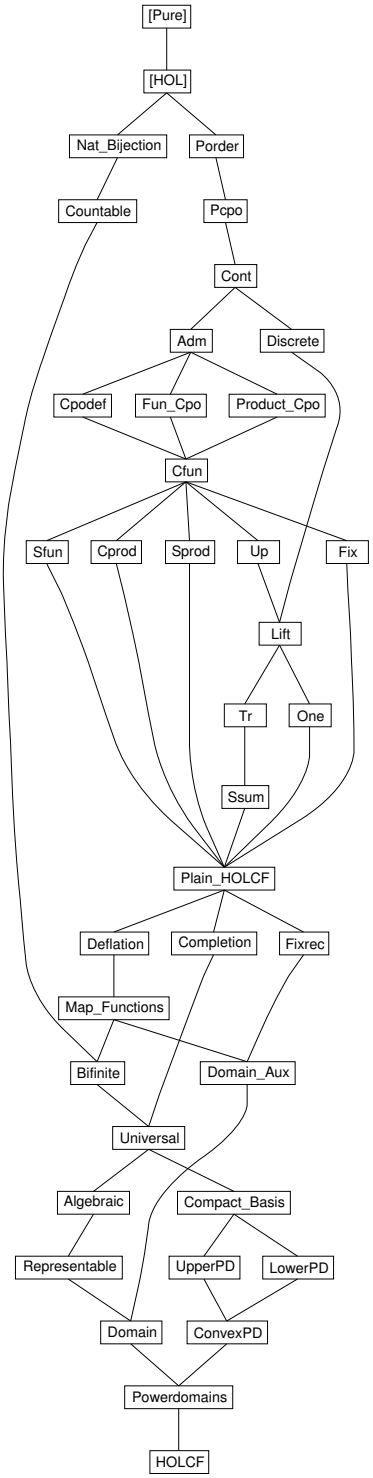
Isabelle/HOLCF — Higher-Order Logic of Computable Functions

August 27, 2014

Contents


[Pure]
[HOL]
Nat_Bijection
Porder
Countable
Pcpo
Cont
Adm
Discrete
Cpodef
Fun_Cpo
Product_Cpo
Cfun
Sfun
Cprod
Sprod
Up
Fix
Lift
Tr
One
Ssum
Plain_HOLCF
Deflation
Completion
Fixrec
Map_Functions
Bifinite
Domain_Aux
Universal
Algebraic
Compact_Basis
Representable
UpperPD
LowerPD
Domain
ConvexPD
Powerdomains
HOLCF

1 Porder: Partial orders

theory *Porder*
imports *Main*
begin

1.1 Type class for partial orders

class *below* =
  fixes *below* :: $'a \Rightarrow 'a \Rightarrow bool$
begin

notation
  below (**infix** $<<$ *50*)

notation (*xsymbols*)
  below (**infix** $\sqsubseteq$ *50*)

abbreviation
  not-below :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sim<<$ *50*)
  where *not-below x y* $\equiv \neg$ *below x y*

notation (*xsymbols*)
  not-below (**infix** $\not\sqsubseteq$ *50*)

lemma *below-eq-trans*: $[\![a \sqsubseteq b; b = c]\!] \Longrightarrow a \sqsubseteq c$
  ⟨*proof*⟩

lemma *eq-below-trans*: $[\![a = b; b \sqsubseteq c]\!] \Longrightarrow a \sqsubseteq c$
  ⟨*proof*⟩

end

class *po* = *below* +
  assumes *below-refl* [*iff*]: $x \sqsubseteq x$
  assumes *below-trans*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
  assumes *below-antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$
begin

lemma *eq-imp-below*: $x = y \Longrightarrow x \sqsubseteq y$
  ⟨*proof*⟩

lemma *box-below*: $a \sqsubseteq b \Longrightarrow c \sqsubseteq a \Longrightarrow b \sqsubseteq d \Longrightarrow c \sqsubseteq d$
  ⟨*proof*⟩

lemma *po-eq-conv*: $x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$
  ⟨*proof*⟩

lemma *rev-below-trans*: $y \sqsubseteq z \Longrightarrow x \sqsubseteq y \Longrightarrow x \sqsubseteq z$
  ⟨*proof*⟩

lemma *not-below2not-eq*: $x \not\sqsubseteq y \Longrightarrow x \neq y$
⟨*proof*⟩

end

lemmas *HOLCF-trans-rules* [*trans*] =
below-trans
below-antisym
below-eq-trans
eq-below-trans

context *po*
begin

1.2 Upper bounds

definition *is-ub* :: $'a\ set \Rightarrow 'a \Rightarrow bool$ (**infix** $<|$ *55*) **where**
$S <| x \longleftrightarrow (\forall y{\in}S.\ y \sqsubseteq x)$

lemma *is-ubI*: $(\bigwedge x.\ x \in S \Longrightarrow x \sqsubseteq u) \Longrightarrow S <| u$
⟨*proof*⟩

lemma *is-ubD*: $[\![S <| u; x \in S]\!] \Longrightarrow x \sqsubseteq u$
⟨*proof*⟩

lemma *ub-imageI*: $(\bigwedge x.\ x \in S \Longrightarrow f\,x \sqsubseteq u) \Longrightarrow (\lambda x.\ f\,x)\ `\ S <| u$
⟨*proof*⟩

lemma *ub-imageD*: $[\![f\ `\ S <| u; x \in S]\!] \Longrightarrow f\,x \sqsubseteq u$
⟨*proof*⟩

lemma *ub-rangeI*: $(\bigwedge i.\ S\,i \sqsubseteq x) \Longrightarrow range\ S <| x$
⟨*proof*⟩

lemma *ub-rangeD*: $range\ S <| x \Longrightarrow S\,i \sqsubseteq x$
⟨*proof*⟩

lemma *is-ub-empty* [*simp*]: $\{\} <| u$
⟨*proof*⟩

lemma *is-ub-insert* [*simp*]: $(insert\ x\ A) <| y = (x \sqsubseteq y \wedge A <| y)$
⟨*proof*⟩

lemma *is-ub-upward*: $[\![S <| x; x \sqsubseteq y]\!] \Longrightarrow S <| y$
⟨*proof*⟩

1.3 Least upper bounds

definition *is-lub* :: $'a\ set \Rightarrow 'a \Rightarrow bool$ (**infix** $<<|$ *55*) **where**

$S <<| x \longleftrightarrow S <| x \wedge (\forall u.\ S <| u \longrightarrow x \sqsubseteq u)$

definition *lub* :: *'a set* ⇒ *'a* **where**
lub S = (*THE x*. *S <<| x*)

end

syntax
-BLub :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* ((*3LUB -:-./ -*) [*0*,*0*, *10*] *10*)

syntax (*xsymbols*)
-BLub :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* ((*3⨆ -∈-./ -*) [*0*,*0*, *10*] *10*)

translations
LUB x:*A*. *t* == *CONST lub* ((%*x*. *t*) ‘ *A*)

context *po*
begin

abbreviation
Lub (**binder** *LUB* *10*) **where**
LUB n. *t n* == *lub* (*range t*)

notation (*xsymbols*)
Lub (**binder** ⨆ *10*)

access to some definition as inference rule

lemma *is-lubD1*: *S <<| x* ⟹ *S <| x*
⟨*proof*⟩

lemma *is-lubD2*: ⟦*S <<| x*; *S <| u*⟧ ⟹ *x* ⊑ *u*
⟨*proof*⟩

lemma *is-lubI*: ⟦*S <| x*; ⋀*u*. *S <| u* ⟹ *x* ⊑ *u*⟧ ⟹ *S <<| x*
⟨*proof*⟩

lemma *is-lub-below-iff*: *S <<| x* ⟹ *x* ⊑ *u* ⟷ *S <| u*
⟨*proof*⟩

lubs are unique

lemma *is-lub-unique*: ⟦*S <<| x*; *S <<| y*⟧ ⟹ *x* = *y*
⟨*proof*⟩

technical lemmas about *lub* and *op <<|*

lemma *is-lub-lub*: *M <<| x* ⟹ *M <<| lub M*
⟨*proof*⟩

lemma *lub-eqI*: *M <<| l* ⟹ *lub M* = *l*
⟨*proof*⟩

lemma *is-lub-singleton*: $\{x\} <<| x$
⟨*proof*⟩

lemma *lub-singleton* [*simp*]: $lub\ \{x\} = x$
⟨*proof*⟩

lemma *is-lub-bin*: $x \sqsubseteq y \Longrightarrow \{x, y\} <<| y$
⟨*proof*⟩

lemma *lub-bin*: $x \sqsubseteq y \Longrightarrow lub\ \{x, y\} = y$
⟨*proof*⟩

lemma *is-lub-maximal*: $[\![S <| x; x \in S]\!] \Longrightarrow S <<| x$
⟨*proof*⟩

lemma *lub-maximal*: $[\![S <| x; x \in S]\!] \Longrightarrow lub\ S = x$
⟨*proof*⟩

1.4 Countable chains

definition *chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$ **where**
— Here we use countable chains and I prefer to code them as functions!
$chain\ Y = (\forall i.\ Y\ i \sqsubseteq Y\ (Suc\ i))$

lemma *chainI*: $(\bigwedge i.\ Y\ i \sqsubseteq Y\ (Suc\ i)) \Longrightarrow chain\ Y$
⟨*proof*⟩

lemma *chainE*: $chain\ Y \Longrightarrow Y\ i \sqsubseteq Y\ (Suc\ i)$
⟨*proof*⟩

chains are monotone functions

lemma *chain-mono-less*: $[\![chain\ Y; i < j]\!] \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

lemma *chain-mono*: $[\![chain\ Y; i \leq j]\!] \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

lemma *chain-shift*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ Y\ (i + j))$
⟨*proof*⟩

technical lemmas about (least) upper bounds of chains

lemma *is-lub-rangeD1*: $range\ S <<| x \Longrightarrow S\ i \sqsubseteq x$
⟨*proof*⟩

lemma *is-ub-range-shift*:
$chain\ S \Longrightarrow range\ (\lambda i.\ S\ (i + j)) <| x = range\ S <| x$
⟨*proof*⟩

lemma *is-lub-range-shift*:
chain S $\Longrightarrow$ *range* $(\lambda i.\ S\ (i + j))$ $<<|$ *x* = *range S* $<<|$ *x*
⟨*proof*⟩

the lub of a constant chain is the constant

lemma *chain-const* [*simp*]: *chain* $(\lambda i.\ c)$
⟨*proof*⟩

lemma *is-lub-const*: *range* $(\lambda x.\ c)$ $<<|$ *c*
⟨*proof*⟩

lemma *lub-const* [*simp*]: $(\bigsqcup i.\ c) = c$
⟨*proof*⟩

1.5 Finite chains

definition *max-in-chain* :: *nat* ⇒ (*nat* ⇒ *'a*) ⇒ *bool* **where**
— finite chains, needed for monotony of continuous functions
max-in-chain i C $\longleftrightarrow$ $(\forall j.\ i \leq j \longrightarrow C\ i = C\ j)$

definition *finite-chain* :: (*nat* ⇒ *'a*) ⇒ *bool* **where**
finite-chain C = (*chain C* ∧ $(\exists i.$ *max-in-chain i C*))

results about finite chains

lemma *max-in-chainI*: $(\bigwedge j.\ i \leq j \Longrightarrow Y\ i = Y\ j) \Longrightarrow$ *max-in-chain i Y*
⟨*proof*⟩

lemma *max-in-chainD*: ⟦*max-in-chain i Y*; *i* ≤ *j*⟧ $\Longrightarrow$ *Y i* = *Y j*
⟨*proof*⟩

lemma *finite-chainI*:
⟦*chain C*; *max-in-chain i C*⟧ $\Longrightarrow$ *finite-chain C*
⟨*proof*⟩

lemma *finite-chainE*:
⟦*finite-chain C*; $\bigwedge i.$ ⟦*chain C*; *max-in-chain i C*⟧ $\Longrightarrow$ *R*⟧ $\Longrightarrow$ *R*
⟨*proof*⟩

lemma *lub-finch1*: ⟦*chain C*; *max-in-chain i C*⟧ $\Longrightarrow$ *range C* $<<|$ *C i*
⟨*proof*⟩

lemma *lub-finch2*:
finite-chain C $\Longrightarrow$ *range C* $<<|$ *C* (*LEAST i. max-in-chain i C*)
⟨*proof*⟩

lemma *finch-imp-finite-range*: *finite-chain Y* $\Longrightarrow$ *finite* (*range Y*)
⟨*proof*⟩

lemma *finite-range-has-max*:

fixes $f :: nat \Rightarrow 'a$ **and** $r :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *mono*: $\bigwedge i\ j.\ i \leq j \Longrightarrow r\ (f\ i)\ (f\ j)$
assumes *finite-range*: *finite* (*range* f)
shows $\exists k.\ \forall i.\ r\ (f\ i)\ (f\ k)$
⟨*proof*⟩

lemma *finite-range-imp-finch*:
$[\![$*chain* Y; *finite* (*range* Y)$]\!] \Longrightarrow$ *finite-chain* Y
⟨*proof*⟩

lemma *bin-chain*: $x \sqsubseteq y \Longrightarrow$ *chain* $(\lambda i.$ *if* $i=0$ *then* x *else* $y)$
⟨*proof*⟩

lemma *bin-chainmax*:
$x \sqsubseteq y \Longrightarrow$ *max-in-chain* (*Suc* 0) $(\lambda i.$ *if* $i=0$ *then* x *else* $y)$
⟨*proof*⟩

lemma *is-lub-bin-chain*:
$x \sqsubseteq y \Longrightarrow$ *range* $(\lambda i{::}nat.$ *if* $i=0$ *then* x *else* $y) <<|\ y$
⟨*proof*⟩

the maximal element in a chain is its lub

lemma *lub-chain-maxelem*: $[\![Y\ i = c; \forall i.\ Y\ i \sqsubseteq c]\!] \Longrightarrow$ *lub* (*range* Y) $= c$
⟨*proof*⟩

end

end

2 Pcpo: Classes cpo and pcpo

theory *Pcpo*
imports *Porder*
begin

2.1 Complete partial orders

The class cpo of chain complete partial orders

class *cpo* = *po* +
assumes *cpo*: *chain* $S \Longrightarrow \exists x.$ *range* $S <<|\ x$
begin

in cpo's everthing equal to THE lub has lub properties for every chain

lemma *cpo-lubI*: *chain* $S \Longrightarrow$ *range* $S <<|\ (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *thelubE*: $[\![$*chain* S; $(\bigsqcup i.\ S\ i) = l]\!] \Longrightarrow$ *range* $S <<|\ l$
⟨*proof*⟩

Properties of the lub

lemma *is-ub-thelub*: $chain\ S \Longrightarrow S\ x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *is-lub-thelub*:
$[\![chain\ S;\ range\ S <|\ x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

lemma *lub-below-iff*: $chain\ S \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x \longleftrightarrow (\forall i.\ S\ i \sqsubseteq x)$
⟨*proof*⟩

lemma *lub-below*: $[\![chain\ S;\ \bigwedge i.\ S\ i \sqsubseteq x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

lemma *below-lub*: $[\![chain\ S;\ x \sqsubseteq S\ i]\!] \Longrightarrow x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *lub-range-mono*:
$[\![range\ X \subseteq range\ Y;\ chain\ Y;\ chain\ X]\!]$
$\Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *lub-range-shift*:
$chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ (i + j)) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *maxinch-is-thelub*:
$chain\ Y \Longrightarrow max\text{-}in\text{-}chain\ i\ Y = ((\bigsqcup i.\ Y\ i) = Y\ i)$
⟨*proof*⟩

the $\sqsubseteq$ relation between two chains is preserved by their lubs

lemma *lub-mono*:
$[\![chain\ X;\ chain\ Y;\ \bigwedge i.\ X\ i \sqsubseteq Y\ i]\!]$
$\Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

the $=$ relation between two chains is preserved by their lubs

lemma *lub-eq*:
$(\bigwedge i.\ X\ i = Y\ i) \Longrightarrow (\bigsqcup i.\ X\ i) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *ch2ch-lub*:
assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
shows $chain\ (\lambda i.\ \bigsqcup j.\ Y\ i\ j)$
⟨*proof*⟩

lemma *diag-lub*:
assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$

assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
shows $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup i.\ Y\ i\ i)$
⟨*proof*⟩

lemma *ex-lub*:
assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
shows $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup j.\ \bigsqcup i.\ Y\ i\ j)$
⟨*proof*⟩

end

2.2 Pointed cpos

The class pcpo of pointed cpos

class *pcpo* = *cpo* +
assumes *least*: $\exists x.\ \forall y.\ x \sqsubseteq y$
begin

definition *bottom* :: *'a*
where *bottom* = $(THE\ x.\ \forall y.\ x \sqsubseteq y)$

notation (*xsymbols*)
bottom ($\bot$)

lemma *minimal* [*iff*]: $\bot \sqsubseteq x$
⟨*proof*⟩

end

Old "UU" syntax:

syntax *UU* :: *logic*

translations *UU* => *CONST bottom*

Simproc to rewrite $\bot = x$ to $x = \bot$.

⟨*ML*⟩
useful lemmas about $\bot$

lemma *below-bottom-iff* [*simp*]: $(x \sqsubseteq \bot) = (x = \bot)$
⟨*proof*⟩

lemma *eq-bottom-iff*: $(x = \bot) = (x \sqsubseteq \bot)$
⟨*proof*⟩

lemma *bottomI*: $x \sqsubseteq \bot \Longrightarrow x = \bot$
⟨*proof*⟩

lemma *lub-eq-bottom-iff*: *chain* $Y \Longrightarrow (\bigsqcup i.\ Y\ i) = \bot \longleftrightarrow (\forall i.\ Y\ i = \bot)$
⟨*proof*⟩

2.3 Chain-finite and flat cpos

further useful classes for HOLCF domains

class *chfin* = *po* +
  assumes *chfin*: *chain* $Y \Longrightarrow \exists n.$ *max-in-chain* $n\ Y$
begin

subclass *cpo*
⟨*proof*⟩

lemma *chfin2finch*: *chain* $Y \Longrightarrow$ *finite-chain* Y
  ⟨*proof*⟩

end

class *flat* = *pcpo* +
  assumes *ax-flat*: $x \sqsubseteq y \Longrightarrow x = \bot \vee x = y$
begin

subclass *chfin*
⟨*proof*⟩

lemma *flat-below-iff*:
  shows $(x \sqsubseteq y) = (x = \bot \vee x = y)$
  ⟨*proof*⟩

lemma *flat-eq*: $a \neq \bot \Longrightarrow a \sqsubseteq b = (a = b)$
  ⟨*proof*⟩

end

2.4 Discrete cpos

class *discrete-cpo* = *below* +
  assumes *discrete-cpo* [*simp*]: $x \sqsubseteq y \longleftrightarrow x = y$
begin

subclass *po*
⟨*proof*⟩

In a discrete cpo, every chain is constant

lemma *discrete-chain-const*:
  assumes S: *chain* S
  shows $\exists x.\ S = (\lambda i.\ x)$
⟨*proof*⟩

subclass *chfin*
⟨*proof*⟩

end

end

3 Cont: Continuity and monotonicity

theory *Cont*
imports *Pcpo*
begin

Now we change the default class! Form now on all untyped type variables are of default class po

default-sort *po*

3.1 Definitions

definition
monofun :: $('a \Rightarrow 'b) \Rightarrow bool$ — monotonicity **where**
$monofun\ f = (\forall x\ y.\ x \sqsubseteq y \longrightarrow f\ x \sqsubseteq f\ y)$

definition
cont :: $('a{::}cpo \Rightarrow 'b{::}cpo) \Rightarrow bool$
where
$cont\ f = (\forall\ Y.\ chain\ Y \longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i))$

lemma *contI*:
$[\![\bigwedge Y.\ chain\ Y \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)]\!] \Longrightarrow cont\ f$
⟨*proof*⟩

lemma *contE*:
$[\![cont\ f;\ chain\ Y]\!] \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *monofunI*:
$[\![\bigwedge x\ y.\ x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y]\!] \Longrightarrow monofun\ f$
⟨*proof*⟩

lemma *monofunE*:
$[\![monofun\ f;\ x \sqsubseteq y]\!] \Longrightarrow f\ x \sqsubseteq f\ y$
⟨*proof*⟩

3.2 Equivalence of alternate definition

monotone functions map chains to chains

lemma *ch2ch-monofun*: $[\![monofun\ f;\ chain\ Y]\!] \Longrightarrow chain\ (\lambda i.\ f\ (Y\ i))$

⟨*proof*⟩

monotone functions map upper bound to upper bounds

lemma *ub2ub-monofun*:
  ⟦*monofun f*; *range Y* <| *u*⟧ ⟹ *range* (λ*i*. *f* (*Y i*)) <| *f u*
⟨*proof*⟩

a lemma about binary chains

lemma *binchain-cont*:
  ⟦*cont f*; *x* ⊑ *y*⟧ ⟹ *range* (λ*i*::*nat*. *f* (*if i* = *0 then x else y*)) <<| *f y*
⟨*proof*⟩

continuity implies monotonicity

lemma *cont2mono*: *cont f* ⟹ *monofun f*
⟨*proof*⟩

lemmas *cont2monofunE* = *cont2mono* [*THEN monofunE*]

lemmas *ch2ch-cont* = *cont2mono* [*THEN ch2ch-monofun*]

continuity implies preservation of lubs

lemma *cont2contlubE*:
  ⟦*cont f*; *chain Y*⟧ ⟹ *f* (⨆ *i*. *Y i*) = (⨆ *i*. *f* (*Y i*))
⟨*proof*⟩

lemma *contI2*:
  fixes *f* :: *'a*::*cpo* ⇒ *'b*::*cpo*
  assumes *mono*: *monofun f*
  assumes *below*: ⋀*Y*. ⟦*chain Y*; *chain* (λ*i*. *f* (*Y i*))⟧
    ⟹ *f* (⨆ *i*. *Y i*) ⊑ (⨆ *i*. *f* (*Y i*))
  shows *cont f*
⟨*proof*⟩

3.3 Collection of continuity rules

⟨*ML*⟩

3.4 Continuity of basic functions

The identity function is continuous

lemma *cont-id* [*simp*, *cont2cont*]: *cont* (λ*x*. *x*)
⟨*proof*⟩

constant functions are continuous

lemma *cont-const* [*simp*, *cont2cont*]: *cont* (λ*x*. *c*)
  ⟨*proof*⟩

application of functions is continuous

lemma *cont-apply*:
fixes $f :: 'a::cpo \Rightarrow 'b::cpo \Rightarrow 'c::cpo$ **and** $t :: 'a \Rightarrow 'b$
assumes *1*: *cont* $(\lambda x.\ t\ x)$
assumes *2*: $\bigwedge x.$ *cont* $(\lambda y.\ f\ x\ y)$
assumes *3*: $\bigwedge y.$ *cont* $(\lambda x.\ f\ x\ y)$
shows *cont* $(\lambda x.\ (f\ x)\ (t\ x))$
⟨*proof*⟩

lemma *cont-compose*:
$[\![$*cont* c; *cont* $(\lambda x.\ f\ x)]\!] \Longrightarrow$ *cont* $(\lambda x.\ c\ (f\ x))$
⟨*proof*⟩

Least upper bounds preserve continuity

lemma *cont2cont-lub* [*simp*]:
assumes *chain*: $\bigwedge x.$ *chain* $(\lambda i.\ F\ i\ x)$ **and** *cont*: $\bigwedge i.$ *cont* $(\lambda x.\ F\ i\ x)$
shows *cont* $(\lambda x.\ \bigsqcup i.\ F\ i\ x)$
⟨*proof*⟩

if-then-else is continuous

lemma *cont-if* [*simp*, *cont2cont*]:
$[\![$*cont* f; *cont* $g]\!] \Longrightarrow$ *cont* $(\lambda x.$ *if* b *then* $f\ x$ *else* $g\ x)$
⟨*proof*⟩

3.5 Finite chains and flat pcpos

Monotone functions map finite chains to finite chains.

lemma *monofun-finch2finch*:
$[\![$*monofun* f; *finite-chain* $Y]\!] \Longrightarrow$ *finite-chain* $(\lambda n.\ f\ (Y\ n))$
⟨*proof*⟩

The same holds for continuous functions.

lemma *cont-finch2finch*:
$[\![$*cont* f; *finite-chain* $Y]\!] \Longrightarrow$ *finite-chain* $(\lambda n.\ f\ (Y\ n))$
⟨*proof*⟩

All monotone functions with chain-finite domain are continuous.

lemma *chfindom-monofun2cont*: *monofun* $f \Longrightarrow$ *cont* $(f::'a::chfin \Rightarrow 'b::cpo)$
⟨*proof*⟩

All strict functions with flat domain are continuous.

lemma *flatdom-strict2mono*: $f\ \bot = \bot \Longrightarrow$ *monofun* $(f::'a::flat \Rightarrow 'b::pcpo)$
⟨*proof*⟩

lemma *flatdom-strict2cont*: $f\ \bot = \bot \Longrightarrow$ *cont* $(f::'a::flat \Rightarrow 'b::pcpo)$
⟨*proof*⟩

All functions with discrete domain are continuous.

lemma *cont-discrete-cpo* [*simp*, *cont2cont*]: *cont* (f::$'a$::*discrete-cpo* ⇒ $'b$::*cpo*)
⟨*proof*⟩

end

4 Adm: Admissibility and compactness

theory *Adm*
imports *Cont*
begin

default-sort *cpo*

4.1 Definitions

definition
adm :: ($'a$::*cpo* ⇒ *bool*) ⇒ *bool* **where**
adm P = (∀ *Y*. *chain Y* ⟶ (∀ *i*. *P* (*Y i*)) ⟶ *P* (⨆ *i*. *Y i*))

lemma *admI*:
(⋀ *Y*. ⟦*chain Y*; ∀ *i*. *P* (*Y i*)⟧ ⟹ *P* (⨆ *i*. *Y i*)) ⟹ *adm P*
⟨*proof*⟩

lemma *admD*: ⟦*adm P*; *chain Y*; ⋀*i*. *P* (*Y i*)⟧ ⟹ *P* (⨆ *i*. *Y i*)
⟨*proof*⟩

lemma *admD2*: ⟦*adm* (λ*x*. ¬ *P x*); *chain Y*; *P* (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *P* (*Y i*)
⟨*proof*⟩

lemma *triv-admI*: ∀ *x*. *P x* ⟹ *adm P*
⟨*proof*⟩

4.2 Admissibility on chain-finite types

For chain-finite (easy) types every formula is admissible.

lemma *adm-chfin* [*simp*]: *adm* (*P*::$'a$::*chfin* ⇒ *bool*)
⟨*proof*⟩

4.3 Admissibility of special formulae and propagation

lemma *adm-const* [*simp*]: *adm* (λ*x*. *t*)
⟨*proof*⟩

lemma *adm-conj* [*simp*]:
⟦*adm* (λ*x*. *P x*); *adm* (λ*x*. *Q x*)⟧ ⟹ *adm* (λ*x*. *P x* ∧ *Q x*)
⟨*proof*⟩

lemma *adm-all* [*simp*]:

$(\bigwedge y.\ adm\ (\lambda x.\ P\ x\ y)) \Longrightarrow adm\ (\lambda x.\ \forall y.\ P\ x\ y)$
⟨*proof*⟩

lemma *adm-ball* [*simp*]:
$(\bigwedge y.\ y \in A \Longrightarrow adm\ (\lambda x.\ P\ x\ y)) \Longrightarrow adm\ (\lambda x.\ \forall y{\in}A.\ P\ x\ y)$
⟨*proof*⟩

Admissibility for disjunction is hard to prove. It requires 2 lemmas.

lemma *adm-disj-lemma1*:
assumes *adm*: $adm\ P$
assumes *chain*: $chain\ Y$
assumes *P*: $\forall i.\ \exists j{\geq}i.\ P\ (Y\ j)$
shows $P\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *adm-disj-lemma2*:
$\forall n{::}nat.\ P\ n \vee Q\ n \Longrightarrow (\forall i.\ \exists j{\geq}i.\ P\ j) \vee (\forall i.\ \exists j{\geq}i.\ Q\ j)$
⟨*proof*⟩

lemma *adm-disj* [*simp*]:
$[\![adm\ (\lambda x.\ P\ x);\ adm\ (\lambda x.\ Q\ x)]\!] \Longrightarrow adm\ (\lambda x.\ P\ x \vee Q\ x)$
⟨*proof*⟩

lemma *adm-imp* [*simp*]:
$[\![adm\ (\lambda x.\ \neg\ P\ x);\ adm\ (\lambda x.\ Q\ x)]\!] \Longrightarrow adm\ (\lambda x.\ P\ x \longrightarrow Q\ x)$
⟨*proof*⟩

lemma *adm-iff* [*simp*]:
$[\![adm\ (\lambda x.\ P\ x \longrightarrow Q\ x);\ adm\ (\lambda x.\ Q\ x \longrightarrow P\ x)]\!]$
$\Longrightarrow adm\ (\lambda x.\ P\ x = Q\ x)$
⟨*proof*⟩

admissibility and continuity

lemma *adm-below* [*simp*]:
$[\![cont\ (\lambda x.\ u\ x);\ cont\ (\lambda x.\ v\ x)]\!] \Longrightarrow adm\ (\lambda x.\ u\ x \sqsubseteq v\ x)$
⟨*proof*⟩

lemma *adm-eq* [*simp*]:
$[\![cont\ (\lambda x.\ u\ x);\ cont\ (\lambda x.\ v\ x)]\!] \Longrightarrow adm\ (\lambda x.\ u\ x = v\ x)$
⟨*proof*⟩

lemma *adm-subst*: $[\![cont\ (\lambda x.\ t\ x);\ adm\ P]\!] \Longrightarrow adm\ (\lambda x.\ P\ (t\ x))$
⟨*proof*⟩

lemma *adm-not-below* [*simp*]: $cont\ (\lambda x.\ t\ x) \Longrightarrow adm\ (\lambda x.\ t\ x \not\sqsubseteq u)$
⟨*proof*⟩

4.4 Compactness

definition
compact :: *'a*::*cpo* ⇒ *bool* **where**
compact k = *adm* (λ*x*. *k* ⋢ *x*)

lemma *compactI*: *adm* (λ*x*. *k* ⋢ *x*) ⟹ *compact k*
⟨*proof*⟩

lemma *compactD*: *compact k* ⟹ *adm* (λ*x*. *k* ⋢ *x*)
⟨*proof*⟩

lemma *compactI2*:
(⋀*Y*. ⟦*chain Y*; *x* ⊑ (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *x* ⊑ *Y i*) ⟹ *compact x*
⟨*proof*⟩

lemma *compactD2*:
⟦*compact x*; *chain Y*; *x* ⊑ (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *x* ⊑ *Y i*
⟨*proof*⟩

lemma *compact-below-lub-iff*:
⟦*compact x*; *chain Y*⟧ ⟹ *x* ⊑ (⨆ *i*. *Y i*) ⟷ (∃ *i*. *x* ⊑ *Y i*)
⟨*proof*⟩

lemma *compact-chfin* [*simp*]: *compact* (*x*::*'a*::*chfin*)
⟨*proof*⟩

lemma *compact-imp-max-in-chain*:
⟦*chain Y*; *compact* (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *max-in-chain i Y*
⟨*proof*⟩

admissibility and compactness

lemma *adm-compact-not-below* [*simp*]:
⟦*compact k*; *cont* (λ*x*. *t x*)⟧ ⟹ *adm* (λ*x*. *k* ⋢ *t x*)
⟨*proof*⟩

lemma *adm-neq-compact* [*simp*]:
⟦*compact k*; *cont* (λ*x*. *t x*)⟧ ⟹ *adm* (λ*x*. *t x* ≠ *k*)
⟨*proof*⟩

lemma *adm-compact-neq* [*simp*]:
⟦*compact k*; *cont* (λ*x*. *t x*)⟧ ⟹ *adm* (λ*x*. *k* ≠ *t x*)
⟨*proof*⟩

lemma *compact-bottom* [*simp*, *intro*]: *compact* ⊥
⟨*proof*⟩

Any upward-closed predicate is admissible.

lemma *adm-upward*:
assumes *P*: ⋀*x y*. ⟦*P x*; *x* ⊑ *y*⟧ ⟹ *P y*

shows *adm P*
⟨*proof*⟩

lemmas *adm-lemmas* =
adm-const adm-conj adm-all adm-ball adm-disj adm-imp adm-iff
adm-below adm-eq adm-not-below
adm-compact-not-below adm-compact-neq adm-neq-compact

end

5 Cpodef: Subtypes of pcpos

theory *Cpodef*
imports *Adm*
keywords *pcpodef cpodef* :: *thy-goal*
begin

5.1 Proving a subtype is a partial order

A subtype of a partial order is itself a partial order, if the ordering is defined in the standard way.

⟨*ML*⟩

theorem *typedef-po*:
fixes *Abs* :: *'a*::*po* ⇒ *'b*::*type*
assumes *type*: *type-definition Rep Abs A*
and *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *OFCLASS*(*'b*, *po-class*)
⟨*proof*⟩

⟨*ML*⟩

5.2 Proving a subtype is finite

lemma *typedef-finite-UNIV*:
fixes *Abs* :: *'a*::*type* ⇒ *'b*::*type*
assumes *type*: *type-definition Rep Abs A*
shows *finite A* ⟹ *finite* (*UNIV* :: *'b set*)
⟨*proof*⟩

5.3 Proving a subtype is chain-finite

lemma *ch2ch-Rep*:
assumes *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *chain S* ⟹ *chain* (λ*i*. *Rep* (*S i*))
⟨*proof*⟩

theorem *typedef-chfin*:

fixes *Abs* :: *'a*::*chfin* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *OFCLASS*(*'b*, *chfin-class*)
⟨*proof*⟩

5.4 Proving a subtype is complete

A subtype of a cpo is itself a cpo if the ordering is defined in the standard way, and the defining subset is closed with respect to limits of chains. A set is closed if and only if membership in the set is an admissible predicate.

lemma *typedef-is-lubI*:
assumes *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *range* (λ*i*. *Rep* (*S i*)) <<| *Rep x* ⟹ *range S* <<| *x*
⟨*proof*⟩

lemma *Abs-inverse-lub-Rep*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x*. *x* ∈ *A*)
shows *chain S* ⟹ *Rep* (*Abs* (⨆ *i*. *Rep* (*S i*))) = (⨆ *i*. *Rep* (*S i*))
⟨*proof*⟩

theorem *typedef-is-lub*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x*. *x* ∈ *A*)
shows *chain S* ⟹ *range S* <<| *Abs* (⨆ *i*. *Rep* (*S i*))
⟨*proof*⟩

lemmas *typedef-lub* = *typedef-is-lub* [*THEN lub-eqI*]

theorem *typedef-cpo*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x*. *x* ∈ *A*)
shows *OFCLASS*(*'b*, *cpo-class*)
⟨*proof*⟩

5.4.1 Continuity of *Rep* and *Abs*

For any sub-cpo, the *Rep* function is continuous.

theorem *typedef-cont-Rep*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
assumes *type*: *type-definition Rep Abs A*

and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: $adm\ (\lambda x.\ x \in A)$
shows $cont\ (\lambda x.\ f\ x) \Longrightarrow cont\ (\lambda x.\ Rep\ (f\ x))$
⟨*proof*⟩

For a sub-cpo, we can make the *Abs* function continuous only if we restrict its domain to the defining subset by composing it with another continuous function.

theorem *typedef-cont-Abs*:
fixes $Abs :: 'a{::}cpo \Rightarrow 'b{::}cpo$
fixes $f :: 'c{::}cpo \Rightarrow 'a{::}cpo$
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: $adm\ (\lambda x.\ x \in A)$
and *f-in-A*: $\bigwedge x.\ f\ x \in A$
shows $cont\ f \Longrightarrow cont\ (\lambda x.\ Abs\ (f\ x))$
⟨*proof*⟩

5.5 Proving subtype elements are compact

theorem *typedef-compact*:
fixes $Abs :: 'a{::}cpo \Rightarrow 'b{::}cpo$
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: $adm\ (\lambda x.\ x \in A)$
shows $compact\ (Rep\ k) \Longrightarrow compact\ k$
⟨*proof*⟩

5.6 Proving a subtype is pointed

A subtype of a cpo has a least element if and only if the defining subset has a least element.

theorem *typedef-pcpo-generic*:
fixes $Abs :: 'a{::}cpo \Rightarrow 'b{::}cpo$
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *z-in-A*: $z \in A$
and *z-least*: $\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x$
shows *OFCLASS*('*b*, *pcpo-class*)
⟨*proof*⟩

As a special case, a subtype of a pcpo has a least element if the defining subset contains $\bot$.

theorem *typedef-pcpo*:
fixes $Abs :: 'a{::}pcpo \Rightarrow 'b{::}cpo$
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$

shows *OFCLASS*('*b*, *pcpo-class*)
⟨*proof*⟩

5.6.1 Strictness of *Rep* and *Abs*

For a sub-pcpo where $\bot$ is a member of the defining subset, *Rep* and *Abs* are both strict.

theorem *typedef-Abs-strict*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $Abs\ \bot = \bot$
⟨*proof*⟩

theorem *typedef-Rep-strict*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $Rep\ \bot = \bot$
⟨*proof*⟩

theorem *typedef-Abs-bottom-iff*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $x \in A \Longrightarrow (Abs\ x = \bot) = (x = \bot)$
⟨*proof*⟩

theorem *typedef-Rep-bottom-iff*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $(Rep\ x = \bot) = (x = \bot)$
⟨*proof*⟩

5.7 Proving a subtype is flat

theorem *typedef-flat*:
fixes *Abs* :: $'a$::*flat* $\Rightarrow$ $'b$::*pcpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows *OFCLASS*('*b*, *flat-class*)
⟨*proof*⟩

5.8 HOLCF type definition package

⟨*ML*⟩

end

6 Fun-Cpo: Class instances for the full function space

theory *Fun-Cpo*
imports *Adm*
begin

6.1 Full function space is a partial order

instantiation *fun* :: (*type*, *below*) *below*
begin

definition
below-fun-def: $(op \sqsubseteq) \equiv (\lambda f\ g.\ \forall x.\ f\ x \sqsubseteq g\ x)$

instance ⟨*proof*⟩
end

instance *fun* :: (*type*, *po*) *po*
⟨*proof*⟩

lemma *fun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\ x \sqsubseteq g\ x)$
⟨*proof*⟩

lemma *fun-belowI*: $(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

lemma *fun-belowD*: $f \sqsubseteq g \Longrightarrow f\ x \sqsubseteq g\ x$
⟨*proof*⟩

6.2 Full function space is chain complete

Properties of chains of functions.

lemma *fun-chain-iff*: *chain* $S \longleftrightarrow (\forall x.\ chain\ (\lambda i.\ S\ i\ x))$
⟨*proof*⟩

lemma *ch2ch-fun*: *chain* $S \Longrightarrow chain\ (\lambda i.\ S\ i\ x)$
⟨*proof*⟩

lemma *ch2ch-lambda*: $(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow chain\ S$
⟨*proof*⟩

Type $'a \Rightarrow 'b$ is chain complete

lemma *is-lub-lambda*:
$(\bigwedge x.\ range\ (\lambda i.\ Y\ i\ x) <<|\ f\ x) \Longrightarrow range\ Y <<|\ f$
⟨*proof*⟩

lemma *is-lub-fun*:
  chain (*S*::*nat* ⇒ 'a::*type* ⇒ 'b::*cpo*)
    ⟹ *range S* <<| (λx. ⨆ i. *S i x*)
⟨*proof*⟩

lemma *lub-fun*:
  chain (*S*::*nat* ⇒ 'a::*type* ⇒ 'b::*cpo*)
    ⟹ (⨆ i. *S i*) = (λx. ⨆ i. *S i x*)
⟨*proof*⟩

instance *fun* :: (*type*, *cpo*) *cpo*
⟨*proof*⟩

instance *fun* :: (*type*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

6.3 Full function space is pointed

lemma *minimal-fun*: (λx. ⊥) ⊑ *f*
⟨*proof*⟩

instance *fun* :: (*type*, *pcpo*) *pcpo*
⟨*proof*⟩

lemma *inst-fun-pcpo*: ⊥ = (λx. ⊥)
⟨*proof*⟩

lemma *app-strict* [*simp*]: ⊥ *x* = ⊥
⟨*proof*⟩

lemma *lambda-strict*: (λx. ⊥) = ⊥
⟨*proof*⟩

6.4 Propagation of monotonicity and continuity

The lub of a chain of monotone functions is monotone.

lemma *adm-monofun*: *adm monofun*
⟨*proof*⟩

The lub of a chain of continuous functions is continuous.

lemma *adm-cont*: *adm cont*
⟨*proof*⟩

Function application preserves monotonicity and continuity.

lemma *mono2mono-fun*: *monofun f* ⟹ *monofun* (λx. *f x y*)
⟨*proof*⟩

lemma *cont2cont-fun*: *cont f* $\Longrightarrow$ *cont* $(\lambda x.\ f\ x\ y)$
⟨*proof*⟩

lemma *cont-fun*: *cont* $(\lambda f.\ f\ x)$
⟨*proof*⟩

Lambda abstraction preserves monotonicity and continuity. (Note $(\lambda x.\ \lambda y.\ f\ x\ y) = f$.)

lemma *mono2mono-lambda*:
assumes *f*: $\bigwedge y.$ *monofun* $(\lambda x.\ f\ x\ y)$ **shows** *monofun f*
⟨*proof*⟩

lemma *cont2cont-lambda* [*simp*]:
assumes *f*: $\bigwedge y.$ *cont* $(\lambda x.\ f\ x\ y)$ **shows** *cont f*
⟨*proof*⟩

What D.A.Schmidt calls continuity of abstraction; never used here

lemma *contlub-lambda*:
$(\bigwedge x{::}'a{::}type.\ chain\ (\lambda i.\ S\ i\ x{::}'b{::}cpo))$
$\Longrightarrow (\lambda x.\ \bigsqcup i.\ S\ i\ x) = (\bigsqcup i.\ (\lambda x.\ S\ i\ x))$
⟨*proof*⟩

end

7 Product-Cpo: The cpo of cartesian products

theory *Product-Cpo*
imports *Adm*
begin

default-sort *cpo*

7.1 Unit type is a pcpo

instantiation *unit* :: *discrete-cpo*
begin

definition
below-unit-def [*simp*]: $x \sqsubseteq (y{::}unit) \longleftrightarrow True$

instance ⟨*proof*⟩

end

instance *unit* :: *pcpo*
⟨*proof*⟩

7.2 Product type is a partial order

instantiation *prod* :: (*below*, *below*) *below*
begin

definition
below-prod-def: $(op \sqsubseteq) \equiv \lambda p1\ p2.\ (fst\ p1 \sqsubseteq fst\ p2 \wedge snd\ p1 \sqsubseteq snd\ p2)$

instance ⟨*proof*⟩
end

instance *prod* :: (*po*, *po*) *po*
⟨*proof*⟩

7.3 Monotonicity of *Pair*, *fst*, *snd*

lemma *prod-belowI*: $[\![fst\ p \sqsubseteq fst\ q;\ snd\ p \sqsubseteq snd\ q]\!] \Longrightarrow p \sqsubseteq q$
⟨*proof*⟩

lemma *Pair-below-iff* [*simp*]: $(a,\ b) \sqsubseteq (c,\ d) \longleftrightarrow a \sqsubseteq c \wedge b \sqsubseteq d$
⟨*proof*⟩

Pair (-,-) is monotone in both arguments

lemma *monofun-pair1*: *monofun* $(\lambda x.\ (x,\ y))$
⟨*proof*⟩

lemma *monofun-pair2*: *monofun* $(\lambda y.\ (x,\ y))$
⟨*proof*⟩

lemma *monofun-pair*:
$[\![x1 \sqsubseteq x2;\ y1 \sqsubseteq y2]\!] \Longrightarrow (x1,\ y1) \sqsubseteq (x2,\ y2)$
⟨*proof*⟩

lemma *ch2ch-Pair* [*simp*]:
chain $X \Longrightarrow$ *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ (X\ i,\ Y\ i))$
⟨*proof*⟩

fst and *snd* are monotone

lemma *fst-monofun*: $x \sqsubseteq y \Longrightarrow fst\ x \sqsubseteq fst\ y$
⟨*proof*⟩

lemma *snd-monofun*: $x \sqsubseteq y \Longrightarrow snd\ x \sqsubseteq snd\ y$
⟨*proof*⟩

lemma *monofun-fst*: *monofun fst*
⟨*proof*⟩

lemma *monofun-snd*: *monofun snd*
⟨*proof*⟩

lemmas *ch2ch-fst* [*simp*] = *ch2ch-monofun* [*OF monofun-fst*]

lemmas *ch2ch-snd* [*simp*] = *ch2ch-monofun* [*OF monofun-snd*]

lemma *prod-chain-cases*:
assumes *chain Y*
obtains *A B*
where *chain A* **and** *chain B* **and** $Y = (\lambda i.\ (A\ i,\ B\ i))$
⟨*proof*⟩

7.4 Product type is a cpo

lemma *is-lub-Pair*:
$[\![range\ A <<|\ x;\ range\ B <<|\ y]\!] \Longrightarrow range\ (\lambda i.\ (A\ i,\ B\ i)) <<|\ (x,\ y)$
⟨*proof*⟩

lemma *lub-Pair*:
$[\![chain\ (A{::}nat \Rightarrow 'a{::}cpo);\ chain\ (B{::}nat \Rightarrow 'b{::}cpo)]\!]$
$\Longrightarrow (\bigsqcup i.\ (A\ i,\ B\ i)) = (\bigsqcup i.\ A\ i,\ \bigsqcup i.\ B\ i)$
⟨*proof*⟩

lemma *is-lub-prod*:
fixes $S :: nat \Rightarrow ('a{::}cpo \times 'b{::}cpo)$
assumes *S*: *chain S*
shows *range S* <<| $(\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
⟨*proof*⟩

lemma *lub-prod*:
chain $(S{::}nat \Rightarrow 'a{::}cpo \times 'b{::}cpo)$
$\Longrightarrow (\bigsqcup i.\ S\ i) = (\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
⟨*proof*⟩

instance *prod* :: (*cpo*, *cpo*) *cpo*
⟨*proof*⟩

instance *prod* :: (*discrete-cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

7.5 Product type is pointed

lemma *minimal-prod*: $(\bot,\ \bot) \sqsubseteq p$
⟨*proof*⟩

instance *prod* :: (*pcpo*, *pcpo*) *pcpo*
⟨*proof*⟩

lemma *inst-prod-pcpo*: $\bot = (\bot,\ \bot)$
⟨*proof*⟩

lemma *Pair-bottom-iff* [*simp*]: $(x,\ y) = \bot \longleftrightarrow x = \bot \wedge y = \bot$

⟨*proof*⟩

lemma *fst-strict* [*simp*]: *fst* ⊥ = ⊥
⟨*proof*⟩

lemma *snd-strict* [*simp*]: *snd* ⊥ = ⊥
⟨*proof*⟩

lemma *Pair-strict* [*simp*]: (⊥, ⊥) = ⊥
⟨*proof*⟩

lemma *split-strict* [*simp*]: *split f* ⊥ = *f* ⊥ ⊥
⟨*proof*⟩

7.6 Continuity of *Pair*, *fst*, *snd*

lemma *cont-pair1*: *cont* (λ*x*. (*x*, *y*))
⟨*proof*⟩

lemma *cont-pair2*: *cont* (λ*y*. (*x*, *y*))
⟨*proof*⟩

lemma *cont-fst*: *cont fst*
⟨*proof*⟩

lemma *cont-snd*: *cont snd*
⟨*proof*⟩

lemma *cont2cont-Pair* [*simp*, *cont2cont*]:
assumes *f*: *cont* (λ*x*. *f x*)
assumes *g*: *cont* (λ*x*. *g x*)
shows *cont* (λ*x*. (*f x*, *g x*))
⟨*proof*⟩

lemmas *cont2cont-fst* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-fst*]

lemmas *cont2cont-snd* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-snd*]

lemma *cont2cont-case-prod*:
assumes *f1*: ⋀*a b*. *cont* (λ*x*. *f x a b*)
assumes *f2*: ⋀*x b*. *cont* (λ*a*. *f x a b*)
assumes *f3*: ⋀*x a*. *cont* (λ*b*. *f x a b*)
assumes *g*: *cont* (λ*x*. *g x*)
shows *cont* (λ*x*. *case g x of* (*a*, *b*) ⇒ *f x a b*)
⟨*proof*⟩

lemma *prod-contI*:
assumes *f1*: ⋀*y*. *cont* (λ*x*. *f* (*x*, *y*))
assumes *f2*: ⋀*x*. *cont* (λ*y*. *f* (*x*, *y*))

shows *cont f*
⟨*proof*⟩

lemma *prod-cont-iff*:
cont f ⟷ (∀ *y*. *cont* (λ*x*. *f* (*x*, *y*))) ∧ (∀ *x*. *cont* (λ*y*. *f* (*x*, *y*)))
⟨*proof*⟩

lemma *cont2cont-case-prod'* [*simp*, *cont2cont*]:
assumes *f*: *cont* (λ*p*. *f* (*fst p*) (*fst* (*snd p*)) (*snd* (*snd p*)))
assumes *g*: *cont* (λ*x*. *g x*)
shows *cont* (λ*x*. *case-prod* (*f x*) (*g x*))
⟨*proof*⟩

The simple version (due to Joachim Breitner) is needed if either element type of the pair is not a cpo.

lemma *cont2cont-split-simple* [*simp*, *cont2cont*]:
assumes ⋀*a b*. *cont* (λ*x*. *f x a b*)
shows *cont* (λ*x*. *case p of* (*a*, *b*) ⇒ *f x a b*)
⟨*proof*⟩

Admissibility of predicates on product types.

lemma *adm-case-prod* [*simp*]:
assumes *adm* (λ*x*. *P x* (*fst* (*f x*)) (*snd* (*f x*)))
shows *adm* (λ*x*. *case f x of* (*a*, *b*) ⇒ *P x a b*)
⟨*proof*⟩

7.7 Compactness and chain-finiteness

lemma *fst-below-iff*: *fst* (*x*::'*a* × '*b*) ⊑ *y* ⟷ *x* ⊑ (*y*, *snd x*)
⟨*proof*⟩

lemma *snd-below-iff*: *snd* (*x*::'*a* × '*b*) ⊑ *y* ⟷ *x* ⊑ (*fst x*, *y*)
⟨*proof*⟩

lemma *compact-fst*: *compact x* ⟹ *compact* (*fst x*)
⟨*proof*⟩

lemma *compact-snd*: *compact x* ⟹ *compact* (*snd x*)
⟨*proof*⟩

lemma *compact-Pair*: ⟦*compact x*; *compact y*⟧ ⟹ *compact* (*x*, *y*)
⟨*proof*⟩

lemma *compact-Pair-iff* [*simp*]: *compact* (*x*, *y*) ⟷ *compact x* ∧ *compact y*
⟨*proof*⟩

instance *prod* :: (*chfin*, *chfin*) *chfin*
⟨*proof*⟩

end

8 Cfun: The type of continuous functions

theory *Cfun*
imports *Cpodef Fun-Cpo Product-Cpo*
begin

default-sort *cpo*

8.1 Definition of continuous function type

definition *cfun* = $\{f::'a \Rightarrow 'b.\ cont\ f\}$

cpodef ('*a*, '*b*) *cfun* (**infixr** −> *0*) = *cfun* :: ('*a* => '*b*) *set*
⟨*proof*⟩

type-notation (*xsymbols*)
cfun ((- →/ -) [*1*, *0*] *0*)

notation
Rep-cfun ((-$/-) [*999*,*1000*] *999*)

notation (*xsymbols*)
Rep-cfun ((-·/-) [*999*,*1000*] *999*)

notation (*HTML* **output**)
Rep-cfun ((-·/-) [*999*,*1000*] *999*)

8.2 Syntax for continuous lambda abstraction

syntax *-cabs* :: [*logic*, *logic*] ⇒ *logic*

⟨*ML*⟩

Syntax for nested abstractions

syntax
-Lambda :: [*cargs*, *logic*] ⇒ *logic* ((*3LAM* -./ -) [*1000*, *10*] *10*)

syntax (*xsymbols*)
-Lambda :: [*cargs*, *logic*] ⇒ *logic* ((*3*Λ -./ -) [*1000*, *10*] *10*)

⟨*ML*⟩

Dummy patterns for continuous abstraction

translations
Λ -. *t* => *CONST Abs-cfun* (λ -. *t*)

8.3 Continuous function space is pointed

lemma *bottom-cfun*: $\bot \in$ *cfun*
⟨*proof*⟩

instance *cfun* :: (*cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

instance *cfun* :: (*cpo*, *pcpo*) *pcpo*
⟨*proof*⟩

lemmas *Rep-cfun-strict* =
typedef-Rep-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

lemmas *Abs-cfun-strict* =
typedef-Abs-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

function application is strict in its first argument

lemma *Rep-cfun-strict1* [*simp*]: $\bot \cdot x = \bot$
⟨*proof*⟩

lemma *LAM-strict* [*simp*]: $(\Lambda\ x.\ \bot) = \bot$
⟨*proof*⟩

for compatibility with old HOLCF-Version

lemma *inst-cfun-pcpo*: $\bot = (\Lambda\ x.\ \bot)$
⟨*proof*⟩

8.4 Basic properties of continuous functions

Beta-equality for continuous functions

lemma *Abs-cfun-inverse2*: *cont f* $\Longrightarrow$ *Rep-cfun* (*Abs-cfun f*) = *f*
⟨*proof*⟩

lemma *beta-cfun*: *cont f* $\Longrightarrow (\Lambda\ x.\ f\ x)\cdot u = f\ u$
⟨*proof*⟩

Beta-reduction simproc

Given the term $(\Lambda\ x.\ f\ x)\cdot y$, the procedure tries to construct the theorem $(\Lambda\ x.\ f\ x)\cdot y \equiv f\ y$. If this theorem cannot be completely solved by the cont2cont rules, then the procedure returns the ordinary conditional *beta-cfun* rule.

The simproc does not solve any more goals that would be solved by using *beta-cfun* as a simp rule. The advantage of the simproc is that it can avoid deeply-nested calls to the simplifier that would otherwise be caused by large continuity side conditions.

Update: The simproc now uses rule *Abs-cfun-inverse2* instead of *beta-cfun*, to avoid problems with eta-contraction.

⟨*ML*⟩

Eta-equality for continuous functions

lemma *eta-cfun*: $(\Lambda\ x.\ f{\cdot}x) = f$
⟨*proof*⟩

Extensionality for continuous functions

lemma *cfun-eq-iff*: $f = g \longleftrightarrow (\forall\, x.\ f{\cdot}x = g{\cdot}x)$
⟨*proof*⟩

lemma *cfun-eqI*: $(\bigwedge x.\ f{\cdot}x = g{\cdot}x) \Longrightarrow f = g$
⟨*proof*⟩

Extensionality wrt. ordering for continuous functions

lemma *cfun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall\, x.\ f{\cdot}x \sqsubseteq g{\cdot}x)$
⟨*proof*⟩

lemma *cfun-belowI*: $(\bigwedge x.\ f{\cdot}x \sqsubseteq g{\cdot}x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

Congruence for continuous function application

lemma *cfun-cong*: $[\![f = g;\ x = y]\!] \Longrightarrow f{\cdot}x = g{\cdot}y$
⟨*proof*⟩

lemma *cfun-fun-cong*: $f = g \Longrightarrow f{\cdot}x = g{\cdot}x$
⟨*proof*⟩

lemma *cfun-arg-cong*: $x = y \Longrightarrow f{\cdot}x = f{\cdot}y$
⟨*proof*⟩

8.5 Continuity of application

lemma *cont-Rep-cfun1*: *cont* $(\lambda f.\ f{\cdot}x)$
⟨*proof*⟩

lemma *cont-Rep-cfun2*: *cont* $(\lambda x.\ f{\cdot}x)$
⟨*proof*⟩

lemmas *monofun-Rep-cfun* = *cont-Rep-cfun* [*THEN cont2mono*]

lemmas *monofun-Rep-cfun1* = *cont-Rep-cfun1* [*THEN cont2mono*]
lemmas *monofun-Rep-cfun2* = *cont-Rep-cfun2* [*THEN cont2mono*]

contlub, cont properties of *Rep-cfun* in each argument

lemma *contlub-cfun-arg*: *chain* $Y \Longrightarrow f{\cdot}(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *contlub-cfun-fun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i){\cdot}x = (\bigsqcup i.\ F\ i{\cdot}x)$
⟨*proof*⟩

monotonicity of application

lemma *monofun-cfun-fun*: $f \sqsubseteq g \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}x$
⟨*proof*⟩

lemma *monofun-cfun-arg*: $x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq f{\cdot}y$
⟨*proof*⟩

lemma *monofun-cfun*: $[\![f \sqsubseteq g;\ x \sqsubseteq y]\!] \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}y$
⟨*proof*⟩

ch2ch - rules for the type $'a \to 'b$

lemma *chain-monofun*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-Rep-cfunR*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-Rep-cfunL*: *chain* $F \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}x)$
⟨*proof*⟩

lemma *ch2ch-Rep-cfun* [*simp*]:
$[\![$*chain* F; *chain* $Y]\!] \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-LAM* [*simp*]:
$[\![\bigwedge x.$ *chain* $(\lambda i.\ S\ i\ x);\ \bigwedge i.$ *cont* $(\lambda x.\ S\ i\ x)]\!] \Longrightarrow$ *chain* $(\lambda i.\ \Lambda\ x.\ S\ i\ x)$
⟨*proof*⟩

contlub, cont properties of *Rep-cfun* in both arguments

lemma *lub-APP*:
$[\![$*chain* F; *chain* $Y]\!] \Longrightarrow (\bigsqcup i.\ F\ i{\cdot}(Y\ i)) = (\bigsqcup i.\ F\ i){\cdot}(\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *lub-LAM*:
$[\![\bigwedge x.$ *chain* $(\lambda i.\ F\ i\ x);\ \bigwedge i.$ *cont* $(\lambda x.\ F\ i\ x)]\!]$
$\Longrightarrow (\bigsqcup i.\ \Lambda\ x.\ F\ i\ x) = (\Lambda\ x.\ \bigsqcup i.\ F\ i\ x)$
⟨*proof*⟩

lemmas *lub-distribs* = *lub-APP lub-LAM*

strictness

lemma *strictI*: $f{\cdot}x = \bot \Longrightarrow f{\cdot}\bot = \bot$
⟨*proof*⟩

type $'a \to 'b$ is chain complete

lemma *lub-cfun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i) = (\Lambda\ x.\ \bigsqcup i.\ F\ i{\cdot}x)$
⟨*proof*⟩

8.6 Continuity simplification procedure

cont2cont lemma for *Rep-cfun*

lemma *cont2cont-APP* [*simp*, *cont2cont*]:
assumes f: *cont* $(\lambda x.\ f\ x)$
assumes t: *cont* $(\lambda x.\ t\ x)$
shows *cont* $(\lambda x.\ (f\ x)\cdot(t\ x))$
⟨*proof*⟩

Two specific lemmas for the combination of LCF and HOL terms. These lemmas are needed in theories that use types like $'a \to 'b \Rightarrow 'c$.

lemma *cont-APP-app* [*simp*]: ⟦*cont f*; *cont g*⟧ ⟹ *cont* $(\lambda x.\ ((f\ x)\cdot(g\ x))\ s)$
⟨*proof*⟩

lemma *cont-APP-app-app* [*simp*]: ⟦*cont f*; *cont g*⟧ ⟹ *cont* $(\lambda x.\ ((f\ x)\cdot(g\ x))\ s\ t)$
⟨*proof*⟩

cont2mono Lemma for $\lambda x.\ \Lambda\ y.\ c1\ x\ y$

lemma *cont2mono-LAM*:
⟦$\bigwedge x.$ *cont* $(\lambda y.\ f\ x\ y)$; $\bigwedge y.$ *monofun* $(\lambda x.\ f\ x\ y)$⟧
⟹ *monofun* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

cont2cont Lemma for $\lambda x.\ \Lambda\ y.\ f\ x\ y$

Not suitable as a cont2cont rule, because on nested lambdas it causes exponential blow-up in the number of subgoals.

lemma *cont2cont-LAM*:
assumes *f1*: $\bigwedge x.$ *cont* $(\lambda y.\ f\ x\ y)$
assumes *f2*: $\bigwedge y.$ *cont* $(\lambda x.\ f\ x\ y)$
shows *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

This version does work as a cont2cont rule, since it has only a single subgoal.

lemma *cont2cont-LAM'* [*simp*, *cont2cont*]:
fixes f :: $'a$::*cpo* ⇒ $'b$::*cpo* ⇒ $'c$::*cpo*
assumes f: *cont* $(\lambda p.\ f\ (fst\ p)\ (snd\ p))$
shows *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

lemma *cont2cont-LAM-discrete* [*simp*, *cont2cont*]:
$(\bigwedge y::'a::$*discrete-cpo*. *cont* $(\lambda x.\ f\ x\ y)) \Longrightarrow$ *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

8.7 Miscellaneous

Monotonicity of *Abs-cfun*

lemma *monofun-LAM*:
$[\![cont\ f;\ cont\ g;\ \bigwedge x.\ f\ x \sqsubseteq g\ x]\!] \Longrightarrow (\Lambda\ x.\ f\ x) \sqsubseteq (\Lambda\ x.\ g\ x)$
⟨*proof*⟩

some lemmata for functions with flat/chfin domain/range types

lemma *chfin-Rep-cfunR*: *chain* $(Y{::}nat \Rightarrow {'a}{::}cpo \mathrel{-\!\!>} {'b}{::}chfin)$
$\Longrightarrow\ !s.\ ?\ n.\ (LUB\ i.\ Y\ i)\$s = Y\ n\$s$
⟨*proof*⟩

lemma *adm-chfindom*: *adm* $(\lambda(u{::}{'a}{::}cpo \to {'b}{::}chfin).\ P(u{\cdot}s))$
⟨*proof*⟩

8.8 Continuous injection-retraction pairs

Continuous retractions are strict.

lemma *retraction-strict*:
$\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *injection-eq*:
$\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow (g{\cdot}x = g{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *injection-below*:
$\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow (g{\cdot}x \sqsubseteq g{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *injection-defined-rev*:
$[\![\forall x.\ f{\cdot}(g{\cdot}x) = x;\ g{\cdot}z = \bot]\!] \Longrightarrow z = \bot$
⟨*proof*⟩

lemma *injection-defined*:
$[\![\forall x.\ f{\cdot}(g{\cdot}x) = x;\ z \neq \bot]\!] \Longrightarrow g{\cdot}z \neq \bot$
⟨*proof*⟩

a result about functions with flat codomain

lemma *flat-eqI*: $[\![(x{::}{'a}{::}flat) \sqsubseteq y;\ x \neq \bot]\!] \Longrightarrow x = y$
⟨*proof*⟩

lemma *flat-codom*:
$f{\cdot}x = (c{::}{'b}{::}flat) \Longrightarrow f{\cdot}\bot = \bot \vee (\forall z.\ f{\cdot}z = c)$
⟨*proof*⟩

8.9 Identity and composition

definition
$ID :: {'a} \to {'a}$ **where**
$ID = (\Lambda\ x.\ x)$

definition
cfcomp :: $('b \to 'c) \to ('a \to 'b) \to 'a \to 'c$ **where**
oo-def: $cfcomp = (\Lambda\ f\ g\ x.\ f{\cdot}(g{\cdot}x))$

abbreviation
cfcomp-syn :: $['b \to 'c,\ 'a \to 'b] \Rightarrow 'a \to 'c$ (**infixr** *oo 100*) **where**
$f\ oo\ g == cfcomp{\cdot}f{\cdot}g$

lemma *ID1* [*simp*]: $ID{\cdot}x = x$
⟨*proof*⟩

lemma *cfcomp1*: $(f\ oo\ g) = (\Lambda\ x.\ f{\cdot}(g{\cdot}x))$
⟨*proof*⟩

lemma *cfcomp2* [*simp*]: $(f\ oo\ g){\cdot}x = f{\cdot}(g{\cdot}x)$
⟨*proof*⟩

lemma *cfcomp-LAM*: $cont\ g \Longrightarrow f\ oo\ (\Lambda\ x.\ g\ x) = (\Lambda\ x.\ f{\cdot}(g\ x))$
⟨*proof*⟩

lemma *cfcomp-strict* [*simp*]: $\bot\ oo\ f = \bot$
⟨*proof*⟩

Show that interpretation of (pcpo,-—>-) is a category. The class of objects is interpretation of syntactical class pcpo. The class of arrows between objects $'a$ and $'b$ is interpret. of $'a \to 'b$. The identity arrow is interpretation of *ID*. The composition of f and g is interpretation of *oo*.

lemma *ID2* [*simp*]: $f\ oo\ ID = f$
⟨*proof*⟩

lemma *ID3* [*simp*]: $ID\ oo\ f = f$
⟨*proof*⟩

lemma *assoc-oo*: $f\ oo\ (g\ oo\ h) = (f\ oo\ g)\ oo\ h$
⟨*proof*⟩

8.10 Strictified functions

default-sort *pcpo*

definition
seq :: $'a \to 'b \to 'b$ **where**
$seq = (\Lambda\ x.\ if\ x = \bot\ then\ \bot\ else\ ID)$

lemma *cont2cont-if-bottom* [*cont2cont*, *simp*]:
assumes *f*: $cont\ (\lambda x.\ f\ x)$ **and** *g*: $cont\ (\lambda x.\ g\ x)$
shows $cont\ (\lambda x.\ if\ f\ x = \bot\ then\ \bot\ else\ g\ x)$
⟨*proof*⟩

lemma *seq-conv-if*: $seq{\cdot}x = (\mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ ID)$
⟨*proof*⟩

lemma *seq-simps* [*simp*]:
$seq{\cdot}\bot = \bot$
$seq{\cdot}x{\cdot}\bot = \bot$
$x \neq \bot \Longrightarrow seq{\cdot}x = ID$
⟨*proof*⟩

definition
$strictify :: ('a \to 'b) \to 'a \to 'b$ **where**
$strictify = (\Lambda\ f\ x.\ seq{\cdot}x{\cdot}(f{\cdot}x))$

lemma *strictify-conv-if*: $strictify{\cdot}f{\cdot}x = (\mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ f{\cdot}x)$
⟨*proof*⟩

lemma *strictify1* [*simp*]: $strictify{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *strictify2* [*simp*]: $x \neq \bot \Longrightarrow strictify{\cdot}f{\cdot}x = f{\cdot}x$
⟨*proof*⟩

8.11 Continuity of let-bindings

lemma *cont2cont-Let*:
assumes *f*: $\mathit{cont}\ (\lambda x.\ f\ x)$
assumes *g1*: $\bigwedge y.\ \mathit{cont}\ (\lambda x.\ g\ x\ y)$
assumes *g2*: $\bigwedge x.\ \mathit{cont}\ (\lambda y.\ g\ x\ y)$
shows $\mathit{cont}\ (\lambda x.\ \mathit{let}\ y = f\ x\ \mathit{in}\ g\ x\ y)$
⟨*proof*⟩

lemma *cont2cont-Let'* [*simp*, *cont2cont*]:
assumes *f*: $\mathit{cont}\ (\lambda x.\ f\ x)$
assumes *g*: $\mathit{cont}\ (\lambda p.\ g\ (\mathit{fst}\ p)\ (\mathit{snd}\ p))$
shows $\mathit{cont}\ (\lambda x.\ \mathit{let}\ y = f\ x\ \mathit{in}\ g\ x\ y)$
⟨*proof*⟩

The simple version (suggested by Joachim Breitner) is needed if the type of the defined term is not a cpo.

lemma *cont2cont-Let-simple* [*simp*, *cont2cont*]:
assumes $\bigwedge y.\ \mathit{cont}\ (\lambda x.\ g\ x\ y)$
shows $\mathit{cont}\ (\lambda x.\ \mathit{let}\ y = t\ \mathit{in}\ g\ x\ y)$
⟨*proof*⟩

end

9 Sfun: The Strict Function Type

theory *Sfun*
imports *Cfun*
begin

pcpodef $('a, 'b)$ *sfun* (**infixr** $-\!>!$ *0*)
$= \{f :: 'a \to 'b.\ f{\cdot}\bot = \bot\}$
⟨*proof*⟩

type-notation (*xsymbols*)
sfun (**infixr** $\to!$ *0*)

TODO: Define nice syntax for abstraction, application.

definition
sfun-abs :: $('a \to 'b) \to ('a \to! 'b)$
where
sfun-abs $= (\Lambda\ f.$ *Abs-sfun* $(strictify{\cdot}f))$

definition
sfun-rep :: $('a \to! 'b) \to 'a \to 'b$
where
sfun-rep $= (\Lambda\ f.$ *Rep-sfun* $f)$

lemma *sfun-rep-beta*: *sfun-rep*$\cdot f =$ *Rep-sfun* f
⟨*proof*⟩

lemma *sfun-rep-strict1* [*simp*]: *sfun-rep*$\cdot\bot = \bot$
⟨*proof*⟩

lemma *sfun-rep-strict2* [*simp*]: *sfun-rep*$\cdot f\cdot\bot = \bot$
⟨*proof*⟩

lemma *strictify-cancel*: $f{\cdot}\bot = \bot \Longrightarrow strictify{\cdot}f = f$
⟨*proof*⟩

lemma *sfun-abs-sfun-rep* [*simp*]: *sfun-abs*$\cdot$(*sfun-rep*$\cdot f$) $= f$
⟨*proof*⟩

lemma *sfun-rep-sfun-abs* [*simp*]: *sfun-rep*$\cdot$(*sfun-abs*$\cdot f$) $=$ *strictify*$\cdot f$
⟨*proof*⟩

lemma *sfun-eq-iff*: $f = g \longleftrightarrow$ *sfun-rep*$\cdot f =$ *sfun-rep*$\cdot g$
⟨*proof*⟩

lemma *sfun-below-iff*: $f \sqsubseteq g \longleftrightarrow$ *sfun-rep*$\cdot f \sqsubseteq$ *sfun-rep*$\cdot g$
⟨*proof*⟩

end

10 Cprod: The cpo of cartesian products

theory *Cprod*
imports *Cfun*
begin

default-sort *cpo*

10.1 Continuous case function for unit type

definition
unit-when :: $'a \to unit \to 'a$ **where**
unit-when = $(\Lambda\ a\ \text{-}.\ a)$

translations
$\Lambda().\ t == CONST\ unit\text{-}when{\cdot}t$

lemma *unit-when* [*simp*]: $unit\text{-}when{\cdot}a{\cdot}u = a$
⟨*proof*⟩

10.2 Continuous version of split function

definition
csplit :: $('a \to 'b \to 'c) \to ('a * 'b) \to 'c$ **where**
csplit = $(\Lambda\ f\ p.\ f{\cdot}(fst\ p){\cdot}(snd\ p))$

translations
$\Lambda(CONST\ Pair\ x\ y).\ t == CONST\ csplit{\cdot}(\Lambda\ x\ y.\ t)$

abbreviation
cfst :: $'a \times 'b \to 'a$ **where**
cfst ≡ *Abs-cfun fst*

abbreviation
csnd :: $'a \times 'b \to 'b$ **where**
csnd ≡ *Abs-cfun snd*

10.3 Convert all lemmas to the continuous versions

lemma *csplit1* [*simp*]: $csplit{\cdot}f{\cdot}\bot = f{\cdot}\bot{\cdot}\bot$
⟨*proof*⟩

lemma *csplit-Pair* [*simp*]: $csplit{\cdot}f{\cdot}(x,\ y) = f{\cdot}x{\cdot}y$
⟨*proof*⟩

end

11 Sprod: The type of strict products

theory *Sprod*

imports *Cfun*
begin

default-sort *pcpo*

11.1 Definition of strict product type

definition $sprod = \{p::'a \times 'b.\ p = \bot \vee (fst\ p \neq \bot \wedge snd\ p \neq \bot)\}$

pcpodef $('a, 'b)\ sprod$ (**infixr** $**$ *20*) $= sprod :: ('a \times 'b)\ set$
⟨*proof*⟩

instance $sprod :: (\{chfin,pcpo\}, \{chfin,pcpo\})\ chfin$
⟨*proof*⟩

type-notation (*xsymbols*)
$sprod$ $((- \otimes/ -)\ [21,20]\ 20)$
type-notation (*HTML* **output**)
$sprod$ $((- \otimes/ -)\ [21,20]\ 20)$

11.2 Definitions of constants

definition
$sfst :: ('a ** 'b) \to 'a$ **where**
$sfst = (\Lambda\ p.\ fst\ (Rep\text{-}sprod\ p))$

definition
$ssnd :: ('a ** 'b) \to 'b$ **where**
$ssnd = (\Lambda\ p.\ snd\ (Rep\text{-}sprod\ p))$

definition
$spair :: 'a \to 'b \to ('a ** 'b)$ **where**
$spair = (\Lambda\ a\ b.\ Abs\text{-}sprod\ (seq{\cdot}b{\cdot}a,\ seq{\cdot}a{\cdot}b))$

definition
$ssplit :: ('a \to 'b \to 'c) \to ('a ** 'b) \to 'c$ **where**
$ssplit = (\Lambda\ f\ p.\ seq{\cdot}p{\cdot}(f{\cdot}(sfst{\cdot}p){\cdot}(ssnd{\cdot}p)))$

syntax
$-stuple :: [logic,\ args] \Rightarrow logic$ $((1'(:-,/\ -:')))$

translations
$(:x,\ y,\ z:) == (:x,\ (:y,\ z:):)$
$(:x,\ y:) \quad == CONST\ spair{\cdot}x{\cdot}y$

translations
$\Lambda(CONST\ spair{\cdot}x{\cdot}y).\ t == CONST\ ssplit{\cdot}(\Lambda\ x\ y.\ t)$

11.3 Case analysis

lemma *spair-sprod*: $(seq{\cdot}b{\cdot}a,\ seq{\cdot}a{\cdot}b) \in sprod$
⟨*proof*⟩

lemma *Rep-sprod-spair*: $Rep\text{-}sprod\ (:a,\ b:) = (seq{\cdot}b{\cdot}a,\ seq{\cdot}a{\cdot}b)$
⟨*proof*⟩

lemmas *Rep-sprod-simps* =
Rep-sprod-inject [*symmetric*] *below-sprod-def*
prod-eq-iff below-prod-def
Rep-sprod-strict Rep-sprod-spair

lemma *sprodE* [*case-names bottom spair*, *cases type*: *sprod*]:
obtains $p = \bot \mid x\ y$ **where** $p = (:x,\ y:)$ **and** $x \neq \bot$ **and** $y \neq \bot$
⟨*proof*⟩

lemma *sprod-induct* [*case-names bottom spair*, *induct type*: *sprod*]:
$[\![P\ \bot;\ \bigwedge x\ y.\ [\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow P\ (:x,\ y:)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

11.4 Properties of *spair*

lemma *spair-strict1* [*simp*]: $(:\bot,\ y:) = \bot$
⟨*proof*⟩

lemma *spair-strict2* [*simp*]: $(:x,\ \bot:) = \bot$
⟨*proof*⟩

lemma *spair-bottom-iff* [*simp*]: $((:x,\ y:) = \bot) = (x = \bot \vee y = \bot)$
⟨*proof*⟩

lemma *spair-below-iff*:
$((:a,\ b:) \sqsubseteq (:c,\ d:)) = (a = \bot \vee b = \bot \vee (a \sqsubseteq c \wedge b \sqsubseteq d))$
⟨*proof*⟩

lemma *spair-eq-iff*:
$((:a,\ b:) = (:c,\ d:)) =$
$(a = c \wedge b = d \vee (a = \bot \vee b = \bot) \wedge (c = \bot \vee d = \bot))$
⟨*proof*⟩

lemma *spair-strict*: $x = \bot \vee y = \bot \Longrightarrow (:x,\ y:) = \bot$
⟨*proof*⟩

lemma *spair-strict-rev*: $(:x,\ y:) \neq \bot \Longrightarrow x \neq \bot \wedge y \neq \bot$
⟨*proof*⟩

lemma *spair-defined*: $[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \neq \bot$
⟨*proof*⟩

lemma *spair-defined-rev*: $(:x,\ y:) = \bot \Longrightarrow x = \bot \vee y = \bot$
⟨*proof*⟩

lemma *spair-below*:
$[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \sqsubseteq (:a,\ b:) = (x \sqsubseteq a \wedge y \sqsubseteq b)$
⟨*proof*⟩

lemma *spair-eq*:
$[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow ((:x,\ y:) = (:a,\ b:)) = (x = a \wedge y = b)$
⟨*proof*⟩

lemma *spair-inject*:
$[\![x \neq \bot;\ y \neq \bot;\ (:x,\ y:) = (:a,\ b:)]\!] \Longrightarrow x = a \wedge y = b$
⟨*proof*⟩

lemma *inst-sprod-pcpo2*: $\bot = (:\bot,\ \bot:)$
⟨*proof*⟩

lemma *sprodE2*: $(\bigwedge x\ y.\ p = (:x,\ y:) \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

11.5 Properties of *sfst* and *ssnd*

lemma *sfst-strict* [*simp*]: $sfst{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *ssnd-strict* [*simp*]: $ssnd{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sfst-spair* [*simp*]: $y \neq \bot \Longrightarrow sfst{\cdot}(:x,\ y:) = x$
⟨*proof*⟩

lemma *ssnd-spair* [*simp*]: $x \neq \bot \Longrightarrow ssnd{\cdot}(:x,\ y:) = y$
⟨*proof*⟩

lemma *sfst-bottom-iff* [*simp*]: $(sfst{\cdot}p = \bot) = (p = \bot)$
⟨*proof*⟩

lemma *ssnd-bottom-iff* [*simp*]: $(ssnd{\cdot}p = \bot) = (p = \bot)$
⟨*proof*⟩

lemma *sfst-defined*: $p \neq \bot \Longrightarrow sfst{\cdot}p \neq \bot$
⟨*proof*⟩

lemma *ssnd-defined*: $p \neq \bot \Longrightarrow ssnd{\cdot}p \neq \bot$
⟨*proof*⟩

lemma *spair-sfst-ssnd*: $(:sfst{\cdot}p,\ ssnd{\cdot}p:) = p$
⟨*proof*⟩

lemma *below-sprod*: $(x \sqsubseteq y) = (sfst{\cdot}x \sqsubseteq sfst{\cdot}y \wedge ssnd{\cdot}x \sqsubseteq ssnd{\cdot}y)$
⟨*proof*⟩

lemma *eq-sprod*: $(x = y) = (sfst{\cdot}x = sfst{\cdot}y \wedge ssnd{\cdot}x = ssnd{\cdot}y)$
⟨*proof*⟩

lemma *sfst-below-iff*: $sfst{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:y,\ ssnd{\cdot}x:)$
⟨*proof*⟩

lemma *ssnd-below-iff*: $ssnd{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:sfst{\cdot}x,\ y:)$
⟨*proof*⟩

11.6 Compactness

lemma *compact-sfst*: $compact\ x \Longrightarrow compact\ (sfst{\cdot}x)$
⟨*proof*⟩

lemma *compact-ssnd*: $compact\ x \Longrightarrow compact\ (ssnd{\cdot}x)$
⟨*proof*⟩

lemma *compact-spair*: $[\![compact\ x;\ compact\ y]\!] \Longrightarrow compact\ (:x,\ y:)$
⟨*proof*⟩

lemma *compact-spair-iff*:
$compact\ (:x,\ y:) = (x = \bot \vee y = \bot \vee (compact\ x \wedge compact\ y))$
⟨*proof*⟩

11.7 Properties of *ssplit*

lemma *ssplit1* [*simp*]: $ssplit{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *ssplit2* [*simp*]: $[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow ssplit{\cdot}f{\cdot}(:x,\ y:) = f{\cdot}x{\cdot}y$
⟨*proof*⟩

lemma *ssplit3* [*simp*]: $ssplit{\cdot}spair{\cdot}z = z$
⟨*proof*⟩

11.8 Strict product preserves flatness

instance *sprod* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

end

12 Discrete: Discrete cpo types

theory *Discrete*

imports *Cont*
begin

datatype *'a discr = Discr 'a :: type*

12.1 Discrete cpo class instance

instantiation *discr :: (type) discrete-cpo*
begin

definition
(op ⊑ :: 'a discr ⇒ 'a discr ⇒ bool) = (op =)

instance
⟨*proof*⟩

end

12.2 *undiscr*

definition
undiscr :: ('a::type)discr => 'a **where**
undiscr x = (case x of Discr y => y)

lemma *undiscr-Discr* [*simp*]: *undiscr (Discr x) = x*
⟨*proof*⟩

lemma *Discr-undiscr* [*simp*]: *Discr (undiscr y) = y*
⟨*proof*⟩

end

13 Up: The type of lifted values

theory *Up*
imports *Cfun*
begin

default-sort *cpo*

13.1 Definition of new type for lifting

datatype *'a u = Ibottom | Iup 'a*

type-notation (*xsymbols*)
u ((-⊥) [*1000*] *999*)

primrec *Ifup :: ('a → 'b::pcpo) ⇒ 'a u ⇒ 'b* **where**
Ifup f Ibottom = ⊥

| *Ifup f (Iup x) = f·x*

13.2 Ordering on lifted cpo

instantiation *u* :: (*cpo*) *below*
begin

definition
below-up-def:
(*op* ⊑) ≡ (λ*x y*. *case x of Ibottom* ⇒ *True* | *Iup a* ⇒
(*case y of Ibottom* ⇒ *False* | *Iup b* ⇒ *a* ⊑ *b*))

instance ⟨*proof*⟩
end

lemma *minimal-up* [*iff*]: *Ibottom* ⊑ *z*
⟨*proof*⟩

lemma *not-Iup-below* [*iff*]: *Iup x* ⋢ *Ibottom*
⟨*proof*⟩

lemma *Iup-below* [*iff*]: (*Iup x* ⊑ *Iup y*) = (*x* ⊑ *y*)
⟨*proof*⟩

13.3 Lifted cpo is a partial order

instance *u* :: (*cpo*) *po*
⟨*proof*⟩

13.4 Lifted cpo is a cpo

lemma *is-lub-Iup*:
range S <<| *x* ⟹ *range* (λ*i*. *Iup* (*S i*)) <<| *Iup x*
⟨*proof*⟩

lemma *up-chain-lemma*:
assumes *Y*: *chain Y* **obtains** ∀ *i*. *Y i* = *Ibottom*
| *A k* **where** ∀ *i*. *Iup* (*A i*) = *Y* (*i* + *k*) **and** *chain A* **and** *range Y* <<| *Iup* (⨆ *i*. *A i*)
⟨*proof*⟩

instance *u* :: (*cpo*) *cpo*
⟨*proof*⟩

13.5 Lifted cpo is pointed

instance *u* :: (*cpo*) *pcpo*
⟨*proof*⟩

for compatibility with old HOLCF-Version

lemma *inst-up-pcpo*: $\bot = Ibottom$
⟨*proof*⟩

13.6 Continuity of *Iup* and *Ifup*

continuity for *Iup*

lemma *cont-Iup*: *cont Iup*
⟨*proof*⟩

continuity for *Ifup*

lemma *cont-Ifup1*: *cont* $(\lambda f.\ Ifup\ f\ x)$
⟨*proof*⟩

lemma *monofun-Ifup2*: *monofun* $(\lambda x.\ Ifup\ f\ x)$
⟨*proof*⟩

lemma *cont-Ifup2*: *cont* $(\lambda x.\ Ifup\ f\ x)$
⟨*proof*⟩

13.7 Continuous versions of constants

definition
up :: $'a \to 'a\ u$ **where**
$up = (\Lambda\ x.\ Iup\ x)$

definition
fup :: $('a \to 'b{::}pcpo) \to 'a\ u \to 'b$ **where**
$fup = (\Lambda\ f\ p.\ Ifup\ f\ p)$

translations
$case\ l\ of\ XCONST\ up{\cdot}x \Rightarrow t == CONST\ fup{\cdot}(\Lambda\ x.\ t){\cdot}l$
$case\ l\ of\ (XCONST\ up :: 'a){\cdot}x \Rightarrow t => CONST\ fup{\cdot}(\Lambda\ x.\ t){\cdot}l$
$\Lambda(XCONST\ up{\cdot}x).\ t == CONST\ fup{\cdot}(\Lambda\ x.\ t)$

continuous versions of lemmas for $'a_\bot$

lemma *Exh-Up*: $z = \bot \vee (\exists\, x.\ z = up{\cdot}x)$
⟨*proof*⟩

lemma *up-eq* [*simp*]: $(up{\cdot}x = up{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *up-inject*: $up{\cdot}x = up{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

lemma *up-defined* [*simp*]: $up{\cdot}x \neq \bot$
⟨*proof*⟩

lemma *not-up-less-UU*: $up{\cdot}x \not\sqsubseteq \bot$

⟨*proof*⟩

lemma *up-below* [*simp*]: $up{\cdot}x \sqsubseteq up{\cdot}y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *upE* [*case-names bottom up*, *cases type*: *u*]:
$[\![p = \bot \Longrightarrow Q; \bigwedge x.\ p = up{\cdot}x \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *up-induct* [*case-names bottom up*, *induct type*: *u*]:
$[\![P\ \bot; \bigwedge x.\ P\ (up{\cdot}x)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

lifting preserves chain-finiteness

lemma *up-chain-cases*:
assumes *Y*: *chain Y* **obtains** $\forall i.\ Y\ i = \bot$
| *A k* **where** $\forall i.\ up{\cdot}(A\ i) = Y\ (i + k)$ **and** *chain A* **and** $(\bigsqcup i.\ Y\ i) = up{\cdot}(\bigsqcup i.\ A\ i)$
⟨*proof*⟩

lemma *compact-up*: *compact x* $\Longrightarrow$ *compact* $(up{\cdot}x)$
⟨*proof*⟩

lemma *compact-upD*: *compact* $(up{\cdot}x) \Longrightarrow$ *compact x*
⟨*proof*⟩

lemma *compact-up-iff* [*simp*]: *compact* $(up{\cdot}x) =$ *compact x*
⟨*proof*⟩

instance *u* :: (*chfin*) *chfin*
⟨*proof*⟩

properties of fup

lemma *fup1* [*simp*]: $fup{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *fup2* [*simp*]: $fup{\cdot}f{\cdot}(up{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

lemma *fup3* [*simp*]: $fup{\cdot}up{\cdot}x = x$
⟨*proof*⟩

end

14 Lift: Lifting types of class type to flat pcpo's

theory *Lift*
imports *Discrete Up*

begin

default-sort *type*

pcpodef *'a lift = UNIV :: 'a discr u set*
⟨*proof*⟩

lemmas *inst-lift-pcpo = Abs-lift-strict* [*symmetric*]

definition
Def :: 'a ⇒ 'a lift **where**
Def x = Abs-lift (up·(Discr x))

14.1 Lift as a datatype

lemma *lift-induct*: $[\![P\ \bot;\ \bigwedge x.\ P\ (Def\ x)]\!] \Longrightarrow P\ y$
⟨*proof*⟩

rep-datatype $\bot$*::'a lift Def*
⟨*proof*⟩

$\bot$ and *Def*

lemma *not-Undef-is-Def*: $(x \neq \bot) = (\exists\, y.\ x = Def\ y)$
⟨*proof*⟩

lemma *lift-definedE*: $[\![x \neq \bot;\ \bigwedge a.\ x = Def\ a \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

For $x \neq \bot$ in assumptions *defined* replaces x by *Def a* in conclusion.

⟨*ML*⟩

lemma *DefE*: $Def\ x = \bot \Longrightarrow R$
⟨*proof*⟩

lemma *DefE2*: $[\![x = Def\ s;\ x = \bot]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *Def-below-Def*: $Def\ x \sqsubseteq Def\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *Def-below-iff* [*simp*]: $Def\ x \sqsubseteq y \longleftrightarrow Def\ x = y$
⟨*proof*⟩

14.2 Lift is flat

instance *lift* :: (*type*) *flat*
⟨*proof*⟩

14.3 Continuity of *case-lift*

lemma *case-lift-eq*: *case-lift* $\bot$ *f x* = *fup*·(Λ *y*. *f* (*undiscr y*))·(*Rep-lift x*)
⟨*proof*⟩

lemma *cont2cont-case-lift* [*simp*]:
⟦⋀*y*. *cont* (λ*x*. *f x y*); *cont g*⟧ ⟹ *cont* (λ*x*. *case-lift* $\bot$ (*f x*) (*g x*))
⟨*proof*⟩

14.4 Further operations

definition
flift1 :: ('*a* ⇒ '*b*::*pcpo*) ⇒ ('*a lift* → '*b*) (**binder** *FLIFT* 10) **where**
flift1 = (λ*f*. (Λ *x*. *case-lift* $\bot$ *f x*))

translations
Λ(*XCONST Def x*). *t* => *CONST flift1* (λ*x*. *t*)
Λ(*CONST Def x*). *FLIFT y*. *t* <= *FLIFT x y*. *t*
Λ(*CONST Def x*). *t* <= *FLIFT x*. *t*

definition
flift2 :: ('*a* ⇒ '*b*) ⇒ ('*a lift* → '*b lift*) **where**
flift2 f = (*FLIFT x*. *Def* (*f x*))

lemma *flift1-Def* [*simp*]: *flift1 f*·(*Def x*) = (*f x*)
⟨*proof*⟩

lemma *flift2-Def* [*simp*]: *flift2 f*·(*Def x*) = *Def* (*f x*)
⟨*proof*⟩

lemma *flift1-strict* [*simp*]: *flift1 f*·$\bot$ = $\bot$
⟨*proof*⟩

lemma *flift2-strict* [*simp*]: *flift2 f*·$\bot$ = $\bot$
⟨*proof*⟩

lemma *flift2-defined* [*simp*]: *x* ≠ $\bot$ ⟹ (*flift2 f*)·*x* ≠ $\bot$
⟨*proof*⟩

lemma *flift2-bottom-iff* [*simp*]: (*flift2 f*·*x* = $\bot$) = (*x* = $\bot$)
⟨*proof*⟩

lemma *FLIFT-mono*:
(⋀*x*. *f x* ⊑ *g x*) ⟹ (*FLIFT x*. *f x*) ⊑ (*FLIFT x*. *g x*)
⟨*proof*⟩

lemma *cont2cont-flift1* [*simp*, *cont2cont*]:
⟦⋀*y*. *cont* (λ*x*. *f x y*)⟧ ⟹ *cont* (λ*x*. *FLIFT y*. *f x y*)
⟨*proof*⟩

end

15 Tr: The type of lifted booleans

theory *Tr*
imports *Lift*
begin

15.1 Type definition and constructors

type-synonym
tr = *bool lift*

translations
(*type*) *tr* <= (*type*) *bool lift*

definition
TT :: *tr* **where**
TT = *Def True*

definition
FF :: *tr* **where**
FF = *Def False*

Exhaustion and Elimination for type *tr*

lemma *Exh-tr*: $t = \bot \vee t = TT \vee t = FF$
⟨*proof*⟩

lemma *trE* [*case-names bottom TT FF*, *cases type*: *tr*]:
$[\![p = \bot \Longrightarrow Q; p = TT \Longrightarrow Q; p = FF \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *tr-induct* [*case-names bottom TT FF*, *induct type*: *tr*]:
$[\![P\ \bot; P\ TT; P\ FF]\!] \Longrightarrow P\ x$
⟨*proof*⟩

distinctness for type *tr*

lemma *dist-below-tr* [*simp*]:
$TT \not\sqsubseteq \bot\ FF \not\sqsubseteq \bot\ TT \not\sqsubseteq FF\ FF \not\sqsubseteq TT$
⟨*proof*⟩

lemma *dist-eq-tr* [*simp*]:
$TT \neq \bot\ FF \neq \bot\ TT \neq FF\ \bot \neq TT\ \bot \neq FF\ FF \neq TT$
⟨*proof*⟩

lemma *TT-below-iff* [*simp*]: $TT \sqsubseteq x \longleftrightarrow x = TT$
⟨*proof*⟩

lemma *FF-below-iff* [*simp*]: $FF \sqsubseteq x \longleftrightarrow x = FF$
⟨*proof*⟩

lemma *not-below-TT-iff* [*simp*]: $x \not\sqsubseteq TT \longleftrightarrow x = FF$
⟨*proof*⟩

lemma *not-below-FF-iff* [*simp*]: $x \not\sqsubseteq FF \longleftrightarrow x = TT$
⟨*proof*⟩

15.2 Case analysis

default-sort *pcpo*

definition *tr-case* :: $'a \to 'a \to tr \to 'a$ **where**
tr-case = $(\Lambda\ t\ e\ (Def\ b).$ *if b then t else e*$)$

abbreviation
cifte-syn :: $[tr, 'c, 'c] \Rightarrow 'c$ ((*If* (-)/ *then* (-)/ *else* (-)) [*0*, *0*, *60*] *60*)
where
If b then e1 else e2 == *tr-case*·*e1*·*e2*·*b*

translations
$\Lambda\ (XCONST\ TT).\ t$ == *CONST tr-case*·*t*·$\bot$
$\Lambda\ (XCONST\ FF).\ t$ == *CONST tr-case*·$\bot$·*t*

lemma *ifte-thms* [*simp*]:
If $\bot$ *then e1 else e2* = $\bot$
If FF then e1 else e2 = *e2*
If TT then e1 else e2 = *e1*
⟨*proof*⟩

15.3 Boolean connectives

definition
trand :: $tr \to tr \to tr$ **where**
andalso-def: *trand* = $(\Lambda\ x\ y.$ *If x then y else FF*$)$
abbreviation
andalso-syn :: $tr \Rightarrow tr \Rightarrow tr$ (- *andalso* - [*36*,*35*] *35*) **where**
x andalso y == *trand*·*x*·*y*

definition
tror :: $tr \to tr \to tr$ **where**
orelse-def: *tror* = $(\Lambda\ x\ y.$ *If x then TT else y*$)$
abbreviation
orelse-syn :: $tr \Rightarrow tr \Rightarrow tr$ (- *orelse* - [*31*,*30*] *30*) **where**
x orelse y == *tror*·*x*·*y*

definition
neg :: $tr \to tr$ **where**
neg = *flift2 Not*

definition
If2 :: [*tr*, *'c*, *'c*] ⇒ *'c* **where**
If2 Q x y = (*If Q then x else y*)

tactic for tr-thms with case split

lemmas *tr-defs* = *andalso-def orelse-def neg-def tr-case-def TT-def FF-def*

lemmas about andalso, orelse, neg and if

lemma *andalso-thms* [*simp*]:
(*TT andalso y*) = *y*
(*FF andalso y*) = *FF*
(⊥ *andalso y*) = ⊥
(*y andalso TT*) = *y*
(*y andalso y*) = *y*
⟨*proof*⟩

lemma *orelse-thms* [*simp*]:
(*TT orelse y*) = *TT*
(*FF orelse y*) = *y*
(⊥ *orelse y*) = ⊥
(*y orelse FF*) = *y*
(*y orelse y*) = *y*
⟨*proof*⟩

lemma *neg-thms* [*simp*]:
neg·*TT* = *FF*
neg·*FF* = *TT*
neg·⊥ = ⊥
⟨*proof*⟩

split-tac for If via If2 because the constant has to be a constant

lemma *split-If2*:
P (*If2 Q x y*) = ((*Q* = ⊥ ⟶ *P* ⊥) ∧ (*Q* = *TT* ⟶ *P x*) ∧ (*Q* = *FF* ⟶ *P y*))
⟨*proof*⟩

⟨*ML*⟩

15.4 Rewriting of HOLCF operations to HOL functions

lemma *andalso-or*:
t ≠ ⊥ ⟹ ((*t andalso s*) = *FF*) = (*t* = *FF* ∨ *s* = *FF*)
⟨*proof*⟩

lemma *andalso-and*:
t ≠ ⊥ ⟹ ((*t andalso s*) ≠ *FF*) = (*t* ≠ *FF* ∧ *s* ≠ *FF*)
⟨*proof*⟩

lemma *Def-bool1* [*simp*]: $(Def\ x \neq FF) = x$
⟨*proof*⟩

lemma *Def-bool2* [*simp*]: $(Def\ x = FF) = (\neg\ x)$
⟨*proof*⟩

lemma *Def-bool3* [*simp*]: $(Def\ x = TT) = x$
⟨*proof*⟩

lemma *Def-bool4* [*simp*]: $(Def\ x \neq TT) = (\neg\ x)$
⟨*proof*⟩

lemma *If-and-if*:
$(If\ Def\ P\ then\ A\ else\ B) = (if\ P\ then\ A\ else\ B)$
⟨*proof*⟩

15.5 Compactness

lemma *compact-TT*: *compact TT*
⟨*proof*⟩

lemma *compact-FF*: *compact FF*
⟨*proof*⟩

end

16 Ssum: The type of strict sums

theory *Ssum*
imports *Tr*
begin

default-sort *pcpo*

16.1 Definition of strict sum type

definition
$ssum =$
$\{p :: tr \times ('a \times 'b).\ p = \bot \vee$
$(fst\ p = TT \wedge fst\ (snd\ p) \neq \bot \wedge snd\ (snd\ p) = \bot) \vee$
$(fst\ p = FF \wedge fst\ (snd\ p) = \bot \wedge snd\ (snd\ p) \neq \bot)\}$

pcpodef $('a, 'b)\ ssum$ (**infixr** ++ *10*) $= ssum :: (tr \times 'a \times 'b)\ set$
⟨*proof*⟩

instance $ssum :: (\{chfin,pcpo\}, \{chfin,pcpo\})\ chfin$
⟨*proof*⟩

type-notation (*xsymbols*)
ssum ((- ⊕/ -) [*21*, *20*] *20*)
type-notation (*HTML* **output**)
ssum ((- ⊕/ -) [*21*, *20*] *20*)

16.2 Definitions of constructors

definition
sinl :: $'a \to ('a \mathbin{++} 'b)$ **where**
$sinl = (\Lambda\ a.\ Abs\text{-}ssum\ (seq{\cdot}a{\cdot}TT,\ a,\ \bot))$

definition
sinr :: $'b \to ('a \mathbin{++} 'b)$ **where**
$sinr = (\Lambda\ b.\ Abs\text{-}ssum\ (seq{\cdot}b{\cdot}FF,\ \bot,\ b))$

lemma *sinl-ssum*: $(seq{\cdot}a{\cdot}TT,\ a,\ \bot) \in ssum$
⟨*proof*⟩

lemma *sinr-ssum*: $(seq{\cdot}b{\cdot}FF,\ \bot,\ b) \in ssum$
⟨*proof*⟩

lemma *Rep-ssum-sinl*: $Rep\text{-}ssum\ (sinl{\cdot}a) = (seq{\cdot}a{\cdot}TT,\ a,\ \bot)$
⟨*proof*⟩

lemma *Rep-ssum-sinr*: $Rep\text{-}ssum\ (sinr{\cdot}b) = (seq{\cdot}b{\cdot}FF,\ \bot,\ b)$
⟨*proof*⟩

lemmas *Rep-ssum-simps* =
Rep-ssum-inject [*symmetric*] *below-ssum-def*
prod-eq-iff below-prod-def
Rep-ssum-strict Rep-ssum-sinl Rep-ssum-sinr

16.3 Properties of *sinl* and *sinr*

Ordering

lemma *sinl-below* [*simp*]: $(sinl{\cdot}x \sqsubseteq sinl{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *sinr-below* [*simp*]: $(sinr{\cdot}x \sqsubseteq sinr{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *sinl-below-sinr* [*simp*]: $(sinl{\cdot}x \sqsubseteq sinr{\cdot}y) = (x = \bot)$
⟨*proof*⟩

lemma *sinr-below-sinl* [*simp*]: $(sinr{\cdot}x \sqsubseteq sinl{\cdot}y) = (x = \bot)$
⟨*proof*⟩

Equality

lemma *sinl-eq* [*simp*]: $(sinl{\cdot}x = sinl{\cdot}y) = (x = y)$

⟨*proof*⟩

lemma *sinr-eq* [*simp*]: $(sinr{\cdot}x = sinr{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *sinl-eq-sinr* [*simp*]: $(sinl{\cdot}x = sinr{\cdot}y) = (x = \bot \wedge y = \bot)$
⟨*proof*⟩

lemma *sinr-eq-sinl* [*simp*]: $(sinr{\cdot}x = sinl{\cdot}y) = (x = \bot \wedge y = \bot)$
⟨*proof*⟩

lemma *sinl-inject*: $sinl{\cdot}x = sinl{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

lemma *sinr-inject*: $sinr{\cdot}x = sinr{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

Strictness

lemma *sinl-strict* [*simp*]: $sinl{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sinr-strict* [*simp*]: $sinr{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sinl-bottom-iff* [*simp*]: $(sinl{\cdot}x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *sinr-bottom-iff* [*simp*]: $(sinr{\cdot}x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *sinl-defined*: $x \neq \bot \Longrightarrow sinl{\cdot}x \neq \bot$
⟨*proof*⟩

lemma *sinr-defined*: $x \neq \bot \Longrightarrow sinr{\cdot}x \neq \bot$
⟨*proof*⟩

Compactness

lemma *compact-sinl*: $compact\ x \Longrightarrow compact\ (sinl{\cdot}x)$
⟨*proof*⟩

lemma *compact-sinr*: $compact\ x \Longrightarrow compact\ (sinr{\cdot}x)$
⟨*proof*⟩

lemma *compact-sinlD*: $compact\ (sinl{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-sinrD*: $compact\ (sinr{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-sinl-iff* [*simp*]: *compact* (*sinl*·*x*) = *compact x*
⟨*proof*⟩

lemma *compact-sinr-iff* [*simp*]: *compact* (*sinr*·*x*) = *compact x*
⟨*proof*⟩

16.4 Case analysis

lemma *ssumE* [*case-names bottom sinl sinr*, *cases type*: *ssum*]:
obtains $p = \bot$
| x **where** $p = sinl{\cdot}x$ **and** $x \neq \bot$
| y **where** $p = sinr{\cdot}y$ **and** $y \neq \bot$
⟨*proof*⟩

lemma *ssum-induct* [*case-names bottom sinl sinr*, *induct type*: *ssum*]:
$[\![P\ \bot;$
$\bigwedge x.\ x \neq \bot \Longrightarrow P\ (sinl{\cdot}x);$
$\bigwedge y.\ y \neq \bot \Longrightarrow P\ (sinr{\cdot}y)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

lemma *ssumE2* [*case-names sinl sinr*]:
$[\![\bigwedge x.\ p = sinl{\cdot}x \Longrightarrow Q;\ \bigwedge y.\ p = sinr{\cdot}y \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *below-sinlD*: $p \sqsubseteq sinl{\cdot}x \Longrightarrow \exists\, y.\ p = sinl{\cdot}y \wedge y \sqsubseteq x$
⟨*proof*⟩

lemma *below-sinrD*: $p \sqsubseteq sinr{\cdot}x \Longrightarrow \exists\, y.\ p = sinr{\cdot}y \wedge y \sqsubseteq x$
⟨*proof*⟩

16.5 Case analysis combinator

definition
sscase :: $('a \to 'c) \to ('b \to 'c) \to ('a \mathbin{++} 'b) \to 'c$ **where**
sscase = $(\Lambda\ f\ g\ s.\ (\lambda(t,\ x,\ y).\ If\ t\ then\ f{\cdot}x\ else\ g{\cdot}y)\ (Rep\text{-}ssum\ s))$

translations
case s of XCONST sinl·*x* ⇒ *t1* | *XCONST sinr*·*y* ⇒ *t2* == *CONST sscase*·(Λ *x. t1*)·(Λ *y. t2*)·*s*
case s of (*XCONST sinl* :: *'a*)·*x* ⇒ *t1* | *XCONST sinr*·*y* ⇒ *t2* => *CONST sscase*·(Λ *x. t1*)·(Λ *y. t2*)·*s*

translations
Λ(*XCONST sinl*·*x*). *t* == *CONST sscase*·(Λ *x. t*)·⊥
Λ(*XCONST sinr*·*y*). *t* == *CONST sscase*·⊥·(Λ *y. t*)

lemma *beta-sscase*:
sscase·*f*·*g*·*s* = $(\lambda(t,\ x,\ y).\ If\ t\ then\ f{\cdot}x\ else\ g{\cdot}y)\ (Rep\text{-}ssum\ s)$
⟨*proof*⟩

lemma *sscase1* [*simp*]: $sscase{\cdot}f{\cdot}g{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sscase2* [*simp*]: $x \neq \bot \Longrightarrow sscase{\cdot}f{\cdot}g{\cdot}(sinl{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

lemma *sscase3* [*simp*]: $y \neq \bot \Longrightarrow sscase{\cdot}f{\cdot}g{\cdot}(sinr{\cdot}y) = g{\cdot}y$
⟨*proof*⟩

lemma *sscase4* [*simp*]: $sscase{\cdot}sinl{\cdot}sinr{\cdot}z = z$
⟨*proof*⟩

16.6 Strict sum preserves flatness

instance *ssum* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

end

17 One: The unit domain

theory *One*
imports *Lift*
begin

type-synonym
one = *unit lift*

translations
(*type*) *one* <= (*type*) *unit lift*

definition *ONE* :: *one*
where *ONE* == *Def* ()

Exhaustion and Elimination for type *one*

lemma *Exh-one*: $t = \bot \vee t = ONE$
⟨*proof*⟩

lemma *oneE* [*case-names bottom ONE*]: $[\![p = \bot \Longrightarrow Q;\ p = ONE \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *one-induct* [*case-names bottom ONE*]: $[\![P\ \bot;\ P\ ONE]\!] \Longrightarrow P\ x$
⟨*proof*⟩

lemma *dist-below-one* [*simp*]: $ONE \not\sqsubseteq \bot$
⟨*proof*⟩

lemma *below-ONE* [*simp*]: $x \sqsubseteq ONE$
⟨*proof*⟩

lemma *ONE-below-iff* [*simp*]: $ONE \sqsubseteq x \longleftrightarrow x = ONE$
⟨*proof*⟩

lemma *ONE-defined* [*simp*]: $ONE \neq \bot$
⟨*proof*⟩

lemma *one-neq-iffs* [*simp*]:
$x \neq ONE \longleftrightarrow x = \bot$
$ONE \neq x \longleftrightarrow x = \bot$
$x \neq \bot \longleftrightarrow x = ONE$
$\bot \neq x \longleftrightarrow x = ONE$
⟨*proof*⟩

lemma *compact-ONE*: *compact ONE*
⟨*proof*⟩

Case analysis function for type *one*

definition
one-case :: $'a$::*pcpo* → *one* → $'a$ **where**
one-case = (Λ *a x*. *seq*·*x*·*a*)

translations
case x of XCONST ONE ⇒ *t* == *CONST one-case*·*t*·*x*
case x of XCONST ONE :: $'a$ ⇒ *t* => *CONST one-case*·*t*·*x*
Λ (*XCONST ONE*). *t* == *CONST one-case*·*t*

lemma *one-case1* [*simp*]: (*case* ⊥ *of ONE* ⇒ *t*) = ⊥
⟨*proof*⟩

lemma *one-case2* [*simp*]: (*case ONE of ONE* ⇒ *t*) = *t*
⟨*proof*⟩

lemma *one-case3* [*simp*]: (*case x of ONE* ⇒ *ONE*) = *x*
⟨*proof*⟩

end

18 Fix: Fixed point operator and admissibility

theory *Fix*
imports *Cfun*
begin

default-sort *pcpo*

18.1 Iteration

primrec *iterate* :: *nat* ⇒ ($'a$::*cpo* → $'a$) → ($'a$ → $'a$) **where**

iterate 0 = (Λ F x. x)
| *iterate (Suc n) = (Λ F x. F·(iterate n·F·x))*

Derive inductive properties of iterate from primitive recursion

lemma *iterate-0* [*simp*]: *iterate 0·F·x = x*
⟨*proof*⟩

lemma *iterate-Suc* [*simp*]: *iterate (Suc n)·F·x = F·(iterate n·F·x)*
⟨*proof*⟩

declare *iterate.simps* [*simp del*]

lemma *iterate-Suc2*: *iterate (Suc n)·F·x = iterate n·F·(F·x)*
⟨*proof*⟩

lemma *iterate-iterate*:
iterate m·F·(iterate n·F·x) = iterate (m + n)·F·x
⟨*proof*⟩

The sequence of function iterations is a chain.

lemma *chain-iterate* [*simp*]: *chain (λi. iterate i·F·⊥)*
⟨*proof*⟩

18.2 Least fixed point operator

definition
fix :: *('a → 'a) → 'a* **where**
fix = (Λ F. ⨆ i. iterate i·F·⊥)

Binder syntax for *fix*

abbreviation
fix-syn :: *('a ⇒ 'a) ⇒ 'a* (**binder** *FIX* *10*) **where**
fix-syn (λx. f x) ≡ fix·(Λ x. f x)

notation (*xsymbols*)
fix-syn (**binder** *μ* *10*)

Properties of *fix*

direct connection between *fix* and iteration

lemma *fix-def2*: *fix·F = (⨆ i. iterate i·F·⊥)*
⟨*proof*⟩

lemma *iterate-below-fix*: *iterate n·f·⊥ ⊑ fix·f*
⟨*proof*⟩

Kleene's fixed point theorems for continuous functions in pointed omega cpo's

lemma *fix-eq*: $fix{\cdot}F = F{\cdot}(fix{\cdot}F)$
⟨*proof*⟩

lemma *fix-least-below*: $F{\cdot}x \sqsubseteq x \Longrightarrow fix{\cdot}F \sqsubseteq x$
⟨*proof*⟩

lemma *fix-least*: $F{\cdot}x = x \Longrightarrow fix{\cdot}F \sqsubseteq x$
⟨*proof*⟩

lemma *fix-eqI*:
assumes *fixed*: $F{\cdot}x = x$ **and** *least*: $\bigwedge z.\ F{\cdot}z = z \Longrightarrow x \sqsubseteq z$
shows $fix{\cdot}F = x$
⟨*proof*⟩

lemma *fix-eq2*: $f \equiv fix{\cdot}F \Longrightarrow f = F{\cdot}f$
⟨*proof*⟩

lemma *fix-eq3*: $f \equiv fix{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
⟨*proof*⟩

lemma *fix-eq4*: $f = fix{\cdot}F \Longrightarrow f = F{\cdot}f$
⟨*proof*⟩

lemma *fix-eq5*: $f = fix{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
⟨*proof*⟩

strictness of *fix*

lemma *fix-bottom-iff*: $(fix{\cdot}F = \bot) = (F{\cdot}\bot = \bot)$
⟨*proof*⟩

lemma *fix-strict*: $F{\cdot}\bot = \bot \Longrightarrow fix{\cdot}F = \bot$
⟨*proof*⟩

lemma *fix-defined*: $F{\cdot}\bot \neq \bot \Longrightarrow fix{\cdot}F \neq \bot$
⟨*proof*⟩

fix applied to identity and constant functions

lemma *fix-id*: $(\mu\ x.\ x) = \bot$
⟨*proof*⟩

lemma *fix-const*: $(\mu\ x.\ c) = c$
⟨*proof*⟩

18.3 Fixed point induction

lemma *fix-ind*: $[\![adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)]\!] \Longrightarrow P\ (fix{\cdot}F)$
⟨*proof*⟩

lemma *cont-fix-ind*:

⟦*cont F*; *adm P*; *P* ⊥; ⋀*x*. *P x* ⟹ *P* (*F x*)⟧ ⟹ *P* (*fix*·(*Abs-cfun F*))
⟨*proof*⟩

lemma *def-fix-ind*:
⟦*f* ≡ *fix*·*F*; *adm P*; *P* ⊥; ⋀*x*. *P x* ⟹ *P* (*F*·*x*)⟧ ⟹ *P f*
⟨*proof*⟩

lemma *fix-ind2*:
assumes *adm*: *adm P*
assumes *0*: *P* ⊥ **and** *1*: *P* (*F*·⊥)
assumes *step*: ⋀*x*. ⟦*P x*; *P* (*F*·*x*)⟧ ⟹ *P* (*F*·(*F*·*x*))
shows *P* (*fix*·*F*)
⟨*proof*⟩

lemma *parallel-fix-ind*:
assumes *adm*: *adm* (λ*x*. *P* (*fst x*) (*snd x*))
assumes *base*: *P* ⊥ ⊥
assumes *step*: ⋀*x y*. *P x y* ⟹ *P* (*F*·*x*) (*G*·*y*)
shows *P* (*fix*·*F*) (*fix*·*G*)
⟨*proof*⟩

lemma *cont-parallel-fix-ind*:
assumes *cont F* **and** *cont G*
assumes *adm* (λ*x*. *P* (*fst x*) (*snd x*))
assumes *P* ⊥ ⊥
assumes ⋀*x y*. *P x y* ⟹ *P* (*F x*) (*G y*)
shows *P* (*fix*·(*Abs-cfun F*)) (*fix*·(*Abs-cfun G*))
⟨*proof*⟩

18.4 Fixed-points on product types

Bekic's Theorem: Simultaneous fixed points over pairs can be written in terms of separate fixed points.

lemma *fix-cprod*:
fix·(*F*::'*a* × '*b* → '*a* × '*b*) =
(μ *x*. *fst* (*F*·(*x*, μ *y*. *snd* (*F*·(*x*, *y*)))),
μ *y*. *snd* (*F*·(μ *x*. *fst* (*F*·(*x*, μ *y*. *snd* (*F*·(*x*, *y*)))), *y*)))
(**is** *fix*·*F* = (*?x*, *?y*))
⟨*proof*⟩

end

19 Plain-HOLCF: Plain HOLCF

theory *Plain-HOLCF*
imports *Cfun Sfun Cprod Sprod Ssum Up Discrete Lift One Tr Fix*
begin

Basic HOLCF concepts and types; does not include definition packages.

end

20 Fixrec: Package for defining recursive functions in HOLCF

theory *Fixrec*
imports *Plain-HOLCF*
keywords *fixrec* :: *thy-decl*
begin

20.1 Pattern-match monad

default-sort *cpo*

pcpodef *'a match* = *UNIV*::(*one* ++ *'a u*) *set*
⟨*proof*⟩

definition
fail :: *'a match* **where**
fail = *Abs-match* (*sinl*·*ONE*)

definition
succeed :: *'a* → *'a match* **where**
succeed = (Λ *x*. *Abs-match* (*sinr*·(*up*·*x*)))

lemma *matchE* [*case-names bottom fail succeed*, *cases type*: *match*]:
⟦*p* = ⊥ ⟹ *Q*; *p* = *fail* ⟹ *Q*; ⋀*x*. *p* = *succeed*·*x* ⟹ *Q*⟧ ⟹ *Q*
⟨*proof*⟩

lemma *succeed-defined* [*simp*]: *succeed*·*x* ≠ ⊥
⟨*proof*⟩

lemma *fail-defined* [*simp*]: *fail* ≠ ⊥
⟨*proof*⟩

lemma *succeed-eq* [*simp*]: (*succeed*·*x* = *succeed*·*y*) = (*x* = *y*)
⟨*proof*⟩

lemma *succeed-neq-fail* [*simp*]:
succeed·*x* ≠ *fail* *fail* ≠ *succeed*·*x*
⟨*proof*⟩

20.1.1 Run operator

definition
run :: *'a match* → *'a*::*pcpo* **where**
run = (Λ *m*. *sscase*·⊥·(*fup*·*ID*)·(*Rep-match m*))

rewrite rules for run

lemma *run-strict* [*simp*]: $run{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *run-fail* [*simp*]: $run{\cdot}fail = \bot$
⟨*proof*⟩

lemma *run-succeed* [*simp*]: $run{\cdot}(succeed{\cdot}x) = x$
⟨*proof*⟩

20.1.2 Monad plus operator

definition
mplus :: *'a match* → *'a match* → *'a match* **where**
$mplus = (\Lambda\ m1\ m2.\ sscase{\cdot}(\Lambda\ \text{-}.\ m2){\cdot}(\Lambda\ \text{-}.\ m1){\cdot}(Rep\text{-}match\ m1))$

abbreviation
mplus-syn :: [*'a match*, *'a match*] ⇒ *'a match* (**infixr** +++ *65*) **where**
$m1 +\!\!+\!\!+ m2 == mplus{\cdot}m1{\cdot}m2$

rewrite rules for mplus

lemma *mplus-strict* [*simp*]: $\bot +\!\!+\!\!+ m = \bot$
⟨*proof*⟩

lemma *mplus-fail* [*simp*]: $fail +\!\!+\!\!+ m = m$
⟨*proof*⟩

lemma *mplus-succeed* [*simp*]: $succeed{\cdot}x +\!\!+\!\!+ m = succeed{\cdot}x$
⟨*proof*⟩

lemma *mplus-fail2* [*simp*]: $m +\!\!+\!\!+ fail = m$
⟨*proof*⟩

lemma *mplus-assoc*: $(x +\!\!+\!\!+ y) +\!\!+\!\!+ z = x +\!\!+\!\!+ (y +\!\!+\!\!+ z)$
⟨*proof*⟩

20.2 Match functions for built-in types

default-sort *pcpo*

definition
match-bottom :: *'a* → *'c match* → *'c match*
where
$match\text{-}bottom = (\Lambda\ x\ k.\ seq{\cdot}x{\cdot}fail)$

definition
match-Pair :: *'a*::*cpo* × *'b*::*cpo* → (*'a* → *'b* → *'c match*) → *'c match*
where
$match\text{-}Pair = (\Lambda\ x\ k.\ csplit{\cdot}k{\cdot}x)$

definition
match-spair :: $'a \otimes 'b \to ('a \to 'b \to 'c\ match) \to 'c\ match$
where
$match\text{-}spair = (\Lambda\ x\ k.\ ssplit{\cdot}k{\cdot}x)$

definition
match-sinl :: $'a \oplus 'b \to ('a \to 'c\ match) \to 'c\ match$
where
$match\text{-}sinl = (\Lambda\ x\ k.\ sscase{\cdot}k{\cdot}(\Lambda\ b.\ fail){\cdot}x)$

definition
match-sinr :: $'a \oplus 'b \to ('b \to 'c\ match) \to 'c\ match$
where
$match\text{-}sinr = (\Lambda\ x\ k.\ sscase{\cdot}(\Lambda\ a.\ fail){\cdot}k{\cdot}x)$

definition
match-up :: $'a{::}cpo\ u \to ('a \to 'c\ match) \to 'c\ match$
where
$match\text{-}up = (\Lambda\ x\ k.\ fup{\cdot}k{\cdot}x)$

definition
match-ONE :: $one \to 'c\ match \to 'c\ match$
where
$match\text{-}ONE = (\Lambda\ ONE\ k.\ k)$

definition
match-TT :: $tr \to 'c\ match \to 'c\ match$
where
$match\text{-}TT = (\Lambda\ x\ k.\ If\ x\ then\ k\ else\ fail)$

definition
match-FF :: $tr \to 'c\ match \to 'c\ match$
where
$match\text{-}FF = (\Lambda\ x\ k.\ If\ x\ then\ fail\ else\ k)$

lemma *match-bottom-simps* [*simp*]:
$match\text{-}bottom{\cdot}x{\cdot}k = (if\ x = \bot\ then\ \bot\ else\ fail)$
⟨*proof*⟩

lemma *match-Pair-simps* [*simp*]:
$match\text{-}Pair{\cdot}(x,\ y){\cdot}k = k{\cdot}x{\cdot}y$
⟨*proof*⟩

lemma *match-spair-simps* [*simp*]:
$[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow match\text{-}spair{\cdot}(:x,\ y:){\cdot}k = k{\cdot}x{\cdot}y$
$match\text{-}spair{\cdot}\bot{\cdot}k = \bot$
⟨*proof*⟩

lemma *match-sinl-simps* [*simp*]:
$x \neq \bot \Longrightarrow \mathit{match\text{-}sinl}\cdot(\mathit{sinl}\cdot x)\cdot k = k\cdot x$
$y \neq \bot \Longrightarrow \mathit{match\text{-}sinl}\cdot(\mathit{sinr}\cdot y)\cdot k = \mathit{fail}$
$\mathit{match\text{-}sinl}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-sinr-simps* [*simp*]:
$x \neq \bot \Longrightarrow \mathit{match\text{-}sinr}\cdot(\mathit{sinl}\cdot x)\cdot k = \mathit{fail}$
$y \neq \bot \Longrightarrow \mathit{match\text{-}sinr}\cdot(\mathit{sinr}\cdot y)\cdot k = k\cdot y$
$\mathit{match\text{-}sinr}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-up-simps* [*simp*]:
$\mathit{match\text{-}up}\cdot(\mathit{up}\cdot x)\cdot k = k\cdot x$
$\mathit{match\text{-}up}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-ONE-simps* [*simp*]:
$\mathit{match\text{-}ONE}\cdot\mathit{ONE}\cdot k = k$
$\mathit{match\text{-}ONE}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-TT-simps* [*simp*]:
$\mathit{match\text{-}TT}\cdot\mathit{TT}\cdot k = k$
$\mathit{match\text{-}TT}\cdot\mathit{FF}\cdot k = \mathit{fail}$
$\mathit{match\text{-}TT}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

lemma *match-FF-simps* [*simp*]:
$\mathit{match\text{-}FF}\cdot\mathit{FF}\cdot k = k$
$\mathit{match\text{-}FF}\cdot\mathit{TT}\cdot k = \mathit{fail}$
$\mathit{match\text{-}FF}\cdot\bot\cdot k = \bot$
⟨*proof*⟩

20.3 Mutual recursion

The following rules are used to prove unfolding theorems from fixed-point definitions of mutually recursive functions.

lemma *Pair-equalI*: $[\![x \equiv \mathit{fst}\ p;\ y \equiv \mathit{snd}\ p]\!] \Longrightarrow (x,\ y) \equiv p$
⟨*proof*⟩

lemma *Pair-eqD1*: $(x,\ y) = (x',\ y') \Longrightarrow x = x'$
⟨*proof*⟩

lemma *Pair-eqD2*: $(x,\ y) = (x',\ y') \Longrightarrow y = y'$
⟨*proof*⟩

lemma *def-cont-fix-eq*:
$[\![f \equiv \mathit{fix}\cdot(\mathit{Abs\text{-}cfun}\ F);\ \mathit{cont}\ F]\!] \Longrightarrow f = F\ f$

⟨*proof*⟩

lemma *def-cont-fix-ind*:
  ⟦$f \equiv$ *fix*·(*Abs-cfun F*); *cont F*; *adm P*; *P* ⊥; ⋀*x*. *P x* ⟹ *P* (*F x*)⟧ ⟹ *P f*
⟨*proof*⟩

lemma for proving rewrite rules

lemma *ssubst-lhs*: ⟦*t* = *s*; *P s* = *Q*⟧ ⟹ *P t* = *Q*
⟨*proof*⟩

20.4 Initializing the fixrec package

⟨*ML*⟩

hide-const (**open**) *succeed fail run*

end

21 Deflation: Continuous deflations and ep-pairs

theory *Deflation*
imports *Plain-HOLCF*
begin

default-sort *cpo*

21.1 Continuous deflations

locale *deflation* =
  fixes *d* :: *'a* → *'a*
  assumes *idem*: ⋀*x*. *d*·(*d*·*x*) = *d*·*x*
  assumes *below*: ⋀*x*. *d*·*x* ⊑ *x*
begin

lemma *below-ID*: *d* ⊑ *ID*
⟨*proof*⟩

The set of fixed points is the same as the range.

lemma *fixes-eq-range*: {*x*. *d*·*x* = *x*} = *range* (λ*x*. *d*·*x*)
⟨*proof*⟩

lemma *range-eq-fixes*: *range* (λ*x*. *d*·*x*) = {*x*. *d*·*x* = *x*}
⟨*proof*⟩

The pointwise ordering on deflation functions coincides with the subset ordering of their sets of fixed-points.

lemma *belowI*:
  assumes *f*: ⋀*x*. *d*·*x* = *x* ⟹ *f*·*x* = *x* **shows** *d* ⊑ *f*

⟨*proof*⟩

lemma *belowD*: ⟦$f \sqsubseteq d$; $f{\cdot}x = x$⟧ ⟹ $d{\cdot}x = x$
⟨*proof*⟩

end

lemma *deflation-strict*: *deflation* d ⟹ $d{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *adm-deflation*: *adm* ($\lambda d.$ *deflation* d)
⟨*proof*⟩

lemma *deflation-ID*: *deflation ID*
⟨*proof*⟩

lemma *deflation-bottom*: *deflation* $\bot$
⟨*proof*⟩

lemma *deflation-below-iff*:
⟦*deflation* p; *deflation* q⟧ ⟹ $p \sqsubseteq q \longleftrightarrow (\forall x.\ p{\cdot}x = x \longrightarrow q{\cdot}x = x)$
⟨*proof*⟩

The composition of two deflations is equal to the lesser of the two (if they are comparable).

lemma *deflation-below-comp1*:
assumes *deflation* f
assumes *deflation* g
shows $f \sqsubseteq g$ ⟹ $f{\cdot}(g{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

lemma *deflation-below-comp2*:
⟦*deflation* f; *deflation* g; $f \sqsubseteq g$⟧ ⟹ $g{\cdot}(f{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

21.2 Deflations with finite range

lemma *finite-range-imp-finite-fixes*:
finite (*range* f) ⟹ *finite* $\{x.\ f\,x = x\}$
⟨*proof*⟩

locale *finite-deflation* = *deflation* +
assumes *finite-fixes*: *finite* $\{x.\ d{\cdot}x = x\}$
begin

lemma *finite-range*: *finite* (*range* ($\lambda x.\ d{\cdot}x$))
⟨*proof*⟩

lemma *finite-image*: *finite* (($\lambda x.\ d{\cdot}x$) ‘ A)

⟨*proof*⟩

lemma *compact*: *compact* $(d{\cdot}x)$
⟨*proof*⟩

end

lemma *finite-deflation-intro*:
deflation $d \Longrightarrow$ *finite* $\{x.\ d{\cdot}x = x\} \Longrightarrow$ *finite-deflation* d
⟨*proof*⟩

lemma *finite-deflation-imp-deflation*:
finite-deflation $d \Longrightarrow$ *deflation* d
⟨*proof*⟩

lemma *finite-deflation-bottom*: *finite-deflation* $\bot$
⟨*proof*⟩

21.3 Continuous embedding-projection pairs

locale *ep-pair* =
fixes $e :: 'a \to 'b$ **and** $p :: 'b \to 'a$
assumes *e-inverse* [*simp*]: $\bigwedge x.\ p{\cdot}(e{\cdot}x) = x$
and *e-p-below*: $\bigwedge y.\ e{\cdot}(p{\cdot}y) \sqsubseteq y$
begin

lemma *e-below-iff* [*simp*]: $e{\cdot}x \sqsubseteq e{\cdot}y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *e-eq-iff* [*simp*]: $e{\cdot}x = e{\cdot}y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *p-eq-iff*:
$[\![e{\cdot}(p{\cdot}x) = x;\ e{\cdot}(p{\cdot}y) = y]\!] \Longrightarrow p{\cdot}x = p{\cdot}y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *p-inverse*: $(\exists\, x.\ y = e{\cdot}x) = (e{\cdot}(p{\cdot}y) = y)$
⟨*proof*⟩

lemma *e-below-iff-below-p*: $e{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq p{\cdot}y$
⟨*proof*⟩

lemma *compact-e-rev*: *compact* $(e{\cdot}x) \Longrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-e*: *compact* $x \Longrightarrow$ *compact* $(e{\cdot}x)$
⟨*proof*⟩

lemma *compact-e-iff*: *compact* $(e{\cdot}x) \longleftrightarrow$ *compact* x

⟨*proof*⟩

Deflations from ep-pairs

lemma *deflation-e-p*: *deflation* (*e oo p*)
⟨*proof*⟩

lemma *deflation-e-d-p*:
assumes *deflation d*
shows *deflation* (*e oo d oo p*)
⟨*proof*⟩

lemma *finite-deflation-e-d-p*:
assumes *finite-deflation d*
shows *finite-deflation* (*e oo d oo p*)
⟨*proof*⟩

lemma *deflation-p-d-e*:
assumes *deflation d*
assumes *d*: $\bigwedge x.\ d{\cdot}x \sqsubseteq e{\cdot}(p{\cdot}x)$
shows *deflation* (*p oo d oo e*)
⟨*proof*⟩

lemma *finite-deflation-p-d-e*:
assumes *finite-deflation d*
assumes *d*: $\bigwedge x.\ d{\cdot}x \sqsubseteq e{\cdot}(p{\cdot}x)$
shows *finite-deflation* (*p oo d oo e*)
⟨*proof*⟩

end

21.4 Uniqueness of ep-pairs

lemma *ep-pair-unique-e-lemma*:
assumes *1*: *ep-pair e1 p* **and** *2*: *ep-pair e2 p*
shows *e1* $\sqsubseteq$ *e2*
⟨*proof*⟩

lemma *ep-pair-unique-e*:
⟦*ep-pair e1 p*; *ep-pair e2 p*⟧ $\Longrightarrow$ *e1* = *e2*
⟨*proof*⟩

lemma *ep-pair-unique-p-lemma*:
assumes *1*: *ep-pair e p1* **and** *2*: *ep-pair e p2*
shows *p1* $\sqsubseteq$ *p2*
⟨*proof*⟩

lemma *ep-pair-unique-p*:
⟦*ep-pair e p1*; *ep-pair e p2*⟧ $\Longrightarrow$ *p1* = *p2*
⟨*proof*⟩

21.5 Composing ep-pairs

lemma *ep-pair-ID-ID*: *ep-pair ID ID*
⟨*proof*⟩

lemma *ep-pair-comp*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*e2 oo e1*) (*p1 oo p2*)
⟨*proof*⟩

locale *pcpo-ep-pair* = *ep-pair e p*
for *e* :: $'a$::*pcpo* → $'b$::*pcpo*
and *p* :: $'b$::*pcpo* → $'a$::*pcpo*
begin

lemma *e-strict* [*simp*]: $e{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *e-bottom-iff* [*simp*]: $e{\cdot}x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *e-defined*: $x \neq \bot \Longrightarrow e{\cdot}x \neq \bot$
⟨*proof*⟩

lemma *p-strict* [*simp*]: $p{\cdot}\bot = \bot$
⟨*proof*⟩

lemmas *stricts* = *e-strict p-strict*

end

end

22 Map-Functions: Map functions for various types

theory *Map-Functions*
imports *Deflation*
begin

22.1 Map operator for continuous function space

default-sort *cpo*

definition
cfun-map :: $('b \to 'a) \to ('c \to 'd) \to ('a \to 'c) \to ('b \to 'd)$
where
cfun-map = $(\Lambda\ a\ b\ f\ x.\ b{\cdot}(f{\cdot}(a{\cdot}x)))$

lemma *cfun-map-beta* [*simp*]: *cfun-map*$\cdot a \cdot b \cdot f \cdot x = b{\cdot}(f{\cdot}(a{\cdot}x))$

⟨*proof*⟩

lemma *cfun-map-ID*: *cfun-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *cfun-map-map*:
cfun-map·*f1*·*g1*·(*cfun-map*·*f2*·*g2*·*p*) =
cfun-map·(Λ *x*. *f2*·(*f1*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
⟨*proof*⟩

lemma *ep-pair-cfun-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*cfun-map*·*p1*·*e2*) (*cfun-map*·*e1*·*p2*)
⟨*proof*⟩

lemma *deflation-cfun-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-range-cfun-map*:
assumes *a*: *finite* (*range* (λ*x*. *a*·*x*))
assumes *b*: *finite* (*range* (λ*y*. *b*·*y*))
shows *finite* (*range* (λ*f*. *cfun-map*·*a*·*b*·*f*)) (**is** *finite* (*range* ?*h*))
⟨*proof*⟩

lemma *finite-deflation-cfun-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

Finite deflations are compact elements of the function space

lemma *finite-deflation-imp-compact*: *finite-deflation d* ⟹ *compact d*
⟨*proof*⟩

22.2 Map operator for product type

definition
prod-map :: ('*a* → '*b*) → ('*c* → '*d*) → '*a* × '*c* → '*b* × '*d*
where
prod-map = (Λ *f g p*. (*f*·(*fst p*), *g*·(*snd p*)))

lemma *prod-map-Pair* [*simp*]: *prod-map*·*f*·*g*·(*x*, *y*) = (*f*·*x*, *g*·*y*)
⟨*proof*⟩

lemma *prod-map-ID*: *prod-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *prod-map-map*:

prod-map·f1·g1·(prod-map·f2·g2·p) =
prod-map·(Λ x. f1·(f2·x))·(Λ x. g1·(g2·x))·p
⟨*proof*⟩

lemma *ep-pair-prod-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*prod-map·e1·e2*) (*prod-map·p1·p2*)
⟨*proof*⟩

lemma *deflation-prod-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*prod-map·d1·d2*)
⟨*proof*⟩

lemma *finite-deflation-prod-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*prod-map·d1·d2*)
⟨*proof*⟩

22.3 Map function for lifted cpo

definition
u-map :: ('*a* → '*b*) → '*a u* → '*b u*
where
u-map = (Λ *f*. *fup·*(*up oo f*))

lemma *u-map-strict* [*simp*]: *u-map·f·*⊥ = ⊥
⟨*proof*⟩

lemma *u-map-up* [*simp*]: *u-map·f·*(*up·x*) = *up·*(*f·x*)
⟨*proof*⟩

lemma *u-map-ID*: *u-map·ID* = *ID*
⟨*proof*⟩

lemma *u-map-map*: *u-map·f·*(*u-map·g·p*) = *u-map·*(Λ *x*. *f·*(*g·x*))·*p*
⟨*proof*⟩

lemma *u-map-oo*: *u-map·*(*f oo g*) = *u-map·f oo u-map·g*
⟨*proof*⟩

lemma *ep-pair-u-map*: *ep-pair e p* ⟹ *ep-pair* (*u-map·e*) (*u-map·p*)
⟨*proof*⟩

lemma *deflation-u-map*: *deflation d* ⟹ *deflation* (*u-map·d*)
⟨*proof*⟩

lemma *finite-deflation-u-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*u-map·d*)

⟨*proof*⟩

22.4 Map function for strict products

default-sort *pcpo*

definition
sprod-map :: $('a \to 'b) \to ('c \to 'd) \to 'a \otimes 'c \to 'b \otimes 'd$
where
sprod-map = $(\Lambda\ f\ g.\ ssplit{\cdot}(\Lambda\ x\ y.\ (:f{\cdot}x,\ g{\cdot}y:)))$

lemma *sprod-map-strict* [*simp*]: *sprod-map*·*a*·*b*·⊥ = ⊥
⟨*proof*⟩

lemma *sprod-map-spair* [*simp*]:
$x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow$ *sprod-map*·*f*·*g*·(:*x*, *y*:) = (:*f*·*x*, *g*·*y*:)
⟨*proof*⟩

lemma *sprod-map-spair'*:
$f{\cdot}\bot = \bot \Longrightarrow g{\cdot}\bot = \bot \Longrightarrow$ *sprod-map*·*f*·*g*·(:*x*, *y*:) = (:*f*·*x*, *g*·*y*:)
⟨*proof*⟩

lemma *sprod-map-ID*: *sprod-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *sprod-map-map*:
⟦*f1*·⊥ = ⊥; *g1*·⊥ = ⊥⟧ ⟹
sprod-map·*f1*·*g1*·(*sprod-map*·*f2*·*g2*·*p*) =
sprod-map·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
⟨*proof*⟩

lemma *ep-pair-sprod-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*sprod-map*·*e1*·*e2*) (*sprod-map*·*p1*·*p2*)
⟨*proof*⟩

lemma *deflation-sprod-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*sprod-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-deflation-sprod-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*sprod-map*·*d1*·*d2*)
⟨*proof*⟩

22.5 Map function for strict sums

definition
ssum-map :: $('a \to 'b) \to ('c \to 'd) \to 'a \oplus 'c \to 'b \oplus 'd$

where
ssum-map = (Λ *f g*. *sscase*·(*sinl oo f*)·(*sinr oo g*))

lemma *ssum-map-strict* [*simp*]: *ssum-map*·*f*·*g*·⊥ = ⊥
⟨*proof*⟩

lemma *ssum-map-sinl* [*simp*]: *x* ≠ ⊥ ⟹ *ssum-map*·*f*·*g*·(*sinl*·*x*) = *sinl*·(*f*·*x*)
⟨*proof*⟩

lemma *ssum-map-sinr* [*simp*]: *x* ≠ ⊥ ⟹ *ssum-map*·*f*·*g*·(*sinr*·*x*) = *sinr*·(*g*·*x*)
⟨*proof*⟩

lemma *ssum-map-sinl'*: *f*·⊥ = ⊥ ⟹ *ssum-map*·*f*·*g*·(*sinl*·*x*) = *sinl*·(*f*·*x*)
⟨*proof*⟩

lemma *ssum-map-sinr'*: *g*·⊥ = ⊥ ⟹ *ssum-map*·*f*·*g*·(*sinr*·*x*) = *sinr*·(*g*·*x*)
⟨*proof*⟩

lemma *ssum-map-ID*: *ssum-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *ssum-map-map*:
⟦*f1*·⊥ = ⊥; *g1*·⊥ = ⊥⟧ ⟹
ssum-map·*f1*·*g1*·(*ssum-map*·*f2*·*g2*·*p*) =
ssum-map·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
⟨*proof*⟩

lemma *ep-pair-ssum-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*ssum-map*·*e1*·*e2*) (*ssum-map*·*p1*·*p2*)
⟨*proof*⟩

lemma *deflation-ssum-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*ssum-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-deflation-ssum-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*ssum-map*·*d1*·*d2*)
⟨*proof*⟩

22.6 Map operator for strict function space

definition
sfun-map :: ('*b* → '*a*) → ('*c* → '*d*) → ('*a* →! '*c*) → ('*b* →! '*d*)
where
sfun-map = (Λ *a b*. *sfun-abs oo cfun-map*·*a*·*b oo sfun-rep*)

lemma *sfun-map-ID*: *sfun-map·ID·ID = ID*
⟨*proof*⟩

lemma *sfun-map-map*:
assumes *f2·⊥ = ⊥* **and** *g2·⊥ = ⊥* **shows**
sfun-map·f1·g1·(sfun-map·f2·g2·p) =
sfun-map·(Λ x. f2·(f1·x))·(Λ x. g1·(g2·x))·p
⟨*proof*⟩

lemma *ep-pair-sfun-map*:
assumes *1*: *ep-pair e1 p1*
assumes *2*: *ep-pair e2 p2*
shows *ep-pair* (*sfun-map·p1·e2*) (*sfun-map·e1·p2*)
⟨*proof*⟩

lemma *deflation-sfun-map*:
assumes *1*: *deflation d1*
assumes *2*: *deflation d2*
shows *deflation* (*sfun-map·d1·d2*)
⟨*proof*⟩

lemma *finite-deflation-sfun-map*:
assumes *1*: *finite-deflation d1*
assumes *2*: *finite-deflation d2*
shows *finite-deflation* (*sfun-map·d1·d2*)
⟨*proof*⟩

end

23 Bifinite: Profinite and bifinite cpos

theory *Bifinite*
imports *Map-Functions* ~~/*src*/*HOL*/*Library*/*Countable*
begin

default-sort *cpo*

23.1 Chains of finite deflations

locale *approx-chain* =
fixes *approx* :: *nat* ⇒ *'a* → *'a*
assumes *chain-approx* [*simp*]: *chain* (λ*i*. *approx i*)
assumes *lub-approx* [*simp*]: (⨆ *i*. *approx i*) = *ID*
assumes *finite-deflation-approx* [*simp*]: ⋀*i*. *finite-deflation* (*approx i*)
begin

lemma *deflation-approx*: *deflation* (*approx i*)
⟨*proof*⟩

lemma *approx-idem*: $approx\ i{\cdot}(approx\ i{\cdot}x) = approx\ i{\cdot}x$
⟨*proof*⟩

lemma *approx-below*: $approx\ i{\cdot}x \sqsubseteq x$
⟨*proof*⟩

lemma *finite-range-approx*: $finite\ (range\ (\lambda x.\ approx\ i{\cdot}x))$
⟨*proof*⟩

lemma *compact-approx* [*simp*]: $compact\ (approx\ n{\cdot}x)$
⟨*proof*⟩

lemma *compact-eq-approx*: $compact\ x \Longrightarrow \exists\, i.\ approx\ i{\cdot}x = x$
⟨*proof*⟩

end

23.2 Omega-profinite and bifinite domains

class *bifinite* = *pcpo* +
  assumes *bifinite*: $\exists\,(a{::}nat \Rightarrow {'a} \to {'a}).\ approx\text{-}chain\ a$

class *profinite* = *cpo* +
  assumes *profinite*: $\exists\,(a{::}nat \Rightarrow {'a_\bot} \to {'a_\bot}).\ approx\text{-}chain\ a$

23.3 Building approx chains

lemma *approx-chain-iso*:
  assumes *a*: *approx-chain a*
  assumes [*simp*]: $\bigwedge x.\ f{\cdot}(g{\cdot}x) = x$
  assumes [*simp*]: $\bigwedge y.\ g{\cdot}(f{\cdot}y) = y$
  shows $approx\text{-}chain\ (\lambda i.\ f\ oo\ a\ i\ oo\ g)$
⟨*proof*⟩

lemma *approx-chain-u-map*:
  assumes *approx-chain a*
  shows $approx\text{-}chain\ (\lambda i.\ u\text{-}map{\cdot}(a\ i))$
  ⟨*proof*⟩

lemma *approx-chain-sfun-map*:
  assumes *approx-chain a* **and** *approx-chain b*
  shows $approx\text{-}chain\ (\lambda i.\ sfun\text{-}map{\cdot}(a\ i){\cdot}(b\ i))$
  ⟨*proof*⟩

lemma *approx-chain-sprod-map*:
  assumes *approx-chain a* **and** *approx-chain b*
  shows $approx\text{-}chain\ (\lambda i.\ sprod\text{-}map{\cdot}(a\ i){\cdot}(b\ i))$
  ⟨*proof*⟩

lemma *approx-chain-ssum-map*:

assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *ssum-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-cfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *cfun-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-prod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* (λ*i*. *prod-map*·(*a i*)·(*b i*))
⟨*proof*⟩

Approx chains for countable discrete types.

definition *discr-approx* :: *nat* ⇒ *'a*::*countable discr u* → *'a discr u*
where *discr-approx* = (λ*i*. Λ(*up*·*x*). *if to-nat* (*undiscr x*) < *i then up*·*x else* ⊥)

lemma *chain-discr-approx* [*simp*]: *chain discr-approx*
⟨*proof*⟩

lemma *lub-discr-approx* [*simp*]: (⨆ *i*. *discr-approx i*) = *ID*
⟨*proof*⟩

lemma *inj-on-undiscr* [*simp*]: *inj-on undiscr A*
⟨*proof*⟩

lemma *finite-deflation-discr-approx*: *finite-deflation* (*discr-approx i*)
⟨*proof*⟩

lemma *discr-approx*: *approx-chain discr-approx*
⟨*proof*⟩

23.4 Class instance proofs

instance *bifinite* ⊆ *profinite*
⟨*proof*⟩

instance *u* :: (*profinite*) *bifinite*
⟨*proof*⟩

Types *'a* → *'b* and *'a*$_\perp$ →! *'b* are isomorphic.

definition *encode-cfun* = (Λ *f*. *sfun-abs*·(*fup*·*f*))

definition *decode-cfun* = (Λ *g x*. *sfun-rep*·*g*·(*up*·*x*))

lemma *decode-encode-cfun* [*simp*]: *decode-cfun*·(*encode-cfun*·*x*) = *x*
⟨*proof*⟩

lemma *encode-decode-cfun* [*simp*]: *encode-cfun*·(*decode-cfun*·*y*) = *y*
⟨*proof*⟩

instance *cfun* :: (*profinite*, *bifinite*) *bifinite*
⟨*proof*⟩

Types $('a \times 'b)_\perp$ and $'a_\perp \otimes 'b_\perp$ are isomorphic.

definition *encode-prod-u* = (Λ(*up*·(*x*, *y*)). (:*up*·*x*, *up*·*y*:))

definition *decode-prod-u* = (Λ(:*up*·*x*, *up*·*y*:). *up*·(*x*, *y*))

lemma *decode-encode-prod-u* [*simp*]: *decode-prod-u*·(*encode-prod-u*·*x*) = *x*
⟨*proof*⟩

lemma *encode-decode-prod-u* [*simp*]: *encode-prod-u*·(*decode-prod-u*·*y*) = *y*
⟨*proof*⟩

instance *prod* :: (*profinite*, *profinite*) *profinite*
⟨*proof*⟩

instance *prod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *sfun* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *sprod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *ssum* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

lemma *approx-chain-unit*: *approx-chain* (⊥ :: *nat* ⇒ *unit* → *unit*)
⟨*proof*⟩

instance *unit* :: *bifinite*
⟨*proof*⟩

instance *discr* :: (*countable*) *profinite*
⟨*proof*⟩

instance *lift* :: (*countable*) *bifinite*
⟨*proof*⟩

end

24 Completion: Defining algebraic domains by ideal completion

theory *Completion*
imports *Plain-HOLCF*
begin

24.1 Ideals over a preorder

locale *preorder* =
  fixes $r :: 'a::type \Rightarrow 'a \Rightarrow bool$ (**infix** $\preceq$ *50*)
  assumes *r-refl*: $x \preceq x$
  assumes *r-trans*: $[\![x \preceq y; y \preceq z]\!] \Longrightarrow x \preceq z$
begin

definition
  ideal :: $'a\ set \Rightarrow bool$ **where**
  $ideal\ A = ((\exists x.\ x \in A) \wedge (\forall x{\in}A.\ \forall y{\in}A.\ \exists z{\in}A.\ x \preceq z \wedge y \preceq z) \wedge$
    $(\forall x\ y.\ x \preceq y \longrightarrow y \in A \longrightarrow x \in A))$

lemma *idealI*:
  assumes $\exists x.\ x \in A$
  assumes $\bigwedge x\ y.\ [\![x \in A; y \in A]\!] \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
  assumes $\bigwedge x\ y.\ [\![x \preceq y; y \in A]\!] \Longrightarrow x \in A$
  shows *ideal A*
⟨*proof*⟩

lemma *idealD1*:
  $ideal\ A \Longrightarrow \exists x.\ x \in A$
⟨*proof*⟩

lemma *idealD2*:
  $[\![ideal\ A; x \in A; y \in A]\!] \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
⟨*proof*⟩

lemma *idealD3*:
  $[\![ideal\ A; x \preceq y; y \in A]\!] \Longrightarrow x \in A$
⟨*proof*⟩

lemma *ideal-principal*: $ideal\ \{x.\ x \preceq z\}$
⟨*proof*⟩

lemma *ex-ideal*: $\exists A.\ A \in \{A.\ ideal\ A\}$
⟨*proof*⟩

The set of ideals is a cpo

lemma *ideal-UN*:
  fixes $A :: nat \Rightarrow 'a\ set$
  assumes *ideal-A*: $\bigwedge i.\ ideal\ (A\ i)$

assumes *chain-A*: $\bigwedge i\ j.\ i \leq j \Longrightarrow A\ i \subseteq A\ j$
shows *ideal* $(\bigcup i.\ A\ i)$
⟨*proof*⟩

lemma *typedef-ideal-po*:
fixes *Abs* :: $'a\ set \Rightarrow 'b$::*below*
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
shows *OFCLASS*($'b$, *po-class*)
⟨*proof*⟩

lemma
fixes *Abs* :: $'a\ set \Rightarrow 'b$::*po*
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
assumes *S*: *chain S*
shows *typedef-ideal-lub*: *range S* <<| *Abs* $(\bigcup i.\ Rep\ (S\ i))$
and *typedef-ideal-rep-lub*: *Rep* $(\bigsqcup i.\ S\ i) = (\bigcup i.\ Rep\ (S\ i))$
⟨*proof*⟩

lemma *typedef-ideal-cpo*:
fixes *Abs* :: $'a\ set \Rightarrow 'b$::*po*
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
shows *OFCLASS*($'b$, *cpo-class*)
⟨*proof*⟩

end

interpretation *below*: *preorder below* :: $'a$::*po* $\Rightarrow 'a \Rightarrow$ *bool*
⟨*proof*⟩

24.2 Lemmas about least upper bounds

lemma *is-ub-thelub-ex*: $[\![\exists u.\ S <<|\ u;\ x \in S]\!] \Longrightarrow x \sqsubseteq lub\ S$
⟨*proof*⟩

lemma *is-lub-thelub-ex*: $[\![\exists u.\ S <<|\ u;\ S <|\ x]\!] \Longrightarrow lub\ S \sqsubseteq x$
⟨*proof*⟩

24.3 Locale for ideal completion

locale *ideal-completion* = *preorder* +
fixes *principal* :: $'a$::*type* $\Rightarrow 'b$::*cpo*
fixes *rep* :: $'b$::*cpo* $\Rightarrow 'a$::*type set*
assumes *ideal-rep*: $\bigwedge x.\ ideal\ (rep\ x)$
assumes *rep-lub*: $\bigwedge Y.\ chain\ Y \Longrightarrow rep\ (\bigsqcup i.\ Y\ i) = (\bigcup i.\ rep\ (Y\ i))$
assumes *rep-principal*: $\bigwedge a.\ rep\ (principal\ a) = \{b.\ b \preceq a\}$
assumes *belowI*: $\bigwedge x\ y.\ rep\ x \subseteq rep\ y \Longrightarrow x \sqsubseteq y$
assumes *countable*: $\exists f::'a \Rightarrow nat.\ inj\ f$

begin

lemma *rep-mono*: $x \sqsubseteq y \Longrightarrow rep\ x \subseteq rep\ y$
⟨*proof*⟩

lemma *below-def*: $x \sqsubseteq y \longleftrightarrow rep\ x \subseteq rep\ y$
⟨*proof*⟩

lemma *principal-below-iff-mem-rep*: $principal\ a \sqsubseteq x \longleftrightarrow a \in rep\ x$
⟨*proof*⟩

lemma *principal-below-iff* [*simp*]: $principal\ a \sqsubseteq principal\ b \longleftrightarrow a \preceq b$
⟨*proof*⟩

lemma *principal-eq-iff*: $principal\ a = principal\ b \longleftrightarrow a \preceq b \wedge b \preceq a$
⟨*proof*⟩

lemma *eq-iff*: $x = y \longleftrightarrow rep\ x = rep\ y$
⟨*proof*⟩

lemma *principal-mono*: $a \preceq b \Longrightarrow principal\ a \sqsubseteq principal\ b$
⟨*proof*⟩

lemma *ch2ch-principal* [*simp*]:
$\forall i.\ Y\ i \preceq Y\ (Suc\ i) \Longrightarrow chain\ (\lambda i.\ principal\ (Y\ i))$
⟨*proof*⟩

24.3.1 Principal ideals approximate all elements

lemma *compact-principal* [*simp*]: $compact\ (principal\ a)$
⟨*proof*⟩

Construct a chain whose lub is the same as a given ideal

lemma *obtain-principal-chain*:
obtains Y **where** $\forall i.\ Y\ i \preceq Y\ (Suc\ i)$ **and** $x = (\bigsqcup i.\ principal\ (Y\ i))$
⟨*proof*⟩

lemma *principal-induct*:
assumes *adm*: $adm\ P$
assumes P: $\bigwedge a.\ P\ (principal\ a)$
shows $P\ x$
⟨*proof*⟩

lemma *compact-imp-principal*: $compact\ x \Longrightarrow \exists a.\ x = principal\ a$
⟨*proof*⟩

24.4 Defining functions in terms of basis elements

definition

extension :: (*'a*::*type* ⇒ *'c*::*cpo*) ⇒ *'b* → *'c* **where**
extension = (λ*f*. (Λ *x*. *lub* (*f* ‘ *rep x*)))

lemma *extension-lemma*:
fixes *f* :: *'a*::*type* ⇒ *'c*::*cpo*
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
shows ∃ *u*. *f* ‘ *rep x* <<| *u*
⟨*proof*⟩

lemma *extension-beta*:
fixes *f* :: *'a*::*type* ⇒ *'c*::*cpo*
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
shows *extension f*·*x* = *lub* (*f* ‘ *rep x*)
⟨*proof*⟩

lemma *extension-principal*:
fixes *f* :: *'a*::*type* ⇒ *'c*::*cpo*
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
shows *extension f*·(*principal a*) = *f a*
⟨*proof*⟩

lemma *extension-mono*:
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
assumes *g-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *g a* ⊑ *g b*
assumes *below*: ⋀*a*. *f a* ⊑ *g a*
shows *extension f* ⊑ *extension g*
⟨*proof*⟩

lemma *cont-extension*:
assumes *f-mono*: ⋀*a b x*. *a* ≼ *b* ⟹ *f x a* ⊑ *f x b*
assumes *f-cont*: ⋀*a*. *cont* (λ*x*. *f x a*)
shows *cont* (λ*x*. *extension* (λ*a*. *f x a*))
⟨*proof*⟩

end

lemma (**in** *preorder*) *typedef-ideal-completion*:
fixes *Abs* :: *'a set* ⇒ *'b*::*cpo*
assumes *type*: *type-definition Rep Abs* {*S*. *ideal S*}
assumes *below*: ⋀*x y*. *x* ⊑ *y* ⟷ *Rep x* ⊆ *Rep y*
assumes *principal*: ⋀*a*. *principal a* = *Abs* {*b*. *b* ≼ *a*}
assumes *countable*: ∃ *f*::*'a* ⇒ *nat*. *inj f*
shows *ideal-completion r principal Rep*
⟨*proof*⟩

end

25 Universal: A universal bifinite domain

theory *Universal*
imports *Bifinite Completion* ~~/*src*/*HOL*/*Library*/*Nat-Bijection*
begin

25.1 Basis for universal domain

25.1.1 Basis datatype

type-synonym *ubasis* = *nat*

definition
  node :: *nat* ⇒ *ubasis* ⇒ *ubasis set* ⇒ *ubasis*
where
  node i a S = *Suc* (*prod-encode* (*i*, *prod-encode* (*a*, *set-encode S*)))

lemma *node-not-0* [*simp*]: *node i a S* ≠ *0*
⟨*proof*⟩

lemma *node-gt-0* [*simp*]: *0* < *node i a S*
⟨*proof*⟩

lemma *node-inject* [*simp*]:
  ⟦*finite S*; *finite T*⟧
    ⟹ *node i a S* = *node j b T* ⟷ *i* = *j* ∧ *a* = *b* ∧ *S* = *T*
⟨*proof*⟩

lemma *node-gt0*: *i* < *node i a S*
⟨*proof*⟩

lemma *node-gt1*: *a* < *node i a S*
⟨*proof*⟩

lemma *nat-less-power2*: *n* < *2*^*n*
⟨*proof*⟩

lemma *node-gt2*: ⟦*finite S*; *b* ∈ *S*⟧ ⟹ *b* < *node i a S*
⟨*proof*⟩

lemma *eq-prod-encode-pairI*:
  ⟦*fst* (*prod-decode x*) = *a*; *snd* (*prod-decode x*) = *b*⟧ ⟹ *x* = *prod-encode* (*a*, *b*)
⟨*proof*⟩

lemma *node-cases*:
  assumes *1*: *x* = *0* ⟹ *P*
  assumes *2*: ⋀*i a S*. ⟦*finite S*; *x* = *node i a S*⟧ ⟹ *P*
  shows *P*
⟨*proof*⟩

lemma *node-induct*:
assumes *1*: $P\ 0$
assumes *2*: $\bigwedge i\ a\ S.\ [\![P\ a;\ \mathit{finite}\ S;\ \forall\, b{\in}S.\ P\ b]\!] \Longrightarrow P\ (\mathit{node}\ i\ a\ S)$
shows $P\ x$
⟨*proof*⟩

25.1.2 Basis ordering

inductive
ubasis-le :: $\mathit{nat} \Rightarrow \mathit{nat} \Rightarrow \mathit{bool}$
where
ubasis-le-refl: *ubasis-le a a*
| *ubasis-le-trans*:
$[\![\mathit{ubasis\text{-}le}\ a\ b;\ \mathit{ubasis\text{-}le}\ b\ c]\!] \Longrightarrow \mathit{ubasis\text{-}le}\ a\ c$
| *ubasis-le-lower*:
$\mathit{finite}\ S \Longrightarrow \mathit{ubasis\text{-}le}\ a\ (\mathit{node}\ i\ a\ S)$
| *ubasis-le-upper*:
$[\![\mathit{finite}\ S;\ b \in S;\ \mathit{ubasis\text{-}le}\ a\ b]\!] \Longrightarrow \mathit{ubasis\text{-}le}\ (\mathit{node}\ i\ a\ S)\ b$

lemma *ubasis-le-minimal*: *ubasis-le 0 x*
⟨*proof*⟩

interpretation *udom*: *preorder ubasis-le*
⟨*proof*⟩

25.1.3 Generic take function

function
ubasis-until :: $(\mathit{ubasis} \Rightarrow \mathit{bool}) \Rightarrow \mathit{ubasis} \Rightarrow \mathit{ubasis}$
where
ubasis-until P 0 = *0*
| $\mathit{finite}\ S \Longrightarrow \mathit{ubasis\text{-}until}\ P\ (\mathit{node}\ i\ a\ S) =$
(*if P* (*node i a S*) *then node i a S else ubasis-until P a*)
⟨*proof*⟩

termination *ubasis-until*
⟨*proof*⟩

lemma *ubasis-until*: $P\ 0 \Longrightarrow P\ (\mathit{ubasis\text{-}until}\ P\ x)$
⟨*proof*⟩

lemma *ubasis-until'*: $0 < \mathit{ubasis\text{-}until}\ P\ x \Longrightarrow P\ (\mathit{ubasis\text{-}until}\ P\ x)$
⟨*proof*⟩

lemma *ubasis-until-same*: $P\ x \Longrightarrow \mathit{ubasis\text{-}until}\ P\ x = x$
⟨*proof*⟩

lemma *ubasis-until-idem*:
$P\ 0 \Longrightarrow \mathit{ubasis\text{-}until}\ P\ (\mathit{ubasis\text{-}until}\ P\ x) = \mathit{ubasis\text{-}until}\ P\ x$
⟨*proof*⟩

lemma *ubasis-until-0*:
$\forall x.\ x \neq 0 \longrightarrow \neg\ P\ x \Longrightarrow$ *ubasis-until* $P\ x = 0$
⟨*proof*⟩

lemma *ubasis-until-less*: *ubasis-le* (*ubasis-until* $P\ x$) x
⟨*proof*⟩

lemma *ubasis-until-chain*:
assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows *ubasis-le* (*ubasis-until* $P\ x$) (*ubasis-until* $Q\ x$)
⟨*proof*⟩

lemma *ubasis-until-mono*:
assumes $\bigwedge i\ a\ S\ b.\ [\![$*finite* S; P (*node* $i\ a\ S$); $b \in S$; *ubasis-le* $a\ b]\!] \Longrightarrow P\ b$
shows *ubasis-le* $a\ b \Longrightarrow$ *ubasis-le* (*ubasis-until* $P\ a$) (*ubasis-until* $P\ b$)
⟨*proof*⟩

lemma *finite-range-ubasis-until*:
finite $\{x.\ P\ x\} \Longrightarrow$ *finite* (*range* (*ubasis-until* P))
⟨*proof*⟩

25.2 Defining the universal domain by ideal completion

typedef *udom* = $\{S.$ *udom.ideal* $S\}$
⟨*proof*⟩

instantiation *udom* :: *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow$ *Rep-udom* $x \subseteq$ *Rep-udom* y

instance ⟨*proof*⟩
end

instance *udom* :: *po*
⟨*proof*⟩

instance *udom* :: *cpo*
⟨*proof*⟩

definition
udom-principal :: *nat* ⇒ *udom* **where**
udom-principal $t =$ *Abs-udom* $\{u.$ *ubasis-le* $u\ t\}$

lemma *ubasis-countable*: $\exists f$::*ubasis* ⇒ *nat*. *inj* f
⟨*proof*⟩

interpretation *udom*:
ideal-completion ubasis-le udom-principal Rep-udom
⟨*proof*⟩

Universal domain is pointed

lemma *udom-minimal*: *udom-principal 0* ⊑ *x*
⟨*proof*⟩

instance *udom* :: *pcpo*
⟨*proof*⟩

lemma *inst-udom-pcpo*: ⊥ = *udom-principal 0*
⟨*proof*⟩

25.3 Compact bases of domains

typedef *'a compact-basis* = {*x*::*'a*::*pcpo*. *compact x*}
⟨*proof*⟩

lemma *Rep-compact-basis'* [*simp*]: *compact* (*Rep-compact-basis a*)
⟨*proof*⟩

lemma *Abs-compact-basis-inverse'* [*simp*]:
compact x ⟹ *Rep-compact-basis* (*Abs-compact-basis x*) = *x*
⟨*proof*⟩

instantiation *compact-basis* :: (*pcpo*) *below*
begin

definition
compact-le-def:
(*op* ⊑) ≡ (λ*x y*. *Rep-compact-basis x* ⊑ *Rep-compact-basis y*)

instance ⟨*proof*⟩
end

instance *compact-basis* :: (*pcpo*) *po*
⟨*proof*⟩

definition
approximants :: *'a* ⇒ *'a compact-basis set* **where**
approximants = (λ*x*. {*a*. *Rep-compact-basis a* ⊑ *x*})

definition
compact-bot :: *'a*::*pcpo compact-basis* **where**
compact-bot = *Abs-compact-basis* ⊥

lemma *Rep-compact-bot* [*simp*]: *Rep-compact-basis compact-bot* = ⊥
⟨*proof*⟩

lemma *compact-bot-minimal* [*simp*]: *compact-bot* $\sqsubseteq$ *a*
⟨*proof*⟩

25.4 Universality of *udom*

We use a locale to parameterize the construction over a chain of approx functions on the type to be embedded.

locale *bifinite-approx-chain* =
approx-chain approx **for** *approx* :: *nat* ⇒ *'a*::*bifinite* → *'a*
begin

25.4.1 Choosing a maximal element from a finite set

lemma *finite-has-maximal*:
fixes *A* :: *'a compact-basis set*
shows ⟦*finite A*; *A* ≠ {}⟧ ⟹ ∃ *x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*
⟨*proof*⟩

definition
choose :: *'a compact-basis set* ⇒ *'a compact-basis*
where
choose A = (*SOME x*. *x* ∈ {*x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*})

lemma *choose-lemma*:
⟦*finite A*; *A* ≠ {}⟧ ⟹ *choose A* ∈ {*x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*}
⟨*proof*⟩

lemma *maximal-choose*:
⟦*finite A*; *y* ∈ *A*; *choose A* ⊑ *y*⟧ ⟹ *choose A* = *y*
⟨*proof*⟩

lemma *choose-in*: ⟦*finite A*; *A* ≠ {}⟧ ⟹ *choose A* ∈ *A*
⟨*proof*⟩

function
choose-pos :: *'a compact-basis set* ⇒ *'a compact-basis* ⇒ *nat*
where
choose-pos A x =
(*if finite A* ∧ *x* ∈ *A* ∧ *x* ≠ *choose A*
then Suc (*choose-pos* (*A* − {*choose A*}) *x*) *else 0*)
⟨*proof*⟩

termination *choose-pos*
⟨*proof*⟩

declare *choose-pos.simps* [*simp del*]

lemma *choose-pos-choose*: *finite A* ⟹ *choose-pos A* (*choose A*) = *0*

⟨*proof*⟩

lemma *inj-on-choose-pos* [*OF refl*]:
⟦*card A* = *n*; *finite A*⟧ ⟹ *inj-on* (*choose-pos A*) *A*
⟨*proof*⟩

lemma *choose-pos-bounded* [*OF refl*]:
⟦*card A* = *n*; *finite A*; *x* ∈ *A*⟧ ⟹ *choose-pos A x* < *n*
⟨*proof*⟩

lemma *choose-pos-lessD*:
⟦*choose-pos A x* < *choose-pos A y*; *finite A*; *x* ∈ *A*; *y* ∈ *A*⟧ ⟹ *x* ⋢ *y*
⟨*proof*⟩

25.4.2 Compact basis take function

primrec
cb-take :: *nat* ⇒ *'a compact-basis* ⇒ *'a compact-basis* **where**
cb-take 0 = (λ*x*. *compact-bot*)
| *cb-take* (*Suc n*) = (λ*a*. *Abs-compact-basis* (*approx n*·(*Rep-compact-basis a*)))

declare *cb-take.simps* [*simp del*]

lemma *cb-take-zero* [*simp*]: *cb-take 0 a* = *compact-bot*
⟨*proof*⟩

lemma *Rep-cb-take*:
Rep-compact-basis (*cb-take* (*Suc n*) *a*) = *approx n*·(*Rep-compact-basis a*)
⟨*proof*⟩

lemmas *approx-Rep-compact-basis* = *Rep-cb-take* [*symmetric*]

lemma *cb-take-covers*: ∃ *n*. *cb-take n x* = *x*
⟨*proof*⟩

lemma *cb-take-less*: *cb-take n x* ⊑ *x*
⟨*proof*⟩

lemma *cb-take-idem*: *cb-take n* (*cb-take n x*) = *cb-take n x*
⟨*proof*⟩

lemma *cb-take-mono*: *x* ⊑ *y* ⟹ *cb-take n x* ⊑ *cb-take n y*
⟨*proof*⟩

lemma *cb-take-chain-le*: *m* ≤ *n* ⟹ *cb-take m x* ⊑ *cb-take n x*
⟨*proof*⟩

lemma *finite-range-cb-take*: *finite* (*range* (*cb-take n*))
⟨*proof*⟩

25.4.3 Rank of basis elements

definition
rank :: *'a compact-basis* ⇒ *nat*
where
rank x = (*LEAST n*. *cb-take n x* = *x*)

lemma *compact-approx-rank*: *cb-take* (*rank x*) *x* = *x*
⟨*proof*⟩

lemma *rank-leD*: *rank x* ≤ *n* ⟹ *cb-take n x* = *x*
⟨*proof*⟩

lemma *rank-leI*: *cb-take n x* = *x* ⟹ *rank x* ≤ *n*
⟨*proof*⟩

lemma *rank-le-iff*: *rank x* ≤ *n* ⟷ *cb-take n x* = *x*
⟨*proof*⟩

lemma *rank-compact-bot* [*simp*]: *rank compact-bot* = *0*
⟨*proof*⟩

lemma *rank-eq-0-iff* [*simp*]: *rank x* = *0* ⟷ *x* = *compact-bot*
⟨*proof*⟩

definition
rank-le :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-le x = {*y*. *rank y* ≤ *rank x*}

definition
rank-lt :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-lt x = {*y*. *rank y* < *rank x*}

definition
rank-eq :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-eq x = {*y*. *rank y* = *rank x*}

lemma *rank-eq-cong*: *rank x* = *rank y* ⟹ *rank-eq x* = *rank-eq y*
⟨*proof*⟩

lemma *rank-lt-cong*: *rank x* = *rank y* ⟹ *rank-lt x* = *rank-lt y*
⟨*proof*⟩

lemma *rank-eq-subset*: *rank-eq x* ⊆ *rank-le x*
⟨*proof*⟩

lemma *rank-lt-subset*: *rank-lt x* ⊆ *rank-le x*

⟨*proof*⟩

lemma *finite-rank-le*: *finite* (*rank-le x*)
⟨*proof*⟩

lemma *finite-rank-eq*: *finite* (*rank-eq x*)
⟨*proof*⟩

lemma *finite-rank-lt*: *finite* (*rank-lt x*)
⟨*proof*⟩

lemma *rank-lt-Int-rank-eq*: *rank-lt x* ∩ *rank-eq x* = {}
⟨*proof*⟩

lemma *rank-lt-Un-rank-eq*: *rank-lt x* ∪ *rank-eq x* = *rank-le x*
⟨*proof*⟩

25.4.4 Sequencing basis elements

definition
place :: *'a compact-basis* ⇒ *nat*
where
place x = *card* (*rank-lt x*) + *choose-pos* (*rank-eq x*) *x*

lemma *place-bounded*: *place x* < *card* (*rank-le x*)
⟨*proof*⟩

lemma *place-ge*: *card* (*rank-lt x*) ≤ *place x*
⟨*proof*⟩

lemma *place-rank-mono*:
fixes *x y* :: *'a compact-basis*
shows *rank x* < *rank y* ⟹ *place x* < *place y*
⟨*proof*⟩

lemma *place-eqD*: *place x* = *place y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *inj-place*: *inj place*
⟨*proof*⟩

25.4.5 Embedding and projection on basis elements

definition
sub :: *'a compact-basis* ⇒ *'a compact-basis*
where
sub x = (*case rank x of 0* ⇒ *compact-bot* | *Suc k* ⇒ *cb-take k x*)

lemma *rank-sub-less*: *x* ≠ *compact-bot* ⟹ *rank* (*sub x*) < *rank x*
⟨*proof*⟩

lemma *place-sub-less*: $x \neq \mathit{compact\text{-}bot} \Longrightarrow \mathit{place}\ (\mathit{sub}\ x) < \mathit{place}\ x$
⟨*proof*⟩

lemma *sub-below*: $\mathit{sub}\ x \sqsubseteq x$
⟨*proof*⟩

lemma *rank-less-imp-below-sub*: $[\![x \sqsubseteq y;\ \mathit{rank}\ x < \mathit{rank}\ y]\!] \Longrightarrow x \sqsubseteq \mathit{sub}\ y$
⟨*proof*⟩

function
basis-emb :: *'a compact-basis* ⇒ *ubasis*
where
basis-emb x = (*if x* = *compact-bot then 0 else*
node (*place x*) (*basis-emb* (*sub x*))
(*basis-emb* ‘ {*y*. *place y* < *place x* ∧ *x* ⊑ *y*}))
⟨*proof*⟩

termination *basis-emb*
⟨*proof*⟩

declare *basis-emb.simps* [*simp del*]

lemma *basis-emb-compact-bot* [*simp*]: *basis-emb compact-bot* = *0*
⟨*proof*⟩

lemma *fin1*: *finite* {*y*. *place y* < *place x* ∧ *x* ⊑ *y*}
⟨*proof*⟩

lemma *fin2*: *finite* (*basis-emb* ‘ {*y*. *place y* < *place x* ∧ *x* ⊑ *y*})
⟨*proof*⟩

lemma *rank-place-mono*:
[*place x* < *place y*; *x* ⊑ *y*] ⟹ *rank x* < *rank y*
⟨*proof*⟩

lemma *basis-emb-mono*:
x ⊑ *y* ⟹ *ubasis-le* (*basis-emb x*) (*basis-emb y*)
⟨*proof*⟩

lemma *inj-basis-emb*: *inj basis-emb*
⟨*proof*⟩

definition
basis-prj :: *ubasis* ⇒ *'a compact-basis*
where
basis-prj x = *inv basis-emb*
(*ubasis-until* (λ*x*. *x* ∈ *range* (*basis-emb* :: *'a compact-basis* ⇒ *ubasis*)) *x*)

lemma *basis-prj-basis-emb*: $\bigwedge x.\ \mathit{basis\text{-}prj}\ (\mathit{basis\text{-}emb}\ x) = x$
⟨*proof*⟩

lemma *basis-prj-node*:
$[\![\mathit{finite}\ S;\ \mathit{node}\ i\ a\ S \notin \mathit{range}\ (\mathit{basis\text{-}emb} :: 'a\ \mathit{compact\text{-}basis} \Rightarrow \mathit{nat})]\!]$
$\Longrightarrow \mathit{basis\text{-}prj}\ (\mathit{node}\ i\ a\ S) = (\mathit{basis\text{-}prj}\ a :: 'a\ \mathit{compact\text{-}basis})$
⟨*proof*⟩

lemma *basis-prj-0*: *basis-prj 0 = compact-bot*
⟨*proof*⟩

lemma *node-eq-basis-emb-iff*:
finite $S \Longrightarrow \mathit{node}\ i\ a\ S = \mathit{basis\text{-}emb}\ x \longleftrightarrow$
$x \neq \mathit{compact\text{-}bot} \wedge i = \mathit{place}\ x \wedge a = \mathit{basis\text{-}emb}\ (\mathit{sub}\ x) \wedge$
$S = \mathit{basis\text{-}emb}\ `\ \{y.\ \mathit{place}\ y < \mathit{place}\ x \wedge x \sqsubseteq y\}$
⟨*proof*⟩

lemma *basis-prj-mono*: *ubasis-le a b* $\Longrightarrow$ *basis-prj a* $\sqsubseteq$ *basis-prj b*
⟨*proof*⟩

lemma *basis-emb-prj-less*: *ubasis-le* (*basis-emb* (*basis-prj x*)) *x*
⟨*proof*⟩

lemma *ideal-completion*:
ideal-completion below Rep-compact-basis (*approximants* :: $'a \Rightarrow$ -)
⟨*proof*⟩

end

interpretation *compact-basis*!:
ideal-completion below Rep-compact-basis
approximants :: $'a$::*bifinite* $\Rightarrow$ $'a$ *compact-basis set*
⟨*proof*⟩

25.4.6 EP-pair from any bifinite domain into *udom*

context *bifinite-approx-chain* **begin**

definition
udom-emb :: $'a \to$ *udom*
where
udom-emb = *compact-basis.extension* ($\lambda x.$ *udom-principal* (*basis-emb x*))

definition
udom-prj :: *udom* $\to$ $'a$
where
udom-prj = *udom.extension* ($\lambda x.$ *Rep-compact-basis* (*basis-prj x*))

lemma *udom-emb-principal*:

udom-emb·(*Rep-compact-basis x*) = *udom-principal* (*basis-emb x*)
⟨*proof*⟩

lemma *udom-prj-principal*:
udom-prj·(*udom-principal x*) = *Rep-compact-basis* (*basis-prj x*)
⟨*proof*⟩

lemma *ep-pair-udom*: *ep-pair udom-emb udom-prj*
⟨*proof*⟩

end

abbreviation *udom-emb* ≡ *bifinite-approx-chain.udom-emb*
abbreviation *udom-prj* ≡ *bifinite-approx-chain.udom-prj*

lemmas *ep-pair-udom* =
bifinite-approx-chain.ep-pair-udom [*unfolded bifinite-approx-chain-def*]

25.5 Chain of approx functions for type *udom*

definition
udom-approx :: *nat* ⇒ *udom* → *udom*
where
udom-approx i =
udom.extension (λ*x*. *udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*))

lemma *udom-approx-mono*:
ubasis-le a b ⟹
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *a*) ⊑
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *b*)
⟨*proof*⟩

lemma *adm-mem-finite*: ⟦*cont f*; *finite S*⟧ ⟹ *adm* (λ*x*. *f x* ∈ *S*)
⟨*proof*⟩

lemma *udom-approx-principal*:
udom-approx i·(*udom-principal x*) =
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*)
⟨*proof*⟩

lemma *finite-deflation-udom-approx*: *finite-deflation* (*udom-approx i*)
⟨*proof*⟩

interpretation *udom-approx*: *finite-deflation udom-approx i*
⟨*proof*⟩

lemma *chain-udom-approx* [*simp*]: *chain* (λ*i*. *udom-approx i*)
⟨*proof*⟩

lemma *lub-udom-approx* [*simp*]: $(\bigsqcup i.\ \textit{udom-approx}\ i) = \textit{ID}$
⟨*proof*⟩

lemma *udom-approx* [*simp*]: *approx-chain udom-approx*
⟨*proof*⟩

instance *udom* :: *bifinite*
⟨*proof*⟩

hide-const (**open**) *node*

end

26 Algebraic: Algebraic deflations

theory *Algebraic*
imports *Universal Map-Functions*
begin

default-sort *bifinite*

26.1 Type constructor for finite deflations

typedef *'a fin-defl* = $\{d{::}'a \to 'a.\ \textit{finite-deflation}\ d\}$
⟨*proof*⟩

instantiation *fin-defl* :: (*bifinite*) *below*
begin

definition *below-fin-defl-def*:
$\textit{below} \equiv \lambda x\ y.\ \textit{Rep-fin-defl}\ x \sqsubseteq \textit{Rep-fin-defl}\ y$

instance ⟨*proof*⟩
end

instance *fin-defl* :: (*bifinite*) *po*
⟨*proof*⟩

lemma *finite-deflation-Rep-fin-defl*: *finite-deflation* (*Rep-fin-defl d*)
⟨*proof*⟩

lemma *deflation-Rep-fin-defl*: *deflation* (*Rep-fin-defl d*)
⟨*proof*⟩

interpretation *Rep-fin-defl*: *finite-deflation Rep-fin-defl d*
⟨*proof*⟩

lemma *fin-defl-belowI*:
$(\bigwedge x.\ \textit{Rep-fin-defl}\ a{\cdot}x = x \Longrightarrow \textit{Rep-fin-defl}\ b{\cdot}x = x) \Longrightarrow a \sqsubseteq b$

⟨*proof*⟩

lemma *fin-defl-belowD*:
⟦*a* ⊑ *b*; *Rep-fin-defl a*·*x* = *x*⟧ ⟹ *Rep-fin-defl b*·*x* = *x*
⟨*proof*⟩

lemma *fin-defl-eqI*:
(⋀*x*. *Rep-fin-defl a*·*x* = *x* ⟷ *Rep-fin-defl b*·*x* = *x*) ⟹ *a* = *b*
⟨*proof*⟩

lemma *Rep-fin-defl-mono*: *a* ⊑ *b* ⟹ *Rep-fin-defl a* ⊑ *Rep-fin-defl b*
⟨*proof*⟩

lemma *Abs-fin-defl-mono*:
⟦*finite-deflation a*; *finite-deflation b*; *a* ⊑ *b*⟧
⟹ *Abs-fin-defl a* ⊑ *Abs-fin-defl b*
⟨*proof*⟩

lemma (**in** *finite-deflation*) *compact-belowI*:
assumes ⋀*x*. *compact x* ⟹ *d*·*x* = *x* ⟹ *f*·*x* = *x* **shows** *d* ⊑ *f*
⟨*proof*⟩

lemma *compact-Rep-fin-defl* [*simp*]: *compact* (*Rep-fin-defl a*)
⟨*proof*⟩

26.2 Defining algebraic deflations by ideal completion

typedef *'a defl* = {*S*::*'a fin-defl set*. *below.ideal S*}
⟨*proof*⟩

instantiation *defl* :: (*bifinite*) *below*
begin

definition
x ⊑ *y* ⟷ *Rep-defl x* ⊆ *Rep-defl y*

instance ⟨*proof*⟩
end

instance *defl* :: (*bifinite*) *po*
⟨*proof*⟩

instance *defl* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
defl-principal :: *'a fin-defl* ⇒ *'a defl* **where**
defl-principal t = *Abs-defl* {*u*. *u* ⊑ *t*}

lemma *fin-defl-countable*: $\exists f::'a\ \textit{fin-defl} \Rightarrow \textit{nat}.\ \textit{inj}\ f$
⟨*proof*⟩

interpretation *defl*: *ideal-completion below defl-principal Rep-defl*
⟨*proof*⟩

Algebraic deflations are pointed

lemma *defl-minimal*: *defl-principal* (*Abs-fin-defl* ⊥) ⊑ *x*
⟨*proof*⟩

instance *defl* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-defl-pcpo*: ⊥ = *defl-principal* (*Abs-fin-defl* ⊥)
⟨*proof*⟩

26.3 Applying algebraic deflations

definition
cast :: *'a defl* → *'a* → *'a*
where
cast = *defl.extension Rep-fin-defl*

lemma *cast-defl-principal*:
cast·(*defl-principal a*) = *Rep-fin-defl a*
⟨*proof*⟩

lemma *deflation-cast*: *deflation* (*cast*·*d*)
⟨*proof*⟩

lemma *finite-deflation-cast*:
compact d ⟹ *finite-deflation* (*cast*·*d*)
⟨*proof*⟩

interpretation *cast*: *deflation cast*·*d*
⟨*proof*⟩

declare *cast.idem* [*simp*]

lemma *compact-cast* [*simp*]: *compact d* ⟹ *compact* (*cast*·*d*)
⟨*proof*⟩

lemma *cast-below-cast*: *cast*·*A* ⊑ *cast*·*B* ⟷ *A* ⊑ *B*
⟨*proof*⟩

lemma *compact-cast-iff*: *compact* (*cast*·*d*) ⟷ *compact d*
⟨*proof*⟩

lemma *cast-below-imp-below*: *cast*·*A* ⊑ *cast*·*B* ⟹ *A* ⊑ *B*

⟨*proof*⟩

lemma *cast-eq-imp-eq*: $cast{\cdot}A = cast{\cdot}B \Longrightarrow A = B$
⟨*proof*⟩

lemma *cast-strict1* [*simp*]: $cast{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *cast-strict2* [*simp*]: $cast{\cdot}A{\cdot}\bot = \bot$
⟨*proof*⟩

26.4 Deflation combinators

definition
defl-fun1 e p f =
defl.extension (λ*a*.
defl-principal (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*) *oo p*)))

definition
defl-fun2 e p f =
defl.extension (λ*a*.
defl.extension (λ*b*.
defl-principal (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*) *oo p*))))

lemma *cast-defl-fun1*:
assumes *ep*: *ep-pair e p*
assumes *f*: $\bigwedge a.$ *finite-deflation a* $\Longrightarrow$ *finite-deflation* (*f*·*a*)
shows *cast*·(*defl-fun1 e p f*·*A*) = *e oo f*·(*cast*·*A*) *oo p*
⟨*proof*⟩

lemma *cast-defl-fun2*:
assumes *ep*: *ep-pair e p*
assumes *f*: $\bigwedge a\ b.$ *finite-deflation a* $\Longrightarrow$ *finite-deflation b* $\Longrightarrow$
finite-deflation (*f*·*a*·*b*)
shows *cast*·(*defl-fun2 e p f*·*A*·*B*) = *e oo f*·(*cast*·*A*)·(*cast*·*B*) *oo p*
⟨*proof*⟩

end

27 Representable: Representable domains

theory *Representable*
imports *Algebraic Map-Functions* ~~/*src*/*HOL*/*Library*/*Countable*
begin

default-sort *cpo*

27.1 Class of representable domains

We define a "domain" as a pcpo that is isomorphic to some algebraic deflation over the universal domain; this is equivalent to being omega-bifinite.

A predomain is a cpo that, when lifted, becomes a domain. Predomains are represented by deflations over a lifted universal domain type.

class *predomain-syn* = *cpo* +
  fixes *liftemb* :: $'a_\perp \to udom_\perp$
  fixes *liftprj* :: $udom_\perp \to 'a_\perp$
  fixes *liftdefl* :: *'a itself* ⇒ *udom u defl*

class *predomain* = *predomain-syn* +
  assumes *predomain-ep*: *ep-pair liftemb liftprj*
  assumes *cast-liftdefl*: *cast*·(*liftdefl TYPE*('*a*)) = *liftemb oo liftprj*

syntax -*LIFTDEFL* :: *type* ⇒ *logic* ((1*LIFTDEFL*/(1'(-'))))
translations *LIFTDEFL*('*t*) ⇌ *CONST liftdefl TYPE*('*t*)

definition *liftdefl-of* :: *udom defl* → *udom u defl*
  where *liftdefl-of* = *defl-fun1 ID ID u-map*

lemma *cast-liftdefl-of*: *cast*·(*liftdefl-of*·*t*) = *u-map*·(*cast*·*t*)
⟨*proof*⟩

class *domain* = *predomain-syn* + *pcpo* +
  fixes *emb* :: *'a* → *udom*
  fixes *prj* :: *udom* → *'a*
  fixes *defl* :: *'a itself* ⇒ *udom defl*
  assumes *ep-pair-emb-prj*: *ep-pair emb prj*
  assumes *cast-DEFL*: *cast*·(*defl TYPE*('*a*)) = *emb oo prj*
  assumes *liftemb-eq*: *liftemb* = *u-map*·*emb*
  assumes *liftprj-eq*: *liftprj* = *u-map*·*prj*
  assumes *liftdefl-eq*: *liftdefl TYPE*('*a*) = *liftdefl-of*·(*defl TYPE*('*a*))

syntax -*DEFL* :: *type* ⇒ *logic* ((1*DEFL*/(1'(-'))))
translations *DEFL*('*t*) ⇌ *CONST defl TYPE*('*t*)

instance *domain* ⊆ *predomain*
⟨*proof*⟩

Constants *liftemb* and *liftprj* imply class predomain.

⟨*ML*⟩

interpretation *predomain*: *pcpo-ep-pair liftemb liftprj*
  ⟨*proof*⟩

interpretation *domain*: *pcpo-ep-pair emb prj*
  ⟨*proof*⟩

lemmas *emb-inverse* = *domain.e-inverse*
lemmas *emb-prj-below* = *domain.e-p-below*
lemmas *emb-eq-iff* = *domain.e-eq-iff*
lemmas *emb-strict* = *domain.e-strict*
lemmas *prj-strict* = *domain.p-strict*

27.2 Domains are bifinite

lemma *approx-chain-ep-cast*:
  assumes *ep*: *ep-pair* (*e*::'*a*::*pcpo* → '*b*::*bifinite*) (*p*::'*b* → '*a*)
  assumes *cast-t*: *cast*·*t* = *e oo p*
  shows ∃(*a*::*nat* ⇒ '*a*::*pcpo* → '*a*). *approx-chain a*
⟨*proof*⟩

instance *domain* ⊆ *bifinite*
⟨*proof*⟩

instance *predomain* ⊆ *profinite*
⟨*proof*⟩

27.3 Universal domain ep-pairs

definition *u-emb* = *udom-emb* (λ*i*. *u-map*·(*udom-approx i*))
definition *u-prj* = *udom-prj* (λ*i*. *u-map*·(*udom-approx i*))

definition *prod-emb* = *udom-emb* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))
definition *prod-prj* = *udom-prj* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sprod-emb* = *udom-emb* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sprod-prj* = *udom-prj* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))

definition *ssum-emb* = *udom-emb* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))
definition *ssum-prj* = *udom-prj* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sfun-emb* = *udom-emb* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sfun-prj* = *udom-prj* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))

lemma *ep-pair-u*: *ep-pair u-emb u-prj*
  ⟨*proof*⟩

lemma *ep-pair-prod*: *ep-pair prod-emb prod-prj*
  ⟨*proof*⟩

lemma *ep-pair-sprod*: *ep-pair sprod-emb sprod-prj*
⟨*proof*⟩

lemma *ep-pair-ssum*: *ep-pair ssum-emb ssum-prj*
⟨*proof*⟩

lemma *ep-pair-sfun*: *ep-pair sfun-emb sfun-prj*
⟨*proof*⟩

27.4 Type combinators

definition *u-defl* :: *udom defl* → *udom defl*
where *u-defl* = *defl-fun1 u-emb u-prj u-map*

definition *prod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *prod-defl* = *defl-fun2 prod-emb prod-prj prod-map*

definition *sprod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sprod-defl* = *defl-fun2 sprod-emb sprod-prj sprod-map*

definition *ssum-defl* :: *udom defl* → *udom defl* → *udom defl*
where *ssum-defl* = *defl-fun2 ssum-emb ssum-prj ssum-map*

definition *sfun-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sfun-defl* = *defl-fun2 sfun-emb sfun-prj sfun-map*

lemma *cast-u-defl*:
cast·(*u-defl*·*A*) = *u-emb oo u-map*·(*cast*·*A*) *oo u-prj*
⟨*proof*⟩

lemma *cast-prod-defl*:
cast·(*prod-defl*·*A*·*B*) =
prod-emb oo prod-map·(*cast*·*A*)·(*cast*·*B*) *oo prod-prj*
⟨*proof*⟩

lemma *cast-sprod-defl*:
cast·(*sprod-defl*·*A*·*B*) =
sprod-emb oo sprod-map·(*cast*·*A*)·(*cast*·*B*) *oo sprod-prj*
⟨*proof*⟩

lemma *cast-ssum-defl*:
cast·(*ssum-defl*·*A*·*B*) =
ssum-emb oo ssum-map·(*cast*·*A*)·(*cast*·*B*) *oo ssum-prj*
⟨*proof*⟩

lemma *cast-sfun-defl*:
cast·(*sfun-defl*·*A*·*B*) =
sfun-emb oo sfun-map·(*cast*·*A*)·(*cast*·*B*) *oo sfun-prj*
⟨*proof*⟩

Special deflation combinator for unpointed types.

definition *u-liftdefl* :: *udom u defl* → *udom defl*
where *u-liftdefl* = *defl-fun1 u-emb u-prj ID*

lemma *cast-u-liftdefl*:
cast·(*u-liftdefl*·*A*) = *u-emb oo cast*·*A oo u-prj*
⟨*proof*⟩

lemma *u-liftdefl-liftdefl-of*:
u-liftdefl·(*liftdefl-of*·*A*) = *u-defl*·*A*
⟨*proof*⟩

27.5 Class instance proofs

27.5.1 Universal domain

instantiation *udom* :: *domain*
begin

definition [*simp*]:
emb = (*ID* :: *udom* → *udom*)

definition [*simp*]:
prj = (*ID* :: *udom* → *udom*)

definition
defl (*t*::*udom itself*) = (⨆ *i*. *defl-principal* (*Abs-fin-defl* (*udom-approx i*)))

definition
(*liftemb* :: *udom u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → *udom u*) = *u-map*·*prj*

definition
liftdefl (*t*::*udom itself*) = *liftdefl-of*·*DEFL*(*udom*)

instance ⟨*proof*⟩

end

27.5.2 Lifted cpo

instantiation *u* :: (*predomain*) *domain*
begin

definition
emb = *u-emb oo liftemb*

definition

prj = *liftprj oo u-prj*

definition
defl (*t*::'*a u itself*) = *u-liftdefl*·*LIFTDEFL*('*a*)

definition
(*liftemb* :: '*a u u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a u u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a u itself*) = *liftdefl-of*·*DEFL*('*a u*)

instance ⟨*proof*⟩

end

lemma *DEFL-u*: *DEFL*('*a*::*predomain u*) = *u-liftdefl*·*LIFTDEFL*('*a*)
⟨*proof*⟩

27.5.3 Strict function space

instantiation *sfun* :: (*domain*, *domain*) *domain*
begin

definition
emb = *sfun-emb oo sfun-map*·*prj*·*emb*

definition
prj = *sfun-map*·*emb*·*prj oo sfun-prj*

definition
defl (*t*::('*a* →! '*b*) *itself*) = *sfun-defl*·*DEFL*('*a*)·*DEFL*('*b*)

definition
(*liftemb* :: ('*a* →! '*b*) *u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → ('*a* →! '*b*) *u*) = *u-map*·*prj*

definition
liftdefl (*t*::('*a* →! '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* →! '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-sfun*:

DEFL('*a*::*domain* →! '*b*::*domain*) = *sfun-defl*·*DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

27.5.4 Continuous function space

instantiation *cfun* :: (*predomain*, *domain*) *domain*
begin

definition
emb = *emb oo encode-cfun*

definition
prj = *decode-cfun oo prj*

definition
defl (*t*::('*a* → '*b*) *itself*) = *DEFL*('*a u* →! '*b*)

definition
(*liftemb* :: ('*a* → '*b*) *u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → ('*a* → '*b*) *u*) = *u-map*·*prj*

definition
liftdefl (*t*::('*a* → '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* → '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-cfun*:
DEFL('*a*::*predomain* → '*b*::*domain*) = *DEFL*('*a u* →! '*b*)
⟨*proof*⟩

27.5.5 Strict product

instantiation *sprod* :: (*domain*, *domain*) *domain*
begin

definition
emb = *sprod-emb oo sprod-map*·*emb*·*emb*

definition
prj = *sprod-map*·*prj*·*prj oo sprod-prj*

definition
defl (*t*::('*a* ⊗ '*b*) *itself*) = *sprod-defl*·*DEFL*('*a*)·*DEFL*('*b*)

definition
(*liftemb* :: ('*a* ⊗ '*b*) *u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → (*'a* ⊗ *'b*) *u*) = *u-map*·*prj*

definition
liftdefl (*t*::(*'a* ⊗ *'b*) *itself*) = *liftdefl-of*·*DEFL*(*'a* ⊗ *'b*)

instance ⟨*proof*⟩

end

lemma *DEFL-sprod*:
DEFL(*'a*::*domain* ⊗ *'b*::*domain*) = *sprod-defl*·*DEFL*(*'a*)·*DEFL*(*'b*)
⟨*proof*⟩

27.5.6 Cartesian product

definition *prod-liftdefl* :: *udom u defl* → *udom u defl* → *udom u defl*
where *prod-liftdefl* = *defl-fun2* (*u-map*·*prod-emb oo decode-prod-u*)
(*encode-prod-u oo u-map*·*prod-prj*) *sprod-map*

lemma *cast-prod-liftdefl*:
cast·(*prod-liftdefl*·*a*·*b*) =
(*u-map*·*prod-emb oo decode-prod-u*) *oo sprod-map*·(*cast*·*a*)·(*cast*·*b*) *oo*
(*encode-prod-u oo u-map*·*prod-prj*)
⟨*proof*⟩

instantiation *prod* :: (*predomain*, *predomain*) *predomain*
begin

definition
liftemb = (*u-map*·*prod-emb oo decode-prod-u*) *oo*
(*sprod-map*·*liftemb*·*liftemb oo encode-prod-u*)

definition
liftprj = (*decode-prod-u oo sprod-map*·*liftprj*·*liftprj*) *oo*
(*encode-prod-u oo u-map*·*prod-prj*)

definition
liftdefl (*t*::(*'a* × *'b*) *itself*) = *prod-liftdefl*·*LIFTDEFL*(*'a*)·*LIFTDEFL*(*'b*)

instance ⟨*proof*⟩

end

instantiation *prod* :: (*domain*, *domain*) *domain*
begin

definition

emb = prod-emb oo prod-map·emb·emb

definition
prj = prod-map·prj·prj oo prod-prj

definition
defl (t::('a × 'b) itself) = prod-defl·DEFL('a)·DEFL('b)

instance ⟨*proof*⟩

end

lemma *DEFL-prod*:
DEFL('a::domain × 'b::domain) = prod-defl·DEFL('a)·DEFL('b)
⟨*proof*⟩

lemma *LIFTDEFL-prod*:
LIFTDEFL('a::predomain × 'b::predomain) =
prod-liftdefl·LIFTDEFL('a)·LIFTDEFL('b)
⟨*proof*⟩

27.5.7 Unit type

instantiation *unit* :: *domain*
begin

definition
emb = (⊥ :: unit → udom)

definition
prj = (⊥ :: udom → unit)

definition
defl (t::unit itself) = ⊥

definition
(liftemb :: unit u → udom u) = u-map·emb

definition
(liftprj :: udom u → unit u) = u-map·prj

definition
liftdefl (t::unit itself) = liftdefl-of·DEFL(unit)

instance ⟨*proof*⟩

end

27.5.8 Discrete cpo

instantiation *discr* :: (*countable*) *predomain*
begin

definition
(*liftemb* :: *'a discr u* → *udom u*) = *strictify·up oo udom-emb discr-approx*

definition
(*liftprj* :: *udom u* → *'a discr u*) = *udom-prj discr-approx oo fup·ID*

definition
liftdefl (*t*::*'a discr itself*) =
(⨆ *i*. *defl-principal* (*Abs-fin-defl* (*liftemb oo discr-approx i oo* (*liftprj*::*udom u* → *'a discr u*))))

instance ⟨*proof*⟩

end

27.5.9 Strict sum

instantiation *ssum* :: (*domain*, *domain*) *domain*
begin

definition
emb = *ssum-emb oo ssum-map·emb·emb*

definition
prj = *ssum-map·prj·prj oo ssum-prj*

definition
defl (*t*::(*'a* ⊕ *'b*) *itself*) = *ssum-defl·DEFL*(*'a*)·*DEFL*(*'b*)

definition
(*liftemb* :: (*'a* ⊕ *'b*) *u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → (*'a* ⊕ *'b*) *u*) = *u-map·prj*

definition
liftdefl (*t*::(*'a* ⊕ *'b*) *itself*) = *liftdefl-of·DEFL*(*'a* ⊕ *'b*)

instance ⟨*proof*⟩

end

lemma *DEFL-ssum*:
DEFL(*'a*::*domain* ⊕ *'b*::*domain*) = *ssum-defl·DEFL*(*'a*)·*DEFL*(*'b*)
⟨*proof*⟩

27.5.10 Lifted HOL type

instantiation *lift* :: (*countable*) *domain*
begin

definition
emb = *emb oo* (Λ *x*. *Rep-lift x*)

definition
prj = (Λ *y*. *Abs-lift y*) *oo prj*

definition
defl (*t*::′*a lift itself*) = *DEFL*(′*a discr u*)

definition
(*liftemb* :: ′*a lift u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → ′*a lift u*) = *u-map*·*prj*

definition
liftdefl (*t*::′*a lift itself*) = *liftdefl-of*·*DEFL*(′*a lift*)

instance ⟨*proof*⟩

end

end

28 Domain-Aux: Domain package support

theory *Domain-Aux*
imports *Map-Functions Fixrec*
begin

28.1 Continuous isomorphisms

A locale for continuous isomorphisms

locale *iso* =
fixes *abs* :: ′*a* → ′*b*
fixes *rep* :: ′*b* → ′*a*
assumes *abs-iso* [*simp*]: *rep*·(*abs*·*x*) = *x*
assumes *rep-iso* [*simp*]: *abs*·(*rep*·*y*) = *y*
begin

lemma *swap*: *iso rep abs*
⟨*proof*⟩

lemma *abs-below*: (*abs*·*x* ⊑ *abs*·*y*) = (*x* ⊑ *y*)

⟨*proof*⟩

lemma *rep-below*: $(rep{\cdot}x \sqsubseteq rep{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *abs-eq*: $(abs{\cdot}x = abs{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *rep-eq*: $(rep{\cdot}x = rep{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *abs-strict*: $abs{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *rep-strict*: $rep{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *abs-defin'*: $abs{\cdot}x = \bot \Longrightarrow x = \bot$
⟨*proof*⟩

lemma *rep-defin'*: $rep{\cdot}z = \bot \Longrightarrow z = \bot$
⟨*proof*⟩

lemma *abs-defined*: $z \neq \bot \Longrightarrow abs{\cdot}z \neq \bot$
⟨*proof*⟩

lemma *rep-defined*: $z \neq \bot \Longrightarrow rep{\cdot}z \neq \bot$
⟨*proof*⟩

lemma *abs-bottom-iff*: $(abs{\cdot}x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *rep-bottom-iff*: $(rep{\cdot}x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *casedist-rule*: $rep{\cdot}x = \bot \vee P \Longrightarrow x = \bot \vee P$
⟨*proof*⟩

lemma *compact-abs-rev*: $compact\ (abs{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-rep-rev*: $compact\ (rep{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-abs*: $compact\ x \Longrightarrow compact\ (abs{\cdot}x)$
⟨*proof*⟩

lemma *compact-rep*: $compact\ x \Longrightarrow compact\ (rep{\cdot}x)$
⟨*proof*⟩

lemma *iso-swap*: $(x = abs{\cdot}y) = (rep{\cdot}x = y)$
⟨*proof*⟩

end

28.2 Proofs about take functions

This section contains lemmas that are used in a module that supports the domain isomorphism package; the module contains proofs related to take functions and the finiteness predicate.

lemma *deflation-abs-rep*:
fixes *abs* **and** *rep* **and** *d*
assumes *abs-iso*: $\bigwedge x.\ rep{\cdot}(abs{\cdot}x) = x$
assumes *rep-iso*: $\bigwedge y.\ abs{\cdot}(rep{\cdot}y) = y$
shows *deflation d* $\Longrightarrow$ *deflation* (*abs oo d oo rep*)
⟨*proof*⟩

lemma *deflation-chain-min*:
assumes *chain*: *chain d*
assumes *defl*: $\bigwedge n.$ *deflation* $(d\ n)$
shows $d\ m{\cdot}(d\ n{\cdot}x) = d\ (min\ m\ n){\cdot}x$
⟨*proof*⟩

lemma *lub-ID-take-lemma*:
assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
assumes $\bigwedge n.\ t\ n{\cdot}x = t\ n{\cdot}y$ **shows** $x = y$
⟨*proof*⟩

lemma *lub-ID-reach*:
assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
shows $(\bigsqcup n.\ t\ n{\cdot}x) = x$
⟨*proof*⟩

lemma *lub-ID-take-induct*:
assumes *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
assumes *adm P* **and** $\bigwedge n.\ P\ (t\ n{\cdot}x)$ **shows** $P\ x$
⟨*proof*⟩

28.3 Finiteness

Let a "decisive" function be a deflation that maps every input to either itself or bottom. Then if a domain's take functions are all decisive, then all values in the domain are finite.

definition
decisive :: $('a{::}pcpo \to 'a) \Rightarrow bool$
where
decisive d $\longleftrightarrow (\forall x.\ d{\cdot}x = x \vee d{\cdot}x = \bot)$

lemma *decisiveI*: $(\bigwedge x.\ d{\cdot}x = x \vee d{\cdot}x = \bot) \Longrightarrow$ *decisive d*
⟨*proof*⟩

lemma *decisive-cases*:
assumes *decisive d* **obtains** $d{\cdot}x = x \mid d{\cdot}x = \bot$
⟨*proof*⟩

lemma *decisive-bottom*: *decisive* $\bot$
⟨*proof*⟩

lemma *decisive-ID*: *decisive ID*
⟨*proof*⟩

lemma *decisive-ssum-map*:
assumes *f*: *decisive f*
assumes *g*: *decisive g*
shows *decisive* (*ssum-map*·*f*·*g*)
⟨*proof*⟩

lemma *decisive-sprod-map*:
assumes *f*: *decisive f*
assumes *g*: *decisive g*
shows *decisive* (*sprod-map*·*f*·*g*)
⟨*proof*⟩

lemma *decisive-abs-rep*:
fixes *abs rep*
assumes *iso*: *iso abs rep*
assumes *d*: *decisive d*
shows *decisive* (*abs oo d oo rep*)
⟨*proof*⟩

lemma *lub-ID-finite*:
assumes *chain*: *chain d*
assumes *lub*: $(\bigsqcup n.\ d\ n) = ID$
assumes *decisive*: $\bigwedge n.$ *decisive* $(d\ n)$
shows $\exists n.\ d\ n{\cdot}x = x$
⟨*proof*⟩

lemma *lub-ID-finite-take-induct*:
assumes *chain d* **and** $(\bigsqcup n.\ d\ n) = ID$ **and** $\bigwedge n.$ *decisive* $(d\ n)$
shows $(\bigwedge n.\ P\ (d\ n{\cdot}x)) \Longrightarrow P\ x$
⟨*proof*⟩

28.4 Proofs about constructor functions

Lemmas for proving nchotomy rule:

lemma *ex-one-bottom-iff*:

$(\exists\, x.\ P\ x \wedge x \neq \bot) = P\ ONE$
⟨*proof*⟩

lemma *ex-up-bottom-iff*:
$(\exists\, x.\ P\ x \wedge x \neq \bot) = (\exists\, x.\ P\ (up{\cdot}x))$
⟨*proof*⟩

lemma *ex-sprod-bottom-iff*:
$(\exists\, y.\ P\ y \wedge y \neq \bot) =$
$(\exists\, x\ y.\ (P\ (:x,\ y:) \wedge x \neq \bot) \wedge y \neq \bot)$
⟨*proof*⟩

lemma *ex-sprod-up-bottom-iff*:
$(\exists\, y.\ P\ y \wedge y \neq \bot) =$
$(\exists\, x\ y.\ P\ (:up{\cdot}x,\ y:) \wedge y \neq \bot)$
⟨*proof*⟩

lemma *ex-ssum-bottom-iff*:
$(\exists\, x.\ P\ x \wedge x \neq \bot) =$
$((\exists\, x.\ P\ (sinl{\cdot}x) \wedge x \neq \bot) \vee$
$(\exists\, x.\ P\ (sinr{\cdot}x) \wedge x \neq \bot))$
⟨*proof*⟩

lemma *exh-start*: $p = \bot \vee (\exists\, x.\ p = x \wedge x \neq \bot)$
⟨*proof*⟩

lemmas *ex-bottom-iffs* =
ex-ssum-bottom-iff
ex-sprod-up-bottom-iff
ex-sprod-bottom-iff
ex-up-bottom-iff
ex-one-bottom-iff

Rules for turning nchotomy into exhaust:

lemma *exh-casedist0*: $[\![R;\ R \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *exh-casedist1*: $((P \vee Q \Longrightarrow R) \Longrightarrow S) \equiv ([\![P \Longrightarrow R;\ Q \Longrightarrow R]\!] \Longrightarrow S)$
⟨*proof*⟩

lemma *exh-casedist2*: $(\exists\, x.\ P\ x \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
⟨*proof*⟩

lemma *exh-casedist3*: $(P \wedge Q \Longrightarrow R) \equiv (P \Longrightarrow Q \Longrightarrow R)$
⟨*proof*⟩

lemmas *exh-casedists* = *exh-casedist1 exh-casedist2 exh-casedist3*

Rules for proving constructor properties

lemmas *con-strict-rules* =
sinl-strict sinr-strict spair-strict1 spair-strict2

lemmas *con-bottom-iff-rules* =
sinl-bottom-iff sinr-bottom-iff spair-bottom-iff up-defined ONE-defined

lemmas *con-below-iff-rules* =
sinl-below sinr-below sinl-below-sinr sinr-below-sinl con-bottom-iff-rules

lemmas *con-eq-iff-rules* =
sinl-eq sinr-eq sinl-eq-sinr sinr-eq-sinl con-bottom-iff-rules

lemmas *sel-strict-rules* =
cfcomp2 sscase1 sfst-strict ssnd-strict fup1

lemma *sel-app-extra-rules*:
$sscase{\cdot}ID{\cdot}\bot{\cdot}(sinr{\cdot}x) = \bot$
$sscase{\cdot}ID{\cdot}\bot{\cdot}(sinl{\cdot}x) = x$
$sscase{\cdot}\bot{\cdot}ID{\cdot}(sinl{\cdot}x) = \bot$
$sscase{\cdot}\bot{\cdot}ID{\cdot}(sinr{\cdot}x) = x$
$fup{\cdot}ID{\cdot}(up{\cdot}x) = x$
⟨*proof*⟩

lemmas *sel-app-rules* =
sel-strict-rules sel-app-extra-rules
ssnd-spair sfst-spair up-defined spair-defined

lemmas *sel-bottom-iff-rules* =
cfcomp2 sfst-bottom-iff ssnd-bottom-iff

lemmas *take-con-rules* =
ssum-map-sinl′ ssum-map-sinr′ sprod-map-spair′ u-map-up
deflation-strict deflation-ID ID1 cfcomp2

28.5 ML setup

⟨*ML*⟩

end

29 Domain: Domain package

theory *Domain*
imports *Representable Domain-Aux*
keywords
domaindef :: *thy-decl* **and** *lazy unsafe* **and**
domain-isomorphism domain :: *thy-decl*
begin

default-sort *domain*

29.1 Representations of types

lemma *emb-prj*: $emb{\cdot}((prj{\cdot}x)::'a) = cast{\cdot}DEFL('a){\cdot}x$
⟨*proof*⟩

lemma *emb-prj-emb*:
fixes $x :: 'a$
assumes $DEFL('a) \sqsubseteq DEFL('b)$
shows $emb{\cdot}(prj{\cdot}(emb{\cdot}x) :: 'b) = emb{\cdot}x$
⟨*proof*⟩

lemma *prj-emb-prj*:
assumes $DEFL('a) \sqsubseteq DEFL('b)$
shows $prj{\cdot}(emb{\cdot}(prj{\cdot}x :: 'b)) = (prj{\cdot}x :: 'a)$
⟨*proof*⟩

Isomorphism lemmas used internally by the domain package:

lemma *domain-abs-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: $DEFL('b) = DEFL('a)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows $rep{\cdot}(abs{\cdot}x) = x$
⟨*proof*⟩

lemma *domain-rep-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: $DEFL('b) = DEFL('a)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows $abs{\cdot}(rep{\cdot}x) = x$
⟨*proof*⟩

29.2 Deflations as sets

definition *defl-set* :: $'a::bifinite\ defl \Rightarrow 'a\ set$
where $defl\text{-}set\ A = \{x.\ cast{\cdot}A{\cdot}x = x\}$

lemma *adm-defl-set*: $adm\ (\lambda x.\ x \in defl\text{-}set\ A)$
⟨*proof*⟩

lemma *defl-set-bottom*: $\bot \in defl\text{-}set\ A$
⟨*proof*⟩

lemma *defl-set-cast* [*simp*]: $cast{\cdot}A{\cdot}x \in defl\text{-}set\ A$
⟨*proof*⟩

lemma *defl-set-subset-iff*: *defl-set A* ⊆ *defl-set B* ⟷ *A* ⊑ *B*
⟨*proof*⟩

29.3 Proving a subtype is representable

Temporarily relax type constraints.

⟨*ML*⟩

lemma *typedef-domain-class*:
fixes *Rep* :: *'a*::*pcpo* ⇒ *udom*
fixes *Abs* :: *udom* ⇒ *'a*::*pcpo*
fixes *t* :: *udom defl*
assumes *type*: *type-definition Rep Abs* (*defl-set t*)
assumes *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
assumes *emb*: *emb* ≡ (Λ *x*. *Rep x*)
assumes *prj*: *prj* ≡ (Λ *x*. *Abs* (*cast*·*t*·*x*))
assumes *defl*: *defl* ≡ (λ *a*::*'a itself*. *t*)
assumes *liftemb*: (*liftemb* :: *'a u* → *udom u*) ≡ *u-map*·*emb*
assumes *liftprj*: (*liftprj* :: *udom u* → *'a u*) ≡ *u-map*·*prj*
assumes *liftdefl*: (*liftdefl* :: *'a itself* ⇒ -) ≡ (λ*t*. *liftdefl-of*·*DEFL*(*'a*))
shows *OFCLASS*(*'a*, *domain-class*)
⟨*proof*⟩

lemma *typedef-DEFL*:
assumes *defl* ≡ (λ*a*::*'a*::*pcpo itself*. *t*)
shows *DEFL*(*'a*::*pcpo*) = *t*
⟨*proof*⟩

Restore original typing constraints.

⟨*ML*⟩

29.4 Isomorphic deflations

definition *isodefl* :: (*'a* → *'a*) ⇒ *udom defl* ⇒ *bool*
where *isodefl d t* ⟷ *cast*·*t* = *emb oo d oo prj*

definition *isodefl'* :: (*'a*::*predomain* → *'a*) ⇒ *udom u defl* ⇒ *bool*
where *isodefl' d t* ⟷ *cast*·*t* = *liftemb oo u-map*·*d oo liftprj*

lemma *isodeflI*: (⋀*x*. *cast*·*t*·*x* = *emb*·(*d*·(*prj*·*x*))) ⟹ *isodefl d t*
⟨*proof*⟩

lemma *cast-isodefl*: *isodefl d t* ⟹ *cast*·*t* = (Λ *x*. *emb*·(*d*·(*prj*·*x*)))
⟨*proof*⟩

lemma *isodefl-strict*: *isodefl d t* ⟹ *d*·⊥ = ⊥
⟨*proof*⟩

lemma *isodefl-imp-deflation*:

fixes $d :: 'a \to 'a$
assumes *isodefl d t* **shows** *deflation d*
⟨*proof*⟩

lemma *isodefl-ID-DEFL*: *isodefl* $(ID :: 'a \to 'a)$ *DEFL*$('a)$
⟨*proof*⟩

lemma *isodefl-LIFTDEFL*:
isodefl' $(ID :: 'a \to 'a)$ *LIFTDEFL*$('a{::}predomain)$
⟨*proof*⟩

lemma *isodefl-DEFL-imp-ID*: *isodefl* $(d :: 'a \to 'a)$ *DEFL*$('a) \Longrightarrow d = ID$
⟨*proof*⟩

lemma *isodefl-bottom*: *isodefl* $\bot$ $\bot$
⟨*proof*⟩

lemma *adm-isodefl*:
cont f $\Longrightarrow$ *cont g* $\Longrightarrow$ *adm* $(\lambda x.$ *isodefl* $(f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *isodefl-lub*:
assumes *chain d* **and** *chain t*
assumes $\bigwedge i.$ *isodefl* $(d\ i)\ (t\ i)$
shows *isodefl* $(\bigsqcup i.\ d\ i)\ (\bigsqcup i.\ t\ i)$
⟨*proof*⟩

lemma *isodefl-fix*:
assumes $\bigwedge d\ t.$ *isodefl d t* $\Longrightarrow$ *isodefl* $(f{\cdot}d)\ (g{\cdot}t)$
shows *isodefl* $(fix{\cdot}f)\ (fix{\cdot}g)$
⟨*proof*⟩

lemma *isodefl-abs-rep*:
fixes *abs* **and** *rep* **and** *d*
assumes *DEFL*: *DEFL*$('b)$ = *DEFL*$('a)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows *isodefl d t* $\Longrightarrow$ *isodefl* $(abs\ oo\ d\ oo\ rep)\ t$
⟨*proof*⟩

lemma *isodefl'-liftdefl-of*: *isodefl d t* $\Longrightarrow$ *isodefl'* d $(liftdefl\text{-}of{\cdot}t)$
⟨*proof*⟩

lemma *isodefl-sfun*:
isodefl d1 t1 $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
isodefl $(sfun\text{-}map{\cdot}d1{\cdot}d2)$ $(sfun\text{-}defl{\cdot}t1{\cdot}t2)$
⟨*proof*⟩

lemma *isodefl-ssum*:

isodefl d1 t1 ⟹ isodefl d2 t2 ⟹
isodefl (ssum-map·d1·d2) (ssum-defl·t1·t2)
⟨*proof*⟩

lemma *isodefl-sprod*:
isodefl d1 t1 ⟹ isodefl d2 t2 ⟹
isodefl (sprod-map·d1·d2) (sprod-defl·t1·t2)
⟨*proof*⟩

lemma *isodefl-prod*:
isodefl d1 t1 ⟹ isodefl d2 t2 ⟹
isodefl (prod-map·d1·d2) (prod-defl·t1·t2)
⟨*proof*⟩

lemma *isodefl-u*:
isodefl d t ⟹ isodefl (u-map·d) (u-defl·t)
⟨*proof*⟩

lemma *isodefl-u-liftdefl*:
isodefl′ d t ⟹ isodefl (u-map·d) (u-liftdefl·t)
⟨*proof*⟩

lemma *encode-prod-u-map*:
encode-prod-u·(u-map·(prod-map·f·g)·(decode-prod-u·x))
= sprod-map·(u-map·f)·(u-map·g)·x
⟨*proof*⟩

lemma *isodefl-prod-u*:
assumes *isodefl′ d1 t1* **and** *isodefl′ d2 t2*
shows *isodefl′ (prod-map·d1·d2) (prod-liftdefl·t1·t2)*
⟨*proof*⟩

lemma *encode-cfun-map*:
encode-cfun·(cfun-map·f·g·(decode-cfun·x))
= sfun-map·(u-map·f)·g·x
⟨*proof*⟩

lemma *isodefl-cfun*:
assumes *isodefl (u-map·d1) t1* **and** *isodefl d2 t2*
shows *isodefl (cfun-map·d1·d2) (sfun-defl·t1·t2)*
⟨*proof*⟩

29.5 Setting up the domain package

⟨*ML*⟩

lemmas [*domain-defl-simps*] =
DEFL-cfun DEFL-sfun DEFL-ssum DEFL-sprod DEFL-prod DEFL-u
liftdefl-eq LIFTDEFL-prod u-liftdefl-liftdefl-of

lemmas [*domain-map-ID*] =
cfun-map-ID sfun-map-ID ssum-map-ID sprod-map-ID prod-map-ID u-map-ID

lemmas [*domain-isodefl*] =
isodefl-u isodefl-sfun isodefl-ssum isodefl-sprod
isodefl-cfun isodefl-prod isodefl-prod-u isodefl'-liftdefl-of
isodefl-u-liftdefl

lemmas [*domain-deflation*] =
deflation-cfun-map deflation-sfun-map deflation-ssum-map
deflation-sprod-map deflation-prod-map deflation-u-map

⟨*ML*⟩

end

30 Compact-Basis: A compact basis for powerdomains

theory *Compact-Basis*
imports *Universal*
begin

default-sort *bifinite*

30.1 A compact basis for powerdomains

definition *pd-basis* = {*S*::'*a compact-basis set*. *finite S* ∧ *S* ≠ {}}

typedef '*a pd-basis* = *pd-basis* :: '*a compact-basis set set*
⟨*proof*⟩

lemma *finite-Rep-pd-basis* [*simp*]: *finite* (*Rep-pd-basis u*)
⟨*proof*⟩

lemma *Rep-pd-basis-nonempty* [*simp*]: *Rep-pd-basis u* ≠ {}
⟨*proof*⟩

The powerdomain basis type is countable.

lemma *pd-basis-countable*: ∃*f*::'*a pd-basis* ⇒ *nat*. *inj f*
⟨*proof*⟩

30.2 Unit and plus constructors

definition
PDUnit :: '*a compact-basis* ⇒ '*a pd-basis* **where**
PDUnit = (λ*x*. *Abs-pd-basis* {*x*})

definition
PDPlus :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *'a pd-basis* **where**
PDPlus t u = *Abs-pd-basis* (*Rep-pd-basis t* ∪ *Rep-pd-basis u*)

lemma *Rep-PDUnit*:
Rep-pd-basis (*PDUnit x*) = {*x*}
⟨*proof*⟩

lemma *Rep-PDPlus*:
Rep-pd-basis (*PDPlus u v*) = *Rep-pd-basis u* ∪ *Rep-pd-basis v*
⟨*proof*⟩

lemma *PDUnit-inject* [*simp*]: (*PDUnit a* = *PDUnit b*) = (*a* = *b*)
⟨*proof*⟩

lemma *PDPlus-assoc*: *PDPlus* (*PDPlus t u*) *v* = *PDPlus t* (*PDPlus u v*)
⟨*proof*⟩

lemma *PDPlus-commute*: *PDPlus t u* = *PDPlus u t*
⟨*proof*⟩

lemma *PDPlus-absorb*: *PDPlus t t* = *t*
⟨*proof*⟩

lemma *pd-basis-induct1*:
assumes *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
assumes *PDPlus*: ⋀*a t*. *P t* ⟹ *P* (*PDPlus* (*PDUnit a*) *t*)
shows *P x*
⟨*proof*⟩

lemma *pd-basis-induct*:
assumes *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
assumes *PDPlus*: ⋀*t u*. ⟦*P t*; *P u*⟧ ⟹ *P* (*PDPlus t u*)
shows *P x*
⟨*proof*⟩

30.3 Fold operator

definition
fold-pd ::
(*'a compact-basis* ⇒ *'b*::*type*) ⇒ (*'b* ⇒ *'b* ⇒ *'b*) ⇒ *'a pd-basis* ⇒ *'b*
where *fold-pd g f t* = *semilattice-set.F f* (*g* ‘ *Rep-pd-basis t*)

lemma *fold-pd-PDUnit*:
assumes *semilattice f*
shows *fold-pd g f* (*PDUnit x*) = *g x*
⟨*proof*⟩

lemma *fold-pd-PDPlus*:
  assumes *semilattice f*
  shows *fold-pd g f (PDPlus t u) = f (fold-pd g f t) (fold-pd g f u)*
⟨*proof*⟩

end

31 UpperPD: Upper powerdomain

theory *UpperPD*
imports *Compact-Basis*
begin

31.1 Basis preorder

definition
  upper-le :: *'a pd-basis ⇒ 'a pd-basis ⇒ bool* (**infix** $\leq\sharp$ *50*) **where**
  upper-le = (λu v. ∀ y∈Rep-pd-basis v. ∃ x∈Rep-pd-basis u. x ⊑ y)

lemma *upper-le-refl* [*simp*]: $t \leq\sharp t$
⟨*proof*⟩

lemma *upper-le-trans*: $[\![t \leq\sharp u;\ u \leq\sharp v]\!] \Longrightarrow t \leq\sharp v$
⟨*proof*⟩

interpretation *upper-le*: *preorder upper-le*
⟨*proof*⟩

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* $\leq\sharp$ *t*
⟨*proof*⟩

lemma *PDUnit-upper-mono*: $x \sqsubseteq y \Longrightarrow$ *PDUnit x* $\leq\sharp$ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-upper-mono*: $[\![s \leq\sharp t;\ u \leq\sharp v]\!] \Longrightarrow$ *PDPlus s u* $\leq\sharp$ *PDPlus t v*
⟨*proof*⟩

lemma *PDPlus-upper-le*: *PDPlus t u* $\leq\sharp$ *t*
⟨*proof*⟩

lemma *upper-le-PDUnit-PDUnit-iff* [*simp*]:
  (PDUnit a $\leq\sharp$ *PDUnit b) = (a ⊑ b)*
⟨*proof*⟩

lemma *upper-le-PDPlus-PDUnit-iff*:
  (PDPlus t u $\leq\sharp$ *PDUnit a) = (t* $\leq\sharp$ *PDUnit a ∨ u* $\leq\sharp$ *PDUnit a)*
⟨*proof*⟩

lemma *upper-le-PDPlus-iff*: *(t* $\leq\sharp$ *PDPlus u v) = (t* $\leq\sharp$ *u ∧ t* $\leq\sharp$ *v)*

⟨*proof*⟩

lemma *upper-le-induct* [*induct set*: *upper-le*]:
assumes *le*: $t \leq\sharp\ u$
assumes *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
assumes *2*: $\bigwedge t\ u\ a.\ P\ t\ (PDUnit\ a) \Longrightarrow P\ (PDPlus\ t\ u)\ (PDUnit\ a)$
assumes *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ t\ v]\!] \Longrightarrow P\ t\ (PDPlus\ u\ v)$
shows $P\ t\ u$
⟨*proof*⟩

31.2 Type definition

typedef *'a upper-pd* =
{*S*::*'a pd-basis set*. *upper-le.ideal S*}
⟨*proof*⟩

type-notation (*xsymbols*) *upper-pd* (('(-')♯))

instantiation *upper-pd* :: (*bifinite*) *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow$ *Rep-upper-pd x* $\subseteq$ *Rep-upper-pd y*

instance ⟨*proof*⟩
end

instance *upper-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
upper-principal :: *'a pd-basis* ⇒ *'a upper-pd* **where**
upper-principal t = *Abs-upper-pd* {*u*. $u \leq\sharp\ t$}

interpretation *upper-pd*:
ideal-completion upper-le upper-principal Rep-upper-pd
⟨*proof*⟩

Upper powerdomain is pointed

lemma *upper-pd-minimal*: *upper-principal* (*PDUnit compact-bot*) $\sqsubseteq$ *ys*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-upper-pd-pcpo*: $\bot$ = *upper-principal* (*PDUnit compact-bot*)

⟨*proof*⟩

31.3 Monadic unit and plus

definition
upper-unit :: *'a* → *'a upper-pd* **where**
upper-unit = *compact-basis.extension* (λ*a*. *upper-principal* (*PDUnit a*))

definition
upper-plus :: *'a upper-pd* → *'a upper-pd* → *'a upper-pd* **where**
upper-plus = *upper-pd.extension* (λ*t*. *upper-pd.extension* (λ*u*.
upper-principal (*PDPlus t u*)))

abbreviation
upper-add :: *'a upper-pd* ⇒ *'a upper-pd* ⇒ *'a upper-pd*
(**infixl** ∪♯ *65*) **where**
xs ∪♯ *ys* == *upper-plus*·*xs*·*ys*

syntax
-upper-pd :: *args* ⇒ *logic* ({-}♯)

translations
{*x*,*xs*}♯ == {*x*}♯ ∪♯ {*xs*}♯
{*x*}♯ == *CONST upper-unit*·*x*

lemma *upper-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♯ = *upper-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *upper-plus-principal* [*simp*]:
upper-principal t ∪♯ *upper-principal u* = *upper-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *upper-add*: *semilattice upper-add* ⟨*proof*⟩

lemmas *upper-plus-assoc* = *upper-add.assoc*
lemmas *upper-plus-commute* = *upper-add.commute*
lemmas *upper-plus-absorb* = *upper-add.idem*
lemmas *upper-plus-left-commute* = *upper-add.left-commute*
lemmas *upper-plus-left-absorb* = *upper-add.left-idem*

Useful for *simp add*: *upper-plus-ac*

lemmas *upper-plus-ac* =
upper-plus-assoc upper-plus-commute upper-plus-left-commute

Useful for *simp only*: *upper-plus-aci*

lemmas *upper-plus-aci* =
upper-plus-ac upper-plus-absorb upper-plus-left-absorb

lemma *upper-plus-below1*: $xs \mathbin{\cup\!\sharp} ys \sqsubseteq xs$
⟨*proof*⟩

lemma *upper-plus-below2*: $xs \mathbin{\cup\!\sharp} ys \sqsubseteq ys$
⟨*proof*⟩

lemma *upper-plus-greatest*: $[\![xs \sqsubseteq ys; xs \sqsubseteq zs]\!] \Longrightarrow xs \sqsubseteq ys \mathbin{\cup\!\sharp} zs$
⟨*proof*⟩

lemma *upper-below-plus-iff* [*simp*]:
$xs \sqsubseteq ys \mathbin{\cup\!\sharp} zs \longleftrightarrow xs \sqsubseteq ys \wedge xs \sqsubseteq zs$
⟨*proof*⟩

lemma *upper-plus-below-unit-iff* [*simp*]:
$xs \mathbin{\cup\!\sharp} ys \sqsubseteq \{z\}\sharp \longleftrightarrow xs \sqsubseteq \{z\}\sharp \vee ys \sqsubseteq \{z\}\sharp$
⟨*proof*⟩

lemma *upper-unit-below-iff* [*simp*]: $\{x\}\sharp \sqsubseteq \{y\}\sharp \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemmas *upper-pd-below-simps* =
upper-unit-below-iff
upper-below-plus-iff
upper-plus-below-unit-iff

lemma *upper-unit-eq-iff* [*simp*]: $\{x\}\sharp = \{y\}\sharp \longleftrightarrow x = y$
⟨*proof*⟩

lemma *upper-unit-strict* [*simp*]: $\{\bot\}\sharp = \bot$
⟨*proof*⟩

lemma *upper-plus-strict1* [*simp*]: $\bot \mathbin{\cup\!\sharp} ys = \bot$
⟨*proof*⟩

lemma *upper-plus-strict2* [*simp*]: $xs \mathbin{\cup\!\sharp} \bot = \bot$
⟨*proof*⟩

lemma *upper-unit-bottom-iff* [*simp*]: $\{x\}\sharp = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *upper-plus-bottom-iff* [*simp*]:
$xs \mathbin{\cup\!\sharp} ys = \bot \longleftrightarrow xs = \bot \vee ys = \bot$
⟨*proof*⟩

lemma *compact-upper-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\sharp$
⟨*proof*⟩

lemma *compact-upper-unit-iff* [*simp*]: *compact* $\{x\}\sharp \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-upper-plus* [*simp*]:
⟦*compact xs*; *compact ys*⟧ ⟹ *compact* (*xs* ∪♯ *ys*)
⟨*proof*⟩

31.4 Induction rules

lemma *upper-pd-induct1*:
assumes *P*: *adm P*
assumes *unit*: ⋀*x*. *P* {*x*}♯
assumes *insert*: ⋀*x ys*. ⟦*P* {*x*}♯; *P ys*⟧ ⟹ *P* ({*x*}♯ ∪♯ *ys*)
shows *P* (*xs*::'*a upper-pd*)
⟨*proof*⟩

lemma *upper-pd-induct*
[*case-names adm upper-unit upper-plus*, *induct type*: *upper-pd*]:
assumes *P*: *adm P*
assumes *unit*: ⋀*x*. *P* {*x*}♯
assumes *plus*: ⋀*xs ys*. ⟦*P xs*; *P ys*⟧ ⟹ *P* (*xs* ∪♯ *ys*)
shows *P* (*xs*::'*a upper-pd*)
⟨*proof*⟩

31.5 Monadic bind

definition
upper-bind-basis ::
'*a pd-basis* ⇒ ('*a* → '*b upper-pd*) → '*b upper-pd* **where**
upper-bind-basis = *fold-pd*
(λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
(λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)

lemma *ACI-upper-bind*:
semilattice (λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)
⟨*proof*⟩

lemma *upper-bind-basis-simps* [*simp*]:
upper-bind-basis (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
upper-bind-basis (*PDPlus t u*) =
(Λ *f*. *upper-bind-basis t*·*f* ∪♯ *upper-bind-basis u*·*f*)
⟨*proof*⟩

lemma *upper-bind-basis-mono*:
t ≤♯ *u* ⟹ *upper-bind-basis t* ⊑ *upper-bind-basis u*
⟨*proof*⟩

definition
upper-bind :: '*a upper-pd* → ('*a* → '*b upper-pd*) → '*b upper-pd* **where**
upper-bind = *upper-pd.extension upper-bind-basis*

syntax
-upper-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
((3⋃♯-∈-./ -) [0, 0, 10] 10)

translations
⋃♯x∈xs. e == CONST upper-bind·xs·(Λ x. e)

lemma *upper-bind-principal* [*simp*]:
upper-bind·(*upper-principal t*) = *upper-bind-basis t*
⟨*proof*⟩

lemma *upper-bind-unit* [*simp*]:
upper-bind·{x}♯·f = f·x
⟨*proof*⟩

lemma *upper-bind-plus* [*simp*]:
upper-bind·(xs ∪♯ ys)·f = *upper-bind*·xs·f ∪♯ *upper-bind*·ys·f
⟨*proof*⟩

lemma *upper-bind-strict* [*simp*]: *upper-bind*·⊥·f = f·⊥
⟨*proof*⟩

lemma *upper-bind-bind*:
upper-bind·(*upper-bind*·xs·f)·g = *upper-bind*·xs·(Λ x. *upper-bind*·(f·x)·g)
⟨*proof*⟩

31.6 Map

definition
upper-map :: ('a → 'b) → 'a *upper-pd* → 'b *upper-pd* **where**
upper-map = (Λ f xs. *upper-bind*·xs·(Λ x. {f·x}♯))

lemma *upper-map-unit* [*simp*]:
upper-map·f·{x}♯ = {f·x}♯
⟨*proof*⟩

lemma *upper-map-plus* [*simp*]:
upper-map·f·(xs ∪♯ ys) = *upper-map*·f·xs ∪♯ *upper-map*·f·ys
⟨*proof*⟩

lemma *upper-map-bottom* [*simp*]: *upper-map*·f·⊥ = {f·⊥}♯
⟨*proof*⟩

lemma *upper-map-ident*: *upper-map*·(Λ x. x)·xs = xs
⟨*proof*⟩

lemma *upper-map-ID*: *upper-map*·ID = ID
⟨*proof*⟩

lemma *upper-map-map*:
upper-map·*f*·(*upper-map*·*g*·*xs*) = *upper-map*·(Λ *x*. *f*·(*g*·*x*))·*xs*
⟨*proof*⟩

lemma *upper-bind-map*:
upper-bind·(*upper-map*·*f*·*xs*)·*g* = *upper-bind*·*xs*·(Λ *x*. *g*·(*f*·*x*))
⟨*proof*⟩

lemma *upper-map-bind*:
upper-map·*f*·(*upper-bind*·*xs*·*g*) = *upper-bind*·*xs*·(Λ *x*. *upper-map*·*f*·(*g*·*x*))
⟨*proof*⟩

lemma *ep-pair-upper-map*: *ep-pair e p* ⟹ *ep-pair* (*upper-map*·*e*) (*upper-map*·*p*)
⟨*proof*⟩

lemma *deflation-upper-map*: *deflation d* ⟹ *deflation* (*upper-map*·*d*)
⟨*proof*⟩

lemma *finite-deflation-upper-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*upper-map*·*d*)
⟨*proof*⟩

31.7 Upper powerdomain is bifinite

lemma *approx-chain-upper-map*:
assumes *approx-chain a*
shows *approx-chain* (λ*i*. *upper-map*·(*a i*))
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

31.8 Join

definition
upper-join :: *'a upper-pd upper-pd* → *'a upper-pd* **where**
upper-join = (Λ *xss*. *upper-bind*·*xss*·(Λ *xs*. *xs*))

lemma *upper-join-unit* [*simp*]:
upper-join·{*xs*}♯ = *xs*
⟨*proof*⟩

lemma *upper-join-plus* [*simp*]:
upper-join·(*xss* ∪♯ *yss*) = *upper-join*·*xss* ∪♯ *upper-join*·*yss*
⟨*proof*⟩

lemma *upper-join-bottom* [*simp*]: *upper-join*·⊥ = ⊥
⟨*proof*⟩

lemma *upper-join-map-unit*:
upper-join·(*upper-map*·*upper-unit*·*xs*) = *xs*
⟨*proof*⟩

lemma *upper-join-map-join*:
upper-join·(*upper-map*·*upper-join*·*xsss*) = *upper-join*·(*upper-join*·*xsss*)
⟨*proof*⟩

lemma *upper-join-map-map*:
upper-join·(*upper-map*·(*upper-map*·*f*)·*xss*) =
upper-map·*f*·(*upper-join*·*xss*)
⟨*proof*⟩

end

32 LowerPD: Lower powerdomain

theory *LowerPD*
imports *Compact-Basis*
begin

32.1 Basis preorder

definition
lower-le :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ≤♭ *50*) **where**
lower-le = (λ*u v*. ∀ *x*∈*Rep-pd-basis u*. ∃ *y*∈*Rep-pd-basis v*. *x* ⊑ *y*)

lemma *lower-le-refl* [*simp*]: *t* ≤♭ *t*
⟨*proof*⟩

lemma *lower-le-trans*: ⟦*t* ≤♭ *u*; *u* ≤♭ *v*⟧ ⟹ *t* ≤♭ *v*
⟨*proof*⟩

interpretation *lower-le*: *preorder lower-le*
⟨*proof*⟩

lemma *lower-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♭ *t*
⟨*proof*⟩

lemma *PDUnit-lower-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♭ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-lower-mono*: ⟦*s* ≤♭ *t*; *u* ≤♭ *v*⟧ ⟹ *PDPlus s u* ≤♭ *PDPlus t v*
⟨*proof*⟩

lemma *PDPlus-lower-le*: *t* ≤♭ *PDPlus t u*
⟨*proof*⟩

lemma *lower-le-PDUnit-PDUnit-iff* [*simp*]:

$(PDUnit\ a \leq\flat\ PDUnit\ b) = (a \sqsubseteq b)$
⟨*proof*⟩

lemma *lower-le-PDUnit-PDPlus-iff*:
$(PDUnit\ a \leq\flat\ PDPlus\ t\ u) = (PDUnit\ a \leq\flat\ t \vee PDUnit\ a \leq\flat\ u)$
⟨*proof*⟩

lemma *lower-le-PDPlus-iff*: $(PDPlus\ t\ u \leq\flat\ v) = (t \leq\flat\ v \wedge u \leq\flat\ v)$
⟨*proof*⟩

lemma *lower-le-induct* [*induct set*: *lower-le*]:
assumes *le*: $t \leq\flat\ u$
assumes *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
assumes *2*: $\bigwedge t\ u\ a.\ P\ (PDUnit\ a)\ t \Longrightarrow P\ (PDUnit\ a)\ (PDPlus\ t\ u)$
assumes *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ v;\ P\ u\ v]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ v$
shows $P\ t\ u$
⟨*proof*⟩

32.2 Type definition

typedef *'a lower-pd* =
$\{S::'a\ pd\text{-}basis\ set.\ lower\text{-}le.ideal\ S\}$
⟨*proof*⟩

type-notation (*xsymbols*) *lower-pd* (($'(\text{-}')\flat$))

instantiation *lower-pd* :: (*bifinite*) *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow Rep\text{-}lower\text{-}pd\ x \subseteq Rep\text{-}lower\text{-}pd\ y$

instance ⟨*proof*⟩
end

instance *lower-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
lower-principal :: *'a pd-basis* ⇒ *'a lower-pd* **where**
$lower\text{-}principal\ t = Abs\text{-}lower\text{-}pd\ \{u.\ u \leq\flat\ t\}$

interpretation *lower-pd*:
ideal-completion lower-le lower-principal Rep-lower-pd
⟨*proof*⟩

Lower powerdomain is pointed

lemma *lower-pd-minimal*: *lower-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-lower-pd-pcpo*: ⊥ = *lower-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

32.3 Monadic unit and plus

definition
lower-unit :: *'a* → *'a lower-pd* **where**
lower-unit = *compact-basis.extension* (λ*a*. *lower-principal* (*PDUnit a*))

definition
lower-plus :: *'a lower-pd* → *'a lower-pd* → *'a lower-pd* **where**
lower-plus = *lower-pd.extension* (λ*t*. *lower-pd.extension* (λ*u*.
lower-principal (*PDPlus t u*)))

abbreviation
lower-add :: *'a lower-pd* ⇒ *'a lower-pd* ⇒ *'a lower-pd*
(**infixl** ∪♭ *65*) **where**
xs ∪♭ *ys* == *lower-plus*·*xs*·*ys*

syntax
-lower-pd :: *args* ⇒ *logic* ({-}♭)

translations
{*x*,*xs*}♭ == {*x*}♭ ∪♭ {*xs*}♭
{*x*}♭ == *CONST lower-unit*·*x*

lemma *lower-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♭ = *lower-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *lower-plus-principal* [*simp*]:
lower-principal t ∪♭ *lower-principal u* = *lower-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *lower-add*: *semilattice lower-add* ⟨*proof*⟩

lemmas *lower-plus-assoc* = *lower-add.assoc*
lemmas *lower-plus-commute* = *lower-add.commute*
lemmas *lower-plus-absorb* = *lower-add.idem*
lemmas *lower-plus-left-commute* = *lower-add.left-commute*
lemmas *lower-plus-left-absorb* = *lower-add.left-idem*

Useful for *simp add*: *lower-plus-ac*

lemmas *lower-plus-ac* =
lower-plus-assoc lower-plus-commute lower-plus-left-commute

Useful for *simp only*: *lower-plus-aci*

lemmas *lower-plus-aci* =
lower-plus-ac lower-plus-absorb lower-plus-left-absorb

lemma *lower-plus-below1*: $xs \sqsubseteq xs \mathbin{\cup\!\flat} ys$
⟨*proof*⟩

lemma *lower-plus-below2*: $ys \sqsubseteq xs \mathbin{\cup\!\flat} ys$
⟨*proof*⟩

lemma *lower-plus-least*: $[\![xs \sqsubseteq zs;\ ys \sqsubseteq zs]\!] \Longrightarrow xs \mathbin{\cup\!\flat} ys \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-plus-below-iff* [*simp*]:
$xs \mathbin{\cup\!\flat} ys \sqsubseteq zs \longleftrightarrow xs \sqsubseteq zs \wedge ys \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-unit-below-plus-iff* [*simp*]:
$\{x\}\flat \sqsubseteq ys \mathbin{\cup\!\flat} zs \longleftrightarrow \{x\}\flat \sqsubseteq ys \vee \{x\}\flat \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-unit-below-iff* [*simp*]: $\{x\}\flat \sqsubseteq \{y\}\flat \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemmas *lower-pd-below-simps* =
lower-unit-below-iff
lower-plus-below-iff
lower-unit-below-plus-iff

lemma *lower-unit-eq-iff* [*simp*]: $\{x\}\flat = \{y\}\flat \longleftrightarrow x = y$
⟨*proof*⟩

lemma *lower-unit-strict* [*simp*]: $\{\bot\}\flat = \bot$
⟨*proof*⟩

lemma *lower-unit-bottom-iff* [*simp*]: $\{x\}\flat = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *lower-plus-bottom-iff* [*simp*]:
$xs \mathbin{\cup\!\flat} ys = \bot \longleftrightarrow xs = \bot \wedge ys = \bot$
⟨*proof*⟩

lemma *lower-plus-strict1* [*simp*]: $\bot \mathbin{\cup\!\flat} ys = ys$
⟨*proof*⟩

lemma *lower-plus-strict2* [*simp*]: $xs \mathbin{\cup\!\flat} \bot = xs$

⟨*proof*⟩

lemma *compact-lower-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\flat$
⟨*proof*⟩

lemma *compact-lower-unit-iff* [*simp*]: *compact* $\{x\}\flat \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-lower-plus* [*simp*]:
$[\![$*compact* xs; *compact* $ys]\!] \Longrightarrow$ *compact* $(xs \cup\flat\ ys)$
⟨*proof*⟩

32.4 Induction rules

lemma *lower-pd-induct1*:
assumes P: *adm* P
assumes *unit*: $\bigwedge x.\ P\ \{x\}\flat$
assumes *insert*:
$\bigwedge x\ ys.\ [\![P\ \{x\}\flat;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\flat \cup\flat\ ys)$
shows P $(xs{::}'a$ *lower-pd*$)$
⟨*proof*⟩

lemma *lower-pd-induct*
[*case-names adm lower-unit lower-plus*, *induct type*: *lower-pd*]:
assumes P: *adm* P
assumes *unit*: $\bigwedge x.\ P\ \{x\}\flat$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\flat\ ys)$
shows P $(xs{::}'a$ *lower-pd*$)$
⟨*proof*⟩

32.5 Monadic bind

definition
lower-bind-basis ::
$'a$ *pd-basis* $\Rightarrow$ $('a \to 'b$ *lower-pd*$) \to 'b$ *lower-pd* **where**
lower-bind-basis = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f{\cdot}($*Rep-compact-basis* $a))$
$(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\flat\ y{\cdot}f)$

lemma *ACI-lower-bind*:
semilattice $(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\flat\ y{\cdot}f)$
⟨*proof*⟩

lemma *lower-bind-basis-simps* [*simp*]:
lower-bind-basis $($*PDUnit* $a) =$
$(\Lambda\ f.\ f{\cdot}($*Rep-compact-basis* $a))$
lower-bind-basis $($*PDPlus* $t\ u) =$
$(\Lambda\ f.$ *lower-bind-basis* $t{\cdot}f \cup\flat$ *lower-bind-basis* $u{\cdot}f)$
⟨*proof*⟩

lemma *lower-bind-basis-mono*:
$t \leq\flat u \Longrightarrow$ *lower-bind-basis* $t \sqsubseteq$ *lower-bind-basis* u
⟨*proof*⟩

definition
lower-bind :: $'a$ *lower-pd* $\to$ ($'a \to 'b$ *lower-pd*) $\to$ $'b$ *lower-pd* **where**
lower-bind = *lower-pd.extension lower-bind-basis*

syntax
-lower-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
((3$\bigcup\flat$-∈-./ -) [0, 0, 10] 10)

translations
$\bigcup\flat x{\in}xs.\ e$ == *CONST lower-bind*·*xs*·(Λ *x*. *e*)

lemma *lower-bind-principal* [*simp*]:
lower-bind·(*lower-principal* t) = *lower-bind-basis* t
⟨*proof*⟩

lemma *lower-bind-unit* [*simp*]:
lower-bind·$\{x\}\flat$·$f = f{\cdot}x$
⟨*proof*⟩

lemma *lower-bind-plus* [*simp*]:
lower-bind·(*xs* $\cup\flat$ *ys*)·f = *lower-bind*·*xs*·f $\cup\flat$ *lower-bind*·*ys*·f
⟨*proof*⟩

lemma *lower-bind-strict* [*simp*]: *lower-bind*·$\bot$·$f = f{\cdot}\bot$
⟨*proof*⟩

lemma *lower-bind-bind*:
lower-bind·(*lower-bind*·*xs*·f)·g = *lower-bind*·*xs*·(Λ *x*. *lower-bind*·($f{\cdot}x$)·g)
⟨*proof*⟩

32.6 Map

definition
lower-map :: ($'a \to 'b$) $\to$ $'a$ *lower-pd* $\to$ $'b$ *lower-pd* **where**
lower-map = (Λ f *xs*. *lower-bind*·*xs*·(Λ *x*. $\{f{\cdot}x\}\flat$))

lemma *lower-map-unit* [*simp*]:
lower-map·f·$\{x\}\flat$ = $\{f{\cdot}x\}\flat$
⟨*proof*⟩

lemma *lower-map-plus* [*simp*]:
lower-map·f·(*xs* $\cup\flat$ *ys*) = *lower-map*·f·*xs* $\cup\flat$ *lower-map*·f·*ys*
⟨*proof*⟩

lemma *lower-map-bottom* [*simp*]: *lower-map*·$f{\cdot}\bot = \{f{\cdot}\bot\}\flat$

⟨*proof*⟩

lemma *lower-map-ident*: *lower-map*·(Λ *x*. *x*)·*xs* = *xs*
⟨*proof*⟩

lemma *lower-map-ID*: *lower-map*·*ID* = *ID*
⟨*proof*⟩

lemma *lower-map-map*:
lower-map·*f*·(*lower-map*·*g*·*xs*) = *lower-map*·(Λ *x*. *f*·(*g*·*x*))·*xs*
⟨*proof*⟩

lemma *lower-bind-map*:
lower-bind·(*lower-map*·*f*·*xs*)·*g* = *lower-bind*·*xs*·(Λ *x*. *g*·(*f*·*x*))
⟨*proof*⟩

lemma *lower-map-bind*:
lower-map·*f*·(*lower-bind*·*xs*·*g*) = *lower-bind*·*xs*·(Λ *x*. *lower-map*·*f*·(*g*·*x*))
⟨*proof*⟩

lemma *ep-pair-lower-map*: *ep-pair e p* ⟹ *ep-pair* (*lower-map*·*e*) (*lower-map*·*p*)
⟨*proof*⟩

lemma *deflation-lower-map*: *deflation d* ⟹ *deflation* (*lower-map*·*d*)
⟨*proof*⟩

lemma *finite-deflation-lower-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*lower-map*·*d*)
⟨*proof*⟩

32.7 Lower powerdomain is bifinite

lemma *approx-chain-lower-map*:
assumes *approx-chain a*
shows *approx-chain* (λ*i*. *lower-map*·(*a i*))
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

32.8 Join

definition
lower-join :: *'a lower-pd lower-pd* → *'a lower-pd* **where**
lower-join = (Λ *xss*. *lower-bind*·*xss*·(Λ *xs*. *xs*))

lemma *lower-join-unit* [*simp*]:
lower-join·{*xs*}♭ = *xs*
⟨*proof*⟩

lemma *lower-join-plus* [*simp*]:
lower-join·(*xss* ∪♭ *yss*) = *lower-join*·*xss* ∪♭ *lower-join*·*yss*
⟨*proof*⟩

lemma *lower-join-bottom* [*simp*]: *lower-join*·⊥ = ⊥
⟨*proof*⟩

lemma *lower-join-map-unit*:
lower-join·(*lower-map*·*lower-unit*·*xs*) = *xs*
⟨*proof*⟩

lemma *lower-join-map-join*:
lower-join·(*lower-map*·*lower-join*·*xsss*) = *lower-join*·(*lower-join*·*xsss*)
⟨*proof*⟩

lemma *lower-join-map-map*:
lower-join·(*lower-map*·(*lower-map*·*f*)·*xss*) =
lower-map·*f*·(*lower-join*·*xss*)
⟨*proof*⟩

end

33 ConvexPD: Convex powerdomain

theory *ConvexPD*
imports *UpperPD LowerPD*
begin

33.1 Basis preorder

definition
convex-le :: 'a *pd-basis* ⇒ 'a *pd-basis* ⇒ *bool* (**infix** ≤♮ *50*) **where**
convex-le = (λ*u v*. *u* ≤♯ *v* ∧ *u* ≤♭ *v*)

lemma *convex-le-refl* [*simp*]: *t* ≤♮ *t*
⟨*proof*⟩

lemma *convex-le-trans*: ⟦*t* ≤♮ *u*; *u* ≤♮ *v*⟧ ⟹ *t* ≤♮ *v*
⟨*proof*⟩

interpretation *convex-le*: *preorder convex-le*
⟨*proof*⟩

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♮ *t*
⟨*proof*⟩

lemma *PDUnit-convex-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♮ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-convex-mono*: $[\![s \leq\natural t;\ u \leq\natural v]\!] \Longrightarrow PDPlus\ s\ u \leq\natural PDPlus\ t\ v$
⟨*proof*⟩

lemma *convex-le-PDUnit-PDUnit-iff* [*simp*]:
$(PDUnit\ a \leq\natural PDUnit\ b) = (a \sqsubseteq b)$
⟨*proof*⟩

lemma *convex-le-PDUnit-lemma1*:
$(PDUnit\ a \leq\natural t) = (\forall\, b{\in}Rep\text{-}pd\text{-}basis\ t.\ a \sqsubseteq b)$
⟨*proof*⟩

lemma *convex-le-PDUnit-PDPlus-iff* [*simp*]:
$(PDUnit\ a \leq\natural PDPlus\ t\ u) = (PDUnit\ a \leq\natural t \wedge PDUnit\ a \leq\natural u)$
⟨*proof*⟩

lemma *convex-le-PDUnit-lemma2*:
$(t \leq\natural PDUnit\ b) = (\forall\, a{\in}Rep\text{-}pd\text{-}basis\ t.\ a \sqsubseteq b)$
⟨*proof*⟩

lemma *convex-le-PDPlus-PDUnit-iff* [*simp*]:
$(PDPlus\ t\ u \leq\natural PDUnit\ a) = (t \leq\natural PDUnit\ a \wedge u \leq\natural PDUnit\ a)$
⟨*proof*⟩

lemma *convex-le-PDPlus-lemma*:
assumes *z*: $PDPlus\ t\ u \leq\natural z$
shows $\exists\, v\ w.\ z = PDPlus\ v\ w \wedge t \leq\natural v \wedge u \leq\natural w$
⟨*proof*⟩

lemma *convex-le-induct* [*induct set*: *convex-le*]:
assumes *le*: $t \leq\natural u$
assumes *2*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ u\ v]\!] \Longrightarrow P\ t\ v$
assumes *3*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
assumes *4*: $\bigwedge t\ u\ v\ w.\ [\![P\ t\ v;\ P\ u\ w]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ (PDPlus\ v\ w)$
shows $P\ t\ u$
⟨*proof*⟩

33.2 Type definition

typedef *'a convex-pd* =
$\{S{::}'a\ pd\text{-}basis\ set.\ convex\text{-}le.ideal\ S\}$
⟨*proof*⟩

type-notation (*xsymbols*) *convex-pd* (($'(\text{-}')\natural$))

instantiation *convex-pd* :: (*bifinite*) *below*
begin

definition

$x \sqsubseteq y \longleftrightarrow$ *Rep-convex-pd* $x \subseteq$ *Rep-convex-pd* y

instance ⟨*proof*⟩
end

instance *convex-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
convex-principal :: *'a pd-basis* ⇒ *'a convex-pd* **where**
convex-principal t = *Abs-convex-pd* {*u*. *u* ≤♮ *t*}

interpretation *convex-pd*:
ideal-completion convex-le convex-principal Rep-convex-pd
⟨*proof*⟩

Convex powerdomain is pointed

lemma *convex-pd-minimal*: *convex-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-convex-pd-pcpo*: ⊥ = *convex-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

33.3 Monadic unit and plus

definition
convex-unit :: *'a* → *'a convex-pd* **where**
convex-unit = *compact-basis.extension* (λ*a*. *convex-principal* (*PDUnit a*))

definition
convex-plus :: *'a convex-pd* → *'a convex-pd* → *'a convex-pd* **where**
convex-plus = *convex-pd.extension* (λ*t*. *convex-pd.extension* (λ*u*.
convex-principal (*PDPlus t u*)))

abbreviation
convex-add :: *'a convex-pd* ⇒ *'a convex-pd* ⇒ *'a convex-pd*
(**infixl** ∪♮ *65*) **where**
xs ∪♮ *ys* == *convex-plus*·*xs*·*ys*

syntax
-convex-pd :: *args* ⇒ *logic* ({-}♮)

translations

$\{x,xs\}\natural == \{x\}\natural \cup\natural \{xs\}\natural$
$\{x\}\natural == CONST\ convex\text{-}unit{\cdot}x$

lemma *convex-unit-Rep-compact-basis* [*simp*]:
$\{Rep\text{-}compact\text{-}basis\ a\}\natural = convex\text{-}principal\ (PDUnit\ a)$
⟨*proof*⟩

lemma *convex-plus-principal* [*simp*]:
$convex\text{-}principal\ t \cup\natural\ convex\text{-}principal\ u = convex\text{-}principal\ (PDPlus\ t\ u)$
⟨*proof*⟩

interpretation *convex-add*: *semilattice convex-add* ⟨*proof*⟩

lemmas *convex-plus-assoc* = *convex-add.assoc*
lemmas *convex-plus-commute* = *convex-add.commute*
lemmas *convex-plus-absorb* = *convex-add.idem*
lemmas *convex-plus-left-commute* = *convex-add.left-commute*
lemmas *convex-plus-left-absorb* = *convex-add.left-idem*

Useful for *simp add*: *convex-plus-ac*

lemmas *convex-plus-ac* =
convex-plus-assoc convex-plus-commute convex-plus-left-commute

Useful for *simp only*: *convex-plus-aci*

lemmas *convex-plus-aci* =
convex-plus-ac convex-plus-absorb convex-plus-left-absorb

lemma *convex-unit-below-plus-iff* [*simp*]:
$\{x\}\natural \sqsubseteq ys \cup\natural zs \longleftrightarrow \{x\}\natural \sqsubseteq ys \wedge \{x\}\natural \sqsubseteq zs$
⟨*proof*⟩

lemma *convex-plus-below-unit-iff* [*simp*]:
$xs \cup\natural ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \wedge ys \sqsubseteq \{z\}\natural$
⟨*proof*⟩

lemma *convex-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *convex-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
⟨*proof*⟩

lemma *convex-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
⟨*proof*⟩

lemma *convex-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *compact-convex-unit*: $compact\ x \Longrightarrow compact\ \{x\}\natural$
⟨*proof*⟩

lemma *compact-convex-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-convex-plus* [*simp*]:
$[\![$*compact* xs; *compact* $ys]\!] \Longrightarrow$ *compact* $(xs \cup\natural\ ys)$
⟨*proof*⟩

33.4 Induction rules

lemma *convex-pd-induct1*:
assumes P: *adm* P
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$
assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\natural;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\natural \cup\natural\ ys)$
shows P $(xs::'a$ *convex-pd*$)$
⟨*proof*⟩

lemma *convex-pd-induct*
[*case-names adm convex-unit convex-plus*, *induct type*: *convex-pd*]:
assumes P: *adm* P
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\natural\ ys)$
shows P $(xs::'a$ *convex-pd*$)$
⟨*proof*⟩

33.5 Monadic bind

definition
convex-bind-basis ::
$'a$ *pd-basis* $\Rightarrow$ $('a \to 'b$ *convex-pd*$) \to 'b$ *convex-pd* **where**
convex-bind-basis = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
$(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\natural\ y{\cdot}f)$

lemma *ACI-convex-bind*:
semilattice $(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\natural\ y{\cdot}f)$
⟨*proof*⟩

lemma *convex-bind-basis-simps* [*simp*]:
convex-bind-basis $(PDUnit\ a) =$
$(\Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
convex-bind-basis $(PDPlus\ t\ u) =$
$(\Lambda\ f.$ *convex-bind-basis* $t{\cdot}f \cup\natural$ *convex-bind-basis* $u{\cdot}f)$
⟨*proof*⟩

lemma *convex-bind-basis-mono*:
$t \leq\natural\ u \Longrightarrow$ *convex-bind-basis* $t \sqsubseteq$ *convex-bind-basis* u
⟨*proof*⟩

definition

convex-bind :: *'a convex-pd* → (*'a* → *'b convex-pd*) → *'b convex-pd* **where**
convex-bind = *convex-pd.extension convex-bind-basis*

syntax
-convex-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
((*3*⋃♮-∈-./ -) [*0*, *0*, *10*] *10*)

translations
⋃♮*x*∈*xs*. *e* == *CONST convex-bind*·*xs*·(Λ *x*. *e*)

lemma *convex-bind-principal* [*simp*]:
convex-bind·(*convex-principal t*) = *convex-bind-basis t*
⟨*proof*⟩

lemma *convex-bind-unit* [*simp*]:
convex-bind·{*x*}♮·*f* = *f*·*x*
⟨*proof*⟩

lemma *convex-bind-plus* [*simp*]:
convex-bind·(*xs* ∪♮ *ys*)·*f* = *convex-bind*·*xs*·*f* ∪♮ *convex-bind*·*ys*·*f*
⟨*proof*⟩

lemma *convex-bind-strict* [*simp*]: *convex-bind*·⊥·*f* = *f*·⊥
⟨*proof*⟩

lemma *convex-bind-bind*:
convex-bind·(*convex-bind*·*xs*·*f*)·*g* =
convex-bind·*xs*·(Λ *x*. *convex-bind*·(*f*·*x*)·*g*)
⟨*proof*⟩

33.6 Map

definition
convex-map :: (*'a* → *'b*) → *'a convex-pd* → *'b convex-pd* **where**
convex-map = (Λ *f xs*. *convex-bind*·*xs*·(Λ *x*. {*f*·*x*}♮))

lemma *convex-map-unit* [*simp*]:
convex-map·*f*·{*x*}♮ = {*f*·*x*}♮
⟨*proof*⟩

lemma *convex-map-plus* [*simp*]:
convex-map·*f*·(*xs* ∪♮ *ys*) = *convex-map*·*f*·*xs* ∪♮ *convex-map*·*f*·*ys*
⟨*proof*⟩

lemma *convex-map-bottom* [*simp*]: *convex-map*·*f*·⊥ = {*f*·⊥}♮
⟨*proof*⟩

lemma *convex-map-ident*: *convex-map*·(Λ *x*. *x*)·*xs* = *xs*
⟨*proof*⟩

lemma *convex-map-ID*: *convex-map·ID* = *ID*
⟨*proof*⟩

lemma *convex-map-map*:
convex-map·f·(*convex-map·g·xs*) = *convex-map·*(Λ *x*. *f·*(*g·x*))*·xs*
⟨*proof*⟩

lemma *convex-bind-map*:
convex-bind·(*convex-map·f·xs*)*·g* = *convex-bind·xs·*(Λ *x*. *g·*(*f·x*))
⟨*proof*⟩

lemma *convex-map-bind*:
convex-map·f·(*convex-bind·xs·g*) = *convex-bind·xs·*(Λ *x*. *convex-map·f·*(*g·x*))
⟨*proof*⟩

lemma *ep-pair-convex-map*: *ep-pair e p* ⟹ *ep-pair* (*convex-map·e*) (*convex-map·p*)
⟨*proof*⟩

lemma *deflation-convex-map*: *deflation d* ⟹ *deflation* (*convex-map·d*)
⟨*proof*⟩

lemma *finite-deflation-convex-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*convex-map·d*)
⟨*proof*⟩

33.7 Convex powerdomain is bifinite

lemma *approx-chain-convex-map*:
assumes *approx-chain a*
shows *approx-chain* (λ*i*. *convex-map·*(*a i*))
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

33.8 Join

definition
convex-join :: *'a convex-pd convex-pd* → *'a convex-pd* **where**
convex-join = (Λ *xss*. *convex-bind·xss·*(Λ *xs*. *xs*))

lemma *convex-join-unit* [*simp*]:
convex-join·{*xs*}♮ = *xs*
⟨*proof*⟩

lemma *convex-join-plus* [*simp*]:
convex-join·(*xss* ∪♮ *yss*) = *convex-join·xss* ∪♮ *convex-join·yss*
⟨*proof*⟩

lemma *convex-join-bottom* [*simp*]: *convex-join*·⊥ = ⊥
⟨*proof*⟩

lemma *convex-join-map-unit*:
convex-join·(*convex-map*·*convex-unit*·*xs*) = *xs*
⟨*proof*⟩

lemma *convex-join-map-join*:
convex-join·(*convex-map*·*convex-join*·*xsss*) = *convex-join*·(*convex-join*·*xsss*)
⟨*proof*⟩

lemma *convex-join-map-map*:
convex-join·(*convex-map*·(*convex-map*·*f*)·*xss*) =
convex-map·*f*·(*convex-join*·*xss*)
⟨*proof*⟩

33.9 Conversions to other powerdomains

Convex to upper

lemma *convex-le-imp-upper-le*: *t* ≤♮ *u* ⟹ *t* ≤♯ *u*
⟨*proof*⟩

definition
convex-to-upper :: *'a convex-pd* → *'a upper-pd* **where**
convex-to-upper = *convex-pd.extension upper-principal*

lemma *convex-to-upper-principal* [*simp*]:
convex-to-upper·(*convex-principal t*) = *upper-principal t*
⟨*proof*⟩

lemma *convex-to-upper-unit* [*simp*]:
convex-to-upper·{*x*}♮ = {*x*}♯
⟨*proof*⟩

lemma *convex-to-upper-plus* [*simp*]:
convex-to-upper·(*xs* ∪♮ *ys*) = *convex-to-upper*·*xs* ∪♯ *convex-to-upper*·*ys*
⟨*proof*⟩

lemma *convex-to-upper-bind* [*simp*]:
convex-to-upper·(*convex-bind*·*xs*·*f*) =
upper-bind·(*convex-to-upper*·*xs*)·(*convex-to-upper oo f*)
⟨*proof*⟩

lemma *convex-to-upper-map* [*simp*]:
convex-to-upper·(*convex-map*·*f*·*xs*) = *upper-map*·*f*·(*convex-to-upper*·*xs*)
⟨*proof*⟩

lemma *convex-to-upper-join* [*simp*]:

convex-to-upper·(*convex-join*·*xss*) =
upper-bind·(*convex-to-upper*·*xss*)·*convex-to-upper*
⟨*proof*⟩

Convex to lower

lemma *convex-le-imp-lower-le*: $t \leq\natural u \Longrightarrow t \leq\flat u$
⟨*proof*⟩

definition
convex-to-lower :: *'a convex-pd* → *'a lower-pd* **where**
convex-to-lower = *convex-pd.extension lower-principal*

lemma *convex-to-lower-principal* [*simp*]:
convex-to-lower·(*convex-principal t*) = *lower-principal t*
⟨*proof*⟩

lemma *convex-to-lower-unit* [*simp*]:
convex-to-lower·{*x*}$\natural$ = {*x*}$\flat$
⟨*proof*⟩

lemma *convex-to-lower-plus* [*simp*]:
convex-to-lower·(*xs* ∪$\natural$ *ys*) = *convex-to-lower*·*xs* ∪$\flat$ *convex-to-lower*·*ys*
⟨*proof*⟩

lemma *convex-to-lower-bind* [*simp*]:
convex-to-lower·(*convex-bind*·*xs*·*f*) =
lower-bind·(*convex-to-lower*·*xs*)·(*convex-to-lower oo f*)
⟨*proof*⟩

lemma *convex-to-lower-map* [*simp*]:
convex-to-lower·(*convex-map*·*f*·*xs*) = *lower-map*·*f*·(*convex-to-lower*·*xs*)
⟨*proof*⟩

lemma *convex-to-lower-join* [*simp*]:
convex-to-lower·(*convex-join*·*xss*) =
lower-bind·(*convex-to-lower*·*xss*)·*convex-to-lower*
⟨*proof*⟩

Ordering property

lemma *convex-pd-below-iff*:
(*xs* ⊑ *ys*) =
(*convex-to-upper*·*xs* ⊑ *convex-to-upper*·*ys* ∧
convex-to-lower·*xs* ⊑ *convex-to-lower*·*ys*)
⟨*proof*⟩

lemmas *convex-plus-below-plus-iff* =
convex-pd-below-iff [**where** *xs*=*xs* ∪$\natural$ *ys* **and** *ys*=*zs* ∪$\natural$ *ws*]
for *xs ys zs ws*

lemmas *convex-pd-below-simps* =
convex-unit-below-plus-iff
convex-plus-below-unit-iff
convex-plus-below-plus-iff
convex-unit-below-iff
convex-to-upper-unit
convex-to-upper-plus
convex-to-lower-unit
convex-to-lower-plus
upper-pd-below-simps
lower-pd-below-simps

end

34 Powerdomains: Powerdomains

theory *Powerdomains*
imports *ConvexPD Domain*
begin

34.1 Universal domain embeddings

definition *upper-emb* = *udom-emb* (λ*i*. *upper-map*·(*udom-approx i*))
definition *upper-prj* = *udom-prj* (λ*i*. *upper-map*·(*udom-approx i*))

definition *lower-emb* = *udom-emb* (λ*i*. *lower-map*·(*udom-approx i*))
definition *lower-prj* = *udom-prj* (λ*i*. *lower-map*·(*udom-approx i*))

definition *convex-emb* = *udom-emb* (λ*i*. *convex-map*·(*udom-approx i*))
definition *convex-prj* = *udom-prj* (λ*i*. *convex-map*·(*udom-approx i*))

lemma *ep-pair-upper*: *ep-pair upper-emb upper-prj*
⟨*proof*⟩

lemma *ep-pair-lower*: *ep-pair lower-emb lower-prj*
⟨*proof*⟩

lemma *ep-pair-convex*: *ep-pair convex-emb convex-prj*
⟨*proof*⟩

34.2 Deflation combinators

definition *upper-defl* :: *udom defl* → *udom defl*
where *upper-defl* = *defl-fun1 upper-emb upper-prj upper-map*

definition *lower-defl* :: *udom defl* → *udom defl*
where *lower-defl* = *defl-fun1 lower-emb lower-prj lower-map*

definition *convex-defl* :: *udom defl* → *udom defl*

where *convex-defl* = *defl-fun1 convex-emb convex-prj convex-map*

lemma *cast-upper-defl*:
cast·(*upper-defl*·*A*) = *upper-emb oo upper-map*·(*cast*·*A*) *oo upper-prj*
⟨*proof*⟩

lemma *cast-lower-defl*:
cast·(*lower-defl*·*A*) = *lower-emb oo lower-map*·(*cast*·*A*) *oo lower-prj*
⟨*proof*⟩

lemma *cast-convex-defl*:
cast·(*convex-defl*·*A*) = *convex-emb oo convex-map*·(*cast*·*A*) *oo convex-prj*
⟨*proof*⟩

34.3 Domain class instances

instantiation *upper-pd* :: (*domain*) *domain*
begin

definition
emb = *upper-emb oo upper-map*·*emb*

definition
prj = *upper-map*·*prj oo upper-prj*

definition
defl (*t*::'*a upper-pd itself*) = *upper-defl*·*DEFL*('*a*)

definition
(*liftemb* :: '*a upper-pd u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a upper-pd u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a upper-pd itself*) = *liftdefl-of*·*DEFL*('*a upper-pd*)

instance ⟨*proof*⟩

end

instantiation *lower-pd* :: (*domain*) *domain*
begin

definition
emb = *lower-emb oo lower-map*·*emb*

definition
prj = *lower-map*·*prj oo lower-prj*

definition
defl (*t*::'*a lower-pd itself*) = *lower-defl*·*DEFL*('*a*)

definition
(*liftemb* :: '*a lower-pd u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a lower-pd u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a lower-pd itself*) = *liftdefl-of*·*DEFL*('*a lower-pd*)

instance ⟨*proof*⟩

end

instantiation *convex-pd* :: (*domain*) *domain*
begin

definition
emb = *convex-emb oo convex-map*·*emb*

definition
prj = *convex-map*·*prj oo convex-prj*

definition
defl (*t*::'*a convex-pd itself*) = *convex-defl*·*DEFL*('*a*)

definition
(*liftemb* :: '*a convex-pd u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a convex-pd u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a convex-pd itself*) = *liftdefl-of*·*DEFL*('*a convex-pd*)

instance ⟨*proof*⟩

end

lemma *DEFL-upper*: *DEFL*('*a*::*domain upper-pd*) = *upper-defl*·*DEFL*('*a*)
⟨*proof*⟩

lemma *DEFL-lower*: *DEFL*('*a*::*domain lower-pd*) = *lower-defl*·*DEFL*('*a*)
⟨*proof*⟩

lemma *DEFL-convex*: *DEFL*('*a*::*domain convex-pd*) = *convex-defl*·*DEFL*('*a*)

⟨*proof*⟩

34.4 Isomorphic deflations

lemma *isodefl-upper*:
  isodefl d t ⟹ *isodefl* (*upper-map*·*d*) (*upper-defl*·*t*)
⟨*proof*⟩

lemma *isodefl-lower*:
  isodefl d t ⟹ *isodefl* (*lower-map*·*d*) (*lower-defl*·*t*)
⟨*proof*⟩

lemma *isodefl-convex*:
  isodefl d t ⟹ *isodefl* (*convex-map*·*d*) (*convex-defl*·*t*)
⟨*proof*⟩

34.5 Domain package setup for powerdomains

lemmas [*domain-defl-simps*] = *DEFL-upper DEFL-lower DEFL-convex*
lemmas [*domain-map-ID*] = *upper-map-ID lower-map-ID convex-map-ID*
lemmas [*domain-isodefl*] = *isodefl-upper isodefl-lower isodefl-convex*

lemmas [*domain-deflation*] =
  deflation-upper-map deflation-lower-map deflation-convex-map

⟨*ML*⟩

end

theory *HOLCF*
imports
  Main
  Domain
  Powerdomains
begin

default-sort *domain*

end