The Isabelle/HOL Algebra Library

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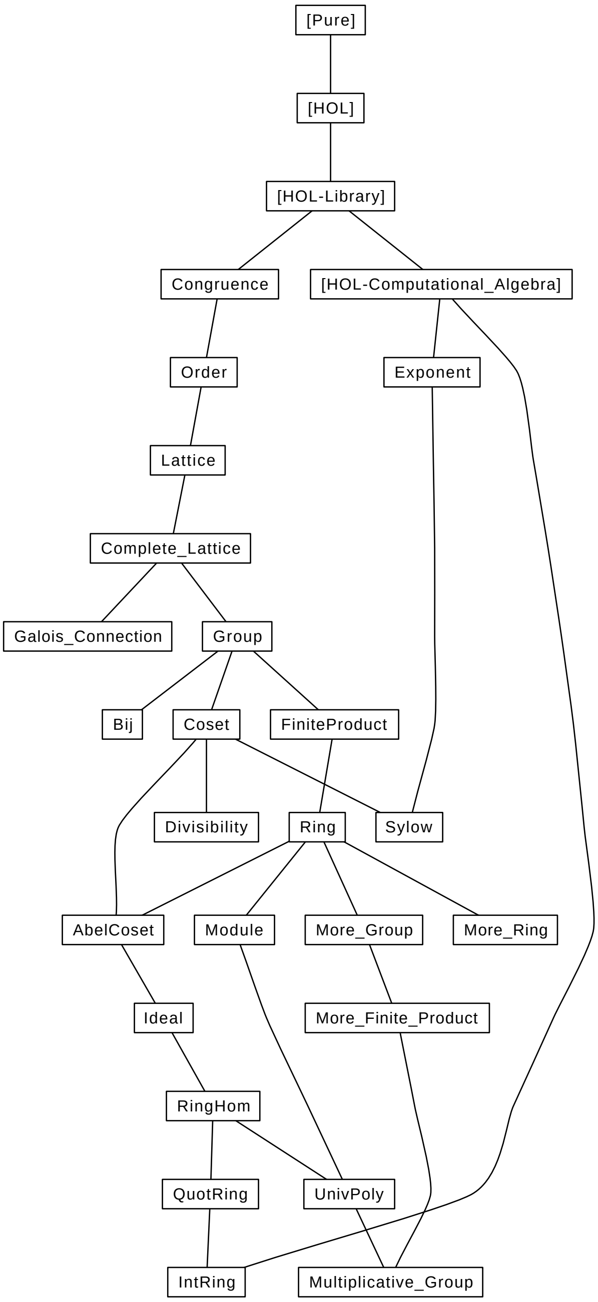
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```
theory Congruence
imports
  Main
  "HOL-Library.FuncSet"
begin
```

1 Objects

1.1 Structure with Carrier Set.

```
record 'a partial_object =
  carrier :: "'a set"

lemma funcset_carrier:
  "⟦ f ∈ carrier X → carrier Y; x ∈ carrier X ⟧ ⟹ f x ∈ carrier Y"
  by (fact funcset_mem)

lemma funcset_carrier':
  "⟦ f ∈ carrier A → carrier A; x ∈ carrier A ⟧ ⟹ f x ∈ carrier A"
  by (fact funcset_mem)
```

1.2 Structure with Carrier and Equivalence Relation `eq`

```
record 'a eq_object = "'a partial_object" +
  eq :: "'a ⇒ 'a ⇒ bool" (infixl ".=ı" 50)

definition
  elem :: "_ ⇒ 'a ⇒ 'a set ⇒ bool" (infixl ".∈ı" 50)
  where "x .∈S A ⟷ (∃y ∈ A. x .=S y)"

definition
  set_eq :: "_ ⇒ 'a set ⇒ 'a set ⇒ bool" (infixl "{.=}ı" 50)
  where "A {.=}S B ⟷ ((∀x ∈ A. x .∈S B) ∧ (∀x ∈ B. x .∈S A))"

definition
  eq_class_of :: "_ ⇒ 'a ⇒ 'a set" ("class'_ofı")
  where "class_ofS x = {y ∈ carrier S. x .=S y}"

definition
  eq_closure_of :: "_ ⇒ 'a set ⇒ 'a set" ("closure'_ofı")
  where "closure_ofS A = {y ∈ carrier S. y .∈S A}"

definition
  eq_is_closed :: "_ ⇒ 'a set ⇒ bool" ("is'_closedı")
  where "is_closedS A ⟷ A ⊆ carrier S ∧ closure_ofS A = A"

abbreviation
```

```
  not_eq :: "_ ⇒ 'a ⇒ 'a ⇒ bool" (infixl ".≠ı" 50)
  where "x .≠S y == ~(x .=S y)"

abbreviation
  not_elem :: "_ ⇒ 'a ⇒ 'a set ⇒ bool" (infixl ".∉ı" 50)
  where "x .∉S A == ~(x .∈S A)"

abbreviation
  set_not_eq :: "_ ⇒ 'a set ⇒ 'a set ⇒ bool" (infixl "{.≠}ı" 50)
  where "A {.≠}S B == ~(A {.=}S B)"

locale equivalence =
  fixes S (structure)
  assumes refl [simp, intro]: "x ∈ carrier S ⟹ x .= x"
    and sym [sym]: "⟦ x .= y; x ∈ carrier S; y ∈ carrier S ⟧ ⟹ y .=
x"
    and trans [trans]:
      "⟦ x .= y; y .= z; x ∈ carrier S; y ∈ carrier S; z ∈ carrier S ⟧
⟹ x .= z"



lemma elemI:
  fixes R (structure)
  assumes "a' ∈ A" and "a .= a'"
  shows "a .∈ A"
unfolding elem_def
using assms
by fast

lemma (in equivalence) elem_exact:
  assumes "a ∈ carrier S" and "a ∈ A"
  shows "a .∈ A"
using assms
by (fast intro: elemI)

lemma elemE:
  fixes S (structure)
  assumes "a .∈ A"
    and "⋀a'. ⟦a' ∈ A; a .= a'⟧ ⟹ P"
  shows "P"
using assms
unfolding elem_def
by fast

lemma (in equivalence) elem_cong_l [trans]:
  assumes cong: "a' .= a"
    and a: "a .∈ A"
    and carr: "a ∈ carrier S"  "a' ∈ carrier S"
```

```
    and Acarr: "A ⊆ carrier S"
  shows "a' .∈ A"
using a
apply (elim elemE, intro elemI)
proof assumption
  fix b
  assume bA: "b ∈ A"
  note [simp] = carr bA[THEN subsetD[OF Acarr]]
  note cong
  also assume "a .= b"
  finally show "a' .= b" by simp
qed

lemma (in equivalence) elem_subsetD:
  assumes "A ⊆ B"
    and aA: "a .∈ A"
  shows "a .∈ B"
using assms
by (fast intro: elemI elim: elemE dest: subsetD)

lemma (in equivalence) mem_imp_elem [simp, intro]:
  "[| x ∈ A; x ∈ carrier S |] ==> x .∈ A"
  unfolding elem_def by blast

lemma set_eqI:
  fixes R (structure)
  assumes ltr: "⋀a. a ∈ A ⟹ a .∈ B"
    and rtl: "⋀b. b ∈ B ⟹ b .∈ A"
  shows "A {.=} B"
unfolding set_eq_def
by (fast intro: ltr rtl)

lemma set_eqI2:
  fixes R (structure)
  assumes ltr: "⋀a b. a ∈ A ⟹ ∃b∈B. a .= b"
    and rtl: "⋀b. b ∈ B ⟹ ∃a∈A. b .= a"
  shows "A {.=} B"
  by (intro set_eqI, unfold elem_def) (fast intro: ltr rtl)+

lemma set_eqD1:
  fixes R (structure)
  assumes AA': "A {.=} A'"
    and "a ∈ A"
  shows "∃a'∈A'. a .= a'"
using assms
unfolding set_eq_def elem_def
by fast

lemma set_eqD2:
```

```
  fixes R (structure)
  assumes AA': "A {.=} A'"
    and "a' ∈ A'"
  shows "∃a∈A. a' .= a"
using assms
unfolding set_eq_def elem_def
by fast

lemma set_eqE:
  fixes R (structure)
  assumes AB: "A {.=} B"
    and r: "⟦∀a∈A. a .∈ B; ∀b∈B. b .∈ A⟧ ⟹ P"
  shows "P"
using AB
unfolding set_eq_def
by (blast dest: r)

lemma set_eqE2:
  fixes R (structure)
  assumes AB: "A {.=} B"
    and r: "⟦∀a∈A. (∃b∈B. a .= b); ∀b∈B. (∃a∈A. b .= a)⟧ ⟹ P"
  shows "P"
using AB
unfolding set_eq_def elem_def
by (blast dest: r)

lemma set_eqE':
  fixes R (structure)
  assumes AB: "A {.=} B"
    and aA: "a ∈ A" and bB: "b ∈ B"
    and r: "⋀a' b'. ⟦a' ∈ A; b .= a'; b' ∈ B; a .= b'⟧ ⟹ P"
  shows "P"
proof -
  from AB aA
      have "∃b'∈B. a .= b'" by (rule set_eqD1)
  from this obtain b'
      where b': "b' ∈ B" "a .= b'" by auto

  from AB bB
      have "∃a'∈A. b .= a'" by (rule set_eqD2)
  from this obtain a'
      where a': "a' ∈ A" "b .= a'" by auto

  from a' b'
      show "P" by (rule r)
qed

lemma (in equivalence) eq_elem_cong_r [trans]:
  assumes a: "a .∈ A"
```

```
    and cong: "A {.=} A'"
    and carr: "a ∈ carrier S"
    and Carr: "A ⊆ carrier S" "A' ⊆ carrier S"
  shows "a .∈ A'"
using a cong
proof (elim elemE set_eqE)
  fix b
  assume bA: "b ∈ A"
     and inA': "∀b∈A. b .∈ A'"
  note [simp] = carr Carr Carr[THEN subsetD] bA
  assume "a .= b"
  also from bA inA'
       have "b .∈ A'" by fast
  finally
       show "a .∈ A'" by simp
qed

lemma (in equivalence) set_eq_sym [sym]:
  assumes "A {.=} B"
    and "A ⊆ carrier S" "B ⊆ carrier S"
  shows "B {.=} A"
using assms
unfolding set_eq_def elem_def
by fast



lemma (in equivalence) equal_set_eq_trans [trans]:
  assumes AB: "A = B" and BC: "B {.=} C"
  shows "A {.=} C"
  using AB BC by simp

lemma (in equivalence) set_eq_equal_trans [trans]:
  assumes AB: "A {.=} B" and BC: "B = C"
  shows "A {.=} C"
  using AB BC by simp


lemma (in equivalence) set_eq_trans [trans]:
  assumes AB: "A {.=} B" and BC: "B {.=} C"
    and carr: "A ⊆ carrier S"  "B ⊆ carrier S"  "C ⊆ carrier S"
  shows "A {.=} C"
proof (intro set_eqI)
  fix a
  assume aA: "a ∈ A"
  with carr have "a ∈ carrier S" by fast
  note [simp] = carr this
```

```
  from aA
       have "a .∈ A" by (simp add: elem_exact)
  also note AB
  also note BC
  finally
       show "a .∈ C" by simp
next
  fix c
  assume cC: "c ∈ C"
  with carr have "c ∈ carrier S" by fast
  note [simp] = carr this

  from cC
       have "c .∈ C" by (simp add: elem_exact)
  also note BC[symmetric]
  also note AB[symmetric]
  finally
       show "c .∈ A" by simp
qed
```

```
lemma (in equivalence) set_eq_pairI:
  assumes xx': "x .= x'"
    and carr: "x ∈ carrier S" "x' ∈ carrier S" "y ∈ carrier S"
  shows "{x, y} {.=} {x', y}"
unfolding set_eq_def elem_def
proof safe
  have "x' ∈ {x', y}" by fast
  with xx' show "∃b∈{x', y}. x .= b" by fast
next
  have "y ∈ {x', y}" by fast
  with carr show "∃b∈{x', y}. y .= b" by fast
next
  have "x ∈ {x, y}" by fast
  with xx'[symmetric] carr
  show "∃a∈{x, y}. x' .= a" by fast
next
  have "y ∈ {x, y}" by fast
  with carr show "∃a∈{x, y}. y .= a" by fast
qed

lemma (in equivalence) is_closedI:
  assumes closed: "!!x y. [| x .= y; x ∈ A; y ∈ carrier S |] ==> y ∈
A"
    and S: "A ⊆ carrier S"
  shows "is_closed A"
```

```
  unfolding eq_is_closed_def eq_closure_of_def elem_def
  using S
  by (blast dest: closed sym)

lemma (in equivalence) closure_of_eq:
  "[| x .= x'; A ⊆ carrier S; x ∈ closure_of A; x ∈ carrier S; x' ∈ carrier
S |] ==> x' ∈ closure_of A"
  unfolding eq_closure_of_def elem_def
  by (blast intro: trans sym)

lemma (in equivalence) is_closed_eq [dest]:
  "[| x .= x'; x ∈ A; is_closed A; x ∈ carrier S; x' ∈ carrier S |] ==>
x' ∈ A"
  unfolding eq_is_closed_def
  using closure_of_eq [where A = A]
  by simp

lemma (in equivalence) is_closed_eq_rev [dest]:
  "[| x .= x'; x' ∈ A; is_closed A; x ∈ carrier S; x' ∈ carrier S |]
==> x ∈ A"
  by (drule sym) (simp_all add: is_closed_eq)

lemma closure_of_closed [simp, intro]:
  fixes S (structure)
  shows "closure_of A ⊆ carrier S"
unfolding eq_closure_of_def
by fast

lemma closure_of_memI:
  fixes S (structure)
  assumes "a .∈ A"
    and "a ∈ carrier S"
  shows "a ∈ closure_of A"
unfolding eq_closure_of_def
using assms
by fast

lemma closure_ofI2:
  fixes S (structure)
  assumes "a .= a'"
    and "a' ∈ A"
    and "a ∈ carrier S"
  shows "a ∈ closure_of A"
unfolding eq_closure_of_def elem_def
using assms
by fast

lemma closure_of_memE:
  fixes S (structure)
```

```
  assumes p: "a ∈ closure_of A"
    and r: "⟦a ∈ carrier S; a .∈ A⟧ ⟹ P"
  shows "P"
proof -
  from p
      have acarr: "a ∈ carrier S"
      and "a .∈ A"
      by (simp add: eq_closure_of_def)+
  thus "P" by (rule r)
qed

lemma closure_ofE2:
  fixes S (structure)
  assumes p: "a ∈ closure_of A"
    and r: "⋀a'. ⟦a ∈ carrier S; a' ∈ A; a .= a'⟧ ⟹ P"
  shows "P"
proof -
  from p have acarr: "a ∈ carrier S" by (simp add: eq_closure_of_def)

  from p have "∃a'∈A. a .= a'" by (simp add: eq_closure_of_def elem_def)
  from this obtain a'
      where "a' ∈ A" and "a .= a'" by auto

  from acarr and this
      show "P" by (rule r)
qed




lemma equivalence_subset:
  assumes "equivalence L" "A ⊆ carrier L"
  shows "equivalence (L⦇ carrier := A ⦈)"
proof -
  interpret L: equivalence L
    by (simp add: assms)
  show ?thesis
    by (unfold_locales, simp_all add: L.sym assms rev_subsetD, meson L.trans
assms(2) contra_subsetD)
qed

end


theory Order
imports
  "HOL-Library.FuncSet"
  Congruence
begin
```

2 Orders

2.1 Partial Orders

```
record 'a gorder = "'a eq_object" +
  le :: "['a, 'a] => bool" (infixl "⊑ı" 50)

abbreviation inv_gorder :: "_ ⇒ 'a gorder" where
  "inv_gorder L ≡
   ⦇ carrier = carrier L,
     eq = op .=_L,
     le = (λ x y. y ⊑_L x) ⦈"

lemma inv_gorder_inv:
  "inv_gorder (inv_gorder L) = L"
  by simp

locale weak_partial_order = equivalence L for L (structure) +
  assumes le_refl [intro, simp]:
      "x ∈ carrier L ==> x ⊑ x"
    and weak_le_antisym [intro]:
      "[| x ⊑ y; y ⊑ x; x ∈ carrier L; y ∈ carrier L |] ==> x .= y"
    and le_trans [trans]:
      "[| x ⊑ y; y ⊑ z; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L
|] ==> x ⊑ z"
    and le_cong:
      "⟦ x .= y; z .= w; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L;
w ∈ carrier L ⟧ ⟹
      x ⊑ z ⟷ y ⊑ w"

definition
  lless :: "[_, 'a, 'a] => bool" (infixl "⊏ı" 50)
  where "x ⊏_L y ⟷ x ⊑_L y & x .≠_L y"
```

2.1.1 The order relation

```
context weak_partial_order
begin

lemma le_cong_l [intro, trans]:
  "⟦ x .= y; y ⊑ z; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L ⟧ ⟹
x ⊑ z"
  by (auto intro: le_cong [THEN iffD2])

lemma le_cong_r [intro, trans]:
  "⟦ x ⊑ y; y .= z; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L ⟧ ⟹
x ⊑ z"
  by (auto intro: le_cong [THEN iffD1])

lemma weak_refl [intro, simp]: "⟦ x .= y; x ∈ carrier L; y ∈ carrier
```

```
L ⟧ ⟹ x ⊑ y"
  by (simp add: le_cong_l)

end

lemma weak_llessI:
  fixes R (structure)
  assumes "x ⊑ y" and "~(x .= y)"
  shows "x ⊏ y"
  using assms unfolding lless_def by simp

lemma lless_imp_le:
  fixes R (structure)
  assumes "x ⊏ y"
  shows "x ⊑ y"
  using assms unfolding lless_def by simp

lemma weak_lless_imp_not_eq:
  fixes R (structure)
  assumes "x ⊏ y"
  shows "¬ (x .= y)"
  using assms unfolding lless_def by simp

lemma weak_llessE:
  fixes R (structure)
  assumes p: "x ⊏ y" and e: "⟦x ⊑ y; ¬ (x .= y)⟧ ⟹ P"
  shows "P"
  using p by (blast dest: lless_imp_le weak_lless_imp_not_eq e)

lemma (in weak_partial_order) lless_cong_l [trans]:
  assumes xx': "x .= x'"
    and xy: "x' ⊏ y"
    and carr: "x ∈ carrier L" "x' ∈ carrier L" "y ∈ carrier L"
  shows "x ⊏ y"
  using assms unfolding lless_def by (auto intro: trans sym)

lemma (in weak_partial_order) lless_cong_r [trans]:
  assumes xy: "x ⊏ y"
    and  yy': "y .= y'"
    and carr: "x ∈ carrier L" "y ∈ carrier L" "y' ∈ carrier L"
  shows "x ⊏ y'"
  using assms unfolding lless_def by (auto intro: trans sym)


lemma (in weak_partial_order) lless_antisym:
  assumes "a ∈ carrier L" "b ∈ carrier L"
    and "a ⊏ b" "b ⊏ a"
  shows "P"
  using assms
```

```
  by (elim weak_llessE) auto

lemma (in weak_partial_order) lless_trans [trans]:
  assumes "a ⊏ b" "b ⊏ c"
    and carr[simp]: "a ∈ carrier L" "b ∈ carrier L" "c ∈ carrier L"
  shows "a ⊏ c"
  using assms unfolding lless_def by (blast dest: le_trans intro: sym)

lemma weak_partial_order_subset:
  assumes "weak_partial_order L" "A ⊆ carrier L"
  shows "weak_partial_order (L⦇ carrier := A ⦈)"
proof -
  interpret L: weak_partial_order L
    by (simp add: assms)
  interpret equivalence "(L⦇ carrier := A ⦈)"
    by (simp add: L.equivalence_axioms assms(2) equivalence_subset)
  show ?thesis
    apply (unfold_locales, simp_all)
    using assms(2) apply auto[1]
    using assms(2) apply auto[1]
    apply (meson L.le_trans assms(2) contra_subsetD)
    apply (meson L.le_cong assms(2) subsetCE)
  done
qed
```

2.1.2 Upper and lower bounds of a set

```
definition
  Upper :: "[_, 'a set] => 'a set"
  where "Upper L A = {u. (ALL x. x ∈ A ∩ carrier L --> x ⊑_L u)} ∩ carrier
L"

definition
  Lower :: "[_, 'a set] => 'a set"
  where "Lower L A = {l. (ALL x. x ∈ A ∩ carrier L --> l ⊑_L x)} ∩ carrier
L"

lemma Upper_closed [intro!, simp]:
  "Upper L A ⊆ carrier L"
  by (unfold Upper_def) clarify

lemma Upper_memD [dest]:
  fixes L (structure)
  shows "[| u ∈ Upper L A; x ∈ A; A ⊆ carrier L |] ==> x ⊑ u ∧ u ∈
carrier L"
  by (unfold Upper_def) blast

lemma (in weak_partial_order) Upper_elemD [dest]:
  "[| u .∈ Upper L A; u ∈ carrier L; x ∈ A; A ⊆ carrier L |] ==> x ⊑
```

```
u"
  unfolding Upper_def elem_def
  by (blast dest: sym)

lemma Upper_memI:
  fixes L (structure)
  shows "[| !! y. y ∈ A ==> y ⊑ x; x ∈ carrier L |] ==> x ∈ Upper L
A"
  by (unfold Upper_def) blast

lemma (in weak_partial_order) Upper_elemI:
  "[| !! y. y ∈ A ==> y ⊑ x; x ∈ carrier L |] ==> x .∈ Upper L A"
  unfolding Upper_def by blast

lemma Upper_antimono:
  "A ⊆ B ==> Upper L B ⊆ Upper L A"
  by (unfold Upper_def) blast

lemma (in weak_partial_order) Upper_is_closed [simp]:
  "A ⊆ carrier L ==> is_closed (Upper L A)"
  by (rule is_closedI) (blast intro: Upper_memI)+

lemma (in weak_partial_order) Upper_mem_cong:
  assumes a'carr: "a' ∈ carrier L" and Acarr: "A ⊆ carrier L"
    and aa': "a .= a'"
    and aelem: "a ∈ Upper L A"
  shows "a' ∈ Upper L A"
proof (rule Upper_memI[OF _ a'carr])
  fix y
  assume yA: "y ∈ A"
  hence "y ⊑ a" by (intro Upper_memD[OF aelem, THEN conjunct1] Acarr)
  also note aa'
  finally
      show "y ⊑ a'"
      by (simp add: a'carr subsetD[OF Acarr yA] subsetD[OF Upper_closed
aelem])
qed

lemma (in weak_partial_order) Upper_cong:
  assumes Acarr: "A ⊆ carrier L" and A'carr: "A' ⊆ carrier L"
    and AA': "A {.=} A'"
  shows "Upper L A = Upper L A'"
unfolding Upper_def
apply rule
 apply (rule, clarsimp) defer 1
 apply (rule, clarsimp) defer 1
proof -
  fix x a'
  assume carr: "x ∈ carrier L" "a' ∈ carrier L"
```

```
    and a'A': "a' ∈ A'"
  assume aLxCond[rule_format]: "∀a. a ∈ A ∧ a ∈ carrier L ⟶ a ⊑ x"

  from AA' and a'A' have "∃a∈A. a' .= a" by (rule set_eqD2)
  from this obtain a
      where aA: "a ∈ A"
      and a'a: "a' .= a"
      by auto
  note [simp] = subsetD[OF Acarr aA] carr

  note a'a
  also have "a ⊑ x" by (simp add: aLxCond aA)
  finally show "a' ⊑ x" by simp
next
  fix x a
  assume carr: "x ∈ carrier L" "a ∈ carrier L"
    and aA: "a ∈ A"
  assume a'LxCond[rule_format]: "∀a'. a' ∈ A' ∧ a' ∈ carrier L ⟶ a'
⊑ x"

  from AA' and aA have "∃a'∈A'. a .= a'" by (rule set_eqD1)
  from this obtain a'
      where a'A': "a' ∈ A'"
      and aa': "a .= a'"
      by auto
  note [simp] = subsetD[OF A'carr a'A'] carr

  note aa'
  also have "a' ⊑ x" by (simp add: a'LxCond a'A')
  finally show "a ⊑ x" by simp
qed

lemma Lower_closed [intro!, simp]:
  "Lower L A ⊆ carrier L"
  by (unfold Lower_def) clarify

lemma Lower_memD [dest]:
  fixes L (structure)
  shows "[| l ∈ Lower L A; x ∈ A; A ⊆ carrier L |] ==> l ⊑ x ∧ l ∈
carrier L"
  by (unfold Lower_def) blast

lemma Lower_memI:
  fixes L (structure)
  shows "[| !! y. y ∈ A ==> x ⊑ y; x ∈ carrier L |] ==> x ∈ Lower L
A"
  by (unfold Lower_def) blast

lemma Lower_antimono:
```

```
  "A ⊆ B ==> Lower L B ⊆ Lower L A"
  by (unfold Lower_def) blast

lemma (in weak_partial_order) Lower_is_closed [simp]:
  "A ⊆ carrier L ⟹ is_closed (Lower L A)"
  by (rule is_closedI) (blast intro: Lower_memI dest: sym)+

lemma (in weak_partial_order) Lower_mem_cong:
  assumes a'carr: "a' ∈ carrier L" and Acarr: "A ⊆ carrier L"
    and aa': "a .= a'"
    and aelem: "a ∈ Lower L A"
  shows "a' ∈ Lower L A"
using assms Lower_closed[of L A]
by (intro Lower_memI) (blast intro: le_cong_l[OF aa'[symmetric]])

lemma (in weak_partial_order) Lower_cong:
  assumes Acarr: "A ⊆ carrier L" and A'carr: "A' ⊆ carrier L"
    and AA': "A {.=} A'"
  shows "Lower L A = Lower L A'"
unfolding Lower_def
apply rule
 apply clarsimp defer 1
 apply clarsimp defer 1
proof -
  fix x a'
  assume carr: "x ∈ carrier L" "a' ∈ carrier L"
    and a'A': "a' ∈ A'"
  assume "∀a. a ∈ A ∧ a ∈ carrier L ⟶ x ⊑ a"
  hence aLxCond: "⋀a. ⟦a ∈ A; a ∈ carrier L⟧ ⟹ x ⊑ a" by fast

  from AA' and a'A' have "∃a∈A. a' .= a" by (rule set_eqD2)
  from this obtain a
      where aA: "a ∈ A"
      and a'a: "a' .= a"
      by auto

  from aA and subsetD[OF Acarr aA]
      have "x ⊑ a" by (rule aLxCond)
  also note a'a[symmetric]
  finally
      show "x ⊑ a'" by (simp add: carr subsetD[OF Acarr aA])
next
  fix x a
  assume carr: "x ∈ carrier L" "a ∈ carrier L"
    and aA: "a ∈ A"
  assume "∀a'. a' ∈ A' ∧ a' ∈ carrier L ⟶ x ⊑ a'"
  hence a'LxCond: "⋀a'. ⟦a' ∈ A'; a' ∈ carrier L⟧ ⟹ x ⊑ a'" by fast+

  from AA' and aA have "∃a'∈A'. a .= a'" by (rule set_eqD1)
```

```
  from this obtain a'
      where a'A': "a' ∈ A'"
      and aa': "a .= a'"
      by auto
  from a'A' and subsetD[OF A'carr a'A']
      have "x ⊑ a'" by (rule a'LxCond)
  also note aa'[symmetric]
  finally show "x ⊑ a" by (simp add: carr subsetD[OF A'carr a'A'])
qed
```

Jacobson: Theorem 8.1

```
lemma Lower_empty [simp]:
  "Lower L {} = carrier L"
  by (unfold Lower_def) simp

lemma Upper_empty [simp]:
  "Upper L {} = carrier L"
  by (unfold Upper_def) simp
```

2.1.3 Least and greatest, as predicate

```
definition
  least :: "[_, 'a, 'a set] => bool"
  where "least L l A ⟷ A ⊆ carrier L & l ∈ A & (ALL x : A. l ⊑L x)"

definition
  greatest :: "[_, 'a, 'a set] => bool"
  where "greatest L g A ⟷ A ⊆ carrier L & g ∈ A & (ALL x : A. x ⊑L
g)"
```

Could weaken these to `l ∈ carrier L ∧ l .∈ A` and `g ∈ carrier L ∧ g .∈ A`.

```
lemma least_closed [intro, simp]:
  "least L l A ==> l ∈ carrier L"
  by (unfold least_def) fast

lemma least_mem:
  "least L l A ==> l ∈ A"
  by (unfold least_def) fast

lemma (in weak_partial_order) weak_least_unique:
  "[| least L x A; least L y A |] ==> x .= y"
  by (unfold least_def) blast

lemma least_le:
  fixes L (structure)
  shows "[| least L x A; a ∈ A |] ==> x ⊑ a"
  by (unfold least_def) fast
```

```
lemma (in weak_partial_order) least_cong:
  "[| x .= x'; x ∈ carrier L; x' ∈ carrier L; is_closed A |] ==> least
L x A = least L x' A"
  by (unfold least_def) (auto dest: sym)

abbreviation is_lub :: "[_, 'a, 'a set] => bool"
where "is_lub L x A ≡ least L x (Upper L A)"
```

`least` is not congruent in the second parameter for `A {.=} A'`

```
lemma (in weak_partial_order) least_Upper_cong_l:
  assumes "x .= x'"
    and "x ∈ carrier L" "x' ∈ carrier L"
    and "A ⊆ carrier L"
  shows "least L x (Upper L A) = least L x' (Upper L A)"
  apply (rule least_cong) using assms by auto

lemma (in weak_partial_order) least_Upper_cong_r:
  assumes Acarrs: "A ⊆ carrier L" "A' ⊆ carrier L"
    and AA': "A {.=} A'"
  shows "least L x (Upper L A) = least L x (Upper L A')"
apply (subgoal_tac "Upper L A = Upper L A'", simp)
by (rule Upper_cong) fact+

lemma least_UpperI:
  fixes L (structure)
  assumes above: "!! x. x ∈ A ==> x ⊑ s"
    and below: "!! y. y ∈ Upper L A ==> s ⊑ y"
    and L: "A ⊆ carrier L"  "s ∈ carrier L"
  shows "least L s (Upper L A)"
proof -
  have "Upper L A ⊆ carrier L" by simp
  moreover from above L have "s ∈ Upper L A" by (simp add: Upper_def)
  moreover from below have "ALL x : Upper L A. s ⊑ x" by fast
  ultimately show ?thesis by (simp add: least_def)
qed

lemma least_Upper_above:
  fixes L (structure)
  shows "[| least L s (Upper L A); x ∈ A; A ⊆ carrier L |] ==> x ⊑ s"
  by (unfold least_def) blast

lemma greatest_closed [intro, simp]:
  "greatest L l A ==> l ∈ carrier L"
  by (unfold greatest_def) fast

lemma greatest_mem:
  "greatest L l A ==> l ∈ A"
  by (unfold greatest_def) fast
```

```
lemma (in weak_partial_order) weak_greatest_unique:
  "[| greatest L x A; greatest L y A |] ==> x .= y"
  by (unfold greatest_def) blast

lemma greatest_le:
  fixes L (structure)
  shows "[| greatest L x A; a ∈ A |] ==> a ⊑ x"
  by (unfold greatest_def) fast

lemma (in weak_partial_order) greatest_cong:
  "[| x .= x’; x ∈ carrier L; x’ ∈ carrier L; is_closed A |] ==>
  greatest L x A = greatest L x’ A"
  by (unfold greatest_def) (auto dest: sym)

abbreviation is_glb :: "[_, ’a, ’a set] => bool"
where "is_glb L x A ≡ greatest L x (Lower L A)"
```

`greatest` is not congruent in the second parameter for `A {.=} A’`

```
lemma (in weak_partial_order) greatest_Lower_cong_l:
  assumes "x .= x’"
    and "x ∈ carrier L" "x’ ∈ carrier L"
    and "A ⊆ carrier L"
  shows "greatest L x (Lower L A) = greatest L x’ (Lower L A)"
  apply (rule greatest_cong) using assms by auto

lemma (in weak_partial_order) greatest_Lower_cong_r:
  assumes Acarrs: "A ⊆ carrier L" "A’ ⊆ carrier L"
    and AA’: "A {.=} A’"
  shows "greatest L x (Lower L A) = greatest L x (Lower L A’)"
apply (subgoal_tac "Lower L A = Lower L A’", simp)
by (rule Lower_cong) fact+

lemma greatest_LowerI:
  fixes L (structure)
  assumes below: "!! x. x ∈ A ==> i ⊑ x"
    and above: "!! y. y ∈ Lower L A ==> y ⊑ i"
    and L: "A ⊆ carrier L"  "i ∈ carrier L"
  shows "greatest L i (Lower L A)"
proof -
  have "Lower L A ⊆ carrier L" by simp
  moreover from below L have "i ∈ Lower L A" by (simp add: Lower_def)
  moreover from above have "ALL x : Lower L A. x ⊑ i" by fast
  ultimately show ?thesis by (simp add: greatest_def)
qed

lemma greatest_Lower_below:
  fixes L (structure)
  shows "[| greatest L i (Lower L A); x ∈ A; A ⊆ carrier L |] ==> i ⊑
x"
```

```
  by (unfold greatest_def) blast

lemma Lower_dual [simp]:
  "Lower (inv_gorder L) A = Upper L A"
  by (simp add:Upper_def Lower_def)

lemma Upper_dual [simp]:
  "Upper (inv_gorder L) A = Lower L A"
  by (simp add:Upper_def Lower_def)

lemma least_dual [simp]:
  "least (inv_gorder L) x A = greatest L x A"
  by (simp add:least_def greatest_def)

lemma greatest_dual [simp]:
  "greatest (inv_gorder L) x A = least L x A"
  by (simp add:least_def greatest_def)

lemma (in weak_partial_order) dual_weak_order:
  "weak_partial_order (inv_gorder L)"
  apply (unfold_locales)
  apply (simp_all)
  apply (metis sym)
  apply (metis trans)
  apply (metis weak_le_antisym)
  apply (metis le_trans)
  apply (metis le_cong_l le_cong_r sym)
done

lemma dual_weak_order_iff:
  "weak_partial_order (inv_gorder A) ⟷ weak_partial_order A"
proof
  assume "weak_partial_order (inv_gorder A)"
  then interpret dpo: weak_partial_order "inv_gorder A"
  rewrites "carrier (inv_gorder A) = carrier A"
  and   "le (inv_gorder A)      = (λ x y. le A y x)"
  and   "eq (inv_gorder A)      = eq A"
    by (simp_all)
  show "weak_partial_order A"
    by (unfold_locales, auto intro: dpo.sym dpo.trans dpo.le_trans)
next
  assume "weak_partial_order A"
  thus "weak_partial_order (inv_gorder A)"
    by (metis weak_partial_order.dual_weak_order)
qed
```

2.1.4 Intervals

definition

```
  at_least_at_most :: "('a, 'c) gorder_scheme ⇒ 'a => 'a => 'a set" ("(1⦃_.._⦄ı)")
  where "⦃l..u⦄_A = {x ∈ carrier A. l ⊑_A x ∧ x ⊑_A u}"

context weak_partial_order
begin

  lemma at_least_at_most_upper [dest]:
    "x ∈ ⦃a..b⦄ ⟹ x ⊑ b"
    by (simp add: at_least_at_most_def)

  lemma at_least_at_most_lower [dest]:
    "x ∈ ⦃a..b⦄ ⟹ a ⊑ x"
    by (simp add: at_least_at_most_def)

  lemma at_least_at_most_closed: "⦃a..b⦄ ⊆ carrier L"
    by (auto simp add: at_least_at_most_def)

  lemma at_least_at_most_member [intro]:
    "⟦ x ∈ carrier L; a ⊑ x; x ⊑ b ⟧ ⟹ x ∈ ⦃a..b⦄"
    by (simp add: at_least_at_most_def)

end
```

2.1.5 Isotone functions

```
definition isotone :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool"
  where
  "isotone A B f ≡
   weak_partial_order A ∧ weak_partial_order B ∧
   (∀x∈carrier A. ∀y∈carrier A. x ⊑_A y ⟶ f x ⊑_B f y)"

lemma isotoneI [intro?]:
  fixes f :: "'a ⇒ 'b"
  assumes "weak_partial_order L1"
          "weak_partial_order L2"
          "(⋀x y. ⟦ x ∈ carrier L1; y ∈ carrier L1; x ⊑_L1 y ⟧
                  ⟹ f x ⊑_L2 f y)"
  shows "isotone L1 L2 f"
  using assms by (auto simp add:isotone_def)

abbreviation Monotone :: "('a, 'b) gorder_scheme ⇒ ('a ⇒ 'a) ⇒ bool"
("Monoı")
  where "Monotone L f ≡ isotone L L f"

lemma use_iso1:
  "⟦isotone A A f; x ∈ carrier A; y ∈ carrier A; x ⊑_A y⟧ ⟹
   f x ⊑_A f y"
  by (simp add: isotone_def)
```

```
lemma use_iso2:
  "⟦isotone A B f; x ∈ carrier A; y ∈ carrier A; x ⊑_A y⟧ ⟹
   f x ⊑_B f y"
  by (simp add: isotone_def)

lemma iso_compose:
  "⟦f ∈ carrier A → carrier B; isotone A B f; g ∈ carrier B → carrier
C; isotone B C g⟧ ⟹
   isotone A C (g ∘ f)"
  by (simp add: isotone_def, safe, metis Pi_iff)

lemma (in weak_partial_order) inv_isotone [simp]:
  "isotone (inv_gorder A) (inv_gorder B) f = isotone A B f"
  by (auto simp add:isotone_def dual_weak_order dual_weak_order_iff)
```

2.1.6 Idempotent functions

```
definition idempotent ::
  "('a, 'b) gorder_scheme ⇒ ('a ⇒ 'a) ⇒ bool" ("Idemı") where
  "idempotent L f ≡ ∀x∈carrier L. f (f x) .=_L f x"

lemma (in weak_partial_order) idempotent:
  "⟦ Idem f; x ∈ carrier L ⟧ ⟹ f (f x) .= f x"
  by (auto simp add: idempotent_def)
```

2.1.7 Order embeddings

```
definition order_emb :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool"
  where
  "order_emb A B f ≡ weak_partial_order A
                    ∧ weak_partial_order B
                    ∧ (∀x∈carrier A. ∀y∈carrier A. f x ⊑_B f y ⟷ x ⊑_A
y )"

lemma order_emb_isotone: "order_emb A B f ⟹ isotone A B f"
  by (auto simp add: isotone_def order_emb_def)
```

2.1.8 Commuting functions

```
definition commuting :: "('a, 'c) gorder_scheme ⇒ ('a ⇒ 'a) ⇒ ('a ⇒
'a) ⇒ bool" where
"commuting A f g = (∀x∈carrier A. (f ∘ g) x .=_A (g ∘ f) x)"
```

2.2 Partial orders where eq is the Equality

```
locale partial_order = weak_partial_order +
  assumes eq_is_equal: "op .= = op ="
begin
```

```
declare weak_le_antisym [rule del]

lemma le_antisym [intro]:
  "[| x ⊑ y; y ⊑ x; x ∈ carrier L; y ∈ carrier L |] ==> x = y"
  using weak_le_antisym unfolding eq_is_equal .

lemma lless_eq:
  "x ⊏ y ⟷ x ⊑ y & x ≠ y"
  unfolding lless_def by (simp add: eq_is_equal)

lemma set_eq_is_eq: "A {.=} B ⟷ A = B"
  by (auto simp add: set_eq_def elem_def eq_is_equal)

end

lemma (in partial_order) dual_order:
  "partial_order (inv_gorder L)"
proof -
  interpret dwo: weak_partial_order "inv_gorder L"
    by (metis dual_weak_order)
  show ?thesis
    by (unfold_locales, simp add:eq_is_equal)
qed

lemma dual_order_iff:
  "partial_order (inv_gorder A) ⟷ partial_order A"
proof
  assume assm:"partial_order (inv_gorder A)"
  then interpret po: partial_order "inv_gorder A"
  rewrites "carrier (inv_gorder A) = carrier A"
  and    "le (inv_gorder A)      = (λ x y. le A y x)"
  and    "eq (inv_gorder A)      = eq A"
    by (simp_all)
  show "partial_order A"
    apply (unfold_locales, simp_all)
    apply (metis po.sym, metis po.trans)
    apply (metis po.weak_le_antisym, metis po.le_trans)
    apply (metis (full_types) po.eq_is_equal, metis po.eq_is_equal)
  done
next
  assume "partial_order A"
  thus "partial_order (inv_gorder A)"
    by (metis partial_order.dual_order)
qed
```

Least and greatest, as predicate

```
lemma (in partial_order) least_unique:
  "[| least L x A; least L y A |] ==> x = y"
```

```
  using weak_least_unique unfolding eq_is_equal .

lemma (in partial_order) greatest_unique:
  "[| greatest L x A; greatest L y A |] ==> x = y"
  using weak_greatest_unique unfolding eq_is_equal .
```

2.3 Bounded Orders

```
definition
  top :: "_ => 'a" ("⊤ı") where
  "⊤L = (SOME x. greatest L x (carrier L))"

definition
  bottom :: "_ => 'a" ("⊥ı") where
  "⊥L = (SOME x. least L x (carrier L))"

locale weak_partial_order_bottom = weak_partial_order L for L (structure)
+
  assumes bottom_exists: "∃ x. least L x (carrier L)"
begin

lemma bottom_least: "least L ⊥ (carrier L)"
proof -
  obtain x where "least L x (carrier L)"
    by (metis bottom_exists)

  thus ?thesis
    by (auto intro:someI2 simp add: bottom_def)
qed

lemma bottom_closed [simp, intro]:
  "⊥ ∈ carrier L"
  by (metis bottom_least least_mem)

lemma bottom_lower [simp, intro]:
  "x ∈ carrier L ⟹ ⊥ ⊑ x"
  by (metis bottom_least least_le)

end

locale weak_partial_order_top = weak_partial_order L for L (structure)
+
  assumes top_exists: "∃ x. greatest L x (carrier L)"
begin

lemma top_greatest: "greatest L ⊤ (carrier L)"
proof -
  obtain x where "greatest L x (carrier L)"
    by (metis top_exists)
```

```
  thus ?thesis
    by (auto intro:someI2 simp add: top_def)
qed

lemma top_closed [simp, intro]:
  "⊤ ∈ carrier L"
  by (metis greatest_mem top_greatest)

lemma top_higher [simp, intro]:
  "x ∈ carrier L ⟹ x ⊑ ⊤"
  by (metis greatest_le top_greatest)

end
```

2.4 Total Orders

```
locale weak_total_order = weak_partial_order +
  assumes total: "⟦ x ∈ carrier L; y ∈ carrier L ⟧ ⟹ x ⊑ y ∨ y ⊑ x"
```

Introduction rule: the usual definition of total order

```
lemma (in weak_partial_order) weak_total_orderI:
  assumes total: "!!x y. ⟦ x ∈ carrier L; y ∈ carrier L ⟧ ⟹ x ⊑ y ∨
y ⊑ x"
  shows "weak_total_order L"
  by unfold_locales (rule total)
```

2.5 Total orders where `eq` is the Equality

```
locale total_order = partial_order +
  assumes total_order_total: "⟦ x ∈ carrier L; y ∈ carrier L ⟧ ⟹ x
⊑ y ∨ y ⊑ x"

sublocale total_order < weak?: weak_total_order
  by unfold_locales (rule total_order_total)
```

Introduction rule: the usual definition of total order

```
lemma (in partial_order) total_orderI:
  assumes total: "!!x y. ⟦ x ∈ carrier L; y ∈ carrier L ⟧ ⟹ x ⊑ y ∨
y ⊑ x"
  shows "total_order L"
  by unfold_locales (rule total)

end


theory Lattice
imports Order
begin
```

3 Lattices

3.1 Supremum and infimum

definition
sup :: "[_, 'a set] => 'a" ("$\bigsqcup_ı$_" [90] 90)
where "$\bigsqcup_L$A = (SOME x. least L x (Upper L A))"

definition
inf :: "[_, 'a set] => 'a" ("$\bigsqcap_ı$_" [90] 90)
where "$\bigsqcap_L$A = (SOME x. greatest L x (Lower L A))"

definition supr ::
"('a, 'b) gorder_scheme ⇒ 'c set ⇒ ('c ⇒ 'a) ⇒ 'a "
where "supr L A f = $\bigsqcup_L$(f ` A)"

definition infi ::
"('a, 'b) gorder_scheme ⇒ 'c set ⇒ ('c ⇒ 'a) ⇒ 'a "
where "infi L A f = $\bigsqcap_L$(f ` A)"

syntax
"_inf1" :: "('a, 'b) gorder_scheme ⇒ pttrns ⇒ 'a ⇒ 'a" ("(3IINFı _./ _)" [0, 10] 10)
"_inf" :: "('a, 'b) gorder_scheme ⇒ pttrn ⇒ 'c set ⇒ 'a ⇒ 'a" ("(3IINFı _:_./ _)" [0, 0, 10] 10)
"_sup1" :: "('a, 'b) gorder_scheme ⇒ pttrns ⇒ 'a ⇒ 'a" ("(3SSUPı _./ _)" [0, 10] 10)
"_sup" :: "('a, 'b) gorder_scheme ⇒ pttrn ⇒ 'c set ⇒ 'a ⇒ 'a" ("(3SSUPı _:_./ _)" [0, 0, 10] 10)

translations
"IINF_L x. B" == "CONST infi L CONST UNIV (%x. B)"
"IINF_L x:A. B" == "CONST infi L A (%x. B)"
"SSUP_L x. B" == "CONST supr L CONST UNIV (%x. B)"
"SSUP_L x:A. B" == "CONST supr L A (%x. B)"

definition
join :: "[_, 'a, 'a] => 'a" (**infixl** "$\sqcup_ı$" 65)
where "x $\sqcup_L$ y = $\bigsqcup_L${x, y}"

definition
meet :: "[_, 'a, 'a] => 'a" (**infixl** "$\sqcap_ı$" 70)
where "x $\sqcap_L$ y = $\bigsqcap_L${x, y}"

definition
LEAST_FP :: "('a, 'b) gorder_scheme ⇒ ('a ⇒ 'a) ⇒ 'a" ("LFPı") **where**
"LEAST_FP L f = $\bigsqcap_L$ {u ∈ carrier L. f u $\sqsubseteq_L$ u}" — least fixed point

definition

```
  GREATEST_FP:: "('a, 'b) gorder_scheme ⇒ ('a ⇒ 'a) ⇒ 'a" ("GFPı")
where
  "GREATEST_FP L f = ⨆L {u ∈ carrier L. u ⊑L f u}"      — greatest fixed
point
```

3.2 Dual operators

```
lemma sup_dual [simp]:
  "⨆inv_gorder LA = ⨅LA"
  by (simp add: sup_def inf_def)

lemma inf_dual [simp]:
  "⨅inv_gorder LA = ⨆LA"
  by (simp add: sup_def inf_def)

lemma join_dual [simp]:
  "p ⊔inv_gorder L q = p ⊓L q"
  by (simp add:join_def meet_def)

lemma meet_dual [simp]:
  "p ⊓inv_gorder L q = p ⊔L q"
  by (simp add:join_def meet_def)

lemma top_dual [simp]:
  "⊤inv_gorder L = ⊥L"
  by (simp add: top_def bottom_def)

lemma bottom_dual [simp]:
  "⊥inv_gorder L = ⊤L"
  by (simp add: top_def bottom_def)

lemma LFP_dual [simp]:
  "LEAST_FP (inv_gorder L) f = GREATEST_FP L f"
  by (simp add:LEAST_FP_def GREATEST_FP_def)

lemma GFP_dual [simp]:
  "GREATEST_FP (inv_gorder L) f = LEAST_FP L f"
  by (simp add:LEAST_FP_def GREATEST_FP_def)
```

3.3 Lattices

```
locale weak_upper_semilattice = weak_partial_order +
  assumes sup_of_two_exists:
    "[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. least L s (Upper L {x,
y})"

locale weak_lower_semilattice = weak_partial_order +
  assumes inf_of_two_exists:
    "[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. greatest L s (Lower
L {x, y})"
```

```
locale weak_lattice = weak_upper_semilattice + weak_lower_semilattice

lemma (in weak_lattice) dual_weak_lattice:
  "weak_lattice (inv_gorder L)"
proof -
  interpret dual: weak_partial_order "inv_gorder L"
    by (metis dual_weak_order)

  show ?thesis
    apply (unfold_locales)
    apply (simp_all add: inf_of_two_exists sup_of_two_exists)
  done
qed
```

3.3.1 Supremum

```
lemma (in weak_upper_semilattice) joinI:
  "[| !!l. least L l (Upper L {x, y}) ==> P l; x ∈ carrier L; y ∈ carrier
L |]
  ==> P (x ⊔ y)"
proof (unfold join_def sup_def)
  assume L: "x ∈ carrier L"  "y ∈ carrier L"
    and P: "!!l. least L l (Upper L {x, y}) ==> P l"
  with sup_of_two_exists obtain s where "least L s (Upper L {x, y})"
by fast
  with L show "P (SOME l. least L l (Upper L {x, y}))"
    by (fast intro: someI2 P)
qed

lemma (in weak_upper_semilattice) join_closed [simp]:
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊔ y ∈ carrier L"
  by (rule joinI) (rule least_closed)

lemma (in weak_upper_semilattice) join_cong_l:
  assumes carr: "x ∈ carrier L" "x' ∈ carrier L" "y ∈ carrier L"
    and xx': "x .= x'"
  shows "x ⊔ y .= x' ⊔ y"
proof (rule joinI, rule joinI)
  fix a b
  from xx' carr
      have seq: "{x, y} {.=} {x', y}" by (rule set_eq_pairI)

  assume leasta: "least L a (Upper L {x, y})"
  assume "least L b (Upper L {x', y})"
  with carr
      have leastb: "least L b (Upper L {x, y})"
      by (simp add: least_Upper_cong_r[OF _ _ seq])
```

```
  from leasta leastb
      show "a .= b" by (rule weak_least_unique)
qed (rule carr)+

lemma (in weak_upper_semilattice) join_cong_r:
  assumes carr: "x ∈ carrier L" "y ∈ carrier L" "y' ∈ carrier L"
    and yy': "y .= y'"
  shows "x ⊔ y .= x ⊔ y'"
proof (rule joinI, rule joinI)
  fix a b
  have "{x, y} = {y, x}" by fast
  also from carr yy'
      have "{y, x} {.=} {y', x}" by (intro set_eq_pairI)
  also have "{y', x} = {x, y'}" by fast
  finally
      have seq: "{x, y} {.=} {x, y'}" .

  assume leasta: "least L a (Upper L {x, y})"
  assume "least L b (Upper L {x, y'})"
  with carr
      have leastb: "least L b (Upper L {x, y})"
      by (simp add: least_Upper_cong_r[OF _ _ seq])

  from leasta leastb
      show "a .= b" by (rule weak_least_unique)
qed (rule carr)+

lemma (in weak_partial_order) sup_of_singletonI:
  "x ∈ carrier L ==> least L x (Upper L {x})"
  by (rule least_UpperI) auto

lemma (in weak_partial_order) weak_sup_of_singleton [simp]:
  "x ∈ carrier L ==> ⨆{x} .= x"
  unfolding sup_def
  by (rule someI2) (auto intro: weak_least_unique sup_of_singletonI)

lemma (in weak_partial_order) sup_of_singleton_closed [simp]:
  "x ∈ carrier L ⟹ ⨆{x} ∈ carrier L"
  unfolding sup_def
  by (rule someI2) (auto intro: sup_of_singletonI)
```

Condition on `A`: supremum exists.

```
lemma (in weak_upper_semilattice) sup_insertI:
  "[| !!s. least L s (Upper L (insert x A)) ==> P s;
  least L a (Upper L A); x ∈ carrier L; A ⊆ carrier L |]
  ==> P (⨆(insert x A))"
proof (unfold sup_def)
  assume L: "x ∈ carrier L"  "A ⊆ carrier L"
    and P: "!!l. least L l (Upper L (insert x A)) ==> P l"
```

```
    and least_a: "least L a (Upper L A)"
  from L least_a have La: "a ∈ carrier L" by simp
  from L sup_of_two_exists least_a
  obtain s where least_s: "least L s (Upper L {a, x})" by blast
  show "P (SOME l. least L l (Upper L (insert x A)))"
  proof (rule someI2)
    show "least L s (Upper L (insert x A))"
    proof (rule least_UpperI)
      fix z
      assume "z ∈ insert x A"
      then show "z ⊑ s"
      proof
        assume "z = x" then show ?thesis
          by (simp add: least_Upper_above [OF least_s] L La)
      next
        assume "z ∈ A"
        with L least_s least_a show ?thesis
          by (rule_tac le_trans [where y = a]) (auto dest: least_Upper_above)
      qed
    next
      fix y
      assume y: "y ∈ Upper L (insert x A)"
      show "s ⊑ y"
      proof (rule least_le [OF least_s], rule Upper_memI)
        fix z
        assume z: "z ∈ {a, x}"
        then show "z ⊑ y"
        proof
          have y': "y ∈ Upper L A"
            apply (rule subsetD [where A = "Upper L (insert x A)"])
             apply (rule Upper_antimono)
             apply blast
            apply (rule y)
            done
          assume "z = a"
          with y' least_a show ?thesis by (fast dest: least_le)
        next
          assume "z ∈ {x}"
          with y L show ?thesis by blast
        qed
      qed (rule Upper_closed [THEN subsetD, OF y])
    next
      from L show "insert x A ⊆ carrier L" by simp
      from least_s show "s ∈ carrier L" by simp
    qed
  qed (rule P)
qed

lemma (in weak_upper_semilattice) finite_sup_least:
```

```
  "[| finite A; A ⊆ carrier L; A ~= {} |] ==> least L (⨆A) (Upper L A)"
proof (induct set: finite)
  case empty
  then show ?case by simp
next
  case (insert x A)
  show ?case
  proof (cases "A = {}")
    case True
    with insert show ?thesis
      by simp (simp add: least_cong [OF weak_sup_of_singleton] sup_of_singletonI)

  next
    case False
    with insert have "least L (⨆A) (Upper L A)" by simp
    with _ show ?thesis
      by (rule sup_insertI) (simp_all add: insert [simplified])
  qed
qed

lemma (in weak_upper_semilattice) finite_sup_insertI:
  assumes P: "!!l. least L l (Upper L (insert x A)) ==> P l"
    and xA: "finite A"  "x ∈ carrier L"  "A ⊆ carrier L"
  shows "P (⨆ (insert x A))"
proof (cases "A = {}")
  case True with P and xA show ?thesis
    by (simp add: finite_sup_least)
next
  case False with P and xA show ?thesis
    by (simp add: sup_insertI finite_sup_least)
qed

lemma (in weak_upper_semilattice) finite_sup_closed [simp]:
  "[| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨆A ∈ carrier L"
proof (induct set: finite)
  case empty then show ?case by simp
next
  case insert then show ?case
    by - (rule finite_sup_insertI, simp_all)
qed

lemma (in weak_upper_semilattice) join_left:
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑ x ⊔ y"
  by (rule joinI [folded join_def]) (blast dest: least_mem)

lemma (in weak_upper_semilattice) join_right:
  "[| x ∈ carrier L; y ∈ carrier L |] ==> y ⊑ x ⊔ y"
  by (rule joinI [folded join_def]) (blast dest: least_mem)
```

```
lemma (in weak_upper_semilattice) sup_of_two_least:
  "[| x ∈ carrier L; y ∈ carrier L |] ==> least L (⨆{x, y}) (Upper L
{x, y})"
proof (unfold sup_def)
  assume L: "x ∈ carrier L"  "y ∈ carrier L"
  with sup_of_two_exists obtain s where "least L s (Upper L {x, y})"
by fast
  with L show "least L (SOME z. least L z (Upper L {x, y})) (Upper L
{x, y})"
  by (fast intro: someI2 weak_least_unique)
qed

lemma (in weak_upper_semilattice) join_le:
  assumes sub: "x ⊑ z"  "y ⊑ z"
    and x: "x ∈ carrier L" and y: "y ∈ carrier L" and z: "z ∈ carrier
L"
  shows "x ⊔ y ⊑ z"
proof (rule joinI [OF _ x y])
  fix s
  assume "least L s (Upper L {x, y})"
  with sub z show "s ⊑ z" by (fast elim: least_le intro: Upper_memI)
qed

lemma (in weak_lattice) weak_le_iff_meet:
  assumes "x ∈ carrier L" "y ∈ carrier L"
  shows "x ⊑ y ⟷ (x ⊔ y) .= y"
  by (meson assms(1) assms(2) join_closed join_le join_left join_right
le_cong_r local.le_refl weak_le_antisym)

lemma (in weak_upper_semilattice) weak_join_assoc_lemma:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "x ⊔ (y ⊔ z) .= ⨆{x, y, z}"
proof (rule finite_sup_insertI)
  — The textbook argument in Jacobson I, p 457
  fix s
  assume sup: "least L s (Upper L {x, y, z})"
  show "x ⊔ (y ⊔ z) .= s"
  proof (rule weak_le_antisym)
    from sup L show "x ⊔ (y ⊔ z) ⊑ s"
      by (fastforce intro!: join_le elim: least_Upper_above)
  next
    from sup L show "s ⊑ x ⊔ (y ⊔ z)"
    by (erule_tac least_le)
      (blast intro!: Upper_memI intro: le_trans join_left join_right join_closed)
  qed (simp_all add: L least_closed [OF sup])
qed (simp_all add: L)
```

Commutativity holds for =.

```
lemma join_comm:
```

```
  fixes L (structure)
  shows "x ⊔ y = y ⊔ x"
  by (unfold join_def) (simp add: insert_commute)

lemma (in weak_upper_semilattice) weak_join_assoc:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "(x ⊔ y) ⊔ z .= x ⊔ (y ⊔ z)"
proof -

  have "(x ⊔ y) ⊔ z = z ⊔ (x ⊔ y)" by (simp only: join_comm)
  also from L have "... .= ⨆{z, x, y}" by (simp add: weak_join_assoc_lemma)
  also from L have "... = ⨆{x, y, z}" by (simp add: insert_commute)
  also from L have "... .= x ⊔ (y ⊔ z)" by (simp add: weak_join_assoc_lemma
[symmetric])
  finally show ?thesis by (simp add: L)
qed
```

3.3.2 Infimum

```
lemma (in weak_lower_semilattice) meetI:
  "[| !!i. greatest L i (Lower L {x, y}) ==> P i;
  x ∈ carrier L; y ∈ carrier L |]
  ==> P (x ⊓ y)"
proof (unfold meet_def inf_def)
  assume L: "x ∈ carrier L"  "y ∈ carrier L"
    and P: "!!g. greatest L g (Lower L {x, y}) ==> P g"
  with inf_of_two_exists obtain i where "greatest L i (Lower L {x, y})"
by fast
  with L show "P (SOME g. greatest L g (Lower L {x, y}))"
  by (fast intro: someI2 weak_greatest_unique P)
qed

lemma (in weak_lower_semilattice) meet_closed [simp]:
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ∈ carrier L"
  by (rule meetI) (rule greatest_closed)

lemma (in weak_lower_semilattice) meet_cong_l:
  assumes carr: "x ∈ carrier L" "x’ ∈ carrier L" "y ∈ carrier L"
    and xx’: "x .= x’"
  shows "x ⊓ y .= x’ ⊓ y"
proof (rule meetI, rule meetI)
  fix a b
  from xx’ carr
      have seq: "{x, y} {.=} {x’, y}" by (rule set_eq_pairI)

  assume greatesta: "greatest L a (Lower L {x, y})"
  assume "greatest L b (Lower L {x’, y})"
  with carr
      have greatestb: "greatest L b (Lower L {x, y})"
```

```
      by (simp add: greatest_Lower_cong_r[OF _ _ seq])

  from greatesta greatestb
      show "a .= b" by (rule weak_greatest_unique)
qed (rule carr)+

lemma (in weak_lower_semilattice) meet_cong_r:
  assumes carr: "x ∈ carrier L" "y ∈ carrier L" "y' ∈ carrier L"
    and yy': "y .= y'"
  shows "x ⊓ y .= x ⊓ y'"
proof (rule meetI, rule meetI)
  fix a b
  have "{x, y} = {y, x}" by fast
  also from carr yy'
      have "{y, x} {.=} {y', x}" by (intro set_eq_pairI)
  also have "{y', x} = {x, y'}" by fast
  finally
      have seq: "{x, y} {.=} {x, y'}" .

  assume greatesta: "greatest L a (Lower L {x, y})"
  assume "greatest L b (Lower L {x, y'})"
  with carr
      have greatestb: "greatest L b (Lower L {x, y})"
      by (simp add: greatest_Lower_cong_r[OF _ _ seq])

  from greatesta greatestb
      show "a .= b" by (rule weak_greatest_unique)
qed (rule carr)+

lemma (in weak_partial_order) inf_of_singletonI:
  "x ∈ carrier L ==> greatest L x (Lower L {x})"
  by (rule greatest_LowerI) auto

lemma (in weak_partial_order) weak_inf_of_singleton [simp]:
  "x ∈ carrier L ==> ⨅{x} .= x"
  unfolding inf_def
  by (rule someI2) (auto intro: weak_greatest_unique inf_of_singletonI)

lemma (in weak_partial_order) inf_of_singleton_closed:
  "x ∈ carrier L ==> ⨅{x} ∈ carrier L"
  unfolding inf_def
  by (rule someI2) (auto intro: inf_of_singletonI)
```

Condition on `A`: infimum exists.

```
lemma (in weak_lower_semilattice) inf_insertI:
  "[| !!i. greatest L i (Lower L (insert x A)) ==> P i;
  greatest L a (Lower L A); x ∈ carrier L; A ⊆ carrier L |]
  ==> P (⨅(insert x A))"
proof (unfold inf_def)
```

```
  assume L: "x ∈ carrier L"  "A ⊆ carrier L"
    and P: "!!g. greatest L g (Lower L (insert x A)) ==> P g"
    and greatest_a: "greatest L a (Lower L A)"
  from L greatest_a have La: "a ∈ carrier L" by simp
  from L inf_of_two_exists greatest_a
  obtain i where greatest_i: "greatest L i (Lower L {a, x})" by blast
  show "P (SOME g. greatest L g (Lower L (insert x A)))"
  proof (rule someI2)
    show "greatest L i (Lower L (insert x A))"
    proof (rule greatest_LowerI)
      fix z
      assume "z ∈ insert x A"
      then show "i ⊑ z"
      proof
        assume "z = x" then show ?thesis
          by (simp add: greatest_Lower_below [OF greatest_i] L La)
      next
        assume "z ∈ A"
        with L greatest_i greatest_a show ?thesis
          by (rule_tac le_trans [where y = a]) (auto dest: greatest_Lower_below)
      qed
    next
      fix y
      assume y: "y ∈ Lower L (insert x A)"
      show "y ⊑ i"
      proof (rule greatest_le [OF greatest_i], rule Lower_memI)
        fix z
        assume z: "z ∈ {a, x}"
        then show "y ⊑ z"
        proof
          have y': "y ∈ Lower L A"
            apply (rule subsetD [where A = "Lower L (insert x A)"])
            apply (rule Lower_antimono)
             apply blast
            apply (rule y)
            done
          assume "z = a"
          with y' greatest_a show ?thesis by (fast dest: greatest_le)
        next
          assume "z ∈ {x}"
          with y L show ?thesis by blast
        qed
      qed (rule Lower_closed [THEN subsetD, OF y])
    next
      from L show "insert x A ⊆ carrier L" by simp
      from greatest_i show "i ∈ carrier L" by simp
    qed
  qed (rule P)
qed
```

```
lemma (in weak_lower_semilattice) finite_inf_greatest:
  "[| finite A; A ⊆ carrier L; A ~= {} |] ==> greatest L (⨅A) (Lower
L A)"
proof (induct set: finite)
  case empty then show ?case by simp
next
  case (insert x A)
  show ?case
  proof (cases "A = {}")
    case True
    with insert show ?thesis
      by simp (simp add: greatest_cong [OF weak_inf_of_singleton]
        inf_of_singleton_closed inf_of_singletonI)
  next
    case False
    from insert show ?thesis
    proof (rule_tac inf_insertI)
      from False insert show "greatest L (⨅A) (Lower L A)" by simp
    qed simp_all
  qed
qed

lemma (in weak_lower_semilattice) finite_inf_insertI:
  assumes P: "!!i. greatest L i (Lower L (insert x A)) ==> P i"
    and xA: "finite A"  "x ∈ carrier L"  "A ⊆ carrier L"
  shows "P (⨅ (insert x A))"
proof (cases "A = {}")
  case True with P and xA show ?thesis
    by (simp add: finite_inf_greatest)
next
  case False with P and xA show ?thesis
    by (simp add: inf_insertI finite_inf_greatest)
qed

lemma (in weak_lower_semilattice) finite_inf_closed [simp]:
  "[| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨅A ∈ carrier L"
proof (induct set: finite)
  case empty then show ?case by simp
next
  case insert then show ?case
    by (rule_tac finite_inf_insertI) (simp_all)
qed

lemma (in weak_lower_semilattice) meet_left:
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ⊑ x"
  by (rule meetI [folded meet_def]) (blast dest: greatest_mem)

lemma (in weak_lower_semilattice) meet_right:
```

```
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ⊑ y"
  by (rule meetI [folded meet_def]) (blast dest: greatest_mem)

lemma (in weak_lower_semilattice) inf_of_two_greatest:
  "[| x ∈ carrier L; y ∈ carrier L |] ==>
  greatest L (⨅{x, y}) (Lower L {x, y})"
proof (unfold inf_def)
  assume L: "x ∈ carrier L"  "y ∈ carrier L"
  with inf_of_two_exists obtain s where "greatest L s (Lower L {x, y})"
by fast
  with L
  show "greatest L (SOME z. greatest L z (Lower L {x, y})) (Lower L {x,
y})"
  by (fast intro: someI2 weak_greatest_unique)
qed

lemma (in weak_lower_semilattice) meet_le:
  assumes sub: "z ⊑ x"  "z ⊑ y"
    and x: "x ∈ carrier L" and y: "y ∈ carrier L" and z: "z ∈ carrier
L"
  shows "z ⊑ x ⊓ y"
proof (rule meetI [OF _ x y])
  fix i
  assume "greatest L i (Lower L {x, y})"
  with sub z show "z ⊑ i" by (fast elim: greatest_le intro: Lower_memI)
qed

lemma (in weak_lattice) weak_le_iff_join:
  assumes "x ∈ carrier L" "y ∈ carrier L"
  shows "x ⊑ y ⟷ x .= (x ⊓ y)"
  by (meson assms(1) assms(2) local.le_refl local.le_trans meet_closed
meet_le meet_left meet_right weak_le_antisym weak_refl)

lemma (in weak_lower_semilattice) weak_meet_assoc_lemma:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "x ⊓ (y ⊓ z) .= ⨅{x, y, z}"
proof (rule finite_inf_insertI)
```

The textbook argument in Jacobson I, p 457

```
  fix i
  assume inf: "greatest L i (Lower L {x, y, z})"
  show "x ⊓ (y ⊓ z) .= i"
  proof (rule weak_le_antisym)
    from inf L show "i ⊑ x ⊓ (y ⊓ z)"
      by (fastforce intro!: meet_le elim: greatest_Lower_below)
  next
    from inf L show "x ⊓ (y ⊓ z) ⊑ i"
    by (erule_tac greatest_le)
      (blast intro!: Lower_memI intro: le_trans meet_left meet_right meet_closed)
```

```
  qed (simp_all add: L greatest_closed [OF inf])
qed (simp_all add: L)

lemma meet_comm:
  fixes L (structure)
  shows "x ⊓ y = y ⊓ x"
  by (unfold meet_def) (simp add: insert_commute)

lemma (in weak_lower_semilattice) weak_meet_assoc:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "(x ⊓ y) ⊓ z .= x ⊓ (y ⊓ z)"
proof -

  have "(x ⊓ y) ⊓ z = z ⊓ (x ⊓ y)" by (simp only: meet_comm)
  also from L have "... .= ⨅ {z, x, y}" by (simp add: weak_meet_assoc_lemma)
  also from L have "... = ⨅ {x, y, z}" by (simp add: insert_commute)
  also from L have "... .= x ⊓ (y ⊓ z)" by (simp add: weak_meet_assoc_lemma
[symmetric])
  finally show ?thesis by (simp add: L)
qed
```

Total orders are lattices.

```
sublocale weak_total_order ⊆ weak?: weak_lattice
proof
  fix x y
  assume L: "x ∈ carrier L"  "y ∈ carrier L"
  show "EX s. least L s (Upper L {x, y})"
  proof -
    note total L
    moreover
    {
      assume "x ⊑ y"
      with L have "least L y (Upper L {x, y})"
        by (rule_tac least_UpperI) auto
    }
    moreover
    {
      assume "y ⊑ x"
      with L have "least L x (Upper L {x, y})"
        by (rule_tac least_UpperI) auto
    }
    ultimately show ?thesis by blast
  qed
next
  fix x y
  assume L: "x ∈ carrier L"  "y ∈ carrier L"
  show "EX i. greatest L i (Lower L {x, y})"
  proof -
    note total L
```

```
    moreover
    {
      assume "y ⊑ x"
      with L have "greatest L y (Lower L {x, y})"
        by (rule_tac greatest_LowerI) auto
    }
    moreover
    {
      assume "x ⊑ y"
      with L have "greatest L x (Lower L {x, y})"
        by (rule_tac greatest_LowerI) auto
    }
    ultimately show ?thesis by blast
  qed
qed
```

3.4 Weak Bounded Lattices

```
locale weak_bounded_lattice =
  weak_lattice +
  weak_partial_order_bottom +
  weak_partial_order_top
begin

lemma bottom_meet: "x ∈ carrier L ⟹ ⊥ ⊓ x .= ⊥"
  by (metis bottom_least least_def meet_closed meet_left weak_le_antisym)

lemma bottom_join: "x ∈ carrier L ⟹ ⊥ ⊔ x .= x"
  by (metis bottom_least join_closed join_le join_right le_refl least_def
weak_le_antisym)

lemma bottom_weak_eq:
  "⟦ b ∈ carrier L; ⋀ x. x ∈ carrier L ⟹ b ⊑ x ⟧ ⟹ b .= ⊥"
  by (metis bottom_closed bottom_lower weak_le_antisym)

lemma top_join: "x ∈ carrier L ⟹ ⊤ ⊔ x .= ⊤"
  by (metis join_closed join_left top_closed top_higher weak_le_antisym)

lemma top_meet: "x ∈ carrier L ⟹ ⊤ ⊓ x .= x"
  by (metis le_refl meet_closed meet_le meet_right top_closed top_higher
weak_le_antisym)

lemma top_weak_eq:  "⟦ t ∈ carrier L; ⋀ x. x ∈ carrier L ⟹ x ⊑ t
⟧ ⟹ t .= ⊤"
  by (metis top_closed top_higher weak_le_antisym)

end

sublocale weak_bounded_lattice ⊆ weak_partial_order ..
```

3.5 Lattices where `eq` is the Equality

```
locale upper_semilattice = partial_order +
  assumes sup_of_two_exists:
    "[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. least L s (Upper L {x,
y})"

sublocale upper_semilattice ⊆ weak?: weak_upper_semilattice
  by unfold_locales (rule sup_of_two_exists)

locale lower_semilattice = partial_order +
  assumes inf_of_two_exists:
    "[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. greatest L s (Lower
L {x, y})"

sublocale lower_semilattice ⊆ weak?: weak_lower_semilattice
  by unfold_locales (rule inf_of_two_exists)

locale lattice = upper_semilattice + lower_semilattice

sublocale lattice ⊆ weak_lattice ..

lemma (in lattice) dual_lattice:
  "lattice (inv_gorder L)"
proof -
  interpret dual: weak_lattice "inv_gorder L"
    by (metis dual_weak_lattice)

  show ?thesis
    apply (unfold_locales)
    apply (simp_all add: inf_of_two_exists sup_of_two_exists)
    apply (simp add:eq_is_equal)
  done
qed

lemma (in lattice) le_iff_join:
  assumes "x ∈ carrier L" "y ∈ carrier L"
  shows "x ⊑ y ⟷ x = (x ⊓ y)"
  by (simp add: assms(1) assms(2) eq_is_equal weak_le_iff_join)

lemma (in lattice) le_iff_meet:
  assumes "x ∈ carrier L" "y ∈ carrier L"
  shows "x ⊑ y ⟷ (x ⊔ y) = y"
  by (simp add: assms(1) assms(2) eq_is_equal weak_le_iff_meet)
```

Total orders are lattices.

```
sublocale total_order ⊆ weak?: lattice
  by standard (auto intro: weak.weak.sup_of_two_exists weak.weak.inf_of_two_exists)
```

Functions that preserve joins and meets

```
definition join_pres :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
"join_pres X Y f ≡ lattice X ∧ lattice Y ∧ (∀ x ∈ carrier X. ∀ y ∈ carrier
X. f (x ⊔X y) = f x ⊔Y f y)"

definition meet_pres :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
"meet_pres X Y f ≡ lattice X ∧ lattice Y ∧ (∀ x ∈ carrier X. ∀ y ∈ carrier
X. f (x ⊓X y) = f x ⊓Y f y)"

lemma join_pres_isotone:
  assumes "f ∈ carrier X → carrier Y" "join_pres X Y f"
  shows "isotone X Y f"
  using assms
  apply (rule_tac isotoneI)
  apply (auto simp add: join_pres_def lattice.le_iff_meet funcset_carrier)
  using lattice_def partial_order_def upper_semilattice_def apply blast
  using lattice_def partial_order_def upper_semilattice_def apply blast
  apply fastforce
done

lemma meet_pres_isotone:
  assumes "f ∈ carrier X → carrier Y" "meet_pres X Y f"
  shows "isotone X Y f"
  using assms
  apply (rule_tac isotoneI)
  apply (auto simp add: meet_pres_def lattice.le_iff_join funcset_carrier)
  using lattice_def partial_order_def upper_semilattice_def apply blast
  using lattice_def partial_order_def upper_semilattice_def apply blast
  apply fastforce
done
```

3.6 Bounded Lattices

```
locale bounded_lattice =
  lattice +
  weak_partial_order_bottom +
  weak_partial_order_top

sublocale bounded_lattice ⊆ weak_bounded_lattice ..

context bounded_lattice
begin

lemma bottom_eq:
  "⟦ b ∈ carrier L; ⋀ x. x ∈ carrier L ⟹ b ⊑ x ⟧ ⟹ b = ⊥"
  by (metis bottom_closed bottom_lower le_antisym)

lemma top_eq:  "⟦ t ∈ carrier L; ⋀ x. x ∈ carrier L ⟹ x ⊑ t ⟧ ⟹
```

```
t = ⊤"
  by (metis le_antisym top_closed top_higher)

end

end


theory Complete_Lattice
imports Lattice
begin
```

4 Complete Lattices

```
locale weak_complete_lattice = weak_partial_order +
  assumes sup_exists:
    "[| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
    and inf_exists:
    "[| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"

sublocale weak_complete_lattice ⊆ weak_lattice
proof
  fix x y
  assume a: "x ∈ carrier L" "y ∈ carrier L"
  thus "∃s. is_lub L s {x, y}"
    by (rule_tac sup_exists[of "{x, y}"], auto)
  from a show "∃s. is_glb L s {x, y}"
    by (rule_tac inf_exists[of "{x, y}"], auto)
qed
```

Introduction rule: the usual definition of complete lattice

```
lemma (in weak_partial_order) weak_complete_latticeI:
  assumes sup_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
    and inf_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"
  shows "weak_complete_lattice L"
  by standard (auto intro: sup_exists inf_exists)

lemma (in weak_complete_lattice) dual_weak_complete_lattice:
  "weak_complete_lattice (inv_gorder L)"
proof -
  interpret dual: weak_lattice "inv_gorder L"
    by (metis dual_weak_lattice)

  show ?thesis
    apply (unfold_locales)
    apply (simp_all add:inf_exists sup_exists)
  done
```

```
qed

lemma (in weak_complete_lattice) supI:
  "[| !!l. least L l (Upper L A) ==> P l; A ⊆ carrier L |]
  ==> P (⨆A)"
proof (unfold sup_def)
  assume L: "A ⊆ carrier L"
    and P: "!!l. least L l (Upper L A) ==> P l"
  with sup_exists obtain s where "least L s (Upper L A)" by blast
  with L show "P (SOME l. least L l (Upper L A))"
  by (fast intro: someI2 weak_least_unique P)
qed

lemma (in weak_complete_lattice) sup_closed [simp]:
  "A ⊆ carrier L ==> ⨆A ∈ carrier L"
  by (rule supI) simp_all

lemma (in weak_complete_lattice) sup_cong:
  assumes "A ⊆ carrier L" "B ⊆ carrier L" "A {.=} B"
  shows "⨆ A .= ⨆ B"
proof -
  have "⋀ x. is_lub L x A ⟷ is_lub L x B"
    by (rule least_Upper_cong_r, simp_all add: assms)
  moreover have "⨆ B ∈ carrier L"
    by (simp add: assms(2))
  ultimately show ?thesis
    by (simp add: sup_def)
qed

sublocale weak_complete_lattice ⊆ weak_bounded_lattice
  apply (unfold_locales)
  apply (metis Upper_empty empty_subsetI sup_exists)
  apply (metis Lower_empty empty_subsetI inf_exists)
done

lemma (in weak_complete_lattice) infI:
  "[| !!i. greatest L i (Lower L A) ==> P i; A ⊆ carrier L |]
  ==> P (⨅A)"
proof (unfold inf_def)
  assume L: "A ⊆ carrier L"
    and P: "!!l. greatest L l (Lower L A) ==> P l"
  with inf_exists obtain s where "greatest L s (Lower L A)" by blast
  with L show "P (SOME l. greatest L l (Lower L A))"
  by (fast intro: someI2 weak_greatest_unique P)
qed

lemma (in weak_complete_lattice) inf_closed [simp]:
  "A ⊆ carrier L ==> ⨅A ∈ carrier L"
  by (rule infI) simp_all
```

```
lemma (in weak_complete_lattice) inf_cong:
  assumes "A ⊆ carrier L" "B ⊆ carrier L" "A {.=} B"
  shows "⨅ A .= ⨅ B"
proof -
  have "⋀ x. is_glb L x A ⟷ is_glb L x B"
    by (rule greatest_Lower_cong_r, simp_all add: assms)
  moreover have "⨅ B ∈ carrier L"
    by (simp add: assms(2))
  ultimately show ?thesis
    by (simp add: inf_def)
qed

theorem (in weak_partial_order) weak_complete_lattice_criterion1:
  assumes top_exists: "EX g. greatest L g (carrier L)"
    and inf_exists:
      "!!A. [| A ⊆ carrier L; A ~= {} |] ==> EX i. greatest L i (Lower
L A)"
  shows "weak_complete_lattice L"
proof (rule weak_complete_latticeI)
  from top_exists obtain top where top: "greatest L top (carrier L)"
..
  fix A
  assume L: "A ⊆ carrier L"
  let ?B = "Upper L A"
  from L top have "top ∈ ?B" by (fast intro!: Upper_memI intro: greatest_le)
  then have B_non_empty: "?B ~= {}" by fast
  have B_L: "?B ⊆ carrier L" by simp
  from inf_exists [OF B_L B_non_empty]
  obtain b where b_inf_B: "greatest L b (Lower L ?B)" ..
  have "least L b (Upper L A)"
apply (rule least_UpperI)
   apply (rule greatest_le [where A = "Lower L ?B"])
    apply (rule b_inf_B)
   apply (rule Lower_memI)
    apply (erule Upper_memD [THEN conjunct1])
     apply assumption
    apply (rule L)
   apply (fast intro: L [THEN subsetD])
  apply (erule greatest_Lower_below [OF b_inf_B])
  apply simp
 apply (rule L)
apply (rule greatest_closed [OF b_inf_B])
done
  then show "EX s. least L s (Upper L A)" ..
next
  fix A
  assume L: "A ⊆ carrier L"
  show "EX i. greatest L i (Lower L A)"
```

```
  proof (cases "A = {}")
    case True then show ?thesis
      by (simp add: top_exists)
  next
    case False with L show ?thesis
      by (rule inf_exists)
  qed
qed
```

Supremum

```
declare (in partial_order) weak_sup_of_singleton [simp del]

lemma (in partial_order) sup_of_singleton [simp]:
  "x ∈ carrier L ==> ⨆{x} = x"
  using weak_sup_of_singleton unfolding eq_is_equal .

lemma (in upper_semilattice) join_assoc_lemma:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "x ⊔ (y ⊔ z) = ⨆{x, y, z}"
  using weak_join_assoc_lemma L unfolding eq_is_equal .

lemma (in upper_semilattice) join_assoc:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "(x ⊔ y) ⊔ z = x ⊔ (y ⊔ z)"
  using weak_join_assoc L unfolding eq_is_equal .
```

Infimum

```
declare (in partial_order) weak_inf_of_singleton [simp del]

lemma (in partial_order) inf_of_singleton [simp]:
  "x ∈ carrier L ==> ⨅{x} = x"
  using weak_inf_of_singleton unfolding eq_is_equal .
```

Condition on `A`: infimum exists.

```
lemma (in lower_semilattice) meet_assoc_lemma:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "x ⊓ (y ⊓ z) = ⨅{x, y, z}"
  using weak_meet_assoc_lemma L unfolding eq_is_equal .

lemma (in lower_semilattice) meet_assoc:
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "(x ⊓ y) ⊓ z = x ⊓ (y ⊓ z)"
  using weak_meet_assoc L unfolding eq_is_equal .
```

4.1 Infimum Laws

```
context weak_complete_lattice
begin
```

```
lemma inf_glb:
  assumes "A ⊆ carrier L"
  shows "greatest L (⨅A) (Lower L A)"
proof -
  obtain i where "greatest L i (Lower L A)"
    by (metis assms inf_exists)

  thus ?thesis
    apply (simp add: inf_def)
    apply (rule someI2[of _ "i"])
    apply (auto)
  done
qed

lemma inf_lower:
  assumes "A ⊆ carrier L" "x ∈ A"
  shows "⨅A ⊑ x"
  by (metis assms greatest_Lower_below inf_glb)

lemma inf_greatest:
  assumes "A ⊆ carrier L" "z ∈ carrier L"
          "(⋀x. x ∈ A ⟹ z ⊑ x)"
  shows "z ⊑ ⨅A"
  by (metis Lower_memI assms greatest_le inf_glb)

lemma weak_inf_empty [simp]: "⨅{} .= ⊤"
  by (metis Lower_empty empty_subsetI inf_glb top_greatest weak_greatest_unique)

lemma weak_inf_carrier [simp]: "⨅carrier L .= ⊥"
  by (metis bottom_weak_eq inf_closed inf_lower subset_refl)

lemma weak_inf_insert [simp]:
  "⟦ a ∈ carrier L; A ⊆ carrier L ⟧ ⟹ ⨅insert a A .= a ⊓ ⨅A"
  apply (rule weak_le_antisym)
  apply (force intro: meet_le inf_greatest inf_lower inf_closed)
  apply (rule inf_greatest)
  apply (force)
  apply (force intro: inf_closed)
  apply (auto)
  apply (metis inf_closed meet_left)
  apply (force intro: le_trans inf_closed meet_right meet_left inf_lower)
done
```

4.2 Supremum Laws

```
lemma sup_lub:
  assumes "A ⊆ carrier L"
  shows "least L (⨆A) (Upper L A)"
    by (metis Upper_is_closed assms least_closed least_cong supI sup_closed
```

```
sup_exists weak_least_unique)

lemma sup_upper:
  assumes "A ⊆ carrier L" "x ∈ A"
  shows "x ⊑ ⨆A"
  by (metis assms least_Upper_above supI)

lemma sup_least:
  assumes "A ⊆ carrier L" "z ∈ carrier L"
          "(⋀x. x ∈ A ⟹ x ⊑ z)"
  shows "⨆A ⊑ z"
  by (metis Upper_memI assms least_le sup_lub)

lemma weak_sup_empty [simp]: "⨆{} .= ⊥"
  by (metis Upper_empty bottom_least empty_subsetI sup_lub weak_least_unique)

lemma weak_sup_carrier [simp]: "⨆carrier L .= ⊤"
  by (metis Lower_closed Lower_empty sup_closed sup_upper top_closed top_higher
weak_le_antisym)

lemma weak_sup_insert [simp]:
  "⟦ a ∈ carrier L; A ⊆ carrier L ⟧ ⟹ ⨆insert a A .= a ⊔ ⨆A"
  apply (rule weak_le_antisym)
  apply (rule sup_least)
  apply (auto)
  apply (metis join_left sup_closed)
  apply (rule le_trans) defer
  apply (rule join_right)
  apply (auto)
  apply (rule join_le)
  apply (auto intro: sup_upper sup_least sup_closed)
done

end
```

4.3 Fixed points of a lattice

```
definition "fps L f = {x ∈ carrier L. f x .=_L x}"

abbreviation "fpl L f ≡ L⦇carrier := fps L f⦈"

lemma (in weak_partial_order)
  use_fps: "x ∈ fps L f ⟹ f x .= x"
  by (simp add: fps_def)

lemma fps_carrier [simp]:
  "fps L f ⊆ carrier L"
  by (auto simp add: fps_def)
```

```
lemma (in weak_complete_lattice) fps_sup_image:
  assumes "f ∈ carrier L → carrier L" "A ⊆ fps L f"
  shows "⨆ (f ‘ A) .= ⨆ A"
proof -
  from assms(2) have AL: "A ⊆ carrier L"
    by (auto simp add: fps_def)

  show ?thesis
  proof (rule sup_cong, simp_all add: AL)
    from assms(1) AL show "f ‘ A ⊆ carrier L"
      by (auto)
    from assms(2) show "f ‘ A {.=} A"
      apply (auto simp add: fps_def)
      apply (rule set_eqI2)
      apply blast
      apply (rename_tac b)
      apply (rule_tac x="f b" in bexI)
      apply (metis (mono_tags, lifting) Ball_Collect assms(1) Pi_iff local.sym)
      apply (auto)
    done
  qed
qed

lemma (in weak_complete_lattice) fps_idem:
  "⟦ f ∈ carrier L → carrier L; Idem f ⟧ ⟹ fps L f {.=} f ‘ carrier
L"
  apply (rule set_eqI2)
  apply (auto simp add: idempotent_def fps_def)
  apply (metis Pi_iff local.sym)
  apply force
done

context weak_complete_lattice
begin

lemma weak_sup_pre_fixed_point:
  assumes "f ∈ carrier L → carrier L" "isotone L L f" "A ⊆ fps L f"
  shows "(⨆L A) ⊑L f (⨆L A)"
proof (rule sup_least)
  from assms(3) show AL: "A ⊆ carrier L"
    by (auto simp add: fps_def)
  thus fA: "f (⨆A) ∈ carrier L"
    by (simp add: assms funcset_carrier[of f L L])
  fix x
  assume xA: "x ∈ A"
  hence "x ∈ fps L f"
    using assms subsetCE by blast
  hence "f x .=L x"
    by (auto simp add: fps_def)
```

```
  moreover have "f x ⊑L f (⨆LA)"
    by (meson AL assms(2) subsetCE sup_closed sup_upper use_iso1 xA)
  ultimately show "x ⊑L f (⨆LA)"
    by (meson AL fA assms(1) funcset_carrier le_cong local.refl subsetCE
xA)
qed

lemma weak_sup_post_fixed_point:
  assumes "f ∈ carrier L → carrier L" "isotone L L f" "A ⊆ fps L f"
  shows "f (⨅L A) ⊑L (⨅L A)"
proof (rule inf_greatest)
  from assms(3) show AL: "A ⊆ carrier L"
    by (auto simp add: fps_def)
  thus fA: "f (⨅A) ∈ carrier L"
    by (simp add: assms funcset_carrier[of f L L])
  fix x
  assume xA: "x ∈ A"
  hence "x ∈ fps L f"
    using assms subsetCE by blast
  hence "f x .=L x"
    by (auto simp add: fps_def)
  moreover have "f (⨅LA) ⊑L f x"
    by (meson AL assms(2) inf_closed inf_lower subsetCE use_iso1 xA)
  ultimately show "f (⨅LA) ⊑L x"
    by (meson AL assms(1) fA funcset_carrier le_cong_r subsetCE xA)
qed
```

4.3.1 Least fixed points

```
lemma LFP_closed [intro, simp]:
  "LFP f ∈ carrier L"
  by (metis (lifting) LEAST_FP_def inf_closed mem_Collect_eq subsetI)

lemma LFP_lowerbound:
  assumes "x ∈ carrier L" "f x ⊑ x"
  shows "LFP f ⊑ x"
  by (auto intro:inf_lower assms simp add:LEAST_FP_def)

lemma LFP_greatest:
  assumes "x ∈ carrier L"
          "(⋀u. ⟦ u ∈ carrier L; f u ⊑ u ⟧ ⟹ x ⊑ u)"
  shows "x ⊑ LFP f"
  by (auto simp add:LEAST_FP_def intro:inf_greatest assms)

lemma LFP_lemma2:
  assumes "Mono f" "f ∈ carrier L → carrier L"
  shows "f (LFP f) ⊑ LFP f"
  using assms
  apply (auto simp add:Pi_def)
```

```
  apply (rule LFP_greatest)
  apply (metis LFP_closed)
  apply (metis LFP_closed LFP_lowerbound le_trans use_iso1)
done

lemma LFP_lemma3:
  assumes "Mono f" "f ∈ carrier L → carrier L"
  shows "LFP f ⊑ f (LFP f)"
  using assms
  apply (auto simp add:Pi_def)
  apply (metis LFP_closed LFP_lemma2 LFP_lowerbound assms(2) use_iso2)
done

lemma LFP_weak_unfold:
  "⟦ Mono f; f ∈ carrier L → carrier L ⟧ ⟹ LFP f .= f (LFP f)"
  by (auto intro: LFP_lemma2 LFP_lemma3 funcset_mem)

lemma LFP_fixed_point [intro]:
  assumes "Mono f" "f ∈ carrier L → carrier L"
  shows "LFP f ∈ fps L f"
proof -
  have "f (LFP f) ∈ carrier L"
    using assms(2) by blast
  with assms show ?thesis
    by (simp add: LFP_weak_unfold fps_def local.sym)
qed

lemma LFP_least_fixed_point:
  assumes "Mono f" "f ∈ carrier L → carrier L" "x ∈ fps L f"
  shows "LFP f ⊑ x"
  using assms by (force intro: LFP_lowerbound simp add: fps_def)

lemma LFP_idem:
  assumes "f ∈ carrier L → carrier L" "Mono f" "Idem f"
  shows "LFP f .= (f ⊥)"
proof (rule weak_le_antisym)
  from assms(1) show fb: "f ⊥ ∈ carrier L"
    by (rule funcset_mem, simp)
  from assms show mf: "LFP f ∈ carrier L"
    by blast
  show "LFP f ⊑ f ⊥"
  proof -
    have "f (f ⊥) .= f ⊥"
      by (auto simp add: fps_def fb assms(3) idempotent)
    moreover have "f (f ⊥) ∈ carrier L"
      by (rule funcset_mem[of f "carrier L"], simp_all add: assms fb)
    ultimately show ?thesis
      by (auto intro: LFP_lowerbound simp add: fb)
  qed
```

```
  show "f ⊥ ⊑ LFP f"
  proof -
    have "f ⊥ ⊑ f (LFP f)"
      by (auto intro: use_iso1[of _ f] simp add: assms)
    moreover have "... .= LFP f"
      using assms(1) assms(2) fps_def by force
    moreover from assms(1) have "f (LFP f) ∈ carrier L"
      by (auto)
    ultimately show ?thesis
      using fb by blast
  qed
qed
```

4.3.2 Greatest fixed points

```
lemma GFP_closed [intro, simp]:
  "GFP f ∈ carrier L"
  by (auto intro:sup_closed simp add:GREATEST_FP_def)

lemma GFP_upperbound:
  assumes "x ∈ carrier L" "x ⊑ f x"
  shows "x ⊑ GFP f"
  by (auto intro:sup_upper assms simp add:GREATEST_FP_def)

lemma GFP_least:
  assumes "x ∈ carrier L"
          "(⋀u. ⟦ u ∈ carrier L; u ⊑ f u ⟧ ⟹ u ⊑ x)"
  shows "GFP f ⊑ x"
  by (auto simp add:GREATEST_FP_def intro:sup_least assms)

lemma GFP_lemma2:
  assumes "Mono f" "f ∈ carrier L → carrier L"
  shows "GFP f ⊑ f (GFP f)"
  using assms
  apply (auto simp add:Pi_def)
  apply (rule GFP_least)
  apply (metis GFP_closed)
  apply (metis GFP_closed GFP_upperbound le_trans use_iso2)
done

lemma GFP_lemma3:
  assumes "Mono f" "f ∈ carrier L → carrier L"
  shows "f (GFP f) ⊑ GFP f"
  by (metis GFP_closed GFP_lemma2 GFP_upperbound assms funcset_mem use_iso2)

lemma GFP_weak_unfold:
  "⟦ Mono f; f ∈ carrier L → carrier L ⟧ ⟹ GFP f .= f (GFP f)"
  by (auto intro: GFP_lemma2 GFP_lemma3 funcset_mem)
```

```
lemma (in weak_complete_lattice) GFP_fixed_point [intro]:
  assumes "Mono f" "f ∈ carrier L → carrier L"
  shows "GFP f ∈ fps L f"
  using assms
proof -
  have "f (GFP f) ∈ carrier L"
    using assms(2) by blast
  with assms show ?thesis
    by (simp add: GFP_weak_unfold fps_def local.sym)
qed

lemma GFP_greatest_fixed_point:
  assumes "Mono f" "f ∈ carrier L → carrier L" "x ∈ fps L f"
  shows "x ⊑ GFP f"
  using assms
  by (rule_tac GFP_upperbound, auto simp add: fps_def, meson PiE local.sym
weak_refl)

lemma GFP_idem:
  assumes "f ∈ carrier L → carrier L" "Mono f" "Idem f"
  shows "GFP f .= (f ⊤)"
proof (rule weak_le_antisym)
  from assms(1) show fb: "f ⊤ ∈ carrier L"
    by (rule funcset_mem, simp)
  from assms show mf: "GFP f ∈ carrier L"
    by blast
  show "f ⊤ ⊑ GFP f"
  proof -
    have "f (f ⊤) .= f ⊤"
      by (auto simp add: fps_def fb assms(3) idempotent)
    moreover have "f (f ⊤) ∈ carrier L"
      by (rule funcset_mem[of f "carrier L"], simp_all add: assms fb)
    ultimately show ?thesis
      by (rule_tac GFP_upperbound, simp_all add: fb local.sym)
  qed
  show "GFP f ⊑ f ⊤"
  proof -
    have "GFP f ⊑ f (GFP f)"
      by (simp add: GFP_lemma2 assms(1) assms(2))
    moreover have "... ⊑ f ⊤"
      by (auto intro: use_iso1[of _ f] simp add: assms)
    moreover from assms(1) have "f (GFP f) ∈ carrier L"
      by (auto)
    ultimately show ?thesis
      using fb local.le_trans by blast
  qed
qed

end
```

4.4 Complete lattices where `eq` is the Equality

```
locale complete_lattice = partial_order +
  assumes sup_exists:
    "[| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
    and inf_exists:
    "[| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"

sublocale complete_lattice ⊆ lattice
proof
  fix x y
  assume a: "x ∈ carrier L" "y ∈ carrier L"
  thus "∃s. is_lub L s {x, y}"
    by (rule_tac sup_exists[of "{x, y}"], auto)
  from a show "∃s. is_glb L s {x, y}"
    by (rule_tac inf_exists[of "{x, y}"], auto)
qed

sublocale complete_lattice ⊆ weak?: weak_complete_lattice
  by standard (auto intro: sup_exists inf_exists)

lemma complete_lattice_lattice [simp]:
  assumes "complete_lattice X"
  shows "lattice X"
proof -
  interpret c: complete_lattice X
    by (simp add: assms)
  show ?thesis
    by (unfold_locales)
qed
```

Introduction rule: the usual definition of complete lattice

```
lemma (in partial_order) complete_latticeI:
  assumes sup_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
    and inf_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"
  shows "complete_lattice L"
  by standard (auto intro: sup_exists inf_exists)

theorem (in partial_order) complete_lattice_criterion1:
  assumes top_exists: "EX g. greatest L g (carrier L)"
    and inf_exists:
      "!!A. [| A ⊆ carrier L; A ~= {} |] ==> EX i. greatest L i (Lower
L A)"
  shows "complete_lattice L"
proof (rule complete_latticeI)
  from top_exists obtain top where top: "greatest L top (carrier L)"
..
  fix A
```

```
  assume L: "A ⊆ carrier L"
  let ?B = "Upper L A"
  from L top have "top ∈ ?B" by (fast intro!: Upper_memI intro: greatest_le)
  then have B_non_empty: "?B ~= {}" by fast
  have B_L: "?B ⊆ carrier L" by simp
  from inf_exists [OF B_L B_non_empty]
  obtain b where b_inf_B: "greatest L b (Lower L ?B)" ..
  have "least L b (Upper L A)"
apply (rule least_UpperI)
   apply (rule greatest_le [where A = "Lower L ?B"])
    apply (rule b_inf_B)
   apply (rule Lower_memI)
    apply (erule Upper_memD [THEN conjunct1])
     apply assumption
    apply (rule L)
   apply (fast intro: L [THEN subsetD])
  apply (erule greatest_Lower_below [OF b_inf_B])
  apply simp
 apply (rule L)
apply (rule greatest_closed [OF b_inf_B])
done
  then show "EX s. least L s (Upper L A)" ..
next
  fix A
  assume L: "A ⊆ carrier L"
  show "EX i. greatest L i (Lower L A)"
  proof (cases "A = {}")
    case True then show ?thesis
      by (simp add: top_exists)
  next
    case False with L show ?thesis
      by (rule inf_exists)
  qed
qed
```

4.5 Fixed points

```
context complete_lattice
begin

lemma LFP_unfold:
  "⟦ Mono f; f ∈ carrier L → carrier L ⟧ ⟹ LFP f = f (LFP f)"
  using eq_is_equal weak.LFP_weak_unfold by auto

lemma LFP_const:
  "t ∈ carrier L ⟹ LFP (λ x. t) = t"
  by (simp add: local.le_antisym weak.LFP_greatest weak.LFP_lowerbound)

lemma LFP_id:
```

```
  "LFP id = ⊥"
  by (simp add: local.le_antisym weak.LFP_lowerbound)

lemma GFP_unfold:
  "⟦ Mono f; f ∈ carrier L → carrier L ⟧ ⟹ GFP f = f (GFP f)"
  using eq_is_equal weak.GFP_weak_unfold by auto

lemma GFP_const:
  "t ∈ carrier L ⟹ GFP (λ x. t) = t"
  by (simp add: local.le_antisym weak.GFP_least weak.GFP_upperbound)

lemma GFP_id:
  "GFP id = ⊤"
  using weak.GFP_upperbound by auto

end
```

4.6 Interval complete lattices

```
context weak_complete_lattice
begin

  lemma at_least_at_most_Sup:
    "⟦ a ∈ carrier L; b ∈ carrier L; a ⊑ b ⟧ ⟹ ⨆ ⦃a..b⦄ .= b"
    apply (rule weak_le_antisym)
    apply (rule sup_least)
    apply (auto simp add: at_least_at_most_closed)
    apply (rule sup_upper)
    apply (auto simp add: at_least_at_most_closed)
  done

  lemma at_least_at_most_Inf:
    "⟦ a ∈ carrier L; b ∈ carrier L; a ⊑ b ⟧ ⟹ ⨅ ⦃a..b⦄ .= a"
    apply (rule weak_le_antisym)
    apply (rule inf_lower)
    apply (auto simp add: at_least_at_most_closed)
    apply (rule inf_greatest)
    apply (auto simp add: at_least_at_most_closed)
  done

end

lemma weak_complete_lattice_interval:
  assumes "weak_complete_lattice L" "a ∈ carrier L" "b ∈ carrier L" "a
⊑L b"
  shows "weak_complete_lattice (L ⦇ carrier := ⦃a..b⦄L ⦈)"
proof -
  interpret L: weak_complete_lattice L
    by (simp add: assms)
```

```
  interpret weak_partial_order "L ⦇ carrier := ⦃a..b⦄_L ⦈"
  proof -
    have "⦃a..b⦄_L ⊆ carrier L"
      by (auto, simp add: at_least_at_most_def)
    thus "weak_partial_order (L⦇carrier := ⦃a..b⦄_L⦈)"
      by (simp add: L.weak_partial_order_axioms weak_partial_order_subset)
  qed

  show ?thesis
  proof
    fix A
    assume a: "A ⊆ carrier (L⦇carrier := ⦃a..b⦄_L⦈)"
    show "∃s. is_lub (L⦇carrier := ⦃a..b⦄_L⦈) s A"
    proof (cases "A = {}")
      case True
      thus ?thesis
        by (rule_tac x="a" in exI, auto simp add: least_def assms)
    next
      case False
      show ?thesis
      proof (rule_tac x="⨆_L A" in exI, rule least_UpperI, simp_all)
        show b:"⋀ x. x ∈ A ⟹ x ⊑_L ⨆_LA"
          using a by (auto intro: L.sup_upper, meson L.at_least_at_most_closed
L.sup_upper subset_trans)
        show "⋀y. y ∈ Upper (L⦇carrier := ⦃a..b⦄_L⦈) A ⟹ ⨆_LA ⊑_L y"
          using a L.at_least_at_most_closed by (rule_tac L.sup_least,
auto intro: funcset_mem simp add: Upper_def)
        from a show "A ⊆ ⦃a..b⦄_L"
          by (auto)
        from a show "⨆_LA ∈ ⦃a..b⦄_L"
          apply (rule_tac L.at_least_at_most_member)
          apply (auto)
          apply (meson L.at_least_at_most_closed L.sup_closed subset_trans)
          apply (meson False L.at_least_at_most_closed L.at_least_at_most_lower
L.le_trans L.sup_closed b all_not_in_conv assms(2) contra_subsetD subset_trans)
          apply (rule L.sup_least)
          apply (auto simp add: assms)
          using L.at_least_at_most_closed apply blast
        done
      qed
    qed
    show "∃s. is_glb (L⦇carrier := ⦃a..b⦄_L⦈) s A"
    proof (cases "A = {}")
      case True
      thus ?thesis
        by (rule_tac x="b" in exI, auto simp add: greatest_def assms)
    next
      case False
      show ?thesis
```

```
      proof (rule_tac x="⨅L A" in exI, rule greatest_LowerI, simp_all)
        show b:"⋀x. x ∈ A ⟹ ⨅LA ⊑L x"
          using a L.at_least_at_most_closed by (force intro!: L.inf_lower)
        show "⋀y. y ∈ Lower (L⦇carrier := ⦃a..b⦄L⦈) A ⟹ y ⊑L ⨅LA"
           using a L.at_least_at_most_closed by (rule_tac L.inf_greatest,
auto intro: funcset_carrier' simp add: Lower_def)
        from a show "A ⊆ ⦃a..b⦄L"
          by (auto)
        from a show "⨅LA ∈ ⦃a..b⦄L"
          apply (rule_tac L.at_least_at_most_member)
          apply (auto)
          apply (meson L.at_least_at_most_closed L.inf_closed subset_trans)
          apply (meson L.at_least_at_most_closed L.at_least_at_most_lower
L.inf_greatest assms(2) set_rev_mp subset_trans)
          apply (meson False L.at_least_at_most_closed L.at_least_at_most_upper
L.inf_closed L.le_trans b all_not_in_conv assms(3) contra_subsetD subset_trans)

        done
      qed
    qed
  qed
qed
```

4.7 Knaster-Tarski theorem and variants

The set of fixed points of a complete lattice is itself a complete lattice

```
theorem Knaster_Tarski:
  assumes "weak_complete_lattice L" "f ∈ carrier L → carrier L" "isotone
L L f"
  shows "weak_complete_lattice (fpl L f)" (is "weak_complete_lattice ?L'")
proof -
  interpret L: weak_complete_lattice L
    by (simp add: assms)
  interpret weak_partial_order ?L'
  proof -
    have "{x ∈ carrier L. f x .=L x} ⊆ carrier L"
      by (auto)
    thus "weak_partial_order ?L'"
      by (simp add: L.weak_partial_order_axioms weak_partial_order_subset)
  qed
  show ?thesis
  proof (unfold_locales, simp_all)
    fix A
    assume A: "A ⊆ fps L f"
    show "∃s. is_lub (fpl L f) s A"
    proof
      from A have AL: "A ⊆ carrier L"
        by (meson fps_carrier subset_eq)
```

```
      let ?w = "⨆L A"
      have w: "f (⨆LA) ∈ carrier L"
        by (rule funcset_mem[of f "carrier L"], simp_all add: AL assms(2))

      have pf_w: "(⨆L A) ⊑L f (⨆L A)"
        by (simp add: A L.weak_sup_pre_fixed_point assms(2) assms(3))

      have f_top_chain: "f ` ⦃?w..⊤L⦄L ⊆ ⦃?w..⊤L⦄L"
      proof (auto simp add: at_least_at_most_def)
        fix x
        assume b: "x ∈ carrier L" "⨆LA ⊑L x"
        from b show fx: "f x ∈ carrier L"
          using assms(2) by blast
        show "⨆LA ⊑L f x"
        proof -
          have "?w ⊑L f ?w"
          proof (rule_tac L.sup_least, simp_all add: AL w)
            fix y
            assume c: "y ∈ A"
            hence y: "y ∈ fps L f"
              using A subsetCE by blast
            with assms have "y .=L f y"
            proof -
              from y have "y ∈ carrier L"
                by (simp add: fps_def)
              moreover hence "f y ∈ carrier L"
                by (rule_tac funcset_mem[of f "carrier L"], simp_all add:
assms)
              ultimately show ?thesis using y
                by (rule_tac L.sym, simp_all add: L.use_fps)
            qed
            moreover have "y ⊑L ⨆LA"
              by (simp add: AL L.sup_upper c(1))
            ultimately show "y ⊑L f (⨆LA)"
              by (meson fps_def AL funcset_mem L.refl L.weak_complete_lattice_axioms
assms(2) assms(3) c(1) isotone_def rev_subsetD weak_complete_lattice.sup_closed
weak_partial_order.le_cong)
          qed
          thus ?thesis
            by (meson AL funcset_mem L.le_trans L.sup_closed assms(2)
assms(3) b(1) b(2) use_iso2)
        qed

        show "f x ⊑L ⊤L"
          by (simp add: fx)
      qed

      let ?L' = "L⦇ carrier := ⦃?w..⊤L⦄L ⦈"
```

```
      interpret L’: weak_complete_lattice ?L’
        by (auto intro: weak_complete_lattice_interval simp add: L.weak_complete_lattice_ax
AL)

      let ?L’’ = "L⦇ carrier := fps L f ⦈"

      show "is_lub ?L’’ (LFP?L’ f) A"
      proof (rule least_UpperI, simp_all)
        fix x
        assume "x ∈ Upper ?L’’ A"
        hence "LFP?L’ f ⊑?L’ x"
          apply (rule_tac L’.LFP_lowerbound)
          apply (auto simp add: Upper_def)
          apply (simp add: A AL L.at_least_at_most_member L.sup_least
set_rev_mp)
          apply (simp add: Pi_iff assms(2) fps_def, rule_tac L.weak_refl)
          apply (auto)
          apply (rule funcset_mem[of f "carrier L"], simp_all add: assms(2))
        done
        thus " LFP?L’ f ⊑L x"
          by (simp)
      next
        fix x
        assume xA: "x ∈ A"
        show "x ⊑L LFP?L’ f"
        proof -
          have "LFP?L’ f ∈ carrier ?L’"
            by blast
          thus ?thesis
            by (simp, meson AL L.at_least_at_most_closed L.at_least_at_most_lower
L.le_trans L.sup_closed L.sup_upper xA subsetCE)
        qed
      next
        show "A ⊆ fps L f"
          by (simp add: A)
      next
        show "LFP?L’ f ∈ fps L f"
        proof (auto simp add: fps_def)
          have "LFP?L’ f ∈ carrier ?L’"
            by (rule L’.LFP_closed)
          thus c:"LFP?L’ f ∈ carrier L"
             by (auto simp add: at_least_at_most_def)
          have "LFP?L’ f .=?L’ f (LFP?L’ f)"
          proof (rule "L’.LFP_weak_unfold", simp_all)
            show "f ∈ ⦃⨆LA..⊤L⦄L → ⦃⨆LA..⊤L⦄L"
              apply (auto simp add: Pi_def at_least_at_most_def)
              using assms(2) apply blast
              apply (meson AL funcset_mem L.le_trans L.sup_closed assms(2)
assms(3) pf_w use_iso2)
```

```
              using assms(2) apply blast
            done
            from assms(3) show "Mono_L⦇carrier := ⦃⨆_LA..⊤_L⦄_L⦈ f"
              apply (auto simp add: isotone_def)
              using L'.weak_partial_order_axioms apply blast
              apply (meson L.at_least_at_most_closed subsetCE)
            done
          qed
          thus "f (LFP_?L' f) .=_L LFP_?L' f"
            by (simp add: L.equivalence_axioms funcset_carrier' c assms(2)
equivalence.sym)
        qed
      qed
    qed
    show "∃i. is_glb (L⦇carrier := fps L f⦈) i A"
    proof
      from A have AL: "A ⊆ carrier L"
        by (meson fps_carrier subset_eq)

      let ?w = "⨅_L A"
      have w: "f (⨅_LA) ∈ carrier L"
        by (simp add: AL funcset_carrier' assms(2))

      have pf_w: "f (⨅_L A) ⊑_L (⨅_L A)"
        by (simp add: A L.weak_sup_post_fixed_point assms(2) assms(3))

      have f_bot_chain: "f ` ⦃⊥_L..?w⦄_L ⊆ ⦃⊥_L..?w⦄_L"
      proof (auto simp add: at_least_at_most_def)
        fix x
        assume b: "x ∈ carrier L" "x ⊑_L ⨅_LA"
        from b show fx: "f x ∈ carrier L"
          using assms(2) by blast
        show "f x ⊑_L ⨅_LA"
        proof -
          have "f ?w ⊑_L ?w"
          proof (rule_tac L.inf_greatest, simp_all add: AL w)
            fix y
            assume c: "y ∈ A"
            with assms have "y .=_L f y"
              by (metis (no_types, lifting) A funcset_carrier'[OF assms(2)]
L.sym fps_def mem_Collect_eq subset_eq)
            moreover have "⨅_LA ⊑_L y"
              by (simp add: AL L.inf_lower c)
            ultimately show "f (⨅_LA) ⊑_L y"
              by (meson AL L.inf_closed L.le_trans c pf_w set_rev_mp w)
          qed
          thus ?thesis
            by (meson AL L.inf_closed L.le_trans assms(3) b(1) b(2) fx
use_iso2 w)
```

```
        qed

        show "⊥L ⊑L f x"
          by (simp add: fx)
      qed

      let ?L’ = "L⦇ carrier := ⦃⊥L..?w⦄L ⦈"

      interpret L’: weak_complete_lattice ?L’
        by (auto intro!: weak_complete_lattice_interval simp add: L.weak_complete_lattice_a
AL)

      let ?L’’ = "L⦇ carrier := fps L f ⦈"

      show "is_glb ?L’’ (GFP?L’ f) A"
      proof (rule greatest_LowerI, simp_all)
        fix x
        assume "x ∈ Lower ?L’’ A"
        hence "x ⊑?L’ GFP?L’ f"
          apply (rule_tac L’.GFP_upperbound)
          apply (auto simp add: Lower_def)
          apply (meson A AL L.at_least_at_most_member L.bottom_lower L.weak_complete_lattic
fps_carrier subsetCE weak_complete_lattice.inf_greatest)
          apply (simp add: funcset_carrier’ L.sym assms(2) fps_def)
        done
        thus "x ⊑L GFP?L’ f"
          by (simp)
      next
        fix x
        assume xA: "x ∈ A"
        show "GFP?L’ f ⊑L x"
        proof -
          have "GFP?L’ f ∈ carrier ?L’"
            by blast
          thus ?thesis
            by (simp, meson AL L.at_least_at_most_closed L.at_least_at_most_upper
L.inf_closed L.inf_lower L.le_trans subsetCE xA)
        qed
      next
        show "A ⊆ fps L f"
          by (simp add: A)
      next
        show "GFP?L’ f ∈ fps L f"
        proof (auto simp add: fps_def)
          have "GFP?L’ f ∈ carrier ?L’"
            by (rule L’.GFP_closed)
          thus c:"GFP?L’ f ∈ carrier L"
             by (auto simp add: at_least_at_most_def)
          have "GFP?L’ f .=?L’ f (GFP?L’ f)"
```

```
          proof (rule "L'.GFP_weak_unfold", simp_all)
            show "f ∈ ⦃⊥_L..?w⦄_L → ⦃⊥_L..?w⦄_L"
              apply (auto simp add: Pi_def at_least_at_most_def)
              using assms(2) apply blast
              apply (simp add: funcset_carrier' assms(2))
              apply (meson AL funcset_carrier L.inf_closed L.le_trans
assms(2) assms(3) pf_w use_iso2)
            done
            from assms(3) show "Mono_L⦇carrier := ⦃⊥_L..?w⦄_L⦈ f"
              apply (auto simp add: isotone_def)
              using L'.weak_partial_order_axioms apply blast
              using L.at_least_at_most_closed apply (blast intro: funcset_carrier')
            done
          qed
          thus "f (GFP_?L' f) .=_L GFP_?L' f"
            by (simp add: L.equivalence_axioms funcset_carrier' c assms(2)
equivalence.sym)
        qed
      qed
    qed
  qed
qed

theorem Knaster_Tarski_top:
  assumes "weak_complete_lattice L" "isotone L L f" "f ∈ carrier L →
carrier L"
  shows "⊤_fpl L f .=_L GFP_L f"
proof -
  interpret L: weak_complete_lattice L
    by (simp add: assms)
  interpret L': weak_complete_lattice "fpl L f"
    by (rule Knaster_Tarski, simp_all add: assms)
  show ?thesis
  proof (rule L.weak_le_antisym, simp_all)
    show "⊤_fpl L f ⊑_L GFP_L f"
      by (rule L.GFP_greatest_fixed_point, simp_all add: assms L'.top_closed[simplified])
    show "GFP_L f ⊑_L ⊤_fpl L f"
    proof -
      have "GFP_L f ∈ fps L f"
        by (rule L.GFP_fixed_point, simp_all add: assms)
      hence "GFP_L f ∈ carrier (fpl L f)"
        by simp
      hence "GFP_L f ⊑_fpl L f ⊤_fpl L f"
        by (rule L'.top_higher)
      thus ?thesis
        by simp
    qed
    show "⊤_fpl L f ∈ carrier L"
    proof -
```

```
      have "carrier (fpl L f) ⊆ carrier L"
        by (auto simp add: fps_def)
      with L'.top_closed show ?thesis
        by blast
    qed
  qed
qed

theorem Knaster_Tarski_bottom:
  assumes "weak_complete_lattice L" "isotone L L f" "f ∈ carrier L →
carrier L"
  shows "⊥⇘fpl L f⇙ .=⇘L⇙ LFP⇘L⇙ f"
proof -
  interpret L: weak_complete_lattice L
    by (simp add: assms)
  interpret L': weak_complete_lattice "fpl L f"
    by (rule Knaster_Tarski, simp_all add: assms)
  show ?thesis
  proof (rule L.weak_le_antisym, simp_all)
    show "LFP⇘L⇙ f ⊑⇘L⇙ ⊥⇘fpl L f⇙"
      by (rule L.LFP_least_fixed_point, simp_all add: assms L'.bottom_closed[simplified])
    show "⊥⇘fpl L f⇙ ⊑⇘L⇙ LFP⇘L⇙ f"
    proof -
      have "LFP⇘L⇙ f ∈ fps L f"
        by (rule L.LFP_fixed_point, simp_all add: assms)
      hence "LFP⇘L⇙ f ∈ carrier (fpl L f)"
        by simp
      hence "⊥⇘fpl L f⇙ ⊑⇘fpl L f⇙ LFP⇘L⇙ f"
        by (rule L'.bottom_lower)
      thus ?thesis
        by simp
    qed
    show "⊥⇘fpl L f⇙ ∈ carrier L"
    proof -
      have "carrier (fpl L f) ⊆ carrier L"
        by (auto simp add: fps_def)
      with L'.bottom_closed show ?thesis
        by blast
    qed
  qed
qed
```

If a function is both idempotent and isotone then the image of the function forms a complete lattice

```
theorem Knaster_Tarski_idem:
  assumes "complete_lattice L" "f ∈ carrier L → carrier L" "isotone
L L f" "idempotent L f"
  shows "complete_lattice (L⦇carrier := f ` carrier L⦈)"
proof -
```

```
  interpret L: complete_lattice L
    by (simp add: assms)
  have "fps L f = f ‘ carrier L"
    using L.weak.fps_idem[OF assms(2) assms(4)]
    by (simp add: L.set_eq_is_eq)
  then interpret L’: weak_complete_lattice "(L⦇carrier := f ‘ carrier
L⦈)"
    by (metis Knaster_Tarski L.weak.weak_complete_lattice_axioms assms(2)
assms(3))
  show ?thesis
    using L’.sup_exists L’.inf_exists
    by (unfold_locales, auto simp add: L.eq_is_equal)
qed

theorem Knaster_Tarski_idem_extremes:
  assumes "weak_complete_lattice L" "isotone L L f" "idempotent L f"
"f ∈ carrier L → carrier L"
  shows "⊤fpl L f .=L f (⊤L)" "⊥fpl L f .=L f (⊥L)"
proof -
  interpret L: weak_complete_lattice "L"
    by (simp_all add: assms)
  interpret L’: weak_complete_lattice "fpl L f"
    by (rule Knaster_Tarski, simp_all add: assms)
  have FA: "fps L f ⊆ carrier L"
    by (auto simp add: fps_def)
  show "⊤fpl L f .=L f (⊤L)"
  proof -
    from FA have "⊤fpl L f ∈ carrier L"
    proof -
      have "⊤fpl L f ∈ fps L f"
        using L’.top_closed by auto
      thus ?thesis
        using FA by blast
    qed
    moreover with assms have "f ⊤L ∈ carrier L"
      by (auto)

    ultimately show ?thesis
      using L.trans[OF Knaster_Tarski_top[of L f] L.GFP_idem[of f]]
      by (simp_all add: assms)
  qed
  show "⊥fpl L f .=L f (⊥L)"
  proof -
    from FA have "⊥fpl L f ∈ carrier L"
    proof -
      have "⊥fpl L f ∈ fps L f"
        using L’.bottom_closed by auto
      thus ?thesis
        using FA by blast
```

```
    qed
    moreover with assms have "f ⊥_{L} ∈ carrier L"
      by (auto)

    ultimately show ?thesis
      using L.trans[OF Knaster_Tarski_bottom[of L f] L.LFP_idem[of f]]
      by (simp_all add: assms)
  qed
qed

theorem Knaster_Tarski_idem_inf_eq:
  assumes "weak_complete_lattice L" "isotone L L f" "idempotent L f"
"f ∈ carrier L → carrier L"
          "A ⊆ fps L f"
  shows "⨅_{fpl L f} A .=_{L} f (⨅_{L} A)"
proof -
  interpret L: weak_complete_lattice "L"
    by (simp_all add: assms)
  interpret L': weak_complete_lattice "fpl L f"
    by (rule Knaster_Tarski, simp_all add: assms)
  have FA: "fps L f ⊆ carrier L"
    by (auto simp add: fps_def)
  have A: "A ⊆ carrier L"
    using FA assms(5) by blast
  have fA: "f (⨅_{L}A) ∈ fps L f"
    by (metis (no_types, lifting) A L.idempotent L.inf_closed PiE assms(3)
assms(4) fps_def mem_Collect_eq)
  have infA: "⨅_{fpl L f}^{A} ∈ fps L f"
    by (rule L'.inf_closed[simplified], simp add: assms)
  show ?thesis
  proof (rule L.weak_le_antisym)
    show ic: "⨅_{fpl L f}^{A} ∈ carrier L"
      using FA infA by blast
    show fc: "f (⨅_{L}A) ∈ carrier L"
      using FA fA by blast
    show "f (⨅_{L}A) ⊑_{L} ⨅_{fpl L f}^{A}"
    proof -
      have "⋀x. x ∈ A ⟹ f (⨅_{L}A) ⊑_{L} x"
        by (meson A FA L.inf_closed L.inf_lower L.le_trans L.weak_sup_post_fixed_point
assms(2) assms(4) assms(5) fA subsetCE)
      hence "f (⨅_{L}A) ⊑_{fpl L f} ⨅_{fpl L f}^{A}"
        by (rule_tac L'.inf_greatest, simp_all add: fA assms(3,5))
      thus ?thesis
        by (simp)
    qed
    show "⨅_{fpl L f}^{A} ⊑_{L} f (⨅_{L}A)"
    proof -
      have "⋀x. x ∈ A ⟹ ⨅_{fpl L f}^{A} ⊑_{fpl L f} x"
        by (rule L'.inf_lower, simp_all add: assms)
```

```
   hence "⨅fpl L fA ⊑L (⨅LA)"
    apply (rule_tac L.inf_greatest, simp_all add: A)
    using FA infA apply blast
    done
   hence 1:"f(⨅fpl L fA) ⊑L f(⨅LA)"
    by (metis (no_types, lifting) A FA L.inf_closed assms(2) infA
subsetCE use_iso1)
   have 2:"⨅fpl L fA ⊑L f (⨅fpl L fA)"
    by (metis (no_types, lifting) FA L.sym L.use_fps L.weak_complete_lattice_axioms
PiE assms(4) infA subsetCE weak_complete_lattice_def weak_partial_order.weak_refl)

   show ?thesis
    using FA fA infA by (auto intro!: L.le_trans[OF 2 1] ic fc, metis
FA PiE assms(4) subsetCE)
  qed
 qed
qed
```

4.8 Examples

4.8.1 The Powerset of a Set is a Complete Lattice

```
theorem powerset_is_complete_lattice:
 "complete_lattice ⦇carrier = Pow A, eq = op =, le = op ⊆⦈"
 (is "complete_lattice ?L")
proof (rule partial_order.complete_latticeI)
 show "partial_order ?L"
  by standard auto
next
 fix B
 assume "B ⊆ carrier ?L"
 then have "least ?L (⋃ B) (Upper ?L B)"
  by (fastforce intro!: least_UpperI simp: Upper_def)
 then show "EX s. least ?L s (Upper ?L B)" ..
next
 fix B
 assume "B ⊆ carrier ?L"
 then have "greatest ?L (⋂ B ∩ A) (Lower ?L B)"
```

`⋂B` is not the infimum of `B`: `⋂{} = UNIV` which is in general bigger than `A`!

```
  by (fastforce intro!: greatest_LowerI simp: Lower_def)
 then show "EX i. greatest ?L i (Lower ?L B)" ..
qed
```

Another example, that of the lattice of subgroups of a group, can be found in Group theory (Section 6.8).

4.9 Limit preserving functions

```
definition weak_sup_pres :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
"weak_sup_pres X Y f ≡ complete_lattice X ∧ complete_lattice Y ∧ (∀ A
⊆ carrier X. A ≠ {} ⟶ f (⨆X A) = (⨆Y (f ‘ A)))"

definition sup_pres :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
"sup_pres X Y f ≡ complete_lattice X ∧ complete_lattice Y ∧ (∀ A ⊆ carrier
X. f (⨆X A) = (⨆Y (f ‘ A)))"

definition weak_inf_pres :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
"weak_inf_pres X Y f ≡ complete_lattice X ∧ complete_lattice Y ∧ (∀ A
⊆ carrier X. A ≠ {} ⟶ f (⨅X A) = (⨅Y (f ‘ A)))"

definition inf_pres :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
"inf_pres X Y f ≡ complete_lattice X ∧ complete_lattice Y ∧ (∀ A ⊆ carrier
X. f (⨅X A) = (⨅Y (f ‘ A)))"

lemma weak_sup_pres:
  "sup_pres X Y f ⟹ weak_sup_pres X Y f"
  by (simp add: sup_pres_def weak_sup_pres_def)

lemma weak_inf_pres:
  "inf_pres X Y f ⟹ weak_inf_pres X Y f"
  by (simp add: inf_pres_def weak_inf_pres_def)

lemma sup_pres_is_join_pres:
  assumes "weak_sup_pres X Y f"
  shows "join_pres X Y f"
  using assms
  apply (simp add: join_pres_def weak_sup_pres_def, safe)
  apply (rename_tac x y)
  apply (drule_tac x="{x, y}" in spec)
  apply (auto simp add: join_def)
done

lemma inf_pres_is_meet_pres:
  assumes "weak_inf_pres X Y f"
  shows "meet_pres X Y f"
  using assms
  apply (simp add: meet_pres_def weak_inf_pres_def, safe)
  apply (rename_tac x y)
  apply (drule_tac x="{x, y}" in spec)
  apply (auto simp add: meet_def)
done
```

```
end

theory Galois_Connection
  imports Complete_Lattice
begin
```

5 Galois connections

5.1 Definition and basic properties

```
record ('a, 'b, 'c, 'd) galcon =
  orderA :: "('a, 'c) gorder_scheme" ("𝒳ı")
  orderB :: "('b, 'd) gorder_scheme" ("𝒴ı")
  lower  :: "'a ⇒ 'b" ("π*ı")
  upper  :: "'b ⇒ 'a" ("π⁎ı")

type_synonym ('a, 'b) galois = "('a, 'b, unit, unit) galcon"

abbreviation "inv_galcon G ≡ ⦇ orderA = inv_gorder 𝒴G, orderB = inv_gorder
𝒳G, lower = upper G, upper = lower G ⦈"

definition comp_galcon :: "('b, 'c) galois ⇒ ('a, 'b) galois ⇒ ('a, 'c)
galois" (infixr "∘g" 85)
  where "G ∘g F = ⦇ orderA = orderA F, orderB = orderB G, lower = lower
G ∘ lower F, upper = upper F ∘ upper G ⦈"

definition id_galcon :: "'a gorder ⇒ ('a, 'a) galois" ("Ig") where
"Ig(A) = ⦇ orderA = A, orderB = A, lower = id, upper = id ⦈"
```

5.2 Well-typed connections

```
locale connection =
  fixes G (structure)
  assumes is_order_A: "partial_order 𝒳"
  and is_order_B: "partial_order 𝒴"
  and lower_closure: "π* ∈ carrier 𝒳 → carrier 𝒴"
  and upper_closure: "π⁎ ∈ carrier 𝒴 → carrier 𝒳"
begin

  lemma lower_closed: "x ∈ carrier 𝒳 ⟹ π* x ∈ carrier 𝒴"
    using lower_closure by auto

  lemma upper_closed: "y ∈ carrier 𝒴 ⟹ π⁎ y ∈ carrier 𝒳"
    using upper_closure by auto

end
```

5.3 Galois connections

```
locale galois_connection = connection +
  assumes galois_property: "⟦x ∈ carrier 𝒳; y ∈ carrier 𝒴⟧ ⟹ π^* x
⊑_𝒴 y ⟷ x ⊑_𝒳 π_* y"
begin

  lemma is_weak_order_A: "weak_partial_order 𝒳"
  proof -
    interpret po: partial_order 𝒳
      by (metis is_order_A)
    show ?thesis ..
  qed

  lemma is_weak_order_B: "weak_partial_order 𝒴"
  proof -
    interpret po: partial_order 𝒴
      by (metis is_order_B)
    show ?thesis ..
  qed

  lemma right: "⟦x ∈ carrier 𝒳; y ∈ carrier 𝒴; π^* x ⊑_𝒴 y⟧ ⟹ x ⊑_𝒳
π_* y"
    by (metis galois_property)

  lemma left: "⟦x ∈ carrier 𝒳; y ∈ carrier 𝒴; x ⊑_𝒳 π_* y⟧ ⟹ π^* x
⊑_𝒴 y"
    by (metis galois_property)

  lemma deflation: "y ∈ carrier 𝒴 ⟹ π^* (π_* y) ⊑_𝒴 y"
    by (metis Pi_iff is_weak_order_A left upper_closure weak_partial_order.le_refl)

  lemma inflation: "x ∈ carrier 𝒳 ⟹ x ⊑_𝒳 π_* (π^* x)"
    by (metis (no_types, lifting) PiE galois_connection.right galois_connection_axioms
is_weak_order_B lower_closure weak_partial_order.le_refl)

  lemma lower_iso: "isotone 𝒳 𝒴 π^*"
  proof (auto simp add:isotone_def)
    show "weak_partial_order 𝒳"
      by (metis is_weak_order_A)
    show "weak_partial_order 𝒴"
      by (metis is_weak_order_B)
    fix x y
    assume a: "x ∈ carrier 𝒳" "y ∈ carrier 𝒳" "x ⊑_𝒳 y"
    have b: "π^* y ∈ carrier 𝒴"
      using a(2) lower_closure by blast
    then have "π_* (π^* y) ∈ carrier 𝒳"
      using upper_closure by blast
    then have "x ⊑_𝒳 π_* (π^* y)"
      by (meson a inflation is_weak_order_A weak_partial_order.le_trans)
```

```
    thus "π^* x ⊑_𝒴 π^* y"
      by (meson b a(1) Pi_iff galois_property lower_closure upper_closure)
  qed

  lemma upper_iso: "isotone 𝒴 𝒳 π_*"
    apply (auto simp add:isotone_def)
    apply (metis is_weak_order_B)
    apply (metis is_weak_order_A)
    apply (metis (no_types, lifting) Pi_mem deflation is_weak_order_B
lower_closure right upper_closure weak_partial_order.le_trans)
  done

  lemma lower_comp: "x ∈ carrier 𝒳 ⟹ π^* (π_* (π^* x)) = π^* x"
    by (meson deflation funcset_mem inflation is_order_B lower_closure
lower_iso partial_order.le_antisym upper_closure use_iso2)

  lemma lower_comp': "x ∈ carrier 𝒳 ⟹ (π^* ∘ π_* ∘ π^*) x = π^* x"
    by (simp add: lower_comp)

  lemma upper_comp: "y ∈ carrier 𝒴 ⟹ π_* (π^* (π_* y)) = π_* y"
  proof -
    assume a1: "y ∈ carrier 𝒴"
    hence f1: "π_* y ∈ carrier 𝒳" using upper_closure by blast
    have f2: "π^* (π_* y) ⊑_𝒴 y" using a1 deflation by blast
    have f3: "π_* (π^* (π_* y)) ∈ carrier 𝒳"
      using f1 lower_closure upper_closure by auto
    have "π^* (π_* y) ∈ carrier 𝒴" using f1 lower_closure by blast
    thus "π_* (π^* (π_* y)) = π_* y"
      by (meson a1 f1 f2 f3 inflation is_order_A partial_order.le_antisym
upper_iso use_iso2)
  qed

  lemma upper_comp': "y ∈ carrier 𝒴 ⟹ (π_* ∘ π^* ∘ π_*) y = π_* y"
    by (simp add: upper_comp)

  lemma adjoint_idem1: "idempotent 𝒴 (π^* ∘ π_*)"
    by (simp add: idempotent_def is_order_B partial_order.eq_is_equal
upper_comp)

  lemma adjoint_idem2: "idempotent 𝒳 (π_* ∘ π^*)"
    by (simp add: idempotent_def is_order_A partial_order.eq_is_equal
lower_comp)

  lemma fg_iso: "isotone 𝒴 𝒴 (π^* ∘ π_*)"
    by (metis iso_compose lower_closure lower_iso upper_closure upper_iso)

  lemma gf_iso: "isotone 𝒳 𝒳 (π_* ∘ π^*)"
    by (metis iso_compose lower_closure lower_iso upper_closure upper_iso)
```

```
lemma semi_inverse1: "x ∈ carrier 𝒳 ⟹ π^* x = π^* (π_* (π^* x))"
  by (metis lower_comp)

lemma semi_inverse2: "x ∈ carrier 𝒴 ⟹ π_* x = π_* (π^* (π_* x))"
  by (metis upper_comp)

theorem lower_by_complete_lattice:
  assumes "complete_lattice 𝒴" "x ∈ carrier 𝒳"
  shows "π^*(x) = ⨅_𝒴 { y ∈ carrier 𝒴. x ⊑_𝒳 π_*(y) }"
proof -
  interpret Y: complete_lattice 𝒴
    by (simp add: assms)

  show ?thesis
  proof (rule Y.le_antisym)
    show x: "π^* x ∈ carrier 𝒴"
      using assms(2) lower_closure by blast
    show "π^* x ⊑_𝒴 ⨅_𝒴{y ∈ carrier 𝒴. x ⊑_𝒳 π_* y}"
    proof (rule Y.weak.inf_greatest)
      show "{y ∈ carrier 𝒴. x ⊑_𝒳 π_* y} ⊆ carrier 𝒴"
        by auto
      show "π^* x ∈ carrier 𝒴" by (fact x)
      fix z
      assume "z ∈ {y ∈ carrier 𝒴. x ⊑_𝒳 π_* y}"
      thus "π^* x ⊑_𝒴 z"
        using assms(2) left by auto
    qed
    show "⨅_𝒴{y ∈ carrier 𝒴. x ⊑_𝒳 π_* y} ⊑_𝒴 π^* x"
    proof (rule Y.weak.inf_lower)
      show "{y ∈ carrier 𝒴. x ⊑_𝒳 π_* y} ⊆ carrier 𝒴"
        by auto
      show "π^* x ∈ {y ∈ carrier 𝒴. x ⊑_𝒳 π_* y}"
      proof (auto)
        show "π^* x ∈ carrier 𝒴" by (fact x)
        show "x ⊑_𝒳 π_* (π^* x)"
          using assms(2) inflation by blast
      qed
    qed
    show "⨅_𝒴{y ∈ carrier 𝒴. x ⊑_𝒳 π_* y} ∈ carrier 𝒴"
     by (auto intro: Y.weak.inf_closed)
  qed
qed

theorem upper_by_complete_lattice:
  assumes "complete_lattice 𝒳" "y ∈ carrier 𝒴"
  shows "π_*(y) = ⨆_𝒳 { x ∈ carrier 𝒳. π^*(x) ⊑_𝒴 y }"
proof -
  interpret X: complete_lattice 𝒳
    by (simp add: assms)
```

```
    show ?thesis
    proof (rule X.le_antisym)
      show y: "π_* y ∈ carrier 𝒳"
        using assms(2) upper_closure by blast
      show "π_* y ⊑_𝒳 ⨆_𝒳{x ∈ carrier 𝒳. π^* x ⊑_𝒴 y}"
      proof (rule X.weak.sup_upper)
        show "{x ∈ carrier 𝒳. π^* x ⊑_𝒴 y} ⊆ carrier 𝒳"
          by auto
        show "π_* y ∈ {x ∈ carrier 𝒳. π^* x ⊑_𝒴 y}"
        proof (auto)
          show "π_* y ∈ carrier 𝒳" by (fact y)
          show "π^* (π_* y) ⊑_𝒴 y"
            by (simp add: assms(2) deflation)
        qed
      qed
      show "⨆_𝒳{x ∈ carrier 𝒳. π^* x ⊑_𝒴 y} ⊑_𝒳 π_* y"
      proof (rule X.weak.sup_least)
        show "{x ∈ carrier 𝒳. π^* x ⊑_𝒴 y} ⊆ carrier 𝒳"
          by auto
        show "π_* y ∈ carrier 𝒳" by (fact y)
        fix z
        assume "z ∈ {x ∈ carrier 𝒳. π^* x ⊑_𝒴 y}"
        thus "z ⊑_𝒳 π_* y"
          by (simp add: assms(2) right)
      qed
      show "⨆_𝒳{x ∈ carrier 𝒳. π^* x ⊑_𝒴 y} ∈ carrier 𝒳"
       by (auto intro: X.weak.sup_closed)
    qed
  qed

end

lemma dual_galois [simp]: " galois_connection ⦇ orderA = inv_gorder B,
orderB = inv_gorder A, lower = f, upper = g ⦈
                          = galois_connection ⦇ orderA = A, orderB = B,
lower = g, upper = f ⦈"
  by (auto simp add: galois_connection_def galois_connection_axioms_def
connection_def dual_order_iff)

definition lower_adjoint :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('a ⇒ 'b) ⇒ bool" where
  "lower_adjoint A B f ≡ ∃g. galois_connection ⦇ orderA = A, orderB =
B, lower = f, upper = g ⦈"

definition upper_adjoint :: "('a, 'c) gorder_scheme ⇒ ('b, 'd) gorder_scheme
⇒ ('b ⇒ 'a) ⇒ bool" where
  "upper_adjoint A B g ≡ ∃f. galois_connection ⦇ orderA = A, orderB =
B, lower = f, upper = g ⦈"
```

```
lemma lower_adjoint_dual [simp]: "lower_adjoint (inv_gorder A) (inv_gorder
B) f = upper_adjoint B A f"
  by (simp add: lower_adjoint_def upper_adjoint_def)

lemma upper_adjoint_dual [simp]: "upper_adjoint (inv_gorder A) (inv_gorder
B) f = lower_adjoint B A f"
  by (simp add: lower_adjoint_def upper_adjoint_def)

lemma lower_type: "lower_adjoint A B f ⟹ f ∈ carrier A → carrier B"
  by (auto simp add:lower_adjoint_def galois_connection_def galois_connection_axioms_def
connection_def)

lemma upper_type: "upper_adjoint A B g ⟹ g ∈ carrier B → carrier A"
  by (auto simp add:upper_adjoint_def galois_connection_def galois_connection_axioms_def
connection_def)
```

5.4 Composition of Galois connections

```
lemma id_galois: "partial_order A ⟹ galois_connection (I_g(A))"
  by (simp add: id_galcon_def galois_connection_def galois_connection_axioms_def
connection_def)

lemma comp_galcon_closed:
  assumes "galois_connection G" "galois_connection F" "𝒴_F = 𝒳_G"
  shows "galois_connection (G ∘_g F)"
proof -
  interpret F: galois_connection F
    by (simp add: assms)
  interpret G: galois_connection G
    by (simp add: assms)

  have "partial_order 𝒳_G ∘g F"
    by (simp add: F.is_order_A comp_galcon_def)
  moreover have "partial_order 𝒴_G ∘g F"
    by (simp add: G.is_order_B comp_galcon_def)
  moreover have "π*_G ∘ π*_F ∈ carrier 𝒳_F → carrier 𝒴_G"
    using F.lower_closure G.lower_closure assms(3) by auto
  moreover have "π⁎_F ∘ π⁎_G ∈ carrier 𝒴_G → carrier 𝒳_F"
    using F.upper_closure G.upper_closure assms(3) by auto
  moreover
  have "⋀ x y. ⟦x ∈ carrier 𝒳_F; y ∈ carrier 𝒴_G ⟧ ⟹
              (π*_G (π*_F x) ⊑_𝒴_G y) = (x ⊑_𝒳_F π⁎_F (π⁎_G y))"
    by (metis F.galois_property F.lower_closure G.galois_property G.upper_closure
assms(3) Pi_iff)
  ultimately show ?thesis
    by (simp add: comp_galcon_def galois_connection_def galois_connection_axioms_def
connection_def)
qed
```

```
lemma comp_galcon_right_unit [simp]: "F ∘g Ig(𝒳F) = F"
  by (simp add: comp_galcon_def id_galcon_def)

lemma comp_galcon_left_unit [simp]: "Ig(𝒴F) ∘g F = F"
  by (simp add: comp_galcon_def id_galcon_def)

lemma galois_connectionI:
  assumes
    "partial_order A" "partial_order B"
    "L ∈ carrier A → carrier B" "R ∈ carrier B → carrier A"
    "isotone A B L" "isotone B A R"
    "⋀ x y. ⟦ x ∈ carrier A; y ∈ carrier B ⟧ ⟹ L x ⊑B y ⟷ x ⊑A R
y"
  shows "galois_connection ⦇ orderA = A, orderB = B, lower = L, upper
= R ⦈"
  using assms by (simp add: galois_connection_def connection_def galois_connection_axioms_d

lemma galois_connectionI':
  assumes
    "partial_order A" "partial_order B"
    "L ∈ carrier A → carrier B" "R ∈ carrier B → carrier A"
    "isotone A B L" "isotone B A R"
    "⋀ X. X ∈ carrier(B) ⟹ L(R(X)) ⊑B X"
    "⋀ X. X ∈ carrier(A) ⟹ X ⊑A R(L(X))"
  shows "galois_connection ⦇ orderA = A, orderB = B, lower = L, upper
= R ⦈"
  using assms
  by (auto simp add: galois_connection_def connection_def galois_connection_axioms_def,
(meson PiE isotone_def weak_partial_order.le_trans)+)
```

5.5 Retracts

```
locale retract = galois_connection +
  assumes retract_property: "x ∈ carrier 𝒳 ⟹ π* (π* x) ⊑𝒳 x"
begin
  lemma retract_inverse: "x ∈ carrier 𝒳 ⟹ π* (π* x) = x"
    by (meson funcset_mem inflation is_order_A lower_closure partial_order.le_antisym
retract_axioms retract_axioms_def retract_def upper_closure)

  lemma retract_injective: "inj_on π* (carrier 𝒳)"
    by (metis inj_onI retract_inverse)
end

theorem comp_retract_closed:
  assumes "retract G" "retract F" "𝒴F = 𝒳G"
  shows "retract (G ∘g F)"
proof -
  interpret f: retract F
    by (simp add: assms)
```

```
  interpret g: retract G
    by (simp add: assms)
  interpret gf: galois_connection "(G ∘g F)"
    by (simp add: assms(1) assms(2) assms(3) comp_galcon_closed retract.axioms(1))
  show ?thesis
  proof
    fix x
    assume "x ∈ carrier 𝒳G ∘g F"
    thus "le 𝒳G ∘g F (π∗G ∘g F (π*G ∘g F x)) x"
      using assms(3) f.inflation f.lower_closed f.retract_inverse g.retract_inverse
by (auto simp add: comp_galcon_def)
  qed
qed
```

5.6 Coretracts

```
locale coretract = galois_connection +
  assumes coretract_property: "y ∈ carrier 𝒴 ⟹ y ⊑𝒴 π* (π∗ y)"
begin
  lemma coretract_inverse: "y ∈ carrier 𝒴 ⟹ π* (π∗ y) = y"
    by (meson coretract_axioms coretract_axioms_def coretract_def deflation
funcset_mem is_order_B lower_closure partial_order.le_antisym upper_closure)

  lemma retract_injective: "inj_on π∗ (carrier 𝒴)"
    by (metis coretract_inverse inj_onI)
end

theorem comp_coretract_closed:
  assumes "coretract G" "coretract F" "𝒴F = 𝒳G"
  shows "coretract (G ∘g F)"
proof -
  interpret f: coretract F
    by (simp add: assms)
  interpret g: coretract G
    by (simp add: assms)
  interpret gf: galois_connection "(G ∘g F)"
    by (simp add: assms(1) assms(2) assms(3) comp_galcon_closed coretract.axioms(1))
  show ?thesis
  proof
    fix y
    assume "y ∈ carrier 𝒴G ∘g F"
    thus "le 𝒴G ∘g F y (π*G ∘g F (π∗G ∘g F y))"
      by (simp add: comp_galcon_def assms(3) f.coretract_inverse g.coretract_property
g.upper_closed)
  qed
qed
```

5.7 Galois Bijections

```
locale galois_bijection = connection +
```

```
  assumes lower_iso: "isotone 𝒳 𝒴 π^*"
  and upper_iso: "isotone 𝒴 𝒳 π_*"
  and lower_inv_eq: "x ∈ carrier 𝒳 ⟹ π_* (π^* x) = x"
  and upper_inv_eq: "y ∈ carrier 𝒴 ⟹ π^* (π_* y) = y"
begin

  lemma lower_bij: "bij_betw π^* (carrier 𝒳) (carrier 𝒴)"
    by (rule bij_betwI[where g="π_*"], auto intro: upper_inv_eq lower_inv_eq
upper_closed lower_closed)

  lemma upper_bij: "bij_betw π_* (carrier 𝒴) (carrier 𝒳)"
    by (rule bij_betwI[where g="π^*"], auto intro: upper_inv_eq lower_inv_eq
upper_closed lower_closed)

sublocale gal_bij_conn: galois_connection
  apply (unfold_locales, auto)
  using lower_closed lower_inv_eq upper_iso use_iso2 apply fastforce
  using lower_iso upper_closed upper_inv_eq use_iso2 apply fastforce
done

sublocale gal_bij_ret: retract
  by (unfold_locales, simp add: gal_bij_conn.is_weak_order_A lower_inv_eq
weak_partial_order.le_refl)

sublocale gal_bij_coret: coretract
  by (unfold_locales, simp add: gal_bij_conn.is_weak_order_B upper_inv_eq
weak_partial_order.le_refl)

end

theorem comp_galois_bijection_closed:
  assumes "galois_bijection G" "galois_bijection F" "𝒴_F = 𝒳_G"
  shows "galois_bijection (G ∘_g F)"
proof -
  interpret f: galois_bijection F
    by (simp add: assms)
  interpret g: galois_bijection G
    by (simp add: assms)
  interpret gf: galois_connection "(G ∘_g F)"
    by (simp add: assms(3) comp_galcon_closed f.gal_bij_conn.galois_connection_axioms
g.gal_bij_conn.galois_connection_axioms galois_connection.axioms(1))
  show ?thesis
  proof
    show "isotone 𝒳_(G ∘_g F) 𝒴_(G ∘_g F) π^*_(G ∘_g F)"
      by (simp add: comp_galcon_def, metis comp_galcon_def galcon.select_convs(1)
galcon.select_convs(2) galcon.select_convs(3) gf.lower_iso)
    show "isotone 𝒴_(G ∘_g F) 𝒳_(G ∘_g F) π_*_(G ∘_g F)"
      by (simp add: gf.upper_iso)
    fix x
```

```
    assume "x ∈ carrier 𝒳G ∘g F"
    thus "π*G ∘g F (π*G ∘g F x) = x"
      using assms(3) f.lower_closed f.lower_inv_eq g.lower_inv_eq by (auto
simp add: comp_galcon_def)
  next
    fix y
    assume "y ∈ carrier 𝒴G ∘g F"
    thus "π*G ∘g F (π*G ∘g F y) = y"
      by (simp add: comp_galcon_def assms(3) f.upper_inv_eq g.upper_closed
g.upper_inv_eq)
  qed
qed

end


theory Group
imports Complete_Lattice "HOL-Library.FuncSet"
begin
```

6 Monoids and Groups

6.1 Definitions

Definitions follow [2].

```
record 'a monoid =  "'a partial_object" +
  mult    :: "['a, 'a] ⇒ 'a" (infixl "⊗ı" 70)
  one     :: 'a ("1ı")

definition
  m_inv :: "('a, 'b) monoid_scheme => 'a => 'a" ("invı _" [81] 80)
  where "invG x = (THE y. y ∈ carrier G & x ⊗G y = 1G & y ⊗G x = 1G)"

definition
  Units :: "_ => 'a set"
  — The set of invertible elements
  where "Units G = {y. y ∈ carrier G & (∃x ∈ carrier G. x ⊗G y = 1G
& y ⊗G x = 1G)}"

consts
  pow :: "[('a, 'm) monoid_scheme, 'a, 'b::semiring_1] => 'a"  (infixr
"'(^')ı" 75)

overloading nat_pow == "pow :: [_, 'a, nat] => 'a"
begin
  definition "nat_pow G a n = rec_nat 1G (%u b. b ⊗G a) n"
end
```

```
overloading int_pow == "pow :: [_, 'a, int] => 'a"
begin
  definition "int_pow G a z =
   (let p = rec_nat 1_G (%u b. b ⊗_G a)
    in if z < 0 then inv_G (p (nat (-z))) else p (nat z))"
end

lemma int_pow_int: "x (^)_G (int n) = x (^)_G n"
by(simp add: int_pow_def nat_pow_def)

locale monoid =
  fixes G (structure)
  assumes m_closed [intro, simp]:
         "⟦x ∈ carrier G; y ∈ carrier G⟧ ⟹ x ⊗ y ∈ carrier G"
      and m_assoc:
         "⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧
          ⟹ (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
      and one_closed [intro, simp]: "1 ∈ carrier G"
      and l_one [simp]: "x ∈ carrier G ⟹ 1 ⊗ x = x"
      and r_one [simp]: "x ∈ carrier G ⟹ x ⊗ 1 = x"

lemma monoidI:
  fixes G (structure)
  assumes m_closed:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier
G"
    and one_closed: "1 ∈ carrier G"
    and m_assoc:
      "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
    and l_one: "!!x. x ∈ carrier G ==> 1 ⊗ x = x"
    and r_one: "!!x. x ∈ carrier G ==> x ⊗ 1 = x"
  shows "monoid G"
  by (fast intro!: monoid.intro intro: assms)

lemma (in monoid) Units_closed [dest]:
  "x ∈ Units G ==> x ∈ carrier G"
  by (unfold Units_def) fast

lemma (in monoid) inv_unique:
  assumes eq: "y ⊗ x = 1"  "x ⊗ y' = 1"
    and G: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
  shows "y = y'"
proof -
  from G eq have "y = y ⊗ (x ⊗ y')" by simp
  also from G have "... = (y ⊗ x) ⊗ y'" by (simp add: m_assoc)
  also from G eq have "... = y'" by simp
  finally show ?thesis .
qed
```

```
lemma (in monoid) Units_m_closed [intro, simp]:
  assumes x: "x ∈ Units G" and y: "y ∈ Units G"
  shows "x ⊗ y ∈ Units G"
proof -
  from x obtain x' where x: "x ∈ carrier G" "x' ∈ carrier G" and xinv:
"x ⊗ x' = 1" "x' ⊗ x = 1"
    unfolding Units_def by fast
  from y obtain y' where y: "y ∈ carrier G" "y' ∈ carrier G" and yinv:
"y ⊗ y' = 1" "y' ⊗ y = 1"
    unfolding Units_def by fast
  from x y xinv yinv have "y' ⊗ (x' ⊗ x) ⊗ y = 1" by simp
  moreover from x y xinv yinv have "x ⊗ (y ⊗ y') ⊗ x' = 1" by simp
  moreover note x y
  ultimately show ?thesis unfolding Units_def
    — Must avoid premature use of hyp_subst_tac.
    apply (rule_tac CollectI)
    apply (rule)
    apply (fast)
    apply (rule bexI [where x = "y' ⊗ x'"])
    apply (auto simp: m_assoc)
    done
qed

lemma (in monoid) Units_one_closed [intro, simp]:
  "1 ∈ Units G"
  by (unfold Units_def) auto

lemma (in monoid) Units_inv_closed [intro, simp]:
  "x ∈ Units G ==> inv x ∈ carrier G"
  apply (unfold Units_def m_inv_def, auto)
  apply (rule theI2, fast)
   apply (fast intro: inv_unique, fast)
  done

lemma (in monoid) Units_l_inv_ex:
  "x ∈ Units G ==> ∃y ∈ carrier G. y ⊗ x = 1"
  by (unfold Units_def) auto

lemma (in monoid) Units_r_inv_ex:
  "x ∈ Units G ==> ∃y ∈ carrier G. x ⊗ y = 1"
  by (unfold Units_def) auto

lemma (in monoid) Units_l_inv [simp]:
  "x ∈ Units G ==> inv x ⊗ x = 1"
  apply (unfold Units_def m_inv_def, auto)
  apply (rule theI2, fast)
   apply (fast intro: inv_unique, fast)
  done
```

```
lemma (in monoid) Units_r_inv [simp]:
  "x ∈ Units G ==> x ⊗ inv x = 1"
  apply (unfold Units_def m_inv_def, auto)
  apply (rule theI2, fast)
   apply (fast intro: inv_unique, fast)
  done

lemma (in monoid) Units_inv_Units [intro, simp]:
  "x ∈ Units G ==> inv x ∈ Units G"
proof -
  assume x: "x ∈ Units G"
  show "inv x ∈ Units G"
    by (auto simp add: Units_def
      intro: Units_l_inv Units_r_inv x Units_closed [OF x])
qed

lemma (in monoid) Units_l_cancel [simp]:
  "[| x ∈ Units G; y ∈ carrier G; z ∈ carrier G |] ==>
   (x ⊗ y = x ⊗ z) = (y = z)"
proof
  assume eq: "x ⊗ y = x ⊗ z"
    and G: "x ∈ Units G"  "y ∈ carrier G"  "z ∈ carrier G"
  then have "(inv x ⊗ x) ⊗ y = (inv x ⊗ x) ⊗ z"
    by (simp add: m_assoc Units_closed del: Units_l_inv)
  with G show "y = z" by simp
next
  assume eq: "y = z"
    and G: "x ∈ Units G"  "y ∈ carrier G"  "z ∈ carrier G"
  then show "x ⊗ y = x ⊗ z" by simp
qed

lemma (in monoid) Units_inv_inv [simp]:
  "x ∈ Units G ==> inv (inv x) = x"
proof -
  assume x: "x ∈ Units G"
  then have "inv x ⊗ inv (inv x) = inv x ⊗ x" by simp
  with x show ?thesis by (simp add: Units_closed del: Units_l_inv Units_r_inv)
qed

lemma (in monoid) inv_inj_on_Units:
  "inj_on (m_inv G) (Units G)"
proof (rule inj_onI)
  fix x y
  assume G: "x ∈ Units G"  "y ∈ Units G" and eq: "inv x = inv y"
  then have "inv (inv x) = inv (inv y)" by simp
  with G show "x = y" by simp
qed
```

```
lemma (in monoid) Units_inv_comm:
  assumes inv: "x ⊗ y = 1"
    and G: "x ∈ Units G"  "y ∈ Units G"
  shows "y ⊗ x = 1"
proof -
  from G have "x ⊗ y ⊗ x = x ⊗ 1" by (auto simp add: inv Units_closed)
  with G show ?thesis by (simp del: r_one add: m_assoc Units_closed)
qed

lemma (in monoid) carrier_not_empty: "carrier G ≠ {}"
by auto
```

Power

```
lemma (in monoid) nat_pow_closed [intro, simp]:
  "x ∈ carrier G ==> x (^) (n::nat) ∈ carrier G"
  by (induct n) (simp_all add: nat_pow_def)

lemma (in monoid) nat_pow_0 [simp]:
  "x (^) (0::nat) = 1"
  by (simp add: nat_pow_def)

lemma (in monoid) nat_pow_Suc [simp]:
  "x (^) (Suc n) = x (^) n ⊗ x"
  by (simp add: nat_pow_def)

lemma (in monoid) nat_pow_one [simp]:
  "1 (^) (n::nat) = 1"
  by (induct n) simp_all

lemma (in monoid) nat_pow_mult:
  "x ∈ carrier G ==> x (^) (n::nat) ⊗ x (^) m = x (^) (n + m)"
  by (induct m) (simp_all add: m_assoc [THEN sym])

lemma (in monoid) nat_pow_pow:
  "x ∈ carrier G ==> (x (^) n) (^) m = x (^) (n * m::nat)"
  by (induct m) (simp, simp add: nat_pow_mult add.commute)
```

6.2 Groups

A group is a monoid all of whose elements are invertible.

```
locale group = monoid +
  assumes Units: "carrier G <= Units G"

lemma (in group) is_group: "group G" by (rule group_axioms)

theorem groupI:
  fixes G (structure)
  assumes m_closed [simp]:
```

```
    "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier
G"
  and one_closed [simp]: "1 ∈ carrier G"
  and m_assoc:
    "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
    (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
  and l_one [simp]: "!!x. x ∈ carrier G ==> 1 ⊗ x = x"
  and l_inv_ex: "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 1"
 shows "group G"
proof -
 have l_cancel [simp]:
   "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (x ⊗ y = x ⊗ z) = (y = z)"
 proof
   fix x y z
   assume eq: "x ⊗ y = x ⊗ z"
     and G: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
   with l_inv_ex obtain x_inv where xG: "x_inv ∈ carrier G"
     and l_inv: "x_inv ⊗ x = 1" by fast
   from G eq xG have "(x_inv ⊗ x) ⊗ y = (x_inv ⊗ x) ⊗ z"
     by (simp add: m_assoc)
   with G show "y = z" by (simp add: l_inv)
 next
   fix x y z
   assume eq: "y = z"
     and G: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
   then show "x ⊗ y = x ⊗ z" by simp
 qed
 have r_one:
   "!!x. x ∈ carrier G ==> x ⊗ 1 = x"
 proof -
   fix x
   assume x: "x ∈ carrier G"
   with l_inv_ex obtain x_inv where xG: "x_inv ∈ carrier G"
     and l_inv: "x_inv ⊗ x = 1" by fast
   from x xG have "x_inv ⊗ (x ⊗ 1) = x_inv ⊗ x"
     by (simp add: m_assoc [symmetric] l_inv)
   with x xG show "x ⊗ 1 = x" by simp
 qed
 have inv_ex:
   "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1"
 proof -
   fix x
   assume x: "x ∈ carrier G"
   with l_inv_ex obtain y where y: "y ∈ carrier G"
     and l_inv: "y ⊗ x = 1" by fast
   from x y have "y ⊗ (x ⊗ y) = y ⊗ 1"
     by (simp add: m_assoc [symmetric] l_inv r_one)
   with x y have r_inv: "x ⊗ y = 1"
```

```
      by simp
    from x y show "∃y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1"
      by (fast intro: l_inv r_inv)
  qed
  then have carrier_subset_Units: "carrier G <= Units G"
    by (unfold Units_def) fast
  show ?thesis
    by standard (auto simp: r_one m_assoc carrier_subset_Units)
qed

lemma (in monoid) group_l_invI:
  assumes l_inv_ex:
    "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 1"
  shows "group G"
  by (rule groupI) (auto intro: m_assoc l_inv_ex)

lemma (in group) Units_eq [simp]:
  "Units G = carrier G"
proof
  show "Units G <= carrier G" by fast
next
  show "carrier G <= Units G" by (rule Units)
qed

lemma (in group) inv_closed [intro, simp]:
  "x ∈ carrier G ==> inv x ∈ carrier G"
  using Units_inv_closed by simp

lemma (in group) l_inv_ex [simp]:
  "x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 1"
  using Units_l_inv_ex by simp

lemma (in group) r_inv_ex [simp]:
  "x ∈ carrier G ==> ∃y ∈ carrier G. x ⊗ y = 1"
  using Units_r_inv_ex by simp

lemma (in group) l_inv [simp]:
  "x ∈ carrier G ==> inv x ⊗ x = 1"
  using Units_l_inv by simp
```

6.3 Cancellation Laws and Basic Properties

```
lemma (in group) l_cancel [simp]:
  "[| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (x ⊗ y = x ⊗ z) = (y = z)"
  using Units_l_inv by simp

lemma (in group) r_inv [simp]:
  "x ∈ carrier G ==> x ⊗ inv x = 1"
```

```
proof -
  assume x: "x ∈ carrier G"
  then have "inv x ⊗ (x ⊗ inv x) = inv x ⊗ 1"
    by (simp add: m_assoc [symmetric])
  with x show ?thesis by (simp del: r_one)
qed

lemma (in group) r_cancel [simp]:
  "[| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (y ⊗ x = z ⊗ x) = (y = z)"
proof
  assume eq: "y ⊗ x = z ⊗ x"
    and G: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
  then have "y ⊗ (x ⊗ inv x) = z ⊗ (x ⊗ inv x)"
    by (simp add: m_assoc [symmetric] del: r_inv Units_r_inv)
  with G show "y = z" by simp
next
  assume eq: "y = z"
    and G: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
  then show "y ⊗ x = z ⊗ x" by simp
qed

lemma (in group) inv_one [simp]:
  "inv 1 = 1"
proof -
  have "inv 1 = 1 ⊗ (inv 1)" by (simp del: r_inv Units_r_inv)
  moreover have "... = 1" by simp
  finally show ?thesis .
qed

lemma (in group) inv_inv [simp]:
  "x ∈ carrier G ==> inv (inv x) = x"
  using Units_inv_inv by simp

lemma (in group) inv_inj:
  "inj_on (m_inv G) (carrier G)"
  using inv_inj_on_Units by simp

lemma (in group) inv_mult_group:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> inv (x ⊗ y) = inv y ⊗ inv x"
proof -
  assume G: "x ∈ carrier G"  "y ∈ carrier G"
  then have "inv (x ⊗ y) ⊗ (x ⊗ y) = (inv y ⊗ inv x) ⊗ (x ⊗ y)"
    by (simp add: m_assoc) (simp add: m_assoc [symmetric])
  with G show ?thesis by (simp del: l_inv Units_l_inv)
qed

lemma (in group) inv_comm:
  "[| x ⊗ y = 1; x ∈ carrier G; y ∈ carrier G |] ==> y ⊗ x = 1"
```

```
  by (rule Units_inv_comm) auto

lemma (in group) inv_equality:
     "[|y ⊗ x = 1; x ∈ carrier G; y ∈ carrier G|] ==> inv x = y"
apply (simp add: m_inv_def)
apply (rule the_equality)
 apply (simp add: inv_comm [of y x])
apply (rule r_cancel [THEN iffD1], auto)
done


lemma (in group) inv_solve_left:
  "⟦ a ∈ carrier G; b ∈ carrier G; c ∈ carrier G ⟧ ⟹ a = inv b ⊗ c
⟷ c = b ⊗ a"
  by (metis inv_equality l_inv_ex l_one m_assoc r_inv)
lemma (in group) inv_solve_right:
  "⟦ a ∈ carrier G; b ∈ carrier G; c ∈ carrier G ⟧ ⟹ a = b ⊗ inv c
⟷ b = a ⊗ c"
  by (metis inv_equality l_inv_ex l_one m_assoc r_inv)
```

Power

```
lemma (in group) int_pow_def2:
  "a (^) (z::int) = (if z < 0 then inv (a (^) (nat (-z))) else a (^) (nat
z))"
  by (simp add: int_pow_def nat_pow_def Let_def)

lemma (in group) int_pow_0 [simp]:
  "x (^) (0::int) = 1"
  by (simp add: int_pow_def2)

lemma (in group) int_pow_one [simp]:
  "1 (^) (z::int) = 1"
  by (simp add: int_pow_def2)



lemma (in group) int_pow_closed [intro, simp]:
  "x ∈ carrier G ==> x (^) (i::int) ∈ carrier G"
  by (simp add: int_pow_def2)

lemma (in group) int_pow_1 [simp]:
  "x ∈ carrier G ⟹ x (^) (1::int) = x"
  by (simp add: int_pow_def2)

lemma (in group) int_pow_neg:
  "x ∈ carrier G ⟹ x (^) (-i::int) = inv (x (^) i)"
  by (simp add: int_pow_def2)

lemma (in group) int_pow_mult:
```

```
  "x ∈ carrier G ⟹ x (^) (i + j::int) = x (^) i ⊗ x (^) j"
proof -
  have [simp]: "-i - j = -j - i" by simp
  assume "x : carrier G" then
  show ?thesis
    by (auto simp add: int_pow_def2 inv_solve_left inv_solve_right nat_add_distrib
[symmetric] nat_pow_mult )
qed

lemma (in group) int_pow_diff:
  "x ∈ carrier G ⟹ x (^) (n - m :: int) = x (^) n ⊗ inv (x (^) m)"
by(simp only: diff_conv_add_uminus int_pow_mult int_pow_neg)

lemma (in group) inj_on_multc: "c ∈ carrier G ⟹ inj_on (λx. x ⊗ c)
(carrier G)"
by(simp add: inj_on_def)

lemma (in group) inj_on_cmult: "c ∈ carrier G ⟹ inj_on (λx. c ⊗ x)
(carrier G)"
by(simp add: inj_on_def)
```

6.4 Subgroups

```
locale subgroup =
  fixes H and G (structure)
  assumes subset: "H ⊆ carrier G"
    and m_closed [intro, simp]: "⟦x ∈ H; y ∈ H⟧ ⟹ x ⊗ y ∈ H"
    and one_closed [simp]: "𝟭 ∈ H"
    and m_inv_closed [intro,simp]: "x ∈ H ⟹ inv x ∈ H"

lemma (in subgroup) is_subgroup:
  "subgroup H G" by (rule subgroup_axioms)

declare (in subgroup) group.intro [intro]

lemma (in subgroup) mem_carrier [simp]:
  "x ∈ H ⟹ x ∈ carrier G"
  using subset by blast

lemma subgroup_imp_subset:
  "subgroup H G ⟹ H ⊆ carrier G"
  by (rule subgroup.subset)

lemma (in subgroup) subgroup_is_group [intro]:
  assumes "group G"
  shows "group (G⦇carrier := H⦈)"
proof -
  interpret group G by fact
  show ?thesis
```

```
    apply (rule monoid.group_l_invI)
    apply (unfold_locales) [1]
    apply (auto intro: m_assoc l_inv mem_carrier)
    done
qed
```

Since `H` is nonempty, it contains some element `x`. Since it is closed under inverse, it contains `inv x`. Since it is closed under product, it contains `x ⊗ inv x = 1`.

```
lemma (in group) one_in_subset:
  "[| H ⊆ carrier G; H ≠ {}; ∀a ∈ H. inv a ∈ H; ∀a∈H. ∀b∈H. a ⊗ b
∈ H |]
   ==> 1 ∈ H"
by force
```

A characterization of subgroups: closed, non-empty subset.

```
lemma (in group) subgroupI:
  assumes subset: "H ⊆ carrier G" and non_empty: "H ≠ {}"
    and inv: "!!a. a ∈ H ⟹ inv a ∈ H"
    and mult: "!!a b. ⟦a ∈ H; b ∈ H⟧ ⟹ a ⊗ b ∈ H"
  shows "subgroup H G"
proof (simp add: subgroup_def assms)
  show "1 ∈ H" by (rule one_in_subset) (auto simp only: assms)
qed

declare monoid.one_closed [iff] group.inv_closed [simp]
  monoid.l_one [simp] monoid.r_one [simp] group.inv_inv [simp]

lemma subgroup_nonempty:
  "~ subgroup {} G"
  by (blast dest: subgroup.one_closed)

lemma (in subgroup) finite_imp_card_positive:
  "finite (carrier G) ==> 0 < card H"
proof (rule classical)
  assume "finite (carrier G)" and a: "~ 0 < card H"
  then have "finite H" by (blast intro: finite_subset [OF subset])
  with is_subgroup a have "subgroup {} G" by simp
  with subgroup_nonempty show ?thesis by contradiction
qed
```

6.5 Direct Products

```
definition
  DirProd :: "_ ⇒ _ ⇒ ('a × 'b) monoid" (infixr "××" 80) where
  "G ×× H =
    ⦇carrier = carrier G × carrier H,
     mult = (λ(g, h) (g', h'). (g ⊗_G g', h ⊗_H h')),
     one = (1_G, 1_H)⦈"
```

```
lemma DirProd_monoid:
  assumes "monoid G" and "monoid H"
  shows "monoid (G ×× H)"
proof -
  interpret G: monoid G by fact
  interpret H: monoid H by fact
  from assms
  show ?thesis by (unfold monoid_def DirProd_def, auto)
qed
```

Does not use the previous result because it's easier just to use auto.

```
lemma DirProd_group:
  assumes "group G" and "group H"
  shows "group (G ×× H)"
proof -
  interpret G: group G by fact
  interpret H: group H by fact
  show ?thesis by (rule groupI)
     (auto intro: G.m_assoc H.m_assoc G.l_inv H.l_inv
           simp add: DirProd_def)
qed

lemma carrier_DirProd [simp]:
     "carrier (G ×× H) = carrier G × carrier H"
  by (simp add: DirProd_def)

lemma one_DirProd [simp]:
     "1_{G ×× H} = (1_{G}, 1_{H})"
  by (simp add: DirProd_def)

lemma mult_DirProd [simp]:
     "(g, h) ⊗_{(G ×× H)} (g', h') = (g ⊗_{G} g', h ⊗_{H} h')"
  by (simp add: DirProd_def)

lemma inv_DirProd [simp]:
  assumes "group G" and "group H"
  assumes g: "g ∈ carrier G"
      and h: "h ∈ carrier H"
  shows "m_inv (G ×× H) (g, h) = (inv_{G} g, inv_{H} h)"
proof -
  interpret G: group G by fact
  interpret H: group H by fact
  interpret Prod: group "G ×× H"
    by (auto intro: DirProd_group group.intro group.axioms assms)
  show ?thesis by (simp add: Prod.inv_equality g h)
qed
```

6.6 Homomorphisms and Isomorphisms

```
definition
  hom :: "_ => _ => ('a => 'b) set" where
  "hom G H =
    {h. h ∈ carrier G → carrier H &
      (∀x ∈ carrier G. ∀y ∈ carrier G. h (x ⊗G y) = h x ⊗H h y)}"

lemma (in group) hom_compose:
  "[|h ∈ hom G H; i ∈ hom H I|] ==> compose (carrier G) i h ∈ hom G I"
by (fastforce simp add: hom_def compose_def)

definition
  iso :: "_ => _ => ('a => 'b) set" (infixr "≅" 60)
  where "G ≅ H = {h. h ∈ hom G H & bij_betw h (carrier G) (carrier H)}"

lemma iso_refl: "(%x. x) ∈ G ≅ G"
by (simp add: iso_def hom_def inj_on_def bij_betw_def Pi_def)

lemma (in group) iso_sym:
     "h ∈ G ≅ H ⟹ inv_into (carrier G) h ∈ H ≅ G"
apply (simp add: iso_def bij_betw_inv_into)
apply (subgoal_tac "inv_into (carrier G) h ∈ carrier H → carrier G")

 prefer 2 apply (simp add: bij_betw_imp_funcset [OF bij_betw_inv_into])

apply (simp add: hom_def bij_betw_def inv_into_f_eq f_inv_into_f Pi_def)
done

lemma (in group) iso_trans:
     "[|h ∈ G ≅ H; i ∈ H ≅ I|] ==> (compose (carrier G) i h) ∈ G ≅ I"
by (auto simp add: iso_def hom_compose bij_betw_compose)

lemma DirProd_commute_iso:
  shows "(λ(x,y). (y,x)) ∈ (G ×× H) ≅ (H ×× G)"
by (auto simp add: iso_def hom_def inj_on_def bij_betw_def)

lemma DirProd_assoc_iso:
  shows "(λ(x,y,z). (x,(y,z))) ∈ (G ×× H ×× I) ≅ (G ×× (H ×× I))"
by (auto simp add: iso_def hom_def inj_on_def bij_betw_def)
```

Basis for homomorphism proofs: we assume two groups `G` and `H`, with a homomorphism `h` between them

```
locale group_hom = G?: group G + H?: group H for G (structure) and H (structure) +
  fixes h
  assumes homh: "h ∈ hom G H"

lemma (in group_hom) hom_mult [simp]:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> h (x ⊗G y) = h x ⊗H h y"
```

```
proof -
  assume "x ∈ carrier G" "y ∈ carrier G"
  with homh [unfolded hom_def] show ?thesis by simp
qed

lemma (in group_hom) hom_closed [simp]:
  "x ∈ carrier G ==> h x ∈ carrier H"
proof -
  assume "x ∈ carrier G"
  with homh [unfolded hom_def] show ?thesis by auto
qed

lemma (in group_hom) one_closed [simp]:
  "h 1 ∈ carrier H"
  by simp

lemma (in group_hom) hom_one [simp]:
  "h 1 = 1H"
proof -
  have "h 1 ⊗H 1H = h 1 ⊗H h 1"
    by (simp add: hom_mult [symmetric] del: hom_mult)
  then show ?thesis by (simp del: r_one)
qed

lemma (in group_hom) inv_closed [simp]:
  "x ∈ carrier G ==> h (inv x) ∈ carrier H"
  by simp

lemma (in group_hom) hom_inv [simp]:
  "x ∈ carrier G ==> h (inv x) = invH (h x)"
proof -
  assume x: "x ∈ carrier G"
  then have "h x ⊗H h (inv x) = 1H"
    by (simp add: hom_mult [symmetric] del: hom_mult)
  also from x have "... = h x ⊗H invH (h x)"
    by (simp add: hom_mult [symmetric] del: hom_mult)
  finally have "h x ⊗H h (inv x) = h x ⊗H invH (h x)" .
  with x show ?thesis by (simp del: H.r_inv H.Units_r_inv)
qed


lemma (in group) int_pow_is_hom:
  "x ∈ carrier G ⟹ (op(^) x) ∈ hom ⦇ carrier = UNIV, mult = op +, one
= 0::int ⦈ G "
  unfolding hom_def by (simp add: int_pow_mult)
```

6.7 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

```
locale comm_monoid = monoid +
  assumes m_comm: "⟦x ∈ carrier G; y ∈ carrier G⟧ ⟹ x ⊗ y = y ⊗ x"

lemma (in comm_monoid) m_lcomm:
  "⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧ ⟹
   x ⊗ (y ⊗ z) = y ⊗ (x ⊗ z)"
proof -
  assume xyz: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
  from xyz have "x ⊗ (y ⊗ z) = (x ⊗ y) ⊗ z" by (simp add: m_assoc)
  also from xyz have "... = (y ⊗ x) ⊗ z" by (simp add: m_comm)
  also from xyz have "... = y ⊗ (x ⊗ z)" by (simp add: m_assoc)
  finally show ?thesis .
qed

lemmas (in comm_monoid) m_ac = m_assoc m_comm m_lcomm

lemma comm_monoidI:
  fixes G (structure)
  assumes m_closed:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier
G"
    and one_closed: "1 ∈ carrier G"
    and m_assoc:
      "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
    and l_one: "!!x. x ∈ carrier G ==> 1 ⊗ x = x"
    and m_comm:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y = y ⊗ x"
  shows "comm_monoid G"
  using l_one
    by (auto intro!: comm_monoid.intro comm_monoid_axioms.intro monoid.intro

             intro: assms simp: m_closed one_closed m_comm)

lemma (in monoid) monoid_comm_monoidI:
  assumes m_comm:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y = y ⊗ x"
  shows "comm_monoid G"
  by (rule comm_monoidI) (auto intro: m_assoc m_comm)



lemma (in comm_monoid) nat_pow_distr:
  "[| x ∈ carrier G; y ∈ carrier G |] ==>
  (x ⊗ y) (^) (n::nat) = x (^) n ⊗ y (^) n"
```

```
  by (induct n) (simp, simp add: m_ac)

locale comm_group = comm_monoid + group

lemma (in group) group_comm_groupI:
  assumes m_comm: "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==>
      x ⊗ y = y ⊗ x"
  shows "comm_group G"
  by standard (simp_all add: m_comm)

lemma comm_groupI:
  fixes G (structure)
  assumes m_closed:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier
G"
    and one_closed: "1 ∈ carrier G"
    and m_assoc:
      "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
    and m_comm:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y = y ⊗ x"
    and l_one: "!!x. x ∈ carrier G ==> 1 ⊗ x = x"
    and l_inv_ex: "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 1"
  shows "comm_group G"
  by (fast intro: group.group_comm_groupI groupI assms)

lemma (in comm_group) inv_mult:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> inv (x ⊗ y) = inv x ⊗ inv y"
  by (simp add: m_ac inv_mult_group)
```

6.8 The Lattice of Subgroups of a Group

```
theorem (in group) subgroups_partial_order:
  "partial_order ⦇carrier = {H. subgroup H G}, eq = op =, le = op ⊆⦈"
  by standard simp_all

lemma (in group) subgroup_self:
  "subgroup (carrier G) G"
  by (rule subgroupI) auto

lemma (in group) subgroup_imp_group:
  "subgroup H G ==> group (G⦇carrier := H⦈)"
  by (erule subgroup.subgroup_is_group) (rule group_axioms)

lemma (in group) is_monoid [intro, simp]:
  "monoid G"
  by (auto intro: monoid.intro m_assoc)

lemma (in group) subgroup_inv_equality:
```

```
  "[| subgroup H G; x ∈ H |] ==> m_inv (G ⦇carrier := H⦈) x = inv x"
apply (rule_tac inv_equality [THEN sym])
  apply (rule group.l_inv [OF subgroup_imp_group, simplified], assumption+)
 apply (rule subsetD [OF subgroup.subset], assumption+)
apply (rule subsetD [OF subgroup.subset], assumption)
apply (rule_tac group.inv_closed [OF subgroup_imp_group, simplified],
assumption+)
done

theorem (in group) subgroups_Inter:
  assumes subgr: "(!!H. H ∈ A ==> subgroup H G)"
    and not_empty: "A ~= {}"
  shows "subgroup (⋂A) G"
proof (rule subgroupI)
  from subgr [THEN subgroup.subset] and not_empty
  show "⋂A ⊆ carrier G" by blast
next
  from subgr [THEN subgroup.one_closed]
  show "⋂A ~= {}" by blast
next
  fix x assume "x ∈ ⋂A"
  with subgr [THEN subgroup.m_inv_closed]
  show "inv x ∈ ⋂A" by blast
next
  fix x y assume "x ∈ ⋂A" "y ∈ ⋂A"
  with subgr [THEN subgroup.m_closed]
  show "x ⊗ y ∈ ⋂A" by blast
qed

theorem (in group) subgroups_complete_lattice:
  "complete_lattice ⦇carrier = {H. subgroup H G}, eq = op =, le = op ⊆⦈"
    (is "complete_lattice ?L")
proof (rule partial_order.complete_lattice_criterion1)
  show "partial_order ?L" by (rule subgroups_partial_order)
next
  have "greatest ?L (carrier G) (carrier ?L)"
    by (unfold greatest_def) (simp add: subgroup.subset subgroup_self)
  then show "∃G. greatest ?L G (carrier ?L)" ..
next
  fix A
  assume L: "A ⊆ carrier ?L" and non_empty: "A ~= {}"
  then have Int_subgroup: "subgroup (⋂A) G"
    by (fastforce intro: subgroups_Inter)
  have "greatest ?L (⋂A) (Lower ?L A)" (is "greatest _ ?Int _")
  proof (rule greatest_LowerI)
    fix H
    assume H: "H ∈ A"
    with L have subgroupH: "subgroup H G" by auto
    from subgroupH have groupH: "group (G ⦇carrier := H⦈)" (is "group
```

```
?H")
      by (rule subgroup_imp_group)
    from groupH have monoidH: "monoid ?H"
      by (rule group.is_monoid)
    from H have Int_subset: "?Int ⊆ H" by fastforce
    then show "le ?L ?Int H" by simp
  next
    fix H
    assume H: "H ∈ Lower ?L A"
    with L Int_subgroup show "le ?L H ?Int"
      by (fastforce simp: Lower_def intro: Inter_greatest)
  next
    show "A ⊆ carrier ?L" by (rule L)
  next
    show "?Int ∈ carrier ?L" by simp (rule Int_subgroup)
  qed
  then show "∃I. greatest ?L I (Lower ?L A)" ..
qed

end


theory FiniteProduct
imports Group
begin
```

6.9 Product Operator for Commutative Monoids

6.9.1 Inductive Definition of a Relation for Products over Sets

Instantiation of locale `LC` of theory `Finite_Set` is not possible, because here we have explicit typing rules like `x ∈ carrier G`. We introduce an explicit argument for the domain `D`.

```
inductive_set
  foldSetD :: "['a set, 'b => 'a => 'a, 'a] => ('b set * 'a) set"
  for D :: "'a set" and f :: "'b => 'a => 'a" and e :: 'a
  where
    emptyI [intro]: "e ∈ D ==> ({}, e) ∈ foldSetD D f e"
  | insertI [intro]: "[| x ~: A; f x y ∈ D; (A, y) ∈ foldSetD D f e |]
==>
                      (insert x A, f x y) ∈ foldSetD D f e"

inductive_cases empty_foldSetDE [elim!]: "({}, x) ∈ foldSetD D f e"

definition
  foldD :: "['a set, 'b => 'a => 'a, 'a, 'b set] => 'a"
  where "foldD D f e A = (THE x. (A, x) ∈ foldSetD D f e)"

lemma foldSetD_closed:
```

```
  "[| (A, z) ∈ foldSetD D f e ; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==>
f x y ∈ D
      |] ==> z ∈ D"
  by (erule foldSetD.cases) auto

lemma Diff1_foldSetD:
  "[| (A - {x}, y) ∈ foldSetD D f e; x ∈ A; f x y ∈ D |] ==>
   (A, f x y) ∈ foldSetD D f e"
  apply (erule insert_Diff [THEN subst], rule foldSetD.intros)
    apply auto
  done

lemma foldSetD_imp_finite [simp]: "(A, x) ∈ foldSetD D f e ==> finite
A"
  by (induct set: foldSetD) auto

lemma finite_imp_foldSetD:
  "[| finite A; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D |] ==>
   EX x. (A, x) ∈ foldSetD D f e"
proof (induct set: finite)
  case empty then show ?case by auto
next
  case (insert x F)
  then obtain y where y: "(F, y) ∈ foldSetD D f e" by auto
  with insert have "y ∈ D" by (auto dest: foldSetD_closed)
  with y and insert have "(insert x F, f x y) ∈ foldSetD D f e"
    by (intro foldSetD.intros) auto
  then show ?case ..
qed
```

Left-Commutative Operations

```
locale LCD =
  fixes B :: "'b set"
  and D :: "'a set"
  and f :: "'b => 'a => 'a"    (infixl "·" 70)
  assumes left_commute:
    "[| x ∈ B; y ∈ B; z ∈ D |] ==> x · (y · z) = y · (x · z)"
  and f_closed [simp, intro!]: "!!x y. [| x ∈ B; y ∈ D |] ==> f x y ∈
D"

lemma (in LCD) foldSetD_closed [dest]:
  "(A, z) ∈ foldSetD D f e ==> z ∈ D"
  by (erule foldSetD.cases) auto

lemma (in LCD) Diff1_foldSetD:
  "[| (A - {x}, y) ∈ foldSetD D f e; x ∈ A; A ⊆ B |] ==>
  (A, f x y) ∈ foldSetD D f e"
  apply (subgoal_tac "x ∈ B")
   prefer 2 apply fast
```

```
  apply (erule insert_Diff [THEN subst], rule foldSetD.intros)
    apply auto
  done

lemma (in LCD) foldSetD_imp_finite [simp]:
  "(A, x) ∈ foldSetD D f e ==> finite A"
  by (induct set: foldSetD) auto

lemma (in LCD) finite_imp_foldSetD:
  "[| finite A; A ⊆ B; e ∈ D |] ==> EX x. (A, x) ∈ foldSetD D f e"
proof (induct set: finite)
  case empty then show ?case by auto
next
  case (insert x F)
  then obtain y where y: "(F, y) ∈ foldSetD D f e" by auto
  with insert have "y ∈ D" by auto
  with y and insert have "(insert x F, f x y) ∈ foldSetD D f e"
    by (intro foldSetD.intros) auto
  then show ?case ..
qed

lemma (in LCD) foldSetD_determ_aux:
  "e ∈ D ==> ∀A x. A ⊆ B & card A < n --> (A, x) ∈ foldSetD D f e -->
    (∀y. (A, y) ∈ foldSetD D f e --> y = x)"
  apply (induct n)
   apply (auto simp add: less_Suc_eq)
  apply (erule foldSetD.cases)
   apply blast
  apply (erule foldSetD.cases)
   apply blast
  apply clarify
```

force simplification of `card A < card (insert ...)`.

```
  apply (erule rev_mp)
  apply (simp add: less_Suc_eq_le)
  apply (rule impI)
  apply (rename_tac xa Aa ya xb Ab yb, case_tac "xa = xb")
   apply (subgoal_tac "Aa = Ab")
    prefer 2 apply (blast elim!: equalityE)
   apply blast
```

case `xa` $\notin$ `xb`.

```
  apply (subgoal_tac "Aa - {xb} = Ab - {xa} & xb ∈ Aa & xa ∈ Ab")
   prefer 2 apply (blast elim!: equalityE)
  apply clarify
  apply (subgoal_tac "Aa = insert xb Ab - {xa}")
   prefer 2 apply blast
  apply (subgoal_tac "card Aa ≤ card Ab")
   prefer 2
```

```
   apply (rule Suc_le_mono [THEN subst])
   apply (simp add: card_Suc_Diff1)
  apply (rule_tac A1 = "Aa - {xb}" in finite_imp_foldSetD [THEN exE])
     apply (blast intro: foldSetD_imp_finite)
    apply best
   apply assumption
  apply (frule (1) Diff1_foldSetD)
   apply best
  apply (subgoal_tac "ya = f xb x")
   prefer 2
   apply (subgoal_tac "Aa ⊆ B")
    prefer 2 apply best
   apply (blast del: equalityCE)
  apply (subgoal_tac "(Ab - {xa}, x) ∈ foldSetD D f e")
   prefer 2 apply simp
  apply (subgoal_tac "yb = f xa x")
   prefer 2
   apply (blast del: equalityCE dest: Diff1_foldSetD)
  apply (simp (no_asm_simp))
  apply (rule left_commute)
    apply assumption
   apply best
  apply best
  done

lemma (in LCD) foldSetD_determ:
  "[| (A, x) ∈ foldSetD D f e; (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B
|]
  ==> y = x"
  by (blast intro: foldSetD_determ_aux [rule_format])

lemma (in LCD) foldD_equality:
  "[| (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |] ==> foldD D f e A = y"
  by (unfold foldD_def) (blast intro: foldSetD_determ)

lemma foldD_empty [simp]:
  "e ∈ D ==> foldD D f e {} = e"
  by (unfold foldD_def) blast

lemma (in LCD) foldD_insert_aux:
  "[| x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
    ((insert x A, v) ∈ foldSetD D f e) =
    (EX y. (A, y) ∈ foldSetD D f e & v = f x y)"
  apply auto
  apply (rule_tac A1 = A in finite_imp_foldSetD [THEN exE])
     apply (fastforce dest: foldSetD_imp_finite)
    apply assumption
   apply assumption
  apply (blast intro: foldSetD_determ)
```

```
  done

lemma (in LCD) foldD_insert:
    "[| finite A; x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
     foldD D f e (insert x A) = f x (foldD D f e A)"
  apply (unfold foldD_def)
  apply (simp add: foldD_insert_aux)
  apply (rule the_equality)
   apply (auto intro: finite_imp_foldSetD
     cong add: conj_cong simp add: foldD_def [symmetric] foldD_equality)
  done

lemma (in LCD) foldD_closed [simp]:
  "[| finite A; e ∈ D; A ⊆ B |] ==> foldD D f e A ∈ D"
proof (induct set: finite)
  case empty then show ?case by simp
next
  case insert then show ?case by (simp add: foldD_insert)
qed

lemma (in LCD) foldD_commute:
  "[| finite A; x ∈ B; e ∈ D; A ⊆ B |] ==>
   f x (foldD D f e A) = foldD D f (f x e) A"
  apply (induct set: finite)
   apply simp
  apply (auto simp add: left_commute foldD_insert)
  done

lemma Int_mono2:
  "[| A ⊆ C; B ⊆ C |] ==> A Int B ⊆ C"
  by blast

lemma (in LCD) foldD_nest_Un_Int:
  "[| finite A; finite C; e ∈ D; A ⊆ B; C ⊆ B |] ==>
   foldD D f (foldD D f e C) A = foldD D f (foldD D f e (A Int C)) (A
Un C)"
  apply (induct set: finite)
   apply simp
  apply (simp add: foldD_insert foldD_commute Int_insert_left insert_absorb
    Int_mono2)
  done

lemma (in LCD) foldD_nest_Un_disjoint:
  "[| finite A; finite B; A Int B = {}; e ∈ D; A ⊆ B; C ⊆ B |]
    ==> foldD D f e (A Un B) = foldD D f (foldD D f e B) A"
  by (simp add: foldD_nest_Un_Int)
```

— Delete rules to do with `foldSetD` relation.

```
declare foldSetD_imp_finite [simp del]
  empty_foldSetDE [rule del]
  foldSetD.intros [rule del]
declare (in LCD)
  foldSetD_closed [rule del]
```

Commutative Monoids

We enter a more restrictive context, with `f :: 'a => 'a => 'a` instead of `'b => 'a => 'a`.

```
locale ACeD =
  fixes D :: "'a set"
    and f :: "'a => 'a => 'a"    (infixl "·" 70)
    and e :: 'a
  assumes ident [simp]: "x ∈ D ==> x · e = x"
    and commute: "[| x ∈ D; y ∈ D |] ==> x · y = y · x"
    and assoc: "[| x ∈ D; y ∈ D; z ∈ D |] ==> (x · y) · z = x · (y · z)"
    and e_closed [simp]: "e ∈ D"
    and f_closed [simp]: "[| x ∈ D; y ∈ D |] ==> x · y ∈ D"

lemma (in ACeD) left_commute:
  "[| x ∈ D; y ∈ D; z ∈ D |] ==> x · (y · z) = y · (x · z)"
proof -
  assume D: "x ∈ D" "y ∈ D" "z ∈ D"
  then have "x · (y · z) = (y · z) · x" by (simp add: commute)
  also from D have "... = y · (z · x)" by (simp add: assoc)
  also from D have "z · x = x · z" by (simp add: commute)
  finally show ?thesis .
qed

lemmas (in ACeD) AC = assoc commute left_commute

lemma (in ACeD) left_ident [simp]: "x ∈ D ==> e · x = x"
proof -
  assume "x ∈ D"
  then have "x · e = x" by (rule ident)
  with ‹x ∈ D› show ?thesis by (simp add: commute)
qed

lemma (in ACeD) foldD_Un_Int:
  "[| finite A; finite B; A ⊆ D; B ⊆ D |] ==>
    foldD D f e A · foldD D f e B =
    foldD D f e (A Un B) · foldD D f e (A Int B)"
  apply (induct set: finite)
   apply (simp add: left_commute LCD.foldD_closed [OF LCD.intro [of D]])
  apply (simp add: AC insert_absorb Int_insert_left
    LCD.foldD_insert [OF LCD.intro [of D]]
    LCD.foldD_closed [OF LCD.intro [of D]]
    Int_mono2)
```

```
  done

lemma (in ACeD) foldD_Un_disjoint:
  "[| finite A; finite B; A Int B = {}; A ⊆ D; B ⊆ D |] ==>
    foldD D f e (A Un B) = foldD D f e A · foldD D f e B"
  by (simp add: foldD_Un_Int
    left_commute LCD.foldD_closed [OF LCD.intro [of D]])
```

6.9.2 Products over Finite Sets

```
definition
  finprod :: "[('b, 'm) monoid_scheme, 'a => 'b, 'a set] => 'b"
  where "finprod G f A =
   (if finite A
    then foldD (carrier G) (mult G o f) 1_G A
    else 1_G)"

syntax
  "_finprod" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨂__∈_. _)" [1000, 0, 51, 10] 10)
translations
  "⨂_Gi∈A. b" ⇌ "CONST finprod G (%i. b) A"
  — Beware of argument permutation!

lemma (in comm_monoid) finprod_empty [simp]:
  "finprod G f {} = 1"
  by (simp add: finprod_def)

lemma (in comm_monoid) finprod_infinite[simp]:
  "¬ finite A ⟹ finprod G f A = 1"
  by (simp add: finprod_def)

declare funcsetI [intro]
  funcset_mem [dest]

context comm_monoid begin

lemma finprod_insert [simp]:
  "[| finite F; a ∉ F; f ∈ F → carrier G; f a ∈ carrier G |] ==>
   finprod G f (insert a F) = f a ⊗ finprod G f F"
  apply (rule trans)
   apply (simp add: finprod_def)
  apply (rule trans)
   apply (rule LCD.foldD_insert [OF LCD.intro [of "insert a F"]])
         apply simp
         apply (rule m_lcomm)
           apply fast
          apply fast
         apply assumption
```

```
        apply fastforce
       apply simp+
   apply fast
  apply (auto simp add: finprod_def)
  done

lemma finprod_one [simp]: "(⨂i∈A. 1) = 1"
proof (induct A rule: infinite_finite_induct)
  case empty show ?case by simp
next
  case (insert a A)
  have "(%i. 1) ∈ A → carrier G" by auto
  with insert show ?case by simp
qed simp

lemma finprod_closed [simp]:
  fixes A
  assumes f: "f ∈ A → carrier G"
  shows "finprod G f A ∈ carrier G"
using f
proof (induct A rule: infinite_finite_induct)
  case empty show ?case by simp
next
  case (insert a A)
  then have a: "f a ∈ carrier G" by fast
  from insert have A: "f ∈ A → carrier G" by fast
  from insert A a show ?case by simp
qed simp

lemma funcset_Int_left [simp, intro]:
  "[| f ∈ A → C; f ∈ B → C |] ==> f ∈ A Int B → C"
  by fast

lemma funcset_Un_left [iff]:
  "(f ∈ A Un B → C) = (f ∈ A → C & f ∈ B → C)"
  by fast

lemma finprod_Un_Int:
  "[| finite A; finite B; g ∈ A → carrier G; g ∈ B → carrier G |] ==>
     finprod G g (A Un B) ⊗ finprod G g (A Int B) =
     finprod G g A ⊗ finprod G g B"
— The reversed orientation looks more natural, but LOOPS as a simprule!
proof (induct set: finite)
  case empty then show ?case by simp
next
  case (insert a A)
  then have a: "g a ∈ carrier G" by fast
  from insert have A: "g ∈ A → carrier G" by fast
  from insert A a show ?case
```

```
    by (simp add: m_ac Int_insert_left insert_absorb Int_mono2)
qed

lemma finprod_Un_disjoint:
  "[| finite A; finite B; A Int B = {};
      g ∈ A → carrier G; g ∈ B → carrier G |]
   ==> finprod G g (A Un B) = finprod G g A ⊗ finprod G g B"
  apply (subst finprod_Un_Int [symmetric])
      apply auto
  done

lemma finprod_multf:
  "[| f ∈ A → carrier G; g ∈ A → carrier G |] ==>
   finprod G (%x. f x ⊗ g x) A = (finprod G f A ⊗ finprod G g A)"
proof (induct A rule: infinite_finite_induct)
  case empty show ?case by simp
next
  case (insert a A) then
  have fA: "f ∈ A → carrier G" by fast
  from insert have fa: "f a ∈ carrier G" by fast
  from insert have gA: "g ∈ A → carrier G" by fast
  from insert have ga: "g a ∈ carrier G" by fast
  from insert have fgA: "(%x. f x ⊗ g x) ∈ A → carrier G"
    by (simp add: Pi_def)
  show ?case
    by (simp add: insert fA fa gA ga fgA m_ac)
qed simp

lemma finprod_cong':
  "[| A = B; g ∈ B → carrier G;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B"
proof -
  assume prems: "A = B" "g ∈ B → carrier G"
    "!!i. i ∈ B ==> f i = g i"
  show ?thesis
  proof (cases "finite B")
    case True
    then have "!!A. [| A = B; g ∈ B → carrier G;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B"
    proof induct
      case empty thus ?case by simp
    next
      case (insert x B)
      then have "finprod G f A = finprod G f (insert x B)" by simp
      also from insert have "... = f x ⊗ finprod G f B"
      proof (intro finprod_insert)
        show "finite B" by fact
      next
        show "x ~: B" by fact
```

```
      next
        assume "x ~: B" "!!i. i ∈ insert x B ⟹ f i = g i"
          "g ∈ insert x B → carrier G"
        thus "f ∈ B → carrier G" by fastforce
      next
        assume "x ~: B" "!!i. i ∈ insert x B ⟹ f i = g i"
          "g ∈ insert x B → carrier G"
        thus "f x ∈ carrier G" by fastforce
      qed
      also from insert have "... = g x ⊗ finprod G g B" by fastforce
      also from insert have "... = finprod G g (insert x B)"
      by (intro finprod_insert [THEN sym]) auto
      finally show ?case .
    qed
    with prems show ?thesis by simp
  next
    case False with prems show ?thesis by simp
  qed
qed

lemma finprod_cong:
  "[| A = B; f ∈ B → carrier G = True;
      !!i. i ∈ B =simp=> f i = g i |] ==> finprod G f A = finprod G g
B"

  by (rule finprod_cong') (auto simp add: simp_implies_def)
```

Usually, if this rule causes a failed congruence proof error, the reason is that the premise `g ∈ B → carrier G` cannot be shown. Adding `Pi_def` to the simpset is often useful. For this reason, `finprod_cong` is not added to the simpset by default.

```
end

declare funcsetI [rule del]
  funcset_mem [rule del]

context comm_monoid begin

lemma finprod_0 [simp]:
  "f ∈ {0::nat} → carrier G ==> finprod G f {..0} = f 0"
by (simp add: Pi_def)

lemma finprod_Suc [simp]:
  "f ∈ {..Suc n} → carrier G ==>
   finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})"
by (simp add: Pi_def atMost_Suc)

lemma finprod_Suc2:
  "f ∈ {..Suc n} → carrier G ==>
```

```
   finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)"
proof (induct n)
  case 0 thus ?case by (simp add: Pi_def)
next
  case Suc thus ?case by (simp add: m_assoc Pi_def)
qed

lemma finprod_mult [simp]:
  "[| f ∈ {..n} → carrier G; g ∈ {..n} → carrier G |] ==>
     finprod G (%i. f i ⊗ g i) {..n::nat} =
     finprod G f {..n} ⊗ finprod G g {..n}"
  by (induct n) (simp_all add: m_ac Pi_def)



lemma finprod_reindex:
  "f : (h ‘ A) → carrier G ⟹
        inj_on h A ==> finprod G f (h ‘ A) = finprod G (%x. f (h x)) A"
proof (induct A rule: infinite_finite_induct)
  case (infinite A)
  hence "¬ finite (h ‘ A)"
    using finite_imageD by blast
  with ‹¬ finite A› show ?case by simp
qed (auto simp add: Pi_def)

lemma finprod_const:
  assumes a [simp]: "a : carrier G"
    shows "finprod G (%x. a) A = a (^) card A"
proof (induct A rule: infinite_finite_induct)
  case (insert b A)
  show ?case
  proof (subst finprod_insert[OF insert(1-2)])
    show "a ⊗ (⨂x∈A. a) = a (^) card (insert b A)"
      by (insert insert, auto, subst m_comm, auto)
  qed auto
qed auto



lemma finprod_singleton:
  assumes i_in_A: "i ∈ A" and fin_A: "finite A" and f_Pi: "f ∈ A →
carrier G"
  shows "(⨂j∈A. if i = j then f j else 1) = f i"
  using i_in_A finprod_insert [of "A - {i}" i "(λj. if i = j then f j
else 1)"]
    fin_A f_Pi finprod_one [of "A - {i}"]
    finprod_cong [of "A - {i}" "A - {i}" "(λj. if i = j then f j else
1)" "(λi. 1)"]
  unfolding Pi_def simp_implies_def by (force simp add: insert_absorb)
```

```
end

end


theory Coset
imports Group
begin
```

7 Cosets and Quotient Groups

```
definition
  r_coset    :: "[_, 'a set, 'a] ⇒ 'a set"    (infixl "#>ı" 60)
  where "H #>G a = (⋃h∈H. {h ⊗G a})"

definition
  l_coset    :: "[_, 'a, 'a set] ⇒ 'a set"    (infixl "<#ı" 60)
  where "a <#G H = (⋃h∈H. {a ⊗G h})"

definition
  RCOSETS  :: "[_, 'a set] ⇒ ('a set)set"   ("rcosetsı _" [81] 80)
  where "rcosetsG H = (⋃a∈carrier G. {H #>G a})"

definition
  set_mult  :: "[_, 'a set ,'a set] ⇒ 'a set" (infixl "<#>ı" 60)
  where "H <#>G K = (⋃h∈H. ⋃k∈K. {h ⊗G k})"

definition
  SET_INV :: "[_,'a set] ⇒ 'a set"  ("set'_invı _" [81] 80)
  where "set_invG H = (⋃h∈H. {invG h})"


locale normal = subgroup + group +
  assumes coset_eq: "(∀x ∈ carrier G. H #> x = x <# H)"

abbreviation
  normal_rel :: "['a set, ('a, 'b) monoid_scheme] ⇒ bool"  (infixl "◁"
60) where
  "H ◁ G ≡ normal H G"
```

7.1 Basic Properties of Cosets

```
lemma (in group) coset_mult_assoc:
     "[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
      ==> (M #> g) #> h = M #> (g ⊗ h)"
by (force simp add: r_coset_def m_assoc)

lemma (in group) coset_mult_one [simp]: "M ⊆ carrier G ==> M #> 1 =
```

```
M"
by (force simp add: r_coset_def)

lemma (in group) coset_mult_inv1:
     "[| M #> (x ⊗ (inv y)) = M;  x ∈ carrier G ; y ∈ carrier G;
         M ⊆ carrier G |] ==> M #> x = M #> y"
apply (erule subst [of concl: "%z. M #> x = z #> y"])
apply (simp add: coset_mult_assoc m_assoc)
done

lemma (in group) coset_mult_inv2:
     "[| M #> x = M #> y;  x ∈ carrier G;  y ∈ carrier G;  M ⊆ carrier
G |]
      ==> M #> (x ⊗ (inv y)) = M "
apply (simp add: coset_mult_assoc [symmetric])
apply (simp add: coset_mult_assoc)
done

lemma (in group) coset_join1:
     "[| H #> x = H;  x ∈ carrier G;  subgroup H G |] ==> x ∈ H"
apply (erule subst)
apply (simp add: r_coset_def)
apply (blast intro: l_one subgroup.one_closed sym)
done

lemma (in group) solve_equation:
    "⟦subgroup H G; x ∈ H; y ∈ H⟧ ⟹ ∃h∈H. y = h ⊗ x"
apply (rule bexI [of _ "y ⊗ (inv x)"])
apply (auto simp add: subgroup.m_closed subgroup.m_inv_closed m_assoc
                      subgroup.subset [THEN subsetD])
done

lemma (in group) repr_independence:
     "⟦y ∈ H #> x;  x ∈ carrier G; subgroup H G⟧ ⟹ H #> x = H #> y"
by (auto simp add: r_coset_def m_assoc [symmetric]
                   subgroup.subset [THEN subsetD]
                   subgroup.m_closed solve_equation)

lemma (in group) coset_join2:
     "⟦x ∈ carrier G;  subgroup H G;  x∈H⟧ ⟹ H #> x = H"
  — Alternative proof is to put x = 1 in repr_independence.
by (force simp add: subgroup.m_closed r_coset_def solve_equation)

lemma (in monoid) r_coset_subset_G:
     "[| H ⊆ carrier G; x ∈ carrier G |] ==> H #> x ⊆ carrier G"
by (auto simp add: r_coset_def)

lemma (in group) rcosI:
     "[| h ∈ H; H ⊆ carrier G; x ∈ carrier G|] ==> h ⊗ x ∈ H #> x"
```

```
by (auto simp add: r_coset_def)

lemma (in group) rcosetsI:
     "⟦H ⊆ carrier G; x ∈ carrier G⟧ ⟹ H #> x ∈ rcosets H"
by (auto simp add: RCOSETS_def)
```

Really needed?

```
lemma (in group) transpose_inv:
     "[| x ⊗ y = z;  x ∈ carrier G;  y ∈ carrier G;  z ∈ carrier G |]
      ==> (inv x) ⊗ z = y"
by (force simp add: m_assoc [symmetric])

lemma (in group) rcos_self: "[| x ∈ carrier G; subgroup H G |] ==> x
∈ H #> x"
apply (simp add: r_coset_def)
apply (blast intro: sym l_one subgroup.subset [THEN subsetD]
                    subgroup.one_closed)
done
```

Opposite of `"repr_independence"`

```
lemma (in group) repr_independenceD:
  assumes "subgroup H G"
  assumes ycarr: "y ∈ carrier G"
      and repr:  "H #> x = H #> y"
  shows "y ∈ H #> x"
proof -
  interpret subgroup H G by fact
  show ?thesis  apply (subst repr)
  apply (intro rcos_self)
   apply (rule ycarr)
   apply (rule is_subgroup)
  done
qed
```

Elements of a right coset are in the carrier

```
lemma (in subgroup) elemrcos_carrier:
  assumes "group G"
  assumes acarr: "a ∈ carrier G"
    and a': "a' ∈ H #> a"
  shows "a' ∈ carrier G"
proof -
  interpret group G by fact
  from subset and acarr
  have "H #> a ⊆ carrier G" by (rule r_coset_subset_G)
  from this and a'
  show "a' ∈ carrier G"
    by fast
qed
```

```
lemma (in subgroup) rcos_const:
  assumes "group G"
  assumes hH: "h ∈ H"
  shows "H #> h = H"
proof -
  interpret group G by fact
  show ?thesis apply (unfold r_coset_def)
    apply rule
    apply rule
    apply clarsimp
    apply (intro subgroup.m_closed)
    apply (rule is_subgroup)
    apply assumption
    apply (rule hH)
    apply rule
    apply simp
  proof -
    fix h'
    assume h'H: "h' ∈ H"
    note carr = hH[THEN mem_carrier] h'H[THEN mem_carrier]
    from carr
    have a: "h' = (h' ⊗ inv h) ⊗ h" by (simp add: m_assoc)
    from h'H hH
    have "h' ⊗ inv h ∈ H" by simp
    from this and a
    show "∃x∈H. h' = x ⊗ h" by fast
  qed
qed
```

Step one for lemma `rcos_module`

```
lemma (in subgroup) rcos_module_imp:
  assumes "group G"
  assumes xcarr: "x ∈ carrier G"
      and x'cos: "x' ∈ H #> x"
  shows "(x' ⊗ inv x) ∈ H"
proof -
  interpret group G by fact
  from xcarr x'cos
      have x'carr: "x' ∈ carrier G"
      by (rule elemrcos_carrier[OF is_group])
  from xcarr
      have ixcarr: "inv x ∈ carrier G"
      by simp
  from x'cos
      have "∃h∈H. x' = h ⊗ x"
      unfolding r_coset_def
      by fast
  from this
      obtain h
```

```
        where hH: "h ∈ H"
        and x': "x' = h ⊗ x"
      by auto
  from hH and subset
      have hcarr: "h ∈ carrier G" by fast
  note carr = xcarr x'carr hcarr
  from x' and carr
      have "x' ⊗ (inv x) = (h ⊗ x) ⊗ (inv x)" by fast
  also from carr
      have "... = h ⊗ (x ⊗ inv x)" by (simp add: m_assoc)
  also from carr
      have "... = h ⊗ 1" by simp
  also from carr
      have "... = h" by simp
  finally
      have "x' ⊗ (inv x) = h" by simp
  from hH this
      show "x' ⊗ (inv x) ∈ H" by simp
qed
```

Step two for lemma `rcos_module`

```
lemma (in subgroup) rcos_module_rev:
  assumes "group G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
      and xixH: "(x' ⊗ inv x) ∈ H"
  shows "x' ∈ H #> x"
proof -
  interpret group G by fact
  from xixH
      have "∃h∈H. x' ⊗ (inv x) = h" by fast
  from this
      obtain h
        where hH: "h ∈ H"
        and hsym: "x' ⊗ (inv x) = h"
      by fast
  from hH subset have hcarr: "h ∈ carrier G" by simp
  note carr = carr hcarr
  from hsym[symmetric] have "h ⊗ x = x' ⊗ (inv x) ⊗ x" by fast
  also from carr
      have "... = x' ⊗ ((inv x) ⊗ x)" by (simp add: m_assoc)
  also from carr
      have "... = x' ⊗ 1" by simp
  also from carr
      have "... = x'" by simp
  finally
      have "h ⊗ x = x'" by simp
  from this[symmetric] and hH
      show "x' ∈ H #> x"
      unfolding r_coset_def
```

```
      by fast
qed
```

Module property of right cosets

```
lemma (in subgroup) rcos_module:
  assumes "group G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
  shows "(x' ∈ H #> x) = (x' ⊗ inv x ∈ H)"
proof -
  interpret group G by fact
  show ?thesis proof  assume "x' ∈ H #> x"
    from this and carr
    show "x' ⊗ inv x ∈ H"
      by (intro rcos_module_imp[OF is_group])
  next
    assume "x' ⊗ inv x ∈ H"
    from this and carr
    show "x' ∈ H #> x"
      by (intro rcos_module_rev[OF is_group])
  qed
qed
```

Right cosets are subsets of the carrier.

```
lemma (in subgroup) rcosets_carrier:
  assumes "group G"
  assumes XH: "X ∈ rcosets H"
  shows "X ⊆ carrier G"
proof -
  interpret group G by fact
  from XH have "∃x∈ carrier G. X = H #> x"
      unfolding RCOSETS_def
      by fast
  from this
      obtain x
        where xcarr: "x∈ carrier G"
        and X: "X = H #> x"
      by fast
  from subset and xcarr
      show "X ⊆ carrier G"
      unfolding X
      by (rule r_coset_subset_G)
qed
```

Multiplication of general subsets

```
lemma (in monoid) set_mult_closed:
  assumes Acarr: "A ⊆ carrier G"
      and Bcarr: "B ⊆ carrier G"
  shows "A <#> B ⊆ carrier G"
apply rule apply (simp add: set_mult_def, clarsimp)
```

```
proof -
  fix a b
  assume "a ∈ A"
  from this and Acarr
      have acarr: "a ∈ carrier G" by fast

  assume "b ∈ B"
  from this and Bcarr
      have bcarr: "b ∈ carrier G" by fast

  from acarr bcarr
      show "a ⊗ b ∈ carrier G" by (rule m_closed)
qed

lemma (in comm_group) mult_subgroups:
  assumes subH: "subgroup H G"
      and subK: "subgroup K G"
  shows "subgroup (H <#> K) G"
apply (rule subgroup.intro)
   apply (intro set_mult_closed subgroup.subset[OF subH] subgroup.subset[OF
subK])
  apply (simp add: set_mult_def) apply clarsimp defer 1
  apply (simp add: set_mult_def) defer 1
  apply (simp add: set_mult_def, clarsimp) defer 1
proof -
  fix ha hb ka kb
  assume haH: "ha ∈ H" and hbH: "hb ∈ H" and kaK: "ka ∈ K" and kbK:
"kb ∈ K"
  note carr = haH[THEN subgroup.mem_carrier[OF subH]] hbH[THEN subgroup.mem_carrier[OF
subH]]
              kaK[THEN subgroup.mem_carrier[OF subK]] kbK[THEN subgroup.mem_carrier[OF
subK]]
  from carr
      have "(ha ⊗ ka) ⊗ (hb ⊗ kb) = ha ⊗ (ka ⊗ hb) ⊗ kb" by (simp add:
m_assoc)
  also from carr
      have "... = ha ⊗ (hb ⊗ ka) ⊗ kb" by (simp add: m_comm)
  also from carr
      have "... = (ha ⊗ hb) ⊗ (ka ⊗ kb)" by (simp add: m_assoc)
  finally
      have eq: "(ha ⊗ ka) ⊗ (hb ⊗ kb) = (ha ⊗ hb) ⊗ (ka ⊗ kb)" .

  from haH hbH have hH: "ha ⊗ hb ∈ H" by (simp add: subgroup.m_closed[OF
subH])
  from kaK kbK have kK: "ka ⊗ kb ∈ K" by (simp add: subgroup.m_closed[OF
subK])

  from hH and kK and eq
      show "∃h'∈H. ∃k'∈K. (ha ⊗ ka) ⊗ (hb ⊗ kb) = h' ⊗ k'" by fast
```

```
next
  have "1 = 1 ⊗ 1" by simp
  from subgroup.one_closed[OF subH] subgroup.one_closed[OF subK] this
      show "∃h∈H. ∃k∈K. 1 = h ⊗ k" by fast
next
  fix h k
  assume hH: "h ∈ H"
     and kK: "k ∈ K"

  from hH[THEN subgroup.mem_carrier[OF subH]] kK[THEN subgroup.mem_carrier[OF
subK]]
      have "inv (h ⊗ k) = inv h ⊗ inv k" by (simp add: inv_mult_group
m_comm)

  from subgroup.m_inv_closed[OF subH hH] and subgroup.m_inv_closed[OF
subK kK] and this
      show "∃ha∈H. ∃ka∈K. inv (h ⊗ k) = ha ⊗ ka" by fast
qed

lemma (in subgroup) lcos_module_rev:
  assumes "group G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
      and xixH: "(inv x ⊗ x') ∈ H"
  shows "x' ∈ x <# H"
proof -
  interpret group G by fact
  from xixH
      have "∃h∈H. (inv x) ⊗ x' = h" by fast
  from this
      obtain h
        where hH: "h ∈ H"
        and hsym: "(inv x) ⊗ x' = h"
      by fast

  from hH subset have hcarr: "h ∈ carrier G" by simp
  note carr = carr hcarr
  from hsym[symmetric] have "x ⊗ h = x ⊗ ((inv x) ⊗ x')" by fast
  also from carr
      have "... = (x ⊗ (inv x)) ⊗ x'" by (simp add: m_assoc[symmetric])
  also from carr
      have "... = 1 ⊗ x'" by simp
  also from carr
      have "... = x'" by simp
  finally
      have "x ⊗ h = x'" by simp

  from this[symmetric] and hH
      show "x' ∈ x <# H"
      unfolding l_coset_def
```

```
    by fast
qed
```

7.2 Normal subgroups

```
lemma normal_imp_subgroup: "H ◁ G ⟹ subgroup H G"
  by (simp add: normal_def subgroup_def)

lemma (in group) normalI:
  "subgroup H G ⟹ (∀x ∈ carrier G. H #> x = x <# H) ⟹ H ◁ G"
  by (simp add: normal_def normal_axioms_def is_group)

lemma (in normal) inv_op_closed1:
     "⟦x ∈ carrier G; h ∈ H⟧ ⟹ (inv x) ⊗ h ⊗ x ∈ H"
apply (insert coset_eq)
apply (auto simp add: l_coset_def r_coset_def)
apply (drule bspec, assumption)
apply (drule equalityD1 [THEN subsetD], blast, clarify)
apply (simp add: m_assoc)
apply (simp add: m_assoc [symmetric])
done

lemma (in normal) inv_op_closed2:
     "⟦x ∈ carrier G; h ∈ H⟧ ⟹ x ⊗ h ⊗ (inv x) ∈ H"
apply (subgoal_tac "inv (inv x) ⊗ h ⊗ (inv x) ∈ H")
apply (simp add: )
apply (blast intro: inv_op_closed1)
done
```

Alternative characterization of normal subgroups

```
lemma (in group) normal_inv_iff:
     "(N ◁ G) =
      (subgroup N G & (∀x ∈ carrier G. ∀h ∈ N. x ⊗ h ⊗ (inv x) ∈ N))"
      (is "_ = ?rhs")
proof
  assume N: "N ◁ G"
  show ?rhs
    by (blast intro: N normal.inv_op_closed2 normal_imp_subgroup)
next
  assume ?rhs
  hence sg: "subgroup N G"
    and closed: "⋀x. x∈carrier G ⟹ ∀h∈N. x ⊗ h ⊗ inv x ∈ N" by auto
  hence sb: "N ⊆ carrier G" by (simp add: subgroup.subset)
  show "N ◁ G"
  proof (intro normalI [OF sg], simp add: l_coset_def r_coset_def, clarify)
    fix x
    assume x: "x ∈ carrier G"
    show "(⋃h∈N. {h ⊗ x}) = (⋃h∈N. {x ⊗ h})"
    proof
```

```
      show "(⋃h∈N. {h ⊗ x}) ⊆ (⋃h∈N. {x ⊗ h})"
      proof clarify
        fix n
        assume n: "n ∈ N"
        show "n ⊗ x ∈ (⋃h∈N. {x ⊗ h})"
        proof
          from closed [of "inv x"]
          show "inv x ⊗ n ⊗ x ∈ N" by (simp add: x n)
          show "n ⊗ x ∈ {x ⊗ (inv x ⊗ n ⊗ x)}"
            by (simp add: x n m_assoc [symmetric] sb [THEN subsetD])
        qed
      qed
    next
      show "(⋃h∈N. {x ⊗ h}) ⊆ (⋃h∈N. {h ⊗ x})"
      proof clarify
        fix n
        assume n: "n ∈ N"
        show "x ⊗ n ∈ (⋃h∈N. {h ⊗ x})"
        proof
          show "x ⊗ n ⊗ inv x ∈ N" by (simp add: x n closed)
          show "x ⊗ n ∈ {x ⊗ n ⊗ inv x ⊗ x}"
            by (simp add: x n m_assoc sb [THEN subsetD])
        qed
      qed
    qed
  qed
qed
```

7.3 More Properties of Cosets

```
lemma (in group) lcos_m_assoc:
     "[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
      ==> g <# (h <# M) = (g ⊗ h) <# M"
by (force simp add: l_coset_def m_assoc)

lemma (in group) lcos_mult_one: "M ⊆ carrier G ==> 1 <# M = M"
by (force simp add: l_coset_def)

lemma (in group) l_coset_subset_G:
     "[| H ⊆ carrier G; x ∈ carrier G |] ==> x <# H ⊆ carrier G"
by (auto simp add: l_coset_def subsetD)

lemma (in group) l_coset_swap:
     "⟦y ∈ x <# H;  x ∈ carrier G;  subgroup H G⟧ ⟹ x ∈ y <# H"
proof (simp add: l_coset_def)
  assume "∃h∈H. y = x ⊗ h"
    and x: "x ∈ carrier G"
    and sb: "subgroup H G"
  then obtain h' where h': "h' ∈ H & x ⊗ h' = y" by blast
```

```
  show "∃h∈H. x = y ⊗ h"
  proof
    show "x = y ⊗ inv h'" using h' x sb
      by (auto simp add: m_assoc subgroup.subset [THEN subsetD])
    show "inv h' ∈ H" using h' sb
      by (auto simp add: subgroup.subset [THEN subsetD] subgroup.m_inv_closed)
  qed
qed

lemma (in group) l_coset_carrier:
     "[| y ∈ x <# H;  x ∈ carrier G;  subgroup H G |] ==> y ∈ carrier
G"
by (auto simp add: l_coset_def m_assoc
                   subgroup.subset [THEN subsetD] subgroup.m_closed)

lemma (in group) l_repr_imp_subset:
  assumes y: "y ∈ x <# H" and x: "x ∈ carrier G" and sb: "subgroup H
G"
  shows "y <# H ⊆ x <# H"
proof -
  from y
  obtain h' where "h' ∈ H" "x ⊗ h' = y" by (auto simp add: l_coset_def)
  thus ?thesis using x sb
    by (auto simp add: l_coset_def m_assoc
                       subgroup.subset [THEN subsetD] subgroup.m_closed)
qed

lemma (in group) l_repr_independence:
  assumes y: "y ∈ x <# H" and x: "x ∈ carrier G" and sb: "subgroup H
G"
  shows "x <# H = y <# H"
proof
  show "x <# H ⊆ y <# H"
    by (rule l_repr_imp_subset,
        (blast intro: l_coset_swap l_coset_carrier y x sb)+)
  show "y <# H ⊆ x <# H" by (rule l_repr_imp_subset [OF y x sb])
qed

lemma (in group) setmult_subset_G:
     "⟦H ⊆ carrier G; K ⊆ carrier G⟧ ⟹ H <#> K ⊆ carrier G"
by (auto simp add: set_mult_def subsetD)

lemma (in group) subgroup_mult_id: "subgroup H G ⟹ H <#> H = H"
apply (auto simp add: subgroup.m_closed set_mult_def Sigma_def)
apply (rule_tac x = x in bexI)
apply (rule bexI [of _ "1"])
apply (auto simp add: subgroup.one_closed subgroup.subset [THEN subsetD])
done
```

7.3.1 Set of Inverses of an `r_coset`.

```
lemma (in normal) rcos_inv:
  assumes x:     "x ∈ carrier G"
  shows "set_inv (H #> x) = H #> (inv x)"
proof (simp add: r_coset_def SET_INV_def x inv_mult_group, safe)
  fix h
  assume h: "h ∈ H"
  show "inv x ⊗ inv h ∈ (⋃j∈H. {j ⊗ inv x})"
  proof
    show "inv x ⊗ inv h ⊗ x ∈ H"
      by (simp add: inv_op_closed1 h x)
    show "inv x ⊗ inv h ∈ {inv x ⊗ inv h ⊗ x ⊗ inv x}"
      by (simp add: h x m_assoc)
  qed
  show "h ⊗ inv x ∈ (⋃j∈H. {inv x ⊗ inv j})"
  proof
    show "x ⊗ inv h ⊗ inv x ∈ H"
      by (simp add: inv_op_closed2 h x)
    show "h ⊗ inv x ∈ {inv x ⊗ inv (x ⊗ inv h ⊗ inv x)}"
      by (simp add: h x m_assoc [symmetric] inv_mult_group)
  qed
qed
```

7.3.2 Theorems for `<#>` with `#>` or `<#`.

```
lemma (in group) setmult_rcos_assoc:
     "⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
      ⟹ H <#> (K #> x) = (H <#> K) #> x"
by (force simp add: r_coset_def set_mult_def m_assoc)

lemma (in group) rcos_assoc_lcos:
     "⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
      ⟹ (H #> x) <#> K = H <#> (x <# K)"
by (force simp add: r_coset_def l_coset_def set_mult_def m_assoc)

lemma (in normal) rcos_mult_step1:
     "⟦x ∈ carrier G; y ∈ carrier G⟧
      ⟹ (H #> x) <#> (H #> y) = (H <#> (x <# H)) #> y"
by (simp add: setmult_rcos_assoc subset
              r_coset_subset_G l_coset_subset_G rcos_assoc_lcos)

lemma (in normal) rcos_mult_step2:
     "⟦x ∈ carrier G; y ∈ carrier G⟧
      ⟹ (H <#> (x <# H)) #> y = (H <#> (H #> x)) #> y"
by (insert coset_eq, simp add: normal_def)

lemma (in normal) rcos_mult_step3:
     "⟦x ∈ carrier G; y ∈ carrier G⟧
      ⟹ (H <#> (H #> x)) #> y = H #> (x ⊗ y)"
```

```
by (simp add: setmult_rcos_assoc coset_mult_assoc
              subgroup_mult_id normal.axioms subset normal_axioms)

lemma (in normal) rcos_sum:
     "⟦x ∈ carrier G; y ∈ carrier G⟧
      ⟹ (H #> x) <#> (H #> y) = H #> (x ⊗ y)"
by (simp add: rcos_mult_step1 rcos_mult_step2 rcos_mult_step3)

lemma (in normal) rcosets_mult_eq: "M ∈ rcosets H ⟹ H <#> M = M"
  — generalizes subgroup_mult_id
  by (auto simp add: RCOSETS_def subset
        setmult_rcos_assoc subgroup_mult_id normal.axioms normal_axioms)
```

7.3.3 An Equivalence Relation

```
definition
  r_congruent :: "[('a,'b)monoid_scheme, 'a set] ⇒ ('a*'a)set"  ("rcongı
_")
  where "rcongG H = {(x,y). x ∈ carrier G & y ∈ carrier G & invG x ⊗G
y ∈ H}"


lemma (in subgroup) equiv_rcong:
   assumes "group G"
   shows "equiv (carrier G) (rcong H)"
proof -
  interpret group G by fact
  show ?thesis
  proof (intro equivI)
    show "refl_on (carrier G) (rcong H)"
      by (auto simp add: r_congruent_def refl_on_def)
  next
    show "sym (rcong H)"
    proof (simp add: r_congruent_def sym_def, clarify)
      fix x y
      assume [simp]: "x ∈ carrier G" "y ∈ carrier G"
         and "inv x ⊗ y ∈ H"
      hence "inv (inv x ⊗ y) ∈ H" by simp
      thus "inv y ⊗ x ∈ H" by (simp add: inv_mult_group)
    qed
  next
    show "trans (rcong H)"
    proof (simp add: r_congruent_def trans_def, clarify)
      fix x y z
      assume [simp]: "x ∈ carrier G" "y ∈ carrier G" "z ∈ carrier G"
         and "inv x ⊗ y ∈ H" and "inv y ⊗ z ∈ H"
      hence "(inv x ⊗ y) ⊗ (inv y ⊗ z) ∈ H" by simp
      hence "inv x ⊗ (y ⊗ inv y) ⊗ z ∈ H"
        by (simp add: m_assoc del: r_inv Units_r_inv)
```

```
    thus "inv x ⊗ z ∈ H" by simp
   qed
  qed
qed
```

Equivalence classes of `rcong` correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

```
lemma (in subgroup) l_coset_eq_rcong:
  assumes "group G"
  assumes a: "a ∈ carrier G"
  shows "a <# H = rcong H ‘‘ {a}"
proof -
  interpret group G by fact
  show ?thesis by (force simp add: r_congruent_def l_coset_def m_assoc
[symmetric] a )
qed
```

7.3.4 Two Distinct Right Cosets are Disjoint

```
lemma (in group) rcos_equation:
  assumes "subgroup H G"
  assumes p: "ha ⊗ a = h ⊗ b" "a ∈ carrier G" "b ∈ carrier G" "h ∈ H"
"ha ∈ H" "hb ∈ H"
  shows "hb ⊗ a ∈ (⋃h∈H. {h ⊗ b})"
proof -
  interpret subgroup H G by fact
  from p show ?thesis apply (rule_tac UN_I [of "hb ⊗ ((inv ha) ⊗ h)"])
    apply (simp add: )
    apply (simp add: m_assoc transpose_inv)
    done
qed

lemma (in group) rcos_disjoint:
  assumes "subgroup H G"
  assumes p: "a ∈ rcosets H" "b ∈ rcosets H" "a≠b"
  shows "a ∩ b = {}"
proof -
  interpret subgroup H G by fact
  from p show ?thesis
    apply (simp add: RCOSETS_def r_coset_def)
    apply (blast intro: rcos_equation assms sym)
    done
qed
```

7.4 Further lemmas for `r_congruent`

The relation is a congruence

```
lemma (in normal) congruent_rcong:
  shows "congruent2 (rcong H) (rcong H) (λa b. a ⊗ b <# H)"
```

```
proof (intro congruent2I[of "carrier G" _ "carrier G" _] equiv_rcong is_group)
  fix a b c
  assume abrcong: "(a, b) ∈ rcong H"
    and ccarr: "c ∈ carrier G"

  from abrcong
      have acarr: "a ∈ carrier G"
        and bcarr: "b ∈ carrier G"
        and abH: "inv a ⊗ b ∈ H"
      unfolding r_congruent_def
      by fast+

  note carr = acarr bcarr ccarr

  from ccarr and abH
      have "inv c ⊗ (inv a ⊗ b) ⊗ c ∈ H" by (rule inv_op_closed1)
  moreover
      from carr and inv_closed
      have "inv c ⊗ (inv a ⊗ b) ⊗ c = (inv c ⊗ inv a) ⊗ (b ⊗ c)"
      by (force cong: m_assoc)
  moreover
      from carr and inv_closed
      have "... = (inv (a ⊗ c)) ⊗ (b ⊗ c)"
      by (simp add: inv_mult_group)
  ultimately
      have "(inv (a ⊗ c)) ⊗ (b ⊗ c) ∈ H" by simp
  from carr and this
     have "(b ⊗ c) ∈ (a ⊗ c) <# H"
     by (simp add: lcos_module_rev[OF is_group])
  from carr and this and is_subgroup
     show "(a ⊗ c) <# H = (b ⊗ c) <# H" by (intro l_repr_independence,
simp+)
next
  fix a b c
  assume abrcong: "(a, b) ∈ rcong H"
    and ccarr: "c ∈ carrier G"

  from ccarr have "c ∈ Units G" by simp
  hence cinvc_one: "inv c ⊗ c = 1" by (rule Units_l_inv)

  from abrcong
      have acarr: "a ∈ carrier G"
       and bcarr: "b ∈ carrier G"
       and abH: "inv a ⊗ b ∈ H"
      by (unfold r_congruent_def, fast+)

  note carr = acarr bcarr ccarr

  from carr and inv_closed
```

```
      have "inv a ⊗ b = inv a ⊗ (1 ⊗ b)" by simp
   also from carr and inv_closed
        have "... = inv a ⊗ (inv c ⊗ c) ⊗ b" by simp
   also from carr and inv_closed
        have "... = (inv a ⊗ inv c) ⊗ (c ⊗ b)" by (force cong: m_assoc)
   also from carr and inv_closed
        have "... = inv (c ⊗ a) ⊗ (c ⊗ b)" by (simp add: inv_mult_group)
   finally
        have "inv a ⊗ b = inv (c ⊗ a) ⊗ (c ⊗ b)" .
   from abH and this
        have "inv (c ⊗ a) ⊗ (c ⊗ b) ∈ H" by simp

   from carr and this
      have "(c ⊗ b) ∈ (c ⊗ a) <# H"
      by (simp add: lcos_module_rev[OF is_group])
   from carr and this and is_subgroup
      show "(c ⊗ a) <# H = (c ⊗ b) <# H" by (intro l_repr_independence,
simp+)
qed
```

7.5 Order of a Group and Lagrange's Theorem

```
definition
  order :: "('a, 'b) monoid_scheme ⇒ nat"
  where "order S = card (carrier S)"

lemma (in monoid) order_gt_0_iff_finite: "0 < order G ⟷ finite (carrier
G)"
by(auto simp add: order_def card_gt_0_iff)

lemma (in group) rcosets_part_G:
  assumes "subgroup H G"
  shows "⋃(rcosets H) = carrier G"
proof -
  interpret subgroup H G by fact
  show ?thesis
    apply (rule equalityI)
    apply (force simp add: RCOSETS_def r_coset_def)
    apply (auto simp add: RCOSETS_def intro: rcos_self assms)
    done
qed

lemma (in group) cosets_finite:
     "⟦c ∈ rcosets H;  H ⊆ carrier G;  finite (carrier G)⟧ ⟹ finite
c"
apply (auto simp add: RCOSETS_def)
apply (simp add: r_coset_subset_G [THEN finite_subset])
done
```

The next two lemmas support the proof of `card_cosets_equal`.

```
lemma (in group) inj_on_f:
    "⟦H ⊆ carrier G;  a ∈ carrier G⟧ ⟹ inj_on (λy. y ⊗ inv a) (H #>
a)"
apply (rule inj_onI)
apply (subgoal_tac "x ∈ carrier G & y ∈ carrier G")
 prefer 2 apply (blast intro: r_coset_subset_G [THEN subsetD])
apply (simp add: subsetD)
done

lemma (in group) inj_on_g:
    "⟦H ⊆ carrier G;  a ∈ carrier G⟧ ⟹ inj_on (λy. y ⊗ a) H"
by (force simp add: inj_on_def subsetD)

lemma (in group) card_cosets_equal:
     "⟦c ∈ rcosets H;  H ⊆ carrier G; finite(carrier G)⟧
      ⟹ card c = card H"
apply (auto simp add: RCOSETS_def)
apply (rule card_bij_eq)
     apply (rule inj_on_f, assumption+)
    apply (force simp add: m_assoc subsetD r_coset_def)
   apply (rule inj_on_g, assumption+)
  apply (force simp add: m_assoc subsetD r_coset_def)
```

The sets `H #> a` and `H` are finite.

```
 apply (simp add: r_coset_subset_G [THEN finite_subset])
apply (blast intro: finite_subset)
done

lemma (in group) rcosets_subset_PowG:
     "subgroup H G  ⟹ rcosets H ⊆ Pow(carrier G)"
apply (simp add: RCOSETS_def)
apply (blast dest: r_coset_subset_G subgroup.subset)
done


theorem (in group) lagrange:
     "⟦finite(carrier G); subgroup H G⟧
      ⟹ card(rcosets H) * card(H) = order(G)"
apply (simp (no_asm_simp) add: order_def rcosets_part_G [symmetric])
apply (subst mult.commute)
apply (rule card_partition)
   apply (simp add: rcosets_subset_PowG [THEN finite_subset])
  apply (simp add: rcosets_part_G)
 apply (simp add: card_cosets_equal subgroup.subset)
apply (simp add: rcos_disjoint)
done
```

7.6 Quotient Groups: Factorization of a Group

```
definition
```

```
  FactGroup :: "[('a,'b) monoid_scheme, 'a set] ⇒ ('a set) monoid" (infixl "Mod" 65)
    — Actually defined for groups rather than monoids
   where "FactGroup G H = (|carrier = rcosets_G H, mult = set_mult G, one
= H|)"

lemma (in normal) setmult_closed:
     "⟦K1 ∈ rcosets H; K2 ∈ rcosets H⟧ ⟹ K1 <#> K2 ∈ rcosets H"
by (auto simp add: rcos_sum RCOSETS_def)

lemma (in normal) setinv_closed:
     "K ∈ rcosets H ⟹ set_inv K ∈ rcosets H"
by (auto simp add: rcos_inv RCOSETS_def)

lemma (in normal) rcosets_assoc:
     "⟦M1 ∈ rcosets H; M2 ∈ rcosets H; M3 ∈ rcosets H⟧
      ⟹ M1 <#> M2 <#> M3 = M1 <#> (M2 <#> M3)"
by (auto simp add: RCOSETS_def rcos_sum m_assoc)

lemma (in subgroup) subgroup_in_rcosets:
  assumes "group G"
  shows "H ∈ rcosets H"
proof -
  interpret group G by fact
  from _ subgroup_axioms have "H #> 1 = H"
    by (rule coset_join2) auto
  then show ?thesis
    by (auto simp add: RCOSETS_def)
qed

lemma (in normal) rcosets_inv_mult_group_eq:
     "M ∈ rcosets H ⟹ set_inv M <#> M = H"
by (auto simp add: RCOSETS_def rcos_inv rcos_sum subgroup.subset normal.axioms
normal_axioms)

theorem (in normal) factorgroup_is_group:
  "group (G Mod H)"
apply (simp add: FactGroup_def)
apply (rule groupI)
    apply (simp add: setmult_closed)
   apply (simp add: normal_imp_subgroup subgroup_in_rcosets [OF is_group])
  apply (simp add: restrictI setmult_closed rcosets_assoc)
 apply (simp add: normal_imp_subgroup
                  subgroup_in_rcosets rcosets_mult_eq)
apply (auto dest: rcosets_inv_mult_group_eq simp add: setinv_closed)
done

lemma mult_FactGroup [simp]: "X ⊗_(G Mod H) X' = X <#>_G X'"
  by (simp add: FactGroup_def)
```

```
lemma (in normal) inv_FactGroup:
     "X ∈ carrier (G Mod H) ⟹ invG Mod H X = set_inv X"
apply (rule group.inv_equality [OF factorgroup_is_group])
apply (simp_all add: FactGroup_def setinv_closed rcosets_inv_mult_group_eq)
done
```

The coset map is a homomorphism from `G` to the quotient group `G Mod H`

```
lemma (in normal) r_coset_hom_Mod:
  "(λa. H #> a) ∈ hom G (G Mod H)"
  by (auto simp add: FactGroup_def RCOSETS_def Pi_def hom_def rcos_sum)
```

7.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

```
definition
  kernel :: "('a, 'm) monoid_scheme ⇒ ('b, 'n) monoid_scheme ⇒  ('a
⇒ 'b) ⇒ 'a set"
    — the kernel of a homomorphism
  where "kernel G H h = {x. x ∈ carrier G & h x = 1H}"

lemma (in group_hom) subgroup_kernel: "subgroup (kernel G H h) G"
apply (rule subgroup.intro)
apply (auto simp add: kernel_def group.intro is_group)
done
```

The kernel of a homomorphism is a normal subgroup

```
lemma (in group_hom) normal_kernel: "(kernel G H h) ◁ G"
apply (simp add: G.normal_inv_iff subgroup_kernel)
apply (simp add: kernel_def)
done

lemma (in group_hom) FactGroup_nonempty:
  assumes X: "X ∈ carrier (G Mod kernel G H h)"
  shows "X ≠ {}"
proof -
  from X
  obtain g where "g ∈ carrier G"
             and "X = kernel G H h #> g"
    by (auto simp add: FactGroup_def RCOSETS_def)
  thus ?thesis
   by (auto simp add: kernel_def r_coset_def image_def intro: hom_one)
qed


lemma (in group_hom) FactGroup_the_elem_mem:
  assumes X: "X ∈ carrier (G Mod (kernel G H h))"
```

```
  shows "the_elem (h`X) ∈ carrier H"
proof -
  from X
  obtain g where g: "g ∈ carrier G"
            and "X = kernel G H h #> g"
    by (auto simp add: FactGroup_def RCOSETS_def)
  hence "h ` X = {h g}" by (auto simp add: kernel_def r_coset_def g intro!:
imageI)
  thus ?thesis by (auto simp add: g)
qed

lemma (in group_hom) FactGroup_hom:
     "(λX. the_elem (h`X)) ∈ hom (G Mod (kernel G H h)) H"
apply (simp add: hom_def FactGroup_the_elem_mem normal.factorgroup_is_group
[OF normal_kernel] group.axioms monoid.m_closed)
proof (intro ballI)
  fix X and X'
  assume X:  "X  ∈ carrier (G Mod kernel G H h)"
     and X': "X' ∈ carrier (G Mod kernel G H h)"
  then
  obtain g and g'
           where "g ∈ carrier G" and "g' ∈ carrier G"
             and "X = kernel G H h #> g" and "X' = kernel G H h #> g'"
    by (auto simp add: FactGroup_def RCOSETS_def)
  hence all: "∀x∈X. h x = h g" "∀x∈X'. h x = h g'"
    and Xsub: "X ⊆ carrier G" and X'sub: "X' ⊆ carrier G"
    by (force simp add: kernel_def r_coset_def image_def)+
  hence "h ` (X <#> X') = {h g ⊗H h g'}" using X X'
    by (auto dest!: FactGroup_nonempty intro!: image_eqI
             simp add: set_mult_def
                       subsetD [OF Xsub] subsetD [OF X'sub])
  then show "the_elem (h ` (X <#> X')) = the_elem (h ` X) ⊗H the_elem
(h ` X')"
    by (auto simp add: all FactGroup_nonempty X X' the_elem_image_unique)
qed
```

Lemma for the following injectivity result

```
lemma (in group_hom) FactGroup_subset:
     "⟦g ∈ carrier G; g' ∈ carrier G; h g = h g'⟧
      ⟹  kernel G H h #> g ⊆ kernel G H h #> g'"
apply (clarsimp simp add: kernel_def r_coset_def)
apply (rename_tac y)
apply (rule_tac x="y ⊗ g ⊗ inv g'" in exI)
apply (simp add: G.m_assoc)
done

lemma (in group_hom) FactGroup_inj_on:
     "inj_on (λX. the_elem (h ` X)) (carrier (G Mod kernel G H h))"
proof (simp add: inj_on_def, clarify)
```

```
  fix X and X’
  assume X:  "X  ∈ carrier (G Mod kernel G H h)"
     and X’: "X’ ∈ carrier (G Mod kernel G H h)"
  then
  obtain g and g’
           where gX: "g ∈ carrier G"  "g’ ∈ carrier G"
              "X = kernel G H h #> g" "X’ = kernel G H h #> g’"
    by (auto simp add: FactGroup_def RCOSETS_def)
  hence all: "∀x∈X. h x = h g" "∀x∈X’. h x = h g’"
    by (force simp add: kernel_def r_coset_def image_def)+
  assume "the_elem (h ‘ X) = the_elem (h ‘ X’)"
  hence h: "h g = h g’"
    by (simp add: all FactGroup_nonempty X X’ the_elem_image_unique)
  show "X=X’" by (rule equalityI) (simp_all add: FactGroup_subset h gX)

qed
```

If the homomorphism `h` is onto `H`, then so is the homomorphism from the quotient group

```
lemma (in group_hom) FactGroup_onto:
  assumes h: "h ‘ carrier G = carrier H"
  shows "(λX. the_elem (h ‘ X)) ‘ carrier (G Mod kernel G H h) = carrier
H"
proof
  show "(λX. the_elem (h ‘ X)) ‘ carrier (G Mod kernel G H h) ⊆ carrier
H"
    by (auto simp add: FactGroup_the_elem_mem)
  show "carrier H ⊆ (λX. the_elem (h ‘ X)) ‘ carrier (G Mod kernel G
H h)"
  proof
    fix y
    assume y: "y ∈ carrier H"
    with h obtain g where g: "g ∈ carrier G" "h g = y"
      by (blast elim: equalityE)
    hence "(⋃x∈kernel G H h #> g. {h x}) = {y}"
      by (auto simp add: y kernel_def r_coset_def)
    with g show "y ∈ (λX. the_elem (h ‘ X)) ‘ carrier (G Mod kernel G
H h)"
      apply (auto intro!: bexI image_eqI simp add: FactGroup_def RCOSETS_def)
      apply (subst the_elem_image_unique)
      apply auto
      done
  qed
qed
```

If `h` is a homomorphism from `G` onto `H`, then the quotient group `G Mod kernel G H h` is isomorphic to `H`.

```
theorem (in group_hom) FactGroup_iso:
  "h ‘ carrier G = carrier H
```

```
   ⟹ (λX. the_elem (h`X)) ∈ (G Mod (kernel G H h)) ≅ H"
by (simp add: iso_def FactGroup_hom FactGroup_inj_on bij_betw_def
              FactGroup_onto)

end
```

```
theory Exponent
imports Main "HOL-Computational_Algebra.Primes"
begin
```

8 Sylow's Theorem

The Combinatorial Argument Underlying the First Sylow Theorem

needed in this form to prove Sylow's theorem

```
corollary (in algebraic_semidom) div_combine:
  "⟦prime_elem p; ¬ p ^ Suc r dvd n; p ^ (a + r) dvd n * k⟧ ⟹ p ^ a
dvd k"
  by (metis add_Suc_right mult.commute prime_elem_power_dvd_cases)

lemma exponent_p_a_m_k_equation:
  fixes p :: nat
  assumes "0 < m" "0 < k" "p ≠ 0" "k < p^a"
    shows "multiplicity p (p^a * m - k) = multiplicity p (p^a - k)"
proof (rule multiplicity_cong [OF iffI])
  fix r
  assume *: "p ^ r dvd p ^ a * m - k"
  show "p ^ r dvd p ^ a - k"
  proof -
    have "k ≤ p ^ a * m" using assms
      by (meson nat_dvd_not_less dvd_triv_left leI mult_pos_pos order.strict_trans)
    then have "r ≤ a"
      by (meson "*" ‹0 < k› ‹k < p^a› dvd_diffD1 dvd_triv_left leI less_imp_le_nat
nat_dvd_not_less power_le_dvd)
    then have "p^r dvd p^a * m" by (simp add: le_imp_power_dvd)
    thus ?thesis
      by (meson ‹k ≤ p ^ a * m› ‹r ≤ a› * dvd_diffD1 dvd_diff_nat le_imp_power_dvd)
  qed
next
  fix r
  assume *: "p ^ r dvd p ^ a - k"
  with assms have "r ≤ a"
    by (metis diff_diff_cancel less_imp_le_nat nat_dvd_not_less nat_le_linear
power_le_dvd zero_less_diff)
  show "p ^ r dvd p ^ a * m - k"
  proof -
    have "p^r dvd p^a*m"
```

```
      by (simp add: ‹r ≤ a› le_imp_power_dvd)
    then show ?thesis
      by (meson assms * dvd_diffD1 dvd_diff_nat le_imp_power_dvd less_imp_le_nat
‹r ≤ a›)
  qed
qed

lemma p_not_div_choose_lemma:
  fixes p :: nat
  assumes eeq: "⋀i. Suc i < K ⟹ multiplicity p (Suc i) = multiplicity
p (Suc (j + i))"
      and "k < K" and p: "prime p"
    shows "multiplicity p (j + k choose k) = 0"
  using ‹k < K›
proof (induction k)
  case 0 then show ?case by simp
next
  case (Suc k)
  then have *: "(Suc (j+k) choose Suc k) > 0" by simp
  then have "multiplicity p ((Suc (j+k) choose Suc k) * Suc k) = multiplicity
p (Suc k)"
    by (subst Suc_times_binomial_eq [symmetric], subst prime_elem_multiplicity_mult_distrib
       (insert p Suc.prems, simp_all add: eeq [symmetric] Suc.IH)
  with p * show ?case
    by (subst (asm) prime_elem_multiplicity_mult_distrib) simp_all
qed
```

The lemma above, with two changes of variables

```
lemma p_not_div_choose:
  assumes "k < K" and "k ≤ n"
      and eeq: "⋀j. ⟦0<j; j<K⟧ ⟹ multiplicity p (n - k + (K - j)) =
multiplicity p (K - j)" "prime p"
    shows "multiplicity p (n choose k) = 0"
apply (rule p_not_div_choose_lemma [of K p "n-k" k, simplified assms nat_minus_add_max
max_absorb1])
apply (metis add_Suc_right eeq diff_diff_cancel order_less_imp_le zero_less_Suc
zero_less_diff)
apply (rule TrueI)+
done

proposition const_p_fac:
  assumes "m>0" and prime: "prime p"
  shows "multiplicity p (p^a * m choose p^a) = multiplicity p m"
proof-
  from assms have p: "0 < p ^ a" "0 < p^a * m" "p^a ≤ p^a * m"
    by (auto simp: prime_gt_0_nat)
  have *: "multiplicity p ((p^a * m - 1) choose (p^a - 1)) = 0"
    apply (rule p_not_div_choose [where K = "p^a"])
    using p exponent_p_a_m_k_equation by (auto simp: diff_le_mono prime)
```

```
  have "multiplicity p ((p ^ a * m choose p ^ a) * p ^ a) = a + multiplicity
p m"
  proof -
    have "(p ^ a * m choose p ^ a) * p ^ a = p ^ a * m * (p ^ a * m -
1 choose (p ^ a - 1))"
      (is "_ = ?rhs") using prime
      by (subst times_binomial_minus1_eq [symmetric]) (auto simp: prime_gt_0_nat)
    also from p have "p ^ a - Suc 0 ≤ p ^ a * m - Suc 0" by linarith
    with prime * p have "multiplicity p ?rhs = multiplicity p (p ^ a
* m)"
      by (subst prime_elem_multiplicity_mult_distrib) auto
    also have "... = a + multiplicity p m"
      using prime p by (subst prime_elem_multiplicity_mult_distrib) simp_all
    finally show ?thesis .
  qed
  then show ?thesis
    using prime p by (subst (asm) prime_elem_multiplicity_mult_distrib)
simp_all
qed

end


theory Sylow
  imports Coset Exponent
begin
```

See also [3].

The combinatorial argument is in theory `Exponent`.

```
lemma le_extend_mult: "⟦0 < c; a ≤ b⟧ ⟹ a ≤ b * c"
  for c :: nat
  by (metis divisors_zero dvd_triv_left leI less_le_trans nat_dvd_not_less
zero_less_iff_neq_zero)

locale sylow = group +
  fixes p and a and m and calM and RelM
  assumes prime_p: "prime p"
    and order_G: "order G = (p^a) * m"
    and finite_G[iff]: "finite (carrier G)"
  defines "calM ≡ {s. s ⊆ carrier G ∧ card s = p^a}"
    and "RelM ≡ {(N1, N2). N1 ∈ calM ∧ N2 ∈ calM ∧ (∃g ∈ carrier G.
N1 = N2 #> g)}"
begin

lemma RelM_refl_on: "refl_on calM RelM"
  by (auto simp: refl_on_def RelM_def calM_def) (blast intro!: coset_mult_one
[symmetric])

lemma RelM_sym: "sym RelM"
```

```
proof (unfold sym_def RelM_def, clarify)
  fix y g
  assume "y ∈ calM"
    and g: "g ∈ carrier G"
  then have "y = y #> g #> (inv g)"
    by (simp add: coset_mult_assoc calM_def)
  then show "∃g’∈carrier G. y = y #> g #> g’"
    by (blast intro: g)
qed

lemma RelM_trans: "trans RelM"
  by (auto simp add: trans_def RelM_def calM_def coset_mult_assoc)

lemma RelM_equiv: "equiv calM RelM"
  unfolding equiv_def by (blast intro: RelM_refl_on RelM_sym RelM_trans)

lemma M_subset_calM_prep: "M’ ∈ calM // RelM  ⟹ M’ ⊆ calM"
  unfolding RelM_def by (blast elim!: quotientE)

end
```

8.1 Main Part of the Proof

```
locale sylow_central = sylow +
  fixes H and M1 and M
  assumes M_in_quot: "M ∈ calM // RelM"
    and not_dvd_M: "¬ (p ^ Suc (multiplicity p m) dvd card M)"
    and M1_in_M: "M1 ∈ M"
  defines "H ≡ {g. g ∈ carrier G ∧ M1 #> g = M1}"
begin

lemma M_subset_calM: "M ⊆ calM"
  by (rule M_in_quot [THEN M_subset_calM_prep])

lemma card_M1: "card M1 = p^a"
  using M1_in_M M_subset_calM calM_def by blast

lemma exists_x_in_M1: "∃x. x ∈ M1"
  using prime_p [THEN prime_gt_Suc_0_nat] card_M1
  by (metis Suc_lessD card_eq_0_iff empty_subsetI equalityI gr_implies_not0
nat_zero_less_power_iff subsetI)

lemma M1_subset_G [simp]: "M1 ⊆ carrier G"
  using M1_in_M M_subset_calM calM_def mem_Collect_eq subsetCE by blast

lemma M1_inj_H: "∃f ∈ H→M1. inj_on f H"
proof -
  from exists_x_in_M1 obtain m1 where m1M: "m1 ∈ M1"..
  have m1: "m1 ∈ carrier G"
```

```
    by (simp add: m1M M1_subset_G [THEN subsetD])
  show ?thesis
  proof
    show "inj_on (λz∈H. m1 ⊗ z) H"
      by (simp add: inj_on_def l_cancel [of m1 x y, THEN iffD1] H_def
m1)
    show "restrict (op ⊗ m1) H ∈ H → M1"
    proof (rule restrictI)
      fix z
      assume zH: "z ∈ H"
      show "m1 ⊗ z ∈ M1"
      proof -
        from zH
        have zG: "z ∈ carrier G" and M1zeq: "M1 #> z = M1"
          by (auto simp add: H_def)
        show ?thesis
          by (rule subst [OF M1zeq]) (simp add: m1M zG rcosI)
      qed
    qed
  qed
qed

end
```

8.2 Discharging the Assumptions of `sylow_central`

```
context sylow
begin

lemma EmptyNotInEquivSet: "{} ∉ calM // RelM"
  by (blast elim!: quotientE dest: RelM_equiv [THEN equiv_class_self])

lemma existsM1inM: "M ∈ calM // RelM ⟹ ∃M1. M1 ∈ M"
  using RelM_equiv equiv_Eps_in by blast

lemma zero_less_o_G: "0 < order G"
  by (simp add: order_def card_gt_0_iff carrier_not_empty)

lemma zero_less_m: "m > 0"
  using zero_less_o_G by (simp add: order_G)

lemma card_calM: "card calM = (p^a) * m choose p^a"
  by (simp add: calM_def n_subsets order_G [symmetric] order_def)

lemma zero_less_card_calM: "card calM > 0"
  by (simp add: card_calM zero_less_binomial le_extend_mult zero_less_m)

lemma max_p_div_calM: "¬ (p ^ Suc (multiplicity p m) dvd card calM)"
proof
```

```
  assume "p ^ Suc (multiplicity p m) dvd card calM"
  with zero_less_card_calM prime_p
  have "Suc (multiplicity p m) ≤ multiplicity p (card calM)"
    by (intro multiplicity_geI) auto
  then have "multiplicity p m < multiplicity p (card calM)" by simp
  also have "multiplicity p m = multiplicity p (card calM)"
    by (simp add: const_p_fac prime_p zero_less_m card_calM)
  finally show False by simp
qed

lemma finite_calM: "finite calM"
  unfolding calM_def by (rule finite_subset [where B = "Pow (carrier
G)"]) auto

lemma lemma_A1: "∃M ∈ calM // RelM. ¬ (p ^ Suc (multiplicity p m) dvd
card M)"
  using RelM_equiv equiv_imp_dvd_card finite_calM max_p_div_calM by blast

end
```

8.2.1 Introduction and Destruct Rules for `H`

```
context sylow_central
begin

lemma H_I: "⟦g ∈ carrier G; M1 #> g = M1⟧ ⟹ g ∈ H"
  by (simp add: H_def)

lemma H_into_carrier_G: "x ∈ H ⟹ x ∈ carrier G"
  by (simp add: H_def)

lemma in_H_imp_eq: "g ∈ H ⟹ M1 #> g = M1"
  by (simp add: H_def)

lemma H_m_closed: "⟦x ∈ H; y ∈ H⟧ ⟹ x ⊗ y ∈ H"
  by (simp add: H_def coset_mult_assoc [symmetric])

lemma H_not_empty: "H ≠ {}"
  apply (simp add: H_def)
  apply (rule exI [of _ 1])
  apply simp
  done

lemma H_is_subgroup: "subgroup H G"
  apply (rule subgroupI)
     apply (rule subsetI)
     apply (erule H_into_carrier_G)
    apply (rule H_not_empty)
   apply (simp add: H_def)
```

```
   apply clarify
   apply (erule_tac P = "λz. lhs z = M1" for lhs in subst)
   apply (simp add: coset_mult_assoc )
  apply (blast intro: H_m_closed)
  done
```

```
lemma rcosetGM1g_subset_G: "⟦g ∈ carrier G; x ∈ M1 #> g⟧ ⟹ x ∈ carrier
G"
  by (blast intro: M1_subset_G [THEN r_coset_subset_G, THEN subsetD])

lemma finite_M1: "finite M1"
  by (rule finite_subset [OF M1_subset_G finite_G])

lemma finite_rcosetGM1g: "g ∈ carrier G ⟹ finite (M1 #> g)"
  using rcosetGM1g_subset_G finite_G M1_subset_G cosets_finite rcosetsI
by blast

lemma M1_cardeq_rcosetGM1g: "g ∈ carrier G ⟹ card (M1 #> g) = card
M1"
  by (simp add: card_cosets_equal rcosetsI)

lemma M1_RelM_rcosetGM1g: "g ∈ carrier G ⟹ (M1, M1 #> g) ∈ RelM"
  apply (simp add: RelM_def calM_def card_M1)
  apply (rule conjI)
   apply (blast intro: rcosetGM1g_subset_G)
  apply (simp add: card_M1 M1_cardeq_rcosetGM1g)
  apply (metis M1_subset_G coset_mult_assoc coset_mult_one r_inv_ex)
  done

end
```

8.3 Equal Cardinalities of `M` and the Set of Cosets

Injections between `M` and `rcosets`$_G$ `H` show that their cardinalities are equal.

```
lemma ElemClassEquiv: "⟦equiv A r; C ∈ A // r⟧ ⟹ ∀x ∈ C. ∀y ∈ C. (x,
y) ∈ r"
  unfolding equiv_def quotient_def sym_def trans_def by blast

context sylow_central
begin

lemma M_elem_map: "M2 ∈ M ⟹ ∃g. g ∈ carrier G ∧ M1 #> g = M2"
  using M1_in_M M_in_quot [THEN RelM_equiv [THEN ElemClassEquiv]]
  by (simp add: RelM_def) (blast dest!: bspec)

lemmas M_elem_map_carrier = M_elem_map [THEN someI_ex, THEN conjunct1]

lemmas M_elem_map_eq = M_elem_map [THEN someI_ex, THEN conjunct2]
```

```
lemma M_funcset_rcosets_H:
  "(λx∈M. H #> (SOME g. g ∈ carrier G ∧ M1 #> g = x)) ∈ M → rcosets
H"
  by (metis (lifting) H_is_subgroup M_elem_map_carrier rcosetsI restrictI
subgroup_imp_subset)

lemma inj_M_GmodH: "∃f ∈ M → rcosets H. inj_on f M"
  apply (rule bexI)
   apply (rule_tac [2] M_funcset_rcosets_H)
  apply (rule inj_onI, simp)
  apply (rule trans [OF _ M_elem_map_eq])
   prefer 2 apply assumption
  apply (rule M_elem_map_eq [symmetric, THEN trans], assumption)
  apply (rule coset_mult_inv1)
     apply (erule_tac [2] M_elem_map_carrier)+
   apply (rule_tac [2] M1_subset_G)
  apply (rule coset_join1 [THEN in_H_imp_eq])
    apply (rule_tac [3] H_is_subgroup)
   prefer 2 apply (blast intro: M_elem_map_carrier)
  apply (simp add: coset_mult_inv2 H_def M_elem_map_carrier subset_eq)
  done

end
```

8.3.1 The Opposite Injection

```
context sylow_central
begin

lemma H_elem_map: "H1 ∈ rcosets H ⟹ ∃g. g ∈ carrier G ∧ H #> g =
H1"
  by (auto simp: RCOSETS_def)

lemmas H_elem_map_carrier = H_elem_map [THEN someI_ex, THEN conjunct1]

lemmas H_elem_map_eq = H_elem_map [THEN someI_ex, THEN conjunct2]

lemma rcosets_H_funcset_M:
  "(λC ∈ rcosets H. M1 #> (@g. g ∈ carrier G ∧ H #> g = C)) ∈ rcosets
H → M"
  apply (simp add: RCOSETS_def)
  apply (fast intro: someI2
      intro!: M1_in_M in_quotient_imp_closed [OF RelM_equiv M_in_quot
_  M1_RelM_rcosetGM1g])
  done
```

Close to a duplicate of `inj_M_GmodH`.

```
lemma inj_GmodH_M: "∃g ∈ rcosets H→M. inj_on g (rcosets H)"
```

```
  apply (rule bexI)
   apply (rule_tac [2] rcosets_H_funcset_M)
  apply (rule inj_onI)
  apply (simp)
  apply (rule trans [OF _ H_elem_map_eq])
   prefer 2 apply assumption
  apply (rule H_elem_map_eq [symmetric, THEN trans], assumption)
  apply (rule coset_mult_inv1)
     apply (erule_tac [2] H_elem_map_carrier)+
   apply (rule_tac [2] H_is_subgroup [THEN subgroup.subset])
  apply (rule coset_join2)
    apply (blast intro: H_elem_map_carrier)
   apply (rule H_is_subgroup)
  apply (simp add: H_I coset_mult_inv2 H_elem_map_carrier)
  done

lemma calM_subset_PowG: "calM ⊆ Pow (carrier G)"
  by (auto simp: calM_def)


lemma finite_M: "finite M"
  by (metis M_subset_calM finite_calM rev_finite_subset)

lemma cardMeqIndexH: "card M = card (rcosets H)"
  apply (insert inj_M_GmodH inj_GmodH_M)
  apply (blast intro: card_bij finite_M H_is_subgroup
      rcosets_subset_PowG [THEN finite_subset]
      finite_Pow_iff [THEN iffD2])
  done

lemma index_lem: "card M * card H = order G"
  by (simp add: cardMeqIndexH lagrange H_is_subgroup)

lemma lemma_leq1: "p^a ≤ card H"
  apply (rule dvd_imp_le)
   apply (rule div_combine [OF prime_imp_prime_elem[OF prime_p] not_dvd_M])
   prefer 2 apply (blast intro: subgroup.finite_imp_card_positive H_is_subgroup)
  apply (simp add: index_lem order_G power_add mult_dvd_mono multiplicity_dvd
zero_less_m)
  done

lemma lemma_leq2: "card H ≤ p^a"
  apply (subst card_M1 [symmetric])
  apply (cut_tac M1_inj_H)
  apply (blast intro!: M1_subset_G intro: card_inj H_into_carrier_G finite_subset
[OF _ finite_G])
  done

lemma card_H_eq: "card H = p^a"
```

```
  by (blast intro: le_antisym lemma_leq1 lemma_leq2)

end

lemma (in sylow) sylow_thm: "∃H. subgroup H G ∧ card H = p^a"
  using lemma_A1
  apply clarify
  apply (frule existsM1inM, clarify)
  apply (subgoal_tac "sylow_central G p a m M1 M")
   apply (blast dest: sylow_central.H_is_subgroup sylow_central.card_H_eq)
  apply (simp add: sylow_central_def sylow_central_axioms_def sylow_axioms
calM_def RelM_def)
  done
```

Needed because the locale's automatic definition refers to `semigroup G` and `Group.group_axioms G` rather than simply to `Group.group G`.

```
lemma sylow_eq: "sylow G p a m ⟷ group G ∧ sylow_axioms G p a m"
  by (simp add: sylow_def group_def)
```

8.4 Sylow's Theorem

```
theorem sylow_thm:
  "⟦prime p; group G; order G = (p^a) * m; finite (carrier G)⟧
    ⟹ ∃H. subgroup H G ∧ card H = p^a"
  by (rule sylow.sylow_thm [of G p a m]) (simp add: sylow_eq sylow_axioms_def)

end


theory Bij
imports Group
begin
```

9 Bijections of a Set, Permutation and Automorphism Groups

```
definition
  Bij :: "'a set ⇒ ('a ⇒ 'a) set"
    — Only extensional functions, since otherwise we get too many.
   where "Bij S = extensional S ∩ {f. bij_betw f S S}"

definition
  BijGroup :: "'a set ⇒ ('a ⇒ 'a) monoid"
  where "BijGroup S =
    ⦇carrier = Bij S,
     mult = λg ∈ Bij S. λf ∈ Bij S. compose S g f,
     one = λx ∈ S. x⦈"
```

```
declare Id_compose [simp] compose_Id [simp]

lemma Bij_imp_extensional: "f ∈ Bij S ⟹ f ∈ extensional S"
  by (simp add: Bij_def)

lemma Bij_imp_funcset: "f ∈ Bij S ⟹ f ∈ S → S"
  by (auto simp add: Bij_def bij_betw_imp_funcset)
```

9.1 Bijections Form a Group

```
lemma restrict_inv_into_Bij: "f ∈ Bij S ⟹ (λx ∈ S. (inv_into S f)
x) ∈ Bij S"
  by (simp add: Bij_def bij_betw_inv_into)

lemma id_Bij: "(λx∈S. x) ∈ Bij S "
  by (auto simp add: Bij_def bij_betw_def inj_on_def)

lemma compose_Bij: "⟦x ∈ Bij S; y ∈ Bij S⟧ ⟹ compose S x y ∈ Bij S"
  by (auto simp add: Bij_def bij_betw_compose)

lemma Bij_compose_restrict_eq:
     "f ∈ Bij S ⟹ compose S (restrict (inv_into S f) S) f = (λx∈S.
x)"
  by (simp add: Bij_def compose_inv_into_id)

theorem group_BijGroup: "group (BijGroup S)"
apply (simp add: BijGroup_def)
apply (rule groupI)
    apply (simp add: compose_Bij)
   apply (simp add: id_Bij)
  apply (simp add: compose_Bij)
  apply (blast intro: compose_assoc [symmetric] dest: Bij_imp_funcset)
 apply (simp add: id_Bij Bij_imp_funcset Bij_imp_extensional, simp)
apply (blast intro: Bij_compose_restrict_eq restrict_inv_into_Bij)
done
```

9.2 Automorphisms Form a Group

```
lemma Bij_inv_into_mem: "⟦ f ∈ Bij S;  x ∈ S⟧ ⟹ inv_into S f x ∈ S"
by (simp add: Bij_def bij_betw_def inv_into_into)

lemma Bij_inv_into_lemma:
 assumes eq: "⋀x y. ⟦x ∈ S; y ∈ S⟧ ⟹ h(g x y) = g (h x) (h y)"
 shows "⟦h ∈ Bij S;  g ∈ S → S → S;  x ∈ S;  y ∈ S⟧
        ⟹ inv_into S h (g x y) = g (inv_into S h x) (inv_into S h y)"
apply (simp add: Bij_def bij_betw_def)
apply (subgoal_tac "∃x'∈S. ∃y'∈S. x = h x' & y = h y'", clarify)
 apply (simp add: eq [symmetric] inv_f_f funcset_mem [THEN funcset_mem],
blast)
```

```
done


definition
  auto :: "('a, 'b) monoid_scheme ⇒ ('a ⇒ 'a) set"
  where "auto G = hom G G ∩ Bij (carrier G)"

definition
  AutoGroup :: "('a, 'c) monoid_scheme ⇒ ('a ⇒ 'a) monoid"
  where "AutoGroup G = BijGroup (carrier G) (|carrier := auto G|)"

lemma (in group) id_in_auto: "(λx ∈ carrier G. x) ∈ auto G"
  by (simp add: auto_def hom_def restrictI group.axioms id_Bij)

lemma (in group) mult_funcset: "mult G ∈ carrier G → carrier G → carrier
G"
  by (simp add:  Pi_I group.axioms)

lemma (in group) restrict_inv_into_hom:
      "⟦h ∈ hom G G; h ∈ Bij (carrier G)⟧
       ⟹ restrict (inv_into (carrier G) h) (carrier G) ∈ hom G G"
  by (simp add: hom_def Bij_inv_into_mem restrictI mult_funcset
                group.axioms Bij_inv_into_lemma)

lemma inv_BijGroup:
     "f ∈ Bij S ⟹ m_inv (BijGroup S) f = (λx ∈ S. (inv_into S f) x)"
apply (rule group.inv_equality)
apply (rule group_BijGroup)
apply (simp_all add:BijGroup_def restrict_inv_into_Bij Bij_compose_restrict_eq)
done

lemma (in group) subgroup_auto:
      "subgroup (auto G) (BijGroup (carrier G))"
proof (rule subgroup.intro)
  show "auto G ⊆ carrier (BijGroup (carrier G))"
    by (force simp add: auto_def BijGroup_def)
next
  fix x y
  assume "x ∈ auto G" "y ∈ auto G"
  thus "x ⊗BijGroup (carrier G) y ∈ auto G"
    by (force simp add: BijGroup_def is_group auto_def Bij_imp_funcset

                       group.hom_compose compose_Bij)
next
  show "𝟭BijGroup (carrier G) ∈ auto G" by (simp add:  BijGroup_def id_in_auto)
next
  fix x
  assume "x ∈ auto G"
  thus "invBijGroup (carrier G) x ∈ auto G"
```

```
  by (simp del: restrict_apply
      add: inv_BijGroup auto_def restrict_inv_into_Bij restrict_inv_into_hom)
qed

theorem (in group) AutoGroup: "group (AutoGroup G)"
by (simp add: AutoGroup_def subgroup.subgroup_is_group subgroup_auto
              group_BijGroup)

end


theory Ring
imports FiniteProduct
begin
```

10 The Algebraic Hierarchy of Rings

10.1 Abelian Groups

```
record 'a ring = "'a monoid" +
  zero :: 'a ("0ı")
  add :: "['a, 'a] => 'a" (infixl "⊕ı" 65)
```

Derived operations.

```
definition
  a_inv :: "[('a, 'm) ring_scheme, 'a ] => 'a" ("⊖ı _" [81] 80)
  where "a_inv R = m_inv ⦇carrier = carrier R, mult = add R, one = zero
R⦈"

definition
  a_minus :: "[('a, 'm) ring_scheme, 'a, 'a] => 'a" (infixl "⊖ı" 65)
  where "[| x ∈ carrier R; y ∈ carrier R |] ==> x ⊖R y = x ⊕R (⊖R y)"

locale abelian_monoid =
  fixes G (structure)
  assumes a_comm_monoid:
     "comm_monoid ⦇carrier = carrier G, mult = add G, one = zero G⦈"

definition
  finsum :: "[('b, 'm) ring_scheme, 'a => 'b, 'a set] => 'b" where
  "finsum G = finprod ⦇carrier = carrier G, mult = add G, one = zero G⦈"

syntax
  "_finsum" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨁__∈_. _)" [1000, 0, 51, 10] 10)
translations
  "⨁Gi∈A. b" ⇌ "CONST finsum G (%i. b) A"
  — Beware of argument permutation!
```

```
locale abelian_group = abelian_monoid +
  assumes a_comm_group:
     "comm_group (|carrier = carrier G, mult = add G, one = zero G|)"
```

10.2 Basic Properties

```
lemma abelian_monoidI:
  fixes R (structure)
  assumes a_closed:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y ∈ carrier
R"
    and zero_closed: "0 ∈ carrier R"
    and a_assoc:
      "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |] ==>
      (x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)"
    and l_zero: "!!x. x ∈ carrier R ==> 0 ⊕ x = x"
    and a_comm:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y = y ⊕ x"
  shows "abelian_monoid R"
  by (auto intro!: abelian_monoid.intro comm_monoidI intro: assms)

lemma abelian_groupI:
  fixes R (structure)
  assumes a_closed:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y ∈ carrier
R"
    and zero_closed: "zero R ∈ carrier R"
    and a_assoc:
      "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |] ==>
      (x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)"
    and a_comm:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y = y ⊕ x"
    and l_zero: "!!x. x ∈ carrier R ==> 0 ⊕ x = x"
    and l_inv_ex: "!!x. x ∈ carrier R ==> EX y : carrier R. y ⊕ x = 0"
  shows "abelian_group R"
  by (auto intro!: abelian_group.intro abelian_monoidI
      abelian_group_axioms.intro comm_monoidI comm_groupI
    intro: assms)

lemma (in abelian_monoid) a_monoid:
  "monoid (|carrier = carrier G, mult = add G, one = zero G|)"
by (rule comm_monoid.axioms, rule a_comm_monoid)

lemma (in abelian_group) a_group:
  "group (|carrier = carrier G, mult = add G, one = zero G|)"
  by (simp add: group_def a_monoid)
    (simp add: comm_group.axioms group.axioms a_comm_group)
```

```
lemmas monoid_record_simps = partial_object.simps monoid.simps
```

Transfer facts from multiplicative structures via interpretation.

```
sublocale abelian_monoid <
  add: monoid "⦇carrier = carrier G, mult = add G, one = zero G⦈"
  rewrites "carrier ⦇carrier = carrier G, mult = add G, one = zero G⦈
= carrier G"
    and "mult ⦇carrier = carrier G, mult = add G, one = zero G⦈ = add
G"
    and "one ⦇carrier = carrier G, mult = add G, one = zero G⦈ = zero
G"
  by (rule a_monoid) auto

context abelian_monoid begin

lemmas a_closed = add.m_closed
lemmas zero_closed = add.one_closed
lemmas a_assoc = add.m_assoc
lemmas l_zero = add.l_one
lemmas r_zero = add.r_one
lemmas minus_unique = add.inv_unique

end

sublocale abelian_monoid <
  add: comm_monoid "⦇carrier = carrier G, mult = add G, one = zero G⦈"
  rewrites "carrier ⦇carrier = carrier G, mult = add G, one = zero G⦈
= carrier G"
    and "mult ⦇carrier = carrier G, mult = add G, one = zero G⦈ = add
G"
    and "one ⦇carrier = carrier G, mult = add G, one = zero G⦈ = zero
G"
    and "finprod ⦇carrier = carrier G, mult = add G, one = zero G⦈ = finsum
G"
  by (rule a_comm_monoid) (auto simp: finsum_def)

context abelian_monoid begin

lemmas a_comm = add.m_comm
lemmas a_lcomm = add.m_lcomm
lemmas a_ac = a_assoc a_comm a_lcomm

lemmas finsum_empty = add.finprod_empty
lemmas finsum_insert = add.finprod_insert
lemmas finsum_zero = add.finprod_one
lemmas finsum_closed = add.finprod_closed
lemmas finsum_Un_Int = add.finprod_Un_Int
lemmas finsum_Un_disjoint = add.finprod_Un_disjoint
lemmas finsum_addf = add.finprod_multf
```

```
lemmas finsum_cong' = add.finprod_cong'
lemmas finsum_0 = add.finprod_0
lemmas finsum_Suc = add.finprod_Suc
lemmas finsum_Suc2 = add.finprod_Suc2
lemmas finsum_add = add.finprod_mult
lemmas finsum_infinite = add.finprod_infinite

lemmas finsum_cong = add.finprod_cong
```

Usually, if this rule causes a failed congruence proof error, the reason is that the premise `g ∈ B → carrier G` cannot be shown. Adding `Pi_def` to the simpset is often useful.

```
lemmas finsum_reindex = add.finprod_reindex



lemmas finsum_singleton = add.finprod_singleton

end

sublocale abelian_group <
  add: group "⦇carrier = carrier G, mult = add G, one = zero G⦈"
  rewrites "carrier ⦇carrier = carrier G, mult = add G, one = zero G⦈
= carrier G"
    and "mult ⦇carrier = carrier G, mult = add G, one = zero G⦈ = add
G"
    and "one ⦇carrier = carrier G, mult = add G, one = zero G⦈ = zero
G"
    and "m_inv ⦇carrier = carrier G, mult = add G, one = zero G⦈ = a_inv
G"
  by (rule a_group) (auto simp: m_inv_def a_inv_def)

context abelian_group
begin

lemmas a_inv_closed = add.inv_closed

lemma minus_closed [intro, simp]:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> x ⊖ y ∈ carrier G"
  by (simp add: a_minus_def)

lemmas a_l_cancel = add.l_cancel
lemmas a_r_cancel = add.r_cancel
lemmas l_neg = add.l_inv [simp del]
lemmas r_neg = add.r_inv [simp del]
lemmas minus_zero = add.inv_one
lemmas minus_minus = add.inv_inv
lemmas a_inv_inj = add.inv_inj
lemmas minus_equality = add.inv_equality
```

```
end

sublocale abelian_group <
  add: comm_group "⦇carrier = carrier G, mult = add G, one = zero G⦈"
  rewrites "carrier ⦇carrier = carrier G, mult = add G, one = zero G⦈
= carrier G"
    and "mult ⦇carrier = carrier G, mult = add G, one = zero G⦈ = add
G"
    and "one ⦇carrier = carrier G, mult = add G, one = zero G⦈ = zero
G"
    and "m_inv ⦇carrier = carrier G, mult = add G, one = zero G⦈ = a_inv
G"
    and "finprod ⦇carrier = carrier G, mult = add G, one = zero G⦈ = finsum
G"
  by (rule a_comm_group) (auto simp: m_inv_def a_inv_def finsum_def)

lemmas (in abelian_group) minus_add = add.inv_mult
```

Derive an `abelian_group` from a `comm_group`

```
lemma comm_group_abelian_groupI:
  fixes G (structure)
  assumes cg: "comm_group ⦇carrier = carrier G, mult = add G, one = zero
G⦈"
  shows "abelian_group G"
proof -
  interpret comm_group "⦇carrier = carrier G, mult = add G, one = zero
G⦈"
    by (rule cg)
  show "abelian_group G" ..
qed
```

10.3 Rings: Basic Definitions

```
locale semiring = abelian_monoid R + monoid R for R (structure) +
  assumes l_distr: "[| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
      ==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
    and r_distr: "[| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
      ==> z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y"
    and l_null[simp]: "x ∈ carrier R ==> 0 ⊗ x = 0"
    and r_null[simp]: "x ∈ carrier R ==> x ⊗ 0 = 0"

locale ring = abelian_group R + monoid R for R (structure) +
  assumes "[| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
      ==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
    and "[| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
      ==> z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y"

locale cring = ring + comm_monoid R
```

```
locale "domain" = cring +
  assumes one_not_zero [simp]: "1 ~= 0"
    and integral: "[| a ⊗ b = 0; a ∈ carrier R; b ∈ carrier R |] ==>
                  a = 0 | b = 0"

locale field = "domain" +
  assumes field_Units: "Units R = carrier R - {0}"
```

10.4 Rings

```
lemma ringI:
  fixes R (structure)
  assumes abelian_group: "abelian_group R"
    and monoid: "monoid R"
    and l_distr: "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier
R |]
      ==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
    and r_distr: "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier
R |]
      ==> z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y"
  shows "ring R"
  by (auto intro: ring.intro
    abelian_group.axioms ring_axioms.intro assms)

context ring begin

lemma is_abelian_group: "abelian_group R" ..

lemma is_monoid: "monoid R"
  by (auto intro!: monoidI m_assoc)

lemma is_ring: "ring R"
  by (rule ring_axioms)

end

lemmas ring_record_simps = monoid_record_simps ring.simps

lemma cringI:
  fixes R (structure)
  assumes abelian_group: "abelian_group R"
    and comm_monoid: "comm_monoid R"
    and l_distr: "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier
R |]
      ==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
  shows "cring R"
proof (intro cring.intro ring.intro)
  show "ring_axioms R"
```

```
    — Right-distributivity follows from left-distributivity and commutativity.
  proof (rule ring_axioms.intro)
    fix x y z
    assume R: "x ∈ carrier R" "y ∈ carrier R" "z ∈ carrier R"
    note [simp] = comm_monoid.axioms [OF comm_monoid]
      abelian_group.axioms [OF abelian_group]
      abelian_monoid.a_closed

    from R have "z ⊗ (x ⊕ y) = (x ⊕ y) ⊗ z"
      by (simp add: comm_monoid.m_comm [OF comm_monoid.intro])
    also from R have "... = x ⊗ z ⊕ y ⊗ z" by (simp add: l_distr)
    also from R have "... = z ⊗ x ⊕ z ⊗ y"
      by (simp add: comm_monoid.m_comm [OF comm_monoid.intro])
    finally show "z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y" .
  qed (rule l_distr)
qed (auto intro: cring.intro
  abelian_group.axioms comm_monoid.axioms ring_axioms.intro assms)
```

```
lemma (in cring) is_cring:
  "cring R" by (rule cring_axioms)
```

10.4.1 Normaliser for Rings

```
lemma (in abelian_group) r_neg2:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> x ⊕ (⊖ x ⊕ y) = y"
proof -
  assume G: "x ∈ carrier G" "y ∈ carrier G"
  then have "(x ⊕ ⊖ x) ⊕ y = y"
    by (simp only: r_neg l_zero)
  with G show ?thesis
    by (simp add: a_ac)
qed

lemma (in abelian_group) r_neg1:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> ⊖ x ⊕ (x ⊕ y) = y"
proof -
  assume G: "x ∈ carrier G" "y ∈ carrier G"
  then have "(⊖ x ⊕ x) ⊕ y = y"
    by (simp only: l_neg l_zero)
  with G show ?thesis by (simp add: a_ac)
qed

context ring begin
```

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89.

```
sublocale semiring
proof -
```

```
  note [simp] = ring_axioms[unfolded ring_def ring_axioms_def]
  show "semiring R"
  proof (unfold_locales)
    fix x
    assume R: "x ∈ carrier R"
    then have "0 ⊗ x ⊕ 0 ⊗ x = (0 ⊕ 0) ⊗ x"
      by (simp del: l_zero r_zero)
    also from R have "... = 0 ⊗ x ⊕ 0" by simp
    finally have "0 ⊗ x ⊕ 0 ⊗ x = 0 ⊗ x ⊕ 0" .
    with R show "0 ⊗ x = 0" by (simp del: r_zero)
    from R have "x ⊗ 0 ⊕ x ⊗ 0 = x ⊗ (0 ⊕ 0)"
      by (simp del: l_zero r_zero)
    also from R have "... = x ⊗ 0 ⊕ 0" by simp
    finally have "x ⊗ 0 ⊕ x ⊗ 0 = x ⊗ 0 ⊕ 0" .
    with R show "x ⊗ 0 = 0" by (simp del: r_zero)
  qed auto
qed

lemma l_minus:
  "[| x ∈ carrier R; y ∈ carrier R |] ==> ⊖ x ⊗ y = ⊖ (x ⊗ y)"
proof -
  assume R: "x ∈ carrier R" "y ∈ carrier R"
  then have "(⊖ x) ⊗ y ⊕ x ⊗ y = (⊖ x ⊕ x) ⊗ y" by (simp add: l_distr)
  also from R have "... = 0" by (simp add: l_neg)
  finally have "(⊖ x) ⊗ y ⊕ x ⊗ y = 0" .
  with R have "(⊖ x) ⊗ y ⊕ x ⊗ y ⊕ ⊖ (x ⊗ y) = 0 ⊕ ⊖ (x ⊗ y)" by
simp
  with R show ?thesis by (simp add: a_assoc r_neg)
qed

lemma r_minus:
  "[| x ∈ carrier R; y ∈ carrier R |] ==> x ⊗ ⊖ y = ⊖ (x ⊗ y)"
proof -
  assume R: "x ∈ carrier R" "y ∈ carrier R"
  then have "x ⊗ (⊖ y) ⊕ x ⊗ y = x ⊗ (⊖ y ⊕ y)" by (simp add: r_distr)
  also from R have "... = 0" by (simp add: l_neg)
  finally have "x ⊗ (⊖ y) ⊕ x ⊗ y = 0" .
  with R have "x ⊗ (⊖ y) ⊕ x ⊗ y ⊕ ⊖ (x ⊗ y) = 0 ⊕ ⊖ (x ⊗ y)" by
simp
  with R show ?thesis by (simp add: a_assoc r_neg )
qed

end

lemma (in abelian_group) minus_eq:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> x ⊖ y = x ⊕ ⊖ y"
  by (simp only: a_minus_def)
```

Setup algebra method: compute distributive normal form in locale contexts

```
ML_file "ringsimp.ML"

attribute_setup algebra = ‹
  Scan.lift ((Args.add >> K true || Args.del >> K false) --| Args.colon
|| Scan.succeed true)
    -- Scan.lift Args.name -- Scan.repeat Args.term
    >> (fn ((b, n), ts) => if b then Ringsimp.add_struct (n, ts) else
Ringsimp.del_struct (n, ts))
› "theorems controlling algebra method"

method_setup algebra = ‹
  Scan.succeed (SIMPLE_METHOD' o Ringsimp.algebra_tac)
› "normalisation of algebraic structure"

lemmas (in semiring) semiring_simprules
  [algebra ring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult R"]
=
  a_closed zero_closed  m_closed one_closed
  a_assoc l_zero  a_comm m_assoc l_one l_distr r_zero
  a_lcomm r_distr l_null r_null

lemmas (in ring) ring_simprules
  [algebra ring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult R"]
=
  a_closed zero_closed a_inv_closed minus_closed m_closed one_closed
  a_assoc l_zero l_neg a_comm m_assoc l_one l_distr minus_eq
  r_zero r_neg r_neg2 r_neg1 minus_add minus_minus minus_zero
  a_lcomm r_distr l_null r_null l_minus r_minus

lemmas (in cring)
  [algebra del: ring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult
R"] =
  -

lemmas (in cring) cring_simprules
  [algebra add: cring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult
R"] =
  a_closed zero_closed a_inv_closed minus_closed m_closed one_closed
  a_assoc l_zero l_neg a_comm m_assoc l_one l_distr m_comm minus_eq
  r_zero r_neg r_neg2 r_neg1 minus_add minus_minus minus_zero
  a_lcomm m_lcomm r_distr l_null r_null l_minus r_minus

lemma (in semiring) nat_pow_zero:
  "(n::nat) ~= 0 ==> 0 (^) n = 0"
  by (induct n) simp_all

context semiring begin

lemma one_zeroD:
```

```
  assumes onezero: "1 = 0"
  shows "carrier R = {0}"
proof (rule, rule)
  fix x
  assume xcarr: "x ∈ carrier R"
  from xcarr have "x = x ⊗ 1" by simp
  with onezero have "x = x ⊗ 0" by simp
  with xcarr have "x = 0" by simp
  then show "x ∈ {0}" by fast
qed fast

lemma one_zeroI:
  assumes carrzero: "carrier R = {0}"
  shows "1 = 0"
proof -
  from one_closed and carrzero
      show "1 = 0" by simp
qed

lemma carrier_one_zero: "(carrier R = {0}) = (1 = 0)"
  apply rule
   apply (erule one_zeroI)
  apply (erule one_zeroD)
  done

lemma carrier_one_not_zero: "(carrier R ≠ {0}) = (1 ≠ 0)"
  by (simp add: carrier_one_zero)

end
```

Two examples for use of method algebra

```
lemma
  fixes R (structure) and S (structure)
  assumes "ring R" "cring S"
  assumes RS: "a ∈ carrier R" "b ∈ carrier R" "c ∈ carrier S" "d ∈ carrier
S"
  shows "a ⊕ ⊖ (a ⊕ ⊖ b) = b & c ⊗S d = d ⊗S c"
proof -
  interpret ring R by fact
  interpret cring S by fact
  from RS show ?thesis by algebra
qed

lemma
  fixes R (structure)
  assumes "ring R"
  assumes R: "a ∈ carrier R" "b ∈ carrier R"
  shows "a ⊖ (a ⊖ b) = b"
proof -
```

```
  interpret ring R by fact
  from R show ?thesis by algebra
qed
```

10.4.2 Sums over Finite Sets

```
lemma (in semiring) finsum_ldistr:
  "[| finite A; a ∈ carrier R; f ∈ A → carrier R |] ==>
   finsum R f A ⊗ a = finsum R (%i. f i ⊗ a) A"
proof (induct set: finite)
  case empty then show ?case by simp
next
  case (insert x F) then show ?case by (simp add: Pi_def l_distr)
qed

lemma (in semiring) finsum_rdistr:
  "[| finite A; a ∈ carrier R; f ∈ A → carrier R |] ==>
   a ⊗ finsum R f A = finsum R (%i. a ⊗ f i) A"
proof (induct set: finite)
  case empty then show ?case by simp
next
  case (insert x F) then show ?case by (simp add: Pi_def r_distr)
qed
```

10.5 Integral Domains

```
context "domain" begin

lemma zero_not_one [simp]:
  "0 ~= 1"
  by (rule not_sym) simp

lemma integral_iff:
  "[| a ∈ carrier R; b ∈ carrier R |] ==> (a ⊗ b = 0) = (a = 0 | b =
0)"
proof
  assume "a ∈ carrier R" "b ∈ carrier R" "a ⊗ b = 0"
  then show "a = 0 | b = 0" by (simp add: integral)
next
  assume "a ∈ carrier R" "b ∈ carrier R" "a = 0 | b = 0"
  then show "a ⊗ b = 0" by auto
qed

lemma m_lcancel:
  assumes prem: "a ~= 0"
    and R: "a ∈ carrier R" "b ∈ carrier R" "c ∈ carrier R"
  shows "(a ⊗ b = a ⊗ c) = (b = c)"
proof
  assume eq: "a ⊗ b = a ⊗ c"
  with R have "a ⊗ (b ⊖ c) = 0" by algebra
```

```
  with R have "a = 0 | (b ⊖ c) = 0" by (simp add: integral_iff)
  with prem and R have "b ⊖ c = 0" by auto
  with R have "b = b ⊖ (b ⊖ c)" by algebra
  also from R have "b ⊖ (b ⊖ c) = c" by algebra
  finally show "b = c" .
next
  assume "b = c" then show "a ⊗ b = a ⊗ c" by simp
qed

lemma m_rcancel:
  assumes prem: "a ~= 0"
    and R: "a ∈ carrier R" "b ∈ carrier R" "c ∈ carrier R"
  shows conc: "(b ⊗ a = c ⊗ a) = (b = c)"
proof -
  from prem and R have "(a ⊗ b = a ⊗ c) = (b = c)" by (rule m_lcancel)
  with R show ?thesis by algebra
qed

end
```

10.6 Fields

Field would not need to be derived from domain, the properties for domain follow from the assumptions of field

```
lemma (in cring) cring_fieldI:
  assumes field_Units: "Units R = carrier R - {0}"
  shows "field R"
proof
  from field_Units have "0 ∉ Units R" by fast
  moreover have "1 ∈ Units R" by fast
  ultimately show "1 ≠ 0" by force
next
  fix a b
  assume acarr: "a ∈ carrier R"
    and bcarr: "b ∈ carrier R"
    and ab: "a ⊗ b = 0"
  show "a = 0 ∨ b = 0"
  proof (cases "a = 0", simp)
    assume "a ≠ 0"
    with field_Units and acarr have aUnit: "a ∈ Units R" by fast
    from bcarr have "b = 1 ⊗ b" by algebra
    also from aUnit acarr have "... = (inv a ⊗ a) ⊗ b" by simp
    also from acarr bcarr aUnit[THEN Units_inv_closed]
    have "... = (inv a) ⊗ (a ⊗ b)" by algebra
    also from ab and acarr bcarr aUnit have "... = (inv a) ⊗ 0" by simp
    also from aUnit[THEN Units_inv_closed] have "... = 0" by algebra
    finally have "b = 0" .
    then show "a = 0 ∨ b = 0" by simp
  qed
```

```
qed (rule field_Units)
```

Another variant to show that something is a field

```
lemma (in cring) cring_fieldI2:
  assumes notzero: "0 ≠ 1"
  and invex: "⋀a. ⟦a ∈ carrier R; a ≠ 0⟧ ⟹ ∃b∈carrier R. a ⊗ b =
1"
  shows "field R"
  apply (rule cring_fieldI, simp add: Units_def)
  apply (rule, clarsimp)
  apply (simp add: notzero)
proof (clarsimp)
  fix x
  assume xcarr: "x ∈ carrier R"
    and "x ≠ 0"
  then have "∃y∈carrier R. x ⊗ y = 1" by (rule invex)
  then obtain y where ycarr: "y ∈ carrier R" and xy: "x ⊗ y = 1" by
fast
  from xy xcarr ycarr have "y ⊗ x = 1" by (simp add: m_comm)
  with ycarr and xy show "∃y∈carrier R. y ⊗ x = 1 ∧ x ⊗ y = 1" by
fast
qed
```

10.7 Morphisms

```
definition
  ring_hom :: "[('a, 'm) ring_scheme, ('b, 'n) ring_scheme] => ('a =>
'b) set"
  where "ring_hom R S =
    {h. h ∈ carrier R → carrier S &
      (ALL x y. x ∈ carrier R & y ∈ carrier R -->
        h (x ⊗R y) = h x ⊗S h y & h (x ⊕R y) = h x ⊕S h y) &
      h 1R = 1S}"

lemma ring_hom_memI:
  fixes R (structure) and S (structure)
  assumes hom_closed: "!!x. x ∈ carrier R ==> h x ∈ carrier S"
    and hom_mult: "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==>
      h (x ⊗ y) = h x ⊗S h y"
    and hom_add: "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==>
      h (x ⊕ y) = h x ⊕S h y"
    and hom_one: "h 1 = 1S"
  shows "h ∈ ring_hom R S"
  by (auto simp add: ring_hom_def assms Pi_def)

lemma ring_hom_closed:
  "[| h ∈ ring_hom R S; x ∈ carrier R |] ==> h x ∈ carrier S"
  by (auto simp add: ring_hom_def funcset_mem)
```

```
lemma ring_hom_mult:
  fixes R (structure) and S (structure)
  shows
    "[| h ∈ ring_hom R S; x ∈ carrier R; y ∈ carrier R |] ==>
    h (x ⊗ y) = h x ⊗S h y"
    by (simp add: ring_hom_def)

lemma ring_hom_add:
  fixes R (structure) and S (structure)
  shows
    "[| h ∈ ring_hom R S; x ∈ carrier R; y ∈ carrier R |] ==>
    h (x ⊕ y) = h x ⊕S h y"
    by (simp add: ring_hom_def)

lemma ring_hom_one:
  fixes R (structure) and S (structure)
  shows "h ∈ ring_hom R S ==> h 1 = 1S"
  by (simp add: ring_hom_def)

locale ring_hom_cring = R?: cring R + S?: cring S
    for R (structure) and S (structure) +
  fixes h
  assumes homh [simp, intro]: "h ∈ ring_hom R S"
  notes hom_closed [simp, intro] = ring_hom_closed [OF homh]
    and hom_mult [simp] = ring_hom_mult [OF homh]
    and hom_add [simp] = ring_hom_add [OF homh]
    and hom_one [simp] = ring_hom_one [OF homh]

lemma (in ring_hom_cring) hom_zero [simp]:
  "h 0 = 0S"
proof -
  have "h 0 ⊕S h 0 = h 0 ⊕S 0S"
    by (simp add: hom_add [symmetric] del: hom_add)
  then show ?thesis by (simp del: S.r_zero)
qed

lemma (in ring_hom_cring) hom_a_inv [simp]:
  "x ∈ carrier R ==> h (⊖ x) = ⊖S h x"
proof -
  assume R: "x ∈ carrier R"
  then have "h x ⊕S h (⊖ x) = h x ⊕S (⊖S h x)"
    by (simp add: hom_add [symmetric] R.r_neg S.r_neg del: hom_add)
  with R show ?thesis by simp
qed

lemma (in ring_hom_cring) hom_finsum [simp]:
  "f ∈ A → carrier R ==>
  h (finsum R f A) = finsum S (h o f) A"
  by (induct A rule: infinite_finite_induct, auto simp: Pi_def)
```

```
lemma (in ring_hom_cring) hom_finprod:
  "f ∈ A → carrier R ==>
  h (finprod R f A) = finprod S (h o f) A"
  by (induct A rule: infinite_finite_induct, auto simp: Pi_def)

declare ring_hom_cring.hom_finprod [simp]

lemma id_ring_hom [simp]:
  "id ∈ ring_hom R R"
  by (auto intro!: ring_hom_memI)

end
```

11 More on groups

```
theory More_Group
imports
  Ring
begin
```

Show that the units in any monoid give rise to a group.

The file Residues.thy provides some infrastructure to use facts about the unit group within the ring locale.

```
definition units_of :: "('a, 'b) monoid_scheme => 'a monoid" where
  "units_of G == (| carrier = Units G,
     Group.monoid.mult = Group.monoid.mult G,
     one  = one G |)"

lemma (in monoid) units_group: "group(units_of G)"
  apply (unfold units_of_def)
  apply (rule groupI)
  apply auto
  apply (subst m_assoc)
  apply auto
  apply (rule_tac x = "inv x" in bexI)
  apply auto
  done

lemma (in comm_monoid) units_comm_group: "comm_group(units_of G)"
  apply (rule group.group_comm_groupI)
  apply (rule units_group)
  apply (insert comm_monoid_axioms)
  apply (unfold units_of_def Units_def comm_monoid_def comm_monoid_axioms_def)
  apply auto
  done

lemma units_of_carrier: "carrier (units_of G) = Units G"
```

```
  unfolding units_of_def by auto

lemma units_of_mult: "mult(units_of G) = mult G"
  unfolding units_of_def by auto

lemma units_of_one: "one(units_of G) = one G"
  unfolding units_of_def by auto

lemma (in monoid) units_of_inv: "x : Units G ==> m_inv (units_of G) x
= m_inv G x"
  apply (rule sym)
  apply (subst m_inv_def)
  apply (rule the1_equality)
  apply (rule ex_ex1I)
  apply (subst (asm) Units_def)
  apply auto
  apply (erule inv_unique)
  apply auto
  apply (rule Units_closed)
  apply (simp_all only: units_of_carrier [symmetric])
  apply (insert units_group)
  apply auto
  apply (subst units_of_mult [symmetric])
  apply (subst units_of_one [symmetric])
  apply (erule group.r_inv, assumption)
  apply (subst units_of_mult [symmetric])
  apply (subst units_of_one [symmetric])
  apply (erule group.l_inv, assumption)
  done

lemma (in group) inj_on_const_mult: "a: (carrier G) ==> inj_on (%x. a
⊗ x) (carrier G)"
  unfolding inj_on_def by auto

lemma (in group) surj_const_mult: "a : (carrier G) ==> (%x. a ⊗ x) `
(carrier G) = (carrier G)"
  apply (auto simp add: image_def)
  apply (rule_tac x = "(m_inv G a) ⊗ x" in bexI)
  apply auto

  apply (subst m_assoc [symmetric])
  apply auto
  done

lemma (in group) l_cancel_one [simp]:
    "x : carrier G ⟹ a : carrier G ⟹ (x ⊗ a = x) = (a = one G)"
  apply auto
  apply (subst l_cancel [symmetric])
  prefer 4
```

```
  apply (erule ssubst)
  apply auto
  done

lemma (in group) r_cancel_one [simp]: "x : carrier G ⟹ a : carrier
G ⟹
    (a ⊗ x = x) = (a = one G)"
  apply auto
  apply (subst r_cancel [symmetric])
  prefer 4
  apply (erule ssubst)
  apply auto
  done


lemma (in group) l_cancel_one' [simp]: "x : carrier G ⟹ a : carrier
G ⟹
    (x = x ⊗ a) = (a = one G)"
  apply (subst eq_commute)
  apply simp
  done

lemma (in group) r_cancel_one' [simp]: "x : carrier G ⟹ a : carrier
G ⟹
    (x = a ⊗ x) = (a = one G)"
  apply (subst eq_commute)
  apply simp
  done



lemma (in comm_group) power_order_eq_one:
  assumes fin [simp]: "finite (carrier G)"
    and a [simp]: "a : carrier G"
  shows "a (^) card(carrier G) = one G"
proof -
  have "(⨂x∈carrier G. x) = (⨂x∈carrier G. a ⊗ x)"
    by (subst (2) finprod_reindex [symmetric],
      auto simp add: Pi_def inj_on_const_mult surj_const_mult)
  also have "... = (⨂x∈carrier G. a) ⊗ (⨂x∈carrier G. x)"
    by (auto simp add: finprod_multf Pi_def)
  also have "(⨂x∈carrier G. a) = a (^) card(carrier G)"
    by (auto simp add: finprod_const)
  finally show ?thesis

    by auto
qed

end
```

12 More on finite products

```
theory More_Finite_Product
imports
  More_Group
begin

lemma (in comm_monoid) finprod_UN_disjoint:
  "finite I ⟹ (ALL i:I. finite (A i)) ⟶ (ALL i:I. ALL j:I. i ~= j
⟶
     (A i) Int (A j) = {}) ⟶
      (ALL i:I. ALL x: (A i). g x : carrier G) ⟶
        finprod G g (UNION I A) = finprod G (%i. finprod G g (A i)) I"
  apply (induct set: finite)
  apply force
  apply clarsimp
  apply (subst finprod_Un_disjoint)
  apply blast
  apply (erule finite_UN_I)
  apply blast
  apply (fastforce)
  apply (auto intro!: funcsetI finprod_closed)
  done

lemma (in comm_monoid) finprod_Union_disjoint:
  "[| finite C; (ALL A:C. finite A & (ALL x:A. f x : carrier G));
      (ALL A:C. ALL B:C. A ~= B --> A Int B = {}) |]
   ==> finprod G f (⋃C) = finprod G (finprod G f) C"
  apply (frule finprod_UN_disjoint [of C id f])
  apply auto
  done

lemma (in comm_monoid) finprod_one:
    "finite A ⟹ (⋀x. x:A ⟹ f x = 1) ⟹ finprod G f A = 1"
  by (induct set: finite) auto




lemma (in cring) sum_zero_eq_neg: "x : carrier R ⟹ y : carrier R ⟹
x ⊕ y = 0 ⟹ x = ⊖ y"
  by (metis minus_equality)

lemma (in domain) square_eq_one:
  fixes x
  assumes [simp]: "x : carrier R"
    and "x ⊗ x = 1"
  shows "x = 1 | x = ⊖1"
```

```
proof -
  have "(x ⊕ 1) ⊗ (x ⊕ ⊖ 1) = x ⊗ x ⊕ ⊖ 1"
    by (simp add: ring_simprules)
  also from ‹x ⊗ x = 1› have "... = 0"
    by (simp add: ring_simprules)
  finally have "(x ⊕ 1) ⊗ (x ⊕ ⊖ 1) = 0" .
  then have "(x ⊕ 1) = 0 | (x ⊕ ⊖ 1) = 0"
    by (intro integral, auto)
  then show ?thesis
    apply auto
    apply (erule notE)
    apply (rule sum_zero_eq_neg)
    apply auto
    apply (subgoal_tac "x = ⊖ (⊖ 1)")
    apply (simp add: ring_simprules)
    apply (rule sum_zero_eq_neg)
    apply auto
    done
qed

lemma (in Ring.domain) inv_eq_self: "x : Units R ⟹ x = inv x ⟹ x
= 1 ∨ x = ⊖1"
  by (metis Units_closed Units_l_inv square_eq_one)
```

The following translates theorems about groups to the facts about the units of a ring. (The list should be expanded as more things are needed.)

```
lemma (in ring) finite_ring_finite_units [intro]: "finite (carrier R)
⟹ finite (Units R)"
  by (rule finite_subset) auto

lemma (in monoid) units_of_pow:
  fixes n :: nat
  shows "x ∈ Units G ⟹ x (^)units_of G n = x (^)G n"
  apply (induct n)
  apply (auto simp add: units_group group.is_monoid
    monoid.nat_pow_0 monoid.nat_pow_Suc units_of_one units_of_mult)
  done

lemma (in cring) units_power_order_eq_one: "finite (Units R) ⟹ a :
Units R
    ⟹ a (^) card(Units R) = 1"
  apply (subst units_of_carrier [symmetric])
  apply (subst units_of_one [symmetric])
  apply (subst units_of_pow [symmetric])
  apply assumption
  apply (rule comm_group.power_order_eq_one)
  apply (rule units_comm_group)
  apply (unfold units_of_def, auto)
  done
```

end

```
theory Module
imports Ring
begin
```

13 Modules over an Abelian Group

13.1 Definitions

```
record ('a, 'b) module = "'b ring" +
  smult :: "['a, 'b] => 'b" (infixl "⊙_ı" 70)

locale module = R?: cring + M?: abelian_group M for M (structure) +
  assumes smult_closed [simp, intro]:
      "[| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙_M x ∈ carrier M"
    and smult_l_distr:
      "[| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊕ b) ⊙_M x = a ⊙_M x ⊕_M b ⊙_M x"
    and smult_r_distr:
      "[| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      a ⊙_M (x ⊕_M y) = a ⊙_M x ⊕_M a ⊙_M y"
    and smult_assoc1:
      "[| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊗ b) ⊙_M x = a ⊙_M (b ⊙_M x)"
    and smult_one [simp]:
      "x ∈ carrier M ==> 1 ⊙_M x = x"

locale algebra = module + cring M +
  assumes smult_assoc2:
      "[| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      (a ⊙_M x) ⊗_M y = a ⊙_M (x ⊗_M y)"

lemma moduleI:
  fixes R (structure) and M (structure)
  assumes cring: "cring R"
    and abelian_group: "abelian_group M"
    and smult_closed:
      "!!a x. [| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙_M x ∈ carrier
M"
    and smult_l_distr:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊕ b) ⊙_M x = (a ⊙_M x) ⊕_M (b ⊙_M x)"
    and smult_r_distr:
      "!!a x y. [| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      a ⊙_M (x ⊕_M y) = (a ⊙_M x) ⊕_M (a ⊙_M y)"
    and smult_assoc1:
```

```
    "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
    (a ⊗ b) ⊙M x = a ⊙M (b ⊙M x)"
  and smult_one:
    "!!x. x ∈ carrier M ==> 1 ⊙M x = x"
shows "module R M"
by (auto intro: module.intro cring.axioms abelian_group.axioms
  module_axioms.intro assms)

lemma algebraI:
  fixes R (structure) and M (structure)
  assumes R_cring: "cring R"
    and M_cring: "cring M"
    and smult_closed:
      "!!a x. [| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙M x ∈ carrier
M"
    and smult_l_distr:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊕ b) ⊙M x = (a ⊙M x) ⊕M (b ⊙M x)"
    and smult_r_distr:
      "!!a x y. [| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      a ⊙M (x ⊕M y) = (a ⊙M x) ⊕M (a ⊙M y)"
    and smult_assoc1:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊗ b) ⊙M x = a ⊙M (b ⊙M x)"
    and smult_one:
      "!!x. x ∈ carrier M ==> (one R) ⊙M x = x"
    and smult_assoc2:
      "!!a x y. [| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      (a ⊙M x) ⊗M y = a ⊙M (x ⊗M y)"
  shows "algebra R M"
apply intro_locales
apply (rule cring.axioms ring.axioms abelian_group.axioms comm_monoid.axioms
assms)+
apply (rule module_axioms.intro)
 apply (simp add: smult_closed)
 apply (simp add: smult_l_distr)
 apply (simp add: smult_r_distr)
 apply (simp add: smult_assoc1)
 apply (simp add: smult_one)
apply (rule cring.axioms ring.axioms abelian_group.axioms comm_monoid.axioms
assms)+
apply (rule algebra_axioms.intro)
 apply (simp add: smult_assoc2)
done

lemma (in algebra) R_cring:
  "cring R"
  ..
```

```
lemma (in algebra) M_cring:
  "cring M"
  ..

lemma (in algebra) module:
  "module R M"
  by (auto intro: moduleI R_cring is_abelian_group
    smult_l_distr smult_r_distr smult_assoc1)
```

13.2 Basic Properties of Algebras

```
lemma (in algebra) smult_l_null [simp]:
  "x ∈ carrier M ==> 0 ⊙_M x = 0_M"
proof -
  assume M: "x ∈ carrier M"
  note facts = M smult_closed [OF R.zero_closed]
  from facts have "0 ⊙_M x = (0 ⊙_M x ⊕_M 0 ⊙_M x) ⊕_M ⊖_M (0 ⊙_M x)" by
algebra
  also from M have "... = (0 ⊕ 0) ⊙_M x ⊕_M ⊖_M (0 ⊙_M x)"
    by (simp add: smult_l_distr del: R.l_zero R.r_zero)
  also from facts have "... = 0_M" apply algebra apply algebra done
  finally show ?thesis .
qed

lemma (in algebra) smult_r_null [simp]:
  "a ∈ carrier R ==> a ⊙_M 0_M = 0_M"
proof -
  assume R: "a ∈ carrier R"
  note facts = R smult_closed
  from facts have "a ⊙_M 0_M = (a ⊙_M 0_M ⊕_M a ⊙_M 0_M) ⊕_M ⊖_M (a ⊙_M 0_M)"
    by algebra
  also from R have "... = a ⊙_M (0_M ⊕_M 0_M) ⊕_M ⊖_M (a ⊙_M 0_M)"
    by (simp add: smult_r_distr del: M.l_zero M.r_zero)
  also from facts have "... = 0_M" by algebra
  finally show ?thesis .
qed

lemma (in algebra) smult_l_minus:
  "[| a ∈ carrier R; x ∈ carrier M |] ==> (⊖a) ⊙_M x = ⊖_M (a ⊙_M x)"
proof -
  assume RM: "a ∈ carrier R" "x ∈ carrier M"
  from RM have a_smult: "a ⊙_M x ∈ carrier M" by simp
  from RM have ma_smult: "⊖a ⊙_M x ∈ carrier M" by simp
  note facts = RM a_smult ma_smult
  from facts have "(⊖a) ⊙_M x = (⊖a ⊙_M x ⊕_M a ⊙_M x) ⊕_M ⊖_M(a ⊙_M x)"
    by algebra
  also from RM have "... = (⊖a ⊕ a) ⊙_M x ⊕_M ⊖_M(a ⊙_M x)"
    by (simp add: smult_l_distr)
  also from facts smult_l_null have "... = ⊖_M(a ⊙_M x)"
```

```
    apply algebra apply algebra done
  finally show ?thesis .
qed

lemma (in algebra) smult_r_minus:
  "[| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙M (⊖Mx) = ⊖M (a ⊙M x)"
proof -
  assume RM: "a ∈ carrier R" "x ∈ carrier M"
  note facts = RM smult_closed
  from facts have "a ⊙M (⊖Mx) = (a ⊙M ⊖Mx ⊕M a ⊙M x) ⊕M ⊖M(a ⊙M x)"
    by algebra
  also from RM have "... = a ⊙M (⊖Mx ⊕M x) ⊕M ⊖M(a ⊙M x)"
    by (simp add: smult_r_distr)
  also from facts smult_r_null have "... = ⊖M(a ⊙M x)" by algebra
  finally show ?thesis .
qed

end


theory AbelCoset
imports Coset Ring
begin
```

13.3 More Lifting from Groups to Abelian Groups

13.3.1 Definitions

Hiding `<+>` from `Sum_Type` until I come up with better syntax here

```
no_notation Sum_Type.Plus (infixr "<+>" 65)

definition
  a_r_coset    :: "[_, 'a set, 'a] ⇒ 'a set"    (infixl "+>ı" 60)
  where "a_r_coset G = r_coset ⦇carrier = carrier G, mult = add G, one
= zero G⦈"

definition
  a_l_coset    :: "[_, 'a, 'a set] ⇒ 'a set"    (infixl "<+ı" 60)
  where "a_l_coset G = l_coset ⦇carrier = carrier G, mult = add G, one
= zero G⦈"

definition
  A_RCOSETS  :: "[_, 'a set] ⇒ ('a set)set"   ("a'_rcosetsı _" [81] 80)
  where "A_RCOSETS G H = RCOSETS ⦇carrier = carrier G, mult = add G,
one = zero G⦈ H"

definition
  set_add  :: "[_, 'a set ,'a set] ⇒ 'a set" (infixl "<+>ı" 60)
  where "set_add G = set_mult ⦇carrier = carrier G, mult = add G, one
```

```
= zero G⦈"

definition
  A_SET_INV :: "[_,'a set] ⇒ 'a set"  ("a'_set'_invı _" [81] 80)
  where "A_SET_INV G H = SET_INV ⦇carrier = carrier G, mult = add G,
one = zero G⦈ H"

definition
  a_r_congruent :: "[('a,'b)ring_scheme, 'a set] ⇒ ('a*'a)set"  ("racongı")
  where "a_r_congruent G = r_congruent ⦇carrier = carrier G, mult = add
G, one = zero G⦈"

definition
  A_FactGroup :: "[('a,'b) ring_scheme, 'a set] ⇒ ('a set) monoid" (in-
fixl "A'_Mod" 65)
    — Actually defined for groups rather than monoids
  where "A_FactGroup G H = FactGroup ⦇carrier = carrier G, mult = add
G, one = zero G⦈ H"

definition
  a_kernel :: "('a, 'm) ring_scheme ⇒ ('b, 'n) ring_scheme ⇒  ('a ⇒
'b) ⇒ 'a set"
    — the kernel of a homomorphism (additive)
  where "a_kernel G H h =
    kernel ⦇carrier = carrier G, mult = add G, one = zero G⦈
      ⦇carrier = carrier H, mult = add H, one = zero H⦈ h"

locale abelian_group_hom = G?: abelian_group G + H?: abelian_group H
    for G (structure) and H (structure) +
  fixes h
  assumes a_group_hom: "group_hom ⦇carrier = carrier G, mult = add G,
one = zero G⦈
                                  ⦇carrier = carrier H, mult = add H,
one = zero H⦈ h"

lemmas a_r_coset_defs =
  a_r_coset_def r_coset_def

lemma a_r_coset_def':
  fixes G (structure)
  shows "H +> a ≡ ⋃h∈H. {h ⊕ a}"
unfolding a_r_coset_defs
by simp

lemmas a_l_coset_defs =
  a_l_coset_def l_coset_def

lemma a_l_coset_def':
  fixes G (structure)
```

```
  shows "a <+ H ≡ ⋃h∈H. {a ⊕ h}"
unfolding a_l_coset_defs
by simp

lemmas A_RCOSETS_defs =
  A_RCOSETS_def RCOSETS_def

lemma A_RCOSETS_def':
  fixes G (structure)
  shows "a_rcosets H ≡ ⋃a∈carrier G. {H +> a}"
unfolding A_RCOSETS_defs
by (fold a_r_coset_def, simp)

lemmas set_add_defs =
  set_add_def set_mult_def

lemma set_add_def':
  fixes G (structure)
  shows "H <+> K ≡ ⋃h∈H. ⋃k∈K. {h ⊕ k}"
unfolding set_add_defs
by simp

lemmas A_SET_INV_defs =
  A_SET_INV_def SET_INV_def

lemma A_SET_INV_def':
  fixes G (structure)
  shows "a_set_inv H ≡ ⋃h∈H. {⊖ h}"
unfolding A_SET_INV_defs
by (fold a_inv_def)
```

13.3.2 Cosets

```
lemma (in abelian_group) a_coset_add_assoc:
     "[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
      ==> (M +> g) +> h = M +> (g ⊕ h)"
by (rule group.coset_mult_assoc [OF a_group,
    folded a_r_coset_def, simplified monoid_record_simps])

lemma (in abelian_group) a_coset_add_zero [simp]:
  "M ⊆ carrier G ==> M +> 𝟬 = M"
by (rule group.coset_mult_one [OF a_group,
    folded a_r_coset_def, simplified monoid_record_simps])

lemma (in abelian_group) a_coset_add_inv1:
     "[| M +> (x ⊕ (⊖ y)) = M;  x ∈ carrier G ; y ∈ carrier G;
         M ⊆ carrier G |] ==> M +> x = M +> y"
by (rule group.coset_mult_inv1 [OF a_group,
    folded a_r_coset_def a_inv_def, simplified monoid_record_simps])
```

lemma (**in** abelian_group) a_coset_add_inv2:
"[| M +> x = M +> y; x ∈ carrier G; y ∈ carrier G; M ⊆ carrier G |]
==> M +> (x ⊕ (⊖ y)) = M"
by (rule group.coset_mult_inv2 [OF a_group,
folded a_r_coset_def a_inv_def, simplified monoid_record_simps])

lemma (**in** abelian_group) a_coset_join1:
"[| H +> x = H; x ∈ carrier G; subgroup H ⦇carrier = carrier G, mult = add G, one = zero G⦈ |] ==> x ∈ H"
by (rule group.coset_join1 [OF a_group,
folded a_r_coset_def, simplified monoid_record_simps])

lemma (**in** abelian_group) a_solve_equation:
"⟦subgroup H ⦇carrier = carrier G, mult = add G, one = zero G⦈; x ∈ H; y ∈ H⟧ ⟹ ∃h∈H. y = h ⊕ x"
by (rule group.solve_equation [OF a_group,
folded a_r_coset_def, simplified monoid_record_simps])

lemma (**in** abelian_group) a_repr_independence:
"⟦y ∈ H +> x; x ∈ carrier G; subgroup H ⦇carrier = carrier G, mult = add G, one = zero G⦈ ⟧ ⟹ H +> x = H +> y"
by (rule group.repr_independence [OF a_group,
folded a_r_coset_def, simplified monoid_record_simps])

lemma (**in** abelian_group) a_coset_join2:
"⟦x ∈ carrier G; subgroup H ⦇carrier = carrier G, mult = add G, one = zero G⦈; x∈H⟧ ⟹ H +> x = H"
by (rule group.coset_join2 [OF a_group,
folded a_r_coset_def, simplified monoid_record_simps])

lemma (**in** abelian_monoid) a_r_coset_subset_G:
"[| H ⊆ carrier G; x ∈ carrier G |] ==> H +> x ⊆ carrier G"
by (rule monoid.r_coset_subset_G [OF a_monoid,
folded a_r_coset_def, simplified monoid_record_simps])

lemma (**in** abelian_group) a_rcosI:
"[| h ∈ H; H ⊆ carrier G; x ∈ carrier G|] ==> h ⊕ x ∈ H +> x"
by (rule group.rcosI [OF a_group,
folded a_r_coset_def, simplified monoid_record_simps])

lemma (**in** abelian_group) a_rcosetsI:
"⟦H ⊆ carrier G; x ∈ carrier G⟧ ⟹ H +> x ∈ a_rcosets H"
by (rule group.rcosetsI [OF a_group,
folded a_r_coset_def A_RCOSETS_def, simplified monoid_record_simps])

Really needed?

lemma (**in** abelian_group) a_transpose_inv:

```
     "[| x ⊕ y = z;  x ∈ carrier G;  y ∈ carrier G;  z ∈ carrier G |]
      ==> (⊖ x) ⊕ z = y"
by (rule group.transpose_inv [OF a_group,
    folded a_r_coset_def a_inv_def, simplified monoid_record_simps])
```

13.3.3 Subgroups

```
locale additive_subgroup =
  fixes H and G (structure)
  assumes a_subgroup: "subgroup H (|carrier = carrier G, mult = add G,
one = zero G|)"

lemma (in additive_subgroup) is_additive_subgroup:
  shows "additive_subgroup H G"
by (rule additive_subgroup_axioms)

lemma additive_subgroupI:
  fixes G (structure)
  assumes a_subgroup: "subgroup H (|carrier = carrier G, mult = add G,
one = zero G|)"
  shows "additive_subgroup H G"
by (rule additive_subgroup.intro) (rule a_subgroup)

lemma (in additive_subgroup) a_subset:
     "H ⊆ carrier G"
by (rule subgroup.subset[OF a_subgroup,
    simplified monoid_record_simps])

lemma (in additive_subgroup) a_closed [intro, simp]:
     "⟦x ∈ H; y ∈ H⟧ ⟹ x ⊕ y ∈ H"
by (rule subgroup.m_closed[OF a_subgroup,
    simplified monoid_record_simps])

lemma (in additive_subgroup) zero_closed [simp]:
     "𝟬 ∈ H"
by (rule subgroup.one_closed[OF a_subgroup,
    simplified monoid_record_simps])

lemma (in additive_subgroup) a_inv_closed [intro,simp]:
     "x ∈ H ⟹ ⊖ x ∈ H"
by (rule subgroup.m_inv_closed[OF a_subgroup,
    folded a_inv_def, simplified monoid_record_simps])
```

13.3.4 Additive subgroups are normal

Every subgroup of an `abelian_group` is normal

```
locale abelian_subgroup = additive_subgroup + abelian_group G +
  assumes a_normal: "normal H (|carrier = carrier G, mult = add G, one
= zero G|)"
```

```
lemma (in abelian_subgroup) is_abelian_subgroup:
  shows "abelian_subgroup H G"
by (rule abelian_subgroup_axioms)

lemma abelian_subgroupI:
  assumes a_normal: "normal H ⦇carrier = carrier G, mult = add G, one
= zero G⦈"
      and a_comm: "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊕G
y = y ⊕G x"
  shows "abelian_subgroup H G"
proof -
  interpret normal "H" "⦇carrier = carrier G, mult = add G, one = zero
G⦈"
    by (rule a_normal)

  show "abelian_subgroup H G"
    by standard (simp add: a_comm)
qed

lemma abelian_subgroupI2:
  fixes G (structure)
  assumes a_comm_group: "comm_group ⦇carrier = carrier G, mult = add
G, one = zero G⦈"
      and a_subgroup: "subgroup H ⦇carrier = carrier G, mult = add G,
one = zero G⦈"
  shows "abelian_subgroup H G"
proof -
  interpret comm_group "⦇carrier = carrier G, mult = add G, one = zero
G⦈"
    by (rule a_comm_group)
  interpret subgroup "H" "⦇carrier = carrier G, mult = add G, one = zero
G⦈"
    by (rule a_subgroup)

  show "abelian_subgroup H G"
    apply unfold_locales
  proof (simp add: r_coset_def l_coset_def, clarsimp)
    fix x
    assume xcarr: "x ∈ carrier G"
    from a_subgroup have Hcarr: "H ⊆ carrier G"
      unfolding subgroup_def by simp
    from xcarr Hcarr show "(⋃h∈H. {h ⊕G x}) = (⋃h∈H. {x ⊕G h})"
      using m_comm [simplified] by fastforce
  qed
qed

lemma abelian_subgroupI3:
  fixes G (structure)
```

```
  assumes asg: "additive_subgroup H G"
      and ag: "abelian_group G"
  shows "abelian_subgroup H G"
apply (rule abelian_subgroupI2)
 apply (rule abelian_group.a_comm_group[OF ag])
apply (rule additive_subgroup.a_subgroup[OF asg])
done

lemma (in abelian_subgroup) a_coset_eq:
     "(∀x ∈ carrier G. H +> x = x <+ H)"
by (rule normal.coset_eq[OF a_normal,
    folded a_r_coset_def a_l_coset_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_inv_op_closed1:
  shows "⟦x ∈ carrier G; h ∈ H⟧ ⟹ (⊖ x) ⊕ h ⊕ x ∈ H"
by (rule normal.inv_op_closed1 [OF a_normal,
    folded a_inv_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_inv_op_closed2:
  shows "⟦x ∈ carrier G; h ∈ H⟧ ⟹ x ⊕ h ⊕ (⊖ x) ∈ H"
by (rule normal.inv_op_closed2 [OF a_normal,
    folded a_inv_def, simplified monoid_record_simps])
```

Alternative characterization of normal subgroups

```
lemma (in abelian_group) a_normal_inv_iff:
     "(N ◁ ⦇carrier = carrier G, mult = add G, one = zero G⦈) =
      (subgroup N ⦇carrier = carrier G, mult = add G, one = zero G⦈ &
(∀x ∈ carrier G. ∀h ∈ N. x ⊕ h ⊕ (⊖ x) ∈ N))"
      (is "_ = ?rhs")
by (rule group.normal_inv_iff [OF a_group,
    folded a_inv_def, simplified monoid_record_simps])

lemma (in abelian_group) a_lcos_m_assoc:
     "[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
      ==> g <+ (h <+ M) = (g ⊕ h) <+ M"
by (rule group.lcos_m_assoc [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])

lemma (in abelian_group) a_lcos_mult_one:
     "M ⊆ carrier G ==> 0 <+ M = M"
by (rule group.lcos_mult_one [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])


lemma (in abelian_group) a_l_coset_subset_G:
     "[| H ⊆ carrier G; x ∈ carrier G |] ==> x <+ H ⊆ carrier G"
by (rule group.l_coset_subset_G [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])
```

```
lemma (in abelian_group) a_l_coset_swap:
     "⟦y ∈ x <+ H;  x ∈ carrier G;  subgroup H ⦇carrier = carrier G, mult
= add G, one = zero G⦈⟧ ⟹ x ∈ y <+ H"
by (rule group.l_coset_swap [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])

lemma (in abelian_group) a_l_coset_carrier:
     "[| y ∈ x <+ H;  x ∈ carrier G;  subgroup H ⦇carrier = carrier G,
mult = add G, one = zero G⦈ |] ==> y ∈ carrier G"
by (rule group.l_coset_carrier [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])

lemma (in abelian_group) a_l_repr_imp_subset:
  assumes y: "y ∈ x <+ H" and x: "x ∈ carrier G" and sb: "subgroup H
⦇carrier = carrier G, mult = add G, one = zero G⦈"
  shows "y <+ H ⊆ x <+ H"
apply (rule group.l_repr_imp_subset [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])
apply (rule y)
apply (rule x)
apply (rule sb)
done

lemma (in abelian_group) a_l_repr_independence:
  assumes y: "y ∈ x <+ H" and x: "x ∈ carrier G" and sb: "subgroup H
⦇carrier = carrier G, mult = add G, one = zero G⦈"
  shows "x <+ H = y <+ H"
apply (rule group.l_repr_independence [OF a_group,
    folded a_l_coset_def, simplified monoid_record_simps])
apply (rule y)
apply (rule x)
apply (rule sb)
done

lemma (in abelian_group) setadd_subset_G:
     "⟦H ⊆ carrier G; K ⊆ carrier G⟧ ⟹ H <+> K ⊆ carrier G"
by (rule group.setmult_subset_G [OF a_group,
    folded set_add_def, simplified monoid_record_simps])

lemma (in abelian_group) subgroup_add_id: "subgroup H ⦇carrier = carrier
G, mult = add G, one = zero G⦈ ⟹ H <+> H = H"
by (rule group.subgroup_mult_id [OF a_group,
    folded set_add_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_rcos_inv:
  assumes x:     "x ∈ carrier G"
  shows "a_set_inv (H +> x) = H +> (⊖ x)"
by (rule normal.rcos_inv [OF a_normal,
```

```
  folded a_r_coset_def a_inv_def A_SET_INV_def, simplified monoid_record_simps])
(rule x)

lemma (in abelian_group) a_setmult_rcos_assoc:
     "⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
      ⟹ H <+> (K +> x) = (H <+> K) +> x"
by (rule group.setmult_rcos_assoc [OF a_group,
    folded set_add_def a_r_coset_def, simplified monoid_record_simps])

lemma (in abelian_group) a_rcos_assoc_lcos:
     "⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
      ⟹ (H +> x) <+> K = H <+> (x <+ K)"
by (rule group.rcos_assoc_lcos [OF a_group,
     folded set_add_def a_r_coset_def a_l_coset_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_rcos_sum:
     "⟦x ∈ carrier G; y ∈ carrier G⟧
      ⟹ (H +> x) <+> (H +> y) = H +> (x ⊕ y)"
by (rule normal.rcos_sum [OF a_normal,
    folded set_add_def a_r_coset_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) rcosets_add_eq:
  "M ∈ a_rcosets H ⟹ H <+> M = M"
  — generalizes subgroup_mult_id
by (rule normal.rcosets_mult_eq [OF a_normal,
    folded set_add_def A_RCOSETS_def, simplified monoid_record_simps])
```

13.3.5 Congruence Relation

```
lemma (in abelian_subgroup) a_equiv_rcong:
   shows "equiv (carrier G) (racong H)"
by (rule subgroup.equiv_rcong [OF a_subgroup a_group,
    folded a_r_congruent_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_l_coset_eq_rcong:
  assumes a: "a ∈ carrier G"
  shows "a <+ H = racong H `` {a}"
by (rule subgroup.l_coset_eq_rcong [OF a_subgroup a_group,
    folded a_r_congruent_def a_l_coset_def, simplified monoid_record_simps])
(rule a)

lemma (in abelian_subgroup) a_rcos_equation:
  shows
     "⟦ha ⊕ a = h ⊕ b; a ∈ carrier G;  b ∈ carrier G;
        h ∈ H;  ha ∈ H;  hb ∈ H⟧
      ⟹ hb ⊕ a ∈ (⋃h∈H. {h ⊕ b})"
by (rule group.rcos_equation [OF a_group a_subgroup,
    folded a_r_congruent_def a_l_coset_def, simplified monoid_record_simps])
```

```
lemma (in abelian_subgroup) a_rcos_disjoint:
  shows "⟦a ∈ a_rcosets H; b ∈ a_rcosets H; a≠b⟧ ⟹ a ∩ b = {}"
by (rule group.rcos_disjoint [OF a_group a_subgroup,
    folded A_RCOSETS_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_rcos_self:
  shows "x ∈ carrier G ⟹ x ∈ H +> x"
by (rule group.rcos_self [OF a_group _ a_subgroup,
    folded a_r_coset_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_rcosets_part_G:
  shows "⋃(a_rcosets H) = carrier G"
by (rule group.rcosets_part_G [OF a_group a_subgroup,
    folded A_RCOSETS_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_cosets_finite:
     "⟦c ∈ a_rcosets H;  H ⊆ carrier G;  finite (carrier G)⟧ ⟹ finite
c"
by (rule group.cosets_finite [OF a_group,
    folded A_RCOSETS_def, simplified monoid_record_simps])

lemma (in abelian_group) a_card_cosets_equal:
     "⟦c ∈ a_rcosets H;  H ⊆ carrier G; finite(carrier G)⟧
      ⟹ card c = card H"
by (rule group.card_cosets_equal [OF a_group,
    folded A_RCOSETS_def, simplified monoid_record_simps])

lemma (in abelian_group) rcosets_subset_PowG:
     "additive_subgroup H G  ⟹ a_rcosets H ⊆ Pow(carrier G)"
by (rule group.rcosets_subset_PowG [OF a_group,
    folded A_RCOSETS_def, simplified monoid_record_simps],
    rule additive_subgroup.a_subgroup)

theorem (in abelian_group) a_lagrange:
     "⟦finite(carrier G); additive_subgroup H G⟧
      ⟹ card(a_rcosets H) * card(H) = order(G)"
by (rule group.lagrange [OF a_group,
    folded A_RCOSETS_def, simplified monoid_record_simps order_def, folded
order_def])
    (fast intro!: additive_subgroup.a_subgroup)+
```

13.3.6 Factorization

```
lemmas A_FactGroup_defs = A_FactGroup_def FactGroup_def

lemma A_FactGroup_def':
  fixes G (structure)
  shows "G A_Mod H ≡ ⦇carrier = a_rcosets⇘G⇙ H, mult = set_add G, one =
H⦈"
```

```
unfolding A_FactGroup_defs
by (fold A_RCOSETS_def set_add_def)


lemma (in abelian_subgroup) a_setmult_closed:
     "⟦K1 ∈ a_rcosets H; K2 ∈ a_rcosets H⟧ ⟹ K1 <+> K2 ∈ a_rcosets H"
by (rule normal.setmult_closed [OF a_normal,
    folded A_RCOSETS_def set_add_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_setinv_closed:
     "K ∈ a_rcosets H ⟹ a_set_inv K ∈ a_rcosets H"
by (rule normal.setinv_closed [OF a_normal,
    folded A_RCOSETS_def A_SET_INV_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_rcosets_assoc:
     "⟦M1 ∈ a_rcosets H; M2 ∈ a_rcosets H; M3 ∈ a_rcosets H⟧
      ⟹ M1 <+> M2 <+> M3 = M1 <+> (M2 <+> M3)"
by (rule normal.rcosets_assoc [OF a_normal,
    folded A_RCOSETS_def set_add_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_subgroup_in_rcosets:
     "H ∈ a_rcosets H"
by (rule subgroup.subgroup_in_rcosets [OF a_subgroup a_group,
    folded A_RCOSETS_def, simplified monoid_record_simps])

lemma (in abelian_subgroup) a_rcosets_inv_mult_group_eq:
     "M ∈ a_rcosets H ⟹ a_set_inv M <+> M = H"
by (rule normal.rcosets_inv_mult_group_eq [OF a_normal,
    folded A_RCOSETS_def A_SET_INV_def set_add_def, simplified monoid_record_simps])

theorem (in abelian_subgroup) a_factorgroup_is_group:
  "group (G A_Mod H)"
by (rule normal.factorgroup_is_group [OF a_normal,
    folded A_FactGroup_def, simplified monoid_record_simps])
```

Since the Factorization is based on an *abelian* subgroup, is results in a commutative group

```
theorem (in abelian_subgroup) a_factorgroup_is_comm_group:
  "comm_group (G A_Mod H)"
apply (intro comm_group.intro comm_monoid.intro) prefer 3
  apply (rule a_factorgroup_is_group)
 apply (rule group.axioms[OF a_factorgroup_is_group])
apply (rule comm_monoid_axioms.intro)
apply (unfold A_FactGroup_def FactGroup_def RCOSETS_def, fold set_add_def
a_r_coset_def, clarsimp)
apply (simp add: a_rcos_sum a_comm)
done

lemma add_A_FactGroup [simp]: "X ⊗(G A_Mod H) X' = X <+>G X'"
```

```
by (simp add: A_FactGroup_def set_add_def)

lemma (in abelian_subgroup) a_inv_FactGroup:
     "X ∈ carrier (G A_Mod H) ⟹ invG A_Mod H X = a_set_inv X"
by (rule normal.inv_FactGroup [OF a_normal,
    folded A_FactGroup_def A_SET_INV_def, simplified monoid_record_simps])
```

The coset map is a homomorphism from `G` to the quotient group `G Mod H`

```
lemma (in abelian_subgroup) a_r_coset_hom_A_Mod:
  "(λa. H +> a) ∈ hom ⦇carrier = carrier G, mult = add G, one = zero G⦈
(G A_Mod H)"
by (rule normal.r_coset_hom_Mod [OF a_normal,
    folded A_FactGroup_def a_r_coset_def, simplified monoid_record_simps])
```

The isomorphism theorems have been omitted from lifting, at least for now

13.3.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

```
lemmas a_kernel_defs =
  a_kernel_def kernel_def

lemma a_kernel_def':
  "a_kernel R S h = {x ∈ carrier R. h x = 0S}"
by (rule a_kernel_def[unfolded kernel_def, simplified ring_record_simps])
```

13.3.8 Homomorphisms

```
lemma abelian_group_homI:
  assumes "abelian_group G"
  assumes "abelian_group H"
  assumes a_group_hom: "group_hom ⦇carrier = carrier G, mult = add G,
one = zero G⦈
                                  ⦇carrier = carrier H, mult = add H,
one = zero H⦈ h"
  shows "abelian_group_hom G H h"
proof -
  interpret G: abelian_group G by fact
  interpret H: abelian_group H by fact
  show ?thesis
    apply (intro abelian_group_hom.intro abelian_group_hom_axioms.intro)
      apply fact
     apply fact
    apply (rule a_group_hom)
    done
qed
```

```
lemma (in abelian_group_hom) is_abelian_group_hom:
  "abelian_group_hom G H h"
  ..

lemma (in abelian_group_hom) hom_add [simp]:
  "[| x : carrier G; y : carrier G |]
        ==> h (x ⊕_G y) = h x ⊕_H h y"
by (rule group_hom.hom_mult[OF a_group_hom,
    simplified ring_record_simps])

lemma (in abelian_group_hom) hom_closed [simp]:
  "x ∈ carrier G ⟹ h x ∈ carrier H"
by (rule group_hom.hom_closed[OF a_group_hom,
    simplified ring_record_simps])

lemma (in abelian_group_hom) zero_closed [simp]:
  "h 0 ∈ carrier H"
by (rule group_hom.one_closed[OF a_group_hom,
    simplified ring_record_simps])

lemma (in abelian_group_hom) hom_zero [simp]:
  "h 0 = 0_H"
by (rule group_hom.hom_one[OF a_group_hom,
    simplified ring_record_simps])

lemma (in abelian_group_hom) a_inv_closed [simp]:
  "x ∈ carrier G ==> h (⊖x) ∈ carrier H"
by (rule group_hom.inv_closed[OF a_group_hom,
    folded a_inv_def, simplified ring_record_simps])

lemma (in abelian_group_hom) hom_a_inv [simp]:
  "x ∈ carrier G ==> h (⊖x) = ⊖_H (h x)"
by (rule group_hom.hom_inv[OF a_group_hom,
    folded a_inv_def, simplified ring_record_simps])

lemma (in abelian_group_hom) additive_subgroup_a_kernel:
  "additive_subgroup (a_kernel G H h) G"
apply (rule additive_subgroup.intro)
apply (rule group_hom.subgroup_kernel[OF a_group_hom,
       folded a_kernel_def, simplified ring_record_simps])
done
```

The kernel of a homomorphism is an abelian subgroup

```
lemma (in abelian_group_hom) abelian_subgroup_a_kernel:
  "abelian_subgroup (a_kernel G H h) G"
apply (rule abelian_subgroupI)
apply (rule group_hom.normal_kernel[OF a_group_hom,
       folded a_kernel_def, simplified ring_record_simps])
apply (simp add: G.a_comm)
```

```
done

lemma (in abelian_group_hom) A_FactGroup_nonempty:
  assumes X: "X ∈ carrier (G A_Mod a_kernel G H h)"
  shows "X ≠ {}"
by (rule group_hom.FactGroup_nonempty[OF a_group_hom,
    folded a_kernel_def A_FactGroup_def, simplified ring_record_simps])
(rule X)

lemma (in abelian_group_hom) FactGroup_the_elem_mem:
  assumes X: "X ∈ carrier (G A_Mod (a_kernel G H h))"
  shows "the_elem (h‘X) ∈ carrier H"
by (rule group_hom.FactGroup_the_elem_mem[OF a_group_hom,
    folded a_kernel_def A_FactGroup_def, simplified ring_record_simps])
(rule X)

lemma (in abelian_group_hom) A_FactGroup_hom:
     "(λX. the_elem (h‘X)) ∈ hom (G A_Mod (a_kernel G H h))
          ⦇carrier = carrier H, mult = add H, one = zero H⦈"
by (rule group_hom.FactGroup_hom[OF a_group_hom,
    folded a_kernel_def A_FactGroup_def, simplified ring_record_simps])

lemma (in abelian_group_hom) A_FactGroup_inj_on:
     "inj_on (λX. the_elem (h ‘ X)) (carrier (G A_Mod a_kernel G H h))"
by (rule group_hom.FactGroup_inj_on[OF a_group_hom,
    folded a_kernel_def A_FactGroup_def, simplified ring_record_simps])
```

If the homomorphism `h` is onto `H`, then so is the homomorphism from the quotient group

```
lemma (in abelian_group_hom) A_FactGroup_onto:
  assumes h: "h ‘ carrier G = carrier H"
  shows "(λX. the_elem (h ‘ X)) ‘ carrier (G A_Mod a_kernel G H h) =
carrier H"
by (rule group_hom.FactGroup_onto[OF a_group_hom,
    folded a_kernel_def A_FactGroup_def, simplified ring_record_simps])
(rule h)
```

If `h` is a homomorphism from `G` onto `H`, then the quotient group `G Mod kernel G H h` is isomorphic to `H`.

```
theorem (in abelian_group_hom) A_FactGroup_iso:
  "h ‘ carrier G = carrier H
   ⟹ (λX. the_elem (h‘X)) ∈ (G A_Mod (a_kernel G H h)) ≅
          ⦇carrier = carrier H, mult = add H, one = zero H⦈"
by (rule group_hom.FactGroup_iso[OF a_group_hom,
    folded a_kernel_def A_FactGroup_def, simplified ring_record_simps])
```

13.3.9 Cosets

Not eveything from `CosetExt.thy` is lifted here.

```
lemma (in additive_subgroup) a_Hcarr [simp]:
  assumes hH: "h ∈ H"
  shows "h ∈ carrier G"
by (rule subgroup.mem_carrier [OF a_subgroup,
    simplified monoid_record_simps]) (rule hH)
```

```
lemma (in abelian_subgroup) a_elemrcos_carrier:
  assumes acarr: "a ∈ carrier G"
      and a': "a' ∈ H +> a"
  shows "a' ∈ carrier G"
by (rule subgroup.elemrcos_carrier [OF a_subgroup a_group,
    folded a_r_coset_def, simplified monoid_record_simps]) (rule acarr,
rule a')
```

```
lemma (in abelian_subgroup) a_rcos_const:
  assumes hH: "h ∈ H"
  shows "H +> h = H"
by (rule subgroup.rcos_const [OF a_subgroup a_group,
    folded a_r_coset_def, simplified monoid_record_simps]) (rule hH)
```

```
lemma (in abelian_subgroup) a_rcos_module_imp:
  assumes xcarr: "x ∈ carrier G"
      and x'cos: "x' ∈ H +> x"
  shows "(x' ⊕ ⊖x) ∈ H"
by (rule subgroup.rcos_module_imp [OF a_subgroup a_group,
    folded a_r_coset_def a_inv_def, simplified monoid_record_simps]) (rule
xcarr, rule x'cos)
```

```
lemma (in abelian_subgroup) a_rcos_module_rev:
  assumes "x ∈ carrier G" "x' ∈ carrier G"
      and "(x' ⊕ ⊖x) ∈ H"
  shows "x' ∈ H +> x"
using assms
by (rule subgroup.rcos_module_rev [OF a_subgroup a_group,
    folded a_r_coset_def a_inv_def, simplified monoid_record_simps])
```

```
lemma (in abelian_subgroup) a_rcos_module:
  assumes "x ∈ carrier G" "x' ∈ carrier G"
  shows "(x' ∈ H +> x) = (x' ⊕ ⊖x ∈ H)"
using assms
by (rule subgroup.rcos_module [OF a_subgroup a_group,
    folded a_r_coset_def a_inv_def, simplified monoid_record_simps])
```

— variant

```
lemma (in abelian_subgroup) a_rcos_module_minus:
  assumes "ring G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
  shows "(x' ∈ H +> x) = (x' ⊖ x ∈ H)"
```

```
proof -
  interpret G: ring G by fact
  from carr
  have "(x' ∈ H +> x) = (x' ⊕ ⊖x ∈ H)" by (rule a_rcos_module)
  with carr
  show "(x' ∈ H +> x) = (x' ⊖ x ∈ H)"
    by (simp add: minus_eq)
qed

lemma (in abelian_subgroup) a_repr_independence':
  assumes y: "y ∈ H +> x"
      and xcarr: "x ∈ carrier G"
  shows "H +> x = H +> y"
  apply (rule a_repr_independence)
    apply (rule y)
   apply (rule xcarr)
  apply (rule a_subgroup)
  done

lemma (in abelian_subgroup) a_repr_independenceD:
  assumes ycarr: "y ∈ carrier G"
      and repr:  "H +> x = H +> y"
  shows "y ∈ H +> x"
by (rule group.repr_independenceD [OF a_group a_subgroup,
    folded a_r_coset_def, simplified monoid_record_simps]) (rule ycarr,
rule repr)


lemma (in abelian_subgroup) a_rcosets_carrier:
  "X ∈ a_rcosets H ⟹ X ⊆ carrier G"
by (rule subgroup.rcosets_carrier [OF a_subgroup a_group,
    folded A_RCOSETS_def, simplified monoid_record_simps])
```

13.3.10 Addition of Subgroups

```
lemma (in abelian_monoid) set_add_closed:
  assumes Acarr: "A ⊆ carrier G"
      and Bcarr: "B ⊆ carrier G"
  shows "A <+> B ⊆ carrier G"
by (rule monoid.set_mult_closed [OF a_monoid,
    folded set_add_def, simplified monoid_record_simps]) (rule Acarr,
rule Bcarr)

lemma (in abelian_group) add_additive_subgroups:
  assumes subH: "additive_subgroup H G"
      and subK: "additive_subgroup K G"
  shows "additive_subgroup (H <+> K) G"
apply (rule additive_subgroup.intro)
apply (unfold set_add_def)
```

```
apply (intro comm_group.mult_subgroups)
  apply (rule a_comm_group)
 apply (rule additive_subgroup.a_subgroup[OF subH])
apply (rule additive_subgroup.a_subgroup[OF subK])
done

end


theory Ideal
imports Ring AbelCoset
begin
```

14 Ideals

14.1 Definitions

14.1.1 General definition

```
locale ideal = additive_subgroup I R + ring R for I and R (structure) +
  assumes I_l_closed: "⟦a ∈ I; x ∈ carrier R⟧ ⟹ x ⊗ a ∈ I"
    and I_r_closed: "⟦a ∈ I; x ∈ carrier R⟧ ⟹ a ⊗ x ∈ I"

sublocale ideal ⊆ abelian_subgroup I R
  apply (intro abelian_subgroupI3 abelian_group.intro)
    apply (rule ideal.axioms, rule ideal_axioms)
   apply (rule abelian_group.axioms, rule ring.axioms, rule ideal.axioms,
rule ideal_axioms)
  apply (rule abelian_group.axioms, rule ring.axioms, rule ideal.axioms,
rule ideal_axioms)
  done

lemma (in ideal) is_ideal: "ideal I R"
  by (rule ideal_axioms)

lemma idealI:
  fixes R (structure)
  assumes "ring R"
  assumes a_subgroup: "subgroup I ⦇carrier = carrier R, mult = add R,
one = zero R⦈"
    and I_l_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ x ⊗ a ∈ I"
    and I_r_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ a ⊗ x ∈ I"
  shows "ideal I R"
proof -
  interpret ring R by fact
  show ?thesis  apply (intro ideal.intro ideal_axioms.intro additive_subgroupI)
     apply (rule a_subgroup)
    apply (rule is_ring)
   apply (erule (1) I_l_closed)
```

```
  apply (erule (1) I_r_closed)
  done
qed
```

14.1.2 Ideals Generated by a Subset of `carrier R`

```
definition genideal :: "_ ⇒ 'a set ⇒ 'a set"  ("Idlı _" [80] 79)
  where "genideal R S = ⋂{I. ideal I R ∧ S ⊆ I}"
```

14.1.3 Principal Ideals

```
locale principalideal = ideal +
  assumes generate: "∃i ∈ carrier R. I = Idl {i}"

lemma (in principalideal) is_principalideal: "principalideal I R"
  by (rule principalideal_axioms)

lemma principalidealI:
  fixes R (structure)
  assumes "ideal I R"
    and generate: "∃i ∈ carrier R. I = Idl {i}"
  shows "principalideal I R"
proof -
  interpret ideal I R by fact
  show ?thesis
    by (intro principalideal.intro principalideal_axioms.intro)
      (rule is_ideal, rule generate)
qed
```

14.1.4 Maximal Ideals

```
locale maximalideal = ideal +
  assumes I_notcarr: "carrier R ≠ I"
    and I_maximal: "⟦ideal J R; I ⊆ J; J ⊆ carrier R⟧ ⟹ J = I ∨ J =
carrier R"

lemma (in maximalideal) is_maximalideal: "maximalideal I R"
  by (rule maximalideal_axioms)

lemma maximalidealI:
  fixes R
  assumes "ideal I R"
    and I_notcarr: "carrier R ≠ I"
    and I_maximal: "⋀J. ⟦ideal J R; I ⊆ J; J ⊆ carrier R⟧ ⟹ J = I
∨ J = carrier R"
  shows "maximalideal I R"
proof -
  interpret ideal I R by fact
  show ?thesis
    by (intro maximalideal.intro maximalideal_axioms.intro)
```

```
      (rule is_ideal, rule I_notcarr, rule I_maximal)
qed
```

14.1.5 Prime Ideals

```
locale primeideal = ideal + cring +
  assumes I_notcarr: "carrier R ≠ I"
    and I_prime: "⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈ I⟧ ⟹ a ∈
I ∨ b ∈ I"

lemma (in primeideal) primeideal: "primeideal I R"
  by (rule primeideal_axioms)

lemma primeidealI:
  fixes R (structure)
  assumes "ideal I R"
    and "cring R"
    and I_notcarr: "carrier R ≠ I"
    and I_prime: "⋀a b. ⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈ I⟧ ⟹
a ∈ I ∨ b ∈ I"
  shows "primeideal I R"
proof -
  interpret ideal I R by fact
  interpret cring R by fact
  show ?thesis
    by (intro primeideal.intro primeideal_axioms.intro)
      (rule is_ideal, rule is_cring, rule I_notcarr, rule I_prime)
qed

lemma primeidealI2:
  fixes R (structure)
  assumes "additive_subgroup I R"
    and "cring R"
    and I_l_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ x ⊗ a ∈ I"
    and I_r_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ a ⊗ x ∈ I"
    and I_notcarr: "carrier R ≠ I"
    and I_prime: "⋀a b. ⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈ I⟧ ⟹
a ∈ I ∨ b ∈ I"
  shows "primeideal I R"
proof -
  interpret additive_subgroup I R by fact
  interpret cring R by fact
  show ?thesis apply (intro_locales)
    apply (intro ideal_axioms.intro)
    apply (erule (1) I_l_closed)
    apply (erule (1) I_r_closed)
    apply (intro primeideal_axioms.intro)
    apply (rule I_notcarr)
    apply (erule (2) I_prime)
```

```
    done
qed
```

14.2 Special Ideals

```
lemma (in ring) zeroideal: "ideal {0} R"
  apply (intro idealI subgroup.intro)
        apply (rule is_ring)
       apply simp+
    apply (fold a_inv_def, simp)
   apply simp+
  done

lemma (in ring) oneideal: "ideal (carrier R) R"
  by (rule idealI) (auto intro: is_ring add.subgroupI)

lemma (in "domain") zeroprimeideal: "primeideal {0} R"
  apply (intro primeidealI)
     apply (rule zeroideal)
    apply (rule domain.axioms, rule domain_axioms)
   defer 1
   apply (simp add: integral)
proof (rule ccontr, simp)
  assume "carrier R = {0}"
  then have "1 = 0" by (rule one_zeroI)
  with one_not_zero show False by simp
qed
```

14.3 General Ideal Properies

```
lemma (in ideal) one_imp_carrier:
  assumes I_one_closed: "1 ∈ I"
  shows "I = carrier R"
  apply (rule)
  apply (rule)
  apply (rule a_Hcarr, simp)
proof
  fix x
  assume xcarr: "x ∈ carrier R"
  with I_one_closed have "x ⊗ 1 ∈ I" by (intro I_l_closed)
  with xcarr show "x ∈ I" by simp
qed

lemma (in ideal) Icarr:
  assumes iI: "i ∈ I"
  shows "i ∈ carrier R"
  using iI by (rule a_Hcarr)
```

14.4 Intersection of Ideals

Intersection of two ideals The intersection of any two ideals is again an ideal in `R`

```
lemma (in ring) i_intersect:
  assumes "ideal I R"
  assumes "ideal J R"
  shows "ideal (I ∩ J) R"
proof -
  interpret ideal I R by fact
  interpret ideal J R by fact
  show ?thesis
    apply (intro idealI subgroup.intro)
          apply (rule is_ring)
         apply (force simp add: a_subset)
        apply (simp add: a_inv_def[symmetric])
       apply simp
      apply (simp add: a_inv_def[symmetric])
     apply (clarsimp, rule)
      apply (fast intro: ideal.I_l_closed ideal.intro assms)+
    apply (clarsimp, rule)
     apply (fast intro: ideal.I_r_closed ideal.intro assms)+
    done
qed
```

The intersection of any Number of Ideals is again an Ideal in `R`

```
lemma (in ring) i_Intersect:
  assumes Sideals: "⋀I. I ∈ S ⟹ ideal I R"
    and notempty: "S ≠ {}"
  shows "ideal (⋂S) R"
  apply (unfold_locales)
  apply (simp_all add: Inter_eq)
        apply rule unfolding mem_Collect_eq defer 1
        apply rule defer 1
        apply rule defer 1
        apply (fold a_inv_def, rule) defer 1
        apply rule defer 1
        apply rule defer 1
proof -
  fix x y
  assume "∀I∈S. x ∈ I"
  then have xI: "⋀I. I ∈ S ⟹ x ∈ I" by simp
  assume "∀I∈S. y ∈ I"
  then have yI: "⋀I. I ∈ S ⟹ y ∈ I" by simp

  fix J
  assume JS: "J ∈ S"
  interpret ideal J R by (rule Sideals[OF JS])
  from xI[OF JS] and yI[OF JS] show "x ⊕ y ∈ J" by (rule a_closed)
```

```
next
  fix J
  assume JS: "J ∈ S"
  interpret ideal J R by (rule Sideals[OF JS])
  show "0 ∈ J" by simp
next
  fix x
  assume "∀I∈S. x ∈ I"
  then have xI: "⋀I. I ∈ S ⟹ x ∈ I" by simp

  fix J
  assume JS: "J ∈ S"
  interpret ideal J R by (rule Sideals[OF JS])

  from xI[OF JS] show "⊖ x ∈ J" by (rule a_inv_closed)
next
  fix x y
  assume "∀I∈S. x ∈ I"
  then have xI: "⋀I. I ∈ S ⟹ x ∈ I" by simp
  assume ycarr: "y ∈ carrier R"

  fix J
  assume JS: "J ∈ S"
  interpret ideal J R by (rule Sideals[OF JS])

  from xI[OF JS] and ycarr show "y ⊗ x ∈ J" by (rule I_l_closed)
next
  fix x y
  assume "∀I∈S. x ∈ I"
  then have xI: "⋀I. I ∈ S ⟹ x ∈ I" by simp
  assume ycarr: "y ∈ carrier R"

  fix J
  assume JS: "J ∈ S"
  interpret ideal J R by (rule Sideals[OF JS])

  from xI[OF JS] and ycarr show "x ⊗ y ∈ J" by (rule I_r_closed)
next
  fix x
  assume "∀I∈S. x ∈ I"
  then have xI: "⋀I. I ∈ S ⟹ x ∈ I" by simp

  from notempty have "∃I0. I0 ∈ S" by blast
  then obtain I0 where I0S: "I0 ∈ S" by auto

  interpret ideal I0 R by (rule Sideals[OF I0S])

  from xI[OF I0S] have "x ∈ I0" .
  with a_subset show "x ∈ carrier R" by fast
```

```
next

qed
```

14.5 Addition of Ideals

```
lemma (in ring) add_ideals:
  assumes idealI: "ideal I R"
      and idealJ: "ideal J R"
  shows "ideal (I <+> J) R"
  apply (rule ideal.intro)
    apply (rule add_additive_subgroups)
     apply (intro ideal.axioms[OF idealI])
    apply (intro ideal.axioms[OF idealJ])
   apply (rule is_ring)
  apply (rule ideal_axioms.intro)
   apply (simp add: set_add_defs, clarsimp) defer 1
   apply (simp add: set_add_defs, clarsimp) defer 1
proof -
  fix x i j
  assume xcarr: "x ∈ carrier R"
    and iI: "i ∈ I"
    and jJ: "j ∈ J"
  from xcarr ideal.Icarr[OF idealI iI] ideal.Icarr[OF idealJ jJ]
  have c: "(i ⊕ j) ⊗ x = (i ⊗ x) ⊕ (j ⊗ x)"
    by algebra
  from xcarr and iI have a: "i ⊗ x ∈ I"
    by (simp add: ideal.I_r_closed[OF idealI])
  from xcarr and jJ have b: "j ⊗ x ∈ J"
    by (simp add: ideal.I_r_closed[OF idealJ])
  from a b c show "∃ha∈I. ∃ka∈J. (i ⊕ j) ⊗ x = ha ⊕ ka"
    by fast
next
  fix x i j
  assume xcarr: "x ∈ carrier R"
    and iI: "i ∈ I"
    and jJ: "j ∈ J"
  from xcarr ideal.Icarr[OF idealI iI] ideal.Icarr[OF idealJ jJ]
  have c: "x ⊗ (i ⊕ j) = (x ⊗ i) ⊕ (x ⊗ j)" by algebra
  from xcarr and iI have a: "x ⊗ i ∈ I"
    by (simp add: ideal.I_l_closed[OF idealI])
  from xcarr and jJ have b: "x ⊗ j ∈ J"
    by (simp add: ideal.I_l_closed[OF idealJ])
  from a b c show "∃ha∈I. ∃ka∈J. x ⊗ (i ⊕ j) = ha ⊕ ka"
    by fast
qed
```

14.6 Ideals generated by a subset of `carrier R`

`genideal` generates an ideal

```
lemma (in ring) genideal_ideal:
  assumes Scarr: "S ⊆ carrier R"
  shows "ideal (Idl S) R"
unfolding genideal_def
proof (rule i_Intersect, fast, simp)
  from oneideal and Scarr
  show "∃I. ideal I R ∧ S ≤ I" by fast
qed

lemma (in ring) genideal_self:
  assumes "S ⊆ carrier R"
  shows "S ⊆ Idl S"
  unfolding genideal_def by fast

lemma (in ring) genideal_self':
  assumes carr: "i ∈ carrier R"
  shows "i ∈ Idl {i}"
proof -
  from carr have "{i} ⊆ Idl {i}" by (fast intro!: genideal_self)
  then show "i ∈ Idl {i}" by fast
qed
```

`genideal` generates the minimal ideal

```
lemma (in ring) genideal_minimal:
  assumes a: "ideal I R"
    and b: "S ⊆ I"
  shows "Idl S ⊆ I"
  unfolding genideal_def by rule (elim InterD, simp add: a b)
```

Generated ideals and subsets

```
lemma (in ring) Idl_subset_ideal:
  assumes Iideal: "ideal I R"
    and Hcarr: "H ⊆ carrier R"
  shows "(Idl H ⊆ I) = (H ⊆ I)"
proof
  assume a: "Idl H ⊆ I"
  from Hcarr have "H ⊆ Idl H" by (rule genideal_self)
  with a show "H ⊆ I" by simp
next
  fix x
  assume "H ⊆ I"
  with Iideal have "I ∈ {I. ideal I R ∧ H ⊆ I}" by fast
  then show "Idl H ⊆ I" unfolding genideal_def by fast
qed

lemma (in ring) subset_Idl_subset:
```

```
  assumes Icarr: "I ⊆ carrier R"
    and HI: "H ⊆ I"
  shows "Idl H ⊆ Idl I"
proof -
  from HI and genideal_self[OF Icarr] have HIdlI: "H ⊆ Idl I"
    by fast

  from Icarr have Iideal: "ideal (Idl I) R"
    by (rule genideal_ideal)
  from HI and Icarr have "H ⊆ carrier R"
    by fast
  with Iideal have "(H ⊆ Idl I) = (Idl H ⊆ Idl I)"
    by (rule Idl_subset_ideal[symmetric])

  with HIdlI show "Idl H ⊆ Idl I" by simp
qed

lemma (in ring) Idl_subset_ideal':
  assumes acarr: "a ∈ carrier R" and bcarr: "b ∈ carrier R"
  shows "(Idl {a} ⊆ Idl {b}) = (a ∈ Idl {b})"
  apply (subst Idl_subset_ideal[OF genideal_ideal[of "{b}"], of "{a}"])
    apply (fast intro: bcarr, fast intro: acarr)
  apply fast
  done

lemma (in ring) genideal_zero: "Idl {0} = {0}"
  apply rule
   apply (rule genideal_minimal[OF zeroideal], simp)
  apply (simp add: genideal_self')
  done

lemma (in ring) genideal_one: "Idl {1} = carrier R"
proof -
  interpret ideal "Idl {1}" "R" by (rule genideal_ideal) fast
  show "Idl {1} = carrier R"
  apply (rule, rule a_subset)
  apply (simp add: one_imp_carrier genideal_self')
  done
qed
```

Generation of Principal Ideals in Commutative Rings

```
definition cgenideal :: "_ ⇒ 'a ⇒ 'a set"  ("PIdlı _" [80] 79)
  where "cgenideal R a = {x ⊗R a | x. x ∈ carrier R}"
```

genhideal (?) really generates an ideal

```
lemma (in cring) cgenideal_ideal:
  assumes acarr: "a ∈ carrier R"
  shows "ideal (PIdl a) R"
  apply (unfold cgenideal_def)
```

```
  apply (rule idealI[OF is_ring])
     apply (rule subgroup.intro)
        apply simp_all
        apply (blast intro: acarr)
        apply clarsimp defer 1
        defer 1
        apply (fold a_inv_def, clarsimp) defer 1
        apply clarsimp defer 1
        apply clarsimp defer 1
proof -
  fix x y
  assume xcarr: "x ∈ carrier R"
    and ycarr: "y ∈ carrier R"
  note carr = acarr xcarr ycarr

  from carr have "x ⊗ a ⊕ y ⊗ a = (x ⊕ y) ⊗ a"
    by (simp add: l_distr)
  with carr show "∃z. x ⊗ a ⊕ y ⊗ a = z ⊗ a ∧ z ∈ carrier R"
    by fast
next
  from l_null[OF acarr, symmetric] and zero_closed
  show "∃x. 0 = x ⊗ a ∧ x ∈ carrier R" by fast
next
  fix x
  assume xcarr: "x ∈ carrier R"
  note carr = acarr xcarr

  from carr have "⊖ (x ⊗ a) = (⊖ x) ⊗ a"
    by (simp add: l_minus)
  with carr show "∃z. ⊖ (x ⊗ a) = z ⊗ a ∧ z ∈ carrier R"
    by fast
next
  fix x y
  assume xcarr: "x ∈ carrier R"
     and ycarr: "y ∈ carrier R"
  note carr = acarr xcarr ycarr

  from carr have "y ⊗ a ⊗ x = (y ⊗ x) ⊗ a"
    by (simp add: m_assoc) (simp add: m_comm)
  with carr show "∃z. y ⊗ a ⊗ x = z ⊗ a ∧ z ∈ carrier R"
    by fast
next
  fix x y
  assume xcarr: "x ∈ carrier R"
     and ycarr: "y ∈ carrier R"
  note carr = acarr xcarr ycarr

  from carr have "x ⊗ (y ⊗ a) = (x ⊗ y) ⊗ a"
    by (simp add: m_assoc)
```

```
  with carr show "∃z. x ⊗ (y ⊗ a) = z ⊗ a ∧ z ∈ carrier R"
    by fast
qed

lemma (in ring) cgenideal_self:
  assumes icarr: "i ∈ carrier R"
  shows "i ∈ PIdl i"
  unfolding cgenideal_def
proof simp
  from icarr have "i = 𝟭 ⊗ i"
    by simp
  with icarr show "∃x. i = x ⊗ i ∧ x ∈ carrier R"
    by fast
qed
```

`cgenideal` is minimal

```
lemma (in ring) cgenideal_minimal:
  assumes "ideal J R"
  assumes aJ: "a ∈ J"
  shows "PIdl a ⊆ J"
proof -
  interpret ideal J R by fact
  show ?thesis
    unfolding cgenideal_def
    apply rule
    apply clarify
    using aJ
    apply (erule I_l_closed)
    done
qed

lemma (in cring) cgenideal_eq_genideal:
  assumes icarr: "i ∈ carrier R"
  shows "PIdl i = Idl {i}"
  apply rule
   apply (intro cgenideal_minimal)
    apply (rule genideal_ideal, fast intro: icarr)
   apply (rule genideal_self', fast intro: icarr)
  apply (intro genideal_minimal)
   apply (rule cgenideal_ideal [OF icarr])
  apply (simp, rule cgenideal_self [OF icarr])
  done

lemma (in cring) cgenideal_eq_rcos: "PIdl i = carrier R #> i"
  unfolding cgenideal_def r_coset_def by fast

lemma (in cring) cgenideal_is_principalideal:
  assumes icarr: "i ∈ carrier R"
  shows "principalideal (PIdl i) R"
```

```
  apply (rule principalidealI)
  apply (rule cgenideal_ideal [OF icarr])
proof -
  from icarr have "PIdl i = Idl {i}"
    by (rule cgenideal_eq_genideal)
  with icarr show "∃i'∈carrier R. PIdl i = Idl {i'}"
    by fast
qed
```

14.7 Union of Ideals

```
lemma (in ring) union_genideal:
  assumes idealI: "ideal I R"
    and idealJ: "ideal J R"
  shows "Idl (I ∪ J) = I <+> J"
  apply rule
   apply (rule ring.genideal_minimal)
     apply (rule is_ring)
    apply (rule add_ideals[OF idealI idealJ])
   apply (rule)
   apply (simp add: set_add_defs) apply (elim disjE) defer 1 defer 1
   apply (rule) apply (simp add: set_add_defs genideal_def) apply clarsimp
defer 1
proof -
  fix x
  assume xI: "x ∈ I"
  have ZJ: "0 ∈ J"
    by (intro additive_subgroup.zero_closed) (rule ideal.axioms[OF idealJ])
  from ideal.Icarr[OF idealI xI] have "x = x ⊕ 0"
    by algebra
  with xI and ZJ show "∃h∈I. ∃k∈J. x = h ⊕ k"
    by fast
next
  fix x
  assume xJ: "x ∈ J"
  have ZI: "0 ∈ I"
    by (intro additive_subgroup.zero_closed, rule ideal.axioms[OF idealI])
  from ideal.Icarr[OF idealJ xJ] have "x = 0 ⊕ x"
    by algebra
  with ZI and xJ show "∃h∈I. ∃k∈J. x = h ⊕ k"
    by fast
next
  fix i j K
  assume iI: "i ∈ I"
    and jJ: "j ∈ J"
    and idealK: "ideal K R"
    and IK: "I ⊆ K"
    and JK: "J ⊆ K"
  from iI and IK have iK: "i ∈ K" by fast
```

```
  from jJ and JK have jK: "j ∈ K" by fast
  from iK and jK show "i ⊕ j ∈ K"
    by (intro additive_subgroup.a_closed) (rule ideal.axioms[OF idealK])
qed
```

14.8 Properties of Principal Ideals

0 generates the zero ideal

```
lemma (in ring) zero_genideal: "Idl {0} = {0}"
  apply rule
  apply (simp add: genideal_minimal zeroideal)
  apply (fast intro!: genideal_self)
  done
```

1 generates the unit ideal

```
lemma (in ring) one_genideal: "Idl {1} = carrier R"
proof -
  have "1 ∈ Idl {1}"
    by (simp add: genideal_self')
  then show "Idl {1} = carrier R"
    by (intro ideal.one_imp_carrier) (fast intro: genideal_ideal)
qed
```

The zero ideal is a principal ideal

```
corollary (in ring) zeropideal: "principalideal {0} R"
  apply (rule principalidealI)
   apply (rule zeroideal)
  apply (blast intro!: zero_genideal[symmetric])
  done
```

The unit ideal is a principal ideal

```
corollary (in ring) onepideal: "principalideal (carrier R) R"
  apply (rule principalidealI)
   apply (rule oneideal)
  apply (blast intro!: one_genideal[symmetric])
  done
```

Every principal ideal is a right coset of the carrier

```
lemma (in principalideal) rcos_generate:
  assumes "cring R"
  shows "∃x∈I. I = carrier R #> x"
proof -
  interpret cring R by fact
  from generate obtain i where icarr: "i ∈ carrier R" and I1: "I = Idl
{i}"
    by fast+

  from icarr and genideal_self[of "{i}"] have "i ∈ Idl {i}"
```

```
    by fast
  then have iI: "i ∈ I" by (simp add: I1)

  from I1 icarr have I2: "I = PIdl i"
    by (simp add: cgenideal_eq_genideal)

  have "PIdl i = carrier R #> i"
    unfolding cgenideal_def r_coset_def by fast

  with I2 have "I = carrier R #> i"
    by simp

  with iI show "∃x∈I. I = carrier R #> x"
    by fast
qed
```

14.9 Prime Ideals

```
lemma (in ideal) primeidealCD:
  assumes "cring R"
  assumes notprime: "¬ primeideal I R"
  shows "carrier R = I ∨ (∃a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗
b ∈ I ∧ a ∉ I ∧ b ∉ I)"
proof (rule ccontr, clarsimp)
  interpret cring R by fact
  assume InR: "carrier R ≠ I"
    and "∀a. a ∈ carrier R ⟶ (∀b. a ⊗ b ∈ I ⟶ b ∈ carrier R ⟶
a ∈ I ∨ b ∈ I)"
  then have I_prime: "⋀ a b. ⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈
I⟧ ⟹ a ∈ I ∨ b ∈ I"
    by simp
  have "primeideal I R"
    apply (rule primeideal.intro [OF is_ideal is_cring])
    apply (rule primeideal_axioms.intro)
     apply (rule InR)
    apply (erule (2) I_prime)
    done
  with notprime show False by simp
qed

lemma (in ideal) primeidealCE:
  assumes "cring R"
  assumes notprime: "¬ primeideal I R"
  obtains "carrier R = I"
    | "∃a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉ I ∧ b
∉ I"
proof -
  interpret R: cring R by fact
  assume "carrier R = I ==> thesis"
```

```
    and "∃a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉ I ∧
b ∉ I ⟹ thesis"
  then show thesis using primeidealCD [OF R.is_cring notprime] by blast
qed
```

If {0} is a prime ideal of a commutative ring, the ring is a domain

```
lemma (in cring) zeroprimeideal_domainI:
  assumes pi: "primeideal {0} R"
  shows "domain R"
  apply (rule domain.intro, rule is_cring)
  apply (rule domain_axioms.intro)
proof (rule ccontr, simp)
  interpret primeideal "{0}" "R" by (rule pi)
  assume "1 = 0"
  then have "carrier R = {0}" by (rule one_zeroD)
  from this[symmetric] and I_notcarr show False
    by simp
next
  interpret primeideal "{0}" "R" by (rule pi)
  fix a b
  assume ab: "a ⊗ b = 0" and carr: "a ∈ carrier R" "b ∈ carrier R"
  from ab have abI: "a ⊗ b ∈ {0}"
    by fast
  with carr have "a ∈ {0} ∨ b ∈ {0}"
    by (rule I_prime)
  then show "a = 0 ∨ b = 0" by simp
qed

corollary (in cring) domain_eq_zeroprimeideal: "domain R = primeideal {0}
R"
  apply rule
   apply (erule domain.zeroprimeideal)
  apply (erule zeroprimeideal_domainI)
  done
```

14.10 Maximal Ideals

```
lemma (in ideal) helper_I_closed:
  assumes carr: "a ∈ carrier R" "x ∈ carrier R" "y ∈ carrier R"
    and axI: "a ⊗ x ∈ I"
  shows "a ⊗ (x ⊗ y) ∈ I"
proof -
  from axI and carr have "(a ⊗ x) ⊗ y ∈ I"
    by (simp add: I_r_closed)
  also from carr have "(a ⊗ x) ⊗ y = a ⊗ (x ⊗ y)"
    by (simp add: m_assoc)
  finally show "a ⊗ (x ⊗ y) ∈ I" .
qed
```

```
lemma (in ideal) helper_max_prime:
  assumes "cring R"
  assumes acarr: "a ∈ carrier R"
  shows "ideal {x∈carrier R. a ⊗ x ∈ I} R"
proof -
  interpret cring R by fact
  show ?thesis apply (rule idealI)
    apply (rule cring.axioms[OF is_cring])
    apply (rule subgroup.intro)
    apply (simp, fast)
    apply clarsimp apply (simp add: r_distr acarr)
    apply (simp add: acarr)
    apply (simp add: a_inv_def[symmetric], clarify) defer 1
    apply clarsimp defer 1
    apply (fast intro!: helper_I_closed acarr)
  proof -
    fix x
    assume xcarr: "x ∈ carrier R"
      and ax: "a ⊗ x ∈ I"
    from ax and acarr xcarr
    have "⊖(a ⊗ x) ∈ I" by simp
    also from acarr xcarr
    have "⊖(a ⊗ x) = a ⊗ (⊖x)" by algebra
    finally show "a ⊗ (⊖x) ∈ I" .
    from acarr have "a ⊗ 0 = 0" by simp
  next
    fix x y
    assume xcarr: "x ∈ carrier R"
      and ycarr: "y ∈ carrier R"
      and ayI: "a ⊗ y ∈ I"
    from ayI and acarr xcarr ycarr have "a ⊗ (y ⊗ x) ∈ I"
      by (simp add: helper_I_closed)
    moreover
    from xcarr ycarr have "y ⊗ x = x ⊗ y"
      by (simp add: m_comm)
    ultimately
    show "a ⊗ (x ⊗ y) ∈ I" by simp
  qed
qed
```

In a cring every maximal ideal is prime

```
lemma (in cring) maximalideal_prime:
  assumes "maximalideal I R"
  shows "primeideal I R"
proof -
  interpret maximalideal I R by fact
  show ?thesis apply (rule ccontr)
    apply (rule primeidealCE)
    apply (rule is_cring)
```

```
    apply assumption
    apply (simp add: I_notcarr)
  proof -
    assume "∃a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉
I ∧ b ∉ I"
    then obtain a b where
      acarr: "a ∈ carrier R" and
      bcarr: "b ∈ carrier R" and
      abI: "a ⊗ b ∈ I" and
      anI: "a ∉ I" and
      bnI: "b ∉ I" by fast
    define J where "J = {x∈carrier R. a ⊗ x ∈ I}"

    from is_cring and acarr have idealJ: "ideal J R"
      unfolding J_def by (rule helper_max_prime)

    have IsubJ: "I ⊆ J"
    proof
      fix x
      assume xI: "x ∈ I"
      with acarr have "a ⊗ x ∈ I"
        by (intro I_l_closed)
      with xI[THEN a_Hcarr] show "x ∈ J"
        unfolding J_def by fast
    qed

    from abI and acarr bcarr have "b ∈ J"
      unfolding J_def by fast
    with bnI have JnI: "J ≠ I" by fast
    from acarr
    have "a = a ⊗ 1" by algebra
    with anI have "a ⊗ 1 ∉ I" by simp
    with one_closed have "1 ∉ J"
      unfolding J_def by fast
    then have Jncarr: "J ≠ carrier R" by fast

    interpret ideal J R by (rule idealJ)

    have "J = I ∨ J = carrier R"
      apply (intro I_maximal)
      apply (rule idealJ)
      apply (rule IsubJ)
      apply (rule a_subset)
      done

    with JnI and Jncarr show False by simp
  qed
qed
```

14.11 Derived Theorems

— A non-zero cring that has only the two trivial ideals is a field

```
lemma (in cring) trivialideals_fieldI:
  assumes carrnzero: "carrier R ≠ {0}"
    and haveideals: "{I. ideal I R} = {{0}, carrier R}"
  shows "field R"
  apply (rule cring_fieldI)
  apply (rule, rule, rule)
   apply (erule Units_closed)
  defer 1
    apply rule
  defer 1
proof (rule ccontr, simp)
  assume zUnit: "0 ∈ Units R"
  then have a: "0 ⊗ inv 0 = 1" by (rule Units_r_inv)
  from zUnit have "0 ⊗ inv 0 = 0"
    by (intro l_null) (rule Units_inv_closed)
  with a[symmetric] have "1 = 0" by simp
  then have "carrier R = {0}" by (rule one_zeroD)
  with carrnzero show False by simp
next
  fix x
  assume xcarr': "x ∈ carrier R - {0}"
  then have xcarr: "x ∈ carrier R" by fast
  from xcarr' have xnZ: "x ≠ 0" by fast
  from xcarr have xIdl: "ideal (PIdl x) R"
    by (intro cgenideal_ideal) fast

  from xcarr have "x ∈ PIdl x"
    by (intro cgenideal_self) fast
  with xnZ have "PIdl x ≠ {0}" by fast
  with haveideals have "PIdl x = carrier R"
    by (blast intro!: xIdl)
  then have "1 ∈ PIdl x" by simp
  then have "∃y. 1 = y ⊗ x ∧ y ∈ carrier R"
    unfolding cgenideal_def by blast
  then obtain y where ycarr: " y ∈ carrier R" and ylinv: "1 = y ⊗ x"
    by fast+
  from ylinv and xcarr ycarr have yrinv: "1 = x ⊗ y"
    by (simp add: m_comm)
  from ycarr and ylinv[symmetric] and yrinv[symmetric]
  have "∃y ∈ carrier R. y ⊗ x = 1 ∧ x ⊗ y = 1" by fast
  with xcarr show "x ∈ Units R"
    unfolding Units_def by fast
qed

lemma (in field) all_ideals: "{I. ideal I R} = {{0}, carrier R}"
  apply (rule, rule)
proof -
```

```
  fix I
  assume a: "I ∈ {I. ideal I R}"
  then interpret ideal I R by simp

  show "I ∈ {{0}, carrier R}"
  proof (cases "∃a. a ∈ I - {0}")
    case True
    then obtain a where aI: "a ∈ I" and anZ: "a ≠ 0"
      by fast+
    from aI[THEN a_Hcarr] anZ have aUnit: "a ∈ Units R"
      by (simp add: field_Units)
    then have a: "a ⊗ inv a = 1" by (rule Units_r_inv)
    from aI and aUnit have "a ⊗ inv a ∈ I"
      by (simp add: I_r_closed del: Units_r_inv)
    then have oneI: "1 ∈ I" by (simp add: a[symmetric])

    have "carrier R ⊆ I"
    proof
      fix x
      assume xcarr: "x ∈ carrier R"
      with oneI have "1 ⊗ x ∈ I" by (rule I_r_closed)
      with xcarr show "x ∈ I" by simp
    qed
    with a_subset have "I = carrier R" by fast
    then show "I ∈ {{0}, carrier R}" by fast
  next
    case False
    then have IZ: "⋀a. a ∈ I ⟹ a = 0" by simp

    have a: "I ⊆ {0}"
    proof
      fix x
      assume "x ∈ I"
      then have "x = 0" by (rule IZ)
      then show "x ∈ {0}" by fast
    qed

    have "0 ∈ I" by simp
    then have "{0} ⊆ I" by fast

    with a have "I = {0}" by fast
    then show "I ∈ {{0}, carrier R}" by fast
  qed
qed (simp add: zeroideal oneideal)

— Jacobson Theorem 2.2
lemma (in cring) trivialideals_eq_field:
  assumes carrnzero: "carrier R ≠ {0}"
  shows "({I. ideal I R} = {{0}, carrier R}) = field R"
```

```
  by (fast intro!: trivialideals_fieldI[OF carrnzero] field.all_ideals)
```

Like zeroprimeideal for domains

```
lemma (in field) zeromaximalideal: "maximalideal {0} R"
  apply (rule maximalidealI)
    apply (rule zeroideal)
proof-
  from one_not_zero have "1 ∉ {0}" by simp
  with one_closed show "carrier R ≠ {0}" by fast
next
  fix J
  assume Jideal: "ideal J R"
  then have "J ∈ {I. ideal I R}" by fast
  with all_ideals show "J = {0} ∨ J = carrier R"
    by simp
qed

lemma (in cring) zeromaximalideal_fieldI:
  assumes zeromax: "maximalideal {0} R"
  shows "field R"
  apply (rule trivialideals_fieldI, rule maximalideal.I_notcarr[OF zeromax])
  apply rule apply clarsimp defer 1
   apply (simp add: zeroideal oneideal)
proof -
  fix J
  assume Jn0: "J ≠ {0}"
    and idealJ: "ideal J R"
  interpret ideal J R by (rule idealJ)
  have "{0} ⊆ J" by (rule ccontr) simp
  from zeromax and idealJ and this and a_subset
  have "J = {0} ∨ J = carrier R"
    by (rule maximalideal.I_maximal)
  with Jn0 show "J = carrier R"
    by simp
qed

lemma (in cring) zeromaximalideal_eq_field: "maximalideal {0} R = field
R"
  apply rule
   apply (erule zeromaximalideal_fieldI)
  apply (erule field.zeromaximalideal)
  done

end


theory RingHom
imports Ideal
begin
```

15 Homomorphisms of Non-Commutative Rings

Lifting existing lemmas in a `ring_hom_ring` locale

```
locale ring_hom_ring = R?: ring R + S?: ring S
    for R (structure) and S (structure) +
  fixes h
  assumes homh: "h ∈ ring_hom R S"
  notes hom_mult [simp] = ring_hom_mult [OF homh]
    and hom_one [simp] = ring_hom_one [OF homh]

sublocale ring_hom_cring ⊆ ring: ring_hom_ring
  by standard (rule homh)

sublocale ring_hom_ring ⊆ abelian_group?: abelian_group_hom R S
apply (rule abelian_group_homI)
  apply (rule R.is_abelian_group)
 apply (rule S.is_abelian_group)
apply (intro group_hom.intro group_hom_axioms.intro)
  apply (rule R.a_group)
 apply (rule S.a_group)
apply (insert homh, unfold hom_def ring_hom_def)
apply simp
done

lemma (in ring_hom_ring) is_ring_hom_ring:
  "ring_hom_ring R S h"
  by (rule ring_hom_ring_axioms)

lemma ring_hom_ringI:
  fixes R (structure) and S (structure)
  assumes "ring R" "ring S"
  assumes
          hom_closed: "!!x. x ∈ carrier R ==> h x ∈ carrier S"
      and compatible_mult: "!!x y. [| x : carrier R; y : carrier R |]
==> h (x ⊗ y) = h x ⊗S h y"
      and compatible_add: "!!x y. [| x : carrier R; y : carrier R |] ==>
h (x ⊕ y) = h x ⊕S h y"
      and compatible_one: "h 1 = 1S"
  shows "ring_hom_ring R S h"
proof -
  interpret ring R by fact
  interpret ring S by fact
  show ?thesis apply unfold_locales
apply (unfold ring_hom_def, safe)
   apply (simp add: hom_closed Pi_def)
  apply (erule (1) compatible_mult)
 apply (erule (1) compatible_add)
apply (rule compatible_one)
done
```

```
qed

lemma ring_hom_ringI2:
  assumes "ring R" "ring S"
  assumes h: "h ∈ ring_hom R S"
  shows "ring_hom_ring R S h"
proof -
  interpret R: ring R by fact
  interpret S: ring S by fact
  show ?thesis apply (intro ring_hom_ring.intro ring_hom_ring_axioms.intro)
    apply (rule R.is_ring)
    apply (rule S.is_ring)
    apply (rule h)
    done
qed

lemma ring_hom_ringI3:
  fixes R (structure) and S (structure)
  assumes "abelian_group_hom R S h" "ring R" "ring S"
  assumes compatible_mult: "!!x y. [| x : carrier R; y : carrier R |]
==> h (x ⊗ y) = h x ⊗_S h y"
      and compatible_one: "h 1 = 1_S"
  shows "ring_hom_ring R S h"
proof -
  interpret abelian_group_hom R S h by fact
  interpret R: ring R by fact
  interpret S: ring S by fact
  show ?thesis apply (intro ring_hom_ring.intro ring_hom_ring_axioms.intro,
rule R.is_ring, rule S.is_ring)
    apply (insert group_hom.homh[OF a_group_hom])
    apply (unfold hom_def ring_hom_def, simp)
    apply safe
    apply (erule (1) compatible_mult)
    apply (rule compatible_one)
    done
qed

lemma ring_hom_cringI:
  assumes "ring_hom_ring R S h" "cring R" "cring S"
  shows "ring_hom_cring R S h"
proof -
  interpret ring_hom_ring R S h by fact
  interpret R: cring R by fact
  interpret S: cring S by fact
  show ?thesis by (intro ring_hom_cring.intro ring_hom_cring_axioms.intro)
    (rule R.is_cring, rule S.is_cring, rule homh)
qed
```

15.1 The Kernel of a Ring Homomorphism

— the kernel of a ring homomorphism is an ideal

```
lemma (in ring_hom_ring) kernel_is_ideal:
  shows "ideal (a_kernel R S h) R"
apply (rule idealI)
   apply (rule R.is_ring)
  apply (rule additive_subgroup.a_subgroup[OF additive_subgroup_a_kernel])
 apply (unfold a_kernel_def', simp+)
done
```

Elements of the kernel are mapped to zero

```
lemma (in abelian_group_hom) kernel_zero [simp]:
  "i ∈ a_kernel R S h ⟹ h i = 0S"
by (simp add: a_kernel_defs)
```

15.2 Cosets

Cosets of the kernel correspond to the elements of the image of the homomorphism

```
lemma (in ring_hom_ring) rcos_imp_homeq:
  assumes acarr: "a ∈ carrier R"
      and xrcos: "x ∈ a_kernel R S h +> a"
  shows "h x = h a"
proof -
  interpret ideal "a_kernel R S h" "R" by (rule kernel_is_ideal)

  from xrcos
      have "∃i ∈ a_kernel R S h. x = i ⊕ a" by (simp add: a_r_coset_defs)
  from this obtain i
      where iker: "i ∈ a_kernel R S h"
        and x: "x = i ⊕ a"
      by fast+
  note carr = acarr iker[THEN a_Hcarr]

  from x
      have "h x = h (i ⊕ a)" by simp
  also from carr
      have "... = h i ⊕S h a" by simp
  also from iker
      have "... = 0S ⊕S h a" by simp
  also from carr
      have "... = h a" by simp
  finally
      show "h x = h a" .
qed

lemma (in ring_hom_ring) homeq_imp_rcos:
  assumes acarr: "a ∈ carrier R"
```

```
      and xcarr: "x ∈ carrier R"
      and hx: "h x = h a"
  shows "x ∈ a_kernel R S h +> a"
proof -
  interpret ideal "a_kernel R S h" "R" by (rule kernel_is_ideal)

  note carr = acarr xcarr
  note hcarr = acarr[THEN hom_closed] xcarr[THEN hom_closed]

  from hx and hcarr
      have a: "h x ⊕_S ⊖_Sh a = 0_S" by algebra
  from carr
      have "h x ⊕_S ⊖_Sh a = h (x ⊕ ⊖a)" by simp
  from a and this
      have b: "h (x ⊕ ⊖a) = 0_S" by simp

  from carr have "x ⊕ ⊖a ∈ carrier R" by simp
  from this and b
      have "x ⊕ ⊖a ∈ a_kernel R S h"
      unfolding a_kernel_def'
      by fast

  from this and carr
      show "x ∈ a_kernel R S h +> a" by (simp add: a_rcos_module_rev)
qed

corollary (in ring_hom_ring) rcos_eq_homeq:
  assumes acarr: "a ∈ carrier R"
  shows "(a_kernel R S h) +> a = {x ∈ carrier R. h x = h a}"
apply rule defer 1
apply clarsimp defer 1
proof
  interpret ideal "a_kernel R S h" "R" by (rule kernel_is_ideal)

  fix x
  assume xrcos: "x ∈ a_kernel R S h +> a"
  from acarr and this
      have xcarr: "x ∈ carrier R"
      by (rule a_elemrcos_carrier)

  from xrcos
      have "h x = h a" by (rule rcos_imp_homeq[OF acarr])
  from xcarr and this
      show "x ∈ {x ∈ carrier R. h x = h a}" by fast
next
  interpret ideal "a_kernel R S h" "R" by (rule kernel_is_ideal)

  fix x
  assume xcarr: "x ∈ carrier R"
```

```
    and hx: "h x = h a"
  from acarr xcarr hx
      show "x ∈ a_kernel R S h +> a" by (rule homeq_imp_rcos)
qed

end


theory UnivPoly
imports Module RingHom
begin
```

16 Univariate Polynomials

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record `up_ring`). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

16.1 The Constructor for Univariate Polynomials

Functions with finite support.

```
locale bound =
  fixes z :: 'a
    and n :: nat
    and f :: "nat => 'a"
  assumes bound: "!!m. n < m ⟹ f m = z"

declare bound.intro [intro!]
  and bound.bound [dest]

lemma bound_below:
  assumes bound: "bound z m f" and nonzero: "f n ≠ z" shows "n ≤ m"
proof (rule classical)
  assume "~ ?thesis"
  then have "m < n" by arith
  with bound have "f n = z" ..
  with nonzero show ?thesis by contradiction
qed

record ('a, 'p) up_ring = "('a, 'p) module" +
  monom :: "['a, nat] => 'p"
  coeff :: "['p, nat] => 'a"
```

```
definition
  up :: "('a, 'm) ring_scheme => (nat => 'a) set"
  where "up R = {f. f ∈ UNIV → carrier R & (EX n. bound 𝟬_R n f)}"

definition UP :: "('a, 'm) ring_scheme => ('a, nat => 'a) up_ring"
  where "UP R = ⦇
   carrier = up R,
   mult = (λp∈up R. λq∈up R. λn. ⨁_R i ∈ {..n}. p i ⊗_R q (n-i)),
   one = (λi. if i=0 then 𝟭_R else 𝟬_R),
   zero = (λi. 𝟬_R),
   add = (λp∈up R. λq∈up R. λi. p i ⊕_R q i),
   smult = (λa∈carrier R. λp∈up R. λi. a ⊗_R p i),
   monom = (λa∈carrier R. λn i. if i=n then a else 𝟬_R),
   coeff = (λp∈up R. λn. p n)⦈"
```

Properties of the set of polynomials up.

```
lemma mem_upI [intro]:
  "[| !!n. f n ∈ carrier R; EX n. bound (zero R) n f |] ==> f ∈ up R"
  by (simp add: up_def Pi_def)

lemma mem_upD [dest]:
  "f ∈ up R ==> f n ∈ carrier R"
  by (simp add: up_def Pi_def)

context ring
begin

lemma bound_upD [dest]: "f ∈ up R ==> EX n. bound 𝟬 n f" by (simp add:
up_def)

lemma up_one_closed: "(λn. if n = 0 then 𝟭 else 𝟬) ∈ up R" using up_def
by force

lemma up_smult_closed: "[| a ∈ carrier R; p ∈ up R |] ==> (λi. a ⊗ p
i) ∈ up R" by force

lemma up_add_closed:
  "[| p ∈ up R; q ∈ up R |] ==> (λi. p i ⊕ q i) ∈ up R"
proof
  fix n
  assume "p ∈ up R" and "q ∈ up R"
  then show "p n ⊕ q n ∈ carrier R"
    by auto
next
  assume UP: "p ∈ up R" "q ∈ up R"
  show "EX n. bound 𝟬 n (λi. p i ⊕ q i)"
  proof -
    from UP obtain n where boundn: "bound 𝟬 n p" by fast
    from UP obtain m where boundm: "bound 𝟬 m q" by fast
```

```
    have "bound 0 (max n m) (λi. p i ⊕ q i)"
    proof
      fix i
      assume "max n m < i"
      with boundn and boundm and UP show "p i ⊕ q i = 0" by fastforce
    qed
    then show ?thesis ..
  qed
qed

lemma up_a_inv_closed:
  "p ∈ up R ==> (λi. ⊖ (p i)) ∈ up R"
proof
  assume R: "p ∈ up R"
  then obtain n where "bound 0 n p" by auto
  then have "bound 0 n (λi. ⊖ p i)" by auto
  then show "EX n. bound 0 n (λi. ⊖ p i)" by auto
qed auto

lemma up_minus_closed:
  "[| p ∈ up R; q ∈ up R |] ==> (λi. p i ⊖ q i) ∈ up R"
  using mem_upD [of p R] mem_upD [of q R] up_add_closed up_a_inv_closed
a_minus_def [of _ R]
  by auto

lemma up_mult_closed:
  "[| p ∈ up R; q ∈ up R |] ==>
  (λn. ⨁i ∈ {..n}. p i ⊗ q (n-i)) ∈ up R"
proof
  fix n
  assume "p ∈ up R" "q ∈ up R"
  then show "(⨁i ∈ {..n}. p i ⊗ q (n-i)) ∈ carrier R"
    by (simp add: mem_upD  funcsetI)
next
  assume UP: "p ∈ up R" "q ∈ up R"
  show "EX n. bound 0 n (λn. ⨁i ∈ {..n}. p i ⊗ q (n-i))"
  proof -
    from UP obtain n where boundn: "bound 0 n p" by fast
    from UP obtain m where boundm: "bound 0 m q" by fast
    have "bound 0 (n + m) (λn. ⨁i ∈ {..n}. p i ⊗ q (n - i))"
    proof
      fix k assume bound: "n + m < k"
      {
        fix i
        have "p i ⊗ q (k-i) = 0"
        proof (cases "n < i")
          case True
          with boundn have "p i = 0" by auto
          moreover from UP have "q (k-i) ∈ carrier R" by auto
```

```
          ultimately show ?thesis by simp
        next
          case False
          with bound have "m < k-i" by arith
          with boundm have "q (k-i) = 0" by auto
          moreover from UP have "p i ∈ carrier R" by auto
          ultimately show ?thesis by simp
        qed
      }
      then show "(⨁i ∈ {..k}. p i ⊗ q (k-i)) = 0"
        by (simp add: Pi_def)
    qed
    then show ?thesis by fast
  qed
qed

end
```

16.2 Effect of Operations on Coefficients

```
locale UP =
  fixes R (structure) and P (structure)
  defines P_def: "P == UP R"

locale UP_ring = UP + R?: ring R

locale UP_cring = UP + R?: cring R

sublocale UP_cring < UP_ring
  by intro_locales [1] (rule P_def)

locale UP_domain = UP + R?: "domain" R

sublocale UP_domain < UP_cring
  by intro_locales [1] (rule P_def)

context UP
begin
```

Temporarily declare `P ≡ UP R` as simp rule.

```
declare P_def [simp]

lemma up_eqI:
  assumes prem: "!!n. coeff P p n = coeff P q n" and R: "p ∈ carrier
P" "q ∈ carrier P"
  shows "p = q"
proof
  fix x
  from prem and R show "p x = q x" by (simp add: UP_def)
```

```
qed

lemma coeff_closed [simp]:
  "p ∈ carrier P ==> coeff P p n ∈ carrier R" by (auto simp add: UP_def)

end

context UP_ring
begin
```

```
lemma coeff_monom [simp]:
  "a ∈ carrier R ==> coeff P (monom P a m) n = (if m=n then a else 0)"
proof -
  assume R: "a ∈ carrier R"
  then have "(λn. if n = m then a else 0) ∈ up R"
    using up_def by force
  with R show ?thesis by (simp add: UP_def)
qed

lemma coeff_zero [simp]: "coeff P 0_P n = 0" by (auto simp add: UP_def)

lemma coeff_one [simp]: "coeff P 1_P n = (if n=0 then 1 else 0)"
  using up_one_closed by (simp add: UP_def)

lemma coeff_smult [simp]:
  "[| a ∈ carrier R; p ∈ carrier P |] ==> coeff P (a ⊙_P p) n = a ⊗ coeff
P p n"
  by (simp add: UP_def up_smult_closed)

lemma coeff_add [simp]:
  "[| p ∈ carrier P; q ∈ carrier P |] ==> coeff P (p ⊕_P q) n = coeff
P p n ⊕ coeff P q n"
  by (simp add: UP_def up_add_closed)

lemma coeff_mult [simp]:
  "[| p ∈ carrier P; q ∈ carrier P |] ==> coeff P (p ⊗_P q) n = (⨁i ∈
{..n}. coeff P p i ⊗ coeff P q (n-i))"
  by (simp add: UP_def up_mult_closed)

end
```

16.3 Polynomials Form a Ring.

```
context UP_ring
begin
```

Operations are closed over `P`.

```
lemma UP_mult_closed [simp]:
  "[| p ∈ carrier P; q ∈ carrier P |] ==> p ⊗P q ∈ carrier P" by (simp
add: UP_def up_mult_closed)

lemma UP_one_closed [simp]:
  "1P ∈ carrier P" by (simp add: UP_def up_one_closed)

lemma UP_zero_closed [intro, simp]:
  "0P ∈ carrier P" by (auto simp add: UP_def)

lemma UP_a_closed [intro, simp]:
  "[| p ∈ carrier P; q ∈ carrier P |] ==> p ⊕P q ∈ carrier P" by (simp
add: UP_def up_add_closed)

lemma monom_closed [simp]:
  "a ∈ carrier R ==> monom P a n ∈ carrier P" by (auto simp add: UP_def
up_def Pi_def)

lemma UP_smult_closed [simp]:
  "[| a ∈ carrier R; p ∈ carrier P |] ==> a ⊙P p ∈ carrier P" by (simp
add: UP_def up_smult_closed)

end

declare (in UP) P_def [simp del]
```

Algebraic ring properties

```
context UP_ring
begin

lemma UP_a_assoc:
  assumes R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
  shows "(p ⊕P q) ⊕P r = p ⊕P (q ⊕P r)" by (rule up_eqI, simp add:
a_assoc R, simp_all add: R)

lemma UP_l_zero [simp]:
  assumes R: "p ∈ carrier P"
  shows "0P ⊕P p = p" by (rule up_eqI, simp_all add: R)

lemma UP_l_neg_ex:
  assumes R: "p ∈ carrier P"
  shows "EX q : carrier P. q ⊕P p = 0P"
proof -
  let ?q = "λi. ⊖ (p i)"
  from R have closed: "?q ∈ carrier P"
    by (simp add: UP_def P_def up_a_inv_closed)
  from R have coeff: "!!n. coeff P ?q n = ⊖ (coeff P p n)"
    by (simp add: UP_def P_def up_a_inv_closed)
  show ?thesis
```

```
  proof
    show "?q ⊕_P p = 0_P"
      by (auto intro!: up_eqI simp add: R closed coeff R.l_neg)
  qed (rule closed)
qed

lemma UP_a_comm:
  assumes R: "p ∈ carrier P" "q ∈ carrier P"
  shows "p ⊕_P q = q ⊕_P p" by (rule up_eqI, simp add: a_comm R, simp_all
add: R)

lemma UP_m_assoc:
  assumes R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
  shows "(p ⊗_P q) ⊗_P r = p ⊗_P (q ⊗_P r)"
proof (rule up_eqI)
  fix n
  {
    fix k and a b c :: "nat=>'a"
    assume R: "a ∈ UNIV → carrier R" "b ∈ UNIV → carrier R"
      "c ∈ UNIV → carrier R"
    then have "k <= n ==>
      (⨁j ∈ {..k}. (⨁i ∈ {..j}. a i ⊗ b (j-i)) ⊗ c (n-j)) =
      (⨁j ∈ {..k}. a j ⊗ (⨁i ∈ {..k-j}. b i ⊗ c (n-j-i)))"
      (is "_ ⟹ ?eq k")
    proof (induct k)
      case 0 then show ?case by (simp add: Pi_def m_assoc)
    next
      case (Suc k)
      then have "k <= n" by arith
      from this R have "?eq k" by (rule Suc)
      with R show ?case
        by (simp cong: finsum_cong
             add: Suc_diff_le Pi_def l_distr r_distr m_assoc)
           (simp cong: finsum_cong add: Pi_def a_ac finsum_ldistr m_assoc)
    qed
  }
  with R show "coeff P ((p ⊗_P q) ⊗_P r) n = coeff P (p ⊗_P (q ⊗_P r))
n"
    by (simp add: Pi_def)
qed (simp_all add: R)

lemma UP_r_one [simp]:
  assumes R: "p ∈ carrier P" shows "p ⊗_P 1_P = p"
proof (rule up_eqI)
  fix n
  show "coeff P (p ⊗_P 1_P) n = coeff P p n"
  proof (cases n)
    case 0
    {
```

```
      with R show ?thesis by simp
    }
  next
    case Suc
    {

      fix nn assume Succ: "n = Suc nn"
      have "coeff P (p ⊗P 1P) (Suc nn) = coeff P p (Suc nn)"
      proof -
        have "coeff P (p ⊗P 1P) (Suc nn) = (⨁i∈{..Suc nn}. coeff P
p i ⊗ (if Suc nn ≤ i then 1 else 0))" using R by simp
        also have "... = coeff P p (Suc nn) ⊗ (if Suc nn ≤ Suc nn then
1 else 0) ⊕ (⨁i∈{..nn}. coeff P p i ⊗ (if Suc nn ≤ i then 1 else 0))"
          using finsum_Suc [of "(λi::nat. coeff P p i ⊗ (if Suc nn ≤
i then 1 else 0))" "nn"] unfolding Pi_def using R by simp
        also have "... = coeff P p (Suc nn) ⊗ (if Suc nn ≤ Suc nn then
1 else 0)"
        proof -
          have "(⨁i∈{..nn}. coeff P p i ⊗ (if Suc nn ≤ i then 1 else
0)) = (⨁i∈{..nn}. 0)"
            using finsum_cong [of "{..nn}" "{..nn}" "(λi::nat. coeff P
p i ⊗ (if Suc nn ≤ i then 1 else 0))" "(λi::nat. 0)"] using R
            unfolding Pi_def by simp
          also have "... = 0" by simp
          finally show ?thesis using r_zero R by simp
        qed
        also have "... = coeff P p (Suc nn)" using R by simp
        finally show ?thesis by simp
      qed
      then show ?thesis using Succ by simp
    }
  qed
qed (simp_all add: R)

lemma UP_l_one [simp]:
  assumes R: "p ∈ carrier P"
  shows "1P ⊗P p = p"
proof (rule up_eqI)
  fix n
  show "coeff P (1P ⊗P p) n = coeff P p n"
  proof (cases n)
    case 0 with R show ?thesis by simp
  next
    case Suc with R show ?thesis
      by (simp del: finsum_Suc add: finsum_Suc2 Pi_def)
  qed
qed (simp_all add: R)

lemma UP_l_distr:
```

```
  assumes R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
  shows "(p ⊕P q) ⊗P r = (p ⊗P r) ⊕P (q ⊗P r)"
  by (rule up_eqI) (simp add: l_distr R Pi_def, simp_all add: R)

lemma UP_r_distr:
  assumes R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
  shows "r ⊗P (p ⊕P q) = (r ⊗P p) ⊕P (r ⊗P q)"
  by (rule up_eqI) (simp add: r_distr R Pi_def, simp_all add: R)

theorem UP_ring: "ring P"
  by (auto intro!: ringI abelian_groupI monoidI UP_a_assoc)
    (auto intro: UP_a_comm UP_l_neg_ex UP_m_assoc UP_l_distr UP_r_distr)

end
```

16.4 Polynomials Form a Commutative Ring.

```
context UP_cring
begin

lemma UP_m_comm:
  assumes R1: "p ∈ carrier P" and R2: "q ∈ carrier P" shows "p ⊗P q
= q ⊗P p"
proof (rule up_eqI)
  fix n
  {
    fix k and a b :: "nat=>'a"
    assume R: "a ∈ UNIV → carrier R" "b ∈ UNIV → carrier R"
    then have "k <= n ==>
      (⨁i ∈ {..k}. a i ⊗ b (n-i)) = (⨁i ∈ {..k}. a (k-i) ⊗ b (i+n-k))"
      (is "_ ⟹ ?eq k")
    proof (induct k)
      case 0 then show ?case by (simp add: Pi_def)
    next
      case (Suc k) then show ?case
        by (subst (2) finsum_Suc2) (simp add: Pi_def a_comm)+
    qed
  }
  note l = this
  from R1 R2 show "coeff P (p ⊗P q) n =  coeff P (q ⊗P p) n"
    unfolding coeff_mult [OF R1 R2, of n]
    unfolding coeff_mult [OF R2 R1, of n]
    using l [of "(λi. coeff P p i)" "(λi. coeff P q i)" "n"] by (simp
add: Pi_def m_comm)
qed (simp_all add: R1 R2)
```

16.5 Polynomials over a commutative ring for a commutative ring

```
theorem UP_cring:
  "cring P" using UP_ring unfolding cring_def by (auto intro!: comm_monoidI
UP_m_assoc UP_m_comm)

end

context UP_ring
begin

lemma UP_a_inv_closed [intro, simp]:
  "p ∈ carrier P ==> ⊖P p ∈ carrier P"
  by (rule abelian_group.a_inv_closed [OF ring.is_abelian_group [OF UP_ring]])

lemma coeff_a_inv [simp]:
  assumes R: "p ∈ carrier P"
  shows "coeff P (⊖P p) n = ⊖ (coeff P p n)"
proof -
  from R coeff_closed UP_a_inv_closed have
    "coeff P (⊖P p) n = ⊖ coeff P p n ⊕ (coeff P p n ⊕ coeff P (⊖P p)
n)"
    by algebra
  also from R have "... =  ⊖ (coeff P p n)"
    by (simp del: coeff_add add: coeff_add [THEN sym]
      abelian_group.r_neg [OF ring.is_abelian_group [OF UP_ring]])
  finally show ?thesis .
qed

end

sublocale UP_ring < P?: ring P using UP_ring .
sublocale UP_cring < P?: cring P using UP_cring .
```

16.6 Polynomials Form an Algebra

```
context UP_ring
begin

lemma UP_smult_l_distr:
  "[| a ∈ carrier R; b ∈ carrier R; p ∈ carrier P |] ==>
  (a ⊕ b) ⊙P p = a ⊙P p ⊕P b ⊙P p"
  by (rule up_eqI) (simp_all add: R.l_distr)

lemma UP_smult_r_distr:
  "[| a ∈ carrier R; p ∈ carrier P; q ∈ carrier P |] ==>
  a ⊙P (p ⊕P q) = a ⊙P p ⊕P a ⊙P q"
  by (rule up_eqI) (simp_all add: R.r_distr)
```

```
lemma UP_smult_assoc1:
      "[| a ∈ carrier R; b ∈ carrier R; p ∈ carrier P |] ==>
      (a ⊗ b) ⊙P p = a ⊙P (b ⊙P p)"
  by (rule up_eqI) (simp_all add: R.m_assoc)

lemma UP_smult_zero [simp]:
      "p ∈ carrier P ==> 0 ⊙P p = 0P"
  by (rule up_eqI) simp_all

lemma UP_smult_one [simp]:
      "p ∈ carrier P ==> 1 ⊙P p = p"
  by (rule up_eqI) simp_all

lemma UP_smult_assoc2:
  "[| a ∈ carrier R; p ∈ carrier P; q ∈ carrier P |] ==>
  (a ⊙P p) ⊗P q = a ⊙P (p ⊗P q)"
  by (rule up_eqI) (simp_all add: R.finsum_rdistr R.m_assoc Pi_def)

end
```

Interpretation of lemmas from `algebra`.

```
lemma (in cring) cring:
  "cring R" ..

lemma (in UP_cring) UP_algebra:
  "algebra R P" by (auto intro!: algebraI R.cring UP_cring UP_smult_l_distr
UP_smult_r_distr
    UP_smult_assoc1 UP_smult_assoc2)

sublocale UP_cring < algebra R P using UP_algebra .
```

16.7 Further Lemmas Involving Monomials

```
context UP_ring
begin

lemma monom_zero [simp]:
  "monom P 0 n = 0P" by (simp add: UP_def P_def)

lemma monom_mult_is_smult:
  assumes R: "a ∈ carrier R" "p ∈ carrier P"
  shows "monom P a 0 ⊗P p = a ⊙P p"
proof (rule up_eqI)
  fix n
  show "coeff P (monom P a 0 ⊗P p) n = coeff P (a ⊙P p) n"
  proof (cases n)
    case 0 with R show ?thesis by simp
  next
    case Suc with R show ?thesis
```

```
      using R.finsum_Suc2 by (simp del: R.finsum_Suc add: Pi_def)
  qed
qed (simp_all add: R)

lemma monom_one [simp]:
  "monom P 1 0 = 1P"
  by (rule up_eqI) simp_all

lemma monom_add [simp]:
  "[| a ∈ carrier R; b ∈ carrier R |] ==>
  monom P (a ⊕ b) n = monom P a n ⊕P monom P b n"
  by (rule up_eqI) simp_all

lemma monom_one_Suc:
  "monom P 1 (Suc n) = monom P 1 n ⊗P monom P 1 1"
proof (rule up_eqI)
  fix k
  show "coeff P (monom P 1 (Suc n)) k = coeff P (monom P 1 n ⊗P monom
P 1 1) k"
  proof (cases "k = Suc n")
    case True show ?thesis
    proof -
      fix m
      from True have less_add_diff:
        "!!i. [| n < i; i <= n + m |] ==> n + m - i < m" by arith
      from True have "coeff P (monom P 1 (Suc n)) k = 1" by simp
      also from True
      have "... = (⨁i ∈ {..<n} ∪ {n}. coeff P (monom P 1 n) i ⊗
        coeff P (monom P 1 1) (k - i))"
        by (simp cong: R.finsum_cong add: Pi_def)
      also have "... = (⨁i ∈  {..n}. coeff P (monom P 1 n) i ⊗
        coeff P (monom P 1 1) (k - i))"
        by (simp only: ivl_disj_un_singleton)
      also from True
      have "... = (⨁i ∈ {..n} ∪ {n<..k}. coeff P (monom P 1 n) i ⊗
        coeff P (monom P 1 1) (k - i))"
        by (simp cong: R.finsum_cong add: R.finsum_Un_disjoint ivl_disj_int_one
          order_less_imp_not_eq Pi_def)
      also from True have "... = coeff P (monom P 1 n ⊗P monom P 1 1)
k"
        by (simp add: ivl_disj_un_one)
      finally show ?thesis .
    qed
  next
    case False
    note neq = False
    let ?s =
      "λi. (if n = i then 1 else 0) ⊗ (if Suc 0 = k - i then 1 else 0)"
    from neq have "coeff P (monom P 1 (Suc n)) k = 0" by simp
```

```
    also have "... = (⨁i ∈ {..k}. ?s i)"
    proof -
      have f1: "(⨁i ∈ {..<n}. ?s i) = 0"
        by (simp cong: R.finsum_cong add: Pi_def)
      from neq have f2: "(⨁i ∈ {n}. ?s i) = 0"
        by (simp cong: R.finsum_cong add: Pi_def) arith
      have f3: "n < k ==> (⨁i ∈ {n<..k}. ?s i) = 0"
        by (simp cong: R.finsum_cong add: order_less_imp_not_eq Pi_def)
      show ?thesis
      proof (cases "k < n")
        case True then show ?thesis by (simp cong: R.finsum_cong add:
Pi_def)
      next
        case False then have n_le_k: "n <= k" by arith
        show ?thesis
        proof (cases "n = k")
          case True
          then have "0 = (⨁i ∈ {..<n} ∪ {n}. ?s i)"
            by (simp cong: R.finsum_cong add: Pi_def)
          also from True have "... = (⨁i ∈ {..k}. ?s i)"
            by (simp only: ivl_disj_un_singleton)
          finally show ?thesis .
        next
          case False with n_le_k have n_less_k: "n < k" by arith
          with neq have "0 = (⨁i ∈ {..<n} ∪ {n}. ?s i)"
            by (simp add: R.finsum_Un_disjoint f1 f2 Pi_def del: Un_insert_right)
          also have "... = (⨁i ∈ {..n}. ?s i)"
            by (simp only: ivl_disj_un_singleton)
          also from n_less_k neq have "... = (⨁i ∈ {..n} ∪ {n<..k}.
?s i)"
            by (simp add: R.finsum_Un_disjoint f3 ivl_disj_int_one Pi_def)
          also from n_less_k have "... = (⨁i ∈ {..k}. ?s i)"
            by (simp only: ivl_disj_un_one)
          finally show ?thesis .
        qed
      qed
    qed
    also have "... = coeff P (monom P 1 n ⊗p monom P 1 1) k" by simp
    finally show ?thesis .
  qed
qed (simp_all)

lemma monom_one_Suc2:
  "monom P 1 (Suc n) = monom P 1 1 ⊗p monom P 1 n"
proof (induct n)
  case 0 show ?case by simp
next
  case Suc
  {
```

```
    fix k:: nat
    assume hypo: "monom P 1 (Suc k) = monom P 1 1 ⊗P monom P 1 k"
    then show "monom P 1 (Suc (Suc k)) = monom P 1 1 ⊗P monom P 1 (Suc
k)"
    proof -
      have lhs: "monom P 1 (Suc (Suc k)) = monom P 1 1 ⊗P monom P 1 k
⊗P monom P 1 1"
        unfolding monom_one_Suc [of "Suc k"] unfolding hypo ..
      note cl = monom_closed [OF R.one_closed, of 1]
      note clk = monom_closed [OF R.one_closed, of k]
      have rhs: "monom P 1 1 ⊗P monom P 1 (Suc k) = monom P 1 1 ⊗P monom
P 1 k ⊗P monom P 1 1"
        unfolding monom_one_Suc [of k] unfolding sym [OF m_assoc  [OF
cl clk cl]] ..
      from lhs rhs show ?thesis by simp
    qed
  }
qed
```

The following corollary follows from lemmas `monom P 1 (Suc ?n) = monom P 1 ?n ⊗P monom P 1 1` and `monom P 1 (Suc ?n) = monom P 1 1 ⊗P monom P 1 ?n`, and is trivial in `UP_cring`

```
corollary monom_one_comm: shows "monom P 1 k ⊗P monom P 1 1 = monom P
1 1 ⊗P monom P 1 k"
  unfolding monom_one_Suc [symmetric] monom_one_Suc2 [symmetric] ..

lemma monom_mult_smult:
  "[| a ∈ carrier R; b ∈ carrier R |] ==> monom P (a ⊗ b) n = a ⊙P monom
P b n"
  by (rule up_eqI) simp_all

lemma monom_one_mult:
  "monom P 1 (n + m) = monom P 1 n ⊗P monom P 1 m"
proof (induct n)
  case 0 show ?case by simp
next
  case Suc then show ?case
    unfolding add_Suc unfolding monom_one_Suc unfolding Suc.hyps
    using m_assoc monom_one_comm [of m] by simp
qed

lemma monom_one_mult_comm: "monom P 1 n ⊗P monom P 1 m = monom P 1 m
⊗P monom P 1 n"
  unfolding monom_one_mult [symmetric] by (rule up_eqI) simp_all

lemma monom_mult [simp]:
  assumes a_in_R: "a ∈ carrier R" and b_in_R: "b ∈ carrier R"
  shows "monom P (a ⊗ b) (n + m) = monom P a n ⊗P monom P b m"
proof (rule up_eqI)
```

```
  fix k
  show "coeff P (monom P (a ⊗ b) (n + m)) k = coeff P (monom P a n ⊗P
monom P b m) k"
  proof (cases "n + m = k")
    case True
    {
      show ?thesis
        unfolding True [symmetric]
          coeff_mult [OF monom_closed [OF a_in_R, of n] monom_closed [OF
b_in_R, of m], of "n + m"]
          coeff_monom [OF a_in_R, of n] coeff_monom [OF b_in_R, of m]
        using R.finsum_cong [of "{.. n + m}" "{.. n + m}" "(λi. (if n
= i then a else 0) ⊗ (if m = n + m - i then b else 0))"
          "(λi. if n = i then a ⊗ b else 0)"]
          a_in_R b_in_R
        unfolding simp_implies_def
        using R.finsum_singleton [of n "{.. n + m}" "(λi. a ⊗ b)"]
        unfolding Pi_def by auto
    }
  next
    case False
    {
      show ?thesis
        unfolding coeff_monom [OF R.m_closed [OF a_in_R b_in_R], of "n
+ m" k] apply (simp add: False)
        unfolding coeff_mult [OF monom_closed [OF a_in_R, of n] monom_closed
[OF b_in_R, of m], of k]
        unfolding coeff_monom [OF a_in_R, of n] unfolding coeff_monom
[OF b_in_R, of m] using False
        using R.finsum_cong [of "{..k}" "{..k}" "(λi. (if n = i then a
else 0) ⊗ (if m = k - i then b else 0))" "(λi. 0)"]
        unfolding Pi_def simp_implies_def using a_in_R b_in_R by force
    }
  qed
qed (simp_all add: a_in_R b_in_R)

lemma monom_a_inv [simp]:
  "a ∈ carrier R ==> monom P (⊖ a) n = ⊖P monom P a n"
  by (rule up_eqI) simp_all

lemma monom_inj:
  "inj_on (λa. monom P a n) (carrier R)"
proof (rule inj_onI)
  fix x y
  assume R: "x ∈ carrier R" "y ∈ carrier R" and eq: "monom P x n = monom
P y n"
  then have "coeff P (monom P x n) n = coeff P (monom P y n) n" by simp
  with R show "x = y" by simp
qed
```

end

16.8 The Degree Function

```
definition
  deg :: "[('a, 'm) ring_scheme, nat => 'a] => nat"
  where "deg R p = (LEAST n. bound 0_R n (coeff (UP R) p))"

context UP_ring
begin

lemma deg_aboveI:
  "[| (!!m. n < m ==> coeff P p m = 0); p ∈ carrier P |] ==> deg R p <=
n"
  by (unfold deg_def P_def) (fast intro: Least_le)



lemma deg_aboveD:
  assumes "deg R p < m" and "p ∈ carrier P"
  shows "coeff P p m = 0"
proof -
  from ‹p ∈ carrier P› obtain n where "bound 0 n (coeff P p)"
    by (auto simp add: UP_def P_def)
  then have "bound 0 (deg R p) (coeff P p)"
    by (auto simp: deg_def P_def dest: LeastI)
  from this and ‹deg R p < m› show ?thesis ..
qed

lemma deg_belowI:
  assumes non_zero: "n ~= 0 ==> coeff P p n ~= 0"
    and R: "p ∈ carrier P"
  shows "n <= deg R p"
— Logically, this is a slightly stronger version of deg_aboveD
proof (cases "n=0")
  case True then show ?thesis by simp
next
  case False then have "coeff P p n ~= 0" by (rule non_zero)
  then have "~ deg R p < n" by (fast dest: deg_aboveD intro: R)
  then show ?thesis by arith
qed

lemma lcoeff_nonzero_deg:
  assumes deg: "deg R p ~= 0" and R: "p ∈ carrier P"
  shows "coeff P p (deg R p) ~= 0"
proof -
  from R obtain m where "deg R p <= m" and m_coeff: "coeff P p m ~=
0"
```

```
  proof -
    have minus: "!!(n::nat) m. n ~= 0 ==> (n - Suc 0 < m) = (n <= m)"
      by arith
    from deg have "deg R p - 1 < (LEAST n. bound 0 n (coeff P p))"
      by (unfold deg_def P_def) simp
    then have "~ bound 0 (deg R p - 1) (coeff P p)" by (rule not_less_Least)
    then have "EX m. deg R p - 1 < m & coeff P p m ~= 0"
      by (unfold bound_def) fast
    then have "EX m. deg R p <= m & coeff P p m ~= 0" by (simp add: deg
minus)
    then show ?thesis by (auto intro: that)
  qed
  with deg_belowI R have "deg R p = m" by fastforce
  with m_coeff show ?thesis by simp
qed

lemma lcoeff_nonzero_nonzero:
  assumes deg: "deg R p = 0" and nonzero: "p ~= 0P" and R: "p ∈ carrier
P"
  shows "coeff P p 0 ~= 0"
proof -
  have "EX m. coeff P p m ~= 0"
  proof (rule classical)
    assume "~ ?thesis"
    with R have "p = 0P" by (auto intro: up_eqI)
    with nonzero show ?thesis by contradiction
  qed
  then obtain m where coeff: "coeff P p m ~= 0" ..
  from this and R have "m <= deg R p" by (rule deg_belowI)
  then have "m = 0" by (simp add: deg)
  with coeff show ?thesis by simp
qed

lemma lcoeff_nonzero:
  assumes neq: "p ~= 0P" and R: "p ∈ carrier P"
  shows "coeff P p (deg R p) ~= 0"
proof (cases "deg R p = 0")
  case True with neq R show ?thesis by (simp add: lcoeff_nonzero_nonzero)
next
  case False with neq R show ?thesis by (simp add: lcoeff_nonzero_deg)
qed

lemma deg_eqI:
  "[| !!m. n < m ==> coeff P p m = 0;
      !!n. n ~= 0 ==> coeff P p n ~= 0; p ∈ carrier P |] ==> deg R p =
n"
by (fast intro: le_antisym deg_aboveI deg_belowI)
```

Degree and polynomial operations

```
lemma deg_add [simp]:
  "p ∈ carrier P ⟹ q ∈ carrier P ⟹
  deg R (p ⊕_P q) <= max (deg R p) (deg R q)"
by(rule deg_aboveI)(simp_all add: deg_aboveD)

lemma deg_monom_le:
  "a ∈ carrier R ==> deg R (monom P a n) <= n"
  by (intro deg_aboveI) simp_all

lemma deg_monom [simp]:
  "[| a ~= 0; a ∈ carrier R |] ==> deg R (monom P a n) = n"
  by (fastforce intro: le_antisym deg_aboveI deg_belowI)

lemma deg_const [simp]:
  assumes R: "a ∈ carrier R" shows "deg R (monom P a 0) = 0"
proof (rule le_antisym)
  show "deg R (monom P a 0) <= 0" by (rule deg_aboveI) (simp_all add:
R)
next
  show "0 <= deg R (monom P a 0)" by (rule deg_belowI) (simp_all add:
R)
qed

lemma deg_zero [simp]:
  "deg R 0_P = 0"
proof (rule le_antisym)
  show "deg R 0_P <= 0" by (rule deg_aboveI) simp_all
next
  show "0 <= deg R 0_P" by (rule deg_belowI) simp_all
qed

lemma deg_one [simp]:
  "deg R 1_P = 0"
proof (rule le_antisym)
  show "deg R 1_P <= 0" by (rule deg_aboveI) simp_all
next
  show "0 <= deg R 1_P" by (rule deg_belowI) simp_all
qed

lemma deg_uminus [simp]:
  assumes R: "p ∈ carrier P" shows "deg R (⊖_P p) = deg R p"
proof (rule le_antisym)
  show "deg R (⊖_P p) <= deg R p" by (simp add: deg_aboveI deg_aboveD
R)
next
  show "deg R p <= deg R (⊖_P p)"
    by (simp add: deg_belowI lcoeff_nonzero_deg
      inj_on_eq_iff [OF R.a_inv_inj, of _ "0", simplified] R)
qed
```

The following lemma is later *overwritten* by the most specific one for domains, `deg_smult`.

```
lemma deg_smult_ring [simp]:
  "[| a ∈ carrier R; p ∈ carrier P |] ==>
  deg R (a ⊙P p) <= (if a = 0 then 0 else deg R p)"
  by (cases "a = 0") (simp add: deg_aboveI deg_aboveD)+

end

context UP_domain
begin

lemma deg_smult [simp]:
  assumes R: "a ∈ carrier R" "p ∈ carrier P"
  shows "deg R (a ⊙P p) = (if a = 0 then 0 else deg R p)"
proof (rule le_antisym)
  show "deg R (a ⊙P p) <= (if a = 0 then 0 else deg R p)"
    using R by (rule deg_smult_ring)
next
  show "(if a = 0 then 0 else deg R p) <= deg R (a ⊙P p)"
  proof (cases "a = 0")
  qed (simp, simp add: deg_belowI lcoeff_nonzero_deg integral_iff R)
qed

end

context UP_ring
begin

lemma deg_mult_ring:
  assumes R: "p ∈ carrier P" "q ∈ carrier P"
  shows "deg R (p ⊗P q) <= deg R p + deg R q"
proof (rule deg_aboveI)
  fix m
  assume boundm: "deg R p + deg R q < m"
  {
    fix k i
    assume boundk: "deg R p + deg R q < k"
    then have "coeff P p i ⊗ coeff P q (k - i) = 0"
    proof (cases "deg R p < i")
      case True then show ?thesis by (simp add: deg_aboveD R)
    next
      case False with boundk have "deg R q < k - i" by arith
      then show ?thesis by (simp add: deg_aboveD R)
    qed
  }
  with boundm R show "coeff P (p ⊗P q) m = 0" by simp
qed (simp add: R)
```

```
end

context UP_domain
begin

lemma deg_mult [simp]:
  "[| p ~= 0P; q ~= 0P; p ∈ carrier P; q ∈ carrier P |] ==>
  deg R (p ⊗P q) = deg R p + deg R q"
proof (rule le_antisym)
  assume "p ∈ carrier P" " q ∈ carrier P"
  then show "deg R (p ⊗P q) <= deg R p + deg R q" by (rule deg_mult_ring)
next
  let ?s = "(λi. coeff P p i ⊗ coeff P q (deg R p + deg R q - i))"
  assume R: "p ∈ carrier P" "q ∈ carrier P" and nz: "p ~= 0P" "q ~=
0P"
  have less_add_diff: "!!(k::nat) n m. k < n ==> m < n + m - k" by arith
  show "deg R p + deg R q <= deg R (p ⊗P q)"
  proof (rule deg_belowI, simp add: R)
    have "(⨁i ∈ {.. deg R p + deg R q}. ?s i)
      = (⨁i ∈ {..< deg R p} ∪ {deg R p .. deg R p + deg R q}. ?s i)"
      by (simp only: ivl_disj_un_one)
    also have "... = (⨁i ∈ {deg R p .. deg R p + deg R q}. ?s i)"
      by (simp cong: R.finsum_cong add: R.finsum_Un_disjoint ivl_disj_int_one
        deg_aboveD less_add_diff R Pi_def)
    also have "...= (⨁i ∈ {deg R p} ∪ {deg R p <.. deg R p + deg R q}.
?s i)"
      by (simp only: ivl_disj_un_singleton)
    also have "... = coeff P p (deg R p) ⊗ coeff P q (deg R q)"
      by (simp cong: R.finsum_cong add: deg_aboveD R Pi_def)
    finally have "(⨁i ∈ {.. deg R p + deg R q}. ?s i)
      = coeff P p (deg R p) ⊗ coeff P q (deg R q)" .
    with nz show "(⨁i ∈ {.. deg R p + deg R q}. ?s i) ~= 0"
      by (simp add: integral_iff lcoeff_nonzero R)
  qed (simp add: R)
qed

end
```

The following lemmas also can be lifted to `UP_ring`.

```
context UP_ring
begin

lemma coeff_finsum:
  assumes fin: "finite A"
  shows "p ∈ A → carrier P ==>
    coeff P (finsum P p A) k = (⨁i ∈ A. coeff P (p i) k)"
  using fin by induct (auto simp: Pi_def)

lemma up_repr:
```

```
  assumes R: "p ∈ carrier P"
  shows "(⨁P i ∈ {..deg R p}. monom P (coeff P p i) i) = p"
proof (rule up_eqI)
  let ?s = "(λi. monom P (coeff P p i) i)"
  fix k
  from R have RR: "!!i. (if i = k then coeff P p i else 0) ∈ carrier
R"
    by simp
  show "coeff P (⨁P i ∈ {..deg R p}. ?s i) k = coeff P p k"
  proof (cases "k <= deg R p")
    case True
    hence "coeff P (⨁P i ∈ {..deg R p}. ?s i) k =
          coeff P (⨁P i ∈ {..k} ∪ {k<..deg R p}. ?s i) k"
      by (simp only: ivl_disj_un_one)
    also from True
    have "... = coeff P (⨁P i ∈ {..k}. ?s i) k"
      by (simp cong: R.finsum_cong add: R.finsum_Un_disjoint
        ivl_disj_int_one order_less_imp_not_eq2 coeff_finsum R RR Pi_def)
    also
    have "... = coeff P (⨁P i ∈ {..<k} ∪ {k}. ?s i) k"
      by (simp only: ivl_disj_un_singleton)
    also have "... = coeff P p k"
      by (simp cong: R.finsum_cong add: coeff_finsum deg_aboveD R RR Pi_def)
    finally show ?thesis .
  next
    case False
    hence "coeff P (⨁P i ∈ {..deg R p}. ?s i) k =
          coeff P (⨁P i ∈ {..<deg R p} ∪ {deg R p}. ?s i) k"
      by (simp only: ivl_disj_un_singleton)
    also from False have "... = coeff P p k"
      by (simp cong: R.finsum_cong add: coeff_finsum deg_aboveD R Pi_def)
    finally show ?thesis .
  qed
qed (simp_all add: R Pi_def)

lemma up_repr_le:
  "[| deg R p <= n; p ∈ carrier P |] ==>
  (⨁P i ∈ {..n}. monom P (coeff P p i) i) = p"
proof -
  let ?s = "(λi. monom P (coeff P p i) i)"
  assume R: "p ∈ carrier P" and "deg R p <= n"
  then have "finsum P ?s {..n} = finsum P ?s ({..deg R p} ∪ {deg R p<..n})"
    by (simp only: ivl_disj_un_one)
  also have "... = finsum P ?s {..deg R p}"
    by (simp cong: P.finsum_cong add: P.finsum_Un_disjoint ivl_disj_int_one
      deg_aboveD R Pi_def)
  also have "... = p" using R by (rule up_repr)
  finally show ?thesis .
qed
```

```
end
```

16.9 Polynomials over Integral Domains

```
lemma domainI:
  assumes cring: "cring R"
    and one_not_zero: "one R ~= zero R"
    and integral: "!!a b. [| mult R a b = zero R; a ∈ carrier R;
      b ∈ carrier R |] ==> a = zero R | b = zero R"
  shows "domain R"
  by (auto intro!: domain.intro domain_axioms.intro cring.axioms assms
    del: disjCI)

context UP_domain
begin

lemma UP_one_not_zero:
  "1P ~= 0P"
proof
  assume "1P = 0P"
  hence "coeff P 1P 0 = (coeff P 0P 0)" by simp
  hence "1 = 0" by simp
  with R.one_not_zero show "False" by contradiction
qed

lemma UP_integral:
  "[| p ⊗P q = 0P; p ∈ carrier P; q ∈ carrier P |] ==> p = 0P | q = 0P"
proof -
  fix p q
  assume pq: "p ⊗P q = 0P" and R: "p ∈ carrier P" "q ∈ carrier P"
  show "p = 0P | q = 0P"
  proof (rule classical)
    assume c: "~ (p = 0P | q = 0P)"
    with R have "deg R p + deg R q = deg R (p ⊗P q)" by simp
    also from pq have "... = 0" by simp
    finally have "deg R p + deg R q = 0" .
    then have f1: "deg R p = 0 & deg R q = 0" by simp
    from f1 R have "p = (⨁P i ∈ {..0}. monom P (coeff P p i) i)"
      by (simp only: up_repr_le)
    also from R have "... = monom P (coeff P p 0) 0" by simp
    finally have p: "p = monom P (coeff P p 0) 0" .
    from f1 R have "q = (⨁P i ∈ {..0}. monom P (coeff P q i) i)"
      by (simp only: up_repr_le)
    also from R have "... = monom P (coeff P q 0) 0" by simp
    finally have q: "q = monom P (coeff P q 0) 0" .
    from R have "coeff P p 0 ⊗ coeff P q 0 = coeff P (p ⊗P q) 0" by
simp
    also from pq have "... = 0" by simp
```

```
    finally have "coeff P p 0 ⊗ coeff P q 0 = 0" .
    with R have "coeff P p 0 = 0 | coeff P q 0 = 0"
      by (simp add: R.integral_iff)
    with p q show "p = 0P | q = 0P" by fastforce
  qed
qed

theorem UP_domain:
  "domain P"
  by (auto intro!: domainI UP_cring UP_one_not_zero UP_integral del: disjCI)

end
```

Interpretation of theorems from `domain`.

```
sublocale UP_domain < "domain" P
  by intro_locales (rule domain.axioms UP_domain)+
```

16.10 The Evaluation Homomorphism and Universal Property

```
lemma (in abelian_monoid) boundD_carrier:
  "[| bound 0 n f; n < m |] ==> f m ∈ carrier G"
  by auto

context ring
begin

theorem diagonal_sum:
  "[| f ∈ {..n + m::nat} → carrier R; g ∈ {..n + m} → carrier R |] ==>
  (⨁k ∈ {..n + m}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
  (⨁k ∈ {..n + m}. ⨁i ∈ {..n + m - k}. f k ⊗ g i)"
proof -
  assume Rf: "f ∈ {..n + m} → carrier R" and Rg: "g ∈ {..n + m} →
carrier R"
  {
    fix j
    have "j <= n + m ==>
      (⨁k ∈ {..j}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
      (⨁k ∈ {..j}. ⨁i ∈ {..j - k}. f k ⊗ g i)"
    proof (induct j)
      case 0 from Rf Rg show ?case by (simp add: Pi_def)
    next
      case (Suc j)
      have R6: "!!i k. [| k <= j; i <= Suc j - k |] ==> g i ∈ carrier
R"
        using Suc by (auto intro!: funcset_mem [OF Rg])
      have R8: "!!i k. [| k <= Suc j; i <= k |] ==> g (k - i) ∈ carrier
R"
        using Suc by (auto intro!: funcset_mem [OF Rg])
```

```
      have R9: "!!i k. [| k <= Suc j |] ==> f k ∈ carrier R"
        using Suc by (auto intro!: funcset_mem [OF Rf])
      have R10: "!!i k. [| k <= Suc j; i <= Suc j - k |] ==> g i ∈ carrier
R"
        using Suc by (auto intro!: funcset_mem [OF Rg])
      have R11: "g 0 ∈ carrier R"
        using Suc by (auto intro!: funcset_mem [OF Rg])
      from Suc show ?case
        by (simp cong: finsum_cong add: Suc_diff_le a_ac
          Pi_def R6 R8 R9 R10 R11)
    qed
  }
  then show ?thesis by fast
qed

theorem cauchy_product:
  assumes bf: "bound 0 n f" and bg: "bound 0 m g"
    and Rf: "f ∈ {..n} → carrier R" and Rg: "g ∈ {..m} → carrier R"
  shows "(⨁k ∈ {..n + m}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
    (⨁i ∈ {..n}. f i) ⊗ (⨁i ∈ {..m}. g i)"
proof -
  have f: "!!x. f x ∈ carrier R"
  proof -
    fix x
    show "f x ∈ carrier R"
      using Rf bf boundD_carrier by (cases "x <= n") (auto simp: Pi_def)
  qed
  have g: "!!x. g x ∈ carrier R"
  proof -
    fix x
    show "g x ∈ carrier R"
      using Rg bg boundD_carrier by (cases "x <= m") (auto simp: Pi_def)
  qed
  from f g have "(⨁k ∈ {..n + m}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
      (⨁k ∈ {..n + m}. ⨁i ∈ {..n + m - k}. f k ⊗ g i)"
    by (simp add: diagonal_sum Pi_def)
  also have "... = (⨁k ∈ {..n} ∪ {n<..n + m}. ⨁i ∈ {..n + m - k}.
f k ⊗ g i)"
    by (simp only: ivl_disj_un_one)
  also from f g have "... = (⨁k ∈ {..n}. ⨁i ∈ {..n + m - k}. f k ⊗
g i)"
    by (simp cong: finsum_cong
      add: bound.bound [OF bf] finsum_Un_disjoint ivl_disj_int_one Pi_def)
  also from f g
  have "... = (⨁k ∈ {..n}. ⨁i ∈ {..m} ∪ {m<..n + m - k}. f k ⊗ g i)"
    by (simp cong: finsum_cong add: ivl_disj_un_one le_add_diff Pi_def)
  also from f g have "... = (⨁k ∈ {..n}. ⨁i ∈ {..m}. f k ⊗ g i)"
    by (simp cong: finsum_cong
      add: bound.bound [OF bg] finsum_Un_disjoint ivl_disj_int_one Pi_def)
```

```
  also from f g have "... = (⨁i ∈ {..n}. f i) ⊗ (⨁i ∈ {..m}. g i)"
    by (simp add: finsum_ldistr diagonal_sum Pi_def,
      simp cong: finsum_cong add: finsum_rdistr Pi_def)
  finally show ?thesis .
qed

end

lemma (in UP_ring) const_ring_hom:
  "(λa. monom P a 0) ∈ ring_hom R P"
  by (auto intro!: ring_hom_memI intro: up_eqI simp: monom_mult_is_smult)

definition
  eval :: "[('a, 'm) ring_scheme, ('b, 'n) ring_scheme,
           'a => 'b, 'b, nat => 'a] => 'b"
  where "eval R S phi s = (λp ∈ carrier (UP R).
    ⨁Si ∈ {..deg R p}. phi (coeff (UP R) p i) ⊗S s (^)S i)"

context UP
begin

lemma eval_on_carrier:
  fixes S (structure)
  shows "p ∈ carrier P ==>
  eval R S phi s p = (⨁S i ∈ {..deg R p}. phi (coeff P p i) ⊗S s (^)S
i)"
  by (unfold eval_def, fold P_def) simp

lemma eval_extensional:
  "eval R S phi p ∈ extensional (carrier P)"
  by (unfold eval_def, fold P_def) simp

end
```

The universal property of the polynomial ring

```
locale UP_pre_univ_prop = ring_hom_cring + UP_cring

locale UP_univ_prop = UP_pre_univ_prop +
  fixes s and Eval
  assumes indet_img_carrier [simp, intro]: "s ∈ carrier S"
  defines Eval_def: "Eval == eval R S h s"
```

JE: I have moved the following lemma from Ring.thy and lifted then to the locale `ring_hom_ring` from `ring_hom_cring`.

JE: I was considering using it in `eval_ring_hom`, but that property does not hold for non commutative rings, so maybe it is not that necessary.

```
lemma (in ring_hom_ring) hom_finsum [simp]:
  "f ∈ A → carrier R ==>
```

```
  h (finsum R f A) = finsum S (h o f) A"
  by (induct A rule: infinite_finite_induct, auto simp: Pi_def)

context UP_pre_univ_prop
begin

theorem eval_ring_hom:
  assumes S: "s ∈ carrier S"
  shows "eval R S h s ∈ ring_hom P S"
proof (rule ring_hom_memI)
  fix p
  assume R: "p ∈ carrier P"
  then show "eval R S h s p ∈ carrier S"
    by (simp only: eval_on_carrier) (simp add: S Pi_def)
next
  fix p q
  assume R: "p ∈ carrier P" "q ∈ carrier P"
  then show "eval R S h s (p ⊕P q) = eval R S h s p ⊕S eval R S h s
q"
  proof (simp only: eval_on_carrier P.a_closed)
    from S R have
      "(⨁S i∈{..deg R (p ⊕P q)}. h (coeff P (p ⊕P q) i) ⊗S s (^)S i)
=
      (⨁S i∈{..deg R (p ⊕P q)} ∪ {deg R (p ⊕P q)<..max (deg R p) (deg
R q)}.
        h (coeff P (p ⊕P q) i) ⊗S s (^)S i)"
      by (simp cong: S.finsum_cong
        add: deg_aboveD S.finsum_Un_disjoint ivl_disj_int_one Pi_def del:
coeff_add)
    also from R have "... =
        (⨁S i ∈ {..max (deg R p) (deg R q)}.
          h (coeff P (p ⊕P q) i) ⊗S s (^)S i)"
      by (simp add: ivl_disj_un_one)
    also from R S have "... =
      (⨁Si∈{..max (deg R p) (deg R q)}. h (coeff P p i) ⊗S s (^)S i)
⊕S
      (⨁Si∈{..max (deg R p) (deg R q)}. h (coeff P q i) ⊗S s (^)S i)"
      by (simp cong: S.finsum_cong
        add: S.l_distr deg_aboveD ivl_disj_int_one Pi_def)
    also have "... =
        (⨁S i ∈ {..deg R p} ∪ {deg R p<..max (deg R p) (deg R q)}.
          h (coeff P p i) ⊗S s (^)S i) ⊕S
        (⨁S i ∈ {..deg R q} ∪ {deg R q<..max (deg R p) (deg R q)}.
          h (coeff P q i) ⊗S s (^)S i)"
      by (simp only: ivl_disj_un_one max.cobounded1 max.cobounded2)
    also from R S have "... =
      (⨁S i ∈ {..deg R p}. h (coeff P p i) ⊗S s (^)S i) ⊕S
      (⨁S i ∈ {..deg R q}. h (coeff P q i) ⊗S s (^)S i)"
      by (simp cong: S.finsum_cong
```

```
        add: deg_aboveD S.finsum_Un_disjoint ivl_disj_int_one Pi_def)
    finally show
      "(⨁Si ∈ {..deg R (p ⊕P q)}. h (coeff P (p ⊕P q) i) ⊗S s (^)S
i) =
      (⨁Si ∈ {..deg R p}. h (coeff P p i) ⊗S s (^)S i) ⊕S
      (⨁Si ∈ {..deg R q}. h (coeff P q i) ⊗S s (^)S i)" .
  qed
next
  show "eval R S h s 1P = 1S"
    by (simp only: eval_on_carrier UP_one_closed) simp
next
  fix p q
  assume R: "p ∈ carrier P" "q ∈ carrier P"
  then show "eval R S h s (p ⊗P q) = eval R S h s p ⊗S eval R S h s
q"
  proof (simp only: eval_on_carrier UP_mult_closed)
    from R S have
      "(⨁S i ∈ {..deg R (p ⊗P q)}. h (coeff P (p ⊗P q) i) ⊗S s (^)S
i) =
      (⨁S i ∈ {..deg R (p ⊗P q)} ∪ {deg R (p ⊗P q)<..deg R p + deg
R q}.
        h (coeff P (p ⊗P q) i) ⊗S s (^)S i)"
      by (simp cong: S.finsum_cong
        add: deg_aboveD S.finsum_Un_disjoint ivl_disj_int_one Pi_def
        del: coeff_mult)
    also from R have "... =
      (⨁S i ∈ {..deg R p + deg R q}. h (coeff P (p ⊗P q) i) ⊗S s (^)S
i)"
      by (simp only: ivl_disj_un_one deg_mult_ring)
    also from R S have "... =
      (⨁S i ∈ {..deg R p + deg R q}.
         ⨁S k ∈ {..i}.
           h (coeff P p k) ⊗S h (coeff P q (i - k)) ⊗S
           (s (^)S k ⊗S s (^)S (i - k)))"
      by (simp cong: S.finsum_cong add: S.nat_pow_mult Pi_def
        S.m_ac S.finsum_rdistr)
    also from R S have "... =
      (⨁S i∈{..deg R p}. h (coeff P p i) ⊗S s (^)S i) ⊗S
      (⨁S i∈{..deg R q}. h (coeff P q i) ⊗S s (^)S i)"
      by (simp add: S.cauchy_product [THEN sym] bound.intro deg_aboveD
S.m_ac
        Pi_def)
    finally show
      "(⨁S i ∈ {..deg R (p ⊗P q)}. h (coeff P (p ⊗P q) i) ⊗S s (^)S
i) =
      (⨁S i ∈ {..deg R p}. h (coeff P p i) ⊗S s (^)S i) ⊗S
      (⨁S i ∈ {..deg R q}. h (coeff P q i) ⊗S s (^)S i)" .
  qed
qed
```

The following lemma could be proved in `UP_cring` with the additional assumption that `h` is closed.

```
lemma (in UP_pre_univ_prop) eval_const:
  "[| s ∈ carrier S; r ∈ carrier R |] ==> eval R S h s (monom P r 0) =
h r"
  by (simp only: eval_on_carrier monom_closed) simp
```

Further properties of the evaluation homomorphism.

The following proof is complicated by the fact that in arbitrary rings one might have **1** = **0**.

```
lemma (in UP_pre_univ_prop) eval_monom1:
  assumes S: "s ∈ carrier S"
  shows "eval R S h s (monom P 1 1) = s"
proof (simp only: eval_on_carrier monom_closed R.one_closed)
   from S have
    "(⨁S i∈{..deg R (monom P 1 1)}. h (coeff P (monom P 1 1) i) ⊗S s
(^)S i) =
    (⨁S i∈{..deg R (monom P 1 1)} ∪ {deg R (monom P 1 1)<..1}.
      h (coeff P (monom P 1 1) i) ⊗S s (^)S i)"
    by (simp cong: S.finsum_cong del: coeff_monom
      add: deg_aboveD S.finsum_Un_disjoint ivl_disj_int_one Pi_def)
  also have "... =
    (⨁S i ∈ {..1}. h (coeff P (monom P 1 1) i) ⊗S s (^)S i)"
    by (simp only: ivl_disj_un_one deg_monom_le R.one_closed)
  also have "... = s"
  proof (cases "s = 0S")
    case True then show ?thesis by (simp add: Pi_def)
  next
    case False then show ?thesis by (simp add: S Pi_def)
  qed
  finally show "(⨁S i ∈ {..deg R (monom P 1 1)}.
    h (coeff P (monom P 1 1) i) ⊗S s (^)S i) = s" .
qed

end
```

Interpretation of ring homomorphism lemmas.

```
sublocale UP_univ_prop < ring_hom_cring P S Eval
  unfolding Eval_def
  by unfold_locales (fast intro: eval_ring_hom)

lemma (in UP_cring) monom_pow:
  assumes R: "a ∈ carrier R"
  shows "(monom P a n) (^)P m = monom P (a (^) m) (n * m)"
proof (induct m)
  case 0 from R show ?case by simp
next
```

```
  case Suc with R show ?case
    by (simp del: monom_mult add: monom_mult [THEN sym] add.commute)
qed

lemma (in ring_hom_cring) hom_pow [simp]:
  "x ∈ carrier R ==> h (x (^) n) = h x (^)S (n::nat)"
  by (induct n) simp_all

lemma (in UP_univ_prop) Eval_monom:
  "r ∈ carrier R ==> Eval (monom P r n) = h r ⊗S s (^)S n"
proof -
  assume R: "r ∈ carrier R"
  from R have "Eval (monom P r n) = Eval (monom P r 0 ⊗P (monom P 1 1)
(^)P n)"
    by (simp del: monom_mult add: monom_mult [THEN sym] monom_pow)
  also
  from R eval_monom1 [where s = s, folded Eval_def]
  have "... = h r ⊗S s (^)S n"
    by (simp add: eval_const [where s = s, folded Eval_def])
  finally show ?thesis .
qed

lemma (in UP_pre_univ_prop) eval_monom:
  assumes R: "r ∈ carrier R" and S: "s ∈ carrier S"
  shows "eval R S h s (monom P r n) = h r ⊗S s (^)S n"
proof -
  interpret UP_univ_prop R S h P s "eval R S h s"
    using UP_pre_univ_prop_axioms P_def R S
    by (auto intro: UP_univ_prop.intro UP_univ_prop_axioms.intro)
  from R
  show ?thesis by (rule Eval_monom)
qed

lemma (in UP_univ_prop) Eval_smult:
  "[| r ∈ carrier R; p ∈ carrier P |] ==> Eval (r ⊙P p) = h r ⊗S Eval
p"
proof -
  assume R: "r ∈ carrier R" and P: "p ∈ carrier P"
  then show ?thesis
    by (simp add: monom_mult_is_smult [THEN sym]
      eval_const [where s = s, folded Eval_def])
qed

lemma ring_hom_cringI:
  assumes "cring R"
    and "cring S"
    and "h ∈ ring_hom R S"
  shows "ring_hom_cring R S h"
  by (fast intro: ring_hom_cring.intro ring_hom_cring_axioms.intro
```

```
    cring.axioms assms)

context UP_pre_univ_prop
begin

lemma UP_hom_unique:
  assumes "ring_hom_cring P S Phi"
  assumes Phi: "Phi (monom P 𝟭 (Suc 0)) = s"
      "!!r. r ∈ carrier R ==> Phi (monom P r 0) = h r"
  assumes "ring_hom_cring P S Psi"
  assumes Psi: "Psi (monom P 𝟭 (Suc 0)) = s"
      "!!r. r ∈ carrier R ==> Psi (monom P r 0) = h r"
    and P: "p ∈ carrier P" and S: "s ∈ carrier S"
  shows "Phi p = Psi p"
proof -
  interpret ring_hom_cring P S Phi by fact
  interpret ring_hom_cring P S Psi by fact
  have "Phi p =
      Phi (⨁P i ∈ {..deg R p}. monom P (coeff P p i) 0 ⊗P monom P 𝟭
1 (^)P i)"
    by (simp add: up_repr P monom_mult [THEN sym] monom_pow del: monom_mult)
  also
  have "... =
      Psi (⨁P i∈{..deg R p}. monom P (coeff P p i) 0 ⊗P monom P 𝟭 1
(^)P i)"
    by (simp add: Phi Psi P Pi_def comp_def)
  also have "... = Psi p"
    by (simp add: up_repr P monom_mult [THEN sym] monom_pow del: monom_mult)
  finally show ?thesis .
qed

lemma ring_homD:
  assumes Phi: "Phi ∈ ring_hom P S"
  shows "ring_hom_cring P S Phi"
  by unfold_locales (rule Phi)

theorem UP_universal_property:
  assumes S: "s ∈ carrier S"
  shows "∃!Phi. Phi ∈ ring_hom P S ∩ extensional (carrier P) &
    Phi (monom P 𝟭 1) = s &
    (ALL r : carrier R. Phi (monom P r 0) = h r)"
  using S eval_monom1
  apply (auto intro: eval_ring_hom eval_const eval_extensional)
  apply (rule extensionalityI)
  apply (auto intro: UP_hom_unique ring_homD)
  done

end
```

JE: The following lemma was added by me; it might be even lifted to a simpler locale

```
context monoid
begin

lemma nat_pow_eone[simp]: assumes x_in_G: "x ∈ carrier G" shows "x
(^) (1::nat) = x"
  using nat_pow_Suc [of x 0] unfolding nat_pow_0 [of x] unfolding l_one
[OF x_in_G] by simp

end

context UP_ring
begin

abbreviation lcoeff :: "(nat =>'a) => 'a" where "lcoeff p == coeff P
p (deg R p)"

lemma lcoeff_nonzero2: assumes p_in_R: "p ∈ carrier P" and p_not_zero:
"p ≠ 0P" shows "lcoeff p ≠ 0"
  using lcoeff_nonzero [OF p_not_zero p_in_R] .
```

16.11 The long division algorithm: some previous facts.

```
lemma coeff_minus [simp]:
  assumes p: "p ∈ carrier P" and q: "q ∈ carrier P" shows "coeff P (p
⊖P q) n = coeff P p n ⊖ coeff P q n"
  unfolding a_minus_def [OF p q] unfolding coeff_add [OF p a_inv_closed
[OF q]] unfolding coeff_a_inv [OF q]
  using coeff_closed [OF p, of n] using coeff_closed [OF q, of n] by algebra

lemma lcoeff_closed [simp]: assumes p: "p ∈ carrier P" shows "lcoeff
p ∈ carrier R"
  using coeff_closed [OF p, of "deg R p"] by simp

lemma deg_smult_decr: assumes a_in_R: "a ∈ carrier R" and f_in_P: "f
∈ carrier P" shows "deg R (a ⊙P f) ≤ deg R f"
  using deg_smult_ring [OF a_in_R f_in_P] by (cases "a = 0", auto)

lemma coeff_monom_mult: assumes R: "c ∈ carrier R" and P: "p ∈ carrier
P"
  shows "coeff P (monom P c n ⊗P p) (m + n) = c ⊗ (coeff P p m)"
proof -
  have "coeff P (monom P c n ⊗P p) (m + n) = (⨁i∈{..m + n}. (if n =
i then c else 0) ⊗ coeff P p (m + n - i))"
    unfolding coeff_mult [OF monom_closed [OF R, of n] P, of "m + n"]
unfolding coeff_monom [OF R, of n] by simp
  also have "(⨁i∈{..m + n}. (if n = i then c else 0) ⊗ coeff P p (m
+ n - i)) =
    (⨁i∈{..m + n}. (if n = i then c ⊗ coeff P p (m + n - i) else 0))"
```

```
    using  R.finsum_cong [of "{..m + n}" "{..m + n}" "(λi::nat. (if n
= i then c else 0) ⊗ coeff P p (m + n - i))"
      "(λi::nat. (if n = i then c ⊗ coeff P p (m + n - i) else 0))"]
    using coeff_closed [OF P] unfolding Pi_def simp_implies_def using
R by auto
  also have "... = c ⊗ coeff P p m" using R.finsum_singleton [of n "{..m
+ n}" "(λi. c ⊗ coeff P p (m + n - i))"]
    unfolding Pi_def using coeff_closed [OF P] using P R by auto
  finally show ?thesis by simp
qed

lemma deg_lcoeff_cancel:
  assumes p_in_P: "p ∈ carrier P" and q_in_P: "q ∈ carrier P" and r_in_P:
"r ∈ carrier P"
  and deg_r_nonzero: "deg R r ≠ 0"
  and deg_R_p: "deg R p ≤ deg R r" and deg_R_q: "deg R q ≤ deg R r"
  and coeff_R_p_eq_q: "coeff P p (deg R r) = ⊖R (coeff P q (deg R r))"
  shows "deg R (p ⊕P q) < deg R r"
proof -
  have deg_le: "deg R (p ⊕P q) ≤ deg R r"
  proof (rule deg_aboveI)
    fix m
    assume deg_r_le: "deg R r < m"
    show "coeff P (p ⊕P q) m = 0"
    proof -
      have slp: "deg R p < m" and "deg R q < m" using deg_R_p deg_R_q
using deg_r_le by auto
      then have max_sl: "max (deg R p) (deg R q) < m" by simp
      then have "deg R (p ⊕P q) < m" using deg_add [OF p_in_P q_in_P]
by arith
      with deg_R_p deg_R_q show ?thesis using coeff_add [OF p_in_P q_in_P,
of m]
        using deg_aboveD [of "p ⊕P q" m] using p_in_P q_in_P by simp
    qed
  qed (simp add: p_in_P q_in_P)
  moreover have deg_ne: "deg R (p ⊕P q) ≠ deg R r"
  proof (rule ccontr)
    assume nz: "¬ deg R (p ⊕P q) ≠ deg R r" then have deg_eq: "deg
R (p ⊕P q) = deg R r" by simp
    from deg_r_nonzero have r_nonzero: "r ≠ 0P" by (cases "r = 0P",
simp_all)
    have "coeff P (p ⊕P q) (deg R r) = 0R" using coeff_add [OF p_in_P
q_in_P, of "deg R r"] using coeff_R_p_eq_q
      using coeff_closed [OF p_in_P, of "deg R r"] coeff_closed [OF q_in_P,
of "deg R r"] by algebra
    with lcoeff_nonzero [OF r_nonzero r_in_P]  and deg_eq show False
using lcoeff_nonzero [of "p ⊕P q"] using p_in_P q_in_P
      using deg_r_nonzero by (cases "p ⊕P q ≠ 0P", auto)
  qed
```

```
  ultimately show ?thesis by simp
qed

lemma monom_deg_mult:
  assumes f_in_P: "f ∈ carrier P" and g_in_P: "g ∈ carrier P" and deg_le:
"deg R g ≤ deg R f"
  and a_in_R: "a ∈ carrier R"
  shows "deg R (g ⊗P monom P a (deg R f - deg R g)) ≤ deg R f"
  using deg_mult_ring [OF g_in_P monom_closed [OF a_in_R, of "deg R f
- deg R g"]]
  apply (cases "a = 0") using g_in_P apply simp
  using deg_monom [OF _ a_in_R, of "deg R f - deg R g"] using deg_le by
simp

lemma deg_zero_impl_monom:
  assumes f_in_P: "f ∈ carrier P" and deg_f: "deg R f = 0"
  shows "f = monom P (coeff P f 0) 0"
  apply (rule up_eqI) using coeff_monom [OF coeff_closed [OF f_in_P],
of 0 0]
  using f_in_P deg_f using deg_aboveD [of f _] by auto

end
```

16.12 The long division proof for commutative rings

```
context UP_cring
begin

lemma exI3: assumes exist: "Pred x y z"
  shows "∃ x y z. Pred x y z"
  using exist by blast
```

Jacobson's Theorem 2.14

```
lemma long_div_theorem:
  assumes g_in_P [simp]: "g ∈ carrier P" and f_in_P [simp]: "f ∈ carrier
P"
  and g_not_zero: "g ≠ 0P"
  shows "∃ q r (k::nat). (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ (lcoeff
g)(^)Rk ⊙P f = g ⊗P q ⊕P r ∧ (r = 0P | deg R r < deg R g)"
  using f_in_P
proof (induct "deg R f" arbitrary: "f" rule: nat_less_induct)
  case (1 f)
  note f_in_P [simp] = "1.prems"
  let ?pred = "(λ q r (k::nat).
    (q ∈ carrier P) ∧ (r ∈ carrier P)
    ∧ (lcoeff g)(^)Rk ⊙P f = g ⊗P q ⊕P r ∧ (r = 0P | deg R r < deg R
g))"
  let ?lg = "lcoeff g" and ?lf = "lcoeff f"
  show ?case
```

```
  proof (cases "deg R f < deg R g")
    case True
    have "?pred 0_P f 0" using True by force
    then show ?thesis by blast
  next
    case False then have deg_g_le_deg_f: "deg R g ≤ deg R f" by simp
    {
      let ?k = "1::nat"
      let ?f1 = "(g ⊗_P (monom P (?lf) (deg R f - deg R g))) ⊕_P ⊖_P (?lg
⊙_P f)"
      let ?q = "monom P (?lf) (deg R f - deg R g)"
      have f1_in_carrier: "?f1 ∈ carrier P" and q_in_carrier: "?q ∈ carrier
P" by simp_all
      show ?thesis
      proof (cases "deg R f = 0")
        case True
        {
          have deg_g: "deg R g = 0" using True using deg_g_le_deg_f by
simp
          have "?pred f 0_P 1"
            using deg_zero_impl_monom [OF g_in_P deg_g]
            using sym [OF monom_mult_is_smult [OF coeff_closed [OF g_in_P,
of 0] f_in_P]]
            using deg_g by simp
          then show ?thesis by blast
        }
      next
        case False note deg_f_nzero = False
        {
          have exist: "lcoeff g (^) ?k ⊙_P f = g ⊗_P ?q ⊕_P ⊖_P ?f1"
            by (simp add: minus_add r_neg sym [
              OF a_assoc [of "g ⊗_P ?q" "⊖_P (g ⊗_P ?q)" "lcoeff g ⊙_P f"]])
          have deg_remainder_l_f: "deg R (⊖_P ?f1) < deg R f"
          proof (unfold deg_uminus [OF f1_in_carrier])
            show "deg R ?f1 < deg R f"
            proof (rule deg_lcoeff_cancel)
              show "deg R (⊖_P (?lg ⊙_P f)) ≤ deg R f"
                using deg_smult_ring [of ?lg f]
                using lcoeff_nonzero2 [OF g_in_P g_not_zero] by simp
              show "deg R (g ⊗_P ?q) ≤ deg R f"
                by (simp add: monom_deg_mult [OF f_in_P g_in_P deg_g_le_deg_f,
of ?lf])
              show "coeff P (g ⊗_P ?q) (deg R f) = ⊖ coeff P (⊖_P (?lg
⊙_P f)) (deg R f)"
                unfolding coeff_mult [OF g_in_P monom_closed
                  [OF lcoeff_closed [OF f_in_P],
                    of "deg R f - deg R g"], of "deg R f"]
                unfolding coeff_monom [OF lcoeff_closed
                  [OF f_in_P], of "(deg R f - deg R g)"]
```

```
        using R.finsum_cong' [of "{..deg R f}" "{..deg R f}"
         "(λi. coeff P g i ⊗ (if deg R f - deg R g = deg R f
- i then ?lf else 0))"
         "(λi. if deg R g = i then coeff P g i ⊗ ?lf else 0)"]
        using R.finsum_singleton [of "deg R g" "{.. deg R f}"
"(λi. coeff P g i ⊗ ?lf)"]
        unfolding Pi_def using deg_g_le_deg_f by force
      qed (simp_all add: deg_f_nzero)
     qed
     then obtain q' r' k'
      where rem_desc: "?lg (^) (k'::nat) ⊙P (⊖P ?f1) = g ⊗P q'
⊕P r'"
      and rem_deg: "(r' = 0P ∨ deg R r' < deg R g)"
      and q'_in_carrier: "q' ∈ carrier P" and r'_in_carrier: "r'
∈ carrier P"
      using "1.hyps" using f1_in_carrier by blast
     show ?thesis
     proof (rule exI3 [of _ "((?lg (^) k') ⊙P ?q ⊕P q')" r' "Suc
k'"], intro conjI)
      show "(?lg (^) (Suc k')) ⊙P f = g ⊗P ((?lg (^) k') ⊙P ?q
⊕P q') ⊕P r'"
      proof -
       have "(?lg (^) (Suc k')) ⊙P f = (?lg (^) k') ⊙P (g ⊗P
?q ⊕P ⊖P ?f1)"
        using smult_assoc1 [OF _ _ f_in_P] using exist by simp
       also have "... = (?lg (^) k') ⊙P (g ⊗P ?q) ⊕P ((?lg (^)
k') ⊙P ( ⊖P ?f1))"
        using UP_smult_r_distr by simp
       also have "... = (?lg (^) k') ⊙P (g ⊗P ?q) ⊕P (g ⊗P q'
⊕P r')"
        unfolding rem_desc ..
       also have "... = (?lg (^) k') ⊙P (g ⊗P ?q) ⊕P g ⊗P q' ⊕P
r'"
        using sym [OF a_assoc [of "?lg (^) k' ⊙P (g ⊗P ?q)" "g
⊗P q'" "r'"]]
        using r'_in_carrier q'_in_carrier by simp
       also have "... = (?lg (^) k') ⊙P (?q ⊗P g) ⊕P q' ⊗P g ⊕P
r'"
        using q'_in_carrier by (auto simp add: m_comm)
       also have "... = (((?lg (^) k') ⊙P ?q) ⊗P g) ⊕P q' ⊗P g
⊕P r'"
        using smult_assoc2 q'_in_carrier "1.prems" by auto
       also have "... = ((?lg (^) k') ⊙P ?q ⊕P q') ⊗P g ⊕P r'"
        using sym [OF l_distr] and q'_in_carrier by auto
       finally show ?thesis using m_comm q'_in_carrier by auto
      qed
     qed (simp_all add: rem_deg q'_in_carrier r'_in_carrier)
    }
   qed
```

```
    }
  qed
qed

end
```

The remainder theorem as corollary of the long division theorem.

```
context UP_cring
begin

lemma deg_minus_monom:
  assumes a: "a ∈ carrier R"
  and R_not_trivial: "(carrier R ≠ {0})"
  shows "deg R (monom P 1_R 1 ⊖_P monom P a 0) = 1"
  (is "deg R ?g = 1")
proof -
  have "deg R ?g ≤ 1"
  proof (rule deg_aboveI)
    fix m
    assume "(1::nat) < m"
    then show "coeff P ?g m = 0"
      using coeff_minus using a by auto algebra
  qed (simp add: a)
  moreover have "deg R ?g ≥ 1"
  proof (rule deg_belowI)
    show "coeff P ?g 1 ≠ 0"
      using a using R.carrier_one_not_zero R_not_trivial by simp algebra
  qed (simp add: a)
  ultimately show ?thesis by simp
qed

lemma lcoeff_monom:
  assumes a: "a ∈ carrier R" and R_not_trivial: "(carrier R ≠ {0})"
  shows "lcoeff (monom P 1_R 1 ⊖_P monom P a 0) = 1"
  using deg_minus_monom [OF a R_not_trivial]
  using coeff_minus a by auto algebra

lemma deg_nzero_nzero:
  assumes deg_p_nzero: "deg R p ≠ 0"
  shows "p ≠ 0_P"
  using deg_zero deg_p_nzero by auto

lemma deg_monom_minus:
  assumes a: "a ∈ carrier R"
  and R_not_trivial: "carrier R ≠ {0}"
  shows "deg R (monom P 1_R 1 ⊖_P monom P a 0) = 1"
  (is "deg R ?g = 1")
proof -
  have "deg R ?g ≤ 1"
```

```
  proof (rule deg_aboveI)
    fix m::nat assume "1 < m" then show "coeff P ?g m = 𝟬"
      using coeff_minus [OF monom_closed [OF R.one_closed, of 1] monom_closed
[OF a, of 0], of m]
      using coeff_monom [OF R.one_closed, of 1 m] using coeff_monom [OF
a, of 0 m] by auto algebra
  qed (simp add: a)
  moreover have "1 ≤ deg R ?g"
  proof (rule deg_belowI)
    show "coeff P ?g 1 ≠ 𝟬"
      using coeff_minus [OF monom_closed [OF R.one_closed, of 1] monom_closed
[OF a, of 0], of 1]
      using coeff_monom [OF R.one_closed, of 1 1] using coeff_monom [OF
a, of 0 1]
      using R_not_trivial using R.carrier_one_not_zero
      by auto algebra
  qed (simp add: a)
  ultimately show ?thesis by simp
qed

lemma eval_monom_expr:
  assumes a: "a ∈ carrier R"
  shows "eval R R id a (monom P 𝟭_R 1 ⊖_P monom P a 0) = 𝟬"
  (is "eval R R id a ?g = _")
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  have eval_ring_hom: "eval R R id a ∈ ring_hom P R" using eval_ring_hom
[OF a] by simp
  interpret ring_hom_cring P R "eval R R id a" by unfold_locales (rule
eval_ring_hom)
  have mon1_closed: "monom P 𝟭_R 1 ∈ carrier P"
    and mon0_closed: "monom P a 0 ∈ carrier P"
    and min_mon0_closed: "⊖_P monom P a 0 ∈ carrier P"
    using a R.a_inv_closed by auto
  have "eval R R id a ?g = eval R R id a (monom P 𝟭 1) ⊖ eval R R id
a (monom P a 0)"
    unfolding P.minus_eq [OF mon1_closed mon0_closed]
    unfolding hom_add [OF mon1_closed min_mon0_closed]
    unfolding hom_a_inv [OF mon0_closed]
    using R.minus_eq [symmetric] mon1_closed mon0_closed by auto
  also have "... = a ⊖ a"
    using eval_monom [OF R.one_closed a, of 1] using eval_monom [OF a
a, of 0] using a by simp
  also have "... = 𝟬"
    using a by algebra
  finally show ?thesis by simp
qed

lemma remainder_theorem_exist:
```

```
  assumes f: "f ∈ carrier P" and a: "a ∈ carrier R"
  and R_not_trivial: "carrier R ≠ {0}"
  shows "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ f = (monom P 1R
1 ⊖P monom P a 0) ⊗P q ⊕P r ∧ (deg R r = 0)"
  (is "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ f = ?g ⊗P q ⊕P r ∧
(deg R r = 0)")
proof -
  let ?g = "monom P 1R 1 ⊖P monom P a 0"
  from deg_minus_monom [OF a R_not_trivial]
  have deg_g_nzero: "deg R ?g ≠ 0" by simp
  have "∃q r (k::nat). q ∈ carrier P ∧ r ∈ carrier P ∧
    lcoeff ?g (^) k ⊙P f = ?g ⊗P q ⊕P r ∧ (r = 0P ∨ deg R r < deg R
?g)"
    using long_div_theorem [OF _ f deg_nzero_nzero [OF deg_g_nzero]] a
    by auto
  then show ?thesis
    unfolding lcoeff_monom [OF a R_not_trivial]
    unfolding deg_monom_minus [OF a R_not_trivial]
    using smult_one [OF f] using deg_zero by force
qed

lemma remainder_theorem_expression:
  assumes f [simp]: "f ∈ carrier P" and a [simp]: "a ∈ carrier R"
  and q [simp]: "q ∈ carrier P" and r [simp]: "r ∈ carrier P"
  and R_not_trivial: "carrier R ≠ {0}"
  and f_expr: "f = (monom P 1R 1 ⊖P monom P a 0) ⊗P q ⊕P r"
  (is "f = ?g ⊗P q ⊕P r" is "f = ?gq ⊕P r")
    and deg_r_0: "deg R r = 0"
    shows "r = monom P (eval R R id a f) 0"
proof -
  interpret UP_pre_univ_prop R R id P by standard simp
  have eval_ring_hom: "eval R R id a ∈ ring_hom P R"
    using eval_ring_hom [OF a] by simp
  have "eval R R id a f = eval R R id a ?gq ⊕R eval R R id a r"
    unfolding f_expr using ring_hom_add [OF eval_ring_hom] by auto
  also have "... = ((eval R R id a ?g) ⊗ (eval R R id a q)) ⊕R eval R
R id a r"
    using ring_hom_mult [OF eval_ring_hom] by auto
  also have "... = 0 ⊕ eval R R id a r"
    unfolding eval_monom_expr [OF a] using eval_ring_hom
    unfolding ring_hom_def using q unfolding Pi_def by simp
  also have "... = eval R R id a r"
    using eval_ring_hom unfolding ring_hom_def using r unfolding Pi_def
by simp
  finally have eval_eq: "eval R R id a f = eval R R id a r" by simp
  from deg_zero_impl_monom [OF r deg_r_0]
  have "r = monom P (coeff P r 0) 0" by simp
  with eval_const [OF a, of "coeff P r 0"] eval_eq
  show ?thesis by auto
```

```
qed

corollary remainder_theorem:
  assumes f [simp]: "f ∈ carrier P" and a [simp]: "a ∈ carrier R"
  and R_not_trivial: "carrier R ≠ {0}"
  shows "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧
     f = (monom P 1R 1 ⊖P monom P a 0) ⊗P q ⊕P monom P (eval R R id a
f) 0"
  (is "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ f = ?g ⊗P q ⊕P monom
P (eval R R id a f) 0")
proof -
  from remainder_theorem_exist [OF f a R_not_trivial]
  obtain q r
    where q_r: "q ∈ carrier P ∧ r ∈ carrier P ∧ f = ?g ⊗P q ⊕P r"
    and deg_r: "deg R r = 0" by force
  with remainder_theorem_expression [OF f a _ _ R_not_trivial, of q r]
  show ?thesis by auto
qed

end
```

16.13 Sample Application of Evaluation Homomorphism

```
lemma UP_pre_univ_propI:
  assumes "cring R"
    and "cring S"
    and "h ∈ ring_hom R S"
  shows "UP_pre_univ_prop R S h"
  using assms
  by (auto intro!: UP_pre_univ_prop.intro ring_hom_cring.intro
    ring_hom_cring_axioms.intro UP_cring.intro)

definition
  INTEG :: "int ring"
  where "INTEG = (|carrier = UNIV, mult = op *, one = 1, zero = 0, add
= op +|)"

lemma INTEG_cring: "cring INTEG"
  by (unfold INTEG_def) (auto intro!: cringI abelian_groupI comm_monoidI
    left_minus distrib_right)

lemma INTEG_id_eval:
  "UP_pre_univ_prop INTEG INTEG id"
  by (fast intro: UP_pre_univ_propI INTEG_cring id_ring_hom)
```

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between `INTEG` and `UP INTEG` globally.

```
interpretation INTEG: UP_pre_univ_prop INTEG INTEG id "UP INTEG"
  using INTEG_id_eval by simp_all
```

```
lemma INTEG_closed [intro, simp]:
  "z ∈ carrier INTEG"
  by (unfold INTEG_def) simp

lemma INTEG_mult [simp]:
  "mult INTEG z w = z * w"
  by (unfold INTEG_def) simp

lemma INTEG_pow [simp]:
  "pow INTEG z n = z ^ n"
  by (induct n) (simp_all add: INTEG_def nat_pow_def)

lemma "eval INTEG INTEG id 10 (monom (UP INTEG) 5 2) = 500"
  by (simp add: INTEG.eval_monom)

end


theory Multiplicative_Group
imports
  Complex_Main
  Group
  More_Group
  More_Finite_Product
  Coset
  UnivPoly
begin
```

17 Simplification Rules for Polynomials

```
lemma (in ring_hom_cring) hom_sub[simp]:
  assumes "x ∈ carrier R" "y ∈ carrier R"
  shows "h (x ⊖ y) = h x ⊖S h y"
  using assms by (simp add: R.minus_eq S.minus_eq)

context UP_ring begin

lemma deg_nzero_nzero:
  assumes deg_p_nzero: "deg R p ≠ 0"
  shows "p ≠ 0P"
  using deg_zero deg_p_nzero by auto

lemma deg_add_eq:
  assumes c: "p ∈ carrier P" "q ∈ carrier P"
  assumes "deg R q ≠ deg R p"
  shows "deg R (p ⊕P q) = max (deg R p) (deg R q)"
proof -
  let ?m = "max (deg R p) (deg R q)"
```

```
  from assms have "coeff P p ?m = 0 ⟷ coeff P q ?m ≠ 0"
    by (metis deg_belowI lcoeff_nonzero[OF deg_nzero_nzero] linear max.absorb_iff2
max.absorb1)
  then have "coeff P (p ⊕P q) ?m ≠ 0"
    using assms by auto
  then have "deg R (p ⊕P q) ≥ ?m"
    using assms by (blast intro: deg_belowI)
  with deg_add[OF c] show ?thesis by arith
qed

lemma deg_minus_eq:
  assumes "p ∈ carrier P" "q ∈ carrier P" "deg R q ≠ deg R p"
  shows "deg R (p ⊖P q) = max (deg R p) (deg R q)"
  using assms by (simp add: deg_add_eq a_minus_def)

end

context UP_cring begin

lemma evalRR_add:
  assumes "p ∈ carrier P" "q ∈ carrier P"
  assumes x:"x ∈ carrier R"
  shows "eval R R id x (p ⊕P q) = eval R R id x p ⊕ eval R R id x q"
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  interpret ring_hom_cring P R "eval R R id x" by unfold_locales (rule
eval_ring_hom[OF x])
  show ?thesis using assms by simp
qed

lemma evalRR_sub:
  assumes "p ∈ carrier P" "q ∈ carrier P"
  assumes x:"x ∈ carrier R"
  shows "eval R R id x (p ⊖P q) = eval R R id x p ⊖ eval R R id x q"
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  interpret ring_hom_cring P R "eval R R id x" by unfold_locales (rule
eval_ring_hom[OF x])
  show ?thesis using assms by simp
qed

lemma evalRR_mult:
  assumes "p ∈ carrier P" "q ∈ carrier P"
  assumes x:"x ∈ carrier R"
  shows "eval R R id x (p ⊗P q) = eval R R id x p ⊗ eval R R id x q"
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  interpret ring_hom_cring P R "eval R R id x" by unfold_locales (rule
eval_ring_hom[OF x])
```

```
  show ?thesis using assms by simp
qed

lemma evalRR_monom:
  assumes a: "a ∈ carrier R" and x: "x ∈ carrier R"
  shows "eval R R id x (monom P a d) = a ⊗ x (^) d"
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  show ?thesis using assms by (simp add: eval_monom)
qed

lemma evalRR_one:
  assumes x: "x ∈ carrier R"
  shows "eval R R id x 1_P = 1"
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  interpret ring_hom_cring P R "eval R R id x" by unfold_locales (rule
eval_ring_hom[OF x])
  show ?thesis using assms by simp
qed

lemma carrier_evalRR:
  assumes x: "x ∈ carrier R" and "p ∈ carrier P"
  shows "eval R R id x p ∈ carrier R"
proof -
  interpret UP_pre_univ_prop R R id by unfold_locales simp
  interpret ring_hom_cring P R "eval R R id x" by unfold_locales (rule
eval_ring_hom[OF x])
  show ?thesis using assms by simp
qed

lemmas evalRR_simps = evalRR_add evalRR_sub evalRR_mult evalRR_monom
evalRR_one carrier_evalRR

end
```

18 Properties of the Euler φ-function

In this section we prove that for every positive natural number the equation $\sum_{d|n}^{n} \varphi(d) = n$ holds.

```
lemma dvd_div_ge_1 :
  fixes a b :: nat
  assumes "a ≥ 1" "b dvd a"
  shows "a div b ≥ 1"
proof -
  from ‹b dvd a› obtain c where "a = b * c" ..
  with ‹a ≥ 1› show ?thesis by simp
qed
```

```
lemma dvd_nat_bounds :
 fixes n p :: nat
 assumes "p > 0" "n dvd p"
 shows "n > 0 ∧ n ≤ p"
 using assms by (simp add: dvd_pos_nat dvd_imp_le)


definition phi' :: "nat => nat"
  where "phi' m = card {x. 1 ≤ x ∧ x ≤ m ∧ gcd x m = 1}"

notation (latex output)
  phi' ("φ _")

lemma phi'_nonzero :
  assumes "m > 0"
  shows "phi' m > 0"
proof -
  have "1 ∈ {x. 1 ≤ x ∧ x ≤ m ∧ gcd x m = 1}" using assms by simp
  hence "card {x. 1 ≤ x ∧ x ≤ m ∧ gcd x m = 1} > 0" by (auto simp: card_gt_0_iff)
  thus ?thesis unfolding phi'_def by simp
qed

lemma dvd_div_eq_1:
  fixes a b c :: nat
  assumes "c dvd a" "c dvd b" "a div c = b div c"
  shows "a = b" using assms dvd_mult_div_cancel[OF ‘c dvd a‘] dvd_mult_div_cancel[OF
‘c dvd b‘]
              by presburger

lemma dvd_div_eq_2:
  fixes a b c :: nat
  assumes "c>0" "a dvd c" "b dvd c" "c div a = c div b"
  shows "a = b"
  proof -
  have "a > 0" "a ≤ c" using dvd_nat_bounds[OF assms(1-2)] by auto
  have "a*(c div a) = c" using assms dvd_mult_div_cancel by fastforce
  also have "... = b*(c div a)" using assms dvd_mult_div_cancel by fastforce
  finally show "a = b" using ‘c>0‘ dvd_div_ge_1[OF _ ‘a dvd c‘] by fastforce
qed

lemma div_mult_mono:
  fixes a b c :: nat
  assumes "a > 0" "a≤d"
  shows "a * b div d ≤ b"
proof -
  have "a*b div d ≤ b*a div a" using assms div_le_mono2 mult.commute[of
a b] by presburger
  thus ?thesis using assms by force
```

qed

We arrive at the main result of this section: For every positive natural number the equation $\sum_{d|n}^{n} \varphi(d) = n$ holds.

The outline of the proof for this lemma is as follows: We count the n fractions $1/n$, …, $(n-1)/n$, n/n. We analyze the reduced form $a/d = m/n$ for any of those fractions. We want to know how many fractions m/n have the reduced form denominator d. The condition $1 \leq m \leq n$ is equivalent to the condition $1 \leq a \leq d$. Therefore we want to know how many a with $1 \leq a \leq d$ exist, s.t. `coprime a d`. This number is exactly `φ d`.

Finally, by counting the fractions m/n according to their reduced form denominator, we get:

```
(∑d | d dvd n. φ d) = n
```

. To formalize this proof in Isabelle, we analyze for an arbitrary divisor d of n

- the set of reduced form numerators `{a. 1 ≤ a ∧ a ≤ d ∧ coprime a d}`
- the set of numerators m, for which m/n has the reduced form denominator d, i.e. the set `{m ∈ {1..n}. n div gcd m n = d}`

We show that `λa. a * n div d` with the inverse `λa. a div gcd a n` is a bijection between theses sets, thus yielding the equality

```
φ d = card {m ∈ {1..n}. n div gcd m n = d}
```

This gives us

```
(∑d | d dvd n. φ d) = card (⋃d∈{d. d dvd n} {m ∈ {1..n}. n div gcd m
n = d})
```

and by showing `{1..n} ⊆ (⋃d∈{d. d dvd n} {m ∈ {1..n}. n div gcd m n = d})` (this is our counting argument) the thesis follows.

```
lemma sum_phi'_factors :
 fixes n :: nat
 assumes "n > 0"
 shows "(∑d | d dvd n. phi' d) = n"
proof -
  { fix d assume "d dvd n" then obtain q where q: "n = d * q" ..
    have "card {a. 1 ≤ a ∧ a ≤ d ∧ coprime a d} = card {m ∈ {1 .. n}.
n div gcd m n = d}"
         (is "card ?RF = card ?F")
    proof (rule card_bij_eq)
      { fix a b assume "a * n div d = b * n div d"
```

```
        hence "a * (n div d) = b * (n div d)"
          using dvd_div_mult[OF `d dvd n`] by (fastforce simp add: mult.commute)
        hence "a = b" using dvd_div_ge_1[OF _ `d dvd n`] `n>0`
          by (simp add: mult.commute nat_mult_eq_cancel1)
      } thus "inj_on (λa. a*n div d) ?RF" unfolding inj_on_def by blast
      { fix a assume a:"a∈?RF"
        hence "a * (n div d) ≥ 1" using `n>0` dvd_div_ge_1[OF _ `d dvd
n`] by simp
        hence ge_1:"a * n div d ≥ 1" by (simp add: `d dvd n` div_mult_swap)
        have le_n:"a * n div d ≤ n" using div_mult_mono a by simp
        have "gcd (a * n div d) n = n div d * gcd a d"
          by (simp add: gcd_mult_distrib_nat q ac_simps)
        hence "n div gcd (a * n div d) n = d*n div (d*(n div d))" us-
ing a by simp
        hence "a * n div d ∈ ?F"
          using ge_1 le_n by (fastforce simp add: `d dvd n` dvd_mult_div_cancel)
      } thus "(λa. a*n div d) ` ?RF ⊆ ?F" by blast
      { fix m l assume A: "m ∈ ?F" "l ∈ ?F" "m div gcd m n = l div gcd
l n"
        hence "gcd m n = gcd l n" using dvd_div_eq_2[OF assms] by fastforce
        hence "m = l" using dvd_div_eq_1[of "gcd m n" m l] A(3) by fastforce
      } thus "inj_on (λa. a div gcd a n) ?F" unfolding inj_on_def by
blast
      { fix m assume "m ∈ ?F"
        hence "m div gcd m n ∈ ?RF" using dvd_div_ge_1
          by (fastforce simp add: div_le_mono div_gcd_coprime)
      } thus "(λa. a div gcd a n) ` ?F ⊆ ?RF" by blast
    qed force+
  } hence phi'_eq:"⋀d. d dvd n ⟹ phi' d = card {m ∈ {1 .. n}. n div
gcd m n = d}"
      unfolding phi'_def by presburger
  have fin:"finite {d. d dvd n}" using dvd_nat_bounds[OF `n>0`] by force
  have "(∑d | d dvd n. phi' d)
                 = card (⋃d ∈ {d. d dvd n}. {m ∈ {1 .. n}. n div gcd
m n = d})"
    using card_UN_disjoint[OF fin, of "(λd. {m ∈ {1 .. n}. n div gcd m
n = d})"] phi'_eq
    by fastforce
  also have "(⋃d ∈ {d. d dvd n}. {m ∈ {1 .. n}. n div gcd m n = d}) =
{1 .. n}" (is "?L = ?R")
  proof
    show "?L ⊇ ?R"
    proof
      fix m assume m: "m ∈ ?R"
      thus "m ∈ ?L" using dvd_triv_right[of "n div gcd m n" "gcd m n"]
        by (simp add: dvd_mult_div_cancel)
    qed
  qed fastforce
  finally show ?thesis by force
```

```
qed
```

19 Order of an Element of a Group

```
context group begin

lemma pow_eq_div2 :
  fixes m n :: nat
  assumes x_car: "x ∈ carrier G"
  assumes pow_eq: "x (^) m = x (^) n"
  shows "x (^) (m - n) = 1"
proof (cases "m < n")
  case False
  have "1 ⊗ x (^) m = x (^) m" by (simp add: x_car)
  also have "... = x (^) (m - n) ⊗ x (^) n"
    using False by (simp add: nat_pow_mult x_car)
  also have "... = x (^) (m - n) ⊗ x (^) m"
    by (simp add: pow_eq)
  finally show ?thesis by (simp add: x_car)
qed simp

definition ord where "ord a = Min {d ∈ {1 .. order G} . a (^) d = 1}"

lemma
  assumes finite:"finite (carrier G)"
  assumes a:"a ∈ carrier G"
  shows ord_ge_1: "1 ≤ ord a" and ord_le_group_order: "ord a ≤ order
G"
    and pow_ord_eq_1: "a (^) ord a = 1"
proof -
  have "¬inj_on (λx. a (^) x) {0 .. order G}"
  proof (rule notI)
    assume A: "inj_on (λx. a (^) x) {0 .. order G}"
    have "order G + 1 = card {0 .. order G}" by simp
    also have "... = card ((λx. a (^) x) ‘ {0 .. order G})" (is "_ = card
?S")
      using A by (simp add: card_image)
    also have "?S = {a (^) x | x. x ∈ {0 .. order G}}" by blast
    also have "... ⊆ carrier G" (is "?S ⊆ _") using a by blast
    then have "card ?S ≤ order G" unfolding order_def
      by (rule card_mono[OF finite])
    finally show False by arith
  qed

  then obtain x y where x_y:"x ≠ y" "x ∈ {0 .. order G}" "y ∈ {0 ..
order G}"
                        "a (^) x = a (^) y" unfolding inj_on_def by blast
  obtain d where "1 ≤ d" "a (^) d = 1" "d ≤ order G"
  proof cases
```

```
    assume "y < x" with x_y show ?thesis
      by (intro that[where d="x - y"]) (auto simp add: pow_eq_div2[OF
a])
  next
    assume "¬y < x" with x_y show ?thesis
      by (intro that[where d="y - x"]) (auto simp add: pow_eq_div2[OF
a])
  qed
  hence "ord a ∈ {d ∈ {1 .. order G} . a (^) d = 1}"
    unfolding ord_def using Min_in[of "{d ∈ {1 .. order G} . a (^) d =
1}"]
    by fastforce
  then show "1 ≤ ord a" and "ord a ≤ order G" and "a (^) ord a = 1"
    by (auto simp: order_def)
qed

lemma finite_group_elem_finite_ord :
  assumes "finite (carrier G)" "x ∈ carrier G"
  shows "∃ d::nat. d ≥ 1 ∧ x (^) d = 1"
  using assms ord_ge_1 pow_ord_eq_1 by auto

lemma ord_min:
  assumes  "finite (carrier G)" "1 ≤ d" "a ∈ carrier G" "a (^) d = 1"
shows "ord a ≤ d"
proof -
  def Ord ≡ "{d ∈ {1..order G}. a (^) d = 1}"
  have fin: "finite Ord" by (auto simp: Ord_def)
  have in_ord: "ord a ∈ Ord"
    using assms pow_ord_eq_1 ord_ge_1 ord_le_group_order by (auto simp:
Ord_def)
  then have "Ord ≠ {}" by auto

  show ?thesis
  proof (cases "d ≤ order G")
    case True
    then have "d ∈ Ord" using assms by (auto simp: Ord_def)
    with fin in_ord show ?thesis
      unfolding ord_def Ord_def[symmetric] by simp
  next
    case False
    then show ?thesis using in_ord by (simp add: Ord_def)
  qed
qed

lemma ord_inj :
  assumes finite: "finite (carrier G)"
  assumes a: "a ∈ carrier G"
  shows "inj_on (λ x . a (^) x) {0 .. ord a - 1}"
proof (rule inj_onI, rule ccontr)
```

```
  fix x y assume A: "x ∈ {0 .. ord a - 1}" "y ∈ {0 .. ord a - 1}" "a
(^) x= a (^) y" "x ≠ y"

  have "finite {d ∈ {1..order G}. a (^) d = 1}" by auto

  { fix x y assume A: "x < y" "x ∈ {0 .. ord a - 1}" "y ∈ {0 .. ord a
- 1}"
        "a (^) x = a (^) y"
    hence "y - x < ord a" by auto
    also have "... ≤ order G" using assms by (simp add: ord_le_group_order)
    finally have y_x_range:"y - x ∈ {1 .. order G}" using A by force
    have "a (^) (y-x) = 1" using a A by (simp add: pow_eq_div2)

    hence y_x:"y - x ∈ {d ∈ {1.. order G}. a (^) d = 1}" using y_x_range
by blast
    have "min (y - x) (ord a) = ord a"
      using Min.in_idem[OF ‘finite {d ∈ {1 .. order G} . a (^) d = 1}‘
y_x] ord_def by auto
    with ‘y - x < ord a‘ have False by linarith
  }
  note X = this

  { assume "x < y" with A X have False by blast }
  moreover
  { assume "x > y" with A X  have False by metis }
  moreover
  { assume "x = y" then have False using A by auto}
  ultimately
  show False by fastforce
qed

lemma ord_inj' :
  assumes finite: "finite (carrier G)"
  assumes a: "a ∈ carrier G"
  shows "inj_on (λ x . a (^) x) {1 .. ord a}"
proof (rule inj_onI, rule ccontr)
  fix x y :: nat
  assume A:"x ∈ {1 .. ord a}" "y ∈ {1 .. ord a}" "a (^) x = a (^) y"
"x≠y"
  { assume "x < ord a" "y < ord a"
    hence False using ord_inj[OF assms] A unfolding inj_on_def by fastforce
  }
  moreover
  { assume "x = ord a" "y < ord a"
    hence "a (^) y = a (^) (0::nat)" using pow_ord_eq_1[OF assms] A by
auto
    hence "y=0" using ord_inj[OF assms] ‘y < ord a‘ unfolding inj_on_def
by force
    hence False using A by fastforce
```

```
  }
  moreover
  { assume "y = ord a" "x < ord a"
    hence "a (^) x = a (^) (0::nat)" using pow_ord_eq_1[OF assms] A by
auto
    hence "x=0" using ord_inj[OF assms] ‘x < ord a‘ unfolding inj_on_def
by force
    hence False using A by fastforce
  }
  ultimately show False using A  by force
qed

lemma ord_elems :
  assumes "finite (carrier G)" "a ∈ carrier G"
  shows "{a(^)x | x. x ∈ (UNIV :: nat set)} = {a(^)x | x. x ∈ {0 .. ord
a - 1}}" (is "?L = ?R")
proof
  show "?R ⊆ ?L" by blast
  { fix y assume "y ∈ ?L"
    then obtain x::nat where x:"y = a(^)x" by auto
    def r ≡ "x mod ord a"
    then obtain q where q:"x = q * ord a + r" using mod_eqD by atomize_elim
presburger
    hence "y = (a(^)ord a)(^)q ⊗ a(^)r"
      using x assms by (simp add: mult.commute nat_pow_mult nat_pow_pow)
    hence "y = a(^)r" using assms by (simp add: pow_ord_eq_1)
    have "r < ord a" using ord_ge_1[OF assms] by (simp add: r_def)
    hence "r ∈ {0 .. ord a - 1}" by (force simp: r_def)
    hence "y ∈ {a(^)x | x. x ∈ {0 .. ord a - 1}}" using ‘y=a(^)r‘ by
blast
  }
  thus "?L ⊆ ?R" by auto
qed

lemma ord_dvd_pow_eq_1 :
  assumes "finite (carrier G)" "a ∈ carrier G" "a (^) k = 1"
  shows "ord a dvd k"
proof -
  def r ≡ "k mod ord a"
  then obtain q where q:"k = q*ord a + r" using mod_eqD by atomize_elim
presburger
  hence "a(^)k = (a(^)ord a)(^)q ⊗ a(^)r"
      using assms by (simp add: mult.commute nat_pow_mult nat_pow_pow)
  hence "a(^)k = a(^)r" using assms by (simp add: pow_ord_eq_1)
  hence "a(^)r = 1" using assms(3) by simp
  have "r < ord a" using ord_ge_1[OF assms(1-2)] by (simp add: r_def)
  hence "r = 0" using ‘a(^)r = 1‘ ord_def[of a] ord_min[of r a] assms(1-2)
by linarith
  thus ?thesis using q by simp
```

```
qed

lemma dvd_gcd :
  fixes a b :: nat
  obtains q where "a * (b div gcd a b) = b*q"
proof
  have "a * (b div gcd a b) = (a div gcd a b) * b" by (simp add:  div_mult_swap
dvd_div_mult)
  also have "... = b * (a div gcd a b)" by simp
  finally show "a * (b div gcd a b) = b * (a div gcd a b) " .
qed

lemma ord_pow_dvd_ord_elem :
  assumes finite[simp]: "finite (carrier G)"
  assumes a[simp]:"a ∈ carrier G"
  shows "ord (a(^)n) = ord a div gcd n (ord a)"
proof -
  have "(a(^)n) (^) ord a = (a (^) ord a) (^) n"
    by (simp add: mult.commute nat_pow_pow)
  hence "(a(^)n) (^) ord a = 1" by (simp add: pow_ord_eq_1)
  obtain q where "n * (ord a div gcd n (ord a)) = ord a * q" by (rule
dvd_gcd)
  hence "(a(^)n) (^) (ord a div gcd n (ord a)) = (a (^) ord a)(^)q"  by
(simp add : nat_pow_pow)
  hence pow_eq_1: "(a(^)n) (^) (ord a div gcd n (ord a)) = 1"
     by (auto simp add : pow_ord_eq_1[of a])
  have "ord a ≥ 1" using ord_ge_1 by simp
  have ge_1:"ord a div gcd n (ord a) ≥ 1"
  proof -
    have "gcd n (ord a) dvd ord a" by blast
    thus ?thesis by (rule dvd_div_ge_1[OF ‘ord a ≥ 1‘])
  qed
  have "ord a ≤ order G" by (simp add: ord_le_group_order)
  have "ord a div gcd n (ord a) ≤ order G"
  proof -
    have "ord a div gcd n (ord a) ≤ ord a" by simp
    thus ?thesis using ‘ord a ≤ order G‘ by linarith
  qed
  hence ord_gcd_elem:"ord a div gcd n (ord a) ∈ {d ∈ {1..order G}. (a(^)n)
(^) d = 1}"
    using ge_1 pow_eq_1 by force
  { fix d :: nat
    assume d_elem:"d ∈ {d ∈ {1..order G}. (a(^)n) (^) d = 1}"
    assume d_lt:"d < ord a div gcd n (ord a)"
    hence pow_nd:"a(^)(n*d)  = 1" using d_elem
      by (simp add : nat_pow_pow)
    hence "ord a dvd n*d" using assms by (auto simp add : ord_dvd_pow_eq_1)
    then obtain q where "ord a * q = n*d" by (metis dvd_mult_div_cancel)
    hence prod_eq:"(ord a div gcd n (ord a)) * q = (n div gcd n (ord a))
```

```
* d"
      by (simp add: dvd_div_mult)
    have cp:"coprime (ord a div gcd n (ord a)) (n div gcd n (ord a))"
    proof -
      have "coprime (n div gcd n (ord a)) (ord a div gcd n (ord a))"
        using div_gcd_coprime[of n "ord a"] ge_1 by fastforce
      thus ?thesis by (simp add: gcd.commute)
    qed
    have dvd_d:"(ord a div gcd n (ord a)) dvd d"
    proof -
      have "ord a div gcd n (ord a) dvd (n div gcd n (ord a)) * d" us-
ing prod_eq
        by (metis dvd_triv_right mult.commute)
      hence "ord a div gcd n (ord a) dvd d * (n div gcd n (ord a))"
        by (simp add:  mult.commute)
      thus ?thesis using coprime_dvd_mult[OF cp, of d] by fastforce
    qed
    have "d > 0" using d_elem by simp
    hence "ord a div gcd n (ord a) ≤ d" using dvd_d by (simp add : Nat.dvd_imp_le)
    hence False using d_lt by simp
  } hence ord_gcd_min: "⋀ d . d ∈ {d ∈ {1..order G}. (a(^)n) (^) d =
1}
                        ⟹ d≥ord a div gcd n (ord a)" by fastforce
  have fin:"finite {d ∈ {1..order G}. (a(^)n) (^) d = 1}" by auto
  thus ?thesis using Min_eqI[OF fin ord_gcd_min ord_gcd_elem]
    unfolding ord_def by simp
qed

lemma ord_1_eq_1 :
  assumes "finite (carrier G)"
  shows "ord 1 = 1"
 using assms ord_ge_1 ord_min[of 1 1] by force

theorem lagrange_dvd:
 assumes "finite(carrier G)" "subgroup H G" shows "(card H) dvd (order
G)"
 using assms by (simp add: lagrange[symmetric])

lemma element_generates_subgroup:
  assumes finite[simp]: "finite (carrier G)"
  assumes a[simp]: "a ∈ carrier G"
  shows "subgroup {a (^) i | i. i ∈ {0 .. ord a - 1}} G"
proof
  show "{a(^)i | i. i ∈ {0 .. ord a - 1} } ⊆ carrier G" by auto
next
  fix x y
  assume A: "x ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}" "y ∈ {a(^)i | i.
i ∈ {0 .. ord a - 1}}"
  obtain i::nat where i:"x = a(^)i" and i2:"i ∈ UNIV" using A by auto
```

```
  obtain j::nat where j:"y = a(^)j" and j2:"j ∈ UNIV" using A by auto
  have "a(^)(i+j) ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}" using ord_elems[OF
assms] A by auto
  thus "x ⊗ y ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}"
    using i j a ord_elems assms by (auto simp add: nat_pow_mult)
next
  show "1 ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}" by force
next
  fix x assume x: "x ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}"
  hence x_in_carrier: "x ∈ carrier G" by auto
  then obtain d::nat where d:"x (^) d = 1" and "d≥1"
    using finite_group_elem_finite_ord by auto
  have inv_1:"x(^)(d - 1) ⊗ x = 1" using ‘d≥1‘ d nat_pow_Suc[of x "d
- 1"] by simp
  have elem:"x (^) (d - 1) ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}"
  proof -
    obtain i::nat where i:"x = a(^)i" using x by auto
    hence "x(^)(d - 1) ∈ {a(^)i | i. i ∈ (UNIV::nat set)}" by (auto simp
add: nat_pow_pow)
    thus ?thesis using ord_elems[of a] by auto
  qed
  have inv:"inv x = x(^)(d - 1)" using inv_equality[OF inv_1] x_in_carrier
by blast
  thus "inv x ∈ {a(^)i | i. i ∈ {0 .. ord a - 1}}" using elem inv by
auto
qed

lemma ord_dvd_group_order :
  assumes finite[simp]: "finite (carrier G)"
  assumes a[simp]: "a ∈ carrier G"
  shows "ord a dvd order G"
proof -
  have card_dvd:"card {a(^)i | i. i ∈ {0 .. ord a - 1}} dvd card (carrier
G)"
    using lagrange_dvd element_generates_subgroup unfolding order_def
by simp
  have "inj_on (λ i . a(^)i) {0..ord a - 1}" using ord_inj by simp
  hence cards_eq:"card ( (λ i . a(^)i) ‘ {0..ord a - 1}) = card {0..ord
a - 1}"
    using card_image[of "λ i . a(^)i" "{0..ord a - 1}"] by auto
  have "(λ i . a(^)i) ‘ {0..ord a - 1} = {a(^)i | i. i ∈ {0..ord a -
1}}" by auto
  hence "card {a(^)i | i. i ∈ {0..ord a - 1}} = card {0..ord a - 1}"
using cards_eq by simp
  also have "... = ord a" using ord_ge_1[of a] by simp
  finally show ?thesis using card_dvd by (simp add: order_def)
qed

end
```

20 Number of Roots of a Polynomial

```
definition mult_of :: "('a, 'b) ring_scheme ⇒ 'a monoid" where
  "mult_of R ≡ (| carrier = carrier R - {0_R}, mult = mult R, one = 1_R|)"

lemma carrier_mult_of: "carrier (mult_of R) = carrier R - {0_R}"
  by (simp add: mult_of_def)

lemma mult_mult_of: "mult (mult_of R) = mult R"
 by (simp add: mult_of_def)

lemma nat_pow_mult_of: "op (^)_mult_of R = (op (^)_R :: _ ⇒ nat ⇒ _)"
  by (simp add: mult_of_def fun_eq_iff nat_pow_def)

lemma one_mult_of: "1_mult_of R = 1_R"
  by (simp add: mult_of_def)

lemmas mult_of_simps = carrier_mult_of mult_mult_of nat_pow_mult_of one_mult_of

context field begin

lemma field_mult_group :
  shows "group (mult_of R)"
  apply (rule groupI)
  apply (auto simp: mult_of_simps m_assoc dest: integral)
  by (metis Diff_iff Units_inv_Units Units_l_inv field_Units singletonE)

lemma finite_mult_of: "finite (carrier R) ⟹ finite (carrier (mult_of
R))"
  by (auto simp: mult_of_simps)

lemma order_mult_of: "finite (carrier R) ⟹ order (mult_of R) = order
R - 1"
  unfolding order_def carrier_mult_of by (simp add: card.remove)

end



lemma (in monoid) Units_pow_closed :
  fixes d :: nat
  assumes "x ∈ Units G"
  shows "x (^) d ∈ Units G"
    by (metis assms group.is_monoid monoid.nat_pow_closed units_group
units_of_carrier units_of_pow)

lemma (in comm_monoid) is_monoid:
  shows "monoid G" by unfold_locales
```

```
declare comm_monoid.is_monoid[intro?]

lemma (in ring) r_right_minus_eq[simp]:
  assumes "a ∈ carrier R" "b ∈ carrier R"
  shows "a ⊖ b = 0 ⟷ a = b"
  using assms by (metis a_minus_def add.inv_closed minus_equality r_neg)

context UP_cring begin

lemma is_UP_cring:"UP_cring R" by (unfold_locales)
lemma is_UP_ring :
  shows "UP_ring R" by (unfold_locales)

end

context UP_domain begin


lemma roots_bound:
  assumes f [simp]: "f ∈ carrier P"
  assumes f_not_zero: "f ≠ 0P"
  assumes finite: "finite (carrier R)"
  shows "finite {a ∈ carrier R . eval R R id a f = 0} ∧
         card {a ∈ carrier R . eval R R id a f = 0} ≤ deg R f" using
f f_not_zero
proof (induction "deg R f" arbitrary: f)
  case 0
  have "⋀x. eval R R id x f ≠ 0"
  proof -
    fix x
    have "(⨁i∈{..deg R f}. id (coeff P f i) ⊗ x (^) i) ≠ 0"
      using 0 lcoeff_nonzero_nonzero[where p = f] by simp
    thus "eval R R id x f ≠ 0" using 0 unfolding eval_def P_def by simp
  qed
  then have *: "{a ∈ carrier R. eval R R (λa. a) a f = 0} = {}"
    by (auto simp: id_def)
  show ?case by (simp add: *)
next
  case (Suc x)
  show ?case
  proof (cases "∃ a ∈ carrier R . eval R R id a f = 0")
    case True
    then obtain a where a_carrier[simp]: "a ∈ carrier R" and a_root:"eval
R R id a f = 0" by blast
    have R_not_triv: "carrier R ≠ {0}"
      by (metis R.one_zeroI R.zero_not_one)
    obtain q  where q:"(q ∈ carrier P)" and
      f:"f = (monom P 1R 1 ⊖ P monom P a 0) ⊗P q ⊕P monom P (eval R R
id a f) 0"
```

```
     using remainder_theorem[OF Suc.prems(1) a_carrier R_not_triv] by
auto
    hence lin_fac: "f = (monom P 1_R 1 ⊖ P monom P a 0) ⊗_P q" using q
by (simp add: a_root)
    have deg:"deg R (monom P 1_R 1 ⊖ P monom P a 0) = 1"
      using a_carrier by (simp add: deg_minus_eq)
    hence mon_not_zero:"(monom P 1_R 1 ⊖ P monom P a 0) ≠ 0_P"
      by (fastforce simp del: r_right_minus_eq)
    have q_not_zero:"q ≠ 0_P" using Suc by (auto simp add : lin_fac)
    hence "deg R q = x" using Suc deg deg_mult[OF mon_not_zero q_not_zero
_ q]
      by (simp add : lin_fac)
    hence q_IH:"finite {a ∈ carrier R . eval R R id a q = 0}
                ∧ card {a ∈ carrier R . eval R R id a q = 0} ≤ x" us-
ing Suc q q_not_zero by blast
    have subs:"{a ∈ carrier R . eval R R id a f = 0}
                ⊆ {a ∈ carrier R . eval R R id a q = 0} ∪ {a}" (is "?L
⊆ ?R ∪ {a}")
      using a_carrier ‘q ∈ _‘
      by (auto simp: evalRR_simps lin_fac R.integral_iff)
    have "{a ∈ carrier R . eval R R id a f = 0} ⊆ insert a {a ∈ carrier
R . eval R R id a q = 0}"
     using subs by auto
    hence "card {a ∈ carrier R . eval R R id a f = 0} ≤
           card (insert a {a ∈ carrier R . eval R R id a q = 0})" us-
ing q_IH by (blast intro: card_mono)
    also have "... ≤ deg R f" using q_IH ‘Suc x = _‘
      by (simp add: card_insert_if)
    finally show ?thesis using q_IH ‘Suc x = _‘ using finite by force
  next
    case False
    hence "card {a ∈ carrier R. eval R R id a f = 0} = 0" using finite
by auto
    also have "... ≤  deg R f" by simp
    finally show ?thesis using finite by auto
  qed
qed

end

lemma (in domain) num_roots_le_deg :
  fixes p d :: nat
  assumes finite:"finite (carrier R)"
  assumes d_neq_zero : "d ≠ 0"
  shows "card {x ∈ carrier R. x (^) d = 1} ≤ d"
proof -
  let ?f = "monom (UP R) 1_R d ⊖ (UP R) monom (UP R) 1_R 0"
  have one_in_carrier:"1 ∈ carrier R" by simp
  interpret R: UP_domain R "UP R" by (unfold_locales)
```

```
  have "deg R ?f = d"
    using d_neq_zero by (simp add: R.deg_minus_eq)
  hence f_not_zero:"?f ≠ 0UP R" using  d_neq_zero by (auto simp add :
R.deg_nzero_nzero)
  have roots_bound:"finite {a ∈ carrier R . eval R R id a ?f = 0} ∧
                  card {a ∈ carrier R . eval R R id a ?f = 0} ≤ deg
R ?f"
                  using finite by (intro R.roots_bound[OF _ f_not_zero])
simp
  have subs:"{x ∈ carrier R. x (^) d = 1} ⊆ {a ∈ carrier R . eval R R
id a ?f = 0}"
    by (auto simp: R.evalRR_simps)
  then have "card {x ∈ carrier R. x (^) d = 1} ≤
        card {a ∈ carrier R. eval R R id a ?f = 0}" using finite by (simp
add : card_mono)
  thus ?thesis using ‘deg R ?f = d‘ roots_bound by linarith
qed
```

21 The Multiplicative Group of a Field

In this section we show that the multiplicative group of a finite field is generated by a single element, i.e. it is cyclic. The proof is inspired by the first proof given in the survey [**?**].

```
lemma (in group) pow_order_eq_1:
  assumes "finite (carrier G)" "x ∈ carrier G" shows "x (^) order G =
1"
  using assms by (metis nat_pow_pow ord_dvd_group_order pow_ord_eq_1 dvdE
nat_pow_one)


lemma nat_div_eq: "a ≠ 0 ⟹ (a :: nat) div b = a ⟷ b = 1"
  apply rule
  apply (cases "b = 0")
  apply simp_all
  apply (metis (full_types) One_nat_def Suc_lessI div_less_dividend less_not_refl3)
  done

lemma (in group)
  assumes finite': "finite (carrier G)"
  assumes "a ∈ carrier G"
  shows pow_ord_eq_ord_iff: "group.ord G (a (^) k) = ord a ⟷ coprime
k (ord a)" (is "?L ⟷ ?R")
proof
  assume A: ?L then show ?R
    using assms ord_ge_1[OF assms] by (auto simp: nat_div_eq ord_pow_dvd_ord_elem)
next
  assume ?R then show ?L
    using ord_pow_dvd_ord_elem[OF assms, of k] by auto
```

```
qed

context field begin

lemma num_elems_of_ord_eq_phi':
  assumes finite: "finite (carrier R)" and dvd: "d dvd order (mult_of
R)"
      and exists: "∃a∈carrier (mult_of R). group.ord (mult_of R) a =
d"
  shows "card {a ∈ carrier (mult_of R). group.ord (mult_of R) a = d}
= phi' d"
proof -
  note mult_of_simps[simp]
  have finite': "finite (carrier (mult_of R))" using finite by (rule
finite_mult_of)

  interpret G:group "mult_of R" rewrites "op (^)mult_of R = (op (^) :: _
⇒ nat ⇒ _)" and "1mult_of R = 1"
    by (rule field_mult_group) simp_all

  from exists
  obtain a where a:"a ∈ carrier (mult_of R)" and ord_a: "group.ord (mult_of
R) a = d"
    by (auto simp add: card_gt_0_iff)

  have set_eq1:"{a(^)n| n. n ∈ {1 .. d}} = {x ∈ carrier (mult_of R).
x (^) d = 1}"
  proof (rule card_seteq)
    show "finite {x ∈ carrier (mult_of R). x (^) d = 1}" using finite
by auto

    show "{a(^)n| n. n ∈ {1 ..d}} ⊆ {x ∈ carrier (mult_of R). x(^)d
= 1}"
    proof
      fix x assume "x ∈ {a(^)n | n. n ∈ {1 .. d}}"
      then obtain n where n:"x = a(^)n ∧ n ∈ {1 .. d}" by auto
      have "x(^)d =(a(^)d)(^)n" using n a ord_a by (simp add:nat_pow_pow
mult.commute)
      hence "x(^)d = 1" using ord_a G.pow_ord_eq_1[OF finite' a] by fastforce
      thus "x ∈ {x ∈ carrier (mult_of R). x(^)d = 1}" using G.nat_pow_closed[OF
a] n by blast
    qed

    show "card {x ∈ carrier (mult_of R). x (^) d = 1} ≤ card {a(^)n
| n. n ∈ {1 .. d}}"
    proof -
      have *:"{a(^)n | n. n ∈ {1 .. d }} = ((λ n. a(^)n) ` {1 .. d})"
by auto
      have "0 < order (mult_of R)" unfolding order_mult_of[OF finite]
```

```
       using card_mono[OF finite, of "{0, 1}"] by (simp add: order_def)
     have "card {x ∈ carrier (mult_of R). x (^) d = 1} ≤ card {x ∈ carrier
R. x (^) d = 1}"
       using finite by (auto intro: card_mono)
     also have "... ≤ d" using ‘0 < order (mult_of R)‘ num_roots_le_deg[OF
finite, of d]
       by (simp add : dvd_pos_nat[OF _ ‘d dvd order (mult_of R)‘])
     finally show ?thesis using G.ord_inj'[OF finite' a] ord_a * by (simp
add: card_image)
   qed
  qed

  have set_eq2:"{x ∈ carrier (mult_of R) . group.ord (mult_of R) x =
d}
              = (λ n . a(^)n) ‘ {n ∈ {1 .. d}. group.ord (mult_of R)
(a(^)n) = d}" (is "?L = ?R")
  proof
    { fix x assume x:"x ∈ (carrier (mult_of R)) ∧ group.ord (mult_of
R) x = d"
     hence "x ∈ {x ∈ carrier (mult_of R). x (^) d = 1}"
       by (simp add: G.pow_ord_eq_1[OF finite', of x, symmetric])
     then obtain n where n:"x = a(^)n ∧ n ∈ {1 .. d}" using set_eq1
by blast
     hence "x ∈ ?R" using x by fast
    } thus "?L ⊆ ?R" by blast
   show "?R ⊆ ?L" using a by (auto simp add: carrier_mult_of[symmetric]
simp del: carrier_mult_of)
  qed
  have "inj_on (λ n . a(^)n) {n ∈ {1 .. d}. group.ord (mult_of R) (a(^)n)
= d}"
   using G.ord_inj'[OF finite' a, unfolded ord_a] unfolding inj_on_def
by fast
  hence "card ((λn. a(^)n) ‘ {n ∈ {1 .. d}. group.ord (mult_of R) (a(^)n)
= d})
        = card {k ∈ {1 .. d}. group.ord (mult_of R) (a(^)k) = d}"
        using card_image by blast
  thus ?thesis using set_eq2 G.pow_ord_eq_ord_iff[OF finite' ‘a ∈ _‘,
unfolded ord_a]
   by (simp add: phi'_def)
qed

end


theorem (in field) finite_field_mult_group_has_gen :
  assumes finite:"finite (carrier R)"
  shows "∃ a ∈ carrier (mult_of R) . carrier (mult_of R) = {a(^)i | i::nat
. i ∈ UNIV}"
proof -
```

```
  note mult_of_simps[simp]
  have finite': "finite (carrier (mult_of R))" using finite by (rule
finite_mult_of)

  interpret G: group "mult_of R" rewrites
      "op (^)mult_of R = (op (^) :: _ ⇒ nat ⇒ _)" and "1mult_of R = 1"
    by (rule field_mult_group) (simp_all add: fun_eq_iff nat_pow_def)

  let ?N = "λ x . card {a ∈ carrier (mult_of R). group.ord (mult_of R)
a  = x}"
  have "0 < order R - 1" unfolding order_def using card_mono[OF finite,
of "{0, 1}"] by simp
  then have *: "0 < order (mult_of R)" using assms by (simp add: order_mult_of)
  have fin: "finite {d. d dvd order (mult_of R) }" using dvd_nat_bounds[OF
*] by force

  have "(∑d | d dvd order (mult_of R). ?N d)
      = card (UN d:{d . d dvd order (mult_of R) }. {a ∈ carrier (mult_of
R). group.ord (mult_of R) a  = d})"
      (is "_ = card ?U")
    using fin finite by (subst card_UN_disjoint) auto
  also have "?U = carrier (mult_of R)"
  proof
    { fix x assume x:"x ∈ carrier (mult_of R)"
      hence x':"x∈carrier (mult_of R)" by simp
      then have "group.ord (mult_of R) x dvd order (mult_of R)"
          using finite' G.ord_dvd_group_order[OF _ x'] by (simp add: order_mult_of)
      hence "x ∈ ?U" using dvd_nat_bounds[of "order (mult_of R)" "group.ord
(mult_of R) x"] x by blast
    } thus "carrier (mult_of R) ⊆ ?U" by blast
  qed auto
  also have "card ... = order (mult_of R)"
    using order_mult_of finite' by (simp add: order_def)
  finally have sum_Ns_eq: "(∑d | d dvd order (mult_of R). ?N d) = order
(mult_of R)" .

  { fix d assume d:"d dvd order (mult_of R)"
    have "card {a ∈ carrier (mult_of R). group.ord (mult_of R) a = d}
≤ phi' d"
    proof cases
      assume "card {a ∈ carrier (mult_of R). group.ord (mult_of R) a
= d} = 0" thus ?thesis by presburger
      next
      assume "card {a ∈ carrier (mult_of R). group.ord (mult_of R) a
= d} ≠ 0"
      hence "∃a ∈ carrier (mult_of R). group.ord (mult_of R) a = d" by
(auto simp: card_eq_0_iff)
      thus ?thesis using num_elems_of_ord_eq_phi'[OF finite d] by auto
    qed
```

```
  }
  hence all_le:"⋀i. i ∈ {d. d dvd order (mult_of R) }
    ⟹ (λi. card {a ∈ carrier (mult_of R). group.ord (mult_of R)
a = i}) i ≤ (λi. phi' i) i" by fast
  hence le:"(∑i | i dvd order (mult_of R). ?N i)
      ≤ (∑i | i dvd order (mult_of R). phi' i)"
      using sum_mono[of "{d .  d dvd order (mult_of R)}"
         "λi. card {a ∈ carrier (mult_of R). group.ord (mult_of
R) a = i}"] by presburger
  have "order (mult_of R) = (∑d | d dvd order (mult_of R). phi' d)"
using *
    by (simp add: sum_phi'_factors)
  hence eq:"(∑i | i dvd order (mult_of R). ?N i)
     = (∑i | i dvd order (mult_of R). phi' i)" using le sum_Ns_eq
by presburger
  have "⋀i. i ∈ {d. d dvd order (mult_of R) } ⟹ ?N i = (λi. phi' i)
i"
  proof (rule ccontr)
    fix i
    assume i1:"i ∈ {d. d dvd order (mult_of R)}" and "?N i ≠ phi' i"
    hence "?N i = 0"
      using num_elems_of_ord_eq_phi'[OF finite, of i] by (auto simp: card_eq_0_iff)
    moreover  have "0 < i" using * i1 by (simp add: dvd_nat_bounds[of
"order (mult_of R)" i])
    ultimately have "?N i < phi' i" using phi'_nonzero by presburger
    hence "(∑i | i dvd order (mult_of R). ?N i)
        < (∑i | i dvd order (mult_of R). phi' i)"
      using sum_strict_mono_ex1[OF fin, of "?N" "λ i . phi' i"]
         i1 all_le by auto
    thus False using eq by force
  qed
  hence "?N (order (mult_of R)) > 0" using * by (simp add: phi'_nonzero)
  then obtain a where a:"a ∈ carrier (mult_of R)" and a_ord:"group.ord
(mult_of R) a = order (mult_of R)"
    by (auto simp add: card_gt_0_iff)
  hence set_eq:"{a(^)i | i::nat. i ∈ UNIV} = (λx. a(^)x) ` {0 .. group.ord
(mult_of R) a - 1}"
    using G.ord_elems[OF finite'] by auto
  have card_eq:"card ((λx. a(^)x) ` {0 .. group.ord (mult_of R) a - 1})
= card {0 .. group.ord (mult_of R) a - 1}"
    by (intro card_image G.ord_inj finite' a)
  hence "card ((λ x . a(^)x) ` {0 .. group.ord (mult_of R) a - 1}) = card
{0 ..order (mult_of R) - 1}"
    using assms by (simp add: card_eq a_ord)
  hence card_R_minus_1:"card {a(^)i | i::nat. i ∈ UNIV} =  order (mult_of
R)"
    using * by (subst set_eq) auto
  have **:"{a(^)i | i::nat. i ∈ UNIV} ⊆ carrier (mult_of R)"
    using G.nat_pow_closed[OF a] by auto
```

```
  with _ have "carrier (mult_of R) = {a(^)i|i::nat. i ∈ UNIV}"
    by (rule card_seteq[symmetric]) (simp_all add: card_R_minus_1 finite
order_def del: UNIV_I)
  thus ?thesis using a by blast
qed

end
```

22 Divisibility in monoids and rings

```
theory Divisibility
  imports "HOL-Library.Permutation" Coset Group
begin
```

23 Factorial Monoids

23.1 Monoids with Cancellation Law

```
locale monoid_cancel = monoid +
  assumes l_cancel: "⟦c ⊗ a = c ⊗ b; a ∈ carrier G; b ∈ carrier G; c
∈ carrier G⟧ ⟹ a = b"
    and r_cancel: "⟦a ⊗ c = b ⊗ c; a ∈ carrier G; b ∈ carrier G; c ∈
carrier G⟧ ⟹ a = b"

lemma (in monoid) monoid_cancelI:
  assumes l_cancel: "⋀a b c. ⟦c ⊗ a = c ⊗ b; a ∈ carrier G; b ∈ carrier
G; c ∈ carrier G⟧ ⟹ a = b"
    and r_cancel: "⋀a b c. ⟦a ⊗ c = b ⊗ c; a ∈ carrier G; b ∈ carrier
G; c ∈ carrier G⟧ ⟹ a = b"
  shows "monoid_cancel G"
    by standard fact+

lemma (in monoid_cancel) is_monoid_cancel: "monoid_cancel G" ..

sublocale group ⊆ monoid_cancel
  by standard simp_all


locale comm_monoid_cancel = monoid_cancel + comm_monoid

lemma comm_monoid_cancelI:
  fixes G (structure)
  assumes "comm_monoid G"
  assumes cancel: "⋀a b c. ⟦a ⊗ c = b ⊗ c; a ∈ carrier G; b ∈ carrier
G; c ∈ carrier G⟧ ⟹ a = b"
  shows "comm_monoid_cancel G"
proof -
  interpret comm_monoid G by fact
```

```
  show "comm_monoid_cancel G"
    by unfold_locales (metis assms(2) m_ac(2))+
qed

lemma (in comm_monoid_cancel) is_comm_monoid_cancel: "comm_monoid_cancel
G"
  by intro_locales

sublocale comm_group ⊆ comm_monoid_cancel ..
```

23.2 Products of Units in Monoids

```
lemma (in monoid) Units_m_closed[simp, intro]:
  assumes h1unit: "h1 ∈ Units G"
    and h2unit: "h2 ∈ Units G"
  shows "h1 ⊗ h2 ∈ Units G"
  unfolding Units_def
  using assms
  by auto (metis Units_inv_closed Units_l_inv Units_m_closed Units_r_inv)

lemma (in monoid) prod_unit_l:
  assumes abunit[simp]: "a ⊗ b ∈ Units G"
    and aunit[simp]: "a ∈ Units G"
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "b ∈ Units G"
proof -
  have c: "inv (a ⊗ b) ⊗ a ∈ carrier G" by simp

  have "(inv (a ⊗ b) ⊗ a) ⊗ b = inv (a ⊗ b) ⊗ (a ⊗ b)"
    by (simp add: m_assoc)
  also have "... = 1" by simp
  finally have li: "(inv (a ⊗ b) ⊗ a) ⊗ b = 1" .

  have "1 = inv a ⊗ a" by (simp add: Units_l_inv[symmetric])
  also have "... = inv a ⊗ 1 ⊗ a" by simp
  also have "... = inv a ⊗ ((a ⊗ b) ⊗ inv (a ⊗ b)) ⊗ a"
    by (simp add: Units_r_inv[OF abunit, symmetric] del: Units_r_inv)
  also have "... = ((inv a ⊗ a) ⊗ b) ⊗ inv (a ⊗ b) ⊗ a"
    by (simp add: m_assoc del: Units_l_inv)
  also have "... = b ⊗ inv (a ⊗ b) ⊗ a" by simp
  also have "... = b ⊗ (inv (a ⊗ b) ⊗ a)" by (simp add: m_assoc)
  finally have ri: "b ⊗ (inv (a ⊗ b) ⊗ a) = 1 " by simp

  from c li ri show "b ∈ Units G" by (auto simp: Units_def)
qed

lemma (in monoid) prod_unit_r:
  assumes abunit[simp]: "a ⊗ b ∈ Units G"
    and bunit[simp]: "b ∈ Units G"
```

```
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "a ∈ Units G"
proof -
  have c: "b ⊗ inv (a ⊗ b) ∈ carrier G" by simp

  have "a ⊗ (b ⊗ inv (a ⊗ b)) = (a ⊗ b) ⊗ inv (a ⊗ b)"
    by (simp add: m_assoc del: Units_r_inv)
  also have "... = 1" by simp
  finally have li: "a ⊗ (b ⊗ inv (a ⊗ b)) = 1" .

  have "1 = b ⊗ inv b" by (simp add: Units_r_inv[symmetric])
  also have "... = b ⊗ 1 ⊗ inv b" by simp
  also have "... = b ⊗ (inv (a ⊗ b) ⊗ (a ⊗ b)) ⊗ inv b"
    by (simp add: Units_l_inv[OF abunit, symmetric] del: Units_l_inv)
  also have "... = (b ⊗ inv (a ⊗ b) ⊗ a) ⊗ (b ⊗ inv b)"
    by (simp add: m_assoc del: Units_l_inv)
  also have "... = b ⊗ inv (a ⊗ b) ⊗ a" by simp
  finally have ri: "(b ⊗ inv (a ⊗ b)) ⊗ a = 1 " by simp

  from c li ri show "a ∈ Units G" by (auto simp: Units_def)
qed

lemma (in comm_monoid) unit_factor:
  assumes abunit: "a ⊗ b ∈ Units G"
    and [simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "a ∈ Units G"
  using abunit[simplified Units_def]
proof clarsimp
  fix i
  assume [simp]: "i ∈ carrier G"

  have carr': "b ⊗ i ∈ carrier G" by simp

  have "(b ⊗ i) ⊗ a = (i ⊗ b) ⊗ a" by (simp add: m_comm)
  also have "... = i ⊗ (b ⊗ a)" by (simp add: m_assoc)
  also have "... = i ⊗ (a ⊗ b)" by (simp add: m_comm)
  also assume "i ⊗ (a ⊗ b) = 1"
  finally have li': "(b ⊗ i) ⊗ a = 1" .

  have "a ⊗ (b ⊗ i) = a ⊗ b ⊗ i" by (simp add: m_assoc)
  also assume "a ⊗ b ⊗ i = 1"
  finally have ri': "a ⊗ (b ⊗ i) = 1" .

  from carr' li' ri'
  show "a ∈ Units G" by (simp add: Units_def, fast)
qed
```

23.3 Divisibility and Association

23.3.1 Function definitions

```
definition factor :: "[_, 'a, 'a] ⇒ bool" (infix "dividesı" 65)
  where "a divides_G b ⟷ (∃c∈carrier G. b = a ⊗_G c)"

definition associated :: "[_, 'a, 'a] ⇒ bool" (infix "∼ı" 55)
  where "a ∼_G b ⟷ a divides_G b ∧ b divides_G a"

abbreviation "division_rel G ≡ ⦇carrier = carrier G, eq = op ∼_G, le =
op divides_G⦈"

definition properfactor :: "[_, 'a, 'a] ⇒ bool"
  where "properfactor G a b ⟷ a divides_G b ∧ ¬(b divides_G a)"

definition irreducible :: "[_, 'a] ⇒ bool"
  where "irreducible G a ⟷ a ∉ Units G ∧ (∀b∈carrier G. properfactor
G b a ⟶ b ∈ Units G)"

definition prime :: "[_, 'a] ⇒ bool"
  where "prime G p ⟷
    p ∉ Units G ∧
    (∀a∈carrier G. ∀b∈carrier G. p divides_G (a ⊗_G b) ⟶ p divides_G
a ∨ p divides_G b)"
```

23.3.2 Divisibility

```
lemma dividesI:
  fixes G (structure)
  assumes carr: "c ∈ carrier G"
    and p: "b = a ⊗ c"
  shows "a divides b"
  unfolding factor_def using assms by fast

lemma dividesI' [intro]:
  fixes G (structure)
  assumes p: "b = a ⊗ c"
    and carr: "c ∈ carrier G"
  shows "a divides b"
  using assms by (fast intro: dividesI)

lemma dividesD:
  fixes G (structure)
  assumes "a divides b"
  shows "∃c∈carrier G. b = a ⊗ c"
  using assms unfolding factor_def by fast

lemma dividesE [elim]:
  fixes G (structure)
```

```
  assumes d: "a divides b"
    and elim: "⋀c. ⟦b = a ⊗ c; c ∈ carrier G⟧ ⟹ P"
  shows "P"
proof -
  from dividesD[OF d] obtain c where "c ∈ carrier G" and "b = a ⊗ c"
by auto
  then show P by (elim elim)
qed

lemma (in monoid) divides_refl[simp, intro!]:
  assumes carr: "a ∈ carrier G"
  shows "a divides a"
  by (intro dividesI[of "1"]) (simp_all add: carr)

lemma (in monoid) divides_trans [trans]:
  assumes dvds: "a divides b"  "b divides c"
    and acarr: "a ∈ carrier G"
  shows "a divides c"
  using dvds[THEN dividesD] by (blast intro: dividesI m_assoc acarr)

lemma (in monoid) divides_mult_lI [intro]:
  assumes ab: "a divides b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "(c ⊗ a) divides (c ⊗ b)"
  using ab
  apply (elim dividesE)
  apply (simp add: m_assoc[symmetric] carr)
  apply (fast intro: dividesI)
  done

lemma (in monoid_cancel) divides_mult_l [simp]:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "(c ⊗ a) divides (c ⊗ b) = a divides b"
  apply safe
   apply (elim dividesE, intro dividesI, assumption)
   apply (rule l_cancel[of c])
      apply (simp add: m_assoc carr)+
  apply (fast intro: carr)
  done

lemma (in comm_monoid) divides_mult_rI [intro]:
  assumes ab: "a divides b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "(a ⊗ c) divides (b ⊗ c)"
  using carr ab
  apply (simp add: m_comm[of a c] m_comm[of b c])
  apply (rule divides_mult_lI, assumption+)
  done
```

```
lemma (in comm_monoid_cancel) divides_mult_r [simp]:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "(a ⊗ c) divides (b ⊗ c) = a divides b"
  using carr by (simp add: m_comm[of a c] m_comm[of b c])

lemma (in monoid) divides_prod_r:
  assumes ab: "a divides b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "a divides (b ⊗ c)"
  using ab carr by (fast intro: m_assoc)

lemma (in comm_monoid) divides_prod_l:
  assumes carr[intro]: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier
G"
    and ab: "a divides b"
  shows "a divides (c ⊗ b)"
  using ab carr
  apply (simp add: m_comm[of c b])
  apply (fast intro: divides_prod_r)
  done

lemma (in monoid) unit_divides:
  assumes uunit: "u ∈ Units G"
    and acarr: "a ∈ carrier G"
  shows "u divides a"
proof (intro dividesI[of "(inv u) ⊗ a"], fast intro: uunit acarr)
  from uunit acarr have xcarr: "inv u ⊗ a ∈ carrier G" by fast
  from uunit acarr have "u ⊗ (inv u ⊗ a) = (u ⊗ inv u) ⊗ a"
    by (fast intro: m_assoc[symmetric])
  also have "... = 1 ⊗ a" by (simp add: Units_r_inv[OF uunit])
  also from acarr have "... = a" by simp
  finally show "a = u ⊗ (inv u ⊗ a)" ..
qed

lemma (in comm_monoid) divides_unit:
  assumes udvd: "a divides u"
    and  carr: "a ∈ carrier G"  "u ∈ Units G"
  shows "a ∈ Units G"
  using udvd carr by (blast intro: unit_factor)

lemma (in comm_monoid) Unit_eq_dividesone:
  assumes ucarr: "u ∈ carrier G"
  shows "u ∈ Units G = u divides 1"
  using ucarr by (fast dest: divides_unit intro: unit_divides)
```

23.3.3 Association

```
lemma associatedI:
  fixes G (structure)
```

```
  assumes "a divides b"  "b divides a"
  shows "a ∼ b"
  using assms by (simp add: associated_def)

lemma (in monoid) associatedI2:
  assumes uunit[simp]: "u ∈ Units G"
    and a: "a = b ⊗ u"
    and bcarr[simp]: "b ∈ carrier G"
  shows "a ∼ b"
  using uunit bcarr
  unfolding a
  apply (intro associatedI)
   apply (rule dividesI[of "inv u"], simp)
   apply (simp add: m_assoc Units_closed)
  apply fast
  done

lemma (in monoid) associatedI2':
  assumes "a = b ⊗ u"
    and "u ∈ Units G"
    and "b ∈ carrier G"
  shows "a ∼ b"
  using assms by (intro associatedI2)

lemma associatedD:
  fixes G (structure)
  assumes "a ∼ b"
  shows "a divides b"
  using assms by (simp add: associated_def)

lemma (in monoid_cancel) associatedD2:
  assumes assoc: "a ∼ b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "∃u∈Units G. a = b ⊗ u"
  using assoc
  unfolding associated_def
proof clarify
  assume "b divides a"
  then obtain u where ucarr: "u ∈ carrier G" and a: "a = b ⊗ u"
    by (rule dividesE)

  assume "a divides b"
  then obtain u' where u'carr: "u' ∈ carrier G" and b: "b = a ⊗ u'"
    by (rule dividesE)
  note carr = carr ucarr u'carr

  from carr have "a ⊗ 1 = a" by simp
  also have "... = b ⊗ u" by (simp add: a)
  also have "... = a ⊗ u' ⊗ u" by (simp add: b)
```

```
  also from carr have "... = a ⊗ (u' ⊗ u)" by (simp add: m_assoc)
  finally have "a ⊗ 1 = a ⊗ (u' ⊗ u)" .
  with carr have u1: "1 = u' ⊗ u" by (fast dest: l_cancel)

  from carr have "b ⊗ 1 = b" by simp
  also have "... = a ⊗ u'" by (simp add: b)
  also have "... = b ⊗ u ⊗ u'" by (simp add: a)
  also from carr have "... = b ⊗ (u ⊗ u')" by (simp add: m_assoc)
  finally have "b ⊗ 1 = b ⊗ (u ⊗ u')" .
  with carr have u2: "1 = u ⊗ u'" by (fast dest: l_cancel)

  from u'carr u1[symmetric] u2[symmetric] have "∃u'∈carrier G. u' ⊗
u = 1 ∧ u ⊗ u' = 1"
    by fast
  then have "u ∈ Units G"
    by (simp add: Units_def ucarr)
  with ucarr a show "∃u∈Units G. a = b ⊗ u" by fast
qed

lemma associatedE:
  fixes G (structure)
  assumes assoc: "a ∼ b"
    and e: "⟦a divides b; b divides a⟧ ⟹ P"
  shows "P"
proof -
  from assoc have "a divides b" "b divides a"
    by (simp_all add: associated_def)
  then show P by (elim e)
qed

lemma (in monoid_cancel) associatedE2:
  assumes assoc: "a ∼ b"
    and e: "⋀u. ⟦a = b ⊗ u; u ∈ Units G⟧ ⟹ P"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "P"
proof -
  from assoc and carr have "∃u∈Units G. a = b ⊗ u"
    by (rule associatedD2)
  then obtain u where "u ∈ Units G"  "a = b ⊗ u"
    by auto
  then show P by (elim e)
qed

lemma (in monoid) associated_refl [simp, intro!]:
  assumes "a ∈ carrier G"
  shows "a ∼ a"
  using assms by (fast intro: associatedI)

lemma (in monoid) associated_sym [sym]:
```

```
  assumes "a ∼ b"
    and "a ∈ carrier G"  "b ∈ carrier G"
  shows "b ∼ a"
  using assms by (iprover intro: associatedI elim: associatedE)

lemma (in monoid) associated_trans [trans]:
  assumes "a ∼ b"  "b ∼ c"
    and "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "a ∼ c"
  using assms by (iprover intro: associatedI divides_trans elim: associatedE)

lemma (in monoid) division_equiv [intro, simp]: "equivalence (division_rel
G)"
  apply unfold_locales
    apply simp_all
   apply (metis associated_def)
  apply (iprover intro: associated_trans)
  done
```

23.3.4 Division and associativity

```
lemma divides_antisym:
  fixes G (structure)
  assumes "a divides b"  "b divides a"
    and "a ∈ carrier G"  "b ∈ carrier G"
  shows "a ∼ b"
  using assms by (fast intro: associatedI)

lemma (in monoid) divides_cong_l [trans]:
  assumes "x ∼ x'"
    and "x' divides y"
    and [simp]: "x ∈ carrier G"  "x' ∈ carrier G"  "y ∈ carrier G"
  shows "x divides y"
proof -
  from assms(1) have "x divides x'" by (simp add: associatedD)
  also note assms(2)
  finally show "x divides y" by simp
qed

lemma (in monoid) divides_cong_r [trans]:
  assumes "x divides y"
    and "y ∼ y'"
    and [simp]: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
  shows "x divides y'"
proof -
  note assms(1)
  also from assms(2) have "y divides y'" by (simp add: associatedD)
  finally show "x divides y'" by simp
qed
```

```
lemma (in monoid) division_weak_partial_order [simp, intro!]:
  "weak_partial_order (division_rel G)"
  apply unfold_locales
        apply simp_all
      apply (simp add: associated_sym)
     apply (blast intro: associated_trans)
    apply (simp add: divides_antisym)
   apply (blast intro: divides_trans)
  apply (blast intro: divides_cong_l divides_cong_r associated_sym)
  done
```

23.3.5 Multiplication and associativity

```
lemma (in monoid_cancel) mult_cong_r:
  assumes "b ∼ b'"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "b' ∈ carrier G"
  shows "a ⊗ b ∼ a ⊗ b'"
  using assms
  apply (elim associatedE2, intro associatedI2)
      apply (auto intro: m_assoc[symmetric])
  done

lemma (in comm_monoid_cancel) mult_cong_l:
  assumes "a ∼ a'"
    and carr: "a ∈ carrier G"  "a' ∈ carrier G"  "b ∈ carrier G"
  shows "a ⊗ b ∼ a' ⊗ b"
  using assms
  apply (elim associatedE2, intro associatedI2)
      apply assumption
     apply (simp add: m_assoc Units_closed)
     apply (simp add: m_comm Units_closed)
    apply simp_all
  done

lemma (in monoid_cancel) assoc_l_cancel:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "b' ∈ carrier G"
    and "a ⊗ b ∼ a ⊗ b'"
  shows "b ∼ b'"
  using assms
  apply (elim associatedE2, intro associatedI2)
      apply assumption
     apply (rule l_cancel[of a])
        apply (simp add: m_assoc Units_closed)
       apply fast+
  done

lemma (in comm_monoid_cancel) assoc_r_cancel:
  assumes "a ⊗ b ∼ a' ⊗ b"
```

```
    and carr: "a ∈ carrier G"  "a' ∈ carrier G"  "b ∈ carrier G"
  shows "a ∼ a'"
  using assms
  apply (elim associatedE2, intro associatedI2)
      apply assumption
     apply (rule r_cancel[of a b])
        apply (metis Units_closed assms(3) assms(4) m_ac)
       apply fast+
  done
```

23.3.6 Units

```
lemma (in monoid_cancel) assoc_unit_l [trans]:
  assumes "a ∼ b"
    and "b ∈ Units G"
    and "a ∈ carrier G"
  shows "a ∈ Units G"
  using assms by (fast elim: associatedE2)

lemma (in monoid_cancel) assoc_unit_r [trans]:
  assumes aunit: "a ∈ Units G"
    and asc: "a ∼ b"
    and bcarr: "b ∈ carrier G"
  shows "b ∈ Units G"
  using aunit bcarr associated_sym[OF asc] by (blast intro: assoc_unit_l)

lemma (in comm_monoid) Units_cong:
  assumes aunit: "a ∈ Units G" and asc: "a ∼ b"
    and bcarr: "b ∈ carrier G"
  shows "b ∈ Units G"
  using assms by (blast intro: divides_unit elim: associatedE)

lemma (in monoid) Units_assoc:
  assumes units: "a ∈ Units G"  "b ∈ Units G"
  shows "a ∼ b"
  using units by (fast intro: associatedI unit_divides)

lemma (in monoid) Units_are_ones: "Units G {.=}(division_rel G) {1}"
  apply (simp add: set_eq_def elem_def, rule, simp_all)
proof clarsimp
  fix a
  assume aunit: "a ∈ Units G"
  show "a ∼ 1"
    apply (rule associatedI)
     apply (fast intro: dividesI[of "inv a"] aunit Units_r_inv[symmetric])
    apply (fast intro: dividesI[of "a"] l_one[symmetric] Units_closed[OF
aunit])
    done
next
```

```
  have "1 ∈ Units G" by simp
  moreover have "1 ∼ 1" by simp
  ultimately show "∃a ∈ Units G. 1 ∼ a" by fast
qed

lemma (in comm_monoid) Units_Lower: "Units G = Lower (division_rel G)
(carrier G)"
  apply (simp add: Units_def Lower_def)
  apply (rule, rule)
   apply clarsimp
   apply (rule unit_divides)
    apply (unfold Units_def, fast)
   apply assumption
  apply clarsimp
  apply (metis Unit_eq_dividesone Units_r_inv_ex m_ac(2) one_closed)
  done
```

23.3.7 Proper factors

```
lemma properfactorI:
  fixes G (structure)
  assumes "a divides b"
    and "¬(b divides a)"
  shows "properfactor G a b"
  using assms unfolding properfactor_def by simp

lemma properfactorI2:
  fixes G (structure)
  assumes advdb: "a divides b"
    and neq: "¬(a ∼ b)"
  shows "properfactor G a b"
proof (rule properfactorI, rule advdb, rule notI)
  assume "b divides a"
  with advdb have "a ∼ b" by (rule associatedI)
  with neq show "False" by fast
qed

lemma (in comm_monoid_cancel) properfactorI3:
  assumes p: "p = a ⊗ b"
    and nunit: "b ∉ Units G"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "p ∈ carrier G"
  shows "properfactor G a p"
  unfolding p
  using carr
  apply (intro properfactorI, fast)
proof (clarsimp, elim dividesE)
  fix c
  assume ccarr: "c ∈ carrier G"
  note [simp] = carr ccarr
```

```
  have "a ⊗ 1 = a" by simp
  also assume "a = a ⊗ b ⊗ c"
  also have "... = a ⊗ (b ⊗ c)" by (simp add: m_assoc)
  finally have "a ⊗ 1 = a ⊗ (b ⊗ c)" .

  then have rinv: "1 = b ⊗ c" by (intro l_cancel[of "a" "1" "b ⊗ c"],
simp+)
  also have "... = c ⊗ b" by (simp add: m_comm)
  finally have linv: "1 = c ⊗ b" .

  from ccarr linv[symmetric] rinv[symmetric] have "b ∈ Units G"
    unfolding Units_def by fastforce
  with nunit show False ..
qed

lemma properfactorE:
  fixes G (structure)
  assumes pf: "properfactor G a b"
    and r: "⟦a divides b; ¬(b divides a)⟧ ⟹ P"
  shows "P"
  using pf unfolding properfactor_def by (fast intro: r)

lemma properfactorE2:
  fixes G (structure)
  assumes pf: "properfactor G a b"
    and elim: "⟦a divides b; ¬(a ∼ b)⟧ ⟹ P"
  shows "P"
  using pf unfolding properfactor_def by (fast elim: elim associatedE)

lemma (in monoid) properfactor_unitE:
  assumes uunit: "u ∈ Units G"
    and pf: "properfactor G a u"
    and acarr: "a ∈ carrier G"
  shows "P"
  using pf unit_divides[OF uunit acarr] by (fast elim: properfactorE)

lemma (in monoid) properfactor_divides:
  assumes pf: "properfactor G a b"
  shows "a divides b"
  using pf by (elim properfactorE)

lemma (in monoid) properfactor_trans1 [trans]:
  assumes dvds: "a divides b"  "properfactor G b c"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G a c"
  using dvds carr
  apply (elim properfactorE, intro properfactorI)
   apply (iprover intro: divides_trans)+
```

```
  done

lemma (in monoid) properfactor_trans2 [trans]:
  assumes dvds: "properfactor G a b"  "b divides c"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G a c"
  using dvds carr
  apply (elim properfactorE, intro properfactorI)
   apply (iprover intro: divides_trans)+
  done

lemma properfactor_lless:
  fixes G (structure)
  shows "properfactor G = lless (division_rel G)"
  apply (rule ext)
  apply (rule ext)
  apply rule
   apply (fastforce elim: properfactorE2 intro: weak_llessI)
  apply (fastforce elim: weak_llessE intro: properfactorI2)
  done

lemma (in monoid) properfactor_cong_l [trans]:
  assumes x'x: "x' ∼ x"
    and pf: "properfactor G x y"
    and carr: "x ∈ carrier G"  "x' ∈ carrier G"  "y ∈ carrier G"
  shows "properfactor G x' y"
  using pf
  unfolding properfactor_lless
proof -
  interpret weak_partial_order "division_rel G" ..
  from x'x have "x' .=division_rel G x" by simp
  also assume "x ⊏division_rel G y"
  finally show "x' ⊏division_rel G y" by (simp add: carr)
qed

lemma (in monoid) properfactor_cong_r [trans]:
  assumes pf: "properfactor G x y"
    and yy': "y ∼ y'"
    and carr: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
  shows "properfactor G x y'"
  using pf
  unfolding properfactor_lless
proof -
  interpret weak_partial_order "division_rel G" ..
  assume "x ⊏division_rel G y"
  also from yy'
  have "y .=division_rel G y'" by simp
  finally show "x ⊏division_rel G y'" by (simp add: carr)
qed
```

```
lemma (in monoid_cancel) properfactor_mult_lI [intro]:
  assumes ab: "properfactor G a b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G (c ⊗ a) (c ⊗ b)"
  using ab carr by (fastforce elim: properfactorE intro: properfactorI)

lemma (in monoid_cancel) properfactor_mult_l [simp]:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G (c ⊗ a) (c ⊗ b) = properfactor G a b"
  using carr by (fastforce elim: properfactorE intro: properfactorI)

lemma (in comm_monoid_cancel) properfactor_mult_rI [intro]:
  assumes ab: "properfactor G a b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G (a ⊗ c) (b ⊗ c)"
  using ab carr by (fastforce elim: properfactorE intro: properfactorI)

lemma (in comm_monoid_cancel) properfactor_mult_r [simp]:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G (a ⊗ c) (b ⊗ c) = properfactor G a b"
  using carr by (fastforce elim: properfactorE intro: properfactorI)

lemma (in monoid) properfactor_prod_r:
  assumes ab: "properfactor G a b"
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G a (b ⊗ c)"
  by (intro properfactor_trans2[OF ab] divides_prod_r) simp_all

lemma (in comm_monoid) properfactor_prod_l:
  assumes ab: "properfactor G a b"
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G a (c ⊗ b)"
  by (intro properfactor_trans2[OF ab] divides_prod_l) simp_all
```

23.4 Irreducible Elements and Primes

23.4.1 Irreducible elements

```
lemma irreducibleI:
  fixes G (structure)
  assumes "a ∉ Units G"
    and "⋀b. ⟦b ∈ carrier G; properfactor G b a⟧ ⟹ b ∈ Units G"
  shows "irreducible G a"
  using assms unfolding irreducible_def by blast

lemma irreducibleE:
  fixes G (structure)
  assumes irr: "irreducible G a"
    and elim: "⟦a ∉ Units G; ∀b. b ∈ carrier G ∧ properfactor G b a ⟶
```

```
b ∈ Units G⟧ ⟹ P"
  shows "P"
  using assms unfolding irreducible_def by blast

lemma irreducibleD:
  fixes G (structure)
  assumes irr: "irreducible G a"
    and pf: "properfactor G b a"
    and bcarr: "b ∈ carrier G"
  shows "b ∈ Units G"
  using assms by (fast elim: irreducibleE)

lemma (in monoid_cancel) irreducible_cong [trans]:
  assumes irred: "irreducible G a"
    and aa': "a ∼ a'"
    and carr[simp]: "a ∈ carrier G"  "a' ∈ carrier G"
  shows "irreducible G a'"
  using assms
  apply (elim irreducibleE, intro irreducibleI)
   apply simp_all
   apply (metis assms(2) assms(3) assoc_unit_l)
  apply (metis assms(2) assms(3) assms(4) associated_sym properfactor_cong_r)
  done

lemma (in monoid) irreducible_prod_rI:
  assumes airr: "irreducible G a"
    and bunit: "b ∈ Units G"
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "irreducible G (a ⊗ b)"
  using airr carr bunit
  apply (elim irreducibleE, intro irreducibleI, clarify)
   apply (subgoal_tac "a ∈ Units G", simp)
   apply (intro prod_unit_r[of a b] carr bunit, assumption)
  apply (metis assms(2,3) associatedI2 m_closed properfactor_cong_r)
  done

lemma (in comm_monoid) irreducible_prod_lI:
  assumes birr: "irreducible G b"
    and aunit: "a ∈ Units G"
    and carr [simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "irreducible G (a ⊗ b)"
  apply (subst m_comm, simp+)
  apply (intro irreducible_prod_rI assms)
  done

lemma (in comm_monoid_cancel) irreducible_prodE [elim]:
  assumes irr: "irreducible G (a ⊗ b)"
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
    and e1: "⟦irreducible G a; b ∈ Units G⟧ ⟹ P"
```

```
    and e2: "⟦a ∈ Units G; irreducible G b⟧ ⟹ P"
  shows P
  using irr
proof (elim irreducibleE)
  assume abnunit: "a ⊗ b ∉ Units G"
    and isunit[rule_format]: "∀ba. ba ∈ carrier G ∧ properfactor G ba
(a ⊗ b) ⟶ ba ∈ Units G"
  show P
  proof (cases "a ∈ Units G")
    case aunit: True
    have "irreducible G b"
    proof (rule irreducibleI, rule notI)
      assume "b ∈ Units G"
      with aunit have "(a ⊗ b) ∈ Units G" by fast
      with abnunit show "False" ..
    next
      fix c
      assume ccarr: "c ∈ carrier G"
        and "properfactor G c b"
      then have "properfactor G c (a ⊗ b)" by (simp add: properfactor_prod_l[of
c b a])
      with ccarr show "c ∈ Units G" by (fast intro: isunit)
    qed
    with aunit show "P" by (rule e2)
  next
    case anunit: False
    with carr have "properfactor G b (b ⊗ a)" by (fast intro: properfactorI3)
    then have bf: "properfactor G b (a ⊗ b)" by (subst m_comm[of a b],
simp+)
    then have bunit: "b ∈ Units G" by (intro isunit, simp)

    have "irreducible G a"
    proof (rule irreducibleI, rule notI)
      assume "a ∈ Units G"
      with bunit have "(a ⊗ b) ∈ Units G" by fast
      with abnunit show "False" ..
    next
      fix c
      assume ccarr: "c ∈ carrier G"
        and "properfactor G c a"
      then have "properfactor G c (a ⊗ b)"
        by (simp add: properfactor_prod_r[of c a b])
      with ccarr show "c ∈ Units G" by (fast intro: isunit)
    qed
    from this bunit show "P" by (rule e1)
  qed
qed
```

23.4.2 Prime elements

```
lemma primeI:
  fixes G (structure)
  assumes "p ∉ Units G"
    and "⋀a b. ⟦a ∈ carrier G; b ∈ carrier G; p divides (a ⊗ b)⟧ ⟹
p divides a ∨ p divides b"
  shows "prime G p"
  using assms unfolding prime_def by blast

lemma primeE:
  fixes G (structure)
  assumes pprime: "prime G p"
    and e: "⟦p ∉ Units G; ∀a∈carrier G. ∀b∈carrier G.
      p divides a ⊗ b ⟶ p divides a ∨ p divides b⟧ ⟹ P"
  shows "P"
  using pprime unfolding prime_def by (blast dest: e)

lemma (in comm_monoid_cancel) prime_divides:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
    and pprime: "prime G p"
    and pdvd: "p divides a ⊗ b"
  shows "p divides a ∨ p divides b"
  using assms by (blast elim: primeE)

lemma (in monoid_cancel) prime_cong [trans]:
  assumes pprime: "prime G p"
    and pp': "p ∼ p'"
    and carr[simp]: "p ∈ carrier G"  "p' ∈ carrier G"
  shows "prime G p'"
  using pprime
  apply (elim primeE, intro primeI)
   apply (metis assms(2) assms(3) assoc_unit_l)
  apply (metis assms(2) assms(3) assms(4) associated_sym divides_cong_l
m_closed)
  done
```

23.5 Factorization and Factorial Monoids

23.5.1 Function definitions

```
definition factors :: "[_, 'a list, 'a] ⇒ bool"
  where "factors G fs a ⟷ (∀x ∈ (set fs). irreducible G x) ∧ foldr
(op ⊗G) fs 1G = a"

definition wfactors ::"[_, 'a list, 'a] ⇒ bool"
  where "wfactors G fs a ⟷ (∀x ∈ (set fs). irreducible G x) ∧ foldr
(op ⊗G) fs 1G ∼G a"

abbreviation list_assoc :: "('a,_) monoid_scheme ⇒ 'a list ⇒ 'a list
```

```
⇒ bool" (infix "[∼]ı" 44)
  where "list_assoc G ≡ list_all2 (op ∼G)"

definition essentially_equal :: "[_, 'a list, 'a list] ⇒ bool"
  where "essentially_equal G fs1 fs2 ⟷ (∃fs1'. fs1 <~~> fs1' ∧ fs1'
[∼]G fs2)"


locale factorial_monoid = comm_monoid_cancel +
  assumes factors_exist: "⟦a ∈ carrier G; a ∉ Units G⟧ ⟹ ∃fs. set fs
⊆ carrier G ∧ factors G fs a"
    and factors_unique:
      "⟦factors G fs a; factors G fs' a; a ∈ carrier G; a ∉ Units G;
        set fs ⊆ carrier G; set fs' ⊆ carrier G⟧ ⟹ essentially_equal
G fs fs'"
```

23.5.2 Comparing lists of elements

Association on lists

```
lemma (in monoid) listassoc_refl [simp, intro]:
  assumes "set as ⊆ carrier G"
  shows "as [∼] as"
  using assms by (induct as) simp_all

lemma (in monoid) listassoc_sym [sym]:
  assumes "as [∼] bs"
    and "set as ⊆ carrier G"
    and "set bs ⊆ carrier G"
  shows "bs [∼] as"
  using assms
proof (induct as arbitrary: bs, simp)
  case Cons
  then show ?case
    apply (induct bs)
     apply simp
    apply clarsimp
    apply (iprover intro: associated_sym)
    done
qed

lemma (in monoid) listassoc_trans [trans]:
  assumes "as [∼] bs" and "bs [∼] cs"
    and "set as ⊆ carrier G" and "set bs ⊆ carrier G" and "set cs ⊆
carrier G"
  shows "as [∼] cs"
  using assms
  apply (simp add: list_all2_conv_all_nth set_conv_nth, safe)
  apply (rule associated_trans)
      apply (subgoal_tac "as ! i ∼ bs ! i", assumption)
```

```
      apply (simp, simp)
    apply blast+
  done

lemma (in monoid_cancel) irrlist_listassoc_cong:
  assumes "∀a∈set as. irreducible G a"
    and "as [∼] bs"
    and "set as ⊆ carrier G" and "set bs ⊆ carrier G"
  shows "∀a∈set bs. irreducible G a"
  using assms
  apply (clarsimp simp add: list_all2_conv_all_nth set_conv_nth)
  apply (blast intro: irreducible_cong)
  done
```

Permutations

```
lemma perm_map [intro]:
  assumes p: "a <~~> b"
  shows "map f a <~~> map f b"
  using p by induct auto

lemma perm_map_switch:
  assumes m: "map f a = map f b" and p: "b <~~> c"
  shows "∃d. a <~~> d ∧ map f d = map f c"
  using p m by (induct arbitrary: a) (simp, force, force, blast)

lemma (in monoid) perm_assoc_switch:
  assumes a:"as [∼] bs" and p: "bs <~~> cs"
  shows "∃bs'. as <~~> bs' ∧ bs' [∼] cs"
  using p a
  apply (induct bs cs arbitrary: as, simp)
    apply (clarsimp simp add: list_all2_Cons2, blast)
   apply (clarsimp simp add: list_all2_Cons2)
   apply blast
  apply blast
  done

lemma (in monoid) perm_assoc_switch_r:
  assumes p: "as <~~> bs" and a:"bs [∼] cs"
  shows "∃bs'. as [∼] bs' ∧ bs' <~~> cs"
  using p a
  apply (induct as bs arbitrary: cs, simp)
    apply (clarsimp simp add: list_all2_Cons1, blast)
   apply (clarsimp simp add: list_all2_Cons1)
   apply blast
  apply blast
  done

declare perm_sym [sym]
```

```
lemma perm_setP:
  assumes perm: "as <~~> bs"
    and as: "P (set as)"
  shows "P (set bs)"
proof -
  from perm have "mset as = mset bs"
    by (simp add: mset_eq_perm)
  then have "set as = set bs"
    by (rule mset_eq_setD)
  with as show "P (set bs)"
    by simp
qed

lemmas (in monoid) perm_closed = perm_setP[of _ _ "λas. as ⊆ carrier
G"]

lemmas (in monoid) irrlist_perm_cong = perm_setP[of _ _ "λas. ∀a∈as.
irreducible G a"]
```

Essentially equal factorizations

```
lemma (in monoid) essentially_equalI:
  assumes ex: "fs1 <~~> fs1'"  "fs1' [∼] fs2"
  shows "essentially_equal G fs1 fs2"
  using ex unfolding essentially_equal_def by fast

lemma (in monoid) essentially_equalE:
  assumes ee: "essentially_equal G fs1 fs2"
    and e: "⋀fs1'. ⟦fs1 <~~> fs1'; fs1' [∼] fs2⟧ ⟹ P"
  shows "P"
  using ee unfolding essentially_equal_def by (fast intro: e)

lemma (in monoid) ee_refl [simp,intro]:
  assumes carr: "set as ⊆ carrier G"
  shows "essentially_equal G as as"
  using carr by (fast intro: essentially_equalI)

lemma (in monoid) ee_sym [sym]:
  assumes ee: "essentially_equal G as bs"
    and carr: "set as ⊆ carrier G"  "set bs ⊆ carrier G"
  shows "essentially_equal G bs as"
  using ee
proof (elim essentially_equalE)
  fix fs
  assume "as <~~> fs"  "fs [∼] bs"
  from perm_assoc_switch_r [OF this] obtain fs' where a: "as [∼] fs'"
and p: "fs' <~~> bs"
    by blast
  from p have "bs <~~> fs'" by (rule perm_sym)
  with a[symmetric] carr show ?thesis
```

```
    by (iprover intro: essentially_equalI perm_closed)
qed

lemma (in monoid) ee_trans [trans]:
  assumes ab: "essentially_equal G as bs" and bc: "essentially_equal
G bs cs"
    and ascarr: "set as ⊆ carrier G"
    and bscarr: "set bs ⊆ carrier G"
    and cscarr: "set cs ⊆ carrier G"
  shows "essentially_equal G as cs"
  using ab bc
proof (elim essentially_equalE)
  fix abs bcs
  assume "abs [∼] bs" and pb: "bs <~~> bcs"
  from perm_assoc_switch [OF this] obtain bs' where p: "abs <~~> bs'"
and a: "bs' [∼] bcs"
    by blast

  assume "as <~~> abs"
  with p have pp: "as <~~> bs'" by fast

  from pp ascarr have c1: "set bs' ⊆ carrier G" by (rule perm_closed)
  from pb bscarr have c2: "set bcs ⊆ carrier G" by (rule perm_closed)
  note a
  also assume "bcs [∼] cs"
  finally (listassoc_trans) have "bs' [∼] cs" by (simp add: c1 c2 cscarr)
  with pp show ?thesis
    by (rule essentially_equalI)
qed
```

23.5.3 Properties of lists of elements

Multiplication of factors in a list

```
lemma (in monoid) multlist_closed [simp, intro]:
  assumes ascarr: "set fs ⊆ carrier G"
  shows "foldr (op ⊗) fs 1 ∈ carrier G"
  using ascarr by (induct fs) simp_all

lemma  (in comm_monoid) multlist_dividesI :
  assumes "f ∈ set fs" and "f ∈ carrier G" and "set fs ⊆ carrier G"
  shows "f divides (foldr (op ⊗) fs 1)"
  using assms
  apply (induct fs)
   apply simp
  apply (case_tac "f = a")
   apply simp
   apply (fast intro: dividesI)
  apply clarsimp
  apply (metis assms(2) divides_prod_l multlist_closed)
```

```
  done

lemma (in comm_monoid_cancel) multlist_listassoc_cong:
  assumes "fs [∼] fs'"
    and "set fs ⊆ carrier G" and "set fs' ⊆ carrier G"
  shows "foldr (op ⊗) fs 1 ∼ foldr (op ⊗) fs' 1"
  using assms
proof (induct fs arbitrary: fs', simp)
  case (Cons a as fs')
  then show ?case
    apply (induct fs', simp)
  proof clarsimp
    fix b bs
    assume "a ∼ b"
      and acarr: "a ∈ carrier G" and bcarr: "b ∈ carrier G"
      and ascarr: "set as ⊆ carrier G"
    then have p: "a ⊗ foldr op ⊗ as 1 ∼ b ⊗ foldr op ⊗ as 1"
      by (fast intro: mult_cong_l)
    also
    assume "as [∼] bs"
      and bscarr: "set bs ⊆ carrier G"
      and "⋀fs'. ⟦as [∼] fs'; set fs' ⊆ carrier G⟧ ⟹ foldr op ⊗ as
1 ∼ foldr op ⊗ fs' 1"
    then have "foldr op ⊗ as 1 ∼ foldr op ⊗ bs 1" by simp
    with ascarr bscarr bcarr have "b ⊗ foldr op ⊗ as 1 ∼ b ⊗ foldr
op ⊗ bs 1"
      by (fast intro: mult_cong_r)
    finally show "a ⊗ foldr op ⊗ as 1 ∼ b ⊗ foldr op ⊗ bs 1"
      by (simp add: ascarr bscarr acarr bcarr)
  qed
qed

lemma (in comm_monoid) multlist_perm_cong:
  assumes prm: "as <~~> bs"
    and ascarr: "set as ⊆ carrier G"
  shows "foldr (op ⊗) as 1 = foldr (op ⊗) bs 1"
  using prm ascarr
  apply (induct, simp, clarsimp simp add: m_ac, clarsimp)
proof clarsimp
  fix xs ys zs
  assume "xs <~~> ys"  "set xs ⊆ carrier G"
  then have "set ys ⊆ carrier G" by (rule perm_closed)
  moreover assume "set ys ⊆ carrier G ⟹ foldr op ⊗ ys 1 = foldr op
⊗ zs 1"
  ultimately show "foldr op ⊗ ys 1 = foldr op ⊗ zs 1" by simp
qed

lemma (in comm_monoid_cancel) multlist_ee_cong:
  assumes "essentially_equal G fs fs'"
```

```
    and "set fs ⊆ carrier G" and "set fs' ⊆ carrier G"
  shows "foldr (op ⊗) fs 𝟭 ∼ foldr (op ⊗) fs' 𝟭"
  using assms
  apply (elim essentially_equalE)
  apply (simp add: multlist_perm_cong multlist_listassoc_cong perm_closed)
  done
```

23.5.4 Factorization in irreducible elements

```
lemma wfactorsI:
  fixes G (structure)
  assumes "∀f∈set fs. irreducible G f"
    and "foldr (op ⊗) fs 𝟭 ∼ a"
  shows "wfactors G fs a"
  using assms unfolding wfactors_def by simp

lemma wfactorsE:
  fixes G (structure)
  assumes wf: "wfactors G fs a"
    and e: "⟦∀f∈set fs. irreducible G f; foldr (op ⊗) fs 𝟭 ∼ a⟧ ⟹
P"
  shows "P"
  using wf unfolding wfactors_def by (fast dest: e)

lemma (in monoid) factorsI:
  assumes "∀f∈set fs. irreducible G f"
    and "foldr (op ⊗) fs 𝟭 = a"
  shows "factors G fs a"
  using assms unfolding factors_def by simp

lemma factorsE:
  fixes G (structure)
  assumes f: "factors G fs a"
    and e: "⟦∀f∈set fs. irreducible G f; foldr (op ⊗) fs 𝟭 = a⟧ ⟹ P"
  shows "P"
  using f unfolding factors_def by (simp add: e)

lemma (in monoid) factors_wfactors:
  assumes "factors G as a" and "set as ⊆ carrier G"
  shows "wfactors G as a"
  using assms by (blast elim: factorsE intro: wfactorsI)

lemma (in monoid) wfactors_factors:
  assumes "wfactors G as a" and "set as ⊆ carrier G"
  shows "∃a'. factors G as a' ∧ a' ∼ a"
  using assms by (blast elim: wfactorsE intro: factorsI)

lemma (in monoid) factors_closed [dest]:
  assumes "factors G fs a" and "set fs ⊆ carrier G"
```

```
  shows "a ∈ carrier G"
  using assms by (elim factorsE, clarsimp)

lemma (in monoid) nunit_factors:
  assumes anunit: "a ∉ Units G"
    and fs: "factors G as a"
  shows "length as > 0"
proof -
  from anunit Units_one_closed have "a ≠ 1" by auto
  with fs show ?thesis by (auto elim: factorsE)
qed

lemma (in monoid) unit_wfactors [simp]:
  assumes aunit: "a ∈ Units G"
  shows "wfactors G [] a"
  using aunit by (intro wfactorsI) (simp, simp add: Units_assoc)

lemma (in comm_monoid_cancel) unit_wfactors_empty:
  assumes aunit: "a ∈ Units G"
    and wf: "wfactors G fs a"
    and carr[simp]: "set fs ⊆ carrier G"
  shows "fs = []"
proof (cases fs)
  case Nil
  then show ?thesis .
next
  case fs: (Cons f fs')
  from carr have fcarr[simp]: "f ∈ carrier G" and carr'[simp]: "set
fs' ⊆ carrier G"
    by (simp_all add: fs)

  from fs wf have "irreducible G f" by (simp add: wfactors_def)
  then have fnunit: "f ∉ Units G" by (fast elim: irreducibleE)

  from fs wf have a: "f ⊗ foldr (op ⊗) fs' 1 ∼ a" by (simp add: wfactors_def)

  note aunit
  also from fs wf
  have a: "f ⊗ foldr (op ⊗) fs' 1 ∼ a" by (simp add: wfactors_def)
  have "a ∼ f ⊗ foldr (op ⊗) fs' 1"
    by (simp add: Units_closed[OF aunit] a[symmetric])
  finally have "f ⊗ foldr (op ⊗) fs' 1 ∈ Units G" by simp
  then have "f ∈ Units G" by (intro unit_factor[of f], simp+)
  with fnunit show ?thesis by contradiction
qed
```

Comparing wfactors

```
lemma (in comm_monoid_cancel) wfactors_listassoc_cong_l:
  assumes fact: "wfactors G fs a"
```

```
    and asc: "fs [∼] fs'"
    and carr: "a ∈ carrier G"  "set fs ⊆ carrier G"  "set fs' ⊆ carrier
G"
  shows "wfactors G fs' a"
  using fact
  apply (elim wfactorsE, intro wfactorsI)
   apply (metis assms(2) assms(4) assms(5) irrlist_listassoc_cong)
proof -
  from asc[symmetric] have "foldr op ⊗ fs' 1 ∼ foldr op ⊗ fs 1"
    by (simp add: multlist_listassoc_cong carr)
  also assume "foldr op ⊗ fs 1 ∼ a"
  finally show "foldr op ⊗ fs' 1 ∼ a" by (simp add: carr)
qed

lemma (in comm_monoid) wfactors_perm_cong_l:
  assumes "wfactors G fs a"
    and "fs <~~> fs'"
    and "set fs ⊆ carrier G"
  shows "wfactors G fs' a"
  using assms
  apply (elim wfactorsE, intro wfactorsI)
   apply (rule irrlist_perm_cong, assumption+)
  apply (simp add: multlist_perm_cong[symmetric])
  done

lemma (in comm_monoid_cancel) wfactors_ee_cong_l [trans]:
  assumes ee: "essentially_equal G as bs"
    and bfs: "wfactors G bs b"
    and carr: "b ∈ carrier G"  "set as ⊆ carrier G"  "set bs ⊆ carrier
G"
  shows "wfactors G as b"
  using ee
proof (elim essentially_equalE)
  fix fs
  assume prm: "as <~~> fs"
  with carr have fscarr: "set fs ⊆ carrier G" by (simp add: perm_closed)

  note bfs
  also assume [symmetric]: "fs [∼] bs"
  also (wfactors_listassoc_cong_l)
  note prm[symmetric]
  finally (wfactors_perm_cong_l)
  show "wfactors G as b" by (simp add: carr fscarr)
qed

lemma (in monoid) wfactors_cong_r [trans]:
  assumes fac: "wfactors G fs a" and aa': "a ∼ a'"
    and carr[simp]: "a ∈ carrier G"  "a' ∈ carrier G"  "set fs ⊆ carrier
G"
```

```
  shows "wfactors G fs a'"
  using fac
proof (elim wfactorsE, intro wfactorsI)
  assume "foldr op ⊗ fs 1 ∼ a" also note aa'
  finally show "foldr op ⊗ fs 1 ∼ a'" by simp
qed
```

23.5.5 Essentially equal factorizations

```
lemma (in comm_monoid_cancel) unitfactor_ee:
  assumes uunit: "u ∈ Units G"
    and carr: "set as ⊆ carrier G"
  shows "essentially_equal G (as[0 := (as!0 ⊗ u)]) as"
    (is "essentially_equal G ?as' as")
  using assms
  apply (intro essentially_equalI[of _ ?as'], simp)
  apply (cases as, simp)
  apply (clarsimp, fast intro: associatedI2[of u])
  done

lemma (in comm_monoid_cancel) factors_cong_unit:
  assumes uunit: "u ∈ Units G"
    and anunit: "a ∉ Units G"
    and afs: "factors G as a"
    and ascarr: "set as ⊆ carrier G"
  shows "factors G (as[0 := (as!0 ⊗ u)]) (a ⊗ u)"
    (is "factors G ?as' ?a'")
  using assms
  apply (elim factorsE, clarify)
  apply (cases as)
   apply (simp add: nunit_factors)
  apply clarsimp
  apply (elim factorsE, intro factorsI)
   apply (clarsimp, fast intro: irreducible_prod_rI)
  apply (simp add: m_ac Units_closed)
  done

lemma (in comm_monoid) perm_wfactorsD:
  assumes prm: "as <~~> bs"
    and afs: "wfactors G as a"
    and bfs: "wfactors G bs b"
    and [simp]: "a ∈ carrier G"  "b ∈ carrier G"
    and ascarr [simp]: "set as ⊆ carrier G"
  shows "a ∼ b"
  using afs bfs
proof (elim wfactorsE)
  from prm have [simp]: "set bs ⊆ carrier G" by (simp add: perm_closed)
  assume "foldr op ⊗ as 1 ∼ a"
  then have "a ∼ foldr op ⊗ as 1" by (rule associated_sym, simp+)
```

```
  also from prm
  have "foldr op ⊗ as 1 = foldr op ⊗ bs 1" by (rule multlist_perm_cong,
simp)
  also assume "foldr op ⊗ bs 1 ∼ b"
  finally show "a ∼ b" by simp
qed

lemma (in comm_monoid_cancel) listassoc_wfactorsD:
  assumes assoc: "as [∼] bs"
    and afs: "wfactors G as a"
    and bfs: "wfactors G bs b"
    and [simp]: "a ∈ carrier G"  "b ∈ carrier G"
    and [simp]: "set as ⊆ carrier G"  "set bs ⊆ carrier G"
  shows "a ∼ b"
  using afs bfs
proof (elim wfactorsE)
  assume "foldr op ⊗ as 1 ∼ a"
  then have "a ∼ foldr op ⊗ as 1" by (rule associated_sym, simp+)
  also from assoc
  have "foldr op ⊗ as 1 ∼ foldr op ⊗ bs 1" by (rule multlist_listassoc_cong,
simp+)
  also assume "foldr op ⊗ bs 1 ∼ b"
  finally show "a ∼ b" by simp
qed

lemma (in comm_monoid_cancel) ee_wfactorsD:
  assumes ee: "essentially_equal G as bs"
    and afs: "wfactors G as a" and bfs: "wfactors G bs b"
    and [simp]: "a ∈ carrier G"  "b ∈ carrier G"
    and ascarr[simp]: "set as ⊆ carrier G" and bscarr[simp]: "set bs
⊆ carrier G"
  shows "a ∼ b"
  using ee
proof (elim essentially_equalE)
  fix fs
  assume prm: "as <~~> fs"
  then have as'carr[simp]: "set fs ⊆ carrier G"
    by (simp add: perm_closed)
  from afs prm have afs': "wfactors G fs a"
    by (rule wfactors_perm_cong_l) simp
  assume "fs [∼] bs"
  from this afs' bfs show "a ∼ b"
    by (rule listassoc_wfactorsD) simp_all
qed

lemma (in comm_monoid_cancel) ee_factorsD:
  assumes ee: "essentially_equal G as bs"
    and afs: "factors G as a" and bfs:"factors G bs b"
    and "set as ⊆ carrier G"  "set bs ⊆ carrier G"
```

```
  shows "a ∼ b"
  using assms by (blast intro: factors_wfactors dest: ee_wfactorsD)

lemma (in factorial_monoid) ee_factorsI:
  assumes ab: "a ∼ b"
    and afs: "factors G as a" and anunit: "a ∉ Units G"
    and bfs: "factors G bs b" and bnunit: "b ∉ Units G"
    and ascarr: "set as ⊆ carrier G" and bscarr: "set bs ⊆ carrier G"
  shows "essentially_equal G as bs"
proof -
  note carr[simp] = factors_closed[OF afs ascarr] ascarr[THEN subsetD]
    factors_closed[OF bfs bscarr] bscarr[THEN subsetD]

  from ab carr obtain u where uunit: "u ∈ Units G" and a: "a = b ⊗ u"
    by (elim associatedE2)

  from uunit bscarr have ee: "essentially_equal G (bs[0 := (bs!0 ⊗ u)])
bs"
    (is "essentially_equal G ?bs' bs")
    by (rule unitfactor_ee)

  from bscarr uunit have bs'carr: "set ?bs' ⊆ carrier G"
    by (cases bs) (simp_all add: Units_closed)

  from uunit bnunit bfs bscarr have fac: "factors G ?bs' (b ⊗ u)"
    by (rule factors_cong_unit)

  from afs fac[simplified a[symmetric]] ascarr bs'carr anunit
  have "essentially_equal G as ?bs'"
    by (blast intro: factors_unique)
  also note ee
  finally show "essentially_equal G as bs"
    by (simp add: ascarr bscarr bs'carr)
qed

lemma (in factorial_monoid) ee_wfactorsI:
  assumes asc: "a ∼ b"
    and asf: "wfactors G as a" and bsf: "wfactors G bs b"
    and acarr[simp]: "a ∈ carrier G" and bcarr[simp]: "b ∈ carrier G"
    and ascarr[simp]: "set as ⊆ carrier G" and bscarr[simp]: "set bs
⊆ carrier G"
  shows "essentially_equal G as bs"
  using assms
proof (cases "a ∈ Units G")
  case aunit: True
  also note asc
  finally have bunit: "b ∈ Units G" by simp

  from aunit asf ascarr have e: "as = []"
```

```
    by (rule unit_wfactors_empty)
  from bunit bsf bscarr have e': "bs = []"
    by (rule unit_wfactors_empty)

  have "essentially_equal G [] []"
    by (fast intro: essentially_equalI)
  then show ?thesis
    by (simp add: e e')
next
  case anunit: False
  have bnunit: "b ∉ Units G"
  proof clarify
    assume "b ∈ Units G"
    also note asc[symmetric]
    finally have "a ∈ Units G" by simp
    with anunit show False ..
  qed

  from wfactors_factors[OF asf ascarr] obtain a' where fa': "factors
G as a'" and a': "a' ∼ a"
    by blast
  from fa' ascarr have a'carr[simp]: "a' ∈ carrier G"
    by fast

  have a'nunit: "a' ∉ Units G"
  proof clarify
    assume "a' ∈ Units G"
    also note a'
    finally have "a ∈ Units G" by simp
    with anunit
    show "False" ..
  qed

  from wfactors_factors[OF bsf bscarr] obtain b' where fb': "factors
G bs b'" and b': "b' ∼ b"
    by blast
  from fb' bscarr have b'carr[simp]: "b' ∈ carrier G"
    by fast

  have b'nunit: "b' ∉ Units G"
  proof clarify
    assume "b' ∈ Units G"
    also note b'
    finally have "b ∈ Units G" by simp
    with bnunit show False ..
  qed

  note a'
  also note asc
```

```
  also note b'[symmetric]
  finally have "a' ∼ b'" by simp
  from this fa' a'nunit fb' b'nunit ascarr bscarr show "essentially_equal
G as bs"
    by (rule ee_factorsI)
qed

lemma (in factorial_monoid) ee_wfactors:
  assumes asf: "wfactors G as a"
    and bsf: "wfactors G bs b"
    and acarr: "a ∈ carrier G" and bcarr: "b ∈ carrier G"
    and ascarr: "set as ⊆ carrier G" and bscarr: "set bs ⊆ carrier G"
  shows asc: "a ∼ b = essentially_equal G as bs"
  using assms by (fast intro: ee_wfactorsI ee_wfactorsD)

lemma (in factorial_monoid) wfactors_exist [intro, simp]:
  assumes acarr[simp]: "a ∈ carrier G"
  shows "∃fs. set fs ⊆ carrier G ∧ wfactors G fs a"
proof (cases "a ∈ Units G")
  case True
  then have "wfactors G [] a" by (rule unit_wfactors)
  then show ?thesis by (intro exI) force
next
  case False
  with factors_exist [OF acarr] obtain fs where fscarr: "set fs ⊆ carrier
G" and f: "factors G fs a"
    by blast
  from f have "wfactors G fs a" by (rule factors_wfactors) fact
  with fscarr show ?thesis by fast
qed

lemma (in monoid) wfactors_prod_exists [intro, simp]:
  assumes "∀a ∈ set as. irreducible G a" and "set as ⊆ carrier G"
  shows "∃a. a ∈ carrier G ∧ wfactors G as a"
  unfolding wfactors_def using assms by blast

lemma (in factorial_monoid) wfactors_unique:
  assumes "wfactors G fs a"
    and "wfactors G fs' a"
    and "a ∈ carrier G"
    and "set fs ⊆ carrier G"
    and "set fs' ⊆ carrier G"
  shows "essentially_equal G fs fs'"
  using assms by (fast intro: ee_wfactorsI[of a a])

lemma (in monoid) factors_mult_single:
  assumes "irreducible G a" and "factors G fb b" and "a ∈ carrier G"
  shows "factors G (a # fb) (a ⊗ b)"
  using assms unfolding factors_def by simp
```

```
lemma (in monoid_cancel) wfactors_mult_single:
  assumes f: "irreducible G a"  "wfactors G fb b"
    "a ∈ carrier G"  "b ∈ carrier G"  "set fb ⊆ carrier G"
  shows "wfactors G (a # fb) (a ⊗ b)"
  using assms unfolding wfactors_def by (simp add: mult_cong_r)

lemma (in monoid) factors_mult:
  assumes factors: "factors G fa a"  "factors G fb b"
    and ascarr: "set fa ⊆ carrier G"
    and bscarr: "set fb ⊆ carrier G"
  shows "factors G (fa @ fb) (a ⊗ b)"
  using assms
  unfolding factors_def
  apply safe
   apply force
  apply hypsubst_thin
  apply (induct fa)
   apply simp
  apply (simp add: m_assoc)
  done

lemma (in comm_monoid_cancel) wfactors_mult [intro]:
  assumes asf: "wfactors G as a" and bsf:"wfactors G bs b"
    and acarr: "a ∈ carrier G" and bcarr: "b ∈ carrier G"
    and ascarr: "set as ⊆ carrier G" and bscarr:"set bs ⊆ carrier G"
  shows "wfactors G (as @ bs) (a ⊗ b)"
  using wfactors_factors[OF asf ascarr] and wfactors_factors[OF bsf bscarr]
proof clarsimp
  fix a' b'
  assume asf': "factors G as a'" and a'a: "a' ∼ a"
    and bsf': "factors G bs b'" and b'b: "b' ∼ b"
  from asf' have a'carr: "a' ∈ carrier G" by (rule factors_closed) fact
  from bsf' have b'carr: "b' ∈ carrier G" by (rule factors_closed) fact

  note carr = acarr bcarr a'carr b'carr ascarr bscarr

  from asf' bsf' have "factors G (as @ bs) (a' ⊗ b')"
    by (rule factors_mult) fact+

  with carr have abf': "wfactors G (as @ bs) (a' ⊗ b')"
    by (intro factors_wfactors) simp_all
  also from b'b carr have trb: "a' ⊗ b' ∼ a' ⊗ b"
    by (intro mult_cong_r)
  also from a'a carr have tra: "a' ⊗ b ∼ a ⊗ b"
    by (intro mult_cong_l)
  finally show "wfactors G (as @ bs) (a ⊗ b)"
    by (simp add: carr)
qed
```

```
lemma (in comm_monoid) factors_dividesI:
  assumes "factors G fs a"
    and "f ∈ set fs"
    and "set fs ⊆ carrier G"
  shows "f divides a"
  using assms by (fast elim: factorsE intro: multlist_dividesI)

lemma (in comm_monoid) wfactors_dividesI:
  assumes p: "wfactors G fs a"
    and fscarr: "set fs ⊆ carrier G" and acarr: "a ∈ carrier G"
    and f: "f ∈ set fs"
  shows "f divides a"
  using wfactors_factors[OF p fscarr]
proof clarsimp
  fix a’
  assume fsa’: "factors G fs a’" and a’a: "a’ ∼ a"
  with fscarr have a’carr: "a’ ∈ carrier G"
    by (simp add: factors_closed)

  from fsa’ fscarr f have "f divides a’"
    by (fast intro: factors_dividesI)
  also note a’a
  finally show "f divides a"
    by (simp add: f fscarr[THEN subsetD] acarr a’carr)
qed
```

23.5.6 Factorial monoids and wfactors

```
lemma (in comm_monoid_cancel) factorial_monoidI:
  assumes wfactors_exists: "⋀a. a ∈ carrier G ⟹ ∃fs. set fs ⊆ carrier
G ∧ wfactors G fs a"
    and wfactors_unique:
      "⋀a fs fs’. ⟦a ∈ carrier G; set fs ⊆ carrier G; set fs’ ⊆ carrier
G;
        wfactors G fs a; wfactors G fs’ a⟧ ⟹ essentially_equal G fs
fs’"
  shows "factorial_monoid G"
proof
  fix a
  assume acarr: "a ∈ carrier G" and anunit: "a ∉ Units G"

  from wfactors_exists[OF acarr]
  obtain as where ascarr: "set as ⊆ carrier G" and afs: "wfactors G
as a"
    by blast
  from wfactors_factors [OF afs ascarr] obtain a’ where afs’: "factors
G as a’" and a’a: "a’ ∼ a"
    by blast
```

```
  from afs' ascarr have a'carr: "a' ∈ carrier G"
    by fast
  have a'nunit: "a' ∉ Units G"
  proof clarify
    assume "a' ∈ Units G"
    also note a'a
    finally have "a ∈ Units G" by (simp add: acarr)
    with anunit show False ..
  qed

  from a'carr acarr a'a obtain u where uunit: "u ∈ Units G" and a':
"a' = a ⊗ u"
    by (blast elim: associatedE2)

  note [simp] = acarr Units_closed[OF uunit] Units_inv_closed[OF uunit]

  have "a = a ⊗ 1" by simp
  also have "... = a ⊗ (u ⊗ inv u)" by (simp add: uunit)
  also have "... = a' ⊗ inv u" by (simp add: m_assoc[symmetric] a'[symmetric])
  finally have a: "a = a' ⊗ inv u" .

  from ascarr uunit have cr: "set (as[0:=(as!0 ⊗ inv u)]) ⊆ carrier
G"
    by (cases as) auto

  from afs' uunit a'nunit acarr ascarr have "factors G (as[0:=(as!0 ⊗
inv u)]) a"
    by (simp add: a factors_cong_unit)
  with cr show "∃fs. set fs ⊆ carrier G ∧ factors G fs a"
    by fast
qed (blast intro: factors_wfactors wfactors_unique)
```

23.6 Factorizations as Multisets

Gives useful operations like intersection

```
abbreviation "assocs G x ≡ eq_closure_of (division_rel G) {x}"

definition "fmset G as = mset (map (λa. assocs G a) as)"
```

Helper lemmas

```
lemma (in monoid) assocs_repr_independence:
  assumes "y ∈ assocs G x"
    and "x ∈ carrier G"
  shows "assocs G x = assocs G y"
  using assms
  apply safe
   apply (elim closure_ofE2, intro closure_ofI2[of _ _ y])
     apply (clarsimp, iprover intro: associated_trans associated_sym,
simp+)
```

```
  apply (elim closure_ofE2, intro closure_ofI2[of _ _ x])
    apply (clarsimp, iprover intro: associated_trans, simp+)
  done

lemma (in monoid) assocs_self:
  assumes "x ∈ carrier G"
  shows "x ∈ assocs G x"
  using assms by (fastforce intro: closure_ofI2)

lemma (in monoid) assocs_repr_independenceD:
  assumes repr: "assocs G x = assocs G y"
    and ycarr: "y ∈ carrier G"
  shows "y ∈ assocs G x"
  unfolding repr using ycarr by (intro assocs_self)

lemma (in comm_monoid) assocs_assoc:
  assumes "a ∈ assocs G b"
    and "b ∈ carrier G"
  shows "a ∼ b"
  using assms by (elim closure_ofE2) simp

lemmas (in comm_monoid) assocs_eqD = assocs_repr_independenceD[THEN assocs_assoc]
```

23.6.1 Comparing multisets

```
lemma (in monoid) fmset_perm_cong:
  assumes prm: "as <~~> bs"
  shows "fmset G as = fmset G bs"
  using perm_map[OF prm] unfolding mset_eq_perm fmset_def by blast

lemma (in comm_monoid_cancel) eqc_listassoc_cong:
  assumes "as [∼] bs"
    and "set as ⊆ carrier G" and "set bs ⊆ carrier G"
  shows "map (assocs G) as = map (assocs G) bs"
  using assms
  apply (induct as arbitrary: bs, simp)
  apply (clarsimp simp add: Cons_eq_map_conv list_all2_Cons1, safe)
   apply (clarsimp elim!: closure_ofE2) defer 1
   apply (clarsimp elim!: closure_ofE2) defer 1
proof -
  fix a x z
  assume carr[simp]: "a ∈ carrier G"  "x ∈ carrier G"  "z ∈ carrier
G"
  assume "x ∼ a"
  also assume "a ∼ z"
  finally have "x ∼ z" by simp
  with carr show "x ∈ assocs G z"
    by (intro closure_ofI2) simp_all
next
```

```
  fix a x z
  assume carr[simp]: "a ∈ carrier G"  "x ∈ carrier G"  "z ∈ carrier
G"
  assume "x ∼ z"
  also assume [symmetric]: "a ∼ z"
  finally have "x ∼ a" by simp
  with carr show "x ∈ assocs G a"
    by (intro closure_ofI2) simp_all
qed

lemma (in comm_monoid_cancel) fmset_listassoc_cong:
  assumes "as [∼] bs"
    and "set as ⊆ carrier G" and "set bs ⊆ carrier G"
  shows "fmset G as = fmset G bs"
  using assms unfolding fmset_def by (simp add: eqc_listassoc_cong)

lemma (in comm_monoid_cancel) ee_fmset:
  assumes ee: "essentially_equal G as bs"
    and ascarr: "set as ⊆ carrier G" and bscarr: "set bs ⊆ carrier G"
  shows "fmset G as = fmset G bs"
  using ee
proof (elim essentially_equalE)
  fix as'
  assume prm: "as <~~> as'"
  from prm ascarr have as'carr: "set as' ⊆ carrier G"
    by (rule perm_closed)

  from prm have "fmset G as = fmset G as'"
    by (rule fmset_perm_cong)
  also assume "as' [∼] bs"
  with as'carr bscarr have "fmset G as' = fmset G bs"
    by (simp add: fmset_listassoc_cong)
  finally show "fmset G as = fmset G bs" .
qed

lemma (in monoid_cancel) fmset_ee__hlp_induct:
  assumes prm: "cas <~~> cbs"
    and cdef: "cas = map (assocs G) as"  "cbs = map (assocs G) bs"
  shows "∀as bs. (cas <~~> cbs ∧ cas = map (assocs G) as ∧
    cbs = map (assocs G) bs) ⟶ (∃as'. as <~~> as' ∧ map (assocs G)
as' = cbs)"
  apply (rule perm.induct[of cas cbs], rule prm)
     apply safe
     apply (simp_all del: mset_map)
    apply (simp add: map_eq_Cons_conv)
    apply blast
   apply force
proof -
  fix ys as bs
```

```
  assume p1: "map (assocs G) as <~~> ys"
    and r1[rule_format]:
      "∀asa bs. map (assocs G) as = map (assocs G) asa ∧ ys = map (assocs
G) bs
        ⟶ (∃as'. asa <~~> as' ∧ map (assocs G) as' = map (assocs G)
bs)"
    and p2: "ys <~~> map (assocs G) bs"
    and r2[rule_format]: "∀as bsa. ys = map (assocs G) as ∧ map (assocs
G) bs = map (assocs G) bsa
      ⟶ (∃as'. as <~~> as' ∧ map (assocs G) as' = map (assocs G) bsa)"
    and p3: "map (assocs G) as <~~> map (assocs G) bs"

  from p1 have "mset (map (assocs G) as) = mset ys"
    by (simp add: mset_eq_perm del: mset_map)
  then have setys: "set (map (assocs G) as) = set ys"
    by (rule mset_eq_setD)

  have "set (map (assocs G) as) = {assocs G x | x. x ∈ set as}" by auto
  with setys have "set ys ⊆ { assocs G x | x. x ∈ set as}" by simp
  then have "∃yy. ys = map (assocs G) yy"
  proof (induct ys)
    case Nil
    then show ?case by simp
  next
    case Cons
    then show ?case
    proof clarsimp
      fix yy x
      show "∃yya. assocs G x # map (assocs G) yy = map (assocs G) yya"
        by (rule exI[of _ "x#yy"]) simp
    qed
  qed
  then obtain yy where ys: "ys = map (assocs G) yy" ..

  from p1 ys have "∃as'. as <~~> as' ∧ map (assocs G) as' = map (assocs
G) yy"
    by (intro r1) simp
  then obtain as' where asas': "as <~~> as'" and as'yy: "map (assocs
G) as' = map (assocs G) yy"
    by auto

  from p2 ys have "∃as'. yy <~~> as' ∧ map (assocs G) as' = map (assocs
G) bs"
    by (intro r2) simp
  then obtain as'' where yyas'': "yy <~~> as''" and as''bs: "map (assocs
G) as'' = map (assocs G) bs"
    by auto

  from perm_map_switch [OF as'yy yyas'']
```

```
  obtain cs where as'cs: "as' <~~> cs" and csas'': "map (assocs G) cs
= map (assocs G) as''"
    by blast

  from asas' and as'cs have ascs: "as <~~> cs"
    by fast
  from csas'' and as''bs have "map (assocs G) cs = map (assocs G) bs"
    by simp
  with ascs show "∃as'. as <~~> as' ∧ map (assocs G) as' = map (assocs
G) bs"
    by fast
qed

lemma (in comm_monoid_cancel) fmset_ee:
  assumes mset: "fmset G as = fmset G bs"
    and ascarr: "set as ⊆ carrier G" and bscarr: "set bs ⊆ carrier G"
  shows "essentially_equal G as bs"
proof -
  from mset have mpp: "map (assocs G) as <~~> map (assocs G) bs"
    by (simp add: fmset_def mset_eq_perm del: mset_map)

  define cas where "cas = map (assocs G) as"
  define cbs where "cbs = map (assocs G) bs"

  from cas_def cbs_def mpp have [rule_format]:
    "∀as bs. (cas <~~> cbs ∧ cas = map (assocs G) as ∧ cbs = map (assocs
G) bs)
      ⟶ (∃as'. as <~~> as' ∧ map (assocs G) as' = cbs)"
    by (intro fmset_ee__hlp_induct, simp+)
  with mpp cas_def cbs_def have "∃as'. as <~~> as' ∧ map (assocs G) as'
= map (assocs G) bs"
    by simp

  then obtain as' where tp: "as <~~> as'" and tm: "map (assocs G) as'
= map (assocs G) bs"
    by auto
  from tm have lene: "length as' = length bs"
    by (rule map_eq_imp_length_eq)
  from tp have "set as = set as'"
    by (simp add: mset_eq_perm mset_eq_setD)
  with ascarr have as'carr: "set as' ⊆ carrier G"
    by simp

  from tm as'carr[THEN subsetD] bscarr[THEN subsetD] have "as' [∼] bs"
    by (induct as' arbitrary: bs) (simp, fastforce dest: assocs_eqD[THEN
associated_sym])
  with tp show "essentially_equal G as bs"
    by (fast intro: essentially_equalI)
qed
```

```
lemma (in comm_monoid_cancel) ee_is_fmset:
  assumes "set as ⊆ carrier G" and "set bs ⊆ carrier G"
  shows "essentially_equal G as bs = (fmset G as = fmset G bs)"
  using assms by (fast intro: ee_fmset fmset_ee)
```

23.6.2 Interpreting multisets as factorizations

```
lemma (in monoid) mset_fmsetEx:
  assumes elems: "⋀X. X ∈ set_mset Cs ⟹ ∃x. P x ∧ X = assocs G x"
  shows "∃cs. (∀c ∈ set cs. P c) ∧ fmset G cs = Cs"
proof -
  from surjE[OF surj_mset] obtain Cs' where Cs: "Cs = mset Cs'"
    by blast
  have "∃cs. (∀c ∈ set cs. P c) ∧ mset (map (assocs G) cs) = Cs"
    using elems
    unfolding Cs
    apply (induct Cs', simp)
  proof (clarsimp simp del: mset_map)
    fix a Cs' cs
    assume ih: "⋀X. X = a ∨ X ∈ set Cs' ⟹ ∃x. P x ∧ X = assocs G
x"
      and csP: "∀x∈set cs. P x"
      and mset: "mset (map (assocs G) cs) = mset Cs'"
    from ih obtain c where cP: "P c" and a: "a = assocs G c"
      by auto
    from cP csP have tP: "∀x∈set (c#cs). P x"
      by simp
    from mset a have "mset (map (assocs G) (c#cs)) = add_mset a (mset
Cs')"
      by simp
    with tP show "∃cs. (∀x∈set cs. P x) ∧ mset (map (assocs G) cs) =
add_mset a (mset Cs')"
      by fast
  qed
  then show ?thesis by (simp add: fmset_def)
qed

lemma (in monoid) mset_wfactorsEx:
  assumes elems: "⋀X. X ∈ set_mset Cs ⟹ ∃x. (x ∈ carrier G ∧ irreducible
G x) ∧ X = assocs G x"
  shows "∃c cs. c ∈ carrier G ∧ set cs ⊆ carrier G ∧ wfactors G cs c
∧ fmset G cs = Cs"
proof -
  have "∃cs. (∀c∈set cs. c ∈ carrier G ∧ irreducible G c) ∧ fmset G
cs = Cs"
    by (intro mset_fmsetEx, rule elems)
  then obtain cs where p[rule_format]: "∀c∈set cs. c ∈ carrier G ∧ irreducible
G c"
```

```
    and Cs[symmetric]: "fmset G cs = Cs" by auto
  from p have cscarr: "set cs ⊆ carrier G" by fast
  from p have "∃c. c ∈ carrier G ∧ wfactors G cs c"
    by (intro wfactors_prod_exists) auto
  then obtain c where ccarr: "c ∈ carrier G" and cfs: "wfactors G cs
c" by auto
  with cscarr Cs show ?thesis by fast
qed
```

23.6.3 Multiplication on multisets

```
lemma (in factorial_monoid) mult_wfactors_fmset:
  assumes afs: "wfactors G as a"
    and bfs: "wfactors G bs b"
    and cfs: "wfactors G cs (a ⊗ b)"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"
              "set as ⊆ carrier G"  "set bs ⊆ carrier G"  "set cs ⊆ carrier
G"
  shows "fmset G cs = fmset G as + fmset G bs"
proof -
  from assms have "wfactors G (as @ bs) (a ⊗ b)"
    by (intro wfactors_mult)
  with carr cfs have "essentially_equal G cs (as@bs)"
    by (intro ee_wfactorsI[of "a⊗b" "a⊗b"]) simp_all
  with carr have "fmset G cs = fmset G (as@bs)"
    by (intro ee_fmset) simp_all
  also have "fmset G (as@bs) = fmset G as + fmset G bs"
    by (simp add: fmset_def)
  finally show "fmset G cs = fmset G as + fmset G bs" .
qed

lemma (in factorial_monoid) mult_factors_fmset:
  assumes afs: "factors G as a"
    and bfs: "factors G bs b"
    and cfs: "factors G cs (a ⊗ b)"
    and "set as ⊆ carrier G"  "set bs ⊆ carrier G"  "set cs ⊆ carrier
G"
  shows "fmset G cs = fmset G as + fmset G bs"
  using assms by (blast intro: factors_wfactors mult_wfactors_fmset)

lemma (in comm_monoid_cancel) fmset_wfactors_mult:
  assumes mset: "fmset G cs = fmset G as + fmset G bs"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
      "set as ⊆ carrier G"  "set bs ⊆ carrier G"  "set cs ⊆ carrier G"
    and fs: "wfactors G as a"  "wfactors G bs b"  "wfactors G cs c"
  shows "c ∼ a ⊗ b"
proof -
  from carr fs have m: "wfactors G (as @ bs) (a ⊗ b)"
    by (intro wfactors_mult)
```

```
  from mset have "fmset G cs = fmset G (as@bs)"
    by (simp add: fmset_def)
  then have "essentially_equal G cs (as@bs)"
    by (rule fmset_ee) (simp_all add: carr)
  then show "c ∼ a ⊗ b"
    by (rule ee_wfactorsD[of "cs" "as@bs"]) (simp_all add: assms m)
qed
```

23.6.4 Divisibility on multisets

```
lemma (in factorial_monoid) divides_fmsubset:
  assumes ab: "a divides b"
    and afs: "wfactors G as a"
    and bfs: "wfactors G bs b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "set as ⊆ carrier G"
"set bs ⊆ carrier G"
  shows "fmset G as ⊆# fmset G bs"
  using ab
proof (elim dividesE)
  fix c
  assume ccarr: "c ∈ carrier G"
  from wfactors_exist [OF this]
  obtain cs where cscarr: "set cs ⊆ carrier G" and cfs: "wfactors G
cs c"
    by blast
  note carr = carr ccarr cscarr

  assume "b = a ⊗ c"
  with afs bfs cfs carr have "fmset G bs = fmset G as + fmset G cs"
    by (intro mult_wfactors_fmset[OF afs cfs]) simp_all
  then show ?thesis by simp
qed

lemma (in comm_monoid_cancel) fmsubset_divides:
  assumes msubset: "fmset G as ⊆# fmset G bs"
    and afs: "wfactors G as a"
    and bfs: "wfactors G bs b"
    and acarr: "a ∈ carrier G"
    and bcarr: "b ∈ carrier G"
    and ascarr: "set as ⊆ carrier G"
    and bscarr: "set bs ⊆ carrier G"
  shows "a divides b"
proof -
  from afs have airr: "∀a ∈ set as. irreducible G a" by (fast elim:
wfactorsE)
  from bfs have birr: "∀b ∈ set bs. irreducible G b" by (fast elim:
wfactorsE)
```

```
  have "∃c cs. c ∈ carrier G ∧ set cs ⊆ carrier G ∧ wfactors G cs c
∧ fmset G cs = fmset G bs - fmset G as"
  proof (intro mset_wfactorsEx, simp)
    fix X
    assume "X ∈# fmset G bs - fmset G as"
    then have "X ∈# fmset G bs" by (rule in_diffD)
    then have "X ∈ set (map (assocs G) bs)" by (simp add: fmset_def)
    then have "∃x. x ∈ set bs ∧ X = assocs G x" by (induct bs) auto
    then obtain x where xbs: "x ∈ set bs" and X: "X = assocs G x" by
auto
    with bscarr have xcarr: "x ∈ carrier G" by fast
    from xbs birr have xirr: "irreducible G x" by simp

    from xcarr and xirr and X show "∃x. x ∈ carrier G ∧ irreducible
G x ∧ X = assocs G x"
      by fast
  qed
  then obtain c cs
    where ccarr: "c ∈ carrier G"
      and cscarr: "set cs ⊆ carrier G"
      and csf: "wfactors G cs c"
      and csmset: "fmset G cs = fmset G bs - fmset G as" by auto

  from csmset msubset
  have "fmset G bs = fmset G as + fmset G cs"
    by (simp add: multiset_eq_iff subseteq_mset_def)
  then have basc: "b ∼ a ⊗ c"
    by (rule fmset_wfactors_mult) fact+
  then show ?thesis
  proof (elim associatedE2)
    fix u
    assume "u ∈ Units G"  "b = a ⊗ c ⊗ u"
    with acarr ccarr show "a divides b"
      by (fast intro: dividesI[of "c ⊗ u"] m_assoc)
  qed (simp_all add: acarr bcarr ccarr)
qed

lemma (in factorial_monoid) divides_as_fmsubset:
  assumes "wfactors G as a"
    and "wfactors G bs b"
    and "a ∈ carrier G"
    and "b ∈ carrier G"
    and "set as ⊆ carrier G"
    and "set bs ⊆ carrier G"
  shows "a divides b = (fmset G as ⊆# fmset G bs)"
  using assms
  by (blast intro: divides_fmsubset fmsubset_divides)
```

Proper factors on multisets

```
lemma (in factorial_monoid) fmset_properfactor:
  assumes asubb: "fmset G as ⊆# fmset G bs"
    and anb: "fmset G as ≠ fmset G bs"
    and "wfactors G as a"
    and "wfactors G bs b"
    and "a ∈ carrier G"
    and "b ∈ carrier G"
    and "set as ⊆ carrier G"
    and "set bs ⊆ carrier G"
  shows "properfactor G a b"
  apply (rule properfactorI)
   apply (rule fmsubset_divides[of as bs], fact+)
proof
  assume "b divides a"
  then have "fmset G bs ⊆# fmset G as"
    by (rule divides_fmsubset) fact+
  with asubb have "fmset G as = fmset G bs"
    by (rule subset_mset.antisym)
  with anb show False ..
qed

lemma (in factorial_monoid) properfactor_fmset:
  assumes pf: "properfactor G a b"
    and "wfactors G as a"
    and "wfactors G bs b"
    and "a ∈ carrier G"
    and "b ∈ carrier G"
    and "set as ⊆ carrier G"
    and "set bs ⊆ carrier G"
  shows "fmset G as ⊆# fmset G bs ∧ fmset G as ≠ fmset G bs"
  using pf
  apply (elim properfactorE)
  apply rule
   apply (intro divides_fmsubset, assumption)
        apply (rule assms)+
  using assms(2,3,4,6,7) divides_as_fmsubset
  apply auto
  done
```

23.7 Irreducible Elements are Prime

```
lemma (in factorial_monoid) irreducible_prime:
  assumes pirr: "irreducible G p"
    and pcarr: "p ∈ carrier G"
  shows "prime G p"
  using pirr
proof (elim irreducibleE, intro primeI)
  fix a b
  assume acarr: "a ∈ carrier G"  and bcarr: "b ∈ carrier G"
```

```
    and pdvdab: "p divides (a ⊗ b)"
    and pnunit: "p ∉ Units G"
  assume irreduc[rule_format]:
    "∀b. b ∈ carrier G ∧ properfactor G b p ⟶ b ∈ Units G"
  from pdvdab obtain c where ccarr: "c ∈ carrier G" and abpc: "a ⊗ b
= p ⊗ c"
    by (rule dividesE)

  from wfactors_exist [OF acarr]
  obtain as where ascarr: "set as ⊆ carrier G" and afs: "wfactors G
as a"
    by blast

  from wfactors_exist [OF bcarr]
  obtain bs where bscarr: "set bs ⊆ carrier G" and bfs: "wfactors G
bs b"
    by auto

  from wfactors_exist [OF ccarr]
  obtain cs where cscarr: "set cs ⊆ carrier G" and cfs: "wfactors G
cs c"
    by auto

  note carr[simp] = pcarr acarr bcarr ccarr ascarr bscarr cscarr

  from afs and bfs have abfs: "wfactors G (as @ bs) (a ⊗ b)"
    by (rule wfactors_mult) fact+

  from pirr cfs have pcfs: "wfactors G (p # cs) (p ⊗ c)"
    by (rule wfactors_mult_single) fact+
  with abpc have abfs’: "wfactors G (p # cs) (a ⊗ b)"
    by simp

  from abfs’ abfs have "essentially_equal G (p # cs) (as @ bs)"
    by (rule wfactors_unique) simp+

  then obtain ds where "p # cs <~~> ds" and dsassoc: "ds [∼] (as @ bs)"
    by (fast elim: essentially_equalE)
  then have "p ∈ set ds"
    by (simp add: perm_set_eq[symmetric])
  with dsassoc obtain p’ where "p’ ∈ set (as@bs)" and pp’: "p ∼ p’"
    unfolding list_all2_conv_all_nth set_conv_nth by force
  then consider "p’ ∈ set as" | "p’ ∈ set bs" by auto
  then show "p divides a ∨ p divides b"
  proof cases
    case 1
    with ascarr have [simp]: "p’ ∈ carrier G" by fast

    note pp’
```

```
    also from afs
    have "p’ divides a" by (rule wfactors_dividesI) fact+
    finally have "p divides a" by simp
    then show ?thesis ..
  next
    case 2
    with bscarr have [simp]: "p’ ∈ carrier G" by fast

    note pp’
    also from bfs
    have "p’ divides b" by (rule wfactors_dividesI) fact+
    finally have "p divides b" by simp
    then show ?thesis ..
  qed
qed
```

— A version using `factors`, more complicated

```
lemma (in factorial_monoid) factors_irreducible_prime:
  assumes pirr: "irreducible G p"
    and pcarr: "p ∈ carrier G"
  shows "prime G p"
  using pirr
  apply (elim irreducibleE, intro primeI)
   apply assumption
proof -
  fix a b
  assume acarr: "a ∈ carrier G"
    and bcarr: "b ∈ carrier G"
    and pdvdab: "p divides (a ⊗ b)"
  assume irreduc[rule_format]: "∀b. b ∈ carrier G ∧ properfactor G b
p ⟶ b ∈ Units G"
  from pdvdab obtain c where ccarr: "c ∈ carrier G" and abpc: "a ⊗ b
= p ⊗ c"
    by (rule dividesE)
  note [simp] = pcarr acarr bcarr ccarr

  show "p divides a ∨ p divides b"
  proof (cases "a ∈ Units G")
    case aunit: True

    note pdvdab
    also have "a ⊗ b = b ⊗ a" by (simp add: m_comm)
    also from aunit have bab: "b ⊗ a ∼ b"
      by (intro associatedI2[of "a"], simp+)
    finally have "p divides b" by simp
    then show ?thesis ..
  next
    case anunit: False
```

```
    show ?thesis
    proof (cases "b ∈ Units G")
      case bunit: True
      note pdvdab
      also from bunit
      have baa: "a ⊗ b ∼ a"
        by (intro associatedI2[of "b"], simp+)
      finally have "p divides a" by simp
      then show ?thesis ..
    next
      case bnunit: False
      have cnunit: "c ∉ Units G"
      proof
        assume cunit: "c ∈ Units G"
        from bnunit have "properfactor G a (a ⊗ b)"
          by (intro properfactorI3[of _ _ b], simp+)
        also note abpc
        also from cunit have "p ⊗ c ∼ p"
          by (intro associatedI2[of c], simp+)
        finally have "properfactor G a p" by simp
        with acarr have "a ∈ Units G" by (fast intro: irreduc)
        with anunit show False ..
      qed

      have abnunit: "a ⊗ b ∉ Units G"
      proof clarsimp
        assume "a ⊗ b ∈ Units G"
        then have "a ∈ Units G" by (rule unit_factor) fact+
        with anunit show False ..
      qed

      from factors_exist [OF acarr anunit]
      obtain as where ascarr: "set as ⊆ carrier G" and afac: "factors
G as a"
        by blast

      from factors_exist [OF bcarr bnunit]
      obtain bs where bscarr: "set bs ⊆ carrier G" and bfac: "factors
G bs b"
        by blast

      from factors_exist [OF ccarr cnunit]
      obtain cs where cscarr: "set cs ⊆ carrier G" and cfac: "factors
G cs c"
        by auto

      note [simp] = ascarr bscarr cscarr

      from afac and bfac have abfac: "factors G (as @ bs) (a ⊗ b)"
```

```
        by (rule factors_mult) fact+

      from pirr cfac have pcfac: "factors G (p # cs) (p ⊗ c)"
        by (rule factors_mult_single) fact+
      with abpc have abfac': "factors G (p # cs) (a ⊗ b)"
        by simp

      from abfac' abfac have "essentially_equal G (p # cs) (as @ bs)"
        by (rule factors_unique) (fact | simp)+
      then obtain ds where "p # cs <~~> ds" and dsassoc: "ds [∼] (as
@ bs)"
        by (fast elim: essentially_equalE)
      then have "p ∈ set ds"
        by (simp add: perm_set_eq[symmetric])
      with dsassoc obtain p' where "p' ∈ set (as@bs)" and pp': "p ∼
p'"
        unfolding list_all2_conv_all_nth set_conv_nth by force
      then consider "p' ∈ set as" | "p' ∈ set bs" by auto
      then show "p divides a ∨ p divides b"
      proof cases
        case 1
        with ascarr have [simp]: "p' ∈ carrier G" by fast

        note pp'
        also from afac 1 have "p' divides a" by (rule factors_dividesI)
fact+
        finally have "p divides a" by simp
        then show ?thesis ..
      next
        case 2
        with bscarr have [simp]: "p' ∈ carrier G" by fast

        note pp'
        also from bfac
        have "p' divides b" by (rule factors_dividesI) fact+
        finally have "p divides b" by simp
        then show ?thesis ..
      qed
    qed
  qed
qed
```

23.8 Greatest Common Divisors and Lowest Common Multiples

23.8.1 Definitions

```
definition isgcd :: "[('a,_) monoid_scheme, 'a, 'a, 'a] ⇒ bool"  ("(_
gcdofı _ _)" [81,81,81] 80)
  where "x gcdof_G a b ⟷ x divides_G a ∧ x divides_G b ∧
```

```
    (∀y∈carrier G. (y dividesG a ∧ y dividesG b ⟶ y dividesG x))"

definition islcm :: "[_, 'a, 'a, 'a] ⇒ bool"  ("(_ lcmofı _ _)" [81,81,81]
80)
  where "x lcmofG a b ⟷ a dividesG x ∧ b dividesG x ∧
    (∀y∈carrier G. (a dividesG y ∧ b dividesG y ⟶ x dividesG y))"

definition somegcd :: "('a,_) monoid_scheme ⇒ 'a ⇒ 'a ⇒ 'a"
  where "somegcd G a b = (SOME x. x ∈ carrier G ∧ x gcdofG a b)"

definition somelcm :: "('a,_) monoid_scheme ⇒ 'a ⇒ 'a ⇒ 'a"
  where "somelcm G a b = (SOME x. x ∈ carrier G ∧ x lcmofG a b)"

definition "SomeGcd G A = inf (division_rel G) A"


locale gcd_condition_monoid = comm_monoid_cancel +
  assumes gcdof_exists: "⟦a ∈ carrier G; b ∈ carrier G⟧ ⟹ ∃c. c ∈ carrier
G ∧ c gcdof a b"

locale primeness_condition_monoid = comm_monoid_cancel +
  assumes irreducible_prime: "⟦a ∈ carrier G; irreducible G a⟧ ⟹ prime
G a"

locale divisor_chain_condition_monoid = comm_monoid_cancel +
  assumes division_wellfounded: "wf {(x, y). x ∈ carrier G ∧ y ∈ carrier
G ∧ properfactor G x y}"
```

23.8.2 Connections to `Lattice.thy`

```
lemma gcdof_greatestLower:
  fixes G (structure)
  assumes carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "(x ∈ carrier G ∧ x gcdof a b) = greatest (division_rel G) x
(Lower (division_rel G) {a, b})"
  by (auto simp: isgcd_def greatest_def Lower_def elem_def)

lemma lcmof_leastUpper:
  fixes G (structure)
  assumes carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  shows "(x ∈ carrier G ∧ x lcmof a b) = least (division_rel G) x (Upper
(division_rel G) {a, b})"
  by (auto simp: islcm_def least_def Upper_def elem_def)

lemma somegcd_meet:
  fixes G (structure)
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "somegcd G a b = meet (division_rel G) a b"
  by (simp add: somegcd_def meet_def inf_def gcdof_greatestLower[OF carr])
```

```
lemma (in monoid) isgcd_divides_l:
  assumes "a divides b"
    and "a ∈ carrier G"  "b ∈ carrier G"
  shows "a gcdof a b"
  using assms unfolding isgcd_def by fast

lemma (in monoid) isgcd_divides_r:
  assumes "b divides a"
    and "a ∈ carrier G"  "b ∈ carrier G"
  shows "b gcdof a b"
  using assms unfolding isgcd_def by fast
```

23.8.3 Existence of gcd and lcm

```
lemma (in factorial_monoid) gcdof_exists:
  assumes acarr: "a ∈ carrier G"
    and bcarr: "b ∈ carrier G"
  shows "∃c. c ∈ carrier G ∧ c gcdof a b"
proof -
  from wfactors_exist [OF acarr]
  obtain as where ascarr: "set as ⊆ carrier G" and afs: "wfactors G
as a"
    by blast
  from afs have airr: "∀a ∈ set as. irreducible G a"
    by (fast elim: wfactorsE)

  from wfactors_exist [OF bcarr]
  obtain bs where bscarr: "set bs ⊆ carrier G" and bfs: "wfactors G
bs b"
    by blast
  from bfs have birr: "∀b ∈ set bs. irreducible G b"
    by (fast elim: wfactorsE)

  have "∃c cs. c ∈ carrier G ∧ set cs ⊆ carrier G ∧ wfactors G cs c
∧
    fmset G cs = fmset G as ∩# fmset G bs"
  proof (intro mset_wfactorsEx)
    fix X
    assume "X ∈# fmset G as ∩# fmset G bs"
    then have "X ∈# fmset G as" by simp
    then have "X ∈ set (map (assocs G) as)"
      by (simp add: fmset_def)
    then have "∃x. X = assocs G x ∧ x ∈ set as"
      by (induct as) auto
    then obtain x where X: "X = assocs G x" and xas: "x ∈ set as"
      by blast
    with ascarr have xcarr: "x ∈ carrier G"
      by blast
```

```
    from xas airr have xirr: "irreducible G x"
      by simp
    from xcarr and xirr and X show "∃x. (x ∈ carrier G ∧ irreducible
G x) ∧ X = assocs G x"
      by blast
  qed
  then obtain c cs
    where ccarr: "c ∈ carrier G"
      and cscarr: "set cs ⊆ carrier G"
      and csirr: "wfactors G cs c"
      and csmset: "fmset G cs = fmset G as ∩# fmset G bs"
    by auto

  have "c gcdof a b"
  proof (simp add: isgcd_def, safe)
    from csmset
    have "fmset G cs ⊆# fmset G as"
      by (simp add: multiset_inter_def subset_mset_def)
    then show "c divides a" by (rule fmsubset_divides) fact+
  next
    from csmset have "fmset G cs ⊆# fmset G bs"
      by (simp add: multiset_inter_def subseteq_mset_def, force)
    then show "c divides b"
      by (rule fmsubset_divides) fact+
  next
    fix y
    assume "y ∈ carrier G"
    from wfactors_exist [OF this]
    obtain ys where yscarr: "set ys ⊆ carrier G" and yfs: "wfactors
G ys y"
      by blast

    assume "y divides a"
    then have ya: "fmset G ys ⊆# fmset G as"
      by (rule divides_fmsubset) fact+

    assume "y divides b"
    then have yb: "fmset G ys ⊆# fmset G bs"
      by (rule divides_fmsubset) fact+

    from ya yb csmset have "fmset G ys ⊆# fmset G cs"
      by (simp add: subset_mset_def)
    then show "y divides c"
      by (rule fmsubset_divides) fact+
  qed
  with ccarr show "∃c. c ∈ carrier G ∧ c gcdof a b"
    by fast
qed
```

```
lemma (in factorial_monoid) lcmof_exists:
  assumes acarr: "a ∈ carrier G"
    and bcarr: "b ∈ carrier G"
  shows "∃c. c ∈ carrier G ∧ c lcmof a b"
proof -
  from wfactors_exist [OF acarr]
  obtain as where ascarr: "set as ⊆ carrier G" and afs: "wfactors G
as a"
    by blast
  from afs have airr: "∀a ∈ set as. irreducible G a"
    by (fast elim: wfactorsE)

  from wfactors_exist [OF bcarr]
  obtain bs where bscarr: "set bs ⊆ carrier G" and bfs: "wfactors G
bs b"
    by blast
  from bfs have birr: "∀b ∈ set bs. irreducible G b"
    by (fast elim: wfactorsE)

  have "∃c cs. c ∈ carrier G ∧ set cs ⊆ carrier G ∧ wfactors G cs c
∧
    fmset G cs = (fmset G as - fmset G bs) + fmset G bs"
  proof (intro mset_wfactorsEx)
    fix X
    assume "X ∈# (fmset G as - fmset G bs) + fmset G bs"
    then have "X ∈# fmset G as ∨ X ∈# fmset G bs"
      by (auto dest: in_diffD)
    then consider "X ∈ set_mset (fmset G as)" | "X ∈ set_mset (fmset
G bs)"
      by fast
    then show "∃x. (x ∈ carrier G ∧ irreducible G x) ∧ X = assocs G
x"
    proof cases
      case 1
      then have "X ∈ set (map (assocs G) as)" by (simp add: fmset_def)
      then have "∃x. x ∈ set as ∧ X = assocs G x" by (induct as) auto
      then obtain x where xas: "x ∈ set as" and X: "X = assocs G x"
by auto
      with ascarr have xcarr: "x ∈ carrier G" by fast
      from xas airr have xirr: "irreducible G x" by simp
      from xcarr and xirr and X show ?thesis by fast
    next
      case 2
      then have "X ∈ set (map (assocs G) bs)" by (simp add: fmset_def)
      then have "∃x. x ∈ set bs ∧ X = assocs G x" by (induct as) auto
      then obtain x where xbs: "x ∈ set bs" and X: "X = assocs G x"
by auto
      with bscarr have xcarr: "x ∈ carrier G" by fast
      from xbs birr have xirr: "irreducible G x" by simp
```

```
      from xcarr and xirr and X show ?thesis by fast
    qed
  qed
  then obtain c cs
    where ccarr: "c ∈ carrier G"
      and cscarr: "set cs ⊆ carrier G"
      and csirr: "wfactors G cs c"
      and csmset: "fmset G cs = fmset G as - fmset G bs + fmset G bs"
    by auto

  have "c lcmof a b"
  proof (simp add: islcm_def, safe)
    from csmset have "fmset G as ⊆# fmset G cs"
      by (simp add: subseteq_mset_def, force)
    then show "a divides c"
      by (rule fmsubset_divides) fact+
  next
    from csmset have "fmset G bs ⊆# fmset G cs"
      by (simp add: subset_mset_def)
    then show "b divides c"
      by (rule fmsubset_divides) fact+
  next
    fix y
    assume "y ∈ carrier G"
    from wfactors_exist [OF this]
    obtain ys where yscarr: "set ys ⊆ carrier G" and yfs: "wfactors
G ys y"
      by blast

    assume "a divides y"
    then have ya: "fmset G as ⊆# fmset G ys"
      by (rule divides_fmsubset) fact+

    assume "b divides y"
    then have yb: "fmset G bs ⊆# fmset G ys"
      by (rule divides_fmsubset) fact+

    from ya yb csmset have "fmset G cs ⊆# fmset G ys"
      apply (simp add: subseteq_mset_def, clarify)
      apply (case_tac "count (fmset G as) a < count (fmset G bs) a")
       apply simp
      apply simp
      done
    then show "c divides y"
      by (rule fmsubset_divides) fact+
  qed
  with ccarr show "∃c. c ∈ carrier G ∧ c lcmof a b"
    by fast
qed
```

23.9 Conditions for Factoriality

23.9.1 Gcd condition

```
lemma (in gcd_condition_monoid) division_weak_lower_semilattice [simp]:
  "weak_lower_semilattice (division_rel G)"
proof -
  interpret weak_partial_order "division_rel G" ..
  show ?thesis
    apply (unfold_locales, simp_all)
  proof -
    fix x y
    assume carr: "x ∈ carrier G"  "y ∈ carrier G"
    from gcdof_exists [OF this] obtain z where zcarr: "z ∈ carrier G"
and isgcd: "z gcdof x y"
      by blast
    with carr have "greatest (division_rel G) z (Lower (division_rel
G) {x, y})"
      by (subst gcdof_greatestLower[symmetric], simp+)
    then show "∃z. greatest (division_rel G) z (Lower (division_rel G)
{x, y})"
      by fast
  qed
qed

lemma (in gcd_condition_monoid) gcdof_cong_l:
  assumes a'a: "a' ∼ a"
    and agcd: "a gcdof b c"
    and a'carr: "a' ∈ carrier G" and carr': "a ∈ carrier G"  "b ∈ carrier
G"  "c ∈ carrier G"
  shows "a' gcdof b c"
proof -
  note carr = a'carr carr'
  interpret weak_lower_semilattice "division_rel G" by simp
  have "a' ∈ carrier G ∧ a' gcdof b c"
    apply (simp add: gcdof_greatestLower carr')
    apply (subst greatest_Lower_cong_l[of _ a])
        apply (simp add: a'a)
       apply (simp add: carr)
      apply (simp add: carr)
     apply (simp add: carr)
    apply (simp add: gcdof_greatestLower[symmetric] agcd carr)
    done
  then show ?thesis ..
qed

lemma (in gcd_condition_monoid) gcd_closed [simp]:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "somegcd G a b ∈ carrier G"
proof -
```

```
  interpret weak_lower_semilattice "division_rel G" by simp
  show ?thesis
    apply (simp add: somegcd_meet[OF carr])
    apply (rule meet_closed[simplified], fact+)
    done
qed

lemma (in gcd_condition_monoid) gcd_isgcd:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "(somegcd G a b) gcdof a b"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  from carr have "somegcd G a b ∈ carrier G ∧ (somegcd G a b) gcdof a
b"
    apply (subst gcdof_greatestLower, simp, simp)
    apply (simp add: somegcd_meet[OF carr] meet_def)
    apply (rule inf_of_two_greatest[simplified], assumption+)
    done
  then show "(somegcd G a b) gcdof a b"
    by simp
qed

lemma (in gcd_condition_monoid) gcd_exists:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "∃x∈carrier G. x = somegcd G a b"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show ?thesis
    by (metis carr(1) carr(2) gcd_closed)
qed

lemma (in gcd_condition_monoid) gcd_divides_l:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "(somegcd G a b) divides a"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show ?thesis
    by (metis carr(1) carr(2) gcd_isgcd isgcd_def)
qed

lemma (in gcd_condition_monoid) gcd_divides_r:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "(somegcd G a b) divides b"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
```

```
  show ?thesis
    by (metis carr gcd_isgcd isgcd_def)
qed

lemma (in gcd_condition_monoid) gcd_divides:
  assumes sub: "z divides x"  "z divides y"
    and L: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
  shows "z divides (somegcd G x y)"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show ?thesis
    by (metis gcd_isgcd isgcd_def assms)
qed

lemma (in gcd_condition_monoid) gcd_cong_l:
  assumes xx': "x ∼ x'"
    and carr: "x ∈ carrier G"  "x' ∈ carrier G"  "y ∈ carrier G"
  shows "somegcd G x y ∼ somegcd G x' y"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show ?thesis
    apply (simp add: somegcd_meet carr)
    apply (rule meet_cong_l[simplified], fact+)
    done
qed

lemma (in gcd_condition_monoid) gcd_cong_r:
  assumes carr: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
    and yy': "y ∼ y'"
  shows "somegcd G x y ∼ somegcd G x y'"
proof -
  interpret weak_lower_semilattice "division_rel G" by simp
  show ?thesis
    apply (simp add: somegcd_meet carr)
    apply (rule meet_cong_r[simplified], fact+)
    done
qed



lemma (in gcd_condition_monoid) gcdI:
  assumes dvd: "a divides b"  "a divides c"
    and others: "∀y∈carrier G. y divides b ∧ y divides c ⟶ y divides
a"
    and acarr: "a ∈ carrier G" and bcarr: "b ∈ carrier G" and ccarr:
"c ∈ carrier G"
  shows "a ∼ somegcd G b c"
```

```
  apply (simp add: somegcd_def)
  apply (rule someI2_ex)
   apply (rule exI[of _ a], simp add: isgcd_def)
   apply (simp add: assms)
  apply (simp add: isgcd_def assms, clarify)
  apply (insert assms, blast intro: associatedI)
  done

lemma (in gcd_condition_monoid) gcdI2:
  assumes "a gcdof b c" and "a ∈ carrier G" and "b ∈ carrier G" and
"c ∈ carrier G"
  shows "a ∼ somegcd G b c"
  using assms unfolding isgcd_def by (blast intro: gcdI)

lemma (in gcd_condition_monoid) SomeGcd_ex:
  assumes "finite A"  "A ⊆ carrier G"  "A ≠ {}"
  shows "∃x∈ carrier G. x = SomeGcd G A"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show ?thesis
    apply (simp add: SomeGcd_def)
    apply (rule finite_inf_closed[simplified], fact+)
    done
qed

lemma (in gcd_condition_monoid) gcd_assoc:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "somegcd G (somegcd G a b) c ∼ somegcd G a (somegcd G b c)"
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show ?thesis
    apply (subst (2 3) somegcd_meet, (simp add: carr)+)
    apply (simp add: somegcd_meet carr)
    apply (rule weak_meet_assoc[simplified], fact+)
    done
qed

lemma (in gcd_condition_monoid) gcd_mult:
  assumes acarr: "a ∈ carrier G" and bcarr: "b ∈ carrier G" and ccarr:
"c ∈ carrier G"
  shows "c ⊗ somegcd G a b ∼ somegcd G (c ⊗ a) (c ⊗ b)"
proof -
  let ?d = "somegcd G a b"
  let ?e = "somegcd G (c ⊗ a) (c ⊗ b)"
  note carr[simp] = acarr bcarr ccarr
  have dcarr: "?d ∈ carrier G" by simp
  have ecarr: "?e ∈ carrier G" by simp
```

```
  note carr = carr dcarr ecarr

  have "?d divides a" by (simp add: gcd_divides_l)
  then have cd'ca: "c ⊗ ?d divides (c ⊗ a)" by (simp add: divides_mult_lI)

  have "?d divides b" by (simp add: gcd_divides_r)
  then have cd'cb: "c ⊗ ?d divides (c ⊗ b)" by (simp add: divides_mult_lI)

  from cd'ca cd'cb have cd'e: "c ⊗ ?d divides ?e"
    by (rule gcd_divides) simp_all
  then obtain u where ucarr[simp]: "u ∈ carrier G" and e_cdu: "?e =
c ⊗ ?d ⊗ u"
    by blast

  note carr = carr ucarr

  have "?e divides c ⊗ a" by (rule gcd_divides_l) simp_all
  then obtain x where xcarr: "x ∈ carrier G" and ca_ex: "c ⊗ a = ?e
⊗ x"
    by blast
  with e_cdu have ca_cdux: "c ⊗ a = c ⊗ ?d ⊗ u ⊗ x"
    by simp

  from ca_cdux xcarr have "c ⊗ a = c ⊗ (?d ⊗ u ⊗ x)"
    by (simp add: m_assoc)
  then have "a = ?d ⊗ u ⊗ x"
    by (rule l_cancel[of c a]) (simp add: xcarr)+
  then have du'a: "?d ⊗ u divides a"
    by (rule dividesI[OF xcarr])

  have "?e divides c ⊗ b" by (intro gcd_divides_r) simp_all
  then obtain x where xcarr: "x ∈ carrier G" and cb_ex: "c ⊗ b = ?e
⊗ x"
    by blast
  with e_cdu have cb_cdux: "c ⊗ b = c ⊗ ?d ⊗ u ⊗ x"
    by simp

  from cb_cdux xcarr have "c ⊗ b = c ⊗ (?d ⊗ u ⊗ x)"
    by (simp add: m_assoc)
  with xcarr have "b = ?d ⊗ u ⊗ x"
    by (intro l_cancel[of c b]) simp_all
  then have du'b: "?d ⊗ u divides b"
    by (intro dividesI[OF xcarr])

  from du'a du'b carr have du'd: "?d ⊗ u divides ?d"
    by (intro gcd_divides) simp_all
  then have uunit: "u ∈ Units G"
  proof (elim dividesE)
    fix v
```

```
    assume vcarr[simp]: "v ∈ carrier G"
    assume d: "?d = ?d ⊗ u ⊗ v"
    have "?d ⊗ 1 = ?d ⊗ u ⊗ v" by simp fact
    also have "?d ⊗ u ⊗ v = ?d ⊗ (u ⊗ v)" by (simp add: m_assoc)
    finally have "?d ⊗ 1 = ?d ⊗ (u ⊗ v)" .
    then have i2: "1 = u ⊗ v" by (rule l_cancel) simp_all
    then have i1: "1 = v ⊗ u" by (simp add: m_comm)
    from vcarr i1[symmetric] i2[symmetric] show "u ∈ Units G"
      by (auto simp: Units_def)
  qed

  from e_cdu uunit have "somegcd G (c ⊗ a) (c ⊗ b) ∼ c ⊗ somegcd G
a b"
    by (intro associatedI2[of u]) simp_all
  from this[symmetric] show "c ⊗ somegcd G a b ∼ somegcd G (c ⊗ a)
(c ⊗ b)"
    by simp
qed

lemma (in monoid) assoc_subst:
  assumes ab: "a ∼ b"
    and cP: "∀a b. a ∈ carrier G ∧ b ∈ carrier G ∧ a ∼ b
      ⟶ f a ∈ carrier G ∧ f b ∈ carrier G ∧ f a ∼ f b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"
  shows "f a ∼ f b"
  using assms by auto

lemma (in gcd_condition_monoid) relprime_mult:
  assumes abrelprime: "somegcd G a b ∼ 1"
    and acrelprime: "somegcd G a c ∼ 1"
    and carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "somegcd G a (b ⊗ c) ∼ 1"
proof -
  have "c = c ⊗ 1" by simp
  also from abrelprime[symmetric]
  have "... ∼ c ⊗ somegcd G a b"
    by (rule assoc_subst) (simp add: mult_cong_r)+
  also have "... ∼ somegcd G (c ⊗ a) (c ⊗ b)"
    by (rule gcd_mult) fact+
  finally have c: "c ∼ somegcd G (c ⊗ a) (c ⊗ b)"
    by simp

  from carr have a: "a ∼ somegcd G a (c ⊗ a)"
    by (fast intro: gcdI divides_prod_l)

  have "somegcd G a (b ⊗ c) ∼ somegcd G a (c ⊗ b)"
    by (simp add: m_comm)
  also from a have "... ∼ somegcd G (somegcd G a (c ⊗ a)) (c ⊗ b)"
    by (rule assoc_subst) (simp add: gcd_cong_l)+
```

```
  also from gcd_assoc have "... ∼ somegcd G a (somegcd G (c ⊗ a) (c ⊗
b))"
    by (rule assoc_subst) simp+
  also from c[symmetric] have "... ∼ somegcd G a c"
    by (rule assoc_subst) (simp add: gcd_cong_r)+
  also note acrelprime
  finally show "somegcd G a (b ⊗ c) ∼ 1"
    by simp
qed

lemma (in gcd_condition_monoid) primeness_condition: "primeness_condition_monoid
G"
  apply unfold_locales
  apply (rule primeI)
   apply (elim irreducibleE, assumption)
proof -
  fix p a b
  assume pcarr: "p ∈ carrier G" and acarr: "a ∈ carrier G" and bcarr:
"b ∈ carrier G"
    and pirr: "irreducible G p"
    and pdvdab: "p divides a ⊗ b"
  from pirr have pnunit: "p ∉ Units G"
    and r[rule_format]: "∀b. b ∈ carrier G ∧ properfactor G b p ⟶ b
∈ Units G"
    by (fast elim: irreducibleE)+

  show "p divides a ∨ p divides b"
  proof (rule ccontr, clarsimp)
    assume npdvda: "¬ p divides a"
    with pcarr acarr have "1 ∼ somegcd G p a"
      apply (intro gcdI, simp, simp, simp)
           apply (fast intro: unit_divides)
          apply (fast intro: unit_divides)
         apply (clarsimp simp add: Unit_eq_dividesone[symmetric])
         apply (rule r, rule, assumption)
         apply (rule properfactorI, assumption)
    proof
      fix y
      assume ycarr: "y ∈ carrier G"
      assume "p divides y"
      also assume "y divides a"
      finally have "p divides a"
        by (simp add: pcarr ycarr acarr)
      with npdvda show False ..
    qed simp_all
    with pcarr acarr have pa: "somegcd G p a ∼ 1"
      by (fast intro: associated_sym[of "1"] gcd_closed)

    assume npdvdb: "¬ p divides b"
```

```
    with pcarr bcarr have "1 ∼ somegcd G p b"
      apply (intro gcdI, simp, simp, simp)
           apply (fast intro: unit_divides)
          apply (fast intro: unit_divides)
         apply (clarsimp simp add: Unit_eq_dividesone[symmetric])
         apply (rule r, rule, assumption)
         apply (rule properfactorI, assumption)
    proof
      fix y
      assume ycarr: "y ∈ carrier G"
      assume "p divides y"
      also assume "y divides b"
      finally have "p divides b" by (simp add: pcarr ycarr bcarr)
      with npdvdb
      show "False" ..
    qed simp_all
    with pcarr bcarr have pb: "somegcd G p b ∼ 1"
      by (fast intro: associated_sym[of "1"] gcd_closed)

    from pcarr acarr bcarr pdvdab have "p gcdof p (a ⊗ b)"
      by (fast intro: isgcd_divides_l)
    with pcarr acarr bcarr have "p ∼ somegcd G p (a ⊗ b)"
      by (fast intro: gcdI2)
    also from pa pb pcarr acarr bcarr have "somegcd G p (a ⊗ b) ∼ 1"
      by (rule relprime_mult)
    finally have "p ∼ 1"
      by (simp add: pcarr acarr bcarr)
    with pcarr have "p ∈ Units G"
      by (fast intro: assoc_unit_l)
    with pnunit show False ..
  qed
qed

sublocale gcd_condition_monoid ⊆ primeness_condition_monoid
  by (rule primeness_condition)
```

23.9.2 Divisor chain condition

```
lemma (in divisor_chain_condition_monoid) wfactors_exist:
  assumes acarr: "a ∈ carrier G"
  shows "∃as. set as ⊆ carrier G ∧ wfactors G as a"
proof -
  have r[rule_format]: "a ∈ carrier G ⟶ (∃as. set as ⊆ carrier G ∧
wfactors G as a)"
  proof (rule wf_induct[OF division_wellfounded])
    fix x
    assume ih: "∀y. (y, x) ∈ {(x, y). x ∈ carrier G ∧ y ∈ carrier G
∧ properfactor G x y}
                    ⟶ y ∈ carrier G ⟶ (∃as. set as ⊆ carrier G ∧
```

```
wfactors G as y)"

    show "x ∈ carrier G ⟶ (∃as. set as ⊆ carrier G ∧ wfactors G as
x)"
      apply clarify
      apply (cases "x ∈ Units G")
       apply (rule exI[of _ "[]"], simp)
      apply (cases "irreducible G x")
       apply (rule exI[of _ "[x]"], simp add: wfactors_def)
    proof -
      assume xcarr: "x ∈ carrier G"
        and xnunit: "x ∉ Units G"
        and xnirr: "¬ irreducible G x"
      then have "∃y. y ∈ carrier G ∧ properfactor G y x ∧ y ∉ Units
G"
        apply -
        apply (rule ccontr)
        apply simp
        apply (subgoal_tac "irreducible G x", simp)
        apply (rule irreducibleI, simp, simp)
        done
      then obtain y where ycarr: "y ∈ carrier G" and ynunit: "y ∉ Units
G"
        and pfyx: "properfactor G y x"
        by blast

      have ih': "⋀y. ⟦y ∈ carrier G; properfactor G y x⟧
          ⟹ ∃as. set as ⊆ carrier G ∧ wfactors G as y"
        by (rule ih[rule_format, simplified]) (simp add: xcarr)+

      from ih' [OF ycarr pfyx]
      obtain ys where yscarr: "set ys ⊆ carrier G" and yfs: "wfactors
G ys y"
        by blast

      from pfyx have "y divides x" and nyx: "¬ y ∼ x"
        by (fast elim: properfactorE2)+
      then obtain z where zcarr: "z ∈ carrier G" and x: "x = y ⊗ z"
        by blast

      from zcarr ycarr have "properfactor G z x"
        apply (subst x)
        apply (intro properfactorI3[of _ _ y])
            apply (simp add: m_comm)
           apply (simp add: ynunit)+
        done
      from ih' [OF zcarr this]
      obtain zs where zscarr: "set zs ⊆ carrier G" and zfs: "wfactors
G zs z"
```

```
        by blast
      from yscarr zscarr have xscarr: "set (ys@zs) ⊆ carrier G"
        by simp
      from yfs zfs ycarr zcarr yscarr zscarr have "wfactors G (ys@zs)
(y⊗z)"
        by (rule wfactors_mult)
      then have "wfactors G (ys@zs) x"
        by (simp add: x)
      with xscarr show "∃xs. set xs ⊆ carrier G ∧ wfactors G xs x"
        by fast
    qed
  qed
  from acarr show ?thesis by (rule r)
qed
```

23.9.3 Primeness condition

```
lemma (in comm_monoid_cancel) multlist_prime_pos:
  assumes carr: "a ∈ carrier G"  "set as ⊆ carrier G"
    and aprime: "prime G a"
    and "a divides (foldr (op ⊗) as 1)"
  shows "∃i<length as. a divides (as!i)"
proof -
  have r[rule_format]: "set as ⊆ carrier G ∧ a divides (foldr (op ⊗)
as 1)
    ⟶ (∃i. i < length as ∧ a divides (as!i))"
    apply (induct as)
     apply clarsimp defer 1
     apply clarsimp defer 1
  proof -
    assume "a divides 1"
    with carr have "a ∈ Units G"
      by (fast intro: divides_unit[of a 1])
    with aprime show False
      by (elim primeE, simp)
  next
    fix aa as
    assume ih[rule_format]: "a divides foldr op ⊗ as 1 ⟶ (∃i<length
as. a divides as ! i)"
      and carr': "aa ∈ carrier G"  "set as ⊆ carrier G"
      and "a divides aa ⊗ foldr op ⊗ as 1"
    with carr aprime have "a divides aa ∨ a divides foldr op ⊗ as 1"
      by (intro prime_divides) simp+
    then show "∃i<Suc (length as). a divides (aa # as) ! i"
    proof
      assume "a divides aa"
      then have p1: "a divides (aa#as)!0" by simp
      have "0 < Suc (length as)" by simp
      with p1 show ?thesis by fast
```

```
    next
      assume "a divides foldr op ⊗ as 1"
      from ih [OF this] obtain i where "a divides as ! i" and len: "i
< length as" by auto
      then have p1: "a divides (aa#as) ! (Suc i)" by simp
      from len have "Suc i < Suc (length as)" by simp
      with p1 show ?thesis by force
   qed
  qed
  from assms show ?thesis
    by (intro r) auto
qed

lemma (in primeness_condition_monoid) wfactors_unique__hlp_induct:
  "∀a as'. a ∈ carrier G ∧ set as ⊆ carrier G ∧ set as' ⊆ carrier G
∧
           wfactors G as a ∧ wfactors G as' a ⟶ essentially_equal G
as as'"
proof (induct as)
  case Nil
  show ?case
  proof auto
    fix a as'
    assume a: "a ∈ carrier G"
    assume "wfactors G [] a"
    then obtain "1 ∼ a" by (auto elim: wfactorsE)
    with a have "a ∈ Units G" by (auto intro: assoc_unit_r)
    moreover assume "wfactors G as' a"
    moreover assume "set as' ⊆ carrier G"
    ultimately have "as' = []" by (rule unit_wfactors_empty)
    then show "essentially_equal G [] as'" by simp
  qed
next
  case (Cons ah as)
  then show ?case
  proof clarsimp
    fix a as'
    assume ih [rule_format]:
      "∀a as'. a ∈ carrier G ∧ set as' ⊆ carrier G ∧ wfactors G as a
∧
        wfactors G as' a ⟶ essentially_equal G as as'"
      and acarr: "a ∈ carrier G" and ahcarr: "ah ∈ carrier G"
      and ascarr: "set as ⊆ carrier G" and as'carr: "set as' ⊆ carrier
G"
      and afs: "wfactors G (ah # as) a"
      and afs': "wfactors G as' a"
    then have ahdvda: "ah divides a"
      by (intro wfactors_dividesI[of "ah#as" "a"]) simp_all
    then obtain a' where a'carr: "a' ∈ carrier G" and a: "a = ah ⊗ a'"
```

```
      by blast
    have a'fs: "wfactors G as a'"
      apply (rule wfactorsE[OF afs], rule wfactorsI, simp)
      apply (simp add: a)
      apply (insert ascarr a'carr)
      apply (intro assoc_l_cancel[of ah _ a'] multlist_closed ahcarr,
assumption+)
      done
    from afs have ahirr: "irreducible G ah"
      by (elim wfactorsE) simp
    with ascarr have ahprime: "prime G ah"
      by (intro irreducible_prime ahcarr)

    note carr [simp] = acarr ahcarr ascarr as'carr a'carr

    note ahdvda
    also from afs' have "a divides (foldr (op ⊗) as' 1)"
      by (elim wfactorsE associatedE, simp)
    finally have "ah divides (foldr (op ⊗) as' 1)"
      by simp
    with ahprime have "∃i<length as'. ah divides as'!i"
      by (intro multlist_prime_pos) simp_all
    then obtain i where len: "i<length as'" and ahdvd: "ah divides as'!i"
      by blast
    from afs' carr have irrasi: "irreducible G (as'!i)"
      by (fast intro: nth_mem[OF len] elim: wfactorsE)
    from len carr have asicarr[simp]: "as'!i ∈ carrier G"
      unfolding set_conv_nth by force
    note carr = carr asicarr

    from ahdvd obtain x where "x ∈ carrier G" and asi: "as'!i = ah ⊗
x"
      by blast
    with carr irrasi[simplified asi] have asiah: "as'!i ∼ ah"
      apply -
      apply (elim irreducible_prodE[of "ah" "x"], assumption+)
       apply (rule associatedI2[of x], assumption+)
      apply (rule irreducibleE[OF ahirr], simp)
      done

    note setparts = set_take_subset[of i as'] set_drop_subset[of "Suc
i" as']
    note partscarr [simp] = setparts[THEN subset_trans[OF _ as'carr]]
    note carr = carr partscarr

    have "∃aa_1. aa_1 ∈ carrier G ∧ wfactors G (take i as') aa_1"
      apply (intro wfactors_prod_exists)
      using setparts afs'
       apply (fast elim: wfactorsE)
```

```
      apply simp
      done
    then obtain aa_1 where aa1carr: "aa_1 ∈ carrier G" and aa1fs: "wfactors
G (take i as') aa_1"
      by auto

    have "∃aa_2. aa_2 ∈ carrier G ∧ wfactors G (drop (Suc i) as') aa_2"
      apply (intro wfactors_prod_exists)
      using setparts afs'
       apply (fast elim: wfactorsE)
      apply simp
      done
    then obtain aa_2 where aa2carr: "aa_2 ∈ carrier G"
      and aa2fs: "wfactors G (drop (Suc i) as') aa_2"
      by auto

    note carr = carr aa1carr[simp] aa2carr[simp]

    from aa1fs aa2fs
    have v1: "wfactors G (take i as' @ drop (Suc i) as') (aa_1 ⊗ aa_2)"
      by (intro wfactors_mult, simp+)
    then have v1': "wfactors G (as'!i # take i as' @ drop (Suc i) as')
(as'!i ⊗ (aa_1 ⊗ aa_2))"
      apply (intro wfactors_mult_single)
      using setparts afs'
          apply (fast intro: nth_mem[OF len] elim: wfactorsE)
         apply simp_all
      done

    from aa2carr carr aa1fs aa2fs have "wfactors G (as'!i # drop (Suc
i) as') (as'!i ⊗ aa_2)"
      by (metis irrasi wfactors_mult_single)
    with len carr aa1carr aa2carr aa1fs
    have v2: "wfactors G (take i as' @ as'!i # drop (Suc i) as') (aa_1
⊗ (as'!i ⊗ aa_2))"
      apply (intro wfactors_mult)
           apply fast
          apply (simp, (fast intro: nth_mem[OF len])?)+
      done

    from len have as': "as' = (take i as' @ as'!i # drop (Suc i) as')"
      by (simp add: Cons_nth_drop_Suc)
    with carr have eer: "essentially_equal G (take i as' @ as'!i # drop
(Suc i) as') as'"
      by simp
    with v2 afs' carr aa1carr aa2carr nth_mem[OF len] have "aa_1 ⊗ (as'!i
⊗ aa_2) ∼ a"
      by (metis as' ee_wfactorsD m_closed)
    then have t1: "as'!i ⊗ (aa_1 ⊗ aa_2) ∼ a"
```

```
      by (metis aa1carr aa2carr asicarr m_lcomm)
    from carr asiah have "ah ⊗ (aa_1 ⊗ aa_2) ∼ as'!i ⊗ (aa_1 ⊗ aa_2)"
      by (metis associated_sym m_closed mult_cong_l)
    also note t1
    finally have "ah ⊗ (aa_1 ⊗ aa_2) ∼ a" by simp

    with carr aa1carr aa2carr a'carr nth_mem[OF len] have a': "aa_1 ⊗
aa_2 ∼ a'"
      by (simp add: a, fast intro: assoc_l_cancel[of ah _ a'])

    note v1
    also note a'
    finally have "wfactors G (take i as' @ drop (Suc i) as') a'"
      by simp

    from a'fs this carr have "essentially_equal G as (take i as' @ drop
(Suc i) as')"
      by (intro ih[of a']) simp
    then have ee1: "essentially_equal G (ah # as) (ah # take i as' @
drop (Suc i) as')"
      by (elim essentially_equalE) (fastforce intro: essentially_equalI)

    from carr have ee2: "essentially_equal G (ah # take i as' @ drop
(Suc i) as')
      (as' ! i # take i as' @ drop (Suc i) as')"
    proof (intro essentially_equalI)
      show "ah # take i as' @ drop (Suc i) as' <~~> ah # take i as' @
drop (Suc i) as'"
        by simp
    next
      show "ah # take i as' @ drop (Suc i) as' [∼] as' ! i # take i as'
@ drop (Suc i) as'"
        by (simp add: list_all2_append) (simp add: asiah[symmetric])
    qed

    note ee1
    also note ee2
    also have "essentially_equal G (as' ! i # take i as' @ drop (Suc i)
as')
      (take i as' @ as' ! i # drop (Suc i) as')"
      apply (intro essentially_equalI)
       apply (subgoal_tac "as' ! i # take i as' @ drop (Suc i) as' <~~>
          take i as' @ as' ! i # drop (Suc i) as'")
        apply simp
       apply (rule perm_append_Cons)
      apply simp
      done
    finally have "essentially_equal G (ah # as) (take i as' @ as' ! i #
drop (Suc i) as')"
```

```
   by simp
  then show "essentially_equal G (ah # as) as'"
   by (subst as')
 qed
qed

lemma (in primeness_condition_monoid) wfactors_unique:
 assumes "wfactors G as a"  "wfactors G as' a"
  and "a ∈ carrier G"  "set as ⊆ carrier G"  "set as' ⊆ carrier G"
 shows "essentially_equal G as as'"
 by (rule wfactors_unique__hlp_induct[rule_format, of a]) (simp add:
assms)
```

23.9.4 Application to factorial monoids

Number of factors for wellfoundedness

```
definition factorcount :: "_ ⇒ 'a ⇒ nat"
 where "factorcount G a =
  (THE c. ∀as. set as ⊆ carrier G ∧ wfactors G as a ⟶ c = length
as)"

lemma (in monoid) ee_length:
 assumes ee: "essentially_equal G as bs"
 shows "length as = length bs"
 by (rule essentially_equalE[OF ee]) (metis list_all2_conv_all_nth perm_length)

lemma (in factorial_monoid) factorcount_exists:
 assumes carr[simp]: "a ∈ carrier G"
 shows "∃c. ∀as. set as ⊆ carrier G ∧ wfactors G as a ⟶ c = length
as"
proof -
 have "∃as. set as ⊆ carrier G ∧ wfactors G as a"
  by (intro wfactors_exist) simp
 then obtain as where ascarr[simp]: "set as ⊆ carrier G" and afs: "wfactors
G as a"
  by (auto simp del: carr)
 have "∀as'. set as' ⊆ carrier G ∧ wfactors G as' a ⟶ length as =
length as'"
  by (metis afs ascarr assms ee_length wfactors_unique)
 then show "∃c. ∀as'. set as' ⊆ carrier G ∧ wfactors G as' a ⟶ c
= length as'" ..
qed

lemma (in factorial_monoid) factorcount_unique:
 assumes afs: "wfactors G as a"
  and acarr[simp]: "a ∈ carrier G" and ascarr[simp]: "set as ⊆ carrier
G"
 shows "factorcount G a = length as"
proof -
```

```
  have "∃ac. ∀as. set as ⊆ carrier G ∧ wfactors G as a ⟶ ac = length
as"
    by (rule factorcount_exists) simp
  then obtain ac where alen: "∀as. set as ⊆ carrier G ∧ wfactors G as
a ⟶ ac = length as"
    by auto
  have ac: "ac = factorcount G a"
    apply (simp add: factorcount_def)
    apply (rule theI2)
      apply (rule alen)
     apply (metis afs alen ascarr)+
    done
  from ascarr afs have "ac = length as"
    by (iprover intro: alen[rule_format])
  with ac show ?thesis
    by simp
qed

lemma (in factorial_monoid) divides_fcount:
  assumes dvd: "a divides b"
    and acarr: "a ∈ carrier G"
    and bcarr:"b ∈ carrier G"
  shows "factorcount G a ≤ factorcount G b"
proof (rule dividesE[OF dvd])
  fix c
  from assms have "∃as. set as ⊆ carrier G ∧ wfactors G as a"
    by blast
  then obtain as where ascarr: "set as ⊆ carrier G" and afs: "wfactors
G as a"
    by blast
  with acarr have fca: "factorcount G a = length as"
    by (intro factorcount_unique)

  assume ccarr: "c ∈ carrier G"
  then have "∃cs. set cs ⊆ carrier G ∧ wfactors G cs c"
    by blast
  then obtain cs where cscarr: "set cs ⊆ carrier G" and cfs: "wfactors
G cs c"
    by blast

  note [simp] = acarr bcarr ccarr ascarr cscarr

  assume b: "b = a ⊗ c"
  from afs cfs have "wfactors G (as@cs) (a ⊗ c)"
    by (intro wfactors_mult) simp_all
  with b have "wfactors G (as@cs) b"
    by simp
  then have "factorcount G b = length (as@cs)"
    by (intro factorcount_unique) simp_all
```

```
  then have "factorcount G b = length as + length cs"
    by simp
  with fca show ?thesis
    by simp
qed

lemma (in factorial_monoid) associated_fcount:
  assumes acarr: "a ∈ carrier G"
    and bcarr: "b ∈ carrier G"
    and asc: "a ∼ b"
  shows "factorcount G a = factorcount G b"
  apply (rule associatedE[OF asc])
  apply (drule divides_fcount[OF _ acarr bcarr])
  apply (drule divides_fcount[OF _ bcarr acarr])
  apply simp
  done

lemma (in factorial_monoid) properfactor_fcount:
  assumes acarr: "a ∈ carrier G" and bcarr:"b ∈ carrier G"
    and pf: "properfactor G a b"
  shows "factorcount G a < factorcount G b"
proof (rule properfactorE[OF pf], elim dividesE)
  fix c
  from assms have "∃as. set as ⊆ carrier G ∧ wfactors G as a"
    by blast
  then obtain as where ascarr: "set as ⊆ carrier G" and afs: "wfactors
G as a"
    by blast
  with acarr have fca: "factorcount G a = length as"
    by (intro factorcount_unique)

  assume ccarr: "c ∈ carrier G"
  then have "∃cs. set cs ⊆ carrier G ∧ wfactors G cs c"
    by blast
  then obtain cs where cscarr: "set cs ⊆ carrier G" and cfs: "wfactors
G cs c"
    by blast

  assume b: "b = a ⊗ c"

  have "wfactors G (as@cs) (a ⊗ c)"
    by (rule wfactors_mult) fact+
  with b have "wfactors G (as@cs) b"
    by simp
  with ascarr cscarr bcarr have "factorcount G b = length (as@cs)"
    by (simp add: factorcount_unique)
  then have fcb: "factorcount G b = length as + length cs"
    by simp
```

```
  assume nbdvda: "¬ b divides a"
  have "c ∉ Units G"
  proof
    assume cunit:"c ∈ Units G"
    have "b ⊗ inv c = a ⊗ c ⊗ inv c"
      by (simp add: b)
    also from ccarr acarr cunit have "... = a ⊗ (c ⊗ inv c)"
      by (fast intro: m_assoc)
    also from ccarr cunit have "... = a ⊗ 1" by simp
    also from acarr have "... = a" by simp
    finally have "a = b ⊗ inv c" by simp
    with ccarr cunit have "b divides a"
      by (fast intro: dividesI[of "inv c"])
    with nbdvda show False by simp
  qed
  with cfs have "length cs > 0"
    apply -
    apply (rule ccontr, simp)
    apply (metis Units_one_closed ccarr cscarr l_one one_closed properfactorI3
properfactor_fmset unit_wfactors)
    done
  with fca fcb show ?thesis
    by simp
qed

sublocale factorial_monoid ⊆ divisor_chain_condition_monoid
  apply unfold_locales
  apply (rule wfUNIVI)
  apply (rule measure_induct[of "factorcount G"])
  apply simp
  apply (metis properfactor_fcount)
  done

sublocale factorial_monoid ⊆ primeness_condition_monoid
  by standard (rule irreducible_prime)


lemma (in factorial_monoid) primeness_condition: "primeness_condition_monoid
G" ..

lemma (in factorial_monoid) gcd_condition [simp]: "gcd_condition_monoid
G"
  by standard (rule gcdof_exists)

sublocale factorial_monoid ⊆ gcd_condition_monoid
  by standard (rule gcdof_exists)

lemma (in factorial_monoid) division_weak_lattice [simp]: "weak_lattice
(division_rel G)"
```

```
proof -
  interpret weak_lower_semilattice "division_rel G"
    by simp
  show "weak_lattice (division_rel G)"
  proof (unfold_locales, simp_all)
    fix x y
    assume carr: "x ∈ carrier G"  "y ∈ carrier G"
    from lcmof_exists [OF this] obtain z where zcarr: "z ∈ carrier G"
and isgcd: "z lcmof x y"
      by blast
    with carr have "least (division_rel G) z (Upper (division_rel G)
{x, y})"
      by (simp add: lcmof_leastUpper[symmetric])
    then show "∃z. least (division_rel G) z (Upper (division_rel G) {x,
y})"
      by blast
  qed
qed
```

23.10 Factoriality Theorems

```
theorem factorial_condition_one:
  "divisor_chain_condition_monoid G ∧ primeness_condition_monoid G ⟷
factorial_monoid G"
proof (rule iffI, clarify)
  assume dcc: "divisor_chain_condition_monoid G"
    and pc: "primeness_condition_monoid G"
  interpret divisor_chain_condition_monoid "G" by (rule dcc)
  interpret primeness_condition_monoid "G" by (rule pc)
  show "factorial_monoid G"
    by (fast intro: factorial_monoidI wfactors_exist wfactors_unique)
next
  assume "factorial_monoid G"
  then interpret factorial_monoid "G" .
  show "divisor_chain_condition_monoid G ∧ primeness_condition_monoid
G"
    by rule unfold_locales
qed

theorem factorial_condition_two:
  "divisor_chain_condition_monoid G ∧ gcd_condition_monoid G ⟷ factorial_monoid
G"
proof (rule iffI, clarify)
  assume dcc: "divisor_chain_condition_monoid G"
    and gc: "gcd_condition_monoid G"
  interpret divisor_chain_condition_monoid "G" by (rule dcc)
  interpret gcd_condition_monoid "G" by (rule gc)
  show "factorial_monoid G"
    by (simp add: factorial_condition_one[symmetric], rule, unfold_locales)
```

```
next
  assume "factorial_monoid G"
  then interpret factorial_monoid "G" .
  show "divisor_chain_condition_monoid G ∧ gcd_condition_monoid G"
    by rule unfold_locales
qed

end


theory QuotRing
imports RingHom
begin
```

24 Quotient Rings

24.1 Multiplication on Cosets

```
definition rcoset_mult :: "[('a, _) ring_scheme, 'a set, 'a set, 'a set]
⇒ 'a set"
    ("[mod _:] _ ⨂ı _" [81,81,81] 80)
  where "rcoset_mult R I A B = (⋃a∈A. ⋃b∈B. I +>R (a ⊗R b))"
```

`rcoset_mult` fulfils the properties required by congruences

```
lemma (in ideal) rcoset_mult_add:
    "x ∈ carrier R ⟹ y ∈ carrier R ⟹ [mod I:] (I +> x) ⨂ (I +> y)
= I +> (x ⊗ y)"
  apply rule
  apply (rule, simp add: rcoset_mult_def, clarsimp)
  defer 1
  apply (rule, simp add: rcoset_mult_def)
  defer 1
proof -
  fix z x' y'
  assume carr: "x ∈ carrier R" "y ∈ carrier R"
    and x'rcos: "x' ∈ I +> x"
    and y'rcos: "y' ∈ I +> y"
    and zrcos: "z ∈ I +> x' ⊗ y'"

  from x'rcos have "∃h∈I. x' = h ⊕ x"
    by (simp add: a_r_coset_def r_coset_def)
  then obtain hx where hxI: "hx ∈ I" and x': "x' = hx ⊕ x"
    by fast+

  from y'rcos have "∃h∈I. y' = h ⊕ y"
    by (simp add: a_r_coset_def r_coset_def)
  then obtain hy where hyI: "hy ∈ I" and y': "y' = hy ⊕ y"
    by fast+
```

```
  from zrcos have "∃h∈I. z = h ⊕ (x' ⊗ y')"
    by (simp add: a_r_coset_def r_coset_def)
  then obtain hz where hzI: "hz ∈ I" and z: "z = hz ⊕ (x' ⊗ y')"
    by fast+

  note carr = carr hxI[THEN a_Hcarr] hyI[THEN a_Hcarr] hzI[THEN a_Hcarr]

  from z have "z = hz ⊕ (x' ⊗ y')" .
  also from x' y' have "... = hz ⊕ ((hx ⊕ x) ⊗ (hy ⊕ y))" by simp
  also from carr have "... = (hz ⊕ (hx ⊗ (hy ⊕ y)) ⊕ x ⊗ hy) ⊕ x ⊗
y" by algebra
  finally have z2: "z = (hz ⊕ (hx ⊗ (hy ⊕ y)) ⊕ x ⊗ hy) ⊕ x ⊗ y" .

  from hxI hyI hzI carr have "hz ⊕ (hx ⊗ (hy ⊕ y)) ⊕ x ⊗ hy ∈ I"
    by (simp add: I_l_closed I_r_closed)

  with z2 have "∃h∈I. z = h ⊕ x ⊗ y" by fast
  then show "z ∈ I +> x ⊗ y" by (simp add: a_r_coset_def r_coset_def)
next
  fix z
  assume xcarr: "x ∈ carrier R"
    and ycarr: "y ∈ carrier R"
    and zrcos: "z ∈ I +> x ⊗ y"
  from xcarr have xself: "x ∈ I +> x" by (intro a_rcos_self)
  from ycarr have yself: "y ∈ I +> y" by (intro a_rcos_self)
  show "∃a∈I +> x. ∃b∈I +> y. z ∈ I +> a ⊗ b"
    using xself and yself and zrcos by fast
qed
```

24.2 Quotient Ring Definition

```
definition FactRing :: "[('a,'b) ring_scheme, 'a set] ⇒ ('a set) ring"
    (infixl "Quot" 65)
  where "FactRing R I =
    ⦇carrier = a_rcosets⇘R⇙ I, mult = rcoset_mult R I,
      one = (I +>⇘R⇙ 𝟭⇘R⇙), zero = I, add = set_add R⦈"
```

24.3 Factorization over General Ideals

The quotient is a ring

```
lemma (in ideal) quotient_is_ring: "ring (R Quot I)"
apply (rule ringI)
   — abelian group
   apply (rule comm_group_abelian_groupI)
   apply (simp add: FactRing_def)
   apply (rule a_factorgroup_is_comm_group[unfolded A_FactGroup_def'])
  — mult monoid
  apply (rule monoidI)
      apply (simp_all add: FactRing_def A_RCOSETS_def RCOSETS_def
```

```
            a_r_coset_def[symmetric])
      — mult closed
      apply (clarify)
      apply (simp add: rcoset_mult_add, fast)
     — mult one_closed
     apply force
    — mult assoc
    apply clarify
    apply (simp add: rcoset_mult_add m_assoc)
   — mult one
   apply clarify
   apply (simp add: rcoset_mult_add)
  apply clarify
  apply (simp add: rcoset_mult_add)
 — distr
 apply clarify
 apply (simp add: rcoset_mult_add a_rcos_sum l_distr)
apply clarify
apply (simp add: rcoset_mult_add a_rcos_sum r_distr)
done
```

This is a ring homomorphism

```
lemma (in ideal) rcos_ring_hom: "(op +> I) ∈ ring_hom R (R Quot I)"
apply (rule ring_hom_memI)
   apply (simp add: FactRing_def a_rcosetsI[OF a_subset])
  apply (simp add: FactRing_def rcoset_mult_add)
 apply (simp add: FactRing_def a_rcos_sum)
apply (simp add: FactRing_def)
done

lemma (in ideal) rcos_ring_hom_ring: "ring_hom_ring R (R Quot I) (op
+> I)"
apply (rule ring_hom_ringI)
     apply (rule is_ring, rule quotient_is_ring)
   apply (simp add: FactRing_def a_rcosetsI[OF a_subset])
  apply (simp add: FactRing_def rcoset_mult_add)
 apply (simp add: FactRing_def a_rcos_sum)
apply (simp add: FactRing_def)
done
```

The quotient of a cring is also commutative

```
lemma (in ideal) quotient_is_cring:
  assumes "cring R"
  shows "cring (R Quot I)"
proof -
  interpret cring R by fact
  show ?thesis
    apply (intro cring.intro comm_monoid.intro comm_monoid_axioms.intro)
      apply (rule quotient_is_ring)
```

```
     apply (rule ring.axioms[OF quotient_is_ring])
    apply (simp add: FactRing_def A_RCOSETS_defs a_r_coset_def[symmetric])
    apply clarify
    apply (simp add: rcoset_mult_add m_comm)
    done
qed
```

Cosets as a ring homomorphism on crings

```
lemma (in ideal) rcos_ring_hom_cring:
  assumes "cring R"
  shows "ring_hom_cring R (R Quot I) (op +> I)"
proof -
  interpret cring R by fact
  show ?thesis
    apply (rule ring_hom_cringI)
      apply (rule rcos_ring_hom_ring)
     apply (rule is_cring)
    apply (rule quotient_is_cring)
   apply (rule is_cring)
   done
qed
```

24.4 Factorization over Prime Ideals

The quotient ring generated by a prime ideal is a domain

```
lemma (in primeideal) quotient_is_domain: "domain (R Quot I)"
  apply (rule domain.intro)
   apply (rule quotient_is_cring, rule is_cring)
  apply (rule domain_axioms.intro)
   apply (simp add: FactRing_def) defer 1
    apply (simp add: FactRing_def A_RCOSETS_defs a_r_coset_def[symmetric],
clarify)
    apply (simp add: rcoset_mult_add) defer 1
proof (rule ccontr, clarsimp)
  assume "I +> 1 = I"
  then have "1 ∈ I" by (simp only: a_coset_join1 one_closed a_subgroup)
  then have "carrier R ⊆ I" by (subst one_imp_carrier, simp, fast)
  with a_subset have "I = carrier R" by fast
  with I_notcarr show False by fast
next
  fix x y
  assume carr: "x ∈ carrier R" "y ∈ carrier R"
    and a: "I +> x ⊗ y = I"
    and b: "I +> y ≠ I"

  have ynI: "y ∉ I"
  proof (rule ccontr, simp)
    assume "y ∈ I"
    then have "I +> y = I" by (rule a_rcos_const)
```

```
    with b show False by simp
  qed

  from carr have "x ⊗ y ∈ I +> x ⊗ y" by (simp add: a_rcos_self)
  then have xyI: "x ⊗ y ∈ I" by (simp add: a)

  from xyI and carr have xI: "x ∈ I ∨ y ∈ I" by (simp add: I_prime)
  with ynI have "x ∈ I" by fast
  then show "I +> x = I" by (rule a_rcos_const)
qed
```

Generating right cosets of a prime ideal is a homomorphism on commutative rings

```
lemma (in primeideal) rcos_ring_hom_cring: "ring_hom_cring R (R Quot
I) (op +> I)"
  by (rule rcos_ring_hom_cring) (rule is_cring)
```

24.5 Factorization over Maximal Ideals

In a commutative ring, the quotient ring over a maximal ideal is a field. The proof follows "W. Adkins, S. Weintraub: Algebra – An Approach via Module Theory"

```
lemma (in maximalideal) quotient_is_field:
  assumes "cring R"
  shows "field (R Quot I)"
proof -
  interpret cring R by fact
  show ?thesis
    apply (intro cring.cring_fieldI2)
      apply (rule quotient_is_cring, rule is_cring)
     defer 1
     apply (simp add: FactRing_def A_RCOSETS_defs a_r_coset_def[symmetric],
clarsimp)
     apply (simp add: rcoset_mult_add) defer 1
  proof (rule ccontr, simp)
    — Quotient is not empty
    assume "0_{R Quot I} = 1_{R Quot I}"
    then have II1: "I = I +> 1" by (simp add: FactRing_def)
    from a_rcos_self[OF one_closed] have "1 ∈ I"
      by (simp add: II1[symmetric])
    then have "I = carrier R" by (rule one_imp_carrier)
    with I_notcarr show False by simp
  next
    — Existence of Inverse
    fix a
    assume IanI: "I +> a ≠ I" and acarr: "a ∈ carrier R"

    — Helper ideal J
```

```
    define J :: "'a set" where "J = (carrier R #> a) <+> I"
    have idealJ: "ideal J R"
      apply (unfold J_def, rule add_ideals)
       apply (simp only: cgenideal_eq_rcos[symmetric], rule cgenideal_ideal,
rule acarr)
      apply (rule is_ideal)
      done

    — Showing J not smaller than I
    have IinJ: "I ⊆ J"
    proof (rule, simp add: J_def r_coset_def set_add_defs)
      fix x
      assume xI: "x ∈ I"
      have Zcarr: "𝟬 ∈ carrier R" by fast
      from xI[THEN a_Hcarr] acarr
      have "x = 𝟬 ⊗ a ⊕ x" by algebra
      with Zcarr and xI show "∃xa∈carrier R. ∃k∈I. x = xa ⊗ a ⊕ k"
by fast
    qed

    — Showing J ≠ I
    have anI: "a ∉ I"
    proof (rule ccontr, simp)
      assume "a ∈ I"
      then have "I +> a = I" by (rule a_rcos_const)
      with IanI show False by simp
    qed

    have aJ: "a ∈ J"
    proof (simp add: J_def r_coset_def set_add_defs)
      from acarr
      have "a = 𝟭 ⊗ a ⊕ 𝟬" by algebra
      with one_closed and additive_subgroup.zero_closed[OF is_additive_subgroup]
      show "∃x∈carrier R. ∃k∈I. a = x ⊗ a ⊕ k" by fast
    qed

    from aJ and anI have JnI: "J ≠ I" by fast

    — Deducing J = carrier R because I is maximal
    from idealJ and IinJ have "J = I ∨ J = carrier R"
    proof (rule I_maximal, unfold J_def)
      have "carrier R #> a ⊆ carrier R"
        using subset_refl acarr by (rule r_coset_subset_G)
      then show "carrier R #> a <+> I ⊆ carrier R"
        using a_subset by (rule set_add_closed)
    qed

    with JnI have Jcarr: "J = carrier R" by simp
```

```
    — Calculating an inverse for a
    from one_closed[folded Jcarr]
    have "∃r∈carrier R. ∃i∈I. 1 = r ⊗ a ⊕ i"
      by (simp add: J_def r_coset_def set_add_defs)
    then obtain r i where rcarr: "r ∈ carrier R"
      and iI: "i ∈ I" and one: "1 = r ⊗ a ⊕ i" by fast
    from one and rcarr and acarr and iI[THEN a_Hcarr]
    have rai1: "a ⊗ r = ⊖i ⊕ 1" by algebra

    — Lifting to cosets
    from iI have "⊖i ⊕ 1 ∈ I +> 1"
      by (intro a_rcosI, simp, intro a_subset, simp)
    with rai1 have "a ⊗ r ∈ I +> 1" by simp
    then have "I +> 1 = I +> a ⊗ r"
      by (rule a_repr_independence, simp) (rule a_subgroup)

    from rcarr and this[symmetric]
    show "∃r∈carrier R. I +> a ⊗ r = I +> 1" by fast
  qed
qed

end
```

```
theory IntRing
imports "HOL-Computational_Algebra.Primes" QuotRing Lattice HOL.Int
begin
```

25 The Ring of Integers

25.1 Some properties of `int`

```
lemma dvds_eq_abseq:
  fixes k :: int
  shows "l dvd k ∧ k dvd l ⟷ |l| = |k|"
apply rule
 apply (simp add: zdvd_antisym_abs)
apply (simp add: dvd_if_abs_eq)
done
```

25.2 𝒵: The Set of Integers as Algebraic Structure

```
abbreviation int_ring :: "int ring" ("𝒵")
  where "int_ring ≡ ⦇carrier = UNIV, mult = op *, one = 1, zero = 0,
add = op +⦈"

lemma int_Zcarr [intro!, simp]: "k ∈ carrier 𝒵"
  by simp
```

```
lemma int_is_cring: "cring 𝒵"
apply (rule cringI)
  apply (rule abelian_groupI, simp_all)
  defer 1
  apply (rule comm_monoidI, simp_all)
 apply (rule distrib_right)
apply (fast intro: left_minus)
done
```

25.3 Interpretations

Since definitions of derived operations are global, their interpretation needs to be done as early as possible — that is, with as few assumptions as possible.

```
interpretation int: monoid 𝒵
  rewrites "carrier 𝒵 = UNIV"
    and "mult 𝒵 x y = x * y"
    and "one 𝒵 = 1"
    and "pow 𝒵 x n = x^n"
proof -
  — Specification
  show "monoid 𝒵" by standard auto
  then interpret int: monoid 𝒵 .

  — Carrier
  show "carrier 𝒵 = UNIV" by simp

  — Operations
  { fix x y show "mult 𝒵 x y = x * y" by simp }
  show "one 𝒵 = 1" by simp
  show "pow 𝒵 x n = x^n" by (induct n) simp_all
qed

interpretation int: comm_monoid 𝒵
  rewrites "finprod 𝒵 f A = prod f A"
proof -
  — Specification
  show "comm_monoid 𝒵" by standard auto
  then interpret int: comm_monoid 𝒵 .

  — Operations
  { fix x y have "mult 𝒵 x y = x * y" by simp }
  note mult = this
  have one: "one 𝒵 = 1" by simp
  show "finprod 𝒵 f A = prod f A"
    by (induct A rule: infinite_finite_induct, auto)
qed

interpretation int: abelian_monoid 𝒵
  rewrites int_carrier_eq: "carrier 𝒵 = UNIV"
```

```
    and int_zero_eq: "zero 𝒵 = 0"
    and int_add_eq: "add 𝒵 x y = x + y"
    and int_finsum_eq: "finsum 𝒵 f A = sum f A"
proof -
  — Specification
  show "abelian_monoid 𝒵" by standard auto
  then interpret int: abelian_monoid 𝒵 .

  — Carrier
  show "carrier 𝒵 = UNIV" by simp

  — Operations
  { fix x y show "add 𝒵 x y = x + y" by simp }
  note add = this
  show zero: "zero 𝒵 = 0"
    by simp
  show "finsum 𝒵 f A = sum f A"
    by (induct A rule: infinite_finite_induct, auto)
qed

interpretation int: abelian_group 𝒵


  rewrites "carrier 𝒵 = UNIV"
    and "zero 𝒵 = 0"
    and "add 𝒵 x y = x + y"
    and "finsum 𝒵 f A = sum f A"
    and int_a_inv_eq: "a_inv 𝒵 x = - x"
    and int_a_minus_eq: "a_minus 𝒵 x y = x - y"
proof -
  — Specification
  show "abelian_group 𝒵"
  proof (rule abelian_groupI)
    fix x
    assume "x ∈ carrier 𝒵"
    then show "∃y ∈ carrier 𝒵. y ⊕𝒵 x = 𝟬𝒵"
      by simp arith
  qed auto
  then interpret int: abelian_group 𝒵 .
  — Operations
  { fix x y have "add 𝒵 x y = x + y" by simp }
  note add = this
  have zero: "zero 𝒵 = 0" by simp
  {
    fix x
    have "add 𝒵 (- x) x = zero 𝒵"
      by (simp add: add zero)
    then show "a_inv 𝒵 x = - x"
      by (simp add: int.minus_equality)
```

```
  }
  note a_inv = this
  show "a_minus 𝒵 x y = x - y"
    by (simp add: int.minus_eq add a_inv)
qed (simp add: int_carrier_eq int_zero_eq int_add_eq int_finsum_eq)+

interpretation int: "domain" 𝒵
  rewrites "carrier 𝒵 = UNIV"
    and "zero 𝒵 = 0"
    and "add 𝒵 x y = x + y"
    and "finsum 𝒵 f A = sum f A"
    and "a_inv 𝒵 x = - x"
    and "a_minus 𝒵 x y = x - y"
proof -
  show "domain 𝒵"
    by unfold_locales (auto simp: distrib_right distrib_left)
qed (simp add: int_carrier_eq int_zero_eq int_add_eq int_finsum_eq int_a_inv_eq
int_a_minus_eq)+
```

Removal of occurrences of `UNIV` in interpretation result — experimental.

```
lemma UNIV:
  "x ∈ UNIV ⟷ True"
  "A ⊆ UNIV ⟷ True"
  "(∀x ∈ UNIV. P x) ⟷ (∀x. P x)"
  "(EX x : UNIV. P x) ⟷ (EX x. P x)"
  "(True ⟶ Q) ⟷ Q"
  "(True ⟹ PROP R) ≡ PROP R"
  by simp_all

interpretation int :
  partial_order "⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈"
  rewrites "carrier ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ =
UNIV"
    and "le ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ x y = (x
≤ y)"
    and "lless ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ x y =
(x < y)"
proof -
  show "partial_order ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈"
    by standard simp_all
  show "carrier ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ = UNIV"
    by simp
  show "le ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ x y = (x ≤
y)"
    by simp
  show "lless ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ x y = (x
< y)"
    by (simp add: lless_def) auto
qed
```

```
interpretation int  :
  lattice "⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈"
  rewrites "join ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ x y =
max x y"
    and "meet ⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈ x y = min
x y"
proof -
  let ?Z = "⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈"
  show "lattice ?Z"
    apply unfold_locales
    apply (simp add: least_def Upper_def)
    apply arith
    apply (simp add: greatest_def Lower_def)
    apply arith
    done
  then interpret int: lattice "?Z" .
  show "join ?Z x y = max x y"
    apply (rule int.joinI)
    apply (simp_all add: least_def Upper_def)
    apply arith
    done
  show "meet ?Z x y = min x y"
    apply (rule int.meetI)
    apply (simp_all add: greatest_def Lower_def)
    apply arith
    done
qed

interpretation int  :
  total_order "⦇carrier = UNIV::int set, eq = op =, le = op ≤⦈"
  by standard clarsimp
```

25.4 Generated Ideals of $\mathcal{Z}$

```
lemma int_Idl: "Idl𝒵 {a} = {x * a | x. True}"
  apply (subst int.cgenideal_eq_genideal[symmetric]) apply simp
  apply (simp add: cgenideal_def)
  done

lemma multiples_principalideal: "principalideal {x * a | x. True } 𝒵"
  by (metis UNIV_I int.cgenideal_eq_genideal int.cgenideal_is_principalideal
int_Idl)

lemma prime_primeideal:
  assumes prime: "prime p"
  shows "primeideal (Idl𝒵 {p}) 𝒵"
apply (rule primeidealI)
   apply (rule int.genideal_ideal, simp)
```

```
  apply (rule int_is_cring)
 apply (simp add: int.cgenideal_eq_genideal[symmetric] cgenideal_def)
 apply clarsimp defer 1
 apply (simp add: int.cgenideal_eq_genideal[symmetric] cgenideal_def)
 apply (elim exE)
proof -
  fix a b x
  assume "a * b = x * p"
  then have "p dvd a * b" by simp
  then have "p dvd a ∨ p dvd b"
    by (metis prime prime_dvd_mult_eq_int)
  then show "(∃x. a = x * p) ∨ (∃x. b = x * p)"
    by (metis dvd_def mult.commute)
next
  assume "UNIV = {uu. EX x. uu = x * p}"
  then obtain x where "1 = x * p" by best
  then have "|p * x| = 1" by (simp add: mult.commute)
  then show False using prime
    by (auto dest!: abs_zmult_eq_1 simp: prime_def)
qed
```

25.5 Ideals and Divisibility

```
lemma int_Idl_subset_ideal: "Idl𝒵 {k} ⊆ Idl𝒵 {l} = (k ∈ Idl𝒵 {l})"
  by (rule int.Idl_subset_ideal') simp_all

lemma Idl_subset_eq_dvd: "Idl𝒵 {k} ⊆ Idl𝒵 {l} ⟷ l dvd k"
  apply (subst int_Idl_subset_ideal, subst int_Idl, simp)
  apply (rule, clarify)
  apply (simp add: dvd_def)
  apply (simp add: dvd_def ac_simps)
  done

lemma dvds_eq_Idl: "l dvd k ∧ k dvd l ⟷ Idl𝒵 {k} = Idl𝒵 {l}"
proof -
  have a: "l dvd k ⟷ (Idl𝒵 {k} ⊆ Idl𝒵 {l})"
    by (rule Idl_subset_eq_dvd[symmetric])
  have b: "k dvd l ⟷ (Idl𝒵 {l} ⊆ Idl𝒵 {k})"
    by (rule Idl_subset_eq_dvd[symmetric])

  have "l dvd k ∧ k dvd l ⟷ Idl𝒵 {k} ⊆ Idl𝒵 {l} ∧ Idl𝒵 {l} ⊆ Idl𝒵
{k}"
    by (subst a, subst b, simp)
  also have "Idl𝒵 {k} ⊆ Idl𝒵 {l} ∧ Idl𝒵 {l} ⊆ Idl𝒵 {k} ⟷ Idl𝒵 {k}
= Idl𝒵 {l}"
    by blast
  finally show ?thesis .
qed
```

```
lemma Idl_eq_abs: "Idl𝒵 {k} = Idl𝒵 {l} ⟷ |l| = |k|"
  apply (subst dvds_eq_abseq[symmetric])
  apply (rule dvds_eq_Idl[symmetric])
  done
```

25.6 Ideals and the Modulus

```
definition ZMod :: "int ⇒ int ⇒ int set"
  where "ZMod k r = (Idl𝒵 {k}) +>𝒵 r"

lemmas ZMod_defs =
  ZMod_def genideal_def

lemma rcos_zfact:
  assumes kIl: "k ∈ ZMod l r"
  shows "∃x. k = x * l + r"
proof -
  from kIl[unfolded ZMod_def] have "∃xl∈Idl𝒵 {l}. k = xl + r"
    by (simp add: a_r_coset_defs)
  then obtain xl where xl: "xl ∈ Idl𝒵 {l}" and k: "k = xl + r"
    by auto
  from xl obtain x where "xl = x * l"
    by (auto simp: int_Idl)
  with k have "k = x * l + r"
    by simp
  then show "∃x. k = x * l + r" ..
qed

lemma ZMod_imp_zmod:
  assumes zmods: "ZMod m a = ZMod m b"
  shows "a mod m = b mod m"
proof -
  interpret ideal "Idl𝒵 {m}" 𝒵
    by (rule int.genideal_ideal) fast
  from zmods have "b ∈ ZMod m a"
    unfolding ZMod_def by (simp add: a_repr_independenceD)
  then have "∃x. b = x * m + a"
    by (rule rcos_zfact)
  then obtain x where "b = x * m + a"
    by fast
  then have "b mod m = (x * m + a) mod m"
    by simp
  also have "... = ((x * m) mod m) + (a mod m)"
    by (simp add: mod_add_eq)
  also have "... = a mod m"
    by simp
  finally have "b mod m = a mod m" .
  then show "a mod m = b mod m" ..
qed
```

```
lemma ZMod_mod: "ZMod m a = ZMod m (a mod m)"
proof -
  interpret ideal "Idlℤ {m}" ℤ
    by (rule int.genideal_ideal) fast
  show ?thesis
    unfolding ZMod_def
    apply (rule a_repr_independence'[symmetric])
    apply (simp add: int_Idl a_r_coset_defs)
  proof -
    have "a = m * (a div m) + (a mod m)"
      by (simp add: mult_div_mod_eq [symmetric])
    then have "a = (a div m) * m + (a mod m)"
      by simp
    then show "∃h. (∃x. h = x * m) ∧ a = h + a mod m"
      by fast
  qed simp
qed

lemma zmod_imp_ZMod:
  assumes modeq: "a mod m = b mod m"
  shows "ZMod m a = ZMod m b"
proof -
  have "ZMod m a = ZMod m (a mod m)"
    by (rule ZMod_mod)
  also have "... = ZMod m (b mod m)"
    by (simp add: modeq[symmetric])
  also have "... = ZMod m b"
    by (rule ZMod_mod[symmetric])
  finally show ?thesis .
qed

corollary ZMod_eq_mod: "ZMod m a = ZMod m b ⟷ a mod m = b mod m"
  apply (rule iffI)
  apply (erule ZMod_imp_zmod)
  apply (erule zmod_imp_ZMod)
  done
```

25.7 Factorization

```
definition ZFact :: "int ⇒ int set ring"
  where "ZFact k = ℤ Quot (Idlℤ {k})"

lemmas ZFact_defs = ZFact_def FactRing_def

lemma ZFact_is_cring: "cring (ZFact k)"
  apply (unfold ZFact_def)
  apply (rule ideal.quotient_is_cring)
   apply (intro ring.genideal_ideal)
```

```
    apply (simp add: cring.axioms[OF int_is_cring] ring.intro)
   apply simp
  apply (rule int_is_cring)
  done

lemma ZFact_zero: "carrier (ZFact 0) = (⋃a. {{a}})"
  apply (insert int.genideal_zero)
  apply (simp add: ZFact_defs A_RCOSETS_defs r_coset_def)
  done

lemma ZFact_one: "carrier (ZFact 1) = {UNIV}"
  apply (simp only: ZFact_defs A_RCOSETS_defs r_coset_def ring_record_simps)
  apply (subst int.genideal_one)
  apply (rule, rule, clarsimp)
   apply (rule, rule, clarsimp)
   apply (rule, clarsimp, arith)
  apply (rule, clarsimp)
  apply (rule exI[of _ "0"], clarsimp)
  done

lemma ZFact_prime_is_domain:
  assumes pprime: "prime p"
  shows "domain (ZFact p)"
  apply (unfold ZFact_def)
  apply (rule primeideal.quotient_is_domain)
  apply (rule prime_primeideal[OF pprime])
  done

end
```

26 More on rings etc.

```
theory More_Ring
imports
  Ring
begin

lemma (in cring) field_intro2: "0_R ~= 1_R ⟹ ∀x ∈ carrier R - {0_R}.
x ∈ Units R ⟹ field R"
  apply (unfold_locales)
  apply (insert cring_axioms, auto)
  apply (rule trans)
  apply (subgoal_tac "a = (a ⊗ b) ⊗ inv b")
  apply assumption
  apply (subst m_assoc)
  apply auto
  apply (unfold Units_def)
  apply auto
  done
```

```
lemma (in monoid) inv_char: "x : carrier G ⟹ y : carrier G ⟹
    x ⊗ y = 1 ⟹ y ⊗ x = 1 ⟹ inv x = y"
  apply (subgoal_tac "x : Units G")
  apply (subgoal_tac "y = inv x ⊗ 1")
  apply simp
  apply (erule subst)
  apply (subst m_assoc [symmetric])
  apply auto
  apply (unfold Units_def)
  apply auto
  done

lemma (in comm_monoid) comm_inv_char: "x : carrier G ⟹ y : carrier
G ⟹
  x ⊗ y = 1 ⟹ inv x = y"
  apply (rule inv_char)
  apply auto
  apply (subst m_comm, auto)
  done

lemma (in ring) inv_neg_one [simp]: "inv (⊖ 1) = ⊖ 1"
  apply (rule inv_char)
  apply (auto simp add: l_minus r_minus)
  done

lemma (in monoid) inv_eq_imp_eq: "x : Units G ⟹ y : Units G ⟹
    inv x = inv y ⟹ x = y"
  apply (subgoal_tac "inv(inv x) = inv(inv y)")
  apply (subst (asm) Units_inv_inv)+
  apply auto
  done

lemma (in ring) Units_minus_one_closed [intro]: "⊖ 1 : Units R"
  apply (unfold Units_def)
  apply auto
  apply (rule_tac x = "⊖ 1" in bexI)
  apply auto
  apply (simp add: l_minus r_minus)
  done

lemma (in monoid) inv_one [simp]: "inv 1 = 1"
  apply (rule inv_char)
  apply auto
  done

lemma (in ring) inv_eq_neg_one_eq: "x : Units R ⟹ (inv x = ⊖ 1) =
(x = ⊖ 1)"
  apply auto
```

```
  apply (subst Units_inv_inv [symmetric])
  apply auto
  done

lemma (in monoid) inv_eq_one_eq: "x : Units G ⟹ (inv x = 1) = (x =
1)"
  by (metis Units_inv_inv inv_one)

end
```

References

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