

The UNITY Formalism

Sidi Ehmety and Lawrence C. Paulson

April 19, 2009

Contents

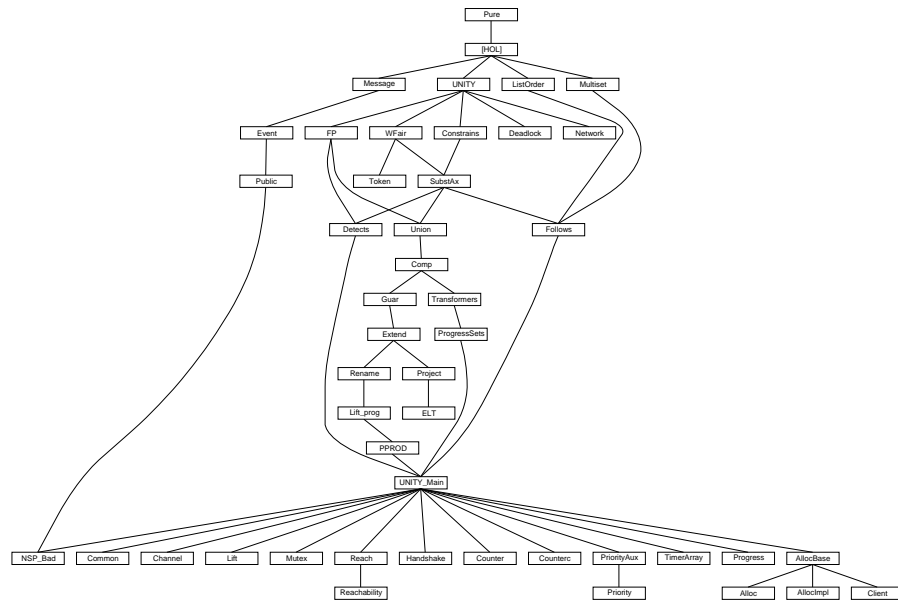
1	The Basic UNITY Theory	7
1.0.1	The abstract type of programs	7
1.0.2	Inspectors for type "program"	8
1.0.3	Equality for UNITY programs	8
1.0.4	co	9
1.0.5	Union	9
1.0.6	Intersection	10
1.0.7	unless	10
1.0.8	stable	10
1.0.9	Union	11
1.0.10	Intersection	11
1.0.11	invariant	12
1.0.12	increasing	12
1.0.13	Theoretical Results from Section 6	12
1.0.14	Ad-hoc set-theory rules	13
1.1	Partial versus Total Transitions	13
1.1.1	Basic properties	13
1.2	Rules for Lazy Definition Expansion	14
1.2.1	Inspectors for type "program"	15
2	Fixed Point of a Program	15
3	Progress	16
3.1	transient	17
3.2	ensures	18
3.3	leadsTo	19
3.4	PSP: Progress-Safety-Progress	23
3.5	Proving the induction rules	24
3.6	wlt	25
3.7	Completion: Binary and General Finite versions	26
4	Weak Safety	28
4.1	traces and reachable	29
4.2	Co	29
4.3	Stable	31
4.4	Increasing	32
4.5	The Elimination Theorem	33

4.6	Specialized laws for handling Always	33
4.7	"Co" rules involving Always	34
4.8	Totalize	35
5	Weak Progress	35
5.1	Specialized laws for handling invariants	36
5.2	Introduction rules: Basis, Trans, Union	36
5.3	Derived rules	37
5.4	PSP: Progress-Safety-Progress	40
5.5	Induction rules	41
5.6	Completion: Binary and General Finite versions	42
6	The Detects Relation	43
7	Unions of Programs	45
7.1	SKIP	45
7.2	SKIP and safety properties	46
7.3	Join	46
7.4	JN	46
7.5	Algebraic laws	47
7.6	Laws Governing \sqcup	47
7.7	Safety: co, stable, FP	48
7.8	Progress: transient, ensures	49
7.9	the ok and OK relations	50
7.10	Allowed	51
7.11	<i>safety_prop</i> , for reasoning about given instances of "ok"	51
8	Composition: Basic Primitives	52
8.1	The component relation	53
8.2	The preserves property	54
9	Guarantees Specifications	57
9.1	Existential Properties	58
9.2	Universal Properties	59
9.3	Guarantees	59
9.4	Distributive Laws. Re-Orient to Perform Miniscoping	60
9.5	Guarantees: Additional Laws (by lcp)	61
9.6	Guarantees Laws for Breaking Down the Program (by lcp)	62
10	Extending State Sets	65
10.1	Restrict	66
10.2	Trivial properties of f, g, h	68
10.3	<i>extend_set</i> : basic properties	68
10.4	<i>project_set</i> : basic properties	69
10.5	More laws	69
10.6	<i>extend_act</i>	70
10.7	extend	71
10.8	Safety: co, stable	73
10.9	Weak safety primitives: Co, Stable	74
10.10	Progress: transient, ensures	76

10.11	Proving the converse takes some doing!	76
10.12	preserves	77
10.13	Guarantees	78
11	Renaming of State Sets	79
11.1	inverse properties	79
11.2	the lattice operations	82
11.3	Strong Safety: co, stable	82
11.4	Weak Safety: Co, Stable	82
11.5	Progress: transient, ensures	83
11.6	"image" versions of the rules, for lifting "guarantees" properties	84
12	Replication of Components	86
12.1	Injectiveness proof	87
12.2	Surjectiveness proof	87
12.3	The Operator <i>lift_set</i>	89
12.4	The Lattice Operations	89
12.5	Safety: constrains, stable, invariant	89
12.6	Progress: transient, ensures	90
12.7	Lemmas to Handle Function Composition (o) More Consistently	92
12.8	More lemmas about extend and project	92
12.9	OK and "lift"	93
13	The Prefix Ordering on Lists	96
13.1	preliminary lemmas	97
13.2	genPrefix is a partial order	98
13.3	recursion equations	99
13.4	The type of lists is partially ordered	101
13.5	pfixLe, pfixGe: properties inherited from the translations	103
14	Multisets	104
14.1	The type of multisets	104
14.2	Algebraic properties	106
14.2.1	Union	106
14.2.2	Difference	107
14.2.3	Count of elements	107
14.2.4	Set of elements	107
14.2.5	Size	108
14.2.6	Equality of multisets	108
14.2.7	Intersection	110
14.2.8	Comprehension (filter)	111
14.3	Induction and case splits	111
14.4	Orderings	113
14.4.1	Well-foundedness	113
14.4.2	Closure-free presentation	115
14.4.3	Partial-order properties	117
14.4.4	Monotonicity of multiset union	118
14.5	Link with lists	119
14.6	Pointwise ordering induced by count	122
14.7	Strong induction and subset induction for multisets	125

14.8 The fold combinator	127
14.9 Image	130
14.10 Termination proofs with multiset orders	131
15 The Follows Relation of Charpentier and Sivilotte	134
15.1 Destruction rules	135
15.2 Union properties (with the subset ordering)	136
15.3 Multiset union properties (with the multiset ordering)	137
16 Predicate Transformers	139
16.1 Defining the Predicate Transformers wp , awp and $wens$	139
16.2 Defining the Weakest Ensures Set	141
16.3 Properties Involving Program Union	142
16.4 The Set $wens_set\ F\ B$ for a Single-Assignment Program	144
17 Progress Sets	148
17.1 Complete Lattices and the Operator $c1$	148
17.2 Progress Sets and the Main Lemma	150
17.3 The Progress Set Union Theorem	153
17.4 Some Progress Sets	153
17.4.1 Lattices and Relations	154
17.4.2 Decoupling Theorems	155
17.5 Composition Theorems Based on Monotonicity and Commutativity	156
17.5.1 Commutativity of $c1\ L$ and assignment.	156
17.5.2 Commutativity of Functions and Relation	157
17.6 Monotonicity	159
18 Comprehensive UNITY Theory	159
19 The Token Ring	164
19.1 Definitions	164
19.2 Progress under Weak Fairness	165
19.3 Progress	171
20 Analyzing the Needham-Schroeder Public-Key Protocol in UNITY	190
20.1 Inductive Proofs about ns_public	191
20.2 Authenticity properties obtained from NS2	192
20.3 Authenticity properties obtained from NS2	194
21 A Family of Similar Counters: Original Version	197
22 A Family of Similar Counters: Version with Compatibility	199
23 The priority system	204
23.1 Component correctness proofs	205
23.2 System properties	206
23.3 The main result: above set decreases	208

24 Progress Set Examples	210
24.1 The Composition of Two Single-Assignment Programs	210
24.1.1 Calculating <i>wens_set</i> <i>FF</i> { <i>k..</i> }	211
24.1.2 Proving <i>FF</i> \in <i>UNIV</i> <i>leadsTo</i> { <i>k..</i> }	211
25 Projections of State Sets	212
25.1 Safety	213
25.2 "projecting" and union/intersection (no converses)	214
25.3 Reachability and project	216
25.4 Converse results for weak safety: benefits of the argument <i>C</i> . . .	216
25.5 A lot of redundant theorems: all are proved to facilitate reasoning about guarantees.	218
25.6 <i>leadsETo</i> in the precondition (??)	219
25.6.1 transient	219
25.6.2 ensures – a primitive combining progress with safety . . .	219
25.7 Towards the theorem <i>project_Ensures_D</i>	221
25.8 Guarantees	222
25.9 guarantees corollaries	223
25.9.1 Some could be deleted: the required versions are easy to prove	223
25.9.2 Guarantees with a <i>leadsTo</i> postcondition	223
26 Progress Under Allowable Sets	224
27 Common Declarations for Chandy and Charpentier's Allocator	235
27.1 State definitions. OUTPUT variables are locals	237
27.1.1 bijectivity of <i>sysOfClient</i>	243
27.1.2 bijectivity of <i>client_map</i>	243
27.2 o-simprules for <i>sysOfAlloc</i> [MUST BE AUTOMATED]	244
27.3 o-simprules for <i>sysOfClient</i> [MUST BE AUTOMATED]	244
27.4 Components Lemmas [MUST BE AUTOMATED]	248
27.5 Proof of the safety property (1)	252
27.6 Proof of the progress property (2)	253
28 Implementation of a multiple-client allocator from a single- client allocator	256
28.1 Theorems for Merge	259
28.2 Theorems for Distributor	260
28.3 Theorems for Allocator	261
29 Distributed Resource Management System: the Client	262



1 The Basic UNITY Theory

theory UNITY imports Main begin

```

typedef (Program)
  'a program = "{(init:: 'a set, acts :: ('a * 'a)set set,
               allowed :: ('a * 'a)set set). Id ∈ acts & Id: allowed}"

by blast

constdefs
  Acts :: "'a program => ('a * 'a)set set"
    "Acts F == %(init, acts, allowed). acts) (Rep_Program F)"

  "constrains" :: "[ 'a set, 'a set] => 'a program set" (infixl "co" 60)
    "A co B == {F. ∀ act ∈ Acts F. act 'A ⊆ B}"

  unless :: "[ 'a set, 'a set] => 'a program set" (infixl "unless" 60)
    "A unless B == (A-B) co (A ∪ B)"

  mk_program :: "('a set * ('a * 'a)set set * ('a * 'a)set set)
               => 'a program"
    "mk_program == %(init, acts, allowed).
                  Abs_Program (init, insert Id acts, insert Id allowed)"

  Init :: "'a program => 'a set"
    "Init F == %(init, acts, allowed). init) (Rep_Program F)"

  AllowedActs :: "'a program => ('a * 'a)set set"
    "AllowedActs F == %(init, acts, allowed). allowed) (Rep_Program F)"

  Allowed :: "'a program => 'a program set"
    "Allowed F == {G. Acts G ⊆ AllowedActs F}"

  stable :: "'a set => 'a program set"
    "stable A == A co A"

  strongest_rhs :: "[ 'a program, 'a set] => 'a set"
    "strongest_rhs F A == Inter {B. F ∈ A co B}"

  invariant :: "'a set => 'a program set"
    "invariant A == {F. Init F ⊆ A} ∩ stable A"

  increasing :: "[ 'a => 'b::order] => 'a program set"
    — Polymorphic in both states and the meaning of ≤
    "increasing f == ⋂ z. stable {s. z ≤ f s}"

```

Perhaps HOL shouldn't add this in the first place!

```
declare image_Collect [simp del]
```

1.0.1 The abstract type of programs

```

lemmas program_typedef =
  Rep_Program Rep_Program_inverse Abs_Program_inverse

```

Program_def Init_def Acts_def AllowedActs_def mk_program_def

```
lemma Id_in_Acts [iff]: "Id ∈ Acts F"
  apply (cut_tac x = F in Rep_Program)
  apply (auto simp add: program_typedef)
  done
```

```
lemma insert_Id_Acts [iff]: "insert Id (Acts F) = Acts F"
  by (simp add: insert_absorb Id_in_Acts)
```

```
lemma Acts_nonempty [simp]: "Acts F ≠ {}"
  by auto
```

```
lemma Id_in_AllowedActs [iff]: "Id ∈ AllowedActs F"
  apply (cut_tac x = F in Rep_Program)
  apply (auto simp add: program_typedef)
  done
```

```
lemma insert_Id_AllowedActs [iff]: "insert Id (AllowedActs F) = AllowedActs F"
  by (simp add: insert_absorb Id_in_AllowedActs)
```

1.0.2 Inspectors for type "program"

```
lemma Init_eq [simp]: "Init (mk_program (init,acts,allowed)) = init"
  by (simp add: program_typedef)
```

```
lemma Acts_eq [simp]: "Acts (mk_program (init,acts,allowed)) = insert Id
  acts"
  by (simp add: program_typedef)
```

```
lemma AllowedActs_eq [simp]:
  "AllowedActs (mk_program (init,acts,allowed)) = insert Id allowed"
  by (simp add: program_typedef)
```

1.0.3 Equality for UNITY programs

```
lemma surjective_mk_program [simp]:
  "mk_program (Init F, Acts F, AllowedActs F) = F"
  apply (cut_tac x = F in Rep_Program)
  apply (auto simp add: program_typedef)
  apply (drule_tac f = Abs_Program in arg_cong)+
  apply (simp add: program_typedef insert_absorb)
  done
```

```
lemma program_equalityI:
  "[| Init F = Init G; Acts F = Acts G; AllowedActs F = AllowedActs G |]

  ==> F = G"
  apply (rule_tac t = F in surjective_mk_program [THEN subst])
  apply (rule_tac t = G in surjective_mk_program [THEN subst], simp)
  done
```

```
lemma program_equalityE:
  "[| F = G;
```



```

    [| Init F = Init G; Acts F = Acts G; AllowedActs F = AllowedActs G
  |]
  ==> P [|] ==> P"
by simp

lemma program_equality_iff:
  "(F=G) =
    (Init F = Init G & Acts F = Acts G & AllowedActs F = AllowedActs G)"
by (blast intro: program_equalityI program_equalityE)

```

1.0.4 co

```

lemma constrainsI:
  "(!act s s'. [| act: Acts F; (s,s') ∈ act; s ∈ A |] ==> s': A')
  ==> F ∈ A co A'"
by (simp add: constrains_def, blast)

```

```

lemma constrainsD:
  "[| F ∈ A co A'; act: Acts F; (s,s'): act; s ∈ A |] ==> s': A'"
by (unfold constrains_def, blast)

```

```

lemma constrains_empty [iff]: "F ∈ {} co B"
by (unfold constrains_def, blast)

```

```

lemma constrains_empty2 [iff]: "(F ∈ A co {}) = (A={})"
by (unfold constrains_def, blast)

```

```

lemma constrains_UNIV [iff]: "(F ∈ UNIV co B) = (B = UNIV)"
by (unfold constrains_def, blast)

```

```

lemma constrains_UNIV2 [iff]: "F ∈ A co UNIV"
by (unfold constrains_def, blast)

```

monotonic in 2nd argument

```

lemma constrains_weaken_R:
  "[| F ∈ A co A'; A' ≤ B' |] ==> F ∈ A co B'"
by (unfold constrains_def, blast)

```

anti-monotonic in 1st argument

```

lemma constrains_weaken_L:
  "[| F ∈ A co A'; B ⊆ A |] ==> F ∈ B co A'"
by (unfold constrains_def, blast)

```

```

lemma constrains_weaken:
  "[| F ∈ A co A'; B ⊆ A; A' ≤ B' |] ==> F ∈ B co B'"
by (unfold constrains_def, blast)

```

1.0.5 Union

```

lemma constrains_Un:
  "[| F ∈ A co A'; F ∈ B co B' |] ==> F ∈ (A ∪ B) co (A' ∪ B')"
by (unfold constrains_def, blast)

```

```

lemma constrains_UN:

```

```

      "(!!i. i ∈ I ==> F ∈ (A i) co (A' i))
      ==> F ∈ (⋃ i ∈ I. A i) co (⋃ i ∈ I. A' i)"
by (unfold constrains_def, blast)

```

```

lemma constrains_Un_distrib: "(A ∪ B) co C = (A co C) ∩ (B co C)"
by (unfold constrains_def, blast)

```

```

lemma constrains_UN_distrib: "(⋃ i ∈ I. A i) co B = (⋂ i ∈ I. A i co B)"
by (unfold constrains_def, blast)

```

```

lemma constrains_Int_distrib: "C co (A ∩ B) = (C co A) ∩ (C co B)"
by (unfold constrains_def, blast)

```

```

lemma constrains_INT_distrib: "A co (⋂ i ∈ I. B i) = (⋂ i ∈ I. A co B i)"
by (unfold constrains_def, blast)

```

1.0.6 Intersection

```

lemma constrains_Int:
  "[| F ∈ A co A'; F ∈ B co B' |] ==> F ∈ (A ∩ B) co (A' ∩ B')"
by (unfold constrains_def, blast)

```

```

lemma constrains_INT:
  "(!!i. i ∈ I ==> F ∈ (A i) co (A' i))
  ==> F ∈ (⋂ i ∈ I. A i) co (⋂ i ∈ I. A' i)"
by (unfold constrains_def, blast)

```

```

lemma constrains_imp_subset: "F ∈ A co A' ==> A ⊆ A'"
by (unfold constrains_def, auto)

```

The reasoning is by subsets since "co" refers to single actions only. So this rule isn't that useful.

```

lemma constrains_trans:
  "[| F ∈ A co B; F ∈ B co C |] ==> F ∈ A co C"
by (unfold constrains_def, blast)

```

```

lemma constrains_cancel:
  "[| F ∈ A co (A' ∪ B); F ∈ B co B' |] ==> F ∈ A co (A' ∪ B')"
by (unfold constrains_def, clarify, blast)

```

1.0.7 unless

```

lemma unlessI: "F ∈ (A-B) co (A ∪ B) ==> F ∈ A unless B"
by (unfold unless_def, assumption)

```

```

lemma unlessD: "F ∈ A unless B ==> F ∈ (A-B) co (A ∪ B)"
by (unfold unless_def, assumption)

```

1.0.8 stable

```

lemma stableI: "F ∈ A co A ==> F ∈ stable A"
by (unfold stable_def, assumption)

```

```

lemma stableD: "F ∈ stable A ==> F ∈ A co A"

```

```
by (unfold stable_def, assumption)
```

```
lemma stable_UNIV [simp]: "stable UNIV = UNIV"
by (unfold stable_def constrains_def, auto)
```

1.0.9 Union

```
lemma stable_Un:
  "[| F ∈ stable A; F ∈ stable A' |] ==> F ∈ stable (A ∪ A')"
```

```
apply (unfold stable_def)
apply (blast intro: constrains_Un)
done
```

```
lemma stable_UN:
  "(!!i. i ∈ I ==> F ∈ stable (A i)) ==> F ∈ stable (⋃ i ∈ I. A i)"
apply (unfold stable_def)
apply (blast intro: constrains_UN)
done
```

```
lemma stable_Union:
  "(!!A. A ∈ X ==> F ∈ stable A) ==> F ∈ stable (⋃ X)"
by (unfold stable_def constrains_def, blast)
```

1.0.10 Intersection

```
lemma stable_Int:
  "[| F ∈ stable A; F ∈ stable A' |] ==> F ∈ stable (A ∩ A')"
```

```
apply (unfold stable_def)
apply (blast intro: constrains_Int)
done
```

```
lemma stable_INT:
  "(!!i. i ∈ I ==> F ∈ stable (A i)) ==> F ∈ stable (⋂ i ∈ I. A i)"
apply (unfold stable_def)
apply (blast intro: constrains_INT)
done
```

```
lemma stable_Inter:
  "(!!A. A ∈ X ==> F ∈ stable A) ==> F ∈ stable (⋂ X)"
by (unfold stable_def constrains_def, blast)
```

```
lemma stable_constrains_Un:
  "[| F ∈ stable C; F ∈ A co (C ∪ A') |] ==> F ∈ (C ∪ A) co (C ∪ A')"
```

```
by (unfold stable_def constrains_def, blast)
```

```
lemma stable_constrains_Int:
  "[| F ∈ stable C; F ∈ (C ∩ A) co A' |] ==> F ∈ (C ∩ A) co (C ∩ A')"
```

```
by (unfold stable_def constrains_def, blast)
```

```
lemmas stable_constrains_stable = stable_constrains_Int[THEN stableI, standard]
```

1.0.11 invariant

```
lemma invariantI: "[| Init F  $\subseteq$  A; F  $\in$  stable A |] ==> F  $\in$  invariant A"
by (simp add: invariant_def)
```

Could also say $\text{invariant } A \cap \text{invariant } B \subseteq \text{invariant } (A \cap B)$

```
lemma invariant_Int:
  "[| F  $\in$  invariant A; F  $\in$  invariant B |] ==> F  $\in$  invariant (A  $\cap$  B)"
by (auto simp add: invariant_def stable_Int)
```

1.0.12 increasing

```
lemma increasingD:
  "F  $\in$  increasing f ==> F  $\in$  stable {s. z  $\subseteq$  f s}"
by (unfold increasing_def, blast)
```

```
lemma increasing_constant [iff]: "F  $\in$  increasing (%s. c)"
by (unfold increasing_def stable_def, auto)
```

```
lemma mono_increasing_o:
  "mono g ==> increasing f  $\subseteq$  increasing (g o f)"
apply (unfold increasing_def stable_def constrains_def, auto)
apply (blast intro: monoD order_trans)
done
```

```
lemma strict_increasingD:
  "!!z::nat. F  $\in$  increasing f ==> F  $\in$  stable {s. z < f s}"
by (simp add: increasing_def Suc_le_eq [symmetric])
```

```
lemma elimination:
  "[|  $\forall m \in M. F \in \{s. s \ x = m\}$  co (B m) |]
  ==> F  $\in \{s. s \ x \in M\}$  co ( $\bigcup_{m \in M. B \ m}$ )"
by (unfold constrains_def, blast)
```

As above, but for the trivial case of a one-variable state, in which the state is identified with its one variable.

```
lemma elimination_sing:
  "( $\forall m \in M. F \in \{m\}$  co (B m)) ==> F  $\in M$  co ( $\bigcup_{m \in M. B \ m}$ )"
by (unfold constrains_def, blast)
```

1.0.13 Theoretical Results from Section 6

```
lemma constrains_strongest_rhs:
  "F  $\in A$  co (strongest_rhs F A)"
by (unfold constrains_def strongest_rhs_def, blast)
```

```
lemma strongest_rhs_is_strongest:
  "F  $\in A$  co B ==> strongest_rhs F A  $\subseteq B$ "
by (unfold constrains_def strongest_rhs_def, blast)
```

1.0.14 Ad-hoc set-theory rules

```
lemma Un_Diff_Diff [simp]: "A  $\cup$  B - (A - B) = B"
by blast
```

```
lemma Int_Union_Union: "Union(B)  $\cap$  A = Union(( $\%C$ . C  $\cap$  A) 'B)"
by blast
```

Needed for WF reasoning in WFair.thy

```
lemma Image_less_than [simp]: "less_than ' ' {k} = greaterThan k"
by blast
```

```
lemma Image_inverse_less_than [simp]: "less_than^-1 ' ' {k} = lessThan k"
by blast
```

1.1 Partial versus Total Transitions

constdefs

```
totalize_act :: "('a * 'a)set => ('a * 'a)set"
"totalize_act act == act  $\cup$  Id_on (-(Domain act))"
```

```
totalize :: "'a program => 'a program"
"totalize F == mk_program (Init F,
                           totalize_act ' Acts F,
                           AllowedActs F)"
```

```
mk_total_program :: "('a set * ('a * 'a)set set * ('a * 'a)set set)
=> 'a program"
"mk_total_program args == totalize (mk_program args)"
```

```
all_total :: "'a program => bool"
"all_total F ==  $\forall$  act  $\in$  Acts F. Domain act = UNIV"
```

```
lemma insert_Id_image_Acts: "f Id = Id ==> insert Id (f 'Acts F) = f ' Acts F"
by (blast intro: sym [THEN image_eqI])
```

1.1.1 Basic properties

```
lemma totalize_act_Id [simp]: "totalize_act Id = Id"
by (simp add: totalize_act_def)
```

```
lemma Domain_totalize_act [simp]: "Domain (totalize_act act) = UNIV"
by (auto simp add: totalize_act_def)
```

```
lemma Init_totalize [simp]: "Init (totalize F) = Init F"
by (unfold totalize_def, auto)
```

```
lemma Acts_totalize [simp]: "Acts (totalize F) = (totalize_act ' Acts F)"
by (simp add: totalize_def insert_Id_image_Acts)
```

```
lemma AllowedActs_totalize [simp]: "AllowedActs (totalize F) = AllowedActs F"
by (simp add: totalize_def)
```

```

lemma totalize_constrains_iff [simp]: "(totalize F ∈ A co B) = (F ∈ A co B)"
by (simp add: totalize_def totalize_act_def constrains_def, blast)

lemma totalize_stable_iff [simp]: "(totalize F ∈ stable A) = (F ∈ stable A)"
by (simp add: stable_def)

lemma totalize_invariant_iff [simp]:
  "(totalize F ∈ invariant A) = (F ∈ invariant A)"
by (simp add: invariant_def)

lemma all_total_totalize: "all_total (totalize F)"
by (simp add: totalize_def all_total_def)

lemma Domain_iff_totalize_act: "(Domain act = UNIV) = (totalize_act act = act)"
by (force simp add: totalize_act_def)

lemma all_total_imp_totalize: "all_total F ==> (totalize F = F)"
apply (simp add: all_total_def totalize_def)
apply (rule program_equalityI)
  apply (simp_all add: Domain_iff_totalize_act image_def)
done

lemma all_total_iff_totalize: "all_total F = (totalize F = F)"
apply (rule iffI)
  apply (erule all_total_imp_totalize)
  apply (erule subst)
  apply (rule all_total_totalize)
done

lemma mk_total_program_constrains_iff [simp]:
  "(mk_total_program args ∈ A co B) = (mk_program args ∈ A co B)"
by (simp add: mk_total_program_def)

```

1.2 Rules for Lazy Definition Expansion

They avoid expanding the full program, which is a large expression

```

lemma def_prg_Init:
  "F == mk_total_program (init,acts,allowed) ==> Init F = init"
by (simp add: mk_total_program_def)

lemma def_prg_Acts:
  "F == mk_total_program (init,acts,allowed)
  ==> Acts F = insert Id (totalize_act ' acts)"
by (simp add: mk_total_program_def)

lemma def_prg_AllowedActs:
  "F == mk_total_program (init,acts,allowed)
  ==> AllowedActs F = insert Id allowed"
by (simp add: mk_total_program_def)

```

An action is expanded if a pair of states is being tested against it

```

lemma def_act_simp:
  "act == {(s,s'). P s s'} ==> ((s,s') ∈ act) = P s s'"
by (simp add: mk_total_program_def)

```

A set is expanded only if an element is being tested against it

```

lemma def_set_simp: "A == B ==> (x ∈ A) = (x ∈ B)"
by (simp add: mk_total_program_def)

```

1.2.1 Inspectors for type "program"

```

lemma Init_total_eq [simp]:
  "Init (mk_total_program (init,acts,allowed)) = init"
by (simp add: mk_total_program_def)

lemma Acts_total_eq [simp]:
  "Acts(mk_total_program(init,acts,allowed)) = insert Id (totalize_act'acts)"
by (simp add: mk_total_program_def)

lemma AllowedActs_total_eq [simp]:
  "AllowedActs (mk_total_program (init,acts,allowed)) = insert Id allowed"
by (auto simp add: mk_total_program_def)

end

```

2 Fixed Point of a Program

```

theory FP imports UNITY begin

```

```

constdefs

```

```

  FP_Orig :: "'a program => 'a set"
  "FP_Orig F == Union{A. ALL B. F : stable (A Int B)}"

  FP :: "'a program => 'a set"
  "FP F == {s. F : stable {s}}

```

```

lemma stable_FP_Orig_Int: "F : stable (FP_Orig F Int B)"
apply (simp only: FP_Orig_def stable_def Int_Union2)
apply (blast intro: constrains_UN)
done

```

```

lemma FP_Orig_weakest:
  "(!!B. F : stable (A Int B)) ==> A <= FP_Orig F"
by (simp add: FP_Orig_def stable_def, blast)

```

```

lemma stable_FP_Int: "F : stable (FP F Int B)"
apply (subgoal_tac "FP F Int B = (UN x:B. FP F Int {x}) ")
prefer 2 apply blast
apply (simp (no_asm_simp) add: Int_insert_right)
apply (simp add: FP_def stable_def)
apply (rule constrains_UN)
apply (simp (no_asm))

```

done

```
lemma FP_equivalence: "FP F = FP_Orig F"
apply (rule equalityI)
  apply (rule stable_FP_Int [THEN FP_Orig_weakest])
apply (simp add: FP_Orig_def FP_def, clarify)
apply (drule_tac x = "{x}" in spec)
apply (simp add: Int_insert_right)
done
```

```
lemma FP_weakest:
  "(!!B. F : stable (A Int B)) ==> A <= FP F"
by (simp add: FP_equivalence FP_Orig_weakest)
```

```
lemma Compl_FP:
  "~(FP F) = (UN act: Acts F. ~{s. act '{s} <= {s}})"
by (simp add: FP_def stable_def constrains_def, blast)
```

```
lemma Diff_FP: "A - (FP F) = (UN act: Acts F. A - {s. act '{s} <= {s}})"
by (simp add: Diff_eq Compl_FP)
```

```
lemma totalize_FP [simp]: "FP (totalize F) = FP F"
by (simp add: FP_def)
```

end

3 Progress

theory *WFair* imports *UNITY* begin

The original version of this theory was based on weak fairness. (Thus, the entire UNITY development embodied this assumption, until February 2003.) Weak fairness states that if a command is enabled continuously, then it is eventually executed. Ernie Cohen suggested that I instead adopt unconditional fairness: every command is executed infinitely often.

In fact, Misra's paper on "Progress" seems to be ambiguous about the correct interpretation, and says that the two forms of fairness are equivalent. They differ only on their treatment of partial transitions, which under unconditional fairness behave magically. That is because if there are partial transitions then there may be no fair executions, making all leads-to properties hold vacuously.

Unconditional fairness has some great advantages. By distinguishing partial transitions from total ones that are the identity on part of their domain, it is more expressive. Also, by simplifying the definition of the transient property, it simplifies many proofs. A drawback is that some laws only hold under the assumption that all transitions are total. The best-known of these is the impossibility law for leads-to.

constdefs

— This definition specifies conditional fairness. The rest of the theory is generic to all forms of fairness. To get weak fairness, conjoin the inclusion below with $A \subseteq \text{Domain } \text{act}$, which specifies that the action is enabled over all of A .


```

transient :: "'a set => 'a program set"
"transient A == {F.  $\exists \text{act} \in \text{Acts } F. \text{act} \subseteq -A$ }"

ensures :: "[ 'a set, 'a set ] => 'a program set"      (infixl "ensures" 60)
"A ensures B == (A-B co A  $\cup$  B)  $\cap$  transient (A-B)"

inductive_set
  leads :: "'a program => ('a set * 'a set) set"
  — LEADS-TO constant for the inductive definition
  for F :: "'a program"
  where

    Basis: "F  $\in$  A ensures B ==> (A,B)  $\in$  leads F"

    | Trans: "[| (A,B)  $\in$  leads F; (B,C)  $\in$  leads F |] ==> (A,C)  $\in$  leads F"

    | Union: " $\forall A \in S. (A,B) \in$  leads F ==> (Union S, B)  $\in$  leads F"

constdefs

  leadsTo :: "[ 'a set, 'a set ] => 'a program set"      (infixl "leadsTo" 60)
  — visible version of the LEADS-TO relation
  "A leadsTo B == {F. (A,B)  $\in$  leads F}"

  wlt :: "[ 'a program, 'a set ] => 'a set"
  — predicate transformer: the largest set that leads to B
  "wlt F B == Union {A. F  $\in$  A leadsTo B}"

syntax (xsymbols)
  "op leadsTo" :: "[ 'a set, 'a set ] => 'a program set" (infixl " $\longrightarrow$ " 60)

```

3.1 transient

```

lemma stable_transient:
  "[| F  $\in$  stable A; F  $\in$  transient A |] ==>  $\exists \text{act} \in \text{Acts } F. A \subseteq -(\text{Domain act})$ "
apply (simp add: stable_def constrains_def transient_def, clarify)
apply (rule rev_bexI, auto)
done

lemma stable_transient_empty:
  "[| F  $\in$  stable A; F  $\in$  transient A; all_total F |] ==> A = {}"
apply (drule stable_transient, assumption)
apply (simp add: all_total_def)
done

lemma transient_strengthen:
  "[| F  $\in$  transient A; B  $\subseteq$  A |] ==> F  $\in$  transient B"
apply (unfold transient_def, clarify)
apply (blast intro!: rev_bexI)
done

```

```

lemma transientI:
  "[| act: Acts F; act 'A  $\subseteq$  -A |] ==> F  $\in$  transient A"
by (unfold transient_def, blast)

```

```

lemma transientE:
  "[| F  $\in$  transient A;
    !!act. [| act: Acts F; act 'A  $\subseteq$  -A |] ==> P |]
  ==> P"
by (unfold transient_def, blast)

```

```

lemma transient_empty [simp]: "transient {} = UNIV"
by (unfold transient_def, auto)

```

This equation recovers the notion of weak fairness. A totaled program satisfies a transient assertion just if the original program contains a suitable action that is also enabled.

```

lemma totalize_transient_iff:
  "(totalize F  $\in$  transient A) = ( $\exists$  act  $\in$  Acts F. A  $\subseteq$  Domain act & act 'A  $\subseteq$ 
  -A)"
apply (simp add: totalize_def totalize_act_def transient_def
  Un_Image Un_subset_iff, safe)
apply (blast intro!: rev_bexI)+
done

```

```

lemma totalize_transientI:
  "[| act: Acts F; A  $\subseteq$  Domain act; act 'A  $\subseteq$  -A |]
  ==> totalize F  $\in$  transient A"
by (simp add: totalize_transient_iff, blast)

```

3.2 ensures

```

lemma ensuresI:
  "[| F  $\in$  (A-B) co (A  $\cup$  B); F  $\in$  transient (A-B) |] ==> F  $\in$  A ensures B"
by (unfold ensures_def, blast)

```

```

lemma ensuresD:
  "F  $\in$  A ensures B ==> F  $\in$  (A-B) co (A  $\cup$  B) & F  $\in$  transient (A-B)"
by (unfold ensures_def, blast)

```

```

lemma ensures_weaken_R:
  "[| F  $\in$  A ensures A'; A'  $\leq$  B' |] ==> F  $\in$  A ensures B'"
apply (unfold ensures_def)
apply (blast intro: constrains_weaken transient_strengthen)
done

```

The L-version (precondition strengthening) fails, but we have this

```

lemma stable_ensures_Int:
  "[| F  $\in$  stable C; F  $\in$  A ensures B |]
  ==> F  $\in$  (C  $\cap$  A) ensures (C  $\cap$  B)"
apply (unfold ensures_def)
apply (auto simp add: ensures_def Int_Un_distrib [symmetric] Diff_Int_distrib
  [symmetric])
prefer 2 apply (blast intro: transient_strengthen)

```

```

apply (blast intro: stable_constrains_Int constrains_weaken)
done

lemma stable_transient_ensures:
  "[| F ∈ stable A; F ∈ transient C; A ⊆ B ∪ C |] ==> F ∈ A ensures
  B"
apply (simp add: ensures_def stable_def)
apply (blast intro: constrains_weaken transient_strengthen)
done

lemma ensures_eq: "(A ensures B) = (A unless B) ∩ transient (A-B)"
by (simp (no_asm) add: ensures_def unless_def)

```

3.3 leadsTo

```

lemma leadsTo_Basis [intro]: "F ∈ A ensures B ==> F ∈ A leadsTo B"
apply (unfold leadsTo_def)
apply (blast intro: leads.Basis)
done

lemma leadsTo_Trans:
  "[| F ∈ A leadsTo B; F ∈ B leadsTo C |] ==> F ∈ A leadsTo C"
apply (unfold leadsTo_def)
apply (blast intro: leads.Trans)
done

lemma leadsTo_Basis':
  "[| F ∈ A co A ∪ B; F ∈ transient A |] ==> F ∈ A leadsTo B"
apply (drule_tac B = "A-B" in constrains_weaken_L)
apply (drule_tac [2] B = "A-B" in transient_strengthen)
apply (rule_tac [3] ensuresI [THEN leadsTo_Basis])
apply (blast+)
done

lemma transient_imp_leadsTo: "F ∈ transient A ==> F ∈ A leadsTo (-A)"
by (simp (no_asm_simp) add: leadsTo_Basis ensuresI Compl_partition)

Useful with cancellation, disjunction

lemma leadsTo_Un_duplicate: "F ∈ A leadsTo (A' ∪ A') ==> F ∈ A leadsTo
A'"
by (simp add: Un_ac)

lemma leadsTo_Un_duplicate2:
  "F ∈ A leadsTo (A' ∪ C ∪ C) ==> F ∈ A leadsTo (A' ∪ C)"
by (simp add: Un_ac)

The Union introduction rule as we should have liked to state it

lemma leadsTo_Union:
  "(!!A. A ∈ S ==> F ∈ A leadsTo B) ==> F ∈ (Union S) leadsTo B"
apply (unfold leadsTo_def)
apply (blast intro: leads.Union)
done

lemma leadsTo_Union_Int:

```

```

"(!A. A ∈ S ==> F ∈ (A ∩ C) leadsTo B) ==> F ∈ (Union S ∩ C) leadsTo B"
apply (unfold leadsTo_def)
apply (simp only: Int_Union_Union)
apply (blast intro: leads.Union)
done

```

```

lemma leadsTo_UN:
  "(!i. i ∈ I ==> F ∈ (A i) leadsTo B) ==> F ∈ (⋃ i ∈ I. A i) leadsTo B"
  apply (subst Union_image_eq [symmetric])
  apply (blast intro: leadsTo_Union)
done

```

Binary union introduction rule

```

lemma leadsTo_Un:
  "[| F ∈ A leadsTo C; F ∈ B leadsTo C |] ==> F ∈ (A ∪ B) leadsTo C"
  apply (subst Un_eq_Union)
  apply (blast intro: leadsTo_Union)
done

```

```

lemma single_leadsTo_I:
  "(!x. x ∈ A ==> F ∈ {x} leadsTo B) ==> F ∈ A leadsTo B"
  by (subst UN_singleton [symmetric], rule leadsTo_UN, blast)

```

The INDUCTION rule as we should have liked to state it

```

lemma leadsTo_induct:
  "[| F ∈ za leadsTo zb;
    !!A B. F ∈ A ensures B ==> P A B;
    !!A B C. [| F ∈ A leadsTo B; P A B; F ∈ B leadsTo C; P B C |]
      ==> P A C;
    !!B S. ∀ A ∈ S. F ∈ A leadsTo B & P A B ==> P (Union S) B
  |] ==> P za zb"
  apply (unfold leadsTo_def)
  apply (drule CollectD, erule leads.induct)
  apply (blast+)
done

```

```

lemma subset_imp_ensures: "A ⊆ B ==> F ∈ A ensures B"
  by (unfold ensures_def constrains_def transient_def, blast)

```

```

lemmas subset_imp_leadsTo = subset_imp_ensures [THEN leadsTo_Basis, standard]

```

```

lemmas leadsTo_refl = subset_refl [THEN subset_imp_leadsTo, standard]

```

```

lemmas empty_leadsTo = empty_subsetI [THEN subset_imp_leadsTo, standard,
simp]

```

```

lemmas leadsTo_UNIV = subset_UNIV [THEN subset_imp_leadsTo, standard, simp]

```

Lemma is the weak version: can't see how to do it in one step

```

lemma leadsTo_induct_pre_lemma:
  "[| F ∈ za leadsTo zb;

```

```

P zb;
!!A B. [| F ∈ A ensures B; P B |] ==> P A;
!!S. ∀ A ∈ S. P A ==> P (Union S)
|] ==> P za"

```

by induction on this formula

```
apply (subgoal_tac "P zb --> P za")
```

now solve first subgoal: this formula is sufficient

```

apply (blast intro: leadsTo_refl)
apply (erule leadsTo_induct)
apply (blast+)
done

```

```
lemma leadsTo_induct_pre:
```

```

"[| F ∈ za leadsTo zb;
P zb;
!!A B. [| F ∈ A ensures B; F ∈ B leadsTo zb; P B |] ==> P A;
!!S. ∀ A ∈ S. F ∈ A leadsTo zb & P A ==> P (Union S)
|] ==> P za"

```

```

apply (subgoal_tac "F ∈ za leadsTo zb & P za")
apply (erule conjunct2)
apply (erule leadsTo_induct_pre_lemma)
prefer 3 apply (blast intro: leadsTo_Union)
prefer 2 apply (blast intro: leadsTo_Trans)
apply (blast intro: leadsTo_refl)
done

```

```
lemma leadsTo_weaken_R: "[| F ∈ A leadsTo A'; A' ≤ B' |] ==> F ∈ A leadsTo B'"

```

```
by (blast intro: subset_imp_leadsTo leadsTo_Trans)
```

```
lemma leadsTo_weaken_L [rule_format]:
```

```
"[| F ∈ A leadsTo A'; B ⊆ A |] ==> F ∈ B leadsTo A'"

```

```
by (blast intro: leadsTo_Trans subset_imp_leadsTo)
```

Distributes over binary unions

```
lemma leadsTo_Un_distrib:
```

```
"F ∈ (A ∪ B) leadsTo C = (F ∈ A leadsTo C & F ∈ B leadsTo C)"

```

```
by (blast intro: leadsTo_Un leadsTo_weaken_L)
```

```
lemma leadsTo_UN_distrib:
```

```
"F ∈ (⋃ i ∈ I. A i) leadsTo B = (∀ i ∈ I. F ∈ (A i) leadsTo B)"

```

```
by (blast intro: leadsTo_UN leadsTo_weaken_L)
```

```
lemma leadsTo_Union_distrib:
```

```
"F ∈ (Union S) leadsTo B = (∀ A ∈ S. F ∈ A leadsTo B)"

```

```
by (blast intro: leadsTo_Union leadsTo_weaken_L)
```

```
lemma leadsTo_weaken:
```

```
"[| F ∈ A leadsTo A'; B ⊆ A; A' ≤ B' |] ==> F ∈ B leadsTo B'"

```

```
by (blast intro: leadsTo_weaken_R leadsTo_weaken_L leadsTo_Trans)
```

Set difference: maybe combine with `leadsTo_weaken_L??`

```
lemma leadsTo_Diff:
  "[| F ∈ (A-B) leadsTo C; F ∈ B leadsTo C |] ==> F ∈ A leadsTo C"
by (blast intro: leadsTo_Un leadsTo_weaken)
```

```
lemma leadsTo_UN_UN:
  "(!! i. i ∈ I ==> F ∈ (A i) leadsTo (A' i))
   ==> F ∈ (⋃ i ∈ I. A i) leadsTo (⋃ i ∈ I. A' i)"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: leadsTo_Union leadsTo_weaken_R)
done
```

Binary union version

```
lemma leadsTo_Un_Un:
  "[| F ∈ A leadsTo A'; F ∈ B leadsTo B' |]
   ==> F ∈ (A ∪ B) leadsTo (A' ∪ B')"
by (blast intro: leadsTo_Un leadsTo_weaken_R)
```

```
lemma leadsTo_cancel2:
  "[| F ∈ A leadsTo (A' ∪ B); F ∈ B leadsTo B' |]
   ==> F ∈ A leadsTo (A' ∪ B')"
by (blast intro: leadsTo_Un_Un subset_imp_leadsTo leadsTo_Trans)
```

```
lemma leadsTo_cancel_Diff2:
  "[| F ∈ A leadsTo (A' ∪ B); F ∈ (B-A') leadsTo B' |]
   ==> F ∈ A leadsTo (A' ∪ B')"
apply (rule leadsTo_cancel2)
prefer 2 apply assumption
apply (simp_all (no_asm_simp))
done
```

```
lemma leadsTo_cancel1:
  "[| F ∈ A leadsTo (B ∪ A'); F ∈ B leadsTo B' |]
   ==> F ∈ A leadsTo (B' ∪ A')"
apply (simp add: Un_commute)
apply (blast intro!: leadsTo_cancel2)
done
```

```
lemma leadsTo_cancel_Diff1:
  "[| F ∈ A leadsTo (B ∪ A'); F ∈ (B-A') leadsTo B' |]
   ==> F ∈ A leadsTo (B' ∪ A')"
apply (rule leadsTo_cancel1)
prefer 2 apply assumption
apply (simp_all (no_asm_simp))
done
```

The impossibility law

```
lemma leadsTo_empty: "[| F ∈ A leadsTo {}; all_total F |] ==> A={}"
apply (erule leadsTo_induct_pre)
apply (simp_all add: ensures_def constrains_def transient_def all_total_def,
clarify)
```

```

apply (drule bspec, assumption)+
apply blast
done

```

3.4 PSP: Progress-Safety-Progress

Special case of PSP: Misra's "stable conjunction"

```

lemma psp_stable:
  "[| F ∈ A leadsTo A'; F ∈ stable B |]
   ==> F ∈ (A ∩ B) leadsTo (A' ∩ B)"
apply (unfold stable_def)
apply (erule leadsTo_induct)
prefer 3 apply (blast intro: leadsTo_Union_Int)
prefer 2 apply (blast intro: leadsTo_Trans)
apply (rule leadsTo_Basis)
apply (simp add: ensures_def Diff_Int_distrib2 [symmetric] Int_Un_distrib2
[symmetric])
apply (blast intro: transient_strengthen constrains_Int)
done

```

```

lemma psp_stable2:
  "[| F ∈ A leadsTo A'; F ∈ stable B |] ==> F ∈ (B ∩ A) leadsTo (B ∩ A')"
by (simp add: psp_stable Int_ac)

```

```

lemma psp Ensures:
  "[| F ∈ A ensures A'; F ∈ B co B' |]
   ==> F ∈ (A ∩ B') ensures ((A' ∩ B) ∪ (B' - B))"
apply (unfold ensures_def constrains_def, clarify)
apply (blast intro: transient_strengthen)
done

```

```

lemma psp:
  "[| F ∈ A leadsTo A'; F ∈ B co B' |]
   ==> F ∈ (A ∩ B') leadsTo ((A' ∩ B) ∪ (B' - B))"
apply (erule leadsTo_induct)
prefer 3 apply (blast intro: leadsTo_Union_Int)

```

Basis case

```

apply (blast intro: psp Ensures)

```

Transitivity case has a delicate argument involving "cancellation"

```

apply (rule leadsTo_Un_duplicate2)
apply (erule leadsTo_cancel_Diff1)
apply (simp add: Int_Diff Diff_triv)
apply (blast intro: leadsTo_weaken_L dest: constrains_imp_subset)
done

```

```

lemma psp2:
  "[| F ∈ A leadsTo A'; F ∈ B co B' |]
   ==> F ∈ (B' ∩ A) leadsTo ((B ∩ A') ∪ (B' - B))"
by (simp (no_asm_simp) add: psp Int_ac)

```

```

lemma psp_unless:

```

```
"[| F ∈ A leadsTo A'; F ∈ B unless B' |]
==> F ∈ (A ∩ B) leadsTo ((A' ∩ B) ∪ B')"
```

```
apply (unfold unless_def)
apply (drule psp, assumption)
apply (blast intro: leadsTo_weaken)
done
```

3.5 Proving the induction rules

```
lemma leadsTo_wf_induct_lemma:
  "[| wf r;
    ∀m. F ∈ (A ∩ f-'{m}) leadsTo
      ((A ∩ f-'(r^-1 '' {m})) ∪ B) |]
  ==> F ∈ (A ∩ f-'{m}) leadsTo B"
apply (erule_tac a = m in wf_induct)
apply (subgoal_tac "F ∈ (A ∩ (f -' (r^-1 '' {x}))) leadsTo B")
  apply (blast intro: leadsTo_cancel1 leadsTo_Un_duplicate)
  apply (subst vimage_eq_UN)
  apply (simp only: UN_simps [symmetric])
  apply (blast intro: leadsTo_UN)
done
```

```
lemma leadsTo_wf_induct:
  "[| wf r;
    ∀m. F ∈ (A ∩ f-'{m}) leadsTo
      ((A ∩ f-'(r^-1 '' {m})) ∪ B) |]
  ==> F ∈ A leadsTo B"
apply (rule_tac t = A in subst)
  defer 1
  apply (rule leadsTo_UN)
  apply (erule leadsTo_wf_induct_lemma)
  apply assumption
  apply fast
done
```

```
lemma bounded_induct:
  "[| wf r;
    ∀m ∈ I. F ∈ (A ∩ f-'{m}) leadsTo
      ((A ∩ f-'(r^-1 '' {m})) ∪ B) |]
  ==> F ∈ A leadsTo ((A - (f-{'I})) ∪ B)"
apply (erule leadsTo_wf_induct, safe)
apply (case_tac "m ∈ I")
  apply (blast intro: leadsTo_weaken)
  apply (blast intro: subset_imp_leadsTo)
done
```

```
lemma lessThan_induct:
  "[| !!m::nat. F ∈ (A ∩ f-'{m}) leadsTo ((A ∩ f-'{..<m}) ∪ B) |]
```



```

    ==> F ∈ A leadsTo B"
  apply (rule wf_less_than [THEN leadsTo_wf_induct])
  apply (simp (no_asm_simp))
  apply blast
done

lemma lessThan_bounded_induct:
  "(!!l::nat. [| ∀m ∈ greaterThan l.
    F ∈ (A ∩ f-'{m}) leadsTo ((A ∩ f-'{lessThan m}) ∪ B) |]
    ==> F ∈ A leadsTo ((A ∩ (f-'{atMost l})) ∪ B))"
  apply (simp only: Diff_eq [symmetric] vimage_Compl Compl_greaterThan [symmetric])
  apply (rule wf_less_than [THEN bounded_induct])
  apply (simp (no_asm_simp))
done

lemma greaterThan_bounded_induct:
  "(!!l::nat. ∀m ∈ lessThan l.
    F ∈ (A ∩ f-'{m}) leadsTo ((A ∩ f-'{greaterThan m}) ∪ B))
    ==> F ∈ A leadsTo ((A ∩ (f-'{atLeast l})) ∪ B)"
  apply (rule_tac f = f and f1 = "%k. l - k"
    in wf_less_than [THEN wf_inv_image, THEN leadsTo_wf_induct])
  apply (simp (no_asm) add: Image_singleton)
  apply clarify
  apply (case_tac "m < l")
  apply (blast intro: leadsTo_weaken_R diff_less_mono2)
  apply (blast intro: not_leE subset_imp_leadsTo)
done

```

3.6 wlt

Misra's property W3

```

lemma wlt_leadsTo: "F ∈ (wlt F B) leadsTo B"
  apply (unfold wlt_def)
  apply (blast intro!: leadsTo_Union)
done

```

```

lemma leadsTo_subset: "F ∈ A leadsTo B ==> A ⊆ wlt F B"
  apply (unfold wlt_def)
  apply (blast intro!: leadsTo_Union)
done

```

Misra's property W2

```

lemma leadsTo_eq_subset_wlt: "F ∈ A leadsTo B = (A ⊆ wlt F B)"
  by (blast intro!: leadsTo_subset wlt_leadsTo [THEN leadsTo_weaken_L])

```

Misra's property W4

```

lemma wlt_increasing: "B ⊆ wlt F B"
  apply (simp (no_asm_simp) add: leadsTo_eq_subset_wlt [symmetric] subset_imp_leadsTo)
done

```

Used in the Trans case below

```

lemma lemma1:

```

```

"[/ B  $\subseteq$  A2;
  F  $\in$  (A1 - B) co (A1  $\cup$  B);
  F  $\in$  (A2 - C) co (A2  $\cup$  C) /]
==> F  $\in$  (A1  $\cup$  A2 - C) co (A1  $\cup$  A2  $\cup$  C)"
by (unfold constrains_def, clarify, blast)

```

Lemma (1,2,3) of Misra's draft book, Chapter 4, "Progress"

```

lemma leadsTo_123:
  "F  $\in$  A leadsTo A'
  ==>  $\exists B. A \subseteq B$  & F  $\in$  B leadsTo A' & F  $\in$  (B-A') co (B  $\cup$  A')"
apply (erule leadsTo_induct)

```

Basis

```

  apply (blast dest: ensuresD)

```

Trans

```

  apply clarify
  apply (rule_tac x = "Ba  $\cup$  Bb" in exI)
  apply (blast intro: lemma1 leadsTo_Un_Un leadsTo_cancel1 leadsTo_Un_duplicate)

```

Union

```

  apply (clarify dest!: ball_conj_distrib [THEN iffD1] bchoice)
  apply (rule_tac x = " $\bigcup A \in S. f A$ " in exI)
  apply (auto intro: leadsTo_UN)

  apply (rule_tac I1=S and A1="%i. f i - B" and A'1="%i. f i  $\cup$  B"
    in constrains_UN [THEN constrains_weaken], auto)
done

```

Misra's property W5

```

lemma wlt_constrains_wlt: "F  $\in$  (wlt F B - B) co (wlt F B)"
proof -
  from wlt_leadsTo [of F B, THEN leadsTo_123]
  show ?thesis
  proof (elim exE conjE)

    fix C
    assume wlt: "wlt F B  $\subseteq$  C"
    and lt: "F  $\in$  C leadsTo B"
    and co: "F  $\in$  C - B co C  $\cup$  B"
    have eq: "C = wlt F B"
    proof -
      from lt and wlt show ?thesis
      by (blast dest: leadsTo_eq_subset_wlt [THEN iffD1])
    qed
    from co show ?thesis by (simp add: eq wlt_increasing Un_absorb2)
  qed
qed

```

3.7 Completion: Binary and General Finite versions

```

lemma completion_lemma :
  "[/ W = wlt F (B'  $\cup$  C);

```

```

      F ∈ A leadsTo (A' ∪ C); F ∈ A' co (A' ∪ C);
      F ∈ B leadsTo (B' ∪ C); F ∈ B' co (B' ∪ C) []
    ==> F ∈ (A ∩ B) leadsTo ((A' ∩ B') ∪ C)"
  apply (subgoal_tac "F ∈ (W-C) co (W ∪ B' ∪ C) ")
  prefer 2
  apply (blast intro: wlt_constrains_wlt [THEN [2] constrains_Un,
                                           THEN constrains_weaken])

  apply (subgoal_tac "F ∈ (W-C) co W")
  prefer 2
  apply (simp add: wlt_increasing Un_assoc Un_absorb2)
  apply (subgoal_tac "F ∈ (A ∩ W - C) leadsTo (A' ∩ W ∪ C) ")
  prefer 2 apply (blast intro: wlt_leadsTo psp [THEN leadsTo_weaken])

  apply (subgoal_tac "F ∈ (A' ∩ W ∪ C) leadsTo (A' ∩ B' ∪ C) ")
  prefer 2
  apply (rule leadsTo_Un_duplicate2)
  apply (blast intro: leadsTo_Un_Un wlt_leadsTo
                  [THEN psp2, THEN leadsTo_weaken] leadsTo_refl)
  apply (drule leadsTo_Diff)
  apply (blast intro: subset_imp_leadsTo)
  apply (subgoal_tac "A ∩ B ⊆ A ∩ W")
  prefer 2
  apply (blast dest!: leadsTo_subset intro!: subset_refl [THEN Int_mono])
  apply (blast intro: leadsTo_Trans subset_imp_leadsTo)
  done

lemmas completion = completion_lemma [OF refl]

lemma finite_completion_lemma:
  "finite I ==> (∀ i ∈ I. F ∈ (A i) leadsTo (A' i ∪ C)) -->
    (∀ i ∈ I. F ∈ (A' i) co (A' i ∪ C)) -->
    F ∈ (⋂ i ∈ I. A i) leadsTo ((⋂ i ∈ I. A' i) ∪ C)"
  apply (erule finite_induct, auto)
  apply (rule completion)
  prefer 4
  apply (simp only: INT_simps [symmetric])
  apply (rule constrains_INT, auto)
  done

lemma finite_completion:
  "[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) leadsTo (A' i ∪ C);
    !!i. i ∈ I ==> F ∈ (A' i) co (A' i ∪ C) |]
  ==> F ∈ (⋂ i ∈ I. A i) leadsTo ((⋂ i ∈ I. A' i) ∪ C)"
  by (blast intro: finite_completion_lemma [THEN mp, THEN mp])

lemma stable_completion:
  "[| F ∈ A leadsTo A'; F ∈ stable A';
    F ∈ B leadsTo B'; F ∈ stable B' |]
  ==> F ∈ (A ∩ B) leadsTo (A' ∩ B' )"
  apply (unfold stable_def)
  apply (rule_tac C1 = "{}" in completion [THEN leadsTo_weaken_R])
  apply (force+)
  done

```

```

lemma finite_stable_completion:
  "[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) leadsTo (A' i);
    !!i. i ∈ I ==> F ∈ stable (A' i) |]
  ==> F ∈ (⋂ i ∈ I. A i) leadsTo (⋂ i ∈ I. A' i)"
apply (unfold stable_def)
apply (rule_tac C1 = "{}" in finite_completion [THEN leadsTo_weaken_R])
apply (simp_all (no_asm_simp))
apply blast+
done

end

```

4 Weak Safety

theory Constrains imports UNITY begin

```

inductive_set
  traces :: "'a set, ('a * 'a)set set] => ('a * 'a list) set"
  for init :: "'a set" and acts :: "('a * 'a)set set"
  where

    Init: "s ∈ init ==> (s,[]) ∈ traces init acts"

    | Acts: "[| act: acts; (s,evs) ∈ traces init acts; (s,s'): act |]
      ==> (s', s#evs) ∈ traces init acts"

inductive_set
  reachable :: "'a program => 'a set"
  for F :: "'a program"
  where

    Init: "s ∈ Init F ==> s ∈ reachable F"

    | Acts: "[| act: Acts F; s ∈ reachable F; (s,s'): act |]
      ==> s' ∈ reachable F"

constdefs
  Constrains :: "'a set, 'a set] => 'a program set" (infixl "Co" 60)
    "A Co B == {F. F ∈ (reachable F ∩ A) co B}"

  Unless :: "'a set, 'a set] => 'a program set" (infixl "Unless" 60)
    "A Unless B == (A-B) Co (A ∪ B)"

  Stable :: "'a set => 'a program set"
    "Stable A == A Co A"

  Always :: "'a set => 'a program set"

```

```
"Always A == {F. Init F  $\subseteq$  A}  $\cap$  Stable A"
```

```
Increasing :: "[a => 'b::{order}] => 'a program set"
"Increasing f ==  $\bigcap$  z. Stable {s. z  $\leq$  f s}"
```

4.1 traces and reachable

```
lemma reachable_equiv_traces:
  "reachable F = {s.  $\exists$  evs. (s, evs)  $\in$  traces (Init F) (Acts F)}"
apply safe
apply (erule_tac [2] traces.induct)
apply (erule reachable.induct)
apply (blast intro: reachable.intros traces.intros)+
done
```

```
lemma Init_subset_reachable: "Init F  $\subseteq$  reachable F"
by (blast intro: reachable.intros)
```

```
lemma stable_reachable [intro!, simp]:
  "Acts G  $\subseteq$  Acts F ==> G  $\in$  stable (reachable F)"
by (blast intro: stableI constrainsI reachable.intros)
```

```
lemma invariant_reachable: "F  $\in$  invariant (reachable F)"
apply (simp add: invariant_def)
apply (blast intro: reachable.intros)
done
```

```
lemma invariant_includes_reachable: "F  $\in$  invariant A ==> reachable F  $\subseteq$  A"
apply (simp add: stable_def constrains_def invariant_def)
apply (rule subsetI)
apply (erule reachable.induct)
apply (blast intro: reachable.intros)+
done
```

4.2 Co

```
lemmas constrains_reachable_Int =
  subset_refl [THEN stable_reachable [unfolded stable_def],
    THEN constrains_Int, standard]
```

```
lemma Constrains_eq_constrains:
  "A Co B = {F. F  $\in$  (reachable F  $\cap$  A) co (reachable F  $\cap$  B)}"
apply (unfold Constrains_def)
apply (blast dest: constrains_reachable_Int intro: constrains_weaken)
done
```

```
lemma constrains_imp_Constrains: "F  $\in$  A co A' ==> F  $\in$  A Co A'"
apply (unfold Constrains_def)
apply (blast intro: constrains_weaken_L)
done
```

```

lemma stable_imp_Stable: "F ∈ stable A ==> F ∈ Stable A"
apply (unfold stable_def Stable_def)
apply (erule constrains_imp_Constrains)
done

lemma ConstrainsI:
  "(!!act s s'. [| act: Acts F; (s,s') ∈ act; s ∈ A |] ==> s': A')
  ==> F ∈ A Co A'"
apply (rule constrains_imp_Constrains)
apply (blast intro: constrainsI)
done

lemma Constrains_empty [iff]: "F ∈ {} Co B"
by (unfold Constrains_def constrains_def, blast)

lemma Constrains_UNIV [iff]: "F ∈ A Co UNIV"
by (blast intro: ConstrainsI)

lemma Constrains_weaken_R:
  "[| F ∈ A Co A'; A' ≤ B' |] ==> F ∈ A Co B'"
apply (unfold Constrains_def)
apply (blast intro: constrains_weaken_R)
done

lemma Constrains_weaken_L:
  "[| F ∈ A Co A'; B ⊆ A |] ==> F ∈ B Co A'"
apply (unfold Constrains_def)
apply (blast intro: constrains_weaken_L)
done

lemma Constrains_weaken:
  "[| F ∈ A Co A'; B ⊆ A; A' ≤ B' |] ==> F ∈ B Co B'"
apply (unfold Constrains_def)
apply (blast intro: constrains_weaken)
done

lemma Constrains_Un:
  "[| F ∈ A Co A'; F ∈ B Co B' |] ==> F ∈ (A ∪ B) Co (A' ∪ B')"
apply (unfold Constrains_def)
apply (blast intro: constrains_Un [THEN constrains_weaken])
done

lemma Constrains_UN:
  assumes Co: "!!i. i ∈ I ==> F ∈ (A i) Co (A' i)"
  shows "F ∈ (⋃ i ∈ I. A i) Co (⋃ i ∈ I. A' i)"
apply (unfold Constrains_def)
apply (rule CollectI)
apply (rule Co [unfolded Constrains_def, THEN CollectD, THEN constrains_UN,
  THEN constrains_weaken], auto)
done

```

```

lemma Constrains_Int:
  "[| F ∈ A Co A'; F ∈ B Co B' |] ==> F ∈ (A ∩ B) Co (A' ∩ B')"
apply (unfold Constrains_def)
apply (blast intro: constrains_Int [THEN constrains_weaken])
done

lemma Constrains_INT:
  assumes Co: "!!i. i ∈ I ==> F ∈ (A i) Co (A' i)"
  shows "F ∈ (⋂ i ∈ I. A i) Co (⋂ i ∈ I. A' i)"
apply (unfold Constrains_def)
apply (rule CollectI)
apply (rule Co [unfolded Constrains_def, THEN CollectD, THEN constrains_INT,
               THEN constrains_weaken], auto)
done

lemma Constrains_imp_subset: "F ∈ A Co A' ==> reachable F ∩ A ⊆ A'"
by (simp add: constrains_imp_subset Constrains_def)

lemma Constrains_trans: "[| F ∈ A Co B; F ∈ B Co C |] ==> F ∈ A Co C"
apply (simp add: Constrains_eq_constrains)
apply (blast intro: constrains_trans constrains_weaken)
done

lemma Constrains_cancel:
  "[| F ∈ A Co (A' ∪ B); F ∈ B Co B' |] ==> F ∈ A Co (A' ∪ B')"
by (simp add: Constrains_eq_constrains constrains_def, blast)

```

4.3 Stable

```

lemma Stable_eq: "[| F ∈ Stable A; A = B |] ==> F ∈ Stable B"
by blast

lemma Stable_eq_stable: "(F ∈ Stable A) = (F ∈ stable (reachable F ∩ A))"
by (simp add: Stable_def Constrains_eq_constrains stable_def)

lemma StableI: "F ∈ A Co A ==> F ∈ Stable A"
by (unfold Stable_def, assumption)

lemma StableD: "F ∈ Stable A ==> F ∈ A Co A"
by (unfold Stable_def, assumption)

lemma Stable_Un:
  "[| F ∈ Stable A; F ∈ Stable A' |] ==> F ∈ Stable (A ∪ A')"
apply (unfold Stable_def)
apply (blast intro: Constrains_Un)
done

lemma Stable_Int:
  "[| F ∈ Stable A; F ∈ Stable A' |] ==> F ∈ Stable (A ∩ A')"
apply (unfold Stable_def)

```

```

apply (blast intro: Constrains_Int)
done

lemma Stable_Constrains_Un:
  "[| F ∈ Stable C; F ∈ A Co (C ∪ A') |]
   ==> F ∈ (C ∪ A) Co (C ∪ A')"
apply (unfold Stable_def)
apply (blast intro: Constrains_Un [THEN Constrains_weaken])
done

lemma Stable_Constrains_Int:
  "[| F ∈ Stable C; F ∈ (C ∩ A) Co A' |]
   ==> F ∈ (C ∩ A) Co (C ∩ A')"
apply (unfold Stable_def)
apply (blast intro: Constrains_Int [THEN Constrains_weaken])
done

lemma Stable_UN:
  "(!!i. i ∈ I ==> F ∈ Stable (A i)) ==> F ∈ Stable (⋃ i ∈ I. A i)"
by (simp add: Stable_def Constrains_UN)

lemma Stable_INT:
  "(!!i. i ∈ I ==> F ∈ Stable (A i)) ==> F ∈ Stable (⋂ i ∈ I. A i)"
by (simp add: Stable_def Constrains_INT)

lemma Stable_reachable: "F ∈ Stable (reachable F)"
by (simp add: Stable_eq_stable)

```

4.4 Increasing

```

lemma IncreasingD:
  "F ∈ Increasing f ==> F ∈ Stable {s. x ≤ f s}"
by (unfold Increasing_def, blast)

lemma mono_Increasing_o:
  "mono g ==> Increasing f ⊆ Increasing (g o f)"
apply (simp add: Increasing_def Stable_def Constrains_def stable_def
               constrains_def)
apply (blast intro: monoD order_trans)
done

lemma strict_IncreasingD:
  "!!z::nat. F ∈ Increasing f ==> F ∈ Stable {s. z < f s}"
by (simp add: Increasing_def Suc_le_eq [symmetric])

lemma increasing_imp_Increasing:
  "F ∈ increasing f ==> F ∈ Increasing f"
apply (unfold increasing_def Increasing_def)
apply (blast intro: stable_imp_Stable)
done

lemmas Increasing_constant =
  increasing_constant [THEN increasing_imp_Increasing, standard, iff]

```


4.5 The Elimination Theorem

```
lemma Elimination:
  "[|  $\forall m. F \in \{s. s\ x = m\} \text{ Co } (B\ m)$  |]
   ==>  $F \in \{s. s\ x \in M\} \text{ Co } (\bigcup m \in M. B\ m)$ "
by (unfold Constrains_def constrains_def, blast)
```

```
lemma Elimination_sing:
  " $(\forall m. F \in \{m\} \text{ Co } (B\ m)) \implies F \in M \text{ Co } (\bigcup m \in M. B\ m)$ "
by (unfold Constrains_def constrains_def, blast)
```

4.6 Specialized laws for handling Always

```
lemma AlwaysI: "[| Init  $F \subseteq A$ ;  $F \in \text{Stable } A$  |] ==>  $F \in \text{Always } A$ "
by (simp add: Always_def)
```

```
lemma AlwaysD: " $F \in \text{Always } A \implies \text{Init } F \subseteq A \ \& \ F \in \text{Stable } A$ "
by (simp add: Always_def)
```

```
lemmas AlwaysE = AlwaysD [THEN conjE, standard]
lemmas Always_imp_Stable = AlwaysD [THEN conjunct2, standard]
```

```
lemma Always_includes_reachable: " $F \in \text{Always } A \implies \text{reachable } F \subseteq A$ "
apply (simp add: Stable_def Constrains_def constrains_def Always_def)
apply (rule subsetI)
apply (erule reachable.induct)
apply (blast intro: reachable.intros)+
done
```

```
lemma invariant_imp_Always:
  " $F \in \text{invariant } A \implies F \in \text{Always } A$ "
apply (unfold Always_def invariant_def Stable_def stable_def)
apply (blast intro: constrains_imp_Constrains)
done
```

```
lemmas Always_reachable =
  invariant_reachable [THEN invariant_imp_Always, standard]
```

```
lemma Always_eq_invariant_reachable:
  " $\text{Always } A = \{F. F \in \text{invariant } (\text{reachable } F \cap A)\}$ "
apply (simp add: Always_def invariant_def Stable_def Constrains_eq_constrains
  stable_def)
apply (blast intro: reachable.intros)
done
```

```
lemma Always_eq_includes_reachable: " $\text{Always } A = \{F. \text{reachable } F \subseteq A\}$ "
by (auto dest: invariant_includes_reachable simp add: Int_absorb2 invariant_reachable
  Always_eq_invariant_reachable)
```

```
lemma Always_UNIV_eq [simp]: " $\text{Always UNIV} = \text{UNIV}$ "
by (auto simp add: Always_eq_includes_reachable)
```

```

lemma UNIV_AlwaysI: "UNIV  $\subseteq$  A ==> F  $\in$  Always A"
by (auto simp add: Always_eq_includes_reachable)

lemma Always_eq_UN_invariant: "Always A = ( $\bigcup$  I  $\in$  Pow A. invariant I)"
apply (simp add: Always_eq_includes_reachable)
apply (blast intro: invariantI Init_subset_reachable [THEN subsetD]
             invariant_includes_reachable [THEN subsetD])
done

lemma Always_weaken: "[| F  $\in$  Always A; A  $\subseteq$  B |] ==> F  $\in$  Always B"
by (auto simp add: Always_eq_includes_reachable)

```

4.7 "Co" rules involving Always

```

lemma Always_Constrains_pre:
  "F  $\in$  Always INV ==> (F  $\in$  (INV  $\cap$  A) Co A') = (F  $\in$  A Co A')"
by (simp add: Always_includes_reachable [THEN Int_absorb2] Constrains_def
      Int_assoc [symmetric])

lemma Always_Constrains_post:
  "F  $\in$  Always INV ==> (F  $\in$  A Co (INV  $\cap$  A')) = (F  $\in$  A Co A')"
by (simp add: Always_includes_reachable [THEN Int_absorb2]
      Constrains_eq_constrains Int_assoc [symmetric])

lemmas Always_ConstrainsI = Always_Constrains_pre [THEN iffD1, standard]

lemmas Always_ConstrainsD = Always_Constrains_post [THEN iffD2, standard]

lemma Always_Constrains_weaken:
  "[| F  $\in$  Always C; F  $\in$  A Co A';
    C  $\cap$  B  $\subseteq$  A; C  $\cap$  A'  $\subseteq$  B' |]
  ==> F  $\in$  B Co B'"
apply (rule Always_ConstrainsI, assumption)
apply (drule Always_ConstrainsD, assumption)
apply (blast intro: Constrains_weaken)
done

lemma Always_Int_distrib: "Always (A  $\cap$  B) = Always A  $\cap$  Always B"
by (auto simp add: Always_eq_includes_reachable)

lemma Always_INT_distrib: "Always (INTER I A) = ( $\bigcap$  i  $\in$  I. Always (A i))"
by (auto simp add: Always_eq_includes_reachable)

lemma Always_Int_I:
  "[| F  $\in$  Always A; F  $\in$  Always B |] ==> F  $\in$  Always (A  $\cap$  B)"
by (simp add: Always_Int_distrib)

```

```

lemma Always_Compl_Un_eq:
  "F ∈ Always A ==> (F ∈ Always (¬A ∪ B)) = (F ∈ Always B)"
by (auto simp add: Always_eq_includes_reachable)

```

```

lemmas Always_thin = thin_rl [of "F ∈ Always A", standard]

```

4.8 Totalize

```

lemma reachable_imp_reachable_tot:
  "s ∈ reachable F ==> s ∈ reachable (totalize F)"
apply (erule reachable.induct)
  apply (rule reachable.Init)
  apply simp
apply (rule_tac act = "totalize_act act" in reachable.Acts)
apply (auto simp add: totalize_act_def)
done

lemma reachable_tot_imp_reachable:
  "s ∈ reachable (totalize F) ==> s ∈ reachable F"
apply (erule reachable.induct)
  apply (rule reachable.Init, simp)
apply (force simp add: totalize_act_def intro: reachable.Acts)
done

lemma reachable_tot_eq [simp]: "reachable (totalize F) = reachable F"
by (blast intro: reachable_imp_reachable_tot reachable_tot_imp_reachable)

lemma totalize_Constrains_iff [simp]: "(totalize F ∈ A Co B) = (F ∈ A Co B)"
by (simp add: Constrains_def)

lemma totalize_Stable_iff [simp]: "(totalize F ∈ Stable A) = (F ∈ Stable A)"
by (simp add: Stable_def)

lemma totalize_Always_iff [simp]: "(totalize F ∈ Always A) = (F ∈ Always A)"
by (simp add: Always_def)

end

```

5 Weak Progress

```

theory SubstAx imports WFair Constrains begin

```

```

constdefs
  Ensures :: "[ 'a set, 'a set ] => 'a program set"    (infixl "Ensures" 60)
  "A Ensures B == {F. F ∈ (reachable F ∩ A) ensures B}"

```

```

    LeadsTo :: "[ 'a set, 'a set ] => 'a program set" (infixl "LeadsTo" 60)
    "A LeadsTo B == {F. F ∈ (reachable F ∩ A) leadsTo B}"

syntax (xsymbols)
  "op LeadsTo" :: "[ 'a set, 'a set ] => 'a program set" (infixl "  $\longrightarrow_w$  " 60)

```

Resembles the previous definition of LeadsTo

```

lemma LeadsTo_eq_leadsTo:
  "A LeadsTo B = {F. F ∈ (reachable F ∩ A) leadsTo (reachable F ∩ B)}"
apply (unfold LeadsTo_def)
apply (blast dest: psp_stable2 intro: leadsTo_weaken)
done

```

5.1 Specialized laws for handling invariants

```

lemma Always_LeadsTo_pre:
  "F ∈ Always INV ==> (F ∈ (INV ∩ A) LeadsTo A') = (F ∈ A LeadsTo A')"
by (simp add: LeadsTo_def Always_eq_includes_reachable Int_absorb2
    Int_assoc [symmetric])

```

```

lemma Always_LeadsTo_post:
  "F ∈ Always INV ==> (F ∈ A LeadsTo (INV ∩ A')) = (F ∈ A LeadsTo A')"
by (simp add: LeadsTo_eq_leadsTo Always_eq_includes_reachable Int_absorb2
    Int_assoc [symmetric])

```

```

lemmas Always_LeadsToI = Always_LeadsTo_pre [THEN iffD1, standard]

```

```

lemmas Always_LeadsToD = Always_LeadsTo_post [THEN iffD2, standard]

```

5.2 Introduction rules: Basis, Trans, Union

```

lemma leadsTo_imp_LeadsTo: "F ∈ A leadsTo B ==> F ∈ A LeadsTo B"
apply (simp add: LeadsTo_def)
apply (blast intro: leadsTo_weaken_L)
done

```

```

lemma LeadsTo_Trans:
  "[| F ∈ A LeadsTo B; F ∈ B LeadsTo C |] ==> F ∈ A LeadsTo C"
apply (simp add: LeadsTo_eq_leadsTo)
apply (blast intro: leadsTo_Trans)
done

```

```

lemma LeadsTo_Union:
  "(!!A. A ∈ S ==> F ∈ A LeadsTo B) ==> F ∈ (Union S) LeadsTo B"
apply (simp add: LeadsTo_def)
apply (subst Int_Union)
apply (blast intro: leadsTo_UN)
done

```

5.3 Derived rules

```
lemma LeadsTo_UNIV [simp]: "F ∈ A LeadsTo UNIV"
by (simp add: LeadsTo_def)
```

Useful with cancellation, disjunction

```
lemma LeadsTo_Un_duplicate:
  "F ∈ A LeadsTo (A' ∪ A') ==> F ∈ A LeadsTo A'"
by (simp add: Un_ac)
```

```
lemma LeadsTo_Un_duplicate2:
  "F ∈ A LeadsTo (A' ∪ C ∪ C) ==> F ∈ A LeadsTo (A' ∪ C)"
by (simp add: Un_ac)
```

```
lemma LeadsTo_UN:
  "(!!i. i ∈ I ==> F ∈ (A i) LeadsTo B) ==> F ∈ (⋃ i ∈ I. A i) LeadsTo B"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: LeadsTo_Union)
done
```

Binary union introduction rule

```
lemma LeadsTo_Un:
  "[| F ∈ A LeadsTo C; F ∈ B LeadsTo C |] ==> F ∈ (A ∪ B) LeadsTo C"
apply (subst Un_eq_Union)
apply (blast intro: LeadsTo_Union)
done
```

Lets us look at the starting state

```
lemma single_LeadsTo_I:
  "(!!s. s ∈ A ==> F ∈ {s} LeadsTo B) ==> F ∈ A LeadsTo B"
by (subst UN_singleton [symmetric], rule LeadsTo_UN, blast)
```

```
lemma subset_imp_LeadsTo: "A ⊆ B ==> F ∈ A LeadsTo B"
apply (simp add: LeadsTo_def)
apply (blast intro: subset_imp_leadsTo)
done
```

```
lemmas empty_LeadsTo = empty_subsetI [THEN subset_imp_LeadsTo, standard, simp]
```

```
lemma LeadsTo_weaken_R [rule_format]:
  "[| F ∈ A LeadsTo A'; A' ⊆ B' |] ==> F ∈ A LeadsTo B'"
apply (simp add: LeadsTo_def)
apply (blast intro: leadsTo_weaken_R)
done
```

```
lemma LeadsTo_weaken_L [rule_format]:
  "[| F ∈ A LeadsTo A'; B ⊆ A |]
  ==> F ∈ B LeadsTo A'"
apply (simp add: LeadsTo_def)
apply (blast intro: leadsTo_weaken_L)
done
```

```

lemma LeadsTo_weaken:
  "[| F ∈ A LeadsTo A';
    B ⊆ A;  A' ⊆ B' |]"
  ==> F ∈ B LeadsTo B'"
by (blast intro: LeadsTo_weaken_R LeadsTo_weaken_L LeadsTo_Trans)

lemma Always_LeadsTo_weaken:
  "[| F ∈ Always C;  F ∈ A LeadsTo A';
    C ∩ B ⊆ A;  C ∩ A' ⊆ B' |]"
  ==> F ∈ B LeadsTo B'"
by (blast dest: Always_LeadsToI intro: LeadsTo_weaken intro: Always_LeadsToD)

lemma LeadsTo_Un_post: "F ∈ A LeadsTo B ==> F ∈ (A ∪ B) LeadsTo B"
by (blast intro: LeadsTo_Un subset_imp_LeadsTo)

lemma LeadsTo_Trans_Un:
  "[| F ∈ A LeadsTo B;  F ∈ B LeadsTo C |]"
  ==> F ∈ (A ∪ B) LeadsTo C"
by (blast intro: LeadsTo_Un subset_imp_LeadsTo LeadsTo_weaken_L LeadsTo_Trans)

lemma LeadsTo_Un_distrib:
  "(F ∈ (A ∪ B) LeadsTo C) = (F ∈ A LeadsTo C & F ∈ B LeadsTo C)"
by (blast intro: LeadsTo_Un LeadsTo_weaken_L)

lemma LeadsTo_UN_distrib:
  "(F ∈ (⋃ i ∈ I. A i) LeadsTo B) = (∀ i ∈ I. F ∈ (A i) LeadsTo B)"
by (blast intro: LeadsTo_UN LeadsTo_weaken_L)

lemma LeadsTo_Union_distrib:
  "(F ∈ (Union S) LeadsTo B) = (∀ A ∈ S. F ∈ A LeadsTo B)"
by (blast intro: LeadsTo_Union LeadsTo_weaken_L)

lemma LeadsTo_Basis: "F ∈ A Ensures B ==> F ∈ A LeadsTo B"
by (simp add: Ensures_def LeadsTo_def leadsTo_Basis)

lemma EnsuresI:
  "[| F ∈ (A-B) Co (A ∪ B);  F ∈ transient (A-B) |]"
  ==> F ∈ A Ensures B"
apply (simp add: Ensures_def Constrains_eq_constrains)
apply (blast intro: ensuresI constrains_weaken transient_strengthen)
done

lemma Always_LeadsTo_Basis:
  "[| F ∈ Always INV;
    F ∈ (INV ∩ (A-A')) Co (A ∪ A');
    F ∈ transient (INV ∩ (A-A')) |]"

```

```

    ==> F ∈ A LeadsTo A'"
  apply (rule Always_LeadsToI, assumption)
  apply (blast intro: EnsuresI LeadsTo_Basis Always_ConstrainsD [THEN Constrains_weaken]
    transient_strengthen)
done

```

Set difference: maybe combine with `leadsTo_weaken_L`?? This is the most useful form of the "disjunction" rule

```

lemma LeadsTo_Diff:
  "[| F ∈ (A-B) LeadsTo C; F ∈ (A ∩ B) LeadsTo C |]
   ==> F ∈ A LeadsTo C"
by (blast intro: LeadsTo_Un LeadsTo_weaken)

```

```

lemma LeadsTo_UN_UN:
  "(!! i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i))
   ==> F ∈ (⋃ i ∈ I. A i) LeadsTo (⋃ i ∈ I. A' i)"
  apply (simp only: Union_image_eq [symmetric])
  apply (blast intro: LeadsTo_Union LeadsTo_weaken_R)
done

```

Version with no index set

```

lemma LeadsTo_UN_UN_noindex:
  "(!! i. F ∈ (A i) LeadsTo (A' i)) ==> F ∈ (⋃ i. A i) LeadsTo (⋃ i. A' i)"
by (blast intro: LeadsTo_UN_UN)

```

Version with no index set

```

lemma all_LeadsTo_UN_UN:
  "∀ i. F ∈ (A i) LeadsTo (A' i)
   ==> F ∈ (⋃ i. A i) LeadsTo (⋃ i. A' i)"
by (blast intro: LeadsTo_UN_UN)

```

Binary union version

```

lemma LeadsTo_Un_Un:
  "[| F ∈ A LeadsTo A'; F ∈ B LeadsTo B' |]
   ==> F ∈ (A ∪ B) LeadsTo (A' ∪ B')"
by (blast intro: LeadsTo_Un LeadsTo_weaken_R)

```

```

lemma LeadsTo_cancel2:
  "[| F ∈ A LeadsTo (A' ∪ B); F ∈ B LeadsTo B' |]
   ==> F ∈ A LeadsTo (A' ∪ B')"
by (blast intro: LeadsTo_Un_Un subset_imp_LeadsTo LeadsTo_Trans)

```

```

lemma LeadsTo_cancel_Diff2:
  "[| F ∈ A LeadsTo (A' ∪ B); F ∈ (B-A') LeadsTo B' |]
   ==> F ∈ A LeadsTo (A' ∪ B')"
  apply (rule LeadsTo_cancel2)
  prefer 2 apply assumption
  apply (simp_all (no_asm_simp))

```

done

```
lemma LeadsTo_cancel1:
  "[| F ∈ A LeadsTo (B ∪ A'); F ∈ B LeadsTo B' |]
   ==> F ∈ A LeadsTo (B' ∪ A')"
```

apply (simp add: Un_commute)

apply (blast intro!: LeadsTo_cancel2)

done

```
lemma LeadsTo_cancel_Diff1:
  "[| F ∈ A LeadsTo (B ∪ A'); F ∈ (B-A') LeadsTo B' |]
   ==> F ∈ A LeadsTo (B' ∪ A')"
```

apply (rule LeadsTo_cancel1)

prefer 2 apply assumption

apply (simp_all (no_asm_simp))

done

The impossibility law

The set "A" may be non-empty, but it contains no reachable states

```
lemma LeadsTo_empty: "[|F ∈ A LeadsTo {}; all_total F|] ==> F ∈ Always (-A)"
```

apply (simp add: LeadsTo_def Always_eq_includes_reachable)

apply (drule leadsTo_empty, auto)

done

5.4 PSP: Progress-Safety-Progress

Special case of PSP: Misra's "stable conjunction"

```
lemma PSP_Stable:
  "[| F ∈ A LeadsTo A'; F ∈ Stable B |]
   ==> F ∈ (A ∩ B) LeadsTo (A' ∩ B)"
```

apply (simp add: LeadsTo_eq_leadsTo Stable_eq_stable)

apply (drule psp_stable, assumption)

apply (simp add: Int_ac)

done

```
lemma PSP_Stable2:
  "[| F ∈ A LeadsTo A'; F ∈ Stable B |]
   ==> F ∈ (B ∩ A) LeadsTo (B ∩ A')"
```

by (simp add: PSP_Stable Int_ac)

```
lemma PSP:
  "[| F ∈ A LeadsTo A'; F ∈ B Co B' |]
   ==> F ∈ (A ∩ B') LeadsTo ((A' ∩ B) ∪ (B' - B))"
```

apply (simp add: LeadsTo_def Constrains_eq_constrains)

apply (blast dest: psp intro: leadsTo_weaken)

done

```
lemma PSP2:
  "[| F ∈ A LeadsTo A'; F ∈ B Co B' |]
   ==> F ∈ (B' ∩ A) LeadsTo ((B ∩ A') ∪ (B' - B))"
```

by (simp add: PSP Int_ac)


```

lemma PSP_Unless:
  "[| F ∈ A LeadsTo A'; F ∈ B Unless B' |]
   ==> F ∈ (A ∩ B) LeadsTo ((A' ∩ B) ∪ B')]"
apply (unfold Unless_def)
apply (drule PSP, assumption)
apply (blast intro: LeadsTo_Diff LeadsTo_weaken subset_imp_LeadsTo)
done

lemma Stable_transient_Always_LeadsTo:
  "[| F ∈ Stable A; F ∈ transient C;
     F ∈ Always (¬A ∪ B ∪ C) |] ==> F ∈ A LeadsTo B"
apply (erule Always_LeadsTo_weaken)
apply (rule LeadsTo_Diff)
  prefer 2
  apply (erule
    transient_imp_leadsTo [THEN leadsTo_imp_LeadsTo, THEN PSP_Stable2])
  apply (blast intro: subset_imp_LeadsTo)+
done

```

5.5 Induction rules

```

lemma LeadsTo_wf_induct:
  "[| wf r;
     ∀m. F ∈ (A ∩ f-'{m}) LeadsTo
              ((A ∩ f-'(r^-1 '' {m})) ∪ B) |]
   ==> F ∈ A LeadsTo B"
apply (simp add: LeadsTo_eq_leadsTo)
apply (erule leadsTo_wf_induct)
apply (blast intro: leadsTo_weaken)
done

```

```

lemma Bounded_induct:
  "[| wf r;
     ∀m ∈ I. F ∈ (A ∩ f-'{m}) LeadsTo
                  ((A ∩ f-'(r^-1 '' {m})) ∪ B) |]
   ==> F ∈ A LeadsTo ((A - (f-`I)) ∪ B)"
apply (erule LeadsTo_wf_induct, safe)
apply (case_tac "m ∈ I")
apply (blast intro: LeadsTo_weaken)
apply (blast intro: subset_imp_LeadsTo)
done

```

```

lemma LessThan_induct:
  "(!!m::nat. F ∈ (A ∩ f-'{m}) LeadsTo ((A ∩ f-'(lessThan m)) ∪ B))
   ==> F ∈ A LeadsTo B"
by (rule wf_less_than [THEN LeadsTo_wf_induct], auto)

```

Integer version. Could generalize from 0 to any lower bound

```

lemma integ_0_le_induct:
  "[| F ∈ Always {s. (0::int) ≤ f s};
     !! z. F ∈ (A ∩ {s. f s = z}) LeadsTo

```

```

      ((A ∩ {s. f s < z}) ∪ B) ∩]
    ==> F ∈ A LeadsTo B"
  apply (rule_tac f = "nat o f" in LessThan_induct)
  apply (simp add: vimage_def)
  apply (rule Always_LeadsTo_weaken, assumption+)
  apply (auto simp add: nat_eq_iff nat_less_iff)
done

lemma LessThan_bounded_induct:
  "!!l::nat. ∀m ∈ greaterThan l.
    F ∈ (A ∩ f-'{m}) LeadsTo ((A ∩ f-'(lessThan m)) ∪ B)
    ==> F ∈ A LeadsTo ((A ∩ (f-'(atMost l))) ∪ B)"
  apply (simp only: Diff_eq [symmetric] vimage_Compl
    Compl_greaterThan [symmetric])
  apply (rule wf_less_than [THEN Bounded_induct], simp)
done

lemma GreaterThan_bounded_induct:
  "!!l::nat. ∀m ∈ lessThan l.
    F ∈ (A ∩ f-'{m}) LeadsTo ((A ∩ f-'(greaterThan m)) ∪ B)
    ==> F ∈ A LeadsTo ((A ∩ (f-'(atLeast l))) ∪ B)"
  apply (rule_tac f = f and f1 = "%k. l - k"
    in wf_less_than [THEN wf_inv_image, THEN LeadsTo_wf_induct])
  apply (simp add: Image_singleton, clarify)
  apply (case_tac "m < l")
  apply (blast intro: LeadsTo_weaken_R diff_less_mono2)
  apply (blast intro: not_leE subset_imp_LeadsTo)
done

```

5.6 Completion: Binary and General Finite versions

```

lemma Completion:
  "[| F ∈ A LeadsTo (A' ∪ C); F ∈ A' Co (A' ∪ C);
    F ∈ B LeadsTo (B' ∪ C); F ∈ B' Co (B' ∪ C) |]
  ==> F ∈ (A ∩ B) LeadsTo ((A' ∩ B') ∪ C)"
  apply (simp add: LeadsTo_eq_leadsTo Constrains_eq_constrains Int_Un_distrib)
  apply (blast intro: completion leadsTo_weaken)
done

lemma Finite_completion_lemma:
  "finite I
  ==> (∀i ∈ I. F ∈ (A i) LeadsTo (A' i ∪ C)) -->
    (∀i ∈ I. F ∈ (A' i) Co (A' i ∪ C)) -->
    F ∈ (⋂i ∈ I. A i) LeadsTo ((⋂i ∈ I. A' i) ∪ C)"
  apply (erule finite_induct, auto)
  apply (rule Completion)
  prefer 4
  apply (simp only: INT_simps [symmetric])
  apply (rule Constrains_INT, auto)
done

lemma Finite_completion:
  "[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i ∪ C);

```

```

      !!i. i ∈ I ==> F ∈ (A' i) Co (A' i ∪ C) []
    ==> F ∈ (⋂ i ∈ I. A i) LeadsTo ((⋂ i ∈ I. A' i) ∪ C)"
  by (blast intro: Finite_completion_lemma [THEN mp, THEN mp])

lemma Stable_completion:
  "[| F ∈ A LeadsTo A'; F ∈ Stable A';
    F ∈ B LeadsTo B'; F ∈ Stable B' |]
  ==> F ∈ (A ∩ B) LeadsTo (A' ∩ B')"
  apply (unfold Stable_def)
  apply (rule_tac C1 = "{}" in Completion [THEN LeadsTo_weaken_R])
  apply (force+)
  done

lemma Finite_stable_completion:
  "[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i);
    !!i. i ∈ I ==> F ∈ Stable (A' i) |]
  ==> F ∈ (⋂ i ∈ I. A i) LeadsTo (⋂ i ∈ I. A' i)"
  apply (unfold Stable_def)
  apply (rule_tac C1 = "{}" in Finite_completion [THEN LeadsTo_weaken_R])
  apply (simp_all, blast+)
  done

end

```

6 The Detects Relation

theory Detects imports FP SubstAx begin

consts

```

  op_Detects :: "[ 'a set, 'a set ] => 'a program set" (infixl "Detects" 60)
  op_Equality :: "[ 'a set, 'a set ] => 'a set" (infixl "<==>" 60)

```

defs

```

  Detects_def: "A Detects B == (Always (¬A ∪ B)) ∩ (B LeadsTo A)"
  Equality_def: "A <==> B == (¬A ∪ B) ∩ (A ∪ ¬B)"

```

lemma Always_at_FP:

```

  "[| F ∈ A LeadsTo B; all_total F |] ==> F ∈ Always (¬((FP F) ∩ A ∩ ¬B))"
  apply (rule LeadsTo_empty)
  apply (subgoal_tac "F ∈ (FP F ∩ A ∩ ¬B) LeadsTo (B ∩ (FP F ∩ ¬B))")
  apply (subgoal_tac [2] " (FP F ∩ A ∩ ¬B) = (A ∩ (FP F ∩ ¬B))")
  apply (subgoal_tac "(B ∩ (FP F ∩ ¬B)) = {}")
  apply auto
  apply (blast intro: PSP_Stable stable_imp_Stable stable_FP_Int)
  done

```

lemma Detects_Trans:

```

      "[| F ∈ A Detects B; F ∈ B Detects C |] ==> F ∈ A Detects C"
    apply (unfold Detects_def Int_def)
    apply (simp (no_asm))
    apply safe
    apply (rule_tac [2] LeadsTo_Trans, auto)
    apply (subgoal_tac "F ∈ Always ((-A ∪ B) ∩ (-B ∪ C))")
      apply (blast intro: Always_weaken)
    apply (simp add: Always_Int_distrib)
  done

lemma Detects_refl: "F ∈ A Detects A"
  apply (unfold Detects_def)
  apply (simp (no_asm) add: Un_commute Compl_partition subset_imp_LeadsTo)
done

lemma Detects_eq_Un: "(A<==>B) = (A ∩ B) ∪ (-A ∩ -B)"
  by (unfold Equality_def, blast)

lemma Detects_antisym:
  "[| F ∈ A Detects B; F ∈ B Detects A |] ==> F ∈ Always (A <==> B)"
  apply (unfold Detects_def Equality_def)
  apply (simp add: Always_Int_I Un_commute)
done

lemma Detects_Always:
  "[| F ∈ A Detects B; all_total F |] ==> F ∈ Always (-(FP F) ∪ (A <==> B))"
  apply (unfold Detects_def Equality_def)
  apply (simp add: Un_Int_distrib Always_Int_distrib)
  apply (blast dest: Always_at_FP intro: Always_weaken)
done

lemma Detects_Imp_LeadstoEQ:
  "F ∈ A Detects B ==> F ∈ UNIV LeadsTo (A <==> B)"
  apply (unfold Detects_def Equality_def)
  apply (rule_tac B = B in LeadsTo_Diff)
  apply (blast intro: Always_LeadsToI subset_imp_LeadsTo)
  apply (blast intro: Always_LeadsTo_weaken)
done

end

```

7 Unions of Programs

theory Union imports SubstAx FP begin

constdefs

```

ok :: "[ 'a program, 'a program ] => bool"      (infixl "ok" 65)
  "F ok G == Acts F  $\subseteq$  AllowedActs G &
    Acts G  $\subseteq$  AllowedActs F"

OK :: "[ 'a set, 'a => 'b program ] => bool"
  "OK I F == ( $\forall i \in I. \forall j \in I - \{i\}. Acts (F i) \subseteq AllowedActs (F j)$ )"

JOIN :: "[ 'a set, 'a => 'b program ] => 'b program"
  "JOIN I F == mk_program ( $\bigcap i \in I. Init (F i), \bigcup i \in I. Acts (F i),$ 
     $\bigcap i \in I. AllowedActs (F i)$ )"

Join :: "[ 'a program, 'a program ] => 'a program"      (infixl "Join" 65)
  "F Join G == mk_program (Init F  $\cap$  Init G, Acts F  $\cup$  Acts G,
    AllowedActs F  $\cap$  AllowedActs G)"

SKIP :: "'a program"
  "SKIP == mk_program (UNIV, {}, UNIV)"

safety_prop :: "'a program set => bool"
  "safety_prop X == SKIP: X & ( $\forall G. Acts G \subseteq UNION X Acts \rightarrow G \in X$ )"

```

syntax

```

"@JOIN1"      :: "[pttrns, 'b set] => 'b set"      ("(3JN _./ _)" 10)
"@JOIN"       :: "[pttrn, 'a set, 'b set] => 'b set" ("(3JN _:_./ _)" 10)

```

translations

```

"JN x : A. B"  == "JOIN A (%x. B)"
"JN x y. B"    == "JN x. JN y. B"
"JN x. B"      == "JOIN CONST UNIV (%x. B)"

```

syntax (xsymbols)

```

SKIP      :: "'a program"      ("⊥")
Join      :: "[ 'a program, 'a program ] => 'a program" (infixl "⊔" 65)
"@JOIN1"  :: "[pttrns, 'b set] => 'b set"      ("(3⊔ _./ _)" 10)
"@JOIN"   :: "[pttrn, 'a set, 'b set] => 'b set" ("(3⊔ _:_./ _)" 10)

```

7.1 SKIP

```

lemma Init_SKIP [simp]: "Init SKIP = UNIV"
by (simp add: SKIP_def)

```

```

lemma Acts_SKIP [simp]: "Acts SKIP = {Id}"
by (simp add: SKIP_def)

```

```

lemma AllowedActs_SKIP [simp]: "AllowedActs SKIP = UNIV"

```

by (auto simp add: SKIP_def)

lemma reachable_SKIP [simp]: "reachable SKIP = UNIV"
by (force elim: reachable.induct intro: reachable.intros)

7.2 SKIP and safety properties

lemma SKIP_in_constrains_iff [iff]: "(SKIP \in A co B) = (A \subseteq B)"
by (unfold constrains_def, auto)

lemma SKIP_in_Constrains_iff [iff]: "(SKIP \in A Co B) = (A \subseteq B)"
by (unfold Constrains_def, auto)

lemma SKIP_in_stable [iff]: "SKIP \in stable A"
by (unfold stable_def, auto)

declare SKIP_in_stable [THEN stable_imp_Stable, iff]

7.3 Join

lemma Init_Join [simp]: "Init (F \sqcup G) = Init F \cap Init G"
by (simp add: Join_def)

lemma Acts_Join [simp]: "Acts (F \sqcup G) = Acts F \cup Acts G"
by (auto simp add: Join_def)

lemma AllowedActs_Join [simp]:
"AllowedActs (F \sqcup G) = AllowedActs F \cap AllowedActs G"
by (auto simp add: Join_def)

7.4 JN

lemma JN_empty [simp]: "($\bigsqcup_{i \in \{ \}$ }. F i) = SKIP"
by (unfold JOIN_def SKIP_def, auto)

lemma JN_insert [simp]: "($\bigsqcup_{i \in \text{insert } a \text{ } I$ }. F i) = (F a) \sqcup ($\bigsqcup_{i \in I$ }. F i)"
apply (rule program_equalityI)
apply (auto simp add: JOIN_def Join_def)
done

lemma Init_JN [simp]: "Init ($\bigsqcup_{i \in I$ }. F i) = ($\bigcap_{i \in I$ }. Init (F i))"
by (simp add: JOIN_def)

lemma Acts_JN [simp]: "Acts ($\bigsqcup_{i \in I$ }. F i) = insert Id ($\bigcup_{i \in I$ }. Acts (F i))"
by (auto simp add: JOIN_def)

lemma AllowedActs_JN [simp]:
"AllowedActs ($\bigsqcup_{i \in I$ }. F i) = ($\bigcap_{i \in I$ }. AllowedActs (F i))"
by (auto simp add: JOIN_def)

lemma JN_cong [cong]:

```

"[/ I=J; !!i. i ∈ J ==> F i = G i /] ==> (⋈ i ∈ I. F i) = (⋈ i ∈ J.
G i)"
by (simp add: JOIN_def)

```

7.5 Algebraic laws

```

lemma Join_commute: "F⋈G = G⋈F"
by (simp add: Join_def Un_commute Int_commute)

```

```

lemma Join_assoc: "(F⋈G)⋈H = F⋈(G⋈H)"
by (simp add: Un_ac Join_def Int_assoc insert_absorb)

```

```

lemma Join_left_commute: "A⋈(B⋈C) = B⋈(A⋈C)"
by (simp add: Un_ac Int_ac Join_def insert_absorb)

```

```

lemma Join_SKIP_left [simp]: "SKIP⋈F = F"
apply (unfold Join_def SKIP_def)
apply (rule program_equalityI)
apply (simp_all (no_asm) add: insert_absorb)
done

```

```

lemma Join_SKIP_right [simp]: "F⋈SKIP = F"
apply (unfold Join_def SKIP_def)
apply (rule program_equalityI)
apply (simp_all (no_asm) add: insert_absorb)
done

```

```

lemma Join_absorb [simp]: "F⋈F = F"
apply (unfold Join_def)
apply (rule program_equalityI, auto)
done

```

```

lemma Join_left_absorb: "F⋈(F⋈G) = F⋈G"
apply (unfold Join_def)
apply (rule program_equalityI, auto)
done

```

```

lemmas Join_ac = Join_assoc Join_left_absorb Join_commute Join_left_commute

```

7.6 Laws Governing \sqcup

```

lemma JN_absorb: "k ∈ I ==> F k⋈(⋈ i ∈ I. F i) = (⋈ i ∈ I. F i)"
by (auto intro!: program_equalityI)

```

```

lemma JN_Un: "(⋈ i ∈ I ∪ J. F i) = ((⋈ i ∈ I. F i)⋈(⋈ i ∈ J. F i))"
by (auto intro!: program_equalityI)

```

```

lemma JN_constant: "(⋈ i ∈ I. c) = (if I={} then SKIP else c)"
by (rule program_equalityI, auto)

```

```

lemma JN_Join_distrib:
  "(⋈ i ∈ I. F i⋈G i) = (⋈ i ∈ I. F i) ⋈ (⋈ i ∈ I. G i)"
by (auto intro!: program_equalityI)

```

```

lemma JN_Join_miniscope:
  "i ∈ I ==> (⋒ i ∈ I. F i ⊔ G) = ((⋒ i ∈ I. F i) ⊔ G)"
by (auto simp add: JN_Join_distrib JN_constant)

```

```

lemma JN_Join_diff: "i ∈ I ==> F i ⊔ JOIN (I - {i}) F = JOIN I F"
apply (unfold JOIN_def Join_def)
apply (rule program_equalityI, auto)
done

```

7.7 Safety: co, stable, FP

```

lemma JN_constrains:
  "i ∈ I ==> (⋒ i ∈ I. F i) ∈ A co B = (∀ i ∈ I. F i ∈ A co B)"
by (simp add: constrains_def JOIN_def, blast)

```

```

lemma Join_constrains [simp]:
  "(F ⊔ G) ∈ A co B = (F ∈ A co B & G ∈ A co B)"
by (auto simp add: constrains_def Join_def)

```

```

lemma Join_unless [simp]:
  "(F ⊔ G) ∈ A unless B = (F ∈ A unless B & G ∈ A unless B)"
by (simp add: Join_constrains unless_def)

```

```

lemma Join_constrains_weaken:
  "[| F ∈ A co A'; G ∈ B co B' |]
   ==> F ⊔ G ∈ (A ∩ B) co (A' ∪ B')"
by (simp, blast intro: constrains_weaken)

```

```

lemma JN_constrains_weaken:
  "[| ∀ i ∈ I. F i ∈ A i co A' i; i ∈ I |]
   ==> (⋒ i ∈ I. F i) ∈ (⋒ i ∈ I. A i) co (⋒ i ∈ I. A' i)"
apply (simp (no_asm_simp) add: JN_constrains)
apply (blast intro: constrains_weaken)
done

```

```

lemma JN_stable: "(⋒ i ∈ I. F i) ∈ stable A = (∀ i ∈ I. F i ∈ stable A)"
by (simp add: stable_def constrains_def JOIN_def)

```

```

lemma invariant_JN_I:
  "[| !!i. i ∈ I ==> F i ∈ invariant A; i ∈ I |]
   ==> (⋒ i ∈ I. F i) ∈ invariant A"
by (simp add: invariant_def JN_stable, blast)

```

```

lemma Join_stable [simp]:
  "(F ⊔ G) ∈ stable A =
   (F ∈ stable A & G ∈ stable A)"
by (simp add: stable_def)

```



```

lemma Join_increasing [simp]:
  "(F⊔G ∈ increasing f) =
   (F ∈ increasing f & G ∈ increasing f)"
by (simp add: increasing_def Join_stable, blast)

lemma invariant_JoinI:
  "[| F ∈ invariant A; G ∈ invariant A |]
   ==> F⊔G ∈ invariant A"
by (simp add: invariant_def, blast)

lemma FP_JN: "FP (⊔ i ∈ I. F i) = (⊓ i ∈ I. FP (F i))"
by (simp add: FP_def JN_stable INTER_def)

```

7.8 Progress: transient, ensures

```

lemma JN_transient:
  "i ∈ I ==>
   (⊔ i ∈ I. F i) ∈ transient A = (∃ i ∈ I. F i ∈ transient A)"
by (auto simp add: transient_def JOIN_def)

lemma Join_transient [simp]:
  "F⊔G ∈ transient A =
   (F ∈ transient A | G ∈ transient A)"
by (auto simp add: bex_Un transient_def Join_def)

lemma Join_transient_I1: "F ∈ transient A ==> F⊔G ∈ transient A"
by (simp add: Join_transient)

lemma Join_transient_I2: "G ∈ transient A ==> F⊔G ∈ transient A"
by (simp add: Join_transient)

lemma JN Ensures:
  "i ∈ I ==>
   (⊔ i ∈ I. F i) ∈ A ensures B =
   ((∀ i ∈ I. F i ∈ (A-B) co (A ∪ B)) & (∃ i ∈ I. F i ∈ A ensures B))"
by (auto simp add: ensures_def JN_constrains JN_transient)

lemma Join Ensures:
  "F⊔G ∈ A ensures B =
   (F ∈ (A-B) co (A ∪ B) & G ∈ (A-B) co (A ∪ B) &
    (F ∈ transient (A-B) | G ∈ transient (A-B)))"
by (auto simp add: ensures_def Join_transient)

lemma stable_Join_constrains:
  "[| F ∈ stable A; G ∈ A co A' |]
   ==> F⊔G ∈ A co A'"
apply (unfold stable_def constrains_def Join_def)
apply (simp add: ball_Un, blast)
done

lemma stable_Join_Always1:
  "[| F ∈ stable A; G ∈ invariant A |] ==> F⊔G ∈ Always A"

```

```

apply (simp (no_asm_use) add: Always_def invariant_def Stable_eq_stable)
apply (force intro: stable_Int)
done

```

```

lemma stable_Join_Always2:
  "[| F ∈ invariant A; G ∈ stable A |] ==> F ⊔ G ∈ Always A"
apply (subst Join_commute)
apply (blast intro: stable_Join_Always1)
done

```

```

lemma stable_Join_ensures1:
  "[| F ∈ stable A; G ∈ A ensures B |] ==> F ⊔ G ∈ A ensures B"
apply (simp (no_asm_simp) add: Join_ensures)
apply (simp add: stable_def ensures_def)
apply (erule constrains_weaken, auto)
done

```

```

lemma stable_Join_ensures2:
  "[| F ∈ A ensures B; G ∈ stable A |] ==> F ⊔ G ∈ A ensures B"
apply (subst Join_commute)
apply (blast intro: stable_Join_ensures1)
done

```

7.9 the ok and OK relations

```

lemma ok_SKIP1 [iff]: "SKIP ok F"
by (simp add: ok_def)

```

```

lemma ok_SKIP2 [iff]: "F ok SKIP"
by (simp add: ok_def)

```

```

lemma ok_Join_commute:
  "(F ok G & (F ⊔ G) ok H) = (G ok H & F ok (G ⊔ H))"
by (auto simp add: ok_def)

```

```

lemma ok_commute: "(F ok G) = (G ok F)"
by (auto simp add: ok_def)

```

```

lemmas ok_sym = ok_commute [THEN iffD1, standard]

```

```

lemma ok_iff_OK:
  "OK {(0::int,F),(1,G),(2,H)} snd = (F ok G & (F ⊔ G) ok H)"
apply (simp add: Ball_def conj_disj_distribR ok_def Join_def OK_def insert_absorb
  all_conj_distrib)
apply blast
done

```

```

lemma ok_Join_iff1 [iff]: "F ok (G ⊔ H) = (F ok G & F ok H)"
by (auto simp add: ok_def)

```

```

lemma ok_Join_iff2 [iff]: "(G ⊔ H) ok F = (G ok F & H ok F)"
by (auto simp add: ok_def)

```

```
lemma ok_Join_commute_I: "[| F ok G; (F ⊔ G) ok H |] ==> F ok (G ⊔ H)"
by (auto simp add: ok_def)
```

```
lemma ok_JN_iff1 [iff]: "F ok (JOIN I G) = (∀ i ∈ I. F ok G i)"
by (auto simp add: ok_def)
```

```
lemma ok_JN_iff2 [iff]: "(JOIN I G) ok F = (∀ i ∈ I. G i ok F)"
by (auto simp add: ok_def)
```

```
lemma OK_iff_ok: "OK I F = (∀ i ∈ I. ∀ j ∈ I - {i}. (F i) ok (F j))"
by (auto simp add: ok_def OK_def)
```

```
lemma OK_imp_ok: "[| OK I F; i ∈ I; j ∈ I; i ≠ j |] ==> (F i) ok (F j)"
by (auto simp add: OK_iff_ok)
```

7.10 Allowed

```
lemma Allowed_SKIP [simp]: "Allowed SKIP = UNIV"
by (auto simp add: Allowed_def)
```

```
lemma Allowed_Join [simp]: "Allowed (F ⊔ G) = Allowed F ∩ Allowed G"
by (auto simp add: Allowed_def)
```

```
lemma Allowed_JN [simp]: "Allowed (JOIN I F) = (⋂ i ∈ I. Allowed (F i))"
by (auto simp add: Allowed_def)
```

```
lemma ok_iff_Allowed: "F ok G = (F ∈ Allowed G & G ∈ Allowed F)"
by (simp add: ok_def Allowed_def)
```

```
lemma OK_iff_Allowed: "OK I F = (∀ i ∈ I. ∀ j ∈ I - {i}. F i ∈ Allowed(F j))"
by (auto simp add: OK_iff_ok ok_iff_Allowed)
```

7.11 safety_prop, for reasoning about given instances of "ok"

```
lemma safety_prop_Acts_iff:
  "safety_prop X ==> (Acts G ⊆ insert Id (UNION X Acts)) = (G ∈ X)"
by (auto simp add: safety_prop_def)
```

```
lemma safety_prop_AllowedActs_iff_Allowed:
  "safety_prop X ==> (UNION X Acts ⊆ AllowedActs F) = (X ⊆ Allowed F)"
by (auto simp add: Allowed_def safety_prop_Acts_iff [symmetric])
```

```
lemma Allowed_eq:
  "safety_prop X ==> Allowed (mk_program (init, acts, UNION X Acts)) = X"
by (simp add: Allowed_def safety_prop_Acts_iff)
```

```
lemma safety_prop_constrains [iff]: "safety_prop (A co B) = (A ⊆ B)"
by (simp add: safety_prop_def constrains_def, blast)
```

```
lemma safety_prop_stable [iff]: "safety_prop (stable A)"
by (simp add: stable_def)
```

```

lemma safety_prop_Int [simp]:
  "[| safety_prop X; safety_prop Y |] ==> safety_prop (X  $\cap$  Y)"
by (simp add: safety_prop_def, blast)

lemma safety_prop_INTER1 [simp]:
  "(!i. safety_prop (X i)) ==> safety_prop ( $\bigcap$  i. X i)"
by (auto simp add: safety_prop_def, blast)

lemma safety_prop_INTER [simp]:
  "(!i. i  $\in$  I ==> safety_prop (X i)) ==> safety_prop ( $\bigcap$  i  $\in$  I. X i)"
by (auto simp add: safety_prop_def, blast)

lemma def_prg_Allowed:
  "[| F == mk_program (init, acts, UNION X Acts) ; safety_prop X |]
   ==> Allowed F = X"
by (simp add: Allowed_eq)

lemma Allowed_totalize [simp]: "Allowed (totalize F) = Allowed F"
by (simp add: Allowed_def)

lemma def_total_prg_Allowed:
  "[| F == mk_total_program (init, acts, UNION X Acts) ; safety_prop X |]
   ==> Allowed F = X"
by (simp add: mk_total_program_def def_prg_Allowed)

lemma def_UNION_ok_iff:
  "[| F == mk_program(init,acts,UNION X Acts); safety_prop X |]
   ==> F ok G = (G  $\in$  X & acts  $\subseteq$  AllowedActs G)"
by (auto simp add: ok_def safety_prop_Acts_iff)

The union of two total programs is total.

lemma totalize_Join: "totalize F  $\sqcup$  totalize G = totalize (F  $\sqcup$  G)"
by (simp add: program_equalityI totalize_def Join_def image_Un)

lemma all_total_Join: "[|all_total F; all_total G|] ==> all_total (F  $\sqcup$  G)"
by (simp add: all_total_def, blast)

lemma totalize_JN: "( $\bigcup$  i  $\in$  I. totalize (F i)) = totalize( $\bigcup$  i  $\in$  I. F i)"
by (simp add: program_equalityI totalize_def JOIN_def image_UN)

lemma all_total_JN: "(!i. i  $\in$  I ==> all_total (F i)) ==> all_total( $\bigcup$  i  $\in$  I. F i)"
by (simp add: all_total_iff_totalize totalize_JN [symmetric])

end

```

8 Composition: Basic Primitives

theory *Comp* imports *Union* begin

```
instance program :: (type) ord ..
```

```
defs
```

```
  component_def:          "F ≤ H == ∃ G. F ⊔ G = H"
  strict_component_def:    "(F < (H :: 'a program)) == (F ≤ H & F ≠ H)"
```

```
constdefs
```

```
  component_of :: "'a program => 'a program => bool"
                    (infixl "component'_of" 50)
  "F component_of H == ∃ G. F ok G & F ⊔ G = H"

  strict_component_of :: "'a program => 'a program => bool"
                    (infixl "strict'_component'_of" 50)
  "F strict_component_of H == F component_of H & F ≠ H"

  preserves :: "('a => 'b) => 'a program set"
  "preserves v == ⋂ z. stable {s. v s = z}"

  localize :: "('a => 'b) => 'a program => 'a program"
  "localize v F == mk_program (Init F, Acts F,
    AllowedActs F ∩ (⋃ G ∈ preserves v. Acts G))"

  funPair      :: "['a => 'b, 'a => 'c, 'a] => 'b * 'c"
  "funPair f g == %x. (f x, g x)"
```

8.1 The component relation

```
lemma componentI: "H ≤ F | H ≤ G ==> H ≤ (F ⊔ G)"
```

```
apply (unfold component_def, auto)
apply (rule_tac x = "G ⊔ Ga" in exI)
apply (rule_tac [2] x = "G ⊔ F" in exI)
apply (auto simp add: Join_ac)
done
```

```
lemma component_eq_subset:
```

```
  "(F ≤ G) =
    (Init G ⊆ Init F & Acts F ⊆ Acts G & AllowedActs G ⊆ AllowedActs F)"
  apply (unfold component_def)
  apply (force intro!: exI program_equalityI)
  done
```

```
lemma component_SKIP [iff]: "SKIP ≤ F"
```

```
apply (unfold component_def)
apply (force intro: Join_SKIP_left)
done
```

```
lemma component_refl [iff]: "F ≤ (F :: 'a program)"
```

```
apply (unfold component_def)
apply (blast intro: Join_SKIP_right)
done
```

```
lemma SKIP_minimal: "F ≤ SKIP ==> F = SKIP"
```

```
by (auto intro!: program_equalityI simp add: component_eq_subset)
```

```

lemma component_Join1: "F ≤ (F⊔G)"
by (unfold component_def, blast)

lemma component_Join2: "G ≤ (F⊔G)"
apply (unfold component_def)
apply (simp add: Join_commute, blast)
done

lemma Join_absorb1: "F ≤ G ==> F⊔G = G"
by (auto simp add: component_def Join_left_absorb)

lemma Join_absorb2: "G ≤ F ==> F⊔G = F"
by (auto simp add: Join_ac component_def)

lemma JN_component_iff: "((JOIN I F) ≤ H) = (∀ i ∈ I. F i ≤ H)"
by (simp add: component_eq_subset, blast)

lemma component_JN: "i ∈ I ==> (F i) ≤ (⊔ i ∈ I. (F i))"
apply (unfold component_def)
apply (blast intro: JN_absorb)
done

lemma component_trans: "[| F ≤ G; G ≤ H |] ==> F ≤ (H :: 'a program)"
apply (unfold component_def)
apply (blast intro: Join_assoc [symmetric])
done

lemma component_antisym: "[| F ≤ G; G ≤ F |] ==> F = (G :: 'a program)"
apply (simp (no_asm_use) add: component_eq_subset)
apply (blast intro!: program_equalityI)
done

lemma Join_component_iff: "((F⊔G) ≤ H) = (F ≤ H & G ≤ H)"
by (simp add: component_eq_subset, blast)

lemma component_constrains: "[| F ≤ G; G ∈ A co B |] ==> F ∈ A co B"
by (auto simp add: constrains_def component_eq_subset)

lemma component_stable: "[| F ≤ G; G ∈ stable A |] ==> F ∈ stable A"
by (auto simp add: stable_def component_constrains)

lemmas program_less_le = strict_component_def [THEN meta_eq_to_obj_eq]

```

8.2 The preserves property

```

lemma preservesI: "(!!z. F ∈ stable {s. v s = z}) ==> F ∈ preserves v"
by (unfold preserves_def, blast)

lemma preserves_imp_eq:
  "[| F ∈ preserves v; act ∈ Acts F; (s,s') ∈ act |] ==> v s = v s'"
by (unfold preserves_def stable_def constrains_def, force)

```

```

lemma Join_preserves [iff]:
  "(F ⊔ G ∈ preserves v) = (F ∈ preserves v & G ∈ preserves v)"
by (unfold preserves_def, auto)

lemma JN_preserves [iff]:
  "(JOIN I F ∈ preserves v) = (∀ i ∈ I. F i ∈ preserves v)"
by (simp add: JN_stable preserves_def, blast)

lemma SKIP_preserves [iff]: "SKIP ∈ preserves v"
by (auto simp add: preserves_def)

lemma funPair_apply [simp]: "(funPair f g) x = (f x, g x)"
by (simp add: funPair_def)

lemma preserves_funPair: "preserves (funPair v w) = preserves v ∩ preserves w"
by (auto simp add: preserves_def stable_def constrains_def, blast)

declare preserves_funPair [THEN eqset_imp_iff, iff]

lemma funPair_o_distrib: "(funPair f g) o h = funPair (f o h) (g o h)"
by (simp add: funPair_def o_def)

lemma fst_o_funPair [simp]: "fst o (funPair f g) = f"
by (simp add: funPair_def o_def)

lemma snd_o_funPair [simp]: "snd o (funPair f g) = g"
by (simp add: funPair_def o_def)

lemma subset_preserves_o: "preserves v ⊆ preserves (w o v)"
by (force simp add: preserves_def stable_def constrains_def)

lemma preserves_subset_stable: "preserves v ⊆ stable {s. P (v s)}"
apply (auto simp add: preserves_def stable_def constrains_def)
apply (rename_tac s' s)
apply (subgoal_tac "v s = v s'")
apply (force+)
done

lemma preserves_subset_increasing: "preserves v ⊆ increasing v"
by (auto simp add: preserves_subset_stable [THEN subsetD] increasing_def)

lemma preserves_id_subset_stable: "preserves id ⊆ stable A"
by (force simp add: preserves_def stable_def constrains_def)

lemma safety_prop_preserves [iff]: "safety_prop (preserves v)"
by (auto intro: safety_prop_INTER1 simp add: preserves_def)

```

```

lemma stable_localTo_stable2:
  "[| F ∈ stable {s. P (v s) (w s)};
    G ∈ preserves v; G ∈ preserves w |]
   ==> F ⊔ G ∈ stable {s. P (v s) (w s)}"
apply simp
apply (subgoal_tac "G ∈ preserves (funPair v w) ")
  prefer 2 apply simp
apply (drule_tac P1 = "split ?Q" in preserves_subset_stable [THEN subsetD],
      auto)
done

lemma Increasing_preserves_Stable:
  "[| F ∈ stable {s. v s ≤ w s}; G ∈ preserves v; F ⊔ G ∈ Increasing w
  |]
   ==> F ⊔ G ∈ Stable {s. v s ≤ w s}"
apply (auto simp add: stable_def Stable_def Increasing_def Constrains_def
all_conj_distrib)
apply (blast intro: constrains_weaken)

apply (auto simp add: preserves_def stable_def constrains_def)

apply (erule_tac V = "∀ act ∈ Acts F. ?P act" in thin_rl)
apply (erule_tac V = "∀ z. ∀ act ∈ Acts F. ?P z act" in thin_rl)
apply (subgoal_tac "v x = v xa")
  apply auto
apply (erule order_trans, blast)
done

lemma component_of_imp_component: "F component_of H ==> F ≤ H"
by (unfold component_def component_of_def, blast)

lemma component_of_refl [simp]: "F component_of F"
apply (unfold component_of_def)
apply (rule_tac x = SKIP in exI, auto)
done

lemma component_of_SKIP [simp]: "SKIP component_of F"
by (unfold component_of_def, auto)

lemma component_of_trans:
  "[| F component_of G; G component_of H |] ==> F component_of H"
apply (unfold component_of_def)
apply (blast intro: Join_assoc [symmetric])
done

lemmas strict_component_of_eq =

```



```

strict_component_of_def [THEN meta_eq_to_obj_eq, standard]

lemma localize_Init_eq [simp]: "Init (localize v F) = Init F"
by (simp add: localize_def)

lemma localize_Acts_eq [simp]: "Acts (localize v F) = Acts F"
by (simp add: localize_def)

lemma localize_AllowedActs_eq [simp]:
  "AllowedActs (localize v F) = AllowedActs F  $\cap$  ( $\bigcup G \in \text{preserves } v. \text{ Acts } G$ )"
by (unfold localize_def, auto)

end

```

9 Guarantees Specifications

```

theory Guar
imports Comp
begin

instance program :: (type) order
proof qed (auto simp add: program_less_le dest: component_antisym intro: component_refl
component_trans)

```

Existential and Universal properties. I formalize the two-program case, proving equivalence with Chandy and Sanders's n-ary definitions

constdefs

```

ex_prop  :: "'a program set => bool"
"ex_prop X ==  $\forall F G. F \text{ ok } G \rightarrow F \in X \mid G \in X \rightarrow (F \sqcup G) \in X$ "

strict_ex_prop  :: "'a program set => bool"
"strict_ex_prop X ==  $\forall F G. F \text{ ok } G \rightarrow (F \in X \mid G \in X) = (F \sqcup G \in X)$ "

uv_prop  :: "'a program set => bool"
"uv_prop X ==  $SKIP \in X \ \& \ (\forall F G. F \text{ ok } G \rightarrow F \in X \ \& \ G \in X \rightarrow (F \sqcup G) \in X)$ "

strict_uv_prop  :: "'a program set => bool"
"strict_uv_prop X ==
   $SKIP \in X \ \& \ (\forall F G. F \text{ ok } G \rightarrow (F \in X \ \& \ G \in X) = (F \sqcup G \in X))$ "

```

Guarantees properties

constdefs

```

guar :: "[ 'a program set, 'a program set ] => 'a program set"
(infixl "guarantees" 55)
"X guarantees Y == {F.  $\forall G. F \text{ ok } G \rightarrow F \sqcup G \in X \rightarrow F \sqcup G \in Y$ }"

```

```

wg :: "[ 'a program, 'a program set ] => 'a program set"
"wg F Y == Union({X. F ∈ X guarantees Y})"

wx :: "( 'a program) set => ( 'a program)set"
"wx X == Union({Y. Y ⊆ X & ex_prop Y})"

welldef :: "'a program set"
"welldef == {F. Init F ≠ {}}"

refines :: "[ 'a program, 'a program, 'a program set ] => bool"
          ("(3_ refines _ wrt _)" [10,10,10] 10)
"G refines F wrt X ==
  ∀H. (F ok H & G ok H & F ⊔ H ∈ welldef ∩ X) -->
      (G ⊔ H ∈ welldef ∩ X)"

iso_refines :: "[ 'a program, 'a program, 'a program set ] => bool"
              ("(3_ iso_refines _ wrt _)" [10,10,10] 10)
"G iso_refines F wrt X ==
  F ∈ welldef ∩ X --> G ∈ welldef ∩ X"

```

```

lemma OK_insert_iff:
  "(OK (insert i I) F) =
    (if i ∈ I then OK I F else OK I F & (F i ok JOIN I F))"
by (auto intro: ok_sym simp add: OK_iff_ok)

```

9.1 Existential Properties

```

lemma ex1 [rule_format]:
  "[| ex_prop X; finite GG |] ==>
    GG ∩ X ≠ {} --> OK GG (%G. G) --> (⋃ G ∈ GG. G) ∈ X"
apply (unfold ex_prop_def)
apply (erule finite_induct)
apply (auto simp add: OK_insert_iff Int_insert_left)
done

```

```

lemma ex2:
  "∀ GG. finite GG & GG ∩ X ≠ {} --> OK GG (%G. G) --> (⋃ G ∈ GG. G) ∈ X"
  ==> ex_prop X"
apply (unfold ex_prop_def, clarify)
apply (drule_tac x = "{F,G}" in spec)
apply (auto dest: ok_sym simp add: OK_iff_ok)
done

```

```

lemma ex_prop_finite:
  "ex_prop X =
    (∀ GG. finite GG & GG ∩ X ≠ {} & OK GG (%G. G) --> (⋃ G ∈ GG. G) ∈ X)"

```

```

X)"
by (blast intro: ex1 ex2)

```

```

lemma ex_prop_equiv:
  "ex_prop X = ( $\forall G. G \in X = (\forall H. (G \text{ component\_of } H) \rightarrow H \in X)$ )"
apply auto
apply (unfold ex_prop_def component_of_def, safe, blast, blast)
apply (subst Join_commute)
apply (drule ok_sym, blast)
done

```

9.2 Universal Properties

```

lemma uv1 [rule_format]:
  "[| uv_prop X; finite GG |]
   ==> GG  $\subseteq$  X & OK GG (%G. G) --> ( $\bigsqcup G \in GG. G$ )  $\in$  X"
apply (unfold uv_prop_def)
apply (erule finite_induct)
apply (auto simp add: Int_insert_left OK_insert_iff)
done

```

```

lemma uv2:
  " $\forall GG. \text{finite } GG \ \& \ GG \subseteq X \ \& \ \text{OK } GG \ (%G. G) \rightarrow (\bigsqcup G \in GG. G) \in X$ 
   ==> uv_prop X"
apply (unfold uv_prop_def)
apply (rule conjI)
  apply (drule_tac x = "{}" in spec)
  prefer 2
  apply clarify
  apply (drule_tac x = "{F,G}" in spec)
apply (auto dest: ok_sym simp add: OK_iff_ok)
done

```

```

lemma uv_prop_finite:
  "uv_prop X =
   ( $\forall GG. \text{finite } GG \ \& \ GG \subseteq X \ \& \ \text{OK } GG \ (%G. G) \rightarrow (\bigsqcup G \in GG. G) \in X$ )"
by (blast intro: uv1 uv2)

```

9.3 Guarantees

```

lemma guaranteesI:
  "( $\neg !G. [| F \text{ ok } G; F \sqcup G \in X |] \implies F \sqcup G \in Y$ )  $\implies F \in X \text{ guarantees } Y$ "
by (simp add: guar_def component_def)

```

```

lemma guaranteesD:
  "[| F  $\in$  X guarantees Y; F ok G; F  $\sqcup$  G  $\in$  X |]  $\implies F \sqcup G \in Y$ "
by (unfold guar_def component_def, blast)

```

```

lemma component_guaranteesD:
  "[| F  $\in$  X guarantees Y; F  $\sqcup$  G = H; H  $\in$  X; F ok G |]  $\implies H \in Y$ "

```

by (unfold guar_def, blast)

lemma guarantees_weaken:

"[| F ∈ X guarantees X'; Y ⊆ X; X' ⊆ Y' |] ==> F ∈ Y guarantees Y'"

by (unfold guar_def, blast)

lemma subset_imp_guarantees_UNIV: "X ⊆ Y ==> X guarantees Y = UNIV"

by (unfold guar_def, blast)

lemma subset_imp_guarantees: "X ⊆ Y ==> F ∈ X guarantees Y"

by (unfold guar_def, blast)

lemma ex_prop_imp: "ex_prop Y ==> (Y = UNIV guarantees Y)"

apply (simp (no_asm_use) add: guar_def ex_prop_equiv)

apply safe

 apply (drule_tac x = x in spec)

 apply (drule_tac [2] x = x in spec)

 apply (drule_tac [2] sym)

apply (auto simp add: component_of_def)

done

lemma guarantees_imp: "(Y = UNIV guarantees Y) ==> ex_prop(Y)"

by (auto simp add: guar_def ex_prop_equiv component_of_def dest: sym)

lemma ex_prop_equiv2: "(ex_prop Y) = (Y = UNIV guarantees Y)"

apply (rule iffI)

 apply (rule ex_prop_imp)

 apply (auto simp add: guarantees_imp)

done

9.4 Distributive Laws. Re-Orient to Perform Miniscoping

lemma guarantees_UN_left:

"(⋃ i ∈ I. X i) guarantees Y = (⋂ i ∈ I. X i guarantees Y)"

by (unfold guar_def, blast)

lemma guarantees_Un_left:

"(X ∪ Y) guarantees Z = (X guarantees Z) ∩ (Y guarantees Z)"

by (unfold guar_def, blast)

lemma guarantees_INT_right:

"X guarantees (⋂ i ∈ I. Y i) = (⋂ i ∈ I. X guarantees Y i)"

by (unfold guar_def, blast)

lemma guarantees_Int_right:

"Z guarantees (X ∩ Y) = (Z guarantees X) ∩ (Z guarantees Y)"

by (unfold guar_def, blast)

lemma guarantees_Int_right_I:

"[| F ∈ Z guarantees X; F ∈ Z guarantees Y |]"

==> F ∈ Z guarantees (X ∩ Y)"

```
by (simp add: guarantees_Int_right)
```

```
lemma guarantees_INT_right_iff:
```

```
"(F ∈ X guarantees (INTER I Y)) = (∀ i ∈ I. F ∈ X guarantees (Y i))"
```

```
by (simp add: guarantees_INT_right)
```

```
lemma shunting: "(X guarantees Y) = (UNIV guarantees (-X ∪ Y))"
```

```
by (unfold guar_def, blast)
```

```
lemma contrapositive: "(X guarantees Y) = -Y guarantees -X"
```

```
by (unfold guar_def, blast)
```

```
lemma combining1:
```

```
"[| F ∈ V guarantees X; F ∈ (X ∩ Y) guarantees Z |]"
```

```
==> F ∈ (V ∩ Y) guarantees Z"
```

```
by (unfold guar_def, blast)
```

```
lemma combining2:
```

```
"[| F ∈ V guarantees (X ∪ Y); F ∈ Y guarantees Z |]"
```

```
==> F ∈ V guarantees (X ∪ Z)"
```

```
by (unfold guar_def, blast)
```

```
lemma all_guarantees:
```

```
"∀ i ∈ I. F ∈ X guarantees (Y i) ==> F ∈ X guarantees (⋂ i ∈ I. Y i)"
```

```
by (unfold guar_def, blast)
```

```
lemma ex_guarantees:
```

```
"∃ i ∈ I. F ∈ X guarantees (Y i) ==> F ∈ X guarantees (⋃ i ∈ I. Y i)"
```

```
by (unfold guar_def, blast)
```

9.5 Guarantees: Additional Laws (by lcp)

```
lemma guarantees_Join_Int:
```

```
"[| F ∈ U guarantees V; G ∈ X guarantees Y; F ok G |]"
```

```
==> F ⋈ G ∈ (U ∩ X) guarantees (V ∩ Y)"
```

```
apply (simp add: guar_def, safe)
```

```
  apply (simp add: Join_assoc)
```

```
apply (subgoal_tac "F ⋈ G ⋈ Ga = G ⋈ (F ⋈ Ga) ")
```

```
  apply (simp add: ok_commute)
```

```
apply (simp add: Join_ac)
```

```
done
```

```
lemma guarantees_Join_Un:
```

```
"[| F ∈ U guarantees V; G ∈ X guarantees Y; F ok G |]"
```

```
==> F ⋈ G ∈ (U ∪ X) guarantees (V ∪ Y)"
```

```
apply (simp add: guar_def, safe)
```

```
  apply (simp add: Join_assoc)
```

```
apply (subgoal_tac "F ⋈ G ⋈ Ga = G ⋈ (F ⋈ Ga) ")
```

```

  apply (simp add: ok_commute)
  apply (simp add: Join_ac)
  done

lemma guarantees_JN_INT:
  "[|  $\forall i \in I. F\ i \in X\ i$  guarantees  $Y\ i$ ;  $OK\ I\ F$  |]
   ==> (JOIN I F)  $\in$  (INTER I X) guarantees (INTER I Y)"
  apply (unfold guar_def, auto)
  apply (drule bspec, assumption)
  apply (rename_tac "i")
  apply (drule_tac x = "JOIN (I-{i}) F  $\sqcup$  G" in spec)
  apply (auto intro: OK_imp_ok
    simp add: Join_assoc [symmetric] JN_Join_diff JN_absorb)
  done

lemma guarantees_JN_UN:
  "[|  $\forall i \in I. F\ i \in X\ i$  guarantees  $Y\ i$ ;  $OK\ I\ F$  |]
   ==> (JOIN I F)  $\in$  (UNION I X) guarantees (UNION I Y)"
  apply (unfold guar_def, auto)
  apply (drule bspec, assumption)
  apply (rename_tac "i")
  apply (drule_tac x = "JOIN (I-{i}) F  $\sqcup$  G" in spec)
  apply (auto intro: OK_imp_ok
    simp add: Join_assoc [symmetric] JN_Join_diff JN_absorb)
  done

```

9.6 Guarantees Laws for Breaking Down the Program (by lcp)

```

lemma guarantees_Join_I1:
  "[|  $F \in X$  guarantees  $Y$ ;  $F\ ok\ G$  |] ==>  $F \sqcup G \in X$  guarantees  $Y$ "
  by (simp add: guar_def Join_assoc)

lemma guarantees_Join_I2:
  "[|  $G \in X$  guarantees  $Y$ ;  $F\ ok\ G$  |] ==>  $F \sqcup G \in X$  guarantees  $Y$ "
  apply (simp add: Join_commute [of _ G] ok_commute [of _ G])
  apply (blast intro: guarantees_Join_I1)
  done

lemma guarantees_JN_I:
  "[|  $i \in I$ ;  $F\ i \in X$  guarantees  $Y$ ;  $OK\ I\ F$  |]
   ==> ( $\sqcup i \in I. (F\ i)$ )  $\in X$  guarantees  $Y$ "
  apply (unfold guar_def, clarify)
  apply (drule_tac x = "JOIN (I-{i}) F  $\sqcup$  G" in spec)
  apply (auto intro: OK_imp_ok
    simp add: JN_Join_diff JN_Join_diff Join_assoc [symmetric])
  done

```

```

lemma Join_welldef_D1: " $F \sqcup G \in welldef$  ==>  $F \in welldef$ "
  by (unfold welldef_def, auto)

```

```
lemma Join_welldef_D2: "F ⊔ G ∈ welldef ==> G ∈ welldef"
by (unfold welldef_def, auto)
```

```
lemma refines_refl: "F refines F wrt X"
by (unfold refines_def, blast)
```

```
lemma refines_trans:
  "[| H refines G wrt X; G refines F wrt X |] ==> H refines F wrt X"
apply (simp add: refines_def)
oops
```

```
lemma strict_ex_refine_lemma:
  "strict_ex_prop X
  ==> (∀H. F ok H & G ok H & F ⊔ H ∈ X --> G ⊔ H ∈ X)
      = (F ∈ X --> G ∈ X)"
by (unfold strict_ex_prop_def, auto)
```

```
lemma strict_ex_refine_lemma_v:
  "strict_ex_prop X
  ==> (∀H. F ok H & G ok H & F ⊔ H ∈ welldef & F ⊔ H ∈ X --> G ⊔ H ∈ X) =
      (F ∈ welldef ∩ X --> G ∈ X)"
apply (unfold strict_ex_prop_def, safe)
apply (erule_tac x = SKIP and P = "%H. ?PP H --> ?RR H" in allE)
apply (auto dest: Join_welldef_D1 Join_welldef_D2)
done
```

```
lemma ex_refinement_thm:
  "[| strict_ex_prop X;
    ∀H. F ok H & G ok H & F ⊔ H ∈ welldef ∩ X --> G ⊔ H ∈ welldef |]
  ==> (G refines F wrt X) = (G iso_refines F wrt X)"
apply (rule_tac x = SKIP in allE, assumption)
apply (simp add: refines_def iso_refines_def strict_ex_refine_lemma_v)
done
```

```
lemma strict_uv_refine_lemma:
  "strict_uv_prop X ==>
    (∀H. F ok H & G ok H & F ⊔ H ∈ X --> G ⊔ H ∈ X) = (F ∈ X --> G ∈ X)"
by (unfold strict_uv_prop_def, blast)
```

```
lemma strict_uv_refine_lemma_v:
  "strict_uv_prop X
  ==> (∀H. F ok H & G ok H & F ⊔ H ∈ welldef & F ⊔ H ∈ X --> G ⊔ H ∈ X) =
      (F ∈ welldef ∩ X --> G ∈ X)"
apply (unfold strict_uv_prop_def, safe)
apply (erule_tac x = SKIP and P = "%H. ?PP H --> ?RR H" in allE)
apply (auto dest: Join_welldef_D1 Join_welldef_D2)
done
```

```

lemma uv_refinement_thm:
  "[| strict_uv_prop X;
    ∀H. F ok H & G ok H & F ⊔ H ∈ welldef ∩ X -->
      G ⊔ H ∈ welldef |]
  ==> (G refines F wrt X) = (G iso_refines F wrt X)"
apply (rule_tac x = SKIP in allE, assumption)
apply (simp add: refines_def iso_refines_def strict_uv_refine_lemma_v)
done

lemma guarantees_equiv:
  "(F ∈ X guarantees Y) = (∀H. H ∈ X → (F component_of H → H ∈ Y))"
by (unfold guar_def component_of_def, auto)

lemma wg_weakest: "!!X. F ∈ (X guarantees Y) ==> X ⊆ (wg F Y)"
by (unfold wg_def, auto)

lemma wg_guarantees: "F ∈ ((wg F Y) guarantees Y)"
by (unfold wg_def guar_def, blast)

lemma wg_equiv: "(H ∈ wg F X) = (F component_of H --> H ∈ X)"
by (simp add: guarantees_equiv wg_def, blast)

lemma component_of_wg: "F component_of H ==> (H ∈ wg F X) = (H ∈ X)"
by (simp add: wg_equiv)

lemma wg_finite:
  "∀FF. finite FF & FF ∩ X ≠ {} --> OK FF (%F. F)
  --> (∀F∈FF. ((⊔ F ∈ FF. F): wg F X) = ((⊔ F ∈ FF. F):X))"
apply clarify
apply (subgoal_tac "F component_of (⊔ F ∈ FF. F) ")
apply (drule_tac X = X in component_of_wg, simp)
apply (simp add: component_of_def)
apply (rule_tac x = "⊔ F ∈ (FF-{F}) . F" in exI)
apply (auto intro: JN_Join_diff dest: ok_sym simp add: OK_iff_ok)
done

lemma wg_ex_prop: "ex_prop X ==> (F ∈ X) = (∀H. H ∈ wg F X)"
apply (simp (no_asm_use) add: ex_prop_equiv wg_equiv)
apply blast
done

lemma wx_subset: "(wx X) ≤ X"
by (unfold wx_def, auto)

lemma wx_ex_prop: "ex_prop (wx X)"
apply (simp add: wx_def ex_prop_equiv cong: bex_cong, safe, blast)
apply force
done

lemma wx_weakest: "∀Z. Z ≤ X --> ex_prop Z --> Z ⊆ wx X"

```



```
by (auto simp add: wx_def)
```

```
lemma wx'_ex_prop: "ex_prop({F.  $\forall G. F \text{ ok } G \rightarrow F \sqcup G \in X$ })"
apply (unfold ex_prop_def, safe)
  apply (drule_tac x = "G  $\sqcup$  Ga" in spec)
  apply (force simp add: ok_Join_iff1 Join_assoc)
apply (drule_tac x = "F  $\sqcup$  Ga" in spec)
apply (simp add: ok_Join_iff1 ok_commute Join_ac)
done
```

Equivalence with the other definition of wx

```
lemma wx_equiv: "wx X = {F.  $\forall G. F \text{ ok } G \rightarrow (F \sqcup G) \in X$ }"
apply (unfold wx_def, safe)
  apply (simp add: ex_prop_def, blast)
apply (simp (no_asm))
apply (rule_tac x = "{F.  $\forall G. F \text{ ok } G \rightarrow F \sqcup G \in X$ }" in exI, safe)
apply (rule_tac [2] wx'_ex_prop)
apply (drule_tac x = SKIP in spec)+
apply auto
done
```

Propositions 7 to 11 are about this second definition of wx. They are the same as the ones proved for the first definition of wx, by equivalence

```
lemma guarantees_wx_eq: "(X guarantees Y) = wx( $\neg X \cup Y$ )"
by (simp add: guar_def wx_equiv)
```

```
lemma stable_guarantees_Always:
  "Init F  $\subseteq A \Rightarrow F \in (\text{stable } A) \text{ guarantees } (\text{Always } A)"
apply (rule guaranteesI)
apply (simp add: Join_commute)
apply (rule stable_Join_Always1)
apply (simp_all add: invariant_def Join_stable)
done$ 
```

```
lemma constrains_guarantees_leadsTo:
  "F  $\in \text{transient } A \Rightarrow F \in (A \text{ co } A \cup B) \text{ guarantees } (A \text{ leadsTo } (B-A))"
apply (rule guaranteesI)
apply (rule leadsTo_Basis')
  apply (drule constrains_weaken_R)
  prefer 2 apply assumption
  apply blast
apply (blast intro: Join_transient_I1)
done$ 
```

```
end
```

10 Extending State Sets

```
theory Extend imports Guar begin
```

constdefs

```

Restrict :: "[ 'a set, ('a*'b) set] => ('a*'b) set"
  "Restrict A r == r  $\cap$  (A <*> UNIV)"

good_map :: "[ 'a*'b => 'c] => bool"
  "good_map h == surj h & ( $\forall$  x y. fst (inv h (h (x,y))) = x)"

extend_set :: "[ 'a*'b => 'c, 'a set] => 'c set"
  "extend_set h A == h ` (A <*> UNIV)"

project_set :: "[ 'a*'b => 'c, 'c set] => 'a set"
  "project_set h C == {x.  $\exists$  y. h(x,y)  $\in$  C}"

extend_act :: "[ 'a*'b => 'c, ('a*'a) set] => ('c*'c) set"
  "extend_act h == %act.  $\bigcup$  (s,s')  $\in$  act.  $\bigcup$  y. {(h(s,y), h(s',y))}"

project_act :: "[ 'a*'b => 'c, ('c*'c) set] => ('a*'a) set"
  "project_act h act == {(x,x').  $\exists$  y y'. (h(x,y), h(x',y'))  $\in$  act}"

extend :: "[ 'a*'b => 'c, 'a program] => 'c program"
  "extend h F == mk_program (extend_set h (Init F),
                             extend_act h ` Acts F,
                             project_act h -` AllowedActs F)"

project :: "[ 'a*'b => 'c, 'c set, 'c program] => 'a program"
  "project h C F ==
    mk_program (project_set h (Init F),
               project_act h ` Restrict C ` Acts F,
               {act. Restrict (project_set h C) act :
                 project_act h ` Restrict C ` AllowedActs F})"

locale Extend =
  fixes f      :: "'c => 'a"
    and g      :: "'c => 'b"
    and h      :: "'a*'b => 'c"
    and slice :: "'c set, 'b] => 'a set"
  assumes
    good_h: "good_map h"
  defines f_def: "f z == fst (inv h z)"
    and g_def: "g z == snd (inv h z)"
    and slice_def: "slice Z y == {x. h(x,y)  $\in$  Z}"

```

10.1 Restrict

lemma Restrict_iff [iff]: " $((x,y): \text{Restrict } A \ r) = ((x,y): r \ \& \ x \in A)$ "
by (unfold Restrict_def, blast)

lemma Restrict_UNIV [simp]: " $\text{Restrict } UNIV = \text{id}$ "
apply (rule ext)

```

apply (auto simp add: Restrict_def)
done

lemma Restrict_empty [simp]: "Restrict {} r = {}"
by (auto simp add: Restrict_def)

lemma Restrict_Int [simp]: "Restrict A (Restrict B r) = Restrict (A ∩ B)
r"
by (unfold Restrict_def, blast)

lemma Restrict_triv: "Domain r ⊆ A ==> Restrict A r = r"
by (unfold Restrict_def, auto)

lemma Restrict_subset: "Restrict A r ⊆ r"
by (unfold Restrict_def, auto)

lemma Restrict_eq_mono:
  "[| A ⊆ B; Restrict B r = Restrict B s |]
   ==> Restrict A r = Restrict A s"
by (unfold Restrict_def, blast)

lemma Restrict_imageI:
  "[| s ∈ RR; Restrict A r = Restrict A s |]
   ==> Restrict A r ∈ Restrict A ' RR"
by (unfold Restrict_def image_def, auto)

lemma Domain_Restrict [simp]: "Domain (Restrict A r) = A ∩ Domain r"
by blast

lemma Image_Restrict [simp]: "(Restrict A r) `` B = r `` (A ∩ B)"
by blast

lemma good_mapI:
  assumes surj_h: "surj h"
  and prem: "!! x x' y y'. h(x,y) = h(x',y') ==> x=x'"
  shows "good_map h"
apply (simp add: good_map_def)
apply (safe intro!: surj_h)
apply (rule prem)
apply (subst surjective_pairing [symmetric])
apply (subst surj_h [THEN surj_f_inv_f])
apply (rule refl)
done

lemma good_map_is_surj: "good_map h ==> surj h"
by (unfold good_map_def, auto)

lemma fst_inv_equalityI:
  assumes surj_h: "surj h"
  and prem: "!! x y. g (h(x,y)) = x"
  shows "fst (inv h z) = g z"
apply (unfold inv_def)

```

```

apply (rule_tac y1 = z in surj_h [THEN surjD, THEN exE])
apply (rule someI2)
apply (drule_tac [2] f = g in arg_cong)
apply (auto simp add: prem)
done

```

10.2 Trivial properties of f, g, h

```

lemma (in Extend) f_h_eq [simp]: "f(h(x,y)) = x"
by (simp add: f_def good_h [unfolded good_map_def, THEN conjunct2])

lemma (in Extend) h_inject1 [dest]: "h(x,y) = h(x',y') ==> x=x'"
apply (drule_tac f = f in arg_cong)
apply (simp add: f_def good_h [unfolded good_map_def, THEN conjunct2])
done

lemma (in Extend) h_f_g_equiv: "h(f z, g z) == z"
by (simp add: f_def g_def
    good_h [unfolded good_map_def, THEN conjunct1, THEN surj_f_inv_f])

lemma (in Extend) h_f_g_eq: "h(f z, g z) = z"
by (simp add: h_f_g_equiv)

lemma (in Extend) split_extended_all:
  "(!!z. PROP P z) == (!!u y. PROP P (h (u, y)))"
proof
  assume allP: " $\bigwedge z. \text{PROP } P \ z$ "
  fix u y
  show "PROP P (h (u, y))" by (rule allP)
next
  assume allPh: " $\bigwedge u y. \text{PROP } P \ (h(u,y))$ "
  fix z
  have Phfgz: "PROP P (h (f z, g z))" by (rule allPh)
  show "PROP P z" by (rule Phfgz [unfolded h_f_g_equiv])
qed

```

10.3 `extend_set`: basic properties

```

lemma project_set_iff [iff]:
  "(x ∈ project_set h C) = ( $\exists y. h(x,y) \in C$ )"
by (simp add: project_set_def)

lemma extend_set_mono: "A  $\subseteq$  B ==> extend_set h A  $\subseteq$  extend_set h B"
by (unfold extend_set_def, blast)

lemma (in Extend) mem_extend_set_iff [iff]: "z ∈ extend_set h A = (f z ∈ A)"
apply (unfold extend_set_def)
apply (force intro: h_f_g_eq [symmetric])
done

lemma (in Extend) extend_set_strict_mono [iff]:
  "(extend_set h A  $\subseteq$  extend_set h B) = (A  $\subseteq$  B)"

```

```
by (unfold extend_set_def, force)
```

```
lemma extend_set_empty [simp]: "extend_set h {} = {}"
by (unfold extend_set_def, auto)
```

```
lemma (in Extend) extend_set_eq_Collect: "extend_set h {s. P s} = {s. P(f
s)}"
by auto
```

```
lemma (in Extend) extend_set_sing: "extend_set h {x} = {s. f s = x}"
by auto
```

```
lemma (in Extend) extend_set_inverse [simp]:
  "project_set h (extend_set h C) = C"
by (unfold extend_set_def, auto)
```

```
lemma (in Extend) extend_set_project_set:
  "C ⊆ extend_set h (project_set h C)"
apply (unfold extend_set_def)
apply (auto simp add: split_extended_all, blast)
done
```

```
lemma (in Extend) inj_extend_set: "inj (extend_set h)"
apply (rule inj_on_inverseI)
apply (rule extend_set_inverse)
done
```

```
lemma (in Extend) extend_set_UNIV_eq [simp]: "extend_set h UNIV = UNIV"
apply (unfold extend_set_def)
apply (auto simp add: split_extended_all)
done
```

10.4 project_set: basic properties

```
lemma (in Extend) project_set_eq: "project_set h C = f ' C"
by (auto intro: f_h_eq [symmetric] simp add: split_extended_all)
```

```
lemma (in Extend) project_set_I: "!!z. z ∈ C ==> f z ∈ project_set h C"
by (auto simp add: split_extended_all)
```

10.5 More laws

```
lemma (in Extend) project_set_extend_set_Int:
  "project_set h ((extend_set h A) ∩ B) = A ∩ (project_set h B)"
by auto
```

```
lemma (in Extend) project_set_extend_set_Un:
  "project_set h ((extend_set h A) ∪ B) = A ∪ (project_set h B)"
by auto
```

```
lemma project_set_Int_subset:
  "project_set h (A ∩ B) ⊆ (project_set h A) ∩ (project_set h B)"
```

by auto

lemma (in Extend) extend_set_Un_distrib:
 "extend_set h (A \cup B) = extend_set h A \cup extend_set h B"
 by auto

lemma (in Extend) extend_set_Int_distrib:
 "extend_set h (A \cap B) = extend_set h A \cap extend_set h B"
 by auto

lemma (in Extend) extend_set_INT_distrib:
 "extend_set h (INTER A B) = (\bigcap x \in A. extend_set h (B x))"
 by auto

lemma (in Extend) extend_set_Diff_distrib:
 "extend_set h (A - B) = extend_set h A - extend_set h B"
 by auto

lemma (in Extend) extend_set_Union:
 "extend_set h (Union A) = (\bigcup X \in A. extend_set h X)"
 by blast

lemma (in Extend) extend_set_subset_Compl_eq:
 "(extend_set h A \subseteq - extend_set h B) = (A \subseteq - B)"
 by (unfold extend_set_def, auto)

10.6 extend_act

lemma (in Extend) mem_extend_act_iff [iff]:
 "((h(s,y), h(s',y)) \in extend_act h act) = ((s, s') \in act)"
 by (unfold extend_act_def, auto)

lemma (in Extend) extend_act_D:
 "(z, z') \in extend_act h act ==> (f z, f z') \in act"
 by (unfold extend_act_def, auto)

lemma (in Extend) extend_act_inverse [simp]:
 "project_act h (extend_act h act) = act"
 by (unfold extend_act_def project_act_def, blast)

lemma (in Extend) project_act_extend_act_restrict [simp]:
 "project_act h (Restrict C (extend_act h act)) =
 Restrict (project_set h C) act"
 by (unfold extend_act_def project_act_def, blast)

lemma (in Extend) subset_extend_act_D:
 "act' \subseteq extend_act h act ==> project_act h act' \subseteq act"
 by (unfold extend_act_def project_act_def, force)

lemma (in Extend) inj_extend_act: "inj (extend_act h)"
 apply (rule inj_on_inverseI)
 apply (rule extend_act_inverse)
 done

```

lemma (in Extend) extend_act_Image [simp]:
  "extend_act h act ' (extend_set h A) = extend_set h (act ' A)"
by (unfold extend_set_def extend_act_def, force)

lemma (in Extend) extend_act_strict_mono [iff]:
  "(extend_act h act'  $\subseteq$  extend_act h act) = (act'  $\leq$  act)"
by (unfold extend_act_def, auto)

declare (in Extend) inj_extend_act [THEN inj_eq, iff]

lemma Domain_extend_act:
  "Domain (extend_act h act) = extend_set h (Domain act)"
by (unfold extend_set_def extend_act_def, force)

lemma (in Extend) extend_act_Id [simp]:
  "extend_act h Id = Id"
apply (unfold extend_act_def)
apply (force intro: h_f_g_eq [symmetric])
done

lemma (in Extend) project_act_I:
  "!!z z'. (z, z')  $\in$  act ==> (f z, f z')  $\in$  project_act h act"
apply (unfold project_act_def)
apply (force simp add: split_extended_all)
done

lemma (in Extend) project_act_Id [simp]: "project_act h Id = Id"
by (unfold project_act_def, force)

lemma (in Extend) Domain_project_act:
  "Domain (project_act h act) = project_set h (Domain act)"
apply (unfold project_act_def)
apply (force simp add: split_extended_all)
done

```

10.7 extend

Basic properties

```

lemma Init_extend [simp]:
  "Init (extend h F) = extend_set h (Init F)"
by (unfold extend_def, auto)

lemma Init_project [simp]:
  "Init (project h C F) = project_set h (Init F)"
by (unfold project_def, auto)

lemma (in Extend) Acts_extend [simp]:
  "Acts (extend h F) = (extend_act h ' Acts F)"
by (simp add: extend_def insert_Id_image_Acts)

lemma (in Extend) AllowedActs_extend [simp]:

```

```

    "AllowedActs (extend h F) = project_act h -' AllowedActs F"
  by (simp add: extend_def insert_absorb)

lemma Acts_project [simp]:
  "Acts(project h C F) = insert Id (project_act h ' Restrict C ' Acts F)"
  by (auto simp add: project_def image_iff)

lemma (in Extend) AllowedActs_project [simp]:
  "AllowedActs(project h C F) =
    {act. Restrict (project_set h C) act
      ∈ project_act h ' Restrict C ' AllowedActs F}"
  apply (simp (no_asm) add: project_def image_iff)
  apply (subst insert_absorb)
  apply (auto intro!: bexI [of _ Id] simp add: project_act_def)
  done

lemma (in Extend) Allowed_extend:
  "Allowed (extend h F) = project h UNIV -' Allowed F"
  apply (simp (no_asm) add: AllowedActs_extend Allowed_def)
  apply blast
  done

lemma (in Extend) extend_SKIP [simp]: "extend h SKIP = SKIP"
  apply (unfold SKIP_def)
  apply (rule program_equalityI, auto)
  done

lemma project_set_UNIV [simp]: "project_set h UNIV = UNIV"
  by auto

lemma project_set_Union:
  "project_set h (Union A) = ( $\bigcup X \in A.$  project_set h X)"
  by blast

lemma (in Extend) project_act.Restrict_subset:
  "project_act h (Restrict C act)  $\subseteq$ 
    Restrict (project_set h C) (project_act h act)"
  by (auto simp add: project_act_def)

lemma (in Extend) project_act.Restrict_Id_eq:
  "project_act h (Restrict C Id) = Restrict (project_set h C) Id"
  by (auto simp add: project_act_def)

lemma (in Extend) project_extend_eq:
  "project h C (extend h F) =
    mk_program (Init F, Restrict (project_set h C) ' Acts F,
      {act. Restrict (project_set h C) act
        ∈ project_act h ' Restrict C '
          (project_act h -' AllowedActs F)})"
  apply (rule program_equalityI)
  apply simp
  apply (simp add: image_eq_UN)

```



```

apply (simp add: project_def)
done

lemma (in Extend) extend_inverse [simp]:
  "project h UNIV (extend h F) = F"
apply (simp (no_asm_simp) add: project_extend_eq image_eq_UN
  subset_UNIV [THEN subset_trans, THEN Restrict_triv])
apply (rule program_equalityI)
apply (simp_all (no_asm))
apply (subst insert_absorb)
apply (simp (no_asm) add: bexI [of _ Id])
apply auto
apply (rename_tac "act")
apply (rule_tac x = "extend_act h act" in bexI, auto)
done

lemma (in Extend) inj_extend: "inj (extend h)"
apply (rule inj_on_inverseI)
apply (rule extend_inverse)
done

lemma (in Extend) extend_Join [simp]:
  "extend h (F  $\sqcup$  G) = extend h F  $\sqcup$  extend h G"
apply (rule program_equalityI)
apply (simp (no_asm) add: extend_set_Int_distrib)
apply (simp add: image_UN, auto)
done

lemma (in Extend) extend_JN [simp]:
  "extend h (JOIN I F) = ( $\bigsqcup$  i  $\in$  I. extend h (F i))"
apply (rule program_equalityI)
  apply (simp (no_asm) add: extend_set_INT_distrib)
  apply (simp add: image_UN, auto)
done

lemma (in Extend) extend_mono: "F  $\leq$  G ==> extend h F  $\leq$  extend h G"
by (force simp add: component_eq_subset)

lemma (in Extend) project_mono: "F  $\leq$  G ==> project h C F  $\leq$  project h C G"
by (simp add: component_eq_subset, blast)

lemma (in Extend) all_total_extend: "all_total F ==> all_total (extend h F)"
by (simp add: all_total_def Domain_extend_act)

```

10.8 Safety: co, stable

```

lemma (in Extend) extend_constrains:
  "(extend h F  $\in$  (extend_set h A) co (extend_set h B)) =
  (F  $\in$  A co B)"
by (simp add: constrains_def)

```

```

lemma (in Extend) extend_stable:
  "(extend h F ∈ stable (extend_set h A)) = (F ∈ stable A)"
by (simp add: stable_def extend_constrains)

lemma (in Extend) extend_invariant:
  "(extend h F ∈ invariant (extend_set h A)) = (F ∈ invariant A)"
by (simp add: invariant_def extend_stable)

lemma (in Extend) extend_constrains_project_set:
  "extend h F ∈ A co B ==> F ∈ (project_set h A) co (project_set h B)"
by (auto simp add: constrains_def, force)

lemma (in Extend) extend_stable_project_set:
  "extend h F ∈ stable A ==> F ∈ stable (project_set h A)"
by (simp add: stable_def extend_constrains_project_set)

```

10.9 Weak safety primitives: Co, Stable

```

lemma (in Extend) reachable_extend_f:
  "p ∈ reachable (extend h F) ==> f p ∈ reachable F"
apply (erule reachable.induct)
apply (auto intro: reachable.intros simp add: extend_act_def image_iff)
done

lemma (in Extend) h_reachable_extend:
  "h(s,y) ∈ reachable (extend h F) ==> s ∈ reachable F"
by (force dest!: reachable_extend_f)

lemma (in Extend) reachable_extend_eq:
  "reachable (extend h F) = extend_set h (reachable F)"
apply (unfold extend_set_def)
apply (rule equalityI)
apply (force intro: h_f_g_eq [symmetric] dest!: reachable_extend_f, clarify)
apply (erule reachable.induct)
apply (force intro: reachable.intros)+
done

lemma (in Extend) extend_Constrains:
  "(extend h F ∈ (extend_set h A) Co (extend_set h B)) =
   (F ∈ A Co B)"
by (simp add: Constrains_def reachable_extend_eq extend_constrains
  extend_set_Int_distrib [symmetric])

lemma (in Extend) extend_Stable:
  "(extend h F ∈ Stable (extend_set h A)) = (F ∈ Stable A)"
by (simp add: Stable_def extend_Constrains)

lemma (in Extend) extend_Always:
  "(extend h F ∈ Always (extend_set h A)) = (F ∈ Always A)"
by (simp (no_asm_simp) add: Always_def extend_Stable)

```

```

lemma project_act_mono:
  "D ⊆ C ==>
    project_act h (Restrict D act) ⊆ project_act h (Restrict C act)"
by (auto simp add: project_act_def)

lemma (in Extend) project_constrains_mono:
  "[| D ⊆ C; project h C F ∈ A co B |] ==> project h D F ∈ A co B"
apply (auto simp add: constrains_def)
apply (drule project_act_mono, blast)
done

lemma (in Extend) project_stable_mono:
  "[| D ⊆ C; project h C F ∈ stable A |] ==> project h D F ∈ stable A"
by (simp add: stable_def project_constrains_mono)

lemma (in Extend) project_constrains:
  "(project h C F ∈ A co B) =
    (F ∈ (C ∩ extend_set h A) co (extend_set h B) & A ⊆ B)"
apply (unfold constrains_def)
apply (auto intro!: project_act_I simp add: ball_Un)
apply (force intro!: project_act_I dest!: subsetD)

apply (unfold project_act_def)
apply (force dest!: subsetD)
done

lemma (in Extend) project_stable:
  "(project h UNIV F ∈ stable A) = (F ∈ stable (extend_set h A))"
apply (unfold stable_def)
apply (simp (no_asm) add: project_constrains)
done

lemma (in Extend) project_stable_I:
  "F ∈ stable (extend_set h A) ==> project h C F ∈ stable A"
apply (drule project_stable [THEN iffD2])
apply (blast intro: project_stable_mono)
done

lemma (in Extend) Int_extend_set_lemma:
  "A ∩ extend_set h ((project_set h A) ∩ B) = A ∩ extend_set h B"
by (auto simp add: split_extended_all)

lemma project_constrains_project_set:
  "G ∈ C co B ==> project h C G ∈ project_set h C co project_set h B"
by (simp add: constrains_def project_def project_act_def, blast)

lemma project_stable_project_set:
  "G ∈ stable C ==> project h C G ∈ stable (project_set h C)"

```

by (simp add: stable_def project_constrains_project_set)

10.10 Progress: transient, ensures

```
lemma (in Extend) extend_transient:
  "(extend h F ∈ transient (extend_set h A)) = (F ∈ transient A)"
by (auto simp add: transient_def extend_set_subset_Compl_eq Domain_extend_act)

lemma (in Extend) extend_sures:
  "(extend h F ∈ (extend_set h A) ensures (extend_set h B)) =
   (F ∈ A ensures B)"
by (simp add: ensures_def extend_constrains extend_transient
  extend_set_Un_distrib [symmetric] extend_set_Diff_distrib [symmetric])

lemma (in Extend) leadsTo_imp_extend_leadsTo:
  "F ∈ A leadsTo B
   ==> extend h F ∈ (extend_set h A) leadsTo (extend_set h B)"
apply (erule leadsTo_induct)
  apply (simp add: leadsTo_Basis extend_sures)
  apply (blast intro: leadsTo_Trans)
  apply (simp add: leadsTo_UN extend_set_Union)
done
```

10.11 Proving the converse takes some doing!

```
lemma (in Extend) slice_iff [iff]: "(x ∈ slice C y) = (h(x,y) ∈ C)"
by (simp (no_asm) add: slice_def)

lemma (in Extend) slice_Union: "slice (Union S) y = (⋃ x ∈ S. slice x y)"
by auto

lemma (in Extend) slice_extend_set: "slice (extend_set h A) y = A"
by auto

lemma (in Extend) project_set_is_UN_slice:
  "project_set h A = (⋃ y. slice A y)"
by auto

lemma (in Extend) extend_transient_slice:
  "extend h F ∈ transient A ==> F ∈ transient (slice A y)"
by (unfold transient_def, auto)

lemma (in Extend) extend_constrains_slice:
  "extend h F ∈ A co B ==> F ∈ (slice A y) co (slice B y)"
by (auto simp add: constrains_def)

lemma (in Extend) extend_sures_slice:
  "extend h F ∈ A ensures B ==> F ∈ (slice A y) ensures (project_set h
  B)"
  apply (auto simp add: ensures_def extend_constrains extend_transient)
  apply (erule_tac [2] extend_transient_slice [THEN transient_strengthen])
  apply (erule extend_constrains_slice [THEN constrains_weaken], auto)
done
```

```

lemma (in Extend) leadsTo_slice_project_set:
  "∀y. F ∈ (slice B y) leadsTo CU ==> F ∈ (project_set h B) leadsTo CU"
apply (simp (no_asm) add: project_set_is_UN_slice)
apply (blast intro: leadsTo_UN)
done

lemma (in Extend) extend_leadsTo_slice [rule_format]:
  "extend h F ∈ AU leadsTo BU
  ==> ∀y. F ∈ (slice AU y) leadsTo (project_set h BU)"
apply (erule leadsTo_induct)
  apply (blast intro: extend_ensures_slice leadsTo_Basis)
  apply (blast intro: leadsTo_slice_project_set leadsTo_Trans)
apply (simp add: leadsTo_UN slice_Union)
done

lemma (in Extend) extend_leadsTo:
  "(extend h F ∈ (extend_set h A) leadsTo (extend_set h B)) =
  (F ∈ A leadsTo B)"
apply safe
apply (erule_tac [2] leadsTo_imp_extend_leadsTo)
apply (drule extend_leadsTo_slice)
apply (simp add: slice_extend_set)
done

lemma (in Extend) extend_LeadsTo:
  "(extend h F ∈ (extend_set h A) LeadsTo (extend_set h B)) =
  (F ∈ A LeadsTo B)"
by (simp add: LeadsTo_def reachable_extend_eq extend_leadsTo
  extend_set_Int_distrib [symmetric])

```

10.12 preserves

```

lemma (in Extend) project_preserves_I:
  "G ∈ preserves (v o f) ==> project h C G ∈ preserves v"
by (auto simp add: preserves_def project_stable_I extend_set_eq_Collect)

lemma (in Extend) project_preserves_id_I:
  "G ∈ preserves f ==> project h C G ∈ preserves id"
by (simp add: project_preserves_I)

lemma (in Extend) extend_preserves:
  "(extend h G ∈ preserves (v o f)) = (G ∈ preserves v)"
by (auto simp add: preserves_def extend_stable [symmetric]
  extend_set_eq_Collect)

lemma (in Extend) inj_extend_preserves: "inj h ==> (extend h G ∈ preserves
g)"
by (auto simp add: preserves_def extend_def extend_act_def stable_def
  constrains_def g_def)

```

10.13 Guarantees

```
lemma (in Extend) project_extend_Join:
  "project h UNIV ((extend h F)  $\sqcup$  G) = F  $\sqcup$  (project h UNIV G)"
apply (rule program_equalityI)
  apply (simp add: project_set_extend_set_Int)
  apply (simp add: image_eq_UN UN_Un, auto)
done
```

```
lemma (in Extend) extend_Join_eq_extend_D:
  "(extend h F)  $\sqcup$  G = extend h H ==> H = F  $\sqcup$  (project h UNIV G)"
apply (drule_tac f = "project h UNIV" in arg_cong)
apply (simp add: project_extend_Join)
done
```

```
lemma (in Extend) ok_extend_imp_ok_project:
  "extend h F ok G ==> F ok project h UNIV G"
apply (auto simp add: ok_def)
apply (drule subsetD)
apply (auto intro!: rev_image_eqI)
done
```

```
lemma (in Extend) ok_extend_iff: "(extend h F ok extend h G) = (F ok G)"
apply (simp add: ok_def, safe)
apply (force+)
done
```

```
lemma (in Extend) OK_extend_iff: "OK I (%i. extend h (F i)) = (OK I F)"
apply (unfold OK_def, safe)
apply (drule_tac x = i in bspec)
apply (drule_tac [2] x = j in bspec)
apply (force+)
done
```

```
lemma (in Extend) guarantees_imp_extend_guarantees:
  "F  $\in$  X guarantees Y ==>
   extend h F  $\in$  (extend h ' X) guarantees (extend h ' Y)"
apply (rule guaranteesI, clarify)
apply (blast dest: ok_extend_imp_ok_project extend_Join_eq_extend_D
               guaranteesD)
done
```

```
lemma (in Extend) extend_guarantees_imp_guarantees:
  "extend h F  $\in$  (extend h ' X) guarantees (extend h ' Y)
   ==> F  $\in$  X guarantees Y"
apply (auto simp add: guar_def)
apply (drule_tac x = "extend h G" in spec)
apply (simp del: extend_Join
            add: extend_Join [symmetric] ok_extend_iff
            inj_extend [THEN inj_image_mem_iff])
done
```

```

lemma (in Extend) extend_guarantees_eq:
  "(extend h F ∈ (extend h ' X) guarantees (extend h ' Y)) =
   (F ∈ X guarantees Y)"
by (blast intro: guarantees_imp_extend_guarantees
    extend_guarantees_imp_guarantees)

end

```

11 Renaming of State Sets

```
theory Rename imports Extend begin
```

```
constdefs
```

```

  rename :: "[ 'a => 'b, 'a program] => 'b program"
  "rename h == extend (%(x,u::unit). h x)"

```

```
declare image_inv_f_f [simp] image_surj_f_inv_f [simp]
```

```
declare Extend.intro [simp,intro]
```

```

lemma good_map_bij [simp,intro]: "bij h ==> good_map (%(x,u). h x)"
apply (rule good_mapI)
apply (unfold bij_def inj_on_def surj_def, auto)
done

```

```

lemma fst_o_inv_eq_inv: "bij h ==> fst (inv (%(x,u). h x) s) = inv h s"
apply (unfold bij_def split_def, clarify)
apply (subgoal_tac "surj (%p. h (fst p))")
  prefer 2 apply (simp add: surj_def)
apply (erule injD)
apply (simp (no_asm_simp) add: surj_f_inv_f)
apply (erule surj_f_inv_f)
done

```

```

lemma mem_rename_set_iff: "bij h ==> z ∈ h'A = (inv h z ∈ A)"
by (force simp add: bij_is_inj bij_is_surj [THEN surj_f_inv_f])

```

```

lemma extend_set_eq_image [simp]: "extend_set (%(x,u). h x) A = h'A"
by (force simp add: extend_set_def)

```

```

lemma Init_rename [simp]: "Init (rename h F) = h'(Init F)"
by (simp add: rename_def)

```

11.1 inverse properties

```

lemma extend_set_inv:
  "bij h
   ==> extend_set (%(x,u::'c). inv h x) = project_set (%(x,u::'c). h x)"
apply (unfold bij_def)

```

```

apply (rule ext)
apply (force simp add: extend_set_def project_set_def surj_f_inv_f)
done

lemma bij_extend_act_eq_project_act: "bij h
  ==> extend_act (%(x,u::'c). h x) = project_act (%(x,u::'c). inv h x)"
apply (rule ext)
apply (force simp add: extend_act_def project_act_def bij_def surj_f_inv_f)
done

lemma bij_extend_act: "bij h ==> bij (extend_act (%(x,u::'c). h x))"
apply (rule bijI)
apply (rule Extend.inj_extend_act)
apply (auto simp add: bij_extend_act_eq_project_act)
apply (rule surjI)
apply (rule Extend.extend_act_inverse)
apply (blast intro: bij_imp_bij_inv good_map_bij)
done

lemma bij_project_act: "bij h ==> bij (project_act (%(x,u::'c). h x))"
apply (frule bij_imp_bij_inv [THEN bij_extend_act])
apply (simp add: bij_extend_act_eq_project_act bij_imp_bij_inv inv_inv_eq)
done

lemma bij_inv_project_act_eq: "bij h ==> inv (project_act (%(x,u::'c). inv
  h x)) =
  project_act (%(x,u::'c). h x)"
apply (simp (no_asm_simp) add: bij_extend_act_eq_project_act [symmetric])
apply (rule surj_imp_inv_eq)
apply (blast intro: bij_extend_act bij_is_surj)
apply (simp (no_asm_simp) add: Extend.extend_act_inverse)
done

lemma extend_inv: "bij h
  ==> extend (%(x,u::'c). inv h x) = project (%(x,u::'c). h x) UNIV"
apply (frule bij_imp_bij_inv)
apply (rule ext)
apply (rule program_equalityI)
  apply (simp (no_asm_simp) add: extend_set_inv)
  apply (simp add: Extend.project_act_Id Extend.Acts_extend
    insert_Id_image_Acts bij_extend_act_eq_project_act inv_inv_eq)
apply (simp add: Extend.AllowedActs_extend Extend.AllowedActs_project
  bij_project_act bij_vimage_eq_inv_image bij_inv_project_act_eq)
done

lemma rename_inv_rename [simp]: "bij h ==> rename (inv h) (rename h F) =
  F"
by (simp add: rename_def extend_inv Extend.extend_inverse)

lemma rename_rename_inv [simp]: "bij h ==> rename h (rename (inv h) F) =
  F"
apply (frule bij_imp_bij_inv)

```



```

apply (erule inv_inv_eq [THEN subst], erule rename_inv_rename)
done

```

```

lemma rename_inv_eq: "bij h ==> rename (inv h) = inv (rename h)"
by (rule inv_equality [symmetric], auto)

```

```

lemma bij_extend: "bij h ==> bij (extend (%(x,u::'c). h x))"
apply (rule bijI)
apply (blast intro: Extend.inj_extend)
apply (rule_tac f = "extend (% (x,u) . inv h x)" in surjI)
apply (subst (1 2) inv_inv_eq [of h, symmetric], assumption+)
apply (simp add: bij_imp_bij_inv extend_inv [of "inv h"])
apply (simp add: inv_inv_eq)
apply (rule Extend.extend_inverse)
apply (simp add: bij_imp_bij_inv)
done

```

```

lemma bij_project: "bij h ==> bij (project (%(x,u::'c). h x) UNIV)"
apply (subst extend_inv [symmetric])
apply (auto simp add: bij_imp_bij_inv bij_extend)
done

```

```

lemma inv_project_eq:
  "bij h
   ==> inv (project (%(x,u::'c). h x) UNIV) = extend (%(x,u::'c). h x)"
apply (rule inj_imp_inv_eq)
apply (erule bij_project [THEN bij_is_inj])
apply (simp (no_asm_simp) add: Extend.extend_inverse)
done

```

```

lemma Allowed_rename [simp]:
  "bij h ==> Allowed (rename h F) = rename h ' Allowed F"
apply (simp (no_asm_simp) add: rename_def Extend.Allowed_extend)
apply (subst bij_vimage_eq_inv_image)
apply (rule bij_project, blast)
apply (simp (no_asm_simp) add: inv_project_eq)
done

```

```

lemma bij_rename: "bij h ==> bij (rename h)"
apply (simp (no_asm_simp) add: rename_def bij_extend)
done
lemmas surj_rename = bij_rename [THEN bij_is_surj, standard]

```

```

lemma inj_rename_imp_inj: "inj (rename h) ==> inj h"
apply (unfold inj_on_def, auto)
apply (drule_tac x = "mk_program ({x}, {}, {})" in spec)
apply (drule_tac x = "mk_program ({y}, {}, {})" in spec)
apply (auto simp add: program_equality_iff rename_def extend_def)
done

```

```

lemma surj_rename_imp_surj: "surj (rename h) ==> surj h"
apply (unfold surj_def, auto)

```

```

apply (drule_tac x = "mk_program ({y}, {}, {})" in spec)
apply (auto simp add: program_equality_iff rename_def extend_def)
done

lemma bij_rename_imp_bij: "bij (rename h) ==> bij h"
apply (unfold bij_def)
apply (simp (no_asm_simp) add: inj_rename_imp_inj surj_rename_imp_surj)
done

lemma bij_rename_iff [simp]: "bij (rename h) = bij h"
by (blast intro: bij_rename bij_rename_imp_bij)

```

11.2 the lattice operations

```

lemma rename_SKIP [simp]: "bij h ==> rename h SKIP = SKIP"
by (simp add: rename_def Extend.extend_SKIP)

lemma rename_Join [simp]:
  "bij h ==> rename h (F Join G) = rename h F Join rename h G"
by (simp add: rename_def Extend.extend_Join)

lemma rename_JN [simp]:
  "bij h ==> rename h (JOIN I F) = (⋂ i ∈ I. rename h (F i))"
by (simp add: rename_def Extend.extend_JN)

```

11.3 Strong Safety: co, stable

```

lemma rename_constrains:
  "bij h ==> (rename h F ∈ (h'A) co (h'B)) = (F ∈ A co B)"
apply (unfold rename_def)
apply (subst extend_set_eq_image [symmetric])
apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_constrains])
done

lemma rename_stable:
  "bij h ==> (rename h F ∈ stable (h'A)) = (F ∈ stable A)"
apply (simp add: stable_def rename_constrains)
done

lemma rename_invariant:
  "bij h ==> (rename h F ∈ invariant (h'A)) = (F ∈ invariant A)"
apply (simp add: invariant_def rename_stable bij_is_inj [THEN inj_image_subset_iff])
done

lemma rename_increasing:
  "bij h ==> (rename h F ∈ increasing func) = (F ∈ increasing (func o h))"
apply (simp add: increasing_def rename_stable [symmetric] bij_image_Collect_eq
  bij_is_surj [THEN surj_f_inv_f])
done

```

11.4 Weak Safety: Co, Stable

```

lemma reachable_rename_eq:

```

```

    "bij h ==> reachable (rename h F) = h ` (reachable F)"
  apply (simp add: rename_def Extend.reachable_extend_eq)
done

lemma rename_Constrains:
  "bij h ==> (rename h F ∈ (h'A) Co (h'B)) = (F ∈ A Co B)"
by (simp add: Constrains_def reachable_rename_eq rename_constrains
    bij_is_inj image_Int [symmetric])

lemma rename_Stable:
  "bij h ==> (rename h F ∈ Stable (h'A)) = (F ∈ Stable A)"
by (simp add: Stable_def rename_Constrains)

lemma rename_Always: "bij h ==> (rename h F ∈ Always (h'A)) = (F ∈ Always
  A)"
by (simp add: Always_def rename_Stable bij_is_inj [THEN inj_image_subset_iff])

lemma rename_Increasing:
  "bij h ==> (rename h F ∈ Increasing func) = (F ∈ Increasing (func o
  h))"
by (simp add: Increasing_def rename_Stable [symmetric] bij_image_Collect_eq
    bij_is_surj [THEN surj_f_inv_f])

```

11.5 Progress: transient, ensures

```

lemma rename_transient:
  "bij h ==> (rename h F ∈ transient (h'A)) = (F ∈ transient A)"
  apply (unfold rename_def)
  apply (subst extend_set_eq_image [symmetric])
  apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_transient])
done

lemma rename Ensures:
  "bij h ==> (rename h F ∈ (h'A) ensures (h'B)) = (F ∈ A ensures B)"
  apply (unfold rename_def)
  apply (subst extend_set_eq_image [symmetric])
  apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend Ensures])
done

lemma rename_leadsTo:
  "bij h ==> (rename h F ∈ (h'A) leadsTo (h'B)) = (F ∈ A leadsTo B)"
  apply (unfold rename_def)
  apply (subst extend_set_eq_image [symmetric])
  apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_leadsTo])
done

lemma rename_LeadsTo:
  "bij h ==> (rename h F ∈ (h'A) LeadsTo (h'B)) = (F ∈ A LeadsTo B)"
  apply (unfold rename_def)
  apply (subst extend_set_eq_image [symmetric])
  apply (erule good_map_bij [THEN Extend.intro, THEN Extend.extend_LeadsTo])
done

```

```

lemma rename_rename_guarantees_eq:
  "bij h ==> (rename h F ∈ (rename h ' X) guarantees
              (rename h ' Y)) =
              (F ∈ X guarantees Y)"
apply (unfold rename_def)
apply (subst good_map_bij [THEN Extend.intro, THEN Extend.extend_guarantees_eq
[symmetric]], assumption)
apply (simp (no_asm_simp) add: fst_o_inv_eq_inv o_def)
done

lemma rename_guarantees_eq_rename_inv:
  "bij h ==> (rename h F ∈ X guarantees Y) =
              (F ∈ (rename (inv h) ' X) guarantees
              (rename (inv h) ' Y))"
apply (subst rename_rename_guarantees_eq [symmetric], assumption)
apply (simp add: image_eq_UN o_def bij_is_surj [THEN surj_f_inv_f])
done

lemma rename_preserves:
  "bij h ==> (rename h G ∈ preserves v) = (G ∈ preserves (v o h))"
apply (subst good_map_bij [THEN Extend.intro, THEN Extend.extend_preserves
[symmetric]], assumption)
apply (simp add: o_def fst_o_inv_eq_inv rename_def bij_is_surj [THEN surj_f_inv_f])
done

lemma ok_rename_iff [simp]: "bij h ==> (rename h F ok rename h G) = (F ok
G)"
by (simp add: Extend.ok_extend_iff rename_def)

lemma OK_rename_iff [simp]: "bij h ==> OK I (%i. rename h (F i)) = (OK I
F)"
by (simp add: Extend.OK_extend_iff rename_def)

```

11.6 "image" versions of the rules, for lifting "guarantees" properties

```

lemmas bij_eq_rename = surj_rename [THEN surj_f_inv_f, symmetric]

lemma rename_image_constrains:
  "bij h ==> rename h ' (A co B) = (h ' A) co (h ' B)"
apply auto
defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_constrains)
done

lemma rename_image_stable: "bij h ==> rename h ' stable A = stable (h ' A)"
apply auto
defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_stable)
done

```

```

lemma rename_image_increasing:
  "bij h ==> rename h ' increasing func = increasing (func o inv h)"
apply auto
  defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_increasing o_def bij_is_inj)

done

lemma rename_image_invariant:
  "bij h ==> rename h ' invariant A = invariant (h ' A)"
apply auto
  defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_invariant)
done

lemma rename_image_Constrains:
  "bij h ==> rename h ' (A Co B) = (h ' A) Co (h ' B)"
apply auto
  defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_Constrains)
done

lemma rename_image_preserves:
  "bij h ==> rename h ' preserves v = preserves (v o inv h)"
by (simp add: o_def rename_image_stable preserves_def bij_image_INT
    bij_image_Collect_eq)

lemma rename_image_Stable:
  "bij h ==> rename h ' Stable A = Stable (h ' A)"
apply auto
  defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_Stable)
done

lemma rename_image_Increasing:
  "bij h ==> rename h ' Increasing func = Increasing (func o inv h)"
apply auto
  defer 1
  apply (rename_tac F)
  apply (subgoal_tac "∃ G. F = rename h G")
  apply (auto intro!: bij_eq_rename simp add: rename_Increasing o_def bij_is_inj)
done

lemma rename_image_Always: "bij h ==> rename h ' Always A = Always (h ' A)"
apply auto

```

```

defer 1
apply (rename_tac F)
apply (subgoal_tac "∃ G. F = rename h G")
apply (auto intro!: bij_eq_rename simp add: rename_Always)
done

lemma rename_image_leadsTo:
  "bij h ==> rename h ` (A leadsTo B) = (h ` A) leadsTo (h ` B)"
apply auto
defer 1
apply (rename_tac F)
apply (subgoal_tac "∃ G. F = rename h G")
apply (auto intro!: bij_eq_rename simp add: rename_leadsTo)
done

lemma rename_image_LeadsTo:
  "bij h ==> rename h ` (A LeadsTo B) = (h ` A) LeadsTo (h ` B)"
apply auto
defer 1
apply (rename_tac F)
apply (subgoal_tac "∃ G. F = rename h G")
apply (auto intro!: bij_eq_rename simp add: rename_LeadsTo)
done

end

```

12 Replication of Components

theory *Lift_prog* imports *Rename* begin

constdefs

```

insert_map :: "[nat, 'b, nat=>'b] => (nat=>'b)"
  "insert_map i z f k == if k<i then f k
                        else if k=i then z
                        else f(k - 1)"

delete_map :: "[nat, nat=>'b] => (nat=>'b)"
  "delete_map i g k == if k<i then g k else g (Suc k)"

lift_map :: "[nat, 'b * ((nat=>'b) * 'c)] => (nat=>'b) * 'c"
  "lift_map i == %(s,(f,uu)). (insert_map i s f, uu)"

drop_map :: "[nat, (nat=>'b) * 'c] => 'b * ((nat=>'b) * 'c)"
  "drop_map i == %(g, uu). (g i, (delete_map i g, uu))"

lift_set :: "[nat, ('b * ((nat=>'b) * 'c)) set] => ((nat=>'b) * 'c) set"
  "lift_set i A == lift_map i ` A"

lift :: "[nat, ('b * ((nat=>'b) * 'c)) program] => ((nat=>'b) * 'c) program"
  "lift i == rename (lift_map i)"

```

```

sub :: "[ 'a, 'a=>'b ] => 'b"
      "sub == %i f. f i"

declare insert_map_def [simp] delete_map_def [simp]

lemma insert_map_inverse: "delete_map i (insert_map i x f) = f"
by (rule ext, simp)

lemma insert_map_delete_map_eq: "(insert_map i x (delete_map i g)) = g(i:=x)"
apply (rule ext)
apply (auto split add: nat_diff_split)
done

```

12.1 Injectiveness proof

```

lemma insert_map_inject1: "(insert_map i x f) = (insert_map i y g) ==> x=y"
by (drule_tac x = i in fun_cong, simp)

lemma insert_map_inject2: "(insert_map i x f) = (insert_map i y g) ==> f=g"
apply (drule_tac f = "delete_map i" in arg_cong)
apply (simp add: insert_map_inverse)
done

lemma insert_map_inject':
  "(insert_map i x f) = (insert_map i y g) ==> x=y & f=g"
by (blast dest: insert_map_inject1 insert_map_inject2)

lemmas insert_map_inject = insert_map_inject' [THEN conjE, elim!]

lemma lift_map_eq_iff [iff]:
  "(lift_map i (s, (f, uu)) = lift_map i' (s', (f', uu')))
  = (uu = uu' & insert_map i s f = insert_map i' s' f')"
by (unfold lift_map_def, auto)

```

```

lemma drop_map_lift_map_eq [simp]: "!!s. drop_map i (lift_map i s) = s"
apply (unfold lift_map_def drop_map_def)
apply (force intro: insert_map_inverse)
done

lemma inj_lift_map: "inj (lift_map i)"
apply (unfold lift_map_def)
apply (rule inj_onI, auto)
done

```

12.2 Surjectiveness proof

```

lemma lift_map_drop_map_eq [simp]: "!!s. lift_map i (drop_map i s) = s"
apply (unfold lift_map_def drop_map_def)
apply (force simp add: insert_map_delete_map_eq)
done

```

```

lemma drop_map_inject [dest!]: "(drop_map i s) = (drop_map i s') ==> s=s'"
by (drule_tac f = "lift_map i" in arg_cong, simp)

lemma surj_lift_map: "surj (lift_map i)"
apply (rule surjI)
apply (rule lift_map_drop_map_eq)
done

lemma bij_lift_map [iff]: "bij (lift_map i)"
by (simp add: bij_def inj_lift_map surj_lift_map)

lemma inv_lift_map_eq [simp]: "inv (lift_map i) = drop_map i"
by (rule inv_equality, auto)

lemma inv_drop_map_eq [simp]: "inv (drop_map i) = lift_map i"
by (rule inv_equality, auto)

lemma bij_drop_map [iff]: "bij (drop_map i)"
by (simp del: inv_lift_map_eq add: inv_lift_map_eq [symmetric] bij_imp_bij_inv)

lemma sub_apply [simp]: "sub i f = f i"
by (simp add: sub_def)

lemma all_total_lift: "all_total F ==> all_total (lift i F)"
by (simp add: lift_def rename_def Extend.all_total_extend)

lemma insert_map_upd_same: "(insert_map i t f)(i := s) = insert_map i s f"
by (rule ext, auto)

lemma insert_map_upd:
  "(insert_map j t f)(i := s) =
   (if i=j then insert_map i s f
    else if i<j then insert_map j t (f(i:=s))
    else insert_map j t (f(i - Suc 0 := s)))"
apply (rule ext)
apply (simp split add: nat_diff_split)

This simplification is VERY slow

done

lemma insert_map_eq_diff:
  "[| insert_map i s f = insert_map j t g; i≠j |]
   ==> ∃g'. insert_map i s' f = insert_map j t g'"
apply (subst insert_map_upd_same [symmetric])
apply (erule ssubst)
apply (simp only: insert_map_upd if_False split: split_if, blast)
done

lemma lift_map_eq_diff:
  "[| lift_map i (s,(f,uu)) = lift_map j (t,(g,vv)); i≠j |]
   ==> ∃g'. lift_map i (s',(f,uu)) = lift_map j (t,(g',vv))"
apply (unfold lift_map_def, auto)

```



```

apply (blast dest: insert_map_eq_diff)
done

```

12.3 The Operator `lift_set`

```

lemma lift_set_empty [simp]: "lift_set i {} = {}"
by (unfold lift_set_def, auto)

```

```

lemma lift_set_iff: "(lift_map i x ∈ lift_set i A) = (x ∈ A)"
apply (unfold lift_set_def)
apply (rule inj_lift_map [THEN inj_image_mem_iff])
done

```

```

lemma lift_set_iff2 [iff]:
  "((f,uu) ∈ lift_set i A) = ((f i, (delete_map i f, uu)) ∈ A)"
by (simp add: lift_set_def mem_rename_set_iff drop_map_def)

```

```

lemma lift_set_mono: "A ⊆ B ==> lift_set i A ⊆ lift_set i B"
apply (unfold lift_set_def)
apply (erule image_mono)
done

```

```

lemma lift_set_Un_distrib: "lift_set i (A ∪ B) = lift_set i A ∪ lift_set
i B"
by (simp add: lift_set_def image_Un)

```

```

lemma lift_set_Diff_distrib: "lift_set i (A-B) = lift_set i A - lift_set
i B"
apply (unfold lift_set_def)
apply (rule inj_lift_map [THEN image_set_diff])
done

```

12.4 The Lattice Operations

```

lemma bij_lift [iff]: "bij (lift i)"
by (simp add: lift_def)

```

```

lemma lift_SKIP [simp]: "lift i SKIP = SKIP"
by (simp add: lift_def)

```

```

lemma lift_Join [simp]: "lift i (F Join G) = lift i F Join lift i G"
by (simp add: lift_def)

```

```

lemma lift_JN [simp]: "lift j (JOIN I F) = (⋒ i ∈ I. lift j (F i))"
by (simp add: lift_def)

```

12.5 Safety: constrains, stable, invariant

```

lemma lift_constrains:
  "(lift i F ∈ (lift_set i A) co (lift_set i B)) = (F ∈ A co B)"
by (simp add: lift_def lift_set_def rename_constrains)

```

```

lemma lift_stable:
  "(lift i F ∈ stable (lift_set i A)) = (F ∈ stable A)"
by (simp add: lift_def lift_set_def rename_stable)

lemma lift_invariant:
  "(lift i F ∈ invariant (lift_set i A)) = (F ∈ invariant A)"
by (simp add: lift_def lift_set_def rename_invariant)

lemma lift_Constrains:
  "(lift i F ∈ (lift_set i A) Co (lift_set i B)) = (F ∈ A Co B)"
by (simp add: lift_def lift_set_def rename_Constrains)

lemma lift_Stable:
  "(lift i F ∈ Stable (lift_set i A)) = (F ∈ Stable A)"
by (simp add: lift_def lift_set_def rename_Stable)

lemma lift_Always:
  "(lift i F ∈ Always (lift_set i A)) = (F ∈ Always A)"
by (simp add: lift_def lift_set_def rename_Always)

```

12.6 Progress: transient, ensures

```

lemma lift_transient:
  "(lift i F ∈ transient (lift_set i A)) = (F ∈ transient A)"
by (simp add: lift_def lift_set_def rename_transient)

lemma lift Ensures:
  "(lift i F ∈ (lift_set i A) ensures (lift_set i B)) =
   (F ∈ A ensures B)"
by (simp add: lift_def lift_set_def rename Ensures)

lemma lift_leadsTo:
  "(lift i F ∈ (lift_set i A) leadsTo (lift_set i B)) =
   (F ∈ A leadsTo B)"
by (simp add: lift_def lift_set_def rename_leadsTo)

lemma lift_LeadsTo:
  "(lift i F ∈ (lift_set i A) LeadsTo (lift_set i B)) =
   (F ∈ A LeadsTo B)"
by (simp add: lift_def lift_set_def rename_LeadsTo)

lemma lift_lift_guarantees_eq:
  "(lift i F ∈ (lift i ' X) guarantees (lift i ' Y)) =
   (F ∈ X guarantees Y)"
apply (unfold lift_def)
apply (subst bij_lift_map [THEN rename_rename_guarantees_eq, symmetric])
apply (simp add: o_def)
done

lemma lift_guarantees_eq_lift_inv:
  "(lift i F ∈ X guarantees Y) =

```

```

(F ∈ (rename (drop_map i) ' X) guarantees (rename (drop_map i) ' Y))"
by (simp add: bij_lift_map [THEN rename_guarantees_eq_rename_inv] lift_def)

```

```

lemma lift_preserves_snd_I: "F ∈ preserves snd ==> lift i F ∈ preserves
snd"
apply (drule_tac w1=snd in subset_preserves_o [THEN subsetD])
apply (simp add: lift_def rename_preserves)
apply (simp add: lift_map_def o_def split_def del: split_comp_eq)
done

```

```

lemma delete_map_eqE':
  "(delete_map i g) = (delete_map i g') ==> ∃x. g = g'(i:=x)"
apply (drule_tac f = "insert_map i (g i) " in arg_cong)
apply (simp add: insert_map_delete_map_eq)
apply (erule exI)
done

```

```

lemmas delete_map_eqE = delete_map_eqE' [THEN exE, elim!]

```

```

lemma delete_map_neq_apply:
  "[| delete_map j g = delete_map j g'; i ≠ j |] ==> g i = g' i"
by force

```

```

lemma vimage_o_fst_eq [simp]: "(f o fst) -' A = (f-'A) <*> UNIV"
by auto

```

```

lemma vimage_sub_eq_lift_set [simp]:
  "(sub i -'A) <*> UNIV = lift_set i (A <*> UNIV)"
by auto

```

```

lemma mem_lift_act_iff [iff]:
  "((s,s') ∈ extend_act (λ(x,u::unit). lift_map i x) act) =
  ((drop_map i s, drop_map i s') ∈ act)"
apply (unfold extend_act_def, auto)
apply (rule bexI, auto)
done

```

```

lemma preserves_snd_lift_stable:
  "[| F ∈ preserves snd; i ≠ j |]
  ==> lift j F ∈ stable (lift_set i (A <*> UNIV))"
apply (auto simp add: lift_def lift_set_def stable_def constrains_def
  rename_def extend_def mem_rename_set_iff)
apply (auto dest!: preserves_imp_eq simp add: lift_map_def drop_map_def)
apply (drule_tac x = i in fun_cong, auto)
done

```

```

lemma constrains_imp_lift_constrains:
  "[| F i ∈ (A <*> UNIV) co (B <*> UNIV);
  F j ∈ preserves snd |]

```

```

    ==> lift j (F j) ∈ (lift_set i (A <*> UNIV)) co (lift_set i (B <*> UNIV))"
  apply (case_tac "i=j")
  apply (simp add: lift_def lift_set_def rename_constrains)
  apply (erule preserves_snd_lift_stable[THEN stableD, THEN constrains_weaken_R],
    assumption)
  apply (erule constrains_imp_subset [THEN lift_set_mono])
  done

```

```

lemma lift_map_image_Times:
  "lift_map i ' (A <*> UNIV) =
    (⋃ s ∈ A. ⋃ f. {insert_map i s f}) <*> UNIV"
  apply (auto intro!: bexI image_eqI simp add: lift_map_def)
  apply (rule split_conv [symmetric])
  done

```

```

lemma lift_preserves_eq:
  "(lift i F ∈ preserves v) = (F ∈ preserves (v o lift_map i))"
  by (simp add: lift_def rename_preserves)

```

```

lemma lift_preserves_sub:
  "F ∈ preserves snd
    ==> lift i F ∈ preserves (v o sub j o fst) =
      (if i=j then F ∈ preserves (v o fst) else True)"
  apply (drule subset_preserves_o [THEN subsetD])
  apply (simp add: lift_preserves_eq o_def drop_map_lift_map_eq)
  apply (auto cong del: if_weak_cong
    simp add: lift_map_def eq_commute split_def o_def simp del: split_comp_eq)
  done

```

12.7 Lemmas to Handle Function Composition (o) More Consistently

```

lemma o_equiv_assoc: "f o g = h ==> f' o f o g = f' o h"
  by (simp add: expand_fun_eq o_def)

```

```

lemma o_equiv_apply: "f o g = h ==> ∀ x. f(g x) = h x"
  by (simp add: expand_fun_eq o_def)

```

```

lemma fst_o_lift_map: "sub i o fst o lift_map i = fst"
  apply (rule ext)
  apply (auto simp add: o_def lift_map_def sub_def)
  done

```

```

lemma snd_o_lift_map: "snd o lift_map i = snd o snd"
  apply (rule ext)
  apply (auto simp add: o_def lift_map_def)
  done

```

12.8 More lemmas about extend and project

They could be moved to theory Extend or Project

```

lemma extend_act_extend_act:
  "extend_act h' (extend_act h act) =
   extend_act (%(x,(y,y')). h'(h(x,y),y')) act"
apply (auto elim!: rev_bexI simp add: extend_act_def, blast)
done

```

```

lemma project_act_project_act:
  "project_act h (project_act h' act) =
   project_act (%(x,(y,y')). h'(h(x,y),y')) act"
by (auto elim!: rev_bexI simp add: project_act_def)

```

```

lemma project_act_extend_act:
  "project_act h (extend_act h' act) =
   {(x,x').  $\exists s s' y y' z. (s,s') \in \text{act} \ \& \ h(x,y) = h'(s,z) \ \& \ h(x',y') = h'(s',z)}$ "
by (simp add: extend_act_def project_act_def, blast)

```

12.9 OK and "lift"

```

lemma act_in_UNION_preserves_fst:
  "act  $\subseteq \{(x,x'). \text{fst } x = \text{fst } x'\} \implies \text{act} \in \text{UNION } (\text{preserves fst}) \text{ Acts}"
apply (rule_tac a = "mk_program (UNIV,{act},UNIV) " in UN_I)
apply (auto simp add: preserves_def stable_def constrains_def)
done$ 
```

```

lemma UNION_OK_lift_I:
  "[|  $\forall i \in I. F i \in \text{preserves snd};$ 
    $\forall i \in I. \text{UNION } (\text{preserves fst}) \text{ Acts} \subseteq \text{AllowedActs } (F i)$  |]
  ==> OK I (%i. lift i (F i))"
apply (auto simp add: OK_def lift_def rename_def Extend.Acts_extend)
apply (simp add: Extend.AllowedActs_extend project_act_extend_act)
apply (rename_tac "act")
apply (subgoal_tac
  "{(x, x').  $\exists s f u s' f' u'.$ 
    $((s, f, u), s', f', u') \in \text{act} \ \& \$ 
    $\text{lift\_map } j \ x = \text{lift\_map } i \ (s, f, u) \ \& \$ 
    $\text{lift\_map } j \ x' = \text{lift\_map } i \ (s', f', u') \}$ 
    $\subseteq \{(x,x') . \text{fst } x = \text{fst } x'\})")
apply (blast intro: act_in_UNION_preserves_fst, clarify)
apply (drule_tac x = j in fun_cong)+
apply (drule_tac x = i in bspec, assumption)
apply (frule preserves_imp_eq, auto)
done$ 
```

```

lemma OK_lift_I:
  "[|  $\forall i \in I. F i \in \text{preserves snd};$ 
    $\forall i \in I. \text{preserves fst} \subseteq \text{Allowed } (F i)$  |]
  ==> OK I (%i. lift i (F i))"
by (simp add: safety_prop_AllowedActs_iff_Allowed UNION_OK_lift_I)

```

```

lemma Allowed_lift [simp]: "Allowed (lift i F) = lift i ` (Allowed F)"
by (simp add: lift_def Allowed_rename)

```

```

lemma lift_image_preserves:

```

```

    "lift i ' preserves v = preserves (v o drop_map i)"
  by (simp add: rename_image_preserves lift_def inv_lift_map_eq)

end

theory PPROD imports Lift_prog begin

constdefs

  PLam  :: "[nat set, nat => ('b * ((nat=>'b) * 'c)) program]
          => ((nat=>'b) * 'c) program"
  "PLam I F ==  $\bigsqcup_{i \in I. \text{lift } i (F i)}$ "

syntax
  "@PLam" :: "[pttrn, nat set, 'b set] => (nat => 'b) set"
  ("(3plam _:_./ _)" 10)

translations
  "plam x : A. B"    == "PLam A (%x. B)"

lemma Init_PLam [simp]: "Init (PLam I F) = ( $\bigcap_{i \in I. \text{lift\_set } i (\text{Init } (F i))}$ )"
  by (simp add: PLam_def lift_def lift_set_def)

lemma PLam_empty [simp]: "PLam {} F = SKIP"
  by (simp add: PLam_def)

lemma PLam_SKIP [simp]: "(plam i : I. SKIP) = SKIP"
  by (simp add: PLam_def lift_SKIP JN_constant)

lemma PLam_insert: "PLam (insert i I) F = (lift i (F i)) Join (PLam I F)"
  by (unfold PLam_def, auto)

lemma PLam_component_iff: " $((\text{PLam } I F) \leq H) = (\forall i \in I. \text{lift } i (F i) \leq H)$ "
  by (simp add: PLam_def JN_component_iff)

lemma component_PLam: " $i \in I \implies \text{lift } i (F i) \leq (\text{PLam } I F)$ "
  apply (unfold PLam_def)

apply (fast intro: component_JN)
done

lemma PLam_constrains:
  "[| i ∈ I;  ∀j. F j ∈ preserves snd |]
   ==> (PLam I F ∈ (lift_set i (A <*> UNIV)) co"

```

```

      (lift_set i (B <*> UNIV))) =
      (F i ∈ (A <*> UNIV) co (B <*> UNIV))"
apply (simp add: PLam_def JN_constrains)
apply (subst insert_Diff [symmetric], assumption)
apply (simp add: lift_constrains)
apply (blast intro: constrains_imp_lift_constrains)
done

lemma PLam_stable:
  "[| i ∈ I;  ∀j. F j ∈ preserves snd |]
   ==> (PLam I F ∈ stable (lift_set i (A <*> UNIV))) =
        (F i ∈ stable (A <*> UNIV))"
by (simp add: stable_def PLam_constrains)

lemma PLam_transient:
  "i ∈ I ==>
   PLam I F ∈ transient A = (∃i ∈ I. lift i (F i) ∈ transient A)"
by (simp add: JN_transient PLam_def)

This holds because the F j cannot change lift_set i

lemma PLam_ensures:
  "[| i ∈ I;  F i ∈ (A <*> UNIV) ensures (B <*> UNIV);
    ∀j. F j ∈ preserves snd |]
   ==> PLam I F ∈ lift_set i (A <*> UNIV) ensures lift_set i (B <*> UNIV)"
apply (simp add: ensures_def PLam_constrains PLam_transient
  lift_set_Un_distrib [symmetric] lift_set_Diff_distrib [symmetric]
  Times_Un_distrib1 [symmetric] Times_Diff_distrib1 [symmetric])
apply (rule rev_bexI, assumption)
apply (simp add: lift_transient)
done

lemma PLam_leadsTo_Basis:
  "[| i ∈ I;
    F i ∈ ((A <*> UNIV) - (B <*> UNIV)) co
          ((A <*> UNIV) ∪ (B <*> UNIV));
    F i ∈ transient ((A <*> UNIV) - (B <*> UNIV));
    ∀j. F j ∈ preserves snd |]
   ==> PLam I F ∈ lift_set i (A <*> UNIV) leadsTo lift_set i (B <*> UNIV)"
by (rule PLam_ensures [THEN leadsTo_Basis], rule_tac [2] ensuresI)

lemma invariant_imp_PLam_invariant:
  "[| F i ∈ invariant (A <*> UNIV);  i ∈ I;
    ∀j. F j ∈ preserves snd |]
   ==> PLam I F ∈ invariant (lift_set i (A <*> UNIV))"
by (auto simp add: PLam_stable invariant_def)

lemma PLam_preserves_fst [simp]:
  "∀j. F j ∈ preserves snd
   ==> (PLam I F ∈ preserves (v o sub j o fst)) ="

```

```

      (if j ∈ I then F j ∈ preserves (v o fst) else True)"
by (simp add: PLam_def lift_preserves_sub)

```

```

lemma PLam_preserves_snd [simp,intro]:
  "∀ j. F j ∈ preserves snd ==> PLam I F ∈ preserves snd"
by (simp add: PLam_def lift_preserves_snd_I)

```

This rule looks unsatisfactory because it refers to *lift*. One must use *lift_guarantees_eq_lift_inv* to rewrite the first subgoal and something like *lift_preserves_sub* to rewrite the third. However there's no obvious alternative for the third premise.

```

lemma guarantees_PLam_I:
  "[| lift i (F i): X guarantees Y; i ∈ I;
    OK I (%i. lift i (F i)) |]
   ==> (PLam I F) ∈ X guarantees Y"
apply (unfold PLam_def)
apply (simp add: guarantees_JN_I)
done

```

```

lemma Allowed_PLam [simp]:
  "Allowed (PLam I F) = (⋂ i ∈ I. lift i ' Allowed(F i))"
by (simp add: PLam_def)

```

```

lemma PLam_preserves [simp]:
  "(PLam I F) ∈ preserves v = (∀ i ∈ I. F i ∈ preserves (v o lift_map
i))"
by (simp add: PLam_def lift_def rename_preserves)

```

end

13 The Prefix Ordering on Lists

```

theory ListOrder
imports Main
begin

```

```

inductive_set
  genPrefix :: "('a * 'a)set => ('a list * 'a list)set"
  for r :: "('a * 'a)set"
where
  Nil:      "([], []) : genPrefix(r)"

| prepend: "[| (xs,ys) : genPrefix(r); (x,y) : r |] ==>
  (x#xs, y#ys) : genPrefix(r)"

```



```

| append: "(xs,ys) : genPrefix(r) ==> (xs, ys@zs) : genPrefix(r)"

instantiation list :: (type) ord
begin

definition
  prefix_def:          "xs <= zs ⟷ (xs, zs) : genPrefix Id"

definition
  strict_prefix_def: "xs < zs ⟷ xs ≤ zs ∧ ¬ zs ≤ (xs :: 'a list)"

instance ..

end

constdefs
  Le :: "(nat*nat) set"
  "Le == {(x,y). x <= y}"

  Ge :: "(nat*nat) set"
  "Ge == {(x,y). y <= x}"

abbreviation
  pfixLe :: "[nat list, nat list] => bool" (infixl "pfixLe" 50) where
  "xs pfixLe ys == (xs,ys) : genPrefix Le"

abbreviation
  pfixGe :: "[nat list, nat list] => bool" (infixl "pfixGe" 50) where
  "xs pfixGe ys == (xs,ys) : genPrefix Ge"

```

13.1 preliminary lemmas

```

lemma Nil_genPrefix [iff]: "([], xs) : genPrefix r"
by (cut_tac genPrefix.Nil [THEN genPrefix.append], auto)

lemma genPrefix_length_le: "(xs,ys) : genPrefix r ==> length xs <= length
ys"
by (erule genPrefix.induct, auto)

lemma cdlemma:
  "[| (xs', ys') : genPrefix r |]
   ==> (ALL x xs. xs' = x#xs --> (EX y ys. ys' = y#ys & (x,y) : r & (xs,
ys) : genPrefix r))"
apply (erule genPrefix.induct, blast, blast)
apply (force intro: genPrefix.append)
done

lemma cons_genPrefixE [elim!]:
  "[| (x#xs, zs) : genPrefix r;
   !!y ys. [| zs = y#ys; (x,y) : r; (xs, ys) : genPrefix r |] ==> P

```

```

    [] ==> P"
by (drule cdlemma, simp, blast)

lemma Cons_genPrefix_Cons [iff]:
  "((x#xs,y#ys) : genPrefix r) = ((x,y) : r & (xs,ys) : genPrefix r)"
by (blast intro: genPrefix.prepend)

```

13.2 genPrefix is a partial order

```

lemma refl_genPrefix: "refl r ==> refl (genPrefix r)"
apply (unfold refl_on_def, auto)
apply (induct_tac "x")
prefer 2 apply (blast intro: genPrefix.prepend)
apply (blast intro: genPrefix.Nil)
done

```

```

lemma genPrefix_refl [simp]: "refl r ==> (1,1) : genPrefix r"
by (erule refl_onD [OF refl_genPrefix UNIV_I])

```

```

lemma genPrefix_mono: "r<=s ==> genPrefix r <= genPrefix s"
apply clarify
apply (erule genPrefix.induct)
apply (auto intro: genPrefix.append)
done

```

```

lemma append_genPrefix [rule_format]:
  "ALL zs. (xs @ ys, zs) : genPrefix r --> (xs, zs) : genPrefix r"
by (induct_tac "xs", auto)

```

```

lemma genPrefix_trans_0 [rule_format]:
  "(x, y) : genPrefix r
  ==> ALL z. (y,z) : genPrefix s --> (x, z) : genPrefix (s 0 r)"
apply (erule genPrefix.induct)
  prefer 3 apply (blast dest: append_genPrefix)
  prefer 2 apply (blast intro: genPrefix.prepend, blast)
done

```

```

lemma genPrefix_trans [rule_format]:
  "[| (x,y) : genPrefix r; (y,z) : genPrefix r; trans r |]
  ==> (x,z) : genPrefix r"
apply (rule trans_0_subset [THEN genPrefix_mono, THEN subsetD])
  apply assumption
  apply (blast intro: genPrefix_trans_0)
done

```

```

lemma prefix_genPrefix_trans [rule_format]:
  "[| x<=y; (y,z) : genPrefix r |] ==> (x, z) : genPrefix r"
apply (unfold prefix_def)
apply (subst R_0_Id [symmetric], erule genPrefix_trans_0, assumption)

```

done

```
lemma genPrefix_prefix_trans [rule_format]:
  "[/ (x,y) : genPrefix r; y<=z [/ ==> (x,z) : genPrefix r"
apply (unfold prefix_def)
apply (subst Id_0_R [symmetric], erule genPrefix_trans_0, assumption)
done
```

```
lemma trans_genPrefix: "trans r ==> trans (genPrefix r)"
by (blast intro: transI genPrefix_trans)
```

```
lemma genPrefix_antisym [rule_format]:
  "[/ (xs,ys) : genPrefix r; antisym r [/
  ==> (ys,xs) : genPrefix r --> xs = ys"
apply (erule genPrefix.induct)
```

Base case

```
  apply blast
```

prepend case

```
  apply (simp add: antisym_def)
```

append case is the hardest

```
apply clarify
apply (subgoal_tac "length zs = 0", force)
apply (drule genPrefix_length_le)+
apply (simp del: length_0_conv)
done
```

```
lemma antisym_genPrefix: "antisym r ==> antisym (genPrefix r)"
by (blast intro: antisymI genPrefix_antisym)
```

13.3 recursion equations

```
lemma genPrefix_Nil [simp]: "((xs, []) : genPrefix r) = (xs = [])"
apply (induct_tac "xs")
prefer 2 apply blast
apply simp
done
```

```
lemma same_genPrefix_genPrefix [simp]:
  "refl r ==> ((xs@ys, xs@zs) : genPrefix r) = ((ys,zs) : genPrefix r)"
apply (unfold refl_on_def)
apply (induct_tac "xs")
apply (simp_all (no_asm_simp))
done
```

```
lemma genPrefix_Cons:
  "((xs, y#ys) : genPrefix r) =
  (xs=[] | (EX z zs. xs=z#zs & (z,y) : r & (zs,ys) : genPrefix r))"
by (case_tac "xs", auto)
```

```

lemma genPrefix_take_append:
  "[| refl r; (xs,ys) : genPrefix r |]
   ==> (xs@zs, take (length xs) ys @ zs) : genPrefix r"
apply (erule genPrefix.induct)
apply (frule_tac [3] genPrefix_length_le)
apply (simp_all (no_asm_simp) add: diff_is_0_eq [THEN iffD2])
done

```

```

lemma genPrefix_append_both:
  "[| refl r; (xs,ys) : genPrefix r; length xs = length ys |]
   ==> (xs@zs, ys @ zs) : genPrefix r"
apply (drule genPrefix_take_append, assumption)
apply (simp add: take_all)
done

```

```

lemma append_cons_eq: "xs @ y # ys = (xs @ [y]) @ ys"
by auto

```

```

lemma aolemma:
  "[| (xs,ys) : genPrefix r; refl r |]
   ==> length xs < length ys --> (xs @ [ys ! length xs], ys) : genPrefix
r"
apply (erule genPrefix.induct)
  apply blast
  apply simp

```

Append case is hardest

```

apply simp
apply (frule genPrefix_length_le [THEN le_imp_less_or_eq])
apply (erule disjE)
apply (simp_all (no_asm_simp) add: neq_Nil_conv nth_append)
apply (blast intro: genPrefix.append, auto)
apply (subst append_cons_eq, fast intro: genPrefix_append_both genPrefix.append)
done

```

```

lemma append_one_genPrefix:
  "[| (xs,ys) : genPrefix r; length xs < length ys; refl r |]
   ==> (xs @ [ys ! length xs], ys) : genPrefix r"
by (blast intro: aolemma [THEN mp])

```

```

lemma genPrefix_imp_nth [rule_format]:
  "ALL i ys. i < length xs
   --> (xs, ys) : genPrefix r --> (xs ! i, ys ! i) : r"
apply (induct_tac "xs", auto)
apply (case_tac "i", auto)
done

```

```

lemma nth_imp_genPrefix [rule_format]:

```

```

    "ALL ys. length xs <= length ys
    --> (ALL i. i < length xs --> (xs ! i, ys ! i) : r)
    --> (xs, ys) : genPrefix r"
  apply (induct_tac "xs")
  apply (simp_all (no_asm_simp) add: less_Suc_eq_0_disj all_conj_distrib)
  apply clarify
  apply (case_tac "ys")
  apply (force+)
  done

lemma genPrefix_iff_nth:
  "((xs,ys) : genPrefix r) =
   (length xs <= length ys & (ALL i. i < length xs --> (xs!i, ys!i) : r))"
  apply (blast intro: genPrefix_length_le genPrefix_imp_nth nth_imp_genPrefix)
  done

```

13.4 The type of lists is partially ordered

```

declare refl_Id [iff]
        antisym_Id [iff]
        trans_Id [iff]

lemma prefix_refl [iff]: "xs <= (xs::'a list)"
  by (simp add: prefix_def)

lemma prefix_trans: "!!xs::'a list. [| xs <= ys; ys <= zs |] ==> xs <= zs"
  apply (unfold prefix_def)
  apply (blast intro: genPrefix_trans)
  done

lemma prefix_antisym: "!!xs::'a list. [| xs <= ys; ys <= xs |] ==> xs = ys"
  apply (unfold prefix_def)
  apply (blast intro: genPrefix_antisym)
  done

lemma prefix_less_le_not_le: "!!xs::'a list. (xs < zs) = (xs <= zs & ¬ zs
  ≤ xs)"
  by (unfold strict_prefix_def, auto)

instance list :: (type) order
  by (intro_classes,
      (assumption | rule prefix_refl prefix_trans prefix_antisym
        prefix_less_le_not_le)+)

lemma set_mono: "xs <= ys ==> set xs <= set ys"
  apply (unfold prefix_def)
  apply (erule genPrefix.induct, auto)
  done

lemma Nil_prefix [iff]: "[ ] <= xs"

```

```

apply (unfold prefix_def)
apply (simp add: Nil_genPrefix)
done

lemma prefix_Nil [simp]: "(xs <= []) = (xs = [])"
apply (unfold prefix_def)
apply (simp add: genPrefix_Nil)
done

lemma Cons_prefix_Cons [simp]: "(x#xs <= y#ys) = (x=y & xs<=ys)"
by (simp add: prefix_def)

lemma same_prefix_prefix [simp]: "(xs@ys <= xs@zs) = (ys <= zs)"
by (simp add: prefix_def)

lemma append_prefix [iff]: "(xs@ys <= xs) = (ys <= [])"
by (insert same_prefix_prefix [of xs ys "[]", simp])

lemma prefix_appendI [simp]: "xs <= ys ==> xs <= ys@zs"
apply (unfold prefix_def)
apply (erule genPrefix.append)
done

lemma prefix_Cons:
  "(xs <= y#ys) = (xs=[] | (? zs. xs=y#zs & zs <= ys))"
by (simp add: prefix_def genPrefix_Cons)

lemma append_one_prefix:
  "[| xs <= ys; length xs < length ys |] ==> xs @ [ys ! length xs] <= ys"
apply (unfold prefix_def)
apply (simp add: append_one_genPrefix)
done

lemma prefix_length_le: "xs <= ys ==> length xs <= length ys"
apply (unfold prefix_def)
apply (erule genPrefix_length_le)
done

lemma spllemma: "xs<=ys ==> xs~=ys --> length xs < length ys"
apply (unfold prefix_def)
apply (erule genPrefix.induct, auto)
done

lemma strict_prefix_length_less: "xs < ys ==> length xs < length ys"
apply (unfold strict_prefix_def)
apply (blast intro: spllemma [THEN mp])
done

lemma mono_length: "mono length"
by (blast intro: monoI prefix_length_le)

lemma prefix_iff: "(xs <= zs) = (EX ys. zs = xs@ys)"
apply (unfold prefix_def)

```

```

apply (auto simp add: genPrefix_iff_nth nth_append)
apply (rule_tac x = "drop (length xs) zs" in exI)
apply (rule nth_equalityI)
apply (simp_all (no_asm_simp) add: nth_append)
done

lemma prefix_snoc [simp]: "(xs <= ys@[y]) = (xs = ys@[y] | xs <= ys)"
apply (simp add: prefix_iff)
apply (rule iffI)
  apply (erule exE)
  apply (rename_tac "zs")
  apply (rule_tac xs = zs in rev_exhaust)
  apply simp
  apply clarify
  apply (simp del: append_assoc add: append_assoc [symmetric], force)
done

lemma prefix_append_iff:
  "(xs <= ys@zs) = (xs <= ys | (? us. xs = ys@us & us <= zs))"
apply (rule_tac xs = zs in rev_induct)
  apply force
  apply (simp del: append_assoc add: append_assoc [symmetric], force)
done

lemma common_prefix_linear [rule_format]:
  "!!zs::'a list. xs <= zs --> ys <= zs --> xs <= ys | ys <= xs"
by (rule_tac xs = zs in rev_induct, auto)

```

13.5 *pfifLe, pfifGe: properties inherited from the translations*

```

lemma refl_Le [iff]: "refl Le"
by (unfold refl_on_def Le_def, auto)

lemma antisym_Le [iff]: "antisym Le"
by (unfold antisym_def Le_def, auto)

lemma trans_Le [iff]: "trans Le"
by (unfold trans_def Le_def, auto)

lemma pfifLe_refl [iff]: "x pfifLe x"
by simp

lemma pfifLe_trans: "[| x pfifLe y; y pfifLe z |] ==> x pfifLe z"
by (blast intro: genPrefix_trans)

lemma pfifLe_antisym: "[| x pfifLe y; y pfifLe x |] ==> x = y"
by (blast intro: genPrefix_antisym)

lemma prefix_imp_pfifLe: "xs<=ys ==> xs pfifLe ys"
apply (unfold prefix_def Le_def)
apply (blast intro: genPrefix_mono [THEN [2] rev_subsetD])
done

```

```

lemma refl_Ge [iff]: "refl Ge"
by (unfold refl_on_def Ge_def, auto)

lemma antisym_Ge [iff]: "antisym Ge"
by (unfold antisym_def Ge_def, auto)

lemma trans_Ge [iff]: "trans Ge"
by (unfold trans_def Ge_def, auto)

lemma pfixGe_refl [iff]: "x pfixGe x"
by simp

lemma pfixGe_trans: "[| x pfixGe y; y pfixGe z |] ==> x pfixGe z"
by (blast intro: genPrefix_trans)

lemma pfixGe_antisym: "[| x pfixGe y; y pfixGe x |] ==> x = y"
by (blast intro: genPrefix_antisym)

lemma prefix_imp_pfixGe: "xs<=ys ==> xs pfixGe ys"
apply (unfold prefix_def Ge_def)
apply (blast intro: genPrefix_mono [THEN [2] rev_subsetD])
done

end

```

14 Multisets

```

theory Multiset
imports List Main
begin

```

14.1 The type of multisets

```

typedef 'a multiset = "{f::'a => nat. finite {x . f x > 0}}"
proof
  show "(\x. 0::nat) \in ?multiset" by simp
qed

lemmas multiset_typedef [simp] =
  Abs_multiset_inverse Rep_multiset_inverse Rep_multiset
  and [simp] = Rep_multiset_inject [symmetric]

definition Mempty :: "'a multiset" ("{#}") where
  [code del]: "{#} = Abs_multiset (\a. 0)"

definition single :: "'a => 'a multiset" where
  [code del]: "single a = Abs_multiset (\b. if b = a then 1 else 0)"

definition count :: "'a multiset => 'a => nat" where
  "count = Rep_multiset"

```



```

definition MCollect :: "'a multiset => ('a => bool) => 'a multiset" where
  "MCollect M P = Abs_multiset ( $\lambda x$ . if P x then Rep_multiset M x else 0)"

abbreviation Melem :: "'a => 'a multiset => bool"  ("(_/ :# _)" [50, 51] 50)
where
  "a :# M == 0 < count M a"

notation (xsymbols)
  Melem (infix "∈#" 50)

syntax
  "_MCollect" :: "pttrn => 'a multiset => bool => 'a multiset"  ("(1{# _
  :# _./ _#})")
translations
  "{#x :# M. P#}" == "CONST MCollect M ( $\lambda x$ . P)"

definition set_of :: "'a multiset => 'a set" where
  "set_of M = {x. x :# M}"

instantiation multiset :: (type) "{plus, minus, zero, size}"
begin

definition union_def [code del]:
  "M + N = Abs_multiset ( $\lambda a$ . Rep_multiset M a + Rep_multiset N a)"

definition diff_def [code del]:
  "M - N = Abs_multiset ( $\lambda a$ . Rep_multiset M a - Rep_multiset N a)"

definition Zero_multiset_def [simp]:
  "0 = {#}"

definition size_def:
  "size M = setsum (count M) (set_of M)"

instance ..

end

definition
  multiset_inter :: "'a multiset  $\Rightarrow$  'a multiset  $\Rightarrow$  'a multiset" (infixl "#∩"
  70) where
  "multiset_inter A B = A - (A - B)"

Multiset Enumeration

syntax
  "_multiset" :: "args => 'a multiset"  ("({#(_)#}")
translations
  "{#x, xs#}" == "{#x#} + {#xs#}"
  "{#x#}" == "CONST single x"

Preservation of the representing set multiset.

lemma const0_in_multiset: "( $\lambda a$ . 0) ∈ multiset"
by (simp add: multiset_def)

```

```

lemma only1_in_multiset: "(λb. if b = a then 1 else 0) ∈ multiset"
by (simp add: multiset_def)

lemma union_preserves_multiset:
  "M ∈ multiset ==> N ∈ multiset ==> (λa. M a + N a) ∈ multiset"
by (simp add: multiset_def)

lemma diff_preserves_multiset:
  "M ∈ multiset ==> (λa. M a - N a) ∈ multiset"
apply (simp add: multiset_def)
apply (rule finite_subset)
apply auto
done

lemma MCollect_preserves_multiset:
  "M ∈ multiset ==> (λx. if P x then M x else 0) ∈ multiset"
apply (simp add: multiset_def)
apply (rule finite_subset, auto)
done

lemmas in_multiset = const0_in_multiset only1_in_multiset
  union_preserves_multiset diff_preserves_multiset MCollect_preserves_multiset

```

14.2 Algebraic properties

14.2.1 Union

```

lemma union_empty [simp]: "M + {#} = M ∧ {#} + M = M"
by (simp add: union_def Mempty_def in_multiset)

lemma union_commute: "M + N = N + (M::'a multiset)"
by (simp add: union_def add_ac in_multiset)

lemma union_assoc: "(M + N) + K = M + (N + (K::'a multiset))"
by (simp add: union_def add_ac in_multiset)

lemma union_lcomm: "M + (N + K) = N + (M + (K::'a multiset))"
proof -
  have "M + (N + K) = (N + K) + M" by (rule union_commute)
  also have "... = N + (K + M)" by (rule union_assoc)
  also have "K + M = M + K" by (rule union_commute)
  finally show ?thesis .
qed

lemmas union_ac = union_assoc union_commute union_lcomm

instance multiset :: (type) comm_monoid_add
proof
  fix a b c :: "'a multiset"
  show "(a + b) + c = a + (b + c)" by (rule union_assoc)
  show "a + b = b + a" by (rule union_commute)
  show "0 + a = a" by simp

```

qed

14.2.2 Difference

```
lemma diff_empty [simp]: "M - {#} = M ∧ {#} - M = {#}"
by (simp add: Mempty_def diff_def in_multiset)
```

```
lemma diff_union_inverse2 [simp]: "M + {#a#} - {#a#} = M"
by (simp add: union_def diff_def in_multiset)
```

```
lemma diff_cancel: "A - A = {#}"
by (simp add: diff_def Mempty_def)
```

14.2.3 Count of elements

```
lemma count_empty [simp]: "count {#} a = 0"
by (simp add: count_def Mempty_def in_multiset)
```

```
lemma count_single [simp]: "count {#b#} a = (if b = a then 1 else 0)"
by (simp add: count_def single_def in_multiset)
```

```
lemma count_union [simp]: "count (M + N) a = count M a + count N a"
by (simp add: count_def union_def in_multiset)
```

```
lemma count_diff [simp]: "count (M - N) a = count M a - count N a"
by (simp add: count_def diff_def in_multiset)
```

```
lemma count_MCollect [simp]:
  "count {# x:#M. P x #} a = (if P a then count M a else 0)"
by (simp add: count_def MCollect_def in_multiset)
```

14.2.4 Set of elements

```
lemma set_of_empty [simp]: "set_of {#} = {}"
by (simp add: set_of_def)
```

```
lemma set_of_single [simp]: "set_of {#b#} = {b}"
by (simp add: set_of_def)
```

```
lemma set_of_union [simp]: "set_of (M + N) = set_of M ∪ set_of N"
by (auto simp add: set_of_def)
```

```
lemma set_of_eq_empty_iff [simp]: "(set_of M = {}) = (M = {#})"
by (auto simp: set_of_def Mempty_def in_multiset count_def expand_fun_eq [where
  f="Rep_multiset M"])
```

```
lemma mem_set_of_iff [simp]: "(x ∈ set_of M) = (x :# M)"
by (auto simp add: set_of_def)
```

```
lemma set_of_MCollect [simp]: "set_of {# x:#M. P x #} = set_of M ∩ {x. P
  x}"
by (auto simp add: set_of_def)
```

14.2.5 Size

```

lemma size_empty [simp]: "size {#} = 0"
by (simp add: size_def)

lemma size_single [simp]: "size {#b#} = 1"
by (simp add: size_def)

lemma finite_set_of [iff]: "finite (set_of M)"
using Rep_multiset [of M] by (simp add: multiset_def set_of_def count_def)

lemma setsum_count_Int:
  "finite A ==> setsum (count N) (A ∩ set_of N) = setsum (count N) A"
apply (induct rule: finite_induct)
  apply simp
  apply (simp add: Int_insert_left set_of_def)
done

lemma size_union [simp]: "size (M + N::'a multiset) = size M + size N"
apply (unfold size_def)
  apply (subgoal_tac "count (M + N) = (λa. count M a + count N a)")
  prefer 2
  apply (rule ext, simp)
  apply (simp (no_asm_simp) add: setsum_Un_nat setsum_addf setsum_count_Int)
  apply (subst Int_commute)
  apply (simp (no_asm_simp) add: setsum_count_Int)
done

lemma size_eq_0_iff_empty [iff]: "(size M = 0) = (M = {#})"
apply (unfold size_def Mempty_def count_def, auto simp: in_multiset)
  apply (simp add: set_of_def count_def in_multiset expand_fun_eq)
done

lemma nonempty_has_size: "(S ≠ {#}) = (0 < size S)"
by (metis gr0I gr_implies_not0 size_empty size_eq_0_iff_empty)

lemma size_eq_Suc_imp_elem: "size M = Suc n ==> ∃a. a :# M"
apply (unfold size_def)
  apply (drule setsum_SucD)
  apply auto
done

```

14.2.6 Equality of multisets

```

lemma multiset_eq_conv_count_eq: "(M = N) = (∀a. count M a = count N a)"
by (simp add: count_def expand_fun_eq)

lemma single_not_empty [simp]: "{#a#} ≠ {#} ∧ {#} ≠ {#a#}"
by (simp add: single_def Mempty_def in_multiset expand_fun_eq)

lemma single_eq_single [simp]: "{#a#} = {#b#} = (a = b)"
by (auto simp add: single_def in_multiset expand_fun_eq)

lemma union_eq_empty [iff]: "(M + N = {#}) = (M = {#} ∧ N = {#})"
by (auto simp add: union_def Mempty_def in_multiset expand_fun_eq)

```

```

lemma empty_eq_union [iff]: "({#} = M + N) = (M = {#} ∧ N = {#})"
by (auto simp add: union_def Mempty_def in_multiset expand_fun_eq)

lemma union_right_cancel [simp]: "(M + K = N + K) = (M = (N::'a multiset))"
by (simp add: union_def in_multiset expand_fun_eq)

lemma union_left_cancel [simp]: "(K + M = K + N) = (M = (N::'a multiset))"
by (simp add: union_def in_multiset expand_fun_eq)

lemma union_is_single:
  "(M + N = {#a#}) = (M = {#a#} ∧ N = {#} ∨ M = {#} ∧ N = {#a#})"
apply (simp add: Mempty_def single_def union_def in_multiset add_is_1 expand_fun_eq)
apply blast
done

lemma single_is_union:
  "({#a#} = M + N) ↔ ({#a#} = M ∧ N = {#} ∨ M = {#} ∧ {#a#} = N)"
apply (unfold Mempty_def single_def union_def)
apply (simp add: add_is_1 one_is_add in_multiset expand_fun_eq)
apply (blast dest: sym)
done

lemma add_eq_conv_diff:
  "(M + {#a#} = N + {#b#}) =
    (M = N ∧ a = b ∨ M = N - {#a#} + {#b#} ∧ N = M - {#b#} + {#a#})"
using [[simproc del: neq]]
apply (unfold single_def union_def diff_def)
apply (simp (no_asm) add: in_multiset expand_fun_eq)
apply (rule conjI, force, safe, simp_all)
apply (simp add: eq_sym_conv)
done

declare Rep_multiset_inject [symmetric, simp del]

instance multiset :: (type) cancel_ab_semigroup_add
proof
  fix a b c :: "'a multiset"
  show "a + b = a + c ⟹ b = c" by simp
qed

lemma insert_DiffM:
  "x ∈# M ⟹ {#x#} + (M - {#x#}) = M"
by (clarsimp simp: multiset_eq_conv_count_eq)

lemma insert_DiffM2[simp]:
  "x ∈# M ⟹ M - {#x#} + {#x#} = M"
by (clarsimp simp: multiset_eq_conv_count_eq)

lemma multi_union_self_other_eq:
  "(A::'a multiset) + X = A + Y ⟹ X = Y"
by (induct A arbitrary: X Y) auto

lemma multi_self_add_other_not_self[simp]: "(A = A + {#x#}) = False"

```

by (metis single_not_empty union_empty union_left_cancel)

```
lemma insert_noteq_member:
  assumes BC: "B + {#b#} = C + {#c#}"
  and bnotc: "b ≠ c"
  shows "c ∈# B"
proof -
  have "c ∈# C + {#c#}" by simp
  have nc: "¬ c ∈# {#b#}" using bnotc by simp
  then have "c ∈# B + {#b#}" using BC by simp
  then show "c ∈# B" using nc by simp
qed
```

```
lemma add_eq_conv_ex:
  "(M + {#a#} = N + {#b#}) =
   (M = N ∧ a = b ∨ (∃ K. M = K + {#b#} ∧ N = K + {#a#}))"
by (auto simp add: add_eq_conv_diff)
```

```
lemma empty_multiset_count:
  "(∀ x. count A x = 0) = (A = {#})"
by (metis count_empty multiset_eq_conv_count_eq)
```

14.2.7 Intersection

```
lemma multiset_inter_count:
  "count (A #∩ B) x = min (count A x) (count B x)"
by (simp add: multiset_inter_def min_def)
```

```
lemma multiset_inter_commute: "A #∩ B = B #∩ A"
by (simp add: multiset_eq_conv_count_eq multiset_inter_count
  min_max.inf_commute)
```

```
lemma multiset_inter_assoc: "A #∩ (B #∩ C) = A #∩ B #∩ C"
by (simp add: multiset_eq_conv_count_eq multiset_inter_count
  min_max.inf_assoc)
```

```
lemma multiset_inter_left_commute: "A #∩ (B #∩ C) = B #∩ (A #∩ C)"
by (simp add: multiset_eq_conv_count_eq multiset_inter_count min_def)
```

```
lemmas multiset_inter_ac =
  multiset_inter_commute
  multiset_inter_assoc
  multiset_inter_left_commute
```

```
lemma multiset_inter_single: "a ≠ b ⇒ {#a#} #∩ {#b#} = {#}"
by (simp add: multiset_eq_conv_count_eq multiset_inter_count)
```

```
lemma multiset_union_diff_commute: "B #∩ C = {#} ⇒ A + B - C = A - C + B"
```

```
apply (simp add: multiset_eq_conv_count_eq multiset_inter_count min_def
  split_if_asm)
apply clarsimp
```

```

apply (erule_tac x = a in allE)
apply auto
done

```

14.2.8 Comprehension (filter)

```

lemma MCollect_empty [simp]: "MCollect {#} P = {#}"
by (simp add: MCollect_def Mempty_def Abs_multiset_inject
    in_multiset expand_fun_eq)

lemma MCollect_single [simp]:
  "MCollect {#x#} P = (if P x then {#x#} else {#})"
by (simp add: MCollect_def Mempty_def single_def Abs_multiset_inject
    in_multiset expand_fun_eq)

lemma MCollect_union [simp]:
  "MCollect (M+N) f = MCollect M f + MCollect N f"
by (simp add: MCollect_def union_def Abs_multiset_inject
    in_multiset expand_fun_eq)

```

14.3 Induction and case splits

```

lemma setsum_decr:
  "finite F ==> (0::nat) < f a ==>
    setsum (f (a := f a - 1)) F = (if a ∈ F then setsum f F - 1 else setsum
  f F)"
apply (induct rule: finite_induct)
apply auto
apply (drule_tac a = a in mk_disjoint_insert, auto)
done

lemma rep_multiset_induct_aux:
assumes 1: "P (λa. (0::nat))"
  and 2: "!!f b. f ∈ multiset ==> P f ==> P (f (b := f b + 1))"
shows "∀f. f ∈ multiset --> setsum f {x. f x ≠ 0} = n --> P f"
apply (unfold multiset_def)
apply (induct_tac n, simp, clarify)
  apply (subgoal_tac "f = (λa. 0)")
    apply simp
    apply (rule 1)
  apply (rule ext, force, clarify)
apply (frule setsum_SucD, clarify)
apply (rename_tac a)
apply (subgoal_tac "finite {x. (f (a := f a - 1)) x > 0}")
  prefer 2
  apply (rule finite_subset)
    prefer 2
    apply assumption
  apply simp
  apply blast
apply (subgoal_tac "f = (f (a := f a - 1))(a := (f (a := f a - 1)) a + 1)")
  prefer 2
  apply (rule ext)
  apply (simp (no_asm_simp))

```

```

    apply (erule ssubst, rule 2 [unfolded multiset_def], blast)
  apply (erule allE, erule impE, erule_tac [2] mp, blast)
  apply (simp (no_asm_simp) add: setsum_decr del: fun_upd_apply One_nat_def)
  apply (subgoal_tac "{x. x ≠ a --> f x ≠ 0} = {x. f x ≠ 0}")
    prefer 2
  apply blast
  apply (subgoal_tac "{x. x ≠ a ∧ f x ≠ 0} = {x. f x ≠ 0} - {a}")
    prefer 2
  apply blast
  apply (simp add: le_imp_diff_is_add setsum_diff1_nat cong: conj_cong)
done

theorem rep_multiset_induct:
  "f ∈ multiset ==> P (λa. 0) ==>
    (∀!f b. f ∈ multiset ==> P f ==> P (f (b := f b + 1))) ==> P f"
using rep_multiset_induct_aux by blast

theorem multiset_induct [case_names empty add, induct type: multiset]:
  assumes empty: "P {#}"
    and add: "∀!M x. P M ==> P (M + {#x#})"
  shows "P M"
proof -
  note defns = union_def single_def Mempty_def
  show ?thesis
    apply (rule Rep_multiset_inverse [THEN subst])
    apply (rule Rep_multiset [THEN rep_multiset_induct])
    apply (rule empty [unfolded defns])
    apply (subgoal_tac "f(b := f b + 1) = (λa. f a + (if a=b then 1 else 0))")
      prefer 2
    apply (simp add: expand_fun_eq)
    apply (erule ssubst)
    apply (erule Abs_multiset_inverse [THEN subst])
    apply (drule add [unfolded defns, simplified])
    apply (simp add: in_multiset)
  done
qed

lemma multi_nonempty_split: "M ≠ {#} ==> ∃ A a. M = A + {#a#}"
by (induct M) auto

lemma multiset_cases [cases type, case_names empty add]:
  assumes em: "M = {#} ==> P"
  assumes add: "∀!N x. M = N + {#x#} ==> P"
  shows "P"
proof (cases "M = {#}")
  assume "M = {#}" then show ?thesis using em by simp
next
  assume "M ≠ {#}"
  then obtain M' m where "M = M' + {#m#}"
    by (blast dest: multi_nonempty_split)
  then show ?thesis using add by simp
qed

lemma multi_member_split: "x ∈# M ==> ∃ A. M = A + {#x#}"

```



```

apply (cases M)
  apply simp
apply (rule_tac x="M - {#x#}" in exI, simp)
done

lemma multiset_partition: "M = {# x:#M. P x #} + {# x:#M. ¬ P x #}"
apply (subst multiset_eq_conv_count_eq)
apply auto
done

declare multiset_typedef [simp del]

lemma multi_drop_mem_not_eq: "c ∈# B ⟹ B - {#c#} ≠ B"
by (cases "B = {#}" ) (auto dest: multi_member_split)

```

14.4 Orderings

14.4.1 Well-foundedness

```

definition mult1 :: "('a × 'a) set => ('a multiset × 'a multiset) set" where
  [code del]: "mult1 r = {(N, M). ∃ a MO K. M = MO + {#a#} ∧ N = MO + K ∧
    (∀ b. b :# K --> (b, a) ∈ r)}"

definition mult :: "('a × 'a) set => ('a multiset × 'a multiset) set" where
  "mult r = (mult1 r)⁺"

lemma not_less_empty [iff]: "(M, {#}) ∉ mult1 r"
by (simp add: mult1_def)

lemma less_add: "(N, MO + {#a#}) ∈ mult1 r ==>
  (∃ M. (M, MO) ∈ mult1 r ∧ N = M + {#a#}) ∨
  (∃ K. (∀ b. b :# K --> (b, a) ∈ r) ∧ N = MO + K)"
(is "_ ⟹ ?case1 (mult1 r) ∨ ?case2")
proof (unfold mult1_def)
  let ?r = "λK a. ∀ b. b :# K --> (b, a) ∈ r"
  let ?R = "λN M. ∃ a MO K. M = MO + {#a#} ∧ N = MO + K ∧ ?r K a"
  let ?case1 = "?case1 {(N, M). ?R N M}"

  assume "(N, MO + {#a#}) ∈ {(N, M). ?R N M}"
  then have "∃ a' MO' K.
    MO + {#a#} = MO' + {#a'#} ∧ N = MO' + K ∧ ?r K a'" by simp
  then show "?case1 ∨ ?case2"
  proof (elim exE conjE)
    fix a' MO' K
    assume N: "N = MO' + K" and r: "?r K a'"
    assume "MO + {#a#} = MO' + {#a'#}"
    then have "MO = MO' ∧ a = a' ∨
      (∃ K'. MO = K' + {#a'#} ∧ MO' = K' + {#a#})"
    by (simp only: add_eq_conv_ex)
    then show ?thesis
  proof (elim disjE conjE exE)
    assume "MO = MO'" "a = a'"
    with N r have "?r K a ∧ N = MO + K" by simp
    then have ?case2 .. then show ?thesis ..
  end

```

```

next
  fix K'
  assume "MO' = K' + {#a#}"
  with N have n: "N = K' + K + {#a#}" by (simp add: union_ac)

  assume "MO = K' + {#a'#}"
  with r have "?R (K' + K) MO" by blast
  with n have ?case1 by simp then show ?thesis ..
qed
qed
qed

lemma all_accessible: "wf r ==>  $\forall M. M \in \text{acc} (\text{mult1 } r)$ "
proof
  let ?R = "mult1 r"
  let ?W = "acc ?R"
  {
    fix M MO a
    assume MO: "MO  $\in$  ?W"
    and wf_hyp: "!!b. (b, a)  $\in$  r ==> ( $\forall M \in ?W. M + \{#b\} \in ?W$ )"
    and acc_hyp: " $\forall M. (M, MO) \in ?R \rightarrow M + \{#a\} \in ?W$ "
    have "MO + {#a#}  $\in$  ?W"
    proof (rule accI [of "MO + {#a#}"])
      fix N
      assume "(N, MO + {#a#})  $\in$  ?R"
      then have "(( $\exists M. (M, MO) \in ?R \wedge N = M + \{#a\}$ )  $\vee$ 
        ( $\exists K. (\forall b. b :# K \rightarrow (b, a) \in r) \wedge N = MO + K))$ "
        by (rule less_add)
      then show "N  $\in$  ?W"
    proof (elim exE disjE conjE)
      fix M assume "(M, MO)  $\in$  ?R" and N: "N = M + {#a#}"
      from acc_hyp have "(M, MO)  $\in$  ?R  $\rightarrow M + \{#a\} \in ?W$ " ..
      from this and '(M, MO)  $\in$  ?R' have "M + {#a#}  $\in$  ?W" ..
      then show "N  $\in$  ?W" by (simp only: N)
    next
      fix K
      assume N: "N = MO + K"
      assume " $\forall b. b :# K \rightarrow (b, a) \in r$ "
      then have "MO + K  $\in$  ?W"
      proof (induct K)
        case empty
        from MO show "MO + {#}  $\in$  ?W" by simp
      next
        case (add K x)
        from add.prem1 have "(x, a)  $\in$  r" by simp
        with wf_hyp have " $\forall M \in ?W. M + \{#x\} \in ?W$ " by blast
        moreover from add have "MO + K  $\in$  ?W" by simp
        ultimately have "(MO + K) + {#x#}  $\in$  ?W" ..
        then show "MO + (K + {#x#})  $\in$  ?W" by (simp only: union_assoc)
      qed
      then show "N  $\in$  ?W" by (simp only: N)
    qed
  }
qed
qed
} note tedious_reasoning = this

```

```

assume wf: "wf r"
fix M
show "M ∈ ?W"
proof (induct M)
  show "{#} ∈ ?W"
  proof (rule accI)
    fix b assume "(b, {#}) ∈ ?R"
    with not_less_empty show "b ∈ ?W" by contradiction
  qed

fix M a assume "M ∈ ?W"
from wf have "∀M ∈ ?W. M + {#a#} ∈ ?W"
proof induct
  fix a
  assume r: "!!b. (b, a) ∈ r ==> (∀M ∈ ?W. M + {#b#} ∈ ?W)"
  show "∀M ∈ ?W. M + {#a#} ∈ ?W"
  proof
    fix M assume "M ∈ ?W"
    then show "M + {#a#} ∈ ?W"
      by (rule acc_induct) (rule tedious_reasoning [OF _ r])
  qed
qed
from this and 'M ∈ ?W' show "M + {#a#} ∈ ?W" ..
qed
qed

theorem wf_mult1: "wf r ==> wf (mult1 r)"
by (rule acc_wfI) (rule all_accessible)

theorem wf_mult: "wf r ==> wf (mult r)"
unfolding mult_def by (rule wf_trancl) (rule wf_mult1)

```

14.4.2 Closure-free presentation

```

lemma diff_union_single_conv: "a :# J ==> I + J - {#a#} = I + (J - {#a#})"
by (simp add: multiset_eq_conv_count_eq)

```

One direction.

```

lemma mult_implies_one_step:
  "trans r ==> (M, N) ∈ mult r ==>
    ∃ I J K. N = I + J ∧ M = I + K ∧ J ≠ {#} ∧
    (∀k ∈ set_of K. ∃ j ∈ set_of J. (k, j) ∈ r)"
apply (unfold mult_def mult1_def set_of_def)
apply (erule converse_trancl_induct, clarify)
  apply (rule_tac x = M0 in exI, simp, clarify)
apply (case_tac "a :# K")
  apply (rule_tac x = I in exI)
  apply (simp (no_asm))
  apply (rule_tac x = "(K - {#a#}) + Ka" in exI)
  apply (simp (no_asm_simp) add: union_assoc [symmetric])
  apply (drule_tac f = "λM. M - {#a#}" in arg_cong)
  apply (simp add: diff_union_single_conv)
  apply (simp (no_asm_use) add: trans_def)

```

```

    apply blast
  apply (subgoal_tac "a :# I")
  apply (rule_tac x = "I - {#a#}" in exI)
  apply (rule_tac x = "J + {#a#}" in exI)
  apply (rule_tac x = "K + Ka" in exI)
  apply (rule conjI)
  apply (simp add: multiset_eq_conv_count_eq split: nat_diff_split)
  apply (rule conjI)
  apply (drule_tac f = "\M. M - {#a#}" in arg_cong, simp)
  apply (simp add: multiset_eq_conv_count_eq split: nat_diff_split)
  apply (simp (no_asm_use) add: trans_def)
  apply blast
  apply (subgoal_tac "a :# (M0 + {#a#})")
  apply simp
  apply (simp (no_asm))
done

lemma elem_imp_eq_diff_union: "a :# M ==> M = M - {#a#} + {#a#}"
by (simp add: multiset_eq_conv_count_eq)

lemma size_eq_Suc_imp_eq_union: "size M = Suc n ==> ∃ a N. M = N + {#a#}"
apply (erule size_eq_Suc_imp_elem [THEN exE])
apply (drule elem_imp_eq_diff_union, auto)
done

lemma one_step_implies_mult_aux:
  "trans r ==>
    ∀ I J K. (size J = n ∧ J ≠ {#} ∧ (∀ k ∈ set_of K. ∃ j ∈ set_of J. (k,
j) ∈ r))
    --> (I + K, I + J) ∈ mult r"
  apply (induct_tac n, auto)
  apply (frule size_eq_Suc_imp_eq_union, clarify)
  apply (rename_tac "J'", simp)
  apply (erule notE, auto)
  apply (case_tac "J' = {#}")
  apply (simp add: mult_def)
  apply (rule r_into_trancl)
  apply (simp add: mult1_def set_of_def, blast)

Now we know  $J' \neq \{\# \}$ .

  apply (cut_tac M = K and P = "\x. (x, a) ∈ r" in multiset_partition)
  apply (erule_tac P = "∀ k ∈ set_of K. ?P k" in rev_mp)
  apply (erule ssubst)
  apply (simp add: Ball_def, auto)
  apply (subgoal_tac
    "((I + {# x :# K. (x, a) ∈ r #}) + {# x :# K. (x, a) ∉ r #},
    (I + {# x :# K. (x, a) ∈ r #}) + J') ∈ mult r")
  prefer 2
  apply force
  apply (simp (no_asm_use) add: union_assoc [symmetric] mult_def)
  apply (erule trancl_trans)
  apply (rule r_into_trancl)
  apply (simp add: mult1_def set_of_def)
  apply (rule_tac x = a in exI)

```

```

apply (rule_tac x = "I + J'" in exI)
apply (simp add: union_ac)
done

```

```

lemma one_step_implies_mult:
  "trans r ==> J ≠ {} ==> ∀ k ∈ set_of K. ∃ j ∈ set_of J. (k, j) ∈ r
    ==> (I + K, I + J) ∈ mult r"
using one_step_implies_mult_aux by blast

```

14.4.3 Partial-order properties

```

instantiation multiset :: (order) order
begin

```

```

definition less_multiset_def [code del]:
  "M' < M ⟷ (M', M) ∈ mult {(x', x). x' < x}"

```

```

definition le_multiset_def [code del]:
  "M' ≤ M ⟷ M' = M ∨ M' < (M::'a multiset)"

```

```

lemma trans_base_order: "trans {(x', x). x' < (x::'a::order)}"
unfolding trans_def by (blast intro: order_less_trans)

```

Irreflexivity.

```

lemma mult_irrefl_aux:
  "finite A ==> (∀ x ∈ A. ∃ y ∈ A. x < (y::'a::order)) ⟹ A = {}"
by (induct rule: finite_induct) (auto intro: order_less_trans)

```

```

lemma mult_less_not_refl: "¬ M < (M::'a::order multiset)"
apply (unfold less_multiset_def, auto)
apply (drule trans_base_order [THEN mult_implies_one_step], auto)
apply (drule finite_set_of [THEN mult_irrefl_aux [rule_format (no_asm)]]))
apply (simp add: set_of_eq_empty_iff)
done

```

```

lemma mult_less_irrefl [elim!]: "M < (M::'a::order multiset) ==> R"
using insert mult_less_not_refl by fast

```

Transitivity.

```

theorem mult_less_trans: "K < M ==> M < N ==> K < (N::'a::order multiset)"
unfolding less_multiset_def mult_def by (blast intro: trancl_trans)

```

Asymmetry.

```

theorem mult_less_not_sym: "M < N ==> ¬ N < (M::'a::order multiset)"
apply auto
apply (rule mult_less_not_refl [THEN notE])
apply (erule mult_less_trans, assumption)
done

```

```

theorem mult_less_asym:
  "M < N ==> (¬ P ==> N < (M::'a::order multiset)) ==> P"
using mult_less_not_sym by blast

```

```
theorem mult_le_refl [iff]: "M <= (M::'a::order multiset)"
unfolding le_multiset_def by auto
```

Anti-symmetry.

```
theorem mult_le_antisym:
  "M <= N ==> N <= M ==> M = (N::'a::order multiset)"
unfolding le_multiset_def by (blast dest: mult_less_not_sym)
```

Transitivity.

```
theorem mult_le_trans:
  "K <= M ==> M <= N ==> K <= (N::'a::order multiset)"
unfolding le_multiset_def by (blast intro: mult_less_trans)
```

```
theorem mult_less_le: "(M < N) = (M <= N ∧ M ≠ (N::'a::order multiset))"
unfolding le_multiset_def by auto
```

```
instance proof
qed (auto simp add: mult_less_le dest: mult_le_antisym elim: mult_le_trans)

end
```

14.4.4 Monotonicity of multiset union

```
lemma mult1_union:
  "(B, D) ∈ mult1 r ==> trans r ==> (C + B, C + D) ∈ mult1 r"
apply (unfold mult1_def)
apply auto
apply (rule_tac x = a in exI)
apply (rule_tac x = "C + M0" in exI)
apply (simp add: union_assoc)
done

lemma union_less_mono2: "B < D ==> C + B < C + (D::'a::order multiset)"
apply (unfold less_multiset_def mult_def)
apply (erule trancl_induct)
  apply (blast intro: mult1_union transI order_less_trans r_into_trancl)
  apply (blast intro: mult1_union transI order_less_trans r_into_trancl trancl_trans)
done

lemma union_less_mono1: "B < D ==> B + C < D + (C::'a::order multiset)"
apply (subst union_commute [of B C])
apply (subst union_commute [of D C])
apply (erule union_less_mono2)
done

lemma union_less_mono:
  "A < C ==> B < D ==> A + B < C + (D::'a::order multiset)"
by (blast intro!: union_less_mono1 union_less_mono2 mult_less_trans)

lemma union_le_mono:
  "A <= C ==> B <= D ==> A + B <= C + (D::'a::order multiset)"
unfolding le_multiset_def
by (blast intro: union_less_mono union_less_mono1 union_less_mono2)
```

```

lemma empty_leI [iff]: "{#} <= (M::'a::order multiset)"
apply (unfold le_multiset_def less_multiset_def)
apply (case_tac "M = {#}")
  prefer 2
  apply (subgoal_tac "{#} + {#}, {#} + M) ∈ mult (Collect (split op <))")
    prefer 2
    apply (rule one_step_implies_mult)
      apply (simp only: trans_def)
      apply auto
done

lemma union_upper1: "A <= A + (B::'a::order multiset)"
proof -
  have "A + {#} <= A + B" by (blast intro: union_le_mono)
  then show ?thesis by simp
qed

lemma union_upper2: "B <= A + (B::'a::order multiset)"
by (subst union_commute) (rule union_upper1)

instance multiset :: (order) pordered_ab_semigroup_add
apply intro_classes
apply (erule union_le_mono[OF mult_le_refl])
done

```

14.5 Link with lists

```

primrec multiset_of :: "'a list ⇒ 'a multiset" where
  "multiset_of [] = {#}" |
  "multiset_of (a # x) = multiset_of x + {# a #}"

lemma multiset_of_zero_iff[simp]: "(multiset_of x = {#}) = (x = [])"
by (induct x) auto

lemma multiset_of_zero_iff_right[simp]: "({#} = multiset_of x) = (x = [])"
by (induct x) auto

lemma set_of_multiset_of[simp]: "set_of(multiset_of x) = set x"
by (induct x) auto

lemma mem_set_multiset_eq: "x ∈ set xs = (x :# multiset_of xs)"
by (induct xs) auto

lemma multiset_of_append [simp]:
  "multiset_of (xs @ ys) = multiset_of xs + multiset_of ys"
by (induct xs arbitrary: ys) (auto simp: union_ac)

lemma surj_multiset_of: "surj multiset_of"
apply (unfold surj_def)
apply (rule allI)
apply (rule_tac M = y in multiset_induct)
  apply auto
  apply (rule_tac x = "x # xa" in exI)
  apply auto

```

done

lemma set_count_greater_0: "set x = {a. count (multiset_of x) a > 0}"
by (induct x) auto

lemma distinct_count_atmost_1:
"distinct x = (! a. count (multiset_of x) a = (if a ∈ set x then 1 else 0))"
apply (induct x, simp, rule iffI, simp_all)
apply (rule conjI)
apply (simp_all add: set_of_multiset_of [THEN sym] del: set_of_multiset_of)
apply (erule_tac x = a in allE, simp, clarify)
apply (erule_tac x = aa in allE, simp)
done

lemma multiset_of_eq_setD:
"multiset_of xs = multiset_of ys \implies set xs = set ys"
by (rule) (auto simp add: multiset_eq_conv_count_eq set_count_greater_0)

lemma set_eq_iff_multiset_of_eq_distinct:
"distinct x \implies distinct y \implies
(set x = set y) = (multiset_of x = multiset_of y)"
by (auto simp: multiset_eq_conv_count_eq distinct_count_atmost_1)

lemma set_eq_iff_multiset_of_remdups_eq:
"(set x = set y) = (multiset_of (remdups x) = multiset_of (remdups y))"
apply (rule iffI)
apply (simp add: set_eq_iff_multiset_of_eq_distinct [THEN iffD1])
apply (drule distinct_remdups [THEN distinct_remdups
[THEN set_eq_iff_multiset_of_eq_distinct [THEN iffD2]]])
apply simp
done

lemma multiset_of_compl_union [simp]:
"multiset_of [x ← xs. P x] + multiset_of [x ← xs. \neg P x] = multiset_of xs"
by (induct xs) (auto simp: union_ac)

lemma count_filter:
"count (multiset_of xs) x = length [y ← xs. y = x]"
by (induct xs) auto

lemma nth_mem_multiset_of: "i < length ls \implies (ls ! i) :# multiset_of ls"
apply (induct ls arbitrary: i)
apply simp
apply (case_tac i)
apply auto
done

lemma multiset_of_remove1: "multiset_of (remove1 a xs) = multiset_of xs - {#a#}"
by (induct xs) (auto simp add: multiset_eq_conv_count_eq)

lemma multiset_of_eq_length:
assumes "multiset_of xs = multiset_of ys"


```

shows "length xs = length ys"
using assms
proof (induct arbitrary: ys rule: length_induct)
  case (1 xs ys)
  show ?case
  proof (cases xs)
    case Nil with "1.prem" show ?thesis by simp
  next
    case (Cons x xs')
    note xCons = Cons
    show ?thesis
    proof (cases ys)
      case Nil
      with "1.prem" Cons show ?thesis by simp
    next
      case (Cons y ys')
      have x_in_ys: "x = y  $\vee$  x  $\in$  set ys'"
      proof (cases "x = y")
        case True then show ?thesis ..
      next
        case False
        from "1.prem" [symmetric] xCons Cons have "x :# multiset_of ys'
+ {#y#}" by simp
        with False show ?thesis by (simp add: mem_set_multiset_eq)
      qed
      from "1.hyps" have IH: "length xs' < length xs  $\longrightarrow$ 
        ( $\forall x$ . multiset_of xs' = multiset_of x  $\longrightarrow$  length xs' = length x)" by
blast
      from "1.prem" x_in_ys Cons xCons have "multiset_of xs' = multiset_of
(remove1 x (y#ys'))"
        apply -
        apply (simp add: multiset_of_remove1, simp only: add_eq_conv_diff)
        apply fastsimp
        done
      with IH xCons have IH': "length xs' = length (remove1 x (y#ys'))" by
fastsimp
      from x_in_ys have "x  $\neq$  y  $\implies$  length ys' > 0" by auto
      with Cons xCons x_in_ys IH' show ?thesis by (auto simp add: length_remove1)
    qed
  qed
qed

```

This lemma shows which properties suffice to show that a function f with $f\ xs = ys$ behaves like sort.

```

lemma properties_for_sort:
  "multiset_of ys = multiset_of xs  $\implies$  sorted ys  $\implies$  sort xs = ys"
proof (induct xs arbitrary: ys)
  case Nil then show ?case by simp
next
  case (Cons x xs)
  then have "x  $\in$  set ys"
  by (auto simp add: mem_set_multiset_eq intro!: ccontr)
  with Cons.prem Cons.hyps [of "remove1 x ys"] show ?case
  by (simp add: sorted_remove1 multiset_of_remove1 insort_remove1)

```

qed

14.6 Pointwise ordering induced by count

```
definition mset_le :: "'a multiset  $\Rightarrow$  'a multiset  $\Rightarrow$  bool" (infix " $\leq\#$ " 50)
where
```

```
[code del]: "A  $\leq\#$  B  $\longleftrightarrow$  ( $\forall a. \text{count } A \ a \leq \text{count } B \ a$ )"
```

```
definition mset_less :: "'a multiset  $\Rightarrow$  'a multiset  $\Rightarrow$  bool" (infix "<#" 50)
where
```

```
[code del]: "A <# B  $\longleftrightarrow$  A  $\leq\#$  B  $\wedge$  A  $\neq$  B"
```

```
notation mset_le (infix " $\subseteq\#$ " 50)
```

```
notation mset_less (infix " $\subset\#$ " 50)
```

```
lemma mset_le_refl[simp]: "A  $\leq\#$  A"
```

```
unfolding mset_le_def by auto
```

```
lemma mset_le_trans: "A  $\leq\#$  B  $\implies$  B  $\leq\#$  C  $\implies$  A  $\leq\#$  C"
```

```
unfolding mset_le_def by (fast intro: order_trans)
```

```
lemma mset_le_antisym: "A  $\leq\#$  B  $\implies$  B  $\leq\#$  A  $\implies$  A = B"
```

```
apply (unfold mset_le_def)
```

```
apply (rule multiset_eq_conv_count_eq [THEN iffD2])
```

```
apply (blast intro: order_antisym)
```

```
done
```

```
lemma mset_le_exists_conv: "(A  $\leq\#$  B) = ( $\exists C. B = A + C$ )"
```

```
apply (unfold mset_le_def, rule iffI, rule_tac x = "B - A" in exI)
```

```
apply (auto intro: multiset_eq_conv_count_eq [THEN iffD2])
```

```
done
```

```
lemma mset_le_mono_add_right_cancel[simp]: "(A + C  $\leq\#$  B + C) = (A  $\leq\#$  B)"
```

```
unfolding mset_le_def by auto
```

```
lemma mset_le_mono_add_left_cancel[simp]: "(C + A  $\leq\#$  C + B) = (A  $\leq\#$  B)"
```

```
unfolding mset_le_def by auto
```

```
lemma mset_le_mono_add: "[ A  $\leq\#$  B; C  $\leq\#$  D ]  $\implies$  A + C  $\leq\#$  B + D"
```

```
apply (unfold mset_le_def)
```

```
apply auto
```

```
apply (erule_tac x = a in allE)+
```

```
apply auto
```

```
done
```

```
lemma mset_le_add_left[simp]: "A  $\leq\#$  A + B"
```

```
unfolding mset_le_def by auto
```

```
lemma mset_le_add_right[simp]: "B  $\leq\#$  A + B"
```

```
unfolding mset_le_def by auto
```

```
lemma mset_le_single: "a :# B  $\implies$  {#a#}  $\leq\#$  B"
```

```
by (simp add: mset_le_def)
```

```
lemma multiset_diff_union_assoc: "C ≤# B ⇒ A + B - C = A + (B - C)"
by (simp add: multiset_eq_conv_count_eq mset_le_def)
```

```
lemma mset_le_multiset_union_diff_commute:
assumes "B ≤# A"
shows "A - B + C = A + C - B"
proof -
  from mset_le_exists_conv [of "B" "A"] assms have "∃ D. A = B + D" ..
  from this obtain D where "A = B + D" ..
  then show ?thesis
    apply simp
    apply (subst union_commute)
    apply (subst multiset_diff_union_assoc)
    apply simp
    apply (simp add: diff_cancel)
    apply (subst union_assoc)
    apply (subst union_commute [of "B" _])
    apply (subst multiset_diff_union_assoc)
    apply simp
    apply (simp add: diff_cancel)
  done
qed
```

```
lemma multiset_of_remdups_le: "multiset_of (remdups xs) ≤# multiset_of xs"
apply (induct xs)
apply auto
apply (rule mset_le_trans)
apply auto
done
```

```
lemma multiset_of_update:
  "i < length ls ⇒ multiset_of (ls[i := v]) = multiset_of ls - {#ls ! i#}
  + {#v#}"
proof (induct ls arbitrary: i)
  case Nil then show ?case by simp
next
  case (Cons x xs)
  show ?case
  proof (cases i)
    case 0 then show ?thesis by simp
  next
    case (Suc i')
    with Cons show ?thesis
      apply simp
      apply (subst union_assoc)
      apply (subst union_commute [where M = "{#v#}" and N = "{#x#}"])
      apply (subst union_assoc [symmetric])
      apply simp
      apply (rule mset_le_multiset_union_diff_commute)
      apply (simp add: mset_le_single nth_mem_multiset_of)
    done
  qed
qed
```

```

lemma multiset_of_swap:
  "i < length ls  $\implies$  j < length ls  $\implies$ 
    multiset_of (ls[j := ls ! i, i := ls ! j]) = multiset_of ls"
apply (case_tac "i = j")
  apply simp
  apply (simp add: multiset_of_update)
  apply (subst elem_imp_eq_diff_union[symmetric])
  apply (simp add: nth_mem_multiset_of)
  apply simp
done

interpretation mset_order: order "op  $\leq$ " "op <"
proof qed (auto intro: order.intro mset_le_refl mset_le_antisym
  mset_le_trans simp: mset_less_def)

interpretation mset_order_cancel_semigroup:
  pordered_cancel_ab_semigroup_add "op +" "op  $\leq$ " "op <"
proof qed (erule mset_le_mono_add [OF mset_le_refl])

interpretation mset_order_semigroup_cancel:
  pordered_ab_semigroup_add_imp_le "op +" "op  $\leq$ " "op <"
proof qed simp

lemma mset_lessD: "A  $\subset$  B  $\implies$  x  $\in$  A  $\implies$  x  $\in$  B"
apply (clarsimp simp: mset_le_def mset_less_def)
apply (erule_tac x=x in allE)
apply auto
done

lemma mset_leD: "A  $\subseteq$  B  $\implies$  x  $\in$  A  $\implies$  x  $\in$  B"
apply (clarsimp simp: mset_le_def mset_less_def)
apply (erule_tac x = x in allE)
apply auto
done

lemma mset_less_insertD: "(A + {x}  $\subset$  B)  $\implies$  (x  $\in$  B  $\wedge$  A  $\subset$  B)"
apply (rule conjI)
  apply (simp add: mset_lessD)
  apply (clarsimp simp: mset_le_def mset_less_def)
  apply safe
  apply (erule_tac x = a in allE)
  apply (auto split: split_if_asm)
done

lemma mset_le_insertD: "(A + {x}  $\subseteq$  B)  $\implies$  (x  $\in$  B  $\wedge$  A  $\subseteq$  B)"
apply (rule conjI)
  apply (simp add: mset_leD)
  apply (force simp: mset_le_def mset_less_def split: split_if_asm)
done

lemma mset_less_of_empty[simp]: "A  $\subset$  {} = False"
by (induct A) (auto simp: mset_le_def mset_less_def)

```

```

lemma multi_psub_of_add_self[simp]: "A  $\subset\#$  A + {#x#}"
by (auto simp: mset_le_def mset_less_def)

lemma multi_psub_self[simp]: "A  $\subset\#$  A = False"
by (auto simp: mset_le_def mset_less_def)

lemma mset_less_add_bothsides:
  "T + {#x#}  $\subset\#$  S + {#x#}  $\implies$  T  $\subset\#$  S"
by (auto simp: mset_le_def mset_less_def)

lemma mset_less_empty_nonempty: "{#}  $\subset\#$  S = (S  $\neq$  {#})"
by (auto simp: mset_le_def mset_less_def)

lemma mset_less_size: "A  $\subset\#$  B  $\implies$  size A < size B"
proof (induct A arbitrary: B)
  case (empty M)
  then have "M  $\neq$  {#}" by (simp add: mset_less_empty_nonempty)
  then obtain M' x where "M = M' + {#x#}"
    by (blast dest: multi_nonempty_split)
  then show ?case by simp
next
  case (add S x T)
  have IH: " $\bigwedge$  B. S  $\subset\#$  B  $\implies$  size S < size B" by fact
  have SxsubT: "S + {#x#}  $\subset\#$  T" by fact
  then have "x  $\in\#$  T" and "S  $\subset\#$  T" by (auto dest: mset_less_insertD)
  then obtain T' where T: "T = T' + {#x#}"
    by (blast dest: multi_member_split)
  then have "S  $\subset\#$  T'" using SxsubT
    by (blast intro: mset_less_add_bothsides)
  then have "size S < size T'" using IH by simp
  then show ?case using T by simp
qed

lemmas mset_less_trans = mset_order.less_trans

lemma mset_less_diff_self: "c  $\in\#$  B  $\implies$  B - {#c#}  $\subset\#$  B"
by (auto simp: mset_le_def mset_less_def multi_drop_mem_not_eq)

```

14.7 Strong induction and subset induction for multisets

Well-foundedness of proper subset operator:

proper multiset subset

definition

```

mset_less_rel :: "('a multiset * 'a multiset) set" where
  "mset_less_rel = {(A,B). A  $\subset\#$  B}"

```

lemma multiset_add_sub_el_shuffle:

```

assumes "c  $\in\#$  B" and "b  $\neq$  c"
shows "B - {#c#} + {#b#} = B + {#b#} - {#c#}"

```

proof -

```

from 'c  $\in\#$  B' obtain A where B: "B = A + {#c#}"
by (blast dest: multi_member_split)
have "A + {#b#} = A + {#b#} + {#c#} - {#c#}" by simp

```

```

then have "A + {#b#} = A + {#c#} + {#b#} - {#c#}"
  by (simp add: union_ac)
then show ?thesis using B by simp
qed

```

```

lemma wf_mset_less_rel: "wf mset_less_rel"
apply (unfold mset_less_rel_def)
apply (rule wf_measure [THEN wf_subset, where f1=size])
apply (clarsimp simp: measure_def inv_image_def mset_less_size)
done

```

The induction rules:

```

lemma full_multiset_induct [case_names less]:
assumes ih: " $\bigwedge B. \forall A. A \subseteq\# B \longrightarrow P A \Longrightarrow P B$ "
shows "P B"
apply (rule wf_mset_less_rel [THEN wf_induct])
apply (rule ih, auto simp: mset_less_rel_def)
done

```

```

lemma multi_subset_induct [consumes 2, case_names empty add]:
assumes "F  $\subseteq\#$  A"
  and empty: "P {#}"
  and insert: " $\bigwedge a F. a \in\# A \Longrightarrow P F \Longrightarrow P (F + \{a\})$ "
shows "P F"
proof -
  from 'F  $\subseteq\#$  A'
  show ?thesis
  proof (induct F)
    show "P {#}" by fact
  next
    fix x F
    assume P: "F  $\subseteq\#$  A  $\Longrightarrow$  P F" and i: "F + {#x#}  $\subseteq\#$  A"
    show "P (F + {#x#})"
    proof (rule insert)
      from i show "x  $\in\#$  A" by (auto dest: mset_le_insertD)
      from i have "F  $\subseteq\#$  A" by (auto dest: mset_le_insertD)
      with P show "P F" .
    qed
  qed
qed

```

A consequence: Extensionality.

```

lemma multi_count_eq: " $(\forall x. \text{count } A \ x = \text{count } B \ x) = (A = B)$ "
apply (rule iffI)
prefer 2
apply clarsimp
apply (induct A arbitrary: B rule: full_multiset_induct)
apply (rename_tac C)
apply (case_tac B rule: multiset_cases)
  apply (simp add: empty_multiset_count)
  apply simp
  apply (case_tac "x  $\in\#$  C")
  apply (force dest: multi_member_split)
  apply (erule_tac x = x in allE)

```

```

apply simp
done

```

```

lemmas multi_count_ext = multi_count_eq [THEN iffD1, rule_format]

```

14.8 The fold combinator

The intended behaviour is $\text{fold_mset } f \ z \ \{\#x_1, \dots, x_n\} = f \ x_1 \ (\dots (f \ x_n \ z) \dots)$ if f is associative-commutative.

The graph of fold_mset , z : the start element, f : folding function, A : the multiset, y : the result.

inductive

```

fold_msetG :: "('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a multiset ⇒ 'b ⇒ bool"
for f :: "'a ⇒ 'b ⇒ 'b"
and z :: 'b

```

where

```

emptyI [intro]: "fold_msetG f z {#} z"
| insertI [intro]: "fold_msetG f z A y ⇒ fold_msetG f z (A + {#x#}) (f x y)"

```

```

inductive_cases empty_fold_msetGE [elim!]: "fold_msetG f z {#} x"

```

```

inductive_cases insert_fold_msetGE: "fold_msetG f z (A + {#}) y"

```

definition

```

fold_mset :: "('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a multiset ⇒ 'b" where
"fold_mset f z A = (THE x. fold_msetG f z A x)"

```

lemma Diff1_fold_msetG:

```

"fold_msetG f z (A - {#x#}) y ⇒ x ∈# A ⇒ fold_msetG f z A (f x y)"

```

```

apply (frule_tac x = x in fold_msetG.insertI)

```

```

apply auto

```

```

done

```

lemma fold_msetG_nonempty: " $\exists x. \text{fold_msetG } f \ z \ A \ x$ "

```

apply (induct A)

```

```

  apply blast

```

```

  apply clarsimp

```

```

  apply (drule_tac x = x in fold_msetG.insertI)

```

```

  apply auto

```

```

done

```

lemma fold_mset_empty[simp]: " $\text{fold_mset } f \ z \ \{\#\} = z$ "

```

unfolding fold_mset_def by blast

```

locale left_commutative =

```

fixes f :: "'a ⇒ 'b ⇒ 'b"

```

```

assumes left_commute: "f x (f y z) = f y (f x z)"

```

begin

lemma fold_msetG_determ:

```

"fold_msetG f z A x ⇒ fold_msetG f z A y ⇒ y = x"

```

```

proof (induct arbitrary: x y z rule: full_multiset_induct)

```

```

  case (less M x1 x2 Z)

```

```

have IH: "∀ A. A ⊂# M →
  (∀ x x' x''. fold_msetG f x'' A x → fold_msetG f x'' A x'
   → x' = x)" by fact
have Mfoldx1: "fold_msetG f Z M x1" and Mfoldx2: "fold_msetG f Z M x2"
by fact+
show ?case
proof (rule fold_msetG.cases [OF Mfoldx1])
  assume "M = {#}" and "x1 = Z"
  then show ?case using Mfoldx2 by auto
next
fix B b u
assume "M = B + {#b#}" and "x1 = f b u" and Bu: "fold_msetG f Z B u"
then have MBb: "M = B + {#b#}" and x1: "x1 = f b u" by auto
show ?case
proof (rule fold_msetG.cases [OF Mfoldx2])
  assume "M = {#}" "x2 = Z"
  then show ?case using Mfoldx1 by auto
next
fix C c v
assume "M = C + {#c#}" and "x2 = f c v" and Cv: "fold_msetG f Z C
v"
then have MCc: "M = C + {#c#}" and x2: "x2 = f c v" by auto
then have CsubM: "C ⊂# M" by simp
from MBb have BsubM: "B ⊂# M" by simp
show ?case
proof cases
  assume "b=c"
  then moreover have "B = C" using MBb MCc by auto
  ultimately show ?thesis using Bu Cv x1 x2 CsubM IH by auto
next
  assume diff: "b ≠ c"
  let ?D = "B - {#c#}"
  have cinB: "c ∈# B" and binC: "b ∈# C" using MBb MCc diff
  by (auto intro: insert_noteq_member dest: sym)
  have "B - {#c#} ⊂# B" using cinB by (rule mset_less_diff_self)
  then have DsubM: "?D ⊂# M" using BsubM by (blast intro: mset_less_trans)
  from MBb MCc have "B + {#b#} = C + {#c#}" by blast
  then have [simp]: "B + {#b#} - {#c#} = C"
  using MBb MCc binC cinB by auto
  have B: "B = ?D + {#c#}" and C: "C = ?D + {#b#}"
  using MBb MCc diff binC cinB
  by (auto simp: multiset_add_sub_el_shuffle)
  then obtain d where Dfoldd: "fold_msetG f Z ?D d"
  using fold_msetG_nonempty by iprover
  then have "fold_msetG f Z B (f c d)" using cinB
  by (rule Diff1_fold_msetG)
  then have "f c d = u" using IH BsubM Bu by blast
  moreover
  have "fold_msetG f Z C (f b d)" using binC cinB diff Dfoldd
  by (auto simp: multiset_add_sub_el_shuffle
    dest: fold_msetG.insertI [where x=b])
  then have "f b d = v" using IH CsubM Cv by blast
  ultimately show ?thesis using x1 x2
  by (auto simp: left_commute)

```



```

      qed
    qed
  qed
qed

lemma fold_mset_insert_aux:
  "(fold_msetG f z (A + {#x#}) v) =
   (∃y. fold_msetG f z A y ∧ v = f x y)"
apply (rule iffI)
prefer 2
apply blast
apply (rule_tac A=A and f=f in fold_msetG_nonempty [THEN exE, standard])
apply (blast intro: fold_msetG_determ)
done

lemma fold_mset_equality: "fold_msetG f z A y  $\implies$  fold_mset f z A = y"
unfolding fold_mset_def by (blast intro: fold_msetG_determ)

lemma fold_mset_insert:
  "fold_mset f z (A + {#x#}) = f x (fold_mset f z A)"
apply (simp add: fold_mset_def fold_mset_insert_aux union_commute)
apply (rule the_equality)
apply (auto cong add: conj_cong
  simp add: fold_mset_def [symmetric] fold_mset_equality fold_msetG_nonempty)
done

lemma fold_mset_insert_idem:
  "fold_mset f z (A + {#a#}) = f a (fold_mset f z A)"
apply (simp add: fold_mset_def fold_mset_insert_aux)
apply (rule the_equality)
apply (auto cong add: conj_cong
  simp add: fold_mset_def [symmetric] fold_mset_equality fold_msetG_nonempty)
done

lemma fold_mset_commute: "f x (fold_mset f z A) = fold_mset f (f x z) A"
by (induct A) (auto simp: fold_mset_insert left_commute [of x])

lemma fold_mset_single [simp]: "fold_mset f z {#x#} = f x z"
using fold_mset_insert [of z "{#}"] by simp

lemma fold_mset_union [simp]:
  "fold_mset f z (A+B) = fold_mset f (fold_mset f z A) B"
proof (induct A)
  case empty then show ?case by simp
next
  case (add A x)
  have "A + {#x#} + B = (A+B) + {#x#}" by (simp add: union_ac)
  then have "fold_mset f z (A + {#x#} + B) = f x (fold_mset f z (A + B))"

    by (simp add: fold_mset_insert)
  also have "... = fold_mset f (fold_mset f z (A + {#x#})) B"
    by (simp add: fold_mset_commute[of x,symmetric] add fold_mset_insert)
  finally show ?case .
qed

```

```

lemma fold_mset_fusion:
  assumes "left_commutative g"
  shows "( $\bigwedge x y. h (g x y) = f x (h y)$ )  $\implies h (fold\_mset g w A) = fold\_mset f (h w) A$ " (is "PROP ?P")
proof -
  interpret left_commutative g by fact
  show "PROP ?P" by (induct A) auto
qed

```

```

lemma fold_mset_rec:
  assumes "a  $\in$  # A"
  shows "fold_mset f z A = f a (fold_mset f z (A - {#a#}))"
proof -
  from assms obtain A' where "A = A' + {#a#}"
  by (blast dest: multi_member_split)
  then show ?thesis by simp
qed

```

end

A note on code generation: When defining some function containing a sub-term `fold_mset F`, code generation is not automatic. When interpreting locale `left_commutative` with `F`, the would be code thms for `fold_mset` become thms like `fold_mset F z {#} = z` where `F` is not a pattern but contains defined symbols, i.e. is not a code thm. Hence a separate constant with its own code thms needs to be introduced for `F`. See the image operator below.

14.9 Image

```

definition [code del]:
  "image_mset f = fold_mset (op + o single o f) {#}"

```

```

interpretation image_left_comm: left_commutative "op + o single o f"
  proof qed (simp add: union_ac)

```

```

lemma image_mset_empty [simp]: "image_mset f {#} = {#}"
by (simp add: image_mset_def)

```

```

lemma image_mset_single [simp]: "image_mset f {#x#} = {#f x#}"
by (simp add: image_mset_def)

```

```

lemma image_mset_insert:
  "image_mset f (M + {#a#}) = image_mset f M + {#f a#}"
by (simp add: image_mset_def add_ac)

```

```

lemma image_mset_union [simp]:
  "image_mset f (M+N) = image_mset f M + image_mset f N"
apply (induct N)
  apply simp
  apply (simp add: union_assoc [symmetric] image_mset_insert)
done

```

```

lemma size_image_mset [simp]: "size (image_mset f M) = size M"

```

```
by (induct M) simp_all
```

```
lemma image_mset_is_empty_iff [simp]: "image_mset f M = {}  $\longleftrightarrow$  M = {}"
by (cases M) auto
```

```
syntax
```

```
comprehension1_mset :: "'a  $\Rightarrow$  'b  $\Rightarrow$  'b multiset  $\Rightarrow$  'a multiset"
  ("({#_/. _ :# _#})")
```

```
translations
```

```
"{#e. x:#M#}" == "CONST image_mset (%x. e) M"
```

```
syntax
```

```
comprehension2_mset :: "'a  $\Rightarrow$  'b  $\Rightarrow$  'b multiset  $\Rightarrow$  bool  $\Rightarrow$  'a multiset"
  ("({#_/. | _ :# _./ _#})")
```

```
translations
```

```
"{#e | x:#M. P#}" => "{#e. x :# {# x:#M. P#}#}"
```

This allows to write not just filters like $\{x : M. x < c\}$ but also images like $\{x + x. x : M\}$ and $\{x+x/x:#M. x<c\}$, where the latter is currently displayed as $\{x + x. x : \{x : M. x < c\}\#$.

14.10 Termination proofs with multiset orders

```
lemma multi_member_skip: "x  $\in$  XS  $\implies$  x  $\in$  {# y #} + XS"
  and multi_member_this: "x  $\in$  {# x #} + XS"
  and multi_member_last: "x  $\in$  {# x #}"
by auto
```

```
definition "ms_strict = mult pair_less"
```

```
definition [code del]: "ms_weak = ms_strict  $\cup$  Id"
```

```
lemma ms_reduction_pair: "reduction_pair (ms_strict, ms_weak)"
```

```
unfolding reduction_pair_def ms_strict_def ms_weak_def pair_less_def
by (auto intro: wf_mult1 wf_trancl simp: mult_def)
```

```
lemma smsI:
```

```
"(set_of A, set_of B)  $\in$  max_strict  $\implies$  (Z + A, Z + B)  $\in$  ms_strict"
  unfolding ms_strict_def
```

```
by (rule one_step_implies_mult) (auto simp add: max_strict_def pair_less_def
  elim!:max_ext.cases)
```

```
lemma wmsI:
```

```
"(set_of A, set_of B)  $\in$  max_strict  $\vee$  A = {#}  $\wedge$  B = {#}
 $\implies$  (Z + A, Z + B)  $\in$  ms_weak"
```

```
unfolding ms_weak_def ms_strict_def
```

```
by (auto simp add: pair_less_def max_strict_def elim!:max_ext.cases intro:
  one_step_implies_mult)
```

```
inductive pw_leq
```

```
where
```

```
  pw_leq_empty: "pw_leq {#} {#}"
```

```
  | pw_leq_step: "[ (x,y)  $\in$  pair_leq; pw_leq X Y ]  $\implies$  pw_leq ({#x#} + X) ({#y#} + Y)"
```

```

lemma pw_leq_lstep:
  "(x, y) ∈ pair_leq ⟹ pw_leq {#x#} {#y#}"
by (drule pw_leq_step) (rule pw_leq_empty, simp)

lemma pw_leq_split:
  assumes "pw_leq X Y"
  shows "∃ A B Z. X = A + Z ∧ Y = B + Z ∧ ((set_of A, set_of B) ∈ max_strict
  ∨ (B = {#} ∧ A = {#}))"
  using assms
proof (induct)
  case pw_leq_empty thus ?case by auto
next
  case (pw_leq_step x y X Y)
  then obtain A B Z where
    [simp]: "X = A + Z" "Y = B + Z"
    and 1[simp]: "(set_of A, set_of B) ∈ max_strict ∨ (B = {#} ∧ A = {#}))"

    by auto
  from pw_leq_step have "x = y ∨ (x, y) ∈ pair_less"
    unfolding pair_leq_def by auto
  thus ?case
  proof
    assume [simp]: "x = y"
    have
      "{#x#} + X = A + ({#y#}+Z)
      ∧ {#y#} + Y = B + ({#y#}+Z)
      ∧ ((set_of A, set_of B) ∈ max_strict ∨ (B = {#} ∧ A = {#}))"
      by (auto simp: add_ac)
    thus ?case by (intro exI)
  next
    assume A: "(x, y) ∈ pair_less"
    let ?A' = "{#x#} + A" and ?B' = "{#y#} + B"
    have "{#x#} + X = ?A' + Z"
      "{#y#} + Y = ?B' + Z"
      by (auto simp add: add_ac)
    moreover have
      "(set_of ?A', set_of ?B') ∈ max_strict"
      using 1 A unfolding max_strict_def
      by (auto elim!: max_ext.cases)
    ultimately show ?thesis by blast
  qed
qed

lemma
  assumes pwleq: "pw_leq Z Z'"
  shows ms_strictI: "(set_of A, set_of B) ∈ max_strict ⟹ (Z + A, Z' + B)
  ∈ ms_strict"
  and ms_weakI1: "(set_of A, set_of B) ∈ max_strict ⟹ (Z + A, Z' + B)
  ∈ ms_weak"
  and ms_weakI2: "(Z + {#}, Z' + {#}) ∈ ms_weak"
proof -
  from pw_leq_split[OF pwleq]
  obtain A' B' Z''
    where [simp]: "Z = A' + Z''" "Z' = B' + Z''"

```

```

    and mx_or_empty: "(set_of A', set_of B') ∈ max_strict ∨ (A' = {#} ∧
B' = {#})"
  by blast
{
  assume max: "(set_of A, set_of B) ∈ max_strict"
  from mx_or_empty
  have "(Z'' + (A + A'), Z'' + (B + B')) ∈ ms_strict"
  proof
    assume max': "(set_of A', set_of B') ∈ max_strict"
    with max have "(set_of (A + A'), set_of (B + B')) ∈ max_strict"
      by (auto simp: max_strict_def intro: max_ext_additive)
    thus ?thesis by (rule smsI)
  next
    assume [simp]: "A' = {#} ∧ B' = {#}"
    show ?thesis by (rule smsI) (auto intro: max)
  qed
  thus "(Z + A, Z' + B) ∈ ms_strict" by (simp add: add_ac)
  thus "(Z + A, Z' + B) ∈ ms_weak" by (simp add: ms_weak_def)
}
from mx_or_empty
have "(Z'' + A', Z'' + B') ∈ ms_weak" by (rule wmsI)
thus "(Z + {#}, Z' + {#}) ∈ ms_weak" by (simp add: add_ac)
qed

```

```

lemma empty_idemp: "{#} + x = x" "x + {#} = x"

```

```

and nonempty_plus: "{# x #} + rs ≠ {#}"

```

```

and nonempty_single: "{# x #} ≠ {#}"

```

```

by auto

```

```

setup {*

```

```

let

```

```

  fun msetT T = Type ("Multiset.multiset", [T]);

```

```

  fun mk_mset T [] = Const (@{const_name Mempty}, msetT T)

```

```

    | mk_mset T [x] = Const (@{const_name single}, T --> msetT T) $ x

```

```

    | mk_mset T (x :: xs) =

```

```

      Const (@{const_name plus}, msetT T --> msetT T --> msetT T) $

```

```

        mk_mset T [x] $ mk_mset T xs

```

```

  fun mset_member_tac m i =

```

```

    (if m <= 0 then

```

```

      rtac @{thm multi_member_this} i ORELSE rtac @{thm multi_member_last}

```

```

    i

```

```

    else

```

```

      rtac @{thm multi_member_skip} i THEN mset_member_tac (m - 1) i)

```

```

  val mset_nonempty_tac =

```

```

    rtac @{thm nonempty_plus} ORELSE' rtac @{thm nonempty_single}

```

```

  val regroup_union_conv =

```

```

    FundefLib.regroup_conv @{const_name Multiset.Mempty} @{const_name plus}

```

```

    (map (fn t => t RS eq_reflection) (@{thms union_ac} @ @{thms empty_idemp}))

```

```

  fun unfold_pwleq_tac i =

```

```

      (rtac @{thm pw_leq_step} i THEN (fn st => unfold_pwleq_tac (i + 1) st))
      ORELSE (rtac @{thm pw_leq_lstep} i)
      ORELSE (rtac @{thm pw_leq_empty} i)

    val set_of_simps = [{thm set_of_empty}, @{thm set_of_single}, @{thm set_of_union},
      @{thm Un_insert_left}, @{thm Un_empty_left}]
  in
    ScnpReconstruct.multiset_setup (ScnpReconstruct.Multiset
    {
      msetT=msetT, mk_mset=mk_mset, mset_regroup_conv=regroup_munion_conv,
      mset_member_tac=mset_member_tac, mset_nonempty_tac=mset_nonempty_tac,
      mset_pwleq_tac=unfold_pwleq_tac, set_of_simps=set_of_simps,
      smsI' = @{thm ms_strictI}, wmsI2'' = @{thm ms_weakI2}, wmsI1 = @{thm ms_weakI1},
      reduction_pair = @{thm ms_reduction_pair}
    })
  end
  *}

end

```

15 The Follows Relation of Charpentier and Sivilotte

theory Follows imports SubstAx ListOrder Multiset begin

constdefs

```

Follows :: "[a => 'b::{order}, 'a => 'b::{order}] => 'a program set"
  (infixl "Fols" 65)
"f Fols g == Increasing g  $\cap$  Increasing f Int
  Always {s. f s  $\leq$  g s} Int
  ( $\bigcap$  k. {s. k  $\leq$  g s} LeadsTo {s. k  $\leq$  f s})"
```

lemma mono_Always_o:

```

  "mono h ==> Always {s. f s  $\leq$  g s}  $\subseteq$  Always {s. h (f s)  $\leq$  h (g s)}"
apply (simp add: Always_eq_includes_reachable)
apply (blast intro: monoD)
done

```

lemma mono_LeadsTo_o:

```

  "mono (h::'a::order => 'b::order)
  ==> ( $\bigcap$  j. {s. j  $\leq$  g s} LeadsTo {s. j  $\leq$  f s})  $\subseteq$ 
  ( $\bigcap$  k. {s. k  $\leq$  h (g s)} LeadsTo {s. k  $\leq$  h (f s)})"
apply auto
apply (rule single_LeadsTo_I)
apply (drule_tac x = "g s" in spec)
apply (erule LeadsTo_weaken)
apply (blast intro: monoD order_trans)+
done

```

lemma Follows_constant [iff]: "F \in (%s. c) Fols (%s. c)"

```

by (simp add: Follows_def)

lemma mono_Follows_o: "mono h ==> f Fols g  $\subseteq$  (h o f) Fols (h o g)"
by (auto simp add: Follows_def mono_Increasing_o [THEN [2] rev_subsetD]
    mono_Always_o [THEN [2] rev_subsetD]
    mono_LeadsTo_o [THEN [2] rev_subsetD, THEN INT_D])

lemma mono_Follows_apply:
  "mono h ==> f Fols g  $\subseteq$  (%x. h (f x)) Fols (%x. h (g x))"
apply (drule mono_Follows_o)
apply (force simp add: o_def)
done

lemma Follows_trans:
  "[| F  $\in$  f Fols g; F  $\in$  g Fols h |] ==> F  $\in$  f Fols h"
apply (simp add: Follows_def)
apply (simp add: Always_eq_includes_reachable)
apply (blast intro: order_trans LeadsTo_Trans)
done

```

15.1 Destruction rules

```

lemma Follows_Increasing1: "F  $\in$  f Fols g ==> F  $\in$  Increasing f"
by (simp add: Follows_def)

lemma Follows_Increasing2: "F  $\in$  f Fols g ==> F  $\in$  Increasing g"
by (simp add: Follows_def)

lemma Follows_Bounded: "F  $\in$  f Fols g ==> F  $\in$  Always {s. f s  $\leq$  g s}"
by (simp add: Follows_def)

lemma Follows_LeadsTo:
  "F  $\in$  f Fols g ==> F  $\in$  {s. k  $\leq$  g s} LeadsTo {s. k  $\leq$  f s}"
by (simp add: Follows_def)

lemma Follows_LeadsTo_prefixLe:
  "F  $\in$  f Fols g ==> F  $\in$  {s. k prefixLe g s} LeadsTo {s. k prefixLe f s}"
apply (rule single_LeadsTo_I, clarify)
apply (drule_tac k="g s" in Follows_LeadsTo)
apply (erule LeadsTo_weaken)
  apply blast
apply (blast intro: prefixLe_trans prefix_imp_prefixLe)
done

lemma Follows_LeadsTo_prefixGe:
  "F  $\in$  f Fols g ==> F  $\in$  {s. k prefixGe g s} LeadsTo {s. k prefixGe f s}"
apply (rule single_LeadsTo_I, clarify)
apply (drule_tac k="g s" in Follows_LeadsTo)
apply (erule LeadsTo_weaken)
  apply blast
apply (blast intro: prefixGe_trans prefix_imp_prefixGe)
done

```

```

lemma Always_Follows1:
  "[| F ∈ Always {s. f s = f' s}; F ∈ f Fols g |] ==> F ∈ f' Fols g"

apply (simp add: Follows_def Increasing_def Stable_def, auto)
apply (erule_tac [3] Always_LeadsTo_weaken)
apply (erule_tac A = "{s. z ≤ f s}" and A' = "{s. z ≤ f' s}"
      in Always_Constrains_weaken, auto)
apply (drule Always_Int_I, assumption)
apply (force intro: Always_weaken)
done

lemma Always_Follows2:
  "[| F ∈ Always {s. g s = g' s}; F ∈ f Fols g |] ==> F ∈ f Fols g'"
apply (simp add: Follows_def Increasing_def Stable_def, auto)
apply (erule_tac [3] Always_LeadsTo_weaken)
apply (erule_tac A = "{s. z ≤ g s}" and A' = "{s. z ≤ g' s}"
      in Always_Constrains_weaken, auto)
apply (drule Always_Int_I, assumption)
apply (force intro: Always_weaken)
done

```

15.2 Union properties (with the subset ordering)

```

lemma increasing_Un:
  "[| F ∈ increasing f; F ∈ increasing g |]
   ==> F ∈ increasing (%s. (f s) ∪ (g s))"
apply (simp add: increasing_def stable_def constrains_def, auto)
apply (drule_tac x = "f xa" in spec)
apply (drule_tac x = "g xa" in spec)
apply (blast dest!: bspec)
done

lemma Increasing_Un:
  "[| F ∈ Increasing f; F ∈ Increasing g |]
   ==> F ∈ Increasing (%s. (f s) ∪ (g s))"
apply (auto simp add: Increasing_def Stable_def Constrains_def
                    stable_def constrains_def)
apply (drule_tac x = "f xa" in spec)
apply (drule_tac x = "g xa" in spec)
apply (blast dest!: bspec)
done

lemma Always_Un:
  "[| F ∈ Always {s. f' s ≤ f s}; F ∈ Always {s. g' s ≤ g s} |]
   ==> F ∈ Always {s. f' s ∪ g' s ≤ f s ∪ g s}"
by (simp add: Always_eq_includes_reachable, blast)

lemma Follows_Un_lemma:
  "[| F ∈ Increasing f; F ∈ Increasing g;
     F ∈ Increasing g'; F ∈ Always {s. f' s ≤ f s};
     ∀k. F ∈ {s. k ≤ f s} LeadsTo {s. k ≤ f' s} |]
   ==> F ∈ {s. k ≤ f s ∪ g s} LeadsTo {s. k ≤ f' s ∪ g s}"

```



```

apply (rule single_LeadsTo_I)
apply (drule_tac x = "f s" in IncreasingD)
apply (drule_tac x = "g s" in IncreasingD)
apply (rule LeadsTo_weaken)
apply (rule PSP_Stable)
apply (erule_tac x = "f s" in spec)
apply (erule Stable_Int, assumption, blast+)
done

lemma Follows_Un:
  "[| F ∈ f' Fols f; F ∈ g' Fols g |]
   ==> F ∈ (%s. (f' s) ∪ (g' s)) Fols (%s. (f s) ∪ (g s))"
apply (simp add: Follows_def Increasing_Un Always_Un del: Un_subset_iff, auto)
apply (rule LeadsTo_Trans)
apply (blast intro: Follows_Un_lemma)

apply (blast intro: Follows_Un_lemma [THEN LeadsTo_weaken])
done

```

15.3 Multiset union properties (with the multiset ordering)

```

lemma increasing_union:
  "[| F ∈ increasing f; F ∈ increasing g |]
   ==> F ∈ increasing (%s. (f s) + (g s :: ('a::order) multiset))"
apply (simp add: increasing_def stable_def constrains_def, auto)
apply (drule_tac x = "f xa" in spec)
apply (drule_tac x = "g xa" in spec)
apply (drule bspec, assumption)
apply (blast intro: union_le_mono order_trans)
done

lemma Increasing_union:
  "[| F ∈ Increasing f; F ∈ Increasing g |]
   ==> F ∈ Increasing (%s. (f s) + (g s :: ('a::order) multiset))"
apply (auto simp add: Increasing_def Stable_def Constrains_def
  stable_def constrains_def)
apply (drule_tac x = "f xa" in spec)
apply (drule_tac x = "g xa" in spec)
apply (drule bspec, assumption)
apply (blast intro: union_le_mono order_trans)
done

lemma Always_union:
  "[| F ∈ Always {s. f' s ≤ f s}; F ∈ Always {s. g' s ≤ g s} |]
   ==> F ∈ Always {s. f' s + g' s ≤ f s + (g s :: ('a::order) multiset)}"
apply (simp add: Always_eq_includes_reachable)
apply (blast intro: union_le_mono)
done

```

```

lemma Follows_union_lemma:
  "[| F ∈ Increasing f; F ∈ Increasing g;
     F ∈ Increasing g'; F ∈ Always {s. f' s ≤ f s};

```

```

       $\forall k :: ('a :: \text{order}) \text{ multiset.}$ 
       $F \in \{s. k \leq f\ s\} \text{ LeadsTo } \{s. k \leq f'\ s\} \mid$ 
 $\implies F \in \{s. k \leq f\ s + g\ s\} \text{ LeadsTo } \{s. k \leq f'\ s + g\ s\}$ 
    apply (rule single_LeadsTo_I)
    apply (drule_tac x = "f s" in IncreasingD)
    apply (drule_tac x = "g s" in IncreasingD)
    apply (rule LeadsTo_weaken)
    apply (rule PSP_Stable)
    apply (erule_tac x = "f s" in spec)
    apply (erule Stable_Int, assumption, blast)
    apply (blast intro: union_le_mono order_trans)
  done

```

```

lemma Follows_union:
  "!!g g' :: 'b => ('a :: order) multiset.
   [| F ∈ f' Fols f; F ∈ g' Fols g |]
   ==> F ∈ (%s. (f' s) + (g' s)) Fols (%s. (f s) + (g s))"
  apply (simp add: Follows_def)
  apply (simp add: Increasing_union Always_union, auto)
  apply (rule LeadsTo_Trans)
  apply (blast intro: Follows_union_lemma)

  apply (simp add: union_commute)
  apply (blast intro: Follows_union_lemma)
done

```

```

lemma Follows_setsum:
  "!!f :: ['c, 'b] => ('a :: order) multiset.
   [|  $\forall i \in I. F \in f'\ i \text{ Fols } f\ i$ ; finite I |]
   ==> F ∈ (%s.  $\sum i \in I. f'\ i\ s$ ) Fols (%s.  $\sum i \in I. f\ i\ s$ )"
  apply (erule rev_mp)
  apply (erule finite_induct, simp)
  apply (simp add: Follows_union)
done

```

```

lemma Increasing_imp_Stable_prefixGe:
  "F ∈ Increasing func ==> F ∈ Stable {s. h prefixGe (func s)}"
  apply (simp add: Increasing_def Stable_def Constrains_def constrains_def)
  apply (blast intro: trans_Ge [THEN trans_genPrefix, THEN transD]
           prefix_imp_prefixGe)
done

```

```

lemma LeadsTo_le_imp_prefixGe:
  " $\forall z. F \in \{s. z \leq f\ s\} \text{ LeadsTo } \{s. z \leq g\ s\}$ 
   ==> F ∈ {s. z prefixGe f s} LeadsTo {s. z prefixGe g s}"
  apply (rule single_LeadsTo_I)
  apply (drule_tac x = "f s" in spec)
  apply (erule LeadsTo_weaken)
  prefer 2
  apply (blast intro: trans_Ge [THEN trans_genPrefix, THEN transD])

```

```

done
    prefix_imp_prefixGe, blast)
end

```

16 Predicate Transformers

theory *Transformers* imports *Comp* begin

16.1 Defining the Predicate Transformers *wp*, *awp* and *wens*

```

constdefs
  wp :: "[('a*'a) set, 'a set] => 'a set"
    — Dijkstra's weakest-precondition operator (for an individual command)
    "wp act B == - (act-1 ' ' (-B))"

  awp :: "[ 'a program, 'a set] => 'a set"
    — Dijkstra's weakest-precondition operator (for a program)
    "awp F B == (⋂ act ∈ Acts F. wp act B)"

  wens :: "[ 'a program, ('a*'a) set, 'a set] => 'a set"
    — The weakest-ensures transformer
    "wens F act B == gfp(λX. (wp act B ∩ awp F (B ∪ X)) ∪ B)"

```

The fundamental theorem for wp

```

theorem wp_iff: "(A ≤ wp act B) = (act ' ' A ≤ B)"
by (force simp add: wp_def)

```

This lemma is a good deal more intuitive than the definition!

```

lemma in_wp_iff: "(a ∈ wp act B) = (∀ x. (a,x) ∈ act --> x ∈ B)"
by (simp add: wp_def, blast)

```

```

lemma Compl_Domain_subset_wp: "- (Domain act) ⊆ wp act B"
by (force simp add: wp_def)

```

```

lemma wp_empty [simp]: "wp act {} = - (Domain act)"
by (force simp add: wp_def)

```

The identity relation is the skip action

```

lemma wp_Id [simp]: "wp Id B = B"
by (simp add: wp_def)

```

```

lemma wp_totalize_act:
  "wp (totalize_act act) B = (wp act B ∩ Domain act) ∪ (B - Domain act)"
by (simp add: wp_def totalize_act_def, blast)

```

```

lemma awp_subset: "(awp F A ⊆ A)"
by (force simp add: awp_def wp_def)

```

```

lemma awp_Int_eq: "awp F (A ∩ B) = awp F A ∩ awp F B"
by (simp add: awp_def wp_def, blast)

```

The fundamental theorem for awp

```
theorem awp_iff_constrains: "(A <= awp F B) = (F ∈ A co B)"
by (simp add: awp_def constrains_def wp_iff INT_subset_iff)
```

```
lemma awp_iff_stable: "(A ⊆ awp F A) = (F ∈ stable A)"
by (simp add: awp_iff_constrains stable_def)
```

```
lemma stable_imp_awp_ident: "F ∈ stable A ==> awp F A = A"
apply (rule equalityI [OF awp_subset])
apply (simp add: awp_iff_stable)
done
```

```
lemma wp_mono: "(A ⊆ B) ==> wp act A ⊆ wp act B"
by (simp add: wp_def, blast)
```

```
lemma awp_mono: "(A ⊆ B) ==> awp F A ⊆ awp F B"
by (simp add: awp_def wp_def, blast)
```

```
lemma wens_unfold:
  "wens F act B = (wp act B ∩ awp F (B ∪ wens F act B)) ∪ B"
apply (simp add: wens_def)
apply (rule gfp_unfold)
apply (simp add: mono_def wp_def awp_def, blast)
done
```

```
lemma wens_Id [simp]: "wens F Id B = B"
by (simp add: wens_def gfp_def wp_def awp_def Sup_set_eq, blast)
```

These two theorems justify the claim that `wens` returns the weakest assertion satisfying the ensures property

```
lemma ensures_imp_wens: "F ∈ A ensures B ==> ∃ act ∈ Acts F. A ⊆ wens F
act B"
apply (simp add: wens_def ensures_def transient_def, clarify)
apply (rule rev_bexI, assumption)
apply (rule gfp_upperbound)
apply (simp add: constrains_def awp_def wp_def, blast)
done
```

```
lemma wens_ensures: "act ∈ Acts F ==> F ∈ (wens F act B) ensures B"
by (simp add: wens_def gfp_def constrains_def awp_def wp_def
  ensures_def transient_def Sup_set_eq, blast)
```

These two results constitute assertion (4.13) of the thesis

```
lemma wens_mono: "(A ⊆ B) ==> wens F act A ⊆ wens F act B"
apply (simp add: wens_def wp_def awp_def)
apply (rule gfp_mono, blast)
done
```

```
lemma wens_weakening: "B ⊆ wens F act B"
by (simp add: wens_def gfp_def Sup_set_eq, blast)
```

Assertion (6), or 4.16 in the thesis

```

lemma subset_wens: "A-B  $\subseteq$  wp act B  $\cap$  awp F (B  $\cup$  A) ==> A  $\subseteq$  wens F act B"
apply (simp add: wens_def wp_def awp_def)
apply (rule gfp_upperbound, blast)
done

```

Assertion 4.17 in the thesis

```

lemma Diff_wens_constrains: "F  $\in$  (wens F act A - A) co wens F act A"
by (simp add: wens_def gfp_def wp_def awp_def constrains_def Sup_set_eq, blast)
— Proved instantly, yet remarkably fragile. If Un_subset_iff is declared as an
iff-rule, then it's almost impossible to prove. One proof is via meson after expanding
all definitions, but it's slow!

```

Assertion (7): 4.18 in the thesis. NOTE that many of these results hold for an arbitrary action. We often do not require $\text{act} \in \text{Acts } F$

```

lemma stable_wens: "F  $\in$  stable A ==> F  $\in$  stable (wens F act A)"
apply (simp add: stable_def)
apply (drule constrains_Un [OF Diff_wens_constrains [of F act A]])
apply (simp add: Un_Int_distrib2 Compl_partition2)
apply (erule constrains_weaken, blast)
apply (simp add: Un_subset_iff wens_weakening)
done

```

Assertion 4.20 in the thesis.

```

lemma wens_Int_eq_lemma:
  "[|T-B  $\subseteq$  awp F T; act  $\in$  Acts F|]
  ==> T  $\cap$  wens F act B  $\subseteq$  wens F act (T $\cap$ B)"
apply (rule subset_wens)
apply (rule_tac P="λx. ?f x  $\subseteq$  ?b" in ssubst [OF wens_unfold])
apply (simp add: wp_def awp_def, blast)
done

```

Assertion (8): 4.21 in the thesis. Here we indeed require $\text{act} \in \text{Acts } F$

```

lemma wens_Int_eq:
  "[|T-B  $\subseteq$  awp F T; act  $\in$  Acts F|]
  ==> T  $\cap$  wens F act B = T  $\cap$  wens F act (T $\cap$ B)"
apply (rule equalityI)
  apply (simp_all add: Int_lower1 Int_subset_iff)
  apply (rule wens_Int_eq_lemma, assumption+)
apply (rule subset_trans [OF _ wens_mono [of "T $\cap$ B" B]], auto)
done

```

16.2 Defining the Weakest Ensures Set

inductive_set

```

  wens_set :: "[ 'a program, 'a set ] => 'a set set"
  for F :: "'a program" and B :: "'a set"
where

```

```

  Basis: "B  $\in$  wens_set F B"

```

```

| Wens: "[|X  $\in$  wens_set F B; act  $\in$  Acts F|] ==> wens F act X  $\in$  wens_set F B"

```

```

| Union: " $W \neq \{\}$ " ==>  $\forall U \in W. U \in \text{wens\_set } F B \implies \bigcup W \in \text{wens\_set } F B$ "

lemma wens_set_imp_co: " $A \in \text{wens\_set } F B \implies F \in (A-B) \text{ co } A$ "
apply (erule wens_set.induct)
  apply (simp add: constrains_def)
  apply (drule_tac act1=act and A1=X
    in constrains_Un [OF Diff_wens_constrains])
  apply (erule constrains_weaken, blast)
  apply (simp add: Un_subset_iff wens_weakening)
  apply (rule constrains_weaken)
  apply (rule_tac I=W and A=" $\lambda v. v-B$ " and A'=" $\lambda v. v$ " in constrains_UN, blast+)
done

lemma wens_set_imp_leadsTo: " $A \in \text{wens\_set } F B \implies F \in A \text{ leadsTo } B$ "
apply (erule wens_set.induct)
  apply (rule leadsTo_refl)
  apply (blast intro: wens_ensures leadsTo_Trans)
  apply (blast intro: leadsTo_Union)
done

lemma leadsTo_imp_wens_set: " $F \in A \text{ leadsTo } B \implies \exists C \in \text{wens\_set } F B. A \subseteq C$ "
apply (erule leadsTo_induct_pre)
  apply (blast dest!: ensures_imp_wens intro: wens_set.Basis wens_set.Wens)

  apply (clarify, drule ensures_weaken_R, assumption)
  apply (blast dest!: ensures_imp_wens intro: wens_set.Wens)
  apply (case_tac "S={}")
  apply (simp, blast intro: wens_set.Basis)
  apply (clarsimp dest!: bchoice simp: ball_conj_distrib Bex_def)
  apply (rule_tac x = " $\bigcup \{Z. \exists U \in S. Z = f U\}$ " in exI)
  apply (blast intro: wens_set.Union)
done

```

Assertion (9): 4.27 in the thesis.

```

lemma leadsTo_iff_wens_set: " $(F \in A \text{ leadsTo } B) = (\exists C \in \text{wens\_set } F B. A \subseteq C)$ "
by (blast intro: leadsTo_imp_wens_set leadsTo_weaken_L wens_set_imp_leadsTo)

```

This is the result that requires the definition of *wens_set* to require *W* to be non-empty in the Unio case, for otherwise we should always have $\{\} \in \text{wens_set } F B$.

```

lemma wens_set_imp_subset: " $A \in \text{wens\_set } F B \implies B \subseteq A$ "
apply (erule wens_set.induct)
  apply (blast intro: wens_weakening [THEN subsetD])
done

```

16.3 Properties Involving Program Union

Assertion (4.30) of thesis, reoriented

```

lemma awp_Join_eq: " $\text{awp } (F \sqcup G) B = \text{awp } F B \cap \text{awp } G B$ "
by (simp add: awp_def wp_def, blast)

```

```

lemma wens_subset: "wens F act B - B  $\subseteq$  wp act B  $\cap$  awp F (B  $\cup$  wens F act B)"
by (subst wens_unfold, fast)

```

Assertion (4.31)

```

lemma subset_wens_Join:
  "[|A = T  $\cap$  wens F act B; T-B  $\subseteq$  awp F T; A-B  $\subseteq$  awp G (A  $\cup$  B)|]
  ==> A  $\subseteq$  wens (F $\sqcup$ G) act B"
apply (subgoal_tac "(T  $\cap$  wens F act B) - B  $\subseteq$ 
      wp act B  $\cap$  awp F (B  $\cup$  wens F act B)  $\cap$  awp F T")
  apply (rule subset_wens)
  apply (simp add: awp_Join_eq awp_Int_eq Int_subset_iff Un_commute)
  apply (simp add: awp_def wp_def, blast)
apply (insert wens_subset [of F act B], blast)
done

```

Assertion (4.32)

```

lemma wens_Join_subset: "wens (F $\sqcup$ G) act B  $\subseteq$  wens F act B"
apply (simp add: wens_def)
apply (rule gfp_mono)
apply (auto simp add: awp_Join_eq)
done

```

Lemma, because the inductive step is just too messy.

```

lemma wens_Union_inductive_step:
  assumes awpF: "T-B  $\subseteq$  awp F T"
  and awpG: "!!X. X  $\in$  wens_set F B ==> (T $\cap$ X) - B  $\subseteq$  awp G (T $\cap$ X)"
  shows "[|X  $\in$  wens_set F B; act  $\in$  Acts F; Y  $\subseteq$  X; T $\cap$ X = T $\cap$ Y|]
    ==> wens (F $\sqcup$ G) act Y  $\subseteq$  wens F act X  $\wedge$ 
      T  $\cap$  wens F act X = T  $\cap$  wens (F $\sqcup$ G) act Y"
apply (subgoal_tac "wens (F $\sqcup$ G) act Y  $\subseteq$  wens F act X")
  prefer 2
  apply (blast dest: wens_mono intro: wens_Join_subset [THEN subsetD], simp)
apply (rule equalityI)
  prefer 2 apply blast
apply (simp add: Int_lower1 Int_subset_iff)
apply (frule wens_set_imp_subset)
apply (subgoal_tac "T-X  $\subseteq$  awp F T")
  prefer 2 apply (blast intro: awpF [THEN subsetD])
apply (rule_tac B = "wens (F $\sqcup$ G) act (T $\cap$ X)" in subset_trans)
  prefer 2 apply (blast intro!: wens_mono)
apply (subst wens_Int_eq, assumption+)
apply (rule subset_wens_Join [of _ T], simp, blast)
apply (subgoal_tac "T  $\cap$  wens F act (T $\cap$ X)  $\cup$  T $\cap$ X = T  $\cap$  wens F act X")
  prefer 2
  apply (subst wens_Int_eq [symmetric], assumption+)
  apply (blast intro: wens_weakening [THEN subsetD], simp)
apply (blast intro: awpG [THEN subsetD] wens_set.Wens)
done

```

```

theorem wens_Union:
  assumes awpF: "T-B  $\subseteq$  awp F T"

```

```

    and awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"
    and major: "X ∈ wens_set F B"
    shows "∃ Y ∈ wens_set (F⊔G) B. Y ⊆ X & T∩X = T∩Y"
  apply (rule wens_set.induct [OF major])

```

Basis: trivial

```

  apply (blast intro: wens_set.Basis)

```

Inductive step

```

  apply clarify
  apply (rule_tac x = "wens (F⊔G) act Y" in rev_bexI)
  apply (force intro: wens_set.Wens)
  apply (simp add: wens_Union_inductive_step [OF awpF awpG])

```

Union: by Axiom of Choice

```

  apply (simp add: ball_conj_distrib Bex_def)
  apply (clarify dest!: bchoice)
  apply (rule_tac x = "⋃ {Z. ∃ U ∈ W. Z = f U}" in exI)
  apply (blast intro: wens_set.Union)
done

```

theorem leadsTo_Join:

```

  assumes leadsTo: "F ∈ A leadsTo B"
  and awpF: "T-B ⊆ awp F T"
  and awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"
  shows "F⊔G ∈ T∩A leadsTo B"
  apply (rule leadsTo [THEN leadsTo_imp_wens_set, THEN bexE])
  apply (rule wens_Union [THEN bexE])
  apply (rule awpF)
  apply (erule awpG, assumption)
  apply (blast intro: wens_set_imp_leadsTo [THEN leadsTo_weaken_L])
done

```

16.4 The Set $wens_set F B$ for a Single-Assignment Program

Thesis Section 4.3.3

We start by proving laws about single-assignment programs

```

lemma awp_single_eq [simp]:
  "awp (mk_program (init, {act}, allowed)) B = B ∩ wp act B"
by (force simp add: awp_def wp_def)

lemma wp_Un_subset: "wp act A ∪ wp act B ⊆ wp act (A ∪ B)"
by (force simp add: wp_def)

lemma wp_Un_eq: "single_valued act ==> wp act (A ∪ B) = wp act A ∪ wp act B"
  B"
  apply (rule equalityI)
  apply (force simp add: wp_def single_valued_def)
  apply (rule wp_Un_subset)
done

lemma wp_UN_subset: "(⋃ i ∈ I. wp act (A i)) ⊆ wp act (⋃ i ∈ I. A i)"

```


by (force simp add: wp_def)

```
lemma wp_UN_eq:
  "[/single_valued act; I≠{}|]
   ==> wp act (⋃ i∈I. A i) = (⋃ i∈I. wp act (A i))"
apply (rule equalityI)
prefer 2 apply (rule wp_UN_subset)
apply (simp add: wp_def Image_INT_eq)
done
```

```
lemma wens_single_eq:
  "wens (mk_program (init, {act}, allowed)) act B = B ∪ wp act B"
by (simp add: wens_def gfp_def wp_def Sup_set_eq, blast)
```

Next, we express the $wens_set$ for single-assignment programs

```
constdefs
  wens_single_finite :: "[('a*'a) set, 'a set, nat] => 'a set"
  "wens_single_finite act B k == ⋃ i ∈ atMost k. ((wp act)^i) B"

  wens_single :: "[('a*'a) set, 'a set] => 'a set"
  "wens_single act B == ⋃ i. ((wp act)^i) B"

lemma wens_single_Un_eq:
  "single_valued act
   ==> wens_single act B ∪ wp act (wens_single act B) = wens_single act
  B"
apply (rule equalityI)
apply (simp_all add: Un_upper1 Un_subset_iff)
apply (simp add: wens_single_def wp_UN_eq, clarify)
apply (rule_tac a="Suc(i)" in UN_I, auto)
done
```

```
lemma atMost_nat_nonempty: "atMost (k::nat) ≠ {}"
by force
```

```
lemma wens_single_finite_0 [simp]: "wens_single_finite act B 0 = B"
by (simp add: wens_single_finite_def)
```

```
lemma wens_single_finite_Suc:
  "single_valued act
   ==> wens_single_finite act B (Suc k) =
       wens_single_finite act B k ∪ wp act (wens_single_finite act B k)"
apply (simp add: wens_single_finite_def image_def
               wp_UN_eq [OF _ atMost_nat_nonempty])
apply (force elim!: le_SucE)
done
```

```
lemma wens_single_finite_Suc_eq_wens:
  "single_valued act
   ==> wens_single_finite act B (Suc k) =
       wens (mk_program (init, {act}, allowed)) act
       (wens_single_finite act B k)"
by (simp add: wens_single_finite_Suc wens_single_eq)
```

```

lemma def_wens_single_finite_Suc_eq_wens:
  "[/F = mk_program (init, {act}, allowed); single_valued act/]
  ==> wens_single_finite act B (Suc k) =
    wens F act (wens_single_finite act B k)"
by (simp add: wens_single_finite_Suc_eq_wens)

lemma wens_single_finite_Un_eq:
  "single_valued act
  ==> wens_single_finite act B k  $\cup$  wp act (wens_single_finite act B k)
     $\in$  range (wens_single_finite act B)"
by (simp add: wens_single_finite_Suc [symmetric])

lemma wens_single_eq_Union:
  "wens_single act B =  $\bigcup$  range (wens_single_finite act B)"
by (simp add: wens_single_finite_def wens_single_def, blast)

lemma wens_single_finite_eq_Union:
  "wens_single_finite act B n = ( $\bigcup_{k \in \text{atMost } n. \text{ wens\_single\_finite act B } k}$ )"
apply (auto simp add: wens_single_finite_def)
apply (blast intro: le_trans)
done

lemma wens_single_finite_mono:
  " $m \leq n \implies \text{wens\_single\_finite act B } m \subseteq \text{wens\_single\_finite act B } n$ "
by (force simp add: wens_single_finite_eq_Union [of act B n])

lemma wens_single_finite_subset_wens_single:
  "wens_single_finite act B k  $\subseteq$  wens_single act B"
by (simp add: wens_single_eq_Union, blast)

lemma subset_wens_single_finite:
  "[/W  $\subseteq$  wens_single_finite act B ' (atMost k); single_valued act; W  $\neq \{\}$ /]
  ==>  $\exists m. \bigcup W = \text{wens\_single\_finite act B } m$ "
apply (induct k)
apply (rule_tac x=0 in exI, simp, blast)
apply (auto simp add: atMost_Suc)
apply (case_tac "wens_single_finite act B (Suc k)  $\in$  W")
  prefer 2 apply blast
apply (drule_tac x="Suc k" in spec)
apply (erule notE, rule equalityI)
  prefer 2 apply blast
apply (subst wens_single_finite_eq_Union)
apply (simp add: atMost_Suc, blast)
done

lemma for Union case
lemma Union_eq_wens_single:
  "[ $\forall k. \neg W \subseteq \text{wens\_single\_finite act B ' } \{..k\};$ 
  W  $\subseteq$  insert (wens_single act B)
  (range (wens_single_finite act B))
   $\implies \bigcup W = \text{wens\_single act B}$ "
apply (case_tac "wens_single act B  $\in$  W")
apply (blast dest: wens_single_finite_subset_wens_single [THEN subsetD])

```

```

apply (simp add: wens_single_eq_Union)
apply (rule equalityI, blast)
apply (simp add: UN_subset_iff, clarify)
apply (subgoal_tac " $\exists y \in W. \exists n. y = wens\_single\_finite\ act\ B\ n \ \&\ i \leq n$ ")
  apply (blast intro: wens_single_finite_mono [THEN subsetD])
apply (drule_tac x=i in spec)
apply (force simp add: atMost_def)
done

lemma wens_set_subset_single:
  "single_valued act
  ==> wens_set (mk_program (init, {act}, allowed)) B  $\subseteq$ 
    insert (wens_single act B) (range (wens_single_finite act B))"
apply (rule subsetI)
apply (erule wens_set.induct)

Basis

  apply (fastsimp simp add: wens_single_finite_def)

Wens inductive step

  apply (case_tac "acta = Id", simp)
  apply (simp add: wens_single_eq)
  apply (elim disjE)
  apply (simp add: wens_single_Un_eq)
  apply (force simp add: wens_single_finite_Un_eq)

Union inductive step

  apply (case_tac " $\exists k. W \subseteq wens\_single\_finite\ act\ B \ \wedge \ (atMost\ k)$ ")
  apply (blast dest!: subset_wens_single_finite, simp)
  apply (rule disjI1 [OF Union_eq_wens_single], blast+)
done

lemma wens_single_finite_in_wens_set:
  "single_valued act ==>
    wens_single_finite act B k
     $\in wens\_set\ (mk\_program\ (init, \{act\}, allowed))\ B$ "
apply (induct_tac k)
  apply (simp add: wens_single_finite_def wens_set.Basis)
  apply (simp add: wens_set.Wens
    wens_single_finite_Suc_eq_wens [of act B _ init allowed])

done

lemma single_subset_wens_set:
  "single_valued act
  ==> insert (wens_single act B) (range (wens_single_finite act B))  $\subseteq$ 
    wens_set (mk_program (init, {act}, allowed)) B"
apply (simp add: wens_single_eq_Union UN_eq)
apply (blast intro: wens_set.Union wens_single_finite_in_wens_set)
done

```

Theorem (4.29)

```

theorem wens_set_single_eq:
  "[/F = mk_program (init, {act}, allowed); single_valued act/]
  ==> wens_set F B =
    insert (wens_single act B) (range (wens_single_finite act B))"
apply (rule equalityI)
  apply (simp add: wens_set_subset_single)
apply (erule ssubst, erule single_subset_wens_set)
done

```

Generalizing Misra's Fixed Point Union Theorem (4.41)

```

lemma fp_leadsTo_Join:
  "[/T-B ⊆ awp F T; T-B ⊆ FP G; F ∈ A leadsTo B/] ==> F ⊔ G ∈ T ∩ A leadsTo B"
apply (rule leadsTo_Join, assumption, blast)
apply (simp add: FP_def awp_iff_constrains stable_def constrains_def, blast)

done

end

```

17 Progress Sets

theory ProgressSets imports Transformers begin

17.1 Complete Lattices and the Operator c_l

```

constdefs
  lattice :: "'a set set => bool"
  — Meier calls them closure sets, but they are just complete lattices
  "lattice L ==
    (∀M. M ⊆ L --> ⋂ M ∈ L) & (∀M. M ⊆ L --> ⋃ M ∈ L)"

  cl :: "[ 'a set set, 'a set ] => 'a set"
  — short for "closure"
  "cl L r == ⋂ {x. x ∈ L & r ⊆ x}"

lemma UNIV_in_lattice: "lattice L ==> UNIV ∈ L"
by (force simp add: lattice_def)

lemma empty_in_lattice: "lattice L ==> {} ∈ L"
by (force simp add: lattice_def)

lemma Union_in_lattice: "[/M ⊆ L; lattice L/] ==> ⋃ M ∈ L"
by (simp add: lattice_def)

lemma Inter_in_lattice: "[/M ⊆ L; lattice L/] ==> ⋂ M ∈ L"
by (simp add: lattice_def)

lemma UN_in_lattice:
  "[/lattice L; !!i. i ∈ I ==> r i ∈ L/] ==> (⋃ i ∈ I. r i) ∈ L"
apply (simp add: Set.UN_eq)
apply (blast intro: Union_in_lattice)

```

done

```
lemma INT_in_lattice:
  "[/[lattice L; !!i. i ∈ I ==> r i ∈ L/] ==> (⋂ i ∈ I. r i) ∈ L"
apply (simp add: INT_eq)
apply (blast intro: Inter_in_lattice)
done
```

```
lemma Un_in_lattice: "[/x ∈ L; y ∈ L; lattice L/] ==> x ∪ y ∈ L"
apply (simp only: Un_eq_Union)
apply (blast intro: Union_in_lattice)
done
```

```
lemma Int_in_lattice: "[/x ∈ L; y ∈ L; lattice L/] ==> x ∩ y ∈ L"
apply (simp only: Int_eq_Inter)
apply (blast intro: Inter_in_lattice)
done
```

```
lemma lattice_stable: "lattice {X. F ∈ stable X}"
by (simp add: lattice_def stable_def constrains_def, blast)
```

The next three results state that $cl\ L\ r$ is the minimal element of L that includes r .

```
lemma cl_in_lattice: "lattice L ==> cl L r ∈ L"
apply (simp add: lattice_def cl_def)
apply (erule conjE)
apply (drule spec, erule mp, blast)
done
```

```
lemma cl_least: "[/c ∈ L; r ⊆ c/] ==> cl L r ⊆ c"
by (force simp add: cl_def)
```

The next three lemmas constitute assertion (4.61)

```
lemma cl_mono: "r ⊆ r' ==> cl L r ⊆ cl L r'"
by (simp add: cl_def, blast)
```

```
lemma subset_cl: "r ⊆ cl L r"
by (simp add: cl_def, blast)
```

A reformulation of $?r ⊆ cl\ ?L\ ?r$

```
lemma clI: "x ∈ r ==> x ∈ cl L r"
by (simp add: cl_def, blast)
```

A reformulation of $[/?c ∈ ?L; ?r ⊆ ?c] ⇒ cl\ ?L\ ?r ⊆ ?c$

```
lemma clD: "[/c ∈ cl L r; B ∈ L; r ⊆ B/] ==> c ∈ B"
by (force simp add: cl_def)
```

```
lemma cl_UN_subset: "(⋃ i ∈ I. cl L (r i)) ⊆ cl L (⋃ i ∈ I. r i)"
by (simp add: cl_def, blast)
```

```
lemma cl_Un: "lattice L ==> cl L (r ∪ s) = cl L r ∪ cl L s"
apply (rule equalityI)
prefer 2
```

```

    apply (simp add: cl_def, blast)
  apply (rule cl_least)
  apply (blast intro: Un_in_lattice cl_in_lattice)
  apply (blast intro: subset_cl [THEN subsetD])
done

lemma cl_UN: "lattice L ==> cl L ( $\bigcup_{i \in I} r\ i$ ) = ( $\bigcup_{i \in I} cl\ L\ (r\ i)$ )"
apply (rule equalityI)
  prefer 2 apply (simp add: cl_def, blast)
apply (rule cl_least)
  apply (blast intro: UN_in_lattice cl_in_lattice)
  apply (blast intro: subset_cl [THEN subsetD])
done

lemma cl_Int_subset: "cl L (r  $\cap$  s)  $\subseteq$  cl L r  $\cap$  cl L s"
by (simp add: cl_def, blast)

lemma cl_idem [simp]: "cl L (cl L r) = cl L r"
by (simp add: cl_def, blast)

lemma cl_ident: "r  $\in$  L ==> cl L r = r"
by (force simp add: cl_def)

lemma cl_empty [simp]: "lattice L ==> cl L {} = {}"
by (simp add: cl_ident empty_in_lattice)

lemma cl_UNIV [simp]: "lattice L ==> cl L UNIV = UNIV"
by (simp add: cl_ident UNIV_in_lattice)

Assertion (4.62)

lemma cl_ident_iff: "lattice L ==> (cl L r = r) = (r  $\in$  L)"
apply (rule iffI)
  apply (erule subst)
  apply (erule cl_in_lattice)
  apply (erule cl_ident)
done

lemma cl_subset_in_lattice: "[| cl L r  $\subseteq$  r; lattice L |] ==> r  $\in$  L"
by (simp add: cl_ident_iff [symmetric] equalityI subset_cl)

```

17.2 Progress Sets and the Main Lemma

A progress set satisfies certain closure conditions and is a simple way of including the set `wens_set F B`.

```

constdefs
  closed :: "[ 'a program, 'a set, 'a set, 'a set set ] => bool"
    "closed F T B L ==  $\forall M. \forall act \in Acts\ F. B \subseteq M \ \& \ T \cap M \in L \ \rightarrow$ 
       $T \cap (B \cup wp\ act\ M) \in L$ "

  progress_set :: "[ 'a program, 'a set, 'a set ] => 'a set set set"
    "progress_set F T B ==
      {L. lattice L & B  $\in$  L & T  $\in$  L & closed F T B L}"

```

```

lemma closedD:
  "[|closed F T B L; act ∈ Acts F; B ⊆ M; T ∩ M ∈ L|]
   ==> T ∩ (B ∪ wp act M) ∈ L"
by (simp add: closed_def)

```

Note: the formalization below replaces Meier's q by B and m by X .

Part of the proof of the claim at the bottom of page 97. It's proved separately because the argument requires a generalization over all $act \in Acts F$.

```

lemma lattice_awp_lemma:
  assumes TXC: "T ∩ X ∈ C" — induction hypothesis in theorem below
    and BsubX: "B ⊆ X" — holds in inductive step
    and latt: "lattice C"
    and TC: "T ∈ C"
    and BC: "B ∈ C"
    and clos: "closed F T B C"
  shows "T ∩ (B ∪ awp F (X ∪ cl C (T ∩ r))) ∈ C"
apply (simp del: INT_simps add: awp_def INT_extend_simps)
apply (rule INT_in_lattice [OF latt])
apply (erule closedD [OF clos])
apply (simp add: subset_trans [OF BsubX Un_upper1])
apply (subgoal_tac "T ∩ (X ∪ cl C (T ∩ r)) = (T ∩ X) ∪ cl C (T ∩ r)")
  prefer 2 apply (blast intro: TC clD)
apply (erule ssubst)
apply (blast intro: Un_in_lattice latt cl_in_lattice TXC)
done

```

Remainder of the proof of the claim at the bottom of page 97.

```

lemma lattice_lemma:
  assumes TXC: "T ∩ X ∈ C" — induction hypothesis in theorem below
    and BsubX: "B ⊆ X" — holds in inductive step
    and act: "act ∈ Acts F"
    and latt: "lattice C"
    and TC: "T ∈ C"
    and BC: "B ∈ C"
    and clos: "closed F T B C"
  shows "T ∩ (wp act X ∩ awp F (X ∪ cl C (T ∩ r))) ∪ X ∈ C"
apply (subgoal_tac "T ∩ (B ∪ wp act X) ∈ C")
  prefer 2 apply (simp add: closedD [OF clos] act BsubX TXC)
apply (drule Int_in_lattice
  [OF _ lattice_awp_lemma [OF TXC BsubX latt TC BC clos, of r]
  latt])
apply (subgoal_tac
  "T ∩ (B ∪ wp act X) ∩ (T ∩ (B ∪ awp F (X ∪ cl C (T ∩ r)))) =
   T ∩ (B ∪ wp act X ∩ awp F (X ∪ cl C (T ∩ r)))")
  prefer 2 apply blast
apply simp
apply (drule Un_in_lattice [OF _ TXC latt])
apply (subgoal_tac
  "T ∩ (B ∪ wp act X ∩ awp F (X ∪ cl C (T ∩ r))) ∪ T ∩ X =
   T ∩ (wp act X ∩ awp F (X ∪ cl C (T ∩ r))) ∪ X")
  apply simp
apply (blast intro: BsubX [THEN subsetD])
done

```

Induction step for the main lemma

```

lemma progress_induction_step:
  assumes TXC: "T ∩ X ∈ C" — induction hypothesis in theorem below
    and act: "act ∈ Acts F"
    and Xwens: "X ∈ wens_set F B"
    and latt: "lattice C"
    and TC: "T ∈ C"
    and BC: "B ∈ C"
    and clos: "closed F T B C"
    and Fstable: "F ∈ stable T"
  shows "T ∩ wens F act X ∈ C"
proof -
  from Xwens have BsubX: "B ⊆ X"
  by (rule wens_set_imp_subset)
  let ?r = "wens F act X"
  have "?r ⊆ (wp act X ∩ awp F (X ∪ ?r)) ∪ X"
  by (simp add: wens_unfold [symmetric])
  then have "T ∩ ?r ⊆ T ∩ ((wp act X ∩ awp F (X ∪ ?r)) ∪ X)"
  by blast
  then have "T ∩ ?r ⊆ T ∩ ((wp act X ∩ awp F (T ∩ (X ∪ ?r))) ∪ X)"
  by (simp add: awp_Int_eq Fstable stable_imp_awp_ident, blast)
  then have "T ∩ ?r ⊆ T ∩ ((wp act X ∩ awp F (X ∪ cl C (T ∩ ?r))) ∪ X)"
  by (blast intro: awp_mono [THEN [2] rev_subsetD] subset_cl [THEN subsetD])
  then have "cl C (T ∩ ?r) ⊆
    cl C (T ∩ ((wp act X ∩ awp F (X ∪ cl C (T ∩ ?r))) ∪ X))"
  by (rule cl_mono)
  then have "cl C (T ∩ ?r) ⊆
    T ∩ ((wp act X ∩ awp F (X ∪ cl C (T ∩ ?r))) ∪ X)"
  by (simp add: cl_ident lattice_lemma [OF TXC BsubX act latt TC BC clos])
  then have "cl C (T ∩ ?r) ⊆ (wp act X ∩ awp F (X ∪ cl C (T ∩ ?r))) ∪ X"
  by blast
  then have "cl C (T ∩ ?r) ⊆ ?r"
  by (blast intro!: subset_wens)
  then have cl_subset: "cl C (T ∩ ?r) ⊆ T ∩ ?r"
  by (simp add: Int_subset_iff cl_ident TC
    subset_trans [OF cl_mono [OF Int_lower1]])
  show ?thesis
  by (rule cl_subset_in_lattice [OF cl_subset latt])
qed

```

Proved on page 96 of Meier's thesis. The special case when $T = \text{UNIV}$ states that every progress set for the program F and set B includes the set $\text{wens_set } F \ B$.

```

lemma progress_set_lemma:
  "[| C ∈ progress_set F T B; r ∈ wens_set F B; F ∈ stable T |] ==> T ∩ r
  ∈ C"
  apply (simp add: progress_set_def, clarify)
  apply (erule wens_set.induct)

```

Base

```

  apply (simp add: Int_in_lattice)

```

The difficult *wens* case

```

  apply (simp add: progress_induction_step)

```


Disjunctive case

```

apply (subgoal_tac " $(\bigcup U \in W. T \cap U) \in C$ ")
  apply (simp add: Int_Union)
apply (blast intro: UN_in_lattice)
done

```

17.3 The Progress Set Union Theorem

```

lemma closed_mono:
  assumes BB': " $B \subseteq B'$ "
    and TBwp: " $T \cap (B \cup \text{wp act } M) \in C$ "
    and B'C: " $B' \in C$ "
    and TC: " $T \in C$ "
    and latt: "lattice C"
  shows " $T \cap (B' \cup \text{wp act } M) \in C$ "
proof -
  from TBwp have " $(T \cap B) \cup (T \cap \text{wp act } M) \in C$ "
    by (simp add: Int_Un_distrib)
  then have TBBC: " $(T \cap B') \cup ((T \cap B) \cup (T \cap \text{wp act } M)) \in C$ "
    by (blast intro: Int_in_lattice Un_in_lattice TC B'C latt)
  show ?thesis
    by (rule eqelem_imp_iff [THEN iffD1, OF _ TBBC],
        blast intro: BB' [THEN subsetD])
qed

```

```

lemma progress_set_mono:
  assumes BB': " $B \subseteq B'$ "
  shows
    "[|  $B' \in C$ ;  $C \in \text{progress\_set } F \ T \ B$  |]
    ==>  $C \in \text{progress\_set } F \ T \ B'$ "
by (simp add: progress_set_def closed_def closed_mono [OF BB']
    subset_trans [OF BB'])

```

```

theorem progress_set_Union:
  assumes leadsTo: " $F \in A \text{ leadsTo } B'$ "
    and prog: " $C \in \text{progress\_set } F \ T \ B$ "
    and Fstable: " $F \in \text{stable } T$ "
    and BB': " $B \subseteq B'$ "
    and B'C: " $B' \in C$ "
    and Gco: " $\forall X. X \in C \implies G \in X - B \text{ co } X$ "
  shows " $F \sqcup G \in T \cap A \text{ leadsTo } B'$ "
apply (insert prog Fstable)
apply (rule leadsTo_Join [OF leadsTo])
  apply (force simp add: progress_set_def awp_iff_stable [symmetric])
apply (simp add: awp_iff_constrains)
apply (drule progress_set_mono [OF BB' B'C])
apply (blast intro: progress_set_lemma Gco constrains_weaken_L
    BB' [THEN subsetD])
done

```

17.4 Some Progress Sets

```

lemma UNIV_in_progress_set: "UNIV  $\in \text{progress\_set } F \ T \ B$ "

```

by (simp add: progress_set_def lattice_def closed_def)

17.4.1 Lattices and Relations

From Meier's thesis, section 4.5.3

constdefs

```
relcl :: "'a set set => ('a * 'a) set"
  — Derived relation from a lattice
  "relcl L == {(x,y). y ∈ cl L {x}}"
```

```
latticeof :: "('a * 'a) set => 'a set set"
  — Derived lattice from a relation: the set of upwards-closed sets
  "latticeof r == {X. ∀ s t. s ∈ X & (s,t) ∈ r --> t ∈ X}"
```

```
lemma relcl_refl: "(a,a) ∈ relcl L"
by (simp add: relcl_def subset_cl [THEN subsetD])
```

```
lemma relcl_trans:
  "[/ (a,b) ∈ relcl L; (b,c) ∈ relcl L; lattice L /] ==> (a,c) ∈ relcl L"
apply (simp add: relcl_def)
apply (blast intro: clD cl_in_lattice)
done
```

```
lemma refl_relcl: "lattice L ==> refl (relcl L)"
by (simp add: refl_onI relcl_def subset_cl [THEN subsetD])
```

```
lemma trans_relcl: "lattice L ==> trans (relcl L)"
by (blast intro: relcl_trans transI)
```

```
lemma lattice_latticeof: "lattice (latticeof r)"
by (auto simp add: lattice_def latticeof_def)
```

```
lemma lattice_singletonI:
  "[/ lattice L; !!s. s ∈ X ==> {s} ∈ L /] ==> X ∈ L"
apply (cut_tac UN_singleton [of X])
apply (erule subst)
apply (simp only: UN_in_lattice)
done
```

Equation (4.71) of Meier's thesis. He gives no proof.

```
lemma cl_latticeof:
  "[/ refl r; trans r /]
  ==> cl (latticeof r) X = {t. ∃ s. s ∈ X & (s,t) ∈ r}"
apply (rule equalityI)
apply (rule cl_least)
  apply (simp (no_asm_use) add: latticeof_def trans_def, blast)
  apply (simp add: latticeof_def refl_on_def, blast)
apply (simp add: latticeof_def, clarify)
apply (unfold cl_def, blast)
done
```

Related to (4.71).

```

lemma cl_eq_Collect_relcl:
  "lattice L ==> cl L X = {t.  $\exists s. s \in X \ \& \ (s,t) \in \text{relcl } L$ }"
apply (cut_tac UN_singleton [of X])
apply (erule subst)
apply (force simp only: relcl_def cl_UN)
done

Meier's theorem of section 4.5.3

theorem latticeof_relcl_eq: "lattice L ==> latticeof (relcl L) = L"
apply (rule equalityI)
  prefer 2 apply (force simp add: latticeof_def relcl_def cl_def, clarify)

apply (rename_tac X)
apply (rule cl_subset_in_lattice)
  prefer 2 apply assumption
apply (drule cl_ident_iff [OF lattice_latticeof, THEN iffD2])
apply (drule equalityD1)
apply (rule subset_trans)
  prefer 2 apply assumption
apply (thin_tac "?U  $\subseteq$  X")
apply (cut_tac A=X in UN_singleton)
apply (erule subst)
apply (simp only: cl_UN lattice_latticeof
               cl_latticeof [OF refl_relcl trans_relcl])
apply (simp add: relcl_def)
done

theorem relcl_latticeof_eq:
  "[|refl r; trans r|] ==> relcl (latticeof r) = r"
by (simp add: relcl_def cl_latticeof)

```

17.4.2 Decoupling Theorems

```

constdefs
  decoupled :: "[a program, a program] => bool"
  "decoupled F G ==
     $\forall \text{act} \in \text{Acts}. \forall B. G \in \text{stable } B \rightarrow G \in \text{stable } (\text{wp act } B)"$ 
```

Rao's Decoupling Theorem

```

lemma stableco: "F  $\in$  stable A ==> F  $\in$  A-B co A"
by (simp add: stable_def constrains_def, blast)

theorem decoupling:
  assumes leadsTo: "F  $\in$  A leadsTo B"
  and Gstable: "G  $\in$  stable B"
  and dec:      "decoupled F G"
  shows "F  $\sqcup$  G  $\in$  A leadsTo B"
proof -
  have prog: "{X. G  $\in$  stable X}  $\in$  progress_set F UNIV B"
  by (simp add: progress_set_def lattice_stable Gstable closed_def
              stable_Un [OF Gstable] dec [unfolded decoupled_def])
  have "F  $\sqcup$  G  $\in$  (UNIV  $\cap$  A) leadsTo B"
  by (rule progress_set_Union [OF leadsTo prog],
      simp_all add: Gstable stableco)

```

```

    thus ?thesis by simp
qed

```

Rao's Weak Decoupling Theorem

```

theorem weak_decoupling:
  assumes leadsTo: "F ∈ A leadsTo B"
    and stable: "F ⊔ G ∈ stable B"
    and dec:      "decoupled F (F ⊔ G)"
  shows "F ⊔ G ∈ A leadsTo B"
proof -
  have prog: "{X. F ⊔ G ∈ stable X} ∈ progress_set F UNIV B"
    by (simp del: Join_stable
        add: progress_set_def lattice_stable stable_closed_def
            stable_Un [OF stable] dec [unfolded decoupled_def])
  have "F ⊔ G ∈ (UNIV ∩ A) leadsTo B"
    by (rule progress_set_Union [OF leadsTo prog],
        simp_all del: Join_stable add: stable,
        simp add: stableco)
  thus ?thesis by simp
qed

```

The “Decoupling via G' Union Theorem”

```

theorem decoupling_via_aux:
  assumes leadsTo: "F ∈ A leadsTo B"
    and prog: "{X. G' ∈ stable X} ∈ progress_set F UNIV B"
    and GG':   "G ≤ G'"
    — Beware! This is the converse of the refinement relation!
  shows "F ⊔ G ∈ A leadsTo B"
proof -
  from prog have stable: "G' ∈ stable B"
    by (simp add: progress_set_def)
  have "F ⊔ G ∈ (UNIV ∩ A) leadsTo B"
    by (rule progress_set_Union [OF leadsTo prog],
        simp_all add: stable stableco component_stable [OF GG'])
  thus ?thesis by simp
qed

```

17.5 Composition Theorems Based on Monotonicity and Commutativity

17.5.1 Commutativity of $c1\ L$ and assignment.

```

constdefs
  commutes :: "[ 'a program, 'a set, 'a set, 'a set set] => bool"
    "commutes F T B L ==
      ∀ M. ∀ act ∈ Acts F. B ⊆ M -->
        c1 L (T ∩ wp act M) ⊆ T ∩ (B ∪ wp act (c1 L (T ∩ M)))"

```

From Meier's thesis, section 4.5.6

```

lemma commutativity1_lemma:
  assumes commutes: "commutes F T B L"
    and lattice:   "lattice L"
    and BL:        "B ∈ L"

```

```

    and TL: "T ∈ L"
    shows "closed F T B L"
  apply (simp add: closed_def, clarify)
  apply (rule ProgressSets.cl_subset_in_lattice [OF _ lattice])
  apply (simp add: Int_Un_distrib cl_Un [OF lattice] Un_subset_iff
    cl_ident Int_in_lattice [OF TL BL lattice] Un_upper1)
  apply (subgoal_tac "cl L (T ∩ wp act M) ⊆ T ∩ (B ∪ wp act (cl L (T ∩ M)))")

  prefer 2
  apply (cut_tac commutes, simp add: commutes_def)
  apply (erule subset_trans)
  apply (simp add: cl_ident)
  apply (blast intro: rev_subsetD [OF _ wp_mono])
done

```

Version packaged with $\llbracket ?F \in ?A \text{ leadsTo } ?B'; ?C \in \text{progress_set } ?F ?T ?B; ?F \in \text{stable } ?T; ?B \subseteq ?B'; ?B' \in ?C; \bigwedge X. X \in ?C \implies ?G \in X - ?B \text{ co } X \rrbracket \implies ?F \sqcup ?G \in ?T \cap ?A \text{ leadsTo } ?B'$

```

lemma commutativity1:
  assumes leadsTo: "F ∈ A leadsTo B"
    and lattice: "lattice L"
    and BL: "B ∈ L"
    and TL: "T ∈ L"
    and Fstable: "F ∈ stable T"
    and Gco: "!!X. X ∈ L ==> G ∈ X-B co X"
    and commutes: "commutes F T B L"
  shows "F ∪ G ∈ T ∩ A leadsTo B"
by (rule progress_set_Union [OF leadsTo _ Fstable subset_refl BL Gco],
  simp add: progress_set_def commutativity1_lemma commutes lattice BL TL)

```

Possibly move to Relation.thy, after `single_valued`

```

constdefs
  funof :: "[(a*b)set, 'a] => 'b"
  "funof r == (λx. THE y. (x,y) ∈ r)"

lemma funof_eq: "[|single_valued r; (x,y) ∈ r|] ==> funof r x = y"
by (simp add: funof_def single_valued_def, blast)

lemma funof_Pair_in:
  "[|single_valued r; x ∈ Domain r|] ==> (x, funof r x) ∈ r"
by (force simp add: funof_eq)

lemma funof_in:
  "[|r ⊆ {x} × A; single_valued r; x ∈ Domain r|] ==> funof r x ∈ A"
by (force simp add: funof_eq)

lemma funof_imp_wp: "[|funof act t ∈ A; single_valued act|] ==> t ∈ wp act A"
by (force simp add: in_wp_iff funof_eq)

```

17.5.2 Commutativity of Functions and Relation

From Meier's thesis, section 4.5.6

```

lemma commutativity2_lemma:
  assumes dcommutes:
    "∀act ∈ Acts F.
     ∀s ∈ T. ∀t. (s,t) ∈ relcl L -->
      s ∈ B | t ∈ B | (funof act s, funof act t) ∈ relcl
L"
  and determ: "!!act. act ∈ Acts F ==> single_valued act"
  and total: "!!act. act ∈ Acts F ==> Domain act = UNIV"
  and lattice: "lattice L"
  and BL: "B ∈ L"
  and TL: "T ∈ L"
  and Fstable: "F ∈ stable T"
  shows "commutes F T B L"
  apply (simp add: commutes_def del: Int_subset_iff, clarify)
  apply (rename_tac t)
  apply (subgoal_tac "∃s. (s,t) ∈ relcl L & s ∈ T ∩ wp act M")
  prefer 2
  apply (force simp add: cl_eq_Collect_relcl [OF lattice], simp, clarify)
  apply (subgoal_tac "∀u∈L. s ∈ u --> t ∈ u")
  prefer 2
  apply (intro ballI impI)
  apply (subst cl_ident [symmetric], assumption)
  apply (simp add: relcl_def)
  apply (blast intro: cl_mono [THEN [2] rev_subsetD])
  apply (subgoal_tac "funof act s ∈ T ∩ M")
  prefer 2
  apply (cut_tac Fstable)
  apply (force intro!: funof_in
    simp add: wp_def stable_def constrains_def determ total)
  apply (subgoal_tac "s ∈ B | t ∈ B | (funof act s, funof act t) ∈ relcl L")
  prefer 2
  apply (rule dcommutes [rule_format], assumption+)
  apply (subgoal_tac "t ∈ B | funof act t ∈ cl L (T ∩ M)")
  prefer 2
  apply (simp add: relcl_def)
  apply (blast intro: BL cl_mono [THEN [2] rev_subsetD])
  apply (subgoal_tac "t ∈ B | t ∈ wp act (cl L (T ∩ M))")
  prefer 2
  apply (blast intro: funof_imp_wp determ)
  apply (blast intro: TL cl_mono [THEN [2] rev_subsetD])
  done

```

Version packaged with $\llbracket ?F \in ?A \text{ leadsTo } ?B'; ?C \in \text{progress_set } ?F ?T ?B; ?F \in \text{stable } ?T; ?B \subseteq ?B'; ?B' \in ?C; \bigwedge X. X \in ?C \implies ?G \in X - ?B \text{ co } X \rrbracket \implies ?F \sqcup ?G \in ?T \cap ?A \text{ leadsTo } ?B'$

```

lemma commutativity2:
  assumes leadsTo: "F ∈ A leadsTo B"
  and dcommutes:
    "∀act ∈ Acts F.
     ∀s ∈ T. ∀t. (s,t) ∈ relcl L -->
      s ∈ B | t ∈ B | (funof act s, funof act t) ∈ relcl
L"

```

```

    and determ: "!!act. act ∈ Acts F ==> single_valued act"
    and total: "!!act. act ∈ Acts F ==> Domain act = UNIV"
    and lattice: "lattice L"
    and BL: "B ∈ L"
    and TL: "T ∈ L"
    and Fstable: "F ∈ stable T"
    and Gco: "!!X. X ∈ L ==> G ∈ X-B co X"
  shows "F ⊔ G ∈ T ∩ A leadsTo B"
apply (rule commutativity1 [OF leadsTo lattice])
apply (simp_all add: Gco commutativity2_lemma dcommutes determ total
      lattice BL TL Fstable)
done

```

17.6 Monotonicity

From Meier's thesis, section 4.5.7, page 110

end

18 Comprehensive UNITY Theory

```

theory UNITY_Main imports Detects PPROD Follows ProgressSets
uses "UNITY_tactics.ML" begin

```

```

method_setup safety = {*
  Scan.succeed (fn ctxt =>
    SIMPLE_METHOD' (constrains_tac (local_clasimpset_of ctxt))) *}
  "for proving safety properties"

method_setup ensures_tac = {*
  Args.goal_spec -- Scan.lift Args.name_source >>
  (fn (quant, s) => fn ctxt => SIMPLE_METHOD'' quant (ensures_tac (local_clasimpset_of
    ctxt) s))
  *} "for proving progress properties"

end

```

```

theory Deadlock imports UNITY begin

```

```

lemma "[| F ∈ (A ∩ B) co A; F ∈ (B ∩ A) co B |] ==> F ∈ stable (A ∩ B)"
by (unfold constrains_def stable_def, blast)

```

```

lemma Collect_le_Int_equals:
  "(⋂ i ∈ atMost n. A (Suc i) ∩ A i) = (⋂ i ∈ atMost (Suc n). A i)"
apply (induct_tac "n")
apply (auto simp add: atMost_Suc)
done

```

```

lemma UN_Int_Compl_subset:
  " $(\bigcup i \in \text{lessThan } n. A \ i) \cap (\neg A \ n) \subseteq$   

 $(\bigcup i \in \text{lessThan } n. (A \ i) \cap (\neg A \ (\text{Suc } i)))$ "
apply (induct_tac "n", simp)
apply (simp add: lessThan_Suc, blast)
done

```

```

lemma INT_Un_Compl_subset:
  " $(\bigcap i \in \text{lessThan } n. \neg A \ i \cup A \ (\text{Suc } i)) \subseteq$   

 $(\bigcap i \in \text{lessThan } n. \neg A \ i) \cup A \ n$ "
apply (induct_tac "n", simp)
apply (simp add: lessThan_Suc, fast)
done

```

```

lemma INT_le_equals_Int_lemma:
  " $A \ 0 \cap (\neg(A \ n) \cap (\bigcap i \in \text{lessThan } n. \neg A \ i \cup A \ (\text{Suc } i))) = \{\}$ "
by (blast intro: grOI dest: INT_Un_Compl_subset [THEN subsetD])

```

```

lemma INT_le_equals_Int:
  " $(\bigcap i \in \text{atMost } n. A \ i) =$   

 $A \ 0 \cap (\bigcap i \in \text{lessThan } n. \neg A \ i \cup A \ (\text{Suc } i))$ "
apply (induct_tac "n", simp)
apply (simp add: Int_ac Int_Un_distrib Int_Un_distrib2
  INT_le_equals_Int_lemma lessThan_Suc atMost_Suc)
done

```

```

lemma INT_le_Suc_equals_Int:
  " $(\bigcap i \in \text{atMost } (\text{Suc } n). A \ i) =$   

 $A \ 0 \cap (\bigcap i \in \text{atMost } n. \neg A \ i \cup A \ (\text{Suc } i))$ "
by (simp add: lessThan_Suc_atMost INT_le_equals_Int)

```

```

lemma
  assumes zeroprem: " $F \in (A \ 0 \cap A \ (\text{Suc } n)) \text{ co } (A \ 0)$ "
  and allprem:
    " $\forall i. i \in \text{atMost } n \implies F \in (A \ (\text{Suc } i) \cap A \ i) \text{ co } (\neg A \ i \cup A \ (\text{Suc } i))$ "
  shows " $F \in \text{stable } (\bigcap i \in \text{atMost } (\text{Suc } n). A \ i)$ "
apply (unfold stable_def)
apply (rule constrains_Int [THEN constrains_weaken])
  apply (rule zeroprem)
  apply (rule constrains_INT)
  apply (erule allprem)
  apply (simp add: Collect_le_Int_equals Int_assoc INT_absorb)
apply (simp add: INT_le_Suc_equals_Int)
done

```


end

theory Common imports "../UNITY_Main" begin

consts

ftime :: "nat=>nat"

gtime :: "nat=>nat"

axioms

fmono: " $m \leq n \implies \text{ftime } m \leq \text{ftime } n$ "

gmono: " $m \leq n \implies \text{gtime } m \leq \text{gtime } n$ "

fasc: " $m \leq \text{ftime } n$ "

gasc: " $m \leq \text{gtime } n$ "

constdefs

common :: "nat set"

"common == {n. ftime n = n & gtime n = n}"

maxfg :: "nat => nat set"

"maxfg m == {t. t ≤ max (ftime m) (gtime m)}"

lemma common_stable:

"[| $\forall m. F \in \{m\} \text{ Co } (\text{maxfg } m)$; $n \in \text{common}$ |]"

$\implies F \in \text{Stable } (\text{atMost } n)$ "

apply (drule_tac M = "{t. t ≤ n}" in Elimination_sing)

apply (simp add: atMost_def Stable_def common_def maxfg_def le_max_iff_disj)

apply (erule Constrains_weaken_R)

apply (blast intro: order_eq_refl le_trans dest: fmono gmono)

done

lemma common_safety:

"[| Init F ⊆ atMost n;

$\forall m. F \in \{m\} \text{ Co } (\text{maxfg } m)$; $n \in \text{common}$ |]"

$\implies F \in \text{Always } (\text{atMost } n)$ "

by (simp add: AlwaysI common_stable)

lemma "SKIP ∈ {m} co (maxfg m)"

by (simp add: constrains_def maxfg_def le_max_iff_disj fasc)

lemma "mk_total_program

(UNIV, {range(%t.(t,ftime t)), range(%t.(t,gtime t))}, UNIV)

∈ {m} co (maxfg m)"

apply (simp add: mk_total_program_def)

apply (simp add: constrains_def maxfg_def le_max_iff_disj fasc)

done

```

lemma "mk_total_program (UNIV, {range(%t.(t, max (ftime t) (gtime t)))},
UNIV)
  ∈ {m} co (maxfg m)"
apply (simp add: mk_total_program_def)
apply (simp add: constrains_def maxfg_def max_def gasc)
done

```

```

lemma "mk_total_program
  (UNIV, { {(t, Suc t) | t. t < max (ftime t) (gtime t)} }, UNIV)
  ∈ {m} co (maxfg m)"
apply (simp add: mk_total_program_def)
apply (simp add: constrains_def maxfg_def max_def gasc)
done

```

```

declare atMost_Int_atLeast [simp]

```

```

lemma leadsTo_common_lemma:
  "[| ∀m. F ∈ {m} Co (maxfg m);
    ∀m ∈ lessThan n. F ∈ {m} LeadsTo (greaterThan m);
    n ∈ common |]
  ==> F ∈ (atMost n) LeadsTo common"
apply (rule LeadsTo_weaken_R)
apply (rule_tac f = id and l = n in GreaterThan_bounded_induct)
apply (simp_all (no_asm_simp))
apply (rule_tac [2] subset_refl)
apply (blast dest: PSP_Stable2 intro: common_stable LeadsTo_weaken_R)
done

```

```

lemma leadsTo_common:
  "[| ∀m. F ∈ {m} Co (maxfg m);
    ∀m ∈ -common. F ∈ {m} LeadsTo (greaterThan m);
    n ∈ common |]
  ==> F ∈ (atMost (LEAST n. n ∈ common)) LeadsTo common"
apply (rule leadsTo_common_lemma)
apply (simp_all (no_asm_simp))
apply (erule_tac [2] LeastI)
apply (blast dest!: not_less_Least)
done

```

```

end

```

```

theory Network imports UNITY begin

```

```

datatype pvar = Sent | Rcvd | Idle

datatype pname = Aproc | Bproc

types state = "pname * pvar => nat"

locale F_props =
  fixes F
  assumes rsA: "F ∈ stable {s. s(Bproc,Rcvd) ≤ s(Aproc,Sent)}"
    and rsB: "F ∈ stable {s. s(Aproc,Rcvd) ≤ s(Bproc,Sent)}"
    and sent_nondec: "F ∈ stable {s. m ≤ s(proc,Sent)}"
    and rcvd_nondec: "F ∈ stable {s. n ≤ s(proc,Rcvd)}"
    and rcvd_idle: "F ∈ {s. s(proc,Idle) = Suc 0 & s(proc,Rcvd) = m}
      co {s. s(proc,Rcvd) = m --> s(proc,Idle) = Suc 0}"
    and sent_idle: "F ∈ {s. s(proc,Idle) = Suc 0 & s(proc,Sent) = n}
      co {s. s(proc,Sent) = n}"

lemmas (in F_props)
  sent_nondec_A = sent_nondec [of _ Aproc]
  and sent_nondec_B = sent_nondec [of _ Bproc]
  and rcvd_nondec_A = rcvd_nondec [of _ Aproc]
  and rcvd_nondec_B = rcvd_nondec [of _ Bproc]
  and rcvd_idle_A = rcvd_idle [of Aproc]
  and rcvd_idle_B = rcvd_idle [of Bproc]
  and sent_idle_A = sent_idle [of Aproc]
  and sent_idle_B = sent_idle [of Bproc]

  and rs_AB = stable_Int [OF rsA rsB]
  and sent_nondec_AB = stable_Int [OF sent_nondec_A sent_nondec_B]
  and rcvd_nondec_AB = stable_Int [OF rcvd_nondec_A rcvd_nondec_B]
  and rcvd_idle_AB = constrains_Int [OF rcvd_idle_A rcvd_idle_B]
  and sent_idle_AB = constrains_Int [OF sent_idle_A sent_idle_B]
  and nondec_AB = stable_Int [OF sent_nondec_AB rcvd_nondec_AB]
  and idle_AB = constrains_Int [OF rcvd_idle_AB sent_idle_AB]
  and nondec_idle = constrains_Int [OF nondec_AB [unfolded stable_def]
    idle_AB]

lemma (in F_props)
  shows "F ∈ stable {s. s(Aproc,Idle) = Suc 0 & s(Bproc,Idle) = Suc 0 &
    s(Aproc,Sent) = s(Bproc,Rcvd) &
    s(Bproc,Sent) = s(Aproc,Rcvd) &
    s(Aproc,Rcvd) = m & s(Bproc,Rcvd) = n}"

apply (unfold stable_def)
apply (rule constrainsI)
apply (drule constrains_Int [OF rs_AB [unfolded stable_def] nondec_idle,
  THEN constrainsD], assumption)

apply simp_all
apply (blast del: le0, clarify)
apply (subgoal_tac "s' (Aproc, Rcvd) = s (Aproc, Rcvd)")
apply (subgoal_tac "s' (Bproc, Rcvd) = s (Bproc, Rcvd)")

```

```

apply simp
apply (blast intro: order_antisym le_trans eq_imp_le)+
done

end

```

19 The Token Ring

```

theory Token
imports "../WFair"

```

```
begin
```

From Misra, "A Logic for Concurrent Programming" (1994), sections 5.2 and 13.2.

19.1 Definitions

```

datatype pstate = Hungry | Eating | Thinking
    — process states

record state =
  token :: "nat"
  proc  :: "nat => pstate"

constdefs
  HasTok :: "nat => state set"
    "HasTok i == {s. token s = i}"

  H :: "nat => state set"
    "H i == {s. proc s i = Hungry}"

  E :: "nat => state set"
    "E i == {s. proc s i = Eating}"

  T :: "nat => state set"
    "T i == {s. proc s i = Thinking}"

locale Token =
  fixes N and F and nodeOrder and "next"
  defines nodeOrder_def:
    "nodeOrder j == measure(%i. ((j+N)-i) mod N)  $\cap$  {.. $N$ }  $\times$  {.. $N$ }"
  and next_def:
    "next i == (Suc i) mod N"
  assumes N_positive [iff]: "0 < N"
  and TR2: "F  $\in$  (T i) co (T i  $\cup$  H i)"
  and TR3: "F  $\in$  (H i) co (H i  $\cup$  E i)"
  and TR4: "F  $\in$  (H i - HasTok i) co (H i)"
  and TR5: "F  $\in$  (HasTok i) co (HasTok i  $\cup$  -(E i))"
  and TR6: "F  $\in$  (H i  $\cap$  HasTok i) leadsTo (E i)"

```

```

and TR7: "F ∈ (HasTok i) leadsTo (HasTok (next i))"

lemma HasTok_partition: "[| s ∈ HasTok i; s ∈ HasTok j |] ==> i=j"
by (unfold HasTok_def, auto)

lemma not_E_eq: "(s ∉ E i) = (s ∈ H i | s ∈ T i)"
apply (simp add: H_def E_def T_def)
apply (case_tac "proc s i", auto)
done

lemma (in Token) token_stable: "F ∈ stable (-(E i) ∪ (HasTok i))"
apply (unfold stable_def)
apply (rule constrains_weaken)
apply (rule constrains_Un [OF constrains_Un [OF TR2 TR4] TR5])
apply (auto simp add: not_E_eq)
apply (simp_all add: H_def E_def T_def)
done

```

19.2 Progress under Weak Fairness

```

lemma (in Token) wf_nodeOrder: "wf(nodeOrder j)"
apply (unfold nodeOrder_def)
apply (rule wf_measure [THEN wf_subset], blast)
done

lemma (in Token) nodeOrder_eq:
  "[| i<N; j<N |] ==> ((next i, i) ∈ nodeOrder j) = (i ≠ j)"
apply (unfold nodeOrder_def next_def)
apply (auto simp add: mod_Suc mod_geq)
apply (auto split add: nat_diff_split simp add: linorder_neq_iff mod_geq)
done

```

From "A Logic for Concurrent Programming", but not used in Chapter 4. Note the use of `case_tac`. Reasoning about `leadsTo` takes practice!

```

lemma (in Token) TR7_nodeOrder:
  "[| i<N; j<N |] ==>
    F ∈ (HasTok i) leadsTo ({s. (token s, i) ∈ nodeOrder j} ∪ HasTok j)"
apply (case_tac "i=j")
apply (blast intro: subset_imp_leadsTo)
apply (rule TR7 [THEN leadsTo_weaken_R])
apply (auto simp add: HasTok_def nodeOrder_eq)
done

```

Chapter 4 variant, the one actually used below.

```

lemma (in Token) TR7_aux: "[| i<N; j<N; i≠j |]
  ==> F ∈ (HasTok i) leadsTo {s. (token s, i) ∈ nodeOrder j}"
apply (rule TR7 [THEN leadsTo_weaken_R])
apply (auto simp add: HasTok_def nodeOrder_eq)
done

```

```

lemma (in Token) token_lemma:
  "({s. token s < N} ∩ token -' {m}) = (if m<N then token -' {m} else {})"
by auto

```

Misra's TR9: the token reaches an arbitrary node

```
lemma (in Token) leadsTo_j: "j < N ==> F ∈ {s. token s < N} leadsTo (HasTok j)"
apply (rule leadsTo_weaken_R)
apply (rule_tac I = "-{j}" and f = token and B = "{}"
      in wf_nodeOrder [THEN bounded_induct])
apply (simp_all (no_asm_simp) add: token_lemma vimage_Diff HasTok_def)
prefer 2 apply blast
apply clarify
apply (rule TR7_aux [THEN leadsTo_weaken])
apply (auto simp add: HasTok_def nodeOrder_def)
done
```

Misra's TR8: a hungry process eventually eats

```
lemma (in Token) token_progress:
  "j < N ==> F ∈ ({s. token s < N} ∩ H j) leadsTo (E j)"
apply (rule leadsTo_cancel1 [THEN leadsTo_Un_duplicate])
apply (rule_tac [2] TR6)
apply (rule psp [OF leadsTo_j TR3, THEN leadsTo_weaken], blast+)
done
```

end

theory Channel imports "../UNITY_Main" begin

types state = "nat set"

consts

F :: "state program"

constdefs

minSet :: "nat set => nat option"

"minSet A == if A={} then None else Some (LEAST x. x ∈ A)"

axioms

UC1: "F ∈ (minSet -' {Some x}) co (minSet -' (Some 'atLeast x))"

UC2: "F ∈ (minSet -' {Some x}) leadsTo {s. x ∉ s}"

lemma minSet_eq_SomeD: "minSet A = Some x ==> x ∈ A"

```
apply (unfold minSet_def)
apply (simp split: split_if_asm)
apply (fast intro: LeastI)
done
```

lemma minSet_empty [simp]: "minSet {} = None"

```

by (unfold minSet_def, simp)

lemma minSet_nonempty: "x ∈ A ==> minSet A = Some (LEAST x. x ∈ A)"
by (unfold minSet_def, auto)

lemma minSet_greaterThan:
  "F ∈ (minSet -' {Some x}) leadsTo (minSet -' (Some 'greaterThan x))"
apply (rule leadsTo_weaken)
apply (rule_tac x1=x in psp [OF UC2 UC1], safe)
apply (auto dest: minSet_eq_SomeD simp add: linorder_neq_iff)
done

lemma Channel_progress_lemma:
  "F ∈ (UNIV-{{}}) leadsTo (minSet -' (Some 'atLeast y))"
apply (rule leadsTo_weaken_R)
apply (rule_tac l = y and f = "the o minSet" and B = "{}"
  in greaterThan_bounded_induct, safe)
apply (simp_all (no_asm_simp))
apply (drule_tac [2] minSet_nonempty)
  prefer 2 apply simp
apply (rule minSet_greaterThan [THEN leadsTo_weaken], safe)
apply simp_all
apply (drule minSet_nonempty, simp)
done

lemma Channel_progress: "!!y::nat. F ∈ (UNIV-{{}}) leadsTo {s. y ∉ s}"
apply (rule Channel_progress_lemma [THEN leadsTo_weaken_R], clarify)
apply (frule minSet_nonempty)
apply (auto dest: Suc_le_lessD not_less_Least)
done

end

theory Lift
imports "../UNITY_Main"

begin

record state =
  floor :: "int"           — current position of the lift
  open  :: "bool"          — whether the door is opened at floor
  stop  :: "bool"          — whether the lift is stopped at floor
  req   :: "int set"        — for each floor, whether the lift is requested
  up    :: "bool"          — current direction of movement
  move  :: "bool"          — whether moving takes precedence over opening

consts
  Min :: "int"             — least and greatest floors
  Max :: "int"             — least and greatest floors

```

axioms

Min_le_Max [iff]: " $\text{Min} \leq \text{Max}$ "

constdefs

— Abbreviations: the "always" part

above :: "state set"
 "*above* == {*s*. $\exists i. \text{floor } s < i \ \& \ i \leq \text{Max} \ \& \ i \in \text{req } s$ }"

below :: "state set"
 "*below* == {*s*. $\exists i. \text{Min} \leq i \ \& \ i < \text{floor } s \ \& \ i \in \text{req } s$ }"

queueing :: "state set"
 "*queueing* == *above* \cup *below*"

goingup :: "state set"
 "*goingup* == *above* \cap ({*s*. *up s*} \cup \neg *below*)"

goingdown :: "state set"
 "*goingdown* == *below* \cap ({*s*. \sim *up s*} \cup \neg *above*)"

ready :: "state set"
 "*ready* == {*s*. *stop s* $\& \ \sim$ *open s* $\& \ \text{move } s$ }"

— Further abbreviations

moving :: "state set"
 "*moving* == {*s*. \sim *stop s* $\& \ \sim$ *open s*}"

stopped :: "state set"
 "*stopped* == {*s*. *stop s* $\& \ \sim$ *open s* $\& \ \sim$ *move s*}"

opened :: "state set"
 "*opened* == {*s*. *stop s* $\& \ \text{open } s \ \& \ \text{move } s$ }"

closed :: "state set" — but this is the same as *ready*!!
 "*closed* == {*s*. *stop s* $\& \ \sim$ *open s* $\& \ \text{move } s$ }"

atFloor :: "int \Rightarrow state set"
 "*atFloor* *n* == {*s*. *floor s* = *n*}"

Req :: "int \Rightarrow state set"
 "*Req* *n* == {*s*. *n* \in *req s*}"

— The program

request_act :: "(state*state) set"
 "*request_act* == {(*s*,*s'*). *s'* = *s* (\neg *stop*:=True, *move*:=False)
 $\& \ \sim$ *stop s* $\& \ \text{floor } s \in \text{req } s$ }"

open_act :: "(state*state) set"


```

"open_act ==
  {(s,s'). s' = s (/open := True,
    req := req s - {floor s},
    move := True|)
  & stop s & ~ open s & floor s ∈ req s
  & ~(move s & s ∈ queueing)}"

close_act :: "(state*state) set"
"close_act == {(s,s'). s' = s (/open := False|) & open s}"

req_up :: "(state*state) set"
"req_up ==
  {(s,s'). s' = s (/stop := False,
    floor := floor s + 1,
    up := True|)
  & s ∈ (ready ∩ goingup)}"

req_down :: "(state*state) set"
"req_down ==
  {(s,s'). s' = s (/stop := False,
    floor := floor s - 1,
    up := False|)
  & s ∈ (ready ∩ goingdown)}"

move_up :: "(state*state) set"
"move_up ==
  {(s,s'). s' = s (/floor := floor s + 1|)
  & ~ stop s & up s & floor s ∉ req s}"

move_down :: "(state*state) set"
"move_down ==
  {(s,s'). s' = s (/floor := floor s - 1|)
  & ~ stop s & ~ up s & floor s ∉ req s}"

button_press :: "(state*state) set"
— This action is omitted from prior treatments, which therefore are unrealistic:
nobody asks the lift to do anything! But adding this action invalidates many of the
existing progress arguments: various "ensures" properties fail. Maybe it should be
constrained to only allow button presses in the current direction of travel, like in a
real lift.
"button_press ==
  {(s,s'). ∃ n. s' = s (/req := insert n (req s)|)
  & Min ≤ n & n ≤ Max}"

Lift :: "state program"
— for the moment, we OMIT button_press
"Lift == mk_total_program
  ({s. floor s = Min & ~ up s & move s & stop s &
    ~ open s & req s = {}},
  {request_act, open_act, close_act,
   req_up, req_down, move_up, move_down},
  UNIV)"

```

— Invariants

```

bounded :: "state set"
  "bounded == {s. Min ≤ floor s & floor s ≤ Max}"

open_stop :: "state set"
  "open_stop == {s. open s --> stop s}"

open_move :: "state set"
  "open_move == {s. open s --> move s}"

stop_floor :: "state set"
  "stop_floor == {s. stop s & ~ move s --> floor s ∈ req s}"

moving_up :: "state set"
  "moving_up == {s. ~ stop s & up s -->
    (∃ f. floor s ≤ f & f ≤ Max & f ∈ req s)}"

moving_down :: "state set"
  "moving_down == {s. ~ stop s & ~ up s -->
    (∃ f. Min ≤ f & f ≤ floor s & f ∈ req s)}"

metric :: "[int,state] => int"
  "metric ==
    %n s. if floor s < n then (if up s then n - floor s
      else (floor s - Min) + (n-Min))
    else
      if n < floor s then (if up s then (Max - floor s) + (Max-n)
        else floor s - n)
    else 0"

locale Floor =
  fixes n
  assumes Min_le_n [iff]: "Min ≤ n"
  and n_le_Max [iff]: "n ≤ Max"

lemma not_mem_distinct: "[| x ∉ A; y ∈ A |] ==> x ≠ y"
by blast

declare Lift_def [THEN def_prg_Init, simp]

declare request_act_def [THEN def_act_simp, simp]
declare open_act_def [THEN def_act_simp, simp]
declare close_act_def [THEN def_act_simp, simp]
declare req_up_def [THEN def_act_simp, simp]
declare req_down_def [THEN def_act_simp, simp]
declare move_up_def [THEN def_act_simp, simp]
declare move_down_def [THEN def_act_simp, simp]
declare button_press_def [THEN def_act_simp, simp]

declare above_def [THEN def_set_simp, simp]

```

```

declare below_def [THEN def_set_simp, simp]
declare queueing_def [THEN def_set_simp, simp]
declare goingup_def [THEN def_set_simp, simp]
declare goingdown_def [THEN def_set_simp, simp]
declare ready_def [THEN def_set_simp, simp]

declare bounded_def [simp]
      open_stop_def [simp]
      open_move_def [simp]
      stop_floor_def [simp]
      moving_up_def [simp]
      moving_down_def [simp]

lemma open_stop: "Lift ∈ Always open_stop"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
done

lemma stop_floor: "Lift ∈ Always stop_floor"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
done

lemma open_move: "Lift ∈ Always open_move"
apply (cut_tac open_stop)
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
done

lemma moving_up: "Lift ∈ Always moving_up"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
apply (auto dest: order_le_imp_less_or_eq simp add: add1_zle_eq)
done

lemma moving_down: "Lift ∈ Always moving_down"
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety)
apply (blast dest: order_le_imp_less_or_eq)
done

lemma bounded: "Lift ∈ Always bounded"
apply (cut_tac moving_up moving_down)
apply (rule AlwaysI, force)
apply (unfold Lift_def, safety, auto)
apply (drule not_mem_distinct, assumption, arith)+
done

```

19.3 Progress

```

declare moving_def [THEN def_set_simp, simp]
declare stopped_def [THEN def_set_simp, simp]

```

```

declare opened_def [THEN def_set_simp, simp]
declare closed_def [THEN def_set_simp, simp]
declare atFloor_def [THEN def_set_simp, simp]
declare Req_def [THEN def_set_simp, simp]

```

The HUG'93 paper mistakenly omits the Req n from these!

```

lemma E_thm01: "Lift ∈ (stopped ∩ atFloor n) LeadsTo (opened ∩ atFloor
n)"
apply (cut_tac stop_floor)
apply (unfold Lift_def, ensures_tac "open_act")
done

```

```

lemma E_thm02: "Lift ∈ (Req n ∩ stopped - atFloor n) LeadsTo
(Req n ∩ opened - atFloor n)"
apply (cut_tac stop_floor)
apply (unfold Lift_def, ensures_tac "open_act")
done

```

```

lemma E_thm03: "Lift ∈ (Req n ∩ opened - atFloor n) LeadsTo
(Req n ∩ closed - (atFloor n - queueing))"
apply (unfold Lift_def, ensures_tac "close_act")
done

```

```

lemma E_thm04: "Lift ∈ (Req n ∩ closed ∩ (atFloor n - queueing))
LeadsTo (opened ∩ atFloor n)"
apply (unfold Lift_def, ensures_tac "open_act")
done

```

```

lemmas linorder_leI = linorder_not_less [THEN iffD1]

```

```

lemmas (in Floor) le_MinD = Min_le_n [THEN order_antisym]
and Max_leD = n_le_Max [THEN [2] order_antisym]

```

```

declare (in Floor) le_MinD [dest!]
and linorder_leI [THEN le_MinD, dest!]
and Max_leD [dest!]
and linorder_leI [THEN Max_leD, dest!]

```

```

lemma (in Floor) E_thm05c:
  "Lift ∈ (Req n ∩ closed - (atFloor n - queueing))
  LeadsTo ((closed ∩ goingup ∩ Req n) ∪
(closed ∩ goingdown ∩ Req n))"
by (auto intro!: subset_imp_LeadsTo simp add: linorder_neq_iff)

```

```

lemma (in Floor) lift_2: "Lift ∈ (Req n ∩ closed - (atFloor n - queueing))

```

```

      LeadsTo (moving  $\cap$  Req n)"
apply (rule LeadsTo_Trans [OF E_thm05c LeadsTo_Un])
apply (unfold Lift_def)
apply (ensures_tac [2] "req_down")
apply (ensures_tac "req_up", auto)
done

declare split_if_asm [split]

lemma (in Floor) E_thm12a:
  "0 < N ==>
    Lift  $\in$  (moving  $\cap$  Req n  $\cap$  {s. metric n s = N}  $\cap$ 
      {s. floor s  $\notin$  req s}  $\cap$  {s. up s})
    LeadsTo
      (moving  $\cap$  Req n  $\cap$  {s. metric n s < N})"
apply (cut_tac moving_up)
apply (unfold Lift_def, ensures_tac "move_up", safe)

apply (erule linorder_leI [THEN order_antisym, symmetric])
apply (auto simp add: metric_def)
done

lemma (in Floor) E_thm12b: "0 < N ==>
  Lift  $\in$  (moving  $\cap$  Req n  $\cap$  {s. metric n s = N}  $\cap$ 
    {s. floor s  $\notin$  req s} - {s. up s})
    LeadsTo (moving  $\cap$  Req n  $\cap$  {s. metric n s < N})"
apply (cut_tac moving_down)
apply (unfold Lift_def, ensures_tac "move_down", safe)

apply (erule linorder_leI [THEN order_antisym, symmetric])
apply (auto simp add: metric_def)
done

lemma (in Floor) lift_4:
  "0 < N ==> Lift  $\in$  (moving  $\cap$  Req n  $\cap$  {s. metric n s = N}  $\cap$ 
    {s. floor s  $\notin$  req s}) LeadsTo
    (moving  $\cap$  Req n  $\cap$  {s. metric n s < N})"
apply (rule LeadsTo_Trans [OF subset_imp_LeadsTo
  LeadsTo_Un [OF E_thm12a E_thm12b]], auto)
done

```

```

lemma (in Floor) E_thm16a: "0 < N
  ==> Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N} ∩ goingup) LeadsTo
    (moving ∩ Req n ∩ {s. metric n s < N})"
apply (cut_tac bounded)
apply (unfold Lift_def, ensures_tac "req_up")
apply (auto simp add: metric_def)
done

```

```

lemma (in Floor) E_thm16b: "0 < N ==>
  Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N} ∩ goingdown) LeadsTo
    (moving ∩ Req n ∩ {s. metric n s < N})"
apply (cut_tac bounded)
apply (unfold Lift_def, ensures_tac "req_down")
apply (auto simp add: metric_def)
done

```

```

lemma (in Floor) E_thm16c:
  "0 < N ==> Req n ∩ {s. metric n s = N} ⊆ goingup ∪ goingdown"
by (force simp add: metric_def)

```

```

lemma (in Floor) lift_5:
  "0 < N ==> Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N}) LeadsTo
    (moving ∩ Req n ∩ {s. metric n s < N})"
apply (rule LeadsTo_Trans [OF subset_imp_LeadsTo
  LeadsTo_Un [OF E_thm16a E_thm16b]])
apply (drule E_thm16c, auto)
done

```

```

lemma (in Floor) metric_eq_OD [dest]:
  "[| metric n s = 0; Min ≤ floor s; floor s ≤ Max |] ==> floor s =
  n"
by (force simp add: metric_def)

```

```

lemma (in Floor) E_thm11: "Lift ∈ (moving ∩ Req n ∩ {s. metric n s = 0})
  LeadsTo
    (stopped ∩ atFloor n)"
apply (cut_tac bounded)
apply (unfold Lift_def, ensures_tac "request_act", auto)
done

```

```

lemma (in Floor) E_thm13:
  "Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})

  LeadsTo (stopped ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})"
apply (unfold Lift_def, ensures_tac "request_act")
apply (auto simp add: metric_def)
done

lemma (in Floor) E_thm14: "0 < N ==>
  Lift ∈
    (stopped ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})
    LeadsTo (opened ∩ Req n ∩ {s. metric n s = N})"
apply (unfold Lift_def, ensures_tac "open_act")
apply (auto simp add: metric_def)
done

lemma (in Floor) E_thm15: "Lift ∈ (opened ∩ Req n ∩ {s. metric n s = N})

  LeadsTo (closed ∩ Req n ∩ {s. metric n s = N})"
apply (unfold Lift_def, ensures_tac "close_act")
apply (auto simp add: metric_def)
done

lemma (in Floor) lift_3_Req: "0 < N ==>
  Lift ∈
    (moving ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})
    LeadsTo (moving ∩ Req n ∩ {s. metric n s < N})"
apply (blast intro!: E_thm13 E_thm14 E_thm15 lift_5 intro: LeadsTo_Trans)
done

lemma (in Floor) Always_nonneg: "Lift ∈ Always {s. 0 ≤ metric n s}"
apply (rule bounded [THEN Always_weaken])
apply (auto simp add: metric_def)
done

lemmas (in Floor) R_thm11 = Always_LeadsTo_weaken [OF Always_nonneg E_thm11]

lemma (in Floor) lift_3:
  "Lift ∈ (moving ∩ Req n) LeadsTo (stopped ∩ atFloor n)"
apply (rule Always_nonneg [THEN integ_0_le_induct])
apply (case_tac "0 < z")

prefer 2 apply (force intro: R_thm11 order_antisym simp add: linorder_not_less)
apply (rule LeadsTo_weaken_R [OF asm_rl Un_upper1])
apply (rule LeadsTo_Trans [OF subset_imp_LeadsTo
  LeadsTo_Un [OF lift_4 lift_3_Req]], auto)
done

```

```

lemma (in Floor) lift_1: "Lift  $\in$  (Req n) LeadsTo (opened  $\cap$  atFloor n)"
apply (rule LeadsTo_Trans)
prefer 2
apply (rule LeadsTo_Un [OF E_thm04 LeadsTo_Un_post])
apply (rule E_thm01 [THEN [2] LeadsTo_Trans_Un])
apply (rule lift_3 [THEN [2] LeadsTo_Trans_Un])
apply (rule lift_2 [THEN [2] LeadsTo_Trans_Un])
apply (rule LeadsTo_Trans_Un [OF E_thm02 E_thm03])
apply (rule open_move [THEN Always_LeadsToI])
apply (rule Always_LeadsToI [OF open_stop subset_imp_LeadsTo], clarify)

apply (case_tac "open x", auto)
done

end

```

```

theory Mutex imports "../UNITY_Main" begin

```

```

record state =
  p :: bool
  m :: int
  n :: int
  u :: bool
  v :: bool

```

```

types command = "(state*state) set"

```

```

constdefs

```

```

U0 :: command
"U0 == {(s,s'). s' = s (|u:=True, m:=1|) & m s = 0}"

U1 :: command
"U1 == {(s,s'). s' = s (|p:= v s, m:=2|) & m s = 1}"

U2 :: command
"U2 == {(s,s'). s' = s (|m:=3|) & ~ p s & m s = 2}"

U3 :: command
"U3 == {(s,s'). s' = s (|u:=False, m:=4|) & m s = 3}"

U4 :: command
"U4 == {(s,s'). s' = s (|p:=True, m:=0|) & m s = 4}"

V0 :: command

```



```

"V0 == {(s,s'). s' = s (/v:=True, n:=1/) & n s = 0}"

V1 :: command
"V1 == {(s,s'). s' = s (/p:= ~ u s, n:=2/) & n s = 1}"

V2 :: command
"V2 == {(s,s'). s' = s (/n:=3/) & p s & n s = 2}"

V3 :: command
"V3 == {(s,s'). s' = s (/v:=False, n:=4/) & n s = 3}"

V4 :: command
"V4 == {(s,s'). s' = s (/p:=False, n:=0/) & n s = 4}"

Mutex :: "state program"
"Mutex == mk_total_program
    ({s. ~ u s & ~ v s & m s = 0 & n s = 0},
    {U0, U1, U2, U3, U4, V0, V1, V2, V3, V4},
    UNIV)"

IU :: "state set"
"IU == {s. (u s = (1 ≤ m s & m s ≤ 3)) & (m s = 3 --> ~ p s)}"

IV :: "state set"
"IV == {s. (v s = (1 ≤ n s & n s ≤ 3)) & (n s = 3 --> p s)}"

bad_IU :: "state set"
"bad_IU == {s. (u s = (1 ≤ m s & m s ≤ 3)) &
    (3 ≤ m s & m s ≤ 4 --> ~ p s)}"

declare Mutex_def [THEN def_prg_Init, simp]

declare U0_def [THEN def_act_simp, simp]
declare U1_def [THEN def_act_simp, simp]
declare U2_def [THEN def_act_simp, simp]
declare U3_def [THEN def_act_simp, simp]
declare U4_def [THEN def_act_simp, simp]
declare V0_def [THEN def_act_simp, simp]
declare V1_def [THEN def_act_simp, simp]
declare V2_def [THEN def_act_simp, simp]
declare V3_def [THEN def_act_simp, simp]
declare V4_def [THEN def_act_simp, simp]

declare IU_def [THEN def_set_simp, simp]
declare IV_def [THEN def_set_simp, simp]
declare bad_IU_def [THEN def_set_simp, simp]

lemma IU: "Mutex ∈ Always IU"

```

```

apply (rule AlwaysI, force)
apply (unfold Mutex_def, safety)
done

```

```

lemma IV: "Mutex  $\in$  Always IV"
apply (rule AlwaysI, force)
apply (unfold Mutex_def, safety)
done

```

```

lemma mutual_exclusion: "Mutex  $\in$  Always {s.  $\sim$  (m s = 3 & n s = 3)}"
apply (rule Always_weaken)
apply (rule Always_Int_I [OF IU IV], auto)
done

```

```

lemma "Mutex  $\in$  Always bad_IU"
apply (rule AlwaysI, force)
apply (unfold Mutex_def, safety, auto)

```

oops

```

lemma eq_123: "((1::int)  $\leq$  i & i  $\leq$  3) = (i = 1 | i = 2 | i = 3)"
by arith

```

```

lemma U_F0: "Mutex  $\in$  {s. m s=2} Unless {s. m s=3}"
by (unfold Unless_def Mutex_def, safety)

```

```

lemma U_F1: "Mutex  $\in$  {s. m s=1} LeadsTo {s. p s = v s & m s = 2}"
by (unfold Mutex_def, ensures_tac U1)

```

```

lemma U_F2: "Mutex  $\in$  {s.  $\sim$  p s & m s = 2} LeadsTo {s. m s = 3}"
apply (cut_tac IU)
apply (unfold Mutex_def, ensures_tac U2)
done

```

```

lemma U_F3: "Mutex  $\in$  {s. m s = 3} LeadsTo {s. p s}"
apply (rule_tac B = "{s. m s = 4}" in LeadsTo_Trans)
  apply (unfold Mutex_def)
  apply (ensures_tac U3)
  apply (ensures_tac U4)
done

```

```

lemma U_lemma2: "Mutex  $\in$  {s. m s = 2} LeadsTo {s. p s}"
apply (rule LeadsTo_Diff [OF LeadsTo_weaken_L
  Int_lower2 [THEN subset_imp_LeadsTo]])
  apply (rule LeadsTo_Trans [OF U_F2 U_F3], auto)
done

```

```

lemma U_lemma1: "Mutex  $\in \{s. m\ s = 1\}$  LeadsTo  $\{s. p\ s\}$ "
by (rule LeadsTo_Trans [OF U_F1 [THEN LeadsTo_weaken_R] U_lemma2], blast)

lemma U_lemma123: "Mutex  $\in \{s. 1 \leq m\ s \ \& \ m\ s \leq 3\}$  LeadsTo  $\{s. p\ s\}$ "
by (simp add: eq_123 Collect_disj_eq LeadsTo_Un_distrib U_lemma1 U_lemma2
U_F3)

lemma u_Leadsto_p: "Mutex  $\in \{s. u\ s\}$  LeadsTo  $\{s. p\ s\}$ "
by (rule Always_LeadsTo_weaken [OF IU U_lemma123], auto)

lemma V_F0: "Mutex  $\in \{s. n\ s=2\}$  Unless  $\{s. n\ s=3\}$ "
by (unfold Unless_def Mutex_def, safety)

lemma V_F1: "Mutex  $\in \{s. n\ s=1\}$  LeadsTo  $\{s. p\ s = (\sim u\ s) \ \& \ n\ s = 2\}$ "
by (unfold Mutex_def, ensures_tac "V1")

lemma V_F2: "Mutex  $\in \{s. p\ s \ \& \ n\ s = 2\}$  LeadsTo  $\{s. n\ s = 3\}$ "
apply (cut_tac IV)
apply (unfold Mutex_def, ensures_tac "V2")
done

lemma V_F3: "Mutex  $\in \{s. n\ s = 3\}$  LeadsTo  $\{s. \sim p\ s\}$ "
apply (rule_tac B = " $\{s. n\ s = 4\}$ " in LeadsTo_Trans)
  apply (unfold Mutex_def)
  apply (ensures_tac V3)
apply (ensures_tac V4)
done

lemma V_lemma2: "Mutex  $\in \{s. n\ s = 2\}$  LeadsTo  $\{s. \sim p\ s\}$ "
apply (rule LeadsTo_Diff [OF LeadsTo_weaken_L
Int_lower2 [THEN subset_imp_LeadsTo]])
apply (rule LeadsTo_Trans [OF V_F2 V_F3], auto)
done

lemma V_lemma1: "Mutex  $\in \{s. n\ s = 1\}$  LeadsTo  $\{s. \sim p\ s\}$ "
by (rule LeadsTo_Trans [OF V_F1 [THEN LeadsTo_weaken_R] V_lemma2], blast)

lemma V_lemma123: "Mutex  $\in \{s. 1 \leq n\ s \ \& \ n\ s \leq 3\}$  LeadsTo  $\{s. \sim p\ s\}$ "
by (simp add: eq_123 Collect_disj_eq LeadsTo_Un_distrib V_lemma1 V_lemma2
V_F3)

lemma v_Leadsto_not_p: "Mutex  $\in \{s. v\ s\}$  LeadsTo  $\{s. \sim p\ s\}$ "
by (rule Always_LeadsTo_weaken [OF IV V_lemma123], auto)

```

```

lemma m1_Leadsto_3: "Mutex  $\in \{s. m\ s = 1\}$  LeadsTo  $\{s. m\ s = 3\}$ "
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
apply (rule_tac [2] U_F2)
apply (simp add: Collect_conj_eq)
apply (subst Un_commute)
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
  apply (rule_tac [2] PSP_Unless [OF v_Leadsto_not_p U_F0])
apply (rule U_F1 [THEN LeadsTo_weaken_R], auto)
done

```

```

lemma n1_Leadsto_3: "Mutex  $\in \{s. n\ s = 1\}$  LeadsTo  $\{s. n\ s = 3\}$ "
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
apply (rule_tac [2] V_F2)
apply (simp add: Collect_conj_eq)
apply (subst Un_commute)
apply (rule LeadsTo_cancel2 [THEN LeadsTo_Un_duplicate])
  apply (rule_tac [2] PSP_Unless [OF u_Leadsto_p V_F0])
apply (rule V_F1 [THEN LeadsTo_weaken_R], auto)
done

```

```

end

```

```

theory Reach imports "../UNITY_Main" begin

```

```

typedecl vertex

```

```

types    state = "vertex=>bool"

```

```

consts

```

```

  init :: "vertex"

```

```

  edges :: "(vertex*vertex) set"

```

```

constdefs

```

```

  asgt :: "[vertex,vertex] => (state*state) set"
  "asgt u v ==  $\{(s,s').\ s' = s(v := s\ u \mid s\ v)\}$ "

```

```

  Rprg :: "state program"
  "Rprg == mk_total_program ( $\{v. v=init\}$ ,  $\bigcup (u,v) \in edges. \{asgt\ u\ v\}$ , UNIV)"

```

```

  reach_invariant :: "state set"
  "reach_invariant ==  $\{s. (\forall v. s\ v \rightarrow (init, v) \in edges^*) \ \&\ s\ init\}$ "

```

```

  fixedpoint :: "state set"
  "fixedpoint ==  $\{s. \forall (u,v) \in edges. s\ u \rightarrow s\ v\}$ "

```

```

  metric :: "state => nat"
  "metric s == card  $\{v. \sim s\ v\}$ "

```

*We assume that the set of vertices is finite

axioms

finite_graph: "finite (UNIV :: vertex set)"

lemma *ifE [elim!]*:

"[| if P then Q else R;
 [| P; Q |] ==> S;
 [| ~ P; R |] ==> S |] ==> S"

by (simp split: split_if_asm)

declare *Rprg_def* [THEN def_prg_Init, simp]

declare *asgt_def* [THEN def_act_simp, simp]

All vertex sets are finite

declare *finite_subset* [OF subset_UNIV finite_graph, iff]

declare *reach_invariant_def* [THEN def_set_simp, simp]

lemma *reach_invariant*: "Rprg ∈ Always reach_invariant"

apply (rule AlwaysI, force)

apply (unfold Rprg_def, safety)

apply (blast intro: rtrancl_trans)

done

lemma *fixedpoint_invariant_correct*:

"fixedpoint ∩ reach_invariant = { %v. (init, v) ∈ edges^* }"

apply (unfold fixedpoint_def)

apply (rule equalityI)

apply (auto intro!: ext)

apply (erule rtrancl_induct, auto)

done

lemma *lemma1*:

"FP Rprg ⊆ fixedpoint"

apply (simp add: FP_def fixedpoint_def Rprg_def mk_total_program_def)

apply (auto simp add: stable_def constrains_def)

apply (drule bspec, assumption)

apply (simp add: Image_singleton image_iff)

apply (drule fun_cong, auto)

done

lemma *lemma2*:

"fixedpoint ⊆ FP Rprg"

apply (simp add: FP_def fixedpoint_def Rprg_def mk_total_program_def)

```

apply (auto intro!: ext simp add: stable_def constrains_def)
done

```

```

lemma FP_fixedpoint: "FP Rprg = fixedpoint"
by (rule equalityI [OF lemma1 lemma2])

```

```

lemma Compl_fixedpoint: "- fixedpoint = ( $\bigcup (u,v) \in \text{edges}. \{s. s \ u \ \& \ \sim s \ v\}$ )"
apply (simp add: FP_fixedpoint [symmetric] Rprg_def mk_total_program_def)
apply (rule subset_antisym)
apply (auto simp add: Compl_FP UN_UN_flatten del: subset_antisym)
  apply (rule fun_upd_idem, force)
apply (force intro!: rev_bexI simp add: fun_upd_idem_iff)
done

```

```

lemma Diff_fixedpoint:
  "A - fixedpoint = ( $\bigcup (u,v) \in \text{edges}. A \cap \{s. s \ u \ \& \ \sim s \ v\}$ )"
by (simp add: Diff_eq Compl_fixedpoint, blast)

```

```

lemma Suc_metric: "~ s x ==> Suc (metric (s(x:=True))) = metric s"
apply (unfold metric_def)
apply (subgoal_tac "{v. ~ (s(x:=True)) v} = {v. ~ s v} - {x}")
  prefer 2 apply force
apply (simp add: card_Suc_Diff1 del: card_Diff_insert)
done

```

```

lemma metric_less [intro!]: "~ s x ==> metric (s(x:=True)) < metric s"
by (erule Suc_metric [THEN subst], blast)

```

```

lemma metric_le: "metric (s(y:=s x | s y)) ≤ metric s"
apply (case_tac "s x --> s y")
apply (auto intro: less_imp_le simp add: fun_upd_idem)
done

```

```

lemma LeadsTo_Diff_fixedpoint:
  "Rprg ∈ ((metric-'{m}) - fixedpoint) LeadsTo (metric-'{lessThan m})"
apply (simp (no_asm) add: Diff_fixedpoint Rprg_def)
apply (rule LeadsTo_UN, auto)
apply (ensures_tac "asgt a b")
  prefer 2 apply blast
apply (simp (no_asm_use) add: linorder_not_less)
apply (drule metric_le [THEN order_antisym])
apply (auto elim: less_not_refl3 [THEN [2] rev_notE])
done

```

```

lemma LeadsTo_Un_fixedpoint:
  "Rprg ∈ (metric-'{m}) LeadsTo (metric-'{lessThan m} ∪ fixedpoint)"
apply (rule LeadsTo_Diff [OF LeadsTo_Diff_fixedpoint [THEN LeadsTo_weaken_R]
  subset_imp_LeadsTo], auto)

```

done

```
lemma LeadsTo_fixedpoint: "Rprg ∈ UNIV LeadsTo fixedpoint"
apply (rule LessThan_induct, auto)
apply (rule LeadsTo_Un_fixedpoint)
done
```

```
lemma LeadsTo_correct: "Rprg ∈ UNIV LeadsTo { %v. (init, v) ∈ edges^* }"
apply (subst fixedpoint_invariant_correct [symmetric])
apply (rule Always_LeadsTo_weaken [OF reach_invariant LeadsTo_fixedpoint],
auto)
done
```

end

theory Reachability imports "../Detects" Reach begin

```
types edge = "(vertex*vertex)"
```

```
record state =
  reach :: "vertex => bool"
  nmsg :: "edge => nat"
```

```
consts root :: "vertex"
      E :: "edge set"
      V :: "vertex set"
```

```
inductive_set REACHABLE :: "edge set"
  where
    base: "v ∈ V ==> ((v,v) ∈ REACHABLE)"
  | step: "((u,v) ∈ REACHABLE) & (v,w) ∈ E ==> ((u,w) ∈ REACHABLE)"
```

constdefs

```
reachable :: "vertex => state set"
"reachable p == {s. reach s p}"
```

```
nmsg_eq :: "nat => edge => state set"
"nmsg_eq k == %e. {s. nmsg s e = k}"
```

```
nmsg_gt :: "nat => edge => state set"
"nmsg_gt k == %e. {s. k < nmsg s e}"
```

```
nmsg_gte :: "nat => edge => state set"
"nmsg_gte k == %e. {s. k ≤ nmsg s e}"
```

```
nmsg_lte :: "nat => edge => state set"
"nmsg_lte k == %e. {s. nmsg s e ≤ k}"
```

```
final :: "state set"
"final == (⋂ v∈V. reachable v <==> {s. (root, v) ∈ REACHABLE}) ∩
```

(INTER E (nmsg_eq 0))"

axioms

Graph1: "root $\in V$ "

Graph2: " $(v, w) \in E \implies (v \in V) \ \& \ (w \in V)$ "

MA1: " $F \in \text{Always } (\text{reachable } \text{root})$ "

MA2: " $v \in V \implies F \in \text{Always } (\neg \text{reachable } v \cup \{s. ((\text{root}, v) \in \text{REACHABLE})\})$ "

MA3: " $[|v \in V; w \in V|] \implies F \in \text{Always } (\neg(\text{nmsg_gt } 0 \ (v, w)) \cup (\text{reachable } v))$ "

MA4: " $(v, w) \in E \implies$
 $F \in \text{Always } (\neg(\text{reachable } v) \cup (\text{nmsg_gt } 0 \ (v, w)) \cup (\text{reachable } w))$ "

MA5: " $[|v \in V; w \in V|]$
 $\implies F \in \text{Always } (\text{nmsg_gte } 0 \ (v, w) \cap \text{nmsg_lte } (\text{Suc } 0) \ (v, w))$ "

MA6: " $[|v \in V|] \implies F \in \text{Stable } (\text{reachable } v)$ "

MA6b: " $[|v \in V; w \in W|] \implies F \in \text{Stable } (\text{reachable } v \cap \text{nmsg_lte } k \ (v, w))$ "

MA7: " $[|v \in V; w \in V|] \implies F \in \text{UNIV LeadsTo nmsg_eq } 0 \ (v, w)$ "

lemmas E_imp_in_V_L = Graph2 [THEN conjunct1, standard]

lemmas E_imp_in_V_R = Graph2 [THEN conjunct2, standard]

lemma lemma2:

" $(v, w) \in E \implies F \in \text{reachable } v \text{ LeadsTo nmsg_eq } 0 \ (v, w) \cap \text{reachable } v$ "

apply (rule MA7 [THEN PSP_Stable, THEN LeadsTo_weaken_L])

apply (rule_tac [3] MA6)

apply (auto simp add: E_imp_in_V_L E_imp_in_V_R)

done

lemma Induction_base: " $(v, w) \in E \implies F \in \text{reachable } v \text{ LeadsTo reachable } w$ "

apply (rule MA4 [THEN Always_LeadsTo_weaken])

apply (rule_tac [2] lemma2)

apply (auto simp add: nmsg_eq_def nmsg_gt_def)

done

lemma REACHABLE_LeadsTo_reachable:

" $(v, w) \in \text{REACHABLE} \implies F \in \text{reachable } v \text{ LeadsTo reachable } w$ "

apply (erule REACHABLE.induct)

apply (rule subset_imp_LeadsTo, blast)

apply (blast intro: LeadsTo_Trans Induction_base)

done

lemma Detects_part1: " $F \in \{s. (\text{root}, v) \in \text{REACHABLE}\} \text{ LeadsTo reachable } v$ "

apply (rule single_LeadsTo_I)

apply (simp split add: split_if_asm)


```

apply (rule MA1 [THEN Always_LeadsToI])
apply (erule REACHABLE_LeadsTo_reachable [THEN LeadsTo_weaken_L], auto)
done

```

```

lemma Reachability_Detected:
  "v ∈ V ==> F ∈ (reachable v) Detects {s. (root,v) ∈ REACHABLE}"
apply (unfold Detects_def, auto)
  prefer 2 apply (blast intro: MA2 [THEN Always_weaken])
apply (rule Detects_part1 [THEN LeadsTo_weaken_L], blast)
done

```

```

lemma LeadsTo_Reachability:
  "v ∈ V ==> F ∈ UNIV LeadsTo (reachable v <==> {s. (root,v) ∈ REACHABLE})"
by (erule Reachability_Detected [THEN Detects_Imp_LeadstoEQ])

```

```

lemma Eq_lemma1:
  "(reachable v <==> {s. (root,v) ∈ REACHABLE}) =
   {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}"
by (unfold Equality_def, blast)

```

```

lemma Eq_lemma2:
  "(reachable v <==> (if (root,v) ∈ REACHABLE then UNIV else {})) =
   {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}"
by (unfold Equality_def, auto)

```

```

lemma final_lemma1:
  "((⋂ v ∈ V. ⋂ w ∈ V. {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}
    &
      s ∈ nmsg_eq 0 (v,w)))
   ⊆ final"
apply (unfold final_def Equality_def, auto)
apply (frule E_imp_in_V_R)
apply (frule E_imp_in_V_L, blast)
done

```

```

lemma final_lemma2:
  "E ≠ {}
   ==> ((⋂ v ∈ V. ⋂ e ∈ E. {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}
      ∩ nmsg_eq 0 e)
    ⊆ final"
apply (unfold final_def Equality_def)

```

```

apply (auto split add: split_if_asm)
apply (frule E_imp_in_V_L, blast)
done

```

```

lemma final_lemma3:
  "E≠{}
  ==> (⋂ v ∈ V. ⋂ e ∈ E.
    (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 e)
    ⊆ final"
apply (frule final_lemma2)
apply (simp (no_asm_use) add: Eq_lemma2)
done

```

```

lemma final_lemma4:
  "E≠{}
  ==> (⋂ v ∈ V. ⋂ e ∈ E.
    {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))} ∩ nmsg_eq 0
e)
    = final"
apply (rule subset_antisym)
apply (erule final_lemma2)
apply (unfold final_def Equality_def, blast)
done

```

```

lemma final_lemma5:
  "E≠{}
  ==> (⋂ v ∈ V. ⋂ e ∈ E.
    ((reachable v) <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 e)
    = final"
apply (frule final_lemma4)
apply (simp (no_asm_use) add: Eq_lemma2)
done

```

```

lemma final_lemma6:
  "(⋂ v ∈ V. ⋂ w ∈ V.
    (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 (v,w))
    ⊆ final"
apply (simp (no_asm) add: Eq_lemma2 Int_def)
apply (rule final_lemma1)
done

```

```

lemma final_lemma7:
  "final =
    (⋂ v ∈ V. ⋂ w ∈ V.
      ((reachable v) <==> {s. (root,v) ∈ REACHABLE}) ∩
      (-{s. (v,w) ∈ E} ∪ (nmsg_eq 0 (v,w))))"
apply (unfold final_def)
apply (rule subset_antisym, blast)
apply (auto split add: split_if_asm)
apply (blast dest: E_imp_in_V_L E_imp_in_V_R)+

```

done

```
lemma not_REACHABLE_imp_Stable_not_reachable:
  "[| v ∈ V; (root,v) ∉ REACHABLE |] ==> F ∈ Stable (- reachable v)"
apply (drule MA2 [THEN AlwaysD], auto)
done
```

```
lemma Stable_reachable_EQ_R:
  "v ∈ V ==> F ∈ Stable (reachable v <==> {s. (root,v) ∈ REACHABLE})"
apply (simp (no_asm) add: Equality_def Eq_lemma2)
apply (blast intro: MA6 not_REACHABLE_imp_Stable_not_reachable)
done
```

```
lemma lemma4:
  "((nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w)) ∩
    (- nmsg_gt 0 (v,w) ∪ A))
   ⊆ A ∪ nmsg_eq 0 (v,w)"
apply (unfold nmsg_gte_def nmsg_lte_def nmsg_gt_def nmsg_eq_def, auto)
done
```

```
lemma lemma5:
  "reachable v ∩ nmsg_eq 0 (v,w) =
    ((nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w)) ∩
     (reachable v ∩ nmsg_lte 0 (v,w)))"
by (unfold nmsg_gte_def nmsg_lte_def nmsg_gt_def nmsg_eq_def, auto)
```

```
lemma lemma6:
  "- nmsg_gt 0 (v,w) ∪ reachable v ⊆ nmsg_eq 0 (v,w) ∪ reachable v"
apply (unfold nmsg_gte_def nmsg_lte_def nmsg_gt_def nmsg_eq_def, auto)
done
```

```
lemma Always_reachable_OR_nmsg_0:
  "[| v ∈ V; w ∈ V |] ==> F ∈ Always (reachable v ∪ nmsg_eq 0 (v,w))"
apply (rule Always_Int_I [OF MA5 MA3, THEN Always_weaken])
apply (rule_tac [5] lemma4, auto)
done
```

```
lemma Stable_reachable_AND_nmsg_0:
  "[| v ∈ V; w ∈ V |] ==> F ∈ Stable (reachable v ∩ nmsg_eq 0 (v,w))"
apply (subst lemma5)
apply (blast intro: MA5 Always_imp_Stable [THEN Stable_Int] MA6b)
done
```

```
lemma Stable_nmsg_0_OR_reachable:
  "[| v ∈ V; w ∈ V |] ==> F ∈ Stable (nmsg_eq 0 (v,w) ∪ reachable v)"
by (blast intro!: Always_weaken [THEN Always_imp_Stable] lemma6 MA3)
```

```

lemma not_REACHABLE_imp_Stable_not_reachable_AND_nmsg_0:
  "[| v ∈ V; w ∈ V; (root,v) ∉ REACHABLE |]
   ==> F ∈ Stable (- reachable v ∩ nmsg_eq 0 (v,w))"
apply (rule Stable_Int [OF MA2 [THEN Always_imp_Stable]
                          Stable_nmsg_0_OR_reachable,
                          THEN Stable_eq])
prefer 4 apply blast
apply auto
done

lemma Stable_reachable_EQ_R_AND_nmsg_0:
  "[| v ∈ V; w ∈ V |]
   ==> F ∈ Stable ((reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩
                    nmsg_eq 0 (v,w))"
by (simp add: Equality_def Eq_lemma2 Stable_reachable_AND_nmsg_0
    not_REACHABLE_imp_Stable_not_reachable_AND_nmsg_0)

lemma UNIV_lemma: "UNIV ⊆ (⋂ v ∈ V. UNIV)"
by blast

lemmas UNIV_LeadsTo_completion =
  LeadsTo_weaken_L [OF Finite_stable_completion UNIV_lemma]

lemma LeadsTo_final_E_empty: "E={} ==> F ∈ UNIV LeadsTo final"
apply (unfold final_def, simp)
apply (rule UNIV_LeadsTo_completion)
apply safe
apply (erule LeadsTo_Reachability [simplified])
apply (drule Stable_reachable_EQ_R, simp)
done

lemma Leadsto_reachability_AND_nmsg_0:
  "[| v ∈ V; w ∈ V |]
   ==> F ∈ UNIV LeadsTo
      ((reachable v <==> {s. (root,v): REACHABLE}) ∩ nmsg_eq 0 (v,w))"
apply (rule LeadsTo_Reachability [THEN LeadsTo_Trans], blast)
apply (subgoal_tac
  "F ∈ (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩
      UNIV LeadsTo (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩
      nmsg_eq 0 (v,w) ")
apply simp
apply (rule PSP_Stable2)
apply (rule MA7)
apply (rule_tac [3] Stable_reachable_EQ_R, auto)

```

done

```
lemma LeadsTo_final_E_NOT_empty: "E≠{} ==> F ∈ UNIV LeadsTo final"
apply (rule LeadsTo_weaken_L [OF LeadsTo_weaken_R UNIV_lemma])
apply (rule_tac [2] final_lemma6)
apply (rule Finite_stable_completion)
  apply blast
  apply (rule UNIV_LeadsTo_completion)
    apply (blast intro: Stable_INT Stable_reachable_EQ_R_AND_nmsg_0
      Leadsto_reachability_AND_nmsg_0)+
done
```

```
lemma LeadsTo_final: "F ∈ UNIV LeadsTo final"
apply (case_tac "E={}")
  apply (rule_tac [2] LeadsTo_final_E_NOT_empty)
  apply (rule LeadsTo_final_E_empty, auto)
done
```

```
lemma Stable_final_E_empty: "E={ } ==> F ∈ Stable final"
apply (unfold final_def, simp)
apply (rule Stable_INT)
apply (drule Stable_reachable_EQ_R, simp)
done
```

```
lemma Stable_final_E_NOT_empty: "E≠{} ==> F ∈ Stable final"
apply (subst final_lemma7)
apply (rule Stable_INT)
apply (rule Stable_INT)
apply (simp (no_asm) add: Eq_lemma2)
apply safe
apply (rule Stable_eq)
apply (subgoal_tac [2]
  "{s. (s ∈ reachable v) = ((root,v) ∈ REACHABLE) } ∩ nmsg_eq 0 (v,w))"
=
  "{s. (s ∈ reachable v) = ( (root,v) ∈ REACHABLE) } ∩ (¬ UNIV ∪ nmsg_eq
  0 (v,w)))")
prefer 2 apply simp
prefer 2 apply blast
apply (rule Stable_reachable_EQ_R_AND_nmsg_0
  [simplified Eq_lemma2 Collect_const])
apply (blast, blast)
apply (rule Stable_eq)
  apply (rule Stable_reachable_EQ_R [simplified Eq_lemma2 Collect_const])
  apply simp
apply (subgoal_tac
  "{s. (s ∈ reachable v) = ((root,v) ∈ REACHABLE) } =
  ({s. (s ∈ reachable v) = ( (root,v) ∈ REACHABLE) } Int
  (¬ {} ∪ nmsg_eq 0 (v,w)))")
apply blast+
```

done

```
lemma Stable_final: "F ∈ Stable final"
apply (case_tac "E={}")
  prefer 2 apply (blast intro: Stable_final_E_NOT_empty)
apply (blast intro: Stable_final_E_empty)
done
```

end

20 Analyzing the Needham-Schroeder Public-Key Protocol in UNITY

```
theory NSP_Bad imports "../Auth/Public" "../UNITY_Main" begin
```

This is the flawed version, vulnerable to Lowe's attack. From page 260 of Burrows, Abadi and Needham. A Logic of Authentication. Proc. Royal Soc. 426 (1989).

```
types state = "event list"
```

```
constdefs
```

```
Fake :: "(state*state) set"
  "Fake == {(s,s').
    ∃ B X. s' = Says Spy B X # s
    & X ∈ synth (analz (spies s))}"
```

```
NS1 :: "(state*state) set"
  "NS1 == {(s1,s').
    ∃ A1 B NA.
    s' = Says A1 B (Crypt (pubK B) {|Nonce NA, Agent A1|}) # s1
    & Nonce NA ∉ used s1}"
```

```
NS2 :: "(state*state) set"
  "NS2 == {(s2,s').
    ∃ A' A2 B NA NB.
    s' = Says B A2 (Crypt (pubK A2) {|Nonce NA, Nonce NB|}) #
s2
    & Says A' B (Crypt (pubK B) {|Nonce NA, Agent A2|}) ∈ set s2
    & Nonce NB ∉ used s2}"
```

```
NS3 :: "(state*state) set"
  "NS3 == {(s3,s').
    ∃ A3 B' B NA NB.
```

```

      s' = Says A3 B (Crypt (pubK B) (Nonce NB)) # s3
    & Says A3 B (Crypt (pubK B) {|Nonce NA, Agent A3|}) ∈ set
s3
      & Says B' A3 (Crypt (pubK A3) {|Nonce NA, Nonce NB|}) ∈ set
s3}"

```

constdefs

```
Nprg :: "state program"
```

```
"Nprg == mk_total_program({[]}, {Fake, NS1, NS2, NS3}, UNIV)"
```

```
declare spies_partsEs [elim]
```

```
declare analz_into_parts [dest]
```

```
declare Fake_parts_insert_in_Un [dest]
```

For other theories, e.g. Mutex and Lift, using [iff] slows proofs down. Here, it facilitates re-use of the Auth proofs.

```
declare Fake_def [THEN def_act_simp, iff]
```

```
declare NS1_def [THEN def_act_simp, iff]
```

```
declare NS2_def [THEN def_act_simp, iff]
```

```
declare NS3_def [THEN def_act_simp, iff]
```

```
declare Nprg_def [THEN def_prg_Init, simp]
```

A "possibility property": there are traces that reach the end. Replace by LEAD-STO proof!

```
lemma "A ≠ B ==>
```

```
  ∃ NB. ∃ s ∈ reachable Nprg. Says A B (Crypt (pubK B) (Nonce NB)) ∈
```

```
set s"
```

```
apply (intro exI bexI)
```

```
apply (rule_tac [2] act = "totalize_act NS3" in reachable.Acts)
```

```
apply (rule_tac [3] act = "totalize_act NS2" in reachable.Acts)
```

```
apply (rule_tac [4] act = "totalize_act NS1" in reachable.Acts)
```

```
apply (rule_tac [5] reachable.Init)
```

```
apply (simp_all (no_asm_simp) add: Nprg_def totalize_act_def)
```

```
apply auto
```

```
done
```

20.1 Inductive Proofs about *ns_public*

lemma ns_constrainsI:

```
"(!act s s'. [| act ∈ {Id, Fake, NS1, NS2, NS3};
               (s,s') ∈ act; s ∈ A |] ==> s' ∈ A')
```

```
==> Nprg ∈ A co A'"
```

```
apply (simp add: Nprg_def mk_total_program_def)
```

```
apply (rule constrainsI, auto)
```

```
done
```

This ML code does the inductions directly.

```
ML{*
```

```
fun ns_constrains_tac(cs,ss) i =
```

```
  SELECT_GOAL
```

```

      (EVERY [REPEAT (etac @{thm Always_ConstrainsI} 1),
        REPEAT (resolve_tac [@{thm StableI}, @{thm stableI},
          @{thm constrains_imp_Constrains}] 1),
          rtac @{thm ns_constrainsI} 1,
          full_simp_tac ss 1,
          REPEAT (FIRSTGOAL (etac disjE)),
          ALLGOALS (clarify_tac (cs delrules [impI, @{thm impCE}])),
          REPEAT (FIRSTGOAL analz_mono_contra_tac),
          ALLGOALS (asm_simp_tac ss))] i;

(*Tactic for proving secrecy theorems*)
fun ns_induct_tac(cs,ss) =
  (SELECT_GOAL o EVERY)
    [rtac @{thm AlwaysI} 1,
     force_tac (cs,ss) 1,
     (*"reachable" gets in here*)
     rtac (@{thm Always_reachable} RS @{thm Always_ConstrainsI} RS @{thm StableI})
  1,
    ns_constrains_tac(cs,ss) 1];
*}

```

```

method_setup ns_induct = {*
  Scan.succeed (fn ctxt =>
    SIMPLE_METHOD' (ns_induct_tac (local_clasimpset_of ctxt))) *}
  "for inductive reasoning about the Needham-Schroeder protocol"

```

Converts invariants into statements about reachable states

```

lemmas Always_Collect_reachableD =
  Always_includes_reachable [THEN subsetD, THEN CollectD]

```

Spy never sees another agent's private key! (unless it's bad at start)

```

lemma Spy_see_priK:
  "Nprg ∈ Always {s. (Key (priK A) ∈ parts (spies s)) = (A ∈ bad)}"
apply ns_induct
apply blast
done
declare Spy_see_priK [THEN Always_Collect_reachableD, simp]

```

```

lemma Spy_analz_priK:
  "Nprg ∈ Always {s. (Key (priK A) ∈ analz (spies s)) = (A ∈ bad)}"
by (rule Always_reachable [THEN Always_weaken], auto)
declare Spy_analz_priK [THEN Always_Collect_reachableD, simp]

```

20.2 Authenticity properties obtained from NS2

It is impossible to re-use a nonce in both NS1 and NS2 provided the nonce is secret. (Honest users generate fresh nonces.)

```

lemma no_nonce_NS1_NS2:
  "Nprg
    ∈ Always {s. Crypt (pubK C) {|NA', Nonce NA|} ∈ parts (spies s) -->
      Crypt (pubK B) {|Nonce NA, Agent A|} ∈ parts (spies s) -->
      Nonce NA ∈ analz (spies s)}"
apply ns_induct

```



```

apply (blast intro: analz_insertI)+
done

```

Adding it to the claset slows down proofs...

```

lemmas no_nonce_NS1_NS2_reachable =
  no_nonce_NS1_NS2 [THEN Always_Collect_reachableD, rule_format]

```

Unicity for NS1: nonce NA identifies agents A and B

```

lemma unique_NA_lemma:
  "Nprg
   ∈ Always {s. Nonce NA ∉ analz (spies s) -->
    Crypt(pubK B) {|Nonce NA, Agent A|} ∈ parts(spies s) -->
    Crypt(pubK B') {|Nonce NA, Agent A'|} ∈ parts(spies s) -->
    A=A' & B=B'}"

```

```

apply ns_induct
apply auto

```

Fake, NS1 are non-trivial

```

done

```

Unicity for NS1: nonce NA identifies agents A and B

```

lemma unique_NA:
  "[| Crypt(pubK B) {|Nonce NA, Agent A|} ∈ parts(spies s);
    Crypt(pubK B') {|Nonce NA, Agent A'|} ∈ parts(spies s);
    Nonce NA ∉ analz (spies s);
    s ∈ reachable Nprg |]
   ==> A=A' & B=B'"
by (blast dest: unique_NA_lemma [THEN Always_Collect_reachableD])

```

Secrecy: Spy does not see the nonce sent in msg NS1 if A and B are secure

```

lemma Spy_not_see_NA:
  "[| A ∉ bad; B ∉ bad |]
   ==> Nprg ∈ Always
    {s. Says A B (Crypt(pubK B) {|Nonce NA, Agent A|}) ∈ set s
     --> Nonce NA ∉ analz (spies s)}"

```

```

apply ns_induct

```

NS3

```

prefer 4 apply (blast intro: no_nonce_NS1_NS2_reachable)

```

NS2

```

prefer 3 apply (blast dest: unique_NA)

```

NS1

```

prefer 2 apply blast

```

Fake

```

apply spy_analz
done

```

Authentication for A: if she receives message 2 and has used NA to start a run, then B has sent message 2.

```

lemma A_trusts_NS2:
  "[| A ∉ bad; B ∉ bad |]"
  ==> Nprg ∈ Always
    {s. Says A B (Crypt(pubK B) {|Nonce NA, Agent A|}) ∈ set s &
      Crypt(pubK A) {|Nonce NA, Nonce NB|} ∈ parts (knows Spy
s)
    --> Says B A (Crypt(pubK A) {|Nonce NA, Nonce NB|}) ∈ set s}"

apply (insert Spy_not_see_NA [of A B NA], simp, ns_induct)
apply (auto dest: unique_NA)
done

```

If the encrypted message appears then it originated with Alice in NS1

```

lemma B_trusts_NS1:
  "Nprg ∈ Always
    {s. Nonce NA ∉ analz (spies s) -->
      Crypt (pubK B) {|Nonce NA, Agent A|} ∈ parts (spies s)
    --> Says A B (Crypt (pubK B) {|Nonce NA, Agent A|}) ∈ set s}"

apply ns_induct
apply blast
done

```

20.3 Authenticity properties obtained from NS2

Unicity for NS2: nonce NB identifies nonce NA and agent A. Proof closely follows that of *unique_NA*.

```

lemma unique_NB_lemma:
  "Nprg
    ∈ Always {s. Nonce NB ∉ analz (spies s) -->
      Crypt (pubK A) {|Nonce NA, Nonce NB|} ∈ parts (spies s) -->
      Crypt(pubK A'){|Nonce NA', Nonce NB|} ∈ parts(spies s) -->
      A=A' & NA=NA'}"

apply ns_induct
apply auto

```

Fake, NS2 are non-trivial

done

```

lemma unique_NB:
  "[| Crypt(pubK A) {|Nonce NA, Nonce NB|} ∈ parts(spies s);
    Crypt(pubK A'){|Nonce NA', Nonce NB|} ∈ parts(spies s);
    Nonce NB ∉ analz (spies s);
    s ∈ reachable Nprg |]"
  ==> A=A' & NA=NA'"

apply (blast dest: unique_NB_lemma [THEN Always_Collect_reachableD])
done

```

NB remains secret PROVIDED Alice never responds with round 3

```

lemma Spy_not_see_NB:
  "[| A ∉ bad; B ∉ bad |]"
  ==> Nprg ∈ Always
    {s. Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s
&

```

```

      (ALL C. Says A C (Crypt (pubK C) (Nonce NB))  $\notin$  set s)
      --> Nonce NB  $\notin$  analz (spies s)}"
apply ns_induct
apply (simp_all (no_asm_simp) add: all_conj_distrib)

NS3: because NB determines A
prefer 4 apply (blast dest: unique_NB)

NS2: by freshness and unicity of NB
prefer 3 apply (blast intro: no_nonce_NS1_NS2_reachable)

NS1: by freshness
prefer 2 apply blast

Fake

apply spy_analz
done

Authentication for B: if he receives message 3 and has used NB in message 2,
then A has sent message 3—to somebody....

lemma B_trusts_NS3:
  "[| A  $\notin$  bad; B  $\notin$  bad |]
  ==> Nprg  $\in$  Always
    {s. Crypt (pubK B) (Nonce NB)  $\in$  parts (spies s) &
      Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|})  $\in$  set s
    }
  --> ( $\exists$  C. Says A C (Crypt (pubK C) (Nonce NB))  $\in$  set s)}"

apply (insert Spy_not_see_NB [of A B NA NB], simp, ns_induct)
apply (simp_all (no_asm_simp) add: ex_disj_distrib)
apply auto

NS3: because NB determines A. This use of unique_NB is robust.
apply (blast intro: unique_NB [THEN conjunct1])
done

Can we strengthen the secrecy theorem? NO

lemma "[| A  $\notin$  bad; B  $\notin$  bad |]
  ==> Nprg  $\in$  Always
    {s. Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|})  $\in$  set s
      --> Nonce NB  $\notin$  analz (spies s)}"

apply ns_induct
apply auto

Fake

apply spy_analz

NS2: by freshness and unicity of NB
  apply (blast intro: no_nonce_NS1_NS2_reachable)

NS3: unicity of NB identifies A and NA, but not B
apply (frule_tac A'=A in Says_imp_spies [THEN parts.Inj, THEN unique_NB])

```

```

apply (erule Says_imp_spies [THEN parts.Inj], auto)
apply (rename_tac s B' C)

```

This is the attack!

```

1.  $\bigwedge s \ B' \ C.$ 
    $\llbracket A \notin \text{bad}; B \notin \text{bad}; s \in \text{reachable } \text{Nprg};$ 
    $\text{Says } A \ C \ (\text{Crypt } (\text{pubK } C) \ \{\text{Nonce } NA, \text{Agent } A\}) \in \text{set } s;$ 
    $\text{Says } B' \ A \ (\text{Crypt } (\text{pubK } A) \ \{\text{Nonce } NA, \text{Nonce } NB\}) \in \text{set } s;$ 
    $C \in \text{bad};$ 
    $\text{Says } B \ A \ (\text{Crypt } (\text{pubK } A) \ \{\text{Nonce } NA, \text{Nonce } NB\}) \in \text{set } s;$ 
    $\text{Nonce } NB \notin \text{analz } (\text{knows } \text{Spy } s) \rrbracket$ 
 $\implies \text{False}$ 

```

oops

end

```

theory Handshake imports "../UNITY_Main" begin

```

```

record state =
  BB :: bool
  NF :: nat
  NG :: nat

```

```

constdefs

```

```

cmdF :: "(state*state) set"
"cmdF == {(s,s'). s' = s (|NF:= Suc(NF s), BB:=False|) & BB s}"

```

```

F :: "state program"
"F == mk_total_program ({s. NF s = 0 & BB s}, {cmdF}, UNIV)"

```

```

cmdG :: "(state*state) set"
"cmdG == {(s,s'). s' = s (|NG:= Suc(NG s), BB:=True|) & ~ BB s}"

```

```

G :: "state program"
"G == mk_total_program ({s. NG s = 0 & BB s}, {cmdG}, UNIV)"

```

```

invFG :: "state set"
"invFG == {s. NG s <= NF s & NF s <= Suc (NG s) & (BB s = (NF s = NG s))}"

```

```

declare F_def [THEN def_prg_Init, simp]
          G_def [THEN def_prg_Init, simp]

```

```

cmdF_def [THEN def_act_simp, simp]
cmdG_def [THEN def_act_simp, simp]

```

```

    invFG_def [THEN def_set_simp, simp]

lemma invFG: "(F Join G) : Always invFG"
apply (rule AlwaysI)
apply force
apply (rule constrains_imp_Constrains [THEN StableI])
apply auto
  apply (unfold F_def, safety)
  apply (unfold G_def, safety)
done

lemma lemma2_1: "(F Join G) : ({s. NF s = k} - {s. BB s}) LeadsTo
  ({s. NF s = k} Int {s. BB s})"
apply (rule stable_Join_ensures1[THEN leadsTo_Basis, THEN leadsTo_imp_LeadsTo])
  apply (unfold F_def, safety)
  apply (unfold G_def, ensures_tac "cmdG")
done

lemma lemma2_2: "(F Join G) : ({s. NF s = k} Int {s. BB s}) LeadsTo
  {s. k < NF s}"
apply (rule stable_Join_ensures2[THEN leadsTo_Basis, THEN leadsTo_imp_LeadsTo])
  apply (unfold F_def, ensures_tac "cmdF")
  apply (unfold G_def, safety)
done

lemma progress: "(F Join G) : UNIV LeadsTo {s. m < NF s}"
apply (rule LeadsTo_weaken_R)
apply (rule_tac f = "NF" and l = "Suc m" and B = "{}"
  in GreaterThan_bounded_induct)

apply (auto intro!: lemma2_1 lemma2_2
  intro: LeadsTo_Trans LeadsTo_Diff simp add: vimage_def)
done

end

```

21 A Family of Similar Counters: Original Version

```
theory Counter imports "../UNITY_Main" begin
```

```
datatype name = C | c nat
types state = "name=>int"
```

```
consts
  sum  :: "[nat,state]=>int"
  sumj :: "[nat, nat, state]=>int"
```

```
primrec
```

```

"sum 0 s = 0"
"sum (Suc i) s = s (c i) + sum i s"

primrec
  "sumj 0 i s = 0"
  "sumj (Suc n) i s = (if n=i then sum n s else s (c n) + sumj n i s)"

types command = "(state*state)set"

constdefs
  a :: "nat=>command"
  "a i == {(s, s'). s'=s(c i:= s (c i) + 1, C:= s C + 1)}"

  Component :: "nat => state program"
  "Component i ==
    mk_total_program({s. s C = 0 & s (c i) = 0}, {a i},
       $\bigcup G \in \text{preserves } (\%s. s (c i)). \text{ Acts } G$ )"

declare Component_def [THEN def_prg_Init, simp]
declare a_def [THEN def_act_simp, simp]

lemma sum_upd_gt [rule_format]: "\n. I<n --> sum I (s(c n := x)) = sum I s"
by (induct_tac "I", auto)

lemma sum_upd_eq: "sum I (s(c I := x)) = sum I s"
apply (induct_tac "I")
apply (auto simp add: sum_upd_gt [unfolded fun_upd_def])
done

lemma sum_upd_C: "sum I (s(C := x)) = sum I s"
by (induct_tac "I", auto)

lemma sumj_upd_ci: "sumj I i (s(c i := x)) = sumj I i s"
apply (induct_tac "I")
apply (auto simp add: sum_upd_eq [unfolded fun_upd_def])
done

lemma sumj_upd_C: "sumj I i (s(C := x)) = sumj I i s"
apply (induct_tac "I")
apply (auto simp add: sum_upd_C [unfolded fun_upd_def])
done

lemma sumj_sum_gt [rule_format]: "\i. I<i--> (sumj I i s = sum I s)"
by (induct_tac "I", auto)

lemma sumj_sum_eq: "(sumj I I s = sum I s)"
apply (induct_tac "I", auto)
apply (simp (no_asm) add: sumj_sum_gt)
done

```

```

lemma sum_sumj [rule_format]: " $\forall i. i < I \rightarrow (\text{sum } I \text{ } s = s \text{ } (c \text{ } i) + \text{sumj } I \text{ } i \text{ } s)$ "
apply (induct_tac "I")
apply (auto simp add: linorder_neq_iff sumj_sum_eq)
done

```

```

lemma p2: "Component i  $\in$  stable {s. s C = s (c i) + k}"
by (simp add: Component_def, safety)

```

```

lemma p3: "Component i  $\in$  stable {s.  $\forall v. v \neq c \text{ } i \ \& \ v \neq C \rightarrow s \text{ } v = k \text{ } v$ }"
by (simp add: Component_def, safety)

```

```

lemma p2_p3_lemma1:
  " $(\forall k. \text{Component } i \in \text{stable } (\{s. s \text{ } C = s \text{ } (c \text{ } i) + \text{sumj } I \text{ } i \text{ } k\} \cap \{s. \forall v. v \neq c \text{ } i \ \& \ v \neq C \rightarrow s \text{ } v = k \text{ } v\}))$ 
    = (Component i  $\in$  stable {s. s C = s (c i) + sumj I i s})"
apply (simp add: Component_def mk_total_program_def)
apply (auto simp add: constrains_def stable_def sumj_upd_C sumj_upd_ci)
done

```

```

lemma p2_p3_lemma2:
  " $\forall k. \text{Component } i \in \text{stable } (\{s. s \text{ } C = s \text{ } (c \text{ } i) + \text{sumj } I \text{ } i \text{ } k\} \text{ Int } \{s. \forall v. v \neq c \text{ } i \ \& \ v \neq C \rightarrow s \text{ } v = k \text{ } v\})$ "
by (blast intro: stable_Int [OF p2 p3])

```

```

lemma p2_p3: "Component i  $\in$  stable {s. s C = s (c i) + sumj I i s}"
by (auto intro!: p2_p3_lemma2 simp add: p2_p3_lemma1 [symmetric])

```

```

lemma sum_0' [rule_format]: " $(\forall i. i < I \rightarrow s \text{ } (c \text{ } i) = 0) \rightarrow \text{sum } I \text{ } s = 0$ "
by (induct_tac "I", auto)

```

```

lemma safety:
  " $0 < I \Rightarrow (\bigcup i \in \{i. i < I\}. \text{Component } i) \in \text{invariant } \{s. s \text{ } C = \text{sum } I \text{ } s\}$ "
apply (simp (no_asm) add: invariant_def JN_stable sum_sumj)
apply (force intro: p2_p3 sum_0')
done

```

```

end

```

22 A Family of Similar Counters: Version with Compatibility

```

theory CounterC imports "../UNITY_Main" begin

```

```

typedcl state

```

```

consts
  C :: "state=>int"
  c :: "state=>nat=>int"

consts
  sum  :: "[nat,state]=>int"
  sumj :: "[nat, nat, state]=>int"

primrec
  "sum 0 s = 0"
  "sum (Suc i) s = (c s) i + sum i s"

primrec
  "sumj 0 i s = 0"
  "sumj (Suc n) i s = (if n=i then sum n s else (c s) n + sumj n i s)"

types command = "(state*state)set"

constdefs
  a :: "nat=>command"
  "a i == {(s, s'). (c s') i = (c s) i + 1 & (C s') = (C s) + 1}"

  Component :: "nat => state program"
  "Component i == mk_total_program({s. C s = 0 & (c s) i = 0},
    {a i},
     $\bigcup G \in \text{preserves } (\%s. (c s) i). \text{ Acts } G$ )"

declare Component_def [THEN def_prg_Init, simp]
declare Component_def [THEN def_prg_AllowedActs, simp]
declare a_def [THEN def_act_simp, simp]

lemma sum_sumj_eq1 [rule_format]: " $\forall i. I < i \rightarrow (\text{sum } I s = \text{sumj } I i s)$ "
by (induct_tac "I", auto)

lemma sum_sumj_eq2 [rule_format]: " $i < I \rightarrow \text{sum } I s = c s i + \text{sumj } I i s$ "
apply (induct_tac "I")
apply (auto simp add: linorder_neq_iff sum_sumj_eq1)
done

lemma sum_ext [rule_format]:
  " $(\forall i. i < I \rightarrow c s' i = c s i) \rightarrow (\text{sum } I s' = \text{sum } I s)$ "
by (induct_tac "I", auto)

lemma sumj_ext [rule_format]:
  " $(\forall j. j < I \ \& \ j \neq i \rightarrow c s' j = c s j) \rightarrow (\text{sumj } I i s' = \text{sumj } I i s)$ "
apply (induct_tac "I", safe)
apply (auto intro!: sum_ext)
done

lemma sum0 [rule_format]: " $(\forall i. i < I \rightarrow c s i = 0) \rightarrow \text{sum } I s = 0$ "

```



```
by (induct_tac "I", auto)
```

```
lemma Component_ok_iff:
  "(Component i ok G) =
   (G ∈ preserves (%s. c s i) & Component i ∈ Allowed G)"
apply (auto simp add: ok_iff_Allowed Component_def [THEN def_total_prg_Allowed])
done
declare Component_ok_iff [iff]
declare OK_iff_ok [iff]
declare preserves_def [simp]
```

```
lemma p2: "Component i ∈ stable {s. C s = (c s) i + k}"
by (simp add: Component_def, safety)
```

```
lemma p3:
  "[| OK I Component; i ∈ I |]
   ==> Component i ∈ stable {s. ∀ j ∈ I. j ≠ i --> c s j = c k j}"
apply simp
apply (unfold Component_def mk_total_program_def)
apply (simp (no_asm_use) add: stable_def constrains_def)
apply blast
done
```

```
lemma p2_p3_lemma1:
  "[| OK {i. i < I} Component; i < I |] ==>
   ∀ k. Component i ∈ stable ({s. C s = c s i + sum j I i k} Int
                               {s. ∀ j ∈ {i. i < I}. j ≠ i --> c s j = c k j})"
by (blast intro: stable_Int [OF p2 p3])
```

```
lemma p2_p3_lemma2:
  "(∀ k. F ∈ stable ({s. C s = (c s) i + sum j I i k} Int
                     {s. ∀ j ∈ {i. i < I}. j ≠ i --> c s j = c k j}))
   ==> (F ∈ stable {s. C s = c s i + sum j I i s})"
apply (simp add: constrains_def stable_def)
apply (force intro!: sumj_ext)
done
```

```
lemma p2_p3:
  "[| OK {i. i < I} Component; i < I |]
   ==> Component i ∈ stable {s. C s = c s i + sum j I i s}"
by (blast intro: p2_p3_lemma1 [THEN p2_p3_lemma2])
```

```
lemma safety:
  "[| 0 < I; OK {i. i < I} Component |]
   ==> (⋂ i ∈ {i. i < I}. (Component i)) ∈ invariant {s. C s = sum I s}"
apply (simp (no_asm) add: invariant_def JN_stable sum_sumj_eq2)
```

```

apply (auto intro!: sum0 p2_p3)
done

end

theory PriorityAux
imports "../UNITY_Main"
begin

typedcl vertex

constdefs
  symcl :: "(vertex*vertex)set=>(vertex*vertex)set"
  "symcl r == r  $\cup$  (r-1)"
  — symmetric closure: removes the orientation of a relation

  neighbors :: "[vertex, (vertex*vertex)set]=>vertex set"
  "neighbors i r == ((r  $\cup$  r-1)+{i}) - {i}"
  — Neighbors of a vertex i

  R :: "[vertex, (vertex*vertex)set]=>vertex set"
  "R i r == r+{i}"

  A :: "[vertex, (vertex*vertex)set]=>vertex set"
  "A i r == (r-1)+{i}"

  reach :: "[vertex, (vertex*vertex)set]=> vertex set"
  "reach i r == (r+)+{i}"
  — reachable and above vertices: the original notation was R* and A*

  above :: "[vertex, (vertex*vertex)set]=> vertex set"
  "above i r == ((r-1)+)+{i}"

  reverse :: "[vertex, (vertex*vertex) set]=>(vertex*vertex)set"
  "reverse i r == (r - {(x,y). x=i | y=i}  $\cap$  r)  $\cup$  ({(x,y). x=i|y=i}  $\cap$  r)-1"

  derive1 :: "[vertex, (vertex*vertex)set, (vertex*vertex)set]=>bool"
  — The original definition
  "derive1 i r q == symcl r = symcl q &
    ( $\forall k k'. k \neq i \ \& \ k' \neq i \ \rightarrow ((k,k'):r) = ((k,k'):q)$ ) &
    A i r = { } & R i q = { }"

  derive :: "[vertex, (vertex*vertex)set, (vertex*vertex)set]=>bool"
  — Our alternative definition
  "derive i r q == A i r = { } & (q = reverse i r)"

axioms
  finite_vertex_univ: "finite (UNIV :: vertex set)"
  — we assume that the universe of vertices is finite

declare derive_def [simp] derive1_def [simp] symcl_def [simp]
      A_def [simp] R_def [simp]

```

```

above_def [simp] reach_def [simp]
reverse_def [simp] neighbors_def [simp]

```

All vertex sets are finite

```

declare finite_subset [OF subset_UNIV finite_vertex_univ, iff]

```

and relations over vertex are finite too

```

lemmas finite_UNIV_Prod =
  finite_Prod_UNIV [OF finite_vertex_univ finite_vertex_univ]

```

```

declare finite_subset [OF subset_UNIV finite_UNIV_Prod, iff]

```

```

lemma image0_trancl_iff_image0_r: "((r+)+{i} = {j}) = (r+{i} = {j})"
apply auto
apply (erule trancl_induct, auto)
done

```

```

lemma image0_r_iff_image0_trancl: "(r+{i}={j}) = (ALL x. ((i,x):r+) = False)"
apply auto
apply (drule image0_trancl_iff_image0_r [THEN ssubst], auto)
done

```

```

lemma acyclic_eq_wf: "!!r::(vertex*vertex)set. acyclic r = wf r"
by (auto simp add: wf_iff_acyclic_if_finite)

```

```

lemma derive_derive1_eq: "derive i r q = derive1 i r q"
by auto

```

```

lemma lemma1_a:
  "[| x ∈ reach i q; derive1 k r q |] ==> x≠k --> x ∈ reach i r"
apply (unfold reach_def)
apply (erule ImageE)
apply (erule trancl_induct)
  apply (case_tac "i=k", simp_all)
  apply (blast intro: r_into_trancl, blast, clarify)
apply (drule_tac x = y in spec)
apply (drule_tac x = z in spec)
apply (blast dest: r_into_trancl intro: trancl_trans)
done

```

```

lemma reach_lemma: "derive k r q ==> reach i q ⊆ (reach i r ∪ {k})"
apply clarify
apply (drule lemma1_a)
apply (auto simp add: derive_derive1_eq
  simp del: reach_def derive_def derive1_def)
done

```

```

lemma reach_above_lemma:
  "( $\forall i. \text{reach } i \ q \subseteq (\text{reach } i \ r \cup \{k\})$ ) =
   ( $\forall x. x \neq k \rightarrow (\forall i. i \notin \text{above } x \ r \rightarrow i \notin \text{above } x \ q)$ )"
by (auto simp add: trancl_converse)

lemma maximal_converse_image0:
  "( $z, i$ ): $r^+$   $\Rightarrow$  ( $\forall y. (y, z):r \rightarrow (y, i) \notin r^+$ ) = ( $(r^{-1})^+ \{z\} = \{i\}$ )"
apply auto
apply (frule_tac  $r = r$  in trancl_into_trancl2, auto)
done

lemma above_lemma_a:
  " $\text{acyclic } r \Rightarrow A \ i \ r \neq \{i\} \rightarrow (\exists j \in \text{above } i \ r. A \ j \ r = \{i\})$ "
apply (simp add: acyclic_eq_wf wf_eq_minimal)
apply (drule_tac  $x = "(r^{-1})^+ \{i\}"$  in spec)
apply auto
apply (simp add: maximal_converse_image0 trancl_converse)
done

lemma above_lemma_b:
  " $\text{acyclic } r \Rightarrow \text{above } i \ r \neq \{i\} \rightarrow (\exists j \in \text{above } i \ r. \text{above } j \ r = \{i\})$ "
apply (drule above_lemma_a)
apply (auto simp add: image0_trancl_iff_image0_r)
done

end

```

23 The priority system

theory Priority imports PriorityAux begin

From Charpentier and Chandy, Examples of Program Composition Illustrating the Use of Universal Properties In J. Rolim (editor), Parallel and Distributed Processing, Spriner LNCS 1586 (1999), pages 1215-1227.

```

types state = "(vertex*vertex)set"
types command = "vertex=>(state*state)set"

```

```

consts
  init :: "(vertex*vertex)set"
  — the initial state

```

Following the definitions given in section 4.4

```

constdefs
  highest :: "[vertex, (vertex*vertex)set] => bool"
  "highest i r == A i r = {}"
  — i has highest priority in r

  lowest :: "[vertex, (vertex*vertex)set] => bool"
  "lowest i r == R i r = {}"

```

— i has lowest priority in r

```
act :: command
"act i == {(s, s'). s'=reverse i s & highest i s}"

Component :: "vertex=>state program"
"Component i == mk_total_program({init}, {act i}, UNIV)"
  — All components start with the same initial state
```

Some Abbreviations

constdefs

```
Highest :: "vertex=>state set"
"Highest i == {s. highest i s}"

Lowest :: "vertex=>state set"
"Lowest i == {s. lowest i s}"

Acyclic :: "state set"
"Acyclic == {s. acyclic s}"

Maximal :: "state set"
  — Every “above” set has a maximal vertex
"Maximal ==  $\bigcap i. \{s. \sim \text{highest } i \text{ s} \rightarrow (\exists j \in \text{above } i \text{ s. highest } j \text{ s})\}$ "

Maximal' :: "state set"
  — Maximal vertex: equivalent definition
"Maximal' ==  $\bigcap i. \text{Highest } i \text{ Un } (\bigcup j. \{s. j \in \text{above } i \text{ s}\} \text{ Int Highest } j)$ "

Safety :: "state set"
"Safety ==  $\bigcap i. \{s. \text{highest } i \text{ s} \rightarrow (\forall j \in \text{neighbors } i \text{ s. } \sim \text{highest } j \text{ s})\}$ "

system :: "state program"
"system == JN i. Component i"
```

```
declare highest_def [simp] lowest_def [simp]
declare Highest_def [THEN def_set_simp, simp]
  and Lowest_def [THEN def_set_simp, simp]

declare Component_def [THEN def_prg_Init, simp]
declare act_def [THEN def_act_simp, simp]
```

23.1 Component correctness proofs

neighbors is stable

```
lemma Component_neighbors_stable: "Component i  $\in$  stable {s. neighbors k
s = n}"
by (simp add: Component_def, safety, auto)
```

property 4

```
lemma Component_waits_priority: "Component i: {s. ((i,j):s) = b} Int (- Highest
i) co {s. ((i,j):s)=b}"
by (simp add: Component_def, safety)
```

property 5: charpentier and Chandy mistakenly express it as 'transient Highest i'. Consider the case where i has neighbors

```
lemma Component_yields_priority:
  "Component i: {s. neighbors i s ≠ {}} Int Highest i
  ensures - Highest i"
apply (simp add: Component_def)
apply (ensures_tac "act i", blast+)
done
```

or better

```
lemma Component_yields_priority': "Component i ∈ Highest i ensures Lowest
i"
apply (simp add: Component_def)
apply (ensures_tac "act i", blast+)
done
```

property 6: Component doesn't introduce cycle

```
lemma Component_well_behaves: "Component i ∈ Highest i co Highest i Un Lowest
i"
by (simp add: Component_def, safety, fast)
```

property 7: local axiom

```
lemma locality: "Component i ∈ stable {s. ∀ j k. j≠i & k≠i--> ((j,k):s)
= b j k}"
by (simp add: Component_def, safety)
```

23.2 System properties

property 8: strictly universal

```
lemma Safety: "system ∈ stable Safety"
apply (unfold Safety_def)
apply (rule stable_INT)
apply (simp add: system_def, safety, fast)
done
```

property 13: universal

```
lemma p13: "system ∈ {s. s = q} co {s. s=q} Un {s. ∃ i. derive i q s}"
by (simp add: system_def Component_def mk_total_program_def totalize_JN, safety,
blast)
```

property 14: the 'above set' of a Component that hasn't got priority doesn't increase

```
lemma above_not_increase:
  "system ∈ -Highest i Int {s. j∉above i s} co {s. j∉above i s}"
apply (insert reach_lemma [of concl: j])
```

```

apply (simp add: system_def Component_def mk_total_program_def totalize_JN,
        safety)
apply (simp add: trancl_converse, blast)
done

```

```

lemma above_not_increase':
  "system ∈ -Highest i Int {s. above i s = x} co {s. above i s ≤ x}"
apply (insert above_not_increase [of i])
apply (simp add: trancl_converse constrains_def, blast)
done

```

p15: universal property: all Components well behave

```

lemma system_well_behaves [rule_format]:
  "∀i. system ∈ Highest i co Highest i Un Lowest i"
apply clarify
apply (simp add: system_def Component_def mk_total_program_def totalize_JN,
        safety, auto)
done

```

```

lemma Acyclic_eq: "Acyclic = (⋂i. {s. i ∉ above i s})"
by (auto simp add: Acyclic_def acyclic_def trancl_converse)

```

```

lemmas system_co =
  constrains_Un [OF above_not_increase [rule_format] system_well_behaves]

```

```

lemma Acyclic_stable: "system ∈ stable Acyclic"
apply (simp add: stable_def Acyclic_eq)
apply (auto intro!: constrains_INT system_co [THEN constrains_weaken]
        simp add: image0_r_iff_image0_trancl trancl_converse)
done

```

```

lemma Acyclic_subset_Maximal: "Acyclic ≤ Maximal"
apply (unfold Acyclic_def Maximal_def, clarify)
apply (drule above_lemma_b, auto)
done

```

property 17: original one is an invariant

```

lemma Acyclic_Maximal_stable: "system ∈ stable (Acyclic Int Maximal)"
by (simp add: Acyclic_subset_Maximal [THEN Int_absorb2] Acyclic_stable)

```

property 5: existential property

```

lemma Highest_leadsTo_Lowest: "system ∈ Highest i leadsTo Lowest i"
apply (simp add: system_def Component_def mk_total_program_def totalize_JN)
apply (ensures_tac "act i", auto)
done

```

a lowest i can never be in any abover set

```
lemma Lowest_above_subset: "Lowest i <= ( $\bigcap k. \{s. i \notin \text{above } k \ s\}$ )"
by (auto simp add: image0_r_iff_image0_trancl trancl_converse)
```

property 18: a simpler proof than the original, one which uses psp

```
lemma Highest_escapes_above: "system  $\in$  Highest i leadsTo ( $\bigcap k. \{s. i \notin \text{above } k \ s\}$ )"
apply (rule leadsTo_weaken_R)
apply (rule_tac [2] Lowest_above_subset)
apply (rule Highest_leadsTo_Lowest)
done
```

```
lemma Highest_escapes_above':
  "system  $\in$  Highest j Int {s. j  $\in$  above i s} leadsTo {s. j  $\notin$  above i s}"
by (blast intro: leadsTo_weaken [OF Highest_escapes_above Int_lower1 INT_lower])
```

23.3 The main result: above set decreases

The original proof of the following formula was wrong

```
lemma Highest_iff_above0: "Highest i = {s. above i s = {}}"
by (auto simp add: image0_trancl_iff_image0_r)
```

```
lemmas above_decreases_lemma =
  psp [THEN leadsTo_weaken, OF Highest_escapes_above' above_not_increase']
```

```
lemma above_decreases:
  "system  $\in$  ( $\bigcup j. \{s. \text{above } i \ s = x\}$  Int {s. j  $\in$  above i s} Int Highest
  j)
  leadsTo {s. above i s < x}"
apply (rule leadsTo_UN)
apply (rule single_leadsTo_I, clarify)
apply (rule_tac x2 = "above i x" in above_decreases_lemma)
apply (simp_all (no_asm_use) add: Highest_iff_above0)
apply blast+
done
```

```
lemma Maximal_eq_Maximal': "Maximal = Maximal'"
by (unfold Maximal_def Maximal'_def Highest_def, blast)
```

```
lemma Acyclic_subset:
  "x  $\neq$  {} ==>
  Acyclic Int {s. above i s = x} <=
  ( $\bigcup j. \{s. \text{above } i \ s = x\}$  Int {s. j  $\in$  above i s} Int Highest j)"
apply (rule_tac B = "Maximal' Int {s. above i s = x}" in subset_trans)
apply (simp (no_asm) add: Maximal_eq_Maximal' [symmetric])
apply (blast intro: Acyclic_subset_Maximal [THEN subsetD])
apply (simp (no_asm) del: above_def add: Maximal'_def Highest_iff_above0)
apply blast
done
```

```
lemmas above_decreases' = leadsTo_weaken_L [OF above_decreases Acyclic_subset]
```



```
lemmas above_decreases_psp = psp_stable [OF above_decreases' Acyclic_stable]
```

```
lemma above_decreases_psp':
  "x≠{}==> system ∈ Acyclic Int {s. above i s = x} leadsTo
    Acyclic Int {s. above i s < x}"
by (erule above_decreases_psp [THEN leadsTo_weaken], blast, auto)
```

```
lemmas finite_psubset_induct = wf_finite_psubset [THEN leadsTo_wf_induct]
```

```
lemma Progress: "system ∈ Acyclic leadsTo Highest i"
apply (rule_tac f = "%s. above i s" in finite_psubset_induct)
apply (simp del: above_def
  add: Highest_iff_above0 vimage_def finite_psubset_def, clarify)
apply (case_tac "m={}")
apply (rule Int_lower2 [THEN [2] leadsTo_weaken_L])
apply (force simp add: leadsTo_refl)
apply (rule_tac A' = "Acyclic Int {x. above i x < m}" in leadsTo_weaken_R)
apply (blast intro: above_decreases_psp')+
done
```

We have proved all (relevant) theorems given in the paper. We didn't assume any thing about the relation r . It is not necessary that r be a priority relation as assumed in the original proof. It suffices that we start from a state which is finite and acyclic.

end

```
theory TimerArray imports "../UNITY_Main" begin
```

```
types 'a state = "nat * 'a"
```

```
constdefs
```

```
count  :: "'a state => nat"
  "count s == fst s"
```

```
decr   :: "('a state * 'a state) set"
  "decr == UN n uu. {(Suc n, uu), (n,uu)}"
```

```
Timer :: "'a state program"
  "Timer == mk_total_program (UNIV, {decr}, UNIV)"
```

```
declare Timer_def [THEN def_prg_Init, simp]
```

```
declare count_def [simp] decr_def [simp]
```

```
lemma Timer_leadsTo_zero: "Timer : UNIV leadsTo {s. count s = 0}"
apply (rule_tac f = count in lessThan_induct, simp)
apply (case_tac "m")
apply (force intro!: subset_imp_leadsTo)
```

```

apply (unfold Timer_def, ensures_tac "decr")
done

lemma Timer_preserves_snd [iff]: "Timer : preserves snd"
apply (rule preservesI)
apply (unfold Timer_def, safety)
done

declare PLam_stable [simp]

lemma TimerArray_leadsTo_zero:
  "finite I
  ==> (plam i: I. Timer) : UNIV leadsTo {(s,uu). ALL i:I. s i = 0}"
apply (erule_tac A'1 = "%i. lift_set i ({0} <*> UNIV)"
  in finite_stable_completion [THEN leadsTo_weaken])
apply auto

prefer 2
apply (simp add: Timer_def, safety)

apply (rule_tac f = "sub i o fst" in lessThan_induct)
apply (case_tac "m")

apply (auto intro: subset_imp_leadsTo
  simp add: insert_absorb
    lift_set_Un_distrib [symmetric] lessThan_Suc [symmetric]
    Times_Un_distrib1 [symmetric] Times_Diff_distrib1 [symmetric])
apply (rename_tac "n")
apply (rule PLam_leadsTo_Basis)
apply (auto simp add: lessThan_Suc [symmetric])
apply (unfold Timer_def mk_total_program_def, safety)
apply (rule_tac act = decr in totalize_transientI, auto)
done

end

```

24 Progress Set Examples

theory Progress imports "../UNITY_Main" begin

24.1 The Composition of Two Single-Assignment Programs

Thesis Section 4.4.2

```

constdefs
  FF :: "int program"
    "FF == mk_total_program (UNIV, {range ( $\lambda x. (x, x+1)$ )}, UNIV)"

  GG :: "int program"
    "GG == mk_total_program (UNIV, {range ( $\lambda x. (x, 2*x)$ )}, UNIV)"

```

24.1.1 Calculating $wens_set\ FF\ \{k..\}$

```

lemma Domain_actFF: "Domain (range ( $\lambda x::int.$  (x, x + 1))) = UNIV"
by force

lemma FF_eq:
  "FF = mk_program (UNIV, {range ( $\lambda x.$  (x, x+1))}, UNIV)"
by (simp add: FF_def mk_total_program_def totalize_def totalize_act_def
    program_equalityI Domain_actFF)

lemma wp_actFF:
  "wp (range ( $\lambda x::int.$  (x, x + 1))) (atLeast k) = atLeast (k - 1)"
by (force simp add: wp_def)

lemma wens_FF: "wens FF (range ( $\lambda x.$  (x, x+1))) (atLeast k) = atLeast (k - 1)"
by (force simp add: FF_eq wens_single_eq wp_actFF)

lemma single_valued_actFF: "single_valued (range ( $\lambda x::int.$  (x, x + 1)))"
by (force simp add: single_valued_def)

lemma wens_single_finite_FF:
  "wens_single_finite (range ( $\lambda x.$  (x, x+1))) (atLeast k) n =
    atLeast (k - int n)"
apply (induct n, simp)
apply (force simp add: wens_FF
    def_wens_single_finite_Suc_eq_wens [OF FF_eq single_valued_actFF])
done

lemma wens_single_FF_eq_UNIV:
  "wens_single (range ( $\lambda x::int.$  (x, x + 1))) (atLeast k) = UNIV"
apply (auto simp add: wens_single_eq_Union)
apply (rule_tac x="nat (k-x)" in exI)
apply (simp add: wens_single_finite_FF)
done

lemma wens_set_FF:
  "wens_set FF (atLeast k) = insert UNIV (atLeast ' atMost k)"
apply (auto simp add: wens_set_single_eq [OF FF_eq single_valued_actFF]
    wens_single_FF_eq_UNIV wens_single_finite_FF)
apply (erule notE)
apply (rule_tac x="nat (k-xa)" in range_eqI)
apply (simp add: wens_single_finite_FF)
done

```

24.1.2 Proving $FF \in UNIV\ leadsTo\ \{k..\}$

```

lemma atLeast_ensures: "FF  $\in$  atLeast (k - 1) ensures atLeast (k::int)"
apply (simp add: Progress.wens_FF [symmetric] wens_ensures)
apply (simp add: wens_ensures FF_eq)
done

lemma atLeast_leadsTo: "FF  $\in$  atLeast (k - int n) leadsTo atLeast (k::int)"
apply (induct n)
apply (simp_all add: leadsTo_refl)

```

```

apply (rule_tac A = "atLeast (k - int n - 1)" in leadsTo_weaken_L)
apply (blast intro: leadsTo_Trans atLeast_ensures, force)
done

```

```

lemma UN_atLeast_UNIV: " $\bigcup n. \text{atLeast } (k - \text{int } n) = \text{UNIV}$ "
apply auto
apply (rule_tac x = "nat (k - x)" in exI, simp)
done

```

```

lemma FF_leadsTo: "FF  $\in$  UNIV leadsTo atLeast (k::int)"
apply (subst UN_atLeast_UNIV [symmetric])
apply (rule leadsTo_UN [OF atLeast_leadsTo])
done

```

Result (4.39): Applying the leadsTo-Join Theorem

```

theorem "FF  $\sqcup$  GG  $\in$  atLeast 0 leadsTo atLeast (k::int)"
apply (subgoal_tac "FF  $\sqcup$  GG  $\in$  (atLeast 0  $\cap$  atLeast 0) leadsTo atLeast k")
apply simp
apply (rule_tac T = "atLeast 0" in leadsTo_Join)
  apply (rule leadsTo_weaken_L [OF FF_leadsTo], simp)
  apply (simp add: awp_iff_constrains FF_def, safety)
apply (simp add: awp_iff_constrains GG_def wens_set_FF, safety)
done

end

```

25 Projections of State Sets

theory Project imports Extend begin

constdefs

```

projecting :: "'c program => 'c set, 'a*'b => 'c,
              'a program, 'c program set, 'a program set] => bool"
"projecting C h F X' X ==
   $\forall G. \text{extend } h \ F \sqcup G \in X' \rightarrow F \sqcup \text{project } h \ (C \ G) \ G \in X$ "

```

```

extending :: "'c program => 'c set, 'a*'b => 'c, 'a program,
              'c program set, 'a program set] => bool"
"extending C h F Y' Y ==
   $\forall G. \text{extend } h \ F \text{ ok } G \rightarrow F \sqcup \text{project } h \ (C \ G) \ G \in Y$ 
   $\rightarrow \text{extend } h \ F \sqcup G \in Y'$ "

```

```

subset_closed :: "'a set set => bool"
"subset_closed U ==  $\forall A \in U. \text{Pow } A \subseteq U$ "

```

```

lemma (in Extend) project_extend_constrains_I:
  "F  $\in$  A co B ==> project h C (extend h F)  $\in$  A co B"
apply (auto simp add: extend_act_def project_act_def constrains_def)
done

```

25.1 Safety

```

lemma (in Extend) project_unless [rule_format]:
  "[/ G ∈ stable C; project h C G ∈ A unless B [/]
  ==> G ∈ (C ∩ extend_set h A) unless (extend_set h B)"
apply (simp add: unless_def project_constrains)
apply (blast dest: stable_constrains_Int intro: constrains_weaken)
done

lemma (in Extend) Join_project_constrains:
  "(F ⊔ project h C G ∈ A co B) =
   (extend h F ⊔ G ∈ (C ∩ extend_set h A) co (extend_set h B) &
    F ∈ A co B)"
apply (simp (no_asm) add: project_constrains)
apply (blast intro: extend_constrains [THEN iffD2, THEN constrains_weaken]
        dest: constrains_imp_subset)
done

lemma (in Extend) Join_project_stable:
  "extend h F ⊔ G ∈ stable C
  ==> (F ⊔ project h C G ∈ stable A) =
   (extend h F ⊔ G ∈ stable (C ∩ extend_set h A) &
    F ∈ stable A)"
apply (unfold stable_def)
apply (simp only: Join_project_constrains)
apply (blast intro: constrains_weaken dest: constrains_Int)
done

lemma (in Extend) project_constrains_I:
  "extend h F ⊔ G ∈ extend_set h A co extend_set h B
  ==> F ⊔ project h C G ∈ A co B"
apply (simp add: project_constrains extend_constrains)
apply (blast intro: constrains_weaken dest: constrains_imp_subset)
done

lemma (in Extend) project_increasing_I:
  "extend h F ⊔ G ∈ increasing (func o f)
  ==> F ⊔ project h C G ∈ increasing func"
apply (unfold increasing_def stable_def)
apply (simp del: Join_constrains
        add: project_constrains_I extend_set_eq_Collect)
done

lemma (in Extend) Join_project_increasing:
  "(F ⊔ project h UNIV G ∈ increasing func) =
   (extend h F ⊔ G ∈ increasing (func o f))"
apply (rule iffI)
apply (erule_tac [2] project_increasing_I)
apply (simp del: Join_stable
        add: increasing_def Join_project_stable)

```

```

apply (auto simp add: extend_set_eq_Collect extend_stable [THEN iffD1])
done

```

```

lemma (in Extend) project_constrains_D:
  "F ⊔ project h UNIV G ∈ A co B
   ==> extend h F ⊔ G ∈ extend_set h A co extend_set h B"
by (simp add: project_constrains extend_constrains)

```

25.2 "projecting" and union/intersection (no converses)

```

lemma projecting_Int:
  "[| projecting C h F XA' XA; projecting C h F XB' XB |]
   ==> projecting C h F (XA' ∩ XB') (XA ∩ XB)"
by (unfold projecting_def, blast)

```

```

lemma projecting_Un:
  "[| projecting C h F XA' XA; projecting C h F XB' XB |]
   ==> projecting C h F (XA' ∪ XB') (XA ∪ XB)"
by (unfold projecting_def, blast)

```

```

lemma projecting_INT:
  "[| !!i. i ∈ I ==> projecting C h F (X' i) (X i) |]
   ==> projecting C h F (⋂ i ∈ I. X' i) (⋂ i ∈ I. X i)"
by (unfold projecting_def, blast)

```

```

lemma projecting_UN:
  "[| !!i. i ∈ I ==> projecting C h F (X' i) (X i) |]
   ==> projecting C h F (⋃ i ∈ I. X' i) (⋃ i ∈ I. X i)"
by (unfold projecting_def, blast)

```

```

lemma projecting_weaken:
  "[| projecting C h F X' X; U' ≤ X'; X ⊆ U |] ==> projecting C h F U'
U"
by (unfold projecting_def, auto)

```

```

lemma projecting_weaken_L:
  "[| projecting C h F X' X; U' ≤ X' |] ==> projecting C h F U' X"
by (unfold projecting_def, auto)

```

```

lemma extending_Int:
  "[| extending C h F YA' YA; extending C h F YB' YB |]
   ==> extending C h F (YA' ∩ YB') (YA ∩ YB)"
by (unfold extending_def, blast)

```

```

lemma extending_Un:
  "[| extending C h F YA' YA; extending C h F YB' YB |]
   ==> extending C h F (YA' ∪ YB') (YA ∪ YB)"
by (unfold extending_def, blast)

```

```

lemma extending_INT:
  "[| !!i. i ∈ I ==> extending C h F (Y' i) (Y i) |]
   ==> extending C h F (⋂ i ∈ I. Y' i) (⋂ i ∈ I. Y i)"
by (unfold extending_def, blast)

```

```

lemma extending_UN:
  "[| !!i. i ∈ I ==> extending C h F (Y' i) (Y i) |]
   ==> extending C h F (⋃ i ∈ I. Y' i) (⋃ i ∈ I. Y i)"
by (unfold extending_def, blast)

lemma extending_weaken:
  "[| extending C h F Y' Y; Y' <= V'; V ⊆ Y |] ==> extending C h F V' V"
by (unfold extending_def, auto)

lemma extending_weaken_L:
  "[| extending C h F Y' Y; Y' <= V' |] ==> extending C h F V' Y"
by (unfold extending_def, auto)

lemma projecting_UNIV: "projecting C h F X' UNIV"
by (simp add: projecting_def)

lemma (in Extend) projecting_constrains:
  "projecting C h F (extend_set h A co extend_set h B) (A co B)"
apply (unfold projecting_def)
apply (blast intro: project_constrains_I)
done

lemma (in Extend) projecting_stable:
  "projecting C h F (stable (extend_set h A)) (stable A)"
apply (unfold stable_def)
apply (rule projecting_constrains)
done

lemma (in Extend) projecting_increasing:
  "projecting C h F (increasing (func o f)) (increasing func)"
apply (unfold projecting_def)
apply (blast intro: project_increasing_I)
done

lemma (in Extend) extending_UNIV: "extending C h F UNIV Y"
apply (simp (no_asm) add: extending_def)
done

lemma (in Extend) extending_constrains:
  "extending (%G. UNIV) h F (extend_set h A co extend_set h B) (A co B)"
apply (unfold extending_def)
apply (blast intro: project_constrains_D)
done

lemma (in Extend) extending_stable:
  "extending (%G. UNIV) h F (stable (extend_set h A)) (stable A)"
apply (unfold stable_def)
apply (rule extending_constrains)
done

lemma (in Extend) extending_increasing:
  "extending (%G. UNIV) h F (increasing (func o f)) (increasing func)"
by (force simp only: extending_def Join_project_increasing)

```

25.3 Reachability and project

```

lemma (in Extend) reachable_imp_reachable_project:
  "[| reachable (extend h F⊔G) ⊆ C;
    z ∈ reachable (extend h F⊔G) |]"
  ==> f z ∈ reachable (F⊔project h C G)"
apply (erule reachable.induct)
apply (force intro!: reachable.Init simp add: split_extended_all, auto)
apply (rule_tac act = x in reachable.Acts)
apply auto
apply (erule extend_act_D)
apply (rule_tac act1 = "Restrict C act"
  in project_act_I [THEN [3] reachable.Acts], auto)
done

lemma (in Extend) project_Constrains_D:
  "F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B
  ==> extend h F⊔G ∈ (extend_set h A) Co (extend_set h B)"
apply (unfold Constrains_def)
apply (simp del: Join_constrains
  add: Join_project_constrains, clarify)
apply (erule constrains_weaken)
apply (auto intro: reachable_imp_reachable_project)
done

lemma (in Extend) project_Stable_D:
  "F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A
  ==> extend h F⊔G ∈ Stable (extend_set h A)"
apply (unfold Stable_def)
apply (simp (no_asm_simp) add: project_Constrains_D)
done

lemma (in Extend) project_Always_D:
  "F⊔project h (reachable (extend h F⊔G)) G ∈ Always A
  ==> extend h F⊔G ∈ Always (extend_set h A)"
apply (unfold Always_def)
apply (force intro: reachable.Init simp add: project_Stable_D split_extended_all)
done

lemma (in Extend) project_Increasing_D:
  "F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func
  ==> extend h F⊔G ∈ Increasing (func o f)"
apply (unfold Increasing_def, auto)
apply (subst extend_set_eq_Collect [symmetric])
apply (simp (no_asm_simp) add: project_Stable_D)
done

```

25.4 Converse results for weak safety: benefits of the argument C

```

lemma (in Extend) reachable_project_imp_reachable:
  "[| C ⊆ reachable(extend h F⊔G);
    x ∈ reachable (F⊔project h C G) |]"
  ==> ∃y. h(x,y) ∈ reachable (extend h F⊔G)"

```



```

apply (erule reachable.induct)
apply (force intro: reachable.Init)
apply (auto simp add: project_act_def)
apply (force del: Id_in_Acts intro: reachable.Acts extend_act_D)+
done

lemma (in Extend) project_set_reachable_extend_eq:
  "project_set h (reachable (extend h F⊔G)) =
   reachable (F⊔project h (reachable (extend h F⊔G)) G)"
by (auto dest: subset_refl [THEN reachable_imp_reachable_project]
    subset_refl [THEN reachable_project_imp_reachable])

lemma (in Extend) reachable_extend_Join_subset:
  "reachable (extend h F⊔G) ⊆ C
   ==> reachable (extend h F⊔G) ⊆
    extend_set h (reachable (F⊔project h C G))"
apply (auto dest: reachable_imp_reachable_project)
done

lemma (in Extend) project_Constrains_I:
  "extend h F⊔G ∈ (extend_set h A) Co (extend_set h B)
   ==> F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B"
apply (unfold Constrains_def)
apply (simp del: Join_constrains
        add: Join_project_constrains extend_set_Int_distrib)
apply (rule conjI)
  prefer 2
  apply (force elim: constrains_weaken_L
              dest!: extend_constrains_project_set
                  subset_refl [THEN reachable_project_imp_reachable])
apply (blast intro: constrains_weaken_L)
done

lemma (in Extend) project_Stable_I:
  "extend h F⊔G ∈ Stable (extend_set h A)
   ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A"
apply (unfold Stable_def)
apply (simp (no_asm_simp) add: project_Constrains_I)
done

lemma (in Extend) project_Always_I:
  "extend h F⊔G ∈ Always (extend_set h A)
   ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Always A"
apply (unfold Always_def)
apply (auto simp add: project_Stable_I)
apply (unfold extend_set_def, blast)
done

lemma (in Extend) project_Increasing_I:
  "extend h F⊔G ∈ Increasing (func o f)
   ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func"
apply (unfold Increasing_def, auto)
apply (simp (no_asm_simp) add: extend_set_eq_Collect project_Stable_I)

```

done

```
lemma (in Extend) project_Constrains:
  "(F ⊔ project h (reachable (extend h F ⊔ G)) G ∈ A Co B) =
   (extend h F ⊔ G ∈ (extend_set h A) Co (extend_set h B))"
apply (blast intro: project_Constrains_I project_Constrains_D)
done
```

```
lemma (in Extend) project_Stable:
  "(F ⊔ project h (reachable (extend h F ⊔ G)) G ∈ Stable A) =
   (extend h F ⊔ G ∈ Stable (extend_set h A))"
apply (unfold Stable_def)
apply (rule project_Constrains)
done
```

```
lemma (in Extend) project_Increasing:
  "(F ⊔ project h (reachable (extend h F ⊔ G)) G ∈ Increasing func) =
   (extend h F ⊔ G ∈ Increasing (func o f))"
apply (simp (no_asm_simp) add: Increasing_def project_Stable extend_set_eq_Collect)
done
```

25.5 A lot of redundant theorems: all are proved to facilitate reasoning about guarantees.

```
lemma (in Extend) projecting_Constrains:
  "projecting (%G. reachable (extend h F ⊔ G)) h F
   (extend_set h A Co extend_set h B) (A Co B)"
apply (unfold projecting_def)
apply (blast intro: project_Constrains_I)
done
```

```
lemma (in Extend) projecting_Stable:
  "projecting (%G. reachable (extend h F ⊔ G)) h F
   (Stable (extend_set h A)) (Stable A)"
apply (unfold Stable_def)
apply (rule projecting_Constrains)
done
```

```
lemma (in Extend) projecting_Always:
  "projecting (%G. reachable (extend h F ⊔ G)) h F
   (Always (extend_set h A)) (Always A)"
apply (unfold projecting_def)
apply (blast intro: project_Always_I)
done
```

```
lemma (in Extend) projecting_Increasing:
  "projecting (%G. reachable (extend h F ⊔ G)) h F
   (Increasing (func o f)) (Increasing func)"
apply (unfold projecting_def)
apply (blast intro: project_Increasing_I)
done
```

```
lemma (in Extend) extending_Constrains:
```

```

      "extending (%G. reachable (extend h F ⊔ G)) h F
        (extend_set h A Co extend_set h B) (A Co B)"
    apply (unfold extending_def)
    apply (blast intro: project_Constrains_D)
  done

```

```

lemma (in Extend) extending_Stable:
  "extending (%G. reachable (extend h F ⊔ G)) h F
    (Stable (extend_set h A)) (Stable A)"
  apply (unfold extending_def)
  apply (blast intro: project_Stable_D)
done

```

```

lemma (in Extend) extending_Always:
  "extending (%G. reachable (extend h F ⊔ G)) h F
    (Always (extend_set h A)) (Always A)"
  apply (unfold extending_def)
  apply (blast intro: project_Always_D)
done

```

```

lemma (in Extend) extending_Increasing:
  "extending (%G. reachable (extend h F ⊔ G)) h F
    (Increasing (func o f)) (Increasing func)"
  apply (unfold extending_def)
  apply (blast intro: project_Increasing_D)
done

```

25.6 leadsETo in the precondition (??)

25.6.1 transient

```

lemma (in Extend) transient_extend_set_imp_project_transient:
  "[| G ∈ transient (C ∩ extend_set h A); G ∈ stable C |]
    ==> project h C G ∈ transient (project_set h C ∩ A)"
  apply (auto simp add: transient_def Domain_project_act)
  apply (subgoal_tac "act ' (C ∩ extend_set h A) ⊆ - extend_set h A")
  prefer 2
  apply (simp add: stable_def constrains_def, blast)

  apply (erule_tac V = "?AA ⊆ -C ∪ ?BB" in thin_rl)
  apply (drule bspec, assumption)
  apply (simp add: extend_set_def project_act_def, blast)
done

```

```

lemma (in Extend) project_extend_transient_D:
  "project h C (extend h F) ∈ transient (project_set h C ∩ D)
    ==> F ∈ transient (project_set h C ∩ D)"
  apply (simp add: transient_def Domain_project_act, safe)
  apply blast+
done

```

25.6.2 ensures – a primitive combining progress with safety

```

lemma (in Extend) ensures_extend_set_imp_project_ensures:

```

```

    "[| extend h F ∈ stable C;  G ∈ stable C;
      extend h F ⊔ G ∈ A ensures B;  A-B = C ∩ extend_set h D |]
    ==> F ⊔ project h C G
      ∈ (project_set h C ∩ project_set h A) ensures (project_set h B)"
  apply (simp add: ensures_def project_constrains Join_transient extend_transient,
    clarify)
  apply (intro conjI)

  apply (blast intro: extend_stable_project_set
    [THEN stableD, THEN constrains_Int, THEN constrains_weaken]

    dest!: extend_constrains_project_set equalityD1)

  apply (erule stableD [THEN constrains_Int, THEN constrains_weaken])
    apply assumption
    apply (simp (no_asm_use) add: extend_set_def)
    apply blast
  apply (simp add: extend_set_Int_distrib extend_set_Un_distrib)
  apply (blast intro!: extend_set_project_set [THEN subsetD], blast)

  apply auto
  prefer 2
  apply (force dest!: equalityD1
    intro: transient_extend_set_imp_project_transient
      [THEN transient_strengthen])
  apply (simp (no_asm_use) add: Int_Diff)
  apply (force dest!: equalityD1
    intro: transient_extend_set_imp_project_transient
      [THEN project_extend_transient_D, THEN transient_strengthen])
done

```

Transferring a transient property upwards

```

lemma (in Extend) project_transient_extend_set:
  "project h C G ∈ transient (project_set h C ∩ A - B)
  ==> G ∈ transient (C ∩ extend_set h A - extend_set h B)"
  apply (simp add: transient_def project_set_def extend_set_def project_act_def)
  apply (elim disjE bexE)
  apply (rule_tac x=Id in bexI)
  apply (blast intro!: rev_bexI )+
done

```

```

lemma (in Extend) project_unless2 [rule_format]:
  "[| G ∈ stable C;  project h C G ∈ (project_set h C ∩ A) unless B |]
  ==> G ∈ (C ∩ extend_set h A) unless (extend_set h B)"
  by (auto dest: stable_constrains_Int intro: constrains_weaken
    simp add: unless_def project_constrains Diff_eq Int_assoc
      Int_extend_set_lemma)

```

```

lemma (in Extend) extend_unless:
  "[| extend h F ∈ stable C;  F ∈ A unless B |]
  ==> extend h F ∈ C ∩ extend_set h A unless extend_set h B"
  apply (simp add: unless_def stable_def)

```

```

apply (drule constrains_Int)
apply (erule extend_constrains [THEN iffD2])
apply (erule constrains_weaken, blast)
apply blast
done

```

```

lemma (in Extend) Join_project_ensures [rule_format]:
  "[| extend h F⊔G ∈ stable C;
    F⊔project h C G ∈ A ensures B |]"
  ==> extend h F⊔G ∈ (C ∩ extend_set h A) ensures (extend_set h B)"
apply (auto simp add: ensures_eq extend_unless project_unless)
apply (blast intro: extend_transient [THEN iffD2] transient_strengthen)
apply (blast intro: project_transient_extend_set transient_strengthen)
done

```

Lemma useful for both STRONG and WEAK progress, but the transient condition's very strong

```

lemma (in Extend) PLD_lemma:
  "[| extend h F⊔G ∈ stable C;
    F⊔project h C G ∈ (project_set h C ∩ A) leadsTo B |]"
  ==> extend h F⊔G ∈
    C ∩ extend_set h (project_set h C ∩ A) leadsTo (extend_set h B)"
apply (erule leadsTo_induct)
  apply (blast intro: leadsTo_Basis Join_project_ensures)
  apply (blast intro: psp_stable2 [THEN leadsTo_weaken_L] leadsTo_Trans)
apply (simp del: UN_simps add: Int_UN_distrib leadsTo_UN extend_set_Union)
done

```

```

lemma (in Extend) project_leadsTo_D_lemma:
  "[| extend h F⊔G ∈ stable C;
    F⊔project h C G ∈ (project_set h C ∩ A) leadsTo B |]"
  ==> extend h F⊔G ∈ (C ∩ extend_set h A) leadsTo (extend_set h B)"
apply (rule PLD_lemma [THEN leadsTo_weaken])
apply (auto simp add: split_extended_all)
done

```

```

lemma (in Extend) Join_project_LeadsTo:
  "[| C = (reachable (extend h F⊔G));
    F⊔project h C G ∈ A LeadsTo B |]"
  ==> extend h F⊔G ∈ (extend_set h A) LeadsTo (extend_set h B)"
by (simp del: Join_stable add: LeadsTo_def project_leadsTo_D_lemma
    project_set_reachable_extend_eq)

```

25.7 Towards the theorem *project_Ensures_D*

```

lemma (in Extend) project_ensures_D_lemma:
  "[| G ∈ stable ((C ∩ extend_set h A) - (extend_set h B));
    F⊔project h C G ∈ (project_set h C ∩ A) ensures B;
    extend h F⊔G ∈ stable C |]"
  ==> extend h F⊔G ∈ (C ∩ extend_set h A) ensures (extend_set h B)"

apply (auto intro!: project_unless2 [unfolded unless_def]
  intro: project_extend_constrains_I

```

```

simp add: ensures_def)

prefer 2
apply (blast intro: project_transient_extend_set)

apply (force elim!: extend_transient [THEN iffD2, THEN transient_strengthen]
      simp add: split_extended_all)
done

lemma (in Extend) project_ensures_D:
  "[| F ⊔ project h UNIV G ∈ A ensures B;
    G ∈ stable (extend_set h A - extend_set h B) |]"
  ==> extend h F ⊔ G ∈ (extend_set h A) ensures (extend_set h B)"
apply (rule project_ensures_D_lemma [of _ UNIV, THEN revcut_rl], auto)
done

lemma (in Extend) project_Ensures_D:
  "[| F ⊔ project h (reachable (extend h F ⊔ G)) G ∈ A Ensures B;
    G ∈ stable (reachable (extend h F ⊔ G) ∩ extend_set h A -
      extend_set h B) |]"
  ==> extend h F ⊔ G ∈ (extend_set h A) Ensures (extend_set h B)"
apply (unfold Ensures_def)
apply (rule project_ensures_D_lemma [THEN revcut_rl])
apply (auto simp add: project_set_reachable_extend_eq [symmetric])
done

```

25.8 Guarantees

```

lemma (in Extend) project_act.Restrict_subset_project_act:
  "project_act h (Restrict C act) ⊆ project_act h act"
apply (auto simp add: project_act_def)
done

lemma (in Extend) subset_closed_ok_extend_imp_ok_project:
  "[| extend h F ok G; subset_closed (AllowedActs F) |]"
  ==> F ok project h C G"
apply (auto simp add: ok_def)
apply (rename_tac act)
apply (drule subsetD, blast)
apply (rule_tac x = "Restrict C (extend_act h act)" in rev_image_eqI)
apply simp +
apply (cut_tac project_act.Restrict_subset_project_act)
apply (auto simp add: subset_closed_def)
done

```

```

lemma (in Extend) project_guarantees_raw:
  assumes xguary: "F ∈ X guarantees Y"
  and closed: "subset_closed (AllowedActs F)"

```

```

and project: "!!G. extend h F ⊔ G ∈ X'
            ==> F ⊔ project h (C G) G ∈ X"
and extend:  "!!G. [| F ⊔ project h (C G) G ∈ Y |]
            ==> extend h F ⊔ G ∈ Y'"
shows "extend h F ∈ X' guarantees Y'"
apply (rule xguary [THEN guaranteesD, THEN extend, THEN guaranteesI])
apply (blast intro: closed_subset_closed_ok_extend_imp_ok_project)
apply (erule project)
done

```

```

lemma (in Extend) project_guarantees:
  "[| F ∈ X guarantees Y; subset_closed (AllowedActs F);
    projecting C h F X' X; extending C h F Y' Y |]
   ==> extend h F ∈ X' guarantees Y'"
apply (rule guaranteesI)
apply (auto simp add: guaranteesD projecting_def extending_def
  subset_closed_ok_extend_imp_ok_project)
done

```

25.9 guarantees corollaries

25.9.1 Some could be deleted: the required versions are easy to prove

```

lemma (in Extend) extend_guar_increasing:
  "[| F ∈ UNIV guarantees increasing func;
    subset_closed (AllowedActs F) |]
   ==> extend h F ∈ X' guarantees increasing (func o f)"
apply (erule project_guarantees)
apply (rule_tac [3] extending_increasing)
apply (rule_tac [2] projecting_UNIV, auto)
done

```

```

lemma (in Extend) extend_guar_Increasing:
  "[| F ∈ UNIV guarantees Increasing func;
    subset_closed (AllowedActs F) |]
   ==> extend h F ∈ X' guarantees Increasing (func o f)"
apply (erule project_guarantees)
apply (rule_tac [3] extending_Increasing)
apply (rule_tac [2] projecting_UNIV, auto)
done

```

```

lemma (in Extend) extend_guar_Always:
  "[| F ∈ Always A guarantees Always B;
    subset_closed (AllowedActs F) |]
   ==> extend h F
      ∈ Always(extend_set h A) guarantees Always(extend_set h B)"
apply (erule project_guarantees)
apply (rule_tac [3] extending_Always)
apply (rule_tac [2] projecting_Always, auto)
done

```

25.9.2 Guarantees with a leadsTo postcondition

```

lemma (in Extend) project_leadsTo_D:

```

```

      "F⊔project h UNIV G ∈ A leadsTo B
      ==> extend h F⊔G ∈ (extend_set h A) leadsTo (extend_set h B)"
    apply (rule_tac C1 = UNIV in project_leadsTo_D_lemma [THEN leadsTo_weaken],
    auto)
  done

lemma (in Extend) project_LeadsTo_D:
  "F⊔project h (reachable (extend h F⊔G)) G ∈ A LeadsTo B
  ==> extend h F⊔G ∈ (extend_set h A) LeadsTo (extend_set h B)"
  apply (rule refl [THEN Join_project_LeadsTo], auto)
  done

lemma (in Extend) extending_leadsTo:
  "extending (%G. UNIV) h F
   (extend_set h A leadsTo extend_set h B) (A leadsTo B)"
  apply (unfold extending_def)
  apply (blast intro: project_leadsTo_D)
  done

lemma (in Extend) extending_LeadsTo:
  "extending (%G. reachable (extend h F⊔G)) h F
   (extend_set h A LeadsTo extend_set h B) (A LeadsTo B)"
  apply (unfold extending_def)
  apply (blast intro: project_LeadsTo_D)
  done

end

```

26 Progress Under Allowable Sets

theory *ELT* imports *Project* begin

inductive_set

```

  elt :: "[ 'a set set, 'a program ] => ( 'a set * 'a set ) set"
  for CC :: "'a set set" and F :: "'a program"
  where

    Basis: "[| F : A ensures B; A-B : (insert {} CC) |] ==> (A,B) : elt CC
    F"

    | Trans: "[| (A,B) : elt CC F; (B,C) : elt CC F |] ==> (A,C) : elt CC F"

    | Union: "ALL A: S. (A,B) : elt CC F ==> (Union S, B) : elt CC F"

```

constdefs

```

givenBy :: "[ 'a => 'b ] => 'a set set"
"givenBy f == range (%B. f-` B)"

```



```

leadsETO :: "[ 'a set, 'a set set, 'a set ] => 'a program set"
              ("(3_/ leadsTo[_]/ _)" [80,0,80] 80)
"leadsETO A CC B == {F. (A,B) : elt CC F}"

LeadsETO :: "[ 'a set, 'a set set, 'a set ] => 'a program set"
              ("(3_/ LeadsTo[_]/ _)" [80,0,80] 80)
"LeadsETO A CC B ==
  {F. F : (reachable F Int A) leadsTo[(%C. reachable F Int C) ' CC] B}"

lemma givenBy_id [simp]: "givenBy id = UNIV"
by (unfold givenBy_def, auto)

lemma givenBy_eq_all: "(givenBy v) = {A. ALL x:A. ALL y. v x = v y --> y:
A}"
apply (unfold givenBy_def, safe)
apply (rule_tac [2] x = "v ' ?u" in image_eqI, auto)
done

lemma givenByI: "(!!x y. [| x:A; v x = v y |] ==> y: A) ==> A: givenBy v"
by (subst givenBy_eq_all, blast)

lemma givenByD: "[| A: givenBy v; x:A; v x = v y |] ==> y: A"
by (unfold givenBy_def, auto)

lemma empty_mem_givenBy [iff]: "{} : givenBy v"
by (blast intro!: givenByI)

lemma givenBy_imp_eq_Collect: "A: givenBy v ==> EX P. A = {s. P(v s)}"
apply (rule_tac x = "%n. EX s. v s = n & s : A" in exI)
apply (simp (no_asm_use) add: givenBy_eq_all)
apply blast
done

lemma Collect_mem_givenBy: "{s. P(v s)} : givenBy v"
by (unfold givenBy_def, best)

lemma givenBy_eq_Collect: "givenBy v = {A. EX P. A = {s. P(v s)}}"
by (blast intro: Collect_mem_givenBy givenBy_imp_eq_Collect)

lemma preserves_givenBy_imp_stable:
  "[| F : preserves v; D : givenBy v |] ==> F : stable D"
by (force simp add: preserves_subset_stable [THEN subsetD] givenBy_eq_Collect)

lemma givenBy_o_subset: "givenBy (w o v) <= givenBy v"
apply (simp (no_asm) add: givenBy_eq_Collect)
apply best
done

lemma givenBy_DiffI:

```

```

      "[| A : givenBy v; B : givenBy v |] ==> A-B : givenBy v"
    apply (simp (no_asm_use) add: givenBy_eq_Collect)
    apply safe
    apply (rule_tac x = "%z. ?R z & ~ ?Q z" in exI)
    unfolding set_diff_eq
    apply auto
  done

```

```

lemma leadsETo_Basis [intro]:
  "[| F : A ensures B; A-B : insert {} CC |] ==> F : A leadsTo[CC] B"
  apply (unfold leadsETo_def)
  apply (blast intro: elt.Basis)
done

```

```

lemma leadsETo_Trans:
  "[| F : A leadsTo[CC] B; F : B leadsTo[CC] C |] ==> F : A leadsTo[CC] C"
  apply (unfold leadsETo_def)
  apply (blast intro: elt.Trans)
done

```

```

lemma leadsETo_Un_duplicate:
  "F : A leadsTo[CC] (A' Un A') ==> F : A leadsTo[CC] A'"
  by (simp add: Un_ac)

```

```

lemma leadsETo_Un_duplicate2:
  "F : A leadsTo[CC] (A' Un C Un C) ==> F : A leadsTo[CC] (A' Un C)"
  by (simp add: Un_ac)

```

```

lemma leadsETo_Union:
  "(!!A. A : S ==> F : A leadsTo[CC] B) ==> F : (Union S) leadsTo[CC] B"
  apply (unfold leadsETo_def)
  apply (blast intro: elt.Union)
done

```

```

lemma leadsETo_UN:
  "(!!i. i : I ==> F : (A i) leadsTo[CC] B)
   ==> F : (UN i:I. A i) leadsTo[CC] B"
  apply (subst Union_image_eq [symmetric])
  apply (blast intro: leadsETo_Union)
done

```

```

lemma leadsETo_induct:
  "[| F : za leadsTo[CC] zb;
    !!A B. [| F : A ensures B; A-B : insert {} CC |] ==> P A B;
    !!A B C. [| F : A leadsTo[CC] B; P A B; F : B leadsTo[CC] C; P B C |]

```

```

      ==> P A C;
    !!B S. ALL A:S. F : A leadsTo[CC] B & P A B ==> P (Union S) B
  [] ==> P za zb"
apply (unfold leadsETo_def)
apply (drule CollectD)
apply (erule elt.induct, blast+)
done

lemma leadsETo_mono: "CC' <= CC ==> (A leadsTo[CC'] B) <= (A leadsTo[CC]
B)"
apply safe
apply (erule leadsETo_induct)
prefer 3 apply (blast intro: leadsETo_Union)
prefer 2 apply (blast intro: leadsETo_Trans)
apply (blast intro: leadsETo_Basis)
done

lemma leadsETo_Trans_Un:
  "[| F : A leadsTo[CC] B; F : B leadsTo[DD] C |]
   ==> F : A leadsTo[CC Un DD] C"
by (blast intro: leadsETo_mono [THEN subsetD] leadsETo_Trans)

lemma leadsETo_Union_Int:
  "(!!A. A : S ==> F : (A Int C) leadsTo[CC] B)
   ==> F : (Union S Int C) leadsTo[CC] B"
apply (unfold leadsETo_def)
apply (simp only: Int_Union_Union)
apply (blast intro: elt.Union)
done

lemma leadsETo_Un:
  "[| F : A leadsTo[CC] C; F : B leadsTo[CC] C |]
   ==> F : (A Un B) leadsTo[CC] C"
apply (subst Un_eq_Union)
apply (blast intro: leadsETo_Union)
done

lemma single_leadsETo_I:
  "(!!x. x : A ==> F : {x} leadsTo[CC] B) ==> F : A leadsTo[CC] B"
by (subst UN_singleton [symmetric], rule leadsETo_UN, blast)

lemma subset_imp_leadsETo: "A<=B ==> F : A leadsTo[CC] B"
by (simp add: subset_imp_ensures [THEN leadsETo_Basis]
    Diff_eq_empty_iff [THEN iffD2])

lemmas empty_leadsETo = empty_subsetI [THEN subset_imp_leadsETo, simp]

```

```

lemma leadsETo_weaken_R:
  "[| F : A leadsTo[CC] A'; A' <= B' |] ==> F : A leadsTo[CC] B'"
by (blast intro: subset_imp_leadsETo leadsETo_Trans)

lemma leadsETo_weaken_L [rule_format]:
  "[| F : A leadsTo[CC] A'; B <= A |] ==> F : B leadsTo[CC] A'"
by (blast intro: leadsETo_Trans subset_imp_leadsETo)

lemma leadsETo_Un_distrib:
  "F : (A Un B) leadsTo[CC] C =
    (F : A leadsTo[CC] C & F : B leadsTo[CC] C)"
by (blast intro: leadsETo_Un leadsETo_weaken_L)

lemma leadsETo_UN_distrib:
  "F : (UN i:I. A i) leadsTo[CC] B =
    (ALL i : I. F : (A i) leadsTo[CC] B)"
by (blast intro: leadsETo_UN leadsETo_weaken_L)

lemma leadsETo_Union_distrib:
  "F : (Union S) leadsTo[CC] B = (ALL A : S. F : A leadsTo[CC] B)"
by (blast intro: leadsETo_Union leadsETo_weaken_L)

lemma leadsETo_weaken:
  "[| F : A leadsTo[CC'] A'; B <= A; A' <= B'; CC' <= CC |]
  ==> F : B leadsTo[CC] B'"
apply (drule leadsETo_mono [THEN subsetD], assumption)
apply (blast del: subsetCE
  intro: leadsETo_weaken_R leadsETo_weaken_L leadsETo_Trans)
done

lemma leadsETo_givenBy:
  "[| F : A leadsTo[CC] A'; CC <= givenBy v |]
  ==> F : A leadsTo[givenBy v] A'"
by (blast intro: empty_mem_givenBy leadsETo_weaken)

lemma leadsETo_Diff:
  "[| F : (A-B) leadsTo[CC] C; F : B leadsTo[CC] C |]
  ==> F : A leadsTo[CC] C"
by (blast intro: leadsETo_Un leadsETo_weaken)

lemma leadsETo_Un_Un:
  "[| F : A leadsTo[CC] A'; F : B leadsTo[CC] B' |]
  ==> F : (A Un B) leadsTo[CC] (A' Un B')"
by (blast intro: leadsETo_Un leadsETo_weaken_R)

```

```

lemma leadsETo_cancel2:
  "[| F : A leadsTo[CC] (A' Un B); F : B leadsTo[CC] B' |]
   ==> F : A leadsTo[CC] (A' Un B')"
by (blast intro: leadsETo_Un_Un subset_imp_leadsETo leadsETo_Trans)

lemma leadsETo_cancel1:
  "[| F : A leadsTo[CC] (B Un A'); F : B leadsTo[CC] B' |]
   ==> F : A leadsTo[CC] (B' Un A')"
apply (simp add: Un_commute)
apply (blast intro!: leadsETo_cancel2)
done

lemma leadsETo_cancel_Diff1:
  "[| F : A leadsTo[CC] (B Un A'); F : (B-A') leadsTo[CC] B' |]
   ==> F : A leadsTo[CC] (B' Un A')"
apply (rule leadsETo_cancel1)
  prefer 2 apply assumption
apply simp_all
done

lemma e_psp_stable:
  "[| F : A leadsTo[CC] A'; F : stable B; ALL C:CC. C Int B : CC |]
   ==> F : (A Int B) leadsTo[CC] (A' Int B)"
apply (unfold stable_def)
apply (erule leadsETo_induct)
prefer 3 apply (blast intro: leadsETo_Union_Int)
prefer 2 apply (blast intro: leadsETo_Trans)
apply (rule leadsETo_Basis)
prefer 2 apply (force simp add: Diff_Int_distrib2 [symmetric])
apply (simp add: ensures_def Diff_Int_distrib2 [symmetric]
              Int_Un_distrib2 [symmetric])
apply (blast intro: transient_strengthen constrains_Int)
done

lemma e_psp_stable2:
  "[| F : A leadsTo[CC] A'; F : stable B; ALL C:CC. C Int B : CC |]
   ==> F : (B Int A) leadsTo[CC] (B Int A')"
by (simp (no_asm_simp) add: e_psp_stable Int_ac)

lemma e_psp:
  "[| F : A leadsTo[CC] A'; F : B co B';
     ALL C:CC. C Int B Int B' : CC |]
   ==> F : (A Int B') leadsTo[CC] ((A' Int B) Un (B' - B))"
apply (erule leadsETo_induct)
prefer 3 apply (blast intro: leadsETo_Union_Int)

apply (rule_tac [2] leadsETo_Un_duplicate2)
apply (erule_tac [2] leadsETo_cancel_Diff1)
prefer 2

```

```

    apply (simp add: Int_Diff Diff_triv)
    apply (blast intro: leadsETo_weaken_L dest: constrains_imp_subset)

  apply (rule leadsETo_Basis)
  apply (blast intro: psp_ensures)
  apply (subgoal_tac "A Int B' - (Ba Int B Un (B' - B)) = (A - Ba) Int B Int B'")
  apply auto
  done

lemma e_psp2:
  "[| F : A leadsTo[CC] A'; F : B co B';
    ALL C:CC. C Int B Int B' : CC |]
   ==> F : (B' Int A) leadsTo[CC] ((B Int A') Un (B' - B))"
  by (simp add: e_psp Int_ac)

lemma gen_leadsETo_imp_Join_leadsETo:
  "[| F : (A leadsTo[givenBy v] B); G : preserves v;
    F □ G : stable C |]
   ==> F □ G : ((C Int A) leadsTo[(%D. C Int D) ' givenBy v] B)"
  apply (erule leadsETo_induct)
  prefer 3
  apply (subst Int_Union)
  apply (blast intro: leadsETo_UN)
  prefer 2
  apply (blast intro: e_psp_stable2 [THEN leadsETo_weaken_L] leadsETo_Trans)
  apply (rule leadsETo_Basis)
  apply (auto simp add: Diff_eq_empty_iff [THEN iffD2]
    Int_Diff ensures_def givenBy_eq_Collect Join_transient)
  prefer 3 apply (blast intro: transient_strengthen)
  apply (drule_tac [2] P1 = P in preserves_subset_stable [THEN subsetD])
  apply (drule_tac P1 = P in preserves_subset_stable [THEN subsetD])
  apply (unfold stable_def)
  apply (blast intro: constrains_Int [THEN constrains_weaken])+
  done

lemma leadsETo_subset_leadsTo: "(A leadsTo[CC] B) <= (A leadsTo B)"
  apply safe
  apply (erule leadsETo_induct)
  prefer 3 apply (blast intro: leadsTo_Union)
  prefer 2 apply (blast intro: leadsTo_Trans, blast)
  done

lemma leadsETo_UNIV_eq_leadsTo: "(A leadsTo[UNIV] B) = (A leadsTo B)"
  apply safe
  apply (erule leadsETo_subset_leadsTo [THEN subsetD])

```

```

apply (erule leadsTo_induct)
  prefer 3 apply (blast intro: leadsETO_Union)
  prefer 2 apply (blast intro: leadsETO_Trans, blast)
done

```

```

lemma LeadsETO_eq_leadsETO:
  "A LeadsTo[CC] B =
    {F. F : (reachable F Int A) leadsTo[(%C. reachable F Int C) ' CC]
      (reachable F Int B)}"
apply (unfold LeadsETO_def)
apply (blast dest: e_psp_stable2 intro: leadsETO_weaken)
done

```

```

lemma LeadsETO_Trans:
  "[| F : A LeadsTo[CC] B; F : B LeadsTo[CC] C |]
   ==> F : A LeadsTo[CC] C"
apply (simp add: LeadsETO_eq_leadsETO)
apply (blast intro: leadsETO_Trans)
done

```

```

lemma LeadsETO_Union:
  "(!!A. A : S ==> F : A LeadsTo[CC] B) ==> F : (Union S) LeadsTo[CC] B"
apply (simp add: LeadsETO_def)
apply (subst Int_Union)
apply (blast intro: leadsETO_UN)
done

```

```

lemma LeadsETO_UN:
  "(!!i. i : I ==> F : (A i) LeadsTo[CC] B)
   ==> F : (UN i:I. A i) LeadsTo[CC] B"
apply (simp only: Union_image_eq [symmetric])
apply (blast intro: LeadsETO_Union)
done

```

```

lemma LeadsETO_Un:
  "[| F : A LeadsTo[CC] C; F : B LeadsTo[CC] C |]
   ==> F : (A Un B) LeadsTo[CC] C"
apply (subst Un_eq_Union)
apply (blast intro: LeadsETO_Union)
done

```

```

lemma single_LeadsETO_I:
  "(!!s. s : A ==> F : {s} LeadsTo[CC] B) ==> F : A LeadsTo[CC] B"
by (subst UN_singleton [symmetric], rule LeadsETO_UN, blast)

```

```

lemma subset_imp_LeadsETO:
  "A <= B ==> F : A LeadsTo[CC] B"

```

```

apply (simp (no_asm) add: LeadsETo_def)
apply (blast intro: subset_imp_leadsETo)
done

lemmas empty_LeadsETo = empty_subsetI [THEN subset_imp_LeadsETo, standard]

lemma LeadsETo_weaken_R [rule_format]:
  "[| F : A LeadsTo[CC] A'; A' <= B' |] ==> F : A LeadsTo[CC] B'"
apply (simp (no_asm_use) add: LeadsETo_def)
apply (blast intro: leadsETo_weaken_R)
done

lemma LeadsETo_weaken_L [rule_format]:
  "[| F : A LeadsTo[CC] A'; B <= A |] ==> F : B LeadsTo[CC] A'"
apply (simp (no_asm_use) add: LeadsETo_def)
apply (blast intro: leadsETo_weaken_L)
done

lemma LeadsETo_weaken:
  "[| F : A LeadsTo[CC'] A';
    B <= A; A' <= B'; CC' <= CC |]
  ==> F : B LeadsTo[CC] B'"
apply (simp (no_asm_use) add: LeadsETo_def)
apply (blast intro: leadsETo_weaken)
done

lemma LeadsETo_subset_LeadsTo: "(A LeadsTo[CC] B) <= (A LeadsTo B)"
apply (unfold LeadsETo_def LeadsTo_def)
apply (blast intro: leadsETo_subset_leadsTo [THEN subsetD])
done

lemma reachable_ensures:
  "F : A ensures B ==> F : (reachable F Int A) ensures B"
apply (rule stable_ensures_Int [THEN ensures_weaken_R], auto)
done

lemma lel_lemma:
  "F : A leadsTo B ==> F : (reachable F Int A) leadsTo[Pow(reachable F)]
  B"
apply (erule leadsTo_induct)
  apply (blast intro: reachable_ensures leadsETo_Basis)
  apply (blast dest: e_psp_stable2 intro: leadsETo_Trans leadsETo_weaken_L)
  apply (subst Int_Union)
  apply (blast intro: leadsETo_UN)
done

lemma LeadsETo_UNIV_eq_LeadsTo: "(A LeadsTo[UNIV] B) = (A LeadsTo B)"
apply safe
apply (erule LeadsETo_subset_LeadsTo [THEN subsetD])

apply (unfold LeadsETo_def LeadsTo_def)
apply (blast intro: lel_lemma [THEN leadsETo_weaken])
done

```



```

lemma (in Extend) givenBy_o_eq_extend_set:
  "givenBy (v o f) = extend_set h ' (givenBy v)"
apply (simp add: givenBy_eq_Collect)
apply (rule equalityI, best)
apply blast
done

lemma (in Extend) givenBy_eq_extend_set: "givenBy f = range (extend_set h)"
by (simp add: givenBy_eq_Collect, best)

lemma (in Extend) extend_set_givenBy_I:
  "D : givenBy v ==> extend_set h D : givenBy (v o f)"
apply (simp (no_asm_use) add: givenBy_eq_all, blast)
done

lemma (in Extend) leadsETo_imp_extend_leadsETo:
  "F : A leadsTo[CC] B
   ==> extend h F : (extend_set h A) leadsTo[extend_set h ' CC]
                (extend_set h B)"
apply (erule leadsETo_induct)
  apply (force intro: leadsETo_Basis subset_imp_sures
    simp add: extend_sures extend_set_Diff_distrib [symmetric])
  apply (blast intro: leadsETo_Trans)
apply (simp add: leadsETo_UN extend_set_Union)
done

lemma (in Extend) Join_project_sures_strong:
  "[| project h C G ~: transient (project_set h C Int (A-B)) |
   project_set h C Int (A - B) = {};
   extend h F ⊔ G : stable C;
   F ⊔ project h C G : (project_set h C Int A) ensures B |]
  ==> extend h F ⊔ G : (C Int extend_set h A) ensures (extend_set h B)"
apply (subst Int_extend_set_lemma [symmetric])
apply (rule Join_project_sures)
apply (auto simp add: Int_Diff)
done

lemma (in Extend) pli_lemma:
  "[| extend h F ⊔ G : stable C;
   F ⊔ project h C G
   : project_set h C Int project_set h A leadsTo project_set h B |]"

```

```

==> F⊔project h C G
      : project_set h C Int project_set h A leadsTo
        project_set h C Int project_set h B"
apply (rule psp_stable2 [THEN leadsTo_weaken_L])
apply (auto simp add: project_stable_project_set extend_stable_project_set)
done

```

```

lemma (in Extend) project_leadsETO_I_lemma:
  "[| extend h F⊔G : stable C;
    extend h F⊔G :
      (C Int A) leadsTo[(%D. C Int D)'givenBy f] B |]
  ==> F⊔project h C G
      : (project_set h C Int project_set h (C Int A)) leadsTo (project_set h
B)"
apply (erule leadsETO_induct)
  prefer 3
  apply (simp only: Int_UN_distrib project_set_Union)
  apply (blast intro: leadsTo_UN)
  prefer 2 apply (blast intro: leadsTo_Trans pli_lemma)
apply (simp add: givenBy_eq_extend_set)
apply (rule leadsTo_Basis)
apply (blast intro: ensures_extend_set_imp_project_ensures)
done

```

```

lemma (in Extend) project_leadsETO_I:
  "extend h F⊔G : (extend_set h A) leadsTo[givenBy f] (extend_set h B)
  ==> F⊔project h UNIV G : A leadsTo B"
apply (rule project_leadsETO_I_lemma [THEN leadsTo_weaken], auto)
done

```

```

lemma (in Extend) project_LeadsETO_I:
  "extend h F⊔G : (extend_set h A) LeadsTo[givenBy f] (extend_set h B)

  ==> F⊔project h (reachable (extend h F⊔G)) G
      : A LeadsTo B"
apply (simp (no_asm_use) add: LeadsTo_def LeadsETO_def)
apply (rule project_leadsETO_I_lemma [THEN leadsTo_weaken])
apply (auto simp add: project_set_reachable_extend_eq [symmetric])
done

```

```

lemma (in Extend) projecting_leadsTo:
  "projecting (%G. UNIV) h F
    (extend_set h A leadsTo[givenBy f] extend_set h B)
    (A leadsTo B)"
apply (unfold projecting_def)
apply (force dest: project_leadsETO_I)
done

```

```

lemma (in Extend) projecting_LeadsTo:
  "projecting (%G. reachable (extend h F⊔G)) h F
    (extend_set h A LeadsTo[givenBy f] extend_set h B)
    (A LeadsTo B)"
apply (unfold projecting_def)
apply (force dest: project_LeadsETO_I)

```

done

end

27 Common Declarations for Chandy and Charpentier's Allocator

```
theory AllocBase imports "../UNITY_Main" begin
```

consts

```
NbT      :: nat
Nclients :: nat
```

axioms

```
NbT_pos: "0 < NbT"
```

```
consts tokens      :: "nat list => nat"
```

primrec

```
"tokens [] = 0"
"tokens (x#xs) = x + tokens xs"
```

consts

```
bag_of :: "'a list => 'a multiset"
```

primrec

```
"bag_of [] = {#}"
"bag_of (x#xs) = {#x#} + bag_of xs"
```

```
lemma setsum_fun_mono [rule_format]:
```

```
"!!f :: nat=>nat.
  (ALL i. i<n --> f i <= g i) -->
  setsum f (lessThan n) <= setsum g (lessThan n)"
```

```
apply (induct_tac "n")
```

```
apply (auto simp add: lessThan_Suc)
```

done

```
lemma tokens_mono_prefix [rule_format]:
```

```
"ALL xs. xs <= ys --> tokens xs <= tokens ys"
```

```
apply (induct_tac "ys")
```

```
apply (auto simp add: prefix_Cons)
```

done

```
lemma mono_tokens: "mono tokens"
```

```
apply (unfold mono_def)
```

```
apply (blast intro: tokens_mono_prefix)
```

done

```
lemma bag_of_append [simp]: "bag_of (l@l') = bag_of l + bag_of l'"
```

```

apply (induct_tac "l", simp)
  apply (simp add: add_ac)
done

lemma mono_bag_of: "mono (bag_of :: 'a list => ('a::order) multiset)"
apply (rule monoI)
apply (unfold prefix_def)
apply (erule genPrefix.induct, auto)
apply (simp add: union_le_mono)
apply (erule order_trans)
apply (rule union_upper1)
done

declare setsum_cong [cong]

lemma bag_of_sublist_lemma:
  "( $\sum_{i \in A \text{ Int lessThan } k. \{\# \text{if } i < k \text{ then } f \ i \text{ else } g \ i \# \}}$ ) =
   ( $\sum_{i \in A \text{ Int lessThan } k. \{\# f \ i \# \}$ )"
by (rule setsum_cong, auto)

lemma bag_of_sublist:
  "bag_of (sublist l A) =
   ( $\sum_{i \in A \text{ Int lessThan } (\text{length } l). \{\# l!i \# \}}$ )"
apply (rule_tac xs = l in rev_induct, simp)
apply (simp add: sublist_append Int_insert_right lessThan_Suc nth_append
  bag_of_sublist_lemma add_ac)
done

lemma bag_of_sublist_Un_Int:
  "bag_of (sublist l (A Un B)) + bag_of (sublist l (A Int B)) =
   bag_of (sublist l A) + bag_of (sublist l B)"
apply (subgoal_tac "A Int B Int  $\{..<\text{length } l\}$  =
  (A Int  $\{..<\text{length } l\}$ ) Int (B Int  $\{..<\text{length } l\}$ )")
apply (simp add: bag_of_sublist Int_Un_distrib2 setsum_Un_Int, blast)
done

lemma bag_of_sublist_Un_disjoint:
  "A Int B = {}
   ==> bag_of (sublist l (A Un B)) =
    bag_of (sublist l A) + bag_of (sublist l B)"
by (simp add: bag_of_sublist_Un_Int [symmetric])

lemma bag_of_sublist_UN_disjoint [rule_format]:
  "[| finite I; ALL i:I. ALL j:I. i~j --> A i Int A j = {} |]
   ==> bag_of (sublist l (UNION I A)) =
    ( $\sum_{i \in I. \text{bag\_of (sublist } l \ (A \ i))}$ )"
apply (simp del: UN_simps
  add: UN_simps [symmetric] add: bag_of_sublist)
apply (subst setsum_UN_disjoint, auto)
done

```

end

```
theory Alloc
imports AllocBase "../PPROD"
begin
```

27.1 State definitions. OUTPUT variables are locals

```
record clientState =
  giv :: "nat list"   — client's INPUT history: tokens GRANTED
  ask  :: "nat list"   — client's OUTPUT history: tokens REQUESTED
  rel  :: "nat list"   — client's OUTPUT history: tokens RELEASED

record 'a clientState_d =
  clientState +
  dummy :: 'a          — dummy field for new variables

constdefs
  — DUPLICATED FROM Client.thy, but with "tok" removed
  — Maybe want a special theory section to declare such maps
  non_dummy :: "'a clientState_d => clientState"
    "non_dummy s == (|giv = giv s, ask = ask s, rel = rel s|)"

  — Renaming map to put a Client into the standard form
  client_map :: "'a clientState_d => clientState*'a"
    "client_map == funPair non_dummy dummy"

record allocState =
  allocGiv :: "nat => nat list" — OUTPUT history: source of "giv" for i
  allocAsk :: "nat => nat list" — INPUT: allocator's copy of "ask" for i
  allocRel :: "nat => nat list" — INPUT: allocator's copy of "rel" for i

record 'a allocState_d =
  allocState +
  dummy :: 'a          — dummy field for new variables

record 'a systemState =
  allocState +
  client :: "nat => clientState" — states of all clients
  dummy  :: 'a                  — dummy field for new variables

constdefs

  — * Resource allocation system specification *

  — spec (1)
  system_safety :: "'a systemState program set"
    "system_safety ==
     Always {s. (SUM i: lessThan Nclients. (tokens o giv o sub i o client)s)
      ≤ NbT + (SUM i: lessThan Nclients. (tokens o rel o sub i o client)s)}"
```

```

— spec (2)
system_progress :: "'a systemState program set"
"system_progress == INT i : lessThan Nclients.
    INT h.
        {s. h ≤ (ask o sub i o client)s} LeadsTo
        {s. h pfixLe (giv o sub i o client) s}"

system_spec :: "'a systemState program set"
"system_spec == system_safety Int system_progress"

— * Client specification (required) **

— spec (3)
client_increasing :: "'a clientState_d program set"
"client_increasing ==
    UNIV guarantees Increasing ask Int Increasing rel"

— spec (4)
client_bounded :: "'a clientState_d program set"
"client_bounded ==
    UNIV guarantees Always {s. ALL elt : set (ask s). elt ≤ NbT}"

— spec (5)
client_progress :: "'a clientState_d program set"
"client_progress ==
    Increasing giv guarantees
    (INT h. {s. h ≤ giv s & h pfixGe ask s}
        LeadsTo {s. tokens h ≤ (tokens o rel) s})"

— spec: preserves part
client_preserves :: "'a clientState_d program set"
"client_preserves == preserves giv Int preserves clientState_d.dummy"

— environmental constraints
client_allowed_acts :: "'a clientState_d program set"
"client_allowed_acts ==
    {F. AllowedActs F =
        insert Id (UNION (preserves (funPair rel ask)) Acts)}"

client_spec :: "'a clientState_d program set"
"client_spec == client_increasing Int client_bounded Int client_progress
    Int client_allowed_acts Int client_preserves"

— * Allocator specification (required) *

— spec (6)
alloc_increasing :: "'a allocState_d program set"
"alloc_increasing ==
    UNIV guarantees
    (INT i : lessThan Nclients. Increasing (sub i o allocGiv))"

— spec (7)
alloc_safety :: "'a allocState_d program set"

```

```

"alloc_safety ==
  (INT i : lessThan Nclients. Increasing (sub i o allocRel))
  guarantees
  Always {s. (SUM i: lessThan Nclients. (tokens o sub i o allocGiv)s)
    ≤ NbT + (SUM i: lessThan Nclients. (tokens o sub i o allocRel)s)}"

— spec (8)
alloc_progress :: "'a allocState_d program set"
"alloc_progress ==
  (INT i : lessThan Nclients. Increasing (sub i o allocAsk) Int
    Increasing (sub i o allocRel))
  Int
  Always {s. ALL i<Nclients.
    ALL elt : set ((sub i o allocAsk) s). elt ≤ NbT}
  Int
  (INT i : lessThan Nclients.
    INT h. {s. h ≤ (sub i o allocGiv)s & h pfixGe (sub i o allocAsk)s}
    LeadsTo
    {s. tokens h ≤ (tokens o sub i o allocRel)s})
  guarantees
  (INT i : lessThan Nclients.
    INT h. {s. h ≤ (sub i o allocAsk) s}
    LeadsTo
    {s. h pfixLe (sub i o allocGiv) s})"

— spec: preserves part
alloc_preserves :: "'a allocState_d program set"
"alloc_preserves == preserves allocRel Int preserves allocAsk Int
  preserves allocState_d.dummy"

— environmental constraints
alloc_allowed_acts :: "'a allocState_d program set"
"alloc_allowed_acts ==
  {F. AllowedActs F =
    insert Id (UNION (preserves allocGiv) Acts)}"

alloc_spec :: "'a allocState_d program set"
"alloc_spec == alloc_increasing Int alloc_safety Int alloc_progress Int
  alloc_allowed_acts Int alloc_preserves"

— * Network specification *

— spec (9.1)
network_ask :: "'a systemState program set"
"network_ask == INT i : lessThan Nclients.
  Increasing (ask o sub i o client) guarantees
  ((sub i o allocAsk) Fols (ask o sub i o client))"

— spec (9.2)
network_giv :: "'a systemState program set"
"network_giv == INT i : lessThan Nclients.
  Increasing (sub i o allocGiv)

```

```

    guarantees
    ((giv o sub i o client) Fols (sub i o allocGiv))"

— spec (9.3)
network_rel :: "'a systemState program set"
"network_rel == INT i : lessThan Nclients.
    Increasing (rel o sub i o client)
    guarantees
    ((sub i o allocRel) Fols (rel o sub i o client))"

— spec: preserves part
network_preserves :: "'a systemState program set"
"network_preserves ==
    preserves allocGiv Int
    (INT i : lessThan Nclients. preserves (rel o sub i o client) Int
        preserves (ask o sub i o client))"

— environmental constraints
network_allowed_acts :: "'a systemState program set"
"network_allowed_acts ==
    {F. AllowedActs F =
        insert Id
        (UNION (preserves allocRel Int
            (INT i : lessThan Nclients. preserves (giv o sub i o client)))
            Acts)}"

network_spec :: "'a systemState program set"
"network_spec == network_ask Int network_giv Int
    network_rel Int network_allowed_acts Int
    network_preserves"

— * State mappings *
sysOfAlloc :: "(nat => clientState) * 'a allocState_d => 'a systemState"
"sysOfAlloc == %s. let (cl,xtr) = allocState_d.dummy s
    in (| allocGiv = allocGiv s,
        allocAsk = allocAsk s,
        allocRel = allocRel s,
        client   = cl,
        dummy    = xtr|)"

sysOfClient :: "(nat => clientState) * 'a allocState_d => 'a systemState"
"sysOfClient == %(cl,al). (| allocGiv = allocGiv al,
    allocAsk = allocAsk al,
    allocRel = allocRel al,
    client   = cl,
    systemState.dummy = allocState_d.dummy al|)"

consts
Alloc    :: "'a allocState_d program"
Client   :: "'a clientState_d program"
Network  :: "'a systemState program"
System   :: "'a systemState program"

```


axioms

```

Alloc:  "Alloc   : alloc_spec"
Client: "Client  : client_spec"
Network: "Network : network_spec"

```

defs

```

System_def:
  "System == rename sysOfAlloc Alloc Join Network Join
    (rename sysOfClient
      (plam x: lessThan Nclients. rename client_map Client))"

```

```
declare image_Collect [simp del]
```

```

declare subset_preserves_o [THEN [2] rev_subsetD, intro]
declare subset_preserves_o [THEN [2] rev_subsetD, simp]
declare funPair_o_distrib [simp]
declare Always_INT_distrib [simp]
declare o_apply [simp del]

```

```
lemmas [simp] =
```

```

  rename_image_constrains
  rename_image_stable
  rename_image_increasing
  rename_image_invariant
  rename_image_Constrains
  rename_image_Stable
  rename_image_Increasing
  rename_image_Always
  rename_image_leadsTo
  rename_image_LeadsTo
  rename_preserves
  rename_image_preserves
  lift_image_preserves
  bij_image_INT
  bij_is_inj [THEN image_Int]
  bij_image_Collect_eq

```

```
ML {*
```

```
(*Splits up conjunctions & intersections: like CONJUNCTS in the HOL system*)
```

```

fun list_of_Int th =
  (list_of_Int (th RS conjunct1) @ list_of_Int (th RS conjunct2))
  handle THM _ => (list_of_Int (th RS IntD1) @ list_of_Int (th RS IntD2))
  handle THM _ => (list_of_Int (th RS @{thm INT_D}))
  handle THM _ => (list_of_Int (th RS bspec))
  handle THM _ => [th];

```

```
*)
```

```
lemmas lessThanBspec = lessThan_iff [THEN iffD2, THEN [2] bspec]
```

```

attribute_setup normalized = {*
let
  fun normalized th =
    normalized (th RS spec
      handle THM _ => th RS @{thm lessThanBspec}
      handle THM _ => th RS bspec
      handle THM _ => th RS (@{thm guarantees_INT_right_iff} RS iffD1))
    handle THM _ => th;
in
  Scan.succeed (Thm.rule_attribute (K normalized))
end
*} ""

ML {*
fun record_auto_tac (cs, ss) =
  auto_tac (cs addIs [ext] addWrapper record_split_wrapper,
    ss addsimps [@{thm sysOfAlloc_def}, @{thm sysOfClient_def},
      @{thm client_map_def}, @{thm non_dummy_def}, @{thm funPair_def},
      @{thm o_apply}, @{thm Let_def}])
*}

method_setup record_auto = {*
  Scan.succeed (fn ctxt => SIMPLE_METHOD (record_auto_tac (local_clasimpset_of
    ctxt)))
*} ""

lemma inj_sysOfAlloc [iff]: "inj sysOfAlloc"
  apply (unfold sysOfAlloc_def Let_def)
  apply (rule inj_onI)
  apply record_auto
  done

We need the inverse; also having it simplifies the proof of surjectivity

lemma inv_sysOfAlloc_eq [simp]: "!!s. inv sysOfAlloc s =
  (| allocGiv = allocGiv s,
    allocAsk = allocAsk s,
    allocRel = allocRel s,
    allocState_d.dummy = (client s, dummy s) |)"
  apply (rule inj_sysOfAlloc [THEN inv_f_eq])
  apply record_auto
  done

lemma surj_sysOfAlloc [iff]: "surj sysOfAlloc"
  apply (simp add: surj_iff expand_fun_eq o_apply)
  apply record_auto
  done

lemma bij_sysOfAlloc [iff]: "bij sysOfAlloc"
  apply (blast intro: bijI)
  done

```

27.1.1 bijectivity of sysOfClient

```

lemma inj_sysOfClient [iff]: "inj sysOfClient"
  apply (unfold sysOfClient_def)
  apply (rule inj_onI)
  apply record_auto
  done

lemma inv_sysOfClient_eq [simp]: "!!s. inv sysOfClient s =
  (client s,
   (| allocGiv = allocGiv s,
      allocAsk = allocAsk s,
      allocRel = allocRel s,
      allocState_d.dummy = systemState.dummy s|) )"
  apply (rule inj_sysOfClient [THEN inv_f_eq])
  apply record_auto
  done

lemma surj_sysOfClient [iff]: "surj sysOfClient"
  apply (simp add: surj_iff expand_fun_eq o_apply)
  apply record_auto
  done

lemma bij_sysOfClient [iff]: "bij sysOfClient"
  apply (blast intro: bijI)
  done

```

27.1.2 bijectivity of client_map

```

lemma inj_client_map [iff]: "inj client_map"
  apply (unfold inj_on_def)
  apply record_auto
  done

lemma inv_client_map_eq [simp]: "!!s. inv client_map s =
  (%(x,y).(lgiv = giv x, ask = ask x, rel = rel x,
          clientState_d.dummy = y|)) s"
  apply (rule inj_client_map [THEN inv_f_eq])
  apply record_auto
  done

lemma surj_client_map [iff]: "surj client_map"
  apply (simp add: surj_iff expand_fun_eq o_apply)
  apply record_auto
  done

lemma bij_client_map [iff]: "bij client_map"
  apply (blast intro: bijI)
  done

o-simprules for client_map

lemma fst_o_client_map: "fst o client_map = non_dummy"
  apply (unfold client_map_def)
  apply (rule fst_o_funPair)
  done

```

```
ML {* bind_thms ("fst_o_client_map'", make_o_equivs (thm "fst_o_client_map"))
*}
declare fst_o_client_map' [simp]
```

```
lemma snd_o_client_map: "snd o client_map = clientState_d.dummy"
  apply (unfold client_map_def)
  apply (rule snd_o_funPair)
  done
```

```
ML {* bind_thms ("snd_o_client_map'", make_o_equivs (thm "snd_o_client_map"))
*}
declare snd_o_client_map' [simp]
```

27.2 o-simprules for *sysOfAlloc* [MUST BE AUTOMATED]

```
lemma client_o_sysOfAlloc: "client o sysOfAlloc = fst o allocState_d.dummy"
"
  apply record_auto
  done
```

```
ML {* bind_thms ("client_o_sysOfAlloc'", make_o_equivs (thm "client_o_sysOfAlloc"))
*}
declare client_o_sysOfAlloc' [simp]
```

```
lemma allocGiv_o_sysOfAlloc_eq: "allocGiv o sysOfAlloc = allocGiv"
  apply record_auto
  done
```

```
ML {* bind_thms ("allocGiv_o_sysOfAlloc_eq'", make_o_equivs (thm "allocGiv_o_sysOfAlloc_eq"))
*}
declare allocGiv_o_sysOfAlloc_eq' [simp]
```

```
lemma allocAsk_o_sysOfAlloc_eq: "allocAsk o sysOfAlloc = allocAsk"
  apply record_auto
  done
```

```
ML {* bind_thms ("allocAsk_o_sysOfAlloc_eq'", make_o_equivs (thm "allocAsk_o_sysOfAlloc_eq"))
*}
declare allocAsk_o_sysOfAlloc_eq' [simp]
```

```
lemma allocRel_o_sysOfAlloc_eq: "allocRel o sysOfAlloc = allocRel"
  apply record_auto
  done
```

```
ML {* bind_thms ("allocRel_o_sysOfAlloc_eq'", make_o_equivs (thm "allocRel_o_sysOfAlloc_eq"))
*}
declare allocRel_o_sysOfAlloc_eq' [simp]
```

27.3 o-simprules for *sysOfClient* [MUST BE AUTOMATED]

```
lemma client_o_sysOfClient: "client o sysOfClient = fst"
  apply record_auto
  done
```

```

ML {* bind_thms ("client_o_sysOfClient'", make_o_equivs (thm "client_o_sysOfClient"))
*}
declare client_o_sysOfClient' [simp]

lemma allocGiv_o_sysOfClient_eq: "allocGiv o sysOfClient = allocGiv o snd
"
  apply record_auto
  done

ML {* bind_thms ("allocGiv_o_sysOfClient_eq'", make_o_equivs (thm "allocGiv_o_sysOfClient_eq"))
*}
declare allocGiv_o_sysOfClient_eq' [simp]

lemma allocAsk_o_sysOfClient_eq: "allocAsk o sysOfClient = allocAsk o snd
"
  apply record_auto
  done

ML {* bind_thms ("allocAsk_o_sysOfClient_eq'", make_o_equivs (thm "allocAsk_o_sysOfClient_eq"))
*}
declare allocAsk_o_sysOfClient_eq' [simp]

lemma allocRel_o_sysOfClient_eq: "allocRel o sysOfClient = allocRel o snd
"
  apply record_auto
  done

ML {* bind_thms ("allocRel_o_sysOfClient_eq'", make_o_equivs (thm "allocRel_o_sysOfClient_eq"))
*}
declare allocRel_o_sysOfClient_eq' [simp]

lemma allocGiv_o_inv_sysOfAlloc_eq: "allocGiv o inv sysOfAlloc = allocGiv"
  apply (simp add: o_def)
  done

ML {* bind_thms ("allocGiv_o_inv_sysOfAlloc_eq'", make_o_equivs (thm "allocGiv_o_inv_sysOfAlloc_eq"))
*}
declare allocGiv_o_inv_sysOfAlloc_eq' [simp]

lemma allocAsk_o_inv_sysOfAlloc_eq: "allocAsk o inv sysOfAlloc = allocAsk"
  apply (simp add: o_def)
  done

ML {* bind_thms ("allocAsk_o_inv_sysOfAlloc_eq'", make_o_equivs (thm "allocAsk_o_inv_sysOfAlloc_eq"))
*}
declare allocAsk_o_inv_sysOfAlloc_eq' [simp]

lemma allocRel_o_inv_sysOfAlloc_eq: "allocRel o inv sysOfAlloc = allocRel"
  apply (simp add: o_def)
  done

ML {* bind_thms ("allocRel_o_inv_sysOfAlloc_eq'", make_o_equivs (thm "allocRel_o_inv_sysOfAlloc_eq"))
*}

```

```
declare allocRel_o_inv_sysOfAlloc_eq' [simp]
```

```
lemma rel_inv_client_map_drop_map: "(rel o inv client_map o drop_map i o
inv sysOfClient) =
  rel o sub i o client"
  apply (simp add: o_def drop_map_def)
done
```

```
ML {* bind_thms ("rel_inv_client_map_drop_map'", make_o_equivs (thm "rel_inv_client_map_drop_
*})
```

```
declare rel_inv_client_map_drop_map [simp]
```

```
lemma ask_inv_client_map_drop_map: "(ask o inv client_map o drop_map i o
inv sysOfClient) =
  ask o sub i o client"
  apply (simp add: o_def drop_map_def)
done
```

```
ML {* bind_thms ("ask_inv_client_map_drop_map'", make_o_equivs (thm "ask_inv_client_map_drop_
*})
```

```
declare ask_inv_client_map_drop_map [simp]
```

```
declare finite_lessThan [iff]
```

Client : \downarrow unfolded specification \downarrow

```
lemmas client_spec_simps =
  client_spec_def client_increasing_def client_bounded_def
  client_progress_def client_allowed_acts_def client_preserves_def
  guarantees_Int_right
```

```
ML {*
val [Client_Increasing_ask, Client_Increasing_rel,
  Client_Bounded, Client_Progress, Client_AllowedActs,
  Client_preserves_giv, Client_preserves_dummy] =
  @{thm Client} |> simplify (@{simpset} addsimps @{thms client_spec_simps})
  |> list_of_Int;
```

```
bind_thm ("Client_Increasing_ask", Client_Increasing_ask);
bind_thm ("Client_Increasing_rel", Client_Increasing_rel);
bind_thm ("Client_Bounded", Client_Bounded);
bind_thm ("Client_Progress", Client_Progress);
bind_thm ("Client_AllowedActs", Client_AllowedActs);
bind_thm ("Client_preserves_giv", Client_preserves_giv);
bind_thm ("Client_preserves_dummy", Client_preserves_dummy);
*}
```

```
declare
  Client_Increasing_ask [iff]
  Client_Increasing_rel [iff]
  Client_Bounded [iff]
  Client_preserves_giv [iff]
  Client_preserves_dummy [iff]
```

Network : \downarrow unfolded specification \downarrow

```

lemmas network_spec_simps =
  network_spec_def network_ask_def network_giv_def
  network_rel_def network_allowed_acts_def network_preserves_def
  ball_conj_distrib

ML {*
val [Network_Ask, Network_Giv, Network_Rel, Network_AllowedActs,
    Network_preserves_allocGiv, Network_preserves_rel,
    Network_preserves_ask] =
    @{thm Network} |> simplify (@{simpset} addsimps @{thms network_spec_simps})
    |> list_of_Int;

bind_thm ("Network_Ask", Network_Ask);
bind_thm ("Network_Giv", Network_Giv);
bind_thm ("Network_Rel", Network_Rel);
bind_thm ("Network_AllowedActs", Network_AllowedActs);
bind_thm ("Network_preserves_allocGiv", Network_preserves_allocGiv);
bind_thm ("Network_preserves_rel", Network_preserves_rel);
bind_thm ("Network_preserves_ask", Network_preserves_ask);
*}

declare Network_preserves_allocGiv [iff]

declare
  Network_preserves_rel [simp]
  Network_preserves_ask [simp]

declare
  Network_preserves_rel [simplified o_def, simp]
  Network_preserves_ask [simplified o_def, simp]

Alloc :  $\downarrow$ unfolded specification $\downarrow$ 

lemmas alloc_spec_simps =
  alloc_spec_def alloc_increasing_def alloc_safety_def
  alloc_progress_def alloc_allowed_acts_def alloc_preserves_def

ML {*
val [Alloc_Increasing_0, Alloc_Safety, Alloc_Progress, Alloc_AllowedActs,
    Alloc_preserves_allocRel, Alloc_preserves_allocAsk,
    Alloc_preserves_dummy] =
    @{thm Alloc} |> simplify (@{simpset} addsimps @{thms alloc_spec_simps})
    |> list_of_Int;

bind_thm ("Alloc_Increasing_0", Alloc_Increasing_0);
bind_thm ("Alloc_Safety", Alloc_Safety);
bind_thm ("Alloc_Progress", Alloc_Progress);
bind_thm ("Alloc_AllowedActs", Alloc_AllowedActs);
bind_thm ("Alloc_preserves_allocRel", Alloc_preserves_allocRel);
bind_thm ("Alloc_preserves_allocAsk", Alloc_preserves_allocAsk);
bind_thm ("Alloc_preserves_dummy", Alloc_preserves_dummy);
*}

```

Strip off the INT in the guarantees postcondition

```
lemmas Alloc_Increasing = Alloc_Increasing_0 [normalized]
```

```
declare
```

```
  Alloc_preserves_allocRel [iff]
  Alloc_preserves_allocAsk [iff]
  Alloc_preserves_dummy [iff]
```

27.4 Components Lemmas [MUST BE AUTOMATED]

```
lemma Network_component_System: "Network Join
  ((rename sysOfClient
    (plam x: (lessThan Nclients). rename client_map Client)) Join
    rename sysOfAlloc Alloc)
  = System"
by (simp add: System_def Join_ac)
```

```
lemma Client_component_System: "(rename sysOfClient
  (plam x: (lessThan Nclients). rename client_map Client)) Join
  (Network Join rename sysOfAlloc Alloc) = System"
by (simp add: System_def Join_ac)
```

```
lemma Alloc_component_System: "rename sysOfAlloc Alloc Join
  ((rename sysOfClient (plam x: (lessThan Nclients). rename client_map
  Client)) Join
    Network) = System"
by (simp add: System_def Join_ac)
```

```
declare
```

```
  Client_component_System [iff]
  Network_component_System [iff]
  Alloc_component_System [iff]
```

* These preservation laws should be generated automatically *

```
lemma Client_Allowed [simp]: "Allowed Client = preserves rel Int preserves
ask"
by (auto simp add: Allowed_def Client_AllowedActs safety_prop_Acts_iff)
```

```
lemma Network_Allowed [simp]: "Allowed Network =
  preserves allocRel Int
  (INT i: lessThan Nclients. preserves(giv o sub i o client))"
by (auto simp add: Allowed_def Network_AllowedActs safety_prop_Acts_iff)
```

```
lemma Alloc_Allowed [simp]: "Allowed Alloc = preserves allocGiv"
by (auto simp add: Allowed_def Alloc_AllowedActs safety_prop_Acts_iff)
```

needed in `rename_client_map_tac`

```
lemma OK_lift_rename_Client [simp]: "OK I (%i. lift i (rename client_map
Client))"
  apply (rule OK_lift_I)
  apply auto
  apply (drule_tac w1 = rel in subset_preserves_o [THEN [2] rev_subsetD])
  apply (drule_tac [2] w1 = ask in subset_preserves_o [THEN [2] rev_subsetD])
  apply (auto simp add: o_def split_def)
  done
```



```

lemma fst_lift_map_eq_fst [simp]: "fst (lift_map i x) i = fst x"
apply (insert fst_o_lift_map [of i])
apply (drule fun_cong [where x=x])
apply (simp add: o_def)
done

```

```

lemma fst_o_lift_map' [simp]:
  "(f o sub i o fst o lift_map i o g) = f o fst o g"
apply (subst fst_o_lift_map [symmetric])
apply (simp only: o_assoc)
done

```

ML

```

{*
fun rename_client_map_tac ss =
  EVERY [
    simp_tac (ss addsimps [@{thm rename_guarantees_eq_rename_inv}]) 1,
    rtac @{thm guarantees_PLam_I} 1,
    assume_tac 2,
    (*preserves: routine reasoning*)
    asm_simp_tac (ss addsimps [@{thm lift_preserves_sub}]) 2,
    (*the guarantee for "lift i (rename client_map Client)" *)
    asm_simp_tac
      (ss addsimps [@{thm lift_guarantees_eq_lift_inv},
        @{thm rename_guarantees_eq_rename_inv},
        @{thm bij_imp_bij_inv}, @{thm surj_rename} RS @{thm surj_range},
        @{thm inv_inv_eq}]) 1,
    asm_simp_tac
      (@{simpset} addsimps [@{thm o_def}, @{thm non_dummy_def}, @{thm guarantees_Int_right}])
  ]
*}

```

```

method_setup rename_client_map = {
  Scan.succeed (fn ctxt =>
    SIMPLE_METHOD (rename_client_map_tac (local_simpset_of ctxt)))
*} ""

```

Lifting *Client_Increasing* to *systemState*

```

lemma rename_Client_Increasing: "i : I
  ==> rename sysOfClient (plam x: I. rename client_map Client) :
    UNIV guarantees
    Increasing (ask o sub i o client) Int
    Increasing (rel o sub i o client)"
by rename_client_map

```

```

lemma preserves_sub_fst_lift_map: "[| F : preserves w; i ~ j |]
  ==> F : preserves (sub i o fst o lift_map j o funPair v w)"
apply (auto simp add: lift_map_def split_def linorder_neq_iff o_def)
apply (drule_tac [!] subset_preserves_o [THEN [2] rev_subsetD])
apply (auto simp add: o_def)
done

```

```

lemma client_preserves_giv_oo_client_map: "[| i < Nclients; j < Nclients
|]
    ==> Client : preserves (giv o sub i o fst o lift_map j o client_map)"
  apply (case_tac "i=j")
  apply (simp, simp add: o_def non_dummy_def)
  apply (drule Client_preserves_dummy [THEN preserves_sub_fst_lift_map])
  apply (drule_tac [!] subset_preserves_o [THEN [2] rev_subsetD])
  apply (simp add: o_def client_map_def)
  done

```

```

lemma rename_sysOfClient_ok_Network:
  "rename sysOfClient (plam x: lessThan Nclients. rename client_map Client)
   ok Network"
  by (auto simp add: ok_iff_Allowed client_preserves_giv_oo_client_map)

```

```

lemma rename_sysOfClient_ok_Alloc:
  "rename sysOfClient (plam x: lessThan Nclients. rename client_map Client)
   ok rename sysOfAlloc Alloc"
  by (simp add: ok_iff_Allowed)

```

```

lemma rename_sysOfAlloc_ok_Network: "rename sysOfAlloc Alloc ok Network"
  by (simp add: ok_iff_Allowed)

```

```

declare
  rename_sysOfClient_ok_Network [iff]
  rename_sysOfClient_ok_Alloc [iff]
  rename_sysOfAlloc_ok_Network [iff]

```

The "ok" laws, re-oriented. But not sure this works: theorem *ok_commute* is needed below

```

declare
  rename_sysOfClient_ok_Network [THEN ok_sym, iff]
  rename_sysOfClient_ok_Alloc [THEN ok_sym, iff]
  rename_sysOfAlloc_ok_Network [THEN ok_sym]

```

```

lemma System_Increasing: "i < Nclients
    ==> System : Increasing (ask o sub i o client) Int
    Increasing (rel o sub i o client)"
  apply (rule component_guaranteesD [OF rename_Client_Increasing Client_component_System])
  apply auto
  done

```

```

lemmas rename_guarantees_sysOfAlloc_I =
  bij_sysOfAlloc [THEN rename_rename_guarantees_eq, THEN iffD2, standard]

```

```

lemmas rename_Alloc_Increasing =
  Alloc_Increasing
  [THEN rename_guarantees_sysOfAlloc_I,
   simplified surj_rename [THEN surj_range] o_def sub_apply
   rename_image_Increasing bij_sysOfAlloc
   allocGiv_o_inv_sysOfAlloc_eq']

```

```

lemma System_Increasing_allocGiv:
  "i < Nclients ==> System : Increasing (sub i o allocGiv)"
  apply (unfold System_def)
  apply (simp add: o_def)
  apply (rule rename_Alloc_Increasing [THEN guarantees_Join_I1, THEN guaranteesD])
  apply auto
  done

```

```

ML {*
  bind_thms ("System_Increasing'", list_of_Int (thm "System_Increasing"))
  *}

```

```

declare System_Increasing' [intro!]

```

Follows consequences. The "Always (INT ...) formulation expresses the general safety property and allows it to be combined using *Always_Int_rule* below.

```

lemma System_Follows_rel:
  "i < Nclients ==> System : ((sub i o allocRel) Fols (rel o sub i o client))"
  apply (auto intro!: Network_Rel [THEN component_guaranteesD])
  apply (simp add: ok_commute [of Network])
  done

```

```

lemma System_Follows_ask:
  "i < Nclients ==> System : ((sub i o allocAsk) Fols (ask o sub i o client))"
  apply (auto intro!: Network_Ask [THEN component_guaranteesD])
  apply (simp add: ok_commute [of Network])
  done

```

```

lemma System_Follows_allocGiv:
  "i < Nclients ==> System : (giv o sub i o client) Fols (sub i o allocGiv)"
  apply (auto intro!: Network_Giv [THEN component_guaranteesD]
    rename_Alloc_Increasing [THEN component_guaranteesD])
  apply (simp_all add: o_def non_dummy_def ok_commute [of Network])
  apply (auto intro!: rename_Alloc_Increasing [THEN component_guaranteesD])
  done

```

```

lemma Always_giv_le_allocGiv: "System : Always (INT i: lessThan Nclients.
  {s. (giv o sub i o client) s ≤ (sub i o allocGiv) s})"
  apply auto
  apply (erule System_Follows_allocGiv [THEN Follows_Bounded])
  done

```

```

lemma Always_allocAsk_le_ask: "System : Always (INT i: lessThan Nclients.
  {s. (sub i o allocAsk) s ≤ (ask o sub i o client) s})"
  apply auto
  apply (erule System_Follows_ask [THEN Follows_Bounded])
  done

```

```

lemma Always_allocRel_le_rel: "System : Always (INT i: lessThan Nclients.

```

```

      {s. (sub i o allocRel) s ≤ (rel o sub i o client) s})"
  by (auto intro!: Follows_Bounded System_Follows_rel)

```

27.5 Proof of the safety property (1)

safety (1), step 1 is *System_Follows_rel*

safety (1), step 2

```

lemmas System_Increasing_allocRel = System_Follows_rel [THEN Follows_Increasing1,
  standard]

```

safety (1), step 3

```

lemma System_sum_bounded:
  "System : Always {s. (∑ i ∈ lessThan Nclients. (tokens o sub i o allocGiv)
  s)
    ≤ NbT + (∑ i ∈ lessThan Nclients. (tokens o sub i o allocRel)
  s)})"
  apply (simp add: o_apply)
  apply (insert Alloc_Safety [THEN rename_guarantees_sysOfAlloc_I])
  apply (simp add: o_def)
  apply (erule component_guaranteesD)
  apply (auto simp add: System_Increasing_allocRel [simplified sub_apply o_def])
  done

```

Follows reasoning

```

lemma Always_tokens_giv_le_allocGiv: "System : Always (INT i: lessThan Nclients.
  {s. (tokens o giv o sub i o client) s
    ≤ (tokens o sub i o allocGiv) s})"
  apply (rule Always_giv_le_allocGiv [THEN Always_weaken])
  apply (auto intro: tokens_mono_prefix simp add: o_apply)
  done

```

```

lemma Always_tokens_allocRel_le_rel: "System : Always (INT i: lessThan Nclients.
  {s. (tokens o sub i o allocRel) s
    ≤ (tokens o rel o sub i o client) s})"
  apply (rule Always_allocRel_le_rel [THEN Always_weaken])
  apply (auto intro: tokens_mono_prefix simp add: o_apply)
  done

```

safety (1), step 4 (final result!)

```

theorem System_safety: "System : system_safety"
  apply (unfold system_safety_def)
  apply (tactic {* rtac (Always_Int_rule [thm "System_sum_bounded",
    thm "Always_tokens_giv_le_allocGiv", thm "Always_tokens_allocRel_le_rel"]
  RS
    thm "Always_weaken") 1 *}})
  apply auto
  apply (rule setsum_fun_mono [THEN order_trans])
  apply (drule_tac [2] order_trans)
  apply (rule_tac [2] add_le_mono [OF order_refl setsum_fun_mono])
  prefer 3 apply assumption
  apply auto
  done

```

27.6 Proof of the progress property (2)

progress (2), step 1 is *System_Follows_ask* and *System_Follows_rel*

progress (2), step 2; see also *System_Increasing_allocRel*

lemmas *System_Increasing_allocAsk* = *System_Follows_ask* [THEN *Follows_Increasing1*, standard]

progress (2), step 3: lifting *Client_Bounded* to *systemState*

```
lemma rename_Client_Bounded: "i : I
  ==> rename sysOfClient (plam x: I. rename client_map Client) :
    UNIV guarantees
    Always {s. ALL elt : set ((ask o sub i o client) s). elt ≤ NbT}"
by rename_client_map

lemma System_Bounded_ask: "i < Nclients
  ==> System : Always
    {s. ALL elt : set ((ask o sub i o client) s). elt ≤ NbT}"
apply (rule component_guaranteesD [OF rename_Client_Bounded Client_component_System])
apply auto
done
```

```
lemma Collect_all_imp_eq: "{x. ALL y. P y --> Q x y} = (INT y: {y. P y}.
{x. Q x y})"
apply blast
done
```

progress (2), step 4

```
lemma System_Bounded_allocAsk: "System : Always {s. ALL i<Nclients.
  ALL elt : set ((sub i o allocAsk) s). elt ≤ NbT}"
apply (auto simp add: Collect_all_imp_eq)
apply (tactic {* rtac (Always_Int_rule [thm "Always_allocAsk_le_ask",
  thm "System_Bounded_ask"] RS thm "Always_weaken") 1 *} )
apply (auto dest: set_mono)
done
```

progress (2), step 5 is *System_Increasing_allocGiv*

progress (2), step 6

lemmas *System_Increasing_giv* = *System_Follows_allocGiv* [THEN *Follows_Increasing1*, standard]

```
lemma rename_Client_Progress: "i: I
  ==> rename sysOfClient (plam x: I. rename client_map Client)
    : Increasing (giv o sub i o client)
    guarantees
    (INT h. {s. h ≤ (giv o sub i o client) s &
      h prefixGe (ask o sub i o client) s}
      LeadsTo {s. tokens h ≤ (tokens o rel o sub i o client) s})"
apply rename_client_map
apply (simp add: Client_Progress [simplified o_def])
```

```

done

progress (2), step 7

lemma System_Client_Progress:
  "System : (INT i : (lessThan Nclients).
    INT h. {s. h ≤ (giv o sub i o client) s &
      h pfixGe (ask o sub i o client) s}
    LeadsTo {s. tokens h ≤ (tokens o rel o sub i o client) s})"
  apply (rule INT_I)

  apply (rule component_guaranteesD [OF rename_Client_Progress Client_component_System])
  apply (auto simp add: System_Increasing_giv)
done

lemmas System_lemma1 =
  Always_LeadsToD [OF System_Follows_ask [THEN Follows_Bounded]
    System_Follows_allocGiv [THEN Follows_LeadsTo]]

lemmas System_lemma2 =
  PSP_Stable [OF System_lemma1
    System_Follows_ask [THEN Follows_Increasing1, THEN IncreasingD]]

lemma System_lemma3: "i < Nclients
  ==> System : {s. h ≤ (sub i o allocGiv) s &
    h pfixGe (sub i o allocAsk) s}
    LeadsTo
    {s. h ≤ (giv o sub i o client) s &
      h pfixGe (ask o sub i o client) s}"
  apply (rule single_LeadsTo_I)
  apply (rule_tac k6 = "h" and x2 = " (sub i o allocAsk) s"
    in System_lemma2 [THEN LeadsTo_weaken])
  apply auto
  apply (blast intro: trans_Ge [THEN trans_genPrefix, THEN transD] prefix_imp_pfixGe)
done

progress (2), step 8: Client i's "release" action is visible system-wide

lemma System_Alloc_Client_Progress: "i < Nclients
  ==> System : {s. h ≤ (sub i o allocGiv) s &
    h pfixGe (sub i o allocAsk) s}
    LeadsTo {s. tokens h ≤ (tokens o sub i o allocRel) s}"
  apply (rule LeadsTo_Trans)
  prefer 2
  apply (drule System_Follows_rel [THEN
    mono_tokens [THEN mono_Follows_o, THEN [2] rev_subsetD],
    THEN Follows_LeadsTo])
  apply (simp add: o_assoc)
  apply (rule LeadsTo_Trans)
  apply (cut_tac [2] System_Client_Progress)
  prefer 2
  apply (blast intro: LeadsTo_Basis)
  apply (erule System_lemma3)

```

done

Lifting *Alloc_Progress* up to the level of *systemState*

progress (2), step 9

```

lemma System_Alloc_Progress:
  "System : (INT i : (lessThan Nclients).
    INT h. {s. h ≤ (sub i o allocAsk) s}
    LeadsTo {s. h pfixLe (sub i o allocGiv) s})"
  apply (simp only: o_apply sub_def)
  apply (insert Alloc_Progress [THEN rename_guarantees_sysOfAlloc_I])
  apply (simp add: o_def del: Set.INT_iff)
  apply (erule component_guaranteesD)
  apply (auto simp add:
    System_Increasing_allocRel [simplified sub_apply o_def]
    System_Increasing_allocAsk [simplified sub_apply o_def]
    System_Bounded_allocAsk [simplified sub_apply o_def]
    System_Alloc_Client_Progress [simplified sub_apply o_def])
  done

```

progress (2), step 10 (final result!)

```

lemma System_Progress: "System : system_progress"
  apply (unfold system_progress_def)
  apply (cut_tac System_Alloc_Progress)
  apply (blast intro: LeadsTo_Trans
    System_Follows_allocGiv [THEN Follows_LeadsTo_pfixLe]
    System_Follows_ask [THEN Follows_LeadsTo])
  done

```

```

theorem System_correct: "System : system_spec"
  apply (unfold system_spec_def)
  apply (blast intro: System_safety System_Progress)
  done

```

Some obsolete lemmas

```

lemma non_dummy_eq_o_funPair: "non_dummy = (% (g,a,r). (! giv = g, ask =
a, rel = r |)) o

```

```

      (funPair giv (funPair ask rel))"

```

```

  apply (rule ext)
  apply (auto simp add: o_def non_dummy_def)
  done

```

```

lemma preserves_non_dummy_eq: "(preserves non_dummy) =
  (preserves rel Int preserves ask Int preserves giv)"
  apply (simp add: non_dummy_eq_o_funPair)
  apply auto
  apply (drule_tac w1 = rel in subset_preserves_o [THEN [2] rev_subsetD])
  apply (drule_tac [2] w1 = ask in subset_preserves_o [THEN [2] rev_subsetD])
  apply (drule_tac [3] w1 = giv in subset_preserves_o [THEN [2] rev_subsetD])
  apply (auto simp add: o_def)
  done

```

Could go to Extend.ML

```

lemma bij_fst_inv_inv_eq: "bij f ==> fst (inv (%(x, u). inv f x) z) = f z"
  apply (rule fst_inv_equalityI)
  apply (rule_tac f = "%z. (f z, ?h z) " in surjI)
  apply (simp add: bij_is_inj inv_f_f)
  apply (simp add: bij_is_surj surj_f_inv_f)
  done

end

```

28 Implementation of a multiple-client allocator from a single-client allocator

```

theory AllocImpl imports AllocBase Follows PPROD begin

```

```

record 'b merge =
  In   :: "nat => 'b list"
  Out  :: "'b list"
  iOut :: "nat list"

record ('a, 'b) merge_d =
  "'b merge" +
  dummy  :: 'a

constdefs
  non_dummy :: "('a, 'b) merge_d => 'b merge"
    "non_dummy s == (|In = In s, Out = Out s, iOut = iOut s|)"

record 'b distr =
  In   :: "'b list"
  iIn  :: "nat list"
  Out  :: "nat => 'b list"

record ('a, 'b) distr_d =
  "'b distr" +
  dummy  :: 'a

record allocState =
  giv :: "nat list"
  ask  :: "nat list"
  rel  :: "nat list"

record 'a allocState_d =
  allocState +
  dummy      :: 'a

record 'a systemState =
  allocState +
  mergeRel  :: "nat merge"

```



```

mergeAsk :: "nat merge"
distr    :: "nat distr"
dummy    :: 'a

```

constdefs

```

merge_increasing :: "('a,'b) merge_d program set"
"merge_increasing ==
  UNIV guarantees (Increasing merge.Out) Int (Increasing merge.iOut)"

merge_eqOut :: "('a,'b) merge_d program set"
"merge_eqOut ==
  UNIV guarantees
    Always {s. length (merge.Out s) = length (merge.iOut s)}"

merge_bounded :: "('a,'b) merge_d program set"
"merge_bounded ==
  UNIV guarantees
    Always {s.  $\forall \text{elt} \in \text{set } (\text{merge.iOut } s). \text{elt} < \text{Nclients}$ }"

merge_follows :: "('a,'b) merge_d program set"
"merge_follows ==
  ( $\bigcap i \in \text{lessThan Nclients. Increasing (sub } i \text{ o merge.In)}$ )
  guarantees
  ( $\bigcap i \in \text{lessThan Nclients.}$ 
    ( $\%s. \text{sublist (merge.Out } s) \{k. k < \text{size(merge.iOut } s) \ \& \ \text{merge.iOut } s! k = i\}$ )
    Fols (sub i o merge.In))"

merge_preserves :: "('a,'b) merge_d program set"
"merge_preserves == preserves merge.In Int preserves merge_d.dummy"

merge_allowed_acts :: "('a,'b) merge_d program set"
"merge_allowed_acts ==
  {F. AllowedActs F =
    insert Id (UNION (preserves (funPair merge.Out merge.iOut)) Acts)}"

merge_spec :: "('a,'b) merge_d program set"
"merge_spec == merge_increasing Int merge_eqOut Int merge_bounded Int
  merge_follows Int merge_allowed_acts Int merge_preserves"

distr_follows :: "('a,'b) distr_d program set"

```

```

"distr_follows ==
  Increasing distr.In Int Increasing distr.iIn Int
  Always {s.  $\forall \text{elt} \in \text{set} (\text{distr.iIn } s). \text{elt} < \text{Nclients}$ }
  guarantees
  ( $\bigcap i \in \text{lessThan } \text{Nclients}.$ 
    (sub i o distr.Out) Fols
    (%s. sublist (distr.In s)
      {k. k < size(distr.iIn s) & distr.iIn s ! k = i}))"

distr_allowed_acts :: "('a,'b) distr_d program set"
"distr_allowed_acts ==
  {D. AllowedActs D = insert Id (UNION (preserves distr.Out) Acts)}"

distr_spec :: "('a,'b) distr_d program set"
"distr_spec == distr_follows Int distr_allowed_acts"

alloc_increasing :: "'a allocState_d program set"
"alloc_increasing == UNIV guarantees Increasing giv"

alloc_safety :: "'a allocState_d program set"
"alloc_safety ==
  Increasing rel
  guarantees Always {s. tokens (giv s)  $\leq$  NbT + tokens (rel s)}"

alloc_progress :: "'a allocState_d program set"
"alloc_progress ==
  Increasing ask Int Increasing rel Int
  Always {s.  $\forall \text{elt} \in \text{set} (\text{ask } s). \text{elt} \leq \text{NbT}$ }
  Int
  ( $\bigcap h. \{s. h \leq \text{giv } s \ \& \ h \text{ pfixGe } (\text{ask } s)\}$ 
    LeadsTo
    {s. tokens h  $\leq$  tokens (rel s)})
  guarantees ( $\bigcap h. \{s. h \leq \text{ask } s\}$  LeadsTo {s. h pfixLe giv s})"

alloc_preserves :: "'a allocState_d program set"
"alloc_preserves == preserves rel Int
  preserves ask Int
  preserves allocState_d.dummy"

alloc_allowed_acts :: "'a allocState_d program set"
"alloc_allowed_acts ==
  {F. AllowedActs F = insert Id (UNION (preserves giv) Acts)}"

alloc_spec :: "'a allocState_d program set"
"alloc_spec == alloc_increasing Int alloc_safety Int alloc_progress Int
  alloc_allowed_acts Int alloc_preserves"

```

```

locale Merge =
  fixes M :: "('a,'b::order) merge_d program"
  assumes
    Merge_spec: "M ∈ merge_spec"

locale Distrib =
  fixes D :: "('a,'b::order) distr_d program"
  assumes
    Distrib_spec: "D ∈ distr_spec"

```

```

declare subset_preserves_o [THEN subsetD, intro]
declare funPair_o_distrib [simp]
declare Always_INT_distrib [simp]
declare o_apply [simp del]

```

28.1 Theorems for Merge

```

lemma (in Merge) Merge_Allowed:
  "Allowed M = (preserves merge.Out) Int (preserves merge.iOut)"
apply (cut_tac Merge_spec)
apply (auto simp add: merge_spec_def merge_allowed_acts_def Allowed_def
  safety_prop_Acts_iff)
done

```

```

lemma (in Merge) M_ok_iff [iff]:
  "M ok G = (G ∈ preserves merge.Out & G ∈ preserves merge.iOut &
    M ∈ Allowed G)"
by (auto simp add: Merge_Allowed ok_iff_Allowed)

```

```

lemma (in Merge) Merge_Always_Out_eq_iOut:
  "[| G ∈ preserves merge.Out; G ∈ preserves merge.iOut; M ∈ Allowed G
  |]
  ==> M Join G ∈ Always {s. length (merge.Out s) = length (merge.iOut
  s)}"
apply (cut_tac Merge_spec)
apply (force dest: guaranteesD simp add: merge_spec_def merge_eqOut_def)
done

```

```

lemma (in Merge) Merge_Bounded:
  "[| G ∈ preserves merge.iOut; G ∈ preserves merge.Out; M ∈ Allowed G
  |]
  ==> M Join G ∈ Always {s. ∀elt ∈ set (merge.iOut s). elt < Nclients}"
apply (cut_tac Merge_spec)
apply (force dest: guaranteesD simp add: merge_spec_def merge_bounded_def)
done

```

```

lemma (in Merge) Merge_Bag_Follows_lemma:
  "[| G ∈ preserves merge.iOut; G ∈ preserves merge.Out; M ∈ Allowed G

```

```

/]
==> M Join G ∈ Always
      {s. (∑ i ∈ lessThan Nclients. bag_of (sublist (merge.Out s)
                                                    {k. k < length (iOut s) & iOut s ! k = i}))
=
      (bag_of o merge.Out) s}"
apply (rule Always_Compl_Un_eq [THEN iffD1])
apply (blast intro: Always_Int_I [OF Merge_Always_Out_eq_iOut Merge_Bounded])
apply (rule UNIV_AlwaysI, clarify)
apply (subst bag_of_sublist_UN_disjoint [symmetric])
  apply (simp)
  apply blast
apply (simp add: set_conv_nth)
apply (subgoal_tac
      "(∪ i ∈ lessThan Nclients. {k. k < length (iOut x) & iOut x ! k = i})
=
      lessThan (length (iOut x))")
  apply (simp (no_asm_simp) add: o_def)
  apply blast
done

lemma (in Merge) Merge_Bag_Follows:
  "M ∈ (∩ i ∈ lessThan Nclients. Increasing (sub i o merge.In))
    guarantees
      (bag_of o merge.Out) Fols
      (%s. ∑ i ∈ lessThan Nclients. (bag_of o sub i o merge.In) s)"
apply (rule Merge_Bag_Follows_lemma [THEN Always_Follows1, THEN guaranteesI],
auto)
apply (rule Follows_setsum)
apply (cut_tac Merge_spec)
apply (auto simp add: merge_spec_def merge_follows_def o_def)
apply (drule guaranteesD)
  prefer 3
  apply (best intro: mono_bag_of [THEN mono_Follows_apply, THEN subsetD],
auto)
done

```

28.2 Theorems for Distributor

```

lemma (in Distrib) Distr_Increasing_Out:
  "D ∈ Increasing distr.In Int Increasing distr.iIn Int
    Always {s. ∀ elt ∈ set (distr.iIn s). elt < Nclients}
    guarantees
      (∩ i ∈ lessThan Nclients. Increasing (sub i o distr.Out))"
apply (cut_tac Distrib_spec)
apply (simp add: distr_spec_def distr_follows_def)
apply clarify
apply (blast intro: guaranteesI Follows_Increasing1 dest: guaranteesD)
done

lemma (in Distrib) Distr_Bag_Follows_lemma:
  "[| G ∈ preserves distr.Out;
    D Join G ∈ Always {s. ∀ elt ∈ set (distr.iIn s). elt < Nclients}
  |]

```

```

==> D Join G ∈ Always
      {s. (∑ i ∈ lessThan Nclients. bag_of (sublist (distr.In s)
                                                    {k. k < length (iIn s) & iIn s ! k = i}))}
=
      bag_of (sublist (distr.In s) (lessThan (length (iIn s))))}
apply (erule Always_Compl_Un_eq [THEN iffD1])
apply (rule UNIV_AlwaysI, clarify)
apply (subst bag_of_sublist_UN_disjoint [symmetric])
  apply (simp (no_asm))
  apply blast
apply (simp add: set_conv_nth)
apply (subgoal_tac
      "(∪ i ∈ lessThan Nclients. {k. k < length (iIn x) & iIn x ! k = i})
=
      lessThan (length (iIn x))")
  apply (simp (no_asm_simp))
  apply blast
done

lemma (in Distrib) D_ok_iff [iff]:
  "D ok G = (G ∈ preserves distr.Out & D ∈ Allowed G)"
apply (cut_tac Distrib_spec)
apply (auto simp add: distr_spec_def distr_allowed_acts_def Allowed_def
      safety_prop_Acts_iff ok_iff_Allowed)
done

lemma (in Distrib) Distr_Bag_Follows:
  "D ∈ Increasing distr.In Int Increasing distr.iIn Int
  Always {s. ∀ elt ∈ set (distr.iIn s). elt < Nclients}
  guarantees
  (∩ i ∈ lessThan Nclients.
    (%s. ∑ i ∈ lessThan Nclients. (bag_of o sub i o distr.Out) s)
    Fols
    (%s. bag_of (sublist (distr.In s) (lessThan (length(distr.iIn s)))))")
apply (rule guaranteesI, clarify)
apply (rule Distr_Bag_Follows_lemma [THEN Always_Follows2], auto)
apply (rule Follows_setsum)
apply (cut_tac Distrib_spec)
apply (auto simp add: distr_spec_def distr_follows_def o_def)
apply (drule guaranteesD)
  prefer 3
  apply (best intro: mono_bag_of [THEN mono_Follows_apply, THEN subsetD],
  auto)
done

```

28.3 Theorems for Allocator

```

lemma alloc_refinement_lemma:
  "!!f::nat=>nat. (∩ i ∈ lessThan n. {s. f i ≤ g i s})
  ⊆ {s. (SUM x: lessThan n. f x) ≤ (SUM x: lessThan n. g x s)}"
apply (induct_tac "n")
apply (auto simp add: lessThan_Suc)
done

```

```

lemma alloc_refinement:
  "(( $\bigcap i \in \text{lessThan } Nclients. \text{Increasing } (\text{sub } i \text{ o } \text{allocAsk}) \text{ Int}$ 
     $\text{Increasing } (\text{sub } i \text{ o } \text{allocRel}))$ 
    Int
    Always {s.  $\forall i. i < Nclients \rightarrow$ 
      ( $\forall elt \in \text{set } ((\text{sub } i \text{ o } \text{allocAsk}) s). elt \leq NbT$ )})
    Int
    ( $\bigcap i \in \text{lessThan } Nclients.$ 
      ( $\bigcap h. \{s. h \leq (\text{sub } i \text{ o } \text{allocGiv})s \ \& \ h \text{ prefixGe } (\text{sub } i \text{ o } \text{allocAsk})s\}$ 
        LeadsTo {s.  $\text{tokens } h \leq (\text{tokens o } \text{sub } i \text{ o } \text{allocRel})s\}$ )
       $\subseteq$ 
      ( $\bigcap i \in \text{lessThan } Nclients. \text{Increasing } (\text{sub } i \text{ o } \text{allocAsk}) \text{ Int}$ 
         $\text{Increasing } (\text{sub } i \text{ o } \text{allocRel}))$ 
      Int
      Always {s.  $\forall i. i < Nclients \rightarrow$ 
        ( $\forall elt \in \text{set } ((\text{sub } i \text{ o } \text{allocAsk}) s). elt \leq NbT$ )})
      Int
      ( $\bigcap hf. (\bigcap i \in \text{lessThan } Nclients.$ 
        {s.  $hf \ i \leq (\text{sub } i \text{ o } \text{allocGiv})s \ \& \ hf \ i \text{ prefixGe } (\text{sub } i \text{ o } \text{allocAsk})s\}$ 
        LeadsTo {s. ( $\sum i \in \text{lessThan } Nclients. \text{tokens } (hf \ i)$ )  $\leq$ 
          ( $\sum i \in \text{lessThan } Nclients. (\text{tokens o } \text{sub } i \text{ o } \text{allocRel})s$ )})})"
  apply (auto simp add: ball_conj_distrib)
  apply (rename_tac F hf)
  apply (rule LeadsTo_weaken_R [OF Finite_stable_completion alloc_refinement_lemma],
    blast, blast)
  apply (subgoal_tac "F  $\in$  Increasing (tokens o (sub i o allocRel))")
  apply (simp add: Increasing_def o_assoc)
  apply (blast intro: mono_tokens [THEN mono_Increasing_o, THEN subsetD])
done

end

```

29 Distributed Resource Management System: the Client

```
theory Client imports Rename AllocBase begin
```

```
types
```

```
  tokbag = nat      — tokbags could be multisets...or any ordered type?
```

```
record state =
```

```
  giv :: "tokbag list" — input history: tokens granted
  ask :: "tokbag list" — output history: tokens requested
  rel :: "tokbag list" — output history: tokens released
  tok :: tokbag        — current token request
```

```
record 'a state_d =
```

```
  state +
  dummy :: 'a      — new variables
```

constdefs

```

rel_act :: "('a state_d * 'a state_d) set"
"rel_act == {(s,s').
  ∃ nrel. nrel = size (rel s) &
  s' = s (| rel := rel s @ [giv s!nrel] |) &
  nrel < size (giv s) &
  ask s!nrel ≤ giv s!nrel}"

tok_act :: "('a state_d * 'a state_d) set"
"tok_act == {(s,s'). s'=s | s' = s (| tok := Suc (tok s mod NbT) |)}"

ask_act :: "('a state_d * 'a state_d) set"
"ask_act == {(s,s'). s'=s |
  (s' = s (| ask := ask s @ [tok s] |))}"

Client :: "'a state_d program"
"Client ==
  mk_total_program
    ({s. tok s ∈ atMost NbT &
      giv s = [] & ask s = [] & rel s = []},
     {rel_act, tok_act, ask_act},
     ⋃ G ∈ preserves rel Int preserves ask Int preserves tok.
       Acts G)"

non_dummy :: "'a state_d => state"
"non_dummy s == (| giv = giv s, ask = ask s, rel = rel s, tok = tok s |)"

client_map :: "'a state_d => state*'a"
"client_map == funPair non_dummy dummy"

declare Client_def [THEN def_prg_Init, simp]
declare Client_def [THEN def_prg_AllowedActs, simp]
declare rel_act_def [THEN def_act_simp, simp]
declare tok_act_def [THEN def_act_simp, simp]
declare ask_act_def [THEN def_act_simp, simp]

lemma Client_ok_iff [iff]:
  "(Client ok G) =
    (G ∈ preserves rel & G ∈ preserves ask & G ∈ preserves tok &
     Client ∈ Allowed G)"
by (auto simp add: ok_iff_Allowed Client_def [THEN def_total_prg_Allowed])

```

Safety property 1: ask, rel are increasing

```

lemma increasing_ask_rel:
  "Client ∈ UNIV guarantees Increasing ask Int Increasing rel"
apply (auto intro!: increasing_imp_Increasing simp add: guar_def preserves_subset_increasing
[THEN subsetD])
apply (auto simp add: Client_def increasing_def)
apply (safety, auto)+
done

```

```

declare nth_append [simp] append_one_prefix [simp]

```

Safety property 2: the client never requests too many tokens. With no Substitution Axiom, we must prove the two invariants simultaneously.

```

lemma ask_bounded_lemma:
  "Client ok G
  ==> Client Join G ∈
    Always ({s. tok s ≤ NbT} Int
    {s. ∀ elt ∈ set (ask s). elt ≤ NbT})"
apply auto
apply (rule invariantI [THEN stable_Join_Always2], force)
prefer 2
apply (fast elim!: preserves_subset_stable [THEN subsetD] intro!: stable_Int)

apply (simp add: Client_def, safety)
apply (cut_tac m = "tok s" in NbT_pos [THEN mod_less_divisor], auto)
done

```

export version, with no mention of tok in the postcondition, but unfortunately tok must be declared local.

```

lemma ask_bounded:
  "Client ∈ UNIV guarantees Always {s. ∀ elt ∈ set (ask s). elt ≤ NbT}"
apply (rule guaranteesI)
apply (erule ask_bounded_lemma [THEN Always_weaken])
apply (rule Int_lower2)
done

```

**** Towards proving the liveness property ****

```

lemma stable_rel_le_giv: "Client ∈ stable {s. rel s ≤ giv s}"
by (simp add: Client_def, safety, auto)

```

```

lemma Join_Stable_rel_le_giv:
  "[| Client Join G ∈ Increasing giv; G ∈ preserves rel |]
  ==> Client Join G ∈ Stable {s. rel s ≤ giv s}"
by (rule stable_rel_le_giv [THEN Increasing_preserves_Stable], auto)

```

```

lemma Join_Always_rel_le_giv:
  "[| Client Join G ∈ Increasing giv; G ∈ preserves rel |]
  ==> Client Join G ∈ Always {s. rel s ≤ giv s}"
by (force intro: AlwaysI Join_Stable_rel_le_giv)

```

```

lemma transient_lemma:
  "Client ∈ transient {s. rel s = k & k < h & h ≤ giv s & h pfixGe ask s}"
apply (simp add: Client_def mk_total_program_def)
apply (rule_tac act = rel_act in totalize_transientI)

```



```

    apply (auto simp add: Domain_def Client_def)
    apply (blast intro: less_le_trans prefix_length_le strict_prefix_length_less)
    apply (auto simp add: prefix_def genPrefix_iff_nth Ge_def)
    apply (blast intro: strict_prefix_length_less)
  done

lemma induct_lemma:
  "[| Client Join G ∈ Increasing giv; Client ok G |]
  ==> Client Join G ∈ {s. rel s = k & k < h & h ≤ giv s & h pfixGe ask s}
    LeadsTo {s. k < rel s & rel s ≤ giv s &
      h ≤ giv s & h pfixGe ask s}"

apply (rule single_LeadsTo_I)
apply (frule increasing_ask_rel [THEN guaranteesD], auto)
apply (rule transient_lemma [THEN Join_transient_I1, THEN transient_imp_leadsTo,
  THEN leadsTo_imp_LeadsTo, THEN PSP_Stable, THEN LeadsTo_weaken])
  apply (rule Stable_Int [THEN Stable_Int, THEN Stable_Int])
    apply (erule_tac f = giv and x = "giv s" in IncreasingD)
    apply (erule_tac f = ask and x = "ask s" in IncreasingD)
    apply (erule_tac f = rel and x = "rel s" in IncreasingD)
    apply (erule Join_Stable_rel_le_giv, blast)
  apply (blast intro: order_less_imp_le order_trans)
apply (blast intro: sym order_less_le [THEN iffD2] order_trans
  prefix_imp_pfixGe pfixGe_trans)
done

lemma rel_progress_lemma:
  "[| Client Join G ∈ Increasing giv; Client ok G |]
  ==> Client Join G ∈ {s. rel s < h & h ≤ giv s & h pfixGe ask s}
    LeadsTo {s. h ≤ rel s}"

apply (rule_tac f = "%s. size h - size (rel s) " in LessThan_induct)
apply (auto simp add: vimage_def)
apply (rule single_LeadsTo_I)
apply (rule induct_lemma [THEN LeadsTo_weaken], auto)
  apply (blast intro: order_less_le [THEN iffD2] dest: common_prefix_linear)
  apply (drule strict_prefix_length_less)+
  apply arith
done

lemma client_progress_lemma:
  "[| Client Join G ∈ Increasing giv; Client ok G |]
  ==> Client Join G ∈ {s. h ≤ giv s & h pfixGe ask s}
    LeadsTo {s. h ≤ rel s}"

apply (rule Join_Always_rel_le_giv [THEN Always_LeadsToI], simp_all)
apply (rule LeadsTo_Un [THEN LeadsTo_weaken_L])
  apply (blast intro: rel_progress_lemma)
  apply (rule subset_refl [THEN subset_imp_LeadsTo])
  apply (blast intro: order_less_le [THEN iffD2] dest: common_prefix_linear)
done

```

Progress property: all tokens that are given will be released

```
lemma client_progress:
```

```

    "Client  $\in$ 
      Increasing giv guarantees
      (INT h. {s. h  $\leq$  giv s & h prefixGe ask s} LeadsTo {s. h  $\leq$  rel s})"
  apply (rule guaranteesI, clarify)
  apply (blast intro: client_progress_lemma)
done

```

This shows that the Client won't alter other variables in any state that it is combined with

```

lemma client_preserves_dummy: "Client  $\in$  preserves dummy"
by (simp add: Client_def preserves_def, clarify, safety, auto)

```

* Obsolete lemmas from first version of the Client *

```

lemma stable_size_rel_le_giv:
  "Client  $\in$  stable {s. size (rel s)  $\leq$  size (giv s)}"
by (simp add: Client_def, safety, auto)

```

clients return the right number of tokens

```

lemma ok_guar_rel_prefix_giv:
  "Client  $\in$  Increasing giv guarantees Always {s. rel s  $\leq$  giv s}"
  apply (rule guaranteesI)
  apply (rule AlwaysI, force)
  apply (blast intro: Increasing_preserves_Stable stable_rel_le_giv)
done

```

end