leq)$
⟨*proof*⟩

declare *ord.max-def* [*code*]

3.4 Reasoning tools setup

⟨*ML*⟩

Declarations to set up transitivity reasoner of partial and linear orders.

context *order*
begin

declare *less-irrefl* [*THEN notE*, *order add less-reflE*: *order op* = :: *'a* ⇒ *'a* ⇒ *bool op* <= *op* <]

declare *order-refl* [*order add le-refl*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-imp-le* [*order add less-imp-le*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *antisym* [*order add eqI*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *eq-refl* [*order add eqD1*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *sym* [*THEN eq-refl*, *order add eqD2*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-trans* [*order add less-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-le-trans* [*order add less-le-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *le-less-trans* [*order add le-less-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *order-trans* [*order add le-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *le-neq-trans* [*order add le-neq-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *neq-le-trans* [*order add neq-le-trans*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-imp-neq* [*order add less-imp-neq*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *eq-neq-eq-imp-neq* [*order add eq-neq-eq-imp-neq*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *not-sym* [*order add not-sym*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

end

context *linorder*
begin

declare [[*order del*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]]

declare *less-irrefl* [*THEN notE*, *order add less-reflE*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *order-refl* [*order add le-refl*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-imp-le* [*order add less-imp-le*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *not-less* [*THEN iffD2*, *order add not-lessI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *not-le* [*THEN iffD2*, *order add not-leI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *not-less* [*THEN iffD1*, *order add not-lessD*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *not-le* [*THEN iffD1*, *order add not-leD*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *antisym* [*order add eqI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *eq-refl* [*order add eqD1*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *sym* [*THEN eq-refl*, *order add eqD2*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-trans* [*order add less-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-le-trans* [*order add less-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *le-less-trans* [*order add le-less-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *order-trans* [*order add le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *le-neq-trans* [*order add le-neq-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *neq-le-trans* [*order add neq-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-imp-neq* [*order add less-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *eq-neq-eq-imp-neq* [*order add eq-neq-eq-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *not-sym* [*order add not-sym*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <]

end

⟨*ML*⟩

3.5 Name duplicates

lemmas *order-less-le* = *less-le*
lemmas *order-eq-refl* = *preorder-class.eq-refl*
lemmas *order-less-irrefl* = *preorder-class.less-irrefl*
lemmas *order-le-less* = *order-class.le-less*
lemmas *order-le-imp-less-or-eq* = *order-class.le-imp-less-or-eq*
lemmas *order-less-imp-le* = *preorder-class.less-imp-le*
lemmas *order-less-imp-not-eq* = *order-class.less-imp-not-eq*
lemmas *order-less-imp-not-eq2* = *order-class.less-imp-not-eq2*
lemmas *order-neq-le-trans* = *order-class.neq-le-trans*
lemmas *order-le-neq-trans* = *order-class.le-neq-trans*

lemmas *order-antisym* = *antisym*
lemmas *order-less-not-sym* = *preorder-class.less-not-sym*
lemmas *order-less-asym* = *preorder-class.less-asym*
lemmas *order-eq-iff* = *order-class.eq-iff*
lemmas *order-antisym-conv* = *order-class.antisym-conv*
lemmas *order-less-trans* = *preorder-class.less-trans*
lemmas *order-le-less-trans* = *preorder-class.le-less-trans*
lemmas *order-less-le-trans* = *preorder-class.less-le-trans*
lemmas *order-less-imp-not-less* = *preorder-class.less-imp-not-less*
lemmas *order-less-imp-triv* = *preorder-class.less-imp-triv*
lemmas *order-less-asym'* = *preorder-class.less-asym'*

lemmas *linorder-linear* = *linear*
lemmas *linorder-less-linear* = *linorder-class.less-linear*
lemmas *linorder-le-less-linear* = *linorder-class.le-less-linear*
lemmas *linorder-le-cases* = *linorder-class.le-cases*
lemmas *linorder-not-less* = *linorder-class.not-less*
lemmas *linorder-not-le* = *linorder-class.not-le*
lemmas *linorder-neq-iff* = *linorder-class.neq-iff*
lemmas *linorder-neqE* = *linorder-class.neqE*
lemmas *linorder-antisym-conv1* = *linorder-class.antisym-conv1*
lemmas *linorder-antisym-conv2* = *linorder-class.antisym-conv2*
lemmas *linorder-antisym-conv3* = *linorder-class.antisym-conv3*

3.6 Bounded quantifiers

syntax
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->=-./ -*) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≤*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≤*-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≥*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≥*-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HOL*)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<=-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≤*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≤*-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≥*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≥*-./ -*) [*0*, *0*, *10*] *10*)

translations
ALL x<y. P => *ALL x. x < y* ⟶ *P*
EX x<y. P => *EX x. x < y* ∧ *P*
ALL x<=y. P => *ALL x. x <= y* ⟶ *P*
EX x<=y. P => *EX x. x <= y* ∧ *P*
ALL x>y. P => *ALL x. x > y* ⟶ *P*
EX x>y. P => *EX x. x > y* ∧ *P*
ALL x>=y. P => *ALL x. x >= y* ⟶ *P*
EX x>=y. P => *EX x. x >= y* ∧ *P*

⟨*ML*⟩

3.7 Transitivity reasoning

context *ord*
begin

lemma *ord-le-eq-trans*: $a \leq b \Longrightarrow b = c \Longrightarrow a \leq c$
⟨*proof*⟩

lemma *ord-eq-le-trans*: $a = b \Longrightarrow b \leq c \Longrightarrow a \leq c$
⟨*proof*⟩

lemma *ord-less-eq-trans*: $a < b \Longrightarrow b = c \Longrightarrow a < c$
⟨*proof*⟩

lemma *ord-eq-less-trans*: $a = b \Longrightarrow b < c \Longrightarrow a < c$
⟨*proof*⟩

end

lemma *order-less-subst2*: $(a::'a::order) < b ==> f\ b < (c::'c::order) ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *order-less-subst1*: $(a::'a::order) < f\ b ==> (b::'b::order) < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

lemma *order-le-less-subst2*: $(a::'a::order) <= b ==> f\ b < (c::'c::order) ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *order-le-less-subst1*: $(a::'a::order) <= f\ b ==> (b::'b::order) < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

lemma *order-less-le-subst2*: $(a::'a::order) < b ==> f\ b <= (c::'c::order) ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *order-less-le-subst1*: $(a::'a::order) < f\ b ==> (b::'b::order) <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a < f\ c$
⟨*proof*⟩

lemma *order-subst1*: $(a::'a::order) <= f\ b ==> (b::'b::order) <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a <= f\ c$
⟨*proof*⟩

lemma *order-subst2*: $(a::'a::order) <= b ==> f\ b <= (c::'c::order) ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a <= c$
⟨*proof*⟩

lemma *ord-le-eq-subst*: $a <= b ==> f\ b = c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a <= c$
⟨*proof*⟩

lemma *ord-eq-le-subst*: $a = f\ b ==> b <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a <= f\ c$
⟨*proof*⟩

lemma *ord-less-eq-subst*: $a < b ==> f\ b = c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *ord-eq-less-subst*: $a = f\ b ==> b < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

Note that this list of rules is in reverse order of priorities.

lemmas [*trans*] =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp

lemmas (**in** *order*) [*trans*] =
neq-le-trans
le-neq-trans

lemmas (**in** *preorder*) [*trans*] =
less-trans
less-asym'
le-less-trans
less-le-trans

order-trans

lemmas (**in** *order*) [*trans*] =
antisym

lemmas (**in** *ord*) [*trans*] =
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans

lemmas [*trans*] =
trans

lemmas *order-trans-rules* =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp
neq-le-trans
le-neq-trans
less-trans
less-asym'
le-less-trans
less-le-trans
order-trans
antisym
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans
trans

These support proving chains of decreasing inequalities a ¿= b ¿= c ... in Isar proofs.

lemma *xt1*:
$a = b \Longrightarrow b > c \Longrightarrow a > c$

$a > b \Longrightarrow b = c \Longrightarrow a > c$
$a = b \Longrightarrow b >= c \Longrightarrow a >= c$
$a >= b \Longrightarrow b = c \Longrightarrow a >= c$
$(x::'a::order) >= y \Longrightarrow y >= x \Longrightarrow x = y$
$(x::'a::order) >= y \Longrightarrow y >= z \Longrightarrow x >= z$
$(x::'a::order) > y \Longrightarrow y >= z \Longrightarrow x > z$
$(x::'a::order) >= y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) > b \Longrightarrow b > a \Longrightarrow P$
$(x::'a::order) > y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) >= b \Longrightarrow a \sim= b \Longrightarrow a > b$
$(a::'a::order) \sim= b \Longrightarrow a >= b \Longrightarrow a > b$
$a = f\ b \Longrightarrow b > c \Longrightarrow (!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
$a > b \Longrightarrow f\ b = c \Longrightarrow (!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
$a = f\ b \Longrightarrow b >= c \Longrightarrow (!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a >= f\ c$
$a >= b \Longrightarrow f\ b = c \Longrightarrow (!!\ x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a >= c$
⟨*proof*⟩

lemma *xt2*:
$(a::'a::order) >= f\ b \Longrightarrow b >= c \Longrightarrow (!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a >= f\ c$
⟨*proof*⟩

lemma *xt3*: $(a::'a::order) >= b \Longrightarrow (f\ b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a >= c$
⟨*proof*⟩

lemma *xt4*: $(a::'a::order) > f\ b \Longrightarrow (b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a > f\ c$
⟨*proof*⟩

lemma *xt5*: $(a::'a::order) > b \Longrightarrow (f\ b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
⟨*proof*⟩

lemma *xt6*: $(a::'a::order) >= f\ b \Longrightarrow b > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
⟨*proof*⟩

lemma *xt7*: $(a::'a::order) >= b \Longrightarrow (f\ b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a > c$
⟨*proof*⟩

lemma *xt8*: $(a::'a::order) > f\ b \Longrightarrow (b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
⟨*proof*⟩

lemma *xt9*: $(a::'a::order) > b \Longrightarrow (f\ b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
⟨*proof*⟩

lemmas *xtrans* = *xt1 xt2 xt3 xt4 xt5 xt6 xt7 xt8 xt9*

3.8 Monotonicity, least value operator and min/max

context *order*
begin

definition *mono* :: (*'a* ⇒ *'b*::*order*) ⇒ *bool* **where**
mono f ⟷ (∀ *x y*. *x* ≤ *y* ⟶ *f x* ≤ *f y*)

lemma *monoI* [*intro?*]:
fixes *f* :: *'a* ⇒ *'b*::*order*
shows (⋀*x y*. *x* ≤ *y* ⟹ *f x* ≤ *f y*) ⟹ *mono f*
⟨*proof*⟩

lemma *monoD* [*dest?*]:
fixes *f* :: *'a* ⇒ *'b*::*order*
shows *mono f* ⟹ *x* ≤ *y* ⟹ *f x* ≤ *f y*
⟨*proof*⟩

definition *strict-mono* :: (*'a* ⇒ *'b*::*order*) ⇒ *bool* **where**
strict-mono f ⟷ (∀ *x y*. *x* < *y* ⟶ *f x* < *f y*)

lemma *strict-monoI* [*intro?*]:
assumes ⋀*x y*. *x* < *y* ⟹ *f x* < *f y*
shows *strict-mono f*
⟨*proof*⟩

lemma *strict-monoD* [*dest?*]:
strict-mono f ⟹ *x* < *y* ⟹ *f x* < *f y*
⟨*proof*⟩

lemma *strict-mono-mono* [*dest?*]:
assumes *strict-mono f*
shows *mono f*
⟨*proof*⟩

end

context *linorder*
begin

lemma *strict-mono-eq*:
assumes *strict-mono f*
shows *f x* = *f y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *strict-mono-less-eq*:

assumes *strict-mono f*
shows $f\ x \leq f\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *strict-mono-less*:
assumes *strict-mono f*
shows $f\ x < f\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *min-of-mono*:
fixes $f :: 'a \Rightarrow 'b$::*linorder*
shows *mono f* $\Longrightarrow$ *min* $(f\ m)\ (f\ n) = f\ (min\ m\ n)$
⟨*proof*⟩

lemma *max-of-mono*:
fixes $f :: 'a \Rightarrow 'b$::*linorder*
shows *mono f* $\Longrightarrow$ *max* $(f\ m)\ (f\ n) = f\ (max\ m\ n)$
⟨*proof*⟩

end

lemma *min-leastL*: (!!x. *least* <= x) ==> *min least x* = *least*
⟨*proof*⟩

lemma *max-leastL*: (!!x. *least* <= x) ==> *max least x* = x
⟨*proof*⟩

lemma *min-leastR*: $(\bigwedge x::'a::order.\ least \leq x) \Longrightarrow min\ x\ least = least$
⟨*proof*⟩

lemma *max-leastR*: $(\bigwedge x::'a::order.\ least \leq x) \Longrightarrow max\ x\ least = x$
⟨*proof*⟩

3.9 Top and bottom elements

class *top* = *preorder* +
fixes *top* :: $'a$
assumes *top-greatest* [*simp*]: $x \leq top$

class *bot* = *preorder* +
fixes *bot* :: $'a$
assumes *bot-least* [*simp*]: $bot \leq x$

3.10 Dense orders

class *dense-linear-order* = *linorder* +
assumes *gt-ex*: $\exists\, y.\ x < y$
and *lt-ex*: $\exists\, y.\ y < x$
and *dense*: $x < y \Longrightarrow (\exists\, z.\ x < z \wedge z < y)$

3.11 Wellorders

class *wellorder* = *linorder* +
  assumes *less-induct* [*case-names less*]: $(\bigwedge x.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$
begin

lemma *wellorder-Least-lemma*:
  fixes $k :: 'a$
  assumes $P\ k$
  shows $P\ (LEAST\ x.\ P\ x)$ **and** $(LEAST\ x.\ P\ x) \leq k$
⟨*proof*⟩

lemmas *LeastI* = *wellorder-Least-lemma*(*1*)
lemmas *Least-le* = *wellorder-Least-lemma*(*2*)

— The following 3 lemmas are due to Brian Huffman
lemma *LeastI-ex*: $\exists\, x.\ P\ x \Longrightarrow P\ (Least\ P)$
  ⟨*proof*⟩

lemma *LeastI2*:
  $P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
  ⟨*proof*⟩

lemma *LeastI2-ex*:
  $\exists\, a.\ P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
  ⟨*proof*⟩

lemma *not-less-Least*: $k < (LEAST\ x.\ P\ x) \Longrightarrow \neg\ P\ k$
⟨*proof*⟩

end

3.12 Order on bool

instantiation *bool* :: {*order*, *top*, *bot*}
begin

definition
  le-bool-def [*code del*]: $P \leq Q \longleftrightarrow P \longrightarrow Q$

definition
  less-bool-def [*code del*]: $(P::bool) < Q \longleftrightarrow \neg\ P \wedge Q$

definition
  top-bool-eq: $top = True$

definition
  bot-bool-eq: $bot = False$

instance ⟨*proof*⟩

end

lemma *le-boolI*: $(P \Longrightarrow Q) \Longrightarrow P \leq Q$
⟨*proof*⟩

lemma *le-boolI'*: $P \longrightarrow Q \Longrightarrow P \leq Q$
⟨*proof*⟩

lemma *le-boolE*: $P \leq Q \Longrightarrow P \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *le-boolD*: $P \leq Q \Longrightarrow P \longrightarrow Q$
⟨*proof*⟩

lemma [*code*]:
$False \leq b \longleftrightarrow True$
$True \leq b \longleftrightarrow b$
$False < b \longleftrightarrow b$
$True < b \longleftrightarrow False$
⟨*proof*⟩

3.13 Order on functions

instantiation *fun* :: (*type*, *ord*) *ord*
begin

definition
le-fun-def [*code del*]: $f \leq g \longleftrightarrow (\forall x.\ f\ x \leq g\ x)$

definition
less-fun-def [*code del*]: $(f::'a \Rightarrow 'b) < g \longleftrightarrow f \leq g \wedge \neg\ (g \leq f)$

instance ⟨*proof*⟩

end

instance *fun* :: (*type*, *preorder*) *preorder* ⟨*proof*⟩

instance *fun* :: (*type*, *order*) *order* ⟨*proof*⟩

instantiation *fun* :: (*type*, *top*) *top*
begin

definition
top-fun-eq: $top = (\lambda x.\ top)$

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *bot*) *bot*
begin

definition
bot-fun-eq: $bot = (\lambda x.\ bot)$

instance ⟨*proof*⟩

end

lemma *le-funI*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow f \leq g$
⟨*proof*⟩

lemma *le-funE*: $f \leq g \Longrightarrow (f\ x \leq g\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *le-funD*: $f \leq g \Longrightarrow f\ x \leq g\ x$
⟨*proof*⟩

Handy introduction and elimination rules for ≤ on unary and binary predicates

lemma *predicate1I*:
assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $P \leq Q$
⟨*proof*⟩

lemma *predicate1D* [*Pure.dest*, *dest*]: $P \leq Q \Longrightarrow P\ x \Longrightarrow Q\ x$
⟨*proof*⟩

lemma *predicate2I* [*Pure.intro*!, *intro*!]:
assumes *PQ*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$
shows $P \leq Q$
⟨*proof*⟩

lemma *predicate2D* [*Pure.dest*, *dest*]: $P \leq Q \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
⟨*proof*⟩

lemma *rev-predicate1D*: $P\ x$ ==> P <= Q ==> $Q\ x$
⟨*proof*⟩

lemma *rev-predicate2D*: $P\ x\ y$ ==> P <= Q ==> $Q\ x\ y$
⟨*proof*⟩

end

4 Lattices: Abstract lattices

theory *Lattices*
imports *Orderings*
begin

4.1 Lattices

notation
  less-eq (**infix** $\sqsubseteq$ *50*) **and**
  less (**infix** $\sqsubset$ *50*)

class *lower-semilattice* = *order* +
  fixes *inf* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcap$ *70*)
  assumes *inf-le1* [*simp*]: $x \sqcap y \sqsubseteq x$
  and *inf-le2* [*simp*]: $x \sqcap y \sqsubseteq y$
  and *inf-greatest*: $x \sqsubseteq y \Longrightarrow x \sqsubseteq z \Longrightarrow x \sqsubseteq y \sqcap z$

class *upper-semilattice* = *order* +
  fixes *sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcup$ *65*)
  assumes *sup-ge1* [*simp*]: $x \sqsubseteq x \sqcup y$
  and *sup-ge2* [*simp*]: $y \sqsubseteq x \sqcup y$
  and *sup-least*: $y \sqsubseteq x \Longrightarrow z \sqsubseteq x \Longrightarrow y \sqcup z \sqsubseteq x$
begin

Dual lattice

lemma *dual-lattice*:
  lower-semilattice (*op* $\geq$) (*op* $>$) *sup*
⟨*proof*⟩

end

class *lattice* = *lower-semilattice* + *upper-semilattice*

4.1.1 Intro and elim rules

context *lower-semilattice*
begin

lemma *le-infI1*[*intro*]:
  assumes $a \sqsubseteq x$
  shows $a \sqcap b \sqsubseteq x$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infI1*

lemma *le-infI2*[*intro*]:
  assumes $b \sqsubseteq x$
  shows $a \sqcap b \sqsubseteq x$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infI2*

lemma *le-infI*[*intro*!]: $x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcap b$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infI*

lemma *le-infE* [*elim*!]: $x \sqsubseteq a \sqcap b \Longrightarrow (x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infE*

lemma *le-inf-iff* [*simp*]:
$x \sqsubseteq y \sqcap z = (x \sqsubseteq y \wedge x \sqsubseteq z)$
⟨*proof*⟩

lemma *le-iff-inf*: $(x \sqsubseteq y) = (x \sqcap y = x)$
⟨*proof*⟩

lemma *mono-inf*:
fixes $f :: 'a \Rightarrow 'b$::*lower-semilattice*
shows *mono* $f \Longrightarrow f\ (A \sqcap B) \leq f\ A \sqcap f\ B$
⟨*proof*⟩

end

context *upper-semilattice*
begin

lemma *le-supI1*[*intro*]: $x \sqsubseteq a \Longrightarrow x \sqsubseteq a \sqcup b$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supI1*

lemma *le-supI2*[*intro*]: $x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcup b$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supI2*

lemma *le-supI*[*intro*!]: $a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow a \sqcup b \sqsubseteq x$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supI*

lemma *le-supE*[*elim*!]: $a \sqcup b \sqsubseteq x \Longrightarrow (a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supE*

lemma *ge-sup-conv*[*simp*]:
$x \sqcup y \sqsubseteq z = (x \sqsubseteq z \wedge y \sqsubseteq z)$
⟨*proof*⟩

lemma *le-iff-sup*: $(x \sqsubseteq y) = (x \sqcup y = y)$
⟨*proof*⟩

lemma *mono-sup*:
fixes $f :: 'a \Rightarrow 'b$::*upper-semilattice*
shows *mono* $f \Longrightarrow f\ A \sqcup f\ B \leq f\ (A \sqcup B)$
⟨*proof*⟩

end

4.1.2 Equational laws

context *lower-semilattice*
begin

lemma *inf-commute*: $(x \sqcap y) = (y \sqcap x)$
⟨*proof*⟩

lemma *inf-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
⟨*proof*⟩

lemma *inf-idem*[*simp*]: $x \sqcap x = x$
⟨*proof*⟩

lemma *inf-left-idem*[*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
⟨*proof*⟩

lemma *inf-absorb1*: $x \sqsubseteq y \Longrightarrow x \sqcap y = x$
⟨*proof*⟩

lemma *inf-absorb2*: $y \sqsubseteq x \Longrightarrow x \sqcap y = y$
⟨*proof*⟩

lemma *inf-left-commute*: $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$
⟨*proof*⟩

lemmas *inf-ACI* = *inf-commute inf-assoc inf-left-commute inf-left-idem*

end

context *upper-semilattice*
begin

lemma *sup-commute*: $(x \sqcup y) = (y \sqcup x)$
⟨*proof*⟩

lemma *sup-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
⟨*proof*⟩

lemma *sup-idem*[*simp*]: $x \sqcup x = x$
⟨*proof*⟩

lemma *sup-left-idem*[*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
⟨*proof*⟩

lemma *sup-absorb1*: $y \sqsubseteq x \Longrightarrow x \sqcup y = x$
⟨*proof*⟩

lemma *sup-absorb2*: $x \sqsubseteq y \Longrightarrow x \sqcup y = y$
⟨*proof*⟩

lemma *sup-left-commute*: $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
⟨*proof*⟩

lemmas *sup-ACI* = *sup-commute sup-assoc sup-left-commute sup-left-idem*

end

context *lattice*
begin

lemma *inf-sup-absorb*: $x \sqcap (x \sqcup y) = x$
⟨*proof*⟩

lemma *sup-inf-absorb*: $x \sqcup (x \sqcap y) = x$
⟨*proof*⟩

lemmas *ACI* = *inf-ACI sup-ACI*

lemmas *inf-sup-ord* = *inf-le1 inf-le2 sup-ge1 sup-ge2*

Towards distributivity

lemma *distrib-sup-le*: $x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

lemma *distrib-inf-le*: $(x \sqcap y) \sqcup (x \sqcap z) \sqsubseteq x \sqcap (y \sqcup z)$
⟨*proof*⟩

If you have one of them, you have them all.

lemma *distrib-imp1*:
assumes *D*: $!!x\ y\ z.\ x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
shows $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

lemma *distrib-imp2*:
assumes *D*: $!!x\ y\ z.\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

lemma *modular-le*: $x \sqsubseteq z \Longrightarrow x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap z$
⟨*proof*⟩

end

4.2 Distributive lattices

class *distrib-lattice* = *lattice* +
  assumes *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

context *distrib-lattice*
begin

lemma *sup-inf-distrib2*:
 $(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
⟨*proof*⟩

lemma *inf-sup-distrib1*:
 $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

lemma *inf-sup-distrib2*:
 $(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
⟨*proof*⟩

lemmas *distrib* =
  sup-inf-distrib1 sup-inf-distrib2 inf-sup-distrib1 inf-sup-distrib2

end

4.3 Uniqueness of inf and sup

lemma (**in** *lower-semilattice*) *inf-unique*:
  fixes f (**infixl** $\triangle$ *70*)
  assumes *le1*: $\bigwedge x\ y.\ x \triangle y \leq x$ **and** *le2*: $\bigwedge x\ y.\ x \triangle y \leq y$
  and *greatest*: $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \triangle z$
  shows $x \sqcap y = x \triangle y$
⟨*proof*⟩

lemma (**in** *upper-semilattice*) *sup-unique*:
  fixes f (**infixl** ∇ *70*)
  assumes *ge1* [*simp*]: $\bigwedge x\ y.\ x \leq x \nabla y$ **and** *ge2*: $\bigwedge x\ y.\ y \leq x \nabla y$
  and *least*: $\bigwedge x\ y\ z.\ y \leq x \Longrightarrow z \leq x \Longrightarrow y \nabla z \leq x$
  shows $x \sqcup y = x \nabla y$
⟨*proof*⟩

4.4 *min*/*max* on linear orders as special case of *op* ⊓/*op* ⊔

lemma (**in** *linorder*) *distrib-lattice-min-max*:
  distrib-lattice (*op* ≤) (*op* <) *min max*

⟨*proof*⟩

interpretation *min-max*: *distrib-lattice op* ≤ :: *'a*::*linorder* ⇒ *'a* ⇒ *bool op* < *min max*
⟨*proof*⟩

lemma *inf-min*: *inf* = (*min* :: *'a*::{*lower-semilattice*, *linorder*} ⇒ *'a* ⇒ *'a*)
⟨*proof*⟩

lemma *sup-max*: *sup* = (*max* :: *'a*::{*upper-semilattice*, *linorder*} ⇒ *'a* ⇒ *'a*)
⟨*proof*⟩

lemmas *le-maxI1* = *min-max.sup-ge1*
lemmas *le-maxI2* = *min-max.sup-ge2*

lemmas *max-ac* = *min-max.sup-assoc min-max.sup-commute*
mk-left-commute [*of max*, *OF min-max.sup-assoc min-max.sup-commute*]

lemmas *min-ac* = *min-max.inf-assoc min-max.inf-commute*
mk-left-commute [*of min*, *OF min-max.inf-assoc min-max.inf-commute*]

Now we have inherited antisymmetry as an intro-rule on all linear orders. This is a problem because it applies to bool, which is undesirable.

lemmas [*rule del*] = *min-max.le-infI min-max.le-supI*
min-max.le-supE min-max.le-infE min-max.le-supI1 min-max.le-supI2
min-max.le-infI1 min-max.le-infI2

4.5 Bool as lattice

instantiation *bool* :: *distrib-lattice*
begin

definition
inf-bool-eq: *P* ⊓ *Q* ⟷ *P* ∧ *Q*

definition
sup-bool-eq: *P* ⊔ *Q* ⟷ *P* ∨ *Q*

instance
⟨*proof*⟩

end

4.6 Fun as lattice

instantiation *fun* :: (*type*, *lattice*) *lattice*
begin

definition

inf-fun-eq [*code del*]: $f \sqcap g = (\lambda x.\ f\ x \sqcap g\ x)$

definition
sup-fun-eq [*code del*]: $f \sqcup g = (\lambda x.\ f\ x \sqcup g\ x)$

instance
⟨*proof*⟩

end

instance *fun* :: (*type*, *distrib-lattice*) *distrib-lattice*
⟨*proof*⟩

redundant bindings

lemmas *inf-aci* = *inf-ACI*
lemmas *sup-aci* = *sup-ACI*

no-notation
less-eq (**infix** $\sqsubseteq$ *50*) **and**
less (**infix** $\sqsubset$ *50*) **and**
inf (**infixl** $\sqcap$ *70*) **and**
sup (**infixl** $\sqcup$ *65*)

end

5 Set: Set theory for higher-order logic

theory *Set*
imports *Lattices*
begin

A set in HOL is simply a predicate.

5.1 Basic syntax

global

types *'a set* = *'a* => *bool*

consts
Collect :: (*'a* => *bool*) => *'a set* — comprehension
op : :: *'a* => *'a set* => *bool* — membership
insert :: *'a* => *'a set* => *'a set*
Ball :: *'a set* => (*'a* => *bool*) => *bool* — bounded universal quantifiers

Bex :: *'a set* => (*'a* => *bool*) => *bool* — bounded existential quantifiers

Bex1 :: *'a set => ('a => bool) => bool* — bounded unique existential quantifiers
Pow :: *'a set => 'a set set* — powerset
image :: *('a => 'b) => 'a set => 'b set* (**infixr** *‘ 90*)

local

notation
op : (*op :*) **and**
op : (*(-/ : -) [50, 51] 50*)

abbreviation
not-mem x A == ~ (x : A) — non-membership

notation
not-mem (*op ~:*) **and**
not-mem (*(-/ ~: -) [50, 51] 50*)

notation (*xsymbols*)
op : (*op ∈*) **and**
op : (*(-/ ∈ -) [50, 51] 50*) **and**
not-mem (*op ∉*) **and**
not-mem (*(-/ ∉ -) [50, 51] 50*)

notation (*HTML* **output**)
op : (*op ∈*) **and**
op : (*(-/ ∈ -) [50, 51] 50*) **and**
not-mem (*op ∉*) **and**
not-mem (*(-/ ∉ -) [50, 51] 50*)

syntax
@Coll :: *pttrn => bool => 'a set* (*(1{-./ -})*)

translations
{x. P} == *Collect (%x. P)*

definition *empty* :: *'a set* ({}) **where**
empty ≡ {x. False}

definition *UNIV* :: *'a set* **where**
UNIV ≡ {x. True}

syntax
@Finset :: *args => 'a set* ({(-)})

translations
{x, xs} == *insert x {xs}*
{x} == *insert x {}*

definition *Int* :: *'a set* ⇒ *'a set* ⇒ *'a set* (**infixl** *Int 70*) **where**
A Int B ≡ {*x*. *x* ∈ *A* ∧ *x* ∈ *B*}

definition *Un* :: *'a set* ⇒ *'a set* ⇒ *'a set* (**infixl** *Un 65*) **where**
A Un B ≡ {*x*. *x* ∈ *A* ∨ *x* ∈ *B*}

notation (*xsymbols*)
Int (**infixl** ∩ *70*) **and**
Un (**infixl** ∪ *65*)

notation (*HTML* **output**)
Int (**infixl** ∩ *70*) **and**
Un (**infixl** ∪ *65*)

syntax
-Ball :: *pttrn* => *'a set* => *bool* => *bool* (*3ALL -:-./ -*) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* => *'a set* => *bool* => *bool* (*3EX -:-./ -*) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* => *'a set* => *bool* => *bool* (*3EX! -:-./ -*) [*0*, *0*, *10*] *10*)
-Bleast :: *id* => *'a set* => *bool* => *'a* (*3LEAST -:-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HOL*)
-Ball :: *pttrn* => *'a set* => *bool* => *bool* (*3! -:-./ -*) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* => *'a set* => *bool* => *bool* (*3? -:-./ -*) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* => *'a set* => *bool* => *bool* (*3?! -:-./ -*) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-Ball :: *pttrn* => *'a set* => *bool* => *bool* (*3∀ -∈-./ -*) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* => *'a set* => *bool* => *bool* (*3∃ -∈-./ -*) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* => *'a set* => *bool* => *bool* (*3∃!-∈-./ -*) [*0*, *0*, *10*] *10*)
-Bleast :: *id* => *'a set* => *bool* => *'a* (*3LEAST-∈-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-Ball :: *pttrn* => *'a set* => *bool* => *bool* (*3∀ -∈-./ -*) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* => *'a set* => *bool* => *bool* (*3∃ -∈-./ -*) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* => *'a set* => *bool* => *bool* (*3∃!-∈-./ -*) [*0*, *0*, *10*] *10*)

translations
ALL x:*A*. *P* == *Ball A* (%*x*. *P*)
EX x:*A*. *P* == *Bex A* (%*x*. *P*)
EX! *x*:*A*. *P* == *Bex1 A* (%*x*. *P*)
LEAST x:*A*. *P* => *LEAST x*. *x*:*A* & *P*

definition *INTER* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ *'b set* **where**
INTER A B ≡ {*y*. ∀*x*∈*A*. *y* ∈ *B x*}

definition *UNION* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ *'b set* **where**
UNION A B ≡ {*y*. ∃*x*∈*A*. *y* ∈ *B x*}

definition *Inter* :: *'a set set ⇒ 'a set* **where**
Inter S ≡ INTER S (λx. x)

definition *Union* :: *'a set set ⇒ 'a set* **where**
Union S ≡ UNION S (λx. x)

notation (*xsymbols*)
Inter (⋂- [*90*] *90*) **and**
Union (⋃- [*90*] *90*)

5.2 Additional concrete syntax

syntax
@SetCompr :: *'a => idts => bool => 'a set* ((*1*{- |/-./ -}))
@Collect :: *idt => 'a set => bool => 'a set* ((*1*{- :/ -./ -}))
@INTER1 :: *pttrns => 'b set => 'b set* ((*3INT* -./ -) [*0*, *10*] *10*)
@UNION1 :: *pttrns => 'b set => 'b set* ((*3UN* -./ -) [*0*, *10*] *10*)
@INTER :: *pttrn => 'a set => 'b set => 'b set* ((*3INT* -:-./ -) [*0*, *10*] *10*)
@UNION :: *pttrn => 'a set => 'b set => 'b set* ((*3UN* -:-./ -) [*0*, *10*] *10*)

syntax (*xsymbols*)
@Collect :: *idt => 'a set => bool => 'a set* ((*1*{- ∈/ -./ -}))
@INTER1 :: *pttrns => 'b set => 'b set* ((*3*⋂-./ -) [*0*, *10*] *10*)
@UNION1 :: *pttrns => 'b set => 'b set* ((*3*⋃-./ -) [*0*, *10*] *10*)
@INTER :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋂-∈-./ -) [*0*, *10*] *10*)
@UNION :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋃-∈-./ -) [*0*, *10*] *10*)

syntax (*latex* **output**)
@INTER1 :: *pttrns => 'b set => 'b set* ((*3*⋂(*00*-)/ -) [*0*, *10*] *10*)
@UNION1 :: *pttrns => 'b set => 'b set* ((*3*⋃(*00*-)/ -) [*0*, *10*] *10*)
@INTER :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋂(*00*-∈-)/ -) [*0*, *10*] *10*)
@UNION :: *pttrn => 'a set => 'b set => 'b set* ((*3*⋃(*00*-∈-)/ -) [*0*, *10*] *10*)

translations
{x:A. P} => *{x. x:A & P}*
INT x y. B == *INT x. INT y. B*
INT x. B == *CONST INTER CONST UNIV (%x. B)*
INT x. B == *INT x:CONST UNIV. B*
INT x:A. B == *CONST INTER A (%x. B)*
UN x y. B == *UN x. UN y. B*
UN x. B == *CONST UNION CONST UNIV (%x. B)*
UN x. B == *UN x:CONST UNIV. B*
UN x:A. B == *CONST UNION A (%x. B)*

Note the difference between ordinary xsymbol syntax of indexed unions and intersections (e.g. $\bigcup a_1{\in}A_1.\ B$) and their LaTeX rendition: $\bigcup_{a_1 \in A_1} B$. The former does not make the index expression a subscript of the union/intersection

symbol because this leads to problems with nested subscripts in Proof General.

abbreviation
subset :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
subset ≡ *less*

abbreviation
subset-eq :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
subset-eq ≡ *less-eq*

notation (**output**)
subset (*op* <) **and**
subset ((-/ < -) [*50*, *51*] *50*) **and**
subset-eq (*op* <=) **and**
subset-eq ((-/ <= -) [*50*, *51*] *50*)

notation (*xsymbols*)
subset (*op* ⊂) **and**
subset ((-/ ⊂ -) [*50*, *51*] *50*) **and**
subset-eq (*op* ⊆) **and**
subset-eq ((-/ ⊆ -) [*50*, *51*] *50*)

notation (*HTML* **output**)
subset (*op* ⊂) **and**
subset ((-/ ⊂ -) [*50*, *51*] *50*) **and**
subset-eq (*op* ⊆) **and**
subset-eq ((-/ ⊆ -) [*50*, *51*] *50*)

abbreviation (*input*)
supset :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
supset ≡ *greater*

abbreviation (*input*)
supset-eq :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
supset-eq ≡ *greater-eq*

notation (*xsymbols*)
supset (*op* ⊃) **and**
supset ((-/ ⊃ -) [*50*, *51*] *50*) **and**
supset-eq (*op* ⊇) **and**
supset-eq ((-/ ⊇ -) [*50*, *51*] *50*)

abbreviation
range :: (*'a* => *'b*) => *'b set* **where** — of function
range f == *f* ‘ *UNIV*

5.2.1 Bounded quantifiers

syntax (**output**)

-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3EX!* -<=-./ -) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!-⊆-./ -) [*0*, *0*, *10*] *10*)

syntax (*HOL* **output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3?* -<-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<=-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3?* -<=-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3?*! -<=-./ -) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!-⊆-./ -) [*0*, *0*, *10*] *10*)

translations
∀ *A*⊂*B*. *P* => *ALL A*. *A* ⊂ *B* −−> *P*
∃ *A*⊂*B*. *P* => *EX A*. *A* ⊂ *B* & *P*
∀ *A*⊆*B*. *P* => *ALL A*. *A* ⊆ *B* −−> *P*
∃ *A*⊆*B*. *P* => *EX A*. *A* ⊆ *B* & *P*
∃!*A*⊆*B*. *P* => *EX*! *A*. *A* ⊆ *B* & *P*

⟨*ML*⟩

Translate between {*e* | *x1...xn*. *P*} and {*u*. *EX x1..xn*. *u* = *e* & *P*}; {*y*. *EX x1..xn*. *y* = *e* & *P*} is only translated if [*0..n*] *subset bvs*(*e*).

⟨*ML*⟩

5.3 Rules and definitions

Isomorphisms between predicates and sets.

defs
mem-def [*code*]: *x* : *S* == *S x*
Collect-def [*code*]: *Collect P* == *P*

defs

Ball-def: $Ball\ A\ P \quad == ALL\ x.\ x{:}A\ --{>}\ P(x)$
Bex-def: $Bex\ A\ P \quad == EX\ x.\ x{:}A\ \&\ P(x)$
Bex1-def: $Bex1\ A\ P \quad == EX!\ x.\ x{:}A\ \&\ P(x)$

instantiation *fun* :: (*type*, *minus*) *minus*
begin

definition
fun-diff-def: $A - B = (\%x.\ A\ x - B\ x)$

instance ⟨*proof*⟩

end

instantiation *bool* :: *minus*
begin

definition
bool-diff-def: $A - B = (A\ \&\ \sim B)$

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *uminus*) *uminus*
begin

definition
fun-Compl-def: $- A = (\%x.\ - A\ x)$

instance ⟨*proof*⟩

end

instantiation *bool* :: *uminus*
begin

definition
bool-Compl-def: $- A = (\sim A)$

instance ⟨*proof*⟩

end

defs
Pow-def: $Pow\ A \quad == \{B.\ B <= A\}$
insert-def: $insert\ a\ B \quad == \{x.\ x{=}a\}\ Un\ B$
image-def: $f`A \quad == \{y.\ EX\ x{:}A.\ y = f(x)\}$

5.4 Lemmas and proof tool setup

5.4.1 Relating predicates and sets

lemma *mem-Collect-eq* [*iff*]: $(a : \{x.\ P(x)\}) = P(a)$
⟨*proof*⟩

lemma *Collect-mem-eq* [*simp*]: $\{x.\ x{:}A\} = A$
⟨*proof*⟩

lemma *CollectI*: $P(a) \Longrightarrow a : \{x.\ P(x)\}$
⟨*proof*⟩

lemma *CollectD*: $a : \{x.\ P(x)\} \Longrightarrow P(a)$
⟨*proof*⟩

lemma *Collect-cong*: $(!!x.\ P\ x = Q\ x) \Longrightarrow \{x.\ P(x)\} = \{x.\ Q(x)\}$
⟨*proof*⟩

lemmas *CollectE* = *CollectD* [*elim-format*]

5.4.2 Bounded quantifiers

lemma *ballI* [*intro!*]: $(!!x.\ x{:}A \Longrightarrow P\ x) \Longrightarrow ALL\ x{:}A.\ P\ x$
⟨*proof*⟩

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: $ALL\ x{:}A.\ P\ x \Longrightarrow x{:}A \Longrightarrow P\ x$
⟨*proof*⟩

lemma *ballE* [*elim*]: $ALL\ x{:}A.\ P\ x \Longrightarrow (P\ x \Longrightarrow Q) \Longrightarrow (x\ \tilde{}{:}\ A \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

⟨*ML*⟩

This tactic takes assumptions $\forall x{\in}A.\ P\ x$ and $a \in A$; creates assumption $P\ a$.

⟨*ML*⟩
Gives better instantiation for bound:

⟨*ML*⟩

lemma *bexI* [*intro*]: $P\ x \Longrightarrow x{:}A \Longrightarrow EX\ x{:}A.\ P\ x$
— Normally the best argument order: $P\ x$ constrains the choice of $x \in A$.
⟨*proof*⟩

lemma *rev-bexI* [*intro?*]: $x{:}A \Longrightarrow P\ x \Longrightarrow EX\ x{:}A.\ P\ x$
— The best argument order when there is only one $x \in A$.

⟨*proof*⟩

lemma *bexCI*: (*ALL* x:A. ~P x ==> P a) ==> a:A ==> *EX* x:A. P x
⟨*proof*⟩

lemma *bexE* [*elim*!]: *EX* x:A. P x ==> (!!x. x:A ==> P x ==> Q) ==> Q
⟨*proof*⟩

lemma *ball-triv* [*simp*]: (*ALL* x:A. P) = ((*EX* x. x:A) --> P)
— Trival rewrite rule.
⟨*proof*⟩

lemma *bex-triv* [*simp*]: (*EX* x:A. P) = ((*EX* x. x:A) & P)
— Dual form for existentials.
⟨*proof*⟩

lemma *bex-triv-one-point1* [*simp*]: (*EX* x:A. x = a) = (a:A)
⟨*proof*⟩

lemma *bex-triv-one-point2* [*simp*]: (*EX* x:A. a = x) = (a:A)
⟨*proof*⟩

lemma *bex-one-point1* [*simp*]: (*EX* x:A. x = a & P x) = (a:A & P a)
⟨*proof*⟩

lemma *bex-one-point2* [*simp*]: (*EX* x:A. a = x & P x) = (a:A & P a)
⟨*proof*⟩

lemma *ball-one-point1* [*simp*]: (*ALL* x:A. x = a --> P x) = (a:A --> P a)
⟨*proof*⟩

lemma *ball-one-point2* [*simp*]: (*ALL* x:A. a = x --> P x) = (a:A --> P a)
⟨*proof*⟩

⟨*ML*⟩

5.4.3 Congruence rules

lemma *ball-cong*:
A = B ==> (!!x. x:B ==> P x = Q x) ==>
(*ALL* x:A. P x) = (*ALL* x:B. Q x)
⟨*proof*⟩

lemma *strong-ball-cong* [*cong*]:
A = B ==> (!!x. x:B =*simp*=> P x = Q x) ==>
(*ALL* x:A. P x) = (*ALL* x:B. Q x)
⟨*proof*⟩

lemma *bex-cong*:

$A = B$ ==> (!!x. x:B ==> $P\ x = Q\ x$) ==>
($EX\ x$:A. $P\ x$) = ($EX\ x$:B. $Q\ x$)
⟨*proof*⟩

lemma *strong-bex-cong* [*cong*]:
$A = B$ ==> (!!x. x:B =*simp*=> $P\ x = Q\ x$) ==>
($EX\ x$:A. $P\ x$) = ($EX\ x$:B. $Q\ x$)
⟨*proof*⟩

5.4.4 Subsets

lemma *subsetI* [*atp*,*intro*!]: (!!x. x:A ==> x:B) ==> $A \subseteq B$
⟨*proof*⟩

Map the type $'a\ set$ => *anything* to just $'a$; for overloading constants whose first argument has type $'a\ set$.

lemma *subsetD* [*elim*, *intro?*]: $A \subseteq B$ ==> $c \in A$ ==> $c \in B$
— Rule in Modus Ponens style.
⟨*proof*⟩

lemma *rev-subsetD* [*intro?*]: $c \in A$ ==> $A \subseteq B$ ==> $c \in B$
— The same, with reversed premises for use with *erule* – cf *rev-mp*.
⟨*proof*⟩

Converts $A \subseteq B$ to $x \in A \Longrightarrow x \in B$.

⟨*ML*⟩

lemma *subsetCE* [*elim*]: $A \subseteq B$ ==> ($c \notin A$ ==> P) ==> ($c \in B$ ==> P) ==> P
— Classical elimination rule.
⟨*proof*⟩

lemma *subset-eq*: $A \leq B = (\forall x{\in}A.\ x \in B)$ ⟨*proof*⟩

Takes assumptions $A \subseteq B$; $c \in A$ and creates the assumption $c \in B$.

⟨*ML*⟩

lemma *contra-subsetD*: $A \subseteq B$ ==> $c \notin B$ ==> $c \notin A$
⟨*proof*⟩

lemma *subset-refl* [*simp*,*atp*]: $A \subseteq A$
⟨*proof*⟩

lemma *subset-trans*: $A \subseteq B$ ==> $B \subseteq C$ ==> $A \subseteq C$
⟨*proof*⟩

5.4.5 Equality

lemma *set-ext*: **assumes** *prem*: (!!x. (x:A) = (x:B)) **shows** $A = B$
⟨*proof*⟩

lemma *expand-set-eq*: $(A = B) = (ALL\ x.\ (x{:}A) = (x{:}B))$
⟨*proof*⟩

lemma *subset-antisym* [*intro*!]: $A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow A = B$
— Anti-symmetry of the subset relation.
⟨*proof*⟩

Equality rules from ZF set theory – are they appropriate here?

lemma *equalityD1*: $A = B \Longrightarrow A \subseteq B$
⟨*proof*⟩

lemma *equalityD2*: $A = B \Longrightarrow B \subseteq A$
⟨*proof*⟩

Be careful when adding this to the claset as *subset-empty* is in the simpset: $A = \{\}$ goes to $\{\} \subseteq A$ and $A \subseteq \{\}$ and then back to $A = \{\}$!

lemma *equalityE*: $A = B \Longrightarrow (A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *equalityCE* [*elim*]:
$A = B \Longrightarrow (c \in A \Longrightarrow c \in B \Longrightarrow P) \Longrightarrow (c \notin A \Longrightarrow c \notin B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *eqset-imp-iff*: $A = B \Longrightarrow (x : A) = (x : B)$
⟨*proof*⟩

lemma *eqelem-imp-iff*: $x = y \Longrightarrow (x : A) = (y : A)$
⟨*proof*⟩

5.4.6 The universal set – UNIV

lemma *UNIV-I* [*simp*]: $x : UNIV$
⟨*proof*⟩

declare *UNIV-I* [*intro*] — unsafe makes it less likely to cause problems

lemma *UNIV-witness* [*intro?*]: $EX\ x.\ x : UNIV$
⟨*proof*⟩

lemma *subset-UNIV* [*simp*]: $A \subseteq UNIV$
⟨*proof*⟩

Eta-contracting these two rules (to remove P) causes them to be ignored because of their interaction with congruence rules.

lemma *ball-UNIV* [*simp*]: $Ball\ UNIV\ P = All\ P$
⟨*proof*⟩

lemma *bex-UNIV* [*simp*]: $Bex\ UNIV\ P = Ex\ P$
⟨*proof*⟩

lemma *UNIV-eq-I*: $(\bigwedge x.\ x \in A) \Longrightarrow UNIV = A$
⟨*proof*⟩

5.4.7 The empty set

lemma *empty-iff* [*simp*]: $(c : \{\}) = False$
⟨*proof*⟩

lemma *emptyE* [*elim*!]: $a : \{\} \Longrightarrow P$
⟨*proof*⟩

lemma *empty-subsetI* [*iff*]: $\{\} \subseteq A$
— One effect is to delete the ASSUMPTION $\{\} \subseteq A$
⟨*proof*⟩

lemma *equals0I*: $(!!y.\ y \in A \Longrightarrow False) \Longrightarrow A = \{\}$
⟨*proof*⟩

lemma *equals0D*: $A = \{\} \Longrightarrow a \notin A$
— Use for reasoning about disjointness: $A \cap B = \{\}$
⟨*proof*⟩

lemma *ball-empty* [*simp*]: $Ball\ \{\}\ P = True$
⟨*proof*⟩

lemma *bex-empty* [*simp*]: $Bex\ \{\}\ P = False$
⟨*proof*⟩

lemma *UNIV-not-empty* [*iff*]: $UNIV \neq \{\}$
⟨*proof*⟩

5.4.8 The Powerset operator – Pow

lemma *Pow-iff* [*iff*]: $(A \in Pow\ B) = (A \subseteq B)$
⟨*proof*⟩

lemma *PowI*: $A \subseteq B \Longrightarrow A \in Pow\ B$
⟨*proof*⟩

lemma *PowD*: $A \in Pow\ B \Longrightarrow A \subseteq B$
⟨*proof*⟩

lemma *Pow-bottom*: $\{\} \in Pow\ B$
⟨*proof*⟩

lemma *Pow-top*: $A \in Pow\ A$
⟨*proof*⟩

5.4.9 Set complement

lemma *Compl-iff* [*simp*]: $(c \in -A) = (c \notin A)$
⟨*proof*⟩

lemma *ComplI* [*intro!*]: $(c \in A \Longrightarrow False) \Longrightarrow c \in -A$
⟨*proof*⟩

This form, with negated conclusion, works well with the Classical prover. Negated assumptions behave like formulae on the right side of the notional turnstile ...

lemma *ComplD* [*dest!*]: $c : -A \Longrightarrow c^{\sim}{:}A$
⟨*proof*⟩

lemmas *ComplE* = *ComplD* [*elim-format*]

lemma *Compl-eq*: $-A = \{x.\ \sim x : A\}$ ⟨*proof*⟩

5.4.10 Binary union – Un

lemma *Un-iff* [*simp*]: $(c : A\ Un\ B) = (c{:}A \mid c{:}B)$
⟨*proof*⟩

lemma *UnI1* [*elim?*]: $c{:}A \Longrightarrow c : A\ Un\ B$
⟨*proof*⟩

lemma *UnI2* [*elim?*]: $c{:}B \Longrightarrow c : A\ Un\ B$
⟨*proof*⟩

Classical introduction rule: no commitment to A vs B.

lemma *UnCI* [*intro!*]: $(c^{\sim}{:}B \Longrightarrow c{:}A) \Longrightarrow c : A\ Un\ B$
⟨*proof*⟩

lemma *UnE* [*elim!*]: $c : A\ Un\ B \Longrightarrow (c{:}A \Longrightarrow P) \Longrightarrow (c{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

5.4.11 Binary intersection – Int

lemma *Int-iff* [*simp*]: $(c : A\ Int\ B) = (c{:}A\ \&\ c{:}B)$
⟨*proof*⟩

lemma *IntI* [*intro*!]: $c{:}A \Longrightarrow c{:}B \Longrightarrow c : A \mathbin{Int} B$
⟨*proof*⟩

lemma *IntD1*: $c : A \mathbin{Int} B \Longrightarrow c{:}A$
⟨*proof*⟩

lemma *IntD2*: $c : A \mathbin{Int} B \Longrightarrow c{:}B$
⟨*proof*⟩

lemma *IntE* [*elim*!]: $c : A \mathbin{Int} B \Longrightarrow (c{:}A \Longrightarrow c{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

5.4.12 Set difference

lemma *Diff-iff* [*simp*]: $(c : A - B) = (c{:}A \mathbin{\&} c{\sim}{:}B)$
⟨*proof*⟩

lemma *DiffI* [*intro*!]: $c : A \Longrightarrow c \sim : B \Longrightarrow c : A - B$
⟨*proof*⟩

lemma *DiffD1*: $c : A - B \Longrightarrow c : A$
⟨*proof*⟩

lemma *DiffD2*: $c : A - B \Longrightarrow c : B \Longrightarrow P$
⟨*proof*⟩

lemma *DiffE* [*elim*!]: $c : A - B \Longrightarrow (c{:}A \Longrightarrow c{\sim}{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *set-diff-eq*: $A - B = \{x.\ x : A \mathbin{\&} \sim x : B\}$ ⟨*proof*⟩

lemma *Compl-eq-Diff-UNIV*: $-A = (UNIV - A)$
⟨*proof*⟩

5.4.13 Augmenting a set − insert

lemma *insert-iff* [*simp*]: $(a : \mathit{insert}\ b\ A) = (a = b \mid a{:}A)$
⟨*proof*⟩

lemma *insertI1*: $a : \mathit{insert}\ a\ B$
⟨*proof*⟩

lemma *insertI2*: $a : B \Longrightarrow a : \mathit{insert}\ b\ B$
⟨*proof*⟩

lemma *insertE* [*elim*!]: $a : \mathit{insert}\ b\ A \Longrightarrow (a = b \Longrightarrow P) \Longrightarrow (a{:}A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *insertCI* [*intro*!]: $(a^{\sim}{:}B \Longrightarrow a = b) \Longrightarrow a{:}\ \mathit{insert}\ b\ B$
— Classical introduction rule.
⟨*proof*⟩

lemma *subset-insert-iff*: $(A \subseteq \mathit{insert}\ x\ B) = (\mathit{if}\ x{:}A\ \mathit{then}\ A - \{x\} \subseteq B\ \mathit{else}\ A \subseteq B)$
⟨*proof*⟩

lemma *set-insert*:
assumes $x \in A$
obtains B **where** $A = \mathit{insert}\ x\ B$ **and** $x \notin B$
⟨*proof*⟩

lemma *insert-ident*: $x\ {}^{\sim}{:}\ A \Longrightarrow x\ {}^{\sim}{:}\ B \Longrightarrow (\mathit{insert}\ x\ A = \mathit{insert}\ x\ B) = (A = B)$
⟨*proof*⟩

5.4.14 Singletons, using insert

lemma *singletonI* [*intro*!,*noatp*]: $a : \{a\}$
— Redundant? But unlike *insertCI*, it proves the subgoal immediately!
⟨*proof*⟩

lemma *singletonD* [*dest*!,*noatp*]: $b : \{a\} \Longrightarrow b = a$
⟨*proof*⟩

lemmas *singletonE* = *singletonD* [*elim-format*]

lemma *singleton-iff*: $(b : \{a\}) = (b = a)$
⟨*proof*⟩

lemma *singleton-inject* [*dest*!]: $\{a\} = \{b\} \Longrightarrow a = b$
⟨*proof*⟩

lemma *singleton-insert-inj-eq* [*iff*,*noatp*]:
$(\{b\} = \mathit{insert}\ a\ A) = (a = b\ \&\ A \subseteq \{b\})$
⟨*proof*⟩

lemma *singleton-insert-inj-eq′* [*iff*,*noatp*]:
$(\mathit{insert}\ a\ A = \{b\}) = (a = b\ \&\ A \subseteq \{b\})$
⟨*proof*⟩

lemma *subset-singletonD*: $A \subseteq \{x\} \Longrightarrow A = \{\} \mid A = \{x\}$
⟨*proof*⟩

lemma *singleton-conv* [*simp*]: $\{x.\ x = a\} = \{a\}$
⟨*proof*⟩

lemma *singleton-conv2* [*simp*]: $\{x.\ a = x\} = \{a\}$

⟨*proof*⟩

lemma *diff-single-insert*: $A - \{x\} \subseteq B$ ==> $x \in A$ ==> $A \subseteq$ *insert* x B
⟨*proof*⟩

lemma *doubleton-eq-iff*: $(\{a,b\} = \{c,d\}) = (a{=}c \ \&\ b{=}d \mid a{=}d \ \&\ b{=}c)$
⟨*proof*⟩

5.4.15 Unions of families

UN $x{:}A.\ B\ x$ is $\bigcup B$ ‘ A.

declare *UNION-def* [*noatp*]

lemma *UN-iff* [*simp*]: $(b{:}\ (UN\ x{:}A.\ B\ x)) = (EX\ x{:}A.\ b{:}\ B\ x)$
⟨*proof*⟩

lemma *UN-I* [*intro*]: $a{:}A$ ==> $b{:}\ B\ a$ ==> $b{:}\ (UN\ x{:}A.\ B\ x)$
— The order of the premises presupposes that A is rigid; b may be flexible.
⟨*proof*⟩

lemma *UN-E* [*elim*!]: $b : (UN\ x{:}A.\ B\ x)$ ==> (!!$x.\ x{:}A$ ==> $b{:}\ B\ x$ ==> R) ==> R
⟨*proof*⟩

lemma *UN-cong* [*cong*]:
$A = B$ ==> (!!$x.\ x{:}B$ ==> $C\ x = D\ x$) ==> $(UN\ x{:}A.\ C\ x) = (UN\ x{:}B.\ D\ x)$
⟨*proof*⟩

lemma *strong-UN-cong*:
$A = B$ ==> (!!$x.\ x{:}B$ =*simp*=> $C\ x = D\ x$) ==> $(UN\ x{:}A.\ C\ x) = (UN\ x{:}B.\ D\ x)$
⟨*proof*⟩

5.4.16 Intersections of families

INT $x{:}A.\ B\ x$ is $\bigcap B$ ‘ A.

lemma *INT-iff* [*simp*]: $(b{:}\ (INT\ x{:}A.\ B\ x)) = (ALL\ x{:}A.\ b{:}\ B\ x)$
⟨*proof*⟩

lemma *INT-I* [*intro*!]: (!!$x.\ x{:}A$ ==> $b{:}\ B\ x$) ==> $b : (INT\ x{:}A.\ B\ x)$
⟨*proof*⟩

lemma *INT-D* [*elim*]: $b : (INT\ x{:}A.\ B\ x)$ ==> $a{:}A$ ==> $b{:}\ B\ a$
⟨*proof*⟩

lemma *INT-E* [*elim*]: $b : (INT\ x{:}A.\ B\ x)$ ==> $(b{:}\ B\ a$ ==> $R)$ ==> $(a^{\sim}{:}A$ ==> $R)$ ==> R

— "Classical" elimination – by the Excluded Middle on $a \in A$.
⟨*proof*⟩

lemma *INT-cong* [*cong*]:
$A = B \Longrightarrow (!!x.\ x{:}B \Longrightarrow C\ x = D\ x) \Longrightarrow (INT\ x{:}A.\ C\ x) = (INT\ x{:}B.\ D\ x)$
⟨*proof*⟩

5.4.17 Union

lemma *Union-iff* [*simp*,*noatp*]: $(A : Union\ C) = (EX\ X{:}C.\ A{:}X)$
⟨*proof*⟩

lemma *UnionI* [*intro*]: $X{:}C \Longrightarrow A{:}X \Longrightarrow A : Union\ C$
— The order of the premises presupposes that C is rigid; A may be flexible.
⟨*proof*⟩

lemma *UnionE* [*elim*!]: $A : Union\ C \Longrightarrow (!!X.\ A{:}X \Longrightarrow X{:}C \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

5.4.18 Inter

lemma *Inter-iff* [*simp*,*noatp*]: $(A : Inter\ C) = (ALL\ X{:}C.\ A{:}X)$
⟨*proof*⟩

lemma *InterI* [*intro*!]: $(!!X.\ X{:}C \Longrightarrow A{:}X) \Longrightarrow A : Inter\ C$
⟨*proof*⟩

A "destruct" rule – every X in C contains A as an element, but $A \in X$ can hold when $X \in C$ does not! This rule is analogous to *spec*.

lemma *InterD* [*elim*]: $A : Inter\ C \Longrightarrow X{:}C \Longrightarrow A{:}X$
⟨*proof*⟩

lemma *InterE* [*elim*]: $A : Inter\ C \Longrightarrow (X^{\sim}{:}C \Longrightarrow R) \Longrightarrow (A{:}X \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination rule – does not require proving $X \in C$.
⟨*proof*⟩

Image of a set under a function. Frequently b does not have the syntactic form of $f\ x$.

declare *image-def* [*noatp*]

lemma *image-eqI* [*simp*, *intro*]: $b = f\ x \Longrightarrow x{:}A \Longrightarrow b : f`A$
⟨*proof*⟩

lemma *imageI*: $x : A \Longrightarrow f\ x : f\ `\ A$
⟨*proof*⟩

lemma *rev-image-eqI*: $x{:}A \Longrightarrow b = f\,x \Longrightarrow b : f`A$
— This version's more effective when we already have the required x.
⟨*proof*⟩

lemma *imageE* [*elim*!]:
$b : (\%x.\ f\,x)`A \Longrightarrow (!!x.\ b = f\,x \Longrightarrow x{:}A \Longrightarrow P) \Longrightarrow P$
— The eta-expansion gives variable-name preservation.
⟨*proof*⟩

lemma *image-Un*: $f`(A\ Un\ B) = f`A\ Un\ f`B$
⟨*proof*⟩

lemma *image-eq-UN*: $f`A = (UN\ x{:}A.\ \{f\,x\})$
⟨*proof*⟩

lemma *image-iff*: $(z : f`A) = (EX\ x{:}A.\ z = f\,x)$
⟨*proof*⟩

lemma *image-subset-iff*: $(f`A \subseteq B) = (\forall x{\in}A.\ f\,x \in B)$
— This rewrite rule would confuse users if made default.
⟨*proof*⟩

lemma *subset-image-iff*: $(B \subseteq f`A) = (EX\ AA.\ AA \subseteq A\ \&\ B = f`AA)$
⟨*proof*⟩

lemma *image-subsetI*: $(!!x.\ x \in A \Longrightarrow f\,x \in B) \Longrightarrow f`A \subseteq B$
— Replaces the three steps *subsetI*, *imageE*, *hypsubst*, but breaks too many existing proofs.
⟨*proof*⟩

Range of a function – just a translation for image!

lemma *range-eqI*: $b = f\,x \Longrightarrow b \in range\ f$
⟨*proof*⟩

lemma *rangeI*: $f\,x \in range\ f$
⟨*proof*⟩

lemma *rangeE* [*elim?*]: $b \in range\ (\lambda x.\ f\,x) \Longrightarrow (!!x.\ b = f\,x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

5.4.19 Set reasoning tools

Rewrite rules for boolean case-splitting: faster than *split-if* [*split*].

lemma *split-if-eq1*: $((if\ Q\ then\ x\ else\ y) = b) = ((Q \dashrightarrow x = b)\ \&\ (\sim Q \dashrightarrow y = b))$
⟨*proof*⟩

lemma *split-if-eq2*: $(a = (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)) = ((Q \dashrightarrow a = x)\ \&\ (\sim Q \dashrightarrow a = y))$
⟨*proof*⟩

Split ifs on either side of the membership relation. Not for [*simp*] – can cause goals to blow up!

lemma *split-if-mem1*: $((\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) : b) = ((Q \dashrightarrow x : b)\ \&\ (\sim Q \dashrightarrow y : b))$
⟨*proof*⟩

lemma *split-if-mem2*: $(a : (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)) = ((Q \dashrightarrow a : x)\ \&\ (\sim Q \dashrightarrow a : y))$
⟨*proof*⟩

lemmas *split-ifs = if-bool-eq-conj split-if-eq1 split-if-eq2 split-if-mem1 split-if-mem2*

⟨*ML*⟩

5.4.20 The "proper subset" relation

lemma *psubsetI* [*intro*!,*noatp*]: $A \subseteq B \Longrightarrow A \neq B \Longrightarrow A \subset B$
⟨*proof*⟩

lemma *psubsetE* [*elim*!,*noatp*]:
$[|A \subset B;\ [|A \subseteq B;\ \sim (B \subseteq A)|] \Longrightarrow R|] \Longrightarrow R$
⟨*proof*⟩

lemma *psubset-insert-iff*:
$(A \subset \mathit{insert}\ x\ B) = (\mathit{if}\ x \in B\ \mathit{then}\ A \subset B\ \mathit{else}\ \mathit{if}\ x \in A\ \mathit{then}\ A - \{x\} \subset B\ \mathit{else}\ A \subseteq B)$
⟨*proof*⟩

lemma *psubset-eq*: $(A \subset B) = (A \subseteq B\ \&\ A \neq B)$
⟨*proof*⟩

lemma *psubset-imp-subset*: $A \subset B \Longrightarrow A \subseteq B$
⟨*proof*⟩

lemma *psubset-trans*: $[|\ A \subset B;\ B \subset C\ |] \Longrightarrow A \subset C$
⟨*proof*⟩

lemma *psubsetD*: $[|\ A \subset B;\ c \in A\ |] \Longrightarrow c \in B$
⟨*proof*⟩

lemma *psubset-subset-trans*: $A \subset B \Longrightarrow B \subseteq C \Longrightarrow A \subset C$
⟨*proof*⟩

lemma *subset-psubset-trans*: $A \subseteq B \Longrightarrow B \subset C \Longrightarrow A \subset C$
⟨*proof*⟩

lemma *psubset-imp-ex-mem*: $A \subset B \Longrightarrow \exists b.\ b \in (B - A)$
⟨*proof*⟩

lemma *atomize-ball*:
$(!!x.\ x \in A \Longrightarrow P\ x) == \mathit{Trueprop}\ (\forall x{\in}A.\ P\ x)$
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-ball*
and [*symmetric*, *defn*] = *atomize-ball*

5.5 Further set-theory lemmas

5.5.1 Derived rules involving subsets.

insert.

lemma *subset-insertI*: $B \subseteq \mathit{insert}\ a\ B$
⟨*proof*⟩

lemma *subset-insertI2*: $A \subseteq B \Longrightarrow A \subseteq \mathit{insert}\ b\ B$
⟨*proof*⟩

lemma *subset-insert*: $x \notin A \Longrightarrow (A \subseteq \mathit{insert}\ x\ B) = (A \subseteq B)$
⟨*proof*⟩

Big Union – least upper bound of a set.

lemma *Union-upper*: $B \in A \Longrightarrow B \subseteq \mathit{Union}\ A$
⟨*proof*⟩

lemma *Union-least*: $(!!X.\ X \in A \Longrightarrow X \subseteq C) \Longrightarrow \mathit{Union}\ A \subseteq C$
⟨*proof*⟩

General union.

lemma *UN-upper*: $a \in A \Longrightarrow B\ a \subseteq (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *UN-least*: $(!!x.\ x \in A \Longrightarrow B\ x \subseteq C) \Longrightarrow (\bigcup x{\in}A.\ B\ x) \subseteq C$
⟨*proof*⟩

Big Intersection – greatest lower bound of a set.

lemma *Inter-lower*: $B \in A \Longrightarrow \mathit{Inter}\ A \subseteq B$
⟨*proof*⟩

lemma *Inter-subset*:
$[|\ !!X.\ X \in A \Longrightarrow X \subseteq B;\ A \neq \{\}\ |] \Longrightarrow \bigcap A \subseteq B$

⟨*proof*⟩

lemma *Inter-greatest*: $(!!X.\ X \in A ==> C \subseteq X) ==> C \subseteq Inter\ A$
⟨*proof*⟩

lemma *INT-lower*: $a \in A ==> (\bigcap x{\in}A.\ B\ x) \subseteq B\ a$
⟨*proof*⟩

lemma *INT-greatest*: $(!!x.\ x \in A ==> C \subseteq B\ x) ==> C \subseteq (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

Finite Union – the least upper bound of two sets.

lemma *Un-upper1*: $A \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-upper2*: $B \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-least*: $A \subseteq C ==> B \subseteq C ==> A \cup B \subseteq C$
⟨*proof*⟩

Finite Intersection – the greatest lower bound of two sets.

lemma *Int-lower1*: $A \cap B \subseteq A$
⟨*proof*⟩

lemma *Int-lower2*: $A \cap B \subseteq B$
⟨*proof*⟩

lemma *Int-greatest*: $C \subseteq A ==> C \subseteq B ==> C \subseteq A \cap B$
⟨*proof*⟩

Set difference.

lemma *Diff-subset*: $A - B \subseteq A$
⟨*proof*⟩

lemma *Diff-subset-conv*: $(A - B \subseteq C) = (A \subseteq B \cup C)$
⟨*proof*⟩

5.5.2 Equalities involving union, intersection, inclusion, etc.

{}.

lemma *Collect-const* [*simp*]: $\{s.\ P\} = (\textit{if } P \textit{ then } UNIV \textit{ else } \{\})$
— supersedes *Collect-False-empty*
⟨*proof*⟩

lemma *subset-empty* [*simp*]: $(A \subseteq \{\}) = (A = \{\})$
⟨*proof*⟩

lemma *not-psubset-empty* [*iff*]: $\neg\ (A < \{\})$
⟨*proof*⟩

lemma *Collect-empty-eq* [*simp*]: $(Collect\ P = \{\}) = (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *empty-Collect-eq* [*simp*]: $(\{\} = Collect\ P) = (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *Collect-neg-eq*: $\{x.\ \neg\ P\ x\} = -\ \{x.\ P\ x\}$
⟨*proof*⟩

lemma *Collect-disj-eq*: $\{x.\ P\ x\ |\ Q\ x\} = \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-imp-eq*: $\{x.\ P\ x\ -->\ Q\ x\} = -\{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-conj-eq*: $\{x.\ P\ x\ \&\ Q\ x\} = \{x.\ P\ x\} \cap \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-all-eq*: $\{x.\ \forall y.\ P\ x\ y\} = (\bigcap y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-ball-eq*: $\{x.\ \forall y{\in}A.\ P\ x\ y\} = (\bigcap y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-ex-eq* [*noatp*]: $\{x.\ \exists y.\ P\ x\ y\} = (\bigcup y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-bex-eq* [*noatp*]: $\{x.\ \exists y{\in}A.\ P\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

insert.

lemma *insert-is-Un*: *insert a A* = $\{a\}$ *Un A*
— NOT SUITABLE FOR REWRITING since $\{a\}$ == *insert a* $\{\}$
⟨*proof*⟩

lemma *insert-not-empty* [*simp*]: *insert a A* $\neq \{\}$
⟨*proof*⟩

lemmas *empty-not-insert* = *insert-not-empty* [*symmetric*, *standard*]
declare *empty-not-insert* [*simp*]

lemma *insert-absorb*: *a* $\in$ *A* ==> *insert a A* = *A*
— [*simp*] causes recursive calls when there are nested inserts
— with *quadratic* running time
⟨*proof*⟩

lemma *insert-absorb2* [*simp*]: *insert x* (*insert x A*) = *insert x A*
⟨*proof*⟩

lemma *insert-commute*: *insert x* (*insert y A*) = *insert y* (*insert x A*)
⟨*proof*⟩

lemma *insert-subset* [*simp*]: (*insert x A* ⊆ *B*) = (*x* ∈ *B* & *A* ⊆ *B*)
⟨*proof*⟩

lemma *mk-disjoint-insert*: *a* ∈ *A* ==> ∃ *B*. *A* = *insert a B* & *a* ∉ *B*
— use new *B* rather than *A* − {*a*} to avoid infinite unfolding
⟨*proof*⟩

lemma *insert-Collect*: *insert a* (*Collect P*) = {*u*. *u* ≠ *a* −−> *P u*}
⟨*proof*⟩

lemma *UN-insert-distrib*: *u* ∈ *A* ==> (⋃ *x*∈*A*. *insert a* (*B x*)) = *insert a* (⋃ *x*∈*A*. *B x*)
⟨*proof*⟩

lemma *insert-inter-insert*[*simp*]: *insert a A* ∩ *insert a B* = *insert a* (*A* ∩ *B*)
⟨*proof*⟩

lemma *insert-disjoint* [*simp*,*noatp*]:
(*insert a A* ∩ *B* = {}) = (*a* ∉ *B* ∧ *A* ∩ *B* = {})
({} = *insert a A* ∩ *B*) = (*a* ∉ *B* ∧ {} = *A* ∩ *B*)
⟨*proof*⟩

lemma *disjoint-insert* [*simp*,*noatp*]:
(*B* ∩ *insert a A* = {}) = (*a* ∉ *B* ∧ *B* ∩ *A* = {})
({} = *A* ∩ *insert b B*) = (*b* ∉ *A* ∧ {} = *A* ∩ *B*)
⟨*proof*⟩

image.

lemma *image-empty* [*simp*]: *f*‘{} = {}
⟨*proof*⟩

lemma *image-insert* [*simp*]: *f* ‘ *insert a B* = *insert* (*f a*) (*f*‘*B*)
⟨*proof*⟩

lemma *image-constant*: *x* ∈ *A* ==> (λ*x*. *c*) ‘ *A* = {*c*}
⟨*proof*⟩

lemma *image-constant-conv*: (%*x*. *c*) ‘ *A* = (*if A* = {} *then* {} *else* {*c*})
⟨*proof*⟩

lemma *image-image*: *f* ‘ (*g* ‘ *A*) = (λ*x*. *f* (*g x*)) ‘ *A*
⟨*proof*⟩

lemma *insert-image* [*simp*]: $x \in A \Longrightarrow$ *insert* $(f\,x)\ (f`A) = f`A$
⟨*proof*⟩

lemma *image-is-empty* [*iff*]: $(f`A = \{\}) = (A = \{\})$
⟨*proof*⟩

lemma *image-Collect* [*noatp*]: $f`\{x.\ P\ x\} = \{f\,x \mid x.\ P\ x\}$
— NOT suitable as a default simprule: the RHS isn't simpler than the LHS, with its implicit quantifier and conjunction. Also image enjoys better equational properties than does the RHS.
⟨*proof*⟩

lemma *if-image-distrib* [*simp*]:
$(\lambda x.\ \textit{if}\ P\ x\ \textit{then}\ f\ x\ \textit{else}\ g\ x)`S$
$= (f`(S \cap \{x.\ P\ x\})) \cup (g`(S \cap \{x.\ \neg\ P\ x\}))$
⟨*proof*⟩

lemma *image-cong*: $M = N \Longrightarrow (!!x.\ x \in N \Longrightarrow f\,x = g\,x) \Longrightarrow f`M = g`N$
⟨*proof*⟩

range.

lemma *full-SetCompr-eq* [*noatp*]: $\{u.\ \exists x.\ u = f\,x\} = \textit{range}\ f$
⟨*proof*⟩

lemma *range-composition*: $\textit{range}\ (\lambda x.\ f\ (g\ x)) = f`\textit{range}\ g$
⟨*proof*⟩

Int

lemma *Int-absorb* [*simp*]: $A \cap A = A$
⟨*proof*⟩

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
⟨*proof*⟩

lemma *Int-commute*: $A \cap B = B \cap A$
⟨*proof*⟩

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
⟨*proof*⟩

lemma *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$
⟨*proof*⟩

lemmas *Int-ac* = *Int-assoc Int-left-absorb Int-commute Int-left-commute*
— Intersection is an AC-operator

lemma *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
⟨*proof*⟩

lemma *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
⟨*proof*⟩

lemma *Int-empty-left* [*simp*]: $\{\} \cap B = \{\}$
⟨*proof*⟩

lemma *Int-empty-right* [*simp*]: $A \cap \{\} = \{\}$
⟨*proof*⟩

lemma *disjoint-eq-subset-Compl*: $(A \cap B = \{\}) = (A \subseteq -B)$
⟨*proof*⟩

lemma *disjoint-iff-not-equal*: $(A \cap B = \{\}) = (\forall x{\in}A.\ \forall y{\in}B.\ x \neq y)$
⟨*proof*⟩

lemma *Int-UNIV-left* [*simp*]: $UNIV \cap B = B$
⟨*proof*⟩

lemma *Int-UNIV-right* [*simp*]: $A \cap UNIV = A$
⟨*proof*⟩

lemma *Int-eq-Inter*: $A \cap B = \bigcap\{A, B\}$
⟨*proof*⟩

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
⟨*proof*⟩

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
⟨*proof*⟩

lemma *Int-UNIV* [*simp*,*noatp*]: $(A \cap B = UNIV) = (A = UNIV\ \&\ B = UNIV)$
⟨*proof*⟩

lemma *Int-subset-iff* [*simp*]: $(C \subseteq A \cap B) = (C \subseteq A\ \&\ C \subseteq B)$
⟨*proof*⟩

lemma *Int-Collect*: $(x \in A \cap \{x.\ P\ x\}) = (x \in A\ \&\ P\ x)$
⟨*proof*⟩

Un.

lemma *Un-absorb* [*simp*]: $A \cup A = A$
⟨*proof*⟩

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
⟨*proof*⟩

lemma *Un-commute*: $A \cup B = B \cup A$
⟨*proof*⟩

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
⟨*proof*⟩

lemma *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$
⟨*proof*⟩

lemmas *Un-ac = Un-assoc Un-left-absorb Un-commute Un-left-commute*
— Union is an AC-operator

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
⟨*proof*⟩

lemma *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
⟨*proof*⟩

lemma *Un-empty-left* [*simp*]: $\{\} \cup B = B$
⟨*proof*⟩

lemma *Un-empty-right* [*simp*]: $A \cup \{\} = A$
⟨*proof*⟩

lemma *Un-UNIV-left* [*simp*]: $UNIV \cup B = UNIV$
⟨*proof*⟩

lemma *Un-UNIV-right* [*simp*]: $A \cup UNIV = UNIV$
⟨*proof*⟩

lemma *Un-eq-Union*: $A \cup B = \bigcup \{A, B\}$
⟨*proof*⟩

lemma *Un-insert-left* [*simp*]: $(insert\ a\ B) \cup C = insert\ a\ (B \cup C)$
⟨*proof*⟩

lemma *Un-insert-right* [*simp*]: $A \cup (insert\ a\ B) = insert\ a\ (A \cup B)$
⟨*proof*⟩

lemma *Int-insert-left*:
$(insert\ a\ B)\ Int\ C = (if\ a \in C\ then\ insert\ a\ (B \cap C)\ else\ B \cap C)$
⟨*proof*⟩

lemma *Int-insert-right*:
$A \cap (insert\ a\ B) = (if\ a \in A\ then\ insert\ a\ (A \cap B)\ else\ A \cap B)$
⟨*proof*⟩

lemma *Un-Int-distrib*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
⟨*proof*⟩

lemma *Un-Int-distrib2*: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$
⟨*proof*⟩

lemma *Un-Int-crazy*:
$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
⟨*proof*⟩

lemma *subset-Un-eq*: $(A \subseteq B) = (A \cup B = B)$
⟨*proof*⟩

lemma *Un-empty* [*iff*]: $(A \cup B = \{\}) = (A = \{\} \ \& \ B = \{\})$
⟨*proof*⟩

lemma *Un-subset-iff* [*simp*]: $(A \cup B \subseteq C) = (A \subseteq C \ \& \ B \subseteq C)$
⟨*proof*⟩

lemma *Un-Diff-Int*: $(A - B) \cup (A \cap B) = A$
⟨*proof*⟩

lemma *Diff-Int2*: $A \cap C - B \cap C = A \cap C - B$
⟨*proof*⟩

Set complement

lemma *Compl-disjoint* [*simp*]: $A \cap -A = \{\}$
⟨*proof*⟩

lemma *Compl-disjoint2* [*simp*]: $-A \cap A = \{\}$
⟨*proof*⟩

lemma *Compl-partition*: $A \cup -A = UNIV$
⟨*proof*⟩

lemma *Compl-partition2*: $-A \cup A = UNIV$
⟨*proof*⟩

lemma *double-complement* [*simp*]: $-(-A) = (A::'a\ set)$
⟨*proof*⟩

lemma *Compl-Un* [*simp*]: $-(A \cup B) = (-A) \cap (-B)$
⟨*proof*⟩

lemma *Compl-Int* [*simp*]: $-(A \cap B) = (-A) \cup (-B)$
⟨*proof*⟩

lemma *Compl-UN* [*simp*]: $-(\bigcup x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ -B\ x)$
⟨*proof*⟩

lemma *Compl-INT* [*simp*]: $-(\bigcap x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ -B\ x)$

⟨*proof*⟩

lemma *subset-Compl-self-eq*: $(A \subseteq -A) = (A = \{\})$
⟨*proof*⟩

lemma *Un-Int-assoc-eq*: $((A \cap B) \cup C = A \cap (B \cup C)) = (C \subseteq A)$
— Halmos, Naive Set Theory, page 16.
⟨*proof*⟩

lemma *Compl-UNIV-eq* [*simp*]: $-UNIV = \{\}$
⟨*proof*⟩

lemma *Compl-empty-eq* [*simp*]: $-\{\} = UNIV$
⟨*proof*⟩

lemma *Compl-subset-Compl-iff* [*iff*]: $(-A \subseteq -B) = (B \subseteq A)$
⟨*proof*⟩

lemma *Compl-eq-Compl-iff* [*iff*]: $(-A = -B) = (A = (B::'a\ set))$
⟨*proof*⟩

Union.

lemma *Union-empty* [*simp*]: $Union(\{\}) = \{\}$
⟨*proof*⟩

lemma *Union-UNIV* [*simp*]: $Union\ UNIV = UNIV$
⟨*proof*⟩

lemma *Union-insert* [*simp*]: $Union\ (insert\ a\ B) = a \cup \bigcup B$
⟨*proof*⟩

lemma *Union-Un-distrib* [*simp*]: $\bigcup(A\ Un\ B) = \bigcup A \cup \bigcup B$
⟨*proof*⟩

lemma *Union-Int-subset*: $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$
⟨*proof*⟩

lemma *Union-empty-conv* [*simp*,*noatp*]: $(\bigcup A = \{\}) = (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

lemma *empty-Union-conv* [*simp*,*noatp*]: $(\{\} = \bigcup A) = (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

lemma *Union-disjoint*: $(\bigcup C \cap A = \{\}) = (\forall B{\in}C.\ B \cap A = \{\})$
⟨*proof*⟩

Inter.

lemma *Inter-empty* [*simp*]: $\bigcap\{\} = UNIV$

⟨*proof*⟩

lemma *Inter-UNIV* [*simp*]: $\bigcap UNIV = \{\}$
⟨*proof*⟩

lemma *Inter-insert* [*simp*]: $\bigcap(insert\ a\ B) = a \cap \bigcap B$
⟨*proof*⟩

lemma *Inter-Un-subset*: $\bigcap A \cup \bigcap B \subseteq \bigcap(A \cap B)$
⟨*proof*⟩

lemma *Inter-Un-distrib*: $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$
⟨*proof*⟩

lemma *Inter-UNIV-conv* [*simp*,*noatp*]:
$(\bigcap A = UNIV) = (\forall x{\in}A.\ x = UNIV)$
$(UNIV = \bigcap A) = (\forall x{\in}A.\ x = UNIV)$
⟨*proof*⟩

UN and *INT*.

Basic identities:

lemma *UN-empty* [*simp*,*noatp*]: $(\bigcup x{\in}\{\}.\ B\ x) = \{\}$
⟨*proof*⟩

lemma *UN-empty2* [*simp*]: $(\bigcup x{\in}A.\ \{\}) = \{\}$
⟨*proof*⟩

lemma *UN-singleton* [*simp*]: $(\bigcup x{\in}A.\ \{x\}) = A$
⟨*proof*⟩

lemma *UN-absorb*: $k \in I \Longrightarrow A\ k \cup (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *INT-empty* [*simp*]: $(\bigcap x{\in}\{\}.\ B\ x) = UNIV$
⟨*proof*⟩

lemma *INT-absorb*: $k \in I \Longrightarrow A\ k \cap (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *UN-insert* [*simp*]: $(\bigcup x{\in}insert\ a\ A.\ B\ x) = B\ a \cup UNION\ A\ B$
⟨*proof*⟩

lemma *UN-Un*[*simp*]: $(\bigcup i \in A \cup B.\ M\ i) = (\bigcup i{\in}A.\ M\ i) \cup (\bigcup i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B\ y).\ C\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}B\ y.\ C\ x)$
⟨*proof*⟩

lemma *UN-subset-iff*: $((\bigcup i{\in}I.\ A\ i) \subseteq B) = (\forall i{\in}I.\ A\ i \subseteq B)$

⟨*proof*⟩

lemma *INT-subset-iff*: $(B \subseteq (\bigcap i{\in}I.\ A\ i)) = (\forall i{\in}I.\ B \subseteq A\ i)$
⟨*proof*⟩

lemma *INT-insert* [*simp*]: $(\bigcap x \in \mathit{insert}\ a\ A.\ B\ x) = B\ a \cap \mathit{INTER}\ A\ B$
⟨*proof*⟩

lemma *INT-Un*: $(\bigcap i \in A \cup B.\ M\ i) = (\bigcap i \in A.\ M\ i) \cap (\bigcap i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *INT-insert-distrib*:
$u \in A \Longrightarrow (\bigcap x{\in}A.\ \mathit{insert}\ a\ (B\ x)) = \mathit{insert}\ a\ (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *Union-image-eq* [*simp*]: $\bigcup(B`A) = (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *image-Union*: $f\ `\ \bigcup S = (\bigcup x{\in}S.\ f\ `\ x)$
⟨*proof*⟩

lemma *Inter-image-eq* [*simp*]: $\bigcap(B`A) = (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ c)$
⟨*proof*⟩

lemma *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ c)$
⟨*proof*⟩

lemma *UN-eq*: $(\bigcup x{\in}A.\ B\ x) = \bigcup(\{Y.\ \exists x{\in}A.\ Y = B\ x\})$
⟨*proof*⟩

lemma *INT-eq*: $(\bigcap x{\in}A.\ B\ x) = \bigcap(\{Y.\ \exists x{\in}A.\ Y = B\ x\})$
— Look: it has an *existential* quantifier
⟨*proof*⟩

lemma *UNION-empty-conv*[*simp*]:
$(\{\} = (\mathit{UN}\ x{:}A.\ B\ x)) = (\forall x{\in}A.\ B\ x = \{\})$
$((\mathit{UN}\ x{:}A.\ B\ x) = \{\}) = (\forall x{\in}A.\ B\ x = \{\})$
⟨*proof*⟩

lemma *INTER-UNIV-conv*[*simp*]:
$(\mathit{UNIV} = (\mathit{INT}\ x{:}A.\ B\ x)) = (\forall x{\in}A.\ B\ x = \mathit{UNIV})$
$((\mathit{INT}\ x{:}A.\ B\ x) = \mathit{UNIV}) = (\forall x{\in}A.\ B\ x = \mathit{UNIV})$
⟨*proof*⟩

Distributive laws:

lemma *Int-Union*: $A \cap \bigcup B = (\bigcup C{\in}B.\ A \cap C)$

⟨*proof*⟩

lemma *Int-Union2*: $\bigcup B \cap A = (\bigcup C{\in}B.\ C \cap A)$
⟨*proof*⟩

lemma *Un-Union-image*: $(\bigcup x{\in}C.\ A\ x \cup B\ x) = \bigcup (A`C) \cup \bigcup (B`C)$
— Devlin, Fundamentals of Contemporary Set Theory, page 12, exercise 5:
— Union of a family of unions
⟨*proof*⟩

lemma *UN-Un-distrib*: $(\bigcup i{\in}I.\ A\ i \cup B\ i) = (\bigcup i{\in}I.\ A\ i) \cup (\bigcup i{\in}I.\ B\ i)$
— Equivalent version
⟨*proof*⟩

lemma *Un-Inter*: $A \cup \bigcap B = (\bigcap C{\in}B.\ A \cup C)$
⟨*proof*⟩

lemma *Int-Inter-image*: $(\bigcap x{\in}C.\ A\ x \cap B\ x) = \bigcap (A`C) \cap \bigcap (B`C)$
⟨*proof*⟩

lemma *INT-Int-distrib*: $(\bigcap i{\in}I.\ A\ i \cap B\ i) = (\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i)$
— Equivalent version
⟨*proof*⟩

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ B \cap A\ i)$
— Halmos, Naive Set Theory, page 35.
⟨*proof*⟩

lemma *Un-INT-distrib*: $B \cup (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ B \cup A\ i)$
⟨*proof*⟩

lemma *Int-UN-distrib2*: $(\bigcup i{\in}I.\ A\ i) \cap (\bigcup j{\in}J.\ B\ j) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A\ i \cap B\ j)$
⟨*proof*⟩

lemma *Un-INT-distrib2*: $(\bigcap i{\in}I.\ A\ i) \cup (\bigcap j{\in}J.\ B\ j) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A\ i \cup B\ j)$
⟨*proof*⟩

Bounded quantifiers.

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall x \in A \cup B.\ P\ x) = ((\forall x{\in}A.\ P\ x)\ \&\ (\forall x{\in}B.\ P\ x))$
⟨*proof*⟩

lemma *bex-Un*: $(\exists x \in A \cup B.\ P\ x) = ((\exists x{\in}A.\ P\ x)\ |\ (\exists x{\in}B.\ P\ x))$
⟨*proof*⟩

lemma *ball-UN*: $(\forall z \in \mathit{UNION}\ A\ B.\ P\ z) = (\forall x{\in}A.\ \forall z \in B\ x.\ P\ z)$
⟨*proof*⟩

lemma *bex-UN*: $(\exists z \in \mathit{UNION}\ A\ B.\ P\ z) = (\exists x{\in}A.\ \exists z{\in}B\ x.\ P\ z)$
⟨*proof*⟩

Set difference.

lemma *Diff-eq*: $A - B = A \cap (-B)$
⟨*proof*⟩

lemma *Diff-eq-empty-iff* [*simp*,*noatp*]: $(A - B = \{\}) = (A \subseteq B)$
⟨*proof*⟩

lemma *Diff-cancel* [*simp*]: $A - A = \{\}$
⟨*proof*⟩

lemma *Diff-idemp* [*simp*]: $(A - B) - B = A - (B::'a\ set)$
⟨*proof*⟩

lemma *Diff-triv*: $A \cap B = \{\} \Longrightarrow A - B = A$
⟨*proof*⟩

lemma *empty-Diff* [*simp*]: $\{\} - A = \{\}$
⟨*proof*⟩

lemma *Diff-empty* [*simp*]: $A - \{\} = A$
⟨*proof*⟩

lemma *Diff-UNIV* [*simp*]: $A - \mathit{UNIV} = \{\}$
⟨*proof*⟩

lemma *Diff-insert0* [*simp*,*noatp*]: $x \notin A \Longrightarrow A - \mathit{insert}\ x\ B = A - B$
⟨*proof*⟩

lemma *Diff-insert*: $A - \mathit{insert}\ a\ B = A - B - \{a\}$
— NOT SUITABLE FOR REWRITING since $\{a\} == \mathit{insert}\ a\ 0$
⟨*proof*⟩

lemma *Diff-insert2*: $A - \mathit{insert}\ a\ B = A - \{a\} - B$
— NOT SUITABLE FOR REWRITING since $\{a\} == \mathit{insert}\ a\ 0$
⟨*proof*⟩

lemma *insert-Diff-if*: $\mathit{insert}\ x\ A - B = (\mathit{if}\ x \in B\ \mathit{then}\ A - B\ \mathit{else}\ \mathit{insert}\ x\ (A - B))$
⟨*proof*⟩

lemma *insert-Diff1* [*simp*]: $x \in B \Longrightarrow \mathit{insert}\ x\ A - B = A - B$
⟨*proof*⟩

lemma *insert-Diff-single*[*simp*]: *insert a* $(A - \{a\}) =$ *insert a A*
⟨*proof*⟩

lemma *insert-Diff*: $a \in A \Longrightarrow$ *insert a* $(A - \{a\}) = A$
⟨*proof*⟩

lemma *Diff-insert-absorb*: $x \notin A \Longrightarrow (\textit{insert}\ x\ A) - \{x\} = A$
⟨*proof*⟩

lemma *Diff-disjoint* [*simp*]: $A \cap (B - A) = \{\}$
⟨*proof*⟩

lemma *Diff-partition*: $A \subseteq B \Longrightarrow A \cup (B - A) = B$
⟨*proof*⟩

lemma *double-diff*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow B - (C - A) = A$
⟨*proof*⟩

lemma *Un-Diff-cancel* [*simp*]: $A \cup (B - A) = A \cup B$
⟨*proof*⟩

lemma *Un-Diff-cancel2* [*simp*]: $(B - A) \cup A = B \cup A$
⟨*proof*⟩

lemma *Diff-Un*: $A - (B \cup C) = (A - B) \cap (A - C)$
⟨*proof*⟩

lemma *Diff-Int*: $A - (B \cap C) = (A - B) \cup (A - C)$
⟨*proof*⟩

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
⟨*proof*⟩

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
⟨*proof*⟩

lemma *Diff-Int-distrib*: $C \cap (A - B) = (C \cap A) - (C \cap B)$
⟨*proof*⟩

lemma *Diff-Int-distrib2*: $(A - B) \cap C = (A \cap C) - (B \cap C)$
⟨*proof*⟩

lemma *Diff-Compl* [*simp*]: $A - (-B) = A \cap B$
⟨*proof*⟩

lemma *Compl-Diff-eq* [*simp*]: $-(A - B) = -A \cup B$
⟨*proof*⟩

Quantification over type *bool*.

lemma *bool-induct*: $P\ True \Longrightarrow P\ False \Longrightarrow P\ x$
⟨*proof*⟩

lemma *all-bool-eq*: $(\forall b.\ P\ b) \longleftrightarrow P\ True \wedge P\ False$
⟨*proof*⟩

lemma *bool-contrapos*: $P\ x \Longrightarrow \neg\ P\ False \Longrightarrow P\ True$
⟨*proof*⟩

lemma *ex-bool-eq*: $(\exists b.\ P\ b) \longleftrightarrow P\ True \vee P\ False$
⟨*proof*⟩

lemma *Un-eq-UN*: $A \cup B = (\bigcup b.\ if\ b\ then\ A\ else\ B)$
⟨*proof*⟩

lemma *UN-bool-eq*: $(\bigcup b{::}bool.\ A\ b) = (A\ True \cup A\ False)$
⟨*proof*⟩

lemma *INT-bool-eq*: $(\bigcap b{::}bool.\ A\ b) = (A\ True \cap A\ False)$
⟨*proof*⟩

Pow

lemma *Pow-empty* [*simp*]: $Pow\ \{\} = \{\{\}\}$
⟨*proof*⟩

lemma *Pow-insert*: $Pow\ (insert\ a\ A) = Pow\ A \cup (insert\ a\ `\ Pow\ A)$
⟨*proof*⟩

lemma *Pow-Compl*: $Pow\ (-\ A) = \{-B \mid B.\ A \in Pow\ B\}$
⟨*proof*⟩

lemma *Pow-UNIV* [*simp*]: $Pow\ UNIV = UNIV$
⟨*proof*⟩

lemma *Un-Pow-subset*: $Pow\ A \cup Pow\ B \subseteq Pow\ (A \cup B)$
⟨*proof*⟩

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ Pow\ (B\ x)) \subseteq Pow\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *subset-Pow-Union*: $A \subseteq Pow\ (\bigcup A)$
⟨*proof*⟩

lemma *Union-Pow-eq* [*simp*]: $\bigcup(Pow\ A) = A$
⟨*proof*⟩

lemma *Pow-Int-eq* [*simp*]: $Pow\ (A \cap B) = Pow\ A \cap Pow\ B$
⟨*proof*⟩

lemma *Pow-INT-eq*: $Pow\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ Pow\ (B\ x))$
⟨*proof*⟩

Miscellany.

lemma *set-eq-subset*: $(A = B) = (A \subseteq B\ \&\ B \subseteq A)$
⟨*proof*⟩

lemma *subset-iff*: $(A \subseteq B) = (\forall t.\ t \in A \longrightarrow t \in B)$
⟨*proof*⟩

lemma *subset-iff-psubset-eq*: $(A \subseteq B) = ((A \subset B) \mid (A = B))$
⟨*proof*⟩

lemma *all-not-in-conv* [*simp*]: $(\forall x.\ x \notin A) = (A = \{\})$
⟨*proof*⟩

lemma *ex-in-conv*: $(\exists x.\ x \in A) = (A \neq \{\})$
⟨*proof*⟩

lemma *distinct-lemma*: $f\ x \neq f\ y \Longrightarrow x \neq y$
⟨*proof*⟩

Miniscoping: pushing in quantifiers and big Unions and Intersections.

lemma *UN-simps* [*simp*]:
$!!a\ B\ C.\ (UN\ x{:}C.\ insert\ a\ (B\ x)) = (if\ C{=}\{\}\ then\ \{\}\ else\ insert\ a\ (UN\ x{:}C.\ B\ x))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x\ Un\ B) = ((if\ C{=}\{\}\ then\ \{\}\ else\ (UN\ x{:}C.\ A\ x)\ Un\ B))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ Un\ B\ x) = ((if\ C{=}\{\}\ then\ \{\}\ else\ A\ Un\ (UN\ x{:}C.\ B\ x)))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x\ Int\ B) = ((UN\ x{:}C.\ A\ x)\ Int\ B)$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ Int\ B\ x) = (A\ Int\ (UN\ x{:}C.\ B\ x))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x - B) = ((UN\ x{:}C.\ A\ x) - B)$
$!!A\ B\ C.\ (UN\ x{:}C.\ A - B\ x) = (A - (INT\ x{:}C.\ B\ x))$
$!!A\ B.\ (UN\ x{:}\ Union\ A.\ B\ x) = (UN\ y{:}A.\ UN\ x{:}y.\ B\ x)$
$!!A\ B\ C.\ (UN\ z{:}\ UNION\ A\ B.\ C\ z) = (UN\ \ x{:}A.\ UN\ z{:}\ B(x).\ C\ z)$
$!!A\ B\ f.\ (UN\ x{:}f`A.\ B\ x) = (UN\ a{:}A.\ B\ (f\ a))$
⟨*proof*⟩

lemma *INT-simps* [*simp*]:
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x\ Int\ B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (INT\ x{:}C.\ A\ x)\ Int\ B)$
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ Int\ B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A\ Int\ (INT\ x{:}C.\ B\ x))$
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x - B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (INT\ x{:}C.\ A\ x) - B)$
$!!A\ B\ C.\ (INT\ x{:}C.\ A - B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A - (UN\ x{:}C.\ B\ x))$

!!$a\ B\ C.\ (INT\ x{:}C.\ insert\ a\ (B\ x)) = insert\ a\ (INT\ x{:}C.\ B\ x)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x\ Un\ B) = ((INT\ x{:}C.\ A\ x)\ Un\ B)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ Un\ B\ x) = (A\ Un\ (INT\ x{:}C.\ B\ x))$
!!$A\ B.\ (INT\ x{:}\ Union\ A.\ B\ x) = (INT\ y{:}A.\ INT\ x{:}y.\ B\ x)$
!!$A\ B\ C.\ (INT\ z{:}\ UNION\ A\ B.\ C\ z) = (INT\ x{:}A.\ INT\ z{:}\ B(x).\ C\ z)$
!!$A\ B\ f.\ (INT\ x{:}f`A.\ B\ x) = (INT\ a{:}A.\ B\ (f\ a))$
⟨*proof*⟩

lemma *ball-simps* [*simp*,*noatp*]:
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ x\ |\ Q) = ((ALL\ x{:}A.\ P\ x)\ |\ Q)$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ |\ Q\ x) = (P\ |\ (ALL\ x{:}A.\ Q\ x))$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ --{>}\ Q\ x) = (P\ --{>}\ (ALL\ x{:}A.\ Q\ x))$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ x\ --{>}\ Q) = ((EX\ x{:}A.\ P\ x)\ --{>}\ Q)$
!!$P.\ (ALL\ x{:}\{\}.\ P\ x) = True$
!!$P.\ (ALL\ x{:}UNIV.\ P\ x) = (ALL\ x.\ P\ x)$
!!$a\ B\ P.\ (ALL\ x{:}insert\ a\ B.\ P\ x) = (P\ a\ \&\ (ALL\ x{:}B.\ P\ x))$
!!$A\ P.\ (ALL\ x{:}Union\ A.\ P\ x) = (ALL\ y{:}A.\ ALL\ x{:}y.\ P\ x)$
!!$A\ B\ P.\ (ALL\ x{:}\ UNION\ A\ B.\ P\ x) = (ALL\ a{:}A.\ ALL\ x{:}\ B\ a.\ P\ x)$
!!$P\ Q.\ (ALL\ x{:}Collect\ Q.\ P\ x) = (ALL\ x.\ Q\ x\ --{>}\ P\ x)$
!!$A\ P\ f.\ (ALL\ x{:}f`A.\ P\ x) = (ALL\ x{:}A.\ P\ (f\ x))$
!!$A\ P.\ (\sim(ALL\ x{:}A.\ P\ x)) = (EX\ x{:}A.\ \sim P\ x)$
⟨*proof*⟩

lemma *bex-simps* [*simp*,*noatp*]:
!!$A\ P\ Q.\ (EX\ x{:}A.\ P\ x\ \&\ Q) = ((EX\ x{:}A.\ P\ x)\ \&\ Q)$
!!$A\ P\ Q.\ (EX\ x{:}A.\ P\ \&\ Q\ x) = (P\ \&\ (EX\ x{:}A.\ Q\ x))$
!!$P.\ (EX\ x{:}\{\}.\ P\ x) = False$
!!$P.\ (EX\ x{:}UNIV.\ P\ x) = (EX\ x.\ P\ x)$
!!$a\ B\ P.\ (EX\ x{:}insert\ a\ B.\ P\ x) = (P(a)\ |\ (EX\ x{:}B.\ P\ x))$
!!$A\ P.\ (EX\ x{:}Union\ A.\ P\ x) = (EX\ y{:}A.\ EX\ x{:}y.\ P\ x)$
!!$A\ B\ P.\ (EX\ x{:}\ UNION\ A\ B.\ P\ x) = (EX\ a{:}A.\ EX\ x{:}B\ a.\ P\ x)$
!!$P\ Q.\ (EX\ x{:}Collect\ Q.\ P\ x) = (EX\ x.\ Q\ x\ \&\ P\ x)$
!!$A\ P\ f.\ (EX\ x{:}f`A.\ P\ x) = (EX\ x{:}A.\ P\ (f\ x))$
!!$A\ P.\ (\sim(EX\ x{:}A.\ P\ x)) = (ALL\ x{:}A.\ \sim P\ x)$
⟨*proof*⟩

lemma *ball-conj-distrib*:
$(ALL\ x{:}A.\ P\ x\ \&\ Q\ x) = ((ALL\ x{:}A.\ P\ x)\ \&\ (ALL\ x{:}A.\ Q\ x))$
⟨*proof*⟩

lemma *bex-disj-distrib*:
$(EX\ x{:}A.\ P\ x\ |\ Q\ x) = ((EX\ x{:}A.\ P\ x)\ |\ (EX\ x{:}A.\ Q\ x))$
⟨*proof*⟩

Maxiscoping: pulling out big Unions and Intersections.

lemma *UN-extend-simps*:
!!$a\ B\ C.\ insert\ a\ (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ \{a\}\ else\ (UN\ x{:}C.\ insert\ a\ (B\ x)))$
!!$A\ B\ C.\ (UN\ x{:}C.\ A\ x)\ Un\ B = (if\ C{=}\{\}\ then\ B\ else\ (UN\ x{:}C.\ A\ x\ Un\ B))$

!!$A\ B\ C.\ A\ Un\ (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (UN\ x{:}C.\ A\ Un\ B\ x))$
!!$A\ B\ C.\ ((UN\ x{:}C.\ A\ x)\ Int\ B) = (UN\ x{:}C.\ A\ x\ Int\ B)$
!!$A\ B\ C.\ (A\ Int\ (UN\ x{:}C.\ B\ x)) = (UN\ x{:}C.\ A\ Int\ B\ x)$
!!$A\ B\ C.\ ((UN\ x{:}C.\ A\ x) - B) = (UN\ x{:}C.\ A\ x - B)$
!!$A\ B\ C.\ (A - (INT\ x{:}C.\ B\ x)) = (UN\ x{:}C.\ A - B\ x)$
!!$A\ B.\ (UN\ y{:}A.\ UN\ x{:}y.\ B\ x) = (UN\ x{:}\ Union\ A.\ B\ x)$
!!$A\ B\ C.\ (UN\ \ x{:}A.\ UN\ z{:}\ B(x).\ C\ z) = (UN\ z{:}\ UNION\ A\ B.\ C\ z)$
!!$A\ B\ f.\ (UN\ a{:}A.\ B\ (f\ a)) = (UN\ x{:}f`A.\ B\ x)$
⟨*proof*⟩

lemma *INT-extend-simps*:
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x)\ Int\ B = (if\ C{=}\{\}\ then\ B\ else\ (INT\ x{:}C.\ A\ x\ Int\ B))$
!!$A\ B\ C.\ A\ Int\ (INT\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (INT\ x{:}C.\ A\ Int\ B\ x))$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x) - B = (if\ C{=}\{\}\ then\ UNIV{-}B\ else\ (INT\ x{:}C.\ A\ x - B))$
!!$A\ B\ C.\ A - (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (INT\ x{:}C.\ A - B\ x))$
!!$a\ B\ C.\ insert\ a\ (INT\ x{:}C.\ B\ x) = (INT\ x{:}C.\ insert\ a\ (B\ x))$
!!$A\ B\ C.\ ((INT\ x{:}C.\ A\ x)\ Un\ B) = (INT\ x{:}C.\ A\ x\ Un\ B)$
!!$A\ B\ C.\ A\ Un\ (INT\ x{:}C.\ B\ x) = (INT\ x{:}C.\ A\ Un\ B\ x)$
!!$A\ B.\ (INT\ y{:}A.\ INT\ x{:}y.\ B\ x) = (INT\ x{:}\ Union\ A.\ B\ x)$
!!$A\ B\ C.\ (INT\ x{:}A.\ INT\ z{:}\ B(x).\ C\ z) = (INT\ z{:}\ UNION\ A\ B.\ C\ z)$
!!$A\ B\ f.\ (INT\ a{:}A.\ B\ (f\ a)) = (INT\ x{:}f`A.\ B\ x)$
⟨*proof*⟩

5.5.3 Monotonicity of various operations

lemma *image-mono*: $A \subseteq B \Longrightarrow f`A \subseteq f`B$
⟨*proof*⟩

lemma *Pow-mono*: $A \subseteq B \Longrightarrow Pow\ A \subseteq Pow\ B$
⟨*proof*⟩

lemma *Union-mono*: $A \subseteq B \Longrightarrow \bigcup A \subseteq \bigcup B$
⟨*proof*⟩

lemma *Inter-anti-mono*: $B \subseteq A \Longrightarrow \bigcap A \subseteq \bigcap B$
⟨*proof*⟩

lemma *UN-mono*:
$A \subseteq B \Longrightarrow (!!x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow (\bigcup x{\in}A.\ f\ x) \subseteq (\bigcup x{\in}B.\ g\ x)$
⟨*proof*⟩

lemma *INT-anti-mono*:
$B \subseteq A \Longrightarrow (!!x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow (\bigcap x{\in}A.\ f\ x) \subseteq (\bigcap x{\in}A.\ g\ x)$
— The last inclusion is POSITIVE!
⟨*proof*⟩

lemma *insert-mono*: $C \subseteq D \Longrightarrow \mathit{insert}\ a\ C \subseteq \mathit{insert}\ a\ D$
⟨*proof*⟩

lemma *Un-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cup B \subseteq C \cup D$
⟨*proof*⟩

lemma *Int-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cap B \subseteq C \cap D$
⟨*proof*⟩

lemma *Diff-mono*: $A \subseteq C \Longrightarrow D \subseteq B \Longrightarrow A - B \subseteq C - D$
⟨*proof*⟩

lemma *Compl-anti-mono*: $A \subseteq B \Longrightarrow -B \subseteq -A$
⟨*proof*⟩

Monotonicity of implications.

lemma *in-mono*: $A \subseteq B \Longrightarrow x \in A \longrightarrow x \in B$
⟨*proof*⟩

lemma *conj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1\ \&\ P2) \longrightarrow (Q1\ \&\ Q2)$
⟨*proof*⟩

lemma *disj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \mid P2) \longrightarrow (Q1 \mid Q2)$
⟨*proof*⟩

lemma *imp-mono*: $Q1 \longrightarrow P1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \longrightarrow P2) \longrightarrow (Q1 \longrightarrow Q2)$
⟨*proof*⟩

lemma *imp-refl*: $P \longrightarrow P$ ⟨*proof*⟩

lemma *ex-mono*: $(!!x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (EX\ x.\ P\ x) \longrightarrow (EX\ x.\ Q\ x)$
⟨*proof*⟩

lemma *all-mono*: $(!!x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (ALL\ x.\ P\ x) \longrightarrow (ALL\ x.\ Q\ x)$
⟨*proof*⟩

lemma *Collect-mono*: $(!!x.\ P\ x \longrightarrow Q\ x) \Longrightarrow \mathit{Collect}\ P \subseteq \mathit{Collect}\ Q$
⟨*proof*⟩

lemma *Int-Collect-mono*:
$A \subseteq B \Longrightarrow (!!x.\ x \in A \Longrightarrow P\ x \longrightarrow Q\ x) \Longrightarrow A \cap \mathit{Collect}\ P \subseteq B \cap \mathit{Collect}\ Q$
⟨*proof*⟩

lemmas *basic-monos* =
subset-refl imp-refl disj-mono conj-mono

ex-mono Collect-mono in-mono

lemma *eq-to-mono*: $a = b \Longrightarrow c = d \Longrightarrow b \dashrightarrow d \Longrightarrow a \dashrightarrow c$
⟨*proof*⟩

lemma *eq-to-mono2*: $a = b \Longrightarrow c = d \Longrightarrow {}^{\sim} b \dashrightarrow {}^{\sim} d \Longrightarrow {}^{\sim} a \dashrightarrow {}^{\sim} c$
⟨*proof*⟩

5.6 Inverse image of a function

constdefs
vimage :: $('a \Rightarrow 'b) \Rightarrow 'b\ set \Rightarrow 'a\ set$ (**infixr** $-`$ *90*)
[*code del*]: $f\ -`\ B == \{x.\ f\ x : B\}$

5.6.1 Basic rules

lemma *vimage-eq* [*simp*]: $(a : f\ -`\ B) = (f\ a : B)$
⟨*proof*⟩

lemma *vimage-singleton-eq*: $(a : f\ -`\ \{b\}) = (f\ a = b)$
⟨*proof*⟩

lemma *vimageI* [*intro*]: $f\ a = b \Longrightarrow b{:}B \Longrightarrow a : f\ -`\ B$
⟨*proof*⟩

lemma *vimageI2*: $f\ a : A \Longrightarrow a : f\ -`\ A$
⟨*proof*⟩

lemma *vimageE* [*elim*!]: $a{:}\ f\ -`\ B \Longrightarrow (!!x.\ f\ a = x \Longrightarrow x{:}B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *vimageD*: $a : f\ -`\ A \Longrightarrow f\ a : A$
⟨*proof*⟩

5.6.2 Equations

lemma *vimage-empty* [*simp*]: $f\ -`\ \{\} = \{\}$
⟨*proof*⟩

lemma *vimage-Compl*: $f\ -`\ (-A) = -(f\ -`\ A)$
⟨*proof*⟩

lemma *vimage-Un* [*simp*]: $f\ -`\ (A\ Un\ B) = (f\ -`\ A)\ Un\ (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-Int* [*simp*]: $f\ -`\ (A\ Int\ B) = (f\ -`\ A)\ Int\ (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-Union*: $f\ -`\ (Union\ A) = (UN\ X{:}A.\ f\ -`\ X)$

⟨*proof*⟩

lemma *vimage-UN*: $f-`(UN\ x{:}A.\ B\ x) = (UN\ x{:}A.\ f\ -`\ B\ x)$
⟨*proof*⟩

lemma *vimage-INT*: $f-`(INT\ x{:}A.\ B\ x) = (INT\ x{:}A.\ f\ -`\ B\ x)$
⟨*proof*⟩

lemma *vimage-Collect-eq* [*simp*]: $f\ -`\ Collect\ P = \{y.\ P\ (f\ y)\}$
⟨*proof*⟩

lemma *vimage-Collect*: $(!!x.\ P\ (f\ x) = Q\ x) ==> f\ -`\ (Collect\ P) = Collect\ Q$
⟨*proof*⟩

lemma *vimage-insert*: $f-`(insert\ a\ B) = (f-`\{a\})\ Un\ (f-`B)$
— NOT suitable for rewriting because of the recurrence of $\{a\}$.
⟨*proof*⟩

lemma *vimage-Diff*: $f\ -`\ (A - B) = (f\ -`\ A) - (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-UNIV* [*simp*]: $f\ -`\ UNIV = UNIV$
⟨*proof*⟩

lemma *vimage-eq-UN*: $f-`B = (UN\ y{:}\ B.\ f-`\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

lemma *vimage-mono*: $A \subseteq B ==> f\ -`\ A \subseteq f\ -`\ B$
— monotonicity
⟨*proof*⟩

lemma *vimage-image-eq* [*noatp*]: $f\ -`\ (f\ `\ A) = \{y.\ EX\ x{:}A.\ f\ x = f\ y\}$
⟨*proof*⟩

lemma *image-vimage-subset*: $f\ `\ (f\ -`\ A) <= A$
⟨*proof*⟩

lemma *image-vimage-eq* [*simp*]: $f\ `\ (f\ -`\ A) = A\ Int\ range\ f$
⟨*proof*⟩

lemma *image-Int-subset*: $f`(A\ Int\ B) <= f`A\ Int\ f`B$
⟨*proof*⟩

lemma *image-diff-subset*: $f`A - f`B <= f`(A - B)$
⟨*proof*⟩

lemma *image-UN*: $(f\ `\ (UNION\ A\ B)) = (UN\ x{:}A.(f\ `\ (B\ x)))$
⟨*proof*⟩

5.7 Getting the Contents of a Singleton Set

definition *contents* :: $'a\ set \Rightarrow 'a$ **where**
[*code del*]: *contents* $X = (THE\ x.\ X = \{x\})$

lemma *contents-eq* [*simp*]: *contents* $\{x\} = x$
⟨*proof*⟩

5.8 Transitivity rules for calculational reasoning

lemma *set-rev-mp*: $x{:}A \Longrightarrow A \subseteq B \Longrightarrow x{:}B$
⟨*proof*⟩

lemma *set-mp*: $A \subseteq B \Longrightarrow x{:}A \Longrightarrow x{:}B$
⟨*proof*⟩

lemmas *basic-trans-rules* [*trans*] =
order-trans-rules set-rev-mp set-mp

5.9 Least value operator

lemma *Least-mono*:
mono $(f{::}'a{::}order \Rightarrow 'b{::}order) \Longrightarrow EX\ x{:}S.\ ALL\ y{:}S.\ x <= y$
$\Longrightarrow (LEAST\ y.\ y : f\ `\ S) = f\ (LEAST\ x.\ x : S)$
— Courtesy of Stephan Merz
⟨*proof*⟩

5.10 Rudimentary code generation

lemma *empty-code* [*code*]: $\{\}\ x \longleftrightarrow False$
⟨*proof*⟩

lemma *UNIV-code* [*code*]: $UNIV\ x \longleftrightarrow True$
⟨*proof*⟩

lemma *insert-code* [*code*]: *insert* $y\ A\ x \longleftrightarrow y = x \vee A\ x$
⟨*proof*⟩

lemma *inter-code* [*code*]: $(A \cap B)\ x \longleftrightarrow A\ x \wedge B\ x$
⟨*proof*⟩

lemma *union-code* [*code*]: $(A \cup B)\ x \longleftrightarrow A\ x \vee B\ x$
⟨*proof*⟩

lemma *vimage-code* [*code*]: $(f\ -`\ A)\ x = A\ (f\ x)$
⟨*proof*⟩

5.11 Complete lattices

notation

less-eq (**infix** $\sqsubseteq$ *50*) **and**
less (**infix** $\sqsubset$ *50*) **and**
inf (**infixl** $\sqcap$ *70*) **and**
sup (**infixl** $\sqcup$ *65*)

class *complete-lattice* = *lattice* + *bot* + *top* +
fixes *Inf* :: *'a set* $\Rightarrow$ *'a* ($\bigsqcap$- [*900*] *900*)
and *Sup* :: *'a set* $\Rightarrow$ *'a* ($\bigsqcup$- [*900*] *900*)
assumes *Inf-lower*: $x \in A \Longrightarrow \bigsqcap A \sqsubseteq x$
and *Inf-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x) \Longrightarrow z \sqsubseteq \bigsqcap A$
assumes *Sup-upper*: $x \in A \Longrightarrow x \sqsubseteq \bigsqcup A$
and *Sup-least*: $(\bigwedge x.\ x \in A \Longrightarrow x \sqsubseteq z) \Longrightarrow \bigsqcup A \sqsubseteq z$
begin

lemma *Inf-Sup*: $\bigsqcap A = \bigsqcup \{b.\ \forall a \in A.\ b \leq a\}$
⟨*proof*⟩

lemma *Sup-Inf*: $\bigsqcup A = \bigsqcap \{b.\ \forall a \in A.\ a \leq b\}$
⟨*proof*⟩

lemma *Inf-Univ*: $\bigsqcap UNIV = \bigsqcup \{\}$
⟨*proof*⟩

lemma *Sup-Univ*: $\bigsqcup UNIV = \bigsqcap \{\}$
⟨*proof*⟩

lemma *Inf-insert*: $\bigsqcap insert\ a\ A = a \sqcap \bigsqcap A$
⟨*proof*⟩

lemma *Sup-insert*: $\bigsqcup insert\ a\ A = a \sqcup \bigsqcup A$
⟨*proof*⟩

lemma *Inf-singleton* [*simp*]:
$\bigsqcap \{a\} = a$
⟨*proof*⟩

lemma *Sup-singleton* [*simp*]:
$\bigsqcup \{a\} = a$
⟨*proof*⟩

lemma *Inf-insert-simp*:
$\bigsqcap insert\ a\ A = (if\ A = \{\}\ then\ a\ else\ a \sqcap \bigsqcap A)$
⟨*proof*⟩

lemma *Sup-insert-simp*:
$\bigsqcup insert\ a\ A = (if\ A = \{\}\ then\ a\ else\ a \sqcup \bigsqcup A)$
⟨*proof*⟩

lemma *Inf-binary*:

$\bigsqcap\{a, b\} = a \sqcap b$
⟨*proof*⟩

lemma *Sup-binary*:
$\bigsqcup\{a, b\} = a \sqcup b$
⟨*proof*⟩

lemma *bot-def*:
$bot = \bigsqcup\{\}$
⟨*proof*⟩

lemma *top-def*:
$top = \bigsqcap\{\}$
⟨*proof*⟩

lemma *sup-bot* [*simp*]:
$x \sqcup bot = x$
⟨*proof*⟩

lemma *inf-top* [*simp*]:
$x \sqcap top = x$
⟨*proof*⟩

definition *SUPR* :: $'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$ **where**
$SUPR\ A\ f == \bigsqcup\ (f\ `\ A)$

definition *INFI* :: $'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$ **where**
$INFI\ A\ f == \bigsqcap\ (f\ `\ A)$

end

syntax
-SUP1 :: *pttrns* => $'b$ => $'b$ ((*3SUP -./ -*) [*0*, *10*] *10*)
-SUP :: *pttrn* => $'a\ set$ => $'b$ => $'b$ ((*3SUP -:-./ -*) [*0*, *10*] *10*)
-INF1 :: *pttrns* => $'b$ => $'b$ ((*3INF -./ -*) [*0*, *10*] *10*)
-INF :: *pttrn* => $'a\ set$ => $'b$ => $'b$ ((*3INF -:-./ -*) [*0*, *10*] *10*)

translations
SUP x y. B == *SUP x. SUP y. B*
SUP x. B == *CONST SUPR CONST UNIV* (%*x. B*)
SUP x. B == *SUP x*:*CONST UNIV. B*
SUP x:*A. B* == *CONST SUPR A* (%*x. B*)
INF x y. B == *INF x. INF y. B*
INF x. B == *CONST INFI CONST UNIV* (%*x. B*)
INF x. B == *INF x*:*CONST UNIV. B*
INF x:*A. B* == *CONST INFI A* (%*x. B*)

⟨*ML*⟩

context *complete-lattice*
begin

lemma *le-SUPI*: $i : A \Longrightarrow M\ i \leq (SUP\ i{:}A.\ M\ i)$
⟨*proof*⟩

lemma *SUP-leI*: $(\bigwedge i.\ i : A \Longrightarrow M\ i \leq u) \Longrightarrow (SUP\ i{:}A.\ M\ i) \leq u$
⟨*proof*⟩

lemma *INF-leI*: $i : A \Longrightarrow (INF\ i{:}A.\ M\ i) \leq M\ i$
⟨*proof*⟩

lemma *le-INFI*: $(\bigwedge i.\ i : A \Longrightarrow u \leq M\ i) \Longrightarrow u \leq (INF\ i{:}A.\ M\ i)$
⟨*proof*⟩

lemma *SUP-const*[*simp*]: $A \neq \{\} \Longrightarrow (SUP\ i{:}A.\ M) = M$
⟨*proof*⟩

lemma *INF-const*[*simp*]: $A \neq \{\} \Longrightarrow (INF\ i{:}A.\ M) = M$
⟨*proof*⟩

end

5.12 Bool as complete lattice

instantiation *bool* :: *complete-lattice*
begin

definition
Inf-bool-def: $\bigsqcap A \longleftrightarrow (\forall x{\in}A.\ x)$

definition
Sup-bool-def: $\bigsqcup A \longleftrightarrow (\exists x{\in}A.\ x)$

instance
⟨*proof*⟩

end

lemma *Inf-empty-bool* [*simp*]:
$\bigsqcap \{\}$
⟨*proof*⟩

lemma *not-Sup-empty-bool* [*simp*]:
$\neg \bigsqcup \{\}$
⟨*proof*⟩

5.13 Fun as complete lattice

instantiation *fun* :: (*type*, *complete-lattice*) *complete-lattice*
begin

definition
Inf-fun-def [*code del*]: $\bigsqcap A = (\lambda x.\ \bigsqcap \{y.\ \exists f{\in}A.\ y = f\ x\})$

definition
Sup-fun-def [*code del*]: $\bigsqcup A = (\lambda x.\ \bigsqcup \{y.\ \exists f{\in}A.\ y = f\ x\})$

instance
⟨*proof*⟩

end

lemma *Inf-empty-fun*:
$\bigsqcap \{\} = (\lambda\text{-}.\ \bigsqcap \{\})$
⟨*proof*⟩

lemma *Sup-empty-fun*:
$\bigsqcup \{\} = (\lambda\text{-}.\ \bigsqcup \{\})$
⟨*proof*⟩

5.14 Set as lattice

lemma *inf-set-eq*: $A \sqcap B = A \cap B$
⟨*proof*⟩

lemma *sup-set-eq*: $A \sqcup B = A \cup B$
⟨*proof*⟩

lemma *mono-Int*: $mono\ f \Longrightarrow f\ (A \cap B) \subseteq f\ A \cap f\ B$
⟨*proof*⟩

lemma *mono-Un*: $mono\ f \Longrightarrow f\ A \cup f\ B \subseteq f\ (A \cup B)$
⟨*proof*⟩

lemma *Inf-set-eq*: $\bigsqcap S = \bigcap S$
⟨*proof*⟩

lemma *Sup-set-eq*: $\bigsqcup S = \bigcup S$
⟨*proof*⟩

lemma *top-set-eq*: $top = UNIV$
⟨*proof*⟩

lemma *bot-set-eq*: $bot = \{\}$
⟨*proof*⟩

no-notation
less-eq (**infix** $\sqsubseteq$ *50*) **and**
less (**infix** $\sqsubset$ *50*) **and**
inf (**infixl** $\sqcap$ *70*) **and**
sup (**infixl** $\sqcup$ *65*) **and**
Inf ($\bigsqcap$ - [*900*] *900*) **and**
Sup ($\bigsqcup$ - [*900*] *900*)

5.15 Misc theorem and ML bindings

lemmas *equalityI* = *subset-antisym*
lemmas *mem-simps* =
insert-iff empty-iff Un-iff Int-iff Compl-iff Diff-iff
mem-Collect-eq UN-iff Union-iff INT-iff Inter-iff
— Each of these has ALREADY been added [*simp*] above.

⟨*ML*⟩

end

6 Typedef: HOL type definitions

theory *Typedef*
imports *Set*
uses
(*Tools/typedef-package.ML*)
(*Tools/typecopy-package.ML*)
(*Tools/typedef-codegen.ML*)
begin

⟨*ML*⟩

locale *type-definition* =
fixes *Rep* **and** *Abs* **and** *A*
assumes *Rep*: *Rep x* ∈ *A*
and *Rep-inverse*: *Abs* (*Rep x*) = *x*
and *Abs-inverse*: *y* ∈ *A* ==> *Rep* (*Abs y*) = *y*
— This will be axiomatized for each typedef!
begin

lemma *Rep-inject*:
(*Rep x* = *Rep y*) = (*x* = *y*)
⟨*proof*⟩

lemma *Abs-inject*:
assumes *x*: *x* ∈ *A* **and** *y*: *y* ∈ *A*
shows (*Abs x* = *Abs y*) = (*x* = *y*)
⟨*proof*⟩

lemma *Rep-cases* [*cases set*]:
  assumes *y*: $y \in A$
    and *hyp*: $!!x.\ y = Rep\ x ==> P$
  shows P
⟨*proof*⟩

lemma *Abs-cases* [*cases type*]:
  assumes *r*: $!!y.\ x = Abs\ y ==> y \in A ==> P$
  shows P
⟨*proof*⟩

lemma *Rep-induct* [*induct set*]:
  assumes *y*: $y \in A$
    and *hyp*: $!!x.\ P\ (Rep\ x)$
  shows $P\ y$
⟨*proof*⟩

lemma *Abs-induct* [*induct type*]:
  assumes *r*: $!!y.\ y \in A ==> P\ (Abs\ y)$
  shows $P\ x$
⟨*proof*⟩

lemma *Rep-range*: $range\ Rep = A$
⟨*proof*⟩

lemma *Abs-image*: $Abs\ `\ A = UNIV$
⟨*proof*⟩

end

⟨*ML*⟩

end

7 Fun: Notions about functions

theory *Fun*
imports *Set*
begin

As a simplification rule, it replaces all function equalities by first-order equalities.

lemma *expand-fun-eq*: $f = g \longleftrightarrow (\forall x.\ f\ x = g\ x)$
⟨*proof*⟩

lemma *apply-inverse*:
  $f\ x = u \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow g\ (f\ x) = x) \Longrightarrow P\ x \Longrightarrow x = g\ u$

⟨*proof*⟩

7.1 The Identity Function *id*

definition
id :: *'a* ⇒ *'a*
where
id = (λ*x*. *x*)

lemma *id-apply* [*simp*]: *id x* = *x*
⟨*proof*⟩

lemma *image-ident* [*simp*]: (%*x*. *x*) ‘ *Y* = *Y*
⟨*proof*⟩

lemma *image-id* [*simp*]: *id* ‘ *Y* = *Y*
⟨*proof*⟩

lemma *vimage-ident* [*simp*]: (%*x*. *x*) −‘ *Y* = *Y*
⟨*proof*⟩

lemma *vimage-id* [*simp*]: *id* −‘ *A* = *A*
⟨*proof*⟩

7.2 The Composition Operator *f* ∘ *g*

definition
comp :: (*'b* ⇒ *'c*) ⇒ (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'c* (**infixl** *o* 55)
where
f o g = (λ*x*. *f* (*g x*))

notation (*xsymbols*)
comp (**infixl** ∘ 55)

notation (*HTML* **output**)
comp (**infixl** ∘ 55)

compatibility

lemmas *o-def* = *comp-def*

lemma *o-apply* [*simp*]: (*f o g*) *x* = *f* (*g x*)
⟨*proof*⟩

lemma *o-assoc*: *f o* (*g o h*) = *f o g o h*
⟨*proof*⟩

lemma *id-o* [*simp*]: *id o g* = *g*
⟨*proof*⟩

lemma *o-id* [*simp*]: $f\ o\ id = f$
⟨*proof*⟩

lemma *image-compose*: $(f\ o\ g)\ `\ r = f`(g`r)$
⟨*proof*⟩

lemma *UN-o*: $UNION\ A\ (g\ o\ f) = UNION\ (f`A)\ g$
⟨*proof*⟩

7.3 The Forward Composition Operator *fcomp*

definition
$fcomp :: ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'c$ (**infixl** $o>\ 60$)
where
$f\ o>\ g = (\lambda x.\ g\ (f\ x))$

lemma *fcomp-apply*: $(f\ o>\ g)\ x = g\ (f\ x)$
⟨*proof*⟩

lemma *fcomp-assoc*: $(f\ o>\ g)\ o>\ h = f\ o>\ (g\ o>\ h)$
⟨*proof*⟩

lemma *id-fcomp* [*simp*]: $id\ o>\ g = g$
⟨*proof*⟩

lemma *fcomp-id* [*simp*]: $f\ o>\ id = f$
⟨*proof*⟩

no-notation *fcomp* (**infixl** $o>\ 60$)

7.4 Injectivity and Surjectivity

constdefs
$inj\text{-}on :: ['a => 'b, 'a\ set] => bool$ — injective
$inj\text{-}on\ f\ A == !\ x{:}A.\ !\ y{:}A.\ f(x){=}f(y)\ --> x{=}y$

A common special case: functions injective over the entire domain type.

abbreviation
$inj\ f == inj\text{-}on\ f\ UNIV$

definition
$bij\text{-}betw :: ('a => 'b) => 'a\ set => 'b\ set => bool$ **where** — bijective
[*code del*]: $bij\text{-}betw\ f\ A\ B \longleftrightarrow inj\text{-}on\ f\ A\ \&\ f\ `\ A = B$

constdefs
$surj :: ('a => 'b) => bool$
$surj\ f == !\ y.\ ?\ x.\ y{=}f(x)$

$bij :: ('a => 'b) => bool$

bij f == inj f & surj f

lemma *injI*:
assumes $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
shows *inj f*
⟨*proof*⟩

For Proofs in *Tools/datatype-rep-proofs*

lemma *datatype-injI*:
(!! *x*. *ALL y*. *f*(*x*) = *f*(*y*) −−> *x*=*y*) ==> *inj*(*f*)
⟨*proof*⟩

theorem *range-ex1-eq*: *inj f* $\Longrightarrow$ *b* : *range f* = (*EX*! *x*. *b* = *f x*)
⟨*proof*⟩

lemma *injD*: [| *inj*(*f*); *f*(*x*) = *f*(*y*) |] ==> *x*=*y*
⟨*proof*⟩

lemma *inj-eq*: *inj*(*f*) ==> (*f*(*x*) = *f*(*y*)) = (*x*=*y*)
⟨*proof*⟩

lemma *inj-on-id*[*simp*]: *inj-on id A*
⟨*proof*⟩

lemma *inj-on-id2*[*simp*]: *inj-on* (%*x*. *x*) *A*
⟨*proof*⟩

lemma *surj-id*[*simp*]: *surj id*
⟨*proof*⟩

lemma *bij-id*[*simp*]: *bij id*
⟨*proof*⟩

lemma *inj-onI*:
(!! *x y*. [| *x*:*A*; *y*:*A*; *f*(*x*) = *f*(*y*) |] ==> *x*=*y*) ==> *inj-on f A*
⟨*proof*⟩

lemma *inj-on-inverseI*: (!!*x*. *x*:*A* ==> *g*(*f*(*x*)) = *x*) ==> *inj-on f A*
⟨*proof*⟩

lemma *inj-onD*: [| *inj-on f A*; *f*(*x*)=*f*(*y*); *x*:*A*; *y*:*A* |] ==> *x*=*y*
⟨*proof*⟩

lemma *inj-on-iff*: [| *inj-on f A*; *x*:*A*; *y*:*A* |] ==> (*f*(*x*)=*f*(*y*)) = (*x*=*y*)
⟨*proof*⟩

lemma *comp-inj-on*:
[| *inj-on f A*; *inj-on g* (*f*‘*A*) |] ==> *inj-on* (*g o f*) *A*

⟨*proof*⟩

lemma *inj-on-imageI*: *inj-on* (*g o f*) *A* ⟹ *inj-on g* (*f ' A*)
⟨*proof*⟩

lemma *inj-on-image-iff*: ⟦ *ALL x*:*A*. *ALL y*:*A*. (*g*(*f x*) = *g*(*f y*)) = (*g x* = *g y*);
inj-on f A ⟧ ⟹ *inj-on g* (*f ' A*) = *inj-on g A*
⟨*proof*⟩

lemma *inj-on-contraD*: [| *inj-on f A*; ~*x*=*y*; *x*:*A*; *y*:*A* |] ==> ~ *f*(*x*)=*f*(*y*)
⟨*proof*⟩

lemma *inj-singleton*: *inj* (%*s*. {*s*})
⟨*proof*⟩

lemma *inj-on-empty*[*iff*]: *inj-on f* {}
⟨*proof*⟩

lemma *subset-inj-on*: [| *inj-on f B*; *A* <= *B* |] ==> *inj-on f A*
⟨*proof*⟩

lemma *inj-on-Un*:
inj-on f (*A Un B*) =
(*inj-on f A* & *inj-on f B* & *f*'(*A*−*B*) *Int f*'(*B*−*A*) = {})
⟨*proof*⟩

lemma *inj-on-insert*[*iff*]:
inj-on f (*insert a A*) = (*inj-on f A* & *f a* ~: *f*'(*A*−{*a*}))
⟨*proof*⟩

lemma *inj-on-diff*: *inj-on f A* ==> *inj-on f* (*A*−*B*)
⟨*proof*⟩

lemma *surjI*: (!! *x*. *g*(*f x*) = *x*) ==> *surj g*
⟨*proof*⟩

lemma *surj-range*: *surj f* ==> *range f* = *UNIV*
⟨*proof*⟩

lemma *surjD*: *surj f* ==> *EX x*. *y* = *f x*
⟨*proof*⟩

lemma *surjE*: *surj f* ==> (!!*x*. *y* = *f x* ==> *C*) ==> *C*
⟨*proof*⟩

lemma *comp-surj*: [| *surj f*; *surj g* |] ==> *surj* (*g o f*)
⟨*proof*⟩

lemma *bijI*: [| *inj f*; *surj f* |] ==> *bij f*

⟨*proof*⟩

lemma *bij-is-inj*: *bij f ==> inj f*
⟨*proof*⟩

lemma *bij-is-surj*: *bij f ==> surj f*
⟨*proof*⟩

lemma *bij-betw-imp-inj-on*: *bij-betw f A B ⟹ inj-on f A*
⟨*proof*⟩

lemma *bij-betw-inv*: **assumes** *bij-betw f A B* **shows** *EX g. bij-betw g B A*
⟨*proof*⟩

lemma *surj-image-vimage-eq*: *surj f ==> f ‘ (f −‘ A) = A*
⟨*proof*⟩

lemma *inj-vimage-image-eq*: *inj f ==> f −‘ (f ‘ A) = A*
⟨*proof*⟩

lemma *vimage-subsetD*: *surj f ==> f −‘ B <= A ==> B <= f ‘ A*
⟨*proof*⟩

lemma *vimage-subsetI*: *inj f ==> B <= f ‘ A ==> f −‘ B <= A*
⟨*proof*⟩

lemma *vimage-subset-eq*: *bij f ==> (f −‘ B <= A) = (B <= f ‘ A)*
⟨*proof*⟩

lemma *inj-on-image-Int*:
[| *inj-on f C*; *A<=C*; *B<=C* |] *==> f‘(A Int B) = f‘A Int f‘B*
⟨*proof*⟩

lemma *inj-on-image-set-diff*:
[| *inj-on f C*; *A<=C*; *B<=C* |] *==> f‘(A−B) = f‘A − f‘B*
⟨*proof*⟩

lemma *image-Int*: *inj f ==> f‘(A Int B) = f‘A Int f‘B*
⟨*proof*⟩

lemma *image-set-diff*: *inj f ==> f‘(A−B) = f‘A − f‘B*
⟨*proof*⟩

lemma *inj-image-mem-iff*: *inj f ==> (f a : f‘A) = (a : A)*
⟨*proof*⟩

lemma *inj-image-subset-iff*: *inj f ==> (f‘A <= f‘B) = (A<=B)*
⟨*proof*⟩

lemma *inj-image-eq-iff*: *inj f ==> (f`A = f`B) = (A = B)*
⟨*proof*⟩

lemma *image-INT*:
[| inj-on f C; ALL x:A. B x <= C; j:A |]
==> f ` (INTER A B) = (INT x:A. f ` B x)
⟨*proof*⟩

lemma *bij-image-INT*: *bij f ==> f ` (INTER A B) = (INT x:A. f ` B x)*
⟨*proof*⟩

lemma *surj-Compl-image-subset*: *surj f ==> -(f`A) <= f`(-A)*
⟨*proof*⟩

lemma *inj-image-Compl-subset*: *inj f ==> f`(-A) <= -(f`A)*
⟨*proof*⟩

lemma *bij-image-Compl-eq*: *bij f ==> f`(-A) = -(f`A)*
⟨*proof*⟩

7.5 Function Updating

constdefs
fun-upd :: *('a => 'b) => 'a => 'b => ('a => 'b)*
fun-upd f a b == % x. if x=a then b else f x

nonterminals
updbinds updbind
syntax
-updbind :: *['a, 'a] => updbind* (*(2- :=/ -)*)
:: *updbind => updbinds* (-)
-updbinds:: *[updbind, updbinds] => updbinds* (-,/ -)
-Update :: *['a, updbinds] => 'a* (*-/'((-)') [1000,0] 900*)

translations
-Update f (-updbinds b bs) == *-Update (-Update f b) bs*
f(x:=y) == *fun-upd f x y*

lemma *fun-upd-idem-iff*: *(f(x:=y) = f) = (f x = y)*
⟨*proof*⟩

lemmas *fun-upd-idem = fun-upd-idem-iff [THEN iffD2, standard]*

lemmas *fun-upd-triv* = *refl* [*THEN fun-upd-idem*]
declare *fun-upd-triv* [*iff*]

lemma *fun-upd-apply* [*simp*]: $(f(x:=y))z = (\mathit{if}\ z{=}x\ \mathit{then}\ y\ \mathit{else}\ f\ z)$
⟨*proof*⟩

lemma *fun-upd-same*: $(f(x:=y))\ x = y$
⟨*proof*⟩

lemma *fun-upd-other*: $z^{\sim}{=}x \Longrightarrow (f(x:=y))\ z = f\ z$
⟨*proof*⟩

lemma *fun-upd-upd* [*simp*]: $f(x:=y,x:=z) = f(x:=z)$
⟨*proof*⟩

lemma *fun-upd-twist*: $a\ {}^{\sim}{=}\ c \Longrightarrow (m(a:=b))(c:=d) = (m(c:=d))(a:=b)$
⟨*proof*⟩

lemma *inj-on-fun-updI*: $[\![\ \mathit{inj\text{-}on}\ f\ A;\ y \notin f`A\ ]\!] \Longrightarrow \mathit{inj\text{-}on}\ (f(x:=y))\ A$
⟨*proof*⟩

lemma *fun-upd-image*:
$f(x:=y)\ `\ A = (\mathit{if}\ x \in A\ \mathit{then\ insert}\ y\ (f\ `\ (A{-}\{x\}))\ \mathit{else}\ f\ `\ A)$
⟨*proof*⟩

7.6 *override-on*

definition
override-on :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ \mathit{set} \Rightarrow 'a \Rightarrow 'b$
where
$\mathit{override\text{-}on}\ f\ g\ A = (\lambda a.\ \mathit{if}\ a \in A\ \mathit{then}\ g\ a\ \mathit{else}\ f\ a)$

lemma *override-on-emptyset*[*simp*]: *override-on f g* {} = *f*
⟨*proof*⟩

lemma *override-on-apply-notin*[*simp*]: $a\ {}^{\sim}{:}\ A \Longrightarrow (\mathit{override\text{-}on}\ f\ g\ A)\ a = f\ a$
⟨*proof*⟩

lemma *override-on-apply-in*[*simp*]: $a : A \Longrightarrow (\mathit{override\text{-}on}\ f\ g\ A)\ a = g\ a$
⟨*proof*⟩

7.7 *swap*

definition
swap :: $'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)$
where
$\mathit{swap}\ a\ b\ f = f\ (a := f\ b,\ b{:=}\ f\ a)$

lemma *swap-self*: *swap a a f* = *f*

⟨*proof*⟩

lemma *swap-commute*: *swap a b f* = *swap b a f*
⟨*proof*⟩

lemma *swap-nilpotent* [*simp*]: *swap a b* (*swap a b f*) = *f*
⟨*proof*⟩

lemma *inj-on-imp-inj-on-swap*:
[|*inj-on f A*; *a* ∈ *A*; *b* ∈ *A*|] ==> *inj-on* (*swap a b f*) *A*
⟨*proof*⟩

lemma *inj-on-swap-iff* [*simp*]:
assumes *A*: *a* ∈ *A b* ∈ *A* **shows** *inj-on* (*swap a b f*) *A* = *inj-on f A*
⟨*proof*⟩

lemma *surj-imp-surj-swap*: *surj f* ==> *surj* (*swap a b f*)
⟨*proof*⟩

lemma *surj-swap-iff* [*simp*]: *surj* (*swap a b f*) = *surj f*
⟨*proof*⟩

lemma *bij-swap-iff*: *bij* (*swap a b f*) = *bij f*
⟨*proof*⟩

hide (**open**) *const swap*

7.8 Proof tool setup

simplifies terms of the form f(...,x:=y,...,x:=z,...) to f(...,x:=z,...)

⟨*ML*⟩

7.9 Code generator setup

types-code
fun ((- ->/ -))
attach (*term-of*) ⟪
fun term-of-fun-type - aT - bT - = Free (<*function*>, *aT* --> *bT*);
⟫
attach (*test*) ⟪
fun gen-fun-type aF aT bG bT i =
let
val tab = *ref* [];
fun mk-upd (*x*, (-, *y*)) *t* = *Const* (*Fun.fun-upd*,
(*aT* --> *bT*) --> *aT* --> *bT* --> *aT* --> *bT*) $ *t* $ *aF x* $ *y* ()
in
(*fn x* =>
case AList.lookup op = (!*tab*) *x of*
NONE =>

```
        let val p as (y, -) = bG i
        in (tab := (x, p) :: !tab; y) end
    | SOME (y, -) => y,
  fn () => Basics.fold mk-upd (!tab) (Const (HOL.undefined, aT --> bT)))
 end;
⟩⟩

code-const op ∘
 (SML infixl 5 o)
 (Haskell infixr 9 .)

code-const id
 (Haskell id)

end
```

8 Sum-Type: The Disjoint Sum of Two Types

```
theory Sum-Type
imports Typedef Fun
begin
```

The representations of the two injections

```
constdefs
 Inl-Rep :: ['a, 'a, 'b, bool] => bool
 Inl-Rep == (%a. %x y p. x=a & p)

 Inr-Rep :: ['b, 'a, 'b, bool] => bool
 Inr-Rep == (%b. %x y p. y=b & ~p)


global

typedef (Sum)
 ('a, 'b) +          (infixr + 10)
   = {f. (? a. f = Inl-Rep(a::'a)) | (? b. f = Inr-Rep(b::'b))}
 ⟨proof⟩

local
```

abstract constants and syntax

```
constdefs
 Inl :: 'a => 'a + 'b
  Inl == (%a. Abs-Sum(Inl-Rep(a)))

 Inr :: 'b => 'a + 'b
  Inr == (%b. Abs-Sum(Inr-Rep(b)))
```

Plus :: [*'a set*, *'b set*] => (*'a* + *'b*) *set* (**infixr** <+> *65*)
A <+> *B* == (*Inl*‘*A*) *Un* (*Inr*‘*B*)
— disjoint sum for sets; the operator + is overloaded with wrong type!

Part :: [*'a set*, *'b* => *'a*] => *'a set*
Part A h == *A Int* {*x*. *?* *z*. *x* = *h*(*z*)}
— for selecting out the components of a mutually recursive definition

lemma *Inl-RepI*: *Inl-Rep*(*a*) : *Sum*
⟨*proof*⟩

lemma *Inr-RepI*: *Inr-Rep*(*b*) : *Sum*
⟨*proof*⟩

lemma *inj-on-Abs-Sum*: *inj-on Abs-Sum Sum*
⟨*proof*⟩

8.1 Freeness Properties for *Inl* and *Inr*

Distinctness

lemma *Inl-Rep-not-Inr-Rep*: *Inl-Rep*(*a*) ~= *Inr-Rep*(*b*)
⟨*proof*⟩

lemma *Inl-not-Inr* [*iff*]: *Inl*(*a*) ~= *Inr*(*b*)
⟨*proof*⟩

lemmas *Inr-not-Inl* = *Inl-not-Inr* [*THEN not-sym*, *standard*]
declare *Inr-not-Inl* [*iff*]

lemmas *Inl-neq-Inr* = *Inl-not-Inr* [*THEN notE*, *standard*]
lemmas *Inr-neq-Inl* = *sym* [*THEN Inl-neq-Inr*, *standard*]

Injectiveness

lemma *Inl-Rep-inject*: *Inl-Rep*(*a*) = *Inl-Rep*(*c*) ==> *a*=*c*
⟨*proof*⟩

lemma *Inr-Rep-inject*: *Inr-Rep*(*b*) = *Inr-Rep*(*d*) ==> *b*=*d*
⟨*proof*⟩

lemma *inj-Inl* [*simp*]: *inj-on Inl A*
⟨*proof*⟩

lemmas *Inl-inject* = *inj-Inl* [*THEN injD*, *standard*]

lemma *inj-Inr* [*simp*]: *inj-on Inr A*
⟨*proof*⟩

lemmas *Inr-inject* = *inj-Inr* [*THEN injD*, *standard*]

lemma *Inl-eq* [*iff*]: (*Inl*(*x*)=*Inl*(*y*)) = (*x*=*y*)
⟨*proof*⟩

lemma *Inr-eq* [*iff*]: (*Inr*(*x*)=*Inr*(*y*)) = (*x*=*y*)
⟨*proof*⟩

8.2 The Disjoint Sum of Sets

lemma *InlI* [*intro*!]: *a* : *A* ==> *Inl*(*a*) : *A* <+> *B*
⟨*proof*⟩

lemma *InrI* [*intro*!]: *b* : *B* ==> *Inr*(*b*) : *A* <+> *B*
⟨*proof*⟩

lemma *PlusE* [*elim*!]:
[| *u*: *A* <+> *B*;
!!*x*. [| *x*:*A*; *u*=*Inl*(*x*) |] ==> *P*;
!!*y*. [| *y*:*B*; *u*=*Inr*(*y*) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

Exhaustion rule for sums, a degenerate form of induction

lemma *sumE*:
[| !!*x*::'*a*. *s* = *Inl*(*x*) ==> *P*; !!*y*::'*b*. *s* = *Inr*(*y*) ==> *P*
|] ==> *P*
⟨*proof*⟩

lemma *UNIV-Plus-UNIV* [*simp*]: *UNIV* <+> *UNIV* = *UNIV*
⟨*proof*⟩

8.3 The *Part* Primitive

lemma *Part-eqI* [*intro*]: [| *a* : *A*; *a*=*h*(*b*) |] ==> *a* : *Part A h*
⟨*proof*⟩

lemmas *PartI* = *Part-eqI* [*OF - refl*, *standard*]

lemma *PartE* [*elim*!]: [| *a* : *Part A h*; !!*z*. [| *a* : *A*; *a*=*h*(*z*) |] ==> *P* |] ==> *P*
⟨*proof*⟩

lemma *Part-subset*: *Part A h <= A*
⟨*proof*⟩

lemma *Part-mono*: *A<=B ==> Part A h <= Part B h*
⟨*proof*⟩

lemmas *basic-monos = basic-monos Part-mono*

lemma *PartD1*: *a : Part A h ==> a : A*
⟨*proof*⟩

lemma *Part-id*: *Part A (%x. x) = A*
⟨*proof*⟩

lemma *Part-Int*: *Part (A Int B) h = (Part A h) Int (Part B h)*
⟨*proof*⟩

lemma *Part-Collect*: *Part (A Int {x. P x}) h = (Part A h) Int {x. P x}*
⟨*proof*⟩

⟨*ML*⟩

end

9 Inductive: Knaster-Tarski Fixpoint Theorem and inductive definitions

theory *Inductive*
imports *Lattices Sum-Type*
uses
(*Tools/inductive-package.ML*)
Tools/dseq.ML
(*Tools/inductive-codegen.ML*)
(*Tools/datatype-aux.ML*)
(*Tools/datatype-prop.ML*)
(*Tools/datatype-rep-proofs.ML*)
(*Tools/datatype-abs-proofs.ML*)
(*Tools/datatype-case.ML*)
(*Tools/datatype-package.ML*)
(*Tools/old-primrec-package.ML*)
(*Tools/primrec-package.ML*)
(*Tools/datatype-codegen.ML*)
begin

9.1 Least and greatest fixed points

context *complete-lattice*
begin

definition
lfp :: $('a \Rightarrow 'a) \Rightarrow 'a$ **where**
lfp f = *Inf* $\{u.\ f\ u \leq u\}$ — least fixed point

definition
gfp :: $('a \Rightarrow 'a) \Rightarrow 'a$ **where**
gfp f = *Sup* $\{u.\ u \leq f\ u\}$ — greatest fixed point

9.2 Proof of Knaster-Tarski Theorem using *lfp*

lfp f is the least upper bound of the set $\{u.\ f\ u \leq u\}$

lemma *lfp-lowerbound*: $f\ A \leq A \Longrightarrow$ *lfp f* $\leq A$
⟨*proof*⟩

lemma *lfp-greatest*: $(!!u.\ f\ u \leq u \Longrightarrow A \leq u) \Longrightarrow A \leq$ *lfp f*
⟨*proof*⟩

end

lemma *lfp-lemma2*: *mono f* $\Longrightarrow$ *f* (*lfp f*) $\leq$ *lfp f*
⟨*proof*⟩

lemma *lfp-lemma3*: *mono f* $\Longrightarrow$ *lfp f* $\leq$ *f* (*lfp f*)
⟨*proof*⟩

lemma *lfp-unfold*: *mono f* $\Longrightarrow$ *lfp f* = *f* (*lfp f*)
⟨*proof*⟩

lemma *lfp-const*: *lfp* $(\lambda x.\ t) = t$
⟨*proof*⟩

9.3 General induction rules for least fixed points

theorem *lfp-induct*:
assumes *mono*: *mono f* **and** *ind*: *f* (*inf* (*lfp f*) *P*) <= *P*
shows *lfp f* <= *P*
⟨*proof*⟩

lemma *lfp-induct-set*:
assumes *lfp*: *a*: *lfp*(*f*)
and *mono*: *mono*(*f*)
and *indhyp*: !!*x*. [| *x*: *f*(*lfp*(*f*) *Int* {*x*. *P*(*x*)}) |] ==> *P*(*x*)
shows *P*(*a*)
⟨*proof*⟩

lemma *lfp-ordinal-induct*:
fixes f :: $'a$::*complete-lattice* $\Rightarrow$ $'a$
assumes *mono*: *mono f*
and *P-f*: $\bigwedge S.\ P\ S \Longrightarrow P\ (f\ S)$
and *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P\ (Sup\ M)$
shows P (*lfp f*)
⟨*proof*⟩

lemma *lfp-ordinal-induct-set*:
assumes *mono*: *mono f*
and *P-f*: !!$S.\ P\ S$ ==> $P(f\ S)$
and *P-Union*: !!$M.$!S:$M.\ P\ S$ ==> $P(Union\ M)$
shows $P(lfp\ f)$
⟨*proof*⟩

Definition forms of *lfp-unfold* and *lfp-induct*, to control unfolding

lemma *def-lfp-unfold*: [| h==*lfp*(f); *mono*(f) |] ==> $h = f(h)$
⟨*proof*⟩

lemma *def-lfp-induct*:
[| A == *lfp*(f); *mono*(f);
$f\ (inf\ A\ P) \leq P$
|] ==> $A \leq P$
⟨*proof*⟩

lemma *def-lfp-induct-set*:
[| A == *lfp*(f); *mono*(f); a:A;
!!x. [| x: $f(A\ Int\ \{x.\ P(x)\})$ |] ==> $P(x)$
|] ==> $P(a)$
⟨*proof*⟩

lemma *lfp-mono*: (!!$Z.\ f\ Z \leq g\ Z$) ==> *lfp f* $\leq$ *lfp g*
⟨*proof*⟩

9.4 Proof of Knaster-Tarski Theorem using *gfp*

gfp f is the greatest lower bound of the set $\{u.\ u \leq f\ u\}$

lemma *gfp-upperbound*: $X \leq f\ X$ ==> $X \leq$ *gfp f*
⟨*proof*⟩

lemma *gfp-least*: (!!$u.\ u \leq f\ u$ ==> $u \leq X$) ==> *gfp f* $\leq X$
⟨*proof*⟩

lemma *gfp-lemma2*: *mono f* ==> *gfp f* $\leq f$ (*gfp f*)
⟨*proof*⟩

lemma *gfp-lemma3*: *mono f* ==> f (*gfp f*) $\leq$ *gfp f*

⟨*proof*⟩

lemma *gfp-unfold*: *mono f* ==> *gfp f* = *f* (*gfp f*)
⟨*proof*⟩

9.5 Coinduction rules for greatest fixed points

weak version

lemma *weak-coinduct*: [| *a*: *X*; *X* ⊆ *f*(*X*) |] ==> *a* : *gfp*(*f*)
⟨*proof*⟩

lemma *weak-coinduct-image*: !!*X*. [| *a* : *X*; *g*‘*X* ⊆ *f* (*g*‘*X*) |] ==> *g a* : *gfp f*
⟨*proof*⟩

lemma *coinduct-lemma*:
[| *X* ≤ *f* (*sup X* (*gfp f*)); *mono f* |] ==> *sup X* (*gfp f*) ≤ *f* (*sup X* (*gfp f*))
⟨*proof*⟩

strong version, thanks to Coen and Frost

lemma *coinduct-set*: [| *mono*(*f*); *a*: *X*; *X* ⊆ *f*(*X Un gfp*(*f*)) |] ==> *a* : *gfp*(*f*)
⟨*proof*⟩

lemma *coinduct*: [| *mono*(*f*); *X* ≤ *f* (*sup X* (*gfp f*)) |] ==> *X* ≤ *gfp*(*f*)
⟨*proof*⟩

lemma *gfp-fun-UnI2*: [| *mono*(*f*); *a*: *gfp*(*f*) |] ==> *a*: *f*(*X Un gfp*(*f*))
⟨*proof*⟩

9.6 Even Stronger Coinduction Rule, by Martin Coen

Weakens the condition *X* ⊆ *f X* to one expressed using both *lfp* and *gfp*

lemma *coinduct3-mono-lemma*: *mono*(*f*) ==> *mono*(%*x*. *f*(*x*) *Un X Un B*)
⟨*proof*⟩

lemma *coinduct3-lemma*:
[| *X* ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*))); *mono*(*f*) |]
==> *lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*)) ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*)))
⟨*proof*⟩

lemma *coinduct3*:
[| *mono*(*f*); *a*:*X*; *X* ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*))) |] ==> *a* : *gfp*(*f*)
⟨*proof*⟩

Definition forms of *gfp-unfold* and *coinduct*, to control unfolding

lemma *def-gfp-unfold*: [| *A*==*gfp*(*f*); *mono*(*f*) |] ==> *A* = *f*(*A*)
⟨*proof*⟩

lemma *def-coinduct*:

[| A==*gfp*(f); *mono*(f); $X \leq f(sup\ X\ A)$ |] ==> $X \leq A$
⟨*proof*⟩

lemma *def-coinduct-set*:
[| A==*gfp*(f); *mono*(f); a:X; $X \subseteq f(X\ Un\ A)$ |] ==> a: A
⟨*proof*⟩

lemma *def-Collect-coinduct*:
[| A == *gfp*(%w. *Collect*($P(w)$)); *mono*(%w. *Collect*($P(w)$));
a: X; !!z. z: X ==> $P\ (X\ Un\ A)\ z$ |] ==>
a : A
⟨*proof*⟩

lemma *def-coinduct3*:
[| A==*gfp*(f); *mono*(f); a:X; $X \subseteq f(lfp(\%x.\ f(x)\ Un\ X\ Un\ A))$ |] ==> a: A
⟨*proof*⟩

Monotonicity of *gfp*!

lemma *gfp-mono*: (!!Z. $f\ Z \leq g\ Z$) ==> *gfp* $f \leq$ *gfp* g
⟨*proof*⟩

9.7 Inductive predicates and sets

Inversion of injective functions.

constdefs
myinv :: ('a => 'b) => ('b => 'a)
myinv (f :: 'a => 'b) == λy. *THE* x. $f\ x = y$

lemma *myinv-f-f*: *inj* f ==> *myinv* $f\ (f\ x) = x$
⟨*proof*⟩

lemma *f-myinv-f*: *inj* f ==> $y \in$ *range* f ==> $f\ (myinv\ f\ y) = y$
⟨*proof*⟩

hide *const myinv*

Package setup.

theorems *basic-monos* =
subset-refl imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
Collect-mono in-mono vimage-mono
imp-conv-disj not-not de-Morgan-disj de-Morgan-conj
not-all not-ex
Ball-def Bex-def
induct-rulify-fallback

⟨*ML*⟩

theorems [*mono*] =
imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
imp-conv-disj not-not de-Morgan-disj de-Morgan-conj
not-all not-ex
Ball-def Bex-def
induct-rulify-fallback

9.8 Inductive datatypes and primitive recursion

Package setup.

⟨*ML*⟩
Lambda-abstractions with pattern matching:

syntax
-lam-pats-syntax :: *cases-syn* => *'a* => *'b* ((%-) *10*)
syntax (*xsymbols*)
-lam-pats-syntax :: *cases-syn* => *'a* => *'b* ((λ-) *10*)

⟨*ML*⟩

end

10 OrderedGroup: Ordered Groups

theory *OrderedGroup*
imports *Lattices*
uses ~~/*src*/*Provers*/*Arith*/*abel-cancel.ML*
begin

The theory of partially ordered groups is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

⟨*ML*⟩
The rewrites accumulated in *algebra-simps* deal with the classical algebraic structures of groups, rings and family. They simplify terms by multiplying everything out (in case of a ring) and bringing sums and products into a

canonical form (by ordered rewriting). As a result it decides group and ring equalities but also helps with inequalities.

Of course it also works for fields, but it knows nothing about multiplicative inverses or division. This is catered for by *field-simps*.

10.1 Semigroups and Monoids

class *semigroup-add* = *plus* +
  assumes *add-assoc*[*algebra-simps*]: $(a + b) + c = a + (b + c)$

class *ab-semigroup-add* = *semigroup-add* +
  assumes *add-commute*[*algebra-simps*]: $a + b = b + a$
begin

lemma *add-left-commute*[*algebra-simps*]: $a + (b + c) = b + (a + c)$
⟨*proof*⟩

theorems *add-ac* = *add-assoc add-commute add-left-commute*

end

theorems *add-ac* = *add-assoc add-commute add-left-commute*

class *semigroup-mult* = *times* +
  assumes *mult-assoc*[*algebra-simps*]: $(a * b) * c = a * (b * c)$

class *ab-semigroup-mult* = *semigroup-mult* +
  assumes *mult-commute*[*algebra-simps*]: $a * b = b * a$
begin

lemma *mult-left-commute*[*algebra-simps*]: $a * (b * c) = b * (a * c)$
⟨*proof*⟩

theorems *mult-ac* = *mult-assoc mult-commute mult-left-commute*

end

theorems *mult-ac* = *mult-assoc mult-commute mult-left-commute*

class *ab-semigroup-idem-mult* = *ab-semigroup-mult* +
  assumes *mult-idem*[*simp*]: $x * x = x$
begin

lemma *mult-left-idem*[*simp*]: $x * (x * y) = x * y$
  ⟨*proof*⟩

end

class *monoid-add* = *zero* + *semigroup-add* +

assumes *add-0-left* [*simp*]: $0 + a = a$
and *add-0-right* [*simp*]: $a + 0 = a$

lemma *zero-reorient*: $0 = x \longleftrightarrow x = 0$
⟨*proof*⟩

class *comm-monoid-add* = *zero* + *ab-semigroup-add* +
assumes *add-0*: $0 + a = a$
begin

subclass *monoid-add*
⟨*proof*⟩

end

class *monoid-mult* = *one* + *semigroup-mult* +
assumes *mult-1-left* [*simp*]: $1 * a = a$
assumes *mult-1-right* [*simp*]: $a * 1 = a$

lemma *one-reorient*: $1 = x \longleftrightarrow x = 1$
⟨*proof*⟩

class *comm-monoid-mult* = *one* + *ab-semigroup-mult* +
assumes *mult-1*: $1 * a = a$
begin

subclass *monoid-mult*
⟨*proof*⟩

end

class *cancel-semigroup-add* = *semigroup-add* +
assumes *add-left-imp-eq*: $a + b = a + c \Longrightarrow b = c$
assumes *add-right-imp-eq*: $b + a = c + a \Longrightarrow b = c$
begin

lemma *add-left-cancel* [*simp*]:
$a + b = a + c \longleftrightarrow b = c$
⟨*proof*⟩

lemma *add-right-cancel* [*simp*]:
$b + a = c + a \longleftrightarrow b = c$
⟨*proof*⟩

end

class *cancel-ab-semigroup-add* = *ab-semigroup-add* +
assumes *add-imp-eq*: $a + b = a + c \Longrightarrow b = c$
begin

subclass *cancel-semigroup-add*
⟨*proof*⟩

end

class *cancel-comm-monoid-add* = *cancel-ab-semigroup-add* + *comm-monoid-add*

10.2 Groups

class *group-add* = *minus* + *uminus* + *monoid-add* +
  assumes *left-minus* [*simp*]: − *a* + *a* = *0*
  assumes *diff-minus*: *a* − *b* = *a* + (− *b*)
begin

lemma *minus-add-cancel*: − *a* + (*a* + *b*) = *b*
⟨*proof*⟩

lemma *minus-zero* [*simp*]: − *0* = *0*
⟨*proof*⟩

lemma *minus-minus* [*simp*]: − (− *a*) = *a*
⟨*proof*⟩

lemma *right-minus* [*simp*]: *a* + − *a* = *0*
⟨*proof*⟩

lemma *right-minus-eq*: *a* − *b* = *0* ⟷ *a* = *b*
⟨*proof*⟩

lemma *equals-zero-I*:
  assumes *a* + *b* = *0* **shows** − *a* = *b*
⟨*proof*⟩

lemma *diff-self* [*simp*]: *a* − *a* = *0*
⟨*proof*⟩

lemma *diff-0* [*simp*]: *0* − *a* = − *a*
⟨*proof*⟩

lemma *diff-0-right* [*simp*]: *a* − *0* = *a*
⟨*proof*⟩

lemma *diff-minus-eq-add* [*simp*]: *a* − − *b* = *a* + *b*
⟨*proof*⟩

lemma *neg-equal-iff-equal* [*simp*]:
  − *a* = − *b* ⟷ *a* = *b*
⟨*proof*⟩

lemma *neg-equal-0-iff-equal* [*simp*]:
$- a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *neg-0-equal-iff-equal* [*simp*]:
$0 = - a \longleftrightarrow 0 = a$
⟨*proof*⟩

The next two equations can make the simplifier loop!

lemma *equation-minus-iff*:
$a = - b \longleftrightarrow b = - a$
⟨*proof*⟩

lemma *minus-equation-iff*:
$- a = b \longleftrightarrow - b = a$
⟨*proof*⟩

lemma *diff-add-cancel*: $a - b + b = a$
⟨*proof*⟩

lemma *add-diff-cancel*: $a + b - b = a$
⟨*proof*⟩

declare *diff-minus*[*symmetric*, *algebra-simps*]

lemma *eq-neg-iff-add-eq-0*: $a = - b \longleftrightarrow a + b = 0$
⟨*proof*⟩

end

class *ab-group-add* = *minus* + *uminus* + *comm-monoid-add* +
assumes *ab-left-minus*: $- a + a = 0$
assumes *ab-diff-minus*: $a - b = a + (- b)$
begin

subclass *group-add*
⟨*proof*⟩

subclass *cancel-comm-monoid-add*
⟨*proof*⟩

lemma *uminus-add-conv-diff*[*algebra-simps*]:
$- a + b = b - a$
⟨*proof*⟩

lemma *minus-add-distrib* [*simp*]:
$- (a + b) = - a + - b$
⟨*proof*⟩

lemma *minus-diff-eq* [*simp*]:
$-(a - b) = b - a$
⟨*proof*⟩

lemma *add-diff-eq*[*algebra-simps*]: $a + (b - c) = (a + b) - c$
⟨*proof*⟩

lemma *diff-add-eq*[*algebra-simps*]: $(a - b) + c = (a + c) - b$
⟨*proof*⟩

lemma *diff-eq-eq*[*algebra-simps*]: $a - b = c \longleftrightarrow a = c + b$
⟨*proof*⟩

lemma *eq-diff-eq*[*algebra-simps*]: $a = c - b \longleftrightarrow a + b = c$
⟨*proof*⟩

lemma *diff-diff-eq*[*algebra-simps*]: $(a - b) - c = a - (b + c)$
⟨*proof*⟩

lemma *diff-diff-eq2*[*algebra-simps*]: $a - (b - c) = (a + c) - b$
⟨*proof*⟩

lemma *eq-iff-diff-eq-0*: $a = b \longleftrightarrow a - b = 0$
⟨*proof*⟩

lemma *diff-eq-0-iff-eq* [*simp*, *noatp*]: $a - b = 0 \longleftrightarrow a = b$
⟨*proof*⟩

end

10.3 (Partially) Ordered Groups

class *pordered-ab-semigroup-add* = *order* + *ab-semigroup-add* +
assumes *add-left-mono*: $a \leq b \Longrightarrow c + a \leq c + b$
begin

lemma *add-right-mono*:
$a \leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

non-strict, in both arguments

lemma *add-mono*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow a + c \leq b + d$
⟨*proof*⟩

end

class *pordered-cancel-ab-semigroup-add* =

pordered-ab-semigroup-add + *cancel-ab-semigroup-add*
begin

lemma *add-strict-left-mono*:
$a < b \Longrightarrow c + a < c + b$
⟨*proof*⟩

lemma *add-strict-right-mono*:
$a < b \Longrightarrow a + c < b + c$
⟨*proof*⟩

Strict monotonicity in both arguments

lemma *add-strict-mono*:
$a < b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

lemma *add-less-le-mono*:
$a < b \Longrightarrow c \leq d \Longrightarrow a + c < b + d$
⟨*proof*⟩

lemma *add-le-less-mono*:
$a \leq b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

end

class *pordered-ab-semigroup-add-imp-le* =
pordered-cancel-ab-semigroup-add +
assumes *add-le-imp-le-left*: $c + a \leq c + b \Longrightarrow a \leq b$
begin

lemma *add-less-imp-less-left*:
assumes *less*: $c + a < c + b$ **shows** $a < b$
⟨*proof*⟩

lemma *add-less-imp-less-right*:
$a + c < b + c \Longrightarrow a < b$
⟨*proof*⟩

lemma *add-less-cancel-left* [*simp*]:
$c + a < c + b \longleftrightarrow a < b$
⟨*proof*⟩

lemma *add-less-cancel-right* [*simp*]:
$a + c < b + c \longleftrightarrow a < b$
⟨*proof*⟩

lemma *add-le-cancel-left* [*simp*]:
$c + a \leq c + b \longleftrightarrow a \leq b$

⟨*proof*⟩

lemma *add-le-cancel-right* [*simp*]:
$a + c \leq b + c \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *add-le-imp-le-right*:
$a + c \leq b + c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *max-add-distrib-left*:
$max\ x\ y + z = max\ (x + z)\ (y + z)$
⟨*proof*⟩

lemma *min-add-distrib-left*:
$min\ x\ y + z = min\ (x + z)\ (y + z)$
⟨*proof*⟩

end

10.4 Support for reasoning about signs

class *pordered-comm-monoid-add* =
pordered-cancel-ab-semigroup-add + *comm-monoid-add*
begin

lemma *add-pos-nonneg*:
assumes $0 < a$ **and** $0 \leq b$ **shows** $0 < a + b$
⟨*proof*⟩

lemma *add-pos-pos*:
assumes $0 < a$ **and** $0 < b$ **shows** $0 < a + b$
⟨*proof*⟩

lemma *add-nonneg-pos*:
assumes $0 \leq a$ **and** $0 < b$ **shows** $0 < a + b$
⟨*proof*⟩

lemma *add-nonneg-nonneg*:
assumes $0 \leq a$ **and** $0 \leq b$ **shows** $0 \leq a + b$
⟨*proof*⟩

lemma *add-neg-nonpos*:
assumes $a < 0$ **and** $b \leq 0$ **shows** $a + b < 0$
⟨*proof*⟩

lemma *add-neg-neg*:
assumes $a < 0$ **and** $b < 0$ **shows** $a + b < 0$
⟨*proof*⟩

lemma *add-nonpos-neg*:
assumes $a \leq 0$ **and** $b < 0$ **shows** $a + b < 0$
⟨*proof*⟩

lemma *add-nonpos-nonpos*:
assumes $a \leq 0$ **and** $b \leq 0$ **shows** $a + b \leq 0$
⟨*proof*⟩

lemmas *add-sign-intros* =
add-pos-nonneg add-pos-pos add-nonneg-pos add-nonneg-nonneg
add-neg-nonpos add-neg-neg add-nonpos-neg add-nonpos-nonpos

lemma *add-nonneg-eq-0-iff*:
assumes x: $0 \leq x$ **and** y: $0 \leq y$
shows $x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

end

class *pordered-ab-group-add* =
ab-group-add + *pordered-ab-semigroup-add*
begin

subclass *pordered-cancel-ab-semigroup-add* ⟨*proof*⟩

subclass *pordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

subclass *pordered-comm-monoid-add* ⟨*proof*⟩

lemma *max-diff-distrib-left*:
shows $max\ x\ y - z = max\ (x - z)\ (y - z)$
⟨*proof*⟩

lemma *min-diff-distrib-left*:
shows $min\ x\ y - z = min\ (x - z)\ (y - z)$
⟨*proof*⟩

lemma *le-imp-neg-le*:
assumes $a \leq b$ **shows** $-b \leq -a$
⟨*proof*⟩

lemma *neg-le-iff-le* [*simp*]: $- b \leq - a \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *neg-le-0-iff-le* [*simp*]: $- a \leq 0 \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *neg-0-le-iff-le* [*simp*]: $0 \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *neg-less-iff-less* [*simp*]: $- b < - a \longleftrightarrow a < b$
⟨*proof*⟩

lemma *neg-less-0-iff-less* [*simp*]: $- a < 0 \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *neg-0-less-iff-less* [*simp*]: $0 < - a \longleftrightarrow a < 0$
⟨*proof*⟩

The next several equations can make the simplifier loop!

lemma *less-minus-iff*: $a < - b \longleftrightarrow b < - a$
⟨*proof*⟩

lemma *minus-less-iff*: $- a < b \longleftrightarrow - b < a$
⟨*proof*⟩

lemma *le-minus-iff*: $a \leq - b \longleftrightarrow b \leq - a$
⟨*proof*⟩

lemma *minus-le-iff*: $- a \leq b \longleftrightarrow - b \leq a$
⟨*proof*⟩

lemma *less-iff-diff-less-0*: $a < b \longleftrightarrow a - b < 0$
⟨*proof*⟩

lemma *diff-less-eq*[*algebra-simps*]: $a - b < c \longleftrightarrow a < c + b$
⟨*proof*⟩

lemma *less-diff-eq*[*algebra-simps*]: $a < c - b \longleftrightarrow a + b < c$
⟨*proof*⟩

lemma *diff-le-eq*[*algebra-simps*]: $a - b \leq c \longleftrightarrow a \leq c + b$
⟨*proof*⟩

lemma *le-diff-eq*[*algebra-simps*]: $a \leq c - b \longleftrightarrow a + b \leq c$
⟨*proof*⟩

lemma *le-iff-diff-le-0*: $a \leq b \longleftrightarrow a - b \leq 0$
⟨*proof*⟩

Legacy - use *algebra-simps*

lemmas *group-simps*[*noatp*] = *algebra-simps*

end

Legacy - use *algebra-simps*

lemmas *group-simps*[*noatp*] = *algebra-simps*

class *ordered-ab-semigroup-add* =
linorder + *pordered-ab-semigroup-add*

class *ordered-cancel-ab-semigroup-add* =
linorder + *pordered-cancel-ab-semigroup-add*
begin

subclass *ordered-ab-semigroup-add* ⟨*proof*⟩

subclass *pordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

end

class *ordered-ab-group-add* =
linorder + *pordered-ab-group-add*
begin

subclass *ordered-cancel-ab-semigroup-add* ⟨*proof*⟩

lemma *neg-less-eq-nonneg*:
− *a* ≤ *a* ⟷ *0* ≤ *a*
⟨*proof*⟩

lemma *less-eq-neg-nonpos*:
a ≤ − *a* ⟷ *a* ≤ *0*
⟨*proof*⟩

lemma *equal-neg-zero*:
a = − *a* ⟷ *a* = *0*
⟨*proof*⟩

lemma *neg-equal-zero*:
− *a* = *a* ⟷ *a* = *0*
⟨*proof*⟩

end

— FIXME localize the following

lemma *add-increasing*:
fixes *c* :: *'a*::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}
shows [|*0*≤*a*; *b*≤*c*|] ==> *b* ≤ *a* + *c*
⟨*proof*⟩

lemma *add-increasing2*:
fixes *c* :: *'a*::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}

shows $[|0 \leq c;\ b \leq a|] ==> b \leq a + c$
⟨*proof*⟩

lemma *add-strict-increasing*:
fixes $c :: 'a::\{pordered\text{-}ab\text{-}semigroup\text{-}add\text{-}imp\text{-}le,\ comm\text{-}monoid\text{-}add\}$
shows $[|0<a;\ b \leq c|] ==> b < a + c$
⟨*proof*⟩

lemma *add-strict-increasing2*:
fixes $c :: 'a::\{pordered\text{-}ab\text{-}semigroup\text{-}add\text{-}imp\text{-}le,\ comm\text{-}monoid\text{-}add\}$
shows $[|0 \leq a;\ b<c|] ==> b < a + c$
⟨*proof*⟩

class *pordered-ab-group-add-abs* = *pordered-ab-group-add* + *abs* +
assumes *abs-ge-zero* [*simp*]: $|a| \geq 0$
and *abs-ge-self*: $a \leq |a|$
and *abs-leI*: $a \leq b \Longrightarrow -a \leq b \Longrightarrow |a| \leq b$
and *abs-minus-cancel* [*simp*]: $|-a| = |a|$
and *abs-triangle-ineq*: $|a + b| \leq |a| + |b|$
begin

lemma *abs-minus-le-zero*: $-|a| \leq 0$
⟨*proof*⟩

lemma *abs-of-nonneg* [*simp*]:
assumes *nonneg*: $0 \leq a$ **shows** $|a| = a$
⟨*proof*⟩

lemma *abs-idempotent* [*simp*]: $||a|| = |a|$
⟨*proof*⟩

lemma *abs-eq-0* [*simp*]: $|a| = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-zero* [*simp*]: $|0| = 0$
⟨*proof*⟩

lemma *abs-0-eq* [*simp*, *noatp*]: $0 = |a| \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-le-zero-iff* [*simp*]: $|a| \leq 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *zero-less-abs-iff* [*simp*]: $0 < |a| \longleftrightarrow a \neq 0$
⟨*proof*⟩

lemma *abs-not-less-zero* [*simp*]: $\neg |a| < 0$
⟨*proof*⟩

lemma *abs-ge-minus-self*: $-\ a \leq |a|$
⟨*proof*⟩

lemma *abs-minus-commute*:
$|a\ -\ b| = |b\ -\ a|$
⟨*proof*⟩

lemma *abs-of-pos*: $0 < a \Longrightarrow |a| = a$
⟨*proof*⟩

lemma *abs-of-nonpos* [*simp*]:
assumes $a \leq 0$ **shows** $|a| = -\ a$
⟨*proof*⟩

lemma *abs-of-neg*: $a < 0 \Longrightarrow |a| = -\ a$
⟨*proof*⟩

lemma *abs-le-D1*: $|a| \leq b \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *abs-le-D2*: $|a| \leq b \Longrightarrow -\ a \leq b$
⟨*proof*⟩

lemma *abs-le-iff*: $|a| \leq b \longleftrightarrow a \leq b \wedge -\ a \leq b$
⟨*proof*⟩

lemma *abs-triangle-ineq2*: $|a| - |b| \leq |a\ -\ b|$
⟨*proof*⟩

lemma *abs-triangle-ineq3*: $||a| - |b|| \leq |a\ -\ b|$
⟨*proof*⟩

lemma *abs-triangle-ineq4*: $|a\ -\ b| \leq |a| + |b|$
⟨*proof*⟩

lemma *abs-diff-triangle-ineq*: $|a\ +\ b\ -\ (c\ +\ d)| \leq |a\ -\ c| + |b\ -\ d|$
⟨*proof*⟩

lemma *abs-add-abs* [*simp*]:
$||a| + |b|| = |a| + |b|$ (**is** *?L = ?R*)
⟨*proof*⟩

end

10.5 Lattice Ordered (Abelian) Groups

class *lordered-ab-group-add-meet* = *pordered-ab-group-add* + *lower-semilattice*
begin

lemma *add-inf-distrib-left*:
$a + inf\ b\ c = inf\ (a + b)\ (a + c)$
⟨*proof*⟩

lemma *add-inf-distrib-right*:
$inf\ a\ b + c = inf\ (a + c)\ (b + c)$
⟨*proof*⟩

end

class *lordered-ab-group-add-join* = *pordered-ab-group-add* + *upper-semilattice*
begin

lemma *add-sup-distrib-left*:
$a + sup\ b\ c = sup\ (a + b)\ (a + c)$
⟨*proof*⟩

lemma *add-sup-distrib-right*:
$sup\ a\ b + c = sup\ (a+c)\ (b+c)$
⟨*proof*⟩

end

class *lordered-ab-group-add* = *pordered-ab-group-add* + *lattice*
begin

subclass *lordered-ab-group-add-meet* ⟨*proof*⟩
subclass *lordered-ab-group-add-join* ⟨*proof*⟩

lemmas *add-sup-inf-distribs* = *add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left*

lemma *inf-eq-neg-sup*: $inf\ a\ b = -\ sup\ (-a)\ (-b)$
⟨*proof*⟩

lemma *sup-eq-neg-inf*: $sup\ a\ b = -\ inf\ (-a)\ (-b)$
⟨*proof*⟩

lemma *neg-inf-eq-sup*: $-\ inf\ a\ b = sup\ (-a)\ (-b)$
⟨*proof*⟩

lemma *neg-sup-eq-inf*: $-\ sup\ a\ b = inf\ (-a)\ (-b)$
⟨*proof*⟩

lemma *add-eq-inf-sup*: $a + b = sup\ a\ b + inf\ a\ b$
⟨*proof*⟩

10.6 Positive Part, Negative Part, Absolute Value

definition
nprt :: *'a* ⇒ *'a* **where**
nprt x = *inf x 0*

definition
pprt :: *'a* ⇒ *'a* **where**
pprt x = *sup x 0*

lemma *pprt-neg*: *pprt* (− *x*) = − *nprt x*
⟨*proof*⟩

lemma *nprt-neg*: *nprt* (− *x*) = − *pprt x*
⟨*proof*⟩

lemma *prts*: *a* = *pprt a* + *nprt a*
⟨*proof*⟩

lemma *zero-le-pprt*[*simp*]: *0* ≤ *pprt a*
⟨*proof*⟩

lemma *nprt-le-zero*[*simp*]: *nprt a* ≤ *0*
⟨*proof*⟩

lemma *le-eq-neg*: *a* ≤ − *b* ⟷ *a* + *b* ≤ *0* (**is** *?l* = *?r*)
⟨*proof*⟩

lemma *pprt-0*[*simp*]: *pprt 0* = *0* ⟨*proof*⟩
lemma *nprt-0*[*simp*]: *nprt 0* = *0* ⟨*proof*⟩

lemma *pprt-eq-id* [*simp*, *noatp*]: *0* ≤ *x* ⟹ *pprt x* = *x*
⟨*proof*⟩

lemma *nprt-eq-id* [*simp*, *noatp*]: *x* ≤ *0* ⟹ *nprt x* = *x*
⟨*proof*⟩

lemma *pprt-eq-0* [*simp*, *noatp*]: *x* ≤ *0* ⟹ *pprt x* = *0*
⟨*proof*⟩

lemma *nprt-eq-0* [*simp*, *noatp*]: *0* ≤ *x* ⟹ *nprt x* = *0*
⟨*proof*⟩

lemma *sup-0-imp-0*: *sup a* (− *a*) = *0* ⟹ *a* = *0*
⟨*proof*⟩

lemma *inf-0-imp-0*: *inf a* (−*a*) = *0* ⟹ *a* = *0*
⟨*proof*⟩

lemma *inf-0-eq-0* [*simp*, *noatp*]: *inf a* (− *a*) = *0* ⟷ *a* = *0*

⟨*proof*⟩

lemma *sup-0-eq-0* [*simp*, *noatp*]: *sup a* (− *a*) = *0* ⟷ *a* = *0*
⟨*proof*⟩

lemma *zero-le-double-add-iff-zero-le-single-add* [*simp*]:
0 ≤ *a* + *a* ⟷ *0* ≤ *a*
⟨*proof*⟩

lemma *double-zero*: *a* + *a* = *0* ⟷ *a* = *0*
⟨*proof*⟩

lemma *zero-less-double-add-iff-zero-less-single-add*:
0 < *a* + *a* ⟷ *0* < *a*
⟨*proof*⟩

lemma *double-add-le-zero-iff-single-add-le-zero* [*simp*]:
a + *a* ≤ *0* ⟷ *a* ≤ *0*
⟨*proof*⟩

lemma *double-add-less-zero-iff-single-less-zero* [*simp*]:
a + *a* < *0* ⟷ *a* < *0*
⟨*proof*⟩

declare *neg-inf-eq-sup* [*simp*] *neg-sup-eq-inf* [*simp*]

lemma *le-minus-self-iff*: *a* ≤ − *a* ⟷ *a* ≤ *0*
⟨*proof*⟩

lemma *minus-le-self-iff*: − *a* ≤ *a* ⟷ *0* ≤ *a*
⟨*proof*⟩

lemma *zero-le-iff-zero-nprt*: *0* ≤ *a* ⟷ *nprt a* = *0*
⟨*proof*⟩

lemma *le-zero-iff-zero-pprt*: *a* ≤ *0* ⟷ *pprt a* = *0*
⟨*proof*⟩

lemma *le-zero-iff-pprt-id*: *0* ≤ *a* ⟷ *pprt a* = *a*
⟨*proof*⟩

lemma *zero-le-iff-nprt-id*: *a* ≤ *0* ⟷ *nprt a* = *a*
⟨*proof*⟩

lemma *pprt-mono* [*simp*, *noatp*]: *a* ≤ *b* ⟹ *pprt a* ≤ *pprt b*
⟨*proof*⟩

lemma *nprt-mono* [*simp*, *noatp*]: *a* ≤ *b* ⟹ *nprt a* ≤ *nprt b*
⟨*proof*⟩

end

lemmas *add-sup-inf-distribs* = *add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left*

class *lordered-ab-group-add-abs* = *lordered-ab-group-add* + *abs* +
  assumes *abs-lattice*: |*a*| = *sup a* (− *a*)
begin

lemma *abs-prts*: |*a*| = *pprt a* − *nprt a*
⟨*proof*⟩

subclass *pordered-ab-group-add-abs*
⟨*proof*⟩

end

lemma *sup-eq-if*:
  fixes *a* :: 'a::{*lordered-ab-group-add*, *linorder*}
  shows *sup a* (− *a*) = (*if a* < *0 then* − *a else a*)
⟨*proof*⟩

lemma *abs-if-lattice*:
  fixes *a* :: 'a::{*lordered-ab-group-add-abs*, *linorder*}
  shows |*a*| = (*if a* < *0 then* − *a else a*)
⟨*proof*⟩

Needed for abelian cancellation simprocs:

lemma *add-cancel-21*: ((*x*::'a::*ab-group-add*) + (*y* + *z*) = *y* + *u*) = (*x* + *z* = *u*)
⟨*proof*⟩

lemma *add-cancel-end*: (*x* + (*y* + *z*) = *y*) = (*x* = − (*z*::'a::*ab-group-add*))
⟨*proof*⟩

lemma *less-eqI*: (*x*::'a::*pordered-ab-group-add*) − *y* = *x'* − *y'* ⟹ (*x* < *y*) = (*x'* < *y'*)
⟨*proof*⟩

lemma *le-eqI*: (*x*::'a::*pordered-ab-group-add*) − *y* = *x'* − *y'* ⟹ (*y* <= *x*) = (*y'* <= *x'*)
⟨*proof*⟩

lemma *eq-eqI*: (*x*::'a::*ab-group-add*) − *y* = *x'* − *y'* ⟹ (*x* = *y*) = (*x'* = *y'*)
⟨*proof*⟩

lemma *diff-def*: (*x*::'a::*ab-group-add*) − *y* == *x* + (−*y*)
⟨*proof*⟩

lemma *add-minus-cancel*: $(a::'a::ab\text{-}group\text{-}add) + (-a + b) = b$
⟨*proof*⟩

lemma *le-add-right-mono*:
assumes
$a <= b + (c::'a::pordered\text{-}ab\text{-}group\text{-}add)$
$c <= d$
shows $a <= b + d$
⟨*proof*⟩

lemma *estimate-by-abs*:
$a + b <= (c::'a::lordered\text{-}ab\text{-}group\text{-}add\text{-}abs) \Longrightarrow a <= c + abs\ b$
⟨*proof*⟩

10.7 Tools setup

lemma *add-mono-thms-ordered-semiring* [*noatp*]:
fixes $i\ j\ k :: 'a::pordered\text{-}ab\text{-}semigroup\text{-}add$
shows $i \leq j \wedge k \leq l \Longrightarrow i + k \leq j + l$
and $i = j \wedge k \leq l \Longrightarrow i + k \leq j + l$
and $i \leq j \wedge k = l \Longrightarrow i + k \leq j + l$
and $i = j \wedge k = l \Longrightarrow i + k = j + l$
⟨*proof*⟩

lemma *add-mono-thms-ordered-field* [*noatp*]:
fixes $i\ j\ k :: 'a::pordered\text{-}cancel\text{-}ab\text{-}semigroup\text{-}add$
shows $i < j \wedge k = l \Longrightarrow i + k < j + l$
and $i = j \wedge k < l \Longrightarrow i + k < j + l$
and $i < j \wedge k \leq l \Longrightarrow i + k < j + l$
and $i \leq j \wedge k < l \Longrightarrow i + k < j + l$
and $i < j \wedge k < l \Longrightarrow i + k < j + l$
⟨*proof*⟩

Simplification of $x - y < (0::'a)$, etc.

lemmas *diff-less-0-iff-less* [*simp*, *noatp*] = *less-iff-diff-less-0* [*symmetric*]
lemmas *diff-le-0-iff-le* [*simp*, *noatp*] = *le-iff-diff-le-0* [*symmetric*]

⟨*ML*⟩

end

11 Ring-and-Field: (Ordered) Rings and Fields

theory *Ring-and-Field*
imports *OrderedGroup*
begin

The theory of partially ordered rings is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

class *semiring* = *ab-semigroup-add* + *semigroup-mult* +
  assumes *left-distrib*[*algebra-simps*]: $(a + b) * c = a * c + b * c$
  assumes *right-distrib*[*algebra-simps*]: $a * (b + c) = a * b + a * c$
begin

For the *combine-numerals* simproc

lemma *combine-common-factor*:
  $a * e + (b * e + c) = (a + b) * e + c$
⟨*proof*⟩

end

class *mult-zero* = *times* + *zero* +
  assumes *mult-zero-left* [*simp*]: $0 * a = 0$
  assumes *mult-zero-right* [*simp*]: $a * 0 = 0$

class *semiring-0* = *semiring* + *comm-monoid-add* + *mult-zero*

class *semiring-0-cancel* = *semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0*
⟨*proof*⟩

end

class *comm-semiring* = *ab-semigroup-add* + *ab-semigroup-mult* +
  assumes *distrib*: $(a + b) * c = a * c + b * c$
begin

subclass *semiring*
⟨*proof*⟩

end

class *comm-semiring-0* = *comm-semiring* + *comm-monoid-add* + *mult-zero*

begin

subclass *semiring-0* ⟨*proof*⟩

end

class *comm-semiring-0-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *comm-semiring-0* ⟨*proof*⟩

end

class *zero-neq-one* = *zero* + *one* +
  assumes *zero-neq-one* [*simp*]: *0* ≠ *1*
begin

lemma *one-neq-zero* [*simp*]: *1* ≠ *0*
⟨*proof*⟩

end

class *semiring-1* = *zero-neq-one* + *semiring-0* + *monoid-mult*

Abstract divisibility

class *dvd* = *times*
begin

definition *dvd* :: *'a* ⇒ *'a* ⇒ *bool* (**infixl** *dvd 50*) **where**
  [*code del*]: *b dvd a* ⟷ (∃ *k*. *a* = *b* ∗ *k*)

lemma *dvdI* [*intro?*]: *a* = *b* ∗ *k* ⟹ *b dvd a*
  ⟨*proof*⟩

lemma *dvdE* [*elim?*]: *b dvd a* ⟹ (⋀*k*. *a* = *b* ∗ *k* ⟹ *P*) ⟹ *P*
  ⟨*proof*⟩

end

class *comm-semiring-1* = *zero-neq-one* + *comm-semiring-0* + *comm-monoid-mult*
+ *dvd*

begin

subclass *semiring-1* ⟨*proof*⟩

lemma *dvd-refl*[*simp*]: *a dvd a*

⟨*proof*⟩

lemma *dvd-trans*:
assumes *a dvd b* **and** *b dvd c*
shows *a dvd c*
⟨*proof*⟩

lemma *dvd-0-left-iff* [*noatp*, *simp*]: *0 dvd a* ⟷ *a = 0*
⟨*proof*⟩

lemma *dvd-0-right* [*iff*]: *a dvd 0*
⟨*proof*⟩

lemma *one-dvd* [*simp*]: *1 dvd a*
⟨*proof*⟩

lemma *dvd-mult*[*simp*]: *a dvd c* ⟹ *a dvd* (*b* ∗ *c*)
⟨*proof*⟩

lemma *dvd-mult2*[*simp*]: *a dvd b* ⟹ *a dvd* (*b* ∗ *c*)
⟨*proof*⟩

lemma *dvd-triv-right* [*simp*]: *a dvd b* ∗ *a*
⟨*proof*⟩

lemma *dvd-triv-left* [*simp*]: *a dvd a* ∗ *b*
⟨*proof*⟩

lemma *mult-dvd-mono*:
assumes *a dvd b*
and *c dvd d*
shows *a* ∗ *c dvd b* ∗ *d*
⟨*proof*⟩

lemma *dvd-mult-left*: *a* ∗ *b dvd c* ⟹ *a dvd c*
⟨*proof*⟩

lemma *dvd-mult-right*: *a* ∗ *b dvd c* ⟹ *b dvd c*
⟨*proof*⟩

lemma *dvd-0-left*: *0 dvd a* ⟹ *a = 0*
⟨*proof*⟩

lemma *dvd-add*[*simp*]:
assumes *a dvd b* **and** *a dvd c* **shows** *a dvd* (*b* + *c*)
⟨*proof*⟩

end

class *no-zero-divisors* = *zero* + *times* +
  assumes *no-zero-divisors*: $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$

class *semiring-1-cancel* = *semiring* + *cancel-comm-monoid-add*
  + *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *semiring-1* ⟨*proof*⟩

end

class *comm-semiring-1-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
  + *zero-neq-one* + *comm-monoid-mult*
begin

subclass *semiring-1-cancel* ⟨*proof*⟩
subclass *comm-semiring-0-cancel* ⟨*proof*⟩
subclass *comm-semiring-1* ⟨*proof*⟩

end

class *ring* = *semiring* + *ab-group-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

Distribution rules

lemma *minus-mult-left*: $-(a * b) = - a * b$
⟨*proof*⟩

lemma *minus-mult-right*: $-(a * b) = a * - b$
⟨*proof*⟩

Extract signs from products

lemmas *mult-minus-left* [*simp*, *noatp*] = *minus-mult-left* [*symmetric*]
lemmas *mult-minus-right* [*simp*,*noatp*] = *minus-mult-right* [*symmetric*]

lemma *minus-mult-minus* [*simp*]: $- a * - b = a * b$
⟨*proof*⟩

lemma *minus-mult-commute*: $- a * b = a * - b$
⟨*proof*⟩

lemma *right-diff-distrib*[*algebra-simps*]: $a * (b - c) = a * b - a * c$
⟨*proof*⟩

lemma *left-diff-distrib*[*algebra-simps*]: $(a - b) * c = a * c - b * c$
⟨*proof*⟩

lemmas *ring-distribs*[*noatp*] =
right-distrib left-distrib left-diff-distrib right-diff-distrib

Legacy - use *algebra-simps*

lemmas *ring-simps*[*noatp*] = *algebra-simps*

lemma *eq-add-iff1*:
$a * e + c = b * e + d \longleftrightarrow (a - b) * e + c = d$
⟨*proof*⟩

lemma *eq-add-iff2*:
$a * e + c = b * e + d \longleftrightarrow c = (b - a) * e + d$
⟨*proof*⟩

end

lemmas *ring-distribs*[*noatp*] =
right-distrib left-distrib left-diff-distrib right-diff-distrib

class *comm-ring* = *comm-semiring* + *ab-group-add*
begin

subclass *ring* ⟨*proof*⟩
subclass *comm-semiring-0-cancel* ⟨*proof*⟩

end

class *ring-1* = *ring* + *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-1-cancel* ⟨*proof*⟩

end

class *comm-ring-1* = *comm-ring* + *zero-neq-one* + *comm-monoid-mult*

begin

subclass *ring-1* ⟨*proof*⟩
subclass *comm-semiring-1-cancel* ⟨*proof*⟩

lemma *dvd-minus-iff* [*simp*]: x *dvd* $- y \longleftrightarrow x$ *dvd* y
⟨*proof*⟩

lemma *minus-dvd-iff* [*simp*]: $-$ x *dvd* $y \longleftrightarrow x$ *dvd* y
⟨*proof*⟩

lemma *dvd-diff*[*simp*]: $x\ dvd\ y \Longrightarrow x\ dvd\ z \Longrightarrow x\ dvd\ (y - z)$
⟨*proof*⟩

end

class *ring-no-zero-divisors* = *ring* + *no-zero-divisors*
begin

lemma *mult-eq-0-iff* [*simp*]:
shows $a * b = 0 \longleftrightarrow (a = 0 \vee b = 0)$
⟨*proof*⟩

Cancellation of equalities with a common factor

lemma *mult-cancel-right* [*simp*, *noatp*]:
$a * c = b * c \longleftrightarrow c = 0 \vee a = b$
⟨*proof*⟩

lemma *mult-cancel-left* [*simp*, *noatp*]:
$c * a = c * b \longleftrightarrow c = 0 \vee a = b$
⟨*proof*⟩

end

class *ring-1-no-zero-divisors* = *ring-1* + *ring-no-zero-divisors*
begin

lemma *mult-cancel-right1* [*simp*]:
$c = b * c \longleftrightarrow c = 0 \vee b = 1$
⟨*proof*⟩

lemma *mult-cancel-right2* [*simp*]:
$a * c = c \longleftrightarrow c = 0 \vee a = 1$
⟨*proof*⟩

lemma *mult-cancel-left1* [*simp*]:
$c = c * b \longleftrightarrow c = 0 \vee b = 1$
⟨*proof*⟩

lemma *mult-cancel-left2* [*simp*]:
$c * a = c \longleftrightarrow c = 0 \vee a = 1$
⟨*proof*⟩

end

class *idom* = *comm-ring-1* + *no-zero-divisors*
begin

subclass *ring-1-no-zero-divisors* ⟨*proof*⟩

lemma *square-eq-iff*: $a * a = b * b \longleftrightarrow (a = b \vee a = - b)$
⟨*proof*⟩

lemma *dvd-mult-cancel-right* [*simp*]:
$a * c$ *dvd* $b * c \longleftrightarrow c = 0 \vee a$ *dvd* b
⟨*proof*⟩

lemma *dvd-mult-cancel-left* [*simp*]:
$c * a$ *dvd* $c * b \longleftrightarrow c = 0 \vee a$ *dvd* b
⟨*proof*⟩

end

class *division-ring* = *ring-1* + *inverse* +
assumes *left-inverse* [*simp*]: $a \neq 0 \Longrightarrow$ *inverse* $a * a = 1$
assumes *right-inverse* [*simp*]: $a \neq 0 \Longrightarrow a *$ *inverse* $a = 1$
begin

subclass *ring-1-no-zero-divisors*
⟨*proof*⟩

lemma *nonzero-imp-inverse-nonzero*:
$a \neq 0 \Longrightarrow$ *inverse* $a \neq 0$
⟨*proof*⟩

lemma *inverse-zero-imp-zero*:
inverse $a = 0 \Longrightarrow a = 0$
⟨*proof*⟩

lemma *inverse-unique*:
assumes *ab*: $a * b = 1$
shows *inverse* $a = b$
⟨*proof*⟩

lemma *nonzero-inverse-minus-eq*:
$a \neq 0 \Longrightarrow$ *inverse* $(- a) = -$ *inverse* a
⟨*proof*⟩

lemma *nonzero-inverse-inverse-eq*:
$a \neq 0 \Longrightarrow$ *inverse* (*inverse* a) $= a$
⟨*proof*⟩

lemma *nonzero-inverse-eq-imp-eq*:
assumes *inverse* $a =$ *inverse* b **and** $a \neq 0$ **and** $b \neq 0$
shows $a = b$
⟨*proof*⟩

lemma *inverse-1* [*simp*]: *inverse* $1 = 1$

⟨*proof*⟩

lemma *nonzero-inverse-mult-distrib*:
assumes $a \neq 0$ **and** $b \neq 0$
shows $inverse\ (a * b) = inverse\ b * inverse\ a$
⟨*proof*⟩

lemma *division-ring-inverse-add*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ a + inverse\ b = inverse\ a * (a + b) * inverse\ b$
⟨*proof*⟩

lemma *division-ring-inverse-diff*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ a - inverse\ b = inverse\ a * (b - a) * inverse\ b$
⟨*proof*⟩

end

class *field* = *comm-ring-1* + *inverse* +
assumes *field-inverse*: $a \neq 0 \Longrightarrow inverse\ a * a = 1$
assumes *divide-inverse*: $a\ /\ b = a * inverse\ b$
begin

subclass *division-ring*
⟨*proof*⟩

subclass *idom* ⟨*proof*⟩

lemma *right-inverse-eq*: $b \neq 0 \Longrightarrow a\ /\ b = 1 \longleftrightarrow a = b$
⟨*proof*⟩

lemma *nonzero-inverse-eq-divide*: $a \neq 0 \Longrightarrow inverse\ a = 1\ /\ a$
⟨*proof*⟩

lemma *divide-self* [*simp*]: $a \neq 0 \Longrightarrow a\ /\ a = 1$
⟨*proof*⟩

lemma *divide-zero-left* [*simp*]: $0\ /\ a = 0$
⟨*proof*⟩

lemma *inverse-eq-divide*: $inverse\ a = 1\ /\ a$
⟨*proof*⟩

lemma *add-divide-distrib*: $(a+b)\ /\ c = a/c + b/c$
⟨*proof*⟩

There is no slick version using division by zero.

lemma *inverse-add*:
[| $a \neq 0$; $b \neq 0$ |]
==> $inverse\ a + inverse\ b = (a + b) * inverse\ a * inverse\ b$

⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-left* [*simp*, *noatp*]:
assumes [*simp*]: $b \neq 0$ **and** [*simp*]: $c \neq 0$ **shows** $(c*a)/(c*b) = a/b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-right* [*simp*, *noatp*]:
$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (a * c) \ / \ (b * c) = a \ / \ b$
⟨*proof*⟩

lemma *divide-1* [*simp*]: $a \ / \ 1 = a$
⟨*proof*⟩

lemma *times-divide-eq-right*: $a * (b \ / \ c) = (a * b) \ / \ c$
⟨*proof*⟩

lemma *times-divide-eq-left*: $(b \ / \ c) * a = (b * a) \ / \ c$
⟨*proof*⟩

These are later declared as simp rules.

lemmas *times-divide-eq* [*noatp*] = *times-divide-eq-right times-divide-eq-left*

lemma *add-frac-eq*:
assumes $y \neq 0$ **and** $z \neq 0$
shows $x \ / \ y + w \ / \ z = (x * z + w * y) \ / \ (y * z)$
⟨*proof*⟩

Special Cancellation Simprules for Division

lemma *nonzero-mult-divide-cancel-right* [*simp*, *noatp*]:
$b \neq 0 \Longrightarrow a * b \ / \ b = a$
⟨*proof*⟩

lemma *nonzero-mult-divide-cancel-left* [*simp*, *noatp*]:
$a \neq 0 \Longrightarrow a * b \ / \ a = b$
⟨*proof*⟩

lemma *nonzero-divide-mult-cancel-right* [*simp*, *noatp*]:
$[\![a \neq 0;\ b \neq 0]\!] \Longrightarrow b \ / \ (a * b) = 1 \ / \ a$
⟨*proof*⟩

lemma *nonzero-divide-mult-cancel-left* [*simp*, *noatp*]:
$[\![a \neq 0;\ b \neq 0]\!] \Longrightarrow a \ / \ (a * b) = 1 \ / \ b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-left2* [*simp*, *noatp*]:
$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (c * a) \ / \ (b * c) = a \ / \ b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-right2* [*simp*, *noatp*]:

$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (a * c) \ / \ (c * b) = a \ / \ b$
⟨*proof*⟩

lemma *minus-divide-left*: $-\ (a \ / \ b) = (-a) \ / \ b$
⟨*proof*⟩

lemma *nonzero-minus-divide-right*: $b \neq 0 \Longrightarrow -\ (a \ / \ b) = a \ / \ (-\ b)$
⟨*proof*⟩

lemma *nonzero-minus-divide-divide*: $b \neq 0 \Longrightarrow (-a) \ / \ (-b) = a \ / \ b$
⟨*proof*⟩

lemma *divide-minus-left* [*simp*, *noatp*]: $(-a) \ / \ b = -\ (a \ / \ b)$
⟨*proof*⟩

lemma *diff-divide-distrib*: $(a - b) \ / \ c = a \ / \ c - b \ / \ c$
⟨*proof*⟩

lemma *add-divide-eq-iff*:
$z \neq 0 \Longrightarrow x + y \ / \ z = (z * x + y) \ / \ z$
⟨*proof*⟩

lemma *divide-add-eq-iff*:
$z \neq 0 \Longrightarrow x \ / \ z + y = (x + z * y) \ / \ z$
⟨*proof*⟩

lemma *diff-divide-eq-iff*:
$z \neq 0 \Longrightarrow x - y \ / \ z = (z * x - y) \ / \ z$
⟨*proof*⟩

lemma *divide-diff-eq-iff*:
$z \neq 0 \Longrightarrow x \ / \ z - y = (x - z * y) \ / \ z$
⟨*proof*⟩

lemma *nonzero-eq-divide-eq*: $c \neq 0 \Longrightarrow a = b \ / \ c \longleftrightarrow a * c = b$
⟨*proof*⟩

lemma *nonzero-divide-eq-eq*: $c \neq 0 \Longrightarrow b \ / \ c = a \longleftrightarrow b = a * c$
⟨*proof*⟩

lemma *divide-eq-imp*: $c \neq 0 \Longrightarrow b = a * c \Longrightarrow b \ / \ c = a$
⟨*proof*⟩

lemma *eq-divide-imp*: $c \neq 0 \Longrightarrow a * c = b \Longrightarrow a = b \ / \ c$
⟨*proof*⟩

lemmas *field-eq-simps*[*noatp*] = *algebra-simps*

add-divide-eq-iff divide-add-eq-iff

diff-divide-eq-iff divide-diff-eq-iff

nonzero-eq-divide-eq nonzero-divide-eq-eq

An example:

lemma $[\![a{\neq}b;\ c{\neq}d;\ e{\neq}f]\!] \Longrightarrow ((a-b)*(c-d)*(e-f))/((c-d)*(e-f)*(a-b)) = 1$
⟨*proof*⟩

lemma *diff-frac-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x\ /\ y - w\ /\ z = (x * z - w * y)\ /\ (y * z)$
⟨*proof*⟩

lemma *frac-eq-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (x\ /\ y = w\ /\ z) = (x * z = w * y)$
⟨*proof*⟩

end

class *division-by-zero* = *zero* + *inverse* +
assumes *inverse-zero* [*simp*]: *inverse 0 = 0*

lemma *divide-zero* [*simp*]:
$a\ /\ 0 = (0::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-self-if* [*simp*]:
$a\ /\ (a::'a::\{field, division\text{-}by\text{-}zero\}) = (if\ a{=}0\ then\ 0\ else\ 1)$
⟨*proof*⟩

class *mult-mono* = *times* + *zero* + *ord* +
assumes *mult-left-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
assumes *mult-right-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow a * c \leq b * c$

class *pordered-semiring* = *mult-mono* + *semiring-0* + *pordered-ab-semigroup-add*

begin

lemma *mult-mono*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq b \Longrightarrow 0 \leq c$
$\Longrightarrow a * c \leq b * d$
⟨*proof*⟩

lemma *mult-mono'*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq a \Longrightarrow 0 \leq c$
$\Longrightarrow a * c \leq b * d$
⟨*proof*⟩

end

class *pordered-cancel-semiring* = *mult-mono* + *pordered-ab-semigroup-add* + *semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩
subclass *pordered-semiring* ⟨*proof*⟩

lemma *mult-nonneg-nonneg*: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

lemma *mult-nonneg-nonpos*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow a * b \leq 0$
⟨*proof*⟩

lemma *mult-nonpos-nonneg*: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow a * b \leq 0$
⟨*proof*⟩

Legacy - use *mult-nonpos-nonneg*

lemma *mult-nonneg-nonpos2*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow b * a \leq 0$
⟨*proof*⟩

lemma *split-mult-neg-le*: $(0 \leq a \;\&\; b \leq 0) \mid (a \leq 0 \;\&\; 0 \leq b) \Longrightarrow a * b \leq 0$
⟨*proof*⟩

end

class *ordered-semiring* = *semiring* + *comm-monoid-add* + *ordered-cancel-ab-semigroup-add* + *mult-mono*
begin

subclass *pordered-cancel-semiring* ⟨*proof*⟩

subclass *pordered-comm-monoid-add* ⟨*proof*⟩

lemma *mult-left-less-imp-less*:
$c * a < c * b \Longrightarrow 0 \leq c \Longrightarrow a < b$
⟨*proof*⟩

lemma *mult-right-less-imp-less*:
$a * c < b * c \Longrightarrow 0 \leq c \Longrightarrow a < b$
⟨*proof*⟩

end

class *ordered-semiring-strict* = *semiring* + *comm-monoid-add* + *ordered-cancel-ab-semigroup-add* +
assumes *mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
assumes *mult-strict-right-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow a * c < b * c$
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *ordered-semiring*
⟨*proof*⟩

lemma *mult-left-le-imp-le*:
$c * a \leq c * b \Longrightarrow 0 < c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *mult-right-le-imp-le*:
$a * c \leq b * c \Longrightarrow 0 < c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *mult-pos-pos*: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a * b$
⟨*proof*⟩

lemma *mult-pos-neg*: $0 < a \Longrightarrow b < 0 \Longrightarrow a * b < 0$
⟨*proof*⟩

lemma *mult-neg-pos*: $a < 0 \Longrightarrow 0 < b \Longrightarrow a * b < 0$
⟨*proof*⟩

Legacy - use *mult-neg-pos*

lemma *mult-pos-neg2*: $0 < a \Longrightarrow b < 0 \Longrightarrow b * a < 0$
⟨*proof*⟩

lemma *zero-less-mult-pos*:
$0 < a * b \Longrightarrow 0 < a \Longrightarrow 0 < b$
⟨*proof*⟩

lemma *zero-less-mult-pos2*:
$0 < b * a \Longrightarrow 0 < a \Longrightarrow 0 < b$
⟨*proof*⟩

Strict monotonicity in both arguments

lemma *mult-strict-mono*:
assumes $a < b$ **and** $c < d$ **and** $0 < b$ **and** $0 \leq c$
shows $a * c < b * d$
⟨*proof*⟩

This weaker variant has more natural premises

lemma *mult-strict-mono'*:
assumes $a < b$ **and** $c < d$ **and** $0 \leq a$ **and** $0 \leq c$
shows $a * c < b * d$
⟨*proof*⟩

lemma *mult-less-le-imp-less*:
assumes $a < b$ **and** $c \leq d$ **and** $0 \leq a$ **and** $0 < c$
shows $a * c < b * d$

⟨*proof*⟩

lemma *mult-le-less-imp-less*:
assumes $a \leq b$ **and** $c < d$ **and** $0 < a$ **and** $0 \leq c$
shows $a * c < b * d$
⟨*proof*⟩

lemma *mult-less-imp-less-left*:
assumes *less*: $c * a < c * b$ **and** *nonneg*: $0 \leq c$
shows $a < b$
⟨*proof*⟩

lemma *mult-less-imp-less-right*:
assumes *less*: $a * c < b * c$ **and** *nonneg*: $0 \leq c$
shows $a < b$
⟨*proof*⟩

end

class *mult-mono1* = *times* + *zero* + *ord* +
assumes *mult-mono1*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$

class *pordered-comm-semiring* = *comm-semiring-0*
+ *pordered-ab-semigroup-add* + *mult-mono1*
begin

subclass *pordered-semiring*
⟨*proof*⟩

end

class *pordered-cancel-comm-semiring* = *comm-semiring-0-cancel*
+ *pordered-ab-semigroup-add* + *mult-mono1*
begin

subclass *pordered-comm-semiring* ⟨*proof*⟩
subclass *pordered-cancel-semiring* ⟨*proof*⟩

end

class *ordered-comm-semiring-strict* = *comm-semiring-0* + *ordered-cancel-ab-semigroup-add*
+
assumes *mult-strict-left-mono-comm*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
begin

subclass *ordered-semiring-strict*
⟨*proof*⟩

subclass *pordered-cancel-comm-semiring*

⟨*proof*⟩

end

class *pordered-ring* = *ring* + *pordered-cancel-semiring*
begin

subclass *pordered-ab-group-add* ⟨*proof*⟩

Legacy - use *algebra-simps*

lemmas *ring-simps*[*noatp*] = *algebra-simps*

lemma *less-add-iff1*:
$a * e + c < b * e + d \longleftrightarrow (a - b) * e + c < d$
⟨*proof*⟩

lemma *less-add-iff2*:
$a * e + c < b * e + d \longleftrightarrow c < (b - a) * e + d$
⟨*proof*⟩

lemma *le-add-iff1*:
$a * e + c \leq b * e + d \longleftrightarrow (a - b) * e + c \leq d$
⟨*proof*⟩

lemma *le-add-iff2*:
$a * e + c \leq b * e + d \longleftrightarrow c \leq (b - a) * e + d$
⟨*proof*⟩

lemma *mult-left-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow c * a \leq c * b$
⟨*proof*⟩

lemma *mult-right-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow a * c \leq b * c$
⟨*proof*⟩

lemma *mult-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

lemma *split-mult-pos-le*:
$(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

end

class *abs-if* = *minus* + *uminus* + *ord* + *zero* + *abs* +
assumes *abs-if*: $|a| = (\mathit{if}\ a < 0\ \mathit{then} - a\ \mathit{else}\ a)$

class *sgn-if* = *minus* + *uminus* + *zero* + *one* + *ord* + *sgn* +

assumes *sgn-if*: $sgn\ x = (if\ x = 0\ then\ 0\ else\ if\ 0 < x\ then\ 1\ else\ -1)$

lemma (**in** *sgn-if*) *sgn0*[*simp*]: $sgn\ 0 = 0$
⟨*proof*⟩

class *ordered-ring* = *ring* + *ordered-semiring*
+ *ordered-ab-group-add* + *abs-if*
begin

subclass *pordered-ring* ⟨*proof*⟩

subclass *pordered-ab-group-add-abs*
⟨*proof*⟩

end

class *ordered-ring-strict* = *ring* + *ordered-semiring-strict*
+ *ordered-ab-group-add* + *abs-if*
begin

subclass *ordered-ring* ⟨*proof*⟩

lemma *mult-strict-left-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow c * a < c * b$
⟨*proof*⟩

lemma *mult-strict-right-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow a * c < b * c$
⟨*proof*⟩

lemma *mult-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow 0 < a * b$
⟨*proof*⟩

subclass *ring-no-zero-divisors*
⟨*proof*⟩

lemma *zero-less-mult-iff*:
$0 < a * b \longleftrightarrow 0 < a \wedge 0 < b \vee a < 0 \wedge b < 0$
⟨*proof*⟩

lemma *zero-le-mult-iff*:
$0 \leq a * b \longleftrightarrow 0 \leq a \wedge 0 \leq b \vee a \leq 0 \wedge b \leq 0$
⟨*proof*⟩

lemma *mult-less-0-iff*:
$a * b < 0 \longleftrightarrow 0 < a \wedge b < 0 \vee a < 0 \wedge 0 < b$
⟨*proof*⟩

lemma *mult-le-0-iff*:
$a * b \leq 0 \longleftrightarrow 0 \leq a \wedge b \leq 0 \vee a \leq 0 \wedge 0 \leq b$

⟨*proof*⟩

lemma *zero-le-square* [*simp*]: $0 \leq a * a$
⟨*proof*⟩

lemma *not-square-less-zero* [*simp*]: $\neg (a * a < 0)$
⟨*proof*⟩

Cancellation laws for $c * a < c * b$ and $a * c < b * c$, also with the relations $\leq$ and equality.

These "disjunction" versions produce two cases when the comparison is an assumption, but effectively four when the comparison is a goal.

lemma *mult-less-cancel-right-disj*:
$a * c < b * c \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
⟨*proof*⟩

lemma *mult-less-cancel-left-disj*:
$c * a < c * b \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
⟨*proof*⟩

The "conjunction of implication" lemmas produce two cases when the comparison is a goal, but give four when the comparison is an assumption.

lemma *mult-less-cancel-right*:
$a * c < b * c \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
⟨*proof*⟩

lemma *mult-less-cancel-left*:
$c * a < c * b \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
⟨*proof*⟩

lemma *mult-le-cancel-right*:
$a * c \leq b * c \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

lemma *mult-le-cancel-left*:
$c * a \leq c * b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

lemma *mult-le-cancel-left-pos*:
$0 < c \Longrightarrow c * a \leq c * b \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *mult-le-cancel-left-neg*:
$c < 0 \Longrightarrow c * a \leq c * b \longleftrightarrow b \leq a$
⟨*proof*⟩

lemma *mult-less-cancel-left-pos*:
$0 < c \Longrightarrow c * a < c * b \longleftrightarrow a < b$

⟨*proof*⟩

lemma *mult-less-cancel-left-neg*:
  $c < 0 \Longrightarrow c * a < c * b \longleftrightarrow b < a$
⟨*proof*⟩

end

Legacy - use *algebra-simps*

lemmas *ring-simps*[*noatp*] = *algebra-simps*

lemmas *mult-sign-intros* =
  mult-nonneg-nonneg mult-nonneg-nonpos
  mult-nonpos-nonneg mult-nonpos-nonpos
  mult-pos-pos mult-pos-neg
  mult-neg-pos mult-neg-neg

class *pordered-comm-ring* = *comm-ring* + *pordered-comm-semiring*
begin

subclass *pordered-ring* ⟨*proof*⟩
subclass *pordered-cancel-comm-semiring* ⟨*proof*⟩

end

class *ordered-semidom* = *comm-semiring-1-cancel* + *ordered-comm-semiring-strict*
+

  assumes *zero-less-one* [*simp*]: $0 < 1$
begin

lemma *pos-add-strict*:
  shows $0 < a \Longrightarrow b < c \Longrightarrow b < a + c$
  ⟨*proof*⟩

lemma *zero-le-one* [*simp*]: $0 \leq 1$
⟨*proof*⟩

lemma *not-one-le-zero* [*simp*]: $\neg\ 1 \leq 0$
⟨*proof*⟩

lemma *not-one-less-zero* [*simp*]: $\neg\ 1 < 0$
⟨*proof*⟩

lemma *less-1-mult*:
  assumes $1 < m$ **and** $1 < n$
  shows $1 < m * n$
  ⟨*proof*⟩

end

class *ordered-idom* = *comm-ring-1* +
ordered-comm-semiring-strict + *ordered-ab-group-add* +
abs-if + *sgn-if*

begin

subclass *ordered-ring-strict* ⟨*proof*⟩
subclass *pordered-comm-ring* ⟨*proof*⟩
subclass *idom* ⟨*proof*⟩

subclass *ordered-semidom*
⟨*proof*⟩

lemma *linorder-neqE-ordered-idom*:
assumes $x \neq y$ **obtains** $x < y \mid y < x$
⟨*proof*⟩

These cancellation simprules also produce two cases when the comparison is a goal.

lemma *mult-le-cancel-right1*:
$c \leq b * c \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
⟨*proof*⟩

lemma *mult-le-cancel-right2*:
$a * c \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
⟨*proof*⟩

lemma *mult-le-cancel-left1*:
$c \leq c * b \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
⟨*proof*⟩

lemma *mult-le-cancel-left2*:
$c * a \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
⟨*proof*⟩

lemma *mult-less-cancel-right1*:
$c < b * c \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
⟨*proof*⟩

lemma *mult-less-cancel-right2*:
$a * c < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
⟨*proof*⟩

lemma *mult-less-cancel-left1*:
$c < c * b \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
⟨*proof*⟩

lemma *mult-less-cancel-left2*:
$c * a < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
⟨*proof*⟩

lemma *sgn-sgn* [*simp*]:
$sgn\ (sgn\ a) = sgn\ a$
⟨*proof*⟩

lemma *sgn-0-0*:
$sgn\ a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *sgn-1-pos*:
$sgn\ a = 1 \longleftrightarrow a > 0$
⟨*proof*⟩

lemma *sgn-1-neg*:
$sgn\ a = -\ 1 \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *sgn-pos* [*simp*]:
$0 < a \Longrightarrow sgn\ a = 1$
⟨*proof*⟩

lemma *sgn-neg* [*simp*]:
$a < 0 \Longrightarrow sgn\ a = -\ 1$
⟨*proof*⟩

lemma *sgn-times*:
$sgn\ (a * b) = sgn\ a * sgn\ b$
⟨*proof*⟩

lemma *abs-sgn*: $abs\ k = k * sgn\ k$
⟨*proof*⟩

lemma *sgn-greater* [*simp*]:
$0 < sgn\ a \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *sgn-less* [*simp*]:
$sgn\ a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *abs-dvd-iff* [*simp*]: $(abs\ m)\ dvd\ k \longleftrightarrow m\ dvd\ k$
⟨*proof*⟩

lemma *dvd-abs-iff* [*simp*]: $m\ dvd\ (abs\ k) \longleftrightarrow m\ dvd\ k$
⟨*proof*⟩

end

class *ordered-field* = *field* + *ordered-idom*

Simprules for comparisons where common factors can be cancelled.

lemmas *mult-compare-simps*[*noatp*] =
mult-le-cancel-right mult-le-cancel-left
mult-le-cancel-right1 mult-le-cancel-right2
mult-le-cancel-left1 mult-le-cancel-left2
mult-less-cancel-right mult-less-cancel-left
mult-less-cancel-right1 mult-less-cancel-right2
mult-less-cancel-left1 mult-less-cancel-left2
mult-cancel-right mult-cancel-left
mult-cancel-right1 mult-cancel-right2
mult-cancel-left1 mult-cancel-left2

— FIXME continue localization here

lemma *inverse-nonzero-iff-nonzero* [*simp*]:
(*inverse a* = *0*) = (*a* = (*0*::'*a*::{*division-ring*,*division-by-zero*}))
⟨*proof*⟩

lemma *inverse-minus-eq* [*simp*]:
inverse(−*a*) = −*inverse*(*a*::'*a*::{*division-ring*,*division-by-zero*})
⟨*proof*⟩

lemma *inverse-eq-imp-eq*:
inverse a = *inverse b* ==> *a* = (*b*::'*a*::{*division-ring*,*division-by-zero*})
⟨*proof*⟩

lemma *inverse-eq-iff-eq* [*simp*]:
(*inverse a* = *inverse b*) = (*a* = (*b*::'*a*::{*division-ring*,*division-by-zero*}))
⟨*proof*⟩

lemma *inverse-inverse-eq* [*simp*]:
inverse(*inverse* (*a*::'*a*::{*division-ring*,*division-by-zero*})) = *a*
⟨*proof*⟩

This version builds in division by zero while also re-orienting the right-hand side.

lemma *inverse-mult-distrib* [*simp*]:
inverse(*a*∗*b*) = *inverse*(*a*) ∗ *inverse*(*b*::'*a*::{*field*,*division-by-zero*})
⟨*proof*⟩

lemma *inverse-divide* [*simp*]:
inverse (*a*/*b*) = *b* / (*a*::'*a*::{*field*,*division-by-zero*})
⟨*proof*⟩

11.1 Calculations with fractions

There is a whole bunch of simp-rules just for class *field* but none for class *field* and *nonzero-divides* because the latter are covered by a simproc.

lemma *mult-divide-mult-cancel-left*:
$c \neq 0 \Longrightarrow (c*a) \,/\, (c*b) = a \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *mult-divide-mult-cancel-right*:
$c \neq 0 \Longrightarrow (a*c) \,/\, (b*c) = a \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-divide-eq-right* [*simp*,*noatp*]:
$a \,/\, (b/c) = (a*c) \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-divide-eq-left* [*simp*,*noatp*]:
$(a \,/\, b) \,/\, (c::'a::\{field, division\text{-}by\text{-}zero\}) = a \,/\, (b*c)$
⟨*proof*⟩

11.1.1 Special Cancellation Simprules for Division

lemma *mult-divide-mult-cancel-left-if* [*simp*,*noatp*]:
fixes $c :: 'a :: \{field, division\text{-}by\text{-}zero\}$
shows $(c*a) \,/\, (c*b) = (\textit{if } c=0 \textit{ then } 0 \textit{ else } a/b)$
⟨*proof*⟩

11.2 Division and Unary Minus

lemma *minus-divide-right*: $-(a/b) = a \,/\, -(b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-minus-right* [*simp*, *noatp*]:
$a \,/\, -(b::'a::\{field, division\text{-}by\text{-}zero\}) = -(a \,/\, b)$
⟨*proof*⟩

lemma *minus-divide-divide*:
$(-a)/(-b) = a \,/\, (b::'a::\{field, division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *eq-divide-eq*:
$((a::'a::\{field, division\text{-}by\text{-}zero\}) = b/c) = (\textit{if } c \neq 0 \textit{ then } a*c = b \textit{ else } a=0)$
⟨*proof*⟩

lemma *divide-eq-eq*:
$(b/c = (a::'a::\{field, division\text{-}by\text{-}zero\})) = (\textit{if } c \neq 0 \textit{ then } b = a*c \textit{ else } a=0)$
⟨*proof*⟩

11.3 Ordered Fields

lemma *positive-imp-inverse-positive*:
assumes *a-gt-0*: $0 < a$ **shows** $0 < inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *negative-imp-inverse-negative*:
$a < 0 \Longrightarrow inverse\ a < (0::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-le-imp-le*:
assumes *invle*: $inverse\ a \leq inverse\ b$ **and** *apos*: $0 < a$
shows $b \leq (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-positive-imp-positive*:
assumes *inv-gt-0*: $0 < inverse\ a$ **and** *nz*: $a \neq 0$
shows $0 < (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-positive-iff-positive* [*simp*]:
$(0 < inverse\ a) = (0 < (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-negative-imp-negative*:
assumes *inv-less-0*: $inverse\ a < 0$ **and** *nz*: $a \neq 0$
shows $a < (0::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-negative-iff-negative* [*simp*]:
$(inverse\ a < 0) = (a < (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-nonnegative-iff-nonnegative* [*simp*]:
$(0 \leq inverse\ a) = (0 \leq (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-nonpositive-iff-nonpositive* [*simp*]:
$(inverse\ a \leq 0) = (a \leq (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *ordered-field-no-lb*: $\forall\ x.\ \exists\ y.\ y < (x::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *ordered-field-no-ub*: $\forall\ x.\ \exists\ y.\ y > (x::'a::ordered\text{-}field)$
⟨*proof*⟩

11.4 Anti-Monotonicity of *inverse*

lemma *less-imp-inverse-less*:

assumes *less*: $a < b$ **and** *apos*: $0 < a$
shows *inverse* $b <$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

lemma *inverse-less-imp-less*:
$[|$*inverse* $a <$ *inverse* $b; 0 < a|] ==> b < (a::'a::$*ordered-field*$)$
⟨*proof*⟩

Both premises are essential. Consider -1 and 1.

lemma *inverse-less-iff-less* [*simp*,*noatp*]:
$[|0 < a; 0 < b|] ==>$ (*inverse* $a <$ *inverse* $b) = (b < (a::'a::$*ordered-field*$))$
⟨*proof*⟩

lemma *le-imp-inverse-le*:
$[|a \leq b; 0 < a|] ==>$ *inverse* $b \leq$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

lemma *inverse-le-iff-le* [*simp*,*noatp*]:
$[|0 < a; 0 < b|] ==>$ (*inverse* $a \leq$ *inverse* $b) = (b \leq (a::'a::$*ordered-field*$))$
⟨*proof*⟩

These results refer to both operands being negative. The opposite-sign case is trivial, since inverse preserves signs.

lemma *inverse-le-imp-le-neg*:
$[|$*inverse* $a \leq$ *inverse* $b; b < 0|] ==> b \leq (a::'a::$*ordered-field*$)$
⟨*proof*⟩

lemma *less-imp-inverse-less-neg*:
$[|a < b; b < 0|] ==>$ *inverse* $b <$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

lemma *inverse-less-imp-less-neg*:
$[|$*inverse* $a <$ *inverse* $b; b < 0|] ==> b < (a::'a::$*ordered-field*$)$
⟨*proof*⟩

lemma *inverse-less-iff-less-neg* [*simp*,*noatp*]:
$[|a < 0; b < 0|] ==>$ (*inverse* $a <$ *inverse* $b) = (b < (a::'a::$*ordered-field*$))$
⟨*proof*⟩

lemma *le-imp-inverse-le-neg*:
$[|a \leq b; b < 0|] ==>$ *inverse* $b \leq$ *inverse* $(a::'a::$*ordered-field*$)$
⟨*proof*⟩

lemma *inverse-le-iff-le-neg* [*simp*,*noatp*]:
$[|a < 0; b < 0|] ==>$ (*inverse* $a \leq$ *inverse* $b) = (b \leq (a::'a::$*ordered-field*$))$
⟨*proof*⟩

11.5 Inverses and the Number One

lemma *one-less-inverse-iff*:
$(1 < inverse\ x) = (0 < x\ \&\ x < (1::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-eq-1-iff* [*simp*]:
$(inverse\ x = 1) = (x = (1::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *one-le-inverse-iff*:
$(1 \leq inverse\ x) = (0 < x\ \&\ x \leq (1::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-less-1-iff*:
$(inverse\ x < 1) = (x \leq 0 \mid 1 < (x::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-le-1-iff*:
$(inverse\ x \leq 1) = (x \leq 0 \mid 1 \leq (x::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

11.6 Simplification of Inequalities Involving Literal Divisors

lemma *pos-le-divide-eq*: $0 < (c::'a::ordered\text{-}field) ==> (a \leq b/c) = (a*c \leq b)$
⟨*proof*⟩

lemma *neg-le-divide-eq*: $c < (0::'a::ordered\text{-}field) ==> (a \leq b/c) = (b \leq a*c)$
⟨*proof*⟩

lemma *le-divide-eq*:
$(a \leq b/c) =$
$(if\ 0 < c\ then\ a*c \leq b$
$\quad else\ if\ c < 0\ then\ b \leq a*c$
$\quad else\ \ a \leq (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *pos-divide-le-eq*: $0 < (c::'a::ordered\text{-}field) ==> (b/c \leq a) = (b \leq a*c)$
⟨*proof*⟩

lemma *neg-divide-le-eq*: $c < (0::'a::ordered\text{-}field) ==> (b/c \leq a) = (a*c \leq b)$
⟨*proof*⟩

lemma *divide-le-eq*:
$(b/c \leq a) =$
$(if\ 0 < c\ then\ b \leq a*c$
$\quad else\ if\ c < 0\ then\ a*c \leq b$
$\quad else\ 0 \leq (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *pos-less-divide-eq*:
$0 < (c::'a::ordered\text{-}field) ==> (a < b/c) = (a*c < b)$
⟨*proof*⟩

lemma *neg-less-divide-eq*:
$c < (0::'a::ordered\text{-}field) ==> (a < b/c) = (b < a*c)$
⟨*proof*⟩

lemma *less-divide-eq*:
$(a < b/c) =$
$(if\ 0 < c\ then\ a*c < b$
$\quad else\ if\ c < 0\ then\ b < a*c$
$\quad else\ \ a < (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *pos-divide-less-eq*:
$0 < (c::'a::ordered\text{-}field) ==> (b/c < a) = (b < a*c)$
⟨*proof*⟩

lemma *neg-divide-less-eq*:
$c < (0::'a::ordered\text{-}field) ==> (b/c < a) = (a*c < b)$
⟨*proof*⟩

lemma *divide-less-eq*:
$(b/c < a) =$
$(if\ 0 < c\ then\ b < a*c$
$\quad else\ if\ c < 0\ then\ a*c < b$
$\quad else\ 0 < (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

11.7 Field simplification

Lemmas *field-simps* multiply with denominators in in(equations) if they can be proved to be non-zero (for equations) or positive/negative (for inequations). Can be too aggressive and is therefore separate from the more benign *algebra-simps*.

lemmas *field-simps*[*noatp*] = *field-eq-simps*

pos-divide-less-eq neg-divide-less-eq
pos-less-divide-eq neg-less-divide-eq
pos-divide-le-eq neg-divide-le-eq
pos-le-divide-eq neg-le-divide-eq

Lemmas *sign-simps* is a first attempt to automate proofs of positivity/negativity needed for *field-simps*. Have not added *sign-simps* to *field-simps* because the former can lead to case explosions.

lemmas *sign-simps*[*noatp*] = *group-simps*
zero-less-mult-iff *mult-less-0-iff*

11.8 Division and Signs

lemma *zero-less-divide-iff*:
$((0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}) < a/b) = (0 < a \ \& \ 0 < b \mid a < 0 \ \& \ b < 0)$
⟨*proof*⟩

lemma *divide-less-0-iff*:
$(a/b < (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) =$
$(0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b)$
⟨*proof*⟩

lemma *zero-le-divide-iff*:
$((0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}) \leq a/b) =$
$(0 \leq a \ \& \ 0 \leq b \mid a \leq 0 \ \& \ b \leq 0)$
⟨*proof*⟩

lemma *divide-le-0-iff*:
$(a/b \leq (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) =$
$(0 \leq a \ \& \ b \leq 0 \mid a \leq 0 \ \& \ 0 \leq b)$
⟨*proof*⟩

lemma *divide-eq-0-iff* [*simp*,*noatp*]:
$(a/b = 0) = (a=0 \mid b=(0::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *divide-pos-pos*:
$0 < (x::'a::ordered\text{-}field) \Longrightarrow 0 < y \Longrightarrow 0 < x \ / \ y$
⟨*proof*⟩

lemma *divide-nonneg-pos*:
$0 <= (x::'a::ordered\text{-}field) \Longrightarrow 0 < y \Longrightarrow 0 <= x \ / \ y$
⟨*proof*⟩

lemma *divide-neg-pos*:
$(x::'a::ordered\text{-}field) < 0 \Longrightarrow 0 < y \Longrightarrow x \ / \ y < 0$
⟨*proof*⟩

lemma *divide-nonpos-pos*:
$(x::'a::ordered\text{-}field) <= 0 \Longrightarrow 0 < y \Longrightarrow x \ / \ y <= 0$
⟨*proof*⟩

lemma *divide-pos-neg*:
$0 < (x::'a::ordered\text{-}field) \Longrightarrow y < 0 \Longrightarrow x \ / \ y < 0$
⟨*proof*⟩

lemma *divide-nonneg-neg*:
$0 <= (x::'a::ordered\text{-}field) \Longrightarrow y < 0 \Longrightarrow x \ / \ y <= 0$
⟨*proof*⟩

lemma *divide-neg-neg*:
$(x::'a::ordered\text{-}field) < 0 ==> y < 0 ==> 0 < x / y$
⟨*proof*⟩

lemma *divide-nonpos-neg*:
$(x::'a::ordered\text{-}field) <= 0 ==> y < 0 ==> 0 <= x / y$
⟨*proof*⟩

11.9 Cancellation Laws for Division

lemma *divide-cancel-right* [*simp*,*noatp*]:
$(a/c = b/c) = (c = 0 \mid a = (b::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *divide-cancel-left* [*simp*,*noatp*]:
$(c/a = c/b) = (c = 0 \mid a = (b::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

11.10 Division and the Number One

Simplify expressions equated with 1

lemma *divide-eq-1-iff* [*simp*,*noatp*]:
$(a/b = 1) = (b \neq 0 \ \& \ a = (b::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *one-eq-divide-iff* [*simp*,*noatp*]:
$(1 = a/b) = (b \neq 0 \ \& \ a = (b::'a::\{field, division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *zero-eq-1-divide-iff* [*simp*,*noatp*]:
$((0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\}) = 1/a) = (a = 0)$
⟨*proof*⟩

lemma *one-divide-eq-0-iff* [*simp*,*noatp*]:
$(1/a = (0::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) = (a = 0)$
⟨*proof*⟩

Simplify expressions such as $0 < 1/x$ to $0 < x$

lemmas *zero-less-divide-1-iff* = *zero-less-divide-iff* [*of 1*, *simplified*]
lemmas *divide-less-0-1-iff* = *divide-less-0-iff* [*of 1*, *simplified*]
lemmas *zero-le-divide-1-iff* = *zero-le-divide-iff* [*of 1*, *simplified*]
lemmas *divide-le-0-1-iff* = *divide-le-0-iff* [*of 1*, *simplified*]

declare *zero-less-divide-1-iff* [*simp*,*noatp*]
declare *divide-less-0-1-iff* [*simp*,*noatp*]
declare *zero-le-divide-1-iff* [*simp*,*noatp*]
declare *divide-le-0-1-iff* [*simp*,*noatp*]

11.11 Ordering Rules for Division

lemma *divide-strict-right-mono*:
[|*a* < *b*; *0* < *c*|] ==> *a* / *c* < *b* / (*c*::'*a*::*ordered-field*)
⟨*proof*⟩

lemma *divide-right-mono*:
[|*a* ≤ *b*; *0* ≤ *c*|] ==> *a*/*c* ≤ *b*/(*c*::'*a*::{*ordered-field*,*division-by-zero*})
⟨*proof*⟩

lemma *divide-right-mono-neg*: (*a*::'*a*::{*division-by-zero*,*ordered-field*}) <= *b*
==> *c* <= *0* ==> *b* / *c* <= *a* / *c*
⟨*proof*⟩

lemma *divide-strict-right-mono-neg*:
[|*b* < *a*; *c* < *0*|] ==> *a* / *c* < *b* / (*c*::'*a*::*ordered-field*)
⟨*proof*⟩

The last premise ensures that *a* and *b* have the same sign

lemma *divide-strict-left-mono*:
[|*b* < *a*; *0* < *c*; *0* < *a*∗*b*|] ==> *c* / *a* < *c* / (*b*::'*a*::*ordered-field*)
⟨*proof*⟩

lemma *divide-left-mono*:
[|*b* ≤ *a*; *0* ≤ *c*; *0* < *a*∗*b*|] ==> *c* / *a* ≤ *c* / (*b*::'*a*::*ordered-field*)
⟨*proof*⟩

lemma *divide-left-mono-neg*: (*a*::'*a*::{*division-by-zero*,*ordered-field*}) <= *b*
==> *c* <= *0* ==> *0* < *a* ∗ *b* ==> *c* / *a* <= *c* / *b*
⟨*proof*⟩

lemma *divide-strict-left-mono-neg*:
[|*a* < *b*; *c* < *0*; *0* < *a*∗*b*|] ==> *c* / *a* < *c* / (*b*::'*a*::*ordered-field*)
⟨*proof*⟩

Simplify quotients that are compared with the value 1.

lemma *le-divide-eq-1* [*noatp*]:
fixes *a* :: '*a* :: {*ordered-field*,*division-by-zero*}
shows (*1* ≤ *b* / *a*) = ((*0* < *a* & *a* ≤ *b*) | (*a* < *0* & *b* ≤ *a*))
⟨*proof*⟩

lemma *divide-le-eq-1* [*noatp*]:
fixes *a* :: '*a* :: {*ordered-field*,*division-by-zero*}
shows (*b* / *a* ≤ *1*) = ((*0* < *a* & *b* ≤ *a*) | (*a* < *0* & *a* ≤ *b*) | *a*=*0*)
⟨*proof*⟩

lemma *less-divide-eq-1* [*noatp*]:
fixes *a* :: '*a* :: {*ordered-field*,*division-by-zero*}
shows (*1* < *b* / *a*) = ((*0* < *a* & *a* < *b*) | (*a* < *0* & *b* < *a*))
⟨*proof*⟩

lemma *divide-less-eq-1* [*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $(b \ / \ a < 1) = ((0 < a \ \& \ b < a) \mid (a < 0 \ \& \ a < b) \mid a=0)$
⟨*proof*⟩

11.12 Conditional Simplification Rules: No Case Splits

lemma *le-divide-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (1 \leq b/a) = (a \leq b)$
⟨*proof*⟩

lemma *le-divide-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow (1 \leq b/a) = (b \leq a)$
⟨*proof*⟩

lemma *divide-le-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (b/a \leq 1) = (b \leq a)$
⟨*proof*⟩

lemma *divide-le-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow (b/a \leq 1) = (a \leq b)$
⟨*proof*⟩

lemma *less-divide-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (1 < b/a) = (a < b)$
⟨*proof*⟩

lemma *less-divide-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow (1 < b/a) = (b < a)$
⟨*proof*⟩

lemma *divide-less-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (b/a < 1) = (b < a)$
⟨*proof*⟩

lemma *divide-less-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow b/a < 1 <-> a < b$
⟨*proof*⟩

lemma *eq-divide-eq-1* [*simp*,*noatp*]:

fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $(1 = b/a) = ((a \neq 0 \ \& \ a = b))$
⟨*proof*⟩

lemma *divide-eq-eq-1* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $(b/a = 1) = ((a \neq 0 \ \& \ a = b))$
⟨*proof*⟩

11.13 Reasoning about inequalities with division

lemma *mult-right-le-one-le*: *0* <= (*x*::'*a*::*ordered-idom*) ==> *0* <= *y* ==> *y* <= *1*
==> *x* * *y* <= *x*
⟨*proof*⟩

lemma *mult-left-le-one-le*: *0* <= (*x*::'*a*::*ordered-idom*) ==> *0* <= *y* ==> *y* <= *1*
==> *y* * *x* <= *x*
⟨*proof*⟩

lemma *mult-imp-div-pos-le*: *0* < (*y*::'*a*::*ordered-field*) ==> *x* <= *z* * *y* ==>
x / *y* <= *z*
⟨*proof*⟩

lemma *mult-imp-le-div-pos*: *0* < (*y*::'*a*::*ordered-field*) ==> *z* * *y* <= *x* ==>
z <= *x* / *y*
⟨*proof*⟩

lemma *mult-imp-div-pos-less*: *0* < (*y*::'*a*::*ordered-field*) ==> *x* < *z* * *y* ==>
x / *y* < *z*
⟨*proof*⟩

lemma *mult-imp-less-div-pos*: *0* < (*y*::'*a*::*ordered-field*) ==> *z* * *y* < *x* ==>
z < *x* / *y*
⟨*proof*⟩

lemma *frac-le*: (*0*::'*a*::*ordered-field*) <= *x* ==>
x <= *y* ==> *0* < *w* ==> *w* <= *z* ==> *x* / *z* <= *y* / *w*
⟨*proof*⟩

lemma *frac-less*: (*0*::'*a*::*ordered-field*) <= *x* ==>
x < *y* ==> *0* < *w* ==> *w* <= *z* ==> *x* / *z* < *y* / *w*
⟨*proof*⟩

lemma *frac-less2*: (*0*::'*a*::*ordered-field*) < *x* ==>
x <= *y* ==> *0* < *w* ==> *w* < *z* ==> *x* / *z* < *y* / *w*
⟨*proof*⟩

It's not obvious whether these should be simprules or not. Their effect is

to gather terms into one big fraction, like a*b*c / x*y*z. The rationale for that is unclear, but many proofs seem to need them.

declare *times-divide-eq* [*simp*]

11.14 Ordered Fields are Dense

context *ordered-semidom*
begin

lemma *less-add-one*: $a < a + 1$
⟨*proof*⟩

lemma *zero-less-two*: $0 < 1 + 1$
⟨*proof*⟩

end

lemma *less-half-sum*: $a < b \Longrightarrow a < (a+b) / (1+1::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *gt-half-sum*: $a < b \Longrightarrow (a+b)/(1+1::'a::ordered\text{-}field) < b$
⟨*proof*⟩

instance *ordered-field* < *dense-linear-order*
⟨*proof*⟩

11.15 Absolute Value

context *ordered-idom*
begin

lemma *mult-sgn-abs*: $sgn\ x * abs\ x = x$
⟨*proof*⟩

end

lemma *abs-one* [*simp*]: $abs\ 1 = (1::'a::ordered\text{-}idom)$
⟨*proof*⟩

class *pordered-ring-abs* = *pordered-ring* + *pordered-ab-group-add-abs* +
assumes *abs-eq-mult*:
$(0 \leq a \vee a \leq 0) \wedge (0 \leq b \vee b \leq 0) \Longrightarrow |a * b| = |a| * |b|$

class *lordered-ring* = *pordered-ring* + *lordered-ab-group-add-abs*
begin

subclass *lordered-ab-group-add-meet* ⟨*proof*⟩
subclass *lordered-ab-group-add-join* ⟨*proof*⟩

end

lemma *abs-le-mult*: $abs\ (a * b) \leq (abs\ a) * (abs\ (b::'a::lordered\text{-}ring))$
⟨*proof*⟩

instance *lordered-ring* ⊆ *pordered-ring-abs*
⟨*proof*⟩

instance *ordered-idom* ⊆ *pordered-ring-abs*
⟨*proof*⟩

lemma *abs-mult*: $abs\ (a * b) = abs\ a * abs\ (b::'a::ordered\text{-}idom)$
⟨*proof*⟩

lemma *abs-mult-self*: $abs\ a * abs\ a = a * (a::'a::ordered\text{-}idom)$
⟨*proof*⟩

lemma *nonzero-abs-inverse*:
$a \neq 0 \Longrightarrow abs\ (inverse\ (a::'a::ordered\text{-}field)) = inverse\ (abs\ a)$
⟨*proof*⟩

lemma *abs-inverse* [*simp*]:
$abs\ (inverse\ (a::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) =$
$inverse\ (abs\ a)$
⟨*proof*⟩

lemma *nonzero-abs-divide*:
$b \neq 0 \Longrightarrow abs\ (a\ /\ (b::'a::ordered\text{-}field)) = abs\ a\ /\ abs\ b$
⟨*proof*⟩

lemma *abs-divide* [*simp*]:
$abs\ (a\ /\ (b::'a::\{ordered\text{-}field, division\text{-}by\text{-}zero\})) = abs\ a\ /\ abs\ b$
⟨*proof*⟩

lemma *abs-mult-less*:
$[|\ abs\ a < c;\ abs\ b < d\ |] \Longrightarrow abs\ a * abs\ b < c*(d::'a::ordered\text{-}idom)$
⟨*proof*⟩

lemmas *eq-minus-self-iff*[*noatp*] = *equal-neg-zero*

lemma *less-minus-self-iff*: $(a < -a) = (a < (0::'a::ordered\text{-}idom))$
⟨*proof*⟩

lemma *abs-less-iff*: $(abs\ a < b) = (a < b\ \&\ -a < (b::'a::ordered\text{-}idom))$
⟨*proof*⟩

lemma *abs-mult-pos*: $(0::'a::ordered\text{-}idom) <= x \Longrightarrow$
$(abs\ y) * x = abs\ (y * x)$

⟨*proof*⟩

lemma *abs-div-pos*: (*0*::'*a*::{*division-by-zero*,*ordered-field*}) < *y* ==>
abs x / *y* = *abs* (*x* / *y*)
⟨*proof*⟩

11.16 Bounds of products via negative and positive Part

lemma *mult-le-prts*:
assumes
a1 <= (*a*::'*a*::*lordered-ring*)
a <= *a2*
b1 <= *b*
b <= *b2*
shows
a * *b* <= *pprt a2* * *pprt b2* + *pprt a1* * *nprt b2* + *nprt a2* * *pprt b1* + *nprt a1* * *nprt b1*
⟨*proof*⟩

lemma *mult-ge-prts*:
assumes
a1 <= (*a*::'*a*::*lordered-ring*)
a <= *a2*
b1 <= *b*
b <= *b2*
shows
a * *b* >= *nprt a1* * *pprt b2* + *nprt a2* * *nprt b2* + *pprt a1* * *pprt b1* + *pprt a2* * *nprt b1*
⟨*proof*⟩

end

12 Nat: Natural numbers

theory *Nat*
imports *Inductive Ring-and-Field*
uses
~~/*src*/*Tools*/*rat.ML*
~~/*src*/*Provers*/*Arith*/*cancel-sums.ML*
Tools/*arith-data.ML*
(*Tools*/*nat-arith.ML*)
~~/*src*/*Provers*/*Arith*/*fast-lin-arith.ML*
(*Tools*/*lin-arith.ML*)
begin

12.1 Type *ind*

typedecl *ind*

axiomatization
  Zero-Rep :: *ind* **and**
  Suc-Rep :: *ind* => *ind*
where
  — the axiom of infinity in 2 parts
  inj-Suc-Rep: *inj Suc-Rep* **and**
  Suc-Rep-not-Zero-Rep: *Suc-Rep x* ≠ *Zero-Rep*

12.2 Type nat

Type definition

inductive *Nat* :: *ind* ⇒ *bool*
where
    Zero-RepI: *Nat Zero-Rep*
  | *Suc-RepI*: *Nat i* ⟹ *Nat* (*Suc-Rep i*)

global

typedef (**open** *Nat*)
  nat = *Nat*
  ⟨*proof*⟩

constdefs
  Suc :: *nat* => *nat*
  Suc-def: *Suc* == (%*n*. *Abs-Nat* (*Suc-Rep* (*Rep-Nat n*)))

local

instantiation *nat* :: *zero*
begin

definition *Zero-nat-def* [*code del*]:
  0 = *Abs-Nat Zero-Rep*

instance ⟨*proof*⟩

end

lemma *Suc-not-Zero*: *Suc m* ≠ *0*
  ⟨*proof*⟩

lemma *Zero-not-Suc*: *0* ≠ *Suc m*
  ⟨*proof*⟩

rep-datatype *0* :: *nat Suc*
  ⟨*proof*⟩

lemma *nat-induct* [*case-names 0 Suc*, *induct type*: *nat*]:

— for backward compatibility – names of variables differ
fixes n
assumes $P\ 0$
and $\bigwedge n.\ P\ n \Longrightarrow P\ (Suc\ n)$
shows $P\ n$
⟨*proof*⟩

declare *nat.exhaust* [*case-names 0 Suc*, *cases type*: *nat*]

lemmas *nat-rec-0* = *nat.recs*(*1*)
and *nat-rec-Suc* = *nat.recs*(*2*)

lemmas *nat-case-0* = *nat.cases*(*1*)
and *nat-case-Suc* = *nat.cases*(*2*)

Injectiveness and distinctness lemmas

lemma *inj-Suc*[*simp*]: *inj-on Suc N*
⟨*proof*⟩

lemma *Suc-neq-Zero*: $Suc\ m = 0 \Longrightarrow R$
⟨*proof*⟩

lemma *Zero-neq-Suc*: $0 = Suc\ m \Longrightarrow R$
⟨*proof*⟩

lemma *Suc-inject*: $Suc\ x = Suc\ y \Longrightarrow x = y$
⟨*proof*⟩

lemma *n-not-Suc-n*: $n \neq Suc\ n$
⟨*proof*⟩

lemma *Suc-n-not-n*: $Suc\ n \neq n$
⟨*proof*⟩

A special form of induction for reasoning about $m < n$ and $m - n$

lemma *diff-induct*: $(!!x.\ P\ x\ 0) ==> (!!y.\ P\ 0\ (Suc\ y)) ==>$
$(!!x\ y.\ P\ x\ y ==> P\ (Suc\ x)\ (Suc\ y)) ==> P\ m\ n$
⟨*proof*⟩

12.3 Arithmetic operators

instantiation *nat* :: {*minus*, *comm-monoid-add*}
begin

primrec *plus-nat*
where
add-0: $0 + n = (n::nat)$
| *add-Suc*: $Suc\ m + n = Suc\ (m + n)$

lemma *add-0-right* [*simp*]: $m + 0 = (m::nat)$
⟨*proof*⟩

lemma *add-Suc-right* [*simp*]: $m + Suc\ n = Suc\ (m + n)$
⟨*proof*⟩

declare *add-0* [*code*]

lemma *add-Suc-shift* [*code*]: $Suc\ m + n = m + Suc\ n$
⟨*proof*⟩

primrec *minus-nat*
where
diff-0: $m - 0 = (m::nat)$
| *diff-Suc*: $m - Suc\ n = (case\ m - n\ of\ 0 => 0 \mid Suc\ k => k)$

declare *diff-Suc* [*simp del*]
declare *diff-0* [*code*]

lemma *diff-0-eq-0* [*simp*, *code*]: $0 - n = (0::nat)$
⟨*proof*⟩

lemma *diff-Suc-Suc* [*simp*, *code*]: $Suc\ m - Suc\ n = m - n$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *nat* :: *comm-semiring-1-cancel*
begin

definition
One-nat-def [*simp*]: $1 = Suc\ 0$

primrec *times-nat*
where
mult-0: $0 * n = (0::nat)$
| *mult-Suc*: $Suc\ m * n = n + (m * n)$

lemma *mult-0-right* [*simp*]: $(m::nat) * 0 = 0$
⟨*proof*⟩

lemma *mult-Suc-right* [*simp*]: $m * Suc\ n = m + (m * n)$
⟨*proof*⟩

lemma *add-mult-distrib*: $(m + n) * k = (m * k) + ((n * k)::nat)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

12.3.1 Addition

lemma *nat-add-assoc*: $(m + n) + k = m + ((n + k)::nat)$
⟨*proof*⟩

lemma *nat-add-commute*: $m + n = n + (m::nat)$
⟨*proof*⟩

lemma *nat-add-left-commute*: $x + (y + z) = y + ((x + z)::nat)$
⟨*proof*⟩

lemma *nat-add-left-cancel* [*simp*]: $(k + m = k + n) = (m = (n::nat))$
⟨*proof*⟩

lemma *nat-add-right-cancel* [*simp*]: $(m + k = n + k) = (m=(n::nat))$
⟨*proof*⟩

Reasoning about $m + 0 = 0$, etc.

lemma *add-is-0* [*iff*]:
fixes *m n* :: *nat*
shows $(m + n = 0) = (m = 0 \ \& \ n = 0)$
⟨*proof*⟩

lemma *add-is-1*:
$(m+n= Suc\ 0) = (m= Suc\ 0 \ \& \ n=0 \mid m=0 \ \& \ n= Suc\ 0)$
⟨*proof*⟩

lemma *one-is-add*:
$(Suc\ 0 = m + n) = (m = Suc\ 0 \ \& \ n = 0 \mid m = 0 \ \& \ n = Suc\ 0)$
⟨*proof*⟩

lemma *add-eq-self-zero*:
fixes *m n* :: *nat*
shows $m + n = m \Longrightarrow n = 0$
⟨*proof*⟩

lemma *inj-on-add-nat*[*simp*]: *inj-on* (%*n*::*nat*. *n*+*k*) *N*
⟨*proof*⟩

12.3.2 Difference

lemma *diff-self-eq-0* [*simp*]: $(m::nat) - m = 0$
⟨*proof*⟩

lemma *diff-diff-left*: $(i::nat) - j - k = i - (j + k)$
⟨*proof*⟩

lemma *Suc-diff-diff* [*simp*]: $(Suc\ m - n) - Suc\ k = m - n - k$
⟨*proof*⟩

lemma *diff-commute*: $(i::nat) - j - k = i - k - j$
⟨*proof*⟩

lemma *diff-add-inverse*: $(n + m) - n = (m::nat)$
⟨*proof*⟩

lemma *diff-add-inverse2*: $(m + n) - n = (m::nat)$
⟨*proof*⟩

lemma *diff-cancel*: $(k + m) - (k + n) = m - (n::nat)$
⟨*proof*⟩

lemma *diff-cancel2*: $(m + k) - (n + k) = m - (n::nat)$
⟨*proof*⟩

lemma *diff-add-0*: $n - (n + m) = (0::nat)$
⟨*proof*⟩

lemma *diff-Suc-1* [*simp*]: $Suc\ n - 1 = n$
⟨*proof*⟩

Difference distributes over multiplication

lemma *diff-mult-distrib*: $((m::nat) - n) * k = (m * k) - (n * k)$
⟨*proof*⟩

lemma *diff-mult-distrib2*: $k * ((m::nat) - n) = (k * m) - (k * n)$
⟨*proof*⟩

12.3.3 Multiplication

lemma *nat-mult-assoc*: $(m * n) * k = m * ((n * k)::nat)$
⟨*proof*⟩

lemma *nat-mult-commute*: $m * n = n * (m::nat)$
⟨*proof*⟩

lemma *add-mult-distrib2*: $k * (m + n) = (k * m) + ((k * n)::nat)$
⟨*proof*⟩

lemma *mult-is-0* [*simp*]: $((m::nat) * n = 0) = (m=0 \mid n=0)$
⟨*proof*⟩

lemmas *nat-distrib* =
add-mult-distrib add-mult-distrib2 diff-mult-distrib diff-mult-distrib2

lemma *mult-eq-1-iff* [*simp*]: $(m * n = Suc\ 0) = (m = Suc\ 0\ \&\ n = Suc\ 0)$
⟨*proof*⟩

lemma *one-eq-mult-iff* [*simp*,*noatp*]: $(Suc\ 0 = m * n) = (m = Suc\ 0\ \&\ n = Suc\ 0)$
⟨*proof*⟩

lemma *nat-mult-eq-1-iff* [*simp*]: $m * n = (1::nat) \longleftrightarrow m = 1 \wedge n = 1$
⟨*proof*⟩

lemma *nat-1-eq-mult-iff* [*simp*]: $(1::nat) = m * n \longleftrightarrow m = 1 \wedge n = 1$
⟨*proof*⟩

lemma *mult-cancel1* [*simp*]: $(k * m = k * n) = (m = n \mid (k = (0::nat)))$
⟨*proof*⟩

lemma *mult-cancel2* [*simp*]: $(m * k = n * k) = (m = n \mid (k = (0::nat)))$
⟨*proof*⟩

lemma *Suc-mult-cancel1*: $(Suc\ k * m = Suc\ k * n) = (m = n)$
⟨*proof*⟩

12.4 Orders on *nat*

12.4.1 Operation definition

instantiation *nat* :: *linorder*
begin

primrec *less-eq-nat* **where**
$(0::nat) \leq n \longleftrightarrow True$
$\mid Suc\ m \leq n \longleftrightarrow (case\ n\ of\ 0 \Rightarrow False \mid Suc\ n \Rightarrow m \leq n)$

declare *less-eq-nat.simps* [*simp del*]
lemma [*code*]: $(0::nat) \leq n \longleftrightarrow True$ ⟨*proof*⟩
lemma *le0* [*iff*]: $0 \leq (n::nat)$ ⟨*proof*⟩

definition *less-nat* **where**
less-eq-Suc-le: $n < m \longleftrightarrow Suc\ n \leq m$

lemma *Suc-le-mono* [*iff*]: $Suc\ n \leq Suc\ m \longleftrightarrow n \leq m$
⟨*proof*⟩

lemma *Suc-le-eq* [*code*]: $Suc\ m \leq n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *le-0-eq* [*iff*]: $(n::nat) \leq 0 \longleftrightarrow n = 0$
⟨*proof*⟩

lemma *not-less0* [*iff*]: $\neg\ n < (0::nat)$

⟨*proof*⟩

lemma *less-nat-zero-code* [*code*]: $n < (0::nat) \longleftrightarrow False$
⟨*proof*⟩

lemma *Suc-less-eq* [*iff*]: $Suc\ m < Suc\ n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *less-Suc-eq-le* [*code*]: $m < Suc\ n \longleftrightarrow m \leq n$
⟨*proof*⟩

lemma *le-SucI*: $m \leq n \Longrightarrow m \leq Suc\ n$
⟨*proof*⟩

lemma *Suc-leD*: $Suc\ m \leq n \Longrightarrow m \leq n$
⟨*proof*⟩

lemma *less-SucI*: $m < n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

lemma *Suc-lessD*: $Suc\ m < n \Longrightarrow m < n$
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *nat* :: *bot*
begin

definition *bot-nat* :: *nat* **where**
bot-nat = *0*

instance ⟨*proof*⟩

end

12.4.2 Introduction properties

lemma *lessI* [*iff*]: $n < Suc\ n$
⟨*proof*⟩

lemma *zero-less-Suc* [*iff*]: $0 < Suc\ n$
⟨*proof*⟩

12.4.3 Elimination properties

lemma *less-not-refl*: $\sim\ n < (n::nat)$
⟨*proof*⟩

lemma *less-not-refl2*: $n < m \Longrightarrow m \neq (n::nat)$
⟨*proof*⟩

lemma *less-not-refl3*: $(s::nat) < t \Longrightarrow s \neq t$
⟨*proof*⟩

lemma *less-irrefl-nat*: $(n::nat) < n \Longrightarrow R$
⟨*proof*⟩

lemma *less-zeroE*: $(n::nat) < 0 \Longrightarrow R$
⟨*proof*⟩

lemma *less-Suc-eq*: $(m < Suc\ n) = (m < n \mid m = n)$
⟨*proof*⟩

lemma *less-Suc0* [*iff*]: $(n < Suc\ 0) = (n = 0)$
⟨*proof*⟩

lemma *less-one* [*iff*, *noatp*]: $(n < (1::nat)) = (n = 0)$
⟨*proof*⟩

lemma *Suc-mono*: $m < n \Longrightarrow Suc\ m < Suc\ n$
⟨*proof*⟩

"Less than" is antisymmetric, sort of

lemma *less-antisym*: $[\![\neg\ n < m;\ n < Suc\ m\ ]\!] \Longrightarrow m = n$
⟨*proof*⟩

lemma *nat-neq-iff*: $((m::nat) \neq n) = (m < n \mid n < m)$
⟨*proof*⟩

lemma *nat-less-cases*: **assumes** *major*: $(m::nat) < n \Longrightarrow P\ n\ m$
and *eqCase*: $m = n \Longrightarrow P\ n\ m$ **and** *lessCase*: $n<m \Longrightarrow P\ n\ m$
shows $P\ n\ m$
⟨*proof*⟩

12.4.4 Inductive (?) properties

lemma *Suc-lessI*: $m < n \Longrightarrow Suc\ m \neq n \Longrightarrow Suc\ m < n$
⟨*proof*⟩

lemma *lessE*:
assumes *major*: $i < k$
and *p1*: $k = Suc\ i \Longrightarrow P$ **and** *p2*: $!!j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *less-SucE*: **assumes** *major*: $m < Suc\ n$

and *less*: $m < n \Longrightarrow P$ **and** *eq*: $m = n \Longrightarrow P$ **shows** P
⟨*proof*⟩

lemma *Suc-lessE*: **assumes** *major*: $Suc\ i < k$
and *minor*: $!!j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$ **shows** P
⟨*proof*⟩

lemma *Suc-less-SucD*: $Suc\ m < Suc\ n \Longrightarrow m < n$
⟨*proof*⟩

lemma *less-trans-Suc*:
assumes *le*: $i < j$ **shows** $j < k \Longrightarrow Suc\ i < k$
⟨*proof*⟩

Can be used with *less-Suc-eq* to get $n = m \vee n < m$

lemma *not-less-eq*: $\neg\ m < n \longleftrightarrow n < Suc\ m$
⟨*proof*⟩

lemma *not-less-eq-eq*: $\neg\ m \leq n \longleftrightarrow Suc\ n \leq m$
⟨*proof*⟩

Properties of "less than or equal"

lemma *le-imp-less-Suc*: $m \leq n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

lemma *Suc-n-not-le-n*: $\sim Suc\ n \leq n$
⟨*proof*⟩

lemma *le-Suc-eq*: $(m \leq Suc\ n) = (m \leq n \mid m = Suc\ n)$
⟨*proof*⟩

lemma *le-SucE*: $m \leq Suc\ n \Longrightarrow (m \leq n \Longrightarrow R) \Longrightarrow (m = Suc\ n \Longrightarrow R)$
$\Longrightarrow R$
⟨*proof*⟩

lemma *Suc-leI*: $m < n \Longrightarrow Suc(m) \leq n$
⟨*proof*⟩

Stronger version of *Suc-leD*

lemma *Suc-le-lessD*: $Suc\ m \leq n \Longrightarrow m < n$
⟨*proof*⟩

lemma *less-imp-le-nat*: $m < n \Longrightarrow m \leq (n::nat)$
⟨*proof*⟩

For instance, $(Suc\ m < Suc\ n) = (Suc\ m \leq n) = (m < n)$

lemmas *le-simps* = *less-imp-le-nat less-Suc-eq-le Suc-le-eq*

Equivalence of $m \leq n$ and $m < n \vee m = n$

lemma *less-or-eq-imp-le*: $m < n \mid m = n \Longrightarrow m \leq (n::nat)$
⟨*proof*⟩

lemma *le-eq-less-or-eq*: $(m \leq (n::nat)) = (m < n \mid m=n)$
⟨*proof*⟩

Useful with *blast*.

lemma *eq-imp-le*: $(m::nat) = n \Longrightarrow m \leq n$
⟨*proof*⟩

lemma *le-refl*: $n \leq (n::nat)$
⟨*proof*⟩

lemma *le-trans*: $[\!|\ i \leq j; j \leq k\ |\!] \Longrightarrow i \leq (k::nat)$
⟨*proof*⟩

lemma *le-anti-sym*: $[\!|\ m \leq n; n \leq m\ |\!] \Longrightarrow m = (n::nat)$
⟨*proof*⟩

lemma *nat-less-le*: $((m::nat) < n) = (m \leq n \ \&\ m \neq n)$
⟨*proof*⟩

lemma *le-neq-implies-less*: $(m::nat) \leq n \Longrightarrow m \neq n \Longrightarrow m < n$
⟨*proof*⟩

lemma *nat-le-linear*: $(m::nat) \leq n \mid n \leq m$
⟨*proof*⟩

lemmas *linorder-neqE-nat* = *linorder-neqE* [**where** $'a = nat$]

lemma *le-less-Suc-eq*: $m \leq n \Longrightarrow (n < Suc\ m) = (n = m)$
⟨*proof*⟩

lemma *not-less-less-Suc-eq*: $\sim n < m \Longrightarrow (n < Suc\ m) = (n = m)$
⟨*proof*⟩

lemmas *not-less-simps* = *not-less-less-Suc-eq le-less-Suc-eq*

These two rules ease the use of primitive recursion. NOTE USE OF ==

lemma *def-nat-rec-0*: $(!!n.\ f\ n == nat\text{-}rec\ c\ h\ n) \Longrightarrow f\ 0 = c$
⟨*proof*⟩

lemma *def-nat-rec-Suc*: $(!!n.\ f\ n == nat\text{-}rec\ c\ h\ n) \Longrightarrow f\ (Suc\ n) = h\ n\ (f\ n)$
⟨*proof*⟩

lemma *not0-implies-Suc*: $n \neq 0 \Longrightarrow \exists\, m.\ n = Suc\ m$
⟨*proof*⟩

lemma *gr0-implies-Suc*: $n > 0 \Longrightarrow \exists\, m.\ n = Suc\ m$

⟨*proof*⟩

lemma *gr-implies-not0*: **fixes** n :: *nat* **shows** $m<n \Longrightarrow n \neq 0$
⟨*proof*⟩

lemma *neq0-conv*[*iff*]: **fixes** n :: *nat* **shows** $(n \neq 0) = (0 < n)$
⟨*proof*⟩

This theorem is useful with *blast*

lemma *gr0I*: $((n::nat) = 0 \Longrightarrow False) \Longrightarrow 0 < n$
⟨*proof*⟩

lemma *gr0-conv-Suc*: $(0 < n) = (\exists m.\ n = Suc\ m)$
⟨*proof*⟩

lemma *not-gr0* [*iff*,*noatp*]: $!!n::nat.\ (\sim (0 < n)) = (n = 0)$
⟨*proof*⟩

lemma *Suc-le-D*: $(Suc\ n \leq m') \Longrightarrow (?\ m.\ m' = Suc\ m)$
⟨*proof*⟩

Useful in certain inductive arguments

lemma *less-Suc-eq-0-disj*: $(m < Suc\ n) = (m = 0 \mid (\exists j.\ m = Suc\ j\ \&\ j < n))$
⟨*proof*⟩

12.4.5 *min* and *max*

lemma *mono-Suc*: *mono Suc*
⟨*proof*⟩

lemma *min-0L* [*simp*]: $min\ 0\ n = (0::nat)$
⟨*proof*⟩

lemma *min-0R* [*simp*]: $min\ n\ 0 = (0::nat)$
⟨*proof*⟩

lemma *min-Suc-Suc* [*simp*]: $min\ (Suc\ m)\ (Suc\ n) = Suc\ (min\ m\ n)$
⟨*proof*⟩

lemma *min-Suc1*:
$min\ (Suc\ n)\ m = (case\ m\ of\ 0 \Rightarrow 0 \mid Suc\ m' \Rightarrow Suc(min\ n\ m'))$
⟨*proof*⟩

lemma *min-Suc2*:
$min\ m\ (Suc\ n) = (case\ m\ of\ 0 \Rightarrow 0 \mid Suc\ m' \Rightarrow Suc(min\ m'\ n))$
⟨*proof*⟩

lemma *max-0L* [*simp*]: $max\ 0\ n = (n::nat)$
⟨*proof*⟩

lemma *max-0R* [*simp*]: *max n 0* = (*n*::*nat*)
⟨*proof*⟩

lemma *max-Suc-Suc* [*simp*]: *max* (*Suc m*) (*Suc n*) = *Suc*(*max m n*)
⟨*proof*⟩

lemma *max-Suc1*:
max (*Suc n*) *m* = (*case m of 0* => *Suc n* | *Suc m'* => *Suc*(*max n m'*))
⟨*proof*⟩

lemma *max-Suc2*:
max m (*Suc n*) = (*case m of 0* => *Suc n* | *Suc m'* => *Suc*(*max m' n*))
⟨*proof*⟩

12.4.6 Monotonicity of Addition

lemma *Suc-pred* [*simp*]: *n*>*0* ==> *Suc* (*n* − *Suc 0*) = *n*
⟨*proof*⟩

lemma *Suc-diff-1* [*simp*]: *0* < *n* ==> *Suc* (*n* − *1*) = *n*
⟨*proof*⟩

lemma *nat-add-left-cancel-le* [*simp*]: (*k* + *m* ≤ *k* + *n*) = (*m*≤(*n*::*nat*))
⟨*proof*⟩

lemma *nat-add-left-cancel-less* [*simp*]: (*k* + *m* < *k* + *n*) = (*m*<(*n*::*nat*))
⟨*proof*⟩

lemma *add-gr-0* [*iff*]: !!*m*::*nat*. (*m* + *n* > *0*) = (*m*>*0* | *n*>*0*)
⟨*proof*⟩

strict, in 1st argument

lemma *add-less-mono1*: *i* < *j* ==> *i* + *k* < *j* + (*k*::*nat*)
⟨*proof*⟩

strict, in both arguments

lemma *add-less-mono*: [|*i* < *j*; *k* < *l*|] ==> *i* + *k* < *j* + (*l*::*nat*)
⟨*proof*⟩

Deleted *less-natE*; use *less-imp-Suc-add RS exE*

lemma *less-imp-Suc-add*: *m* < *n* ==> (∃ *k*. *n* = *Suc* (*m* + *k*))
⟨*proof*⟩

strict, in 1st argument; proof is by induction on *k* > *0*

lemma *mult-less-mono2*: (*i*::*nat*) < *j* ==> *0*<*k* ==> *k* ∗ *i* < *k* ∗ *j*
⟨*proof*⟩

The naturals form an ordered *comm-semiring-1-cancel*

instance *nat* :: *ordered-semidom*
⟨*proof*⟩

instance *nat* :: *no-zero-divisors*
⟨*proof*⟩

lemma *nat-mult-1*: $(1::nat) * n = n$
⟨*proof*⟩

lemma *nat-mult-1-right*: $n * (1::nat) = n$
⟨*proof*⟩

12.4.7 Additional theorems about *op* ≤

Complete induction, aka course-of-values induction

instance *nat* :: *wellorder* ⟨*proof*⟩

lemma *Least-Suc*:
[| *P n*; ~ *P 0* |] ==> (*LEAST n*. *P n*) = *Suc* (*LEAST m*. *P*(*Suc m*))
⟨*proof*⟩

lemma *Least-Suc2*:
[|*P n*; *Q m*; ~*P 0*; !*k*. *P* (*Suc k*) = *Q k*|] ==> *Least P* = *Suc* (*Least Q*)
⟨*proof*⟩

lemma *ex-least-nat-le*: $\neg P(0) \Longrightarrow P(n::nat) \Longrightarrow \exists k{\leq}n.\ (\forall i{<}k.\ \neg P\ i)\ \&\ P(k)$
⟨*proof*⟩

lemma *ex-least-nat-less*: $\neg P(0) \Longrightarrow P(n::nat) \Longrightarrow \exists k{<}n.\ (\forall i{\leq}k.\ \neg P\ i)\ \&\ P(k+1)$
⟨*proof*⟩

lemma *nat-less-induct*:
assumes !!*n*. ∀ *m*::*nat*. *m* < *n* −−> *P m* ==> *P n* **shows** *P n*
⟨*proof*⟩

lemma *measure-induct-rule* [*case-names less*]:
fixes *f* :: *'a* ⇒ *nat*
assumes *step*: $\bigwedge x.\ (\bigwedge y.\ f\ y < f\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows *P a*
⟨*proof*⟩

old style induction rules:

lemma *measure-induct*:
fixes *f* :: *'a* ⇒ *nat*
shows $(\bigwedge x.\ \forall y.\ f\ y < f\ x \longrightarrow P\ y \Longrightarrow P\ x) \Longrightarrow P\ a$
⟨*proof*⟩

lemma *full-nat-induct*:
assumes *step*: (!!*n*. (*ALL m*. *Suc m* <= *n* −−> *P m*) ==> *P n*)

shows $P\ n$
⟨*proof*⟩

An induction rule for estabilishing binary relations

lemma *less-Suc-induct*:
assumes *less*: $i < j$
and *step*: $!!i.\ P\ i\ (Suc\ i)$
and *trans*: $!!i\ j\ k.\ P\ i\ j \Longrightarrow P\ j\ k \Longrightarrow P\ i\ k$
shows $P\ i\ j$
⟨*proof*⟩

The method of infinite descent, frequently used in number theory. Provided by Roelof Oosterhuis. $P(n)$ is true for all $n \in \mathbb{N}$ if

- case "0": given $n = 0$ prove $P(n)$,
- case "smaller": given $n > 0$ and $\neg P(n)$ prove there exists a smaller integer m such that $\neg P(m)$.

A compact version without explicit base case:

lemma *infinite-descent*:
$[\![\ !!n::nat.\ \neg\ P\ n \Longrightarrow\ \exists\, m<n.\ \neg\ \ P\ m\ ]\!] \Longrightarrow\ \ P\ n$
⟨*proof*⟩

lemma *infinite-descent0*[*case-names 0 smaller*]:
$[\![\ P\ 0;\ !!n.\ n>0 \Longrightarrow \neg\ P\ n \Longrightarrow (\exists\, m::nat.\ m < n \wedge \neg P\ m)\ ]\!] \Longrightarrow P\ n$
⟨*proof*⟩

Infinite descent using a mapping to $\mathbb{N}$: $P(x)$ is true for all $x \in D$ if there exists a $V : D \to \mathbb{N}$ and

- case "0": given $V(x) = 0$ prove $P(x)$,
- case "smaller": given $V(x) > 0$ and $\neg P(x)$ prove there exists a $y \in D$ such that $V(y) < V(x)$ and $\neg P(y)$.

NB: the proof also shows how to use the previous lemma.

corollary *infinite-descent0-measure* [*case-names 0 smaller*]:
assumes *A0*: $!!x.\ V\ x = (0::nat) \Longrightarrow P\ x$
and *A1*: $!!x.\ V\ x > 0 \Longrightarrow \neg P\ x \Longrightarrow (\exists\, y.\ V\ y < V\ x \wedge \neg P\ y)$
shows $P\ x$
⟨*proof*⟩

Again, without explicit base case:

lemma *infinite-descent-measure*:
assumes $!!x.\ \neg\ P\ x \Longrightarrow \exists\, y.\ (V::'a \Rightarrow nat)\ y < V\ x \wedge \neg\ P\ y$ **shows** $P\ x$
⟨*proof*⟩

A [clumsy] way of lifting $<$ monotonicity to $\leq$ monotonicity

lemma *less-mono-imp-le-mono*:
$[\![\; !!i\ j{::}nat.\ i < j \Longrightarrow f\ i < f\ j;\ i \leq j \;]\!] \Longrightarrow f\ i \leq ((f\ j){::}nat)$
⟨*proof*⟩

non-strict, in 1st argument

lemma *add-le-mono1*: $i \leq j \Longrightarrow i + k \leq j + (k{::}nat)$
⟨*proof*⟩

non-strict, in both arguments

lemma *add-le-mono*: $[\!|\ i \leq j;\ \ k \leq l\ |\!] \Longrightarrow i + k \leq j + (l{::}nat)$
⟨*proof*⟩

lemma *le-add2*: $n \leq ((m + n){::}nat)$
⟨*proof*⟩

lemma *le-add1*: $n \leq ((n + m){::}nat)$
⟨*proof*⟩

lemma *less-add-Suc1*: $i < Suc\ (i + m)$
⟨*proof*⟩

lemma *less-add-Suc2*: $i < Suc\ (m + i)$
⟨*proof*⟩

lemma *less-iff-Suc-add*: $(m < n) = (\exists\, k.\ n = Suc\ (m + k))$
⟨*proof*⟩

lemma *trans-le-add1*: $(i{::}nat) \leq j \Longrightarrow i \leq j + m$
⟨*proof*⟩

lemma *trans-le-add2*: $(i{::}nat) \leq j \Longrightarrow i \leq m + j$
⟨*proof*⟩

lemma *trans-less-add1*: $(i{::}nat) < j \Longrightarrow i < j + m$
⟨*proof*⟩

lemma *trans-less-add2*: $(i{::}nat) < j \Longrightarrow i < m + j$
⟨*proof*⟩

lemma *add-lessD1*: $i + j < (k{::}nat) \Longrightarrow i < k$
⟨*proof*⟩

lemma *not-add-less1* [*iff*]: $\sim (i + j < (i{::}nat))$
⟨*proof*⟩

lemma *not-add-less2* [*iff*]: $\sim (j + i < (i{::}nat))$
⟨*proof*⟩

lemma *add-leD1*: $m + k \leq n \Longrightarrow m \leq (n::nat)$
⟨*proof*⟩

lemma *add-leD2*: $m + k \leq n \Longrightarrow k \leq (n::nat)$
⟨*proof*⟩

lemma *add-leE*: $(m::nat) + k \leq n \Longrightarrow (m \leq n \Longrightarrow k \leq n \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

needs $!!k$ for *add-ac* to work

lemma *less-add-eq-less*: $!!k::nat.\ k < l \Longrightarrow m + l = k + n \Longrightarrow m < n$
⟨*proof*⟩

12.4.8 More results about difference

Addition is the inverse of subtraction: if $n \leq m$ then $n + (m - n) = m$.

lemma *add-diff-inverse*: $\sim\ m < n \Longrightarrow n + (m - n) = (m::nat)$
⟨*proof*⟩

lemma *le-add-diff-inverse* [*simp*]: $n \leq m \Longrightarrow n + (m - n) = (m::nat)$
⟨*proof*⟩

lemma *le-add-diff-inverse2* [*simp*]: $n \leq m \Longrightarrow (m - n) + n = (m::nat)$
⟨*proof*⟩

lemma *Suc-diff-le*: $n \leq m \Longrightarrow Suc\ m - n = Suc\ (m - n)$
⟨*proof*⟩

lemma *diff-less-Suc*: $m - n < Suc\ m$
⟨*proof*⟩

lemma *diff-le-self* [*simp*]: $m - n \leq (m::nat)$
⟨*proof*⟩

lemma *le-iff-add*: $(m::nat) \leq n = (\exists k.\ n = m + k)$
⟨*proof*⟩

lemma *less-imp-diff-less*: $(j::nat) < k \Longrightarrow j - n < k$
⟨*proof*⟩

lemma *diff-Suc-less* [*simp*]: $0<n \Longrightarrow n - Suc\ i < n$
⟨*proof*⟩

lemma *diff-add-assoc*: $k \leq (j::nat) \Longrightarrow (i + j) - k = i + (j - k)$
⟨*proof*⟩

lemma *diff-add-assoc2*: $k \leq (j::nat) \Longrightarrow (j + i) - k = (j - k) + i$
⟨*proof*⟩

lemma *le-imp-diff-is-add*: $i \leq (j::nat) \Longrightarrow (j - i = k) = (j = k + i)$
⟨*proof*⟩

lemma *diff-is-0-eq* [*simp*]: $((m::nat) - n = 0) = (m \leq n)$
⟨*proof*⟩

lemma *diff-is-0-eq'* [*simp*]: $m \leq n \Longrightarrow (m::nat) - n = 0$
⟨*proof*⟩

lemma *zero-less-diff* [*simp*]: $(0 < n - (m::nat)) = (m < n)$
⟨*proof*⟩

lemma *less-imp-add-positive*:
assumes $i < j$
shows $\exists k::nat.\ 0 < k\ \&\ i + k = j$
⟨*proof*⟩

a nice rewrite for bounded subtraction

lemma *nat-minus-add-max*:
fixes $n\ m :: nat$
shows $n - m + m = max\ n\ m$
⟨*proof*⟩

lemma *nat-diff-split*:
$P(a - b::nat) = ((a<b \longrightarrow P\ 0)\ \&\ (ALL\ d.\ a = b + d \longrightarrow P\ d))$
— elimination of $-$ on *nat*
⟨*proof*⟩

lemma *nat-diff-split-asm*:
$P(a - b::nat) = (\sim\ (a < b\ \&\ \sim\ P\ 0 \mid (EX\ d.\ a = b + d\ \&\ \sim\ P\ d)))$
— elimination of $-$ on *nat* in assumptions
⟨*proof*⟩

12.4.9 Monotonicity of Multiplication

lemma *mult-le-mono1*: $i \leq (j::nat) \Longrightarrow i * k \leq j * k$
⟨*proof*⟩

lemma *mult-le-mono2*: $i \leq (j::nat) \Longrightarrow k * i \leq k * j$
⟨*proof*⟩

$\leq$ monotonicity, BOTH arguments

lemma *mult-le-mono*: $i \leq (j::nat) \Longrightarrow k \leq l \Longrightarrow i * k \leq j * l$
⟨*proof*⟩

lemma *mult-less-mono1*: $(i::nat) < j \Longrightarrow 0 < k \Longrightarrow i * k < j * k$
⟨*proof*⟩

Differs from the standard *zero-less-mult-iff* in that there are no negative numbers.

lemma *nat-0-less-mult-iff* [*simp*]: $(0 < (m::nat) * n) = (0 < m \ \& \ 0 < n)$
⟨*proof*⟩

lemma *one-le-mult-iff* [*simp*]: $(Suc\ 0 \leq m * n) = (Suc\ 0 \leq m \ \& \ Suc\ 0 \leq n)$
⟨*proof*⟩

lemma *mult-less-cancel2* [*simp*]: $((m::nat) * k < n * k) = (0 < k \ \& \ m < n)$
⟨*proof*⟩

lemma *mult-less-cancel1* [*simp*]: $(k * (m::nat) < k * n) = (0 < k \ \& \ m < n)$
⟨*proof*⟩

lemma *mult-le-cancel1* [*simp*]: $(k * (m::nat) \leq k * n) = (0 < k \longrightarrow m \leq n)$
⟨*proof*⟩

lemma *mult-le-cancel2* [*simp*]: $((m::nat) * k \leq n * k) = (0 < k \longrightarrow m \leq n)$
⟨*proof*⟩

lemma *Suc-mult-less-cancel1*: $(Suc\ k * m < Suc\ k * n) = (m < n)$
⟨*proof*⟩

lemma *Suc-mult-le-cancel1*: $(Suc\ k * m \leq Suc\ k * n) = (m \leq n)$
⟨*proof*⟩

lemma *le-square*: $m \leq m * (m::nat)$
⟨*proof*⟩

lemma *le-cube*: $(m::nat) \leq m * (m * m)$
⟨*proof*⟩

Lemma for *gcd*

lemma *mult-eq-self-implies-10*: $(m::nat) = m * n \Longrightarrow n = 1 \mid m = 0$
⟨*proof*⟩

the lattice order on *nat*

instantiation *nat* :: *distrib-lattice*
begin

definition
$(inf :: nat \Rightarrow nat \Rightarrow nat) = min$

definition
$(sup :: nat \Rightarrow nat \Rightarrow nat) = max$

instance ⟨*proof*⟩

end

12.5 Embedding of the Naturals into any *semiring-1*: *of-nat*

context *semiring-1*
begin

primrec
of-nat :: *nat* ⇒ *'a*
where
of-nat-0: *of-nat 0 = 0*
| *of-nat-Suc*: *of-nat (Suc m) = 1 + of-nat m*

lemma *of-nat-1* [*simp*]: *of-nat 1 = 1*
⟨*proof*⟩

lemma *of-nat-add* [*simp*]: *of-nat (m + n) = of-nat m + of-nat n*
⟨*proof*⟩

lemma *of-nat-mult*: *of-nat (m * n) = of-nat m * of-nat n*
⟨*proof*⟩

primrec *of-nat-aux* :: *('a ⇒ 'a) ⇒ nat ⇒ 'a ⇒ 'a* **where**
of-nat-aux inc 0 i = i
| *of-nat-aux inc (Suc n) i = of-nat-aux inc n (inc i)* — tail recursive

lemma *of-nat-code* [*code*, *code unfold*, *code inline del*]:
of-nat n = of-nat-aux (λi. i + 1) n 0
⟨*proof*⟩

end

Class for unital semirings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *semiring-char-0 = semiring-1 +*
assumes *of-nat-eq-iff* [*simp*]: *of-nat m = of-nat n ⟷ m = n*
begin

Special cases where either operand is zero

lemma *of-nat-0-eq-iff* [*simp*, *noatp*]: *0 = of-nat n ⟷ 0 = n*
⟨*proof*⟩

lemma *of-nat-eq-0-iff* [*simp*, *noatp*]: *of-nat m = 0 ⟷ m = 0*
⟨*proof*⟩

lemma *inj-of-nat*: *inj of-nat*
⟨*proof*⟩

end

context *ordered-semidom*

begin

lemma *zero-le-imp-of-nat*: $0 \leq$ *of-nat* m
⟨*proof*⟩

lemma *less-imp-of-nat-less*: $m < n \Longrightarrow$ *of-nat* $m <$ *of-nat* n
⟨*proof*⟩

lemma *of-nat-less-imp-less*: *of-nat* $m <$ *of-nat* $n \Longrightarrow m < n$
⟨*proof*⟩

lemma *of-nat-less-iff* [*simp*]: *of-nat* $m <$ *of-nat* $n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *of-nat-le-iff* [*simp*]: *of-nat* $m \leq$ *of-nat* $n \longleftrightarrow m \leq n$
⟨*proof*⟩

Every *ordered-semidom* has characteristic zero.

subclass *semiring-char-0*
⟨*proof*⟩

Special cases where either operand is zero

lemma *of-nat-0-le-iff* [*simp*]: $0 \leq$ *of-nat* n
⟨*proof*⟩

lemma *of-nat-le-0-iff* [*simp*, *noatp*]: *of-nat* $m \leq 0 \longleftrightarrow m = 0$
⟨*proof*⟩

lemma *of-nat-0-less-iff* [*simp*]: $0 <$ *of-nat* $n \longleftrightarrow 0 < n$
⟨*proof*⟩

lemma *of-nat-less-0-iff* [*simp*]: $\neg$ *of-nat* $m < 0$
⟨*proof*⟩

end

context *ring-1*
begin

lemma *of-nat-diff*: $n \leq m \Longrightarrow$ *of-nat* $(m - n) =$ *of-nat* $m -$ *of-nat* n
⟨*proof*⟩

end

context *ordered-idom*
begin

lemma *abs-of-nat* [*simp*]: $|$*of-nat* $n| =$ *of-nat* n
⟨*proof*⟩

end

lemma *of-nat-id* [*simp*]: *of-nat n* = *n*
⟨*proof*⟩

lemma *of-nat-eq-id* [*simp*]: *of-nat* = *id*
⟨*proof*⟩

12.6 The Set of Natural Numbers

context *semiring-1*
begin

definition
Nats :: *'a set* **where**
[*code del*]: *Nats* = *range of-nat*

notation (*xsymbols*)
Nats (ℕ)

lemma *of-nat-in-Nats* [*simp*]: *of-nat n* ∈ ℕ
⟨*proof*⟩

lemma *Nats-0* [*simp*]: *0* ∈ ℕ
⟨*proof*⟩

lemma *Nats-1* [*simp*]: *1* ∈ ℕ
⟨*proof*⟩

lemma *Nats-add* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* + *b* ∈ ℕ
⟨*proof*⟩

lemma *Nats-mult* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* ∗ *b* ∈ ℕ
⟨*proof*⟩

end

12.7 Further Arithmetic Facts Concerning the Natural Numbers

lemma *subst-equals*:
assumes *1*: *t* = *s* **and** *2*: *u* = *t*
shows *u* = *s*
⟨*proof*⟩

⟨*ML*⟩

lemmas [*arith-split*] = *nat-diff-split split-min split-max*

context *order*
begin

lemma *lift-Suc-mono-le*:
  assumes *mono*: !!n. $f\ n \leq f(Suc\ n)$ **and** $n \leq n'$
  shows $f\ n \leq f\ n'$
⟨*proof*⟩

lemma *lift-Suc-mono-less*:
  assumes *mono*: !!n. $f\ n < f(Suc\ n)$ **and** $n < n'$
  shows $f\ n < f\ n'$
⟨*proof*⟩

lemma *lift-Suc-mono-less-iff*:
  (!!n. $f\ n < f(Suc\ n)$) $\Longrightarrow f(n) < f(m) \longleftrightarrow n<m$
⟨*proof*⟩

end

lemma *mono-iff-le-Suc*: *mono* $f = (\forall\, n.\ f\ n \leq f\ (Suc\ n))$
⟨*proof*⟩

lemma *mono-nat-linear-lb*:
  (!!$m\ n$::*nat*. $m<n \Longrightarrow f\ m < f\ n) \Longrightarrow f(m)+k \leq f(m+k)$
⟨*proof*⟩

Subtraction laws, mostly by Clemens Ballarin

lemma *diff-less-mono*: [| $a < (b$::*nat*); $c \leq a$ |] ==> $a-c < b-c$
⟨*proof*⟩

lemma *less-diff-conv*: $(i < j-k) = (i+k < (j$::*nat*))
⟨*proof*⟩

lemma *le-diff-conv*: $(j-k \leq (i$::*nat*)) $= (j \leq i+k)$
⟨*proof*⟩

lemma *le-diff-conv2*: $k \leq j$ ==> $(i \leq j-k) = (i+k \leq (j$::*nat*))
⟨*proof*⟩

lemma *diff-diff-cancel* [*simp*]: $i \leq (n$::*nat*) ==> $n - (n - i) = i$
⟨*proof*⟩

lemma *le-add-diff*: $k \leq (n$::*nat*) ==> $m \leq n + m - k$
⟨*proof*⟩

lemma *diff-less*[*simp*]: !!m::*nat*. [| $0<n$; $0<m$ |] ==> $m - n < m$
⟨*proof*⟩

Simplification of relational expressions involving subtraction

lemma *diff-diff-eq*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $((m-k) - (n-k)) = (m-n)$
⟨*proof*⟩

lemma *eq-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k = n-k) = (m=n)$
⟨*proof*⟩

lemma *less-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k < n-k) = (m<n)$
⟨*proof*⟩

lemma *le-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k \leq n-k) = (m \leq n)$
⟨*proof*⟩

(Anti)Monotonicity of subtraction – by Stephan Merz

lemma *diff-le-mono*: $m \leq (n::nat)$ ==> $(m-l) \leq (n-l)$
⟨*proof*⟩

lemma *diff-le-mono2*: $m \leq (n::nat)$ ==> $(l-n) \leq (l-m)$
⟨*proof*⟩

lemma *diff-less-mono2*: [| $m < (n::nat)$; $m<l$ |] ==> $(l-n) < (l-m)$
⟨*proof*⟩

lemma *diffs0-imp-equal*: !!$m::nat$. [| $m-n = 0$; $n-m = 0$ |] ==> $m=n$
⟨*proof*⟩

lemma *min-diff*: $min\ (m - (i::nat))\ (n - i) = min\ m\ n - i$
⟨*proof*⟩

lemma *inj-on-diff-nat*:
assumes *k-le-n*: $\forall n \in N.\ k \leq (n::nat)$
shows *inj-on* $(\lambda n.\ n - k)\ N$
⟨*proof*⟩

Rewriting to pull differences out

lemma *diff-diff-right* [*simp*]: $k \leq j$ --> $i - (j - k) = i + (k::nat) - j$
⟨*proof*⟩

lemma *diff-Suc-diff-eq1* [*simp*]: $k \leq j$ ==> $m - Suc\ (j - k) = m + k - Suc\ j$
⟨*proof*⟩

lemma *diff-Suc-diff-eq2* [*simp*]: $k \leq j$ ==> $Suc\ (j - k) - m = Suc\ j - (k + m)$
⟨*proof*⟩

Lemmas for ex/Factorization

lemma *one-less-mult*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] ==> $Suc\ 0 < m*n$
⟨*proof*⟩

lemma *n-less-m-mult-n*: [| *Suc 0* < *n*; *Suc 0* < *m* |] ==> *n*<*m*∗*n*
⟨*proof*⟩

lemma *n-less-n-mult-m*: [| *Suc 0* < *n*; *Suc 0* < *m* |] ==> *n*<*n*∗*m*
⟨*proof*⟩

Specialized induction principles that work "backwards":

lemma *inc-induct*[*consumes 1*, *case-names base step*]:
assumes *less*: *i* <= *j*
assumes *base*: *P j*
assumes *step*: !!*i*. [| *i* < *j*; *P* (*Suc i*) |] ==> *P i*
shows *P i*
⟨*proof*⟩

lemma *strict-inc-induct*[*consumes 1*, *case-names base step*]:
assumes *less*: *i* < *j*
assumes *base*: !!*i*. *j* = *Suc i* ==> *P i*
assumes *step*: !!*i*. [| *i* < *j*; *P* (*Suc i*) |] ==> *P i*
shows *P i*
⟨*proof*⟩

lemma *zero-induct-lemma*: *P k* ==> (!!*n*. *P* (*Suc n*) ==> *P n*) ==> *P* (*k* − *i*)
⟨*proof*⟩

lemma *zero-induct*: *P k* ==> (!!*n*. *P* (*Suc n*) ==> *P n*) ==> *P 0*
⟨*proof*⟩

lemma *nat-not-singleton*: (∀ *x*. *x* = (*0*::*nat*)) = *False*
⟨*proof*⟩

lemmas *add-diff-assoc* = *diff-add-assoc* [*symmetric*]
lemmas *add-diff-assoc2* = *diff-add-assoc2*[*symmetric*]
declare *diff-diff-left* [*simp*] *add-diff-assoc* [*simp*] *add-diff-assoc2*[*simp*]

At present we prove no analogue of *not-less-Least* or *Least-Suc*, since there appears to be no need.

12.8 size of a datatype value

class *size* =
fixes *size* :: *'a* ⇒ *nat* — see further theory *Wellfounded*

end

13 Product-Type: Cartesian products

theory *Product-Type*

imports *Inductive*
uses
(*Tools/split-rule.ML*)
(*Tools/inductive-set-package.ML*)
(*Tools/inductive-realizer.ML*)
(*Tools/datatype-realizer.ML*)
begin

13.1 *bool* is a datatype

rep-datatype *True False* ⟨*proof*⟩

declare *case-split* [*cases type*: *bool*]
— prefer plain propositional version

lemma
shows [*code*]: *eq-class.eq False P* ⟷ ¬ *P*
and [*code*]: *eq-class.eq True P* ⟷ *P*
and [*code*]: *eq-class.eq P False* ⟷ ¬ *P*
and [*code*]: *eq-class.eq P True* ⟷ *P*
and [*code nbe*]: *eq-class.eq P P* ⟷ *True*
⟨*proof*⟩

code-const *eq-class.eq* :: *bool* ⇒ *bool* ⇒ *bool*
(*Haskell* **infixl** 4 ==)

code-instance *bool* :: *eq*
(*Haskell* −)

13.2 Unit

typedef *unit* = {*True*}
⟨*proof*⟩

definition
Unity :: *unit* ('())
where
() = *Abs-unit True*

lemma *unit-eq* [*noatp*]: *u* = ()
⟨*proof*⟩

Simplification procedure for *unit-eq*. Cannot use this rule directly — it loops!

⟨*ML*⟩

rep-datatype () ⟨*proof*⟩

lemma *unit-all-eq1*: (!!*x*::*unit*. *PROP P x*) == *PROP P* ()

⟨*proof*⟩

lemma *unit-all-eq2*: (!!*x*::*unit*. *PROP P*) == *PROP P*
⟨*proof*⟩

This rewrite counters the effect of *unit-eq-proc* on %*u*::*unit*. *f u*, replacing it by *f* rather than by %*u*. *f* ().

lemma *unit-abs-eta-conv* [*simp*,*noatp*]: (%*u*::*unit*. *f* ()) = *f*
⟨*proof*⟩

code generator setup

instance *unit* :: *eq* ⟨*proof*⟩

lemma [*code*]:
eq-class.eq (*u*::*unit*) *v* ⟷ *True* ⟨*proof*⟩

code-type *unit*
(*SML unit*)
(*OCaml unit*)
(*Haskell* ())

code-instance *unit* :: *eq*
(*Haskell* −)

code-const *eq-class.eq* :: *unit* ⇒ *unit* ⇒ *bool*
(*Haskell* **infixl** 4 ==)

code-const *Unity*
(*SML* ())
(*OCaml* ())
(*Haskell* ())

code-reserved *SML*
unit

code-reserved *OCaml*
unit

13.3 Pairs

13.3.1 Product type, basic operations and concrete syntax

definition
Pair-Rep :: 'a ⇒ 'b ⇒ 'a ⇒ 'b ⇒ *bool*
where
Pair-Rep a b = (λ*x y*. *x* = *a* ∧ *y* = *b*)

global

typedef (*Prod*)
  $('a, 'b) *$ (**infixr** $*$ *20*)
    $= \{f.\ \exists\, a\ b.\ f = Pair\text{-}Rep\ (a{::}'a)\ (b{::}'b)\}$
⟨*proof*⟩

syntax (*xsymbols*)
  $*$ :: [*type*, *type*] => *type* ((- ×/ -) [*21*, *20*] *20*)
syntax (*HTML* **output**)
  $*$ :: [*type*, *type*] => *type* ((- ×/ -) [*21*, *20*] *20*)

consts
  Pair :: $'a \Rightarrow 'b \Rightarrow 'a \times 'b$
  fst :: $'a \times 'b \Rightarrow 'a$
  snd :: $'a \times 'b \Rightarrow 'b$
  split :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'c$
  curry :: $('a \times 'b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'b \Rightarrow 'c$

local

defs
  Pair-def: *Pair a b == Abs-Prod (Pair-Rep a b)*
  fst-def: *fst p == THE a. EX b. p = Pair a b*
  snd-def: *snd p == THE b. EX a. p = Pair a b*
  split-def: *split == (%c p. c (fst p) (snd p))*
  curry-def: *curry == (%c x y. c (Pair x y))*

Patterns – extends pre-defined type *pttrn* used in abstractions.

nonterminals
  tuple-args patterns

syntax
  -tuple :: *'a => tuple-args => 'a * 'b* (($1'(-,/ -')$))
  -tuple-arg :: *'a => tuple-args* (-)
  -tuple-args :: *'a => tuple-args => tuple-args* (-,/ -)
  -pattern :: [*pttrn*, *patterns*] => *pttrn* ('(-,/ -'))
    :: *pttrn => patterns* (-)
  -patterns :: [*pttrn*, *patterns*] => *patterns* (-,/ -)

translations
  (x, y) *== Pair x y*
  -tuple x (-tuple-args y z) == -tuple x (-tuple-arg (-tuple y z))
  %(x,y,zs).b *== split(%x (y,zs).b)*
  %(x,y).b *== split(%x y. b)*
  -abs (Pair x y) t => %(x,y).t

⟨*ML*⟩

Towards a datatype declaration

lemma *surj-pair* [*simp*]: *EX x y. p = (x, y)*
⟨*proof*⟩

lemma *PairE* [*cases type*: ∗]:
obtains *x y* **where** *p = (x, y)*
⟨*proof*⟩

lemma *ProdI*: *Pair-Rep a b ∈ Prod*
⟨*proof*⟩

lemma *Pair-Rep-inject*: *Pair-Rep a b = Pair-Rep a′ b′ ⟹ a = a′ ∧ b = b′*
⟨*proof*⟩

lemma *inj-on-Abs-Prod*: *inj-on Abs-Prod Prod*
⟨*proof*⟩

lemma *Pair-inject*:
assumes *(a, b) = (a′, b′)*
and *a = a′ ==> b = b′ ==> R*
shows *R*
⟨*proof*⟩

rep-datatype (*prod*) *Pair*
⟨*proof*⟩

lemmas *Pair-eq = prod.inject*

lemma *fst-conv* [*simp*, *code*]: *fst (a, b) = a*
⟨*proof*⟩

lemma *snd-conv* [*simp*, *code*]: *snd (a, b) = b*
⟨*proof*⟩

13.3.2 Basic rules and proof tools

lemma *fst-eqD*: *fst (x, y) = a ==> x = a*
⟨*proof*⟩

lemma *snd-eqD*: *snd (x, y) = a ==> y = a*
⟨*proof*⟩

lemma *pair-collapse* [*simp*]: *(fst p, snd p) = p*
⟨*proof*⟩

lemmas *surjective-pairing = pair-collapse* [*symmetric*]

lemma *split-paired-all*: *(!!x. PROP P x) == (!!a b. PROP P (a, b))*
⟨*proof*⟩

The rule *split-paired-all* does not work with the Simplifier because it also affects premises in congrence rules, where this can lead to premises of the form !!*a b*. ... = *?P*(*a*, *b*) which cannot be solved by reflexivity.

lemmas *split-tupled-all* = *split-paired-all unit-all-eq2*

⟨*ML*⟩

lemma *split-paired-All* [*simp*]: (*ALL x*. *P x*) = (*ALL a b*. *P* (*a*, *b*))
— [*iff*] is not a good idea because it makes *blast* loop
⟨*proof*⟩

lemma *split-paired-Ex* [*simp*]: (*EX x*. *P x*) = (*EX a b*. *P* (*a*, *b*))
⟨*proof*⟩

lemma *Pair-fst-snd-eq*: *s* = *t* ⟷ *fst s* = *fst t* ∧ *snd s* = *snd t*
⟨*proof*⟩

lemma *prod-eqI* [*intro?*]: *fst p* = *fst q* ⟹ *snd p* = *snd q* ⟹ *p* = *q*
⟨*proof*⟩

13.3.3 *split* and *curry*

lemma *split-conv* [*simp*, *code*]: *split f* (*a*, *b*) = *f a b*
⟨*proof*⟩

lemma *curry-conv* [*simp*, *code*]: *curry f a b* = *f* (*a*, *b*)
⟨*proof*⟩

lemmas *split* = *split-conv* — for backwards compatibility

lemma *splitI*: *f a b* ⟹ *split f* (*a*, *b*)
⟨*proof*⟩

lemma *splitD*: *split f* (*a*, *b*) ⟹ *f a b*
⟨*proof*⟩

lemma *curryI* [*intro!*]: *f* (*a*, *b*) ⟹ *curry f a b*
⟨*proof*⟩

lemma *curryD* [*dest!*]: *curry f a b* ⟹ *f* (*a*, *b*)
⟨*proof*⟩

lemma *curryE*: *curry f a b* ⟹ (*f* (*a*, *b*) ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

lemma *curry-split* [*simp*]: *curry* (*split f*) = *f*
⟨*proof*⟩

lemma *split-curry* [*simp*]: *split* (*curry f*) = *f*

⟨*proof*⟩

lemma *split-Pair* [*simp*]: $(\lambda(x,\ y).\ (x,\ y)) = id$
⟨*proof*⟩

lemma *split-eta*: $(\lambda(x,\ y).\ f\ (x,\ y)) = f$
— Subsumes the old *split-Pair* when f is the identity function.
⟨*proof*⟩

lemma *split-comp*: $split\ (f \circ g)\ x = f\ (g\ (fst\ x))\ (snd\ x)$
⟨*proof*⟩

lemma *split-twice*: $split\ f\ (split\ g\ p) = split\ (\lambda x\ y.\ split\ f\ (g\ x\ y))\ p$
⟨*proof*⟩

lemma *split-paired-The*: $(THE\ x.\ P\ x) = (THE\ (a,\ b).\ P\ (a,\ b))$
— Can't be added to simpset: loops!
⟨*proof*⟩

lemma *The-split*: $The\ (split\ P) = (THE\ xy.\ P\ (fst\ xy)\ (snd\ xy))$
⟨*proof*⟩

lemma *split-weak-cong*: $p = q \Longrightarrow split\ c\ p = split\ c\ q$
— Prevents simplification of c: much faster
⟨*proof*⟩

lemma *cond-split-eta*: $(!!x\ y.\ f\ x\ y = g\ (x,\ y)) ==> (\%(x,\ y).\ f\ x\ y) = g$
⟨*proof*⟩

Simplification procedure for *cond-split-eta*. Using *split-eta* as a rewrite rule is not general enough, and using *cond-split-eta* directly would render some existing proofs very inefficient; similarly for *split-beta*.

⟨*ML*⟩

lemma *split-beta* [*mono*]: $(\%(x,\ y).\ P\ x\ y)\ z = P\ (fst\ z)\ (snd\ z)$
⟨*proof*⟩

lemma *split-split* [*noatp*]: $R(split\ c\ p) = (ALL\ x\ y.\ p = (x,\ y)\ --> R(c\ x\ y))$
— For use with *split* and the Simplifier.
⟨*proof*⟩

split-split could be declared as [*split*] done after the Splitter has been speeded up significantly; precompute the constants involved and don't do anything unless the current goal contains one of those constants.

lemma *split-split-asm* [*noatp*]: $R\ (split\ c\ p) = (\sim(EX\ x\ y.\ p = (x,\ y)\ \&\ (\sim R\ (c\ x\ y))))$
⟨*proof*⟩

split used as a logical connective or set former.

These rules are for use with *blast*; could instead call *simp* using *split* as rewrite.

lemma *splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *c a b* |] ==> *split c p*
⟨*proof*⟩

lemma *splitI2'*: !!*p*. [| !!*a b*. (*a*, *b*) = *p* ==> *c a b x* |] ==> *split c p x*
⟨*proof*⟩

lemma *splitE*: *split c p* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y* ==> *Q*) ==> *Q*
⟨*proof*⟩

lemma *splitE'*: *split c p z* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y z* ==> *Q*) ==> *Q*
⟨*proof*⟩

lemma *splitE2*:
[| *Q* (*split P z*); !!*x y*. [|*z* = (*x*, *y*); *Q* (*P x y*)|] ==> *R* |] ==> *R*
⟨*proof*⟩

lemma *splitD'*: *split R* (*a*,*b*) *c* ==> *R a b c*
⟨*proof*⟩

lemma *mem-splitI*: *z*: *c a b* ==> *z*: *split c* (*a*, *b*)
⟨*proof*⟩

lemma *mem-splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *z*: *c a b* |] ==> *z*: *split c p*
⟨*proof*⟩

lemma *mem-splitE*:
assumes *major*: *z*: *split c p*
and *cases*: !!*x y*. [| *p* = (*x*,*y*); *z*: *c x y* |] ==> *Q*
shows *Q*
⟨*proof*⟩

declare *mem-splitI2* [*intro*!] *mem-splitI* [*intro*!] *splitI2'* [*intro*!] *splitI2* [*intro*!] *splitI* [*intro*!]
declare *mem-splitE* [*elim*!] *splitE'* [*elim*!] *splitE* [*elim*!]

⟨*ML*⟩

lemma *split-eta-SetCompr* [*simp*,*noatp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P* (*x*, *y*)) = *P*
⟨*proof*⟩

lemma *split-eta-SetCompr2* [*simp*,*noatp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P x y*) = *split P*
⟨*proof*⟩

lemma *split-part* [*simp*]: (%(*a*,*b*). *P* & *Q a b*) = (%*ab*. *P* & *split Q ab*)
— Allows simplifications of nested splits in case of independent predicates.

⟨*proof*⟩

lemma *split-comp-eq*:
fixes $f :: 'a => 'b => 'c$ **and** $g :: 'd => 'a$
shows $(\%u.\ f\ (g\ (fst\ u))\ (snd\ u)) = (split\ (\%x.\ f\ (g\ x)))$
⟨*proof*⟩

lemma *pair-imageI* [*intro*]: $(a,\ b) : A ==> f\ a\ b : (\%(a,\ b).\ f\ a\ b)\ `\ A$
⟨*proof*⟩

lemma *The-split-eq* [*simp*]: $(THE\ (x',y').\ x = x'\ \&\ y = y') = (x,\ y)$
⟨*proof*⟩

Setup of internal *split-rule*.

definition
internal-split :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a \times 'b \Rightarrow 'c$
where
internal-split == *split*

lemma *internal-split-conv*: *internal-split c* $(a,\ b) = c\ a\ b$
⟨*proof*⟩

hide *const internal-split*

⟨*ML*⟩

lemmas *prod-caseI* = *prod.cases* [*THEN iffD2*, *standard*]

lemma *prod-caseI2*: $!!p.\ [|\ !!a\ b.\ p = (a,\ b) ==> c\ a\ b\ |] ==>$ *prod-case c p*
⟨*proof*⟩

lemma *prod-caseI2'*: $!!p.\ [|\ !!a\ b.\ (a,\ b) = p ==> c\ a\ b\ x\ |] ==>$ *prod-case c p x*
⟨*proof*⟩

lemma *prod-caseE*: *prod-case c p* $==> (!!x\ y.\ p = (x,\ y) ==> c\ x\ y ==> Q) ==> Q$
⟨*proof*⟩

lemma *prod-caseE'*: *prod-case c p z* $==> (!!x\ y.\ p = (x,\ y) ==> c\ x\ y\ z ==> Q) ==> Q$
⟨*proof*⟩

lemma *prod-case-unfold*: *prod-case* $= (\%c\ p.\ c\ (fst\ p)\ (snd\ p))$
⟨*proof*⟩

declare *prod-caseI2'* [*intro*!] *prod-caseI2* [*intro*!] *prod-caseI* [*intro*!]
declare *prod-caseE'* [*elim*!] *prod-caseE* [*elim*!]

lemma *prod-case-split*:
prod-case = *split*
⟨*proof*⟩

lemma *prod-case-beta*:
prod-case f p = *f* (*fst p*) (*snd p*)
⟨*proof*⟩

13.4 Further cases/induct rules for tuples

lemma *prod-cases3* [*cases type*]:
obtains (*fields*) *a b c* **where** *y* = (*a*, *b*, *c*)
⟨*proof*⟩

lemma *prod-induct3* [*case-names fields*, *induct type*]:
(!!*a b c*. *P* (*a*, *b*, *c*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases4* [*cases type*]:
obtains (*fields*) *a b c d* **where** *y* = (*a*, *b*, *c*, *d*)
⟨*proof*⟩

lemma *prod-induct4* [*case-names fields*, *induct type*]:
(!!*a b c d*. *P* (*a*, *b*, *c*, *d*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases5* [*cases type*]:
obtains (*fields*) *a b c d e* **where** *y* = (*a*, *b*, *c*, *d*, *e*)
⟨*proof*⟩

lemma *prod-induct5* [*case-names fields*, *induct type*]:
(!!*a b c d e*. *P* (*a*, *b*, *c*, *d*, *e*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases6* [*cases type*]:
obtains (*fields*) *a b c d e f* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*)
⟨*proof*⟩

lemma *prod-induct6* [*case-names fields*, *induct type*]:
(!!*a b c d e f*. *P* (*a*, *b*, *c*, *d*, *e*, *f*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases7* [*cases type*]:
obtains (*fields*) *a b c d e f g* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*)
⟨*proof*⟩

lemma *prod-induct7* [*case-names fields*, *induct type*]:
(!!*a b c d e f g*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*)) ==> *P x*
⟨*proof*⟩

13.4.1 Derived operations

The composition-uncurry combinator.

notation *fcomp* (**infixl** *o>* *60*)

definition
  scomp :: (*'a* ⇒ *'b* × *'c*) ⇒ (*'b* ⇒ *'c* ⇒ *'d*) ⇒ *'a* ⇒ *'d* (**infixl** *o→* *60*)
where
  f o→ g = (λ*x*. *split g* (*f x*))

lemma *scomp-apply*: (*f o→ g*) *x* = *split g* (*f x*)
  ⟨*proof*⟩

lemma *Pair-scomp*: *Pair x o→ f* = *f x*
  ⟨*proof*⟩

lemma *scomp-Pair*: *x o→ Pair* = *x*
  ⟨*proof*⟩

lemma *scomp-scomp*: (*f o→ g*) *o→ h* = *f o→* (λ*x*. *g x o→ h*)
  ⟨*proof*⟩

lemma *scomp-fcomp*: (*f o→ g*) *o> h* = *f o→* (λ*x*. *g x o> h*)
  ⟨*proof*⟩

lemma *fcomp-scomp*: (*f o> g*) *o→ h* = *f o>* (*g o→ h*)
  ⟨*proof*⟩

no-notation *fcomp* (**infixl** *o>* *60*)
no-notation *scomp* (**infixl** *o→* *60*)

prod-fun — action of the product functor upon functions.

definition *prod-fun* :: (*'a* ⇒ *'c*) ⇒ (*'b* ⇒ *'d*) ⇒ *'a* × *'b* ⇒ *'c* × *'d* **where**
  [*code del*]: *prod-fun f g* = (λ(*x*, *y*). (*f x*, *g y*))

lemma *prod-fun* [*simp*, *code*]: *prod-fun f g* (*a*, *b*) = (*f a*, *g b*)
  ⟨*proof*⟩

lemma *prod-fun-compose*: *prod-fun* (*f1 o f2*) (*g1 o g2*) = (*prod-fun f1 g1 o prod-fun f2 g2*)
  ⟨*proof*⟩

lemma *prod-fun-ident* [*simp*]: *prod-fun* (%*x*. *x*) (%*y*. *y*) = (%*z*. *z*)
  ⟨*proof*⟩

lemma *prod-fun-imageI* [*intro*]: (*a*, *b*) : *r* ==> (*f a*, *g b*) : *prod-fun f g* ‘ *r*
  ⟨*proof*⟩

lemma *prod-fun-imageE* [*elim*!]:

assumes *major*: *c*: (*prod-fun f g*)'*r*
and *cases*: !!*x y*. [| *c*=(*f*(*x*),*g*(*y*)); (*x*,*y*):*r* |] ==> *P*
shows *P*
⟨*proof*⟩

definition
apfst :: ('*a* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*c* × '*b*
where
[*code del*]: *apfst f* = *prod-fun f id*

definition
apsnd :: ('*b* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*a* × '*c*
where
[*code del*]: *apsnd f* = *prod-fun id f*

lemma *apfst-conv* [*simp*, *code*]:
apfst f (*x*, *y*) = (*f x*, *y*)
⟨*proof*⟩

lemma *upd-snd-conv* [*simp*, *code*]:
apsnd f (*x*, *y*) = (*x*, *f y*)
⟨*proof*⟩

Disjoint union of a family of sets – Sigma.

definition *Sigma* :: ['*a set*, '*a* => '*b set*] => ('*a* × '*b*) *set* **where**
Sigma-def: *Sigma A B* == *UN x*:*A*. *UN y*:*B x*. {*Pair x y*}

abbreviation
Times :: ['*a set*, '*b set*] => ('*a* * '*b*) *set*
(**infixr** <*> *80*) **where**
A <*> *B* == *Sigma A* (%-. *B*)

notation (*xsymbols*)
Times (**infixr** × *80*)

notation (*HTML* **output**)
Times (**infixr** × *80*)

syntax
@*Sigma* ::[*pttrn*, '*a set*, '*b set*] => ('*a* * '*b*) *set* ((*3SIGMA* -:-./ -) [*0*, *0*, *10*] *10*)

translations
SIGMA x:*A*. *B* == *Product-Type.Sigma A* (%*x*. *B*)

lemma *SigmaI* [*intro*!]: [| *a*:*A*; *b*:*B*(*a*) |] ==> (*a*,*b*) : *Sigma A B*
⟨*proof*⟩

lemma *SigmaE* [*elim*!]:
[| *c*: *Sigma A B*;

!!x y.[| x:A; y:$B(x)$; c=(x,y) |] ==> P
|] ==> P
— The general elimination rule.
⟨*proof*⟩

Elimination of $(a,\ b) \in A \times B$ – introduces no eigenvariables.

lemma *SigmaD1*: $(a,\ b)$: *Sigma* A B ==> a : A
⟨*proof*⟩

lemma *SigmaD2*: $(a,\ b)$: *Sigma* A B ==> b : B a
⟨*proof*⟩

lemma *SigmaE2*:
[| $(a,\ b)$: *Sigma* A B;
[| a:A; b:$B(a)$ |] ==> P
|] ==> P
⟨*proof*⟩

lemma *Sigma-cong*:
⟦$A = B$; !!x. $x \in B \Longrightarrow C\ x = D\ x$⟧
$\Longrightarrow$ (*SIGMA* x: A. $C\ x$) = (*SIGMA* x: B. $D\ x$)
⟨*proof*⟩

lemma *Sigma-mono*: [| A <= C; !!x. x:A ==> $B\ x$ <= $D\ x$ |] ==> *Sigma* A B <= *Sigma* C D
⟨*proof*⟩

lemma *Sigma-empty1* [*simp*]: *Sigma* {} B = {}
⟨*proof*⟩

lemma *Sigma-empty2* [*simp*]: A <*> {} = {}
⟨*proof*⟩

lemma *UNIV-Times-UNIV* [*simp*]: *UNIV* <*> *UNIV* = *UNIV*
⟨*proof*⟩

lemma *Compl-Times-UNIV1* [*simp*]: − (*UNIV* <*> A) = *UNIV* <*> (−A)
⟨*proof*⟩

lemma *Compl-Times-UNIV2* [*simp*]: − (A <*> *UNIV*) = (−A) <*> *UNIV*
⟨*proof*⟩

lemma *mem-Sigma-iff* [*iff*]: ((a,b): *Sigma* A B) = (a:A & b:$B(a)$)
⟨*proof*⟩

lemma *Times-subset-cancel2*: x:C ==> (A <*> C <= B <*> C) = (A <= B)
⟨*proof*⟩

lemma *Times-eq-cancel2*: x:C ==> (A <*> C = B <*> C) = (A = B)

⟨*proof*⟩

lemma *SetCompr-Sigma-eq*:
Collect (*split* (%*x y*. *P x* & *Q x y*)) = (*SIGMA x*:*Collect P*. *Collect* (*Q x*))
⟨*proof*⟩

lemma *Collect-split* [*simp*]: {(*a*,*b*). *P a* & *Q b*} = *Collect P* <*∗*> *Collect Q*
⟨*proof*⟩

lemma *UN-Times-distrib*:
(*UN* (*a*,*b*):(*A* <*∗*> *B*). *E a* <*∗*> *F b*) = (*UNION A E*) <*∗*> (*UNION B F*)
— Suggested by Pierre Chartier
⟨*proof*⟩

lemma *split-paired-Ball-Sigma* [*simp*,*noatp*]:
(*ALL z*: *Sigma A B*. *P z*) = (*ALL x*:*A*. *ALL y*: *B x*. *P*(*x*,*y*))
⟨*proof*⟩

lemma *split-paired-Bex-Sigma* [*simp*,*noatp*]:
(*EX z*: *Sigma A B*. *P z*) = (*EX x*:*A*. *EX y*: *B x*. *P*(*x*,*y*))
⟨*proof*⟩

lemma *Sigma-Un-distrib1*: (*SIGMA i*:*I Un J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) *Un* (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

lemma *Sigma-Un-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *Un B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) *Un* (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

lemma *Sigma-Int-distrib1*: (*SIGMA i*:*I Int J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) *Int* (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

lemma *Sigma-Int-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *Int B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) *Int* (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

lemma *Sigma-Diff-distrib1*: (*SIGMA i*:*I − J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) − (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

lemma *Sigma-Diff-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *− B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) − (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

lemma *Sigma-Union*: *Sigma* (*Union X*) *B* = (*UN A*:*X*. *Sigma A B*)
⟨*proof*⟩

Non-dependent versions are needed to avoid the need for higher-order match-

ing, especially when the rules are re-oriented.

lemma *Times-Un-distrib1*: $(A\ Un\ B) <*> C = (A <*> C)\ Un\ (B <*> C)$
⟨*proof*⟩

lemma *Times-Int-distrib1*: $(A\ Int\ B) <*> C = (A <*> C)\ Int\ (B <*> C)$
⟨*proof*⟩

lemma *Times-Diff-distrib1*: $(A - B) <*> C = (A <*> C) - (B <*> C)$
⟨*proof*⟩

lemma *insert-times-insert*[*simp*]:
$insert\ a\ A \times insert\ b\ B =$
$insert\ (a,b)\ (A \times insert\ b\ B \cup insert\ a\ A \times B)$
⟨*proof*⟩

13.4.2 Code generator setup

instance $*$ :: $(eq, eq)\ eq$ ⟨*proof*⟩

lemma [*code*]:
$eq\text{-}class.eq\ (x1::'a::eq, y1::'b::eq)\ (x2, y2) \longleftrightarrow x1 = x2 \wedge y1 = y2$ ⟨*proof*⟩

lemma *split-case-cert*:
assumes $CASE \equiv split\ f$
shows $CASE\ (a, b) \equiv f\ a\ b$
⟨*proof*⟩

⟨*ML*⟩

code-type $*$
(*SML* **infix** *2* $*$)
(*OCaml* **infix** *2* $*$)
(*Haskell* !((-),/ (-)))

code-instance $*$ :: *eq*
(*Haskell* −)

code-const *eq-class.eq* :: $'a::eq \times 'b::eq \Rightarrow 'a \times 'b \Rightarrow bool$
(*Haskell* **infixl** *4* ==)

code-const *Pair*
(*SML* !((-),/ (-)))
(*OCaml* !((-),/ (-)))
(*Haskell* !((-),/ (-)))

code-const *fst* **and** *snd*
(*Haskell fst* **and** *snd*)

types-code

```
  *     ((- */ -))
attach (term-of) ⟪
fun term-of-id-42 aF aT bF bT (x, y) = HOLogic.pair-const aT bT $ aF x $ bF
y;
⟫
attach (test) ⟪
fun gen-id-42 aG aT bG bT i =
  let
    val (x, t) = aG i;
    val (y, u) = bG i
  in ((x, y), fn () => HOLogic.pair-const aT bT $ t () $ u ()) end;
⟫

consts-code
  Pair    ((-,/ -))
```

⟨*ML*⟩

13.5 Legacy bindings

⟨*ML*⟩

13.6 Further inductive packages

⟨*ML*⟩

end

14 Datatype: Analogues of the Cartesian Product and Disjoint Sum for Datatypes

```
theory Datatype
imports Nat Product-Type
begin

typedef (Node)
  ('a,'b) node = {p. EX f x k. p = (f::nat=>'b+nat, x::'a+nat) & f k = Inr 0}
    — it is a subtype of (nat=>'b+nat) * ('a+nat)
  ⟨proof⟩
```

Datatypes will be represented by sets of type *node*

```
types 'a item        = ('a, unit) node set
      ('a, 'b) dtree = ('a, 'b) node set

consts
  Push      :: [('b + nat), nat => ('b + nat)] => (nat => ('b + nat))
```

Push-Node :: $[('b + nat), ('a, 'b)\ node] \Rightarrow ('a, 'b)\ node$
ndepth :: $('a, 'b)\ node \Rightarrow nat$

Atom :: $('a + nat) \Rightarrow ('a, 'b)\ dtree$
Leaf :: $'a \Rightarrow ('a, 'b)\ dtree$
Numb :: $nat \Rightarrow ('a, 'b)\ dtree$
Scons :: $[('a, 'b)\ dtree, ('a, 'b)\ dtree] \Rightarrow ('a, 'b)\ dtree$
In0 :: $('a, 'b)\ dtree \Rightarrow ('a, 'b)\ dtree$
In1 :: $('a, 'b)\ dtree \Rightarrow ('a, 'b)\ dtree$
Lim :: $('b \Rightarrow ('a, 'b)\ dtree) \Rightarrow ('a, 'b)\ dtree$

ntrunc :: $[nat, ('a, 'b)\ dtree] \Rightarrow ('a, 'b)\ dtree$

uprod :: $[('a, 'b)\ dtree\ set, ('a, 'b)\ dtree\ set] \Rightarrow ('a, 'b)\ dtree\ set$
usum :: $[('a, 'b)\ dtree\ set, ('a, 'b)\ dtree\ set] \Rightarrow ('a, 'b)\ dtree\ set$

Split :: $[[('a, 'b)\ dtree, ('a, 'b)\ dtree] \Rightarrow 'c, ('a, 'b)\ dtree] \Rightarrow 'c$
Case :: $[[('a, 'b)\ dtree] \Rightarrow 'c, [('a, 'b)\ dtree] \Rightarrow 'c, ('a, 'b)\ dtree] \Rightarrow 'c$

dprod :: $[(('a, 'b)\ dtree * ('a, 'b)\ dtree)set, (('a, 'b)\ dtree * ('a, 'b)\ dtree)set] \Rightarrow (('a, 'b)\ dtree * ('a, 'b)\ dtree)set$
dsum :: $[(('a, 'b)\ dtree * ('a, 'b)\ dtree)set, (('a, 'b)\ dtree * ('a, 'b)\ dtree)set] \Rightarrow (('a, 'b)\ dtree * ('a, 'b)\ dtree)set$

defs

Push-Node-def: *Push-Node* == (%*n x*. *Abs-Node* (*apfst* (*Push n*) (*Rep-Node x*)))

Push-def: *Push* == (%*b h*. *nat-case b h*)

Atom-def: *Atom* == (%*x*. {*Abs-Node*((%*k*. *Inr 0*, *x*))})
Scons-def: *Scons M N* == (*Push-Node* (*Inr 1*) ‘ *M*) *Un* (*Push-Node* (*Inr* (*Suc 1*)) ‘ *N*)

Leaf-def: *Leaf* == *Atom o Inl*
Numb-def: *Numb* == *Atom o Inr*

In0-def: *In0*(*M*) == *Scons* (*Numb 0*) *M*
In1-def: *In1*(*M*) == *Scons* (*Numb 1*) *M*

Lim-def: $Lim\ f == Union\ \{z.\ ?\ x.\ z = Push\text{-}Node\ (Inl\ x)\ `\ (f\ x)\}$

ndepth-def: $ndepth(n) == (\%(f,x).\ LEAST\ k.\ f\ k = Inr\ 0)\ (Rep\text{-}Node\ n)$
ntrunc-def: $ntrunc\ k\ N == \{n.\ n{:}N\ \&\ ndepth(n){<}k\}$

uprod-def: $uprod\ A\ B == UN\ x{:}A.\ UN\ y{:}B.\ \{\ Scons\ x\ y\ \}$
usum-def: $usum\ A\ B == In0`A\ Un\ In1`B$

Split-def: $Split\ c\ M == THE\ u.\ EX\ x\ y.\ M = Scons\ x\ y\ \&\ u = c\ x\ y$

Case-def: $Case\ c\ d\ M == THE\ u.\ (EX\ x\ .\ M = In0(x)\ \&\ u = c(x))$
$\quad | (EX\ y\ .\ M = In1(y)\ \&\ u = d(y))$

dprod-def: $dprod\ r\ s == UN\ (x,x'){:}r.\ UN\ (y,y'){:}s.\ \{(Scons\ x\ y,\ Scons\ x'\ y')\}$

dsum-def: $dsum\ r\ s == (UN\ (x,x'){:}r.\ \{(In0(x),In0(x'))\})\ Un$
$\quad (UN\ (y,y'){:}s.\ \{(In1(y),In1(y'))\})$

lemma *apfst-convE*:
$[|\ q = apfst\ f\ p;\ \ !!x\ y.\ [|\ p = (x,y);\ \ q = (f(x),y)\ |] ==> R\ |] ==> R$
⟨*proof*⟩

lemma *Push-inject1*: $Push\ i\ f = Push\ j\ g\ ==> i{=}j$
⟨*proof*⟩

lemma *Push-inject2*: $Push\ i\ f = Push\ j\ g\ ==> f{=}g$
⟨*proof*⟩

lemma *Push-inject*:
$[|\ Push\ i\ f\ {=}Push\ j\ g;\ \ [|\ i{=}j;\ \ f{=}g\ |] ==> P\ |] ==> P$
⟨*proof*⟩

lemma *Push-neq-K0*: $Push\ (Inr\ (Suc\ k))\ f = (\%z.\ Inr\ 0) ==> P$
⟨*proof*⟩

lemmas *Abs-Node-inj* = *Abs-Node-inject* [*THEN* [*2*] *rev-iffD1*, *standard*]

lemma *Node-K0-I*: (%*k*. *Inr 0*, *a*) : *Node*
⟨*proof*⟩

lemma *Node-Push-I*: *p*: *Node* ==> *apfst* (*Push i*) *p* : *Node*
⟨*proof*⟩

14.1 Freeness: Distinctness of Constructors

lemma *Scons-not-Atom* [*iff*]: *Scons M N* ≠ *Atom*(*a*)
⟨*proof*⟩

lemmas *Atom-not-Scons* [*iff*] = *Scons-not-Atom* [*THEN not-sym*, *standard*]

lemma *inj-Atom*: *inj*(*Atom*)
⟨*proof*⟩
lemmas *Atom-inject* = *inj-Atom* [*THEN injD*, *standard*]

lemma *Atom-Atom-eq* [*iff*]: (*Atom*(*a*)=*Atom*(*b*)) = (*a*=*b*)
⟨*proof*⟩

lemma *inj-Leaf*: *inj*(*Leaf*)
⟨*proof*⟩

lemmas *Leaf-inject* [*dest*!] = *inj-Leaf* [*THEN injD*, *standard*]

lemma *inj-Numb*: *inj*(*Numb*)
⟨*proof*⟩

lemmas *Numb-inject* [*dest*!] = *inj-Numb* [*THEN injD*, *standard*]

lemma *Push-Node-inject*:
[| *Push-Node i m* =*Push-Node j n*; [| *i*=*j*; *m*=*n* |] ==> *P*
|] ==> *P*
⟨*proof*⟩

lemma *Scons-inject-lemma1*: *Scons M N* <= *Scons M' N'* ==> *M*<=*M'*

⟨*proof*⟩

lemma *Scons-inject-lemma2*: *Scons M N <= Scons M' N' ==> N<=N'*
⟨*proof*⟩

lemma *Scons-inject1*: *Scons M N = Scons M' N' ==> M=M'*
⟨*proof*⟩

lemma *Scons-inject2*: *Scons M N = Scons M' N' ==> N=N'*
⟨*proof*⟩

lemma *Scons-inject*:
[| *Scons M N = Scons M' N'*; [| *M=M'*; *N=N'* |] ==> *P* |] ==> *P*
⟨*proof*⟩

lemma *Scons-Scons-eq* [*iff*]: (*Scons M N = Scons M' N'*) = (*M=M'* & *N=N'*)
⟨*proof*⟩

lemma *Scons-not-Leaf* [*iff*]: *Scons M N* ≠ *Leaf*(*a*)
⟨*proof*⟩

lemmas *Leaf-not-Scons* [*iff*] = *Scons-not-Leaf* [*THEN not-sym*, *standard*]

lemma *Scons-not-Numb* [*iff*]: *Scons M N* ≠ *Numb*(*k*)
⟨*proof*⟩

lemmas *Numb-not-Scons* [*iff*] = *Scons-not-Numb* [*THEN not-sym*, *standard*]

lemma *Leaf-not-Numb* [*iff*]: *Leaf*(*a*) ≠ *Numb*(*k*)
⟨*proof*⟩

lemmas *Numb-not-Leaf* [*iff*] = *Leaf-not-Numb* [*THEN not-sym*, *standard*]

lemma *ndepth-K0*: *ndepth* (*Abs-Node*(%*k*. *Inr 0*, *x*)) = *0*
⟨*proof*⟩

lemma *ndepth-Push-Node-aux*:

nat-case (*Inr* (*Suc i*)) *f k* = *Inr 0* −−> *Suc*(*LEAST x*. *f x* = *Inr 0*) <= *k*
⟨*proof*⟩

lemma *ndepth-Push-Node*:
ndepth (*Push-Node* (*Inr* (*Suc i*)) *n*) = *Suc*(*ndepth*(*n*))
⟨*proof*⟩

lemma *ntrunc-0* [*simp*]: *ntrunc 0 M* = {}
⟨*proof*⟩

lemma *ntrunc-Atom* [*simp*]: *ntrunc* (*Suc k*) (*Atom a*) = *Atom*(*a*)
⟨*proof*⟩

lemma *ntrunc-Leaf* [*simp*]: *ntrunc* (*Suc k*) (*Leaf a*) = *Leaf*(*a*)
⟨*proof*⟩

lemma *ntrunc-Numb* [*simp*]: *ntrunc* (*Suc k*) (*Numb i*) = *Numb*(*i*)
⟨*proof*⟩

lemma *ntrunc-Scons* [*simp*]:
ntrunc (*Suc k*) (*Scons M N*) = *Scons* (*ntrunc k M*) (*ntrunc k N*)
⟨*proof*⟩

lemma *ntrunc-one-In0* [*simp*]: *ntrunc* (*Suc 0*) (*In0 M*) = {}
⟨*proof*⟩

lemma *ntrunc-In0* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In0 M*) = *In0* (*ntrunc* (*Suc k*) *M*)
⟨*proof*⟩

lemma *ntrunc-one-In1* [*simp*]: *ntrunc* (*Suc 0*) (*In1 M*) = {}
⟨*proof*⟩

lemma *ntrunc-In1* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In1 M*) = *In1* (*ntrunc* (*Suc k*) *M*)
⟨*proof*⟩

14.2 Set Constructions

lemma *uprodI* [*intro*!]: [| *M*:*A*; *N*:*B* |] ==> *Scons M N* : *uprod A B*
⟨*proof*⟩

lemma *uprodE* [*elim*!]:
[| c : *uprod* A B;
!!x y. [| x:A; y:B; c = *Scons* x y |] ==> P
|] ==> P
⟨*proof*⟩

lemma *uprodE2*: [| *Scons* M N : *uprod* A B; [| M:A; N:B |] ==> P |] ==> P
⟨*proof*⟩

lemma *usum-In0I* [*intro*]: M:A ==> *In0*(M) : *usum* A B
⟨*proof*⟩

lemma *usum-In1I* [*intro*]: N:B ==> *In1*(N) : *usum* A B
⟨*proof*⟩

lemma *usumE* [*elim*!]:
[| u : *usum* A B;
!!x. [| x:A; u=*In0*(x) |] ==> P;
!!y. [| y:B; u=*In1*(y) |] ==> P
|] ==> P
⟨*proof*⟩

lemma *In0-not-In1* [*iff*]: *In0*(M) ≠ *In1*(N)
⟨*proof*⟩

lemmas *In1-not-In0* [*iff*] = *In0-not-In1* [*THEN not-sym*, *standard*]

lemma *In0-inject*: *In0*(M) = *In0*(N) ==> M=N
⟨*proof*⟩

lemma *In1-inject*: *In1*(M) = *In1*(N) ==> M=N
⟨*proof*⟩

lemma *In0-eq* [*iff*]: (*In0* M = *In0* N) = (M=N)
⟨*proof*⟩

lemma *In1-eq* [*iff*]: (*In1* M = *In1* N) = (M=N)
⟨*proof*⟩

lemma *inj-In0*: *inj In0*

⟨*proof*⟩

lemma *inj-In1*: *inj In1*
⟨*proof*⟩

lemma *Lim-inject*: *Lim f = Lim g ==> f = g*
⟨*proof*⟩

lemma *ntrunc-subsetI*: *ntrunc k M <= M*
⟨*proof*⟩

lemma *ntrunc-subsetD*: *(!!k. ntrunc k M <= N) ==> M<=N*
⟨*proof*⟩

lemma *ntrunc-equality*: *(!!k. ntrunc k M = ntrunc k N) ==> M=N*
⟨*proof*⟩

lemma *ntrunc-o-equality*:
[| !!k. (ntrunc(k) o h1) = (ntrunc(k) o h2) |] ==> h1=h2
⟨*proof*⟩

lemma *uprod-mono*: *[| A<=A'; B<=B' |] ==> uprod A B <= uprod A' B'*
⟨*proof*⟩

lemma *usum-mono*: *[| A<=A'; B<=B' |] ==> usum A B <= usum A' B'*
⟨*proof*⟩

lemma *Scons-mono*: *[| M<=M'; N<=N' |] ==> Scons M N <= Scons M' N'*
⟨*proof*⟩

lemma *In0-mono*: *M<=N ==> In0(M) <= In0(N)*
⟨*proof*⟩

lemma *In1-mono*: *M<=N ==> In1(M) <= In1(N)*
⟨*proof*⟩

lemma *Split* [*simp*]: *Split c* (*Scons M N*) = *c M N*
⟨*proof*⟩

lemma *Case-In0* [*simp*]: *Case c d* (*In0 M*) = *c*(*M*)
⟨*proof*⟩

lemma *Case-In1* [*simp*]: *Case c d* (*In1 N*) = *d*(*N*)
⟨*proof*⟩

lemma *ntrunc-UN1*: *ntrunc k* (*UN x. f*(*x*)) = (*UN x. ntrunc k* (*f x*))
⟨*proof*⟩

lemma *Scons-UN1-x*: *Scons* (*UN x. f x*) *M* = (*UN x. Scons* (*f x*) *M*)
⟨*proof*⟩

lemma *Scons-UN1-y*: *Scons M* (*UN x. f x*) = (*UN x. Scons M* (*f x*))
⟨*proof*⟩

lemma *In0-UN1*: *In0*(*UN x. f*(*x*)) = (*UN x. In0*(*f*(*x*)))
⟨*proof*⟩

lemma *In1-UN1*: *In1*(*UN x. f*(*x*)) = (*UN x. In1*(*f*(*x*)))
⟨*proof*⟩

lemma *dprodI* [*intro*!]:
[| (*M*,*M'*):*r*; (*N*,*N'*):*s* |] ==> (*Scons M N*, *Scons M' N'*) : *dprod r s*
⟨*proof*⟩

lemma *dprodE* [*elim*!]:
[| *c* : *dprod r s*;
!!*x y x' y'*. [| (*x*,*x'*) : *r*; (*y*,*y'*) : *s*;
c = (*Scons x y*, *Scons x' y'*) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

lemma *dsum-In0I* [*intro*]: (*M*,*M'*):*r* ==> (*In0*(*M*), *In0*(*M'*)) : *dsum r s*
⟨*proof*⟩

lemma *dsum-In1I* [*intro*]: (*N*,*N'*):*s* ==> (*In1*(*N*), *In1*(*N'*)) : *dsum r s*
⟨*proof*⟩

lemma *dsumE* [*elim*!]:
[| *w* : *dsum r s*;
!!*x x'*. [| (*x*,*x'*) : *r*; *w* = (*In0*(*x*), *In0*(*x'*)) |] ==> *P*;
!!*y y'*. [| (*y*,*y'*) : *s*; *w* = (*In1*(*y*), *In1*(*y'*)) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

lemma *dprod-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dprod r s* <= *dprod r' s'*
⟨*proof*⟩

lemma *dsum-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dsum r s* <= *dsum r' s'*
⟨*proof*⟩

lemma *dprod-Sigma*: (*dprod* (*A* <*> *B*) (*C* <*> *D*)) <= (*uprod A C*) <*> (*uprod B D*)
⟨*proof*⟩

lemmas *dprod-subset-Sigma* = *subset-trans* [*OF dprod-mono dprod-Sigma*, *standard*]

lemma *dprod-subset-Sigma2*:
(*dprod* (*Sigma A B*) (*Sigma C D*)) <=
Sigma (*uprod A C*) (*Split* (%*x y*. *uprod* (*B x*) (*D y*)))
⟨*proof*⟩

lemma *dsum-Sigma*: (*dsum* (*A* <*> *B*) (*C* <*> *D*)) <= (*usum A C*) <*> (*usum B D*)
⟨*proof*⟩

lemmas *dsum-subset-Sigma* = *subset-trans* [*OF dsum-mono dsum-Sigma*, *standard*]

hides popular names

hide (**open**) *type node item*
hide (**open**) *const Push Node Atom Leaf Numb Lim Split Case*

15 Datatypes

15.1 Representing sums

rep-datatype (*sum*) *Inl Inr*
⟨*proof*⟩

lemma *sum-case-KK*[*simp*]: *sum-case* (%*x*. *a*) (%*x*. *a*) = (%*x*. *a*)
⟨*proof*⟩

lemma *surjective-sum*: *sum-case* (%*x*::'*a*. *f* (*Inl x*)) (%*y*::'*b*. *f* (*Inr y*)) *s* = *f*(*s*)
⟨*proof*⟩

lemma *sum-case-weak-cong*: *s* = *t* ==> *sum-case f g s* = *sum-case f g t*
— Prevents simplification of *f* and *g*: much faster.
⟨*proof*⟩

lemma *sum-case-inject*:
sum-case f1 f2 = *sum-case g1 g2* ==> (*f1* = *g1* ==> *f2* = *g2* ==> *P*) ==> *P*
⟨*proof*⟩

constdefs
Suml :: ('*a* => '*c*) => '*a* + '*b* => '*c*
Suml == (%*f*. *sum-case f undefined*)

Sumr :: ('*b* => '*c*) => '*a* + '*b* => '*c*
Sumr == *sum-case undefined*

lemma *Suml-inject*: *Suml f* = *Suml g* ==> *f* = *g*
⟨*proof*⟩

lemma *Sumr-inject*: *Sumr f* = *Sumr g* ==> *f* = *g*
⟨*proof*⟩

primrec *Projl* :: '*a* + '*b* => '*a*
where *Projl-Inl*: *Projl* (*Inl x*) = *x*

primrec *Projr* :: '*a* + '*b* => '*b*
where *Projr-Inr*: *Projr* (*Inr x*) = *x*

hide (**open**) *const Suml Sumr Projl Projr*

end

16 Power: Exponentiation

theory *Power*

imports *Nat*
begin

class *power* =
  fixes *power* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ^ *80*)

16.1 Powers for Arbitrary Monoids

class *recpower* = *monoid-mult* + *power* +
  assumes *power-0* [*simp*]: *a ^ 0 = 1*
  assumes *power-Suc* [*simp*]: *a ^ Suc n = a * (a ^ n)*

lemma *power-0-Suc* [*simp*]: *(0::'a::{recpower,semiring-0}) ^ (Suc n) = 0*
  ⟨*proof*⟩

It looks plausible as a simprule, but its effect can be strange.

lemma *power-0-left*: *0^n = (if n=0 then 1 else (0::'a::{recpower,semiring-0}))*
  ⟨*proof*⟩

lemma *power-one* [*simp*]: *1^n = (1::'a::recpower)*
  ⟨*proof*⟩

lemma *power-one-right* [*simp*]: *(a::'a::recpower) ^ 1 = a*
  ⟨*proof*⟩

lemma *power-commutes*: *(a::'a::recpower) ^ n * a = a * a ^ n*
  ⟨*proof*⟩

lemma *power-Suc2*: *(a::'a::recpower) ^ Suc n = a ^ n * a*
  ⟨*proof*⟩

lemma *power-add*: *(a::'a::recpower) ^ (m+n) = (a^m) * (a^n)*
  ⟨*proof*⟩

lemma *power-mult*: *(a::'a::recpower) ^ (m*n) = (a^m) ^ n*
  ⟨*proof*⟩

lemma *power-mult-distrib*: *((a::'a::{recpower,comm-monoid-mult}) * b) ^ n = (a^n) * (b^n)*
  ⟨*proof*⟩

lemma *zero-less-power*[*simp*]:
    0 < (a::'a::{ordered-semidom,recpower}) ==> 0 < a^n
⟨*proof*⟩

lemma *zero-le-power*[*simp*]:
    0 ≤ (a::'a::{ordered-semidom,recpower}) ==> 0 ≤ a^n
⟨*proof*⟩

lemma *one-le-power*[*simp*]:
1 ≤ (*a*::'*a*::{*ordered-semidom*,*recpower*}) ==> *1* ≤ *a* ^ *n*
⟨*proof*⟩

lemma *gt1-imp-ge0*: *1* < *a* ==> *0* ≤ (*a*::'*a*::*ordered-semidom*)
⟨*proof*⟩

lemma *power-gt1-lemma*:
assumes *gt1*: *1* < (*a*::'*a*::{*ordered-semidom*,*recpower*})
shows *1* < *a* * *a* ^ *n*
⟨*proof*⟩

lemma *one-less-power*[*simp*]:
⟦*1* < (*a*::'*a*::{*ordered-semidom*,*recpower*}); *0* < *n*⟧ ⟹ *1* < *a* ^ *n*
⟨*proof*⟩

lemma *power-gt1*:
1 < (*a*::'*a*::{*ordered-semidom*,*recpower*}) ==> *1* < *a* ^ (*Suc n*)
⟨*proof*⟩

lemma *power-le-imp-le-exp*:
assumes *gt1*: (*1*::'*a*::{*recpower*,*ordered-semidom*}) < *a*
shows !!*n*. *a* ^ *m* ≤ *a* ^ *n* ==> *m* ≤ *n*
⟨*proof*⟩

Surely we can strengthen this? It holds for *0*<*a*<*1* too.

lemma *power-inject-exp* [*simp*]:
1 < (*a*::'*a*::{*ordered-semidom*,*recpower*}) ==> (*a* ^ *m* = *a* ^ *n*) = (*m*=*n*)
⟨*proof*⟩

Can relax the first premise to (*0*::'*a*) < *a* in the case of the natural numbers.

lemma *power-less-imp-less-exp*:
[| (*1*::'*a*::{*recpower*,*ordered-semidom*}) < *a*; *a* ^ *m* < *a* ^ *n* |] ==> *m* < *n*
⟨*proof*⟩

lemma *power-mono*:
[|*a* ≤ *b*; (*0*::'*a*::{*recpower*,*ordered-semidom*}) ≤ *a*|] ==> *a* ^ *n* ≤ *b* ^ *n*
⟨*proof*⟩

lemma *power-strict-mono* [*rule-format*]:
[|*a* < *b*; (*0*::'*a*::{*recpower*,*ordered-semidom*}) ≤ *a*|]
==> *0* < *n* --> *a* ^ *n* < *b* ^ *n*
⟨*proof*⟩

lemma *power-eq-0-iff* [*simp*]:
(*a* ^ *n* = *0*) ⟷
(*a* = (*0*::'*a*::{*mult-zero*,*zero-neq-one*,*no-zero-divisors*,*recpower*}) & *n*≠*0*)
⟨*proof*⟩

lemma *field-power-not-zero*:
$a \neq (0::'a::\{ring\text{-}1\text{-}no\text{-}zero\text{-}divisors,recpower\}) ==> a\hat{}n \neq 0$
⟨*proof*⟩

lemma *nonzero-power-inverse*:
fixes $a :: 'a::\{division\text{-}ring,recpower\}$
shows $a \neq 0 ==> inverse\ (a\ \hat{}\ n) = (inverse\ a)\ \hat{}\ n$
⟨*proof*⟩

Perhaps these should be simprules.

lemma *power-inverse*:
fixes $a :: 'a::\{division\text{-}ring,division\text{-}by\text{-}zero,recpower\}$
shows $inverse\ (a\ \hat{}\ n) = (inverse\ a)\ \hat{}\ n$
⟨*proof*⟩

lemma *power-one-over*: $1\ /\ (a::'a::\{field,division\text{-}by\text{-}zero,recpower\})\hat{}n = (1\ /\ a)\hat{}n$
⟨*proof*⟩

lemma *nonzero-power-divide*:
$b \neq 0 ==> (a/b)\ \hat{}\ n = ((a::'a::\{field,recpower\})\ \hat{}\ n)\ /\ (b\ \hat{}\ n)$
⟨*proof*⟩

lemma *power-divide*:
$(a/b)\ \hat{}\ n = ((a::'a::\{field,division\text{-}by\text{-}zero,recpower\})\ \hat{}\ n\ /\ b\ \hat{}\ n)$
⟨*proof*⟩

lemma *power-abs*: $abs(a\ \hat{}\ n) = abs(a::'a::\{ordered\text{-}idom,recpower\})\ \hat{}\ n$
⟨*proof*⟩

lemma *abs-power-minus* [*simp*]:
fixes $a:: 'a::\{ordered\text{-}idom,recpower\}$ **shows** $abs((-a)\ \hat{}\ n) = abs(a\ \hat{}\ n)$
⟨*proof*⟩

lemma *zero-less-power-abs-iff* [*simp*,*noatp*]:
$(0 < (abs\ a)\hat{}n) = (a \neq (0::'a::\{ordered\text{-}idom,recpower\}) \mid n=0)$
⟨*proof*⟩

lemma *zero-le-power-abs* [*simp*]:
$(0::'a::\{ordered\text{-}idom,recpower\}) \leq (abs\ a)\hat{}n$
⟨*proof*⟩

lemma *power-minus*: $(-a)\ \hat{}\ n = (-\ 1)\hat{}n * (a::'a::\{ring\text{-}1,recpower\})\ \hat{}\ n$
⟨*proof*⟩

Lemma for *power-strict-decreasing*

lemma *power-Suc-less*:

$[|(0::'a::\{ordered\text{-}semidom,recpower\}) < a;\ a < 1|]$
$\Longrightarrow a * a\hat{}n < a\hat{}n$
⟨*proof*⟩

lemma *power-strict-decreasing*:
$[|n < N;\ 0 < a;\ a < (1::'a::\{ordered\text{-}semidom,recpower\})|]$
$\Longrightarrow a\hat{}N < a\hat{}n$
⟨*proof*⟩

Proof resembles that of *power-strict-decreasing*

lemma *power-decreasing*:
$[|n \leq N;\ 0 \leq a;\ a \leq (1::'a::\{ordered\text{-}semidom,recpower\})|]$
$\Longrightarrow a\hat{}N \leq a\hat{}n$
⟨*proof*⟩

lemma *power-Suc-less-one*:
$[|\ 0 < a;\ a < (1::'a::\{ordered\text{-}semidom,recpower\})\ |] \Longrightarrow a\ \hat{}\ Suc\ n < 1$
⟨*proof*⟩

Proof again resembles that of *power-strict-decreasing*

lemma *power-increasing*:
$[|n \leq N;\ (1::'a::\{ordered\text{-}semidom,recpower\}) \leq a|] \Longrightarrow a\hat{}n \leq a\hat{}N$
⟨*proof*⟩

Lemma for *power-strict-increasing*

lemma *power-less-power-Suc*:
$(1::'a::\{ordered\text{-}semidom,recpower\}) < a \Longrightarrow a\hat{}n < a * a\hat{}n$
⟨*proof*⟩

lemma *power-strict-increasing*:
$[|n < N;\ (1::'a::\{ordered\text{-}semidom,recpower\}) < a|] \Longrightarrow a\hat{}n < a\hat{}N$
⟨*proof*⟩

lemma *power-increasing-iff* [*simp*]:
$1 < (b::'a::\{ordered\text{-}semidom,recpower\}) \Longrightarrow (b\ \hat{}\ x \leq b\ \hat{}\ y) = (x \leq y)$
⟨*proof*⟩

lemma *power-strict-increasing-iff* [*simp*]:
$1 < (b::'a::\{ordered\text{-}semidom,recpower\}) \Longrightarrow (b\ \hat{}\ x < b\ \hat{}\ y) = (x < y)$
⟨*proof*⟩

lemma *power-le-imp-le-base*:
assumes *le*: $a\ \hat{}\ Suc\ n \leq b\ \hat{}\ Suc\ n$
and *ynonneg*: $(0::'a::\{ordered\text{-}semidom,recpower\}) \leq b$
shows $a \leq b$
⟨*proof*⟩

lemma *power-less-imp-less-base*:
fixes $a\ b :: 'a::\{ordered\text{-}semidom,recpower\}$

assumes *less*: $a \hat{} n < b \hat{} n$
assumes *nonneg*: $0 \leq b$
shows $a < b$
⟨*proof*⟩

lemma *power-inject-base*:
$[\!| a \hat{} Suc\ n = b \hat{} Suc\ n; 0 \leq a; 0 \leq b |\!]$
$\Longrightarrow a = (b::'a::\{ordered\text{-}semidom,recpower\})$
⟨*proof*⟩

lemma *power-eq-imp-eq-base*:
fixes $a\ b :: 'a::\{ordered\text{-}semidom,recpower\}$
shows $[\![a \hat{} n = b \hat{} n; 0 \leq a; 0 \leq b; 0 < n]\!] \Longrightarrow a = b$
⟨*proof*⟩

The divides relation

lemma *le-imp-power-dvd*:
fixes $a :: 'a::\{comm\text{-}semiring\text{-}1,recpower\}$
assumes $m \leq n$ **shows** $a \hat{} m\ dvd\ a \hat{} n$
⟨*proof*⟩

lemma *power-le-dvd*:
fixes $a\ b :: 'a::\{comm\text{-}semiring\text{-}1,recpower\}$
shows $a \hat{} n\ dvd\ b \Longrightarrow m \leq n \Longrightarrow a \hat{} m\ dvd\ b$
⟨*proof*⟩

lemma *dvd-power-same*:
$(x::'a::\{comm\text{-}semiring\text{-}1,recpower\})\ dvd\ y \Longrightarrow x \hat{} n\ dvd\ y \hat{} n$
⟨*proof*⟩

lemma *dvd-power-le*:
$(x::'a::\{comm\text{-}semiring\text{-}1,recpower\})\ dvd\ y \Longrightarrow m >= n \Longrightarrow x \hat{} n\ dvd\ y \hat{} m$
⟨*proof*⟩

lemma *dvd-power* [*simp*]:
$n > 0 \mid (x::'a::\{comm\text{-}semiring\text{-}1,recpower\}) = 1 \Longrightarrow x\ dvd\ x \hat{} n$
⟨*proof*⟩

16.2 Exponentiation for the Natural Numbers

instantiation *nat* :: *recpower*
begin

primrec *power-nat* **where**
$p \hat{} 0 = (1::nat)$
$\mid p \hat{} (Suc\ n) = (p::nat) * (p \hat{} n)$

instance ⟨*proof*⟩

declare *power-nat.simps* [*simp del*]

end

lemma *of-nat-power*:
of-nat (*m* ^ *n*) = (*of-nat m*::'*a*::{*semiring-1*,*recpower*}) ^ *n*
⟨*proof*⟩

lemma *nat-one-le-power* [*simp*]: *Suc 0* ≤ *i* ==> *Suc 0* ≤ *i*^*n*
⟨*proof*⟩

lemma *nat-zero-less-power-iff* [*simp*]: (*x*^*n* > *0*) = (*x* > (*0*::*nat*) | *n*=*0*)
⟨*proof*⟩

lemma *nat-power-eq-Suc-0-iff* [*simp*]:
(((*x*::*nat*)^*m* = *Suc 0*) = (*m* = *0* | *x* = *Suc 0*)
⟨*proof*⟩

lemma *power-Suc-0*[*simp*]: (*Suc 0*)^*n* = *Suc 0*
⟨*proof*⟩

Valid for the naturals, but what if *0*<*i*<*1*? Premises cannot be weakened: consider the case where $i = (0::'a)$, $m = (1::'a)$ and $n = (0::'a)$.

lemma *nat-power-less-imp-less*:
assumes *nonneg*: *0* < (*i*::*nat*)
assumes *less*: *i*^*m* < *i*^*n*
shows *m* < *n*
⟨*proof*⟩

lemma *power-diff*:
assumes *nz*: *a* ~= *0*
shows *n* <= *m* ==> (*a*::'*a*::{*recpower*, *field*}) ^ (*m*−*n*) = (*a*^*m*) / (*a*^*n*)
⟨*proof*⟩

end

17 Finite-Set: Finite sets

theory *Finite-Set*
imports *Nat Product-Type Power*
begin

17.1 Definition and basic properties

inductive *finite* :: '*a set* => *bool*
where
emptyI [*simp*, *intro*!]: *finite* {}

| *insertI* [*simp*, *intro*!]: *finite A ==> finite* (*insert a A*)

lemma *ex-new-if-finite*: — does not depend on def of finite at all
assumes ¬ *finite* (*UNIV* :: *'a set*) **and** *finite A*
shows ∃ *a*::*'a*. *a* ∉ *A*
⟨*proof*⟩

lemma *finite-induct* [*case-names empty insert*, *induct set*: *finite*]:
finite F ==>
P {} ==> (!!*x F*. *finite F ==> x* ∉ *F ==> P F ==> P* (*insert x F*)) ==> *P F*
— Discharging *x* ∉ *F* entails extra work.
⟨*proof*⟩

lemma *finite-ne-induct*[*case-names singleton insert*, *consumes 2*]:
assumes *fin*: *finite F* **shows** *F* ≠ {} ⟹
⟦ ⋀*x*. *P*{*x*};
⋀*x F*. ⟦ *finite F*; *F* ≠ {}; *x* ∉ *F*; *P F* ⟧ ⟹ *P* (*insert x F*) ⟧
⟹ *P F*
⟨*proof*⟩

lemma *finite-subset-induct* [*consumes 2*, *case-names empty insert*]:
assumes *finite F* **and** *F* ⊆ *A*
and *empty*: *P* {}
and *insert*: !!*a F*. *finite F ==> a* ∈ *A ==> a* ∉ *F ==> P F ==> P* (*insert a F*)
shows *P F*
⟨*proof*⟩

A finite choice principle. Does not need the SOME choice operator.

lemma *finite-set-choice*:
finite A ⟹ *ALL x*:*A*. (*EX y*. *P x y*) ⟹ *EX f*. *ALL x*:*A*. *P x* (*f x*)
⟨*proof*⟩

Finite sets are the images of initial segments of natural numbers:

lemma *finite-imp-nat-seg-image-inj-on*:
assumes *fin*: *finite A*
shows ∃ (*n*::*nat*) *f*. *A* = *f* ‘ {*i*. *i*<*n*} & *inj-on f* {*i*. *i*<*n*}
⟨*proof*⟩

lemma *nat-seg-image-imp-finite*:
!!*f A*. *A* = *f* ‘ {*i*::*nat*. *i*<*n*} ⟹ *finite A*
⟨*proof*⟩

lemma *finite-conv-nat-seg-image*:
finite A = (∃ (*n*::*nat*) *f*. *A* = *f* ‘ {*i*::*nat*. *i*<*n*})
⟨*proof*⟩

lemma *finite-Collect-less-nat*[*iff*]: *finite*{*n*::*nat*. *n*<*k*}

⟨*proof*⟩

17.1.1 Finiteness and set theoretic constructions

lemma *finite-UnI*: *finite F* ==> *finite G* ==> *finite* (*F Un G*)
⟨*proof*⟩

lemma *finite-subset*: $A \subseteq B$ ==> *finite B* ==> *finite A*
— Every subset of a finite set is finite.
⟨*proof*⟩

lemma *finite-Un* [*iff*]: *finite* (*F Un G*) = (*finite F* & *finite G*)
⟨*proof*⟩

lemma *finite-Collect-disjI*[*simp*]:
finite{*x*. *P x* | *Q x*} = (*finite*{*x*. *P x*} & *finite*{*x*. *Q x*})
⟨*proof*⟩

lemma *finite-Int* [*simp*, *intro*]: *finite F* | *finite G* ==> *finite* (*F Int G*)
— The converse obviously fails.
⟨*proof*⟩

lemma *finite-Collect-conjI* [*simp*, *intro*]:
finite{*x*. *P x*} | *finite*{*x*. *Q x*} ==> *finite*{*x*. *P x* & *Q x*}
— The converse obviously fails.
⟨*proof*⟩

lemma *finite-Collect-le-nat*[*iff*]: *finite*{*n*::*nat*. *n*<=*k*}
⟨*proof*⟩

lemma *finite-insert* [*simp*]: *finite* (*insert a A*) = *finite A*
⟨*proof*⟩

lemma *finite-Union*[*simp*, *intro*]:
⟦ *finite A*; !!*M*. *M* ∈ *A* ⟹ *finite M* ⟧ ⟹ *finite*(⋃ *A*)
⟨*proof*⟩

lemma *finite-empty-induct*:
assumes *finite A*
and *P A*
and !!*a A*. *finite A* ==> *a*:*A* ==> *P A* ==> *P* (*A* − {*a*})
shows *P* {}
⟨*proof*⟩

lemma *finite-Diff* [*simp*]: *finite A* ==> *finite* (*A* − *B*)
⟨*proof*⟩

lemma *finite-Diff2* [*simp*]:
assumes *finite B* **shows** *finite* (*A* − *B*) = *finite A*

⟨*proof*⟩

lemma *finite-compl*[*simp*]:
finite(*A*::'*a set*) ⟹ *finite*(−*A*) = *finite*(*UNIV*::'*a set*)
⟨*proof*⟩

lemma *finite-Collect-not*[*simp*]:
finite{*x*::'*a*. *P x*} ⟹ *finite*{*x*. ~*P x*} = *finite*(*UNIV*::'*a set*)
⟨*proof*⟩

lemma *finite-Diff-insert* [*iff*]: *finite* (*A* − *insert a B*) = *finite* (*A* − *B*)
⟨*proof*⟩

Image and Inverse Image over Finite Sets

lemma *finite-imageI*[*simp*]: *finite F* ==> *finite* (*h* ' *F*)
— The image of a finite set is finite.
⟨*proof*⟩

lemma *finite-surj*: *finite A* ==> *B* <= *f* ' *A* ==> *finite B*
⟨*proof*⟩

lemma *finite-range-imageI*:
finite (*range g*) ==> *finite* (*range* (%*x*. *f* (*g x*)))
⟨*proof*⟩

lemma *finite-imageD*: *finite* (*f*'*A*) ==> *inj-on f A* ==> *finite A*
⟨*proof*⟩

lemma *inj-vimage-singleton*: *inj f* ==> *f*−'{*a*} ⊆ {*THE x*. *f x* = *a*}
— The inverse image of a singleton under an injective function is included in a singleton.
⟨*proof*⟩

lemma *finite-vimageI*: [|*finite F*; *inj h*|] ==> *finite* (*h* −' *F*)
— The inverse image of a finite set under an injective function is finite.
⟨*proof*⟩

The finite UNION of finite sets

lemma *finite-UN-I*: *finite A* ==> (!!*a*. *a*:*A* ==> *finite* (*B a*)) ==> *finite* (*UN a*:*A*. *B a*)
⟨*proof*⟩

Strengthen RHS to (∀ *x*∈*A*. *finite* (*B x*)) ∧ *finite* {*x* ∈ *A*. *B x* ≠ {}}?
We'd need to prove *finite C* ⟹ ∀ *A B*. *UNION A B* ⊆ *C* ⟶ *finite* {*x* ∈ *A*. *B x* ≠ {}} by induction.

lemma *finite-UN* [*simp*]:
finite A ==> *finite* (*UNION A B*) = (*ALL x*:*A*. *finite* (*B x*))

⟨*proof*⟩

lemma *finite-Collect-bex*[*simp*]: *finite A* ⟹
finite{*x*. *EX y*:*A*. *Q x y*} = (*ALL y*:*A*. *finite*{*x*. *Q x y*})
⟨*proof*⟩

lemma *finite-Collect-bounded-ex*[*simp*]: *finite*{*y*. *P y*} ⟹
finite{*x*. *EX y*. *P y* & *Q x y*} = (*ALL y*. *P y* ⟶ *finite*{*x*. *Q x y*})
⟨*proof*⟩

lemma *finite-Plus*: [| *finite A*; *finite B* |] ==> *finite* (*A* <+> *B*)
⟨*proof*⟩

Sigma of finite sets

lemma *finite-SigmaI* [*simp*]:
finite A ==> (!!*a*. *a*:*A* ==> *finite* (*B a*)) ==> *finite* (*SIGMA a*:*A*. *B a*)
⟨*proof*⟩

lemma *finite-cartesian-product*: [| *finite A*; *finite B* |] ==>
finite (*A* <*> *B*)
⟨*proof*⟩

lemma *finite-Prod-UNIV*:
finite (*UNIV*::'*a set*) ==> *finite* (*UNIV*::'*b set*) ==> *finite* (*UNIV*::('*a* * '*b*) *set*)
⟨*proof*⟩

lemma *finite-cartesian-productD1*:
[| *finite* (*A* <*> *B*); *B* ≠ {} |] ==> *finite A*
⟨*proof*⟩

lemma *finite-cartesian-productD2*:
[| *finite* (*A* <*> *B*); *A* ≠ {} |] ==> *finite B*
⟨*proof*⟩

The powerset of a finite set

lemma *finite-Pow-iff* [*iff*]: *finite* (*Pow A*) = *finite A*
⟨*proof*⟩

lemma *finite-Collect-subsets*[*simp*,*intro*]: *finite A* ⟹ *finite*{*B*. *B* ⊆ *A*}
⟨*proof*⟩

lemma *finite-UnionD*: *finite*(⋃ *A*) ⟹ *finite A*
⟨*proof*⟩

17.2 Class *finite*

⟨*ML*⟩
class *finite* =
  assumes *finite-UNIV*: *finite* (*UNIV* :: *'a set*)
⟨*ML*⟩
hide *const finite*

context *finite*
begin

lemma *finite* [*simp*]: *finite* (*A* :: *'a set*)
  ⟨*proof*⟩

end

lemma *UNIV-unit* [*noatp*]:
  UNIV = {()} ⟨*proof*⟩

instance *unit* :: *finite*
  ⟨*proof*⟩

lemma *UNIV-bool* [*noatp*]:
  UNIV = {*False*, *True*} ⟨*proof*⟩

instance *bool* :: *finite*
  ⟨*proof*⟩

instance ∗ :: (*finite*, *finite*) *finite*
  ⟨*proof*⟩

lemma *inj-graph*: *inj* (%*f*. {(*x*, *y*). *y* = *f x*})
  ⟨*proof*⟩

instance *fun* :: (*finite*, *finite*) *finite*
⟨*proof*⟩

instance + :: (*finite*, *finite*) *finite*
  ⟨*proof*⟩

17.3 A fold functional for finite sets

The intended behaviour is *fold f z* {x_1, ..., x_n} = *f* x_1 (... (*f* x_n *z*)...) if *f* is "left-commutative":

locale *fun-left-comm* =
  fixes *f* :: *'a* ⇒ *'b* ⇒ *'b*
  assumes *fun-left-comm*: *f x* (*f y z*) = *f y* (*f x z*)
begin

On a functional level it looks much nicer:

lemma *fun-comp-comm*: $f\ x \circ f\ y = f\ y \circ f\ x$
⟨*proof*⟩

end

inductive *fold-graph* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b \Rightarrow bool$
for $f :: 'a \Rightarrow 'b \Rightarrow 'b$ **and** $z :: 'b$ **where**
emptyI [*intro*]: *fold-graph f z* {} *z* |
insertI [*intro*]: $x \notin A \Longrightarrow$ *fold-graph f z A y*
$\Longrightarrow$ *fold-graph f z* (*insert x A*) (*f x y*)

inductive-cases *empty-fold-graphE* [*elim*!]: *fold-graph f z* {} *x*

definition *fold* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b$ **where**
[*code del*]: *fold f z A* = (*THE y*. *fold-graph f z A y*)

A tempting alternative for the definiens is *if finite A then THE y*. *fold-graph f z A y else e*. It allows the removal of finiteness assumptions from the theorems *fold-comm*, *fold-reindex* and *fold-distrib*. The proofs become ugly. It is not worth the effort. (???)

lemma *Diff1-fold-graph*:
fold-graph f z (*A* − {*x*}) *y* $\Longrightarrow$ $x \in A \Longrightarrow$ *fold-graph f z A* (*f x y*)
⟨*proof*⟩

lemma *fold-graph-imp-finite*: *fold-graph f z A x* $\Longrightarrow$ *finite A*
⟨*proof*⟩

lemma *finite-imp-fold-graph*: *finite A* $\Longrightarrow \exists x$. *fold-graph f z A x*
⟨*proof*⟩

17.3.1 From *fold-graph* to *fold*

lemma *image-less-Suc*: *h* ‘ {*i*. *i* < *Suc m*} = *insert* (*h m*) (*h* ‘ {*i*. *i* < *m*})
⟨*proof*⟩

lemma *insert-image-inj-on-eq*:
[|*insert* (*h m*) *A* = *h* ‘ {*i*. *i* < *Suc m*}; *h m* ∉ *A*;
inj-on h {*i*. *i* < *Suc m*}|]
==> *A* = *h* ‘ {*i*. *i* < *m*}
⟨*proof*⟩

lemma *insert-inj-onE*:
assumes *aA*: *insert a A* = *h*‘{*i*::*nat*. *i*<*n*} **and** *anot*: *a* ∉ *A*
and *inj-on*: *inj-on h* {*i*::*nat*. *i*<*n*}
shows ∃*hm m*. *inj-on hm* {*i*::*nat*. *i*<*m*} & *A* = *hm* ‘ {*i*. *i*<*m*} & *m* < *n*
⟨*proof*⟩

context *fun-left-comm*

begin

lemma *fold-graph-determ-aux*:
$A = h`\{i::nat.\ i<n\} \Longrightarrow$ *inj-on* $h\ \{i.\ i<n\}$
$\Longrightarrow$ *fold-graph* $f\ z\ A\ x \Longrightarrow$ *fold-graph* $f\ z\ A\ x' \Longrightarrow x' = x$
⟨*proof*⟩

lemma *fold-graph-determ*:
fold-graph $f\ z\ A\ x \Longrightarrow$ *fold-graph* $f\ z\ A\ y \Longrightarrow y = x$
⟨*proof*⟩

lemma *fold-equality*:
fold-graph $f\ z\ A\ y \Longrightarrow$ *fold* $f\ z\ A = y$
⟨*proof*⟩

The base case for *fold*:

lemma (**in** −) *fold-empty* [*simp*]: *fold* $f\ z\ \{\} = z$
⟨*proof*⟩

The various recursion equations for *fold*:

lemma *fold-insert-aux*: $x \notin A$
$\Longrightarrow$ *fold-graph* $f\ z$ (*insert* $x\ A$) $v \longleftrightarrow$
$(\exists y.$ *fold-graph* $f\ z\ A\ y \wedge v = f\ x\ y)$
⟨*proof*⟩

lemma *fold-insert* [*simp*]:
finite A ==> $x \notin A$ ==> *fold* $f\ z$ (*insert* $x\ A$) $= f\ x$ (*fold* $f\ z\ A$)
⟨*proof*⟩

lemma *fold-fun-comm*:
finite $A \Longrightarrow f\ x$ (*fold* $f\ z\ A$) $=$ *fold* f $(f\ x\ z)\ A$
⟨*proof*⟩

lemma *fold-insert2*:
finite $A \Longrightarrow x \notin A \Longrightarrow$ *fold* $f\ z$ (*insert* $x\ A$) $=$ *fold* f $(f\ x\ z)\ A$
⟨*proof*⟩

lemma *fold-rec*:
assumes *finite* A **and** $x \in A$
shows *fold* $f\ z\ A = f\ x$ (*fold* $f\ z$ $(A - \{x\})$)
⟨*proof*⟩

lemma *fold-insert-remove*:
assumes *finite* A
shows *fold* $f\ z$ (*insert* $x\ A$) $= f\ x$ (*fold* $f\ z$ $(A - \{x\})$)
⟨*proof*⟩

end

A simplified version for idempotent functions:

locale *fun-left-comm-idem* = *fun-left-comm* +
  assumes *fun-left-idem*: *f x* (*f x z*) = *f x z*
begin

The nice version:

lemma *fun-comp-idem* : *f x o f x* = *f x*
⟨*proof*⟩

lemma *fold-insert-idem*:
  assumes *fin*: *finite A*
  shows *fold f z* (*insert x A*) = *f x* (*fold f z A*)
⟨*proof*⟩

declare *fold-insert*[*simp del*] *fold-insert-idem*[*simp*]

lemma *fold-insert-idem2*:
  finite A ⟹ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
⟨*proof*⟩

end

17.3.2 The derived combinator *fold-image*

definition *fold-image* :: ('*b* ⇒ '*b* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*) ⇒ '*b* ⇒ '*a set* ⇒ '*b*
where *fold-image f g* = *fold* (%*x y*. *f* (*g x*) *y*)

lemma *fold-image-empty*[*simp*]: *fold-image f g z* {} = *z*
⟨*proof*⟩

context *ab-semigroup-mult*
begin

lemma *fold-image-insert*[*simp*]:
assumes *finite A* **and** *a* ∉ *A*
shows *fold-image times g z* (*insert a A*) = *g a* ∗ (*fold-image times g z A*)
⟨*proof*⟩

lemma *fold-image-reindex*:
assumes *fin*: *finite A*
shows *inj-on h A* ⟹ *fold-image times g z* (*h*‘*A*) = *fold-image times* (*g*∘*h*) *z A*
⟨*proof*⟩

lemma *fold-image-cong*:
  finite A ⟹

(!!x. x:A ==> $g\ x = h\ x$) ==> *fold-image times* $g\ z\ A$ = *fold-image times* $h\ z\ A$
⟨*proof*⟩

end

context *comm-monoid-mult*
begin

lemma *fold-image-Un-Int*:
finite A ==> *finite* B ==>
fold-image times $g\ 1\ A * $ *fold-image times* $g\ 1\ B$ =
fold-image times $g\ 1$ (A *Un* B) $*$ *fold-image times* $g\ 1$ (A *Int* B)
⟨*proof*⟩

corollary *fold-Un-disjoint*:
finite A ==> *finite* B ==> A *Int* $B = \{\}$ ==>
fold-image times $g\ 1$ (A *Un* B) =
fold-image times $g\ 1\ A * $ *fold-image times* $g\ 1\ B$
⟨*proof*⟩

lemma *fold-image-UN-disjoint*:
⟦ *finite* I; *ALL* i:I. *finite* ($A\ i$);
ALL i:I. *ALL* j:I. $i \neq j$ --> $A\ i$ *Int* $A\ j = \{\}$ ⟧
⟹ *fold-image times* $g\ 1$ (*UNION* $I\ A$) =
fold-image times (%i. *fold-image times* $g\ 1$ ($A\ i$)) $1\ I$
⟨*proof*⟩

lemma *fold-image-Sigma*: *finite* A ==> *ALL* x:A. *finite* ($B\ x$) ==>
fold-image times (%x. *fold-image times* ($g\ x$) 1 ($B\ x$)) $1\ A$ =
fold-image times (*split* g) 1 (*SIGMA* x:A. $B\ x$)
⟨*proof*⟩

lemma *fold-image-distrib*: *finite* A ⟹
fold-image times (%x. $g\ x * h\ x$) $1\ A$ =
fold-image times $g\ 1\ A *$ *fold-image times* $h\ 1\ A$
⟨*proof*⟩

lemma *fold-image-related*:
assumes *Re*: $R\ e\ e$
and *Rop*: ∀ *x1 y1 x2 y2*. R *x1 x2* ∧ R *y1 y2* ⟶ R (*x1* $*$ *y1*) (*x2* $*$ *y2*)
and *fS*: *finite* S **and** *Rfg*: ∀ x∈S. R ($h\ x$) ($g\ x$)
shows R (*fold-image* ($op\ *$) $h\ e\ S$) (*fold-image* ($op\ *$) $g\ e\ S$)
⟨*proof*⟩

lemma *fold-image-eq-general*:
assumes *fS*: *finite* S
and h: ∀ y∈S'. ∃!x. x∈ S ∧ $h(x) = y$
and *f12*: ∀ x∈S. $h\ x$ ∈ S' ∧ *f2*($h\ x$) = *f1* x
shows *fold-image* ($op\ *$) *f1* $e\ S$ = *fold-image* ($op\ *$) *f2* $e\ S'$

⟨*proof*⟩

lemma *fold-image-eq-general-inverses*:
assumes *fS*: *finite S*
and *kh*: $\bigwedge y.\ y \in T \Longrightarrow k\ y \in S \wedge h\ (k\ y) = y$
and *hk*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in T \wedge k\ (h\ x) = x\ \wedge\ g\ (h\ x) = f\ x$
shows *fold-image* (*op* ∗) *f e S* = *fold-image* (*op* ∗) *g e T*

⟨*proof*⟩

end

17.4 Generalized summation over a set

interpretation *comm-monoid-add*: *comm-monoid-mult 0*::′*a*::*comm-monoid-add op* +
⟨*proof*⟩

definition *setsum* :: (′*a* => ′*b*) => ′*a set* => ′*b*::*comm-monoid-add*
where *setsum f A* == *if finite A then fold-image* (*op* +) *f 0 A else 0*

abbreviation
Setsum (∑ - [*1000*] *999*) **where**
∑ *A* == *setsum* (%*x*. *x*) *A*

Now: lot's of fancy syntax. First, *setsum* (λ*x*. *e*) *A* is written ∑ *x*∈*A*. *e*.

syntax
-setsum :: *pttrn* => ′*a set* => ′*b* => ′*b*::*comm-monoid-add* ((*3SUM* -:-. -) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-setsum :: *pttrn* => ′*a set* => ′*b* => ′*b*::*comm-monoid-add* ((*3*∑ -∈-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-setsum :: *pttrn* => ′*a set* => ′*b* => ′*b*::*comm-monoid-add* ((*3*∑ -∈-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
SUM i:*A*. *b* == *CONST setsum* (%*i*. *b*) *A*
∑ *i*∈*A*. *b* == *CONST setsum* (%*i*. *b*) *A*

Instead of ∑ *x*∈{*x*. *P*}. *e* we introduce the shorter ∑ *x*|*P*. *e*.

syntax
-qsetsum :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a* ((*3SUM* - |/ -./ -) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-qsetsum :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a* ((*3*∑ - | (-)./ -) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-qsetsum :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a* ((*3*∑ - | (-)./ -) [*0*,*0*,*10*] *10*)

translations

SUM x|P. t => CONST setsum (%x. t) {x. P}
$\sum x|P.\ t => CONST\ setsum\ (\%x.\ t)\ \{x.\ P\}$

⟨*ML*⟩

lemma *setsum-empty* [*simp*]: *setsum f {} = 0*
⟨*proof*⟩

lemma *setsum-insert* [*simp*]:
finite F ==> a ∉ F ==> setsum f (insert a F) = f a + setsum f F
⟨*proof*⟩

lemma *setsum-infinite* [*simp*]: ~ *finite A ==> setsum f A = 0*
⟨*proof*⟩

lemma *setsum-reindex*:
inj-on f B ==> setsum h (f ' B) = setsum (h ∘ f) B
⟨*proof*⟩

lemma *setsum-reindex-id*:
inj-on f B ==> setsum f B = setsum id (f ' B)
⟨*proof*⟩

lemma *setsum-reindex-nonzero*:
assumes *fS*: *finite S*
and *nz*: $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow x \neq y \Longrightarrow f\ x = f\ y \Longrightarrow h\ (f\ x) = 0$
shows *setsum h (f ' S) = setsum (h o f) S*
⟨*proof*⟩

lemma *setsum-cong*:
A = B ==> (!!x. x:B ==> f x = g x) ==> setsum f A = setsum g B
⟨*proof*⟩

lemma *strong-setsum-cong*[*cong*]:
A = B ==> (!!x. x:B =simp=> f x = g x)
==> setsum (%x. f x) A = setsum (%x. g x) B
⟨*proof*⟩

lemma *setsum-cong2*: $[\![\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x]\!] \Longrightarrow setsum\ f\ A = setsum\ g\ A$
⟨*proof*⟩

lemma *setsum-reindex-cong*:
[|inj-on f A; B = f ' A; !!a. a:A ==> g a = h (f a)|]
==> setsum h B = setsum g A
⟨*proof*⟩

lemma *setsum-0*[*simp*]: *setsum (%i. 0) A = 0*

⟨*proof*⟩

lemma *setsum-0'*: *ALL a:A. f a = 0 ==> setsum f A = 0*
⟨*proof*⟩

lemma *setsum-Un-Int*: *finite A ==> finite B ==>*
setsum g (A Un B) + setsum g (A Int B) = setsum g A + setsum g B
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

lemma *setsum-Un-disjoint*: *finite A ==> finite B*
==> A Int B = {} ==> setsum g (A Un B) = setsum g A + setsum g B
⟨*proof*⟩

lemma *setsum-mono-zero-left*:
assumes *fT*: *finite T* **and** *ST*: $S \subseteq T$
and *z*: $\forall i \in T - S.\ f\ i = 0$
shows *setsum f S = setsum f T*
⟨*proof*⟩

lemma *setsum-mono-zero-right*:
finite T $\Longrightarrow$ $S \subseteq T \Longrightarrow \forall i \in T - S.\ f\ i = 0 \Longrightarrow$ *setsum f T = setsum f S*
⟨*proof*⟩

lemma *setsum-mono-zero-cong-left*:
assumes *fT*: *finite T* **and** *ST*: $S \subseteq T$
and *z*: $\forall i \in T - S.\ g\ i = 0$
and *fg*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows *setsum f S = setsum g T*
⟨*proof*⟩

lemma *setsum-mono-zero-cong-right*:
assumes *fT*: *finite T* **and** *ST*: $S \subseteq T$
and *z*: $\forall i \in T - S.\ f\ i = 0$
and *fg*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows *setsum f T = setsum g S*
⟨*proof*⟩

lemma *setsum-delta*:
assumes *fS*: *finite S*
shows *setsum* $(\lambda k.$ *if k=a then b k else 0*$)$ *S* = $($*if a* $\in$ *S then b a else 0*$)$
⟨*proof*⟩
lemma *setsum-delta'*:
assumes *fS*: *finite S* **shows**
setsum $(\lambda k.$ *if a = k then b k else 0*$)$ *S* =
$($*if a*$\in$ *S then b a else 0*$)$
⟨*proof*⟩

lemma *setsum-restrict-set*:

assumes *fA*: *finite A*
shows *setsum f* $(A \cap B)$ = *setsum* $(\lambda x.$ *if* $x \in B$ *then f x else 0*$)$ *A*
⟨*proof*⟩

lemma *setsum-cases*:
assumes *fA*: *finite A*
shows *setsum* $(\lambda x.$ *if* $x \in B$ *then f x else g x*$)$ *A* =
setsum f $(A \cap B)$ + *setsum g* $(A \cap -B)$
⟨*proof*⟩

lemma *setsum-UN-disjoint*:
finite I ==> (*ALL i*:*I*. *finite* (*A i*)) ==>
(*ALL i*:*I*. *ALL j*:*I*. *i* ≠ *j* --> *A i Int A j* = {}) ==>
setsum f (*UNION I A*) = ($\sum$ *i*∈*I*. *setsum f* (*A i*))
⟨*proof*⟩

No need to assume that C is finite. If infinite, the rhs is directly 0, and $\bigcup C$ is also infinite, hence the lhs is also 0.

lemma *setsum-Union-disjoint*:
[| (*ALL A*:*C*. *finite A*);
(*ALL A*:*C*. *ALL B*:*C*. *A* ≠ *B* --> *A Int B* = {}) |]
==> *setsum f* (*Union C*) = *setsum* (*setsum f*) *C*
⟨*proof*⟩

lemma *setsum-Sigma*: *finite A* ==> *ALL x*:*A*. *finite* (*B x*) ==>
($\sum$ *x*∈*A*. ($\sum$ *y*∈*B x*. *f x y*)) = ($\sum$ (*x*,*y*)∈(*SIGMA x*:*A*. *B x*). *f x y*)
⟨*proof*⟩

Here we can eliminate the finiteness assumptions, by cases.

lemma *setsum-cartesian-product*:
($\sum$ *x*∈*A*. ($\sum$ *y*∈*B*. *f x y*)) = ($\sum$ (*x*,*y*) ∈ *A* <*> *B*. *f x y*)
⟨*proof*⟩

lemma *setsum-addf*: *setsum* (%*x*. *f x* + *g x*) *A* = (*setsum f A* + *setsum g A*)
⟨*proof*⟩

17.4.1 Properties in more restricted classes of structures

lemma *setsum-SucD*: *setsum f A* = *Suc n* ==> *EX a*:*A*. *0* < *f a*
⟨*proof*⟩

lemma *setsum-eq-0-iff* [*simp*]:
finite F ==> (*setsum f F* = *0*) = (*ALL a*:*F*. *f a* = (*0*::*nat*))
⟨*proof*⟩

lemma *setsum-eq-Suc0-iff*: *finite A* ⟹

(*setsum f A = Suc 0*) = (*EX a:A. f a = Suc 0 & (ALL b:A. a≠b ⟶ f b = 0*))
⟨*proof*⟩

lemmas *setsum-eq-1-iff = setsum-eq-Suc0-iff*[*simplified One-nat-def*[*symmetric*]]

lemma *setsum-Un-nat*: *finite A ==> finite B ==>*
(*setsum f* (*A Un B*) :: *nat*) = *setsum f A + setsum f B − setsum f* (*A Int B*)
— For the natural numbers, we have subtraction.
⟨*proof*⟩

lemma *setsum-Un*: *finite A ==> finite B ==>*
(*setsum f* (*A Un B*) :: *'a* :: *ab-group-add*) =
setsum f A + setsum f B − setsum f (*A Int B*)
⟨*proof*⟩

lemma (**in** *comm-monoid-mult*) *fold-image-1*: *finite S* ⟹ (∀ *x*∈*S*. *f x = 1*) ⟹ *fold-image op ∗ f 1 S = 1*
⟨*proof*⟩

lemma (**in** *comm-monoid-mult*) *fold-image-Un-one*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *I0*: ∀ *x* ∈ *S*∩*T*. *f x = 1*
shows *fold-image* (*op ∗*) *f 1* (*S* ∪ *T*) = *fold-image* (*op ∗*) *f 1 S ∗ fold-image* (*op ∗*) *f 1 T*
⟨*proof*⟩

lemma *setsum-eq-general-reverses*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *kh*: ⋀*y*. *y* ∈ *T* ⟹ *k y* ∈ *S* ∧ *h* (*k y*) = *y*
and *hk*: ⋀*x*. *x* ∈ *S* ⟹ *h x* ∈ *T* ∧ *k* (*h x*) = *x* ∧ *g* (*h x*) = *f x*
shows *setsum f S = setsum g T*
⟨*proof*⟩

lemma *setsum-Un-zero*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *I0*: ∀ *x* ∈ *S*∩*T*. *f x = 0*
shows *setsum f* (*S* ∪ *T*) = *setsum f S* + *setsum f T*
⟨*proof*⟩

lemma *setsum-UNION-zero*:
assumes *fS*: *finite S* **and** *fSS*: ∀ *T* ∈ *S*. *finite T*
and *f0*: ⋀*T1 T2 x*. *T1*∈*S* ⟹ *T2*∈*S* ⟹ *T1* ≠ *T2* ⟹ *x* ∈ *T1* ⟹ *x* ∈ *T2* ⟹ *f x = 0*
shows *setsum f* (⋃ *S*) = *setsum* (λ*T*. *setsum f T*) *S*
⟨*proof*⟩

lemma *setsum-diff1-nat*: $(setsum\ f\ (A - \{a\}) :: nat) =$
$(if\ a{:}A\ then\ setsum\ f\ A - f\ a\ else\ setsum\ f\ A)$
⟨*proof*⟩

lemma *setsum-diff1*: *finite* $A \Longrightarrow$
$(setsum\ f\ (A - \{a\}) :: ('a{::}ab\text{-}group\text{-}add)) =$
$(if\ a{:}A\ then\ setsum\ f\ A - f\ a\ else\ setsum\ f\ A)$
⟨*proof*⟩

lemma *setsum-diff1'*[*rule-format*]:
finite $A \Longrightarrow a \in A \longrightarrow (\sum\ x \in A.\ f\ x) = f\ a + (\sum\ x \in (A - \{a\}).\ f\ x)$
⟨*proof*⟩

lemma *setsum-diff-nat*:
assumes *finite* B **and** $B \subseteq A$
shows $(setsum\ f\ (A - B) :: nat) = (setsum\ f\ A) - (setsum\ f\ B)$
⟨*proof*⟩

lemma *setsum-diff*:
assumes *le*: *finite* A $B \subseteq A$
shows $setsum\ f\ (A - B) = setsum\ f\ A - ((setsum\ f\ B){::}('a{::}ab\text{-}group\text{-}add))$
⟨*proof*⟩

lemma *setsum-mono*:
assumes *le*: $\bigwedge i.\ i{\in}K \Longrightarrow f\ (i{::}'a) \leq ((g\ i){::}('b{::}\{comm\text{-}monoid\text{-}add, pordered\text{-}ab\text{-}semigroup\text{-}add\}))$
shows $(\sum i{\in}K.\ f\ i) \leq (\sum i{\in}K.\ g\ i)$
⟨*proof*⟩

lemma *setsum-strict-mono*:
fixes $f :: 'a \Rightarrow 'b{::}\{pordered\text{-}cancel\text{-}ab\text{-}semigroup\text{-}add, comm\text{-}monoid\text{-}add\}$
assumes *finite* A $A \neq \{\}$
and $!!x.\ x{:}A \Longrightarrow f\ x < g\ x$
shows $setsum\ f\ A < setsum\ g\ A$
⟨*proof*⟩

lemma *setsum-negf*:
$setsum\ (\%x.\ -\ (f\ x){::}'a{::}ab\text{-}group\text{-}add)\ A = -\ setsum\ f\ A$
⟨*proof*⟩

lemma *setsum-subtractf*:
$setsum\ (\%x.\ ((f\ x){::}'a{::}ab\text{-}group\text{-}add)\ -\ g\ x)\ A =$
$setsum\ f\ A\ -\ setsum\ g\ A$
⟨*proof*⟩

lemma *setsum-nonneg*:
assumes *nn*: $\forall\, x{\in}A.\ (0{::}'a{::}\{pordered\text{-}ab\text{-}semigroup\text{-}add, comm\text{-}monoid\text{-}add\}) \leq$

f x
shows *0 ≤ setsum f A*
⟨*proof*⟩

lemma *setsum-nonpos*:
assumes *np*: ∀ *x*∈*A*. *f x ≤* (*0*::*'a*::{*pordered-ab-semigroup-add*,*comm-monoid-add*})
shows *setsum f A ≤ 0*
⟨*proof*⟩

lemma *setsum-mono2*:
fixes *f* :: *'a ⇒ 'b* :: {*pordered-ab-semigroup-add-imp-le*,*comm-monoid-add*}
assumes *fin*: *finite B* **and** *sub*: *A ⊆ B* **and** *nn*: ⋀*b*. *b* ∈ *B*−*A ⟹ 0 ≤ f b*
shows *setsum f A ≤ setsum f B*
⟨*proof*⟩

lemma *setsum-mono3*: *finite B ==> A <= B ==>*
ALL x: *B − A*.
0 <= ((f x)::*'a*::{*comm-monoid-add*,*pordered-ab-semigroup-add*}) ==>
setsum f A <= setsum f B
⟨*proof*⟩

lemma *setsum-right-distrib*:
fixes *f* :: *'a => ('b*::*semiring-0)*
shows *r * setsum f A = setsum (%n. r * f n) A*
⟨*proof*⟩

lemma *setsum-left-distrib*:
*setsum f A * (r*::*'a*::*semiring-0) = (*∑ *n*∈*A*. *f n * r)*
⟨*proof*⟩

lemma *setsum-divide-distrib*:
setsum f A / (r::*'a*::*field) = (*∑ *n*∈*A*. *f n / r)*
⟨*proof*⟩

lemma *setsum-abs*[*iff*]:
fixes *f* :: *'a => ('b*::*pordered-ab-group-add-abs)*
shows *abs (setsum f A) ≤ setsum (%i. abs(f i)) A*
⟨*proof*⟩

lemma *setsum-abs-ge-zero*[*iff*]:
fixes *f* :: *'a => ('b*::*pordered-ab-group-add-abs)*
shows *0 ≤ setsum (%i. abs(f i)) A*
⟨*proof*⟩

lemma *abs-setsum-abs*[*simp*]:
fixes *f* :: *'a => ('b*::*pordered-ab-group-add-abs)*
shows *abs (*∑ *a*∈*A*. *abs(f a)) = (*∑ *a*∈*A*. *abs(f a))*
⟨*proof*⟩

Commuting outer and inner summation

lemma *swap-inj-on*:
inj-on (%(*i*, *j*). (*j*, *i*)) (*A* × *B*)
⟨*proof*⟩

lemma *swap-product*:
(%(*i*, *j*). (*j*, *i*)) ' (*A* × *B*) = *B* × *A*
⟨*proof*⟩

lemma *setsum-commute*:
(∑ *i*∈*A*. ∑ *j*∈*B*. *f i j*) = (∑ *j*∈*B*. ∑ *i*∈*A*. *f i j*)
⟨*proof*⟩

lemma *setsum-product*:
fixes *f* :: *'a* => (*'b*::*semiring-0*)
shows *setsum f A* * *setsum g B* = (∑ *i*∈*A*. ∑ *j*∈*B*. *f i* * *g j*)
⟨*proof*⟩

17.5 Generalized product over a set

definition *setprod* :: (*'a* => *'b*) => *'a set* => *'b*::*comm-monoid-mult*
where *setprod f A* == *if finite A then fold-image* (*op* *) *f 1 A else 1*

abbreviation
Setprod (∏ - [*1000*] *999*) **where**
∏ *A* == *setprod* (%*x*. *x*) *A*

syntax
-setprod :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*3PROD* -:-. -) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-setprod :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*3*∏ -∈-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-setprod :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*3*∏ -∈-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
PROD i:*A*. *b* == *CONST setprod* (%*i*. *b*) *A*
∏ *i*∈*A*. *b* == *CONST setprod* (%*i*. *b*) *A*

Instead of ∏ *x*∈{*x*. *P*}. *e* we introduce the shorter ∏ *x*|*P*. *e*.

syntax
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3PROD* - |/ -./ -) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*∏ - | (-)./ -) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*∏ - | (-)./ -) [*0*,*0*,*10*] *10*)

translations

PROD x|P. t => CONST setprod (%x. t) {x. P}
$\prod x|P.\ t$ => *CONST setprod (%x. t) {x. P}*

lemma *setprod-empty* [*simp*]: *setprod f {} = 1*
⟨*proof*⟩

lemma *setprod-insert* [*simp*]: [| *finite A*; *a* ∉ *A* |] ==>
*setprod f (insert a A) = f a * setprod f A*
⟨*proof*⟩

lemma *setprod-infinite* [*simp*]: ~ *finite A* ==> *setprod f A = 1*
⟨*proof*⟩

lemma *setprod-reindex*:
inj-on f B ==> *setprod h (f ' B) = setprod (h ∘ f) B*
⟨*proof*⟩

lemma *setprod-reindex-id*: *inj-on f B* ==> *setprod f B = setprod id (f ' B)*
⟨*proof*⟩

lemma *setprod-cong*:
A = B ==> (!!*x*. *x*:*B* ==> *f x = g x*) ==> *setprod f A = setprod g B*
⟨*proof*⟩

lemma *strong-setprod-cong*[*cong*]:
A = B ==> (!!*x*. *x*:*B* =*simp*=> *f x = g x*) ==> *setprod f A = setprod g B*
⟨*proof*⟩

lemma *setprod-reindex-cong*: *inj-on f A* ==>
B = f ' A ==> *g = h ∘ f* ==> *setprod h B = setprod g A*
⟨*proof*⟩

lemma *strong-setprod-reindex-cong*: **assumes** *i*: *inj-on f A*
and *B*: *B = f ' A* **and** *eq*: ⋀*x*. *x* ∈ *A* ⟹ *g x = (h ∘ f) x*
shows *setprod h B = setprod g A*
⟨*proof*⟩

lemma *setprod-Un-one*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *I0*: ∀ *x* ∈ *S*∩*T*. *f x = 1*
shows *setprod f (S ∪ T) = setprod f S * setprod f T*
⟨*proof*⟩

lemma *setprod-1*: *setprod (%i. 1) A = 1*
⟨*proof*⟩

lemma *setprod-1'*: *ALL a*:*F*. *f a = 1* ==> *setprod f F = 1*

⟨*proof*⟩

lemma *setprod-Un-Int*: *finite A* ==> *finite B*
  ==> *setprod g* (*A Un B*) * *setprod g* (*A Int B*) = *setprod g A* * *setprod g B*
⟨*proof*⟩

lemma *setprod-Un-disjoint*: *finite A* ==> *finite B*
 ==> *A Int B* = {} ==> *setprod g* (*A Un B*) = *setprod g A* * *setprod g B*
⟨*proof*⟩

lemma *setprod-mono-one-left*:
 assumes *fT*: *finite T* **and** *ST*: *S* ⊆ *T*
 and *z*: ∀ *i* ∈ *T* − *S*. *f i* = *1*
 shows *setprod f S* = *setprod f T*
⟨*proof*⟩

lemmas *setprod-mono-one-right* = *setprod-mono-one-left* [*THEN sym*]

lemma *setprod-delta*:
 assumes *fS*: *finite S*
 shows *setprod* (λ*k*. *if k*=*a then b k else 1*) *S* = (*if a* ∈ *S then b a else 1*)
⟨*proof*⟩

lemma *setprod-delta'*:
 assumes *fS*: *finite S* **shows**
 setprod (λ*k*. *if a* = *k then b k else 1*) *S* =
  (*if a*∈ *S then b a else 1*)
 ⟨*proof*⟩

lemma *setprod-UN-disjoint*:
  finite I ==> (*ALL i*:*I*. *finite* (*A i*)) ==>
    (*ALL i*:*I*. *ALL j*:*I*. *i* ≠ *j* --> *A i Int A j* = {}) ==>
   setprod f (*UNION I A*) = *setprod* (%*i*. *setprod f* (*A i*)) *I*
⟨*proof*⟩

lemma *setprod-Union-disjoint*:
 [| (*ALL A*:*C*. *finite A*);
    (*ALL A*:*C*. *ALL B*:*C*. *A* ≠ *B* --> *A Int B* = {}) |]
  ==> *setprod f* (*Union C*) = *setprod* (*setprod f*) *C*
⟨*proof*⟩

lemma *setprod-Sigma*: *finite A* ==> *ALL x*:*A*. *finite* (*B x*) ==>
  (∏ *x*∈*A*. (∏ *y*∈ *B x*. *f x y*)) =
  (∏ (*x*,*y*)∈(*SIGMA x*:*A*. *B x*). *f x y*)
⟨*proof*⟩

Here we can eliminate the finiteness assumptions, by cases.

lemma *setprod-cartesian-product*:

$(\prod x{\in}A.\ (\prod y{\in}\ B.\ f\ x\ y)) = (\prod (x,y){\in}(A <*> B).\ f\ x\ y)$
⟨*proof*⟩

lemma *setprod-timesf*:
$setprod\ (\%x.\ f\ x * g\ x)\ A = (setprod\ f\ A * setprod\ g\ A)$
⟨*proof*⟩

17.5.1 Properties in more restricted classes of structures

lemma *setprod-eq-1-iff* [*simp*]:
$finite\ F ==> (setprod\ f\ F = 1) = (ALL\ a{:}F.\ f\ a = (1{::}nat))$
⟨*proof*⟩

lemma *setprod-zero*:
$finite\ A ==> EX\ x{:}\ A.\ f\ x = (0{::}'a{::}comm\text{-}semiring\text{-}1) ==> setprod\ f\ A = 0$
⟨*proof*⟩

lemma *setprod-nonneg* [*rule-format*]:
$(ALL\ x{:}\ A.\ (0{::}'a{::}ordered\text{-}semidom) \le f\ x) --> 0 \le setprod\ f\ A$
⟨*proof*⟩

lemma *setprod-pos* [*rule-format*]: $(ALL\ x{:}\ A.\ (0{::}'a{::}ordered\text{-}semidom) < f\ x)$
$--> 0 < setprod\ f\ A$
⟨*proof*⟩

lemma *setprod-zero-iff* [*simp*]: $finite\ A ==>$
$(setprod\ f\ A = (0{::}'a{::}\{comm\text{-}semiring\text{-}1,no\text{-}zero\text{-}divisors\})) =$
$(EX\ x{:}\ A.\ f\ x = 0)$
⟨*proof*⟩

lemma *setprod-pos-nat*:
$finite\ S ==> (ALL\ x : S.\ f\ x > (0{::}nat)) ==> setprod\ f\ S > 0$
⟨*proof*⟩

lemma *setprod-pos-nat-iff* [*simp*]:
$finite\ S ==> (setprod\ f\ S > 0) = (ALL\ x : S.\ f\ x > (0{::}nat))$
⟨*proof*⟩

lemma *setprod-Un*: $finite\ A ==> finite\ B ==> (ALL\ x{:}\ A\ Int\ B.\ f\ x \neq 0) ==>$
$(setprod\ f\ (A\ Un\ B) :: 'a ::\{field\})$
$= setprod\ f\ A * setprod\ f\ B\ /\ setprod\ f\ (A\ Int\ B)$
⟨*proof*⟩

lemma *setprod-diff1*: $finite\ A ==> f\ a \neq 0 ==>$
$(setprod\ f\ (A - \{a\}) :: 'a :: \{field\}) =$
$(if\ a{:}A\ then\ setprod\ f\ A\ /\ f\ a\ else\ setprod\ f\ A)$
⟨*proof*⟩

lemma *setprod-inversef*: $finite\ A ==>$

ALL x: A. f x ≠ (0::'a::{field,division-by-zero}) ==>
setprod (inverse ∘ f) A = inverse (setprod f A)
⟨*proof*⟩

lemma *setprod-dividef*:
[|*finite A*;
∀x ∈ A. g x ≠ (0::'a::{field,division-by-zero})|]
==> setprod (%x. f x / g x) A = setprod f A / setprod g A
⟨*proof*⟩

lemma *setprod-dvd-setprod* [*rule-format*]:
(ALL x : A. f x dvd g x) ⟶ setprod f A dvd setprod g A
⟨*proof*⟩

lemma *setprod-dvd-setprod-subset*:
finite B ⟹ A <= B ⟹ setprod f A dvd setprod f B
⟨*proof*⟩

lemma *setprod-dvd-setprod-subset2*:
finite B ⟹ A <= B ⟹ ALL x : A. (f x::'a::comm-semiring-1) dvd g x ⟹
setprod f A dvd setprod g B
⟨*proof*⟩

lemma *dvd-setprod*: *finite A ⟹ i:A ⟹*
(f i ::'a::comm-semiring-1) dvd setprod f A
⟨*proof*⟩

lemma *dvd-setsum* [*rule-format*]: *(ALL i : A. d dvd f i) ⟶*
(d::'a::comm-semiring-1) dvd (SUM x : A. f x)
⟨*proof*⟩

17.6 Finite cardinality

This definition, although traditional, is ugly to work with: *card A == LEAST n. EX f. A = {f i | i. i < n}*. But now that we have *setsum* things are easy:

definition *card* :: *'a set ⇒ nat*
where *card A = setsum (λx. 1) A*

lemma *card-empty* [*simp*]: *card {} = 0*
⟨*proof*⟩

lemma *card-infinite* [*simp*]: *~ finite A ==> card A = 0*
⟨*proof*⟩

lemma *card-eq-setsum*: *card A = setsum (%x. 1) A*
⟨*proof*⟩

lemma *card-insert-disjoint* [*simp*]:

finite A ==> x ∉ A ==> card (insert x A) = Suc(card A)
⟨*proof*⟩

lemma *card-insert-if*:
finite A ==> card (insert x A) = (if x:A then card A else Suc(card(A)))
⟨*proof*⟩

lemma *card-0-eq* [*simp*,*noatp*]: *finite A ==> (card A = 0) = (A = {})*
⟨*proof*⟩

lemma *card-eq-0-iff*: *(card A = 0) = (A = {} | ~ finite A)*
⟨*proof*⟩

lemma *card-Suc-Diff1*: *finite A ==> x: A ==> Suc (card (A − {x})) = card A*
⟨*proof*⟩

lemma *card-Diff-singleton*:
finite A ==> x: A ==> card (A − {x}) = card A − 1
⟨*proof*⟩

lemma *card-Diff-singleton-if*:
finite A ==> card (A−{x}) = (if x : A then card A − 1 else card A)
⟨*proof*⟩

lemma *card-Diff-insert*[*simp*]:
assumes *finite A* **and** *a*:*A* **and** *a* ~: *B*
shows *card(A − insert a B) = card(A − B) − 1*
⟨*proof*⟩

lemma *card-insert*: *finite A ==> card (insert x A) = Suc (card (A − {x}))*
⟨*proof*⟩

lemma *card-insert-le*: *finite A ==> card A <= card (insert x A)*
⟨*proof*⟩

lemma *card-mono*: ⟦ *finite B*; *A* ⊆ *B* ⟧ ⟹ *card A* ≤ *card B*
⟨*proof*⟩

lemma *card-seteq*: *finite B ==> (!!A. A <= B ==> card B <= card A ==> A = B)*
⟨*proof*⟩

lemma *psubset-card-mono*: *finite B ==> A < B ==> card A < card B*
⟨*proof*⟩

lemma *card-Un-Int*: *finite A ==> finite B*
==> card A + card B = card (A Un B) + card (A Int B)
⟨*proof*⟩

lemma *card-Un-disjoint*: *finite A* ==> *finite B*
==> *A Int B* = {} ==> *card* (*A Un B*) = *card A* + *card B*
⟨*proof*⟩

lemma *card-Diff-subset*:
finite B ==> *B* <= *A* ==> *card* (*A* − *B*) = *card A* − *card B*
⟨*proof*⟩

lemma *card-Diff1-less*: *finite A* ==> *x*: *A* ==> *card* (*A* − {*x*}) < *card A*
⟨*proof*⟩

lemma *card-Diff2-less*:
finite A ==> *x*: *A* ==> *y*: *A* ==> *card* (*A* − {*x*} − {*y*}) < *card A*
⟨*proof*⟩

lemma *card-Diff1-le*: *finite A* ==> *card* (*A* − {*x*}) <= *card A*
⟨*proof*⟩

lemma *card-psubset*: *finite B* ==> *A* ⊆ *B* ==> *card A* < *card B* ==> *A* < *B*
⟨*proof*⟩

lemma *insert-partition*:
⟦ *x* ∉ *F*; ∀ *c1* ∈ *insert x F*. ∀ *c2* ∈ *insert x F*. *c1* ≠ *c2* ⟶ *c1* ∩ *c2* = {} ⟧
⟹ *x* ∩ ⋃ *F* = {}
⟨*proof*⟩

main cardinality theorem

lemma *card-partition* [*rule-format*]:
finite C ==>
finite (⋃ *C*) −−>
(∀ *c*∈*C*. *card c* = *k*) −−>
(∀ *c1* ∈ *C*. ∀ *c2* ∈ *C*. *c1* ≠ *c2* −−> *c1* ∩ *c2* = {}) −−>
k ∗ *card*(*C*) = *card* (⋃ *C*)
⟨*proof*⟩

The form of a finite set of given cardinality

lemma *card-eq-SucD*:
assumes *card A* = *Suc k*
shows ∃ *b B*. *A* = *insert b B* & *b* ∉ *B* & *card B* = *k* & (*k*=*0* ⟶ *B*={})
⟨*proof*⟩

lemma *card-Suc-eq*:
(*card A* = *Suc k*) =
(∃ *b B*. *A* = *insert b B* & *b* ∉ *B* & *card B* = *k* & (*k*=*0* ⟶ *B*={}))
⟨*proof*⟩

lemma *setsum-constant* [*simp*]: (∑ *x* ∈ *A*. *y*) = *of-nat*(*card A*) ∗ *y*
⟨*proof*⟩

lemma *setprod-constant*: *finite A ==>* $(\prod x{\in}A.\ (y{::}'a{::}\{recpower, comm\text{-}monoid\text{-}mult\}))$
$= y\hat{}(card\ A)$
⟨*proof*⟩

lemma *setprod-gen-delta*:
assumes *fS*: *finite S*
shows *setprod* $(\lambda k.\ if\ k{=}a\ then\ b\ k\ else\ c)\ S = (if\ a \in S\ then\ (b\ a\ {::}'a{::}\{comm\text{-}monoid\text{-}mult, recpower\})\ *\ c\hat{}\ (card\ S - 1)\ else\ c\hat{}\ card\ S)$
⟨*proof*⟩

lemma *setsum-bounded*:
assumes *le*: $\bigwedge i.\ i{\in}A \Longrightarrow f\ i \leq (K{::}'a{::}\{semiring\text{-}1, pordered\text{-}ab\text{-}semigroup\text{-}add\})$
shows *setsum f A* $\leq$ *of-nat*$(card\ A) * K$
⟨*proof*⟩

17.6.1 Cardinality of unions

lemma *card-UN-disjoint*:
finite I ==> $(ALL\ i{:}I.\ finite\ (A\ i))$ *==>*
$(ALL\ i{:}I.\ ALL\ j{:}I.\ i \neq j\ -{-}{>}\ A\ i\ Int\ A\ j = \{\})$
==> card $(UNION\ I\ A) = (\sum i{\in}I.\ card(A\ i))$
⟨*proof*⟩

lemma *card-Union-disjoint*:
finite C ==> $(ALL\ A{:}C.\ finite\ A)$ *==>*
$(ALL\ A{:}C.\ ALL\ B{:}C.\ A \neq B\ -{-}{>}\ A\ Int\ B = \{\})$
==> card $(Union\ C) = setsum\ card\ C$
⟨*proof*⟩

17.6.2 Cardinality of image

The image of a finite set can be expressed using *fold-image*.

lemma *image-eq-fold-image*:
finite A ==> $f\ `\ A = fold\text{-}image\ (op\ Un)\ (\%x.\ \{f\ x\})\ \{\}\ A$
⟨*proof*⟩

lemma *card-image-le*: *finite A ==> card* $(f\ `\ A) <= card\ A$
⟨*proof*⟩

lemma *card-image*: *inj-on f A ==> card* $(f\ `\ A) = card\ A$
⟨*proof*⟩

lemma *endo-inj-surj*: *finite A ==>* $f\ `\ A \subseteq A$ *==> inj-on f A ==>* $f\ `\ A = A$
⟨*proof*⟩

lemma *eq-card-imp-inj-on*:
[| *finite A*; $card(f\ `\ A) = card\ A$ |] *==> inj-on f A*

⟨*proof*⟩

lemma *inj-on-iff-eq-card*:
finite A ==> inj-on f A = (card(f ' A) = card A)
⟨*proof*⟩

lemma *card-inj-on-le*:
[|inj-on f A; f ' A ⊆ B; finite B |] ==> card A ≤ card B
⟨*proof*⟩

lemma *card-bij-eq*:
[|inj-on f A; f ' A ⊆ B; inj-on g B; g ' B ⊆ A;
finite A; finite B |] ==> card A = card B
⟨*proof*⟩

17.6.3 Cardinality of products

lemma *card-SigmaI* [*simp*]:
⟦ finite A; ALL a:A. finite (B a) ⟧
⟹ card (SIGMA x: A. B x) = (∑ a∈A. card (B a))
⟨*proof*⟩

lemma *card-cartesian-product*: *card (A <*> B) = card(A) * card(B)*
⟨*proof*⟩

lemma *card-cartesian-product-singleton*: *card({x} <*> A) = card(A)*
⟨*proof*⟩

17.6.4 Cardinality of sums

lemma *card-Plus*:
assumes *finite A* **and** *finite B*
shows *card (A <+> B) = card A + card B*
⟨*proof*⟩

17.6.5 Cardinality of the Powerset

lemma *card-Pow*: *finite A ==> card (Pow A) = Suc (Suc 0) ^ card A*
⟨*proof*⟩

Relates to equivalence classes. Based on a theorem of F. Kammüller.

lemma *dvd-partition*:
finite (Union C) ==>
ALL c : C. k dvd card c ==>
(ALL c1: C. ALL c2: C. c1 ≠ c2 --> c1 Int c2 = {}) ==>
k dvd card (Union C)
⟨*proof*⟩

17.6.6 Relating injectivity and surjectivity

lemma *finite-surj-inj*: *finite*(*A*) $\Longrightarrow$ *A* <= *f*'*A* $\Longrightarrow$ *inj-on f A*
⟨*proof*⟩

lemma *finite-UNIV-surj-inj*: **fixes** *f* :: *'a* $\Rightarrow$ *'a*
shows *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *surj f* $\Longrightarrow$ *inj f*
⟨*proof*⟩

lemma *finite-UNIV-inj-surj*: **fixes** *f* :: *'a* $\Rightarrow$ *'a*
shows *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *inj f* $\Longrightarrow$ *surj f*
⟨*proof*⟩

corollary *infinite-UNIV-nat*: ~*finite*(*UNIV*::*nat set*)
⟨*proof*⟩

lemma *infinite-UNIV-char-0*:
¬ *finite* (*UNIV*::*'a*::*semiring-char-0 set*)
⟨*proof*⟩

17.7 A fold functional for non-empty sets

Does not require start value.

inductive
fold1Set :: (*'a* => *'a* => *'a*) => *'a set* => *'a* => *bool*
for *f* :: *'a* => *'a* => *'a*
where
fold1Set-insertI [*intro*]:
⟦ *fold-graph f a A x*; *a* ∉ *A* ⟧ $\Longrightarrow$ *fold1Set f* (*insert a A*) *x*

constdefs
fold1 :: (*'a* => *'a* => *'a*) => *'a set* => *'a*
fold1 f A == *THE x*. *fold1Set f A x*

lemma *fold1Set-nonempty*:
fold1Set f A x $\Longrightarrow$ *A* ≠ {}
⟨*proof*⟩

inductive-cases *empty-fold1SetE* [*elim*!]: *fold1Set f* {} *x*

inductive-cases *insert-fold1SetE* [*elim*!]: *fold1Set f* (*insert a X*) *x*

lemma *fold1Set-sing* [*iff*]: (*fold1Set f* {*a*} *b*) = (*a* = *b*)
⟨*proof*⟩

lemma *fold1-singleton* [*simp*]: *fold1 f* {*a*} = *a*
⟨*proof*⟩

lemma *finite-nonempty-imp-fold1Set*:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *EX x*. *fold1Set f A x*
⟨*proof*⟩

First, some lemmas about *fold-graph*.

context *ab-semigroup-mult*
begin

lemma *fun-left-comm*: *fun-left-comm*(*op* ∗)
⟨*proof*⟩

lemma *fold-graph-insert-swap*:
assumes *fold*: *fold-graph times* (*b*::′*a*) *A y* **and** *b* ∉ *A*
shows *fold-graph times z* (*insert b A*) (*z* ∗ *y*)
⟨*proof*⟩

lemma *fold-graph-permute-diff*:
assumes *fold*: *fold-graph times b A x*
shows !!*a*. ⟦*a* ∈ *A*; *b* ∉ *A*⟧ ⟹ *fold-graph times a* (*insert b* (*A*−{*a*})) *x*
⟨*proof*⟩

lemma *fold1-eq-fold*:
assumes *finite A a* ∉ *A* **shows** *fold1 times* (*insert a A*) = *fold times a A*
⟨*proof*⟩

lemma *nonempty-iff*: (*A* ≠ {}) = (∃ *x B*. *A* = *insert x B* & *x* ∉ *B*)
⟨*proof*⟩

lemma *fold1-insert*:
assumes *nonempty*: *A* ≠ {} **and** *A*: *finite A x* ∉ *A*
shows *fold1 times* (*insert x A*) = *x* ∗ *fold1 times A*
⟨*proof*⟩

end

context *ab-semigroup-idem-mult*
begin

lemma *fun-left-comm-idem*: *fun-left-comm-idem*(*op* ∗)
⟨*proof*⟩

lemma *fold1-insert-idem* [*simp*]:
assumes *nonempty*: *A* ≠ {} **and** *A*: *finite A*
shows *fold1 times* (*insert x A*) = *x* ∗ *fold1 times A*
⟨*proof*⟩

lemma *hom-fold1-commute*:
assumes *hom*: !!*x y*. *h* (*x* ∗ *y*) = *h x* ∗ *h y*

and N: *finite N* $N \neq \{\}$ **shows** h (*fold1 times N*) = *fold1 times* (h ' N)
⟨*proof*⟩

end

Now the recursion rules for definitions:

lemma *fold1-singleton-def*: g = *fold1* $f \Longrightarrow g\ \{a\} = a$
⟨*proof*⟩

lemma (**in** *ab-semigroup-mult*) *fold1-insert-def*:
⟦ g = *fold1 times*; *finite* A; $x \notin A$; $A \neq \{\}$ ⟧ $\Longrightarrow g$ (*insert x A*) = $x * g\ A$
⟨*proof*⟩

lemma (**in** *ab-semigroup-idem-mult*) *fold1-insert-idem-def*:
⟦ g = *fold1 times*; *finite* A; $A \neq \{\}$ ⟧ $\Longrightarrow g$ (*insert x A*) = $x * g\ A$
⟨*proof*⟩

17.7.1 Determinacy for *fold1Set*

declare
empty-fold-graphE [*rule del*] *fold-graph.intros* [*rule del*]
empty-fold1SetE [*rule del*] *insert-fold1SetE* [*rule del*]
— No more proofs involve these relations.

17.7.2 Lemmas about *fold1*

context *ab-semigroup-mult*
begin

lemma *fold1-Un*:
assumes A: *finite* A $A \neq \{\}$
shows *finite* $B \Longrightarrow B \neq \{\} \Longrightarrow A\ Int\ B = \{\} \Longrightarrow$
fold1 times ($A\ Un\ B$) = *fold1 times* A $*$ *fold1 times* B
⟨*proof*⟩

lemma *fold1-in*:
assumes A: *finite* (A) $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x,y\}$
shows *fold1 times* $A \in A$
⟨*proof*⟩

end

lemma (**in** *ab-semigroup-idem-mult*) *fold1-Un2*:
assumes A: *finite* A $A \neq \{\}$
shows *finite* $B \Longrightarrow B \neq \{\} \Longrightarrow$
fold1 times ($A\ Un\ B$) = *fold1 times* A $*$ *fold1 times* B
⟨*proof*⟩

17.7.3 Fold1 in lattices with *inf* and *sup*

As an application of *fold1* we define infimum and supremum in (not necessarily complete!) lattices over (non-empty) sets by means of *fold1*.

context *lower-semilattice*
begin

lemma *ab-semigroup-idem-mult-inf*:
ab-semigroup-idem-mult inf
⟨*proof*⟩

lemma *below-fold1-iff*:
assumes *finite A* $A \neq \{\}$
shows $x \leq$ *fold1 inf A* $\longleftrightarrow (\forall a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *fold1-belowI*:
assumes *finite A*
and $a \in A$
shows *fold1 inf A* $\leq a$
⟨*proof*⟩

end

lemma (**in** *upper-semilattice*) *ab-semigroup-idem-mult-sup*:
ab-semigroup-idem-mult sup
⟨*proof*⟩

context *lattice*
begin

definition
Inf-fin :: *'a set* ⇒ *'a* ($\bigsqcap_{fin}$- [*900*] *900*)
where
Inf-fin = *fold1 inf*

definition
Sup-fin :: *'a set* ⇒ *'a* ($\bigsqcup_{fin}$- [*900*] *900*)
where
Sup-fin = *fold1 sup*

lemma *Inf-le-Sup* [*simp*]: ⟦ *finite A*; $A \neq \{\}$ ⟧ ⟹ $\bigsqcap_{fin} A \leq \bigsqcup_{fin} A$
⟨*proof*⟩

lemma *sup-Inf-absorb* [*simp*]:
finite A ⟹ $a \in A$ ⟹ *sup a* ($\bigsqcap_{fin} A$) = a
⟨*proof*⟩

lemma *inf-Sup-absorb* [*simp*]:

finite $A \Longrightarrow a \in A \Longrightarrow inf\ a\ (\bigsqcup_{fin} A) = a$
⟨*proof*⟩

end

context *distrib-lattice*
begin

lemma *sup-Inf1-distrib*:
assumes *finite* A
and $A \neq \{\}$
shows $sup\ x\ (\bigsqcap_{fin} A) = \bigsqcap_{fin} \{sup\ x\ a | a.\ a \in A\}$
⟨*proof*⟩

lemma *sup-Inf2-distrib*:
assumes A: *finite* A $A \neq \{\}$ **and** B: *finite* B $B \neq \{\}$
shows $sup\ (\bigsqcap_{fin} A)\ (\bigsqcap_{fin} B) = \bigsqcap_{fin} \{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

lemma *inf-Sup1-distrib*:
assumes *finite* A **and** $A \neq \{\}$
shows $inf\ x\ (\bigsqcup_{fin} A) = \bigsqcup_{fin} \{inf\ x\ a | a.\ a \in A\}$
⟨*proof*⟩

lemma *inf-Sup2-distrib*:
assumes A: *finite* A $A \neq \{\}$ **and** B: *finite* B $B \neq \{\}$
shows $inf\ (\bigsqcup_{fin} A)\ (\bigsqcup_{fin} B) = \bigsqcup_{fin} \{inf\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

end

context *complete-lattice*
begin

Coincidence on finite sets in complete lattices:

lemma *Inf-fin-Inf*:
assumes *finite* A **and** $A \neq \{\}$
shows $\bigsqcap_{fin} A = Inf\ A$
⟨*proof*⟩

lemma *Sup-fin-Sup*:
assumes *finite* A **and** $A \neq \{\}$
shows $\bigsqcup_{fin} A = Sup\ A$
⟨*proof*⟩

end

17.7.4 Fold1 in linear orders with *min* and *max*

As an application of *fold1* we define minimum and maximum in (not necessarily complete!) linear orders over (non-empty) sets by means of *fold1*.

context *linorder*
begin

lemma *ab-semigroup-idem-mult-min*:
ab-semigroup-idem-mult min
⟨*proof*⟩

lemma *ab-semigroup-idem-mult-max*:
ab-semigroup-idem-mult max
⟨*proof*⟩

lemma *min-lattice*:
lower-semilattice (*op* ≤) (*op* <) *min*
⟨*proof*⟩

lemma *max-lattice*:
lower-semilattice (*op* ≥) (*op* >) *max*
⟨*proof*⟩

lemma *dual-max*:
ord.max (*op* ≥) = *min*
⟨*proof*⟩

lemma *dual-min*:
ord.min (*op* ≥) = *max*
⟨*proof*⟩

lemma *strict-below-fold1-iff*:
assumes *finite A* **and** *A* ≠ {}
shows *x* < *fold1 min A* ⟷ (∀ *a*∈*A*. *x* < *a*)
⟨*proof*⟩

lemma *fold1-below-iff*:
assumes *finite A* **and** *A* ≠ {}
shows *fold1 min A* ≤ *x* ⟷ (∃ *a*∈*A*. *a* ≤ *x*)
⟨*proof*⟩

lemma *fold1-strict-below-iff*:
assumes *finite A* **and** *A* ≠ {}
shows *fold1 min A* < *x* ⟷ (∃ *a*∈*A*. *a* < *x*)
⟨*proof*⟩

lemma *fold1-antimono*:
assumes *A* ≠ {} **and** *A* ⊆ *B* **and** *finite B*
shows *fold1 min B* ≤ *fold1 min A*

⟨*proof*⟩

definition
  Min :: *'a set* ⇒ *'a*
where
  Min = *fold1 min*

definition
  Max :: *'a set* ⇒ *'a*
where
  Max = *fold1 max*

lemmas *Min-singleton* [*simp*] = *fold1-singleton-def* [*OF Min-def*]
lemmas *Max-singleton* [*simp*] = *fold1-singleton-def* [*OF Max-def*]

lemma *Min-insert* [*simp*]:
  assumes *finite A* **and** *A* ≠ {}
  shows *Min* (*insert x A*) = *min x* (*Min A*)
⟨*proof*⟩

lemma *Max-insert* [*simp*]:
  assumes *finite A* **and** *A* ≠ {}
  shows *Max* (*insert x A*) = *max x* (*Max A*)
⟨*proof*⟩

lemma *Min-in* [*simp*]:
  assumes *finite A* **and** *A* ≠ {}
  shows *Min A* ∈ *A*
⟨*proof*⟩

lemma *Max-in* [*simp*]:
  assumes *finite A* **and** *A* ≠ {}
  shows *Max A* ∈ *A*
⟨*proof*⟩

lemma *Min-Un*:
  assumes *finite A* **and** *A* ≠ {} **and** *finite B* **and** *B* ≠ {}
  shows *Min* (*A* ∪ *B*) = *min* (*Min A*) (*Min B*)
⟨*proof*⟩

lemma *Max-Un*:
  assumes *finite A* **and** *A* ≠ {} **and** *finite B* **and** *B* ≠ {}
  shows *Max* (*A* ∪ *B*) = *max* (*Max A*) (*Max B*)
⟨*proof*⟩

lemma *hom-Min-commute*:
  assumes ⋀*x y*. *h* (*min x y*) = *min* (*h x*) (*h y*)
    and *finite N* **and** *N* ≠ {}
  shows *h* (*Min N*) = *Min* (*h* ‘ *N*)

⟨*proof*⟩

lemma *hom-Max-commute*:
assumes $\bigwedge x\ y.\ h\ (max\ x\ y) = max\ (h\ x)\ (h\ y)$
and *finite* N **and** $N \neq \{\}$
shows $h\ (Max\ N) = Max\ (h\ `\ N)$
⟨*proof*⟩

lemma *Min-le* [*simp*]:
assumes *finite* A **and** $x \in A$
shows $Min\ A \leq x$
⟨*proof*⟩

lemma *Max-ge* [*simp*]:
assumes *finite* A **and** $x \in A$
shows $x \leq Max\ A$
⟨*proof*⟩

lemma *Min-ge-iff* [*simp*, *noatp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $x \leq Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *Max-le-iff* [*simp*, *noatp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $Max\ A \leq x \longleftrightarrow (\forall\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

lemma *Min-gr-iff* [*simp*, *noatp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $x < Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x < a)$
⟨*proof*⟩

lemma *Max-less-iff* [*simp*, *noatp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $Max\ A < x \longleftrightarrow (\forall\, a{\in}A.\ a < x)$
⟨*proof*⟩

lemma *Min-le-iff* [*noatp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $Min\ A \leq x \longleftrightarrow (\exists\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

lemma *Max-ge-iff* [*noatp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $x \leq Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *Min-less-iff* [*noatp*]:

assumes *finite A* **and** $A \neq \{\}$
shows $Min\ A < x \longleftrightarrow (\exists\, a{\in}A.\ a < x)$
⟨*proof*⟩

lemma *Max-gr-iff* [*noatp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $x < Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x < a)$
⟨*proof*⟩

lemma *Min-eqI*:
assumes *finite A*
assumes $\bigwedge y.\ y \in A \Longrightarrow y \geq x$
and $x \in A$
shows $Min\ A = x$
⟨*proof*⟩

lemma *Max-eqI*:
assumes *finite A*
assumes $\bigwedge y.\ y \in A \Longrightarrow y \leq x$
and $x \in A$
shows $Max\ A = x$
⟨*proof*⟩

lemma *Min-antimono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows $Min\ N \leq Min\ M$
⟨*proof*⟩

lemma *Max-mono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows $Max\ M \leq Max\ N$
⟨*proof*⟩

lemma *finite-linorder-induct*[*consumes 1*, *case-names empty insert*]:
finite A $\Longrightarrow$ $P\ \{\} \Longrightarrow$
$(!!A\ b.\ finite\ A \Longrightarrow ALL\ a{:}A.\ a < b \Longrightarrow P\ A \Longrightarrow P(insert\ b\ A))$
$\Longrightarrow P\ A$
⟨*proof*⟩

end

context *ordered-ab-semigroup-add*
begin

lemma *add-Min-commute*:
fixes *k*
assumes *finite N* **and** $N \neq \{\}$
shows $k + Min\ N = Min\ \{k + m \mid m.\ m \in N\}$
⟨*proof*⟩

lemma *add-Max-commute*:
fixes k
assumes *finite* N **and** $N \neq \{\}$
shows $k + Max\ N = Max\ \{k + m \mid m.\ m \in N\}$
⟨*proof*⟩

end

end

18 Relation: Relations

theory *Relation*
imports *Datatype Finite-Set*
begin

18.1 Definitions

definition
converse :: $('a * 'b)\ set \Rightarrow ('b * 'a)\ set$
((-^−1) [1000] 999) **where**
$r\hat{}-1 == \{(y, x).\ (x, y) : r\}$

notation (*xsymbols*)
converse ((-$^{-1}$) [1000] 999)

definition
rel-comp :: $[('b * 'c)\ set, ('a * 'b)\ set] \Rightarrow ('a * 'c)\ set$
(**infixr** *O* 75) **where**
$r\ O\ s == \{(x,z).\ EX\ y.\ (x, y) : s\ \&\ (y, z) : r\}$

definition
Image :: $[('a * 'b)\ set, 'a\ set] \Rightarrow 'b\ set$
(**infixl** `` 90) **where**
$r\ ``\ s == \{y.\ EX\ x{:}s.\ (x,y){:}r\}$

definition
Id :: $('a * 'a)\ set$ **where** — the identity relation
$Id == \{p.\ EX\ x.\ p = (x,x)\}$

definition
Id-on :: $'a\ set \Rightarrow ('a * 'a)\ set$ **where** — diagonal: identity over a set
$Id\text{-}on\ A == \bigcup x{\in}A.\ \{(x,x)\}$

definition
Domain :: $('a * 'b)\ set \Rightarrow 'a\ set$ **where**
$Domain\ r == \{x.\ EX\ y.\ (x,y){:}r\}$

definition
Range :: $('a * 'b)$ *set* => $'b$ *set* **where**
Range r == *Domain*$(r\hat{}-1)$

definition
Field :: $('a * 'a)$ *set* => $'a$ *set* **where**
Field r == *Domain* $r \cup$ *Range* r

definition
refl-on :: [$'a$ *set*, $('a * 'a)$ *set*] => *bool* **where** — reflexivity over a set
refl-on A r == $r \subseteq A \times A$ & (*ALL* x: A. (x,x) : r)

abbreviation
refl :: $('a * 'a)$ *set* => *bool* **where** — reflexivity over a type
refl == *refl-on UNIV*

definition
sym :: $('a * 'a)$ *set* => *bool* **where** — symmetry predicate
sym r == *ALL* x y. (x,y): r −−> (y,x): r

definition
antisym :: $('a * 'a)$ *set* => *bool* **where** — antisymmetry predicate
antisym r == *ALL* x y. (x,y):r −−> (y,x):r −−> x=y

definition
trans :: $('a * 'a)$ *set* => *bool* **where** — transitivity predicate
trans r == (*ALL* x y z. (x,y):r −−> (y,z):r −−> (x,z):r)

definition
irrefl :: $('a * 'a)$ *set* => *bool* **where**
irrefl $r \equiv \forall x$. $(x,x) \notin r$

definition
total-on :: $'a$ *set* => $('a * 'a)$ *set* => *bool* **where**
total-on A $r \equiv \forall x{\in}A.\forall y{\in}A.\ x{\neq}y \longrightarrow (x,y){\in}r \vee (y,x){\in}r$

abbreviation *total* $\equiv$ *total-on UNIV*

definition
single-valued :: $('a * 'b)$ *set* => *bool* **where**
single-valued r == *ALL* x y. (x,y):r −−> (*ALL* z. (x,z):r −−> y=z)

definition
inv-image :: $('b * 'b)$ *set* => $('a$ => $'b) => ('a * 'a)$ *set* **where**
inv-image r f == $\{(x,\ y).\ (f\ x,\ f\ y) : r\}$

18.2 The identity relation

lemma *IdI* [*intro*]: $(a, a) : Id$
⟨*proof*⟩

lemma *IdE* [*elim*!]: $p : Id \Longrightarrow (!!x.\ p = (x, x) \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *pair-in-Id-conv* [*iff*]: $((a, b) : Id) = (a = b)$
⟨*proof*⟩

lemma *refl-Id*: *refl Id*
⟨*proof*⟩

lemma *antisym-Id*: *antisym Id*
— A strange result, since *Id* is also symmetric.
⟨*proof*⟩

lemma *sym-Id*: *sym Id*
⟨*proof*⟩

lemma *trans-Id*: *trans Id*
⟨*proof*⟩

18.3 Diagonal: identity over a set

lemma *Id-on-empty* [*simp*]: *Id-on* $\{\} = \{\}$
⟨*proof*⟩

lemma *Id-on-eqI*: $a = b \Longrightarrow a : A \Longrightarrow (a, b) : Id\text{-}on\ A$
⟨*proof*⟩

lemma *Id-onI* [*intro*!,*noatp*]: $a : A \Longrightarrow (a, a) : Id\text{-}on\ A$
⟨*proof*⟩

lemma *Id-onE* [*elim*!]:
$c : Id\text{-}on\ A \Longrightarrow (!!x.\ x : A \Longrightarrow c = (x, x) \Longrightarrow P) \Longrightarrow P$
— The general elimination rule.
⟨*proof*⟩

lemma *Id-on-iff*: $((x, y) : Id\text{-}on\ A) = (x = y\ \&\ x : A)$
⟨*proof*⟩

lemma *Id-on-subset-Times*: $Id\text{-}on\ A \subseteq A \times A$
⟨*proof*⟩

18.4 Composition of two relations

lemma *rel-compI* [*intro*]:
$(a, b) : s \Longrightarrow (b, c) : r \Longrightarrow (a, c) : r\ O\ s$

⟨*proof*⟩

lemma *rel-compE* [*elim*!]: $xz : r\ O\ s \Longrightarrow$
$(!!x\ y\ z.\ xz = (x,\ z) \Longrightarrow (x,\ y) : s \Longrightarrow (y,\ z) : r \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *rel-compEpair*:
$(a,\ c) : r\ O\ s \Longrightarrow (!!y.\ (a,\ y) : s \Longrightarrow (y,\ c) : r \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *R-O-Id* [*simp*]: $R\ O\ Id = R$
⟨*proof*⟩

lemma *Id-O-R* [*simp*]: $Id\ O\ R = R$
⟨*proof*⟩

lemma *rel-comp-empty1*[*simp*]: $\{\}\ O\ R = \{\}$
⟨*proof*⟩

lemma *rel-comp-empty2*[*simp*]: $R\ O\ \{\} = \{\}$
⟨*proof*⟩

lemma *O-assoc*: $(R\ O\ S)\ O\ T = R\ O\ (S\ O\ T)$
⟨*proof*⟩

lemma *trans-O-subset*: $trans\ r \Longrightarrow r\ O\ r \subseteq r$
⟨*proof*⟩

lemma *rel-comp-mono*: $r' \subseteq r \Longrightarrow s' \subseteq s \Longrightarrow (r'\ O\ s') \subseteq (r\ O\ s)$
⟨*proof*⟩

lemma *rel-comp-subset-Sigma*:
$s \subseteq A \times B \Longrightarrow r \subseteq B \times C \Longrightarrow (r\ O\ s) \subseteq A \times C$
⟨*proof*⟩

lemma *rel-comp-distrib*[*simp*]: $R\ O\ (S \cup T) = (R\ O\ S) \cup (R\ O\ T)$
⟨*proof*⟩

lemma *rel-comp-distrib2*[*simp*]: $(S \cup T)\ O\ R = (S\ O\ R) \cup (T\ O\ R)$
⟨*proof*⟩

18.5 Reflexivity

lemma *refl-onI*: $r \subseteq A \times A \Longrightarrow (!!x.\ x : A \Longrightarrow (x,\ x) : r) \Longrightarrow refl\text{-}on\ A\ r$
⟨*proof*⟩

lemma *refl-onD*: $refl\text{-}on\ A\ r \Longrightarrow a : A \Longrightarrow (a,\ a) : r$
⟨*proof*⟩

lemma *refl-onD1*: *refl-on A r* ==> $(x, y) : r$ ==> $x : A$
⟨*proof*⟩

lemma *refl-onD2*: *refl-on A r* ==> $(x, y) : r$ ==> $y : A$
⟨*proof*⟩

lemma *refl-on-Int*: *refl-on A r* ==> *refl-on B s* ==> *refl-on* $(A \cap B)$ $(r \cap s)$
⟨*proof*⟩

lemma *refl-on-Un*: *refl-on A r* ==> *refl-on B s* ==> *refl-on* $(A \cup B)$ $(r \cup s)$
⟨*proof*⟩

lemma *refl-on-INTER*:
ALL x:S. refl-on $(A\ x)$ $(r\ x)$ ==> *refl-on* $(INTER\ S\ A)$ $(INTER\ S\ r)$
⟨*proof*⟩

lemma *refl-on-UNION*:
ALL x:S. refl-on $(A\ x)$ $(r\ x)$ ⟹ *refl-on* $(UNION\ S\ A)$ $(UNION\ S\ r)$
⟨*proof*⟩

lemma *refl-on-empty*[*simp*]: *refl-on* {} {}
⟨*proof*⟩

lemma *refl-on-Id-on*: *refl-on A* (*Id-on A*)
⟨*proof*⟩

18.6 Antisymmetry

lemma *antisymI*:
(!!$x\ y$. $(x, y) : r$ ==> $(y, x) : r$ ==> $x=y$) ==> *antisym r*
⟨*proof*⟩

lemma *antisymD*: *antisym r* ==> $(a, b) : r$ ==> $(b, a) : r$ ==> $a = b$
⟨*proof*⟩

lemma *antisym-subset*: $r \subseteq s$ ==> *antisym s* ==> *antisym r*
⟨*proof*⟩

lemma *antisym-empty* [*simp*]: *antisym* {}
⟨*proof*⟩

lemma *antisym-Id-on* [*simp*]: *antisym* (*Id-on A*)
⟨*proof*⟩

18.7 Symmetry

lemma *symI*: (!!$a\ b$. $(a, b) : r$ ==> $(b, a) : r$) ==> *sym r*
⟨*proof*⟩

lemma *symD*: *sym r* ==> $(a, b) : r$ ==> $(b, a) : r$

⟨*proof*⟩

lemma *sym-Int*: *sym r* ==> *sym s* ==> *sym* (*r* ∩ *s*)
⟨*proof*⟩

lemma *sym-Un*: *sym r* ==> *sym s* ==> *sym* (*r* ∪ *s*)
⟨*proof*⟩

lemma *sym-INTER*: *ALL x*:*S*. *sym* (*r x*) ==> *sym* (*INTER S r*)
⟨*proof*⟩

lemma *sym-UNION*: *ALL x*:*S*. *sym* (*r x*) ==> *sym* (*UNION S r*)
⟨*proof*⟩

lemma *sym-Id-on* [*simp*]: *sym* (*Id-on A*)
⟨*proof*⟩

18.8 Transitivity

lemma *transI*:
(!!*x y z*. (*x*, *y*) : *r* ==> (*y*, *z*) : *r* ==> (*x*, *z*) : *r*) ==> *trans r*
⟨*proof*⟩

lemma *transD*: *trans r* ==> (*a*, *b*) : *r* ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*
⟨*proof*⟩

lemma *trans-Int*: *trans r* ==> *trans s* ==> *trans* (*r* ∩ *s*)
⟨*proof*⟩

lemma *trans-INTER*: *ALL x*:*S*. *trans* (*r x*) ==> *trans* (*INTER S r*)
⟨*proof*⟩

lemma *trans-Id-on* [*simp*]: *trans* (*Id-on A*)
⟨*proof*⟩

lemma *trans-diff-Id*: *trans r* ⟹ *antisym r* ⟹ *trans* (*r*−*Id*)
⟨*proof*⟩

18.9 Irreflexivity

lemma *irrefl-diff-Id*[*simp*]: *irrefl*(*r*−*Id*)
⟨*proof*⟩

18.10 Totality

lemma *total-on-empty*[*simp*]: *total-on* {} *r*
⟨*proof*⟩

lemma *total-on-diff-Id*[*simp*]: *total-on A* (*r*−*Id*) = *total-on A r*
⟨*proof*⟩

18.11 Converse

lemma *converse-iff* [*iff*]: $((a,b): r\hat{}-1) = ((b,a) : r)$
⟨*proof*⟩

lemma *converseI*[*sym*]: $(a,\ b) : r \Longrightarrow (b,\ a) : r\hat{}-1$
⟨*proof*⟩

lemma *converseD*[*sym*]: $(a,b) : r\hat{}-1 \Longrightarrow (b,\ a) : r$
⟨*proof*⟩

lemma *converseE* [*elim*!]:
$yx : r\hat{}-1 \Longrightarrow (!!x\ y.\ yx = (y,\ x) \Longrightarrow (x,\ y) : r \Longrightarrow P) \Longrightarrow P$
— More general than *converseD*, as it "splits" the member of the relation.
⟨*proof*⟩

lemma *converse-converse* [*simp*]: $(r\hat{}-1)\hat{}-1 = r$
⟨*proof*⟩

lemma *converse-rel-comp*: $(r\ O\ s)\hat{}-1 = s\hat{}-1\ O\ r\hat{}-1$
⟨*proof*⟩

lemma *converse-Int*: $(r \cap s)\hat{}-1 = r\hat{}-1 \cap s\hat{}-1$
⟨*proof*⟩

lemma *converse-Un*: $(r \cup s)\hat{}-1 = r\hat{}-1 \cup s\hat{}-1$
⟨*proof*⟩

lemma *converse-INTER*: $(INTER\ S\ r)\hat{}-1 = (INT\ x{:}S.\ (r\ x)\hat{}-1)$
⟨*proof*⟩

lemma *converse-UNION*: $(UNION\ S\ r)\hat{}-1 = (UN\ x{:}S.\ (r\ x)\hat{}-1)$
⟨*proof*⟩

lemma *converse-Id* [*simp*]: $Id\hat{}-1 = Id$
⟨*proof*⟩

lemma *converse-Id-on* [*simp*]: $(Id\text{-}on\ A)\hat{}-1 = Id\text{-}on\ A$
⟨*proof*⟩

lemma *refl-on-converse* [*simp*]: $refl\text{-}on\ A\ (converse\ r) = refl\text{-}on\ A\ r$
⟨*proof*⟩

lemma *sym-converse* [*simp*]: $sym\ (converse\ r) = sym\ r$
⟨*proof*⟩

lemma *antisym-converse* [*simp*]: $antisym\ (converse\ r) = antisym\ r$
⟨*proof*⟩

lemma *trans-converse* [*simp*]: $trans\ (converse\ r) = trans\ r$

⟨*proof*⟩

lemma *sym-conv-converse-eq*: *sym r* = (*r*^−*1* = *r*)
⟨*proof*⟩

lemma *sym-Un-converse*: *sym* (*r* ∪ *r*^−*1*)
⟨*proof*⟩

lemma *sym-Int-converse*: *sym* (*r* ∩ *r*^−*1*)
⟨*proof*⟩

lemma *total-on-converse*[*simp*]: *total-on A* (*r*^−*1*) = *total-on A r*
⟨*proof*⟩

18.12 Domain

declare *Domain-def* [*noatp*]

lemma *Domain-iff*: (*a* : *Domain r*) = (*EX y*. (*a*, *y*) : *r*)
⟨*proof*⟩

lemma *DomainI* [*intro*]: (*a*, *b*) : *r* ==> *a* : *Domain r*
⟨*proof*⟩

lemma *DomainE* [*elim*!]:
a : *Domain r* ==> (!!*y*. (*a*, *y*) : *r* ==> *P*) ==> *P*
⟨*proof*⟩

lemma *Domain-empty* [*simp*]: *Domain* {} = {}
⟨*proof*⟩

lemma *Domain-insert*: *Domain* (*insert* (*a*, *b*) *r*) = *insert a* (*Domain r*)
⟨*proof*⟩

lemma *Domain-Id* [*simp*]: *Domain Id* = *UNIV*
⟨*proof*⟩

lemma *Domain-Id-on* [*simp*]: *Domain* (*Id-on A*) = *A*
⟨*proof*⟩

lemma *Domain-Un-eq*: *Domain*(*A* ∪ *B*) = *Domain*(*A*) ∪ *Domain*(*B*)
⟨*proof*⟩

lemma *Domain-Int-subset*: *Domain*(*A* ∩ *B*) ⊆ *Domain*(*A*) ∩ *Domain*(*B*)
⟨*proof*⟩

lemma *Domain-Diff-subset*: *Domain*(*A*) − *Domain*(*B*) ⊆ *Domain*(*A* − *B*)
⟨*proof*⟩

lemma *Domain-Union*: *Domain* (*Union S*) = (⋃ *A*∈*S*. *Domain A*)
⟨*proof*⟩

lemma *Domain-converse*[*simp*]: *Domain*(*r*^−1) = *Range r*
⟨*proof*⟩

lemma *Domain-mono*: *r* ⊆ *s* ==> *Domain r* ⊆ *Domain s*
⟨*proof*⟩

lemma *fst-eq-Domain*: *fst* ' *R* = *Domain R*
⟨*proof*⟩

lemma *Domain-dprod* [*simp*]: *Domain* (*dprod r s*) = *uprod* (*Domain r*) (*Domain s*)
⟨*proof*⟩

lemma *Domain-dsum* [*simp*]: *Domain* (*dsum r s*) = *usum* (*Domain r*) (*Domain s*)
⟨*proof*⟩

18.13 Range

lemma *Range-iff*: (*a* : *Range r*) = (*EX y*. (*y*, *a*) : *r*)
⟨*proof*⟩

lemma *RangeI* [*intro*]: (*a*, *b*) : *r* ==> *b* : *Range r*
⟨*proof*⟩

lemma *RangeE* [*elim*!]: *b* : *Range r* ==> (!!*x*. (*x*, *b*) : *r* ==> *P*) ==> *P*
⟨*proof*⟩

lemma *Range-empty* [*simp*]: *Range* {} = {}
⟨*proof*⟩

lemma *Range-insert*: *Range* (*insert* (*a*, *b*) *r*) = *insert b* (*Range r*)
⟨*proof*⟩

lemma *Range-Id* [*simp*]: *Range Id* = *UNIV*
⟨*proof*⟩

lemma *Range-Id-on* [*simp*]: *Range* (*Id-on A*) = *A*
⟨*proof*⟩

lemma *Range-Un-eq*: *Range*(*A* ∪ *B*) = *Range*(*A*) ∪ *Range*(*B*)
⟨*proof*⟩

lemma *Range-Int-subset*: *Range*(*A* ∩ *B*) ⊆ *Range*(*A*) ∩ *Range*(*B*)
⟨*proof*⟩

lemma *Range-Diff-subset*: $Range(A) - Range(B) \subseteq Range(A - B)$
⟨*proof*⟩

lemma *Range-Union*: $Range\ (Union\ S) = (\bigcup A{\in}S.\ Range\ A)$
⟨*proof*⟩

lemma *Range-converse*[*simp*]: $Range(r\hat{}{-}1) = Domain\ r$
⟨*proof*⟩

lemma *snd-eq-Range*: $snd\ `\ R = Range\ R$
⟨*proof*⟩

18.14 Field

lemma *mono-Field*: $r \subseteq s \Longrightarrow Field\ r \subseteq Field\ s$
⟨*proof*⟩

lemma *Field-empty*[*simp*]: $Field\ \{\} = \{\}$
⟨*proof*⟩

lemma *Field-insert*[*simp*]: $Field\ (insert\ (a,b)\ r) = \{a,b\} \cup Field\ r$
⟨*proof*⟩

lemma *Field-Un*[*simp*]: $Field\ (r \cup s) = Field\ r \cup Field\ s$
⟨*proof*⟩

lemma *Field-Union*[*simp*]: $Field\ (\bigcup R) = \bigcup (Field\ `\ R)$
⟨*proof*⟩

lemma *Field-converse*[*simp*]: $Field(r\hat{}{-}1) = Field\ r$
⟨*proof*⟩

18.15 Image of a set under a relation

declare *Image-def* [*noatp*]

lemma *Image-iff*: $(b : r``A) = (EX\ x{:}A.\ (x,\ b) : r)$
⟨*proof*⟩

lemma *Image-singleton*: $r``\{a\} = \{b.\ (a,\ b) : r\}$
⟨*proof*⟩

lemma *Image-singleton-iff* [*iff*]: $(b : r``\{a\}) = ((a,\ b) : r)$
⟨*proof*⟩

lemma *ImageI* [*intro*,*noatp*]: $(a,\ b) : r \Longrightarrow a : A \Longrightarrow b : r``A$
⟨*proof*⟩

lemma *ImageE* [*elim*!]:
$b : r\ ``\ A \Longrightarrow (!!x.\ (x,\ b) : r \Longrightarrow x : A \Longrightarrow P) \Longrightarrow P$

⟨*proof*⟩

lemma *rev-ImageI*: $a : A \Longrightarrow (a, b) : r \Longrightarrow b : r \mathbin{``} A$
— This version's more effective when we already have the required a
⟨*proof*⟩

lemma *Image-empty* [*simp*]: $R``\{\} = \{\}$
⟨*proof*⟩

lemma *Image-Id* [*simp*]: $Id \mathbin{``} A = A$
⟨*proof*⟩

lemma *Image-Id-on* [*simp*]: $Id\text{-}on\ A \mathbin{``} B = A \cap B$
⟨*proof*⟩

lemma *Image-Int-subset*: $R \mathbin{``} (A \cap B) \subseteq R \mathbin{``} A \cap R \mathbin{``} B$
⟨*proof*⟩

lemma *Image-Int-eq*:
$single\text{-}valued\ (converse\ R) \Longrightarrow R \mathbin{``} (A \cap B) = R \mathbin{``} A \cap R \mathbin{``} B$
⟨*proof*⟩

lemma *Image-Un*: $R \mathbin{``} (A \cup B) = R \mathbin{``} A \cup R \mathbin{``} B$
⟨*proof*⟩

lemma *Un-Image*: $(R \cup S) \mathbin{``} A = R \mathbin{``} A \cup S \mathbin{``} A$
⟨*proof*⟩

lemma *Image-subset*: $r \subseteq A \times B \Longrightarrow r``C \subseteq B$
⟨*proof*⟩

lemma *Image-eq-UN*: $r``B = (\bigcup y{\in}\ B.\ r``\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

lemma *Image-mono*: $r' \subseteq r \Longrightarrow A' \subseteq A \Longrightarrow (r' \mathbin{``} A') \subseteq (r \mathbin{``} A)$
⟨*proof*⟩

lemma *Image-UN*: $(r \mathbin{``} (UNION\ A\ B)) = (\bigcup x{\in}A.\ r \mathbin{``} (B\ x))$
⟨*proof*⟩

lemma *Image-INT-subset*: $(r \mathbin{``} INTER\ A\ B) \subseteq (\bigcap x{\in}A.\ r \mathbin{``} (B\ x))$
⟨*proof*⟩

Converse inclusion requires some assumptions

lemma *Image-INT-eq*:
$[|single\text{-}valued\ (r^{-1});\ A{\neq}\{\}|] \Longrightarrow r \mathbin{``} INTER\ A\ B = (\bigcap x{\in}A.\ r \mathbin{``} B\ x)$
⟨*proof*⟩

lemma *Image-subset-eq*: $(r``A \subseteq B) = (A \subseteq - ((r\hat{}-1) `` (-B)))$
⟨*proof*⟩

18.16 Single valued relations

lemma *single-valuedI*:
$ALL\ x\ y.\ (x,y):r \longrightarrow (ALL\ z.\ (x,z):r \longrightarrow y=z) \Longrightarrow single\text{-}valued\ r$
⟨*proof*⟩

lemma *single-valuedD*:
$single\text{-}valued\ r \Longrightarrow (x,\ y) : r \Longrightarrow (x,\ z) : r \Longrightarrow y = z$
⟨*proof*⟩

lemma *single-valued-rel-comp*:
$single\text{-}valued\ r \Longrightarrow single\text{-}valued\ s \Longrightarrow single\text{-}valued\ (r\ O\ s)$
⟨*proof*⟩

lemma *single-valued-subset*:
$r \subseteq s \Longrightarrow single\text{-}valued\ s \Longrightarrow single\text{-}valued\ r$
⟨*proof*⟩

lemma *single-valued-Id* [*simp*]: $single\text{-}valued\ Id$
⟨*proof*⟩

lemma *single-valued-Id-on* [*simp*]: $single\text{-}valued\ (Id\text{-}on\ A)$
⟨*proof*⟩

18.17 Graphs given by *Collect*

lemma *Domain-Collect-split* [*simp*]: $Domain\{(x,y).\ P\ x\ y\} = \{x.\ EX\ y.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Range-Collect-split* [*simp*]: $Range\{(x,y).\ P\ x\ y\} = \{y.\ EX\ x.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Image-Collect-split* [*simp*]: $\{(x,y).\ P\ x\ y\} `` A = \{y.\ EX\ x{:}A.\ P\ x\ y\}$
⟨*proof*⟩

18.18 Inverse image

lemma *sym-inv-image*: $sym\ r \Longrightarrow sym\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

lemma *trans-inv-image*: $trans\ r \Longrightarrow trans\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

18.19 Finiteness

lemma *finite-converse* [*iff*]: $finite\ (r\hat{}-1) = finite\ r$
⟨*proof*⟩

Finiteness of transitive closure (Thanks to Sidi Ehmety)

lemma *finite-Field*: *finite r ==> finite* (*Field r*)
— A finite relation has a finite field (= *domain* ∪ *range*.
⟨*proof*⟩

18.20 Version of *lfp-induct* for binary relations

lemmas *lfp-induct2* =
lfp-induct-set [*of* (*a*, *b*), *split-format* (*complete*)]

end

19 Predicate: Predicates as relations and enumerations

theory *Predicate*
imports *Inductive Relation*
begin

notation
inf (**infixl** ⊓ *70*) **and**
sup (**infixl** ⊔ *65*) **and**
Inf (⨅- [*900*] *900*) **and**
Sup (⨆- [*900*] *900*) **and**
top (⊤) **and**
bot (⊥)

19.1 Predicates as (complete) lattices

19.1.1 *op* ⊔ **on** *bool*

lemma *sup-boolI1*:
$P \Longrightarrow P \sqcup Q$
⟨*proof*⟩

lemma *sup-boolI2*:
$Q \Longrightarrow P \sqcup Q$
⟨*proof*⟩

lemma *sup-boolE*:
$P \sqcup Q \Longrightarrow (P \Longrightarrow R) \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

19.1.2 Equality and Subsets

lemma *pred-equals-eq*: $((\lambda x.\ x \in R) = (\lambda x.\ x \in S)) = (R = S)$
⟨*proof*⟩

lemma *pred-equals-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x,\ y) \in R) = (\lambda x\ y.\ (x,\ y) \in S)) = (R = S)$
⟨*proof*⟩

lemma *pred-subset-eq*: $((\lambda x.\ x \in R) <= (\lambda x.\ x \in S)) = (R <= S)$
⟨*proof*⟩

lemma *pred-subset-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x,\ y) \in R) <= (\lambda x\ y.\ (x,\ y) \in S)) = (R <= S)$
⟨*proof*⟩

19.1.3 Top and bottom elements

lemma *top1I* [*intro*!]: *top x*
⟨*proof*⟩

lemma *top2I* [*intro*!]: *top x y*
⟨*proof*⟩

lemma *bot1E* [*elim*!]: *bot x* $\Longrightarrow$ *P*
⟨*proof*⟩

lemma *bot2E* [*elim*!]: *bot x y* $\Longrightarrow$ *P*
⟨*proof*⟩

19.1.4 The empty set

lemma *bot-empty-eq*: *bot* $= (\lambda x.\ x \in \{\})$
⟨*proof*⟩

lemma *bot-empty-eq2*: *bot* $= (\lambda x\ y.\ (x,\ y) \in \{\})$
⟨*proof*⟩

19.1.5 Binary union

lemma *sup1-iff* [*simp*]: *sup A B x* $\longleftrightarrow$ *A x* | *B x*
⟨*proof*⟩

lemma *sup2-iff* [*simp*]: *sup A B x y* $\longleftrightarrow$ *A x y* | *B x y*
⟨*proof*⟩

lemma *sup-Un-eq* [*pred-set-conv*]: *sup* $(\lambda x.\ x \in R)\ (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cup S)$
⟨*proof*⟩

lemma *sup-Un-eq2* [*pred-set-conv*]: *sup* $(\lambda x\ y.\ (x,\ y) \in R)\ (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cup S)$
⟨*proof*⟩

lemma *sup1I1* [*elim?*]: *A x* $\Longrightarrow$ *sup A B x*

⟨*proof*⟩

lemma *sup2I1* [*elim?*]: $A\ x\ y \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

lemma *sup1I2* [*elim?*]: $B\ x \Longrightarrow sup\ A\ B\ x$
⟨*proof*⟩

lemma *sup2I2* [*elim?*]: $B\ x\ y \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

Classical introduction rule: no commitment to A vs B.

lemma *sup1CI* [*intro*!]: $(\sim B\ x \Longrightarrow A\ x) \Longrightarrow sup\ A\ B\ x$
⟨*proof*⟩

lemma *sup2CI* [*intro*!]: $(\sim B\ x\ y \Longrightarrow A\ x\ y) \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

lemma *sup1E* [*elim*!]: $sup\ A\ B\ x \Longrightarrow (A\ x \Longrightarrow P) \Longrightarrow (B\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *sup2E* [*elim*!]: $sup\ A\ B\ x\ y \Longrightarrow (A\ x\ y \Longrightarrow P) \Longrightarrow (B\ x\ y \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

19.1.6 Binary intersection

lemma *inf1-iff* [*simp*]: $inf\ A\ B\ x \longleftrightarrow A\ x \wedge B\ x$
⟨*proof*⟩

lemma *inf2-iff* [*simp*]: $inf\ A\ B\ x\ y \longleftrightarrow A\ x\ y \wedge B\ x\ y$
⟨*proof*⟩

lemma *inf-Int-eq* [*pred-set-conv*]: $inf\ (\lambda x.\ x \in R)\ (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cap S)$
⟨*proof*⟩

lemma *inf-Int-eq2* [*pred-set-conv*]: $inf\ (\lambda x\ y.\ (x,\ y) \in R)\ (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cap S)$
⟨*proof*⟩

lemma *inf1I* [*intro*!]: $A\ x \Longrightarrow B\ x \Longrightarrow inf\ A\ B\ x$
⟨*proof*⟩

lemma *inf2I* [*intro*!]: $A\ x\ y \Longrightarrow B\ x\ y \Longrightarrow inf\ A\ B\ x\ y$
⟨*proof*⟩

lemma *inf1D1*: $inf\ A\ B\ x \Longrightarrow A\ x$
⟨*proof*⟩

lemma *inf2D1*: *inf A B x y ==> A x y*
⟨*proof*⟩

lemma *inf1D2*: *inf A B x ==> B x*
⟨*proof*⟩

lemma *inf2D2*: *inf A B x y ==> B x y*
⟨*proof*⟩

lemma *inf1E* [*elim*!]: *inf A B x ==> (A x ==> B x ==> P) ==> P*
⟨*proof*⟩

lemma *inf2E* [*elim*!]: *inf A B x y ==> (A x y ==> B x y ==> P) ==> P*
⟨*proof*⟩

19.1.7 Unions of families

lemma *SUP1-iff* [*simp*]: *(SUP x:A. B x) b = (EX x:A. B x b)*
⟨*proof*⟩

lemma *SUP2-iff* [*simp*]: *(SUP x:A. B x) b c = (EX x:A. B x b c)*
⟨*proof*⟩

lemma *SUP1-I* [*intro*]: *a : A ==> B a b ==> (SUP x:A. B x) b*
⟨*proof*⟩

lemma *SUP2-I* [*intro*]: *a : A ==> B a b c ==> (SUP x:A. B x) b c*
⟨*proof*⟩

lemma *SUP1-E* [*elim*!]: *(SUP x:A. B x) b ==> (!!x. x : A ==> B x b ==> R) ==> R*
⟨*proof*⟩

lemma *SUP2-E* [*elim*!]: *(SUP x:A. B x) b c ==> (!!x. x : A ==> B x b c ==> R) ==> R*
⟨*proof*⟩

lemma *SUP-UN-eq*: *(SUP i. (λx. x ∈ r i)) = (λx. x ∈ (UN i. r i))*
⟨*proof*⟩

lemma *SUP-UN-eq2*: *(SUP i. (λx y. (x, y) ∈ r i)) = (λx y. (x, y) ∈ (UN i. r i))*
⟨*proof*⟩

19.1.8 Intersections of families

lemma *INF1-iff* [*simp*]: *(INF x:A. B x) b = (ALL x:A. B x b)*
⟨*proof*⟩

lemma *INF2-iff* [*simp*]: *(INF x:A. B x) b c = (ALL x:A. B x b c)*

⟨*proof*⟩

lemma *INF1-I* [*intro*!]: (!!*x*. *x* : *A* ==> *B x b*) ==> (*INF x*:*A*. *B x*) *b*
⟨*proof*⟩

lemma *INF2-I* [*intro*!]: (!!*x*. *x* : *A* ==> *B x b c*) ==> (*INF x*:*A*. *B x*) *b c*
⟨*proof*⟩

lemma *INF1-D* [*elim*]: (*INF x*:*A*. *B x*) *b* ==> *a* : *A* ==> *B a b*
⟨*proof*⟩

lemma *INF2-D* [*elim*]: (*INF x*:*A*. *B x*) *b c* ==> *a* : *A* ==> *B a b c*
⟨*proof*⟩

lemma *INF1-E* [*elim*]: (*INF x*:*A*. *B x*) *b* ==> (*B a b* ==> *R*) ==> (*a* ~: *A* ==> *R*) ==> *R*
⟨*proof*⟩

lemma *INF2-E* [*elim*]: (*INF x*:*A*. *B x*) *b c* ==> (*B a b c* ==> *R*) ==> (*a* ~: *A* ==> *R*) ==> *R*
⟨*proof*⟩

lemma *INF-INT-eq*: (*INF i*. (λ*x*. *x* ∈ *r i*)) = (λ*x*. *x* ∈ (*INT i*. *r i*))
⟨*proof*⟩

lemma *INF-INT-eq2*: (*INF i*. (λ*x y*. (*x*, *y*) ∈ *r i*)) = (λ*x y*. (*x*, *y*) ∈ (*INT i*. *r i*))
⟨*proof*⟩

19.2 Predicates as relations

19.2.1 Composition

inductive
pred-comp :: ['*b* => '*c* => *bool*, '*a* => '*b* => *bool*] => '*a* => '*c* => *bool*
(**infixr** *OO* *75*)
for *r* :: '*b* => '*c* => *bool* **and** *s* :: '*a* => '*b* => *bool*
where
pred-compI [*intro*]: *s a b* ==> *r b c* ==> (*r OO s*) *a c*

inductive-cases *pred-compE* [*elim*!]: (*r OO s*) *a c*

lemma *pred-comp-rel-comp-eq* [*pred-set-conv*]:
((λ*x y*. (*x*, *y*) ∈ *r*) *OO* (λ*x y*. (*x*, *y*) ∈ *s*)) = (λ*x y*. (*x*, *y*) ∈ *r O s*)
⟨*proof*⟩

19.2.2 Converse

inductive
conversep :: ('*a* => '*b* => *bool*) => '*b* => '*a* => *bool*
((-^--*1*) [*1000*] *1000*)

for $r :: 'a \Rightarrow 'b \Rightarrow bool$
where
conversepI: $r\ a\ b \Longrightarrow r\hat{}{-}{-}1\ b\ a$

notation (*xsymbols*)
conversep $((\text{-}^{-1-1})\ [1000]\ 1000)$

lemma *conversepD*:
assumes *ab*: $r\hat{}{-}{-}1\ a\ b$
shows $r\ b\ a$ ⟨*proof*⟩

lemma *conversep-iff* [*iff*]: $r\hat{}{-}{-}1\ a\ b = r\ b\ a$
⟨*proof*⟩

lemma *conversep-converse-eq* [*pred-set-conv*]:
$(\lambda x\ y.\ (x,\ y) \in r)\hat{}{-}{-}1 = (\lambda x\ y.\ (x,\ y) \in r\hat{}{-}1)$
⟨*proof*⟩

lemma *conversep-conversep* [*simp*]: $(r\hat{}{-}{-}1)\hat{}{-}{-}1 = r$
⟨*proof*⟩

lemma *converse-pred-comp*: $(r\ OO\ s)\hat{}{-}{-}1 = s\hat{}{-}{-}1\ OO\ r\hat{}{-}{-}1$
⟨*proof*⟩

lemma *converse-meet*: $(inf\ r\ s)\hat{}{-}{-}1 = inf\ r\hat{}{-}{-}1\ s\hat{}{-}{-}1$
⟨*proof*⟩

lemma *converse-join*: $(sup\ r\ s)\hat{}{-}{-}1 = sup\ r\hat{}{-}{-}1\ s\hat{}{-}{-}1$
⟨*proof*⟩

lemma *conversep-noteq* [*simp*]: $(op\ \tilde{}{=})\hat{}{-}{-}1 = op\ \tilde{}{=}$
⟨*proof*⟩

lemma *conversep-eq* [*simp*]: $(op =)\hat{}{-}{-}1 = op =$
⟨*proof*⟩

19.2.3 Domain

inductive
DomainP $:: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$
for $r :: 'a \Rightarrow 'b \Rightarrow bool$
where
DomainPI [*intro*]: $r\ a\ b \Longrightarrow DomainP\ r\ a$

inductive-cases *DomainPE* [*elim*!]: $DomainP\ r\ a$

lemma *DomainP-Domain-eq* [*pred-set-conv*]: $DomainP\ (\lambda x\ y.\ (x,\ y) \in r) = (\lambda x.\ x \in Domain\ r)$
⟨*proof*⟩

19.2.4 Range

inductive
RangeP :: (*'a => 'b => bool*) => *'b => bool*
for *r* :: *'a => 'b => bool*
where
RangePI [*intro*]: *r a b ==> RangeP r b*

inductive-cases *RangePE* [*elim!*]: *RangeP r b*

lemma *RangeP-Range-eq* [*pred-set-conv*]: *RangeP* (*λx y.* (*x*, *y*) ∈ *r*) = (*λx.* *x* ∈ *Range r*)
⟨*proof*⟩

19.2.5 Inverse image

definition
inv-imagep :: (*'b => 'b => bool*) => (*'a => 'b*) => *'a => 'a => bool* **where**
inv-imagep r f == *%x y. r* (*f x*) (*f y*)

lemma [*pred-set-conv*]: *inv-imagep* (*λx y.* (*x*, *y*) ∈ *r*) *f* = (*λx y.* (*x*, *y*) ∈ *inv-image r f*)
⟨*proof*⟩

lemma *in-inv-imagep* [*simp*]: *inv-imagep r f x y* = *r* (*f x*) (*f y*)
⟨*proof*⟩

19.2.6 Powerset

definition *Powp* :: (*'a ⇒ bool*) ⇒ *'a set ⇒ bool* **where**
Powp A == *λB.* ∀ *x* ∈ *B. A x*

lemma *Powp-Pow-eq* [*pred-set-conv*]: *Powp* (*λx.* *x* ∈ *A*) = (*λx.* *x* ∈ *Pow A*)
⟨*proof*⟩

lemmas *Powp-mono* [*mono*] = *Pow-mono* [*to-pred pred-subset-eq*]

19.2.7 Properties of relations

abbreviation *antisymP* :: (*'a => 'a => bool*) => *bool* **where**
antisymP r == *antisym* {(*x*, *y*). *r x y*}

abbreviation *transP* :: (*'a => 'a => bool*) => *bool* **where**
transP r == *trans* {(*x*, *y*). *r x y*}

abbreviation *single-valuedP* :: (*'a => 'b => bool*) => *bool* **where**
single-valuedP r == *single-valued* {(*x*, *y*). *r x y*}

19.3 Predicates as enumerations

19.3.1 The type of predicate enumerations (a monad)

datatype *'a pred* = *Pred 'a* ⇒ *bool*

primrec *eval* :: *'a pred* ⇒ *'a* ⇒ *bool* **where**
eval-pred: *eval* (*Pred f*) = *f*

lemma *Pred-eval* [*simp*]:
Pred (*eval x*) = *x*
⟨*proof*⟩

lemma *eval-inject*: *eval x* = *eval y* ⟷ *x* = *y*
⟨*proof*⟩

definition *single* :: *'a* ⇒ *'a pred* **where**
single x = *Pred* ((*op* =) *x*)

definition *bind* :: *'a pred* ⇒ (*'a* ⇒ *'b pred*) ⇒ *'b pred* (**infixl** ⤜ *70*) **where**
P ⤜ *f* = *Pred* (λ*x*. (∃ *y*. *eval P y* ∧ *eval* (*f y*) *x*))

instantiation *pred* :: (*type*) *complete-lattice*
begin

definition
P ≤ *Q* ⟷ *eval P* ≤ *eval Q*

definition
P < *Q* ⟷ *eval P* < *eval Q*

definition
⊥ = *Pred* ⊥

definition
⊤ = *Pred* ⊤

definition
P ⊓ *Q* = *Pred* (*eval P* ⊓ *eval Q*)

definition
P ⊔ *Q* = *Pred* (*eval P* ⊔ *eval Q*)

definition
⨅ *A* = *Pred* (*INFI A eval*)

definition
⨆ *A* = *Pred* (*SUPR A eval*)

instance ⟨*proof*⟩

end

lemma *bind-bind*:
$(P \gg\!= Q) \gg\!= R = P \gg\!= (\lambda x.\ Q\ x \gg\!= R)$
⟨*proof*⟩

lemma *bind-single*:
$P \gg\!= single = P$
⟨*proof*⟩

lemma *single-bind*:
$single\ x \gg\!= P = P\ x$
⟨*proof*⟩

lemma *bottom-bind*:
$\bot \gg\!= P = \bot$
⟨*proof*⟩

lemma *sup-bind*:
$(P \sqcup Q) \gg\!= R = P \gg\!= R \sqcup Q \gg\!= R$
⟨*proof*⟩

lemma *Sup-bind*: $(\bigsqcup A \gg\!= f) = \bigsqcup((\lambda x.\ x \gg\!= f)\ `\ A)$
⟨*proof*⟩

lemma *pred-iffI*:
assumes $\bigwedge x.\ eval\ A\ x \Longrightarrow eval\ B\ x$
and $\bigwedge x.\ eval\ B\ x \Longrightarrow eval\ A\ x$
shows $A = B$
⟨*proof*⟩

lemma *singleI*: $eval\ (single\ x)\ x$
⟨*proof*⟩

lemma *singleI-unit*: $eval\ (single\ ())\ x$
⟨*proof*⟩

lemma *singleE*: $eval\ (single\ x)\ y \Longrightarrow (y = x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *singleE'*: $eval\ (single\ x)\ y \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *bindI*: $eval\ P\ x \Longrightarrow eval\ (Q\ x)\ y \Longrightarrow eval\ (P \gg\!= Q)\ y$
⟨*proof*⟩

lemma *bindE*: $eval\ (R \gg\!= Q)\ y \Longrightarrow (\bigwedge x.\ eval\ R\ x \Longrightarrow eval\ (Q\ x)\ y \Longrightarrow P) \Longrightarrow P$

⟨*proof*⟩

lemma *botE*: *eval* ⊥ *x* ⟹ *P*
⟨*proof*⟩

lemma *supI1*: *eval A x* ⟹ *eval* (*A* ⊔ *B*) *x*
⟨*proof*⟩

lemma *supI2*: *eval B x* ⟹ *eval* (*A* ⊔ *B*) *x*
⟨*proof*⟩

lemma *supE*: *eval* (*A* ⊔ *B*) *x* ⟹ (*eval A x* ⟹ *P*) ⟹ (*eval B x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

19.3.2 Derived operations

definition *if-pred* :: *bool* ⇒ *unit pred* **where**
if-pred-eq: *if-pred b* = (*if b then single* () *else* ⊥)

definition *not-pred* :: *unit pred* ⇒ *unit pred* **where**
not-pred-eq: *not-pred P* = (*if eval P* () *then* ⊥ *else single* ())

lemma *if-predI*: *P* ⟹ *eval* (*if-pred P*) ()
⟨*proof*⟩

lemma *if-predE*: *eval* (*if-pred b*) *x* ⟹ (*b* ⟹ *x* = () ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *not-predI*: ¬ *P* ⟹ *eval* (*not-pred* (*Pred* (λ*u*. *P*))) ()
⟨*proof*⟩

lemma *not-predI'*: ¬ *eval P* () ⟹ *eval* (*not-pred P*) ()
⟨*proof*⟩

lemma *not-predE*: *eval* (*not-pred* (*Pred* (λ*u*. *P*))) *x* ⟹ (¬ *P* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

lemma *not-predE'*: *eval* (*not-pred P*) *x* ⟹ (¬ *eval P x* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

19.3.3 Implementation

datatype *'a seq* = *Empty* | *Insert 'a 'a pred* | *Join 'a pred 'a seq*

primrec *pred-of-seq* :: *'a seq* ⇒ *'a pred* **where**
pred-of-seq Empty = ⊥
| *pred-of-seq* (*Insert x P*) = *single x* ⊔ *P*
| *pred-of-seq* (*Join P xq*) = *P* ⊔ *pred-of-seq xq*

definition *Seq* :: (*unit* ⇒ *'a seq*) ⇒ *'a pred* **where**
Seq f = *pred-of-seq* (*f* ())

code-datatype *Seq*

primrec *member* :: *'a seq* ⇒ *'a* ⇒ *bool* **where**
member Empty x ⟷ *False*
| *member* (*Insert y P*) *x* ⟷ *x* = *y* ∨ *eval P x*
| *member* (*Join P xq*) *x* ⟷ *eval P x* ∨ *member xq x*

lemma *eval-member*:
member xq = *eval* (*pred-of-seq xq*)
⟨*proof*⟩

lemma *eval-code* [*code*]: *eval* (*Seq f*) = *member* (*f* ())
⟨*proof*⟩

lemma *single-code* [*code*]:
single x = *Seq* (λ*u*. *Insert x* ⊥)
⟨*proof*⟩

primrec *apply* :: (*'a* ⇒ *'b Predicate.pred*) ⇒ *'a seq* ⇒ *'b seq* **where**
apply f Empty = *Empty*
| *apply f* (*Insert x P*) = *Join* (*f x*) (*Join* (*P* ⤜ *f*) *Empty*)
| *apply f* (*Join P xq*) = *Join* (*P* ⤜ *f*) (*apply f xq*)

lemma *apply-bind*:
pred-of-seq (*apply f xq*) = *pred-of-seq xq* ⤜ *f*
⟨*proof*⟩

lemma *bind-code* [*code*]:
Seq g ⤜ *f* = *Seq* (λ*u*. *apply f* (*g* ()))
⟨*proof*⟩

lemma *bot-set-code* [*code*]:
⊥ = *Seq* (λ*u*. *Empty*)
⟨*proof*⟩

primrec *adjunct* :: *'a pred* ⇒ *'a seq* ⇒ *'a seq* **where**
adjunct P Empty = *Join P Empty*
| *adjunct P* (*Insert x Q*) = *Insert x* (*Q* ⊔ *P*)
| *adjunct P* (*Join Q xq*) = *Join Q* (*adjunct P xq*)

lemma *adjunct-sup*:
pred-of-seq (*adjunct P xq*) = *P* ⊔ *pred-of-seq xq*
⟨*proof*⟩

lemma *sup-code* [*code*]:
Seq f ⊔ *Seq g* = *Seq* (λ*u*. *case f* ()

```
    of Empty ⇒ g ()
     | Insert x P ⇒ Insert x (P ⊔ Seq g)
     | Join P xq ⇒ adjunct (Seq g) (Join P xq))
⟨proof⟩

primrec contained :: 'a seq ⇒ 'a pred ⇒ bool where
    contained Empty Q ⟷ True
  | contained (Insert x P) Q ⟷ eval Q x ∧ P ≤ Q
  | contained (Join P xq) Q ⟷ P ≤ Q ∧ contained xq Q

lemma single-less-eq-eval:
  single x ≤ P ⟷ eval P x
  ⟨proof⟩

lemma contained-less-eq:
  contained xq Q ⟷ pred-of-seq xq ≤ Q
  ⟨proof⟩

lemma less-eq-pred-code [code]:
  Seq f ≤ Q = (case f ()
   of Empty ⇒ True
    | Insert x P ⇒ eval Q x ∧ P ≤ Q
    | Join P xq ⇒ P ≤ Q ∧ contained xq Q)
  ⟨proof⟩

lemma eq-pred-code [code]:
  fixes P Q :: 'a::eq pred
  shows eq-class.eq P Q ⟷ P ≤ Q ∧ Q ≤ P
  ⟨proof⟩

lemma [code]:
  pred-case f P = f (eval P)
  ⟨proof⟩

lemma [code]:
  pred-rec f P = f (eval P)
  ⟨proof⟩

no-notation
  inf (infixl ⊓ 70) and
  sup (infixl ⊔ 65) and
  Inf (⨅- [900] 900) and
  Sup (⨆- [900] 900) and
  top (⊤) and
  bot (⊥) and
  bind (infixl ≫= 70)

hide (open) type pred seq
hide (open) const Pred eval single bind if-pred not-pred
```

Empty Insert Join Seq member pred-of-seq apply adjunct

end

20 Transitive-Closure: Reflexive and Transitive closure of a relation

theory *Transitive-Closure*
imports *Predicate*
uses ~~/*src/Provers/trancl.ML*
begin

rtrancl is reflexive/transitive closure, *trancl* is transitive closure, *reflcl* is reflexive closure.

These postfix operators have *maximum priority*, forcing their operands to be atomic.

inductive-set
  rtrancl :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set* ((-^*) [*1000*] *999*)
  for r :: $('a \times 'a)$ *set*
where
    rtrancl-refl [*intro*!, *Pure.intro*!, *simp*]: $(a, a) : r\hat{}*$
  | *rtrancl-into-rtrancl* [*Pure.intro*]: $(a, b) : r\hat{}* \Longrightarrow (b, c) : r \Longrightarrow (a, c) : r\hat{}*$

inductive-set
  trancl :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set* ((-^+) [*1000*] *999*)
  for r :: $('a \times 'a)$ *set*
where
    r-into-trancl [*intro*, *Pure.intro*]: $(a, b) : r \Longrightarrow (a, b) : r\hat{}+$
  | *trancl-into-trancl* [*Pure.intro*]: $(a, b) : r\hat{}+ \Longrightarrow (b, c) : r \Longrightarrow (a, c) : r\hat{}+$

notation
  rtranclp ((-^**) [*1000*] *1000*) **and**
  tranclp ((-^++) [*1000*] *1000*)

abbreviation
  reflclp :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ((-^==) [*1000*] *1000*)
where
  $r\hat{}== == sup\ r\ op =$

abbreviation
  reflcl :: $('a \times 'a)$ *set* => $('a \times 'a)$ *set* ((-^=) [*1000*] *999*) **where**
  $r\hat{}= == r \cup Id$

notation (*xsymbols*)
  rtranclp ((-**) [*1000*] *1000*) **and**
  tranclp ((-++) [*1000*] *1000*) **and**
  reflclp ((-==) [*1000*] *1000*) **and**

rtrancl ((-*) [*1000*] *999*) **and**
trancl ((-+) [*1000*] *999*) **and**
reflcl ((-=) [*1000*] *999*)

notation (*HTML* **output**)
rtranclp ((-**) [*1000*] *1000*) **and**
tranclp ((-++) [*1000*] *1000*) **and**
reflclp ((-==) [*1000*] *1000*) **and**
rtrancl ((-*) [*1000*] *999*) **and**
trancl ((-+) [*1000*] *999*) **and**
reflcl ((-=) [*1000*] *999*)

20.1 Reflexive closure

lemma *refl-reflcl*[*simp*]: *refl*(*r* ^ =)
⟨*proof*⟩

lemma *antisym-reflcl*[*simp*]: *antisym*(*r* ^ =) = *antisym r*
⟨*proof*⟩

lemma *trans-reflclI*[*simp*]: *trans r* ⟹ *trans*(*r* ^ =)
⟨*proof*⟩

20.2 Reflexive-transitive closure

lemma *reflcl-set-eq* [*pred-set-conv*]: (*sup* (λ*x y*. (*x*, *y*) ∈ *r*) *op* =) = (λ*x y*. (*x*, *y*) ∈ *r Un Id*)
⟨*proof*⟩

lemma *r-into-rtrancl* [*intro*]: !!*p*. *p* ∈ *r* ==> *p* ∈ *r* ^ *
— *rtrancl* of *r* contains *r*
⟨*proof*⟩

lemma *r-into-rtranclp* [*intro*]: *r x y* ==> *r* ^ ** *x y*
— *rtrancl* of *r* contains *r*
⟨*proof*⟩

lemma *rtranclp-mono*: *r* ≤ *s* ==> *r* ^ ** ≤ *s* ^ **
— monotonicity of *rtrancl*
⟨*proof*⟩

lemmas *rtrancl-mono* = *rtranclp-mono* [*to-set*]

theorem *rtranclp-induct* [*consumes 1*, *case-names base step*, *induct set*: *rtranclp*]:
assumes *a*: *r* ^ ** *a b*
and *cases*: *P a* !!*y z*. [| *r* ^ ** *a y*; *r y z*; *P y* |] ==> *P z*
shows *P b*
⟨*proof*⟩

lemmas *rtrancl-induct* [*induct set*: *rtrancl*] = *rtranclp-induct* [*to-set*]

lemmas *rtranclp-induct2* =
rtranclp-induct[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names refl step*]

lemmas *rtrancl-induct2* =
rtrancl-induct[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names refl step*]

lemma *refl-rtrancl*: *refl* (*r*^*)
⟨*proof*⟩

Transitivity of transitive closure.

lemma *trans-rtrancl*: *trans* (*r*^*)
⟨*proof*⟩

lemmas *rtrancl-trans* = *trans-rtrancl* [*THEN transD*, *standard*]

lemma *rtranclp-trans*:
assumes *xy*: *r*^** *x y*
and *yz*: *r*^** *y z*
shows *r*^** *x z* ⟨*proof*⟩

lemma *rtranclE* [*cases set*: *rtrancl*]:
assumes *major*: (*a*::'*a*, *b*) : *r*^*
obtains
(*base*) *a* = *b*
| (*step*) *y* **where** (*a*, *y*) : *r*^* **and** (*y*, *b*) : *r*
— elimination of *rtrancl* – by induction on a special formula
⟨*proof*⟩

lemma *rtrancl-Int-subset*: [| *Id* ⊆ *s*; *r O* (*r*^* ∩ *s*) ⊆ *s*|] ==> *r*^* ⊆ *s*
⟨*proof*⟩

lemma *converse-rtranclp-into-rtranclp*:
r a b ⟹ *r*** *b c* ⟹ *r*** *a c*
⟨*proof*⟩

lemmas *converse-rtrancl-into-rtrancl* = *converse-rtranclp-into-rtranclp* [*to-set*]

More r^* equations and inclusions.

lemma *rtranclp-idemp* [*simp*]: (*r*^**)^** = *r*^**
⟨*proof*⟩

lemmas *rtrancl-idemp* [*simp*] = *rtranclp-idemp* [*to-set*]

lemma *rtrancl-idemp-self-comp* [*simp*]: *R*^* *O R*^* = *R*^*
⟨*proof*⟩

lemma *rtrancl-subset-rtrancl*: $r \subseteq s\hat{}* \Longrightarrow r\hat{}* \subseteq s\hat{}*$
⟨*proof*⟩

lemma *rtranclp-subset*: $R \leq S \Longrightarrow S \leq R\hat{}** \Longrightarrow S\hat{}** = R\hat{}**$
⟨*proof*⟩

lemmas *rtrancl-subset* = *rtranclp-subset* [*to-set*]

lemma *rtranclp-sup-rtranclp*: $(sup\ (R\hat{}**)\ (S\hat{}**))\hat{}** = (sup\ R\ S)\hat{}**$
⟨*proof*⟩

lemmas *rtrancl-Un-rtrancl* = *rtranclp-sup-rtranclp* [*to-set*]

lemma *rtranclp-reflcl* [*simp*]: $(R\hat{}==)\hat{}** = R\hat{}**$
⟨*proof*⟩

lemmas *rtrancl-reflcl* [*simp*] = *rtranclp-reflcl* [*to-set*]

lemma *rtrancl-r-diff-Id*: $(r - Id)\hat{}* = r\hat{}*$
⟨*proof*⟩

lemma *rtranclp-r-diff-Id*: $(inf\ r\ op\ \sim=)\hat{}** = r\hat{}**$
⟨*proof*⟩

theorem *rtranclp-converseD*:
assumes *r*: $(r\hat{}--1)\hat{}**\ x\ y$
shows $r\hat{}**\ y\ x$
⟨*proof*⟩

lemmas *rtrancl-converseD* = *rtranclp-converseD* [*to-set*]

theorem *rtranclp-converseI*:
assumes $r\hat{}**\ y\ x$
shows $(r\hat{}--1)\hat{}**\ x\ y$
⟨*proof*⟩

lemmas *rtrancl-converseI* = *rtranclp-converseI* [*to-set*]

lemma *rtrancl-converse*: $(r\hat{}-1)\hat{}* = (r\hat{}*)\hat{}-1$
⟨*proof*⟩

lemma *sym-rtrancl*: *sym r* ⟹ *sym* $(r\hat{}*)$
⟨*proof*⟩

theorem *converse-rtranclp-induct*[*consumes 1*]:
assumes *major*: $r\hat{}**\ a\ b$
and *cases*: $P\ b$!!$y\ z$. [| $r\ y\ z$; $r\hat{}**\ z\ b$; $P\ z$ |] ⟹ $P\ y$
shows $P\ a$
⟨*proof*⟩

lemmas *converse-rtrancl-induct* = *converse-rtranclp-induct* [*to-set*]

lemmas *converse-rtranclp-induct2* =
converse-rtranclp-induct [*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names refl step*]

lemmas *converse-rtrancl-induct2* =
converse-rtrancl-induct [*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names refl step*]

lemma *converse-rtranclpE*:
assumes *major*: $r\hat{}** x z$
and *cases*: $x=z ==> P$
$!!y. [| r x y; r\hat{}** y z |] ==> P$
shows P
⟨*proof*⟩

lemmas *converse-rtranclE* = *converse-rtranclpE* [*to-set*]

lemmas *converse-rtranclpE2* = *converse-rtranclpE* [*of* - (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemmas *converse-rtranclE2* = *converse-rtranclE* [*of* (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemma *r-comp-rtrancl-eq*: $r\ O\ r\hat{}* = r\hat{}*\ O\ r$
⟨*proof*⟩

lemma *rtrancl-unfold*: $r\hat{}* = Id\ Un\ r\ O\ r\hat{}*$
⟨*proof*⟩

20.3 Transitive closure

lemma *trancl-mono*: $!!p.\ p \in r\hat{}+ ==> r \subseteq s ==> p \in s\hat{}+$
⟨*proof*⟩

lemma *r-into-trancl'*: $!!p.\ p : r ==> p : r\hat{}+$
⟨*proof*⟩

Conversions between *trancl* and *rtrancl*.

lemma *tranclp-into-rtranclp*: $r\hat{}++\ a\ b ==> r\hat{}**\ a\ b$
⟨*proof*⟩

lemmas *trancl-into-rtrancl* = *tranclp-into-rtranclp* [*to-set*]

lemma *rtranclp-into-tranclp1*: **assumes** *r*: $r\hat{}**\ a\ b$
shows $!!c.\ r\ b\ c ==> r\hat{}++\ a\ c$ ⟨*proof*⟩

lemmas *rtrancl-into-trancl1* = *rtranclp-into-tranclp1* [*to-set*]

lemma *rtranclp-into-tranclp2*: [| *r a b*; *r* ^** *b c* |] ==> *r* ^++ *a c*
— intro rule from *r* and *rtrancl*
⟨*proof*⟩

lemmas *rtrancl-into-trancl2* = *rtranclp-into-tranclp2* [*to-set*]

Nice induction rule for *trancl*

lemma *tranclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *tranclp*]:
assumes *r* ^++ *a b*
and *cases*: !!*y*. *r a y* ==> *P y*
!!*y z*. *r* ^++ *a y* ==> *r y z* ==> *P y* ==> *P z*
shows *P b*
⟨*proof*⟩

lemmas *trancl-induct* [*induct set*: *trancl*] = *tranclp-induct* [*to-set*]

lemmas *tranclp-induct2* =
tranclp-induct [*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names base step*]

lemmas *trancl-induct2* =
trancl-induct [*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names base step*]

lemma *tranclp-trans-induct*:
assumes *major*: *r* ^++ *x y*
and *cases*: !!*x y*. *r x y* ==> *P x y*
!!*x y z*. [| *r* ^++ *x y*; *P x y*; *r* ^++ *y z*; *P y z* |] ==> *P x z*
shows *P x y*
— Another induction rule for trancl, incorporating transitivity
⟨*proof*⟩

lemmas *trancl-trans-induct* = *tranclp-trans-induct* [*to-set*]

lemma *tranclE* [*cases set*: *trancl*]:
assumes (*a*, *b*) : *r* ^+
obtains
(*base*) (*a*, *b*) : *r*
| (*step*) *c* **where** (*a*, *c*) : *r* ^+ **and** (*c*, *b*) : *r*
⟨*proof*⟩

lemma *trancl-Int-subset*: [| *r* ⊆ *s*; *r O* (*r* ^+ ∩ *s*) ⊆ *s*|] ==> *r* ^+ ⊆ *s*
⟨*proof*⟩

lemma *trancl-unfold*: *r* ^+ = *r Un r O r* ^+
⟨*proof*⟩

Transitivity of r^+

lemma *trans-trancl* [*simp*]: *trans* (*r* ^+)

⟨*proof*⟩

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*, *standard*]

lemma *tranclp-trans*:
assumes *xy*: $r\hat{}{+}{+}\ x\ y$
and *yz*: $r\hat{}{+}{+}\ y\ z$
shows $r\hat{}{+}{+}\ x\ z$ ⟨*proof*⟩

lemma *trancl-id* [*simp*]: *trans* $r \Longrightarrow r\hat{}{+} = r$
⟨*proof*⟩

lemma *rtranclp-tranclp-tranclp*:
assumes $r\hat{}{*}{*}\ x\ y$
shows $!!z.\ r\hat{}{+}{+}\ y\ z \Longrightarrow r\hat{}{+}{+}\ x\ z$ ⟨*proof*⟩

lemmas *rtrancl-trancl-trancl* = *rtranclp-tranclp-tranclp* [*to-set*]

lemma *tranclp-into-tranclp2*: $r\ a\ b \Longrightarrow r\hat{}{+}{+}\ b\ c \Longrightarrow r\hat{}{+}{+}\ a\ c$
⟨*proof*⟩

lemmas *trancl-into-trancl2* = *tranclp-into-tranclp2* [*to-set*]

lemma *trancl-insert*:
$(\mathit{insert}\ (y,\ x)\ r)\hat{}{+} = r\hat{}{+} \cup \{(a,\ b).\ (a,\ y) \in r\hat{}{*} \wedge (x,\ b) \in r\hat{}{*}\}$
— primitive recursion for *trancl* over finite relations
⟨*proof*⟩

lemma *tranclp-converseI*: $(r\hat{}{+}{+})\hat{}{-}{-}1\ x\ y \Longrightarrow (r\hat{}{-}{-}1)\hat{}{+}{+}\ x\ y$
⟨*proof*⟩

lemmas *trancl-converseI* = *tranclp-converseI* [*to-set*]

lemma *tranclp-converseD*: $(r\hat{}{-}{-}1)\hat{}{+}{+}\ x\ y \Longrightarrow (r\hat{}{+}{+})\hat{}{-}{-}1\ x\ y$
⟨*proof*⟩

lemmas *trancl-converseD* = *tranclp-converseD* [*to-set*]

lemma *tranclp-converse*: $(r\hat{}{-}{-}1)\hat{}{+}{+} = (r\hat{}{+}{+})\hat{}{-}{-}1$
⟨*proof*⟩

lemmas *trancl-converse* = *tranclp-converse* [*to-set*]

lemma *sym-trancl*: *sym* $r \Longrightarrow$ *sym* $(r\hat{}{+})$
⟨*proof*⟩

lemma *converse-tranclp-induct*:
assumes *major*: $r\hat{}{+}{+}\ a\ b$
and *cases*: $!!y.\ r\ y\ b \Longrightarrow P(y)$

$!!y\ z.[|\ r\ y\ z;\ \ r\hat{}{+}{+}\ z\ b;\ \ P(z)\ |] ==> P(y)$
shows $P\ a$
⟨*proof*⟩

lemmas *converse-trancl-induct* = *converse-tranclp-induct* [*to-set*]

lemma *tranclpD*: $R\hat{}{+}{+}\ x\ y ==> EX\ z.\ R\ x\ z \wedge R\hat{}{*}{*}\ z\ y$
⟨*proof*⟩

lemmas *tranclD* = *tranclpD* [*to-set*]

lemma *tranclD2*:
$(x,\ y) \in R^+ \Longrightarrow \exists z.\ (x,\ z) \in R^* \wedge (z,\ y) \in R$
⟨*proof*⟩

lemma *irrefl-tranclI*: $r\hat{}{-}1 \cap r\hat{}{*} = \{\} ==> (x,\ x) \notin r\hat{}{+}$
⟨*proof*⟩

lemma *irrefl-trancl-rD*: $!!X.\ ALL\ x.\ (x,\ x) \notin r\hat{}{+} ==> (x,\ y) \in r ==> x \neq y$
⟨*proof*⟩

lemma *trancl-subset-Sigma-aux*:
$(a,\ b) \in r\hat{}{*} ==> r \subseteq A \times A ==> a = b \vee a \in A$
⟨*proof*⟩

lemma *trancl-subset-Sigma*: $r \subseteq A \times A ==> r\hat{}{+} \subseteq A \times A$
⟨*proof*⟩

lemma *reflcl-tranclp* [*simp*]: $(r\hat{}{+}{+})\hat{}{=}{=} = r\hat{}{*}{*}$
⟨*proof*⟩

lemmas *reflcl-trancl* [*simp*] = *reflcl-tranclp* [*to-set*]

lemma *trancl-reflcl* [*simp*]: $(r\hat{}{=})\hat{}{+} = r\hat{}{*}$
⟨*proof*⟩

lemma *trancl-empty* [*simp*]: $\{\}\hat{}{+} = \{\}$
⟨*proof*⟩

lemma *rtrancl-empty* [*simp*]: $\{\}\hat{}{*} = Id$
⟨*proof*⟩

lemma *rtranclpD*: $R\hat{}{*}{*}\ a\ b ==> a = b \vee a \neq b \wedge R\hat{}{+}{+}\ a\ b$
⟨*proof*⟩

lemmas *rtranclD* = *rtranclpD* [*to-set*]

lemma *rtrancl-eq-or-trancl*:
$(x,y) \in R^* = (x{=}y \vee x{\neq}y \wedge (x,y) \in R^+)$

⟨*proof*⟩

Domain and *Range*

lemma *Domain-rtrancl* [*simp*]: *Domain* $(R\hat{}*) = UNIV$
⟨*proof*⟩

lemma *Range-rtrancl* [*simp*]: *Range* $(R\hat{}*) = UNIV$
⟨*proof*⟩

lemma *rtrancl-Un-subset*: $(R\hat{}* \cup S\hat{}*) \subseteq (R\ Un\ S)\hat{}*$
⟨*proof*⟩

lemma *in-rtrancl-UnI*: $x \in R\hat{}* \vee x \in S\hat{}* ==> x \in (R \cup S)\hat{}*$
⟨*proof*⟩

lemma *trancl-domain* [*simp*]: *Domain* $(r\hat{}+) =$ *Domain* r
⟨*proof*⟩

lemma *trancl-range* [*simp*]: *Range* $(r\hat{}+) =$ *Range* r
⟨*proof*⟩

lemma *Not-Domain-rtrancl*:
$x \sim$: *Domain* $R ==> ((x,\ y) : R\hat{}*) = (x = y)$
⟨*proof*⟩

lemma *trancl-subset-Field2*: $r\hat{}+ <=$ *Field* $r \times$ *Field* r
⟨*proof*⟩

lemma *finite-trancl*: *finite* $(r\hat{}+) =$ *finite* r
⟨*proof*⟩

More about converse *rtrancl* and *trancl*, should be merged with main body.

lemma *single-valued-confluent*:
⟦ *single-valued* r; $(x,y) \in r\hat{}*$; $(x,z) \in r\hat{}*$ ⟧
$\Longrightarrow (y,z) \in r\hat{}* \vee (z,y) \in r\hat{}*$
⟨*proof*⟩

lemma *r-r-into-trancl*: $(a,\ b) \in R ==> (b,\ c) \in R ==> (a,\ c) \in R\hat{}+$
⟨*proof*⟩

lemma *trancl-into-trancl* [*rule-format*]:
$(a,\ b) \in r^{+} ==> (b,\ c) \in r --> (a,c) \in r^{+}$
⟨*proof*⟩

lemma *tranclp-rtranclp-tranclp*:
$r^{++}\ a\ b ==> r^{**}\ b\ c ==> r^{++}\ a\ c$
⟨*proof*⟩

lemmas *trancl-rtrancl-trancl* = *tranclp-rtranclp-tranclp* [*to-set*]

lemmas *transitive-closure-trans* [*trans*] =
r-r-into-trancl trancl-trans rtrancl-trans
trancl.trancl-into-trancl trancl-into-trancl2
rtrancl.rtrancl-into-rtrancl converse-rtrancl-into-rtrancl
rtrancl-trancl-trancl trancl-rtrancl-trancl

lemmas *transitive-closurep-trans'* [*trans*] =
tranclp-trans rtranclp-trans
tranclp.trancl-into-trancl tranclp-into-tranclp2
rtranclp.rtrancl-into-rtrancl converse-rtranclp-into-rtranclp
rtranclp-tranclp-tranclp tranclp-rtranclp-tranclp

declare *trancl-into-rtrancl* [*elim*]

20.4 Setup of transitivity reasoner

⟨*ML*⟩

end

21 Wellfounded: Well-founded Recursion

theory *Wellfounded*
imports *Finite-Set Transitive-Closure Nat*
uses (*Tools/function-package/size.ML*)
begin

21.1 Basic Definitions

inductive
wfrec-rel :: ('*a* * '*a*) *set* => (('*a* => '*b*) => '*a* => '*b*) => '*a* => '*b* => *bool*
for *R* :: ('*a* * '*a*) *set*
and *F* :: ('*a* => '*b*) => '*a* => '*b*
where
wfrecI: *ALL z.* (*z*, *x*) : *R* --> *wfrec-rel R F z* (*g z*) ==>
wfrec-rel R F x (*F g x*)

constdefs
wf :: ('*a* * '*a*)*set* => *bool*
wf(*r*) == (!*P*. (!*x*. (!*y*. (*y*,*x*):*r* --> *P*(*y*)) --> *P*(*x*)) --> (!*x*. *P*(*x*)))

wfP :: ('*a* => '*a* => *bool*) => *bool*
wfP r == *wf* {(*x*, *y*). *r x y*}

acyclic :: ('*a**'*a*)*set* => *bool*
acyclic r == !*x*. (*x*,*x*) ~: *r*^+

cut :: ('a => 'b) => ('a * 'a)*set* => 'a => 'a => 'b
cut f r x == (%*y*. *if* (*y*,*x*):*r then f y else undefined*)

adm-wf :: ('a * 'a) *set* => (('a => 'b) => 'a => 'b) => *bool*
adm-wf R F == *ALL f g x*.
(*ALL z*. (*z*, *x*) : *R* --> *f z* = *g z*) --> *F f x* = *F g x*

wfrec :: ('a * 'a) *set* => (('a => 'b) => 'a => 'b) => 'a => 'b
[*code del*]: *wfrec R F* == %*x*. *THE y*. *wfrec-rel R* (%*f x*. *F* (*cut f R x*) *x*) *x y*

abbreviation *acyclicP* :: ('a => 'a => *bool*) => *bool* **where**
acyclicP r == *acyclic* {(*x*, *y*). *r x y*}

lemma *wfP-wf-eq* [*pred-set-conv*]: *wfP* (λ*x y*. (*x*, *y*) ∈ *r*) = *wf r*
⟨*proof*⟩

lemma *wfUNIVI*:
(!!*P x*. (*ALL x*. (*ALL y*. (*y*,*x*) : *r* --> *P*(*y*)) --> *P*(*x*)) ==> *P*(*x*)) ==> *wf*(*r*)
⟨*proof*⟩

lemmas *wfPUNIVI* = *wfUNIVI* [*to-pred*]

Restriction to domain A and range B. If r is well-founded over their intersection, then *wf r*

lemma *wfI*:
[| *r* ⊆ *A* <*> *B*;
!!*x P*. [|∀ *x*. (∀ *y*. (*y*,*x*) : *r* --> *P y*) --> *P x*; *x* : *A*; *x* : *B* |] ==> *P x* |]
==> *wf r*
⟨*proof*⟩

lemma *wf-induct*:
[| *wf*(*r*);
!!*x*.[| *ALL y*. (*y*,*x*): *r* --> *P*(*y*) |] ==> *P*(*x*)
|] ==> *P*(*a*)
⟨*proof*⟩

lemmas *wfP-induct* = *wf-induct* [*to-pred*]

lemmas *wf-induct-rule* = *wf-induct* [*rule-format*, *consumes 1*, *case-names less*, *induct set*: *wf*]

lemmas *wfP-induct-rule* = *wf-induct-rule* [*to-pred*, *induct set*: *wfP*]

lemma *wf-not-sym*: *wf r* ==> (*a*, *x*) : *r* ==> (*x*, *a*) ~: *r*
⟨*proof*⟩

lemmas *wf-asym* = *wf-not-sym* [*elim-format*]

lemma *wf-not-refl* [*simp*]: *wf r ==> (a, a) ~: r*
⟨*proof*⟩

lemmas *wf-irrefl = wf-not-refl* [*elim-format*]

lemma *wf-wellorderI*:
assumes *wf*: *wf {(x::'a::ord, y). x < y}*
assumes *lin*: *OFCLASS('a::ord, linorder-class)*
shows *OFCLASS('a::ord, wellorder-class)*
⟨*proof*⟩

lemma (**in** *wellorder*) *wf*:
wf {(x, y). x < y}
⟨*proof*⟩

21.2 Basic Results

transitive closure of a well-founded relation is well-founded!

lemma *wf-trancl*:
assumes *wf r*
shows *wf (r^+)*
⟨*proof*⟩

lemmas *wfP-trancl = wf-trancl* [*to-pred*]

lemma *wf-converse-trancl*: *wf (r^-1) ==> wf ((r^+)^-1)*
⟨*proof*⟩

Minimal-element characterization of well-foundedness

lemma *wf-eq-minimal*: *wf r = (∀Q x. x∈Q --> (∃z∈Q. ∀y. (y,z)∈r --> y∉Q))*
⟨*proof*⟩

lemma *wfE-min*:
assumes *wf R x ∈ Q*
obtains *z* **where** *z ∈ Q* ⋀*y. (y, z) ∈ R ⟹ y ∉ Q*
⟨*proof*⟩

lemma *wfI-min*:
(⋀x Q. x ∈ Q ⟹ ∃z∈Q. ∀y. (y, z) ∈ R ⟶ y ∉ Q)
⟹ wf R
⟨*proof*⟩

lemmas *wfP-eq-minimal = wf-eq-minimal* [*to-pred*]

Well-foundedness of subsets

lemma *wf-subset*: [| *wf(r); p<=r* |] ==> *wf(p)*

⟨*proof*⟩

lemmas *wfP-subset* = *wf-subset* [*to-pred*]

Well-foundedness of the empty relation

lemma *wf-empty* [*iff*]: *wf*({})
⟨*proof*⟩

lemmas *wfP-empty* [*iff*] =
wf-empty [*to-pred bot-empty-eq2*, *simplified bot-fun-eq bot-bool-eq*]

lemma *wf-Int1*: *wf r* ==> *wf* (*r Int r'*)
⟨*proof*⟩

lemma *wf-Int2*: *wf r* ==> *wf* (*r' Int r*)
⟨*proof*⟩

Well-foundedness of insert

lemma *wf-insert* [*iff*]: *wf*(*insert* (*y*,*x*) *r*) = (*wf*(*r*) & (*x*,*y*) ~: *r*^*)
⟨*proof*⟩

Well-foundedness of image

lemma *wf-prod-fun-image*: [| *wf r*; *inj f* |] ==> *wf*(*prod-fun f f* ` *r*)
⟨*proof*⟩

21.3 Well-Foundedness Results for Unions

lemma *wf-union-compatible*:
assumes *wf R wf S*
assumes *S O R* ⊆ *R*
shows *wf* (*R* ∪ *S*)
⟨*proof*⟩

Well-foundedness of indexed union with disjoint domains and ranges

lemma *wf-UN*: [| *ALL i*:*I*. *wf*(*r i*);
ALL i:*I*. *ALL j*:*I*. *r i* ~= *r j* --> *Domain*(*r i*) *Int Range*(*r j*) = {}
|] ==> *wf*(*UN i*:*I*. *r i*)
⟨*proof*⟩

lemmas *wfP-SUP* = *wf-UN* [**where** *I*=*UNIV* **and** *r*=λ*i*. {(*x*, *y*). *r i x y*},
to-pred SUP-UN-eq2 bot-empty-eq pred-equals-eq, *simplified*, *standard*]

lemma *wf-Union*:
[| *ALL r*:*R*. *wf r*;
ALL r:*R*. *ALL s*:*R*. *r* ~= *s* --> *Domain r Int Range s* = {}
|] ==> *wf*(*Union R*)
⟨*proof*⟩

lemma *wf-Un*:
[| *wf r*; *wf s*; *Domain r Int Range s* = {} |] ==> *wf*(*r Un s*)
⟨*proof*⟩

lemma *wf-union-merge*:
wf (*R* ∪ *S*) = *wf* (*R O R* ∪ *R O S* ∪ *S*) (**is** *wf ?A* = *wf ?B*)
⟨*proof*⟩

lemma *wf-comp-self*: *wf R* = *wf* (*R O R*) — special case
⟨*proof*⟩

21.3.1 acyclic

lemma *acyclicI*: *ALL x*. (*x*, *x*) ~: *r*^+ ==> *acyclic r*
⟨*proof*⟩

lemma *wf-acyclic*: *wf r* ==> *acyclic r*
⟨*proof*⟩

lemmas *wfP-acyclicP* = *wf-acyclic* [*to-pred*]

lemma *acyclic-insert* [*iff*]:
acyclic(*insert* (*y*,*x*) *r*) = (*acyclic r* & (*x*,*y*) ~: *r*^*)
⟨*proof*⟩

lemma *acyclic-converse* [*iff*]: *acyclic*(*r*^−1) = *acyclic r*
⟨*proof*⟩

lemmas *acyclicP-converse* [*iff*] = *acyclic-converse* [*to-pred*]

lemma *acyclic-impl-antisym-rtrancl*: *acyclic r* ==> *antisym*(*r*^*)
⟨*proof*⟩

lemma *acyclic-subset*: [| *acyclic s*; *r* <= *s* |] ==> *acyclic r*
⟨*proof*⟩

Wellfoundedness of finite acyclic relations

lemma *finite-acyclic-wf* [*rule-format*]: *finite r* ==> *acyclic r* −−> *wf r*
⟨*proof*⟩

lemma *finite-acyclic-wf-converse*: [|*finite r*; *acyclic r*|] ==> *wf* (*r*^−1)
⟨*proof*⟩

lemma *wf-iff-acyclic-if-finite*: *finite r* ==> *wf r* = *acyclic r*
⟨*proof*⟩

21.4 Well-Founded Recursion

cut

lemma *cuts-eq*: $(cut\ f\ r\ x = cut\ g\ r\ x) = (ALL\ y.\ (y,x):r\ --> f(y)=g(y))$
⟨*proof*⟩

lemma *cut-apply*: $(x,a):r ==> (cut\ f\ r\ a)(x) = f(x)$
⟨*proof*⟩

Inductive characterization of wfrec combinator; for details see: John Harrison, "Inductive definitions: automation and application"

lemma *wfrec-unique*: $[|\ adm\text{-}wf\ R\ F;\ wf\ R\ |] ==> EX!\ y.\ wfrec\text{-}rel\ R\ F\ x\ y$
⟨*proof*⟩

lemma *adm-lemma*: $adm\text{-}wf\ R\ (\%f\ x.\ F\ (cut\ f\ R\ x)\ x)$
⟨*proof*⟩

lemma *wfrec*: $wf(r) ==> wfrec\ r\ H\ a = H\ (cut\ (wfrec\ r\ H)\ r\ a)\ a$
⟨*proof*⟩

21.5 Code generator setup

consts-code
wfrec (⟨**module**⟩*wfrec?*)
attach ⟪
fun wfrec f x = f (wfrec f) x;
⟫

21.6 *nat* is well-founded

lemma *less-nat-rel*: $op < = (\lambda m\ n.\ n = Suc\ m)\hat{}++$
⟨*proof*⟩

definition
pred-nat :: $(nat * nat)\ set$ **where**
$pred\text{-}nat = \{(m,\ n).\ n = Suc\ m\}$

definition
less-than :: $(nat * nat)\ set$ **where**
$less\text{-}than = pred\text{-}nat\hat{}+$

lemma *less-eq*: $(m,\ n) \in pred\text{-}nat\hat{}+ \longleftrightarrow m < n$
⟨*proof*⟩

lemma *pred-nat-trancl-eq-le*:
$(m,\ n) \in pred\text{-}nat\hat{}* \longleftrightarrow m \leq n$
⟨*proof*⟩

lemma *wf-pred-nat*: *wf pred-nat*

⟨*proof*⟩

lemma *wf-less-than* [*iff*]: *wf less-than*
⟨*proof*⟩

lemma *trans-less-than* [*iff*]: *trans less-than*
⟨*proof*⟩

lemma *less-than-iff* [*iff*]: ((*x*,*y*): *less-than*) = (*x*<*y*)
⟨*proof*⟩

lemma *wf-less*: *wf* {(*x*, *y*::*nat*). *x* < *y*}
⟨*proof*⟩

21.7 Accessible Part

Inductive definition of the accessible part *acc r* of a relation; see also [**?**].

inductive-set
acc :: ('*a* ∗ '*a*) *set* => '*a set*
for *r* :: ('*a* ∗ '*a*) *set*
where
accI: (!!*y*. (*y*, *x*) : *r* ==> *y* : *acc r*) ==> *x* : *acc r*

abbreviation
termip :: ('*a* => '*a* => *bool*) => '*a* => *bool* **where**
termip r == *accp* (r^{-1-1})

abbreviation
termi :: ('*a* ∗ '*a*) *set* => '*a set* **where**
termi r == *acc* (r^{-1})

lemmas *accpI* = *accp.accI*

Induction rules

theorem *accp-induct*:
assumes *major*: *accp r a*
assumes *hyp*: !!*x*. *accp r x* ==> ∀ *y*. *r y x* −−> *P y* ==> *P x*
shows *P a*
⟨*proof*⟩

theorems *accp-induct-rule* = *accp-induct* [*rule-format*, *induct set*: *accp*]

theorem *accp-downward*: *accp r b* ==> *r a b* ==> *accp r a*
⟨*proof*⟩

lemma *not-accp-down*:
assumes *na*: ¬ *accp R x*
obtains *z* **where** *R z x* **and** ¬ *accp R z*
⟨*proof*⟩

lemma *accp-downwards-aux*: $r^{**}\ b\ a \Longrightarrow accp\ r\ a \longrightarrow accp\ r\ b$
⟨*proof*⟩

theorem *accp-downwards*: $accp\ r\ a \Longrightarrow r^{**}\ b\ a \Longrightarrow accp\ r\ b$
⟨*proof*⟩

theorem *accp-wfPI*: $\forall x.\ accp\ r\ x \Longrightarrow wfP\ r$
⟨*proof*⟩

theorem *accp-wfPD*: $wfP\ r \Longrightarrow accp\ r\ x$
⟨*proof*⟩

theorem *wfP-accp-iff*: $wfP\ r = (\forall x.\ accp\ r\ x)$
⟨*proof*⟩

Smaller relations have bigger accessible parts:

lemma *accp-subset*:
assumes *sub*: $R1 \leq R2$
shows $accp\ R2 \leq accp\ R1$
⟨*proof*⟩

This is a generalized induction theorem that works on subsets of the accessible part.

lemma *accp-subset-induct*:
assumes *subset*: $D \leq accp\ R$
and *dcl*: $\bigwedge x\ z.\ [\![D\ x;\ R\ z\ x]\!] \Longrightarrow D\ z$
and $D\ x$
and *istep*: $\bigwedge x.\ [\![D\ x;\ (\bigwedge z.\ R\ z\ x \Longrightarrow P\ z)]\!] \Longrightarrow P\ x$
shows $P\ x$
⟨*proof*⟩

Set versions of the above theorems

lemmas *acc-induct* = *accp-induct* [*to-set*]

lemmas *acc-induct-rule* = *acc-induct* [*rule-format*, *induct set*: *acc*]

lemmas *acc-downward* = *accp-downward* [*to-set*]

lemmas *not-acc-down* = *not-accp-down* [*to-set*]

lemmas *acc-downwards-aux* = *accp-downwards-aux* [*to-set*]

lemmas *acc-downwards* = *accp-downwards* [*to-set*]

lemmas *acc-wfI* = *accp-wfPI* [*to-set*]

lemmas *acc-wfD* = *accp-wfPD* [*to-set*]

lemmas *wf-acc-iff* = *wfP-accp-iff* [*to-set*]

lemmas *acc-subset* = *accp-subset* [*to-set pred-subset-eq*]

lemmas *acc-subset-induct* = *accp-subset-induct* [*to-set pred-subset-eq*]

21.8 Tools for building wellfounded relations

Inverse Image

lemma *wf-inv-image* [*simp*,*intro*!]: *wf*(*r*) ==> *wf*(*inv-image r* (*f*::'*a*=>'*b*))
⟨*proof*⟩

lemma *in-inv-image*[*simp*]: ((*x*,*y*) : *inv-image r f*) = ((*f x*, *f y*) : *r*)
⟨*proof*⟩

Measure functions into *nat*

definition *measure* :: ('*a* => *nat*) => ('*a* * '*a*)*set*
where *measure* == *inv-image less-than*

lemma *in-measure*[*simp*]: ((*x*,*y*) : *measure f*) = (*f x* < *f y*)
⟨*proof*⟩

lemma *wf-measure* [*iff*]: *wf* (*measure f*)
⟨*proof*⟩

Lexicographic combinations

definition
lex-prod :: [('*a**'*a*)*set*, ('*b**'*b*)*set*] => (('*a**'*b*)*('*a**'*b*))*set*
(**infixr** <**lex**> *80*)
where
ra <**lex**> *rb* == {((*a*,*b*),(*a*',*b*')). (*a*,*a*') : *ra* | *a*=*a*' & (*b*,*b*') : *rb*}

lemma *wf-lex-prod* [*intro*!]: [| *wf*(*ra*); *wf*(*rb*) |] ==> *wf*(*ra* <**lex**> *rb*)
⟨*proof*⟩

lemma *in-lex-prod*[*simp*]:
(((*a*,*b*),(*a*',*b*')): *r* <**lex**> *s*) = ((*a*,*a*'): *r* ∨ (*a* = *a*' ∧ (*b*, *b*') : *s*))
⟨*proof*⟩

op <**lex**> preserves transitivity

lemma *trans-lex-prod* [*intro*!]:
[| *trans R1*; *trans R2* |] ==> *trans* (*R1* <**lex**> *R2*)
⟨*proof*⟩

lexicographic combinations with measure functions

definition
mlex-prod :: ('*a* ⇒ *nat*) ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* (**infixr** <**mlex**> *80*)
where

$f <*mlex*> R = \mathit{inv\text{-}image}\ (\mathit{less\text{-}than} <*lex*> R)\ (\%x.\ (f\ x,\ x))$

lemma *wf-mlex*: $\mathit{wf}\ R \Longrightarrow \mathit{wf}\ (f <*mlex*> R)$
⟨*proof*⟩

lemma *mlex-less*: $f\ x < f\ y \Longrightarrow (x,\ y) \in f <*mlex*> R$
⟨*proof*⟩

lemma *mlex-leq*: $f\ x \leq f\ y \Longrightarrow (x,\ y) \in R \Longrightarrow (x,\ y) \in f <*mlex*> R$
⟨*proof*⟩

proper subset relation on finite sets

definition *finite-psubset* :: $('a\ \mathit{set} * 'a\ \mathit{set})\ \mathit{set}$
where $\mathit{finite\text{-}psubset} == \{(A,B).\ A < B\ \&\ \mathit{finite}\ B\}$

lemma *wf-finite-psubset*[*simp*]: *wf*(*finite-psubset*)
⟨*proof*⟩

lemma *trans-finite-psubset*: *trans finite-psubset*
⟨*proof*⟩

lemma *in-finite-psubset*[*simp*]: $(A,\ B) \in \mathit{finite\text{-}psubset} = (A < B\ \&\ \mathit{finite}\ B)$
⟨*proof*⟩

max- and min-extension of order to finite sets

inductive-set *max-ext* :: $('a \times 'a)\ \mathit{set} \Rightarrow ('a\ \mathit{set} \times 'a\ \mathit{set})\ \mathit{set}$
for R :: $('a \times 'a)\ \mathit{set}$
where
max-extI[*intro*]: $\mathit{finite}\ X \Longrightarrow \mathit{finite}\ Y \Longrightarrow Y \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow \exists\, y{\in}Y.\ (x,\ y) \in R) \Longrightarrow (X,\ Y) \in \mathit{max\text{-}ext}\ R$

lemma *max-ext-wf*:
assumes *wf*: *wf r*
shows *wf* (*max-ext r*)
⟨*proof*⟩

lemma *max-ext-additive*:
$(A,\ B) \in \mathit{max\text{-}ext}\ R \Longrightarrow (C,\ D) \in \mathit{max\text{-}ext}\ R \Longrightarrow$
$(A \cup C,\ B \cup D) \in \mathit{max\text{-}ext}\ R$
⟨*proof*⟩

definition
min-ext :: $('a \times 'a)\ \mathit{set} \Rightarrow ('a\ \mathit{set} \times 'a\ \mathit{set})\ \mathit{set}$
where
[*code del*]: $\mathit{min\text{-}ext}\ r = \{(X,\ Y) \mid X\ Y.\ X \neq \{\} \wedge (\forall\, y \in Y.\ (\exists\, x \in X.\ (x,\ y) \in r))\}$

lemma *min-ext-wf*:

assumes *wf r*
shows *wf* (*min-ext r*)
⟨*proof*⟩

Wellfoundedness of *same-fst*

definition
same-fst :: ('*a* => *bool*) => ('*a* => ('*b* * '*b*)*set*) => (('*a*∗'*b*)∗('*a*∗'*b*))*set*
where
same-fst P R == {((*x*',*y*'),(*x*,*y*)) . *x*'=*x* & *P x* & (*y*',*y*) : *R x*}
— For *rec-def* declarations where the first n parameters stay unchanged in the recursive call.

lemma *same-fstI* [*intro*!]:
[| *P x*; (*y*',*y*) : *R x* |] ==> ((*x*,*y*'),(*x*,*y*)) : *same-fst P R*
⟨*proof*⟩

lemma *wf-same-fst*:
assumes *prem*: (!!*x*. *P x* ==> *wf*(*R x*))
shows *wf*(*same-fst P R*)
⟨*proof*⟩

21.9 Weakly decreasing sequences (w.r.t. some well-founded order) stabilize.

This material does not appear to be used any longer.

lemma *sequence-trans*: [| *ALL i*. (*f* (*Suc i*), *f i*) : *r*^∗ |] ==> (*f* (*i*+*k*), *f i*) : *r*^∗
⟨*proof*⟩

lemma *wf-weak-decr-stable*:
assumes *as*: *ALL i*. (*f* (*Suc i*), *f i*) : *r*^∗ *wf* (*r*^+)
shows *EX i*. *ALL k*. *f* (*i*+*k*) = *f i*
⟨*proof*⟩

lemma *weak-decr-stable*:
ALL i. *f* (*Suc i*) <= ((*f i*)::*nat*) ==> *EX i*. *ALL k*. *f* (*i*+*k*) = *f i*
⟨*proof*⟩

21.10 size of a datatype value

⟨*ML*⟩

lemma *size-bool* [*code*]:
size (*b*::*bool*) = *0* ⟨*proof*⟩

lemma *nat-size* [*simp*, *code*]: *size* (*n*::*nat*) = *n*
⟨*proof*⟩

declare *prod.size* [*noatp*]

lemma [*code*]:
size (*P* :: *'a Predicate.pred*) = *0* ⟨*proof*⟩

lemma [*code*]:
pred-size f P = *0* ⟨*proof*⟩

end

22 FunDef: Function Definitions and Termination Proofs

theory *FunDef*
imports *Wellfounded*
uses
Tools/prop-logic.ML
Tools/sat-solver.ML
(*Tools/function-package/fundef-lib.ML*)
(*Tools/function-package/fundef-common.ML*)
(*Tools/function-package/inductive-wrap.ML*)
(*Tools/function-package/context-tree.ML*)
(*Tools/function-package/fundef-core.ML*)
(*Tools/function-package/sum-tree.ML*)
(*Tools/function-package/mutual.ML*)
(*Tools/function-package/pattern-split.ML*)
(*Tools/function-package/fundef-package.ML*)
(*Tools/function-package/auto-term.ML*)
(*Tools/function-package/measure-functions.ML*)
(*Tools/function-package/lexicographic-order.ML*)
(*Tools/function-package/fundef-datatype.ML*)
(*Tools/function-package/induction-scheme.ML*)
(*Tools/function-package/termination.ML*)
(*Tools/function-package/decompose.ML*)
(*Tools/function-package/descent.ML*)
(*Tools/function-package/scnp-solve.ML*)
(*Tools/function-package/scnp-reconstruct.ML*)
begin

22.1 Definitions with default value.

definition
THE-default :: *'a* ⇒ (*'a* ⇒ *bool*) ⇒ *'a* **where**
THE-default d P = (*if* (∃!*x*. *P x*) *then* (*THE x*. *P x*) *else d*)

lemma *THE-defaultI'*: ∃!*x*. *P x* ⟹ *P* (*THE-default d P*)
⟨*proof*⟩

lemma *THE-default1-equality*:
$[\![\exists!x.\ P\ x;\ P\ a]\!] \Longrightarrow$ *THE-default* $d\ P = a$
⟨*proof*⟩

lemma *THE-default-none*:
$\neg(\exists!x.\ P\ x) \Longrightarrow$ *THE-default* $d\ P = d$
⟨*proof*⟩

lemma *fundef-ex1-existence*:
assumes *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $G\ x\ (f\ x)$
⟨*proof*⟩

lemma *fundef-ex1-uniqueness*:
assumes *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
assumes *elm*: $G\ x\ (h\ x)$
shows $h\ x = f\ x$
⟨*proof*⟩

lemma *fundef-ex1-iff*:
assumes *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $(G\ x\ y) = (f\ x = y)$
⟨*proof*⟩

lemma *fundef-default-value*:
assumes *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *graph*: $\bigwedge x\ y.\ G\ x\ y \Longrightarrow D\ x$
assumes $\neg\ D\ x$
shows $f\ x = d\ x$
⟨*proof*⟩

definition *in-rel-def*[*simp*]:
in-rel $R\ x\ y == (x,\ y) \in R$

lemma *wf-in-rel*:
wf $R \Longrightarrow$ *wfP* (*in-rel* R)
⟨*proof*⟩

⟨*ML*⟩

22.2 Measure Functions

inductive *is-measure* :: $('a \Rightarrow nat) \Rightarrow bool$
where *is-measure-trivial*: *is-measure* f

⟨*ML*⟩

lemma *measure-size*[*measure-function*]: *is-measure size*
⟨*proof*⟩

lemma *measure-fst*[*measure-function*]: *is-measure f* ⟹ *is-measure* (λ*p*. *f* (*fst p*))
⟨*proof*⟩
lemma *measure-snd*[*measure-function*]: *is-measure f* ⟹ *is-measure* (λ*p*. *f* (*snd p*))
⟨*proof*⟩

⟨*ML*⟩

22.3 Congruence Rules

lemma *let-cong* [*fundef-cong*]:
M = *N* ⟹ (⋀*x*. *x* = *N* ⟹ *f x* = *g x*) ⟹ *Let M f* = *Let N g*
⟨*proof*⟩

lemmas [*fundef-cong*] =
if-cong image-cong INT-cong UN-cong
bex-cong ball-cong imp-cong

lemma *split-cong* [*fundef-cong*]:
(⋀*x y*. (*x*, *y*) = *q* ⟹ *f x y* = *g x y*) ⟹ *p* = *q*
⟹ *split f p* = *split g q*
⟨*proof*⟩

lemma *comp-cong* [*fundef-cong*]:
f (*g x*) = *f* ′ (*g* ′ *x* ′) ⟹ (*f o g*) *x* = (*f* ′ *o g* ′) *x* ′
⟨*proof*⟩

22.4 Simp rules for termination proofs

lemma *termination-basic-simps*[*termination-simp*]:
x < (*y*::*nat*) ⟹ *x* < *y* + *z*
x < *z* ⟹ *x* < *y* + *z*
x ≤ *y* ⟹ *x* ≤ *y* + (*z*::*nat*)
x ≤ *z* ⟹ *x* ≤ *y* + (*z*::*nat*)
x < *y* ⟹ *x* ≤ (*y*::*nat*)
⟨*proof*⟩

declare *le-imp-less-Suc*[*termination-simp*]

lemma *prod-size-simp*[*termination-simp*]:
prod-size f g p = *f* (*fst p*) + *g* (*snd p*) + *Suc 0*
⟨*proof*⟩

22.5 Decomposition

lemma *less-by-empty*:
$A = \{\} \Longrightarrow A \subseteq B$
and *union-comp-emptyL*:
$[\![A\ O\ C = \{\};\ B\ O\ C = \{\}]\!] \Longrightarrow (A \cup B)\ O\ C = \{\}$
and *union-comp-emptyR*:
$[\![A\ O\ B = \{\};\ A\ O\ C = \{\}]\!] \Longrightarrow A\ O\ (B \cup C) = \{\}$
and *wf-no-loop*:
$R\ O\ R = \{\} \Longrightarrow wf\ R$
⟨*proof*⟩

22.6 Reduction Pairs

definition
reduction-pair $P = (wf\ (fst\ P) \wedge snd\ P\ O\ fst\ P \subseteq fst\ P)$

lemma *reduction-pairI*[*intro*]: $wf\ R \Longrightarrow S\ O\ R \subseteq R \Longrightarrow$ *reduction-pair* (R, S)
⟨*proof*⟩

lemma *reduction-pair-lemma*:
assumes *rp*: *reduction-pair* P
assumes $R \subseteq fst\ P$
assumes $S \subseteq snd\ P$
assumes $wf\ S$
shows $wf\ (R \cup S)$
⟨*proof*⟩

definition
rp-inv-image $= (\lambda(R,S)\ f.\ (inv\text{-}image\ R\ f,\ inv\text{-}image\ S\ f))$

lemma *rp-inv-image-rp*:
reduction-pair $P \Longrightarrow$ *reduction-pair* $(rp\text{-}inv\text{-}image\ P\ f)$
⟨*proof*⟩

22.7 Concrete orders for SCNP termination proofs

definition *pair-less* $=$ *less-than* $<*lex*>$ *less-than*
definition [*code del*]: *pair-leq* $=$ *pair-less*$\hat{}=$
definition *max-strict* $=$ *max-ext pair-less*
definition [*code del*]: *max-weak* $=$ *max-ext pair-leq* $\cup$ $\{(\{\}, \{\})\}$
definition [*code del*]: *min-strict* $=$ *min-ext pair-less*
definition [*code del*]: *min-weak* $=$ *min-ext pair-leq* $\cup$ $\{(\{\}, \{\})\}$

lemma *wf-pair-less*[*simp*]: *wf pair-less*
⟨*proof*⟩

Introduction rules for *pair-less*/*pair-leq*

lemma *pair-leqI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*
and *pair-leqI2*: $a \leq b \Longrightarrow s \leq t \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*

and *pair-lessI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
and *pair-lessI2*: $a \leq b \Longrightarrow s < t \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
⟨*proof*⟩

Introduction rules for max

lemma *smax-emptyI*:
finite $Y \Longrightarrow Y \neq \{\} \Longrightarrow (\{\}, Y) \in$ *max-strict*
and *smax-insertI*:
$[\![y \in Y; (x, y) \in$ *pair-less*; $(X, Y) \in$ *max-strict*$]\!] \Longrightarrow (insert\ x\ X, Y) \in$ *max-strict*
and *wmax-emptyI*:
finite $X \Longrightarrow (\{\}, X) \in$ *max-weak*
and *wmax-insertI*:
$[\![y \in YS; (x, y) \in$ *pair-leq*; $(XS, YS) \in$ *max-weak*$]\!] \Longrightarrow (insert\ x\ XS, YS) \in$ *max-weak*
⟨*proof*⟩

Introduction rules for min

lemma *smin-emptyI*:
$X \neq \{\} \Longrightarrow (X, \{\}) \in$ *min-strict*
and *smin-insertI*:
$[\![x \in XS; (x, y) \in$ *pair-less*; $(XS, YS) \in$ *min-strict*$]\!] \Longrightarrow (XS, insert\ y\ YS) \in$ *min-strict*
and *wmin-emptyI*:
$(X, \{\}) \in$ *min-weak*
and *wmin-insertI*:
$[\![x \in XS; (x, y) \in$ *pair-leq*; $(XS, YS) \in$ *min-weak*$]\!] \Longrightarrow (XS, insert\ y\ YS) \in$ *min-weak*
⟨*proof*⟩

Reduction Pairs

lemma *max-ext-compat*:
assumes $S\ O\ R \subseteq R$
shows $(max\text{-}ext\ S \cup \{(\{\},\{\})\})\ O\ max\text{-}ext\ R \subseteq max\text{-}ext\ R$
⟨*proof*⟩

lemma *max-rpair-set*: *reduction-pair* (*max-strict*, *max-weak*)
⟨*proof*⟩

lemma *min-ext-compat*:
assumes $S\ O\ R \subseteq R$
shows $(min\text{-}ext\ S \cup \{(\{\},\{\})\})\ O\ min\text{-}ext\ R \subseteq min\text{-}ext\ R$
⟨*proof*⟩

lemma *min-rpair-set*: *reduction-pair* (*min-strict*, *min-weak*)
⟨*proof*⟩

22.8 Tool setup

⟨*ML*⟩

end

23 Record: Extensible records with structural subtyping

theory *Record*
imports *Product-Type*
uses (*Tools/record-package.ML*)
begin

lemma *prop-subst*: $s = t \Longrightarrow PROP\ P\ t \Longrightarrow PROP\ P\ s$
⟨*proof*⟩

lemma *rec-UNIV-I*: $\bigwedge x.\ x{\in}UNIV \equiv True$
⟨*proof*⟩

lemma *rec-True-simp*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

lemma *K-record-comp*: $(\lambda x.\ c) \circ f = (\lambda x.\ c)$
⟨*proof*⟩

23.1 Concrete record syntax

nonterminals
ident field-type field-types field fields update updates
syntax
-constify :: *id* => *ident* (-)
-constify :: *longid* => *ident* (-)

-field-type :: [*ident*, *type*] => *field-type* ((*2*- ::/ -))
:: *field-type* => *field-types* (-)
-field-types :: [*field-type*, *field-types*] => *field-types* (-,/ -)
-record-type :: *field-types* => *type* ((*3*'(| - |')))
-record-type-scheme :: [*field-types*, *type*] => *type* ((*3*'(| -,/ (*2*... ::/ -) |')))

-field :: [*ident*, 'a] => *field* ((*2*- =/ -))
:: *field* => *fields* (-)
-fields :: [*field*, *fields*] => *fields* (-,/ -)
-record :: *fields* => 'a ((*3*'(| - |')))
-record-scheme :: [*fields*, 'a] => 'a ((*3*'(| -,/ (*2*... =/ -) |')))

-update-name :: *idt*
-update :: [*ident*, 'a] => *update* ((*2*- :=/ -))
:: *update* => *updates* (-)
-updates :: [*update*, *updates*] => *updates* (-,/ -)

-record-update :: [*'a*, *updates*] => *'b* (-/(*3'*(| - |')) [*900*,*0*] *900*)

syntax (*xsymbols*)
-record-type :: *field-types* => *type* ((*3*(⦇-))
-record-type-scheme :: [*field-types*, *type*] => *type* ((*3*(⦇-,/ (*2*... ::/ -)⦈)))
-record :: *fields* => *'a* ((*3*(⦇-)))
-record-scheme :: [*fields*, *'a*] => *'a* ((*3*(⦇-,/ (*2*... =/ -)⦈)))
-record-update :: [*'a*, *updates*] => *'b* (-/(*3*(⦇-)) [*900*,*0*] *900*)

⟨*ML*⟩

end

24 Option: Datatype option

theory *Option*
imports *Datatype Finite-Set*
begin

datatype *'a option* = *None* | *Some 'a*

lemma *not-None-eq* [*iff*]: (*x* ~= *None*) = (*EX y*. *x* = *Some y*)
⟨*proof*⟩

lemma *not-Some-eq* [*iff*]: (*ALL y*. *x* ~= *Some y*) = (*x* = *None*)
⟨*proof*⟩

Although it may appear that both of these equalities are helpful only when applied to assumptions, in practice it seems better to give them the uniform iff attribute.

lemma *option-caseE*:
assumes *c*: (*case x of None* => *P* | *Some y* => *Q y*)
obtains
(*None*) *x* = *None* **and** *P*
| (*Some*) *y* **where** *x* = *Some y* **and** *Q y*
⟨*proof*⟩

lemma *insert-None-conv-UNIV*: *insert None* (*range Some*) = *UNIV*
⟨*proof*⟩

instance *option* :: (*finite*) *finite* ⟨*proof*⟩

lemma *inj-Some* [*simp*]: *inj-on Some A*
⟨*proof*⟩

24.0.1 Operations

primrec *the* :: *'a option* => *'a* **where**

the (Some x) = x

primrec *set :: 'a option => 'a set* **where**
set None = {} |
set (Some x) = {x}

lemma *ospec [dest]: (ALL x:set A. P x) ==> A = Some x ==> P x*
⟨*proof*⟩

⟨*ML*⟩

lemma *elem-set [iff]: (x : set xo) = (xo = Some x)*
⟨*proof*⟩

lemma *set-empty-eq [simp]: (set xo = {}) = (xo = None)*
⟨*proof*⟩

definition
map :: ('a ⇒ 'b) ⇒ 'a option ⇒ 'b option
where
[code del]: map = (%f y. case y of None => None | Some x => Some (f x))

lemma *option-map-None [simp, code]: map f None = None*
⟨*proof*⟩

lemma *option-map-Some [simp, code]: map f (Some x) = Some (f x)*
⟨*proof*⟩

lemma *option-map-is-None [iff]:*
(map f opt = None) = (opt = None)
⟨*proof*⟩

lemma *option-map-eq-Some [iff]:*
(map f xo = Some y) = (EX z. xo = Some z & f z = y)
⟨*proof*⟩

lemma *option-map-comp:*
map f (map g opt) = map (f o g) opt
⟨*proof*⟩

lemma *option-map-o-sum-case [simp]:*
map f o sum-case g h = sum-case (map f o g) (map f o h)
⟨*proof*⟩

hide (**open**) *const set map*

24.0.2 Code generator setup

definition
is-none :: *'a option* ⇒ *bool* **where**
is-none-none [*code post*, *symmetric*, *code inline*]: *is-none x* ⟷ *x* = *None*

lemma *is-none-code* [*code*]:
shows *is-none None* ⟷ *True*
and *is-none* (*Some x*) ⟷ *False*
⟨*proof*⟩

hide (**open**) *const is-none*

code-type *option*
(*SML* - *option*)
(*OCaml* - *option*)
(*Haskell Maybe* -)

code-const *None* **and** *Some*
(*SML NONE* **and** *SOME*)
(*OCaml None* **and** *Some* -)
(*Haskell Nothing* **and** *Just*)

code-instance *option* :: *eq*
(*Haskell* −)

code-const *eq-class.eq* :: *'a*::*eq option* ⇒ *'a option* ⇒ *bool*
(*Haskell* **infixl** *4* ==)

code-reserved *SML*
option NONE SOME

code-reserved *OCaml*
option None Some

end

25 Extraction: Program extraction for HOL

theory *Extraction*
imports *Option*
uses *Tools/rewrite-hol-proof.ML*
begin

25.1 Setup

⟨*ML*⟩

lemmas [*extraction-expand*] =
meta-spec atomize-eq atomize-all atomize-imp atomize-conj
allE rev-mp conjE Eq-TrueI Eq-FalseI eqTrueI eqTrueE eq-cong2
notE′ impE′ impE iffE imp-cong simp-thms eq-True eq-False
induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq
induct-forall-def induct-implies-def induct-equal-def induct-conj-def
induct-atomize induct-rulify induct-rulify-fallback
True-implies-equals TrueE

datatype *sumbool* = *Left* | *Right*

25.2 Type of extracted program

extract-type
typeof (*Trueprop P*) ≡ *typeof P*

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*Q*))

typeof Q ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof* (*P* ⟶ *Q*) ≡ *Type* (*TYPE*(*Null*))

typeof P ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*P* ⇒ '*Q*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*(*Null*))) ⟹
typeof (∀ *x*. *P x*) ≡ *Type* (*TYPE*(*Null*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*('*P*))) ⟹
typeof (∀ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a* ⇒ '*P*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*(*Null*))) ⟹
typeof (∃ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*('*P*))) ⟹
typeof (∃ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a* × '*P*))

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*(*Null*)) ⟹
typeof (*P* ∨ *Q*) ≡ *Type* (*TYPE*(*sumbool*))

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*Q option*))

typeof P ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*(*Null*)) ⟹
typeof (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*P option*))

typeof P ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*P* + '*Q*))

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹

$typeof\ (P \wedge Q) \equiv Type\ (TYPE('Q))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P \times 'Q))$

$typeof\ (P = Q) \equiv typeof\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P))$

$typeof\ (x \in P) \equiv typeof\ P$

25.3 Realizability

realizability
$(realizes\ t\ (Trueprop\ P)) \equiv (Trueprop\ (realizes\ t\ P))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (realizes\ Null\ P \longrightarrow realizes\ t\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE('P))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x::'P.\ realizes\ x\ P \longrightarrow realizes\ Null\ Q)$

$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x.\ realizes\ x\ P \longrightarrow realizes\ (t\ x)\ Q)$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ Null\ (P\ x))$

$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ (t\ x)\ (P\ x))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ Null\ (P\ t))$

$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ (snd\ t)\ (P\ (fst\ t)))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Left \Rightarrow realizes\ Null\ P \mid Right \Rightarrow realizes\ Null\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ P \mid Some\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ Q \mid Some\ p \Rightarrow realizes\ p\ P)$

$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Inl\ p \Rightarrow realizes\ p\ P \mid Inr\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ Null\ P \wedge realizes\ t\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ t\ P \wedge realizes\ Null\ Q)$

$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ (fst\ t)\ P \wedge realizes\ (snd\ t)\ Q)$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (\neg\ P) \equiv \neg\ realizes\ Null\ P$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow$
$realizes\ t\ (\neg\ P) \equiv (\forall x{::}'P.\ \neg\ realizes\ x\ P)$

$typeof\ (P{::}bool) \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (P = Q) \equiv realizes\ Null\ P = realizes\ Null\ Q$

$(realizes\ t\ (P = Q)) \equiv (realizes\ t\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P)))$

25.4 Computational content of basic inference rules

theorem *disjE-realizer*:
assumes *r*: $case\ x\ of\ Inl\ p \Rightarrow P\ p \mid Inr\ q \Rightarrow Q\ q$
and *r1*: $\bigwedge p.\ P\ p \Longrightarrow R\ (f\ p)$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
shows $R\ (case\ x\ of\ Inl\ p \Rightarrow f\ p \mid Inr\ q \Rightarrow g\ q)$
⟨*proof*⟩

theorem *disjE-realizer2*:
assumes *r*: $case\ x\ of\ None \Rightarrow P \mid Some\ q \Rightarrow Q\ q$
and *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
shows $R\ (case\ x\ of\ None \Rightarrow f \mid Some\ q \Rightarrow g\ q)$
⟨*proof*⟩

theorem *disjE-realizer3*:
assumes *r*: $case\ x\ of\ Left \Rightarrow P \mid Right \Rightarrow Q$
and *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $Q \Longrightarrow R\ g$
shows $R\ (case\ x\ of\ Left \Rightarrow f \mid Right \Rightarrow g)$
⟨*proof*⟩

theorem *conjI-realizer*:
$P\ p \Longrightarrow Q\ q \Longrightarrow P\ (fst\ (p,\ q)) \wedge Q\ (snd\ (p,\ q))$
⟨*proof*⟩

theorem *exI-realizer*:
$P\ y\ x \Longrightarrow P\ (snd\ (x,\ y))\ (fst\ (x,\ y))$ ⟨*proof*⟩

theorem *exE-realizer*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q\ (f\ x\ y)) \Longrightarrow Q\ (let\ (x,\ y) = p\ in\ f\ x\ y)$
⟨*proof*⟩

theorem *exE-realizer'*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q) \Longrightarrow Q$ ⟨*proof*⟩

⟨*ML*⟩

realizers
impI $(P,\ Q)$: $\lambda pq.\ pq$
$\Lambda\ P\ Q\ pq\ (h\text{: }\text{-}).\ allI \cdot \text{-} \bullet (\Lambda\ x.\ impI \cdot \text{-} \cdot \text{-} \cdot \text{-} \bullet (h \cdot x))$

impI (P): *Null*
$\Lambda\ P\ Q\ (h\text{: }\text{-}).\ allI \cdot \text{-} \bullet (\Lambda\ x.\ impI \cdot \text{-} \cdot \text{-} \cdot \text{-} \bullet (h \cdot x))$

impI (Q): $\lambda q.\ q\ \Lambda\ P\ Q\ q.\ impI \cdot \text{-} \cdot \text{-}$

impI: *Null impI*

mp $(P,\ Q)$: $\lambda pq.\ pq$
$\Lambda\ P\ Q\ pq\ (h\text{: }\text{-})\ p.\ mp \cdot \text{-} \cdot \text{-} \cdot \text{-} \bullet (spec \cdot \text{-} \cdot p \bullet h)$

mp (P): *Null*
$\Lambda\ P\ Q\ (h\text{: }\text{-})\ p.\ mp \cdot \text{-} \cdot \text{-} \cdot \text{-} \bullet (spec \cdot \text{-} \cdot p \bullet h)$

mp (Q): $\lambda q.\ q\ \Lambda\ P\ Q\ q.\ mp \cdot \text{-} \cdot \text{-}$

mp: *Null mp*

allI (P): $\lambda p.\ p\ \Lambda\ P\ p.\ allI \cdot \text{-}$

allI: *Null allI*

spec (P): $\lambda x\ p.\ p\ x\ \Lambda\ P\ x\ p.\ spec \cdot \text{-} \cdot x$

spec: *Null spec*

exI (P): $\lambda x\ p.\ (x,\ p)\ \Lambda\ P\ x\ p.\ exI\text{-}realizer \cdot P \cdot p \cdot x$

exI: $\lambda x.\ x\ \Lambda\ P\ x\ (h\text{: }\text{-}).\ h$

exE $(P,\ Q)$: $\lambda p\ pq.\ let\ (x,\ y) = p\ in\ pq\ x\ y$
$\Lambda\ P\ Q\ p\ (h\text{: }\text{-})\ pq.\ exE\text{-}realizer \cdot P \cdot p \cdot Q \cdot pq \bullet h$

exE (P): *Null*
$\Lambda\ P\ Q\ p.\ exE\text{-}realizer' \cdot \text{-} \cdot \text{-} \cdot \text{-} \cdot \text{-}$

exE (*Q*): $\lambda x\ pq.\ pq\ x$
  $\Lambda\ P\ Q\ x\ (h1\colon \text{-})\ pq\ (h2\colon \text{-}).\ h2 \cdot x \bullet h1$

exE: *Null*
  $\Lambda\ P\ Q\ x\ (h1\colon \text{-})\ (h2\colon \text{-}).\ h2 \cdot x \bullet h1$

conjI (*P*, *Q*): *Pair*
  $\Lambda\ P\ Q\ p\ (h\colon \text{-})\ q.\ conjI\text{-}realizer \cdot P \cdot p \cdot Q \cdot q \bullet h$

conjI (*P*): $\lambda p.\ p$
  $\Lambda\ P\ Q\ p.\ conjI \cdot \text{-} \cdot \text{-}$

conjI (*Q*): $\lambda q.\ q$
  $\Lambda\ P\ Q\ (h\colon \text{-})\ q.\ conjI \cdot \text{-} \cdot \text{-} \bullet h$

conjI: *Null conjI*

conjunct1 (*P*, *Q*): *fst*
  $\Lambda\ P\ Q\ pq.\ conjunct1 \cdot \text{-} \cdot \text{-}$

conjunct1 (*P*): $\lambda p.\ p$
  $\Lambda\ P\ Q\ p.\ conjunct1 \cdot \text{-} \cdot \text{-}$

conjunct1 (*Q*): *Null*
  $\Lambda\ P\ Q\ q.\ conjunct1 \cdot \text{-} \cdot \text{-}$

conjunct1: *Null conjunct1*

conjunct2 (*P*, *Q*): *snd*
  $\Lambda\ P\ Q\ pq.\ conjunct2 \cdot \text{-} \cdot \text{-}$

conjunct2 (*P*): *Null*
  $\Lambda\ P\ Q\ p.\ conjunct2 \cdot \text{-} \cdot \text{-}$

conjunct2 (*Q*): $\lambda p.\ p$
  $\Lambda\ P\ Q\ p.\ conjunct2 \cdot \text{-} \cdot \text{-}$

conjunct2: *Null conjunct2*

disjI1 (*P*, *Q*): *Inl*
  $\Lambda\ P\ Q\ p.\ iffD2 \cdot \text{-} \cdot \text{-} \bullet (sum.cases\text{-}1 \cdot P \cdot \text{-} \cdot p)$

disjI1 (*P*): *Some*
  $\Lambda\ P\ Q\ p.\ iffD2 \cdot \text{-} \cdot \text{-} \bullet (option.cases\text{-}2 \cdot \text{-} \cdot P \cdot p)$

disjI1 (*Q*): *None*
  $\Lambda\ P\ Q.\ iffD2 \cdot \text{-} \cdot \text{-} \bullet (option.cases\text{-}1 \cdot \text{-} \cdot \text{-})$

disjI1: *Left*

$\Lambda\ P\ Q.\ \mathit{iffD2} \cdot - \cdot - \cdot - \bullet (\mathit{sumbool.cases\text{-}1} \cdot - \cdot - \cdot -)$

disjI2 $(P,\ Q)$: *Inr*
$\Lambda\ Q\ P\ q.\ \mathit{iffD2} \cdot - \cdot - \cdot - \bullet (\mathit{sum.cases\text{-}2} \cdot - \cdot - \cdot\ Q \cdot q)$

disjI2 (P): *None*
$\Lambda\ Q\ P.\ \mathit{iffD2} \cdot - \cdot - \cdot - \bullet (\mathit{option.cases\text{-}1} \cdot - \cdot - \cdot -)$

disjI2 (Q): *Some*
$\Lambda\ Q\ P\ q.\ \mathit{iffD2} \cdot - \cdot - \cdot - \bullet (\mathit{option.cases\text{-}2} \cdot - \cdot - \cdot\ Q \cdot q)$

disjI2: *Right*
$\Lambda\ Q\ P.\ \mathit{iffD2} \cdot - \cdot - \cdot - \bullet (\mathit{sumbool.cases\text{-}2} \cdot - \cdot - \cdot -)$

disjE $(P,\ Q,\ R)$: $\lambda pq\ pr\ qr.$
$(\mathit{case}\ pq\ \mathit{of}\ \mathit{Inl}\ p \Rightarrow pr\ p \mid \mathit{Inr}\ q \Rightarrow qr\ q)$
$\Lambda\ P\ Q\ R\ pq\ (h1\text{:}\ -)\ pr\ (h2\text{:}\ -)\ qr.$
$\mathit{disjE\text{-}realizer} \cdot - \cdot - \cdot - \cdot\ pq \cdot R \cdot pr \cdot qr \bullet h1 \bullet h2$

disjE $(Q,\ R)$: $\lambda pq\ pr\ qr.$
$(\mathit{case}\ pq\ \mathit{of}\ \mathit{None} \Rightarrow pr \mid \mathit{Some}\ q \Rightarrow qr\ q)$
$\Lambda\ P\ Q\ R\ pq\ (h1\text{:}\ -)\ pr\ (h2\text{:}\ -)\ qr.$
$\mathit{disjE\text{-}realizer2} \cdot - \cdot - \cdot - \cdot\ pq \cdot R \cdot pr \cdot qr \bullet h1 \bullet h2$

disjE $(P,\ R)$: $\lambda pq\ pr\ qr.$
$(\mathit{case}\ pq\ \mathit{of}\ \mathit{None} \Rightarrow qr \mid \mathit{Some}\ p \Rightarrow pr\ p)$
$\Lambda\ P\ Q\ R\ pq\ (h1\text{:}\ -)\ pr\ (h2\text{:}\ -)\ qr\ (h3\text{:}\ -).$
$\mathit{disjE\text{-}realizer2} \cdot - \cdot - \cdot - \cdot\ pq \cdot R \cdot qr \cdot pr \bullet h1 \bullet h3 \bullet h2$

disjE (R): $\lambda pq\ pr\ qr.$
$(\mathit{case}\ pq\ \mathit{of}\ \mathit{Left} \Rightarrow pr \mid \mathit{Right} \Rightarrow qr)$
$\Lambda\ P\ Q\ R\ pq\ (h1\text{:}\ -)\ pr\ (h2\text{:}\ -)\ qr.$
$\mathit{disjE\text{-}realizer3} \cdot - \cdot - \cdot - \cdot\ pq \cdot R \cdot pr \cdot qr \bullet h1 \bullet h2$

disjE $(P,\ Q)$: *Null*
$\Lambda\ P\ Q\ R\ pq.\ \mathit{disjE\text{-}realizer} \cdot - \cdot - \cdot - \cdot\ pq \cdot (\lambda x.\ R) \cdot - \cdot - \cdot -$

disjE (Q): *Null*
$\Lambda\ P\ Q\ R\ pq.\ \mathit{disjE\text{-}realizer2} \cdot - \cdot - \cdot - \cdot\ pq \cdot (\lambda x.\ R) \cdot - \cdot - \cdot -$

disjE (P): *Null*
$\Lambda\ P\ Q\ R\ pq\ (h1\text{:}\ -)\ (h2\text{:}\ -)\ (h3\text{:}\ -).$
$\mathit{disjE\text{-}realizer2} \cdot - \cdot - \cdot - \cdot\ pq \cdot (\lambda x.\ R) \cdot - \cdot - \cdot - \bullet h1 \bullet h3 \bullet h2$

disjE: *Null*
$\Lambda\ P\ Q\ R\ pq.\ \mathit{disjE\text{-}realizer3} \cdot - \cdot - \cdot - \cdot\ pq \cdot (\lambda x.\ R) \cdot - \cdot - \cdot -$

FalseE (P): *default*
$\Lambda\ P.\ \mathit{FalseE} \cdot -$

FalseE: *Null FalseE*

notI (*P*): *Null*
  Λ *P* (*h*: -). *allI* · - · (Λ *x*. *notI* · - · (*h* · *x*))

notI: *Null notI*

notE (*P*, *R*): λ*p*. *default*
  Λ *P R* (*h*: -) *p*. *notE* · - · - · (*spec* · - · *p* · *h*)

notE (*P*): *Null*
  Λ *P R* (*h*: -) *p*. *notE* · - · - · (*spec* · - · *p* · *h*)

notE (*R*): *default*
  Λ *P R*. *notE* · - · -

notE: *Null notE*

subst (*P*): λ*s t ps*. *ps*
  Λ *s t P* (*h*: -) *ps*. *subst* · *s* · *t* · *P ps* · *h*

subst: *Null subst*

iffD1 (*P*, *Q*): *fst*
  Λ *Q P pq* (*h*: -) *p*.
    mp · - · - · (*spec* · - · *p* · (*conjunct1* · - · - · *h*))

iffD1 (*P*): λ*p*. *p*
  Λ *Q P p* (*h*: -). *mp* · - · - · (*conjunct1* · - · - · *h*)

iffD1 (*Q*): *Null*
  Λ *Q P q1* (*h*: -) *q2*.
    mp · - · - · (*spec* · - · *q2* · (*conjunct1* · - · - · *h*))

iffD1: *Null iffD1*

iffD2 (*P*, *Q*): *snd*
  Λ *P Q pq* (*h*: -) *q*.
    mp · - · - · (*spec* · - · *q* · (*conjunct2* · - · - · *h*))

iffD2 (*P*): λ*p*. *p*
  Λ *P Q p* (*h*: -). *mp* · - · - · (*conjunct2* · - · - · *h*)

iffD2 (*Q*): *Null*
  Λ *P Q q1* (*h*: -) *q2*.
    mp · - · - · (*spec* · - · *q2* · (*conjunct2* · - · - · *h*))

iffD2: *Null iffD2*

iffI (*P*, *Q*): *Pair*
  Λ *P Q pq* (*h1* : -) *qp* (*h2* : -). *conjI-realizer* ·
    (λ*pq*. ∀ *x*. *P x* ⟶ *Q* (*pq x*)) · *pq* ·
    (λ*qp*. ∀ *x*. *Q x* ⟶ *P* (*qp x*)) · *qp* ·
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h1* · *x*))) ·
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h2* · *x*)))

iffI (*P*): λ*p*. *p*
  Λ *P Q* (*h1* : -) *p* (*h2* : -). *conjI* · - · - · -
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h1* · *x*))) ·
    (*impI* · - · - · - · *h2*)

iffI (*Q*): λ*q*. *q*
  Λ *P Q q* (*h1* : -) (*h2* : -). *conjI* · - · - · -
    (*impI* · - · - · - · *h1*) ·
    (*allI* · - · (Λ *x*. *impI* · - · - · - · (*h2* · *x*)))

iffI: *Null iffI*

⟨*ML*⟩

end

26 Divides: The division operators div and mod

theory *Divides*
imports *Nat Power Product-Type*
uses ~~/*src*/*Provers*/*Arith*/*cancel-div-mod.ML*
begin

26.1 Syntactic division operations

class *div* = *dvd* +
  fixes *div* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *div 70*)
    and *mod* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *mod 70*)

26.2 Abstract division in commutative semirings.

class *semiring-div* = *comm-semiring-1-cancel* + *div* +
  assumes *mod-div-equality*: *a div b* ∗ *b* + *a mod b* = *a*
    and *div-by-0* [*simp*]: *a div 0* = *0*
    and *div-0* [*simp*]: *0 div a* = *0*
    and *div-mult-self1* [*simp*]: *b* ≠ *0* ⟹ (*a* + *c* ∗ *b*) *div b* = *c* + *a div b*
begin

op div and *op mod*

lemma *mod-div-equality2*: $b * (a\ div\ b) + a\ mod\ b = a$
⟨*proof*⟩

lemma *mod-div-equality'*: $a\ mod\ b + a\ div\ b * b = a$
⟨*proof*⟩

lemma *div-mod-equality*: $((a\ div\ b) * b + a\ mod\ b) + c = a + c$
⟨*proof*⟩

lemma *div-mod-equality2*: $(b * (a\ div\ b) + a\ mod\ b) + c = a + c$
⟨*proof*⟩

lemma *mod-by-0* [*simp*]: $a\ mod\ 0 = a$
⟨*proof*⟩

lemma *mod-0* [*simp*]: $0\ mod\ a = 0$
⟨*proof*⟩

lemma *div-mult-self2* [*simp*]:
assumes $b \neq 0$
shows $(a + b * c)\ div\ b = c + a\ div\ b$
⟨*proof*⟩

lemma *mod-mult-self1* [*simp*]: $(a + c * b)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

lemma *mod-mult-self2* [*simp*]: $(a + b * c)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

lemma *div-mult-self1-is-id* [*simp*]: $b \neq 0 \Longrightarrow b * a\ div\ b = a$
⟨*proof*⟩

lemma *div-mult-self2-is-id* [*simp*]: $b \neq 0 \Longrightarrow a * b\ div\ b = a$
⟨*proof*⟩

lemma *mod-mult-self1-is-0* [*simp*]: $b * a\ mod\ b = 0$
⟨*proof*⟩

lemma *mod-mult-self2-is-0* [*simp*]: $a * b\ mod\ b = 0$
⟨*proof*⟩

lemma *div-by-1* [*simp*]: $a\ div\ 1 = a$
⟨*proof*⟩

lemma *mod-by-1* [*simp*]: $a\ mod\ 1 = 0$
⟨*proof*⟩

lemma *mod-self* [*simp*]: $a\ mod\ a = 0$

⟨*proof*⟩

lemma *div-self* [*simp*]: $a \neq 0 \Longrightarrow a \ div \ a = 1$
⟨*proof*⟩

lemma *div-add-self1* [*simp*]:
assumes $b \neq 0$
shows $(b + a) \ div \ b = a \ div \ b + 1$
⟨*proof*⟩

lemma *div-add-self2* [*simp*]:
assumes $b \neq 0$
shows $(a + b) \ div \ b = a \ div \ b + 1$
⟨*proof*⟩

lemma *mod-add-self1* [*simp*]:
$(b + a) \ mod \ b = a \ mod \ b$
⟨*proof*⟩

lemma *mod-add-self2* [*simp*]:
$(a + b) \ mod \ b = a \ mod \ b$
⟨*proof*⟩

lemma *mod-div-decomp*:
fixes $a \ b$
obtains $q \ r$ **where** $q = a \ div \ b$ **and** $r = a \ mod \ b$
and $a = q * b + r$
⟨*proof*⟩

lemma *dvd-eq-mod-eq-0* [*code unfold*]: $a \ dvd \ b \longleftrightarrow b \ mod \ a = 0$
⟨*proof*⟩

lemma *mod-div-trivial* [*simp*]: $a \ mod \ b \ div \ b = 0$
⟨*proof*⟩

lemma *mod-mod-trivial* [*simp*]: $a \ mod \ b \ mod \ b = a \ mod \ b$
⟨*proof*⟩

lemma *dvd-imp-mod-0*: $a \ dvd \ b \Longrightarrow b \ mod \ a = 0$
⟨*proof*⟩

lemma *dvd-div-mult-self*: $a \ dvd \ b \Longrightarrow (b \ div \ a) * a = b$
⟨*proof*⟩

lemma *dvd-div-mult*: $a \ dvd \ b \Longrightarrow (b \ div \ a) * c = b * c \ div \ a$
⟨*proof*⟩

lemma *div-dvd-div*[*simp*]:
$a \ dvd \ b \Longrightarrow a \ dvd \ c \Longrightarrow (b \ div \ a \ dvd \ c \ div \ a) = (b \ dvd \ c)$

⟨*proof*⟩

lemma *dvd-mod-imp-dvd*: [| *k dvd m mod n*; *k dvd n* |] ==> *k dvd m*
⟨*proof*⟩

Addition respects modular equivalence.

lemma *mod-add-left-eq*: $(a + b)\ mod\ c = (a\ mod\ c + b)\ mod\ c$
⟨*proof*⟩

lemma *mod-add-right-eq*: $(a + b)\ mod\ c = (a + b\ mod\ c)\ mod\ c$
⟨*proof*⟩

lemma *mod-add-eq*: $(a + b)\ mod\ c = (a\ mod\ c + b\ mod\ c)\ mod\ c$
⟨*proof*⟩

lemma *mod-add-cong*:
assumes $a\ mod\ c = a'\ mod\ c$
assumes $b\ mod\ c = b'\ mod\ c$
shows $(a + b)\ mod\ c = (a' + b')\ mod\ c$
⟨*proof*⟩

lemma *div-add*[*simp*]: *z dvd x* ⟹ *z dvd y*
⟹ $(x + y)\ div\ z = x\ div\ z + y\ div\ z$
⟨*proof*⟩

Multiplication respects modular equivalence.

lemma *mod-mult-left-eq*: $(a * b)\ mod\ c = ((a\ mod\ c) * b)\ mod\ c$
⟨*proof*⟩

lemma *mod-mult-right-eq*: $(a * b)\ mod\ c = (a * (b\ mod\ c))\ mod\ c$
⟨*proof*⟩

lemma *mod-mult-eq*: $(a * b)\ mod\ c = ((a\ mod\ c) * (b\ mod\ c))\ mod\ c$
⟨*proof*⟩

lemma *mod-mult-cong*:
assumes $a\ mod\ c = a'\ mod\ c$
assumes $b\ mod\ c = b'\ mod\ c$
shows $(a * b)\ mod\ c = (a' * b')\ mod\ c$
⟨*proof*⟩

lemma *mod-mod-cancel*:
assumes *c dvd b*
shows $a\ mod\ b\ mod\ c = a\ mod\ c$
⟨*proof*⟩

end

lemma *div-mult-div-if-dvd*: $(y::'a::\{semiring\text{-}div, no\text{-}zero\text{-}divisors\})\ dvd\ x \Longrightarrow$

z dvd w $\Longrightarrow$ (*x div y*) * (*w div z*) = (*x* * *w*) *div* (*y* * *z*)
⟨*proof*⟩

lemma *div-power*: (*y*::'*a*::{*semiring-div*,*no-zero-divisors*,*recpower*}) *dvd x* $\Longrightarrow$
(*x div y*) ^*n* = *x*^*n div y*^*n*
⟨*proof*⟩

class *ring-div* = *semiring-div* + *comm-ring-1*
begin

Negation respects modular equivalence.

lemma *mod-minus-eq*: (− *a*) *mod b* = (− (*a mod b*)) *mod b*
⟨*proof*⟩

lemma *mod-minus-cong*:
assumes *a mod b* = *a'* *mod b*
shows (− *a*) *mod b* = (− *a'*) *mod b*
⟨*proof*⟩

Subtraction respects modular equivalence.

lemma *mod-diff-left-eq*: (*a* − *b*) *mod c* = (*a mod c* − *b*) *mod c*
⟨*proof*⟩

lemma *mod-diff-right-eq*: (*a* − *b*) *mod c* = (*a* − *b mod c*) *mod c*
⟨*proof*⟩

lemma *mod-diff-eq*: (*a* − *b*) *mod c* = (*a mod c* − *b mod c*) *mod c*
⟨*proof*⟩

lemma *mod-diff-cong*:
assumes *a mod c* = *a'* *mod c*
assumes *b mod c* = *b'* *mod c*
shows (*a* − *b*) *mod c* = (*a'* − *b'*) *mod c*
⟨*proof*⟩

lemma *dvd-neg-div*: *y dvd x* $\Longrightarrow$ −*x div y* = − (*x div y*)
⟨*proof*⟩

lemma *dvd-div-neg*: *y dvd x* $\Longrightarrow$ *x div* −*y* = − (*x div y*)
⟨*proof*⟩

end

26.3 Division on *nat*

We define *op div* and *op mod* on *nat* by means of a characteristic relation with two input arguments *m*, *n* and two output arguments *q*(uotient) and *r*(emainder).

definition *divmod-rel* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
divmod-rel m n q r ⟷ *m* = *q* ∗ *n* + *r* ∧ (*if n* > *0 then* 0 ≤ *r* ∧ *r* < *n else q* = *0*)

divmod-rel is total:

lemma *divmod-rel-ex*:
obtains *q r* **where** *divmod-rel m n q r*
⟨*proof*⟩

divmod-rel is injective:

lemma *divmod-rel-unique-div*:
assumes *divmod-rel m n q r*
and *divmod-rel m n q′ r′*
shows *q* = *q′*
⟨*proof*⟩

lemma *divmod-rel-unique-mod*:
assumes *divmod-rel m n q r*
and *divmod-rel m n q′ r′*
shows *r* = *r′*
⟨*proof*⟩

We instantiate divisibility on the natural numbers by means of *divmod-rel*:

instantiation *nat* :: *semiring-div*
begin

definition *divmod* :: *nat* ⇒ *nat* ⇒ *nat* × *nat* **where**
[*code del*]: *divmod m n* = (*THE* (*q*, *r*). *divmod-rel m n q r*)

definition *div-nat* **where**
m div n = *fst* (*divmod m n*)

definition *mod-nat* **where**
m mod n = *snd* (*divmod m n*)

lemma *divmod-div-mod*:
divmod m n = (*m div n*, *m mod n*)
⟨*proof*⟩

lemma *divmod-eq*:
assumes *divmod-rel m n q r*
shows *divmod m n* = (*q*, *r*)
⟨*proof*⟩

lemma *div-eq*:
assumes *divmod-rel m n q r*
shows *m div n* = *q*
⟨*proof*⟩

lemma *mod-eq*:
assumes *divmod-rel m n q r*
shows *m mod n = r*
⟨*proof*⟩

lemma *divmod-rel*: *divmod-rel m n (m div n) (m mod n)*
⟨*proof*⟩

lemma *divmod-zero*:
divmod m 0 = (0, m)
⟨*proof*⟩

lemma *divmod-base*:
assumes *m < n*
shows *divmod m n = (0, m)*
⟨*proof*⟩

lemma *divmod-step*:
assumes *0 < n* **and** *n ≤ m*
shows *divmod m n = (Suc ((m − n) div n), (m − n) mod n)*
⟨*proof*⟩

The "recursion" equations for *op div* and *op mod*

lemma *div-less* [*simp*]:
fixes *m n* :: *nat*
assumes *m < n*
shows *m div n = 0*
⟨*proof*⟩

lemma *le-div-geq*:
fixes *m n* :: *nat*
assumes *0 < n* **and** *n ≤ m*
shows *m div n = Suc ((m − n) div n)*
⟨*proof*⟩

lemma *mod-less* [*simp*]:
fixes *m n* :: *nat*
assumes *m < n*
shows *m mod n = m*
⟨*proof*⟩

lemma *le-mod-geq*:
fixes *m n* :: *nat*
assumes *n ≤ m*
shows *m mod n = (m − n) mod n*
⟨*proof*⟩

instance ⟨*proof*⟩

end

Simproc for cancelling *op div* and *op mod*

⟨*ML*⟩

code generator setup

lemma *divmod-if* [*code*]: *divmod m n* = (*if n* = *0* ∨ *m* < *n then* (*0*, *m*) *else*
let (*q*, *r*) = *divmod* (*m* − *n*) *n in* (*Suc q*, *r*))
⟨*proof*⟩

code-modulename *SML*
Divides Nat

code-modulename *OCaml*
Divides Nat

code-modulename *Haskell*
Divides Nat

26.3.1 Quotient

lemma *div-geq*: *0* < *n* ⟹ ¬ *m* < *n* ⟹ *m div n* = *Suc* ((*m* − *n*) *div n*)
⟨*proof*⟩

lemma *div-if*: *0* < *n* ⟹ *m div n* = (*if m* < *n then 0 else Suc* ((*m* − *n*) *div n*))
⟨*proof*⟩

lemma *div-mult-self-is-m* [*simp*]: *0*<*n* ==> (*m*∗*n*) *div n* = (*m*::*nat*)
⟨*proof*⟩

lemma *div-mult-self1-is-m* [*simp*]: *0*<*n* ==> (*n*∗*m*) *div n* = (*m*::*nat*)
⟨*proof*⟩

26.3.2 Remainder

lemma *mod-less-divisor* [*simp*]:
fixes *m n* :: *nat*
assumes *n* > *0*
shows *m mod n* < (*n*::*nat*)
⟨*proof*⟩

lemma *mod-less-eq-dividend* [*simp*]:
fixes *m n* :: *nat*
shows *m mod n* ≤ *m*
⟨*proof*⟩

lemma *mod-geq*: ¬ *m* < (*n*::*nat*) ⟹ *m mod n* = (*m* − *n*) *mod n*
⟨*proof*⟩

lemma *mod-if*: *m mod* (*n*::*nat*) = (*if m* < *n then m else* (*m* − *n*) *mod n*)

⟨*proof*⟩

lemma *mod-1* [*simp*]: *m mod Suc 0 = 0*
⟨*proof*⟩

lemma *mod-mult-distrib*: *(m mod n) * (k::nat) = (m * k) mod (n * k)*
⟨*proof*⟩

lemma *mod-mult-distrib2*: *(k::nat) * (m mod n) = (k*m) mod (k*n)*
⟨*proof*⟩

lemma *mult-div-cancel*: *(n::nat) * (m div n) = m - (m mod n)*
⟨*proof*⟩

lemma *mod-le-divisor*[*simp*]: *0 < n ⟹ m mod n ≤ (n::nat)*
⟨*proof*⟩

26.3.3 Quotient and Remainder

lemma *divmod-rel-mult1-eq*:
[| divmod-rel b c q r; c > 0 |]
*==> divmod-rel (a*b) c (a*q + a*r div c) (a*r mod c)*
⟨*proof*⟩

lemma *div-mult1-eq*: *(a*b) div c = a*(b div c) + a*(b mod c) div (c::nat)*
⟨*proof*⟩

lemma *divmod-rel-add1-eq*:
[| divmod-rel a c aq ar; divmod-rel b c bq br; c > 0 |]
==> divmod-rel (a + b) c (aq + bq + (ar+br) div c) ((ar + br) mod c)
⟨*proof*⟩

lemma *div-add1-eq*:
(a+b) div (c::nat) = a div c + b div c + ((a mod c + b mod c) div c)
⟨*proof*⟩

lemma *mod-lemma*: *[| (0::nat) < c; r < b |] ==> b * (q mod c) + r < b * c*
⟨*proof*⟩

lemma *divmod-rel-mult2-eq*: *[| divmod-rel a b q r; 0 < b; 0 < c |]*
*==> divmod-rel a (b*c) (q div c) (b*(q mod c) + r)*
⟨*proof*⟩

lemma *div-mult2-eq*: *a div (b*c) = (a div b) div (c::nat)*
⟨*proof*⟩

lemma *mod-mult2-eq*: *a mod (b*c) = b*(a div b mod c) + a mod (b::nat)*

⟨*proof*⟩

26.3.4 Cancellation of Common Factors in Division

lemma *div-mult-mult-lemma*:
[| (0::*nat*) < *b*; 0 < *c* |] ==> (*c*∗*a*) *div* (*c*∗*b*) = *a div b*
⟨*proof*⟩

lemma *div-mult-mult1* [*simp*]: (0::*nat*) < *c* ==> (*c*∗*a*) *div* (*c*∗*b*) = *a div b*
⟨*proof*⟩

lemma *div-mult-mult2* [*simp*]: (0::*nat*) < *c* ==> (*a*∗*c*) *div* (*b*∗*c*) = *a div b*
⟨*proof*⟩

26.3.5 Further Facts about Quotient and Remainder

lemma *div-1* [*simp*]: *m div Suc* 0 = *m*
⟨*proof*⟩

lemma *div-le-mono* [*rule-format*]:
∀ *m*::*nat*. *m* ≤ *n* −−> (*m div k*) ≤ (*n div k*)
⟨*proof*⟩

lemma *div-le-mono2*: !!*m*::*nat*. [| 0<*m*; *m*≤*n* |] ==> (*k div n*) ≤ (*k div m*)
⟨*proof*⟩

lemma *div-le-dividend* [*simp*]: *m div n* ≤ (*m*::*nat*)
⟨*proof*⟩

lemma *div-less-dividend* [*rule-format*]:
!!*n*::*nat*. 1<*n* ==> 0 < *m* −−> *m div n* < *m*
⟨*proof*⟩

lemma *nat-div-eq-0* [*simp*]: (*n*::*nat*) > 0 ==> ((*m div n*) = 0) = (*m* < *n*)
⟨*proof*⟩

lemma *nat-div-gt-0* [*simp*]: (*n*::*nat*) > 0 ==> ((*m div n*) > 0) = (*m* >= *n*)
⟨*proof*⟩

declare *div-less-dividend* [*simp*]

A fact for the mutilated chess board

lemma *mod-Suc*: *Suc*(*m*) *mod n* = (*if Suc*(*m mod n*) = *n then* 0 *else Suc*(*m mod n*))
⟨*proof*⟩

26.3.6 The Divides Relation

lemma *dvd-1-left* [*iff*]: *Suc 0 dvd k*
⟨*proof*⟩

lemma *dvd-1-iff-1* [*simp*]: (*m dvd Suc 0*) = (*m = Suc 0*)
⟨*proof*⟩

lemma *nat-dvd-1-iff-1* [*simp*]: *m dvd* (*1*::*nat*) ⟷ *m = 1*
⟨*proof*⟩

lemma *dvd-anti-sym*: [| *m dvd n*; *n dvd m* |] ==> *m* = (*n*::*nat*)
⟨*proof*⟩

op dvd is a partial order

interpretation *dvd*: *order op dvd* λ*n m* :: *nat*. *n dvd m* ∧ ¬ *m dvd n*
⟨*proof*⟩

lemma *nat-dvd-diff* [*simp*]: [| *k dvd m*; *k dvd n* |] ==> *k dvd* (*m*−*n* :: *nat*)
⟨*proof*⟩

lemma *dvd-diffD*: [| *k dvd m*−*n*; *k dvd n*; *n*≤*m* |] ==> *k dvd* (*m*::*nat*)
⟨*proof*⟩

lemma *dvd-diffD1*: [| *k dvd m*−*n*; *k dvd m*; *n*≤*m* |] ==> *k dvd* (*n*::*nat*)
⟨*proof*⟩

lemma *dvd-reduce*: (*k dvd n* + *k*) = (*k dvd* (*n*::*nat*))
⟨*proof*⟩

lemma *dvd-mod*: !!*n*::*nat*. [| *f dvd m*; *f dvd n* |] ==> *f dvd m mod n*
⟨*proof*⟩

lemma *dvd-mod-iff*: *k dvd n* ==> ((*k*::*nat*) *dvd m mod n*) = (*k dvd m*)
⟨*proof*⟩

lemma *dvd-mult-cancel*: !!*k*::*nat*. [| *k*∗*m dvd k*∗*n*; *0*<*k* |] ==> *m dvd n*
⟨*proof*⟩

lemma *dvd-mult-cancel1*: *0*<*m* ==> (*m*∗*n dvd m*) = (*n* = (*1*::*nat*))
⟨*proof*⟩

lemma *dvd-mult-cancel2*: *0*<*m* ==> (*n*∗*m dvd m*) = (*n* = (*1*::*nat*))
⟨*proof*⟩

lemma *dvd-imp-le*: [| *k dvd n*; *0* < *n* |] ==> *k* ≤ (*n*::*nat*)
⟨*proof*⟩

lemma *nat-dvd-not-less*: (*0*::*nat*) < *m* ⟹ *m* < *n* ⟹ ¬ *n dvd m*
⟨*proof*⟩

lemma *dvd-mult-div-cancel*: n *dvd* m ==> $n * (m$ *div* $n) = (m$::*nat*)
⟨*proof*⟩

lemma *nat-zero-less-power-iff* [*simp*]: $(x\hat{}n > 0) = (x > (0$::*nat*) | $n=0)$
⟨*proof*⟩

lemma *power-dvd-imp-le*: $[|i\hat{}m$ *dvd* $i\hat{}n;\ (1$::*nat*) $< i|]$ ==> $m \leq n$
⟨*proof*⟩

lemma *mod-eq-0-iff*: $(m$ *mod* $d = 0) = (\exists\, q$::*nat*. $m = d*q)$
⟨*proof*⟩

lemmas *mod-eq-0D* [*dest*!] = *mod-eq-0-iff* [*THEN iffD1*]

lemma *mod-eqD*: $(m$ *mod* $d = r)$ ==> $\exists\, q$::*nat*. $m = r + q*d$
⟨*proof*⟩

lemma *split-div*:
$P(n$ *div* k :: *nat*) =
$((k = 0 \longrightarrow P\ 0) \wedge (k \neq 0 \longrightarrow (!i.\ !j<k.\ n = k*i + j \longrightarrow P\ i)))$
(**is** *?P* = *?Q* **is** - = (- ∧ (- ⟶ *?R*)))
⟨*proof*⟩

lemma *split-div-lemma*:
assumes $0 < n$
shows $n * q \leq m \wedge m < n * Suc\ q \longleftrightarrow q = ((m$::*nat*) *div* $n)$ (**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

theorem *split-div'*:
$P\ ((m$::*nat*) *div* $n) = ((n = 0 \wedge P\ 0) \vee$
$(\exists\, q.\ (n * q \leq m \wedge m < n * (Suc\ q)) \wedge P\ q))$
⟨*proof*⟩

lemma *split-mod*:
$P(n$ *mod* k :: *nat*) =
$((k = 0 \longrightarrow P\ n) \wedge (k \neq 0 \longrightarrow (!i.\ !j<k.\ n = k*i + j \longrightarrow P\ j)))$
(**is** *?P* = *?Q* **is** - = (- ∧ (- ⟶ *?R*)))
⟨*proof*⟩

theorem *mod-div-equality'*: $(m$::*nat*$)$ *mod* $n = m - (m$ *div* $n) * n$
⟨*proof*⟩

lemma *div-mod-equality'*:
fixes $m\ n$ :: *nat*
shows m *div* $n * n = m - m$ *mod* n
⟨*proof*⟩

26.3.7 An "induction" law for modulus arithmetic.

lemma *mod-induct-0*:
assumes *step*: $\forall i<p.\ P\ i \longrightarrow P\ ((Suc\ i)\ mod\ p)$
and *base*: $P\ i$ **and** i: $i<p$
shows $P\ 0$
⟨*proof*⟩

lemma *mod-induct*:
assumes *step*: $\forall i<p.\ P\ i \longrightarrow P\ ((Suc\ i)\ mod\ p)$
and *base*: $P\ i$ **and** i: $i<p$ **and** j: $j<p$
shows $P\ j$
⟨*proof*⟩

end

27 Plain: Plain HOL

theory *Plain*
imports *Datatype FunDef Record Extraction Divides*
begin

Plain bootstrap of fundamental HOL tools and packages; does not include *Hilbert-Choice*.

⟨*ML*⟩

end

References